ANNALS OF
K-THEORY

On the cycle map of a finite group

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Let p be an odd prime number. We show that there exists a finite group of order p^{p+3} for which the mod p cycle map from the mod p Chow ring of its classifying space to its ordinary mod p cohomology is not injective.

1. Introduction

The Chow group $\mathrm{CH}^i X$ of a smooth algebraic variety X is the group of finite $\mathbb{Z}$-linear combinations of closed subvarieties of X of codimension i modulo rational equivalence and $\bigoplus_{i \geq 0} \mathrm{CH}^i X$, called the Chow ring of X, is a ring under intersection product. It is an important object of study in algebraic geometry. For a smooth complex algebraic variety, the cycle map is a homomorphism from the Chow ring to the ordinary integral cohomology of the underlying topological space. Thus, the cycle map relates algebraic geometry to algebraic topology. Totaro [1999] considered the Chow ring of the classifying space BG of an algebraic group G. In his recently published book, for each prime number p Totaro [2014] gave an example of a finite group K of order p^{2p+1} such that the mod p cycle map

$$\mathrm{cl} : \mathrm{CH}^2 BK / p \to H^4(BK)$$

is not injective, where $H^*(-)$ is the ordinary mod p cohomology and the finite group K is regarded as a complex algebraic group. Totaro wrote " ... but there are probably smaller examples" in his book.

In this paper, we find a smaller example, possibly the smallest one. To be precise, we construct a finite group H of order p^{p+3} to prove the following result:

Theorem 1.1. *For each prime number p, there exists a finite group H of order p^{p+3} such that the mod p cycle map $\mathrm{cl} : \mathrm{CH}^2 BH / p \to H^4(BH)$ is not injective, where the finite group H is regarded as a complex algebraic group.*

For a complex algebraic group G, the following results were obtained by Totaro [1999, Corollary 3.5] using Merkurjev's theorem:

(1) $\mathrm{CH}^2 BG$ is generated by Chern classes.

(2) $\mathrm{CH}^2 BG \to H^4(BG; \mathbb{Z})$ is injective.

Thus, we may use the ordinary integral cohomology and Chern classes to study the Chow group $\mathrm{CH}^2 BG$. A problem concerning the Chow group $\mathrm{CH}^2 BG$ in algebraic geometry could be viewed as a problem on the Chern subgroup of the ordinary integral cohomology $H^4(BG; \mathbb{Z})$, that is, the subgroup of $H^4(BG; \mathbb{Z})$ generated by Chern classes of complex representations of G, in classical algebraic topology. In what follows, we consider $\mathrm{CH}^2 BG$ as the Chern subgroup of the integral cohomology $H^4(BG; \mathbb{Z})$ and the mod p cycle map $\mathrm{CH}^2 BG/p \to H^4(BG)$ as the homomorphism induced by the mod p reduction $\rho : H^4(BG; \mathbb{Z}) \to H^4(BG)$. Since we consider the ordinary integral and mod p cohomology only, the group G could be a topological group and it need not be a complex algebraic group.

Throughout the rest of this paper, we assume that p is an odd prime number unless otherwise stated explicitly. Let p_+^{1+2} be the extraspecial p-group of order p^3 with exponent p. We consider it as a subgroup of the special unitary group $\mathrm{SU}(p)$. We will define a subgroup H_2 of $\mathrm{SU}(p)$ in Section 2. The group H in Theorem 1.1 is given in terms of p_+^{1+2} and H_2, that is,

$$H = p_+^{1+2} \times H_2/\langle \Delta(\xi) \rangle,$$

where $\langle \Delta(\xi) \rangle$ is a cyclic group in the center of $\mathrm{SU}(p) \times \mathrm{SU}(p)$. We define the group G as

$$G = \mathrm{SU}(p) \times \mathrm{SU}(p)/\langle \Delta(\xi) \rangle.$$

We will give the detail of G, H and H_2 in Section 2. What we prove in this paper is the following theorem:

Theorem 1.2. *Let p be an odd prime number. Let K be a subgroup of*

$$G = \mathrm{SU}(p) \times \mathrm{SU}(p)/\langle \Delta(\xi) \rangle$$

containing

$$H = p_+^{1+2} \times H_2/\langle \Delta(\xi) \rangle.$$

Then the mod p cycle map $\mathrm{cl} : \mathrm{CH}^2 BK/p \to H^4(BK)$ *is not injective.*

The order of the group $p_+^{1+2} \times H_2/\langle \Delta(\xi) \rangle$ is p^{p+3} and it is the group H in Theorem 1.1. Applying Theorem 1.2 to

$$K = p_+^{1+2} \times ((\mathbb{Z}/p^2)^{p-1} \rtimes \mathbb{Z}/p)/\langle \Delta(\xi) \rangle,$$

we obtain the example in [Totaro 2014, Section 15]. Thus our result not only gives a smaller group whose mod p cycle map is not injective but it extends Totaro's result. For $p = 2$, Theorem 1.1 was proved by Totaro [2014, Theorem 15.13]. For $p = 2$, the finite group H is the extraspecial 2-group 2_+^{1+4} of order 2^5. It is not difficult to see that we cannot replace H_2 by the extraspecial p-group p_+^{1+2} in Theorem 1.2. See Remark 6.3. This observation leads us to the following conjecture:

Conjecture 1.3. *Let p be a prime number. For a finite p-group K of order less than p^{p+3}, the mod p cycle map* $\mathrm{cl} : \mathrm{CH}^2 BK/p \to H^4(BK)$ *is injective.*

This paper is organized as follows: In Section 2, we define groups that we use in this paper, including G and H above. In Section 3, we recall the cohomology of the classifying space of the projective unitary group $\mathrm{PU}(p)$ up to degree 5. In Section 3, we prove that the mod p cycle map $\mathrm{CH}^2 BG/p \to H^4(BG)$ is not injective and describe its kernel. In Section 4, we collect some properties of the mod p cohomology of $B\tilde{\pi}(H_2)$, where $\tilde{\pi}$ is the restriction of the projection from $\mathrm{SU}(p)$ to $\mathrm{PU}(p)$. We use the mod p cohomology of $B\tilde{\pi}(H_2)$ in Section 5, where we study the mod p cycle map $\mathrm{CH}^2 BH/p \to H^4(BH)$ to complete the proof of Theorem 1.2.

Throughout the rest of this paper, by abuse of notation, we denote the map between classifying spaces induced by a group homomorphism $f : G \to G'$ by $f : BG \to BG'$.

2. Subgroups and quotient groups

In this section, we define subgroups of the unitary group $U(p)$ and of the product $\mathrm{SU}(p) \times \mathrm{SU}(p)$ of special unitary groups $\mathrm{SU}(p)$. We also define their quotient groups. For a finite subset $\{x_1, \ldots, x_r\}$ of a group, we denote by $\langle x_1, \ldots, x_r \rangle$ the subgroup generated by $\{x_1, \ldots, x_r\}$. As we already mentioned, we assume that p is an odd prime number.

We start with subgroups of the special unitary group $\mathrm{SU}(p)$. Let $\xi = \exp(2\pi i / p)$, $\omega = \exp(2\pi i / p^2)$ and $\delta_{ij} = 1$ if $i \equiv j \bmod p$, $\delta_{ij} = 0$ if $i \not\equiv j \bmod p$. We consider the following matrices in $\mathrm{SU}(p)$:

$$\xi = (\xi \delta_{ij}) = \mathrm{diag}(\xi, \ldots, \xi),$$
$$\alpha = (\xi^{i-1} \delta_{ij}) = \mathrm{diag}(1, \xi, \ldots, \xi^{p-1}),$$
$$\beta = (\delta_{i, j-1}),$$
$$\sigma_1 = \mathrm{diag}(\omega \xi^{p-1}, \omega, \ldots, \omega).$$

Moreover, let σ_k be the diagonal matrix whose (i, i)-entry is $\omega \xi^{p-1}$ for $i = k$ and ω for $i \neq k$. Let us consider the following subgroups of $\mathrm{SU}(p)$:

$$p_+^{1+2} = \langle \alpha, \beta, \xi \rangle,$$
$$H_2 = \langle \beta, \sigma_1, \ldots, \sigma_p \rangle.$$

The group p_+^{1+2} is the extraspecial p-group of order p^3 with exponent p. Since $\sigma_1^p = \cdots = \sigma_p^p = \xi$ and

$$\sigma_2 \sigma_3^2 \cdots \sigma_p^{p-1} = \xi^{(p-1)/2} \alpha^{-1},$$

the group H_2 contains p_+^{1+2} as a subgroup. An element in the subgroup of H_2 generated by $\sigma_1, \ldots, \sigma_p$ could be described as

$$\omega^j \operatorname{diag}(\xi^{i_1}, \ldots, \xi^{i_p}),$$

where $0 \leq j \leq p - 1$, $0 \leq i_1 \leq p - 1$, ..., $0 \leq i_p \leq p - 1$ and $i_1 + \cdots + i_p$ is divisible by p. So, the order of this subgroup is p^p. Since β acts on the subgroup of diagonal matrices as a cyclic permutation, the order of H_2 is p^{p+1}.

We write A_2 for the quotient group $p_+^{1+2}/\langle \xi \rangle$. The group A_2 is an elementary abelian p-group of rank 2. We denote by $\tilde{\pi}$ the obvious projection $\mathrm{SU}(p) \to \mathrm{PU}(p)$ and projections induced by this projection, e.g., $\tilde{\pi} : p_+^{1+2} \to \tilde{\pi}(p_+^{1+2}) = A_2$. We denote the obvious inclusions among p_+^{1+2}, H_2 and $\mathrm{SU}(p)$ and among A_2, $\tilde{\pi}(H_2)$ and $\mathrm{PU}(p)$ by ι.

Let us consider the following maps:

$$\Delta : \mathrm{SU}(p) \to \mathrm{SU}(p) \times \mathrm{SU}(p), \quad m \mapsto \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}.$$

$$\Gamma_1 : \mathrm{SU}(p) \to \mathrm{SU}(p) \times \mathrm{SU}(p), \quad m \mapsto \begin{pmatrix} m & 0 \\ 0 & I \end{pmatrix}.$$

$$\Gamma_2 : \mathrm{SU}(p) \to \mathrm{SU}(p) \times \mathrm{SU}(p), \quad m \mapsto \begin{pmatrix} I & 0 \\ 0 & m \end{pmatrix}.$$

Using these maps and matrices in $\mathrm{SU}(p)$ above, we consider the following groups:

$$
\begin{aligned}
G &= \mathrm{SU}(p) \times \mathrm{SU}(p)/\langle \Delta(\xi) \rangle, \\
H &= \langle \Delta(\alpha), \Delta(\beta), \Delta(\xi), \Gamma_2(\beta), \Gamma_2(\sigma_1), \ldots, \Gamma_2(\sigma_p) \rangle / \langle \Delta(\xi) \rangle, \\
A_3 &= \langle \Delta(\alpha), \Delta(\beta), \Delta(\xi), \Gamma_2(\xi) \rangle / \langle \Delta(\xi) \rangle, \\
A_3' &= \langle \Gamma_1(\alpha), \Gamma_2(\beta), \Delta(\xi), \Gamma_2(\xi) \rangle / \langle \Delta(\xi) \rangle.
\end{aligned}
$$

Since α and β are in H_2, the subgroup

$$\langle \Delta(\alpha), \Delta(\beta), \Delta(\xi), \Gamma_2(\beta), \Gamma_2(\sigma_1), \ldots, \Gamma_2(\sigma_p) \rangle$$

contains

$$\Gamma_1(\alpha) = \Delta(\alpha)\Gamma_2(\alpha^{-1}), \quad \Gamma_1(\beta) = \Delta(\beta)\Gamma_2(\beta^{-1}), \quad \Gamma_1(\xi) = \Delta(\xi)\Gamma_2(\xi^{-1}).$$

Therefore, it is equal to the subgroup

$$p_+^{1+2} \times H_2 = \langle \Gamma_1(\alpha), \Gamma_1(\beta), \Gamma_1(\xi), \Gamma_2(\beta), \Gamma_2(\sigma_1), \ldots, \Gamma_2(\sigma_p) \rangle.$$

Hence, we have

$$H = p_+^{1+2} \times H_2/\langle \Delta(\xi) \rangle.$$

We denote the obvious inclusion of H by $f : H \to G$. It is also clear that A_3 and A_3' are elementary abelian p-subgroups of rank 3. We use the elementary abelian p-subgroup A_3' only in the proof of Proposition 6.4. In the above groups, $\Gamma_1(\xi) = \Gamma_2(\xi)$. We denote by π the obvious projections induced by $\pi : G \to PU(p) \times PU(p)$. It is clear that

$$\pi(H) = H/\langle \Gamma_2(\xi) \rangle = A_2 \times \tilde{\pi}(H_2)$$

and

$$PU(p) \times PU(p) = SU(p) \times SU(p)/\langle \Delta(\xi), \Gamma_2(\xi) \rangle.$$

Moreover, we have the following commutative diagram:

$$
\begin{array}{ccccc}
A_3 & \xrightarrow{\ g\ } & H & \xleftarrow{\ g'\ } & A_3' \\
{\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi'} \\
A_2 & \xrightarrow{\ g\ } & A_2 \times \tilde{\pi}(H_2) & \xleftarrow{\ g'\ } & A_2
\end{array}
$$

where the upper g and g' are the obvious inclusions, $A_2 = \langle \tilde{\pi}(\alpha), \tilde{\pi}(\beta) \rangle$,

$$
\begin{aligned}
\varphi(\Delta(\alpha)) &= \tilde{\pi}(\alpha), & \varphi(\Delta(\beta)) &= \tilde{\pi}(\beta), \\
\varphi'(\Gamma_1(\alpha)) &= \tilde{\pi}(\alpha), & \varphi'(\Gamma_2(\beta)) &= \tilde{\pi}(\beta), \\
g(\tilde{\pi}(\alpha)) &= (\tilde{\pi}(\alpha), \tilde{\pi}(\alpha)), & g(\tilde{\pi}(\beta)) &= (\tilde{\pi}(\beta), \tilde{\pi}(\beta)), \\
g'(\tilde{\pi}(\alpha)) &= (\tilde{\pi}(\alpha), 1), & g'(\tilde{\pi}(\beta)) &= (1, \tilde{\pi}(\beta)).
\end{aligned}
$$

We end this section by considering another subgroup H_2' of the unitary group $U(p)$ and its quotient group $\tilde{\pi}(H_2')$, which is a subgroup of $PU(p)$. We use H_2' and $\tilde{\pi}(H_2')$ only in the proof of Proposition 5.2. Let T^p be the set of all diagonal matrices in $U(p)$, which is a maximal torus of $U(p)$. We define $H_2' = T^p \rtimes \mathbb{Z}/p$ as the subgroup generated by T^p and β. It is clear that $\tilde{\pi}(H_2)$ is a subgroup of $\tilde{\pi}'(H_2') \subset PU(p)$, where we denote by $\tilde{\pi}'$ the obvious projection $U(p) \to PU(p)$.

3. The cohomology of $B\,PU(p)$

In this section, we recall the integral and mod p cohomology of $B\,PU(p)$. Throughout the rest of this paper, we denote the integral cohomology of a space X by $H^*(X; \mathbb{Z})$ and its mod p cohomology by $H^*(X)$. Also, we denote the mod p reduction by

$$\rho : H^*(X; \mathbb{Z}) \to H^*(X).$$

We also define generators $u_2 \in H^2(B\,PU(p))$ and $z_1 \in H^1(B\langle\xi\rangle)$ with $d_2(z_1) = x_1 y_1$, $d_2(z_1) = u_2$ and $\iota^*(u_2) = x_1 y_1$, where $x_1, y_1 \in H^1(BA_2)$ are generators corresponding to α and β in $\pi_1(BA_2) = \langle \tilde{\pi}(\alpha), \tilde{\pi}(\beta) \rangle$, and the d_2 are differentials in

the Leray–Serre spectral sequence associated with the vertical fibrations $\tilde{\pi}$ in

$$
\begin{CD}
Bp_+^{1+2} @>\iota>> B\,\mathrm{SU}(p) \\
@V\tilde{\pi}VV @VV\tilde{\pi}V \\
BA_2 @>\iota>> B\,\mathrm{PU}(p)
\end{CD}
\tag{3.1}
$$

where vertical maps are induced by the obvious projections and horizontal maps are induced by the obvious inclusions.

First, we set up notations related to the spectral sequence. Let

$$\pi : X \to B$$

be a fibration. Since the base space B is usually clear from the context, we write $E_r^{s,t}(X)$ for the Leray–Serre spectral sequence associated with the above fibration converging to the mod p cohomology $H^*(X)$. If it is clear from the context, we write $E_r^{s,t}$ for the Leray–Serre spectral sequence. We denote by

$$H^{s+t}(X) = F^0 H^{s+t}(X) \supseteq F^1 H^{s+t}(X) \supseteq \cdots \supseteq F^{s+t+1} H^{s+t}(X) = \{0\}$$

the filtration on $H^{s+t}(X)$ associated with the spectral sequence. Unless otherwise stated explicitly, by abuse of notation, we denote the cohomology class and the element it represents in the spectral sequence by the same symbol. Usually, it is clear from the context whether we mean the cohomology class or the element in the spectral sequence. Let R be an algebra or a graded algebra. Let $\{x_1, \ldots, x_r\}$ be a finite set. We denote by $R\{x_1, \ldots, x_r\}$ the free R-module spanned by $\{x_1, \ldots, x_r\}$. For a graded module M, we say M is a free R-module up to degree m if the R-module homomorphism

$$f : (R\{x_1, \ldots, x_r\})^i \to M^i$$

is an isomorphism for $i \leq m$ for some finite subset $\{x_1, \ldots, x_r\}$ of M. We say a spectral sequence collapses at the E_r-level up to degree m if $E_r^{s,t} = E_\infty^{s,t}$ for $s+t \leq m$.

Next, we recall the integral and mod p cohomology of $B\,\mathrm{PU}(p)$. The mod 3 cohomology of $B\,\mathrm{PU}(3)$ was computed by Kono, Mimura and Shimada [Kono et al. 1975]. The integral and mod p cohomology of $B\,\mathrm{PU}(p)$ was computed by Vistoli [2007]. The mod p cohomology was computed by Kameko and Yagita [2008] independently. The computation up to degree 5 was also done by Antieau and Williams [2014]. Although the direct computation is not difficult, we prove the following proposition by direct computation because it is slightly different from the one in [Antieau and Williams 2014].

Proposition 3.2. *Up to degree* 5, *the integral cohomology of* $B\,\mathrm{PU}(p)$ *is given by*

$$H^i(B\,\mathrm{PU}(p); \mathbb{Z}) = \{0\} \quad \text{for } i = 1, 2, 5,$$
$$H^i(B\,\mathrm{PU}(p); \mathbb{Z}) = \mathbb{Z}/p \quad \text{for } i = 3,$$
$$H^i(B\,\mathrm{PU}(p); \mathbb{Z}) = \mathbb{Z} \quad \text{for } i = 0, 4.$$

Up to degree 5, *the mod* p *cohomology of* $B\,\mathrm{PU}(p)$ *is given by*

$$H^i(B\,\mathrm{PU}(p)) = \{0\} \quad \text{for } i = 1, 5,$$
$$H^i(B\,\mathrm{PU}(p)) = \mathbb{Z}/p \quad \text{for } i = 0, 2, 3, 4.$$

Proof. Consider the Leray–Serre spectral sequence associated with

$$BU(p) \to B\,\mathrm{PU}(p) \to K(\mathbb{Z}, 3)$$

converging to $H^*(B\,\mathrm{PU}(p); \mathbb{Z})$. The integral cohomology of $BU(p)$ is a polynomial algebra generated by Chern classes, that is, $H^*(BU(p); \mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_p]$, where $\deg c_i = 2i$. The integral cohomology $H^i(K(\mathbb{Z}, 3); \mathbb{Z})$ of the Eilenberg–Mac Lane space $K(\mathbb{Z}, 3)$ is $\mathbb{Z}$ for $i = 0, 3$ and $\{0\}$ for $i = 1, 2, 4, 5$. We fix a generator u_3 of $H^3(K(\mathbb{Z}, 3); \mathbb{Z})$. Up to degree 5, the only nontrivial E_2-terms are

$$E_2^{0,0} = E_2^{0,2} = \mathbb{Z}, \quad E_2^{0,4} = \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad E_2^{3,0} = E_2^{3,2} = \mathbb{Z}.$$

Hence, up to degree 5, the only nontrivial differential is $d_3 : E_3^{0,t} \to E_3^{3,t-2}$, which is given by

$$d_3(c_1) = \alpha_1 u_3, \quad d_3(c_2) = \alpha_2 c_1 u_3,$$

where $\alpha_1, \alpha_2 \in \mathbb{Z}$. Since $B\,\mathrm{PU}(p)$ is simply connected and $\pi_2(B\,\mathrm{PU}(p)) = \mathbb{Z}/p$, by the Hurewicz theorem we have $H_1(B\,\mathrm{PU}(p); \mathbb{Z}) = \{0\}$ and $H_2(B\,\mathrm{PU}(p); \mathbb{Z}) = \mathbb{Z}/p$. By the universal coefficient theorem, we have $H^2(B\,\mathrm{PU}(p); \mathbb{Z}) = \{0\}$ and that $H^3(B\,\mathrm{PU}(p); \mathbb{Z})$ has $\mathbb{Z}/p$ as a direct summand. Therefore, α_1 must be $\pm p$ and $E_3^{3,0} = \mathbb{Z}/p$. The cohomology suspension $\sigma : H^4(BU(p)) \to H^3(U(p))$ maps $\rho(c_2)$ to a nontrivial primitive element in $H^3(U(p))$, but there exists no primitive element in $H^3(\mathrm{PU}(p))$ by the computation due to Baum and Browder [1965]. Hence, in the Leray–Serre spectral sequence $E_r^{s,t}(B\,\mathrm{SU}(p))$, the element $\rho(c_2)$ in $E_2^{0,4}(B\,\mathrm{SU}(p))$ must support a nontrivial differential. Therefore, α_2 is not divisible by p and, up to degree 5, the nontrivial E_3-terms are

$$E_3^{0,0} = E_3^{0,4} = \mathbb{Z}, \quad E_3^{3,0} = \mathbb{Z}/p.$$

As for $E_r^{s,t}(B\,\mathrm{PU}(p))$, we have

$$E_2^{0,0}(B\,\mathrm{PU}(p)) = E_2^{0,2}(B\,\mathrm{PU}(p)) = \mathbb{Z}/p, \quad E_2^{0,4}(B\,\mathrm{PU}(p)) = \mathbb{Z}/p \oplus \mathbb{Z}/p,$$
$$E_2^{3,0}(B\,\mathrm{PU}(p)) = E_2^{3,2}(B\,\mathrm{PU}(p)) = \mathbb{Z}/p,$$

and

$$d_3(\rho(c_1)) = 0, \quad d_3(\rho(c_2)) = \rho(\alpha_2 c_1 u_3) \neq 0.$$

So, we have the desired result. $\square$

With the following proposition, we choose generators

$$z_1 \in H^1(B\langle\xi\rangle), \quad u_2 \in H^2(B\,\mathrm{PU}(p))$$

such that

$$d_2(z_1) = u_2, \quad d_2(z_1) = x_1 y_1$$

in the spectral sequences associated with vertical fiber bundles in (3.1).

Proposition 3.3. *We may choose $u_2 \in H^2(B\,\mathrm{PU}(p))$ such that the induced homomorphism $\iota^* : H^2(B\,\mathrm{PU}(p)) \to H^2(BA_2)$ maps u_2 to $x_1 y_1$.*

Proof. From the commutative diagram (3.1), there exists the induced homomorphism between the Leray–Serre spectral sequences

$$\iota^* : E_r^{s,t}(B\,\mathrm{SU}(p)) \to E_r^{s,t}(Bp_+^{1+2}).$$

Since the group extension

$$\mathbb{Z}/p \to p_+^{1+2} \to A_2$$

corresponds to $x_1 y_1$ in $H^2(BA_2)$, the differential $d_2 : E_2^{0,1}(Bp_+^{1+2}) \to E_2^{2,0}(Bp_+^{1+2})$ is given by

$$d_2(z_1) = x_1 y_1$$

for some $z_1 \in H^1(B\langle\xi\rangle) = \mathbb{Z}/p[z_2] \otimes \Lambda(z_1)$. Hence,

$$d_2 : E_2^{0,1}(B\,\mathrm{SU}(p)) \to E_2^{2,0}(B\,\mathrm{SU}(p))$$

is nontrivial and we may define u_2 by $d_2(z_1)$. Hence, we have the desired result. $\square$

We end this section by computing $H^4(BG; \mathbb{Z})$ for $G = \mathrm{SU}(p) \times \mathrm{SU}(p)/\langle\Delta(\xi)\rangle$. The following computation was done in the proof of [Totaro 2014, Theorem 15.4].

Proposition 3.4. *Consider a homomorphism*

$$\psi : H^4(BG; \mathbb{Z}) \to H^4(B\,\mathrm{PU}(p); \mathbb{Z}) \oplus H^4(B\,\mathrm{SU}(p); \mathbb{Z})$$

sending x to $(\Delta^*(x), \Gamma_2^*(x))$. It is an isomorphism.

Proof. Let $p_1 : \mathrm{PU}(p) \times \mathrm{PU}(p) \to \mathrm{PU}(p)$ be the projection onto the first factor. Then, the fiber of $p_1 \circ \pi$ is $\mathrm{SU}(p)$. Consider the spectral sequence associated with

$$B\,\mathrm{SU}(p) \xrightarrow{\Gamma_2} BG \xrightarrow{p_1 \circ \pi} B\,\mathrm{PU}(p).$$

The E_2-term is $H^s(B\,\mathrm{PU}(p); H^t(B\,\mathrm{SU}(p); \mathbb{Z}))$. By Proposition 3.2, $E_2^{s,t} = \{0\}$ unless $s = 0, 3, 4$ and $t = 0, 4$ up to degree 5. In particular, $E_2^{s,t} = \{0\}$ for $s + t = 5$. The nonzero E_2-terms of total degree 4 are given by

$$E_2^{4,0} = \mathbb{Z}, \quad E_2^{0,4} = \mathbb{Z}.$$

The nonzero E_2-term of total degree 3 is given by

$$E_2^{3,0} = \mathbb{Z}/p.$$

So, for dimensional reasons, we have $E_\infty^{s,t} = E_2^{s,t}$ for $s + t = 4$. Hence, we have $H^4(BG; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ and a short exact sequence

$$0 \to H^4(B\,\mathrm{PU}(p); \mathbb{Z}) \xrightarrow{(p_1 \circ \pi)^*} H^4(BG; \mathbb{Z}) \xrightarrow{\Gamma_2^*} H^4(B\,\mathrm{SU}(p); \mathbb{Z}) \to 0.$$

Since the composition $p_1 \circ \pi \circ \Delta$ is the identity map, this short exact sequence splits and the homomorphism ψ is an isomorphism. □

4. The mod p cycle map for G

Let $G = \mathrm{SU}(p) \times \mathrm{SU}(p)/\langle \Delta(\xi) \rangle$, as in Section 2. In this section, we define a virtual complex representation λ'' of G. Using the Chern class $c_2(\lambda'')$, we prove Theorem 1.2 for $K = G$. To be precise, we show that $c_2(\lambda'')$ is nonzero in $\mathrm{CH}^2 BG/p$ and the mod p reduction maps $c_2(\lambda'')$ to 0 in $H^4(BG)$. Theorem 1.2 for $K = G$ was obtained by Totaro [2014] and by the author in [Kameko 2015] independently. From now on, we denote the Bockstein operation of degree 1 by Q_0 and the Milnor operation of degree $2p - 1$ by Q_1. These are cohomology operations on the mod p cohomology.

Let $\lambda_1 : \mathrm{SU}(p) \to U(p)$ be the tautological representation, so that $\lambda_1(g)(v) = gv$ for $v \in \mathbb{C}^p$. Let

$$\lambda_1^* \otimes \lambda_1 : \mathrm{SU}(p) \times \mathrm{SU}(p) \to U(p^2)$$

be the complex representation defined by

$$(\lambda_1^* \otimes \lambda_1)(g_1, g_2)(v_1^* \otimes v_2) = (v_1^* g_1^{-1}) \otimes (g_2 v_2),$$

where $\mathbb{C}^{p^2} = (\mathbb{C}^p)^* \otimes \mathbb{C}^p$ and $(\mathbb{C}^p)^* = \mathrm{Hom}(\mathbb{C}^p, \mathbb{C})$. The complex representation $\lambda_1^* \otimes \lambda_1$ induces a complex representation $\lambda : G \to U(p^2)$. We define a complex representation λ' by $\lambda \circ \Delta \circ p_1 \circ \pi$. Using the complex representations λ and λ', we define a virtual complex representation λ'' by $\lambda'' = \lambda - \lambda'$. An element in the complex representation ring of G corresponds to an element in the topological K-theory $K^0(BG) = [BG, \mathbb{Z} \times BU]$. By abuse of notation, we denote by $\lambda'' : BG \to \mathbb{Z} \times BU$ a map in the homotopy class corresponding to λ''. It is clear that

$$\Delta^*(\lambda'') = 0 \quad \text{and} \quad \Gamma_2^*(\lambda'') = p\lambda_1$$

in the complex representation ring of G.

We denote by x_4 the cohomology class in $H^4(BG; \mathbb{Z})$ such that

(1) $\Gamma_2^*(x_4) = c_2(\lambda_1)$,

(2) $\Delta^*(x_4) = 0$.

Then $c_2(\lambda'') = px_4$. Hence, $\rho(c_2(\lambda'')) = 0$ in $H^4(BG)$. It is clear from the definition that $c_2(\lambda'') \neq 0$ in $H^4(BG; \mathbb{Z})$. Thus, if we show that the Chern class $c_2(\lambda'')$ is not divisible by p in $\mathrm{CH}^2 BG$, then $c_2(\lambda'')$ represents a nonzero element in $\mathrm{CH}^2 BG/p$ and the mod p cycle map is not injective for BG. We prove it by contradiction: Suppose that the Chern class $c_2(\lambda'')$ is divisible by p, that is, we suppose that there exists a virtual complex representation $\mu : BG \to \mathbb{Z} \times BU$ of G such that $x_4 \in \mathrm{Im}\,\mu^* \subset H^4(BG; \mathbb{Z})$. Then $Q_1\rho(x_4)$ must be zero since $H^{\mathrm{odd}}(\mathbb{Z} \times BU) = \{0\}$. We prove the nonexistence of the above virtual complex representation by showing that $Q_1\rho(x_4) \neq 0$. To show that $Q_1\rho(x_4) \neq 0$, we show that $Q_1(f \circ g)^*(\rho(x_4)) \neq 0$ in $H^*(BA_3)$, where f, g and A_3 are as defined in Section 2. The following Proposition 4.1 completes the proof of Theorem 1.2 for $K = G$.

We proved $(f \circ g)^*(\rho(x_4)) = Q_0(x_1 y_1 z_1)$ in [Kameko 2015]. Because we use a similar but slightly different argument in the proof of Theorem 1.2 for $K = H$, we prove the following weaker form in this paper:

Proposition 4.1. *We have* $Q_1(f \circ g)^*(\rho(x_4)) \neq 0$ *in* $H^{2p+3}(BA_3)$.

To prove Proposition 4.1, we compute the Leray–Serre spectral sequences and the homomorphism $(f \circ g)^*$ induced by the following commutative diagram:

$$
\begin{array}{ccccc}
BA_3 & \xrightarrow{\;f \circ g\;} & BG & \xleftarrow{\;\Gamma_2\;} & B\,\mathrm{SU}(p) \\
\varphi \downarrow & & \pi \downarrow & & \tilde{\pi} \downarrow \\
BA_2 & \xrightarrow{\;f \circ g\;} & B\,\mathrm{PU}(p) \times B\,\mathrm{PU}(p) & \xleftarrow{\;\Gamma_2\;} & B\,\mathrm{PU}(p)
\end{array}
$$

We denote by x_1 and y_1 the generators of the mod p cohomology of BA_3 corresponding to the generators $\Delta(\alpha)$ and $\Delta(\beta)$ of A_3, so that we have $\varphi^*(x_1) = x_1$ and $\varphi^*(y_1) = y_1$. Let z_1 be the element in $H^1(B\langle\Gamma_2(\xi)\rangle)$ such that $\Gamma_2^*(z_1) = -z_1 \in E_2^{0,1}(B\,\mathrm{SU}(p))$. The element z_1 in $E_2^{0,1}(B\,\mathrm{SU}(p))$ and $u_2 \in E_2^{2,0}(B\,\mathrm{SU}(p))$ are defined in Section 3, so that $d_2(z_1) = u_2$ in $E_2^{2,0}(B\,\mathrm{SU}(p))$. We define the generator u_3 of $H^3(B\,\mathrm{PU}(p))$ by $u_3 = Q_0 u_2$. Let us consider the E_2-term of the spectral sequence $E_r^{s,t}(BG)$. The E_2-term is as follows:

$$
E_2^{*,*} = H^*(B\,\mathrm{PU}(p)) \otimes H^*(B\,\mathrm{PU}(p)) \otimes \mathbb{Z}/p[z_2] \otimes \Lambda(z_1).
$$

Since $f \circ g = \Delta \circ \iota$, we have $(f \circ g)^*(1 \otimes u) = (f \circ g)^*(u \otimes 1) = \iota^*(u)$. Moreover, we have $\Gamma_2^*(1 \otimes u) = u$ and $\Gamma_2^*(u \otimes 1) = 0$ for $\deg u > 0$.

Let $a_i = u_i \otimes 1 - 1 \otimes u_i$, $b_i = u_i \otimes 1$. Then, up to degree 6, the E_2-term is a free $\mathbb{Z}/p[a_2, z_2] \otimes \Lambda(z_1)$-module with basis $\{1, b_2, a_3, b_3, b_2^2, a_3 b_3, b_2^3\}$. Since

$(f \circ g)^* d_2(z_1) = 0$ and $\Gamma_2^*(d_2(z_1)) = -u_2$, the first nontrivial differential is given by

$$d_2(z_1) = a_2.$$

So, up to degree 5, the E_3-term is a free $\mathbb{Z}/p[z_2]$-module with basis $\{1, b_2, a_3, b_3, b_2^2\}$. In particular, $a_3 b_2 = 0$ in $E_3^{5,0}$. Since $(f \circ g)^*(d_3(z_2)) = 0$ and $\Gamma_2^*(d_3(z_2)) = -u_3$, the second nontrivial differential is given by

$$d_3(z_2) = a_3.$$

Up to degree 4, the E_4-term is a free $\mathbb{Z}/p$-module with basis $\{1, b_2, b_3, b_2^2, b_2 z_2\}$ and the spectral sequence collapses at the E_4-level. Thus, the E_∞-terms of total degree 4 are as follows:

$$E_\infty^{0,4} = \{0\}, \quad E_\infty^{1,3} = \{0\}, \quad E_\infty^{2,2} = \mathbb{Z}/p\{b_2 z_2\}, \quad E_\infty^{3,1} = \{0\}, \quad E_\infty^{4,0} = \mathbb{Z}/p\{b_2^2\}.$$

The element b_2 is a permanent cocycle. By abuse of notation, we denote by b_2 the cohomology class in $F^2 H^2(BG)$ representing b_2. Since $H^2(B \operatorname{SU}(p)) = \{0\}$, we have

$$\Gamma_2^*(\pi^*(b_2)) = 0.$$

Moreover, $\pi^*\big(H^4(B \operatorname{PU}(p) \times B \operatorname{PU}(p))\big) = \mathbb{Z}/p\{b_2^2\}$. Hence, we have

$$\Gamma_2^*\big(\pi^*\big(H^4(B \operatorname{PU}(p) \times B \operatorname{PU}(p))\big)\big) = \{0\}.$$

On the other hand, $\Gamma_2^* \rho(x_4) = \rho(c_2(\lambda_1)) \neq 0$ in $H^4(B \operatorname{SU}(p))$. Therefore, $\rho(x_4)$ is not in the image of

$$\pi^* : H^4(B \operatorname{PU}(p) \times B \operatorname{PU}(p)) \to H^4(BG).$$

Hence, we have the following result:

Proposition 4.2. *The cohomology class $\rho(x_4)$ represents $\alpha b_2 z_2$ in $E_\infty^{2,2}$ for some $\alpha \neq 0$ in $\mathbb{Z}/p$.*

Now, we complete the proof of Proposition 4.1 using Proposition 4.2.

Proof of Proposition 4.1. Since $(f \circ g)^*(b_2) = x_1 y_1$, we have

$$(f \circ g)^*(b_2 z_2) = x_1 y_1 z_2$$

in the spectral sequence, where $z_2 = Q_0 z_1$ in $H^2(B \langle \Gamma_2(\xi) \rangle)$. Let $x_2 = Q_0 x_1$ and $y_2 = Q_0 y_1$. Then $H^*(B A_3) = \mathbb{Z}/p[x_2, y_2, z_2] \otimes \Lambda(x_1, y_1, z_1)$ and $\varphi^*(H^*(B A_2))$ is the subalgebra generated by x_1, y_1, x_2, y_2. Therefore, we have

$$(f \circ g)^*(\rho(x_4)) = \alpha x_1 y_1 z_2 + u' z_1 + u''$$

for some $u', u'' \in \varphi^*(H^*(BA_2))$. Let M be the $\varphi^*(H^*(BA_2))$-module generated by

$$1, \quad z_1, \quad z_1 z_2, \quad z_2^i \quad \text{and} \quad z_1 z_2^i \quad (i \geq 2),$$

so that

$$H^*(BA_3)/M = \varphi^*(H^*(BA_2))\{z_2\}.$$

Since $Q_1 z_1 = z_2^p$, $Q_1 z_2 = 0$ and Q_1 is a derivation, M is closed under the action of the Milnor operation Q_1. We have

$$(f \circ g)^*(\rho(x_4)) \equiv \alpha x_2^p y_1 z_2 - \alpha x_1 y_2^p z_2 \not\equiv 0 \mod M.$$

This completes the proof of Proposition 4.1. $\hfill\square$

5. The mod p cohomology of $B\tilde{\pi}(H_2)$

In this section, we collect some facts on the mod p cohomology of $B\tilde{\pi}(H_2)$ as Propositions 5.1 and 5.2. We use these facts in the proof of Proposition 6.1.

We begin by defining generators of $H^1(B\tilde{\pi}(H_2))$. Since the commutator subgroup $[\tilde{\pi}(H_2), \tilde{\pi}(H_2)]$ is generated by $\tilde{\pi}(\text{diag}(\xi^{a_1}, \ldots, \xi^{a_p}))$ for $0 \leq a_i \leq p - 1$, $1 \leq i \leq p$, with $a_1 + \cdots + a_p \equiv 0 \mod p$,

$$\tilde{\pi}(H_2)/[\tilde{\pi}(H_2), \tilde{\pi}(H_2)] = \mathbb{Z}/p \oplus \mathbb{Z}/p.$$

This elementary abelian p-group is generated by $\tilde{\pi}(\sigma_1)$ and $\tilde{\pi}(\beta)$. We denote by v_1 and w_1 the generators of $H^1(B\langle\tilde{\pi}(\sigma_1)\rangle)$ and $H^1(B\langle\tilde{\pi}(\beta)\rangle)$ corresponding to $\tilde{\pi}(\sigma_1)$ and $\tilde{\pi}(\beta)$, respectively. By abuse of notation, we denote the corresponding generators in $H^1(B\tilde{\pi}(H_2))$ by the same symbol, so that, for the inclusions

$$\iota_\beta : \langle\tilde{\pi}(\beta)\rangle \to \tilde{\pi}(H_2), \quad \iota_\sigma : \langle\tilde{\pi}(\sigma_1)\rangle \to \tilde{\pi}(H_2),$$

we have $\iota_\beta^*(w_1) = w_1$, $\iota_\beta^*(v_1) = 0$, $\iota_\sigma^*(w_1) = 0$ and $\iota_\sigma^*(v_1) = v_1$. Indeed, we have $H^*(B\langle\tilde{\pi}(\sigma_1)\rangle) = \mathbb{Z}/p[v_2] \otimes \Lambda(v_1)$ and $H^*(B\langle\tilde{\pi}(\beta)\rangle) = \mathbb{Z}/p[w_2] \otimes \Lambda(w_1)$, where $v_2 = Q_0 v_1$ and $w_2 = Q_0 w_1$. We denote the inclusion of $\tilde{\pi}(H_2)$ to $\text{PU}(p)$ by

$$\iota : \tilde{\pi}(H_2) \to \text{PU}(p)$$

and we recall that we defined the generator u_2 of $H^2(B\text{PU}(p))$ in Proposition 3.3.

Proposition 5.1. *In $H^*(B\tilde{\pi}(H_2))$, we have $\iota^*(u_2)v_1 \neq 0$ and $\iota^*(u_2^2) \neq 0$.*

Proof. We consider the Leray–Serre spectral sequences associated with the vertical fibrations in the following commutative diagram:

$$
\begin{array}{ccccc}
B\langle\sigma_1\rangle & \xrightarrow{\iota_\sigma} & BH_2 & \xrightarrow{\iota} & B\,\text{SU}(p) \\
\tilde{\pi} \downarrow & & \tilde{\pi} \downarrow & & \pi \downarrow \\
B\langle\tilde{\pi}(\sigma_1)\rangle & \xrightarrow{\iota_\sigma} & B\tilde{\pi}(H_2) & \xrightarrow{\iota} & B\,\text{PU}(p)
\end{array}
$$

Let $z_1 \in E_2^{0,1}(B\,\mathrm{SU}(p))$ and $u_2 \in E_2^{2,0}(B\,\mathrm{SU}(p))$ be elements defined in Section 3. By abuse of notation, we denote elements $\iota^*(z_1)$ in $E_2^{0,1}(BH_2)$ and $\iota_\sigma^*(\iota^*(z_1))$ in $E_2^{0,1}(B\langle\sigma_1\rangle)$ by z_1. Since $\langle\sigma_1\rangle = \mathbb{Z}/p^2$,

$$d_2(z_1) = \alpha v_2$$

for some $\alpha \neq 0$ in $\mathbb{Z}/p$ in the Leray–Serre spectral sequence $E_2^{2,0}(B\langle\sigma_1\rangle)$. Since $u_2 = d_2(z_1)$ in the Leray–Serre spectral sequence $E_2^{2,0}(B\,\mathrm{SU}(p))$, we have

$$\iota_\sigma^*(\iota^*(u_2)) = d_2(z_1) = \alpha v_2$$

in $H^*\big(B\langle\tilde{\pi}(\sigma_1)\rangle\big) = \mathbb{Z}/p[v_2] \otimes \Lambda(v_1)$. Hence, we have $\iota_\sigma^*(\iota^*(u_2)v_1) = \alpha v_1 v_2 \neq 0$ and $\iota_\sigma^*(\iota^*(u_2^2)) = \alpha^2 v_2^2 \neq 0$. Therefore, we obtain the desired result: $\iota^*(u_2)v_1 \neq 0$ and $\iota^*(u_2^2) \neq 0$ in $H^*(B\tilde{\pi}(H_2))$. $\qquad\square$

Proposition 5.2. *In $H^*(B\tilde{\pi}(H_2))$, we have $\iota^*(u_2)w_1 = 0$.*

To prove Proposition 5.2, at the end of Section 2 we defined the subgroup $H_2' = T^p \rtimes \mathbb{Z}/p$ of the unitary group $U(p)$ generated by diagonal matrices and β. The quotient group $\tilde{\pi}'(H_2')$ contains $\tilde{\pi}(H_2)$ as a subgroup and they are subgroups of the projective unitary group $\mathrm{PU}(p)$. We denote by

$$\iota'' : \tilde{\pi}(H_2) \to \tilde{\pi}'(H_2'), \quad \iota' : \tilde{\pi}'(H_2') \to \mathrm{PU}(p)$$

the inclusions, so that $\iota = \iota' \circ \iota''$. We use the following lemma in the proof of Proposition 5.2:

Lemma 5.3. *In $H^*(B\tilde{\pi}'(H_2'))$, there exists an element $t_2 \in H^2(B\tilde{\pi}'(H_2'))$ such that $H^1(B\tilde{\pi}'(H_2')) = \mathbb{Z}/p\{w_1\}$ and $H^2(B\tilde{\pi}'(H_2')) = \mathbb{Z}/p\{t_2, w_2\}$, where $w_2 = Q_0 w_1$, $(\iota'' \circ \iota_\sigma)^*(t_2) = v_2$ and $(\iota'' \circ \iota_\beta)^*(t_2) = 0$. Moreover, we have $t_2 w_1 = 0$ in $H^*(B\tilde{\pi}'(H_2'))$.*

Now, we prove Proposition 5.2 assuming Lemma 5.3.

Proof of Proposition 5.2. We consider the Leray–Serre spectral sequences associated with the vertical fibrations in the commutative diagram

$$
\begin{array}{ccccc}
B\langle\beta,\xi\rangle & \xrightarrow{\iota_\beta} & BH_2 & \xrightarrow{\iota} & B\,\mathrm{SU}(p) \\
{\scriptstyle\tilde{\pi}}\downarrow & & {\scriptstyle\tilde{\pi}}\downarrow & & {\scriptstyle\tilde{\pi}}\downarrow \\
B\langle\tilde{\pi}(\beta)\rangle & \xrightarrow{\iota_\beta} & B\tilde{\pi}(H_2) & \xrightarrow{\iota} & B\,\mathrm{PU}(p)
\end{array}
$$

Suppose that $\iota'^*(u_2) = \alpha_1 t_2 + \alpha_2 w_2$, where $\alpha_1, \alpha_2 \in \mathbb{Z}/p$. Then, by Lemma 5.3, we have

$$\iota'^*(u_2)w_1 = \alpha_1 t_2 w_1 + \alpha_2 w_1 w_2 = \alpha_2 w_1 w_2.$$

Hence, we have $(\iota \circ \iota_\beta)^*(u_2)w_1 = \alpha_2 w_1 w_2$. On the other hand, since the group extension

$$\langle\xi\rangle \to \langle\beta,\xi\rangle \to \langle\tilde{\pi}(\beta)\rangle$$

is trivial, $d_2 : H^1(B\langle\xi\rangle) \to H^2\big(B\langle\tilde{\pi}(\beta)\rangle\big)$ in $E_2^{s,t}(B\langle\beta,\xi\rangle)$ is zero and

$$(\iota \circ \iota_\beta)^*(u_2) = d_2((\iota \circ \iota_\beta)^*(z_1)) = 0$$

in $H^*\big(B\langle\tilde{\pi}(\beta)\rangle\big) = E_2^{2,0}(B\langle\beta,\xi\rangle)$. Therefore, we have $\alpha_2 = 0$ and $w_1 \iota'^*(u_2) = 0$ in $H^*(B\tilde{\pi}'(H_2'))$. Therefore, we have

$$\iota^*(u_2)w_1 = \iota''^*(\iota'^*(u_2)w_1) = 0$$

in $H^*(B\tilde{\pi}(H_2))$. $\square$

We end this section by proving Lemma 5.3.

Proof of Lemma 5.3. We need to study the mod p cohomology only up to degree 3. We define t_2 by $\iota'^*(u_2)$, where u_2 is the generator of $H^2(B\,\mathrm{PU}(p))$.

We consider the Leray–Serre spectral sequence associated with the following commutative diagram:

$$
\begin{array}{ccc}
BT^p & \xrightarrow{\ \tilde{\pi}'\ } & BT^{p-1} \\
\downarrow & & \downarrow \\
BH_2' & \xrightarrow{\ \tilde{\pi}'\ } & B\tilde{\pi}'(H_2') \\
\downarrow & & \downarrow \\
B\langle\beta\rangle & \xrightarrow{\ \tilde{\pi}\ } & B\langle\tilde{\pi}(\beta)\rangle
\end{array}
$$

We choose a generator $t_2^{(i)} \in H^2(BT^p)$ corresponding to the i-th diagonal entry of T^p, so that $H^2(BT^p) = \mathbb{Z}/p\{t_2^{(1)}, \dots, t_2^{(p)}\}$. The matrix β acts on T^p as the cyclic permutation of diagonal entries, so that it acts on $H^2(BT^p)$ as the cyclic permutation on $t_2^{(1)}, \dots, t_2^{(p)}$. The induced homomorphism $\tilde{\pi}'^* : H^2(BT^{p-1}) \to H^2(BT^p)$ is injective and we may take a basis $\{u_2^{(1)}, \dots, u_2^{(p-1)}\}$ for $H^2(BT^{p-1})$ such that $\tilde{\pi}'^*(u_2^{(i)}) = t_2^{(i)} - t_2^{(i+1)}$ for $i = 1, \dots, p-1$. Hence, $\langle\beta\rangle$ acts on $H^2(BT^{p-1})$ by

$$gu_2^{(i)} = u_2^{(i+1)}$$

for $i = 1, \dots, p-2$ and

$$gu_2^{(p-1)} = -(u_2^{(1)} + \cdots + u_2^{(p-1)})$$

for some generator g of $\langle\beta\rangle$. We consider the Leray–Serre spectral sequence converging to the mod p cohomology of $B\tilde{\pi}'(H_2')$. The E_1-term is additively given as follows:

$$E_1 = \mathbb{Z}/p[u_2^{(1)}, \dots, u_2^{(p-1)}]\{w_2^i, w_1 w_2^i \mid i \geq 0\}.$$

The first nontrivial differential is given by

$$d_1(uw_2^i) = ((1-g)u)w_1 w_2^i, \quad d_1(uw_1 w_2^i) = ((1-g)^{p-1}u)w_2^{i+1},$$

where $u \in \mathbb{Z}/p\,[u_2^{(1)}, \ldots, u_2^{(p-1)}] = E_1^{0,*}$. The kernel of

$$(1-g) : \mathbb{Z}/p\,\{u_2^{(1)}, \ldots, u_2^{(p-1)}\} \to \mathbb{Z}/p\,\{u_2^{(1)}, \ldots, u_2^{(p-1)}\}$$

is spanned by a single element,

$$u_2^{(1)} + 2u_2^{(2)} + \cdots + (p-1)u_2^{(p-1)},$$

and the image of $(1-g)$ is spanned by the $p-2$ elements

$$u_2^{(1)} - u_2^{(2)}, \ldots, \ u_2^{(p-2)} - u_2^{(p-1)}.$$

We denote the generator of the kernel of $(1-g)$ by $\tilde{u}$, that is,

$$\tilde{u} = u_2^{(1)} + 2u_2^{(2)} + \cdots + (p-1)u_2^{(p-1)}.$$

It is easy to see that

$$\tilde{u} \equiv (1 + \cdots + (p-1))u_2^{(p-1)} \equiv \tfrac{1}{2}p(p-1)u_2^{(p-1)} \equiv 0$$

modulo the image of $(1-g)$. By direct calculation, we have $(1-g)^{p-1}(u_2^{(1)}) = 0$ and $\mathrm{Ker}(1-g)^{p-1} = \mathbb{Z}/p\,\{u_2^{(1)}, \ldots, u_2^{(p-1)}\}$. Hence, we have

$$E_2^{0,2} = \mathrm{Ker}(1-g) = \mathbb{Z}/p\,\{\tilde{u}\},$$
$$E_2^{1,2} = \big(\mathrm{Ker}(1-g)^{p-1}/\,\mathrm{Im}(1-g)\big)\{w_1\} = \mathbb{Z}/p\,\{u_2^{(1)}w_1\},$$

respectively. Moreover, we have $E_r^{*,\mathrm{odd}} = \{0\}$ and $E_r^{*,0} = \mathbb{Z}/p\,[w_2] \otimes \Lambda(w_1)$ for $* \geq 0$ and $r \geq 1$. Since the elements in $E_r^{*,0}$ are permanent cocycles, the spectral sequence collapses at the E_2-level up to degree 3. Choose a cohomology class t_2' in $H^2(B\tilde{\pi}'(H_2'))$ representing the generator $\tilde{u}$ of $E_\infty^{0,2} = \mathbb{Z}/p$. Then, $H^2(B\tilde{\pi}'(H_2'))$ is generated by t_2' and w_2. Suppose that

$$\iota'^*(u_2) = \alpha_1 w_2 + \alpha_2 t_2',$$

where $\alpha_1, \alpha_2 \in \mathbb{Z}/p$. Since $(\iota' \circ \iota'' \circ \iota_\sigma)^*(u_2) = v_2$ and $(\iota'' \circ \iota_\sigma)^*(w_2) = 0$,

$$(\iota'' \circ \iota_\sigma)^*(\alpha_2 t_2') = v_2$$

and so $\alpha_2 \neq 0$. Hence, t_2 and w_2 generate $H^2(B\tilde{\pi}'(H_2'))$.

Next, we prove that $t_2 w_1 = 0$. The E_∞-terms of total degree 3 are given by

$$E_\infty^{0,3} = \{0\}, \quad E_\infty^{1,2} = \mathbb{Z}/p\,\{u_2^{(1)}w_1\}, \quad E_\infty^{2,1} = \{0\} \quad \text{and} \quad E_\infty^{3,0} = \mathbb{Z}/p\,\{w_1 w_2\}.$$

Therefore, we have

$$F^2 H^3(B\tilde{\pi}'(H_2')) = F^3 H^3(B\tilde{\pi}'(H_2')) = \mathbb{Z}/p\,\{w_1 w_2\}.$$

Since $\alpha_2 t_2' w_1$ represents $\alpha_2 \tilde{u} w_1$ and $\tilde{u} \in \mathrm{Ker}(1-g)$ is congruent to zero modulo the image of $(1-g)$, we have $\tilde{u} w_1 = 0$ in $E_\infty^{1,2}$. So, we have

$$t_2 w_1 \in F^3 H^3(B\tilde{\pi}'(H_2')) = \mathbb{Z}/p\{w_1 w_2\}.$$

Therefore, $t_2 w_1 = \alpha_3 w_1 w_2$ for some $\alpha_3 \in \mathbb{Z}/p$. We proved that $(\iota'' \circ \iota_\beta)^*(t_2) = (\iota' \circ \iota'' \circ \iota_\beta)^*(u_2) = 0$ in the proof of Proposition 5.2. Thus, we have $(\iota'' \circ \iota_\beta)^*(t_2 w_1) = 0$. On the other hand, we have $(\iota'' \circ \iota_\beta)^*(w_1 w_2) = w_1 w_2 \neq 0$ in $H^*\big(B\langle \tilde{\pi}(\beta)\rangle\big)$. Hence, we obtain $\alpha_3 = 0$. $\qquad\qquad\square$

6. The mod p cycle map for H

In this section, we prove Theorem 1.2. Let G be $\mathrm{SU}(p) \times \mathrm{SU}(p)/\langle \Delta(\xi)\rangle$ and let $H = p_+^{1+2} \times H_2/\Delta(\xi)$, as in Section 3. Let K be a subgroup of G containing H, that is, $H \subset K \subset G$. We proved in Section 4 that the mod p cycle map $\mathrm{CH}^2 BG/p \to H^4(BG)$ is not injective. To be more precise, we defined the virtual complex representation $\lambda'' : BG \to \mathbb{Z} \times BU$ such that the Chern class $c_2(\lambda'') \in \mathrm{CH}^2 BG$ is nontrivial in $\mathrm{CH}^2 BG/p$, that is, $c_2(\lambda'')$ is not divisible by p, and the mod p cycle map maps $c_2(\lambda'')$ to $\rho(c_2(\lambda'')) = 0$. We denote the inclusions by $f' : K \to G$, $f'' : H \to K$ and $f : H \to G$, so that $f = f' \circ f'' : H \to G$. It is clear that $\rho(c_2(\lambda'' \circ f'))$ is zero in $H^4(BK)$. So, in order to prove Theorem 1.2, we need to show that $c_2(\lambda'' \circ f')$ remains nonzero in $\mathrm{CH}^2 BK \subset H^4(BK; \mathbb{Z})$ and that $c_2(\lambda'' \circ f')$ remains not divisible by p in $\mathrm{CH}^2 BK$. These follow immediately from:

(1) $c_2(\lambda'' \circ f) = f''^*(c_2(\lambda'' \circ f'))$ is not zero in $\mathrm{CH}^2 BH \subset H^4(BH; \mathbb{Z})$.

(2) $c_2(\lambda'' \circ f) = f''^*(c_2(\lambda'' \circ f'))$ is not divisible by p in $\mathrm{CH}^2 BH$.

To prove (1) and (2), we consider the spectral sequences associated with the vertical fibrations below and the induced homomorphism between them:

$$
\begin{array}{ccc}
BH & \xrightarrow{\ \ f\ \ } & BG \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
BA_2 \times B\tilde{\pi}(H_2) & \xrightarrow{\ \ f\ \ } & B\,\mathrm{PU}(p) \times B\,\mathrm{PU}(p)
\end{array}
$$

Let $g : BA_2 \to BA_2 \times B\tilde{\pi}(H_2)$ be the map defined in Section 2 by $g(\tilde{\pi}(\alpha)) = (\tilde{\pi}(\alpha), \tilde{\pi}(\alpha))$ and $g(\tilde{\pi}(\beta)) = (\tilde{\pi}(\beta), \tilde{\pi}(\beta))$. Let v_1 and w_1 be the generators of $H^1(B\tilde{\pi}(H_2))$ defined in the previous section; let x_1 and y_1 be those of $H^1(BA_2)$, as defined in Section 3. We denote by x_1, y_1, v_1 and w_1 the corresponding generators of $H^1(BA_2 \times B\tilde{\pi}(H_2))$, so that $g^*(x_1) = x_1$, $g^*(v_1) = 0$ and $g^*(y_1) = g^*(w_1) = y_1$. We denote by z_1 a generator of $H^1(B\langle \Gamma_2(\xi)\rangle) = E_2^{0,1}$ as in Section 4. Let $x_2 = Q_0 x_1$, $y_2 = Q_0 y_1$ and $z_2 = Q_0 z_1$, as usual, so that $H^*(BA_2) = \mathbb{Z}/p[x_2, y_2] \otimes \Lambda(x_1, y_1)$.

Also, let u_2 be the generator of $H^2(B\,\mathrm{PU}(p))$ defined in Section 3, and let $u_3 = Q_0 u_2$, as in Section 4. Let ι be the map induced by the inclusion of $\tilde{\pi}(H_2)$ into $\mathrm{PU}(p)$. We need to compute the spectral sequence up to degree 4. Differentials d_2 and d_3 in the spectral sequence $E_r^{s,t}(BH)$ are given by

$$d_2(z_1) = x_1 y_1 - \iota^*(u_2),$$
$$d_3(z_2) = x_2 y_1 - x_1 y_2 - \iota^*(u_3),$$

since

$$f^*(u_2 \otimes 1 - 1 \otimes u_2) = x_1 y_1 - \iota^*(u_2),$$
$$f^*(u_3 \otimes 1 - 1 \otimes u_3) = x_2 y_1 - x_1 y_2 - \iota^*(u_3),$$

and the differentials d_2 and d_3 in the spectral sequence $E_r^{s,t}(BG)$ are given by $d_2(z_1) = u_2 \otimes 1 - 1 \otimes u_2$ and $d_3(z_2) = u_3 \otimes 1 - 1 \otimes u_3$, as we saw in Section 4.

Proposition 6.1. *The E_∞-terms $E_\infty^{s,t}$ (for $s = 0, 1, 2$ and $s + t = 3, 4$) for the spectral sequence $E_r^{s,t}(BH)$ are given as follows:* $E_\infty^{0,3} = E_\infty^{1,2} = E_\infty^{0,4} = E_\infty^{1,3} = \{0\}$,

$$E_\infty^{2,1} = \mathbb{Z}/p\,\{w_1 x_1 z_1, w_1 y_1 z_1\},$$
$$E_\infty^{2,2} = \mathbb{Z}/p\,\{x_1 y_1 z_2, w_1 x_1 z_2, w_1 y_1 z_2\}.$$

Proof. For the sake of notational simplicity, let

$$R = \mathbb{Z}/p[x_2, y_2] \otimes H^*(B\tilde{\pi}(H_2)),$$

so that

$$H^*(BA_2) \otimes H^*(B\tilde{\pi}(H_2)) = R\{1, x_1, y_1, x_1 y_1\}.$$

The set $\{v_1, w_1\}$ is a basis for $H^1(B\tilde{\pi}(H_2))$. We consider a basis for $H^2(B\tilde{\pi}(H_2))$. By Proposition 5.1, we have $\iota^*(u_2)^2 \neq 0$. We choose a basis $\{m^{(i)}, \iota^*(u_2)\}$ for $H^2(B\tilde{\pi}(H_2))$, where $1 \leq i < \dim H^2(B\tilde{\pi}(H_2))$. Here, we do not exclude the possibility that $\{m^{(i)}\}$ could be the empty set. Then, the set $\{m^{(i)}, \iota^*(u_2), x_2, y_2\}$ is a basis for the subspace of R spanned by elements of degree 2 and $\{m^{(i)}, x_2, y_2\}$ is a basis for the subspace of $R/(\iota^*(u_2))$ spanned by elements of degree 2. The set

$$\{v_1, w_1, x_1, y_1\}$$

is a basis for $E_2^{1,0} = H^1(BA_2 \times B\tilde{\pi}(H_2))$ and

$$\{m^{(i)}, \iota^*(u_2), x_2, y_2, v_1 x_1, v_1 y_1, w_1 x_1, w_1 y_1, x_1 y_1\}$$

is a basis for $E_2^{2,0} = H^2(BA_2 \times B\tilde{\pi}(H_2))$.

First, we compute E_3-terms $E_3^{0,3}$, $E_3^{2,1}$ and $E_3^{1,3}$. Let us consider R-module homomorphisms

$$\mathrm{pr}_2^{(k)} : E_2^{*,2k} = R\{z_2^k, x_1 z_2^k, y_1 z_2^k, x_1 y_1 z_2^k\} \to R\{x_1 z_2^k, y_1 z_2^k, x_1 y_1 z_2^k\}$$

sending z_2^k, $x_1 z_2^k$, $y_1 z_2^k$ and $x_1 y_1 z_2^k$ to 0, $x_1 z_2^k$, $y_1 z_2^k$ and $x_1 y_1 z_2^k$, respectively. Recall that

$$d_2(z_1) = x_1 y_1 - \iota^*(u_2).$$

The E_2-term $E_2^{0,3}$ is spanned by $z_1 z_2$. It is clear from $d_2(z_2) = 0$ that

$$d_2(z_1 z_2) = d_2(z_1) z_2 = (x_1 y_1 - \iota^*(u_2)) z_2 \neq 0.$$

Hence the homomorphism $d_2 : E_2^{0,3} \to E_2^{2,2}$ is injective and we have $E_3^{0,3} = \{0\}$.

The E_2-term $E_2^{2,1}$ is spanned by

$$m^{(i)} z_1, \quad \iota^*(u_2) z_1, \quad x_2 z_1, \quad y_2 z_1, \quad v_1 x_1 z_1, \quad v_1 y_1 z_1, \quad w_1 x_1 z_1, \quad w_1 y_1 z_1, \quad x_1 y_1 z_1$$

and

$$d_2(\alpha_2 z_1) = \alpha_2 d_2(z_1) = \alpha_2 x_1 y_1 - \alpha_2 \iota^*(u_2).$$

for any degree 2 element α_2 in $E_2^{2,0} = H^2(BA_2 \times B\tilde{\pi}(H_2))$ since $d_2(\alpha_2) = 0$. If α_2 is one of $m^{(i)}$, $\iota^*(u_2)$, x_2 or y_2, then $\alpha_2 \iota^*(u_2) \in R\{1\}$ and so $\mathrm{pr}_2^{(0)}(\alpha_2 \iota^*(u_2)) = 0$, by definition. Hence, for $\alpha_2 = m^{(i)}$, $\iota^*(u_2)$, x_2 and y_2, we have

$$\mathrm{pr}_2^{(0)}(d_2(\alpha_2 z_1)) = \alpha_2 x_1 y_1.$$

So, we have

$$\mathrm{pr}_2^{(0)}(d_2(m^{(i)} z_1)) = m^{(i)} x_1 y_1,$$
$$\mathrm{pr}_2^{(0)}(d_2(\iota^*(u_2) z_1)) = \iota^*(u_2) x_1 y_1,$$
$$\mathrm{pr}_2^{(0)}(d_2(x_2 z_1)) = x_2 x_1 y_1,$$
$$\mathrm{pr}_2^{(0)}(d_2(y_2 z_1)) = y_2 x_1 y_1.$$

If α_2 is one of $v_1 x_1$, $v_1 y_1$, $w_1 x_1$, $w_1 y_1$ or $x_1 y_1$, then $\alpha_2 x_1 y_1 = 0$. So, we have

$$d_2(\alpha_2 z_1) = -\alpha_2 \iota^*(u_2) = -\iota^*(u_2) \alpha_2.$$

By [Proposition 5.2](#), $\iota^*(u_2) w_1 = 0$ in $H^*(B\tilde{\pi}(H_2))$. Using this, we have

$$d_2(w_1 x_1 z_1) = -\iota^*(u_2) w_1 x_1 = 0,$$
$$d_2(w_1 y_1 z_1) = -\iota^*(u_2) w_1 y_1 = 0.$$

Also, we have

$$\mathrm{pr}_2^{(0)}(d_2(v_1 x_1 z_1)) = \qquad\qquad -\iota^*(u_2) v_1 x_1,$$
$$\mathrm{pr}_2^{(0)}(d_2(v_1 y_1 z_1)) = \qquad\qquad\qquad -\iota^*(u_2) v_1 y_1,$$
$$\mathrm{pr}_2^{(0)}(d_2(x_1 y_1 z_1)) = -\iota^*(u_2) x_1 y_1.$$

By [Proposition 5.1](#), $\iota^*(u_2) v_1 \neq 0$. So, the kernel of $\mathrm{pr}_2^{(0)} \circ d_2$ is spanned by

$$x_1 y_1 z_1 + \iota^*(u_2) z_1, \quad w_1 x_1 z_1, \quad w_1 y_1 z_1.$$

On the other hand, we have

$$d_2(x_1y_1z_1 + \iota^*(u_2)z_1) = x_1y_1(x_1y_1 - \iota^*(u_2)) + \iota^*(u_2)(x_1y_1 - \iota^*(u_2)) = -\iota^*(u_2)^2,$$

and, since $\iota^*(u_2)^2 \neq 0$ by [Proposition 5.1](#), $x_1y_1z_1 + \iota^*(u_2)z_1$ is not in the kernel of d_2. Hence, the kernel of d_2 is spanned by $w_1x_1z_1$ and $w_1y_1z_1$, and the image of $d_2 : E_2^{0,2} \to E_2^{2,1}$ is trivial since $E_2^{0,2}$ is spanned by z_2 and $d_2(z_2) = 0$. Thus, we have $E_3^{2,1} = \mathbb{Z}/p\{w_1x_1z_1, w_1y_1z_1\}$.

As for the E_2-term $E_2^{1,3}$, we have a basis

$$\{x_1z_1z_2, y_1z_1z_2, v_1z_1z_2, w_1z_1z_2\}$$

and

$$d_2(\alpha_1z_1z_2) = -\alpha_1d_2(z_1)z_2 = -\alpha_1x_1y_1z_2 + \alpha_1\iota^*(u_2)z_2$$

for $\alpha_1 = x_1, y_1, v_1, w_1$, since $d_2(\alpha_1) = d_2(z_2) = 0$. For $\alpha_1 = x_1, y_1$, since $\alpha_1x_1y_1 = 0$ we have

$$d_2(\alpha_1z_1z_2) = \alpha_1\iota^*(u_2)z_2 = \iota^*(u_2)\alpha_1z_2.$$

For $\alpha_1 = v_1, w_1$, since $\alpha_1\iota^*(u_2)z_2 \in R\{z_2\}$, we have $\mathrm{pr}_2^{(1)}(\alpha_1\iota^*(u_2)z_2) = 0$ by definition. Hence, we have

$$\mathrm{pr}_2^{(1)}(d_2(\alpha_1z_1z_2)) = -\alpha_1x_1y_1z_2.$$

Thus, we obtain

$$\mathrm{pr}_2^{(1)}(d_2(x_1z_1z_2)) = \iota^*(u_2)x_1z_2,$$
$$\mathrm{pr}_2^{(1)}(d_2(y_1z_1z_2)) = \iota^*(u_2)y_1z_2,$$
$$\mathrm{pr}_2^{(1)}(d_2(v_1z_1z_2)) = -v_1x_1y_1z_2,$$
$$\mathrm{pr}_2^{(1)}(d_2(w_1z_1z_2)) = -w_1x_1y_1z_2.$$

Hence, it is clear that the composition

$$\mathrm{pr}_2^{(1)} \circ d_2 : E_2^{1,3} \to E_2^{3,2} \to R\{x_1z_2, y_1z_2, x_1y_1z_2\}$$

is injective and so is $d_2 : E_2^{1,3} \to E_2^{3,2}$. Therefore, we have $E_3^{1,3} = \{0\}$.

Next we compute the E_4-terms $E_4^{0,4}$, $E_4^{1,2}$ and $E_4^{2,2}$. In the E_3-term, the relations are given by $x_1y_1 = \iota^*(u_2)$, $\iota^*(u_2)x_1 = 0$ and $\iota^*(u_2)y_1 = 0$. In particular, $\iota^*(u_2)^2 = 0$. For simplicity, we write R' and R'' for $R/(\iota^*(u_2))$ and $R/(\iota^*(u_2)^2)$, respectively. We have

$$E_3^{*,2k} = R'\{x_1z_2^k, y_1z_2^k\} \oplus R''\{z_2^k\}$$

as a graded $\mathbb{Z}/p$-module. Let N be the subspace of $R'\{x_1\}$ spanned by elements of the form xx_1, where x ranges over a basis for $H^*(B\tilde{\pi}(H_2))/(\iota^*(u_2)) \subset R'$. Here, we emphasize that N is not an R-submodule and that $\tilde{x}m^{(i)}x_1$, $\tilde{x}x_1$, $\tilde{x}v_1x_1$

and $\tilde{x} w_1 x_1$ are linearly independent in $R'\{x_1\}/N$, where $\tilde{x}$ ranges over positive-degree monomials in x_2 and y_2. We consider a $\mathbb{Z}/p$-module homomorphism

$$\mathrm{pr}_3 : E_3^{*,0} = R'\{x_1, y_1\} \oplus R''\{1\} \to R'\{x_1\}/N \oplus R''\{1\},$$

sending $r'x_1$, $r'y_1$ and r'' to $r'x_1$, 0 and r'', respectively, where $r' \in R'$ and $r'' \in R''$. Recall that

$$d_3(z_2) = x_2 y_1 - x_1 y_2 - \iota^*(u_3).$$

The E_3-term $E_3^{0,4}$ is spanned by z_2^2 and, since $y_2 x_1 z_2$ is nontrivial in $R'\{x_1 z_2\}$,

$$d_3(z_2^2) = 2d_3(z_2)z_2 = 2x_2 y_1 z_2 - 2x_1 y_2 z_2 - 2\iota^*(u_3)z_2$$
$$= -2y_2 x_1 z_2 + 2x_2 y_1 z_2 - 2\iota^*(u_3)z_2$$

is nontrivial in $E_3^{*,2} = R'\{x_1 z_2, y_1 z_2\} \oplus R''\{z_2\}$. Hence, $d_3 : E_3^{0,4} \to E_3^{3,2}$ is injective and $E_4^{0,4} = \{0\}$.

The E_3-term $E_3^{1,2}$ is spanned by

$$v_1 z_2, \quad w_1 z_2, \quad x_1 z_2, \quad y_1 z_2,$$

since the subspace of R'' spanned by degree 1 elements is equal to $H^1(B\tilde{\pi}(H_2))$ and $H^1(B\tilde{\pi}(H_2))$ is spanned by v_1 and w_1. For $\alpha_1 = v_1, w_1, x_1, y_1$, since $d_3(\alpha_1) = 0$ we have

$$d_3(\alpha_1 z_2) = -\alpha_1 d_3(z_2) = -\alpha_1 x_2 y_1 + \alpha_1 x_1 y_2 + \alpha_1 \iota^*(u_3).$$

Hence, for $\alpha_1 = v_1, w_1$, since $\mathrm{pr}_3(\alpha_1 x_2 y_1) = 0$ by definition, we have

$$\mathrm{pr}_3(d_3(\alpha_1 z_2)) = \alpha_1 x_1 y_2 + \alpha_1 \iota^*(u_3) = y_2 \alpha_1 x_1 + \alpha_1 \iota^*(u_3).$$

For $\alpha_1 = x_1, y_1$, since $x_1^2 = y_1^2 = 0$, $x_1 y_1 = \iota^*(u_2)$ and $y_1 x_1 = -\iota^*(u_2)$, we have

$$d_3(x_1 z_2) = -x_1 x_2 y_1 + x_1 \iota^*(u_3) = -\iota^*(u_3)x_1 - x_2 \iota^*(u_2)$$
$$d_3(y_1 z_2) = y_1 x_1 y_2 + y_1 \iota^*(u_3) = -\iota^*(u_3)y_1 - y_2 \iota^*(u_2).$$

Since $\iota^*(u_3)x_1$ is in N, $\mathrm{pr}_3(\iota^*(u_3)x_1) = 0$. By definition, $\mathrm{pr}_3(\iota^*(u_3)y_1) = 0$. Therefore, we have

$$\mathrm{pr}_3(d_3(v_1 z_2)) = v_1 y_2 x_1 + v_1 \iota^*(u_3),$$
$$\mathrm{pr}_3(d_3(w_1 z_2)) = w_1 y_2 x_1 + w_1 \iota^*(u_3),$$
$$\mathrm{pr}_3(d_3(x_1 z_2)) = \quad\quad - x_2 \iota^*(u_2),$$
$$\mathrm{pr}_3(d_3(y_1 z_2)) = \quad\quad - y_2 \iota^*(u_2).$$

Since $v_1 y_2 x_1$ and $w_1 y_2 x_1$ are linearly independent in $R'\{x_1\}/N$, and $\iota^*(u_2)x_2$ and $\iota^*(u_2)y_2$ are linearly independent in $\mathbb{Z}/p\{x_2, y_2\} \otimes H^2(B\tilde{\pi}(H_2)) \subset R''\{1\}$, the four

elements

$$d_3(v_1 z_2), \quad d_3(w_1 z_2), \quad d_3(x_1 z_2), \quad d_3(y_1 z_2)$$

are linearly independent in $E_3^{*,0} = R'\{x_1, y_1\} \oplus R''\{1\}$. Hence, the homomorphism $d_3 : E_3^{1,2} \to E_3^{4,0}$ is injective. Therefore, we have $E_4^{1,2} = \{0\}$.

The E_3-term $E_3^{2,2}$ is spanned by

$$m^{(i)} z_2, \quad \iota^*(u_2) z_2, \quad x_2 z_2, \quad y_2 z_2, \quad v_1 x_1 z_2, \quad v_1 y_1 z_2, \quad w_1 x_1 z_2, \quad w_1 y_1 z_2.$$

For $\alpha_2 = m^{(i)}, \iota^*(u_2), x_2, y_2, v_1 x_1, v_1 y_1, w_1 x_1, w_1 y_1 \in E_3^{2,0}$, since $d_3(\alpha_2)$ is in $E_3^{5,-2} = \{0\}$ we have

$$d_3(\alpha_2 z_2) = \alpha_2 d_3(z_2) = \alpha_2 x_2 y_1 - \alpha_2 x_1 y_2 - \alpha_2 \iota^*(u_3).$$

For $\alpha_2 = m^{(i)}, \iota^*(u_2), x_2, y_2$, since $\mathrm{pr}_3(\alpha_2 x_2 y_1) = 0$ by definition, we have

$$\mathrm{pr}_3(d_3(\alpha_2 z_2)) = -\alpha_2 y_2 x_1 - \alpha_2 \iota^*(u_3).$$

Thus, we have

$$\mathrm{pr}_3(d_3(m^{(i)} z_2)) = -y_2 m^{(i)} x_1 - m^{(i)} \iota^*(u_3),$$
$$\mathrm{pr}_3(d_3(x_2 z_2)) = -x_2 y_2 x_1 - x_2 \iota^*(u_3),$$
$$\mathrm{pr}_3(d_3(y_2 z_2)) = -y_2^2 x_1 - y_2 \iota^*(u_3).$$

Moreover, since $\iota^*(u_2)\iota^*(u_3) = \iota^*(u_2 u_3) = 0$ in $H^*(B\tilde{\pi}(H_2))$ by Proposition 3.2, and since $\iota^*(u_2) x_1 = \iota^*(u_2) y_1 = 0$ in $R'\{x_1, y_1\}$, we have

$$d_3(\iota^*(u_2) z_2) = 0.$$

For $\alpha_1 = v_1, w_1$, using the relations $x_1^2 = y_1^2 = 0$, $x_1 y_1 = \iota^*(u_2)$ and $y_1 x_1 = -\iota^*(u_2)$ in E_3, we have

$$d_3(\alpha_1 x_1 z_2) = \alpha_1 x_1 x_2 y_1 - \alpha_1 x_1 x_1 y_2 - \alpha_1 x_1 \iota^*(u_3) = \alpha_1 \iota^*(u_3) x_1 + x_2 \alpha_1 \iota^*(u_2)$$
$$d_3(\alpha_1 y_1 z_2) = \alpha_1 y_1 x_2 y_1 - \alpha_1 y_1 x_1 y_2 - \alpha_1 y_1 \iota^*(u_3) = \alpha_1 \iota^*(u_3) y_1 + y_2 \alpha_1 \iota^*(u_2).$$

Since $\alpha_1 \iota^*(u_3) \in H^*(B\tilde{\pi}(H_2))/(\iota^*(u_2))$, we obtain $\alpha_1 \iota^*(u_3) x_1 \equiv 0$ in $R'\{x_1\}/N$, hence $\mathrm{pr}_3(\alpha_1 \iota^*(u_3) x_1) = 0$. Moreover, $\mathrm{pr}_3(\alpha_1 \iota^*(u_3) y_1) = 0$ by definition. So, we have

$$\mathrm{pr}_3(d_3(\alpha_1 x_1 z_2)) = \alpha_1 x_2 \iota^*(u_2) = x_2 \alpha_1 \iota^*(u_2)$$
$$\mathrm{pr}_3(d_3(\alpha_1 y_1 z_2)) = \alpha_1 y_2 \iota^*(u_2) = y_2 \alpha_1 \iota^*(u_2).$$

By Proposition 5.2, $w_1 \iota^*(u_2) = 0$. Hence, we have

$$d_3(w_1 x_1 z_2) = w_1 \iota^*(u_3) x_1$$
$$d_3(w_1 y_1 z_2) = w_1 \iota^*(u_3) y_1.$$

Furthermore, by Proposition 5.2, $Q_0(w_1\iota^*(u_2)) = Q_0 w_1 \cdot \iota^*(u_2) - w_1\iota^*(u_3) = 0$ in $H^*(B\tilde{\pi}(H_2))$, hence $w_1\iota^*(u_3)x_1 = (Q_0 w_1)\iota^*(u_2)x_1 = 0$ in $R'\{x_1, y_1\} \subset E_3^{*,0}$. Thus, we obtain $d_3(w_1 x_1 z_2) = 0$. Similarly, we also have $d_3(w_1 y_1 z_2) = 0$. Thus, we have

$$\mathrm{pr}_3(d_3(v_1 x_1 z_2)) = x_2 v_1 \iota^*(u_2),$$

$$\mathrm{pr}_3(d_3(v_1 y_1 z_2)) = y_2 v_1 \iota^*(u_2),$$

and

$$d_3(w_1 x_1 z_2) = 0,$$

$$d_3(w_1 y_1 z_2) = 0.$$

Since $y_2 m^{(i)} x_1$, $x_2 y_2 x_1$ and $y_2^2 x_1$ are linearly independent in $R'\{x_1\}/N$ and, by Proposition 5.1, $x_2 v_1 \iota^*(u_2)$ and $y_2 v_1 \iota^*(u_2)$ are linearly independent in

$$\mathbb{Z}/p\{x_2, y_2\} \otimes H^3(B\tilde{\pi}(H_2)) \subset R''\{1\},$$

the kernel of $\mathrm{pr}_3 \circ d_3$ is spanned by $\iota^*(u_2)z_2$, $w_1 x_1 z_2$ and $w_1 y_1 z_2$, and, since these are in the kernel of d_3, the kernel of d_3 is spanned by these elements. Moreover, the image $d_3 : E_3^{-1,4} \to E_3^{2,2}$ is trivial. Therefore, we obtain

$$E_4^{2,2} = \mathbb{Z}/p\{\iota^*(u_2)z_2, w_1 x_1 z_2, w_1 y_1 z_2\} = \mathbb{Z}/p\{x_1 y_1 z_2, w_1 x_1 z_2, w_1 y_1 z_2\},$$

where $\iota^*(u_2)z_2 = x_1 y_1 z_2$.

Finally, we compute the E_∞-terms $E_\infty^{0,3}$, $E_\infty^{1,2}$, $E_\infty^{2,1}$ and $E_\infty^{0,4}$, $E_\infty^{1,3}$, $E_\infty^{2,2}$. Since $E_3^{0,3} = E_4^{1,2} = \{0\}$, we have $E_\infty^{0,3} = E_\infty^{1,2} = \{0\}$. Similarly, since $E_4^{0,4} = E_3^{1,3} = \{0\}$, we have $E_\infty^{0,4} = E_\infty^{1,3} = \{0\}$. Since the Leray–Serre spectral sequence is the first quadrant spectral sequence, for $s \leq r - 1$ and $t \leq r - 2$,

$$E_r^{s-r,t+r-1} = E_r^{s+r,t-r+1} = \{0\},$$

and the differentials

$$d_r : E_r^{s-r,t+r-1} \to E_r^{s,t}, \quad d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$$

are trivial. Hence, we have $E_r^{s,t} = E_\infty^{s,t}$ for $s \leq r - 1$ and $t \leq r - 2$. In particular, $E_3^{s,t} = E_\infty^{s,t}$ for $s \leq 2$ and $t \leq 1$, and $E_4^{s,t} = E_\infty^{s,t}$ for $s \leq 3$ and $t \leq 2$. Hence, we have $E_\infty^{2,1} = E_3^{2,1}$ and $E_\infty^{2,2} = E_4^{2,2}$. $\qquad\square$

In Section 4, we defined $x_4 \in H^4(BG; \mathbb{Z})$, so that $c_2(\lambda'') = px_4$ in $H^4(BG; \mathbb{Z})$. Therefore, to show that $c_2(\lambda'' \circ f) \neq 0$ in $H^4(BH; \mathbb{Z})$ it is equivalent to show that $pf^*(x_4) \neq 0$ in $H^4(BH; \mathbb{Z})$. Hence, in order to prove (1), it suffices to show that the mod p reduction $\rho(f^*(x_4)) \in H^4(BH)$ of $f^*(x_4) \in H^4(BH; \mathbb{Z})$ is not in the image of the Bockstein homomorphism. So, we prove the following proposition:

Proposition 6.2. *The cohomology class $f^*(\rho(x_4))$ is not in the image of the Bockstein homomorphism*

$$Q_0 : H^3(BH) \to H^4(BH).$$

Proof. Since $E_\infty^{0,4} = E_\infty^{1,3} = \{0\}$, we have $F^2 H^4(BH) = H^4(BH)$. Similarly, since $E_\infty^{0,3} = E_\infty^{1,2} = \{0\}$, we have $F^2 H^3(BH) = H^3(BH)$. Hence, we have

$$Q_0 H^3(BH) \subset F^2 H^4(BH)$$

and each cohomology class in $Q_0 H^3(BH)$ represents an element in

$$E_\infty^{2,2} = F^2 H^4(BH)/F^3 H^4(BH).$$

Since $E_\infty^{2,1}$ is spanned by $w_1 x_1 z_1$ and $w_1 y_1 z_1$, using the properties of the vertical operation $\beta \wp^0$ constructed by Araki [1957, Corollary 4.1] in the spectral sequence of a fibration, we have that if x is in $Q_0 H^3(BH)$ then x represents a linear combination of $w_1 x_1 z_2$ and $w_1 y_1 z_2$ in $E_\infty^{2,2}$.

On the other hand, by Proposition 4.2, $\rho(x_4) \in H^4(BG)$ represents $\alpha b_2 z_2$ in $E_\infty^{2,2}(BG)$, where $\alpha \neq 0$ is in $\mathbb{Z}/p$. Using Proposition 3.3 and the definition of b_2 in Section 4, we have $f^*(b_2) = x_1 y_1$. Therefore, $f^*(\rho(x_4))$ represents $\alpha x_1 y_1 z_2$ in $E_\infty^{2,2}$. Hence, $f^*(\rho(x_4))$ is not in the image of the Bockstein homomorphism Q_0.
□

Remark 6.3. If we replace H_2 by the extraspecial p-group p_+^{1+2}, then (1) does not hold. To be more precise, $f^*(\rho(x_4))$ is in the image of the Bockstein homomorphism $Q_0 : H^3(Bp_+^{1+4}) \to H^4(Bp_+^{1+4})$ and $c_2(\lambda'' \circ f) = pf^*(x_4) = 0$ in $H^4(Bp_+^{1+4}; \mathbb{Z})$.

Finally, we prove (2) by proving the following proposition:

Proposition 6.4. *There exists no virtual complex representation*

$$\mu : BH \to \mathbb{Z} \times BU$$

such that $c_2(\lambda'' \circ f) \in p \cdot \operatorname{Im} \mu^*$.

Proof. We prove this by contradiction. Suppose that there exists a virtual complex representation

$$\mu : BH \to \mathbb{Z} \times BU$$

such that $c_2(\lambda'' \circ f) \in p \cdot \operatorname{Im} \mu^*$. Then, $p(\mu^*(y_4) - f^*(x_4)) = 0$ for some y_4 in $H^4(\mathbb{Z} \times BU; \mathbb{Z})$. Since Q_1 acts trivially on $H^*(\mathbb{Z} \times BU)$, we have

$$Q_1 \rho(\mu^*(y_4)) = 0.$$

In what follows, we show that

$$Q_1 \rho(\mu^*(y_4)) \neq 0,$$

which proves the proposition.

Since $p(\mu^*(y_4) - f^*(x_4)) = 0$, $\rho(\mu^*(y_4) - f^*(x_4))$ is in the image of the Bockstein homomorphism, that is, as in the proof of Proposition 6.2, $\rho(\mu^*(y_4) - f^*(x_4))$ represents

$$\alpha_1 w_1 x_1 z_2 + \alpha_2 w_1 y_1 z_2$$

in $E_\infty^{2,2}$ for some $\alpha_1, \alpha_2 \in \mathbb{Z}/p$. We already verified that $f^*(\rho(x_4)) = \rho(f^*(x_4))$ represents $\alpha x_1 y_1 z_2 \in E_\infty^{2,2}$, where $\alpha \neq 0$, in the proof of Proposition 6.2. So, $\rho(\mu^*(y_4))$ represents

$$\alpha x_1 y_1 z_2 + \alpha_1 w_1 x_1 z_2 + \alpha_2 w_1 y_1 z_2$$

in $E_\infty^{2,2}$ and $\alpha \neq 0$.

We recall the structure of H_2 defined in Section 2. Also, we recall the diagram

$$\begin{array}{ccccc}
A_3 & \xrightarrow{\ g\ } & H & \xleftarrow{\ g'\ } & A_3' \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\varphi'} \\
A_2 & \xrightarrow{\ g\ } & A_2 \times \tilde{\pi}(H_2) & \xleftarrow{\ g'\ } & A_2
\end{array}$$

where the upper g and g' are the obvious inclusions, $A_2 = \langle \tilde{\pi}(\alpha), \tilde{\pi}(\beta) \rangle$,

$$g(\tilde{\pi}(\alpha)) = (\tilde{\pi}(\alpha), \tilde{\pi}(\alpha)), \quad g(\tilde{\pi}(\beta)) = (\tilde{\pi}(\beta), \tilde{\pi}(\beta)),$$
$$g'(\tilde{\pi}(\alpha)) = (\tilde{\pi}(\alpha), 1), \quad g'(\tilde{\pi}(\beta)) = (1, \tilde{\pi}(\beta)).$$

In Section 5, we defined $w_1 \in H^1(B\tilde{\pi}(H_2))$, so that the induced homomorphism $H^1(B\tilde{\pi}(H_2)) \to H^1(B\langle\tilde{\pi}(\beta)\rangle)$ maps w_1 to the element corresponding to the generator $\tilde{\pi}(\beta)$. So, we see that the induced homomorphisms g^* and g'^* satisfy

$$g^*(x_1) = x_1, \quad g^*(y_1) = y_1, \quad g^*(w_1) = y_1,$$
$$g'^*(x_1) = x_1, \quad g'^*(y_1) = 0, \quad g'^*(w_1) = y_1.$$

Therefore, $g^*(\rho(\mu^*(y_4))) \in H^4(BA_3)$ represents

$$g^*(\alpha x_1 y_1 z_2 + \alpha_1 w_1 x_1 z_2 + \alpha_2 w_1 y_1 z_2) = \alpha x_1 y_1 z_2 + \alpha_1 y_1 x_1 z_2 = (\alpha - \alpha_1) x_1 y_1 z_2$$

in the spectral sequence for $H^*(BA_3)$ and $g'^*(\rho(\mu^*(y_4))) \in H^4(BA_3')$ represents

$$g'^*(\alpha x_1 y_1 z_2 + \alpha_1 w_1 x_1 z_2 + \alpha_2 w_1 y_1 z_2) = \alpha_1 y_1 x_1 z_2 = -\alpha_1 x_1 y_1 z_2$$

in the spectral sequence for $H^*(BA_3')$.

As in the proof of Proposition 4.1, let M be the $\varphi^*(H^*(BA_2))$-submodule of $H^*(BA_3)$ and M' the $\varphi'^*(H^*(BA_2))$-submodule of $H^*(BA_3')$ generated by

$$1, \quad z_1, \quad z_1 z_2, \quad z_2^i, \quad z_1 z_2^i \quad (i \geq 2),$$

where $\varphi : BA_3 \to BA_1$ and $\varphi' : BA'_3 \to BA_2$ are the maps defined in Section 2, so that

$$H^*(BA_3)/M = \varphi^*(H^*(BA_2))\{z_2\} = \mathbb{Z}/p[x_2, y_2] \otimes \Lambda(x_1, y_1)\{z_2\},$$
$$H^*(BA'_3)/M' = \varphi'^*(H^*(BA_2))\{z_2\} = \mathbb{Z}/p[x_2, y_2] \otimes \Lambda(x_1, y_1)\{z_2\},$$

respectively. Since $Q_1 z_1 = z_2^p$, $Q_1 z_2 = 0$, and Q_1 is a derivation, M and M' are closed under the action of Milnor operation Q_1. We have

$$g^*(\rho(\mu^*(y_4))) \equiv (\alpha - \alpha_1)x_1 y_1 z_2 \mod M,$$
$$g'^*(\rho(\mu^*(y_4))) \equiv -\alpha_1 x_1 y_1 z_2 \mod M'.$$

and so

$$Q_1 g^*(\rho(\mu^*(y_4))) \equiv (\alpha - \alpha_1)(x_2^p y_1 - x_1 y_2^p)z_2 \mod M,$$
$$Q_1 g'^*(\rho(\mu^*(y_4))) \equiv -\alpha_1(x_2^p y_1 - x_1 y_2^p)z_2 \mod M'.$$

Since $\alpha \neq 0$, at least one of $\alpha - \alpha_1$ and $-\alpha_1$ is nonzero. Therefore, we have

$$Q_1 \rho(\mu^*(y_4)) \neq 0.$$

This completes the proof. $\square$

Acknowledgements

References

[Antieau and Williams 2014] B. Antieau and B. Williams, "The topological period-index problem over 6-complexes", *J. Topol.* **7**:3 (2014), 617–640. MR Zbl

[Araki 1957] S. Araki, "Steenrod reduced powers in the spectral sequences associated with a fibering, II", *Mem. Fac. Sci. Kyusyu Univ. Ser. A. Math.* **11** (1957), 81–97. MR Zbl

[Baum and Browder 1965] P. F. Baum and W. Browder, "The cohomology of quotients of classical groups", *Topology* **3** (1965), 305–336. MR Zbl

[Kameko 2015] M. Kameko, "On the integral Tate conjecture over finite fields", *Math. Proc. Cambridge Philos. Soc.* **158**:3 (2015), 531–546. MR

[Kameko and Yagita 2008] M. Kameko and N. Yagita, "The Brown–Peterson cohomology of the classifying spaces of the projective unitary groups PU(p) and exceptional Lie groups", *Trans. Amer. Math. Soc.* **360**:5 (2008), 2265–2284. MR Zbl

[Kono et al. 1975] A. Kono, M. Mimura, and N. Shimada, "Cohomology of classifying spaces of certain associative H-spaces", *J. Math. Kyoto Univ.* **15**:3 (1975), 607–617. MR Zbl

[Totaro 1999] B. Totaro, "The Chow ring of a classifying space", pp. 249–281 in *Algebraic K-theory* (Seattle, WA, 1997), edited by W. Raskind and C. Weibel, Proc. Sympos. Pure Math. **67**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Totaro 2014] B. Totaro, *Group cohomology and algebraic cycles*, Cambridge Tracts in Mathematics **204**, Cambridge University Press, 2014. MR Zbl

[Vistoli 2007] A. Vistoli, "On the cohomology and the Chow ring of the classifying space of PGL_p", *J. Reine Angew. Math.* **610** (2007), 181–227. MR Zbl

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