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# On the vanishing of Hochster's $\theta$ invariant

Mark E. Walker

Hochster's theta invariant is defined for a pair of finitely generated modules on a hypersurface ring having only an isolated singularity. Up to a sign, it agrees with the Euler invariant of a pair of matrix factorizations.

Working over the complex numbers, Buchweitz and van Straten have established an interesting connection between Hochster's theta invariant and the classical linking form on the link of the singularity. In particular, they establish the vanishing of the theta invariant if the hypersurface is even-dimensional by exploiting the fact that the (reduced) cohomology of the Milnor fiber is concentrated in odd degrees in this situation.

We give purely algebraic versions of some of these results. In particular, we establish the vanishing of the theta invariant for isolated hypersurface singularities of even dimension in characteristic $p > 0$ under some mild extra assumptions. This confirms, in a large number of cases, a conjecture of Hailong Dao.



## 1. Introduction

In this paper, a *hypersurface* will refer to a ring $R$ that can be expressed as a quotient of a regular (Noetherian) ring $Q$ by a non-zero-divisor $f$; i.e., $R = Q/f$.

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(We do not require $Q$ to be local.) If $M$ and $N$ are finitely generated $R$-modules then, using the standard long exact sequence

$$\cdots \to \mathrm{Tor}^Q_{i-1}(M, N) \to \mathrm{Tor}^R_i(M, N) \to \mathrm{Tor}^R_{i-2}(M, N) \to \mathrm{Tor}^Q_{i-2}(M, N) \to \cdots$$

and the fact that $\mathrm{Tor}^Q_j(M, N) = 0$ for $j \gg 0$ since $Q$ is regular, we conclude that $\mathrm{Tor}^R_*(M, N)$ is eventually two-periodic: there is an isomorphism

$$\mathrm{Tor}^R_i(M, N) \xrightarrow{\cong} \mathrm{Tor}^R_{i+2}(M, N) \quad \text{for } i \gg 0.$$

If we assume, in addition, that $\mathrm{Tor}^R_i(M, N)$ has finite length for all $i \gg 0$, then Hochster's *theta invariant* [1981] of the pair $(M, N)$ is defined to be the integer

$$\theta^R(M, N) = \mathrm{length}\,\mathrm{Tor}^R_{2i}(M, N) - \mathrm{length}\,\mathrm{Tor}^R_{2i+1}(M, N) \quad \text{for } i \gg 0.$$

The modules $\mathrm{Tor}^R_i(M, N)$ will have finite length for $i \gg 0$ if, for example, there exists a maximal ideal $\mathfrak{m}$ of $R$ such that one of the modules, say $N$, is locally of finite projective dimension on the punctured spectrum $\mathrm{Spec}(R) \setminus \mathfrak{m}$. In this situation, $N$ determines a coherent sheaf on the quasiaffine scheme $\mathrm{Spec}(R) \setminus \mathfrak{m}$ that admits a finite resolution by locally free coherent sheaves, and hence $N$ determines a class in $[N] \in K_0(\mathrm{Spec}(R) \setminus \mathfrak{m})$. The module $M$, of course, determines a class $[M] \in G_0(R)$. Hochster [1981, Theorem 1.2] proves that $\theta$ is biadditive for such pairs of modules and hence determines a pairing

$$G_0(R) \times K_0(\mathrm{Spec}\, R \setminus \mathfrak{m}) \to \mathbb{Z}.$$

Moreover, $\theta(R/\mathfrak{m}, -)$ is identically zero and, since $G_0(R) \to G_0(\mathrm{Spec}(R) \setminus \mathfrak{m})$ is surjective with kernel generated by $[R/\mathfrak{m}]$, we obtain an induced pairing

$$\theta = \theta_{(Q,\mathfrak{m},f)} : G_0(\mathrm{Spec}(R) \setminus \mathfrak{m}) \times K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}) \to \mathbb{Z}. \tag{1.1}$$

We will refer to this as *Hochster's $\theta$ pairing associated to the data* $(Q, f, \mathfrak{m})$.

Buchweitz and van Straten [2012] relate Hochster's $\theta$ pairing for isolated hypersurface singularities of the form $R = \mathbb{C}\{x_0, \dots, x_n\}/(f)$, where $\mathbb{C}\{x_0, \dots, x_n\}$ is the ring of convergent power series, to the linking form on the link of the singularity. Using this relationship, they also prove the $\theta$ pairing vanishes when $\dim(R)$ is even.

The goal of this paper is to give a purely algebraic interpretation of some of the results of Buchweitz and van Straten, ones which are also valid in characteristic $p > 0$. In particular, we obtain the vanishing of $\theta$ for isolated hypersurface singularities of even dimension in all characteristics for a large number of rings, confirming in many cases a conjecture of H. Dao [2013, Conjecture 3.15]. We refer the reader to Corollaries 7.16 and 7.17 for the most general statements, but an important special case of our results is given by the following theorem:

**Theorem 1.2.** *Let $k$ be a perfect field, $Q$ a finitely generated and regular $k$-algebra, and $f \in Q$ a non-zero-divisor. Assume the associated morphism of affine varieties*

$$f : \mathrm{Spec}(Q) \to \mathbb{A}^1_k$$

*has only isolated singularities. Set $R = Q/f$.*

*If* $\dim(R)$ *is even, then $\theta^R(M, N) = 0$ for all finitely generated $R$-modules $M$ and $N$.*

The theorem may be extended easily to allow for localizations of smooth algebras, and thus justifies examples such as the following:

**Example 1.3.** Let $k$ be a perfect field, let $f \in k[x_0, \dots, x_n]$ be a polynomial in $n+1$ variables contained in $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$, and assume $\langle \partial f/\partial x_0, \dots, \partial f/\partial x_n \rangle_{\mathfrak{m}}$ is an $\mathfrak{m}$-primary ideal of $k[x_0, \dots, x_n]_{\mathfrak{m}}$. Set $R = k[x_0, \dots, x_n]_{\mathfrak{m}}/f$.

If $n$ is even, then $\theta^R(M, N) = 0$ for all pairs of finitely generated $R$-modules $M$ and $N$.

One technical aspect of this paper may be of independent interest: in Section 3, we prove a slightly weakened form of a conjecture of C. Sherman [2004, §2] concerning his "star pairing" in algebraic $K$-theory.

## 2. Overview

We give an overview of this paper by comparing the details we use with those occurring in the work of Buchweitz and van Straten.

***The Milnor fibration.*** Given a power series $f \in \mathbb{C}\{x_0, \dots, x_n\}$ with positive radius of convergence, we interpret $f$ as defining a holomorphic function defined on an open neighborhood of 0 in $\mathbb{C}^{n+1}$. Let $\bar{B}$, the *Milnor ball*, be a closed ball of radius $\epsilon$ centered at the origin of $\mathbb{C}^{n+1}$ and let $D$ be an open disk of radius $\delta$ centered at the origin in the complex plane $\mathbb{C}$. Here, $\epsilon$ is chosen first and to be sufficiently small, and $\delta$ is chosen to be sufficiently smaller than $\epsilon$: $0 < \delta \ll \epsilon \ll 1$. Set $X = f^{-1}(D) \cap \bar{B}$ and also write $f$ for the restricted function $f : X \to D$.

Let us assume that the fiber $X_0$ of $f$ over $0 \in D$ is an isolated singularity at the origin. It follows that $X_0$ is homeomorphic to the cone over the *link* $L$ of the singularity, defined as $L := X_0 \cap S$, where $S := \partial \bar{B} \cong S^{2n+1}$ is the *Milnor sphere*. The link $L$ is a smooth orientable manifold of real dimension $2n - 1$.

The map away from the singular locus,

$$X^* := (X \setminus X_0) \to (D \setminus 0) =: D^*,$$

induced by $f$ is a fibration, called the *Milnor fibration*. For any $t \neq 0$ in $D$, the fiber $X_t$ of $f$ over $t$ is a smooth manifold with boundary of (real) dimension $2n$. Up to diffeomorphism, $X_t$ is independent of $t$, and it is called the *Milnor fiber* of the singularity. A key result of Milnor [1968] gives that the Milnor fiber has the homotopy type of a bouquet of $n$-dimensional spheres; the number of spheres is the *Milnor number*, often written as $\mu$. In particular, the reduced singular cohomology of the Milnor fiber, $\widetilde{H}^*(X_t)$, is a free abelian group of rank $\mu$ concentrated in degree $n$.

The restriction of $f$ to $S \cap X$ is a fibration and thus, up to diffeomorphism, we may identify the boundary of the Milnor fiber, $\partial X_t = (f|_{X \cap S})^{-1}(t)$, with the link $L = (f|_{X \cap S})^{-1}(0)$. In particular, for each $t \in D^*$ there is a continuous map

$$\rho_t : L \to X^*$$

given by composing the diffeomorphism $L \cong \partial X_t$ with the inclusion $\partial X_t \subseteq X^*$.

***The vanishing of the pairing when n is even.*** With the above notation set, we sketch the proof of Buchweitz and van Straten for the vanishing of $\theta$ when $n$ is even.

Associated to maximal Cohen–Macaulay (MCM) $\mathbb{C}\{x_0, \dots, x_n\}/f$-modules $M$ and $N$, Buchweitz and van Straten associate classes in topological $K$-theory:

$$\alpha(M) \in K^1_{\mathrm{top}}(X^*) \quad \text{and} \quad [N]_{\mathrm{top}} \in K^0_{\mathrm{top}}(L).$$

The class $[N]_{\mathrm{top}}$ is given by the evident restriction to $L$ of the coherent sheaf associated to $N$ and the class $\alpha(M)$ is built from a *matrix factorization* representation of $M$. They prove [Buchweitz and van Straten 2012, Theorem 4.2] that

$$\theta(M, N) = \chi_{\mathrm{top}}\big(\rho_t^*(\alpha(M)) \cup [N]_{\mathrm{top}}\big).$$

Here, $\rho_t^* : K^1_{\mathrm{top}}(X^*) \to K^1_{\mathrm{top}}(L)$ is the map induced by the map $\rho_t : L \rightarrowtail X^*$ defined above, $\cup$ is the product operation for the ring $K^*_{\mathrm{top}}$, and

$$\chi_{\mathrm{top}} : K^1_{\mathrm{top}}(L) \to K^0_{\mathrm{top}}(\mathrm{pt}) \cong \mathbb{Z}$$

is induced by push-forward. (Recall $L$ is odd-dimensional and so $\chi_{\mathrm{top}}$ switches the parity of degrees.)

Notice that $\rho_t^*$ factors through $K^1_{\mathrm{top}}(X_t)$, since $\rho_t$ factors through $X_t$ by its very construction. When $n$ is even, we have $K^1_{\mathrm{top}}(X_t) = 0$, since $X_t$ is a bouquet of $n$-dimensional spheres by Milnor's theorem [1968]. It follows that $\rho_t^*(\alpha(M)) = 0$, and hence

$$\theta(M, N) = \chi_{\mathrm{top}}\big(\rho_t^*(\alpha(M)) \cup [N]_{\mathrm{top}}\big) = 0.$$

***The algebraic analogue of the Milnor fibration.*** The algebraic analogue of the Milnor fibration is constructed as follows: Assume $V$ is a Henselian DVR with algebraically closed residue field $k$ and field of fractions $F$. For example, $V$ could be $k[[t]]$ with $k = \bar{k}$. The affine scheme $\mathrm{Spec}(V)$ is the analogue of $D$, a small open disk in the complex plane. Let $Q$ be a flat $V$-algebra of finite type and assume the associated morphism of affine schemes

$$\operatorname{Spec}(Q) \to \operatorname{Spec}(V)$$

is smooth except at a single point $\mathfrak{m} \in \operatorname{Spec}(Q)$, necessarily belonging to the closed fiber. Let $Q_{\mathfrak{m}}^{\text{hen}}$ denote the Henselization of $Q$ at $\mathfrak{m}$. The affine scheme $\operatorname{Spec}(Q_{\mathfrak{m}}^{\text{hen}})$ is the analogue of $X = f^{-1}(D) \cap \bar{B}$ in the notation above. Thus the morphism

$$X^{\text{alg}} := \operatorname{Spec}(Q_{\mathfrak{m}}^{\text{hen}}) \to \operatorname{Spec}(V) =: D^{\text{alg}}$$

is the algebraic analogue of the analytic map $f : X \to D$ considered above. The generic fiber of $X^{\text{alg}} \to D^{\text{alg}}$,

$$X_{\text{alg}}^{*} := \operatorname{Spec}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V F) \to \operatorname{Spec}(F) =: D_{\text{alg}}^{*},$$

is the analogue of the Milnor fibration, and the geometric generic fiber,

$$X_t^{\text{alg}} := \operatorname{Spec}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}),$$

where $\bar{F}$ is the algebraic closure of $F$, is the analogue of the Milnor fiber.

**Remark 2.1.** It is important to be aware that $Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}$ need not be a Noetherian ring.

The singularity is the closed fiber of $X^{\text{alg}} \to D^{\text{alg}}$,

$$X_0^{\text{alg}} := \operatorname{Spec}(R_{\mathfrak{m}}^{\text{hen}}) = \operatorname{Spec}(Q_{\mathfrak{m}}^{\text{hen}}/f),$$

where $f$ denotes the image in $Q$ of a specified uniformizing local parameter $t \in V$ and we set $R = Q/f$. The algebraic analogue of the link is the punctured spectrum of the singularity,

$$L^{\text{alg}} := \operatorname{Spec}(R_{\mathfrak{m}}^{\text{hen}}) \setminus \mathfrak{m}.$$

We summarize these constructions and introduce a few more analogies in Table 1.

**Remark 2.2.** The algebraic analogue of the Milnor ball and the algebraic analogue of $X$ coincide, in contrast with what occurs in the analytic setting. The analogy could be slightly improved if one does not assume from the start that $V$ is Henselian. Then the algebraic analogue the Milnor ball remains $\operatorname{Spec}(Q_{\mathfrak{m}}^{\text{hen}})$ but the algebraic analogue of $X \to D$ would become $\operatorname{Spec}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V V^{\text{hen}}) \to \operatorname{Spec}(V^{\text{hen}})$, where $V^{\text{hen}}$ denotes the Henselization of $V$ at its unique maximal ideal. For this paper, however, this more general construction has no advantage.

With this notation fixed, let us sketch our proof of the vanishing of the $\theta$ pairing for $R = Q/f$.

Given finitely generated MCM $R$-modules $M$ and $N$, we define classes

$$\alpha^{\text{alg}}(M) \in K_1(X_{\text{alg}}^{*}) = K_1(Q[1/f])$$

| geometric notion | algebraic analogue |
|---|---|
| $D$, a small open disc in complex plane | $D^{\mathrm{alg}} := \mathrm{Spec}(V)$, where $V$ is a Henselian DVR with uniformizing parameter $t$, algebraically closed residue field $k$, and field of fractions $F$ |
| convergent power series $f \in \mathbb{C}\{x_0, \dots, x_n\}$ representing an isolated singularity | a flat, finite-type ring map $V \to Q$ sending $t \in V$ to $f \in Q$ that is smooth away from some point $\mathfrak{m}$ in the closed fiber |
| $X$, the intersection of a small closed ball in $\mathbb{C}^{n+1}$ with $f^{-1}(D)$ | $X^{\mathrm{alg}} = \mathrm{Spec}(Q^{\mathrm{hen}}_{\mathfrak{m}})$ |
| Milnor fibration $X^* \to D^*$ | the generic fiber of $X^{\mathrm{alg}} \to D^{\mathrm{alg}}$; namely, $\mathrm{Spec}(Q^{\mathrm{hen}} \otimes_V F) \to \mathrm{Spec}(F)$ |
| the Milnor fiber $X_t := f^{-1}(t)$, $t \neq 0$ | the geometric generic fiber of $X^{\mathrm{alg}} \to D^{\mathrm{alg}}$; namely, $X^{\mathrm{alg}}_t := \mathrm{Spec}(Q^{\mathrm{hen}}_{\mathfrak{m}} \otimes_V \bar{F})$ |
| the singularity $X_0 = f^{-1}(0)$ | $X^{\mathrm{alg}}_0 := \mathrm{Spec}(R^{\mathrm{hen}}_{\mathfrak{m}})$, where $R = Q/f$ (i.e., the closed fiber of $X^{\mathrm{alg}} \to D^{\mathrm{alg}}$) |
| the link $L$ | the punctured spectrum $\mathrm{Spec}(R^{\mathrm{hen}}_{\mathfrak{m}}) \setminus \mathfrak{m} =: L^{\mathrm{alg}}$ |
| Milnor ball $\bar{B}$ | $\bar{B}^{\mathrm{alg}} = \mathrm{Spec}(Q^{\mathrm{hen}}_{\mathfrak{m}})$ |
| Milnor sphere $S = \partial \bar{B}$ | $S^{\mathrm{alg}} = \mathrm{Spec}(Q^{\mathrm{hen}}_{\mathfrak{m}}) \setminus \mathfrak{m}$ |
| $\rho^*_t : K^1_{\mathrm{top}}(X^*) \to K^1_{\mathrm{top}}(L)$ | the "specialization map" in $K$-theory with finite coefficients. |

**Table 1.** Summary of geometric notions and their algebraic analogues.

and

$$[N] \in K_0(L^{\mathrm{alg}}) = K_0(\mathrm{Spec}(R_{\mathfrak{m}}) \setminus \mathfrak{m}),$$

where $[N]$ is defined as the image of the class of $N$ in $G_0(R)$ under the restriction map $G_0(R) \to G_0(\mathrm{Spec}(R) \setminus \mathfrak{m}) \cong K_0(\mathrm{Spec}(R) \setminus \mathfrak{m})$ and $\alpha^{\mathrm{alg}}(M) = [A] \in K_1(Q[1/f])$ for any matrix factorization $(A, B)$ representation of $M$. Our first key result (see Corollary 4.10) is the equation

$$\theta^R(M, N) = \chi\big(\partial(\alpha(M) \cup [f]) \cup [N]\big). \tag{2.3}$$

Let us explain the components of this formula. Since $f$ is a unit of $Q[1/f]$, it determines a class $[f] \in K_1(Q[1/f])$. The symbol $\cup$ denotes the product operation in $K$-theory. The map $\partial : K_2(Q[1/f]) \to K_1(\mathrm{Spec}(R) \setminus \mathfrak{m})$ is the boundary map in the long exact localization sequence associated to the closed subscheme $L^{\mathrm{alg}} =$

$\operatorname{Spec}(R) \setminus \mathfrak{m}$ of $S^{\text{alg}} = \operatorname{Spec}(Q) \setminus \mathfrak{m}$. (Since $Q[1/f]$ and $\operatorname{Spec}(R) \setminus \mathfrak{m}$ are regular, we use, as we may, $K$-theory in place of $G$-theory.) Finally,

$$\chi : K_1(\operatorname{Spec}(R) \setminus \mathfrak{m}) \to \mathbb{Z}$$

is the composition of the boundary map

$$K_1(\operatorname{Spec}(R) \setminus \mathfrak{m}) \to K_0(\operatorname{Spec}(R/\mathfrak{m}))$$

with the canonical isomorphism $K_0(\operatorname{Spec}(R/\mathfrak{m})) \cong \mathbb{Z}$, sending $[R/\mathfrak{m}]$ to 1.

All of these facts remain true if we use $K$-theory with $\mathbb{Z}/l$ coefficients, where $l$ is a prime distinct from the characteristic of $Q/\mathfrak{m}$ (and, for technical reasons, $l \geq 5$). Moreover, we may replace algebraic $K$-theory with (a version of) étale $K$-theory with $\mathbb{Z}/l$ coefficients, which we write as $K_*^{\text{top}}(-, \mathbb{Z}/l)$.

A portion of the formula (2.3), using $K_*^{\text{top}}(-, \mathbb{Z}/l)$ instead of $K_*(-)$, is given by the mapping

$$K_1^{\text{top}}(X_{\text{top}}^*, \mathbb{Z}/l) = K_1^{\text{top}}(Q[1/f], \mathbb{Z}/l) \to K_1^{\text{top}}(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) = K_1^{\text{top}}(L^{\text{alg}}, \mathbb{Z}/l)$$

that sends $\alpha$ to $\partial(\alpha \cup [f])$. We interpret this map, under some additional hypotheses, as a "specialization map" in $K$-theory, and it is the analogue of the map $\rho_t^*$ occurring in the work of Buchweitz and van Straten. The next key result is that this specialization map factors through

$$K_1^{\text{top}}(X_t^{\text{alg}}, \mathbb{Z}/l) = K_1^{\text{top}}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l).$$

This factorization is the analogue of the factorization of $\rho_t^*$ through the $K$-theory of the Milnor fiber used in the work of Buchweitz and van Straten.

Finally, we combine a theorem of Rosenschon and Østvær [2006, Theorem 4.3] (which is a generalization of a celebrated theorem of Thomason [1985, Theorem 4.1]) and a theorem of Illusie [2003, Corollaire 2.10] to prove

$$K_1^{\text{top}}(X_t^{\text{alg}}, \mathbb{Z}/l) = 0$$

if $n$ is even (and some additional mild hypotheses hold). Illusie's theorem is in fact a good analogue of Milnor's theorem that the Milnor fiber of an isolated singularity is homotopy equivalent to a bouquet of $n$-spheres.

These results combine to prove the vanishing of $\theta$ for $n$ even. See Theorem 7.15 and its corollaries for the precise statement of our vanishing result.

Finally, when $n$ is odd, Buchweitz and van Straten prove that the $\theta$ pairing is induced by the linking form on the homology of the link of the singularity. Our Corollary 4.10 may be interpreted as analogue of this result in algebraic $K$-theory; see the discussion at the end of Section 4.

## 3. On Sherman's star pairing

Suppose $Q$ is a regular ring, $f \in Q$ is a non-zero-divisor, and $(A, B)$ is a *matrix factorization* of $f$. Recall that this means $A$ and $B$ are $m \times m$ matrices with entries in $Q$ such that $AB = fI_m = BA$. Note that $A$ may be regarded as an element of $\mathrm{GL}_m(Q[1/f])$ and hence it determines a class $[A] \in K_1(Q[1/f])$. The main result of this section, Corollary 3.11, gives an explicit description of the image of $[A]$ under the boundary map

$$K_1(Q[1/f]) \xrightarrow{\partial} G_0(Q/f)$$

in the long exact localization sequence in $G$-theory. (Since $Q[1/f]$ is regular, $K_1(Q[1/f]) \cong G_1(Q[1/f])$.) Our description of this image, and its proof, builds on work of C. Sherman, which we review here.

***Pairings in K-theory.*** We will need some results about various pairings in algebraic $K$-theory, including Sherman's *star pairing*.

For an exact category $\mathcal{P}$, an object $M$ of it, and a pair of commuting automorphisms $\alpha$ and $\beta$ of $M$, Sherman [2004] defines an element

$$\alpha \star_{\mathrm{Sh}} \beta \in K_2(\mathcal{P}).$$

In this definition, the group $K_2(\mathcal{P})$ is taken to be $\pi_2|G(\mathcal{P})|$ where $G(\mathcal{P})$ is a simplicial set, the "$G$-construction", defined by Gillet and Grayson [1987]. The reader is referred to their paper for the full definition, but let us recall the definition of zero, one and two simplices. A zero simplex in $G(\mathcal{P})$ is an ordered pair $(X, Y)$ of objects of $\mathcal{P}$. An edge (i.e., a one simplex) connecting $(X, Y)$ to $(X', Y')$ is given by a pair of short exact sequences of the form

$$0 \to X \xrightarrow{i} X' \xrightarrow{p} Z \to 0 \quad \text{and} \quad 0 \to Y \xrightarrow{j} Y' \xrightarrow{q} Z \to 0.$$

(Notice the right-most nonzero object is the same in both sequences.) Equivalently, an edge is a pair of monomorphisms, together with a compatible collection of isomorphisms on the various representations of their cokernels. We typically write an edge as $(X, Y) \xrightarrow{(i,j)} (X', Y')$, leaving $Z$, $p$, and $q$ implicit.

A two simplex is represented by a pair of commutative diagrams

$$\begin{tikzcd}
X_0 \arrow[r, tail] & X_1 \arrow[r, tail] \arrow[d, two heads] & X_2 \arrow[d, two heads] \\
& X_{1/0} \arrow[r, tail, "i"] & X_{2/0} \arrow[d, two heads, "p"] \\
& & X_{2/1}
\end{tikzcd}
\qquad
\begin{tikzcd}
X'_0 \arrow[r, tail] & X'_1 \arrow[r, tail] \arrow[d, two heads] & X'_2 \arrow[d, two heads] \\
& X_{1/0} \arrow[r, tail, "i"] & X_{2/0} \arrow[d, two heads, "p"] \\
& & X_{2/1}
\end{tikzcd}$$

such that $0 \to X_i \to X_j \to X_{i/j} \to 0$ and $0 \to X'_i \to X'_j \to X_{i/j} \to 0$ for all $0 \le i < j \le 2$, and $0 \to X_{1/0} \to X_{2/0} \to X_{2/1} \to 0$ are short exact sequences.

The element $\alpha \star_{\mathrm{Sh}} \beta$ of $K_2(\mathcal{P})$ is represented by the following diagram of simplices in $G(\mathcal{P})$:

$$\begin{tikzcd}
& (M,M) \arrow[r, "{(1,\beta)}"] \arrow[d, "{(\alpha,\alpha)}"'] \arrow[dr, "{(\alpha,\alpha\beta)}"] & (M,M) \arrow[d, "{(\alpha,\alpha)}"] & \\
(0,0) \arrow[ur] \arrow[dr] & & & (0,0) \arrow[ul] \arrow[dl] \\
& (M,M) \arrow[r, "{(1,\beta)}"] & (M,M) &
\end{tikzcd} \tag{3.1}$$

In this diagram, each triangle is a two simplex. The lower-middle triangle is the two simplex

$$\begin{tikzcd}
M \arrow[r, tail, "\alpha"] & M \arrow[r, tail, "="] \arrow[d, two heads] & M \arrow[d, two heads] \\
& 0 \arrow[r, tail] & 0 \arrow[d, two heads] \\
& & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, tail, "\alpha"] & M \arrow[r, tail, "\beta"] \arrow[d, two heads] & M \arrow[d, two heads] \\
& 0 \arrow[r, tail] & 0 \arrow[d, two heads] \\
& & 0
\end{tikzcd}$$

and the others are defined similarly. Since the top and bottom paths in (3.1) represent the same loop in $|G(\mathcal{P})|$, this diagram represents a map from the two-sphere to $|G(\mathcal{P})|$ and hence an element of $K_2(\mathcal{P})$.

Sherman's pairing is functorial in the following sense: if $F : \mathcal{P} \to \mathcal{P}'$ is an exact functor between exact categories, $M \in \operatorname{ob} \mathcal{P}$, and $\alpha$ and $\beta$ are commuting automorphisms of $M$, then $F_* : K_2(\mathcal{P}) \to K_2(\mathcal{P}')$ sends $\alpha \star_{\mathrm{Sh}} \beta$ to $F(\alpha) \star_{\mathrm{Sh}} F(\beta)$.

If $\mathcal{P} = \mathcal{P}(S)$, the category of projective right $S$-modules for some (not necessarily commutative) ring $S$, and $A, B \in \mathrm{GL}_n(S)$ are matrices that commute, we define

$$A \star_{\mathrm{Sh}} B \in K_2(S)$$

by viewing $A$ and $B$ as commuting automorphisms of the right $S$-module $S^n$, with $S^n$ thought of as column vectors and the action of $A$ and $B$ given by multiplication on the left of $S^n$. If $g : S \to S'$ is a ring map and $A$ and $B$ are commuting elements of $\mathrm{GL}_n(S)$, then $g_*(A \star_{\mathrm{Sh}} B) = g(A) \star_{\mathrm{Sh}} g(B)$ holds in $K_2(S')$.

Grayson [1979] has also defined a "star" pairing, defined for pairs of commuting $n \times n$ invertible matrices $(A, B)$ in a (not necessarily commutative) ring $S$. Grayson's definition amounts to

$$A \star_{\mathrm{Gr}} B := D(A) \star_{\mathrm{Mi}} D'(B),$$

where $D(A)$ and $D'(B)$ are the $3n \times 3n$ elementary matrices

$$D(A) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & I_n \end{bmatrix} \quad \text{and} \quad D'(B) = \begin{bmatrix} B & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & B^{-1} \end{bmatrix},$$

and $\star_{\mathrm{Mi}}$ denotes *Milnor's star pairing*. The latter pairing was defined by Milnor [1971] for pairs of commuting *elementary* matrices $E_1$ and $E_2$ (in $E(R) \subset \mathrm{GL}(R) := \mathrm{GL}_\infty(R)$) and is given by

$$E_1 \star_{\mathrm{Mi}} E_2 = [\widetilde{E}_1, \widetilde{E}_2],$$

where $\widetilde{E}_1$ and $\widetilde{E}_2$ are lifts of $E_1$ and $E_2$ to elements of the Steinberg group, $\mathrm{St}(R)$, under the canonical surjection $\mathrm{St}(R) \twoheadrightarrow E(R)$, and $[-,-]$ denotes the commutator of a pair of elements of $\mathrm{St}(R)$.

We also have the multiplication rule for $K_*(S)$ when $S$ is a commutative ring, which gives a pairing we will write as a cup product:

$$- \cup - : K_1(S) \otimes_{\mathbb{Z}} K_1(S) \to K_2(S).$$

There are several equivalent methods of describing this multiplication rule [Weibel 1981]; we use Milnor's [1971] original description of it: For $A \in \mathrm{GL}_m(S)$ and $B \in \mathrm{GL}_n(S)$, we write $[A], [B] \in K_1(R)$ for the associated $K$-theory classes. One then defines

$$[A] \cup [B] = D(A \otimes I_n) \star_{\mathrm{Mi}} D'(I_m \otimes B),$$

where $A \otimes I_n$ and $I_m \otimes B$ are identified with elements of $\mathrm{GL}_{mn}(R)$ by viewing each as an automorphism of $S^m \otimes_S S^n$ and choosing, arbitrarily, a basis of $S^m \otimes_S S^n$.

It is important to realize that $A \star_{\mathrm{Gr}} B$ and $[A] \cup [B]$ are not the same element of $K_2(S)$ in general. This is clear even for diagonal matrices: Say $A$ and $B$ are $2 \times 2$ diagonal matrices with diagonal entries $a_1$, $a_2$ and $b_1$, $b_2$ in a commutative ring $S$. Then $A \star_{\mathrm{Gr}} B = [a_1] \cup [b_1] + [a_2] \cup [b_2] \in K_2(R)$ but $[A] \cup [B] = [a_1] \cup [b_1] + [a_1] \cup [b_2] + [a_2] \cup [b_1] + [a_2] \cup [b_2]$. (We have written the group law for $K_2(S)$ additively here.)

We do have, however, the identity

$$A \star_{\mathrm{Gr}} u I_n = [A] \cup [u]$$

for any commutative ring $S$, unit $u \in S$ and matrix $A \in \mathrm{GL}_n(S)$. In particular,

$$u \star_{\mathrm{Gr}} v = [u] \cup [v] \tag{3.2}$$

for any pair of units $u$ and $v$ in a commutative ring.

It is clear that Grayson's star pairing is also functorial for ring maps: if $g : S \to S'$ is any ring homomorphism and $A$ and $B$ are elements of $\mathrm{GL}_n(S)$ that commute, then

$$g_*(A \star^{S}_{\mathrm{Gr}} B) = g(A) \star^{S'}_{\mathrm{Gr}} g(B),$$

where $g_* : K_2(S) \to K_2(S')$ is the induced map on $K_2$ and the superscript indicates in which ring the star operation is being performed.

Both Grayson's and Sherman's operations are also preserved by Morita equivalence, in the following sense: if we identify an $n \times n$ invertible matrix with entries in $\mathrm{Mat}_m(S)$ with an $nm \times nm$ invertible matrix with entries in $S$ in the usual manner, then

$$\psi(A \star^{\mathrm{Mat}_m(S)}_{\mathrm{Sh}} B) = A \star^{S}_{\mathrm{Sh}} B$$
$$\psi(A \star^{\mathrm{Mat}_m(S)}_{\mathrm{Gr}} B) = A \star^{S}_{\mathrm{Gr}} B,$$

where $\psi : K_2(\mathrm{Mat}_m(S)) \xrightarrow{\cong} K_2(S)$ is the canonical isomorphism induced from the isomorphisms $\psi : \mathrm{St}(\mathrm{Mat}_m(S)) \xrightarrow{\cong} \mathrm{St}(S)$ and $\psi : \mathrm{GL}(\mathrm{Mat}_m(S)) \xrightarrow{\cong} \mathrm{GL}(S)$ (see [Grayson 1979, Lemma 4.5]).

***Sherman's theorem on the connecting homomorphism.*** Sherman's theorem [2004, Theorem 3.6] describes the image of certain elements of the form $\alpha \star_{\mathrm{Sh}} \beta \in K_1(\mathcal{P})$ under the boundary map in a long exact localization sequence. To state it precisely, and for use again in Section 4, we review some technical details of Sherman's work.

In an exact category $\mathcal{A}$, a *mirror image sequence* is a pair $(E, F)$ of short exact sequences on the same three objects, but in the opposite order,

$$E = \big(0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0\big) \quad \text{and} \quad F = \big(0 \to Z \xrightarrow{j} Y \xrightarrow{q} X \to 0\big). \tag{3.3}$$

Sherman [1998, p. 18] associates to a mirror image sequence $(E, F)$ a loop in $|G(\mathcal{A})|$ given by the diagram of one simplices in $G(\mathcal{A})$

$$(0, 0) \to (X, X) \xrightarrow{(\iota_1, i)} (X \oplus Z, Y) \xleftarrow{(\iota_2, j)} (Z, Z) \leftarrow (0, 0),$$

where $\iota_1$ and $\iota_2$ denote inclusions into the first and second summands, and he writes $G(E, F)$ for the associated element of $K_1(\mathcal{A}) = \pi_1|G(\mathcal{A})|$.

Mainly for use in Section 4, we recall here an alternative description of the class $G(E, F)$. A *double short exact sequence* in an exact category $\mathcal{A}$, defined originally by Nenashev [1996], is a pair of short exact sequences involving the same three objects, in the same order:

$$l = \big(0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0,\ 0 \to A \xrightarrow{j} B \xrightarrow{q} C \to 0\big).$$

Nenashev associates to a double short exact sequence $l$ the class $m(l) \in K_1(\mathcal{A})$ represented by the loop

$$(0, 0) \to (A, A) \xrightarrow{(i, j)} (B, B) \leftarrow (0, 0).$$

(Moreover, Nenashev [1996; 1998] proves that $K_1(\mathcal{A})$ is generated by these loops and gives explicit generators for all the relations.)

Associated to a mirror image sequence (3.3), we have the double short exact sequence $l_{(E,F)}$ given by

$$0 \to X \oplus Z \xrightarrow{\begin{bmatrix} 0 & j \\ 0 & 0 \\ 1 & 0 \end{bmatrix}} Y \oplus Z \oplus X \xrightarrow{\begin{bmatrix} 0 & 1 & 0 \\ q & 0 & 0 \end{bmatrix}} Z \oplus X \to 0,$$
$$0 \to X \oplus Z \xrightarrow{\begin{bmatrix} i & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} Y \oplus Z \oplus X \xrightarrow{\begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} Z \oplus X \to 0. \tag{3.4}$$

The following result connects explicitly the two types of elements of $K_1(\mathcal{A})$:

**Proposition 3.5** [Sherman 2013, p. 164]. *For any mirror image sequence* (3.3) *in an exact category* $\mathcal{A}$, *we have*

$$G(E, F) = m(l_{E,F}) - G(Y, -1) \in K_1(\mathcal{A}),$$

*where* $(Y, -1)$ *denotes the mirror image sequence*

$$\big(0 \to Y \xrightarrow{-1} Y \to 0 \to 0,\ 0 \to 0 \to Y \xrightarrow{1} Y \to 0\big).$$

Suppose now that $\mathcal{A}$ is an abelian category, $\mathcal{S} \subset \mathcal{A}$ is a Serre subcategory, and $\mathcal{P} = \mathcal{A}/\mathcal{S}$, the associated quotient abelian category. Recall that Quillen's localization sequence is a long exact sequence of the form

$$\cdots \to K_i(\mathcal{S}) \to K_i(\mathcal{A}) \to K_i(\mathcal{P}) \xrightarrow{\partial} K_{i-1}(\mathcal{S}) \to \cdots.$$

Suppose also that $\alpha$ and $\beta$ are commuting injective endomorphisms of an object $M$ of $\mathcal{A}$ whose cokernels lie in $\mathcal{S}$. These form a commutative diagram

$$\begin{tikzcd} M \arrow[r, tail, "\alpha"] \arrow[d, tail, "\beta"'] \arrow[dr, tail, "\beta\alpha", "\alpha\beta"'] & M \arrow[d, tail, "\beta"] \\ M \arrow[r, tail, "\alpha"] & M \end{tikzcd}$$

and by the "snake chasing its tail lemma", this diagram determines the mirror image sequence $(E, F)$:

$$E := \big(0 \to \operatorname{coker}(\alpha) \xrightarrow{\beta} \operatorname{coker}(\beta\alpha) \longrightarrow \operatorname{coker}(\beta) \to 0\big),$$
$$F := \big(0 \to \operatorname{coker}(\beta) \xrightarrow{\alpha} \operatorname{coker}(\alpha\beta) \longrightarrow \operatorname{coker}(\alpha) \to 0\big). \tag{3.6}$$

Since we are assuming that $\operatorname{coker}(\alpha)$ and $\operatorname{coker}(\beta)$ belong to $\mathcal{S}$, the mirror image sequence $(E, F)$ determines the class $G(E, F) \in K_1(\mathcal{S})$. Also, the morphisms in $\mathcal{P} = \mathcal{A}/\mathcal{S}$ induced by $\alpha$ and $\beta$, which we will write as $\bar{\alpha}$ and $\bar{\beta}$, are commuting automorphisms of $M \in \mathcal{P}$, so that we have the element $\bar{\alpha} \star_{\mathrm{Sh}} \bar{\beta} \in K_2(\mathcal{P})$.

**Theorem 3.7** [Sherman 2004, Theorem 3.6]. *Under these hypotheses,*

$$\partial(\bar{\alpha} \star_{\mathrm{Sh}} \bar{\beta}) = \pm G(E, F) \in K_1(\mathcal{S}), \tag{3.8}$$

*where $(E, F)$ is the mirror image sequence* (3.6).

***A proof of (a weakened form of) a conjecture of Sherman.*** Sherman has conjectured that his pairing $\star_{\mathrm{Sh}}$ and Grayson's pairing $\star_{\mathrm{Gr}}$ coincide up to a sign. We prove here a slightly weaker form of this conjecture:

**Theorem 3.9.** *If $S$ is a (not necessarily commutative) ring and $A$ and $B$ are $n \times n$ invertible matrices with entries in $S$ that commute, then $A \star_{\mathrm{Sh}} B$ and $A \star_{\mathrm{Gr}} B$ agree up to a sign and two-torsion — i.e.,*

$$2A \star_{\mathrm{Sh}} B = \pm 2A \star_{\mathrm{Gr}} B \in K_2(S).$$

*Proof.* We first prove this when $S = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ and $A$ and $B$ are the $1 \times 1$ matrices $x$ and $y$. Since

$$K_2(S) \cong K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z})^{\oplus 2} \oplus K_0(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus (\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z},$$

to prove the theorem in this case it suffices to show the images of $x \star_{\mathrm{Sh}} y$ and $x \star_{\mathrm{Gr}} y$ under the canonical map $K_2(S) \twoheadrightarrow \mathbb{Z}$, given by modding out torsion, are both generators of $\mathbb{Z}$.

We have the localization long exact sequence

$$\cdots \to K_2(\mathbb{Z}[x^{\pm 1}, y]) \to K_2(S) \xrightarrow{\partial} G_1(\mathbb{Z}[x^{\pm 1}, y]/y) \xrightarrow{0} K_1(\mathbb{Z}[x^{\pm 1}, y]) \rightarrowtail K_1(S) \to \cdots .$$

The map labeled as vanishing does so because the next map is injective, and this injectivity holds since

$$K_1(\mathbb{Z}[x^{\pm 1}]) \xrightarrow{\cong} K_1(\mathbb{Z}[x^{\pm 1}, y])$$

is an isomorphism and the map $\mathbb{Z}[x^{\pm 1}] \to S$ splits.

Observe $\mathbb{Z}[x^{\pm 1}, y]/y \cong \mathbb{Z}[x^{\pm 1}]$ and hence

$$G_1(\mathbb{Z}[x^{\pm 1}, y]/y) \cong K_1(\mathbb{Z}[x^{\pm 1}]) \cong K_1(\mathbb{Z}) \oplus K_0(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}.$$

Also,

$$K_2(\mathbb{Z}[x^{\pm 1}, y]) \cong K_2(\mathbb{Z}[x^{\pm 1}]) \cong K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

From these calculations it follows that, in order to prove the theorem in this special case, it suffices to prove $\partial(x \star_{\mathrm{Sh}} y)$ and $\partial(x \star_{\mathrm{Gr}} y)$ each map to a generator of $\mathbb{Z}$ under the canonical surjection $G_1(\mathbb{Z}[x^{\pm 1}, y]/y) \twoheadrightarrow \mathbb{Z}$ given by modding out torsion.

To prove $\partial(x \star_{\mathrm{Sh}} y)$ maps to a generator, we apply Sherman's equation (3.8) with $\mathcal{A} = \mathcal{M}(\mathbb{Z}[x^{\pm 1}, y])$ and $\mathcal{S}$ the Serre subcategory of $y$-torsion modules, so that $\mathcal{A}/\mathcal{S} = \mathcal{M}(\mathbb{Z}[x^{\pm 1}, y^{\pm 1}])$. (In general, $\mathcal{M}(A)$ denotes the abelian category of finitely generated $A$-modules.) This gives $\partial(x \star_{\mathrm{Sh}} y) = \pm G(E, F)$, where $E$ and $F$ are the short exact sequences, in $\mathcal{S}$,

$$E = \big(0 \to 0 \to \mathbb{Z}[x^{\pm 1}] \xrightarrow{=} \mathbb{Z}[x^{\pm 1}] \to 0\big),$$
$$F = \big(0 \to \mathbb{Z}[x^{\pm 1}] \xrightarrow{x} \mathbb{Z}[x^{\pm 1}] \to 0 \to 0\big).$$

Under the isomorphism $K_1(\mathbb{Z}[x^{\pm 1}]) \cong K_1(\mathcal{S})$ that is induced by the inclusion of $\mathcal{P}(\mathbb{Z}[x^{\pm 1}, y]/y)$ into $\mathcal{S}$, the element $G(E, F)$ corresponds to $[x] \in K_1(\mathbb{Z}[x^{\pm 1}])$, which maps to a generator under $K_1(\mathbb{Z}[x^{\pm 1}]) \twoheadrightarrow \mathbb{Z}$.

To prove $\partial(x \star_{\mathrm{Gr}} y)$ also maps to a generator of $\mathbb{Z}$, we first observe that $x \star_{\mathrm{Gr}} y = [x] \cup [y] \in K_2(S)$ by (3.2). Note that $[x]$ lifts to an element of $K_1(\mathbb{Z}[x^{\pm 1}, y])$ and hence, by [Suslin 1983, (1.1)], we have

$$\partial([x] \cup [y]) = \overline{[x]} \cup \partial'([y]),$$

where $\overline{[x]}$ is the image of $[x]$ under the map

$$K_1(\mathbb{Z}[x^{\pm 1}, y]) \to K_1(\mathbb{Z}[x^{\pm 1}, y]/y) \cong K_1(\mathbb{Z}[x^{\pm 1}])$$

and

$$\partial' : K_1(S) \to G_0(\mathbb{Z}[x^{\pm 1}, y]/y) \cong G_0(\mathbb{Z}[x^{\pm 1}]) \cong K_0(\mathbb{Z}[x^{\pm 1}]) \cong \mathbb{Z}$$

is the boundary map in the localization long exact sequence. Now $\partial'([y]) = \operatorname{coker}(\mathbb{Z}[x^{\pm 1}, y] \xrightarrow{y} \mathbb{Z}[x^{\pm 1}, y]) = 1$ in $K_0(\mathbb{Z}[x^{\pm 1}])$ and thus

$$\partial([x] \cup [y]) = \overline{[x]} \cup \partial'([y]) = \overline{[x]},$$

whose image under

$$K_1(\mathbb{Z}[x^{\pm 1}]) \twoheadrightarrow \mathbb{Z}$$

is a generator. This completes the proof of the theorem in this special case.

The general form of the theorem now follows readily by naturality. Let $\mathrm{Mat}_n(S)$ be the ring of $n \times n$ matrices with entries in $S$ and define $g : \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \to \mathrm{Mat}_n(S)$ by $g(x) = A$ and $g(y) = B$. This is well-defined since $A$ and $B$ are assumed to commute. We have proven $2x \star_{\mathrm{Sh}} y = \pm 2x \star_{\mathrm{Gr}} y$. Using the naturality of $\star_{\mathrm{Sh}}$ and $\star_{\mathrm{Gr}}$ for ring maps and Morita equivalence, it follows that

$$\begin{aligned} 2A \star_{\mathrm{Sh}} B &= 2\psi(A \star_{\mathrm{Sh}}^{\mathrm{Mat}_n(S)} B) = 2\psi g_*(x \star_{\mathrm{Sh}} y) = \pm 2\psi g_*(x \star_{\mathrm{Gr}} y) \\ &= \pm 2\psi(A \star_{\mathrm{Gr}}^{\mathrm{Mat}_n(S)} B) = \pm 2A \star_{\mathrm{Gr}} B. \end{aligned}$$ $\square$

**Corollary 3.10.** *Suppose $S$ is a commutative ring, $A$ and $B$ are $n \times n$ invertible matrices with entries in $S$, and $AB = BA = f I_n$ for some (unit) $f \in S$. Then $2A \star_{\mathrm{Gr}} B = 2[A] \cup [f]$ and $2A \star_{\mathrm{Sh}} B = \pm 2[A] \cup [f]$.*

*Proof.* By Theorem 3.9, it suffices to prove the first equation.

Since $B = f A^{-1}$, we have $D'(B) = D'(f I_n) D'(A^{-1}) = D'(f I_n) D'(A)^{-1}$ and hence the bimultiplicativity of $\star_{\mathrm{Mi}}$ (see [Milnor 1971, Lemma 8.1]) gives

$$\begin{aligned} A \star_{\mathrm{Gr}} B = D(A) \star_{\mathrm{Mi}} D'(B) &= D(A) \star_{\mathrm{Mi}} D'(f I_n) + D(A) \star_{\mathrm{Mi}} D'(A^{-1}) \\ &= [A] \cup [f] - D(A) \star_{\mathrm{Mi}} D'(A) \end{aligned}$$

(where, as before, the group law for $K_2$ is written additively).

We now extend slightly an argument of Milnor [1971, Lemma 8.2]. Let

$$P = \begin{bmatrix} -I_n & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix}.$$

Then $P D(A) P^{-1} = D'(A)$ and $P D(B) P^{-1} = D'(A)$. So, using the invariance of $\star_{\mathrm{Mi}}$ under inner automorphisms [Milnor 1971, Lemma 8.1], we have

$$D(A) \star_{\mathrm{Mi}} D'(A) = D'(A) \star_{\mathrm{Mi}} D(A) = -D(A) \star_{\mathrm{Mi}} D'(A).$$

It follows that $2D(A) \star_{\mathrm{Mi}} D'(A) = 0$ and hence

$$2A \star_{\mathrm{Gr}} B = 2[A] \cup [f]. \qquad \square$$

The following result will be used to reinterpret Hochster's theta invariant in the next section. We will apply it in the case when $Q$ is assumed to be regular, but state it here in its natural level of generality.

**Corollary 3.11.** *Let $Q$ be a commutative ring and $f \in Q$ a non-zero-divisor, and set $R = Q/f$. Assume $(A, B)$ is a matrix factorization of $f$ in $Q$; i.e., $A$ and $B$ are $m \times m$ matrices with entries in $Q$ such that $AB = BA = f I_m$.*

*Then, under the composition of*

$$K_2(Q[1/f]) \xrightarrow{\text{can}} G_2(Q[1/f]) \xrightarrow{\partial} G_1(R),$$

*where $\partial$ is the boundary map in the localization long exact sequence, the image of the element $[A] \cup [f] \in K_2(Q[1/f])$ is, up to a sign and two-torsion, equal to $G(E, F) \in G_1(R)$, where $(E, F)$ is the mirror image sequence*

$$\begin{aligned} E &:= (0 \to \operatorname{coker}(A) \xrightarrow{\beta} R^m \to \operatorname{coker}(B) \to 0), \\ F &:= (0 \to \operatorname{coker}(B) \xrightarrow{\alpha} R^m \to \operatorname{coker}(A) \to 0). \end{aligned}$$

*Proof.* This follows immediately from Theorem 3.7 and Corollary 3.10. $\square$

## 4. Reinterpreting theta

In this section, we use the results of the previous section to give a reformulation of the $\theta$ pairing in terms of more familiar $K$-theoretic constructions. Our results here are valid for a general hypersurface ring.

Let $Q$ be a regular (Noetherian) ring, $\mathfrak{m}$ a maximal ideal of $Q$, and $f \in \mathfrak{m}$ a non-zero-divisor. Set $R = Q/f$ and also write $\mathfrak{m}$ for the image of $\mathfrak{m}$ in $R$. Recall that Hochster's theta invariant may be viewed as a bilinear pairing of the form

$$\theta = \theta_{(Q,\mathfrak{m},f)} : G_0(\operatorname{Spec} R \setminus \mathfrak{m}) \times K_0(\operatorname{Spec} R \setminus \mathfrak{m}) \longrightarrow \mathbb{Z},$$

and is determined by the formula

$$(M, N) \mapsto \operatorname{length} \operatorname{Tor}^R_{2j}(M, N) - \operatorname{length} \operatorname{Tor}^R_{2j+1}(M, N), \quad j \gg 0,$$

where $M$ and $N$ are finitely generated $R$-modules such that $\operatorname{pd}_{R_\mathfrak{p}} N_\mathfrak{p} < \infty$ for all $\mathfrak{p} \neq \mathfrak{m}$. We will need to know the $\theta$ pairing factors through localization at $\mathfrak{m}$; in fact, more is true:

**Proposition 4.1.** *Let $Q$, $\mathfrak{m}$, $f$, and $R$ be as above. Suppose $Q \to Q'$ is a flat ring map such that $\mathfrak{m}' := \mathfrak{m}Q'$ is a maximal ideal of $Q'$ and $Q/\mathfrak{m} \xrightarrow{\cong} Q'/\mathfrak{m}'$ is an isomorphism. Let $f' \in Q'$ be the image of $f$ in $Q'$ and set $R' = Q'/f' = Q' \otimes_Q R$. The triangle*

$$\begin{tikzcd} G_0(\operatorname{Spec}(R) \setminus \mathfrak{m}) \times K_0(\operatorname{Spec}(R) \setminus \mathfrak{m}) \arrow[d, "\phi^* \times \phi^*"'] \arrow[dr, "\theta^R"] & \\ G_0(\operatorname{Spec}(R' \setminus \mathfrak{m}')) \times K_0(\operatorname{Spec}(R') \setminus \mathfrak{m}') \arrow[r, "\theta^{R'}"'] & \mathbb{Z} \end{tikzcd}$$

*commutes, where $\phi : \operatorname{Spec}(R') \setminus \mathfrak{m}' \to \operatorname{Spec}(R) \setminus \mathfrak{m}$ is the induced flat morphism.*

*In particular, the $\theta$ pairing factors through the localization at $\mathfrak{m}$; that is,*

$$\begin{tikzcd} G_0(\operatorname{Spec}(R) \setminus \mathfrak{m}) \times K_0(\operatorname{Spec}(R) \setminus \mathfrak{m}) \arrow[d] \arrow[dr, "\theta^R"] & \\ G_0(\operatorname{Spec}(R_\mathfrak{m}) \setminus \mathfrak{m}) \times K_0(\operatorname{Spec}(R_\mathfrak{m}) \setminus \mathfrak{m}) \arrow[r, "\theta^{R_\mathfrak{m}}"'] & \mathbb{Z} \end{tikzcd}$$

*commutes.*

*Proof.* The hypothesis $\mathfrak{m}Q' = \mathfrak{m}'$ implies that $Q \to Q'$ does indeed determine a flat morphism $\phi : \mathrm{Spec}(R) \setminus \mathfrak{m} \to \mathrm{Spec}(R') \setminus \mathfrak{m}'$ of schemes. Since $R \to R'$ is flat, we have an isomorphism

$$\mathrm{Tor}_j^R(M, N) \otimes_R R' \cong \mathrm{Tor}_j^{R'}(M \otimes_R R', N \otimes_R R').$$

Finally, for any finitely generated $R$-module $T$ supported at $\mathfrak{m}$, we have

$$\mathrm{length}_R(T) = \mathrm{length}_{R'}(T \otimes_R R'),$$

since $R' \otimes_R R/\mathfrak{m} \cong R'/\mathfrak{m}R' = R'/\mathfrak{m}'$ by assumption. $\square$

For any Noetherian scheme $U$, we write $K_*(U)$ for the (Quillen) $K$-groups of the exact category of locally free coherent sheaves on $U$ and $G_*(U)$ for the $K$-groups of the abelian category of all coherent sheaves. The groups $K_*(U)$ form a graded ring with the multiplication rule, which we write as a cup product $- \cup -$, induced by the tensor product. The tensor product also defines the cap product pairing

$$- \cap - : K_i(U) \times G_j(U) \to G_{i+j}(U),$$

making $G_*(U)$ into a graded $K_*(U)$-module.

For any integer $l \geq 0$, let $K_*(U, \mathbb{Z}/l)$ and $G_*(U, \mathbb{Z}/l)$ denote $K$-theory and $G$-theory with $\mathbb{Z}/l$ coefficients. These are defined as the homotopy groups of spectra obtains from the $\mathcal{K}$- and $\mathcal{G}$-theory spectra by smashing them with the mod-$l$ Moore space. There are long exact coefficient sequences

$$\cdots \to K_m(U) \xrightarrow{\cdot l} K_m(U) \longrightarrow K_m(U, \mathbb{Z}/l) \longrightarrow K_{m-1}(U) \to \cdots$$

and similarly for $G$-theory.

We also will need the mod-$l$ versions of the cup and cap product pairings. To avoid complications in the multiplication rule for the Moore spaces for small primes, we assume for simplicity that $\mathbb{Z}/l$ has no 2- or 3-torsion; i.e., either $l = 0$ or it is not divisible by 2 or 3. In this case, $K_*(U, \mathbb{Z}/l)$ is a graded ring under cup product and $G_*(U, \mathbb{Z}/l)$ is a module over this ring under cap product. Moreover, $K_*(U) \to K_*(U, \mathbb{Z}/l)$ is a ring map and $G_*(U) \to G_*(U, \mathbb{Z}/l)$ is $K_*(U)$-linear. See [Thomason 1985, Section A.6] for more information.

The open complement of the closed subscheme $\mathrm{Spec}(R) \setminus \mathfrak{m}$ of $\mathrm{Spec}(Q) \setminus \mathfrak{m}$ is $\mathrm{Spec}\, Q[1/f]$ and (using that $Q$ is regular) we have a long exact localization sequence in $K$-theory

$$\begin{aligned} \cdots \to K_{i+1}(\mathrm{Spec}(Q) \setminus \mathfrak{m}, \mathbb{Z}/l) &\longrightarrow K_{i+1}(\mathrm{Spec}\, Q[1/f], \mathbb{Z}/l) \\ &\xrightarrow{\partial} G_i(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \longrightarrow K_i(\mathrm{Spec}(Q) \setminus \mathfrak{m}, \mathbb{Z}/l) \to \cdots . \end{aligned} \tag{4.2}$$

We also have the long exact localization sequence in $G$-theory associated to the closed subscheme $\operatorname{Spec}(R/\mathfrak{m})$ of $\operatorname{Spec}(R)$,

$$\begin{aligned}\cdots \to G_{i+1}(\operatorname{Spec} R, \mathbb{Z}/l) &\longrightarrow G_{i+1}(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \\ &\xrightarrow{\partial'} K_i(R/\mathfrak{m}, \mathbb{Z}/l) \longrightarrow G_i(R, \mathbb{Z}/l) \to \cdots . \quad (4.3)\end{aligned}$$

**Definition 4.4.** Assume $Q$ is a (not necessarily regular) Noetherian ring, $\mathfrak{m}$ is a maximal ideal, $f \in \mathfrak{m}$ is any element, and $l$ is an integer such that $\mathbb{Z}/l$ has no 2- or 3-torsion. Set $R = Q/f$. Let $\partial$ and $\partial'$ be the boundary maps in the $G$-theory localization sequences (4.2) and (4.3).

Define $\tilde{\theta} = \tilde{\theta}_{(Q,f,\mathfrak{m}),l,(i,j)}$ to be the pairing

$$\tilde{\theta} : G_i(Q[1/f], \mathbb{Z}/l) \times K_j(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \to K_{i+j-1}(R/\mathfrak{m}, \mathbb{Z}/l)$$

defined by the formula

$$\tilde{\theta}(\alpha, \gamma) = \partial'\big(\partial(\alpha \cap [f]) \cap \gamma\big),$$

where $[f] \in K_1(Q[1/f])$ denotes the class of the unit $f$ of $Q[1/f]$.

Taking $i = 1$ and $j = 0$, we have in particular the pairing

$$\tilde{\theta} : G_1(Q[1/f], \mathbb{Z}/l) \times K_0(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \to K_0(R/\mathfrak{m}, \mathbb{Z}/l) \cong \mathbb{Z}/l, \quad (4.5)$$

where the isomorphism sends $[R/\mathfrak{m}]$ to $1 \in \mathbb{Z}/l$. The goal of this section is to relate this pairing to Hochster's $\theta$ pairing.

We shall need the following analogue of Proposition 4.1 for $\tilde{\theta}$:

**Proposition 4.6.** *Suppose $(Q, \mathfrak{m}, f, l, R)$ are as in Definition 4.4 and that $Q \to Q'$ is a flat ring map such that $\mathfrak{m}' := \mathfrak{m}Q'$ is a maximal ideal of $Q'$ and $Q/\mathfrak{m} \xrightarrow{\cong} Q'/\mathfrak{m}'$ is an isomorphism. Let $f' \in Q'$ be the image of $f$ in $Q'$ and set $R' = Q'/f' = Q' \otimes_Q R$. Then the square*

$$\begin{array}{ccc} G_i(Q[1/f], \mathbb{Z}/l) \times K_j(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) & \xrightarrow{\tilde{\theta}} & K_{i+1-1}(R/\mathfrak{m}, \mathbb{Z}/l) \\ \downarrow & & \downarrow \cong \\ G_i(Q'[1/f'], \mathbb{Z}/l) \times K_j(\operatorname{Spec}(R') \setminus \mathfrak{m}', \mathbb{Z}/l) & \xrightarrow{\tilde{\theta}} & K_{i+1-1}(R'/\mathfrak{m}', \mathbb{Z}/l) \end{array}$$

*commutes.*

*Proof.* Write $\phi : \operatorname{Spec}(Q') \to \operatorname{Spec}(Q)$ for the associated map of affine schemes. The hypotheses ensure that each square in

$$\begin{array}{ccccccc} \operatorname{Spec}(R') \setminus \mathfrak{m}' & \longrightarrow & \operatorname{Spec}(Q') \setminus \mathfrak{m}' & \longrightarrow & \operatorname{Spec}(Q') & \longleftarrow & \operatorname{Spec}(Q'[1/f']) \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ \operatorname{Spec}(R) \setminus \mathfrak{m} & \longrightarrow & \operatorname{Spec}(Q) \setminus \mathfrak{m} & \longrightarrow & \operatorname{Spec}(Q) & \longleftarrow & \operatorname{Spec}(Q[1/f]) \end{array}$$

is a pull-back. It follows that the diagram

$$\begin{tikzcd}
\mathcal{G}(\mathrm{Spec}(R) \setminus \mathfrak{m}) \arrow[r] \arrow[d, "\phi^*"] & \mathcal{G}(\mathrm{Spec}(Q) \setminus \mathfrak{m}) \arrow[r] \arrow[d, "\phi^*"] & \mathcal{G}(\mathrm{Spec}(Q[1/f])) \arrow[d, "\phi^*"] \\
\mathcal{G}(\mathrm{Spec}(R') \setminus \mathfrak{m}) \arrow[r] & \mathcal{G}(\mathrm{Spec}(Q') \setminus \mathfrak{m}') \arrow[r] & \mathcal{G}(\mathrm{Spec}(Q'[1/f']))
\end{tikzcd}$$

is homotopy commutative. This gives us that $\phi^*$ commutes with the boundary maps $\partial$ in the associated long exact sequence. Whenever $\phi$ is flat, $\phi^*$ commutes with the cap product. This proves that

$$\phi^*\big(\partial(\alpha \cap [f]) \cap \gamma\big) = \partial(\phi^*(\alpha) \cap [f']) \cap \phi^*(\gamma) \in G_{i+j}(\mathrm{Spec}(R') \setminus \mathfrak{m}').$$

Similarly, the diagram

$$\begin{tikzcd}
\mathcal{G}(\mathrm{Spec}(R/m)) \arrow[r] \arrow[d, "\phi^*"] & \mathcal{G}(R) \arrow[r] \arrow[d, "\phi^*"] & \mathcal{G}(\mathrm{Spec}(R) \setminus \mathfrak{m}) \arrow[d, "\phi^*"] \\
\mathcal{G}(\mathrm{Spec}(R'/m')) \arrow[r] & \mathcal{G}(R') \arrow[r] & \mathcal{G}(\mathrm{Spec}(R') \setminus \mathfrak{m}')
\end{tikzcd}$$

is homotopy commutative, so that $\phi^*$ commutes with the boundary maps $\partial'$. $\square$

**Example 4.7.** We will use the previous result, in particular, when $Q' = Q_{\mathfrak{m}}^{\mathrm{hen}}$, the Henselization of $Q$ at the maximal ideal $\mathfrak{m}$. That is, the $\tilde{\theta}$ pairing for $R = Q/f$ factors through its Henselization at $\mathfrak{m}$.

Assume $Q$ is a regular ring and $f \in Q$ is a non-zero-divisor. Given a matrix factorization $(A, B)$ of $f$ in $Q$, the module $\mathrm{coker}(A)$ is annihilated by $f$ and thus may be regarded as an $R$-module, where $R := Q/f$. It is necessarily a MCM $R$-module, as seen by the depth formula and the fact that $\mathrm{pd}_Q(M) = 1$. Moreover, if $Q$ is local, every MCM $R$-module is the cokernel of some matrix factorization. Recall that $A$ is an invertible matrix when regarded as a matrix with entries in $Q[1/f]$ and we write $[A]$ for the class in $K_1(Q[1/f])$ it determines. We write $[A]_l$ for the image of this class in $K_1(Q[1/f], \mathbb{Z}/l)$. (Note that $[A]_0 = [A]$.)

For finitely generated $R$-modules $M$ and $N$ such that $N_{\mathfrak{p}}$ has finite projective dimension for all $\mathfrak{p} \neq \mathfrak{m}$, let $\theta_l(M, N) \in \mathbb{Z}/l$ be the value of $\theta(M, N)$ modulo $l$.

Recall that such a module $N$ determines a coherent sheaf on $\mathrm{Spec}(R) \setminus \mathfrak{m}$ that admits a finite resolution by locally free coherent sheaves, and hence $N$ determines a class $[N'] \in K_0(\mathrm{Spec}(R) \setminus \mathfrak{m})$. We write $[N']_l$ for the image of $[N']$ in $K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l)$. (As before, $[N]_0 = [N]$.)

**Theorem 4.8.** *Assume $Q$ is a regular ring, $\mathfrak{m}$ is a maximal ideal of $Q$ and $f \in \mathfrak{m}$ is a non-zero-divisor. Let $R = Q/f$ and let $\mathfrak{m}$ also denote the image of $\mathfrak{m}$ in $R$. Let $(A, B)$ be a matrix factorization of $f$ in $Q$ and let $M =$* coker$(A)$ *be the associated*

*MCM $R$-module. Let $N$ be a finitely generated $R$-module such that* $\mathrm{pd}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) < \infty$ *for all* $\mathfrak{p} \neq \mathfrak{m}$, *and let* $[N']$ *be the class it determines in* $K_0(\mathrm{Spec}(R) \setminus \mathfrak{m})$.

*For any integer $l$ such that $\mathbb{Z}/l$ has no 2- or 3-torsion,*

$$\theta_l(M, N) = \pm\tilde{\theta}_l([A]_l, [N']_l)$$

*Proof.* The compatibility of the cup and cap products with the canonical maps $K_*(U) \to K_*(U, \mathbb{Z}/l)$ implies that the diagram

$$\begin{array}{ccc}
K_1(Q[1/f]) \times K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}) & \xrightarrow{\tilde{\theta}} & G_0(R/\mathfrak{m}) \\
\downarrow & & \downarrow \\
K_1(Q[1/f], \mathbb{Z}/l) \times K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) & \xrightarrow{\tilde{\theta}_l} & G_0(R/\mathfrak{m}, \mathbb{Z}/l)
\end{array}$$

commutes. It therefore suffices to prove the theorem with integral coefficients.

To shorten notation, set $U = \mathrm{Spec}(Q)$, $U' = U \setminus \mathfrak{m}$, $V = \mathrm{Spec}(R)$ and $V' = V \setminus \mathfrak{m}$. Note that $U'$ is open in $U$, $V$ is closed in $U$, and $V' = V \cap U$.

Let $N'$ denote the coherent sheaf on $V'$ given by the restriction of $N$, viewed as a coherent sheaf on $V$, to $V'$. Since a high enough syzygy for an $R$-free resolution of $N$ will be locally free on $V'$ and since the equation in question depends only on the class of $N'$ in $K_0(V')$, we may assume $N'$ is a locally free coherent sheaf on $V'$.

From Corollary 3.11 we have

$$2\partial([A] \cup [f]) = \pm 2G(E, F) \in G_1(\mathrm{Spec}\, R) = G_1(V),$$

where $E$ and $F$ are the short exact sequences of coherent sheaves on $V = \mathrm{Spec}(R)$

$$E := \big(0 \to \mathrm{coker}(A) \to \mathcal{O}_V^m \to \mathrm{coker}(B) \to 0\big),$$
$$F := \big(0 \to \mathrm{coker}(B) \to \mathcal{O}_V^m \to \mathrm{coker}(A) \to 0\big).$$

(We are assuming $A$ and $B$ are $m \times m$ matrices.) The image of $G(E, F)$ under $G_1(V) \to G_1(V')$ is $G(E', F')$, where

$$E' := \big(0 \to \mathrm{coker}(A') \to \mathrm{coker}\, \mathcal{O}_{V'}^m \to \mathrm{coker}(B') \to 0\big),$$
$$F' := \big(0 \to \mathrm{coker}(B') \to \mathrm{coker}\, \mathcal{O}_{V'}^m \to \mathrm{coker}(A') \to 0\big),$$

and $A'$ and $B'$ are the injective maps $\mathcal{O}_U^m \rightarrowtail \mathcal{O}_U^m$ induced by $A$ and $B$. Thus

$$2\partial([A] \cup [f]) = \pm 2G(E', F') \in G_1(V').$$

We may describe the mirror image sequence $(E, F)$ in the following equivalent way: Let $\bar{A}$ and $\bar{B}$ be the matrices in $R = Q/f$ determined by $A$ and $B$. We have an unbounded, two-periodic exact sequence of locally free $\mathcal{O}_V$-modules

$$\cdots \to \mathcal{O}_V^r \xrightarrow{\bar{A}} \mathcal{O}_V^r \xrightarrow{\bar{B}} \mathcal{O}_V^r \xrightarrow{\bar{A}} \mathcal{O}_V^r \xrightarrow{\bar{B}} \cdots . \tag{4.9}$$

Then $E$ and $F$ are the two canonical short exact sequences involving syzygies coming from this two-periodic exact sequence.

The key result we need is that

$$\partial'(G(E', F') \cap [N']) = \theta(M, N),$$

where $\partial'$ is the boundary map $G_1(V') \to G_0(R/\mathfrak{m}) \cong \mathbb{Z}$. This is essentially proven by Buchweitz and van Straten [2012, Theorem 3.4], but they use the notation of double short exact sequences, and not that of mirror image sequences. We proceed to summarize their proof in the notation we need.

Since $N'$ is locally free on $V'$, the exact sequences $E'$ and $F'$ remain exact upon tensoring with $N'$ and it follows that

$$G(E', F') \cap [N'] = G(E' \otimes_{\mathcal{O}_{V'}} N', F' \otimes_{\mathcal{O}_{V'}} N') \in G_1(V').$$

Tensoring the sequence (4.9) with $N$ yields the two-periodic complex of coherent sheaves on $V$

$$\cdots \xrightarrow{\bar{B}\otimes \mathrm{id}_N} \mathcal{O}_V^r \otimes_{\mathcal{O}_V} N \xrightarrow{\bar{A}\otimes \mathrm{id}_N} \mathcal{O}_V^r \otimes_{\mathcal{O}_V} N \xrightarrow{\bar{B}\otimes \mathrm{id}_N} \mathcal{O}_V^r \otimes_{\mathcal{O}_V} N \xrightarrow{\bar{A}\otimes \mathrm{id}_N} \cdots$$

and, again using that $N'$ is locally free on $V'$, this complex is exact on the open subset $V'$ of $V$. From this complex, one builds what Buchweitz and van Straten call a "cyclic diagram":

$$\zeta := \begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\bar{A} \otimes \mathrm{id}_N) & \longrightarrow & \mathcal{O}_V^r & \longrightarrow & \mathrm{im}(\bar{A} \otimes \mathrm{id}_N) & \longrightarrow & 0 \\ & & \subseteq \big\uparrow & & \big\| & & \big\downarrow \subseteq & & \\ 0 & \longleftarrow & \mathrm{im}(\bar{B} \otimes \mathrm{id}_N) & \longleftarrow & \mathcal{O}_V^r & \longleftarrow & \ker(\bar{B} \otimes \mathrm{id}_N) & \longleftarrow & 0 \end{array}$$

(Note that the diagram does *not* commute.) The left and right vertical arrows in this are the canonical inclusions. Buchweitz and van Straten [2012, p. 247] define $\{\zeta\} \in G_1(R) = G_1(V)$ to be $m(l_\zeta)$, where $l_\zeta$ is a certain double short exact sequence associated to a cyclic diagram $\zeta$; see their paper for the precise formula.

The image of $\{\zeta\}$ under $G_1(V) \to G_1(V')$ is $\{\zeta'\}$, where

$$\zeta' := \begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\bar{A}' \otimes \mathrm{id}_{N'}) & \longrightarrow & \mathcal{O}_{V'}^r & \longrightarrow & \mathrm{im}(\bar{A}' \otimes \mathrm{id}_{N'}) & \longrightarrow & 0 \\ & & \big\| & & \big\| & & \big\| & & \\ 0 & \longleftarrow & \mathrm{im}(\bar{B}' \otimes \mathrm{id}_{N'}) & \longleftarrow & \mathcal{O}_{V'}^r & \longleftarrow & \ker(\bar{B}' \otimes \mathrm{id}_{N'}) & \longleftarrow & 0 \end{array}$$

Observe that the vertical arrows are now all identity maps, and thus we may interpret $\zeta'$ as determining the mirror image sequence $(E'', F'')$, where

$$E'' := \big(0 \to \ker(\bar{A}' \otimes \mathrm{id}_{N'}) \xrightarrow{\subseteq} \mathcal{O}^r_{V'} \xrightarrow{\bar{A}' \otimes \mathrm{id}_{N'}} \mathrm{im}(\bar{A}' \otimes \mathrm{id}_{N'}) \to 0\big),$$
$$F'' := \big(0 \to \mathrm{im}(\bar{A}' \otimes \mathrm{id}_{N'}) \xrightarrow{\subseteq} \mathcal{O}^r_{V'} \xrightarrow{\bar{B}' \otimes \mathrm{id}_{N'}} \ker(\bar{A}' \otimes \mathrm{id}_{N'}) \to 0\big).$$

Since $\ker(\bar{A}') = \mathrm{im}(\bar{B}') \cong \mathrm{coker}(\bar{A}')$ and $\ker(\bar{B}') = \mathrm{im}(\bar{A}') \cong \mathrm{coker}(\bar{B}')$ we have an isomorphism of mirror image sequences

$$(E'', F'') \cong (E' \otimes_{\mathcal{O}_{V'}} N', F' \otimes_{\mathcal{O}_{V'}} N').$$

Moreover, by comparing the definitions $l_{\zeta'}$ and $l_{(E'',F'')}$ (see [loc. cit.] and (3.4)), it is clear that

$$l_{\zeta'} = -l_{(E'',F'')}.$$

Using also that $m(l_{(E'',F'')}) = G(E'', F'') + G(\mathcal{O}^r_{V'}, -1)$ [Sherman 2013, p. 164], we obtain

$$-\{\zeta'\} = (G(E', F') \cap [N']) + G(\mathcal{O}^r_{V'}, -1) \in G_1(V').$$

The class $G(\mathcal{O}^r_{V'}, -1)$ clearly lifts along $G_1(V) \to G_1(V')$ to the class $G(\mathcal{O}^r_V, -1)$ and hence $\partial'(G(\mathcal{O}^r_{V'}, -1)) = 0$ and

$$\partial'(G(E', F') \cap [N']) = \partial'(\zeta').$$

Finally, Buchweitz and van Straten show [2012, proof of Theorem 3.4] that

$$\partial'(\{\zeta'\}) = \theta(M, N).$$

We have proven that

$$2\theta(M, N) = \pm 2\partial'\big(\partial([A] \cup [f]) \cup [N']\big) = \pm 2\tilde{\theta}_l(M, N)$$

and, since this is an equation in $\mathbb{Z}$, we may divide by 2. $\square$

**Corollary 4.10.** *Let $Q$ be a regular ring, $\mathfrak{m}$ a maximal ideal of $Q$, $f \in \mathfrak{m}$ a non-zero-divisor, $R = Q/f$, and $l \geq 0$ an integer such that $\mathbb{Z}/l$ is without 2- and 3-torsion. The diagram*

$$\begin{tikzcd} K_1(Q[1/f], \mathbb{Z}/l) \times K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \arrow[d, "\partial \times \mathrm{id}"] \arrow[dr, "\tilde{\theta}"] & \\ G_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \times K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \arrow[r, "\theta"'] & \mathbb{Z}/l \end{tikzcd}$$

*commutes up to a sign.*

*Proof.* By Propositions 4.1 and 4.6, $\theta$ and $\tilde{\theta}$ factor through the localization of $R$ at $\mathfrak{m}$, and hence we may assume $Q$ is local.

Theorem 4.8 shows that $\tilde{\theta}_l(A, N) = \theta_l(\operatorname{coker}(A), N)$ for all matrix factorizations $(A, B)$ of $f$ in $Q$ and all $N$. An arbitrary class of $K_1(Q[1/f])$ is determined by an invertible matrix $T$ with entries in $Q[1/f]$ (not necessarily coming from a matrix factorization). In $G_0(R)$, we may represent $\partial([T])$ as the difference of the classes of MCM modules. Since $Q$ is assumed to be local, every MCM $R$-module is represented as the cokernel of a matrix factorization. It follows that $K_1(Q[1/f])$ is generated by classes coming from matrix factorizations of $f$ and the image of $K_1(Q) \to K_1(A[1/f])$. It therefore suffices to prove $\tilde{\theta}$ annihilates the image of $K_1(Q)$.

Let, then, $T$ be an invertible matrix with entries in $Q$. Using [Suslin 1983, (1.1)], we have

$$\tilde{\theta}([T], N) = \partial'\big(\partial([T] \cup [f]) \cap [N]\big) = \partial'\big((\overline{[T]} \cap \partial[f]) \cap [N]\big),$$

where $\overline{[T]}$ is the image of $[T]$ under the canonical map $K_1(Q) \to K_1(\operatorname{Spec}(R) \setminus \mathfrak{m})$. The map $\partial$ factors as

$$K_1(Q[1/f]) \xrightarrow{\tilde{\partial}} G_0(R) \xrightarrow{\text{can}} G_0(\operatorname{Spec}(R) \setminus \mathfrak{m}).$$

Since $\tilde{\partial}([f]) = [R]$ we have $\overline{[T]} \cap \partial[f] = \overline{[T]}$ and hence

$$\tilde{\theta}([T], N) = \partial'(\overline{[T]} \cap [N]).$$

Finally, since $\overline{[T]}$ and $[N]$ lift to elements of $K_1(R)$ and $G_0(R)$, respectively, $\overline{[T]} \cap [N] \in G_1(\operatorname{Spec}(R) \setminus \mathfrak{m})$ lifts to an element of $G_1(R)$. Thus, $\partial'(\overline{[T]} \cap [N]) = 0$. $\square$

Let us indicate how the previous result relates to the work of Buchweitz and van Straten, at least on the level of analogy. We start with:

**Proposition 4.11.** *Assume $(Q, \mathfrak{m}, k)$ is a Henselian, regular local ring with algebraically closed residue field $k$ and $0 \neq f \in \mathfrak{m}$. Set $R = Q/f$. Assume $l$ is a positive integer that is relatively prime to* $\operatorname{char}(k)$ *and is such that $[R/\mathfrak{m}] = 0$ in $G_0(R, \mathbb{Z}/l)$.*

*Then the boundary map*

$$K_1(Q[1/f], \mathbb{Z}/l) \xrightarrow{\cong} G_0(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \tag{4.12}$$

*is an isomorphism.*

**Remark 4.13.** Before proving this, let us show $[R/\mathfrak{m}] = 0$ in $G_0(R, \mathbb{Z}/l)$, at least when $\dim(R) > 0$, for all but a finite number of primes $l$, so that the hypotheses of the proposition are often met.

Let $\mathfrak{p}$ be a prime in $R$ of height $\dim(R)-1$ (recall that we assume $\dim(R)>0$) and choose $g\in\mathfrak{m}\setminus\mathfrak{p}$. Then the short exact sequence

$$0\to R/\mathfrak{p}\xrightarrow{g} R/\mathfrak{p}\longrightarrow R/(\mathfrak{p},g)\to 0$$

gives $[R/(\mathfrak{p},g)]=0$ in $G_0(R)$. But $R/(\mathfrak{p},g)$ is a finite length $R$-module, say of length $N$, and hence $[R/(\mathfrak{p},g)]=N[R/\mathfrak{m}]$. This proves $N[R/\mathfrak{m}]=0$ in $G_0(R)$ and thus, if $l$ is a prime such that $l\nmid N$, then $[R/\mathfrak{m}]$ is divisible by $l$ in $G_0(R)$.

*Proof.* The localization sequences for $G$-theory (with $\mathbb{Z}/l$ coefficients), and their naturality, applied to the schemes, closed subschemes, and open complements

$$\begin{aligned}\operatorname{Spec}(R/\mathfrak{m})&\subseteq\operatorname{Spec}(R)\supseteq\operatorname{Spec}(R)\setminus\mathfrak{m},\\ \operatorname{Spec}(Q/\mathfrak{m})&\subseteq\operatorname{Spec}(Q)\supseteq\operatorname{Spec}(Q)\setminus\mathfrak{m},\\ \operatorname{Spec}(R)\setminus\mathfrak{m}&\subseteq\operatorname{Spec}(Q)\setminus\mathfrak{m}\supseteq\operatorname{Spec}(Q[1/f]),\end{aligned}$$

yield the commutative diagram

$$\begin{tikzcd}
G_1(R) \arrow[r] \arrow[d] & G_1(R\setminus\mathfrak{m}) \arrow[r] \arrow[d, two heads] & G_0(R/\mathfrak{m}) \arrow[r, "0"] \arrow[d, "\cong"] & G_0(R) \arrow[d] \\
G_1(Q) \arrow[r, "0"] & G_1(Q\setminus\mathfrak{m}) \arrow[r, "\cong"] \arrow[d, "0"] & G_0(Q/\mathfrak{m}) \arrow[r] & G_0(Q) \\
 & G_1(Q[1/f]) \arrow[d] & & \\
 & G_0(R\setminus\mathfrak{m}) \arrow[d, "0"] & & \\
 & G_0(Q\setminus\mathfrak{m}) \arrow[d, "\cong"] & & \\
 & G_0(Q[1/f]) & &
\end{tikzcd}$$

with exact rows and column. (Here, we have suppressed the coefficients of $\mathbb{Z}/l$ and written $R\setminus\mathfrak{m}$ and $Q\setminus\mathfrak{m}$ for the nonaffine schemes $\operatorname{Spec}(R)\setminus\mathfrak{m}$ and $\operatorname{Spec}(Q)\setminus\mathfrak{m}$.) The proposition follows immediately from this diagram, provided we justify the labels.

The hypothesis that $[R/\mathfrak{m}]=0$ in $G_0(R,\mathbb{Z}/l)$ gives us that $G_0(R/\mathfrak{m},\mathbb{Z}/l)\to G_0(R,\mathbb{Z}/l)$ is the zero map, as labeled. The map $G_1(Q,\mathbb{Z}/l)\to G_1(Q\setminus\mathfrak{m},\mathbb{Z}/l)$ is the zero map, since

$$G_1(Q,\mathbb{Z}/l)\cong K_1(Q,\mathbb{Z}/l)\cong K_1(Q/\mathfrak{m},\mathbb{Z}/l)=0$$

by [Gabber 1992; Suslin 1983]. A diagram chase then gives that $G_1(R\setminus\mathfrak{m},\mathbb{Z}/l)\to G_1(Q\setminus\mathfrak{m},\mathbb{Z}/l)$ is onto and hence that $G_1(Q\setminus\mathfrak{m},\mathbb{Z}/l)\to G_1(Q[1/f],\mathbb{Z}/l)$ is the zero map. The group $G_0(Q\setminus\mathfrak{m},\mathbb{Z}/l)\cong K_0(Q\setminus\mathfrak{m},\mathbb{Z}/l)$ is isomorphic to $\mathbb{Z}/l$ and is generated by the class of the structure sheaf, since $G_0(Q,\mathbb{Z}/l)\cong K_0(Q,\mathbb{Z}/l)\cong\mathbb{Z}/l$ (using that $Q$ is local) and $K_0(Q,\mathbb{Z}/l)\to K_0(Q\setminus\mathfrak{m},\mathbb{Z}/l)$ is onto. It follows that $K_0(Q\setminus\mathfrak{m},\mathbb{Z}/l)\to K_0(Q[1/f],\mathbb{Z}/l)$ is an isomorphism and hence that $G_0(R\setminus\mathfrak{m},\mathbb{Z}/l)\to G_0(Q\setminus\mathfrak{m},\mathbb{Z}/l)$ is the zero map. □

Assuming the hypotheses of the proposition are met and that $l$ is not divisible by 2 or 3, then the mod-$l$ theta pairing

$$\theta_l : G_0(\mathrm{Spec}(R)\setminus\mathfrak{m},\mathbb{Z}/l)\times G_0(\mathrm{Spec}(R)\setminus\mathfrak{m},\mathbb{Z}/l)\to\mathbb{Z}/l$$

may be given by the formula

$$\theta_l(\alpha,\beta)=\langle\gamma(\alpha),\beta\rangle. \tag{4.14}$$

Here, we define

$$\gamma : G_0(\mathrm{Spec}(R)\setminus\mathfrak{m},\mathbb{Z}/l)\to G_1(\mathrm{Spec}(R)\setminus\mathfrak{m},\mathbb{Z}/l) \tag{4.15}$$

as the composition of the inverse of the isomorphism (4.12) and the map

$$K_1(Q[1/f],\mathbb{Z}/l)\xrightarrow{\partial(-\cup[t])} G_1(\mathrm{Spec}(R)\setminus\mathfrak{m},\mathbb{Z}/l). \tag{4.16}$$

(In the next section, we note that the latter map is the "specialization map" in $K$-theory, and we will explore its properties.) The pairing

$$\langle -,-\rangle : G_1(\mathrm{Spec}(R)\setminus\mathfrak{m},\mathbb{Z}/l)\times G_0(\mathrm{Spec}(R)\setminus\mathfrak{m},\mathbb{Z}/l)\to\mathbb{Z}$$

is the composition of the cup product, the boundary map $G_1(\mathrm{Spec}(R)\setminus\mathfrak{m},\mathbb{Z}/l)\xrightarrow{\partial'} G_0(R/\mathfrak{m},\mathbb{Z}/l)$ and the canonical isomorphism $G_0(R/\mathfrak{m})\cong\mathbb{Z}/l$.

The isomorphism (4.12) is the analogue of the Alexander duality isomorphism $H^{2m+1}(S\setminus L)\xrightarrow{\cong} H^{2m}(L)$ and the map (4.16) is the analogue of the map $\rho_t^* : H^{j+1}(S\setminus L)\to H^{j+1}(L)$ induced by the "push aside" map $\rho_t : L\to\partial_t F\subseteq(S\setminus L)$. Thus the map $\gamma$ defined in (4.15) is the analogue of the map

$$\gamma^{\mathrm{top}} : H^j(L)\to H^{j+1}(L)$$

occurring in the work of Buchweitz and van Straten. They prove [2012, Proposition 5.1]

$$\mathrm{link}(\alpha,\beta)=\langle\gamma^{\mathrm{top}}(\alpha),\beta\rangle$$

for classes $\alpha$ and $\beta$ in $H^{n-1}(L)\cong H_n(L)$, where link is the linking form on $H_n(L)$. (The linking form may be defined as the restriction of the Seifert form, defined on the homology of the Milnor fiber $H_n(F_t)$, along the canonical map $H_n(\partial(F_t))\to H_n(F_t)$, using the diffeomorphism $L\cong\partial F_t$.)

We may thus interpret Corollary 4.10 as saying that the mod-$l$ theta pairing is the analogue for algebraic $K$-theory with finite coefficients of the linking form in topology.

## 5. Specialization with finite coefficients (following Suslin)

In the previous section we have given a new interpretation of the $\theta$ pairing in terms of the pairing $\tilde{\theta}$ defined as

$$\tilde{\theta}(\alpha, \gamma) = \partial'\big(\partial(\alpha \cap [f]) \cap \gamma\big).$$

Under some additional assumptions, a portion of this formula, namely the map

$$\sigma : G_i(Q[1/f], \mathbb{Z}/l) \to G_i(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l)$$

sending $\alpha$ to $\partial(\alpha \cap [f])$, can be interpreted as a *specialization map*. The goal of this section is to make this precise and to establish properties of this specialization map. When working with the algebraic analogue of the Milnor fiber, the specialization map is the analogue of the map

$$\rho_t^* : K^1_{\text{top}}(X^*) \to K^1_{\text{top}}(L)$$

occurring in the work of Buchweitz and van Straten. As mentioned, $\rho_t^*$ factors through the topological $K$-theory of the Milnor fiber. We prove the analogous fact here, by showing, under certain hypotheses, that the specialization map $\sigma$ factors through the $G$-theory of the algebraic analogue of the Milnor fiber. Our notion of specialization in $K$- and $G$-theory is based on work of Andrei Suslin [1983].

Define a *pointed curve* to be a pair $(C, c)$, where $C$ is an affine, Noetherian, integral scheme that is regular of dimension one and $c \in C$ is a closed point such that the associated maximal ideal is principal and the residue field $\kappa(c)$ is algebraically closed. Equivalently, a pointed curve is a pair $(V, \mathfrak{n})$ where $V$ is a Dedekind domain and $\mathfrak{n}$ is a principal maximal ideal of $V$ such that $V/\mathfrak{n}$ is an algebraically closed field. We will use both notations $(C, c)$ and $(V, \mathfrak{n})$ interchangeably. We typically write $t$ for a chosen generator of $\mathfrak{n}$, but such a choice is not part of the defining data.

A typical example occurs when $V$ is a DVR with algebraically closed residue field (and hence $t$ is a uniformizing parameter), but we do not limit ourselves to this case.

We will use the exact sequence

$$K_1(C) \xrightarrow{i^*} K_1(C \setminus c) \xrightarrow{\partial} K_0(c),$$

equivalently, the exact sequence

$$K_1(V) \xrightarrow{i^*} K_1(V[1/t]) \xrightarrow{\partial} K_0(V/\mathfrak{n})$$

coming from a portion of the long exact localization sequence in $K$-theory. (Since $V$, $V[1/t]$ and $V/\mathfrak{n}$ are all regular, their $K$- and $G$-theories coincide.) The map $\partial$ sends $[t] \in K_1(V[1/t])$ to $[V/\mathfrak{n}] \in K_0(V/\mathfrak{n})$ and $[V/\mathfrak{n}]$ generates $K_0(V/\mathfrak{n}) \cong \mathbb{Z}$.

Given a pointed curve $(C, c)$ and a morphism of Noetherian schemes $p : X \to C$, write $X_c$ for the fiber over $c \in C$ and $X \setminus X_c$ for its open complement in $X$. Observe that if $V$ is a DVR then $X_c$ is the closed fiber and $X \setminus X_c$ is the generic fiber. If $X = \operatorname{Spec}(Q)$ is affine, so that $p$ is given by a ring map $V \to Q$, then $X_c = \operatorname{Spec}(Q/f)$ and $X \setminus X_c = \operatorname{Spec}(Q[1/f])$, where $f \in Q$ is the image of a chosen generator $t$ of $\mathfrak{n}$.

Similarly, given a ring map $V \to Q$ and a maximal ideal $\mathfrak{m}$ of $Q$ such that $\mathfrak{m} \cap V = \mathfrak{n}$, then, setting $X = \operatorname{Spec}(Q) \setminus \mathfrak{m}$, we have $X_c = \operatorname{Spec}(Q/f) \setminus \mathfrak{m}$ and $X \setminus X_c = \operatorname{Spec}(Q[1/f])$. This is the main example we have in mind but, for most of this section, we allow $X$ to be an arbitrary Noetherian scheme.

**Definition 5.1.** Given a pointed curve $(C, c)$ corresponding to $(V, \mathfrak{n})$, a morphism of Noetherian schemes $p : X \to C$, and a positive integer $l$ such that $\operatorname{char}(\kappa(c)) \nmid l$, define the *specialization map in $G$-theory* (for $X$ with $\mathbb{Z}/l$ coefficients) to be the map

$$\sigma = \sigma_{C,c,p} : G_i(X \setminus X_c, \mathbb{Z}/l) \to G_i(X_c, \mathbb{Z}/l)$$

given as follows: Choose $z$ to be any element of $K_1(V[1/t])$ that maps to $1 \in \mathbb{Z}$ under $\partial : K_1(V[1/t]) \to K_0(V/\mathfrak{n}) = \mathbb{Z}$; for example, $z$ could be $[t]$. Then $\sigma$ is the composition of

$$G_i(X \setminus X_c, \mathbb{Z}/l) \xrightarrow{-\cap z} G_{i+1}(X \setminus X_c, \mathbb{Z}/l) \xrightarrow{\partial} G_i(X_c, \mathbb{Z}/l),$$

where $\partial$ is the boundary map in the localization long exact sequence in $G$-theory associated to the closed subscheme $X_c$ of $X$.

**Lemma 5.2** (Suslin). *The specialization map $\sigma_{C,c,p}$ is independent of the choice of $z$.*

*Proof.* Our proof is basically that of Suslin's, with some minor modifications.

Suppose $z'$ is another element of $K_1(V[1/t])$ with $\partial(z') = 1$, and let

$$\sigma' : G_i(X \setminus X_c, \mathbb{Z}/l) \to G_i(X_c, \mathbb{Z}/l)$$

be the map $\sigma'(\alpha) = \partial(\alpha \cap z')$. The difference $\sigma - \sigma'$ sends $\alpha \in G_i(U \setminus U_c, \mathbb{Z}/l)$ to $\partial(\alpha \cap (z - z'))$. Since $\partial(z - z') = 0$, we have $z - z' = i^*(w)$ for some $w \in V$. Using [Suslin 1983, (1.1)], we have

$$\partial(\alpha \cap i^*(w)) = \partial(\alpha) \cap j^*(w) \in G_i(U_c, \mathbb{Z}/l),$$

where $j^* : K_1(V) \to K_1(V/\mathfrak{n})$ is the map induced by modding out by $\mathfrak{n}$. Since we assume $V/\mathfrak{n}$ is algebraically closed and $\operatorname{char}(V/\mathfrak{n}) \nmid l$, $j^*(w)$ is $l$-divisible, whence $\partial(\alpha) \cap j^*(w) = 0$ since $G_i(U_c, \mathbb{Z}/l)$ is $l$-torsion. $\square$

**Remark 5.3.** In fact, the previous result remains true (and the given proof remains valid) if the field $V/\mathfrak{n}$ is merely assumed to be closed under taking $l$-th roots.

The specialization map is closely related to the map $\tilde{\theta}$. In detail, suppose $(V, \mathfrak{n})$ is a pointed curve with $\mathfrak{n} = (t)$ and $V \to Q$ is a map of rings sending $t$ to $f$. Let $\mathfrak{m}$ be a maximal ideal of $Q$ with $f \in \mathfrak{m}$ such that $\mathfrak{m} \cap V = \mathfrak{n}$. Set $X = \mathrm{Spec}(Q) \setminus \mathfrak{m}$. Then $X \setminus X_c = \mathrm{Spec}(Q[1/f])$, $X_c = \mathrm{Spec}(Q/f) \setminus \mathfrak{m}$, and the specialization map $\sigma : G_i(Q[1/f], \mathbb{Z}/l) \to G_i(\mathrm{Spec}(Q/f) \setminus \mathfrak{m}, \mathbb{Z}/l)$ (defined by setting $z = [t]$) is given by

$$\sigma(\alpha) = \partial(\alpha \cap [f]).$$

We have proven:

**Proposition 5.4.** *Suppose $Q$ is a Noetherian ring, $\mathfrak{m}$ is a maximal ideal, $f \in \mathfrak{m}$ is a non-zero-divisor, and $l$ is a positive integer that is not divisible by* $\mathrm{char}(Q/\mathfrak{m})$, 2 *or* 3. *Set $R = Q/f$. If there exists a pointed curve $(V, \mathfrak{n})$ and a ring map $V \to Q$ that sends some generator $t$ of $\mathfrak{n}$ to $f$, then for all integers $i$ and $j$ the $\tilde{\theta}$ pairing fits into a commutative square*

$$\begin{array}{ccc} G_i(Q[1/f], \mathbb{Z}/l) \times K_j(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) & \xrightarrow{\tilde{\theta}} & G_{i+j-1}(R/\mathfrak{m}, l) \\ \Big\downarrow{\scriptstyle \sigma \times \mathrm{id}} & & \Big\uparrow{\scriptstyle \partial'} \\ G_i(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \times K_j(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) & \xrightarrow[(-\cap-)]{} & G_{i+j}(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \end{array} \tag{5.5}$$

**Example 5.6.** If $Q$ contains a field $k$, then we may take $(V, \mathfrak{n}) = (k[t], (t))$ and define $V \to Q$ to be the unique $k$-algebra map sending $t$ to $f$.

We will need to understand the behavior of the specialization map as we allow the base $(C, c)$ to vary in a suitably nice manner.

**Definition 5.7.** A *morphism of pointed curves*, say from $(C', c')$ to $(C, c)$, is a flat morphism of schemes $\phi : C' \to C$ such that $\phi^{-1}(c) = \{c'\}$ and the induced map on residue fields is an isomorphism $\kappa(c) \xrightarrow{\cong} \kappa(c')$. Equivalently, if $(C, c)$ corresponds to $(V, \mathfrak{n})$ and $(C', c')$ to $(V', \mathfrak{n}')$, then a morphism is a flat ring map $g : V \to V'$ such that $\mathfrak{n}V'$ is $\mathfrak{n}'$-primary and the induced map $V/\mathfrak{n} \xrightarrow{\cong} V'/\mathfrak{n}'$ is an isomorphism.

Since $g$ is necessarily injective, we often just write a morphism as if it were a ring extension $V \subseteq V'$. If $\mathfrak{n} = (t)$ and $\mathfrak{n} = (t')$ then, since $V$ and $V'$ are Dedekind domains, the condition that $\mathfrak{n}V'$ is $\mathfrak{n}'$-primary is equivalent to the existence of an equation $t = u'(t')^n$ for some integer $n \geq 1$ and unit $u' \in (V')^\times$.

**Definition 5.8.** A morphism of pointed curves is *unramified* if $C' \times_C \mathrm{Spec}(\kappa(c)) \cong \kappa(c')$ (via the canonical map), or, equivalently, if $\mathfrak{n}V = \mathfrak{n}'$. If $\mathfrak{n} = (t)$ and $\mathfrak{n}' = (t')$, being unramified is equivalent to $t = u't'$ for some $u' \in (V')^\times$.

A morphism of pointed curves is *finite* if the underlying map of schemes $\phi : C' \to C$ is finite, or, equivalently, if $V'$ is a finitely generated $V$-module. Since $V$

and $V'$ are Dedekind domains, $V \subseteq V'$ is finite if and only if the induced map on fields of fractions $E \subseteq E'$ is finite and $V'$ is the integral closure of $V$ in $E'$.

If $\phi : (C', c') \to (C, c)$ is a morphism of pointed curves and $p : X \to C$ is a morphism of Noetherian schemes, we will write $p' : X' \to C'$ for the pull-back of $p$ along $\phi$. Abusing notation, we write $\phi : X' \to X$ for the induced map, and also use $\phi$ for the induced maps on fibers. Observe that our hypotheses ensure that the induced map on fibers over the marked points is an isomorphism $\phi : X'_{c'} \xrightarrow{\cong} X_c$. Using also that $\phi^{-1}(c) = \{c'\}$, we have that both squares in

$$\begin{tikzcd} X' \setminus X'_{c'} \arrow[r] \arrow[d, "\phi"'] & X' \arrow[d, "\phi"'] & X'_{c'} \arrow[l] \arrow[d, "\phi"', "\cong"] \\ X \setminus X_c \arrow[r] & X & X_c \arrow[l] \end{tikzcd}$$

are Cartesian.

**Lemma 5.9.** *If $\phi : (C', c') \to (C, c)$ is a morphism pointed curves that is either finite or unramified, $p : X \to C$ is any morphism of Noetherian schemes, and $l$ is any integer, the diagram*

$$\begin{tikzcd} G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) \arrow[r, "\partial'"] \arrow[d, "\phi_*"'] & G_{i-1}(X'_{c'}, \mathbb{Z}/l) \arrow[d, "\phi_*"', "\cong"] \\ G_i(X \setminus X_c, \mathbb{Z}/l) \arrow[r, "\partial"] & G_{i-1}(X_c, \mathbb{Z}/l) \end{tikzcd}$$

*commutes.*

*Proof.* This follows from the fact that

$$\begin{tikzcd} \mathcal{G}(X'_{c'}) \arrow[r] \arrow[d, "\phi_*"'] & \mathcal{G}(X') \arrow[r] \arrow[d, "\phi_*"'] & \mathcal{G}(X' \setminus X'_{c'}) \arrow[d, "\phi_*"'] \\ \mathcal{G}(X_c) \arrow[r] & \mathcal{G}(X) \arrow[r] & \mathcal{G}(X \setminus X_c) \end{tikzcd}$$

is a homotopy commutative diagram of spectra. □

**Lemma 5.10.** *If $\phi : (C', c') \to (C, c)$ is either a finite or an unramified morphism of pointed curves, $p : X \to C$ is any morphism of Noetherian schemes, and $l$ is a positive integer not divisible by* $\operatorname{char}(\kappa(c))$, *the diagram*

$$\begin{tikzcd} G_i(X \setminus X_c, \mathbb{Z}/l) \arrow[r, "\sigma"] \arrow[d, "\phi^*"'] & G_i(X_c, \mathbb{Z}/l) \arrow[d, "\phi^*"', "\cong"] \\ G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) \arrow[r, "\sigma'"] & G_i(X'_{c'}, \mathbb{Z}/l) \end{tikzcd}$$

*commutes, where $\sigma = \sigma_{C,c,p}$ and $\sigma' = \sigma_{C',c',p'}$ are the specialization maps.*

*Proof.* Assume $\phi$ is finite. Since $\phi_* : G_i(X_c, \mathbb{Z}/l) \xrightarrow{\cong} G_i(X'_{c'}, \mathbb{Z}/l)$ is the inverse of $\phi^*$, it suffices to prove

$$\begin{tikzcd} G_i(X \setminus X_c, \mathbb{Z}/l) \arrow[r, "\sigma"] \arrow[d, "\phi^*"'] & G_i(X_c, \mathbb{Z}/l) \\ G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) \arrow[r, "\sigma'"] & G_{i-1}(X'_{c'}, \mathbb{Z}/l) \arrow[u, "\phi_*"', "\cong"] \end{tikzcd}$$

commutes. Choose any $z' \in K_1(V'[1/t'])$ as in the definition of the specialization map $\sigma'$. A special case of Lemma 5.9 gives that $\partial\phi_*(z') = 1$, and hence we may choose $z := \phi_*(z')$ in the definition of $\sigma$. Using Lemma 5.9 again gives

$$\phi_*\sigma'(\phi^*\alpha) = \phi_*\partial'(\phi^*(\alpha) \cap z') = \partial\phi_*(\phi^*(\alpha) \cap z').$$

By the projection formula, $\phi_*(\phi^*(\alpha) \cap z') = \alpha \cap \phi_*(z') = \alpha \cap z$, since $z = \phi_*(z')$. The result follows.

Now assume $\phi$ is unramified. Then

$$\begin{tikzcd} X'_{c'} \arrow[r] \arrow[d, "\cong"] & X' \arrow[d] \\ X_c \arrow[r] & X \end{tikzcd}$$

is a Cartesian square, and it follows that

$$\begin{tikzcd} \mathcal{G}(X_c) \arrow[r] \arrow[d, "\phi^*"'] & \mathcal{G}(X) \arrow[r] \arrow[d, "\phi^*"'] & \mathcal{G}(X \setminus X_c) \arrow[d, "\phi^*"'] \\ \mathcal{G}(X'_{c'}) \arrow[r] & \mathcal{G}(X') \arrow[r] & \mathcal{G}(X' \setminus X'_{c'}) \end{tikzcd}$$

is a homotopy commutative diagram. The boundary maps $\partial$ and $\partial'$ in the associated long exact sequences of homotopy groups (with any coefficients) thus commute with $\phi^*$.

Let $t \in \mathfrak{n}$ be any generator. Since $\phi$ is unramified, $t' := \phi(t)$ generates $\mathfrak{n}'$. Using $z = [t]$ and $z' = [t']$ for the definition of $\sigma$ and $\sigma'$, we have, for any $\alpha \in G_i(X \setminus X_c, \mathbb{Z}/l)$,

$$\sigma(\alpha) = \phi^*\partial(\alpha \cap [t]) = \partial'\big(\phi^*(\alpha \cap [t])\big) = \partial'\big(\phi^*(a) \cap [t']\big) = \sigma'(\phi^*(\alpha)),$$

since $\phi^*$ commutes with the cap product and $\phi^*([t]) = [t']$. □

The notion of the Henselization of a ring at a maximal ideal will be important for the rest of this paper. Let us recall its main properties. Given a Noetherian ring $Q$ and a maximal ideal $\mathfrak{m}$ of it, we write $Q^{\text{hen}}_{\mathfrak{m}}$ for the Henselization of $Q$ at $\mathfrak{m}$. We have:

- $Q^{\mathrm{hen}}_{\mathfrak{m}}$ is a Noetherian local ring with maximal ideal $\mathfrak{m}^{\mathrm{hen}}$, and the pair $(Q^{\mathrm{hen}}, \mathfrak{m}^{\mathrm{hen}})$ satisfies Hensel's lemma: given a monic polynomial $p(x) \in Q^{\mathrm{hen}}_{\mathfrak{m}}[x]$, if its image $\bar{p}(x) \in Q^{\mathrm{hen}}_{\mathfrak{m}}/\mathfrak{m}^{\mathrm{hen}}[t]$ has a simple root $a$, then $p(x)$ has a root $\alpha \in Q^{\mathrm{hen}}_{\mathfrak{m}}$ whose image in $Q^{\mathrm{hen}}_{\mathfrak{m}}/\mathfrak{m}^{\mathrm{hen}}$ is $a$.
- There are flat ring maps $Q \to Q_{\mathfrak{m}} \to Q^{\mathrm{hen}}_{\mathfrak{m}}$ and $Q_{\mathfrak{m}} \to Q^{\mathrm{hen}}_{\mathfrak{m}}$ is faithfully flat.
- We have $\mathfrak{m}Q^{\mathrm{hen}}_{\mathfrak{m}} = \mathfrak{m}^{\mathrm{hen}}$, or, in other words, the fiber of $\mathrm{Spec}(Q^{\mathrm{hen}}_{\mathfrak{m}}) \to \mathrm{Spec}(Q)$ over $\mathfrak{m}$ is $\mathrm{Spec}(\kappa(\mathfrak{m}^{\mathrm{hen}})) = \mathrm{Spec}(Q^{\mathrm{hen}}_{\mathfrak{m}}/\mathfrak{m}^{\mathrm{hen}})$.
- $Q_{\mathfrak{m}}$ is regular if and only if $Q^{\mathrm{hen}}_{\mathfrak{m}}$ is regular.
- [EGA IV$_4$ 1967, Théorème (18.6.9)] The fibers of $\mathrm{Spec}(Q^{\mathrm{hen}}_{\mathfrak{m}}) \to \mathrm{Spec}(Q)$ are spectra of finite products of algebraic, separable field extensions. That is, for every $\mathfrak{p} \in \mathrm{Spec}(Q)$, there are a finite number of primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_d$ in $Q^{\mathrm{hen}}_{\mathfrak{m}}$ such that $\mathfrak{q}_i \cap Q = \mathfrak{p}$ and, for each $i$, the induced field map $\kappa(\mathfrak{p}) \rightarrowtail \kappa(\mathfrak{q}_i)$ is separable algebraic.

For any pointed curve $(V, \mathfrak{n})$, let $V^{\mathrm{hen}}$ denote the Henselization of $V$ at $\mathfrak{n}$ and let $\mathfrak{n}^{\mathrm{hen}}$ denote also the maximal ideal of $V^{\mathrm{hen}}$. Then $V^{\mathrm{hen}}$ is also a Dedekind domain and $(V, \mathfrak{n}) \to (V^{\mathrm{hen}}, \mathfrak{n}^{\mathrm{hen}})$ is an unramified morphism of pointed curves.

In the following theorem, given a morphism $p : X \to \mathrm{Spec}(V)$ of Noetherian schemes we will consider the $G$-theory of the geometric generic fiber

$$X_{\overline{F}} := X \times_{\mathrm{Spec}(V)} \mathrm{Spec}(\overline{F}),$$

where $F$ is the field of fractions of a Henselian ring $V$ and $\overline{F}$ is its algebraic closure. Note that $X_{\overline{F}}$ may fail to be Noetherian and $G$-theory is ordinarily defined for just Noetherian schemes. (For non-Noetherian schemes, the category of coherent sheaves may fail to be an abelian category.) However, since $X_L := X \times_{\mathrm{Spec}\, V} \mathrm{Spec}\, L$ is Noetherian for every finite field extension $L$ of $F$, we have that $X_{\overline{F}}$ is a filtered colimit of Noetherian schemes with flat transition maps. We thus take, as the definition, the $G$-theory spectrum of $X_{\overline{F}}$ to be the filtered colimit of spectra

$$\mathcal{G}(X_{\overline{F}}) := \operatorname*{colim}_{L} \mathcal{G}(X_L)$$

indexed by the finite field extensions of $F$ contained in $\overline{F}$. The associated $G$-groups are thus also given by colimits:

$$G_n(X_{\overline{F}}) = \operatorname*{colim}_{L} G_n(X_L).$$

**Remark 5.11.** At least when $X = \mathrm{Spec}(A)$ is affine, $G_n(X_L)$ may be interpreted as the $K$-theory of an abelian category. Indeed, more generally, if $A$ is a filtered colimit $A = \mathrm{colim}_{i \in I} A_i$ of Noetherian rings with flat transition maps, then $G_n(A) := \varinjlim_{i \in I} G_n(A_i)$ is the $K$-theory of the category of finitely presented $A$-modules and the latter forms an abelian category under these assumptions.

In later sections we will be mostly interested in the case where $X_F \to \mathrm{Spec}(F)$ is smooth, so that each transition map in the colimit giving $X_{\overline{F}}$ is a flat morphism of regular rings, and we will also be interested only in the case when $X$ is affine. In this situation, $G$-theory and $K$-theory coincide, as we now explain.

More generally, suppose $A = \mathrm{colim}_{i \in I} A_i$ is a filtered colimit of regular Noetherian rings with flat transition maps. Even if $A$ is not Noetherian, the correct notion of the $K$-theory of $A$ is unambiguous: it is the $K$-theory of the exact category $\mathcal{P}(A)$ of finitely generated projective $A$-modules. Moreover, we have

$$K_n(A) = \varinjlim_{i \in I} K_n(A_i)$$

for all $n$. Since we assume each $A_i$ is regular Noetherian, each natural map $K_n(A_i) \to G_n(A_i)$ is an isomorphism and thus the canonical map

$$K_n(A) \xrightarrow{\cong} G_n(A)$$

is an isomorphism. In particular, $K_n(X_{\overline{F}}) \cong G_n(X_{\overline{F}})$ provided $X_F \to \mathrm{Spec}(F)$ is smooth and $X$ is affine.

**Theorem 5.12.** *Given a pointed curve* $(V, \mathfrak{n})$ *and a morphism* $p : X \to \mathrm{Spec}(V)$ *of Noetherian schemes, let* $X_{\overline{F}} = X \times_{\mathrm{Spec}(V)} \mathrm{Spec}(\overline{F})$, *where* $\overline{F}$ *is the algebraic closure of the field of fractions of the Henselization* $V^{\mathrm{hen}}$ *of* $V$ *at* $\mathfrak{n}$. *For any positive integer* $l$ *such that* $\mathrm{char}(\kappa(c)) \nmid l$, *the specialization map* $\sigma : G_i(X \setminus X_c, \mathbb{Z}/l) \to G_i(X_c, \mathbb{Z}/l)$ *factors through* $G_i(X_{\overline{F}})$; *i.e., there is a commutative triangle*

$$\begin{tikzcd}
G_i(X \setminus X_c, \mathbb{Z}/l) \arrow[rr, "\sigma_{C,c,p}"] \arrow[dr] & & G_i(X_c, \mathbb{Z}/l) \\
& G_i(X_{\overline{F}}, \mathbb{Z}/l) \arrow[ur] &
\end{tikzcd}$$

*Moreover, if* $l$ *is also not divisible by* 2 *or* 3, *then the direct sum indexed over* $i \geq 0$ *of these morphisms are graded* $K_*(V, \mathbb{Z}/l)$*-module homomorphisms.*

*Proof.* As discussed above, the map $(V, \mathfrak{n}) \to (V^{\mathrm{hen}}, \mathfrak{n})$ is unramified and so, using Lemma 5.10, we may assume without loss of generality that $(V, \mathfrak{n})$ is Henselian with field of fractions $F$. Note that the integral closure $\overline{V}$ of $F$ is a filtered colimit of rings of the form $V'$, where $V'$ is the integral closure of $V$ in some finite field extension $F \subseteq F'$ of the field of fractions of $V$. Using Lemma 5.10 again, we have a commutative triangle

$$\begin{tikzcd}
G_i(X \setminus X_c, \mathbb{Z}/l) \arrow[r, "\sigma"] \arrow[d] & G_i(X_c, \mathbb{Z}/l) \\
G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) \arrow[ur, "(\phi^*)^{-1} \circ \sigma'"'] &
\end{tikzcd}$$

for each such $V'$. Moreover, given maps $V \rightarrowtail V' \rightarrowtail V''$ associated to a chain of finite field extensions $F \subseteq F' \subseteq F''$, the diagram

$$\begin{tikzcd}
G_i(X \setminus X_c, \mathbb{Z}/l) \arrow[rr, "\sigma"] \arrow[d] & & G_i(X_c, \mathbb{Z}/l) \\
G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) \arrow[urr, "\sigma'"] \arrow[d] & & \\
G_i(X'' \setminus X''_{c''}, \mathbb{Z}/l) \arrow[uurr, "\sigma''"] & &
\end{tikzcd}$$

commutes. Taking colimits thus gives a commutative triangle

$$\begin{tikzcd}
G_i(X \setminus X_c, \mathbb{Z}/l) \arrow[r, "\sigma"] \arrow[d] & G_i(X_c, \mathbb{Z}/l) \\
\operatorname{colim}_{F \subseteq F'} G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) \arrow[ur] &
\end{tikzcd}$$

The first result follows, since $\operatorname{colim}_{F \subseteq F'}(X' \setminus X'_{c'}) \cong X_{\overline{F}}$ and

$$G_i(X_{\overline{F}}, \mathbb{Z}/l) := \operatorname*{colim}_{F \subseteq F'} G_i(X' \setminus X'_{c'}, \mathbb{Z}/l).$$

For the final claim, note that every morphism occurring in this proof commutes with multiplication by any fixed element of $K_*(V, \mathbb{Z}/l)$. $\square$

**Corollary 5.13.** *Suppose $Q$ is a Noetherian ring, $\mathfrak{m}$ is a maximal ideal, $f \in \mathfrak{m}$ is a non-zero-divisor, and $l$ is a positive integer that is not divisible by* $\operatorname{char}(Q/\mathfrak{m})$, 2 *or* 3. *Set $R = Q/f$. If there exists a pointed curve $(V, \mathfrak{n})$ and a ring map $V \to Q$ that sends some generator $t$ of $\mathfrak{m}$ to $f$, then for all $i$ and $j$ the $\tilde{\theta}$ pairing fits into a commutative square*

$$\begin{tikzcd}
G_i(Q[1/f], \mathbb{Z}/l) \times K_j(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \arrow[r, "\tilde{\theta}"] \arrow[d] & K_{i+j-1}(R/\mathfrak{m}, \mathbb{Z}/l) \\
G_i(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \overline{F}, \mathbb{Z}/l) \times K_j(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \arrow[ur] &
\end{tikzcd}$$

*where $Q^{\text{hen}}_{\mathfrak{m}}$ is the Henselization of $Q$ at $\mathfrak{m}$ and $\overline{F}$ is the algebraic closure of the field of fractions $F$ of $V^{\text{hen}}$. Moreover, the direct sum indexed by $i, j \geq 0$ of these morphisms are graded $K_*(V, \mathbb{Z}/l)$-module homomorphisms.*

*If $F \to Q \otimes_V F$ is smooth, then*

$$G_i(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \overline{F}, \mathbb{Z}/l) \cong K_i(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \overline{F}, \mathbb{Z}/l).$$

*Proof.* Applying Proposition 4.6 to the flat map $Q \to Q^{\mathrm{hen}}_{\mathfrak{m}}$ allows us to reduce to the case where $Q = Q^{\mathrm{hen}}_{\mathfrak{m}}$. The result then follows from Proposition 5.4 and Theorem 5.12. □

**Remark 5.14.** In fact, Corollary 5.13 remains valid even if $l = 0$ (i.e., with $\mathbb{Z}$-coefficients). Since we shall not need this fact in this paper, we omit its proof.

## 6. Étale (Bott inverted) *K*-theory

Recall that the vanishing result of Buchweitz and van Straten uses topological *K*-theory, not algebraic *K*-theory. In a general characteristic setting, the best replacement for topological *K*-theory is étale *K*-theory (with finite coefficients). This leads us to the goal of proving that, under suitable hypotheses, the pairing

$$\tilde{\theta} : G_0(Q[1/f], \mathbb{Z}/l) \times K_1(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \to \mathbb{Z}/l$$

factors through the analogous pairing involving étale $G$- and $K$-theory.

This is roughly what we achieve in this section. But, to avoid some nagging issues in the foundations of étale *K*-theory, it proves simpler to use instead the theory obtained from algebraic *K*-theory (and $G$-theory) with finite coefficients by inverting the so-called "Bott element". Since the resulting theory is closely related to topological *K*-theory, we will write it as $K^{\mathrm{top}}$ (and $G^{\mathrm{top}}$); see Definition 6.1 below. Motivation for our approach is provided by Thomason's theorem [1985, Theorem 4.1] (recently extended by Rosenschon and Østvær [2006, Theorem 4.3]), which says, roughly, that "Bott inverted algebraic *K*-theory with $\mathbb{Z}/l$ coefficients and étale *K*-theory with $\mathbb{Z}/l$ coefficients coincide".

**Definition 6.1.** For any scheme $X$ and prime $l$, let $\mathcal{K}(X, \mathbb{Z}/l)$ be the result of smashing the algebraic *K*-theory spectrum $\mathcal{K}(X)$ of $X$ with the mod-*l* Moore space and, for any Noetherian scheme $X$, let $\mathcal{G}(X, \mathbb{Z}/l)$ be the result of smashing $\mathcal{G}(X)$ with the mod-*l* Moore space.

Let $\mathcal{KU}$ denote the spectrum representing (two-periodic) topological *K*-theory and let $L_{\mathcal{KU}}\mathcal{E}$ denote the Bousfield localization of a spectrum $\mathcal{E}$ at $\mathcal{KU}$.

Finally, for any scheme $X$ and prime $l$, define

$$K^{\mathrm{top}}_n(X, \mathbb{Z}/l) := \pi_n L_{\mathcal{KU}}\mathcal{K}(X, \mathbb{Z}/l), \quad n \in \mathbb{Z},$$

and, for any Noetherian scheme $X$, define

$$G^{\mathrm{top}}_n(X, \mathbb{Z}/l) := \pi_n L_{\mathcal{KU}}\mathcal{G}(X, \mathbb{Z}/l), \quad n \in \mathbb{Z}.$$

Let us bring these definitions down to Earth a bit. Assume for simplicity that $l \geq 5$ and define $\mu_l = e^{2\pi i/l}$, a primitive $l$-th root of unity in $\mathbb{C}^\times$, and consider the ring $\mathbb{Z}[\mu_l]$. Recall (from [Thomason 1985, Appendix A], for example) that the *Bott element* is a certain canonical element $\beta \in K_2(\mathbb{Z}[\mu_l], \mathbb{Z}/l)$, which maps to $[\mu_l] \in K_1(\mathbb{Z}[\mu_l])$ under the boundary map. If $X$ is a scheme over $\mathrm{Spec}(\mathbb{Z}[\mu_l])$, then we obtain by pullback a Bott element $\beta \in K_2(X, \mathbb{Z}/l)$.

For example, if $X = \mathrm{Spec}(k)$ is an algebraically closed field with $\mathrm{char}(k) \neq l$, then there is a map $\mathbb{Z}[\mu_l] \to k$, specified by a choice of a primitive $l$-th root of unity for $k$. Moreover, the boundary map of the long exact coefficients sequence for the $K$-theory of $k$ determines (using [Suslin 1983; 1984]) an isomorphism

$$K_2(k, \mathbb{Z}/l) \xrightarrow{\cong} \mu_l(k),$$

and a Bott element $\beta$ for $k$ maps to the chosen primitive $l$-th root of unity in $k$ under this map. More generally, if $V$ is a Henselian DVR with algebraically closed residue field $k$ such that $\mathrm{char}(k) \neq l$, then we have

$$K_i(V, \mathbb{Z}/l) \cong K_i(k, \mathbb{Z}/l) \cong \begin{cases} \mathbb{Z}/l & \text{if } i \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

by [Suslin 1983; 1984], and a Bott element for $V$ may also be specified by choosing a primitive $l$-th root of unity in $k$.

Given a Bott element $\beta \in K_2(\mathbb{Z}[\mu_l], \mathbb{Z}/l)$ for some prime $l \geq 5$, the element $\beta$ acts on the $K$- and $G$-groups of all schemes over $\mathbb{Z}[\mu_l]$, and this action is compatible with the localization long exact sequences for such schemes and it commutes with the cup and cap products. Thus, formally inverting the action of $\beta$ preserves all the structure needed in this paper.

**Lemma 6.2.** *Let $l \geq 5$ be a prime, $X$ a scheme over $\mathbb{Z}[\mu_l]$, and let $\beta \in K_2(X, \mathbb{Z}/l)$ be the associated Bott element for $X$. There is a natural isomorphism*

$$K_*^{\mathrm{top}}(X, \mathbb{Z}/l) \cong K_*(X, \mathbb{Z}/l)[1/\beta],$$

*where $K_*(X, \mathbb{Z}/l)[1/\beta]$ denotes the graded ring obtained by inverting the homogeneous central element $\beta$ in the graded ring $K_*(X, \mathbb{Z}/l)$.*

*If $X$ is Noetherian, there is a natural isomorphism*

$$G_*^{\mathrm{top}}(X, \mathbb{Z}/l) \cong G_*(X, \mathbb{Z}/l)[1/\beta],$$

*where $G_*(X, \mathbb{Z}/l)[1/\beta]$ is the localization by $\beta$ of the graded $K_*(X, \mathbb{Z}/l)$-module $G_*(X, \mathbb{Z}/l)$.*

*Proof.* See [Thomason 1985, Section A.14]. □

**Remark 6.3.** Since $\beta$ has degree two, one can view $K_*^{\mathrm{top}}$ and $G_*^{\mathrm{top}}$ as $\mathbb{Z}/2$-graded abelian groups.

If $V$ is a Henselian DVR with algebraically closed residue field $k$ such that $\mathrm{char}(k) \neq l$, $Q$ is a $V$-algebra, $\mathfrak{m}$ is a maximal ideal of $Q$, and $f \in \mathfrak{m}$ is a non-zero-divisor, then the family of pairings

$$\tilde{\theta} : G_i(Q[1/f], \mathbb{Z}/l) \times K_j(\mathrm{Spec}(Q/f) \setminus \mathfrak{m}, \mathbb{Z}/l) \to K_{i+j-1}(Q/\mathfrak{m}, \mathbb{Z}/l)$$

forms a pairing of graded $K_*(V, \mathbb{Z}/l)$-modules, and hence, upon inverting the action of the Bott element $\beta \in K_2(V, \mathbb{Z}/l)$, we obtain the pairing

$$\tilde{\theta}^{\mathrm{top}} : G_i^{\mathrm{top}}(Q[1/f], \mathbb{Z}/l) \times K_j^{\mathrm{top}}(\mathrm{Spec}(Q/f) \setminus \mathfrak{m}, \mathbb{Z}/l) \to K_{i+j-1}^{\mathrm{top}}(Q/\mathfrak{m}, \mathbb{Z}/l).$$

If $Q/\mathfrak{m}$ is algebraically closed (e.g., if $Q$ is a finite-type $V$-algebra or the localization of such at a maximal ideal), then by [Suslin 1983; 1984] we have

$$K_*(Q/\mathfrak{m}, \mathbb{Z}/l) = \mathbb{Z}/l[\beta]$$

and hence

$$K_*^{\mathrm{top}}(Q/\mathfrak{m}, \mathbb{Z}/l) = \mathbb{Z}/l[\beta, \beta^{-1}].$$

In particular, $K_0(Q/\mathfrak{m}, \mathbb{Z}/l) \cong K_0^{\mathrm{top}}(Q/\mathfrak{m}, \mathbb{Z}/l) \cong \mathbb{Z}/l$.

**Proposition 6.4.** *Assume $V$ is a DVR with algebraically closed residue field $k$ and field of fractions $F$, $Q$ is a flat $V$-algebra, $t$ is a uniformizing parameter of $V$ that maps to an element $f \in Q$, and $\mathfrak{m}$ is a maximal ideal of $Q$ containing $f$ such that $Q/\mathfrak{m}$ is algebraically closed. Set $R = Q/f$. For any prime $l \geq 5$ not divisible by* $\mathrm{char}(k)$, *there is a commutative diagram*

$$\begin{tikzcd}
G_1(Q[1/f], \mathbb{Z}/l) \times K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \arrow[d] \arrow[dr, "\tilde{\theta}"] & \\
G_1^{\mathrm{top}}(Q[1/f], \mathbb{Z}/l) \times K_0^{\mathrm{top}}(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \arrow[d] \arrow[r, "\tilde{\theta}^{\mathrm{top}}"] & \mathbb{Z}/l \\
G_1^{\mathrm{top}}(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \overline{F}, \mathbb{Z}/l) \times K_0^{\mathrm{top}}(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \arrow[ur] &
\end{tikzcd}$$

*Proof.* This diagram is obtained from the commutative diagrams in Corollaries 4.10 and 5.13 by inverting the action of $\beta \in K_2(V, \mathbb{Z}/l)$. □

**Corollary 6.5.** *With the assumptions of Proposition 6.4, if $G_1^{\mathrm{top}}(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \overline{F}, \mathbb{Z}/l) = 0$ for infinitely many primes $l \geq 5$, then $\theta^R(M, N) = 0$ for all finitely generated $R$-modules $M$ and $N$ such that $N_{\mathfrak{p}}$ has finite projective dimension for all $\mathfrak{p} \neq \mathfrak{m}$.*

*Proof.* The proposition implies that $\theta^R(M, N)$ is a multiple of $l$ for infinitely many primes $l$. □

## 7. Vanishing of $K_1^{\mathrm{top}}$

In this section, we combine a theorem of Rosenschon and Østvær [2006, Theorem 4.3], which is an improvement of [Thomason 1985, Theorem 4.1], with a theorem of Illusie [2003, Corollaire 2.10] to establish the vanishing of the odd-degree topological $K$-groups with $\mathbb{Z}/l$-coefficients of the "algebraic Milnor fiber", $\mathrm{Spec}(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \overline{F})$, in certain cases.

***The Thomason–Rosenschon–Østvær theorem.***

**Definition 7.1.** Fix a prime $l$. The *mod-$l$ étale cohomological dimension* of a scheme $X$, written $\mathrm{cd}_l(X) \in \mathbb{N} \cup \{\infty\}$, is defined as

$$\mathrm{cd}_l(X) = \sup\{i \mid H^i_{\text{ét}}(X, \mathcal{F}) \neq 0 \text{ for some } l\text{-power torsion étale sheaf } \mathcal{F}\}.$$

If $F$ is algebraically closed (or even just separably closed), then $\mathrm{cd}_l(F) = 0$ since the étale topology of such a field is trivial. If $E$ is a field extension of $F$ of transcendence degree $d$, then by [Serre 1994, Proposition II.4.2.11] we have

$$\mathrm{cd}_l(E) \leq \mathrm{cd}_l(F) + d.$$

It follows that if $E$ is a field of transcendence degree $M$ over an algebraically closed field then

$$\mathrm{cd}_l(E) \leq M. \tag{7.2}$$

**Definition 7.3.** For an odd prime $l$, a scheme $X$ is *$l$-good* if $X$ is quasiseparated, quasicompact and of finite Krull dimension, $l$ is a unit of $\Gamma(X, \mathcal{O}_X)$, and there is a uniform finite bound on the mod-$l$ étale cohomological dimensions of all the residue fields of $X$. A commutative ring $A$ is *$l$-good* if $\mathrm{Spec}(A)$ is $l$-good.

**Remark 7.4.** The definition of $l$-good is motivated by the hypotheses of the Rosenschon–Østvær theorem. The correct version of $l$-good for $l = 2$ involves "virtual étale cohomological dimension".

Note that an affine scheme $X = \mathrm{Spec}(A)$ is automatically separated (and hence quasiseparated) and quasicompact. So, $A$ is $l$-good if and only if $A$ has finite Krull dimension, $l$ is a unit of $A$, and there is a uniform finite bound on $\mathrm{cd}_l(\kappa(\mathfrak{p}))$ as $\mathfrak{p}$ ranges over all primes ideals of $A$.

If $X$ is a separated scheme of finite type over an algebraically closed field $k$, then the transcendence degree of each of its residue fields is at most $\dim(X) < \infty$, and hence we have $\mathrm{cd}_l(\kappa(x)) \leq \dim(X)$ for all $x \in X$. It follows that, for such $X$, if $l \neq \mathrm{char}(k)$ then $X$ is $l$-good.

We will need a generalization of this fact that involves Henselizations. Recall that, given a commutative ring $V$, a $V$-algebra $Q$ is *essentially of finite type* over $V$ if $Q$ is the localization of a finitely generated $V$-algebra by some multiplicatively closed subset.

**Lemma 7.5.** *Suppose $V$ is a Noetherian ring, $\mathfrak{p}$ is any prime of $V$, $Q$ is a $V$-algebra essentially of finite type, and $\mathfrak{m}$ is a maximal ideal of $Q$. Let $Q^{\mathrm{hen}}_{\mathfrak{m}}$ denote the Henselization of $Q$ at $\mathfrak{m}$. If $l \neq \mathrm{char}(\kappa(\mathfrak{p}))$, then $Q^{\mathrm{hen}}_{\mathfrak{m}} \otimes_V \overline{\kappa(\mathfrak{p})}$ is $l$-good, where $\overline{\kappa(\mathfrak{p})}$ is the algebraic closure of $\kappa(\mathfrak{p})$.*

**Remark 7.6.** The lemma generalizes to schemes: if $S$ is a Noetherian scheme, $p : X \to S$ is a morphism of schemes essentially of finite type, and $x \in X$ is any closed point, then each geometric fiber of $X_x^{\mathrm{hen}} \to S$ is $l$-good for all $l$ not equal to the characteristic of the fiber.

*Proof.* Since $Q$ is essentially of finite type over $V$ and $V$ is Noetherian, $Q$ is also Noetherian. It follows [EGA IV$_4$ 1967, Théorème (18.6.6)] that $Q_{\mathfrak{m}}^{\mathrm{hen}}$ is Noetherian and, since it is local, $\dim(Q_{\mathfrak{m}}^{\mathrm{hen}}) < \infty$.

Again using that $Q$ is essentially of finite type over $V$, there is a bound $M$ such that for each prime $\mathfrak{q} \in \mathrm{Spec}(Q)$, the transcendence degree of $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{q} \cap V)$ is at most $M$. The residue fields of $Q_{\mathfrak{m}}^{\mathrm{hen}}$ are algebraic (and separable) over the corresponding residue fields of $Q$ by [EGA IV$_4$ 1967, Théorème (18.6.9)]. It follows that for each prime $\mathfrak{q} \in \mathrm{Spec}(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \kappa(\mathfrak{p}))$, the transcendence degree of $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$ is at most $M$.

To simplify notation, we state and prove a more general assertion: if $k$ is any field, $A$ is a commutative $k$-algebra of finite Krull dimension, and there exists a finite bound $M$ such that the transcendence degree of $\kappa(\mathfrak{q})$ over $k$ is at most $M$ for all $\mathfrak{q} \in \mathrm{Spec}(A)$, then $A \otimes_k \bar{k}$ is $l$-good for all $l \neq \mathrm{char}(k)$. The lemma follows from the case $k = \kappa(\mathfrak{p})$ and $A = Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \kappa(\mathfrak{p})$.

The field $\bar{k}$ is the filtered colimit of the finite extensions $L$ of $k$ contained in it, and hence $A \otimes_k \bar{k}$ is the filtered colimit of the collection of rings $\{A \otimes_k L\}$. For each $L$, each residue field of $A \otimes_k L$ is a finite extension of the corresponding residue field of $A$. As with any colimit of rings, the residue field of $A \otimes_k \bar{k}$ at a prime $\mathfrak{q}$ is the filtered colimit of the residue fields $\mathfrak{q} \cap (A \otimes_k L)$. We conclude that for each $\mathfrak{q} \in \mathrm{Spec}(A \otimes_k \bar{k})$ the residue field $\kappa(\mathfrak{q})$ is algebraic over $\kappa(\mathfrak{q} \cap A)$ and hence has transcendence degree at most $M$ over $k$. Since $\kappa(\mathfrak{q})$ contains $\bar{k}$, it has transcendence degree at most $M$ over $\bar{k}$ as well. By (7.2) we have $\mathrm{cd}_l(\kappa(\mathfrak{q})) \leq M$ for all $\mathfrak{q} \in \mathrm{Spec}(A \otimes_k \bar{k})$.

Finally, since $A \subseteq A \otimes_k \bar{k}$ is an integral extension, $\dim(A \otimes_k \bar{k}) = \dim(A) < \infty$. □

The following result is an improvement of a celebrated theorem of Thomason [1985, Theorem 4.1]:

**Theorem 7.7** [Rosenschon and Østvær 2006, Theorem 4.3]. *If $X$ is an $l$-good scheme for a prime $l \geq 5$, there is a strongly convergent, right half-plane spectral sequence*

$$E_2^{p,q} \Longrightarrow K_{q-p}^{\mathrm{top}}(X, \mathbb{Z}/l),$$

*where*

$$E_2^{p,q} = \begin{cases} H_{\text{ét}}^p(X, \mu_l^{\otimes i}) & \text{if } q = 2i, \\ 0 & \text{if } q \text{ is odd}, \end{cases}$$

*and the differential $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$ on the $r$-th page has bidegree $(r, r-1)$.*

**Remarks 7.8.** (1) In their original paper, the abutment of this spectral sequence is $\pi_{q-p}L_{\mathcal{KU}}\mathcal{K}^B(X, \mathbb{Z}/l)$, where $\mathcal{K}^B$ denotes *Bass's algebraic K-theory spectrum*. Since $\mathcal{K} \to \mathcal{K}^B$ induces an isomorphism on nonnegative homotopy groups, the natural map $L_{\mathcal{KU}}\mathcal{K} \xrightarrow{\sim} L_{\mathcal{KU}}\mathcal{K}^B$ is a weak equivalence.

(2) The integer $i$ in $\mu_l^{\otimes i}$ is allowed to be negative. For $i < 0$, $\mu^{\otimes i} = (\mu_l^{-1})^{\otimes|i|}$ where $\mu_l^{-1}$ is the $\mathbb{Z}/l$-linear dual of $\mu_l$.

(3) If $X$ is a scheme over an algebraically closed field of characteristic not equal to $l$, then upon choosing a primitive $l$-th root of unity, we may identify $\mu_l^{\otimes i}$ with $\mathbb{Z}/l$ for all $i \in \mathbb{Z}$.

(4) We have assumed $l \geq 5$ only to avoid some technical complications and because the cases $l = 2, 3$ will not be important for our purposes. But, appropriately interpreted, this theorem remains valid for $l \in \{2, 3\}$.

The following result gives the special case of the theorem that we will need:

**Corollary 7.9.** *Assume $V$ is a Noetherian domain with field of fractions $F$, $Q$ is essentially of finite type over $V$, the generic fiber of* $\operatorname{Spec}(Q) \to \operatorname{Spec}(V)$ (*namely*, $\operatorname{Spec}(Q \otimes_V F) \to \operatorname{Spec}(F)$) *is essentially smooth*, $\mathfrak{m}$ *is a maximal ideal of $Q$, and $l$ is a prime such that $l \geq 5$ and $l \neq$* $\operatorname{char}(F)$. *Then there is a strongly convergent spectral sequence*

$$E_2^{p,q} \Longrightarrow K_{q-p}^{\text{top}}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l),$$

*where*

$$E_2^{p,q} = \begin{cases} H_{\text{ét}}^p(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mu_l^{\otimes i}) & \text{if } q = 2i, \\ 0 & \text{if } q \text{ is odd}, \end{cases}$$

*and the differential on the $r$-th page has bidegree* $(r, r-1)$.

*Proof.* This follows from Lemma 7.5 and Theorem 7.7. □

***Illusie's theorem.*** We will need to make the following assumptions:

**Assumptions 7.10.** Assume $(V, k, F, Q, \mathfrak{m}, f, l)$ satisfy:

(1) $V$ is a Henselian DVR with algebraically closed residue field $k$ and field of fractions $F$.

(2) $Q$ is a regular ring, $\mathfrak{m}$ is a maximal ideal of $Q$, and $f \in \mathfrak{m}$.

(3) There is a flat, finite-type map $\operatorname{Spec}(Q) \to \operatorname{Spec}(V)$ of affine schemes of relative dimension $n$ such that the associated map of rings sends some uniformizing parameter $t \in V$ to $f \in Q$.

(4) The morphism $\operatorname{Spec}(Q) \to \operatorname{Spec}(V)$ is smooth at every point except, possibly, $\mathfrak{m} \in \operatorname{Spec}(Q)$. Notice in particular that the generic fiber $\operatorname{Spec}(Q \otimes_V F) \to \operatorname{Spec}(F)$ is smooth.

(5) The morphism $\mathrm{Spec}(Q) \to \mathrm{Spec}(V)$ is a complete intersection near $\mathfrak{m}$ — that is, for some $g \in Q \setminus \mathfrak{m}$, $Q[1/g]$ the quotient of a smooth $V$-algebra by a regular sequence.

(6) $l$ is prime not equal to 2, 3 or $\mathrm{char}(k)$. Notice this implies $l \neq \mathrm{char}(F)$ too.

**Theorem 7.11** [Illusie 2003, Corollaire 2.10]. *Under Assumptions 7.10,*

$$H^j_{\text{ét}}((Q^{\text{hen}}_{\mathfrak{m}}) \otimes_V \bar{F}, \mathbb{Z}/l) = 0$$

*if $j \notin \{0, n\}$, where $Q^{\text{hen}}_{\mathfrak{m}}$ is the Henselization of $Q$ at $\mathfrak{m}$ and $\bar{F}$ is the algebraic closure of $F$.*

**Remark 7.12.** Illusie's theorem is the analogue of Milnor's theorem, stating that the Milnor fiber of an analytic isolated singularity is homotopy equivalent to a bouquet of $n$-dimensional spheres.

**Corollary 7.13.** *Under Assumptions 7.10, if $n$ is even then*

$$K^{\text{top}}_1(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}, \mathbb{Z}/l) = 0.$$

*Proof.* The assumptions allow us to apply Corollary 7.9, giving a strongly convergent spectral sequence

$$E_2^{p,q} \Longrightarrow K^{\text{top}}_{q-p}(Q \otimes_V \bar{F}, \mathbb{Z}/l),$$

where

$$E_2^{p,q} = \begin{cases} H^p_{\text{ét}}(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}, \mu_l^{\otimes i}) & \text{if } q = 2i, \\ 0 & \text{if } q \text{ is odd,} \end{cases}$$

and the differential on the $r$-th page has bidegree $(r, r-1)$. Since $\bar{F}$ is algebraically closed, $\mu_l \cong \mathbb{Z}/l$ (noncanonically) and thus Illusie's theorem applies to give that the only nonzero $E_2$-terms are $E_2^{n,2i}$ and $E_2^{0,2i}$. Since $n$ is even, these terms only contribute to the even degree part of $K^{\text{top}}_*$. □

**Remark 7.14.** The proof also shows that, when $n$ is even, there exists an exact sequence

$$0 \to H^n_{\text{ét}}(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}, \mathbb{Z}/l) \to K^{\text{top}}_0(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}, \mathbb{Z}/l) \to H^0_{\text{ét}}(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}, \mathbb{Z}/l) \to 0.$$

Similarly, when $n$ is odd, there exists an exact sequence

$$\begin{aligned} 0 \to K^{\text{top}}_0(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}, \mathbb{Z}/l) &\to H^0_{\text{ét}}(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}, \mathbb{Z}/l) \\ &\to H^n_{\text{ét}}(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}, \mathbb{Z}/l) \to K^{\text{top}}_1(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}, \mathbb{Z}/l) \to 0. \end{aligned}$$

**Theorem 7.15.** *If conditions* (1)–(5) *of Assumptions 7.10 hold and $n$ is even, then $\theta^R(M, N) = 0$ for all finitely generated $R$-modules $M$ and $N$, where $R = Q/f$.*

*Proof.* The only singular point of $R/f$ is $\mathfrak{m}$ and hence $\theta^R(M,N)$ is defined for all finitely generated $R$-modules.

The hypotheses ensure that $Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}$ is a filtered colimit of regular (Noetherian) rings with flat transition maps, and so

$$K_1^{\text{top}}(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}, \mathbb{Z}/l) \cong G_1^{\text{top}}(Q^{\text{hen}}_{\mathfrak{m}} \otimes_V \bar{F}, \mathbb{Z}/l).$$

The theorem is thus an immediate consequence of Corollaries 6.5 and 7.13. $\square$

**Corollary 7.16.** *Let $V$ be a Dedekind domain, $\mathfrak{n}$ be a maximal ideal of $V$ such that $V/\mathfrak{n}$ is a perfect field, and $Q$ be a regular, flat $V$-algebra of finite type. Assume the singular locus of the morphism* $\operatorname{Spec}(Q) \to \operatorname{Spec}(V)$ *is a finite set* $\{\mathfrak{m}_1, \dots, \mathfrak{m}_l\}$ *of maximal ideals of $Q$ that lie over $\mathfrak{n}$ and that the morphism* $\operatorname{Spec}(Q) \to \operatorname{Spec}(V)$ *is a complete intersection in an open neighborhood of each of the $\mathfrak{m}_i$.*

*Then $R := Q/f$ is a hypersurface with only isolated singularities and, if* $\dim(R)$ *is even,*

$$\theta^R(M,N) = 0$$

*for all finitely generated $R$-modules $M$ and $N$.*

*Proof.* We may assume $V$ is local and hence a DVR. Then $R$ is the hypersurface $Q/f$, where $f$ is the image in $Q$ of a chosen uniformizing parameter $t$ of $V$. The non-regular locus of $R$ is $\{\mathfrak{m}_1, \dots, \mathfrak{m}_l\}$ and we have

$$\theta^R(M,N) = \sum_{i=1}^{l} \theta^{R_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}, N_{\mathfrak{m}_i})$$

for all finitely generated $R$-modules $M$ and $N$. It suffices to prove $\theta^{R_{\mathfrak{m}_i}} \equiv 0$ for all $i$, and thus, upon replacing $\operatorname{Spec}(Q)$ by a sufficiently small affine open neighborhood of each $\mathfrak{m}_i$, we may assume that $l = 1$, that $\mathfrak{m} := \mathfrak{m}_1$ is the only singular point of the morphism $\operatorname{Spec}(Q) \to \operatorname{Spec}(V)$, and that this morphism is a complete intersection.

Let $V^{\text{sh}}_{\mathfrak{n}}$ denote the *strict Henselization* of $V$ at its maximal ideal $\mathfrak{n}$. Recall from [EGA IV$_4$ 1967, §18.8] that there is a faithfully flat local ring map $V \to V^{\text{sh}}_{\mathfrak{n}}$, that $\mathfrak{n}^{\text{sh}} := \mathfrak{n}V^{\text{sh}}_{\mathfrak{n}}$ is the maximal ideal of $V^{\text{sh}}_{\mathfrak{n}}$, and that the induced map on residue fields $V/\mathfrak{n} \rightarrowtail V^{\text{sh}}_{\mathfrak{n}}/\mathfrak{n}^{\text{sh}}$ is a (initially chosen) separable closure of $V/\mathfrak{n}$. Since we assume $V/\mathfrak{n}$ is perfect, the residue field of $V^{\text{sh}}_{\mathfrak{n}}$ is, in fact, algebraically closed.

Set $Q' = Q \otimes_V V^{\text{sh}}_{\mathfrak{n}}$ and $R' := R \otimes_V V^{\text{sh}}_{\mathfrak{n}} = Q'/f'$, where $f'$ is the image of $f$ under $Q \to Q'$. The fiber of $\operatorname{Spec}(Q') \to \operatorname{Spec}(Q)$ over $\mathfrak{m}$ is

$$Q/\mathfrak{m} \otimes_V V^{\text{sh}}_{\mathfrak{n}} \cong Q/\mathfrak{m} \otimes_{V/\mathfrak{n}} V^{\text{sh}}/\mathfrak{n}^{\text{sh}} = Q/\mathfrak{m} \otimes_k \bar{k}.$$

Since $Q$ has finite type over $V$, $k \rightarrowtail Q/\mathfrak{m}$ is a finite field extension. This shows that the fiber of $\operatorname{Spec}(Q') \to \operatorname{Spec}(Q)$ over $\mathfrak{m}$ consists of a finite number of maximal ideals $\mathfrak{m}'_1, \dots, \mathfrak{m}'_n$ of $Q'$. Since $\operatorname{Spec}(Q) \setminus \mathfrak{m} \to \operatorname{Spec}(V)$ is smooth, so is

$\operatorname{Spec}(Q)\setminus\{\mathfrak{m}'_1, \dots, \mathfrak{m}'_n\} \to \operatorname{Spec}(V^{\mathrm{sh}})$. For each $i$, upon replacing $Q'$ by a suitably small affine open neighborhood of $\mathfrak{m}'_i$, conditions (1)–(5) of Assumptions 7.10 are met, and thus we have $\theta^{R'_{\mathfrak{m}_i}}(-,-) \equiv 0$ by the theorem.

To prove $\theta^R$ vanishes, it suffices to prove the following more general fact: if $R$ is a hypersurface ring having only one singular point, $\mathfrak{m}$, and there is a flat local ring map $(R_{\mathfrak{m}}, \mathfrak{m}) \to (R', \mathfrak{m}')$ such that $R'$ is a local hypersurface with an isolated singularity satisfying $\theta^{R'} \equiv 0$, then $\theta^R \equiv 0$. To prove this, observe that if $T$ is a finitely generated $R$-module supported on $\mathfrak{m}$ then

$$\operatorname{length}_{R'}(T \otimes_R R') = \operatorname{length}_R(T) \cdot \operatorname{length}_{R'}(R'/\mathfrak{m}_R R').$$

It follows that, for any pair of finitely generated $R$-modules $M$ and $N$, we have

$$\begin{aligned}\operatorname{length}_{R'} \operatorname{Tor}_i^{R'}(M \otimes_R R', N \otimes_R R') &= \operatorname{length}_{R'}(\operatorname{Tor}_i^R(M, N) \otimes_R R') \\ &= \operatorname{length}_R(\operatorname{Tor}_i^R(M, N)) \cdot \operatorname{length}_{R'}(R'/\mathfrak{m}_R R')\end{aligned}$$

for $i \gg 0$, and hence

$$\theta^R(M, N) = \frac{\theta^{R'}(M \otimes_R R', N \otimes_R R')}{\operatorname{length}_{R'}(R'/\mathfrak{m}_R R')} = 0. \qquad \square$$

Theorem 1.2 from the introduction follows quickly from the previous corollary by taking $V = k[t]$: Since $f$ is a non-zero-divisor, the map of $k$-algebras $k[t] \to Q$ sending $t$ to $f$ is flat. Since $k$ is perfect, $Q$ is smooth over $k$ and hence $Q[t]$ is smooth over $k[t]$. The morphism $\operatorname{Spec}(Q) \to \operatorname{Spec}(V) = \mathbb{A}^1_k$ is thus a complete intersection because $Q \cong Q[t]/(f - t)$.

We can extend our main vanishing result slightly by allowing localizations:

**Corollary 7.17.** *For any ring $R$ as in Corollary 7.16, $\theta^{S^{-1}R}(M, N) = 0$ for any multiplicatively closed set $S$ disjoint from the singular locus of $R$ and any pair of finitely generated $S^{-1}R$-modules $M$ and $N$.*

*Proof.* More generally, suppose $R$ is any hypersurface whose nonregular locus is $\{\mathfrak{m}_1, \dots, \mathfrak{m}_t\} \subseteq \operatorname{mSpec}(R)$ and $\theta^R \equiv 0$, and let $R' = S^{-1}R$ for any multiplicatively closed set $S$ with $S \cap \mathfrak{m}_i = \varnothing$ for all $i$. We claim that $\theta^{R'} \equiv 0$ too. It is clear $R'$ is also a hypersurface with isolated singularities. Given finitely generated $R'$-modules $M$ and $N$, there exist finitely generated $R$-modules $\widetilde{M}$ and $\widetilde{N}$ such that $\widetilde{M} \otimes_R R' \cong M$ and $\widetilde{N} \otimes_R R' \cong N$. For all $i$ we have

$$S^{-1} \operatorname{Tor}_i^R(\widetilde{M}, \widetilde{N}) \cong \operatorname{Tor}_i^{R'}(M, N)$$

and, for $i \gg 0$, $\operatorname{Tor}_i^R(\widetilde{M}, \widetilde{N})$ is supported on $\{\mathfrak{m}_1, \dots, \mathfrak{m}_t\}$ so that $S^{-1} \operatorname{Tor}_i^R(\widetilde{M}, \widetilde{N}) \cong \operatorname{Tor}_i^R(\widetilde{M}, \widetilde{N})$. It follows that $\theta^{R'}(M, N) = \theta^R(\widetilde{M}, \widetilde{N}) = 0$. $\square$

In particular, Corollary 7.17 justifies Example 1.3 in the introduction.

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# Low-dimensional Milnor–Witt stems over $\mathbb{R}$

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We compute some motivic stable homotopy groups over $\mathbb{R}$. For $0 \leq p - q \leq 3$, we describe the motivic stable homotopy groups $\hat{\pi}_{p,q}$ of a completion of the motivic sphere spectrum. These are the first four Milnor–Witt stems. We start with the known Ext groups over $\mathbb{C}$ and apply the $\rho$-Bockstein spectral sequence to obtain Ext groups over $\mathbb{R}$. This is the input to an Adams spectral sequence, which collapses in our low-dimensional range.

## 1. Introduction

This paper takes place in the context of motivic stable homotopy theory over $\mathbb{R}$. Write $\mathbb{M}_2 = H^{*,*}(\mathbb{R}; \mathbb{F}_2)$ for the bigraded motivic cohomology ring of a point, and write $\mathcal{A}$ for the motivic Steenrod algebra at the prime 2. Our goal is to study the trigraded Adams spectral sequence

$$E_2 = \mathrm{Ext}_{\mathcal{A}}^{*,*,*}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \hat{\pi}_{*,*},$$

where $\hat{\pi}_{*,*}$ represents the stable motivic homotopy groups of a completion of the motivic sphere spectrum over $\mathbb{R}$. This spectral sequence is known to have good convergence properties [Morel 1999; Dugger and Isaksen 2010; Hu et al. 2011a; 2011b]. Specifically, in a range of dimensions we:

(1) compute the Ext groups appearing in the $E_2$-page of the motivic Adams spectral sequence over $\mathbb{R}$;

(2) analyze all Adams differentials;

(3) reconstruct the groups $\hat{\pi}_{*,*}$ from their filtration quotients given by the Adams $E_\infty$-page.

Point (1) is tackled by introducing an auxiliary, purely algebraic spectral sequence that converges to these Ext groups.

To describe our results more specifically we must introduce some notation and terminology related to the three indices in our spectral sequence. We first have the homological degree of the Ext groups, also called the Adams filtration degree —

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*Keywords:* motivic stable homotopy group, motivic Adams spectral sequence, $\rho$-Bockstein spectral sequence, Milnor–Witt stem.

we label this $f$ and simply call it the *filtration*. We then have the internal bidegree $(t, w)$ for $\mathcal{A}$-modules, where $t$ is the usual *topological degree* and $w$ is the *weight*. We introduce the grading $s = t - f$ and call this the *topological stem*, or just the *stem*. The triple $(s, f, w)$ of stem, filtration, and weight will be our main index of reference. Using these variables, the motivic Adams spectral sequence can be written

$$E_2 = \mathrm{Ext}_{\mathcal{A}}^{s,f,w}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \hat{\pi}_{s,w}.$$

Morel [2005] has computed that $\hat{\pi}_{s,w} = 0$ for $s < w$ (in fact, this is true integrally before completion). Write $\Pi_0 = \bigoplus_n \pi_{n,n}$, considered as a $\mathbb{Z}$-graded ring. This is called the *Milnor–Witt ring*, and Morel [2004b] has given a complete description of this via generators and relations. It is convenient to set $\Pi_k = \bigoplus_n \hat{\pi}_{n+k,n}$, as this is a $\mathbb{Z}$-graded module over $\Pi_0$. We call $\Pi_k$ the completed *Milnor–Witt k-stem*. Related to this, the group $\hat{\pi}_{s,w}$ has *Milnor–Witt degree* $s - w$.

The completed Milnor–Witt ring $\Pi_0$ is equal to

$$\mathbb{Z}_2[\rho, \eta]/(\eta^2\rho + 2\eta, \rho^2\eta + 2\rho),$$

where $\eta$ has degree $(1, 1)$ and $\rho$ has degree $(-1, -1)$. Note that $\Pi_0$ is the 2-completion of the Milnor–Witt ring of $\mathbb{R}$ described by Morel [2004b].

We have found that the analysis of the motivic Ext groups over $\mathbb{R}$, and of the Adams spectral sequence, is most conveniently done with respect to the Milnor–Witt degree. In this paper we focus only on the range $s - w \leq 3$, leading to an analysis of the Milnor–Witt stems $\Pi_1$, $\Pi_2$, and $\Pi_3$. The restriction to $s - w \leq 3$ is done for didactic purposes; our methods can be applied to cover a much greater range, but at the expense of more laborious computation. The focus on $s - w \leq 3$ allows us to demonstrate the methods and see examples of the interesting phenomena, while keeping the intensity of the labor down to manageable levels.

**1.1.** ***An algebraic spectral sequence for* Ext.** The main tool in this paper is the $\rho$-Bockstein spectral sequence that computes the groups $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$. This was originally introduced by Hill [2011] and analyzed for the subalgebra $\mathcal{A}(1)$ of $\mathcal{A}$ generated by $\mathrm{Sq}^1$ and $\mathrm{Sq}^2$. Most of our hard work is focused on analyzing the differentials in this spectral sequence, as well as the hidden extensions encountered when passing from the $E_\infty$-page to the true Ext groups.

Over the ground field $\mathbb{R}$, one has $\mathbb{M}_2 = \mathbb{F}_2[\tau, \rho]$, where $\tau$ has bidegree $(0, 1)$ and $\rho$ has bidegree $(1, 1)$. In contrast, over $\mathbb{C}$ one has $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$. The groups $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s,f,w}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$, where $\mathcal{A}^{\mathbb{C}}$ denotes the motivic Steenrod algebra over $\mathbb{C}$, were computed in [Dugger and Isaksen 2010] for $s \leq 34$ and in [Isaksen 2014b] for $s \leq 70$. The $\rho$-Bockstein spectral sequence takes these groups as input, having the form

$$E_1 = \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho] \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2). \tag{1.2}$$

The spectral sequence converges for the simple reason that the $E_1$-page is finite for each degree $(s, f, w)$. The differentials in the spectral sequence are extensive. However, in a large range they can be completely analyzed by a method we describe next.

As an $\mathbb{F}_2[\rho]$-module, $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$ splits as a summand of $\rho$-torsion modules and $\rho$-nontorsion modules; we call the latter *$\rho$-local* modules for short. The first step in our work is to analyze the $\rho$-local part of the Ext groups, and this turns out to have a remarkably simple answer. We prove that

$$\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)[\rho^{-1}] \cong \mathrm{Ext}_{\mathcal{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)[\rho^{\pm 1}],$$

where $\mathcal{A}_{\mathrm{cl}}$ is the classical Steenrod algebra at the prime 2. The isomorphism is highly structured, in the sense that it is compatible with all products and Massey products, and the element $h_i$ in $\mathrm{Ext}_{\mathcal{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ corresponds to the element $h_{i+1}$ in $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)[\rho^{-1}]$ for every $i \geq 0$. In other words, the motivic Ext groups $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$ have a shifted copy of $\mathrm{Ext}_{\mathcal{A}_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ sitting inside them as the $\rho$-local part.

It turns out, that through a large range of dimensions, there is only one pattern of $\rho$-Bockstein differentials that is consistent with the $\rho$-local calculation described in the previous paragraph. This is what allows the analysis of the $\rho$-Bockstein spectral sequence (1.2).

It is not so easy to organize this calculation: the trigraded nature of the spectral sequence, coupled with a fairly irregular pattern of differentials, makes it close to impossible to depict the spectral sequence via the usual charts. See [Hill 2011; Ormsby and Østvær 2013] for calculations of similar complexity.

We analyze what is happening via a collection of charts and tables, but mostly focusing on the tables. A large portion of the present paper is devoted to explaining how to navigate this computation.

Figure 2 on page 198 shows the result, namely $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$ through Milnor–Witt degree 4. Our computations agree with machine computations carried out by Glen Wilson and Knight Fu (personal communication, 2014).

**1.3.** ***Adams differentials.*** Once we have computed $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$, the next step is the analysis of Adams differentials. Identifying even *possible* differentials is again hampered by the trigraded nature of the situation, but we explain the calculus that allows one to accomplish this — it is not as easy as it is for the classical Adams spectral sequence, but it is at least mechanical. In the range $s - w \leq 3$ there are only a few possible differentials for degree reasons. We show via some Toda bracket arguments that in fact all of the differentials are zero.

**1.4.** ***Milnor–Witt modules.*** After analyzing Adams differentials, we obtain the Adams $E_\infty$-page, which is an associated graded object of the motivic stable homotopy groups over $\mathbb{R}$. We convert the associated graded information into the structure of the Milnor–Witt modules $\Pi_1$, $\Pi_2$, and $\Pi_3$, as modules over $\Pi_0$. We must be wary of extensions that are hidden by the Adams spectral sequence, but these turn out to be manageable.

Figure 3 on page 203 describes the results of this process. We draw attention to a curious phenomenon in the 7-stem of $\Pi_3$. Here we see that the third Hopf map $\sigma$ has order 32, not order 16. This indicates that the motivic image of $J$ is not the same as the classical image of $J$. This unexpected behavior suggests that the theory of motivic $v_1$-self maps is not what one might expect. This phenomenon deserve more study. By comparison, it is known that the $C_2$-equivariant Hopf map $\sigma$ also has order 32 [Araki and Iriye 1982].

We also observe that the 1-stem of $\Pi_1$ is consistent with Morel's conjecture on the structure of $\pi_{1,0}$. (See [Ormsby and Østvær 2014, p. 98] for a clearly stated version of the conjecture.) The group $\pi_{1,0}$ is also discussed in [Heller and Ormsby 2016; Röndigs et al. 2014].

Unsurprisingly, our calculations are similar to calculations of $\mathbb{Z}/2$-equivariant stable homotopy groups [Araki and Iriye 1982]. There is a realization functor from motivic homotopy theory over $\mathbb{R}$ to $\mathbb{Z}/2$-equivariant homotopy theory, and this functor induces an isomorphism in stable homotopy groups $\hat{\pi}_{s,w}$ when $s \geq 3w - 5$, and perhaps in a larger range. Details are in [Dugger and Isaksen 2016].

**1.5.** ***Other base fields.*** Although we only work with the base field $\mathbb{R}$ in this article, the phenomena that we study most likely occur for other base fields as well. This is especially true for fields $k$ that are similar to $\mathbb{R}$, such as fields that have an embedding into $\mathbb{R}$.

One might use our calculations to speculate on the structure of $\Pi_1$, $\Pi_2$, and $\Pi_3$ for arbitrary base fields. We leave this to the imagination of the reader.

**1.6.** ***Organization of the paper.*** We begin in Section 2 with a brief reminder of the motivic Steenrod algebra and the motivic Adams spectral sequence. We construct the $\rho$-Bockstein spectral sequence in Section 3, and we perform some preliminary calculations. In Section 4, we consider the effect of inverting $\rho$. Then we return in Section 5 to a detailed analysis of the $\rho$-Bockstein spectral sequence. We resolve extensions that are hidden in the $\rho$-Bockstein spectral sequence in Section 6, and obtain a description of $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$. We show that there are no Adams differentials in Section 7. In Section 8, we convert the associated graded information of the Adams spectral sequence into explicit descriptions of Milnor–Witt modules.

**1.7.** ***Notation.*** For the reader's convenience, we provide a table of notation to be used later:

(1) $\mathbb{M}_2 = \mathbb{F}_2[\tau, \rho]$ is the motivic $\mathbb{F}_2$-cohomology ring of $\mathbb{R}$.

(2) $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$ is the motivic $\mathbb{F}_2$-cohomology ring of $\mathbb{C}$.

(3) $\mathcal{A}$ is the motivic Steenrod algebra over $\mathbb{R}$ at the prime 2.

(4) $\mathcal{A}_*$ is the dual motivic Steenrod algebra over $\mathbb{R}$ at the prime 2.

(5) $\mathcal{A}^{\mathbb{C}}$ is the motivic Steenrod algebra over $\mathbb{C}$ at the prime 2.

(6) $\mathcal{A}_{\mathrm{cl}}$ is the classical Steenrod algebra at the prime 2.

(7) Ext or $\mathrm{Ext}_{\mathbb{R}}$ is the cohomology of $\mathcal{A}$, i.e., $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$.

(8) $\mathrm{Ext}_{\mathbb{C}}$ is the cohomology of $\mathcal{A}^{\mathbb{C}}$, i.e., $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$.

(9) $\mathrm{Ext}_{\mathrm{cl}}$ is the cohomology of $\mathcal{A}^{\mathrm{cl}}$, i.e., $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$.

(10) $\hat{\pi}_{*,*}$ is the bigraded stable homotopy ring of the completion of the motivic sphere spectrum over $\mathbb{R}$ with respect to the motivic Eilenberg–Mac Lane spectrum $H\mathbb{F}_2$.

(11) $\Pi_k = \bigoplus_n \hat{\pi}_{n+k,k}$ is the $k$-th completed Milnor–Witt stem over $\mathbb{R}$.

## 2. Background

This section establishes the basic setting and notation that will be assumed throughout the paper.

Write $\mathbb{M}_2 = H^{*,*}(\mathbb{R}; \mathbb{F}_2)$ for the (bigraded) motivic cohomology ring of $\mathbb{R}$. We use the usual motivic bigrading, where the first index is the topological dimension and the second index is the weight. Recall that $\mathbb{M}_2$ is equal to $\mathbb{F}_2[\tau, \rho]$, where $\tau$ has degree (0, 1) and $\rho$ has degree (1, 1). The class $\rho$ is the element $[-1]$ under the standard isomorphism $\mathbb{M}_2^{1,1} \cong F^*/(F^*)^2$, and $\tau$ is the unique element such that $\mathrm{Sq}^1(\tau) = \rho$.

Let $\mathcal{A}_*$ denote the dual motivic Steenrod algebra over $\mathbb{R}$. The pair $(\mathbb{M}_2, \mathcal{A}_*)$ is a Hopf algebroid; recall from [Voevodsky 2003] (see also [Borghesi 2007; Hoyois et al. 2013; Riou 2012]) that this structure is described by

$$\mathcal{A}_* = \mathbb{M}_2[\tau_0, \tau_1, \ldots, \xi_0, \xi_1, \ldots]/(\xi_0 = 1,\ \tau_k^2 = \tau\xi_{k+1} + \rho\tau_{k+1} + \rho\tau_0\xi_{k+1}),$$
$$\eta_L(\tau) = \tau, \quad \eta_R(\tau) = \tau + \rho\tau_0, \quad \eta_L(\rho) = \eta_R(\rho) = \rho,$$
$$\Delta(\tau_k) = \tau_k \otimes 1 + \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i, \quad \Delta(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i.$$

The Hopf algebroid axioms force $\Delta(\tau) = \tau \otimes 1$ and $\Delta(\rho) = \rho \otimes 1$, but it is useful to record these for reference. The dual $\mathcal{A}_*$ is homologically graded, so $\tau$ has degree $(0, -1)$ and $\rho$ has degree $(-1, -1)$. Moreover, $\tau_k$ has degree $(2^{k+1} - 1, 2^k - 1)$ and $\xi_k$ has degree $(2^{k+1} - 2, 2^k - 1)$.

The groups $\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{M}_2, \mathbb{M}_2)$ are trigraded. There is the homological degree $f$ (the degree on the Ext) and the internal bidegree $(p, q)$ of $\mathcal{A}_*$-comodules. The symbol $f$ comes from "filtration", as this index coincides with the Adams filtration in the Adams spectral sequence. Classical notation would write $\mathrm{Ext}^{f,(p,q)}$ for the corresponding homogeneous piece of the Ext group. In the Adams spectral sequence this Ext group contributes to $\pi_{p-f,q}$. We call $p-f$ the *stem* and will usually denote it by $s$. It turns out to be more convenient to use the indices $(s, f, w)$ of stem, filtration, and weight rather than $(f, p, q)$. So we will write $\mathrm{Ext}^{s,f,w}$ for the group that would classically be denoted $\mathrm{Ext}^{f,(s+f,w)}$. This works very well in practice; in particular, when we draw charts, the group $\mathrm{Ext}^{s,f,w}$ will be located at Cartesian coordinates $(s, f)$.

The motivic Adams spectral sequence takes the form

$$E_2 = \mathrm{Ext}^{s,f,w}_{\mathcal{A}_*}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \hat{\pi}_{s,w},$$

with $d_r\colon \mathrm{Ext}^{s,f,w} \to \mathrm{Ext}^{s-1,f+r,w}$. Here $\hat{\pi}_{*,*}$ is the stable motivic homotopy ring of the completion of the motivic sphere spectrum with respect to the motivic Eilenberg–Mac Lane spectrum $H\mathbb{F}_2$. (According to [Hu et al. 2011b], this completion is also the 2-completion of the motivic sphere spectrum, but this is not essential for our calculations.)

Our methods also require us to consider the motivic cohomology of $\mathbb{C}$ and the motivic Steenrod algebra over $\mathbb{C}$. We write $\mathbb{M}_2^{\mathbb{C}}$ and $\mathcal{A}^{\mathbb{C}}$ for these objects. They are obtained from $\mathbb{M}_2$ and $\mathcal{A}$ by setting $\rho$ equal to zero. More explicitly, $\mathbb{M}_2^{\mathbb{C}}$ equals $\mathbb{F}_2[\tau]$, and the dual motivic Steenrod algebra over $\mathbb{C}$ has relations of the form $\tau_k^2 = \tau\xi_{k+1}$.

We will also use the abbreviations

$$\mathrm{Ext}_{\mathbb{R}} = \mathrm{Ext}_{\mathcal{A}_*}(\mathbb{M}_2, \mathbb{M}_2),$$
$$\mathrm{Ext}_{\mathbb{C}} = \mathrm{Ext}_{\mathcal{A}_*^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}}).$$

**2.1. *Milnor–Witt degree.*** Given a class with an associated stem $s$ and weight $w$, we call $s-w$ the *Milnor–Witt degree* of the class. The terminology comes from the fact that the elements of Milnor–Witt degree zero in the motivic stable homotopy ring constitute Morel's Milnor–Witt $K$-theory ring. More generally, the elements of Milnor–Witt degree $r$ in $\hat{\pi}_{*,*}$ form a module over (2-completed) Milnor–Witt $K$-theory.

Many of the calculations in this paper are handled by breaking things up into the homogeneous Milnor–Witt components. The following lemma about $\mathrm{Ext}_{\mathbb{C}}$ will be particularly useful:

**Lemma 2.2.** *Let $x$ be a nonzero class in $\mathrm{Ext}_{\mathbb{C}}^{s,f,w}$ with Milnor–Witt degree $t$. Then $f \geq s-2t$.*

*Proof.* The motivic May spectral sequence [Dugger and Isaksen 2010] has $E_1$-page generated by classes $h_{ij}$, and converges to $\mathrm{Ext}^{\mathbb{C}}$. All of the classes $h_{ij}$ are readily checked to satisfy the inequality $s+f-2w \geq 0$, and this extends to all products. This inequality is the same as $f \geq s-2(s-w)$, and $t$ equals $s-w$ by definition. $\square$

In practice, Lemma 2.2 tells us where to look for elements of $\mathrm{Ext}_{\mathbb{C}}$ in a given Milnor–Witt degree $t$. All such elements lie above a line of slope 1 on an Adams chart (specifically, the line $f = s-2t$).

## 3. The $\rho$-Bockstein spectral sequence

Our aim is to compute $\mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$. What makes this calculation difficult is the presence of $\rho$. If one formally sets $\rho = 0$ then the formulas become simpler and the calculations more manageable; this is essentially the case that was handled in [Dugger and Isaksen 2010; Isaksen 2014b]. Following ideas of Hill [2011], we use an algebraic spectral sequence for building up the general calculation from the simpler one where $\rho = 0$. This section sets up the spectral sequence and establishes some basic properties.

Let $\mathcal{C}$ be the (unreduced) cobar complex for the Hopf algebroid $(\mathbb{M}_2, \mathcal{A}_*)$. Recall that this is the cochain complex associated to the cosimplicial ring

$$\mathbb{M}_2 \underset{\eta_L}{\overset{\eta_R}{\rightrightarrows}} \mathcal{A}_* \,\substack{\rightarrow\\\rightarrow\\\rightarrow}\, \mathcal{A}_* \otimes_{\mathbb{M}_2} \mathcal{A}_* \,\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\, \mathcal{A}_* \otimes_{\mathbb{M}_2} \mathcal{A}_* \otimes_{\mathbb{M}_2} \mathcal{A}_* \cdots$$

by taking $d_{\mathcal{C}}$ to be the alternating sum of the coface maps. Since we are working over $\mathbb{F}_2$, we do not have to deal with minus signs and can just take the sum of the coface maps. For $u$ an $r$-fold tensor, one has $d^0(u) = 1 \otimes u$, $d^{r+1}(u) = u \otimes 1$, and $d^i(u)$ applies the diagonal of $\mathcal{A}_*$ to the $i$-th tensor factor of $u$. For $u$ in $\mathbb{M}_2$ (i.e., a 0-fold tensor), one has $d^0(u) = \eta_R(u)$ and $d^1(u) = \eta_L(u)$.

The pair $(\mathcal{C}, d_{\mathcal{C}})$ is a differential graded algebra. As usual, we will denote $r$-fold tensors via the bar notation $[x_1|x_2|\cdots|x_r]$.

The element $\xi_1^{2^k}$ is primitive in $\mathcal{A}_*$ for any $k$ because $\xi_1$ is primitive. Hence $[\xi_1^{2^k}]$ is a cycle in the cobar complex that is denoted by $h_{k+1}$. Likewise, $\tau_0$ is primitive, and the cycle $[\tau_0]$ is denoted by $h_0$.

The maps $\eta_L$, $\eta_R$, and $\Delta$ all fix $\rho$, and this implies that all the coface maps are $\rho$-linear. The filtration

$$\mathcal{C} \supseteq \rho\mathcal{C} \supseteq \rho^2\mathcal{C} \supseteq \cdots$$

is therefore a filtration of chain complexes. The associated spectral sequence is called the *$\rho$-Bockstein spectral sequence*.

The $\rho$-Bockstein spectral sequence has the form

$$E_1 = \mathrm{Ext}_{\mathrm{Gr}_\rho \mathcal{A}}(\mathrm{Gr}_\rho \mathbb{M}_2, \mathrm{Gr}_\rho \mathbb{M}_2) \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2),$$

where $\mathrm{Gr}_\rho$ refers to the associated graded of the filtration by powers of $\rho$. Since $\mathbb{M}_2 = \mathbb{F}_2[\tau, \rho]$, we have $\mathrm{Gr}_\rho \mathbb{M}_2 \cong \mathbb{M}_2$. Similarly, it follows easily that there is an isomorphism of Hopf algebroids

$$(\mathrm{Gr}_\rho \mathbb{M}_2, \mathrm{Gr}_\rho \mathcal{A}) \cong (\mathbb{M}_2^{\mathbb{C}}, \mathcal{A}^{\mathbb{C}}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\rho],$$

where $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$ is the motivic cohomology ring of $\mathbb{C}$. The point here is that after taking associated gradeds, the formulas for $\eta_L$ and $\eta_R$ both fix $\tau$, whereas the formulas for $\Delta$ are unchanged; and all of this exactly matches the formulas for $\mathcal{A}^{\mathbb{C}}$.

Our $\rho$-Bockstein spectral sequence takes the form

$$E_1 = \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho] \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$$

because tensoring with $\mathbb{F}_2[\rho]$ commutes with Ext. Alternatively, one can view the identification of the $E_1$-page as a change-of-rings isomorphism. It will be convenient to denote $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$ simply by $\mathrm{Ext}_{\mathbb{C}}$.

The differentials in the $\rho$-Bockstein spectral sequence are essentially truncations of the cobar differential. We make this more precise in Remark 3.1.

Our Ext groups are graded in the form $(s, f, w)$. The filtration by powers of $\rho$ introduces a fourth grading on the $\rho$-Bockstein spectral sequence, which we will not write explicitly since it is apparent in the powers of $\rho$ in any monomial. The $\rho$-Bockstein $d_r$ differential increases the $\rho$ filtration by $r$, decreases $s$ by 1, increases $f$ by 1, and preserves $w$.

We observe two general properties of the $\rho$-Bockstein spectral sequence. First, the element $\rho$ is a permanent cycle because $\rho$ supports no Steenrod operations. Second, the spectral sequence is multiplicative, so the Leibniz rule can be used effectively to compute differentials on decomposable elements.

**Remark 3.1.** Here is a method for deducing $\rho$-Bockstein differentials from explicit cobar calculations. Let $u$ be an element in $\mathcal{C}$, and assume that $u$ is not a multiple of $\rho$. If possible, write $d_{\mathcal{C}}(u) = \rho d_{\mathcal{C}}(u_1) + \rho^2 v_2$, where $u_1$ has a tensor expression that does not involve $\rho$; then the $\rho$-Bockstein differential $d_1(u)$ is zero. Otherwise, $d_1(u)$ equals $d_{\mathcal{C}}(u)$ modulo $\rho^2$.

If $d_1(u)$ is zero, then if possible write $d_{\mathcal{C}}(u) = \rho d_{\mathcal{C}}(u_1) + \rho^2 d_{\mathcal{C}}(u_2) + \rho^3 v_3$, where $u_2$ has a tensor expression that does not involve $\rho$; then $d_2(u)$ is zero. Otherwise, $d_2(u)$ equals $\rho^2 v_2$ modulo $\rho^3$.

Inductively, assume that

$$d_{\mathcal{C}}(u) = \rho d_{\mathcal{C}}(u_1) + \cdots + \rho^{r-1} d_{\mathcal{C}}(u_{r-1}) + \rho^r v_r,$$

where each $u_i$ has a tensor expression that does not involve $\rho$. If possible, write $v_r = d_{\mathcal{C}}(u_r) + \rho v_{r+1}$, where $u_r$ has a tensor expression that does not involve $\rho$; then $d_r(u)$ is zero. Otherwise, $d_r(u)$ equals $\rho^r v_r$ modulo $\rho^{r+1}$.

The method described in Remark 3.1 is mostly not needed in our analysis; in fact, we will eventually show how to deduce most of the differentials in a large range of the spectral sequence by a completely mechanical process. Still, it is often useful to understand that the $\rho$-Bockstein spectral sequence is all about computing $\rho$-truncations of differentials in $\mathcal{C}$. Proposition 3.2 and Example 3.3 illustrate this technique.

**Proposition 3.2.** (1) $d_1(\tau) = \rho h_0$.

(2) $d_{2^k}(\tau^{2^k}) = \rho^{2^k} \tau^{2^{k-1}} h_k$ *for* $k \geq 1$.

Part (2) of Proposition 3.2 implicitly also means that $d_r(\tau^{2^k})$ is zero for all $r < 2^k$.

*Proof.* Note that $d_{\mathcal{C}}(x) = \eta_R(x) - \eta_L(x)$ for $x$ in $\mathbb{M}_2$. In particular, $d_{\mathcal{C}}(\tau) = [\tau + \rho\tau_0] - [\tau] = \rho[\tau_0] = \rho h_0$. Now use Remark 3.1 to deduce that $d_1(\tau) = \rho h_0$.

Next we analyze $d_{\mathcal{C}}(\tau^{2^k})$. Start with

$$d_{\mathcal{C}}(\tau^{2^k}) = \eta_R(\tau^{2^k}) - \eta_L(\tau^{2^k}) = [(\tau + \rho\tau_0)^{2^k}] - [\tau^{2^k}] = \rho^{2^k}[\tau_0^{2^k}].$$

Recall that $\tau_0^2 = \tau\xi_1 + \rho\tau_1 + \rho\tau_0\xi_1$ in $\mathcal{A}_*$, and so $\tau_0^{2^k} = \tau^{2^{k-1}}\xi_1^{2^{k-1}}$ modulo $\rho^{2^{k-1}}$. Thus, $d_{\mathcal{C}}(\tau^{2^k}) = \rho^{2^k}\tau^{2^{k-1}}[\xi_1^{2^{k-1}}]$ modulo $\rho^{2^k+1}$. Remark 3.1 implies that $d_{2^k}(\tau^{2^k}) = \rho^{2^k}\tau^{2^{k-1}}h_k$. □

Proposition 3.2 is essentially the same as the content of [Ormsby and Østvær 2013, Theorem 5.5; Hill 2011, Theorem 3.2].

**Example 3.3.** We will demonstrate that $d_6(\tau^4 h_1) = \rho^6\tau h_2^2$. As in the proof of Proposition 3.2, $d_{\mathcal{C}}(\tau^4) = \rho^4[(\tau\xi_1 + \rho\tau_1 + \rho\tau_0\xi_1)^2]$. Use the relations $\tau_0^2 = \tau\xi_1 + \rho\tau_1 + \rho\tau_0\xi_1$ and $\tau_1^2 = \tau\xi_2 + \rho\tau_2 + \rho\tau_0\xi_2$ to see that this expression equals $\rho^4\tau^2[\xi_1^2] + \rho^6\tau[\xi_2] + \rho^6\tau[\xi_1^3]$ modulo $\rho^7$.

Since $h_1 = [\xi_1]$ is a cycle, we therefore have

$$d_{\mathcal{C}}(\tau^4 h_1) = \rho^4\tau^2[\xi_1^2|\xi_1] + \rho^6\tau\big([\xi_2|\xi_1] + [\xi_1^3|\xi_1]\big)$$

modulo $\rho^7$.

The coproduct on $\xi_2$ implies that $d_{\mathcal{C}}([\xi_2]) = [\xi_1^2|\xi_1]$. We also have that

$$d_{\mathcal{C}}(\tau^2) = \rho^2[\tau_0^2] = \rho^2\tau[\xi_1] + \rho^3[\tau_1] + \rho^2[\tau_0\xi_1],$$

as in the proof of Proposition 3.2. Recall that the cobar complex is a differential graded algebra, so there is a Leibniz rule for $d_{\mathcal{C}}$. Therefore,

$$d_{\mathcal{C}}(\tau^2[\xi_2]) = \rho^2\tau[\xi_1|\xi_2] + \rho^3[\tau_1|\xi_2] + \rho^3[\tau_0\xi_1|\xi_2] + \tau^2[\xi_1^2|\xi_1].$$

We can now write

$$d_{\mathcal{C}}(\tau^4 h_1) = \rho^4 d_{\mathcal{C}}(\tau^2[\xi_2]) + \rho^6\tau\big([\xi_2|\xi_1] + [\xi_1^3|\xi_1] + [\xi_1|\xi_2]\big)$$

modulo $\rho^7$. From Remark 3.1, one has $d_i(\tau^4 h_1) = 0$ for $i < 6$ in the $\rho$-Bockstein spectral sequence, and $d_6(\tau^4 h_1) = \rho^6\tau\big([\xi_2|\xi_1] + [\xi_1^3|\xi_1] + [\xi_1|\xi_2]\big)$.

Finally, the coproduct in $\mathcal{A}_*$ implies that

$$d_C([\xi_2\xi_1]) = [\xi_1^3|\xi_1] + [\xi_1|\xi_2] + [\xi_2|\xi_1] + [\xi_1^2|\xi_1^2].$$

This shows that $[\xi_2|\xi_1] + [\xi_1^3|\xi_1] + [\xi_1|\xi_2] = h_2^2$ in Ext.

The long analysis in Example 3.3 demonstrates that direct work with the cobar complex is not practical. Instead, we will use some clever tricks that take advantage of various algebraic structures. But it is useful to remember what is going on behind the scenes: these computations of differentials are always giving us clues about the cobar differential $d_C$.

The following result is useful in analyzing $\rho$-Bockstein differentials:

**Lemma 3.4.** *If $d_r(x)$ is nontrivial in the $\rho$-Bockstein spectral sequence, then $x$ and $d_r(x)$ are both $\rho$-torsion-free on the $E_r$-page.*

*Proof.* First note that if $y$ is nonzero on the $E_r$-page then $y$ is $\rho$-torsion if and only if $\rho^{r-1}y = 0$. The reason is that the differentials $d_s$ for $s < r$ can only hit $\rho^s$-multiples of $y$.

Now suppose that $d_r(x) = \rho^r y$, where $\rho^r y$ is nonzero on the $E_r$-page. This immediately forces $y$ to be $\rho$-torsion-free. Since $d_r$ is $\rho$-linear, this implies that $x$ must also be $\rho$-torsion-free on the $E_r$-page. □

## 4. $\rho$-localization

The analysis of the $\rho$-Bockstein spectral sequence is best broken up into two pieces. There are a large number of $\rho$-torsion classes in the $E_\infty$-page. If one throws away all of this $\rho$-torsion, then the end result turns out to be fairly simple. In this section we compute this simple piece of $\mathrm{Ext}_\mathbb{R}$. More precisely, we will consider the $\rho$-localization $\mathrm{Ext}_\mathbb{R}[\rho^{-1}]$ of $\mathrm{Ext}_\mathbb{R}$. Inverting $\rho$ annihilates all of the $\rho$-torsion.

The $h_1$-localizations of Ext groups [Andrews and Miller 2015; Guillou and Isaksen 2015a; 2015c] has proven to be an interesting calculation that is useful for understanding global structure. Localization with respect to $\rho$ is similarly useful.

Let $\mathcal{A}^{\mathrm{cl}}$ denote the classical Steenrod algebra (at the prime 2), and write $\mathrm{Ext}_{\mathrm{cl}} = \mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$.

**Theorem 4.1.** *There is an isomorphism from* $\mathrm{Ext}_{\mathrm{cl}}[\rho^{\pm 1}]$ *to* $\mathrm{Ext}_\mathbb{R}[\rho^{-1}]$ *such that*:

(1) *The isomorphism is highly structured, i.e., preserves products, Massey products, and algebraic squaring operations in the sense of* [May 1970].

(2) *The element $h_n$ of* $\mathrm{Ext}_{\mathrm{cl}}$ *corresponds to the element $h_{n+1}$ of* $\mathrm{Ext}_\mathbb{R}$.

(3) *An element in* $\mathrm{Ext}_{\mathrm{cl}}$ *of degree $(s, f)$ corresponds to an element in* $\mathrm{Ext}_\mathbb{R}$ *of degree $(2s + f, f, s + f)$.*

The formula for degrees appears to be more complicated than it is. The idea is that one doubles the internal degree, which is the stem plus the Adams filtration, while leaving the Adams filtration unchanged. Then the weight is always exactly half of the internal degree.

*Proof.* Since localization is exact, we may compute the cohomology of the Hopf algebroid $(\mathbb{M}_2[\rho^{-1}], \mathcal{A}_*[\rho^{-1}])$ to obtain $\mathrm{Ext}_{\mathbb{R}}[\rho^{-1}]$. After localizing at $\rho$, we have $\tau_{k+1} = \rho^{-1}\tau_k^2 + \rho^{-1}\tau\xi_{k+1} + \tau_0\xi_{k+1}$, and so the Hopf algebroid $(\mathbb{M}_2[\rho^{-1}], \mathcal{A}_*[\rho^{-1}])$ is described by

$$\mathcal{A}_*[\rho^{-1}] = \mathbb{M}_2[\rho^{-1}][\tau_0, \xi_0, \xi_1, \dots]/(\xi_0 = 1),$$
$$\eta_L(\tau) = \tau, \quad \eta_R(\tau) = \tau + \rho\tau_0, \quad \eta_L(\rho) = \eta_R(\rho) = \rho,$$
$$\Delta(\tau_0) = \tau_0 \otimes 1 + 1 \otimes \tau_0, \quad \Delta(\xi_k) = \sum \xi_{k-i}^{2^i} \otimes \xi_i.$$

Since these formulas contain no interactions between the $\tau_i$ and the $\xi_j$, there is a splitting

$$(\mathbb{M}_2[\rho^{-1}], \mathcal{A}_*[\rho^{-1}]) \cong (\mathbb{M}_2[\rho^{-1}], \mathcal{A}'_*) \otimes_{\mathbb{F}_2} (\mathbb{F}_2, \mathcal{A}''_*),$$

where $(\mathbb{M}_2[\rho^{-1}], \mathcal{A}'_*)$ is the Hopf algebroid

$$\mathcal{A}'_* = \mathbb{M}_2[\rho^{-1}][\tau_0],$$
$$\eta_L(\tau) = \tau, \quad \eta_R(\tau) = \tau + \rho\tau_0,$$
$$\Delta(\tau_0) = \tau_0 \otimes 1 + 1 \otimes \tau_0,$$

and $(\mathbb{F}_2, \mathcal{A}''_*)$ is the Hopf algebra

$$\mathcal{A}''_* = \mathbb{F}_2[\xi_0, \xi_1, \dots]/(\xi_0 = 1),$$
$$\Delta(\xi_k) = \sum \xi_{k-i}^{2^i} \otimes \xi_i.$$

Notice that $\mathcal{A}''_*$ is equal to the classical dual Steenrod algebra, and so its cohomology is isomorphic to $\mathrm{Ext}_{\mathrm{cl}}$. By careful inspection of the degrees of elements, $\mathrm{Ext}_{\mathrm{cl}}^{s,f}$ contributes to $\mathrm{Ext}_{\mathbb{R}}^{2s+f,f,s+f}$ under this isomorphism.

For $\mathcal{A}'_*$, we can perform the change of variables $x = \rho\tau_0$ since $\rho$ is invertible, yielding

$$(\mathbb{M}_2[\rho^{-1}], \mathcal{A}'_*) \cong \mathbb{F}_2[\rho^{\pm 1}] \otimes_{\mathbb{F}_2} (\mathbb{F}_2[\tau], \mathcal{B}),$$

where $(\mathbb{F}_2[\tau], \mathcal{B})$ is the Hopf algebroid defined in Lemma 4.3 below. The lemma implies that the cohomology of $(\mathbb{M}_2[\rho^{-1}], \mathcal{A}'_*)$ is $\mathbb{F}_2[\rho^{\pm 1}]$, concentrated in homological degree zero. □

**Remark 4.2.** The proof of Theorem 4.1 describes a splitting of the $\rho$-inverted motivic Steenrod algebroid. This $\rho$-inverted splitting occurs more generally in the motivic context over any field of characteristic different from 2.

**Lemma 4.3.** *Let $R = \mathbb{F}_2[t]$ and let $\mathcal{B} = R[x]$, with Hopf algebroid structure on $(R, \mathcal{B})$ given by the formulas*

$$\eta_L(t) = t, \quad \eta_R(t) = t + x, \quad \Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(t) = t \otimes 1.$$

*Then the cohomology of $(R, \mathcal{B})$ is isomorphic to $\mathbb{F}_2$, concentrated in homological degree* 0.

*Proof.* Let $\mathcal{C}_{\mathcal{B}}$ be the cobar complex of $(R, \mathcal{B})$, and filter by powers of $x$. More explicitly, let $F_i\mathcal{C}_{\mathcal{B}}$ be the subcomplex

$$0 \to x^i\mathcal{B} \to \sum_{p+q=i} x^p\mathcal{B} \otimes_R x^q\mathcal{B} \to \sum_{p+q+r=i} x^p\mathcal{B} \otimes_R x^q\mathcal{B} \otimes_R x^r\mathcal{B} \to \cdots .$$

This is indeed a subcomplex, and the associated graded $\mathrm{Gr}_x\,\mathcal{C}_{\mathcal{B}}$ is the cobar complex for $(R, \mathrm{Gr}_x\,\mathcal{B})$. Although $(R, \mathcal{B})$ is a Hopf algebroid because of the nontrivial right unit $\eta_R$, the pair $(R, \mathrm{Gr}_x\,\mathcal{B})$ is isomorphic to a Hopf algebra because $\eta_L(t) = \eta_R(t) = t$ and $\Delta(x) = x \otimes 1 + 1 \otimes x$ modulo higher powers of $x$. The associated cohomology is the infinite polynomial algebra $\mathbb{F}_2[t, h_0, h_1, h_2, \dots]$, where $h_i = [x^{2^i}]$. One easy way to see this is to note that the dual of $\mathrm{Gr}_x\,\mathcal{B}$ is the exterior algebra $\mathbb{F}_2[t](e_0, e_1, e_2, \dots)$, where $e_i$ is dual to $x^{2^i}$.

Our filtered cobar complex gives rise to a multiplicative spectral sequence with $E_1$-page equal to $\mathbb{F}_2[t, h_0, h_1, \dots]$ and converging to the cohomology of $(R, \mathcal{B})$. The classes $h_i$ are all infinite cycles, since $[x^{2^i}]$ is indeed a cocycle in $\mathcal{C}_{\mathcal{B}}$. Essentially the same analysis as in Proposition 3.2 shows that $d_1(t) = h_0$. This shows that the $E_2$-page is $\mathbb{F}_2[t^2, h_1, h_2, \dots]$. The analysis from Proposition 3.2 again shows $d_2(t^2) = h_1$, which implies that the $E_3$-page is $\mathbb{F}_2[t^4, h_2, h_3, \dots]$. Continue inductively, using that $d_{2^i}(t^{2^i}) = h_i$. The $E_\infty$-page is just $\mathbb{F}_2$. $\square$

**Remark 4.4.** We gave a calculational proof of Lemma 4.3. Here is a sketch of a more conceptual proof.

The Hopf algebroid $(R, \mathcal{B})$ has the same information as the presheaf of groupoids which sends an $\mathbb{F}_2$-algebra $S$ to the groupoid with object set $\mathrm{Hom}_{\mathbb{F}_2\text{–alg}}(R, S)$ and morphism set $\mathrm{Hom}_{\mathbb{F}_2\text{–alg}}(\mathcal{B}, S)$. One readily checks that this groupoid is the translation category associated to the abelian group $(S, +)$; very briefly, the image of $x$ in $S$ is the name of the morphism, the image of $t$ is its domain, and therefore $t + x$ is its codomain. Notice that this groupoid is contractible no matter what $S$ is — this is the key observation. By [Hovey 2002, Theorems A and B] it follows that the category of $(R, \mathcal{B})$-comodules is equivalent to the category of comodules for the trivial Hopf algebroid $(\mathbb{F}_2, \mathbb{F}_2)$. In particular, one obtains an isomorphism of Ext groups.

## 5. Analysis of the $\rho$-Bockstein spectral sequence

In this section we determine all differentials in the $\rho$-Bockstein spectral sequence, within a given range of dimensions.

**5.1.** ***Identification of the $E_1$-page.*** From Section 3, the $\rho$-Bockstein spectral sequence takes the form

$$E_1 = \mathrm{Ext}_{\mathbb{C}}[\rho] \Rightarrow \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2).$$

The groups $\mathrm{Ext}_{\mathbb{C}}$ have been computed in [Dugger and Isaksen 2010; Isaksen 2014b] through a large range of dimensions. Figure 1 gives a picture of $\mathrm{Ext}_{\mathbb{C}}$.

Recall that this chart is a two-dimensional representation of a trigraded object. For every black dot $x$ in the chart there are classes $\tau^i x$ for $i \geq 1$ lying behind $x$ (going into the page); in contrast, the red dots are killed by $\tau$. To get the $E_1$-page for the $\rho$-Bockstein spectral sequence, we freely adjoin the class $\rho$ to this chart. With respect to the picture, multiplication by $\rho$ moves one degree to the left and one degree back. So we can regard the same chart as a depiction of our $E_1$-page if we interpret every black dot as representing an entire triangular cone moving back

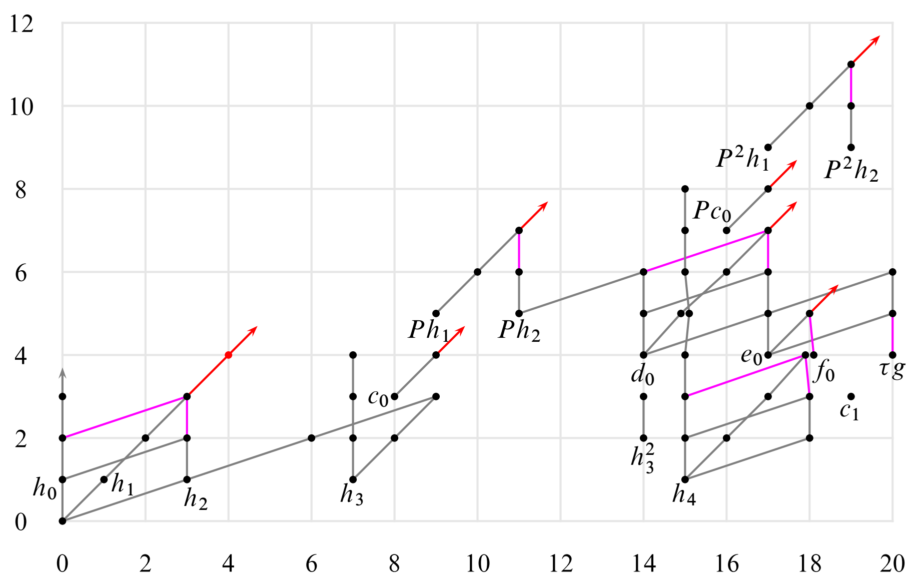



**Figure 1.** $\mathrm{Ext}_{\mathbb{C}} = \mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$. Black dots: copies of $\mathbb{M}_2^{\mathbb{C}}$. Red dots: copies of $\mathbb{M}_2^{\mathbb{C}}/\tau$. Lines indicate multiplications by $h_0$, $h_1$, and $h_2$. Red arrows indicate infinitely many copies of $\mathbb{M}_2^{\mathbb{C}}/\tau$ connected by $h_1$ multiplications. Magenta lines indicate that a multiplication hits $\tau$ times a generator. (For example, $h_0 \cdot h_0 h_2$ equals $\tau h_1^3$.) Data from [Dugger and Isaksen 2010] or [Isaksen 2014a].

(via multiplication by $\tau$) and to the left (via multiplication by $\rho$); and every red dot represents a line of $\rho$-multiples going back and to the left. For example, we must remember that in the (2, 1) spot on the grid there are classes $\rho\tau^i h_2$, $\rho^5\tau^i h_3$, $\rho^{13}\tau^i h_4$, and so forth. In general, when looking at coordinates $(s, f)$ on the chart, one must look horizontally to the right and be aware that $\rho^k x$ is potentially present, where $x$ is a class in $\mathrm{Ext}_{\mathbb{C}}$ at coordinates $(s+k, f)$.

There are so many classes in the $E_1$-page, and it is so difficult to represent the three-dimensional chart, that one of the largest challenges of running the $\rho$-Bockstein spectral sequence is one of organization. We will explain some techniques for managing this.

**5.2.** ***Sorting the $E_1$-page.*** To analyze the $\rho$-Bockstein spectral sequence it is useful to sort the $E_1$-page by the Milnor–Witt degree $s-w$. The $\rho$-Bockstein differentials all have degree $(-1, 1, 0)$ with respect to the $(s, f, w)$-grading, and therefore have degree $-1$ with respect to the Milnor–Witt degree.

Table 1 shows the multiplicative generators for the $\rho$-Bockstein $E_1$-page through Milnor–Witt degree 5. The information in Table 1 was extracted from the $\mathrm{Ext}_{\mathbb{C}}$ chart in Figure 1 in the following manner. Lemma 2.2 says that elements in Milnor–Witt degree $t$ satisfy $f \geq s-2t$. Specifically, elements in Milnor–Witt degree at most 5 lie on or above the line $f = s-10$ of slope 1.

This region is infinite, and in principle could contain generators in very high stems. However, in $\mathrm{Ext}_{\mathbb{C}}$ there is a line of slope $\frac{1}{2}$ above which all elements are multiples of $h_1$ [Guillou and Isaksen 2015b]. The line of slope 1 and the line of slope $\frac{1}{2}$ bound a finite region, which is easily searched exhaustively for generators of Milnor–Witt degree at most 5.

Note that the converse does not hold: some elements bounded by these lines may have Milnor–Witt degree greater than 5.

| $s-w$ | element | $(s, f, w)$ |
|---|---|---|
| 0 | $\rho$ | $(-1, 0, -1)$ |
| 0 | $h_0$ | $(0, 1, 0)$ |
| 0 | $h_1$ | $(1, 1, 1)$ |
| 1 | $\tau$ | $(0, 0, -1)$ |
| 1 | $h_2$ | $(3, 1, 2)$ |
| 3 | $h_3$ | $(7, 1, 4)$ |
| 3 | $c_0$ | $(8, 3, 5)$ |
| 4 | $Ph_1$ | $(9, 5, 5)$ |
| 5 | $Ph_2$ | $(11, 5, 6)$ |

**Table 1.** Multiplicative generators for the $\rho$-Bockstein $E_1$-page.

| 0 | 1 | 2 | 3 | | 4 | |
|---|---|---|---|---|---|---|
| $h_0^k$ | $\tau$ | $\tau^2$ | $\tau^3$ | $h_3$ | $\tau^4$ | $\tau h_0 h_3$ |
| $h_1^k$ | $\tau h_0^k$ | $\tau^2 h_0^k$ | $\tau^3 h_0^k$ | $h_0 h_3$ | $\tau^4 h_0^k$ | $\tau h_0^2 h_3$ |
| | $\tau h_1$ | $\tau^2 h_1$ | $\tau^3 h_1$ | $h_0^2 h_3$ | $\tau^4 h_1$ | $\tau h_0^3 h_3$ |
| | $\tau h_1^2$ | $\tau^2 h_1^2$ | $\tau^3 h_1^2$ | $h_0^3 h_3$ | $\tau^4 h_1^2$ | $\tau h_1 h_3$ |
| | $\tau h_1^3$ | $\tau^2 h_1^3$ | $\tau^3 h_1^3$ | $h_1 h_3$ | $\tau^4 h_1^3$ | $\tau h_1^2 h_3$ |
| | $h_2$ | $\tau h_2$ | $\tau^2 h_2$ | $h_1^2 h_3$ | $\tau^3 h_2$ | $\tau c_0$ |
| | $h_0 h_2$ | $\tau h_0 h_2$ | $\tau^2 h_0 h_2$ | $c_0$ | $\tau^3 h_0 h_2$ | $\tau h_1 c_0$ |
| | | $h_2^2$ | $\tau h_2^2$ | $h_1^k c_0$ | $\tau^2 h_2^2$ | $P h_1$ |
| | | | | | $\tau h_3$ | $h_1^k P h_1$ |

**Table 2.** $\mathbb{F}_2[\rho]$-module generators for the $\rho$-Bockstein $E_1$-page.

Our $E_1$-page is additively generated by all nonvanishing products of the elements from Table 1. Because the Bockstein differentials are $\rho$-linear, it suffices to understand how the differentials behave on products that do not involve $\rho$. Table 2 shows $\mathbb{F}_2[\rho]$-module generators for the $E_1$-page, sorted by Milnor–Witt degree.

**5.3.** ***Bockstein differentials.*** Proposition 3.2 established some $\rho$-Bockstein differentials with a brute force approach via the cobar complex. We will next describe a different technique that computes all differentials in a large range; these are summarized in Table 3.

All our arguments will center on the $\rho$-local calculation of Theorem 4.1, which says that if we invert $\rho$, then the $\rho$-Bockstein spectral sequence converges to a copy of $\mathrm{Ext}_{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\rho, \rho^{-1}]$, with the motivic $h_i$ corresponding to the classical $h_{i-1}$.

When identifying possible $\rho$-Bockstein differentials, there are two useful things to keep in mind:

- Relative to our $\mathrm{Ext}_{\mathbb{C}}$ chart, the differentials all go up one spot and left one spot.
- Relative to Table 2, the differentials all go to the left one column.

| $(s, f, w)$ | $x$ | $d_r$ | $d_r(x)$ | proof |
|---|---|---|---|---|
| $(0, 0, -1)$ | $\tau$ | $d_1$ | $\rho h_0$ | Lemma 5.4 |
| $(0, 0, -2)$ | $\tau^2$ | $d_2$ | $\rho^2 \tau h_1$ | Lemma 5.6 |
| $(0, 0, -4)$ | $\tau^4$ | $d_4$ | $\rho^4 \tau^2 h_2$ | Lemma 5.8 |
| $(1, 1, -3)$ | $\tau^4 h_1$ | $d_6$ | $\rho^6 \tau h_2^2$ | Lemma 5.10 |
| $(2, 2, -2)$ | $\tau^4 h_1^2$ | $d_7$ | $\rho^7 c_0$ | Lemma 5.10 |
| $(7, 4, 3)$ | $\tau h_0^3 h_3$ | $d_4$ | $\rho^4 h_1^2 c_0$ | Lemma 5.8 |
| $(9, 5, 5)$ | $P h_1$ | $d_3$ | $\rho^3 h_1^3 c_0$ | Lemma 5.7 |

**Table 3.** Bockstein differentials.

Combining these two facts (which involves switching back and forth between the chart and table), one can often severely narrow the possibilities for differentials.

**Lemma 5.4.** *The $\rho$-Bockstein $d_1$ differential is completely determined by*:

(1) $d_1(\tau) = \rho h_0$.

(2) *The elements $h_0$, $h_1$, $h_2$, $h_3$, and $c_0$ are all permanent cycles.*

(3) $d_1(Ph_1) = 0$.

*Proof.* The differential $d_1(\tau) = \rho h_0$ was established in Proposition 3.2.

The classes $h_0$ and $h_1$ cannot support differentials because there are no elements in negative Milnor–Witt degrees. The classes $h_2$ and $h_3$ must survive the $\rho$-local spectral sequence, so they cannot support differentials. Comparing chart and table, there are no possibilities for a differential on $c_0$.

Finally, if $d_1(Ph_1)$ is nonzero, then it is of the form $\rho x$ for a class $x$ that does not contain $\rho$. This class $x$ would appear at coordinates (9, 6) in the $\mathrm{Ext}_{\mathbb{C}}$ chart. By inspection, there is no such $x$. □

**Remark 5.5.** We have shown that $Ph_1$ survives to the $E_2$-page, but we have not shown that it is a permanent cycle. The $\mathrm{Ext}_{\mathbb{C}}$ chart shows that $\rho^3 h_1^3 c_0$ is the only potential target for a differential on $Ph_1$. If $Ph_1$ is not a permanent cycle, then the only possibility is that $d_3(Ph_1)$ equals $\rho^3 h_1^3 c_0$. We will see below in Lemma 5.7 that this differential does occur.

Lemma 5.4 allows us to compute all $d_1$-differentials, using the product structure. The resulting $E_2$-page, sorted by Milnor–Witt degree, is displayed in Figure 4, which due to its size appears at the end of the article (pages 206–208).

Table 4 gives $\mathbb{F}_2[\rho]$-module generators for part of the $E_2$-page. Recall from Lemma 3.4 that $\rho$-torsion elements cannot be involved in any further differentials, so we have not included such elements in the table. We have also eliminated the elements that cannot be involved in any differentials because we know they are $\rho$-local by Theorem 4.1.

| 0 | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| | $\tau h_1$ | $\tau^2$ | $\tau^3 h_1$ | $\tau^4$ | $\tau h_1 h_3$ |
| | $\tau h_1^2$ | $\tau^2 h_1$ | $\tau^3 h_1^2$ | $\tau^4 h_1$ | $\tau h_1^2 h_3$ |
| | | $\tau^2 h_1^2$ | $\tau^2 h_2$ | $\tau^4 h_1^2$ | $\tau c_0$ |
| | | $\tau^2 h_1^3$ | $\tau h_2^2$ | $\tau^4 h_1^3$ | $\tau h_1 c_0$ |
| | | | $c_0$ | $\tau^2 h_2^2$ | $Ph_1$ |
| | | | $h_1^k c_0$ | $\tau h_0^3 h_3$ | $h_1^k Ph_1$ |

**Table 4.** Some $\mathbb{F}_2[\rho]$-module generators for the $\rho$-Bockstein $E_2$-page.

Note that $\tau h_1$ is indecomposable in the $E_2$-page, although $\tau h_1^2$ does decompose as $\tau h_1 \cdot h_1$. The multiplicative generators for the $E_2$-page are then

$$\boxed{h_0},\ \boxed{h_1},\ \tau h_1,\ \boxed{h_2},\ \tau^2,\ \tau h_2^2,\ \boxed{h_3},\ \boxed{c_0},\ \tau h_0^3 h_3,\ \tau c_0,\ Ph_1,$$

where boxes indicate classes that we already know are permanent cycles.

**Lemma 5.6.** *The $\rho$-Bockstein $d_2$ differential is completely determined by*:

(1) $d_2(\tau^2) = \rho^2 \cdot \tau h_1$.

(2) *The elements $\tau h_1$, $\tau h_2^2$, and $\tau c_0$ are permanent cycles.*

(3) $d_2(\tau h_0^3 h_3) = 0$.

(4) $d_2(Ph_1) = 0$.

*Proof.* The differential $d_2(\tau^2) = \rho^2 \tau h_1$ was established in Proposition 3.2.

Comparison of chart and table shows that a Bockstein differential on $\tau h_1$ could only hit $h_0^2$ or $\rho^2 h_1^2$. The first is impossible since the target of a $d_2$ differential must be divisible by $\rho^2$, and the second is ruled out by the fact that $h_1^2$ survives $\rho$-localization. So no differential can ever exist on $\tau h_1$.

Similarly, chart and table show that there are no possible differentials on $\tau h_2^2$, and no possible $d_2$ differential on either $\tau h_0^3 h_3$ or $Ph_1$.

It remains to consider $\tau c_0$. The only possibility for a differential is that $d_2(\tau c_0)$ might equal $\rho^2 h_1 c_0$. But if this happened we would also have $d_2(h_1^2 \tau c_0) = \rho^2 h_1^3 c_0$, which contradicts the fact that $\tau h_1^2 c_0$ is zero on the $E_2$-page, while $\rho^2 h_1^3 c_0$ is nonzero. □

Once again, Lemma 5.6 allows the complete computation of the $E_3$-page (in our given range), which is shown in Figure 4 (pages 206–208), sorted by Milnor–Witt degree. Table 5 gives $\mathbb{F}_2[\rho]$-module generators for part of the $E_3$-page. Recall from Lemma 3.4 that $\rho$-torsion elements cannot be involved in any further differentials, so we have not included such elements in the table. We have also eliminated the elements that cannot be involved in any differentials because we know they are $\rho$-local by Theorem 4.1.

| 0 | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|
| | | | $\tau^2 h_2$ | $\tau^4$ | $\tau h_0^3 h_3$ |
| | | | $\tau h_2^2$ | $\tau^4 h_1$ | $\tau c_0$ |
| | | | $c_0$ | $\tau^4 h_1^2$ | $Ph_1$ |
| | | | $h_1^k c_0$ | $\tau^4 h_1^3$ | $h_1^k Ph_1$ |
| | | | | $\tau^2 h_2^2$ | |

**Table 5.** Some $\mathbb{F}_2[\rho]$-module generators for the $\rho$-Bockstein $E_3$-page.

The multiplicative generators for the $E_3$-page are

$$\boxed{h_0},\ \boxed{h_1},\ \boxed{\tau h_1},\ \boxed{h_2},\ \tau^2 h_0,\ \tau^2 h_2,\ \boxed{\tau h_2^2},\ \boxed{h_3},\ \boxed{c_0},\ \tau^4,\ \tau h_0^3 h_3,\ \boxed{\tau c_0},\ P h_1,$$

where boxes indicate classes that we already know are permanent cycles.

**Lemma 5.7.** *The $\rho$-Bockstein $d_3$ differential is completely determined by*:

(1) $d_3(Ph_1) = \rho^3 h_1^3 c_0$.

(2) *The elements $\tau^2 h_0$ and $\tau^2 h_2$ are permanent cycles.*

(3) $d_3(\tau^4) = 0$.

(4) $d_3(\tau h_0^3 h_3) = 0$.

*Proof.* As we saw in Lemma 5.4, $h_1$ and $c_0$ are permanent cycles. Therefore, $h_1^3 c_0$ is a permanent cycle. We know from Theorem 4.1 that $h_1^3 c_0$ does not survive $\rho$-localization. Therefore, some differential hits $\rho^r h_1^3 c_0$. The only possibility is that $d_3(Ph_1)$ equals $\rho^3 h_1^3 c_0$.

Inspection of the $E_3$-page shows that there are no possible values for differentials on $\tau^2 h_0$. For $\tau^2 h_2$, there is a possibility that $d_4(\tau^2 h_2)$ equals $\rho^2 h_2^2$. However, this differential is ruled out by Theorem 4.1.

By inspection, there are no possible values for $d_3$ differentials on $\tau^4$ or $\tau h_0^3 h_3$. □

The $d_3$ differential has a very mild effect on the $E_3$-page of our spectral sequence. In Table 5, the elements $Ph_1$ and $h_1^k Ph_1$ disappear from column four, and the elements $h_1^k c_0$ disappear from column three for $k \geq 3$. Everything else remains the same, so we will not include a separate table for the $E_4$-page. The multiplicative generators are the same as for the $E_3$-page, except that $Ph_1$ is thrown out. Figure 4 depicts the $E_4$-page, sorted by Milnor–Witt degree.

Also, all these generators are permanent cycles except possibly for $\tau^4$ and $\tau h_0^3 h_3$. In particular, every element of the $E_4$-page in Milnor–Witt degrees strictly less than 4 is now known to be a permanent cycle. All the remaining differentials will go from Milnor–Witt degree 4 to Milnor–Witt degree 3.

**Lemma 5.8.** *The $\rho$-Bockstein $d_4$ differential is completely determined by*:

(1) $d_4(\tau^4) = \rho^4 \tau^2 h_2$.

(2) $d_4(\tau h_0^3 h_3) = \rho^4 h_1^2 c_0$.

(3) *The other generators of the $E_4$-page are permanent cycles.*

*Proof.* The differential equation $d_4(\tau^4) = \rho^4 \tau^2 h_2$ was established in Proposition 3.2.

We know that $h_1^2 c_0$ is a permanent cycle, but we also know from Theorem 4.1 that $h_1^2 c_0$ does not survive $\rho$-localization. Therefore, some differential hits $\rho^r h_1^2 c_0$ for some $r$. Looking at the chart, the only possibility is that $d_4(\tau h_0^3 h_3)$ equals $\rho^4 h_1^2 c_0$. □

The multiplicative generators of the $E_5$-page are the permanent cycles we have seen already, together with $\tau^4 h_0$ and $\tau^4 h_1$.

**Lemma 5.9.** *The $\rho$-Bockstein $d_5$ differential is zero.*

*Proof.* We only have to check for possible $d_5$ differentials on $\tau^4 h_0$ and $\tau^4 h_1$. Inspection of the $\mathrm{Ext}_{\mathbb{C}}$ chart shows that there are no classes in the relevant degrees. □

Figure 4 displays the $E_6$-page, sorted by Milnor–Witt degree.

**Lemma 5.10.** *The $\rho$-Bockstein $d_6$ differential is completely determined by*:

(1) $d_6(\tau^4 h_1) = \rho^6 \tau h_2^2$.

(2) *The element $\tau^4 h_0$ is a permanent cycle.*

*Proof.* Lemma 3.4 implies that $\tau^4 h_0$ is a permanent cycle because of the differential $d_1(\tau^5) = \rho\tau^4 h_0$.

By Theorem 4.1, we know that $\tau^4 h_1$ does not survive $\rho$-localization. Since $\rho^r \tau^4 h_1$ cannot be hit by a differential, it follows that $\tau^4 h_1$ supports a differential. The two possibilities are that $d_6(\tau^4 h_1)$ equals $\rho^6 \tau h_2^2$ or $d_8(\tau^4 h_1)$ equals $\rho^8 h_1 h_3$. We know from Theorem 4.1 that $h_1 h_3$ survives $\rho$-localization. Therefore, we must have $d_6(\tau^4 h_1) = \rho^6 \tau h_2^2$. □

The multiplicative generators for the $E_7$-page are $\tau^4 h_1^2$, together with other classes that we already know are permanent cycles. Figure 4 displays the $E_7$-page, sorted by Milnor–Witt degree.

**Lemma 5.11.** *The $\rho$-Bockstein $d_7$ differential is completely determined by*:

(1) $d_7(\tau^4 h_1^2) = \rho^7 c_0$.

*Proof.* By Theorem 4.1, we know that $\tau^4 h_1^2$ does not survive $\rho$-localization, and since $\rho^r \tau^4 h_1^2$ cannot be hit by a differential it follows that $\tau^4 h_1^2$ supports a differential. The two possibilities are that $d_7(\tau^4 h_1^2)$ equals $\rho^7 c_0$ or $d_8(\tau^4 h_1^2)$ equals $\rho^8 h_1^2 h_3$. We know from Theorem 4.1 that $h_1^2 h_3$ survives $\rho$-localization. Therefore, we must have $d_7(\tau^4 h_1^2) = \rho^7 c_0$. □

Finally, once we reach the $E_8$-page, we simply observe that all the multiplicative generators are classes that have already been checked to be permanent cycles.

**5.12.** ***The $\rho$-Bockstein $E_\infty$-page.*** Table 6 describes the $\rho$-Bockstein $E_\infty$-page in the range of interest. The table gives a list of $\mathbb{M}_2$-module generators for the $E_\infty$-page. We write $x$ $(\rho^k)$ if $x$ is killed by $\rho^k$, and we write $x(\mathrm{loc})$ for classes that are nonzero after $\rho$-localization.

The reader is invited to construct a single Adams chart that captures all of this information. We have found that combining all of the Milnor–Witt degrees into

| 0 | 1 | 2 | 3 | | 4 |
|---|---|---|---|---|---|
| $h_0^k(\rho)$ | $\tau h_1(\rho^2)$ | $\tau^2 h_0^k(\rho)$ | $\tau^3 h_1^3(\rho)$ | $h_0^3 h_3(\rho)$ | $\tau^4 h_0^k(\rho)$ |
| $h_1^k(\text{loc})$ | $\tau h_1^2(\rho^2)$ | $\tau^2 h_1^2(\rho^2)$ | $\tau^2 h_2(\rho^4)$ | $h_1 h_3(\text{loc})$ | $\tau^2 h_2^2(\rho^4)$ |
| | $\tau h_1^3(\rho)$ | $\tau^2 h_1^3(\rho^2)$ | $\tau^2 h_0 h_2(\rho)$ | $h_1^2 h_3(\text{loc})$ | $\tau h_1 h_3(\rho^2)$ |
| | $h_2(\text{loc})$ | $h_2^2(\text{loc})$ | $\tau h_2^2(\rho^6)$ | $c_0(\rho^7)$ | $\tau h_1^2 h_3(\rho^2)$ |
| | $h_0 h_2(\rho)$ | | $h_3(\text{loc})$ | $h_1 c_0(\rho^7)$ | $\tau c_0(\rho^3)$ |
| | | | $h_0 h_3(\rho)$ | $h_1^2 c_0(\rho^4)$ | $\tau h_1 c_0(\rho^2)$ |
| | | | $h_0^2 h_3(\rho)$ | $h_1^{k+3} c_0(\rho^3)$ | |

**Table 6.** $\mathbb{F}_2[\rho]$-module generators for the $\rho$-Bockstein $E_\infty$-page.

one picture makes it too difficult to get a feel for what is going on. For example, at coordinates (3, 3), one has six elements $h_1^3$, $\tau h_1^3$, $\tau^2 h_1^3$, $\tau^3 h_1^3$, $\rho^5 c_0$, and $\rho^6 h_1^2 h_3$. Each of them is related by $h_0$, $h_1$, and $\rho$ extensions to other elements.

## 6. From the $\rho$-Bockstein $E_\infty$-page to $\mathrm{Ext}_{\mathbb{R}}$

Having obtained the $E_\infty$-page of the $\rho$-Bockstein spectral sequence, we will now compute all hidden extensions in the range under consideration. The key arguments rely on May's convergence theorem [May 1969] in a slightly unusual way. We use this theorem to argue that certain Massey products $\langle a, b, c\rangle$ cannot be well-defined. We deduce that either $ab$ or $bc$ must be nonzero via a hidden extension.

**Remark 6.1.** As is typical in this kind of analysis, there are issues underlying the naming of classes. An element $x$ of the Bockstein $E_\infty$-page represents a coset of elements of $\mathrm{Ext}_{\mathbb{R}}$, and it is convenient if we can slightly ambiguously use the same symbol $x$ for one particular element from this coset. This selection has to happen on a case-by-case basis, but once done it allows us to use the same symbols for elements of the Bockstein $E_\infty$-page and for elements of $\mathrm{Ext}_{\mathbb{R}}$ that they represent.

For example, the element $h_0$ on the $E_\infty$-page represents two elements of $\mathrm{Ext}_{\mathbb{R}}$, because of the presence of $\rho h_1$ in higher Bockstein filtration. One of these elements is annihilated by $\rho$ and the other is not. We write $h_0$ for the element of $\mathrm{Ext}_{\mathbb{R}}$ that is annihilated by $\rho$.

Table 7 summarizes these ambiguities and gives definitions in terms of $\rho$-torsion.

Once again, careful bookkeeping is critical at this stage. We begin by choosing preferred $\mathbb{F}_2[\rho]$-module generators for $\mathrm{Ext}_{\mathbb{R}}$ up to Milnor–Witt degree 4. First, we choose an ordering of the multiplicative generators of $\mathrm{Ext}_{\mathbb{R}}$:

$$\rho < h_0 < h_1 < \tau h_1 < h_2 < \tau^2 h_0 < \tau^2 h_2 < \tau h_2^2 < h_3 < c_0 < \tau^4 h_0 < \tau c_0.$$

The ordering here is essentially arbitrary, although it is convenient to have elements of low Milnor–Witt degree appear first.

| $(s, f, w)$ | generator | ambiguity | definition |
|---|---|---|---|
| (0, 1, 0) | $h_0$ | $\rho h_1$ | $\rho \cdot h_0 = 0$ |
| (1, 1, 1) | $h_1$ | | |
| (1, 1, 0) | $\tau h_1$ | $\rho^2 h_2$ | $\rho^2 \cdot \tau h_1 = 0$ |
| (3, 1, 2) | $h_2$ | | |
| (0, 1, −2) | $\tau^2 h_0$ | | |
| (3, 1, 0) | $\tau^2 h_2$ | $\rho^4 h_3$ | $\rho^4 \cdot \tau^2 h_2 = 0$ |
| (6, 2, 3) | $\tau h_2^2$ | $\rho^2 h_1 h_3$ | $\rho^6 \cdot \tau h_2^2 = 0$ |
| (7, 1, 4) | $h_3$ | | |
| (8, 3, 5) | $c_0$ | $\rho h_1^2 h_3$ | $\rho^7 \cdot c_0 = 0$ |
| (0, 1, −4) | $\tau^4 h_0$ | | |
| (8, 3, 4) | $\tau c_0$ | | |

**Table 7.** Multiplicative generators of $\mathrm{Ext}_{\mathbb{R}}$.

| 0 | 1 | 2 | 3 | | 4 |
|---|---|---|---|---|---|
| $h_0^k(\rho)$ | $\tau h_1(\rho^2)$ | $\tau^2 h_0(\rho)$ | $\tau^2 h_2(\rho^4)$ | $h_0^3 h_3(\rho)$ | $\tau^4 h_0(\rho)$ |
| $h_1^k(\mathrm{loc})$ | $\tau h_1 \cdot h_1(\rho^2)$ | $\tau^2 h_0 \cdot h_0^k(\rho)$ | $h_0 \cdot \tau^2 h_2(\rho)$ | $h_1 h_3(\mathrm{loc})$ | $\tau^4 h_0 \cdot h_0^k(\rho)$ |
| | $h_2(\mathrm{loc})$ | $(\tau h_1)^2(\rho^2)$ | $h_0^2 \cdot \tau^2 h_2(\rho)$ | $h_1^2 h_3(\mathrm{loc})$ | $\tau^2 h_2 \cdot h_2(\rho^4)$ |
| | $h_0 h_2(\rho)$ | $(\tau h_1^2) h_1(\rho^2)$ | $\tau h_2^2(\rho^6)$ | $c_0(\rho^7)$ | $\tau h_1 \cdot h_3(\rho^2)$ |
| | $h_0^2 h_2(\rho)$ | $h_2^2(\mathrm{loc})$ | $h_3(\mathrm{loc})$ | $h_1 c_0(\rho^7)$ | $\tau h_1 \cdot h_1 h_3(\rho^2)$ |
| | | | $h_0 h_3(\rho)$ | $h_1^2 c_0(\rho^4)$ | $\tau c_0(\rho^3)$ |
| | | | $h_0^2 h_3(\rho)$ | $h_1^{k+3} c_0(\rho^3)$ | $h_1 \cdot \tau c_0(\rho^2)$ |

**Table 8.** $\mathbb{F}_2[\rho]$-module generators for $\mathrm{Ext}_{\mathbb{R}}$.

Next, we choose $\mathbb{F}_2[\rho]$-module generators for $\mathrm{Ext}_{\mathbb{R}}$ that come first in the lexicographic ordering on monomials in these generators. For example, we could choose either $h_0^2 h_2$ or $\tau h_1 \cdot h_1^2$ to be an $\mathbb{F}_2[\rho]$-module generator; we select $h_0^2 h_2$ because $h_0 < h_1$. We do this for each element listed in Table 6.

The results of these choices are displayed in Table 8. This table lists $\mathbb{F}_2[\rho]$-module generators of $\mathrm{Ext}_{\mathbb{R}}$. We write $x\,(\rho^k)$ if $x$ is killed by $\rho^k$, and we write $x(\mathrm{loc})$ for classes that are nonzero after $\rho$-localization.

Our goal is to produce a list of relations for $\mathrm{Ext}_{\mathbb{R}}$ that allows every monomial to be reduced to a linear combination of monomials listed in Table 8. We will begin by considering all pairwise products of generators.

**Lemma 6.2.** *Through Milnor–Witt degree* 4, *Table 9 lists the products of all pairs of multiplicative generators of* $\mathrm{Ext}_{\mathbb{R}}$.

| | $h_0$ | $h_1$ | $\tau h_1$ | $h_2$ | $\tau^2 h_0$ |
|---|---|---|---|---|---|
| $h_0$ | $-$ | | | | |
| $h_1$ | 0 | $-$ | | | |
| $\tau h_1$ | $\rho h_1 \cdot \tau h_1$ | $-$ | $-$ | | |
| $h_2$ | $-$ | 0 | 0 | $-$ | |
| $\tau^2 h_0$ | $-$ | $\rho(\tau h_1)^2$ | $\rho^5 \tau h_2^2$ | $h_0 \cdot \tau^2 h_2$ | $\tau^4 h_0 \cdot h_0$ |
| $\tau^2 h_2$ | $-$ | $\rho^2 \tau h_2^2$ | $\rho^2 \tau^2 h_2 \cdot h_2$ | $-$ | |
| $\tau h_2^2$ | 0 | $\rho c_0$ | $\rho \tau c_0$ | $\tau h_1 \cdot h_1 h_3$ | |
| $h_3$ | $-$ | $-$ | $-$ | 0 | |
| $c_0$ | 0 | $-$ | $h_1 \cdot \tau c_0$ | 0 | |
| $\tau^4 h_0$ | $-$ | 0 | | | |
| $\tau c_0$ | $\rho h_1 \cdot \tau c_0$ | $-$ | | | |

**Table 9.** $\mathrm{Ext}_{\mathbb{R}}$ multiplication table.

In Table 9, the symbol $-$ indicates that the product has no simpler form, i.e., is a monomial listed in Table 8.

*Proof.* Some products are zero because there is no other possibility; for example $h_1 h_2$ is zero because there are no nonzero elements in the appropriate degree.

Some products are zero because we already know that they are annihilated by some power of $\rho$, while the only nonzero elements in the appropriate degree are all $\rho$-local. For example, for degree reasons, it is possible that $h_0 h_1$ equals $\rho h_1^2$. However, we already know that $\rho h_0$ is zero, while $h_1^2$ is $\rho$-local. Therefore, $h_0 h_1$ must be zero. Similar arguments explain all of the pairwise products that are zero in Table 9.

Some of the nonzero pairwise products are not hidden in the $\rho$-Bockstein spectral sequence. For example, consider the product $\tau^2 h_0 \cdot h_2$. Then $\tau^2 h_0 \cdot h_2 + h_0 \cdot \tau^2 h_2$ is zero on the $\rho$-Bockstein $E_\infty$-page, but $\tau^2 h_0 \cdot h_2 + h_0 \cdot \tau^2 h_2$ might equal something of higher $\rho$ filtration in $\mathrm{Ext}_{\mathbb{R}}$. The possible values for this expression in $\mathrm{Ext}_{\mathbb{R}}$ are the linear combinations of $\rho^3 \cdot \tau h_2^2$ and $\rho^5 h_1 h_3$. Both of these elements are nonzero after multiplication by $\rho$, while $\tau^2 h_0 \cdot h_2 + h_0 \cdot \tau^2 h_2$ is annihilated by $\rho$ in $\mathrm{Ext}_{\mathbb{R}}$. Therefore, we must have $\tau^2 h_0 \cdot h_2 + h_0 \cdot \tau^2 h_2 = 0$ in $\mathrm{Ext}_{\mathbb{R}}$.

The same argument applies to the other nonhidden extensions in Table 9, except that they are somewhat easier because there are no possible hidden values.

The remaining nonzero pairwise products are hidden in the $\rho$-Bockstein spectral sequence; they require a more sophisticated argument involving Massey products and May's convergence theorem [May 1969]. This theorem says that when there are no "crossing" differentials, one can compute Massey products in $\mathrm{Ext}_{\mathbb{R}}$ using the $\rho$-Bockstein differentials. The precise definition of a crossing differential is too technical to include here. See [Isaksen 2014b, Theorem 2.2.1] for details.

We will demonstrate how this works for the product $h_0 \cdot \tau h_1$. Consider the Massey product $\langle \rho, h_0, \tau h_1 \rangle$ in $\mathrm{Ext}_{\mathbb{R}}$. If this Massey product were well-defined, then May's convergence theorem and the $\rho$-Bockstein differential $d_1(\tau) = \rho h_0$ would imply that the Massey product contains an element that is detected by $\tau^2 h_1$ in the $\rho$-Bockstein $E_\infty$-page. (Beware that one needs to check that there are no crossing differentials.) The element $\tau^2 h_1$ does not survive to the $E_\infty$-page. Therefore, the Massey product is not well-defined, so $h_0 \cdot \tau h_1$ must be nonzero. The only possible value for the product is $\rho h_1 \cdot \tau h_1$.

The same style of argument works for all of the hidden extensions listed in Table 9, with one additional complication in some cases. Consider the product $h_1 \cdot \tau^2 h_0$. Analysis of the Massey product $\langle \rho, \tau^2 h_0, h_1 \rangle$ implies that the product must be nonzero, since $\tau^3 h_1$ does not survive to the $\rho$-Bockstein $E_\infty$-page. However, there is more than one possible value for $h_1 \cdot \tau^2 h_0$: it could be any linear combination of $\rho(\tau h_1)^2$ and $\rho^5 h_2^2$. We know that $\rho \cdot \tau^2 h_0$ is zero, while $h_2^2$ is $\rho$-local. Therefore, we deduce that $h_1 \cdot \tau^2 h_0$ equals $\rho(\tau h_1)^2$. This type of $\rho$-local analysis allows us to nail down the precise value of each hidden extension in every case where there is more than one possible nonzero value. □

**Proposition 6.3.** *Table 10 gives some relations in* $\mathrm{Ext}_{\mathbb{R}}$ *that are hidden in the $\rho$-Bockstein spectral sequence. Together with the products given in Table 9, they form a complete set of multiplicative relations for* $\mathrm{Ext}_{\mathbb{R}}$ *up to Milnor–Witt degree* 4.

*Proof.* These relations follow from arguments similar to those given in the proof of Lemma 6.2. The most interesting is the relation $h_0^2 \cdot \tau^2 h_2 + (\tau h_1)^3 = \rho^5 c_0$, which follows from an analysis of the matric Massey product

$$\left\langle \rho^2, [\, h_0 \;\; \tau h_1 \,], \begin{bmatrix} h_0 \cdot \tau^2 h_2 \\ (\tau h_1)^2 \end{bmatrix} \right\rangle.$$

If this matric Massey product were defined, then May's convergence theorem and

| $(s, f, w)$ | relation |
|---|---|
| $(3,3,2)$ | $h_0^2 h_2 + \tau h_1 \cdot h_1^2 = 0$ |
| $(3,3,0)$ | $h_0^2 \cdot \tau^2 h_2 + (\tau h_1)^3 = \rho^5 c_0$ |
| $(9,3,6)$ | $h_1^2 h_3 + h_2^3 = 0$ |
| $(6,3,4)$ | $h_0 h_2^2 = 0$ |
| $(3,4,0)$ | $h_0^3 \cdot \tau^2 h_2 = \rho^6 h_1 c_0$ |
| $(7,5,4)$ | $h_0^4 h_3 = \rho^3 h_1^2 c_0$ |
| $(6,3,2)$ | $h_0 h_2 \cdot \tau^2 h_2 = \rho^2 \cdot \tau c_0$ |
| $(10,5,6)$ | $h_1^2 \cdot \tau c_0 = 0$ |

**Table 10.** Some relations in $\mathrm{Ext}_{\mathbb{R}}$.

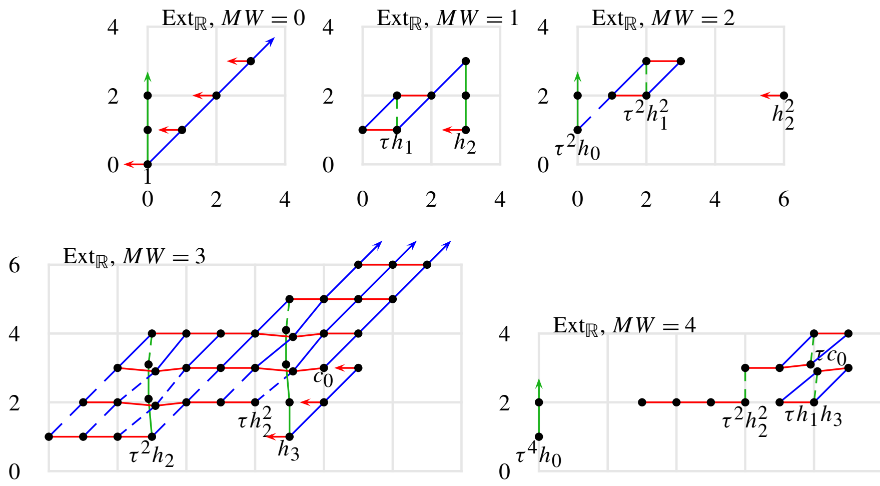



**Figure 2.** $\mathrm{Ext}_{\mathbb{R}} = \mathrm{Ext}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$. Black dots: copies of $\mathbb{F}_2$. Red lines: multiplications by $\rho$. Green lines: multiplications by $h_0$. Blue lines: multiplications by $h_1$. Red (or blue) arrows indicate infinitely many copies of $\mathbb{F}_2$ connected by $\rho$ (or $\eta$) multiplications. Dashed lines indicate $h_0$ or $h_1$ multiplications that are hidden in the $\rho$-Bockstein spectral sequence.

the differential $d_2(\tau^2) = \rho^2 \tau h_1$ would imply that it is detected by $\tau^4 h_1^2$ in the $\rho$-Bockstein $E_\infty$-page. But $\tau^4 h_1^2$ does not survive to the $\rho$-Bockstein $E_\infty$-page.

For every monomial $x$ in Table 8 and every multiplicative generator $y$ of $\mathrm{Ext}_{\mathbb{R}}$, one can check by brute force that the relations in Tables 9 and 10 allow one to identify $xy$ in terms of the monomials in Table 8. □

Figure 2 displays $\mathrm{Ext}_{\mathbb{R}}$, sorted by Milnor–Witt degree. The picture is similar to the $E_\infty$-page shown in Figure 4 (pages 206–208), except that the hidden extensions by $h_0$ and by $h_1$ are indicated with dashed lines.

## 7. The Adams spectral sequence

At this point we have computed the trigraded ring

$$\mathrm{Ext}_{\mathbb{R}} = \mathrm{Ext}_{\mathcal{A}}^{*,*,*}(\mathbb{M}_2, \mathbb{M}_2)$$

up through Milnor–Witt degree four. We will now consider the motivic Adams spectral sequence based on mod 2 motivic cohomology, which takes the form

$$\mathrm{Ext}_{\mathcal{A}}^{s,f,w}(\mathbb{M}_2, \mathbb{M}_2) \Rightarrow \hat{\pi}_{s,w}.$$

This spectral sequence is known to have good convergence properties [Morel 1999;

Dugger and Isaksen 2010; Hu et al. 2011a; 2011b]. Recall that we are writing $\hat{\pi}_{*,*}$ for the motivic stable homotopy groups of the completion of the motivic sphere spectrum with respect to the motivic Eilenberg–Mac Lane spectrum $H\mathbb{F}_2$. The Adams $d_r$ differential takes elements of tridegree $(s, f, w)$ to elements of tridegree $(s-1, f+r, w)$. In particular, the Adams $d_r$ differential decreases the Milnor–Witt degree by 1. So it pays off to once again fracture the $E_2$-page into the different Milnor–Witt degrees.

It turns out that there are no Adams differentials in the range under consideration:

**Proposition 7.1.** *Up through Milnor–Witt degree four, there are no differentials in the motivic Adams spectral sequence.*

*Proof.* The proof uses Table 8 and the $\mathrm{Ext}_{\mathbb{R}}$ charts in Figure 2 to keep track of elements.

The elements $\rho$, $h_0$, and $h_1$ are permanent cycles, as there are no classes in Milnor–Witt degree $-1$. For $\tau^2 h_0$, we observe that there are no classes of Milnor–Witt degree 1 in the range of the possible differentials on $\tau^2 h_0$. Similarly, there are no possible values in Milnor–Witt degree 2 for differentials on $\tau h_2^2$, $h_3$, and $c_0$.

For degree reasons, the only possible values for $d_r(\tau h_1)$ are $h_0^{r+1}$ and $\rho^{r+1} h_1^{r+1}$. However, $h_0^2 \cdot h_0^{r+1}$ is nonzero on the Adams $E_r$-page, while $h_0^2 \cdot \tau h_1$ is zero. Also, $\rho^2 \cdot \rho^{r+1} h_1^{r+1}$ is nonzero on the Adams $E_r$-page, while $\rho^2 \cdot \tau h_1$ is zero. This implies that there are no differentials on $\tau h_1$.

The only possible value for $d_r(h_2)$ is $\rho^{r-1} h_1^{r+1}$. However, $h_1 \cdot \rho^{r-1} h_1^{r+1}$ is nonzero on the Adams $E_r$-page, while $h_1 \cdot h_2$ is zero. This implies that there are no differentials on $h_2$.

The only possibility for a nonzero differential on $\tau^4 h_0$ is that $d_2(\tau^4 h_0)$ might equal $\rho^{10} h_1^2 h_3$. However, $\rho \cdot \tau^4 h_0$ is zero on the Adams $E_2$-page, while $\rho \cdot \rho^{10} h_1^2 h_3$ is not. This implies that there are no differentials on $\tau^4 h_0$.

It remains to show that $\tau^2 h_2$ and $\tau c_0$ are permanent cycles. We handle these more complicated arguments below, in Lemmas 7.3 and 7.6. □

**Lemma 7.2.** *The Massey product $\langle \rho^2, \tau h_1, h_2 \rangle$ contains $\tau^2 h_2$, with indeterminacy generated by $\rho^4 h_3$.*

*Proof.* Apply May's convergence theorem [1969], using the $\rho$-Bockstein differential $d_2(\tau^2) = \rho^2 \cdot \tau h_1$. This shows that $\tau^2 h_2$ or $\tau^2 h_2 + \rho^4 h_3$ is contained in the bracket. By inspection, the indeterminacy is generated by $\rho^4 h_3$. □

**Lemma 7.3.** *The element $\tau^2 h_2$ is a permanent cycle.*

*Proof.* As shown in Table 11, let $\tau\eta$ and $\nu$ be elements of $\hat{\pi}_{1,0}$ and $\hat{\pi}_{3,2}$, respectively, that are detected by $\tau h_1$ and $h_2$. The product $\rho^2 \cdot \tau\eta$ is zero because there is no other possibility. For degree reasons, the product $\tau\eta \cdot \nu$ could possibly equal $\rho^2 \nu^2$. However, $\rho^2 \cdot \tau\eta \cdot \nu$ is zero, while $\rho^2 \cdot \rho^2 \nu^2$ is not. Therefore, $\tau\eta \cdot \nu$ is also zero.

| $(s, w)$ | element | Ext | definition |
|---|---|---|---|
| $(-1,-1)$ | $\rho$ | $\rho$ | $\{\pm 1\} \to (\mathbb{A}^1 - 0)$ |
| $(0,0)$ | $\epsilon$ | $1$ | twist on $S^{1,1} \wedge S^{1,1}$ |
| $(0,0)$ | $\omega$ | $h_0$ | $1-\epsilon$ |
| $(1,1)$ | $\eta$ | $h_1$ | Hopf construction$^{\dagger\ddagger}$ |
| $(1,0)$ | $\tau\eta$ | $\tau h_1$ | $\eta_{\text{top}} + \rho^2\nu$ |
| $(3,2)$ | $\nu$ | $h_2$ | Hopf construction$^{\dagger}$ |
| $(0,-2)$ | $\tau^2\omega$ | $\tau^2 h_0$ | realizes to 2 |
| $(3,0)$ | $\tau^2\nu$ | $\tau^2 h_2$ | $\nu_{\text{top}} + \rho^4\sigma$ |
| $(6,3)$ | $\tau\nu^2$ | $\tau h_2^2$ | |
| $(7,4)$ | $\sigma$ | $h_3$ | Hopf construction$^{\dagger}$ |
| $(8,5)$ | $\bar{\epsilon}$ | $c_0$ | $\rho\bar{\epsilon} = \eta \cdot \tau\nu^2$ |

**Table 11.** Notation for $\hat{\pi}_{*,*}$. $\dagger$ = [Dugger and Isaksen 2013]; $\ddagger$ = [Morel 2004a].

We have just shown that the Toda bracket $\langle \rho^2, \tau\eta, \nu\rangle$ is well-defined. Moss's convergence theorem [1970] then implies that the Massey product $\langle \rho^2, \tau h_1, h_2\rangle$ contains a permanent cycle. We computed this Massey product in Lemma 7.2, so we know that $\tau^2 h_2$ or $\tau^2 h_2 + \rho^4 h_3$ is a permanent cycle. We already know that $\rho^4 h_3$ is a permanent cycle, so $\tau^2 h_2$ is also a permanent cycle. □

For completeness, we will give an alternative proof that $\tau^2 h_2$ is a permanent cycle that has a more geometric flavor. There is a functor from classical homotopy theory to motivic homotopy theory over $\mathbb{R}$ (or over any field) that takes the sphere $S^p$ to $S^{p,0}$. Let $\nu_{\text{top}}$ be the unstable map $S^{7,0} \to S^{4,0}$ that is the image under this functor of the classical Hopf map $S^7 \to S^4$.

**Lemma 7.4.** *The cohomology of the cofiber of $\nu_{\text{top}}$ is a free $\mathbb{M}_2$-module on two generators $x$ and $y$ of degrees $(4,0)$ and $(8,0)$, satisfying $\mathrm{Sq}^4(x) = \tau^2 y$ and $\mathrm{Sq}^8(x) = \rho^4 y$.*

*Proof.* Consider the cofiber sequence

$$S^{7,0} \to S^{4,0} \to C\nu_{\text{top}} \to S^{8,0},$$

where $C\nu_{\text{top}}$ is the cofiber of $\nu_{\text{top}}$. Apply motivic cohomology to obtain a long exact sequence. It follows that the cohomology of $C\nu_{\text{top}}$ is a free $\mathbb{M}_2$-module on two generators $x$ and $y$ of degrees $(4,0)$ and $(8,0)$.

For degree reasons, the only possible nonzero cohomology operations are that $\mathrm{Sq}^4(x)$ and $\mathrm{Sq}^8(x)$ might equal $\tau^2 y$ and $\rho^4 y$, respectively. It follows by comparison to the classical case that $\mathrm{Sq}^4(x) = \tau^2 y$.

The formula for $\mathrm{Sq}^8(x)$ is more difficult. Consider $S^{4,4} \wedge C\nu_{\mathrm{top}}$, which has cells in dimensions $(8,4)$ and $(12,8)$. The cohomology generator in degree $(8,4)$ is the external product $z \wedge x$, where $z$ is the cohomology generator of $S^{4,4}$ in degree $(4,4)$. The cohomology generator in degree $(12,4)$ is $z \wedge y$.

Now we can compute $\mathrm{Sq}^8$ in terms of the cup product $(z \wedge x)^2$. According to [Voevodsky 2003, Lemma 6.8], the cup product $z^2$ equals $\rho^4 z$ in the cohomology of $S^{4,4}$. Also, the cup product $x^2$ equals $y$ in the cohomology of $C\nu_{\mathrm{top}}$ by comparison to the classical case. By the Künneth formula, it follows that $(z \wedge x)^2 = \rho^4(z \wedge y)$ and that $\mathrm{Sq}^8(x) = \rho^4 y$. □

*Another proof of Lemma 7.3.* Lemma 7.4 shows that the stabilization of $\nu_{\mathrm{top}}$ in $\hat{\pi}_{3,0}$ is detected by $\tau^2 h_2 + \rho^4 h_3$ in the motivic Adams spectral sequence. There are elements $\rho$ and $\sigma$ in $\hat{\pi}_{-1,-1}$ and $\hat{\pi}_{7,4}$ detected by $\rho$ and $h_3$ in the motivic Adams spectral sequence. Therefore, $\tau^2 h_2$ is a permanent cycle that detects $\nu_{\mathrm{top}} + \rho^4 \sigma$. □

**Lemma 7.5.** *The Massey product $\langle \tau h_1, h_2, h_0 h_2 \rangle$ contains $\tau c_0$, with indeterminacy generated by $\rho \cdot \tau h_1 \cdot h_1 h_3$.*

*Proof.* Recall that there is a classical Massey product $\langle h_1, h_2, h_0 h_2 \rangle = c_0$. This implies that the motivic Massey product $\langle \tau h_1, h_2, h_0 h_2 \rangle$ contains $\tau c_0$.

By inspection, the indeterminacy is generated by $\rho \cdot \tau h_1 \cdot h_1 h_3$. □

**Lemma 7.6.** *The element $\tau c_0$ is a permanent cycle.*

*Proof.* As shown in Table 11, let $\tau\eta$ and $\omega$ be elements of $\hat{\pi}_{1,0}$ and $\hat{\pi}_{0,0}$ detected by $\tau h_1$ and $h_0$. As in the proof of Lemma 7.3, the product $\tau\eta \cdot \nu$ is zero. Also, the product $\omega\nu^2$ is zero because there are no other possibilities.

We have just shown that the Toda bracket $\langle \tau\eta, \nu, \omega\nu \rangle$ is well-defined. Moss's convergence theorem [1970] then implies that the Massey product $\langle \tau h_1, h_2, h_0 h_2 \rangle$ contains a permanent cycle. We computed this Massey product in Lemma 7.5, so we know that $\tau c_0$ or $\tau c_0 + \rho \cdot \tau h_1 \cdot h_1 h_3$ is a permanent cycle. We already know that $\rho \cdot \tau h_1 \cdot h_1 h_3$ is a permanent cycle, so $\tau c_0$ is also a permanent cycle. □

## 8. Milnor–Witt modules and $\hat{\pi}_{*,*}$

In this section, we will describe how to pass from the Adams $E_\infty$-page to $\hat{\pi}_{*,*}$. We recall certain well-known elements [Dugger and Isaksen 2013; Morel 2004a]:

(1) $\epsilon$ in $\hat{\pi}_{0,0}$ is represented by the twist map on $S^{1,1} \wedge S^{1,1}$.

(2) $\omega = 1 - \epsilon$ in $\hat{\pi}_{0,0}$ is the zeroth Hopf map detected by $h_0$.

(3) $\rho$ in $\hat{\pi}_{-1,-1}$ is represented by the inclusion $\{\pm 1\} \to (\mathbb{A}^1 - 0)$.

(4) $\eta$ in $\hat{\pi}_{1,1}$ is represented by the Hopf construction on the multiplication map $(\mathbb{A}^1 - 0) \times (\mathbb{A}^1 - 0) \to (\mathbb{A}^1 - 0)$.

(5) $\nu$ in $\hat{\pi}_{3,2}$ is represented by the Hopf construction on a version of quaternionic multiplication.

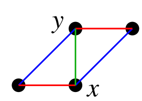
(6) $\sigma$ in $\hat{\pi}_{7,4}$ is represented by the Hopf construction on a version of octonionic multiplication.

The element $1-\epsilon$ is detected in the Adams spectral sequence by $h_0$. Thus it, rather than 2, deserves to be considered the zeroth motivic Hopf map. Because it plays a critical role, it is convenient to name this element $\omega$.

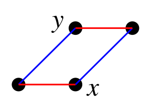
The motivic Adams $E_\infty$-page is the associated graded module of the motivic homotopy groups $\hat{\pi}_{*,*}$ with respect to the adic filtration for the ideal generated by $\omega$ and $\eta$. Note that this ideal also equals $(2, \eta)$ because of the relation $\rho\eta = -1-\epsilon = \omega - 2$.

The elements $\rho$, $h_0$, and $h_1$ detect the homotopy elements $\rho$, $\omega$, and $\eta$ in the zeroth Milnor–Witt stem $\Pi_0$. The relation $2 = \omega - \rho\eta$ implies that $h_0 + \rho h_1$, rather than $h_0$, detects 2. This means that we must be careful when computing the additive structure of Milnor–Witt stems.

In the Adams chart, a parallelogram such as




indicates that 2 times the homotopy elements detected by $x$ are zero (or detected in higher Adams filtration by a hidden extension) because $(h_0 + \rho h_1)x = 0$. On the other hand, a parallelogram such as




indicates that 2 times the homotopy elements detected by $x$ are detected by $y$ because $(h_0 + \rho h_1)x = y$.

We will choose specific homotopy elements to serve as our $\Pi_0$-module generators. Because of the associated graded nature of the Adams $E_\infty$-page, there is some choice in these generators. For the most part, the $\Pi_0$-module structures of the Milnor–Witt modules in our range are insensitive to these choices, so this is not of immediate concern. However, we would like to be as precise as we can to facilitate further study.

These observations allow us to pass from the Adams spectral sequence to the diagrams of $\Pi_0$, $\Pi_1$, $\Pi_2$, and $\Pi_3$ given in Figure 3.

**8.1.** ***The zeroth Milnor–Witt module.*** For $\Pi_0$, the Adams spectral sequence consists of an infinite sequence of dots extending upwards in each stem except for the 0-stem. These dots are all connected by 2 extensions, so they assemble into copies

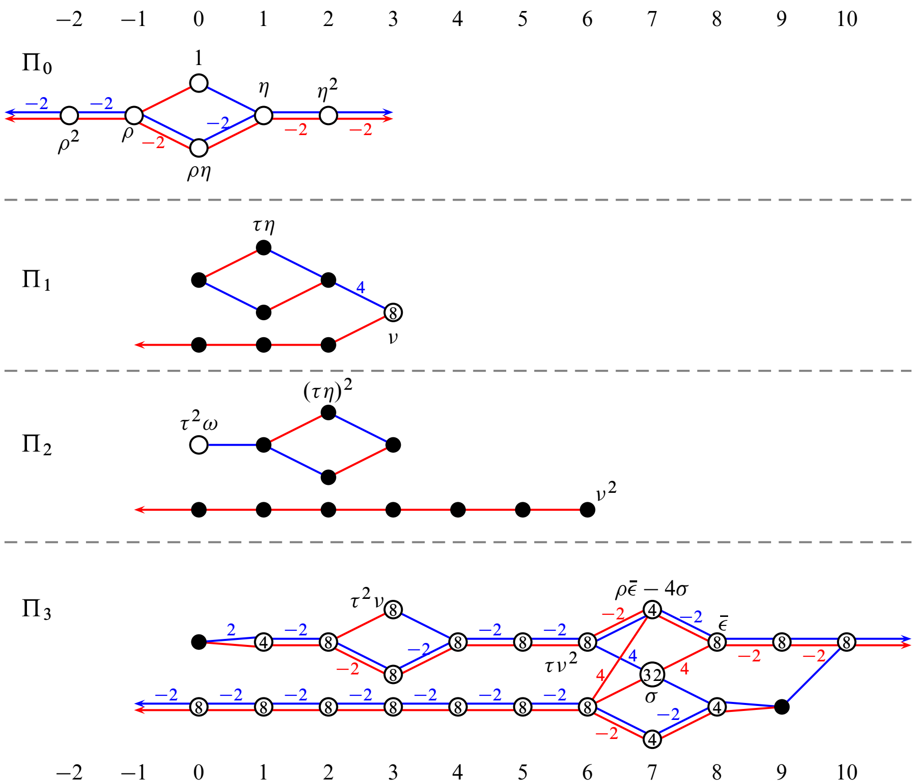



**Figure 3.** Milnor–Witt modules. Empty circles: copies of $\mathbb{Z}_2$. Solid circles: copies of $\mathbb{Z}/2$. Circles with $n$ inside: copies of $\mathbb{Z}/n$. Blue lines: $\eta$-multiplications going to the right. Red lines: $\rho$-multiplications going to the left.

Remaining conventions:

- Lines labeled $n$ indicate that the result of the multiplication is $n$ times the labeled generator; for example, $\rho\cdot\rho\eta=-2\rho$ or $\eta\cdot\rho^3=-2\rho^2$ in $\Pi_0$.

- Two blue (or red lines) with the same source indicate that the multiplication by $\eta$ (or $\rho$) equals a linear combination. For example, $\eta\cdot\tau\nu^2$ equals $(\rho\bar{\epsilon}-4\sigma)+4\sigma=\rho\bar{\epsilon}$ in $\Pi_3$.

- Arrows pointing off the diagram indicate infinitely many multiplications by $\rho$ or by $\eta$.

- Elements in the same topological stem are aligned vertically. For example, $\eta^3$, $\nu$, $\eta(\tau\eta)^2$, and $\rho^3\nu^2$ all belong to the 3-stem. Their weights are 1, 2, 1, and 1, respectively; this can be deduced from their stems and Milnor–Witt degrees.

of $\mathbb{Z}_2$. The 0-stem is somewhat more complicated. Here there are two sequences of dots extending upwards: elements of the form $\rho^k h_1^k$ and elements of the form $(h_0+\rho h_1)^k = h_0^k+\rho^k h_1^k$. The former elements assemble into a copy of $\mathbb{Z}_2$ generated by $\rho\eta$, while the latter elements assemble into a copy of $\mathbb{Z}_2$ generated by 1.

**8.2.** ***The first Milnor–Witt module.*** For $\Pi_1$, there are three elements in the 3-stem of the Adams spectral sequence. These elements assemble into a copy of $\mathbb{Z}/8$ generated by $\nu$; note that $h_0^k h_2 = (h_0+\rho h_1)^k h_2$ because $h_1 h_2 = 0$. The two elements $\tau h_1$ and $\rho\tau h_1^2$ in the 1-stem do not assemble into a copy of $\mathbb{Z}/4$ because $(h_0+\rho h_1)\tau h_1$ is zero.

We will now discuss precise definitions of the $\Pi_0$-module generators of $\Pi_1$.

Recall that there is a functor from classical homotopy theory to motivic homotopy theory over $\mathbb{R}$ that takes a sphere $S^p$ to $S^{p,0}$. Let $\eta_{\mathrm{top}}$ in $\hat{\pi}_{1,0}$ be the image of the classical Hopf map $\eta$. By an argument analogous to the proof of Lemma 7.4, $\eta_{\mathrm{top}}$ is detected by $\tau h_1 + \rho^2 h_2$. Therefore $\eta_{\mathrm{top}} + \rho^2\nu$ is detected by $\tau h_1$.

**Definition 8.3.** Let $\tau\eta$ be the element $\eta_{\mathrm{top}} + \rho^2\nu$ of $\hat{\pi}_{1,0}$.

Another possible approach to defining $\tau\eta$ is to use a Toda bracket to specify a single element. However, the obvious Toda brackets detecting $\tau h_1$ all have indeterminacy, so they are unsuitable for this purpose.

In terms of algebraic formulas we could write

$$\Pi_1 = \Pi_0\langle \tau\eta, \nu\rangle/(2\cdot\tau\eta,\ 8\nu,\ \eta\nu,\ \rho^2\cdot\tau\eta,\ \eta^2\cdot\tau\eta - 4\nu),$$

but we find Figure 3 to be more informative.

**8.4.** ***The second Milnor–Witt module.*** The calculation of $\Pi_2$ involves the same kinds of considerations that we already described for $\Pi_0$ and for $\Pi_1$. The names of the generators $(\tau\eta)^2$ and $\nu^2$ reflect the multiplicative structure of the Milnor–Witt stems.

It remains to specify a choice of generator detected by $\tau^2 h_0$. Recall that there is a realization functor from motivic homotopy theory over $\mathbb{R}$ to classical homotopy theory. (This functor factors through $\mathbb{Z}/2$-equivariant homotopy theory, but we won't use the equivariance for now.)

**Definition 8.5.** Let $\tau^2\omega$ be the element of $\hat{\pi}_{0,-2}$ detected by $\tau^2 h_0$ that realizes to 2 in classical $\pi_0$.

In terms of algebraic formulas we could write $\Pi_2$ as

$$\Pi_0\langle\nu^2\rangle/(2\nu^2,\ \eta\nu^2)\oplus\Pi_0\langle\tau^2\omega,\ (\tau\eta)^2\rangle/(\rho\cdot\tau^2\omega,\ 2(\tau\eta)^2,\ \eta^2(\tau\eta)^2,\ \rho(\tau\eta)^2-\eta\cdot\tau^2\omega).$$

The unreadability of this formula illustrates why the graphical calculus of Figure 3 is so helpful.

**8.6. *The third Milnor–Witt module.*** The structure of $\Pi_3$ is significantly more complicated. We will begin by discussing choices of generators.

**Definition 8.7.** Let $\tau^2\nu$ be the element $\nu_{\mathrm{top}} + \rho^4\sigma$ in $\hat{\pi}_{3,0}$.

A precise definition of the generator $\tau\nu^2$ detected by $\tau h_2^2$ has so far eluded us. For the purposes of this article, it suffices to choose an arbitrary homotopy element detected by $\tau h_2^2$. The distinction between the choices is not relevant in the range under consideration here, but it may be important for an analysis of higher Milnor–Witt stems.

Lemma 8.8 gives a definition of the generator detected by $c_0$, assuming that $\tau\nu^2$ has already been chosen.

**Lemma 8.8.** *There is a unique element $\bar{\epsilon}$ in $\hat{\pi}_{8,5}$ detected by $c_0$ such that $\rho\bar{\epsilon} - \eta\cdot\tau\nu^2$ equals zero.*

*Proof.* Let $\bar{\epsilon}'$ be any element detected by $c_0$. The relation $\rho c_0 + h_1\cdot\tau h_2^2 = 0$ implies that $\rho\bar{\epsilon}' - \eta\cdot\tau\nu^2$ is detected in higher Adams filtration. Note that $\omega$ kills $\rho\bar{\epsilon}' - \eta\cdot\tau\nu^2$ because it kills both $\rho$ and $\eta$. Therefore, $\rho\bar{\epsilon}' - \eta\cdot\tau\nu^2$ cannot be detected by $h_0^3h_3$. If $\rho\bar{\epsilon}' - \eta\cdot\tau\nu^2$ is detected by $\rho^2h_1c_0$, then we can add an element detected by $\rho h_1 c_0$ to $\bar{\epsilon}'$ to obtain our desired element $\bar{\epsilon}$. Similarly, if $\rho\bar{\epsilon}' - \eta\cdot\tau\nu^2$ is detected by $\rho^3h_1^2c_0$, then we can add an element detected by $\rho^2h_1^2c_0$ to $\bar{\epsilon}'$. $\square$

**Remark 8.9.** The classical analogue of $\bar{\epsilon}$ is traditionally called $\epsilon$; we have changed the notation to avoid the unfortunate coincidence with the motivic element $\epsilon$ in $\hat{\pi}_{0,0}$.

Having determined generators for $\Pi_3$, we now proceed to analyze its $\Pi_0$-module structure. For the most part, this analysis follows the same arguments familiar from the earlier Milnor–Witt stems. The 3-stem and the 7-stem present the greatest challenges, so we discuss them in more detail.

In the 3-stem, $\tau^2\nu$ generates a copy of $\mathbb{Z}/8$; these eight elements are detected by $\tau^2h_2$, $h_0\cdot\tau^2h_2 + \rho^3\cdot\tau h_2^2$, and $h_0^2\cdot\tau^2h_2 + \rho^5c_0$. The element $\rho^3\cdot\tau\nu^2$ also generates a copy of $\mathbb{Z}/8$; these eight elements are detected by $\rho^3\cdot\tau h_2^2$, $\rho^5c_0$, and $\rho^6h_1c_0$. Finally, $\rho^4\sigma$ generates a copy of $\mathbb{Z}/8$; these eight elements are detected by $\rho^4h_3$, $\rho^5h_1h_3$, and $\rho^6h_1^2h_3$.

In the 7-stem, $\sigma$ generates a copy of $\mathbb{Z}/32$; these 32 elements are detected by $h_3$, $h_0h_3 + \rho h_1h_3$, $h_0^2h_3 + \rho^2h_1^2h_3$, $h_0^3h_3$, and $\rho^3h_1^2c_0$. The element $\rho\eta\sigma$ generates a copy of $\mathbb{Z}/4$; these four elements are detected by $\rho h_1h_3$ and $\rho^2h_1^2h_3$. The element $\rho\bar{\epsilon} - 4\sigma$ also generates a copy of $\mathbb{Z}/4$; these four elements are detected by $h_0^2h_3 + \rho^2h_1^2h_3 + \rho c_0$ and $h_0^3h_3 + \rho^2h_1c_0$.

The $\eta$ extension from $\eta^2\sigma$ to $\eta^2\bar{\epsilon}$ is hidden in the Adams spectral sequence. This is the same as the analogous hidden extension in the classical situation [Toda 1962]. This is the only hidden extension by $\rho$, $\omega$, or $\eta$ in the range under consideration.

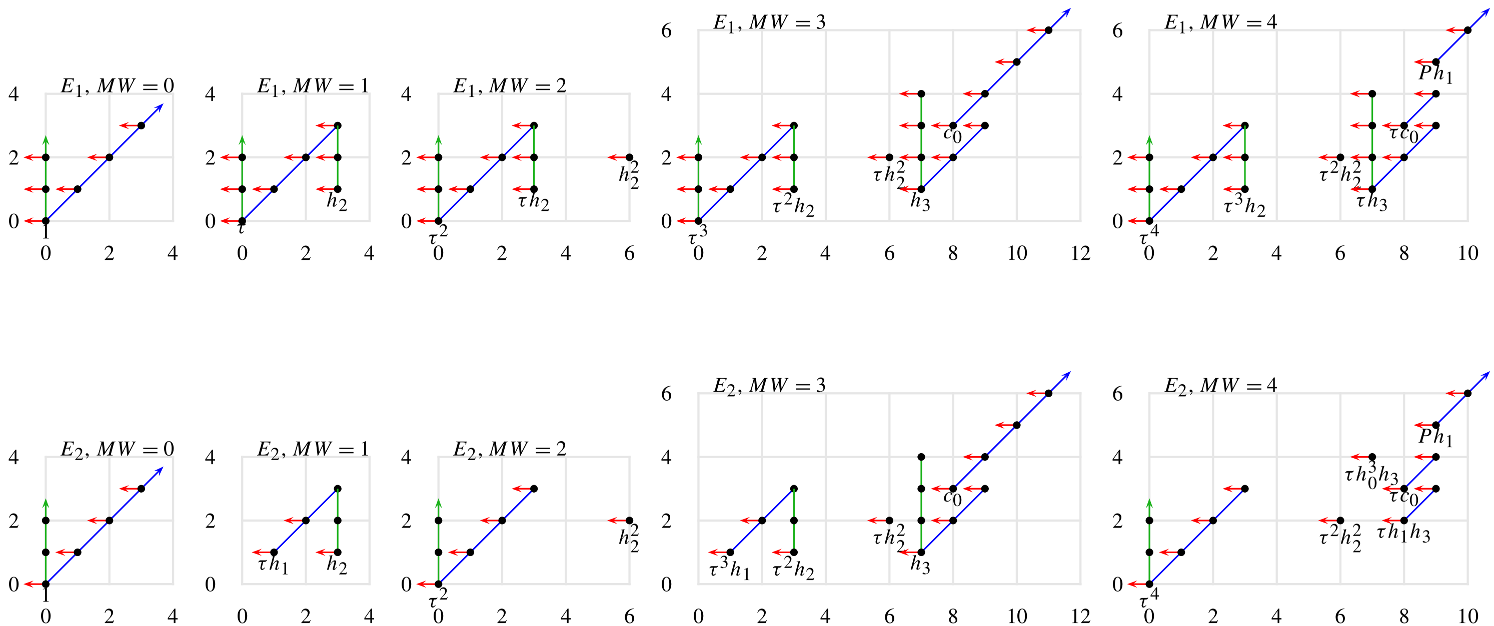
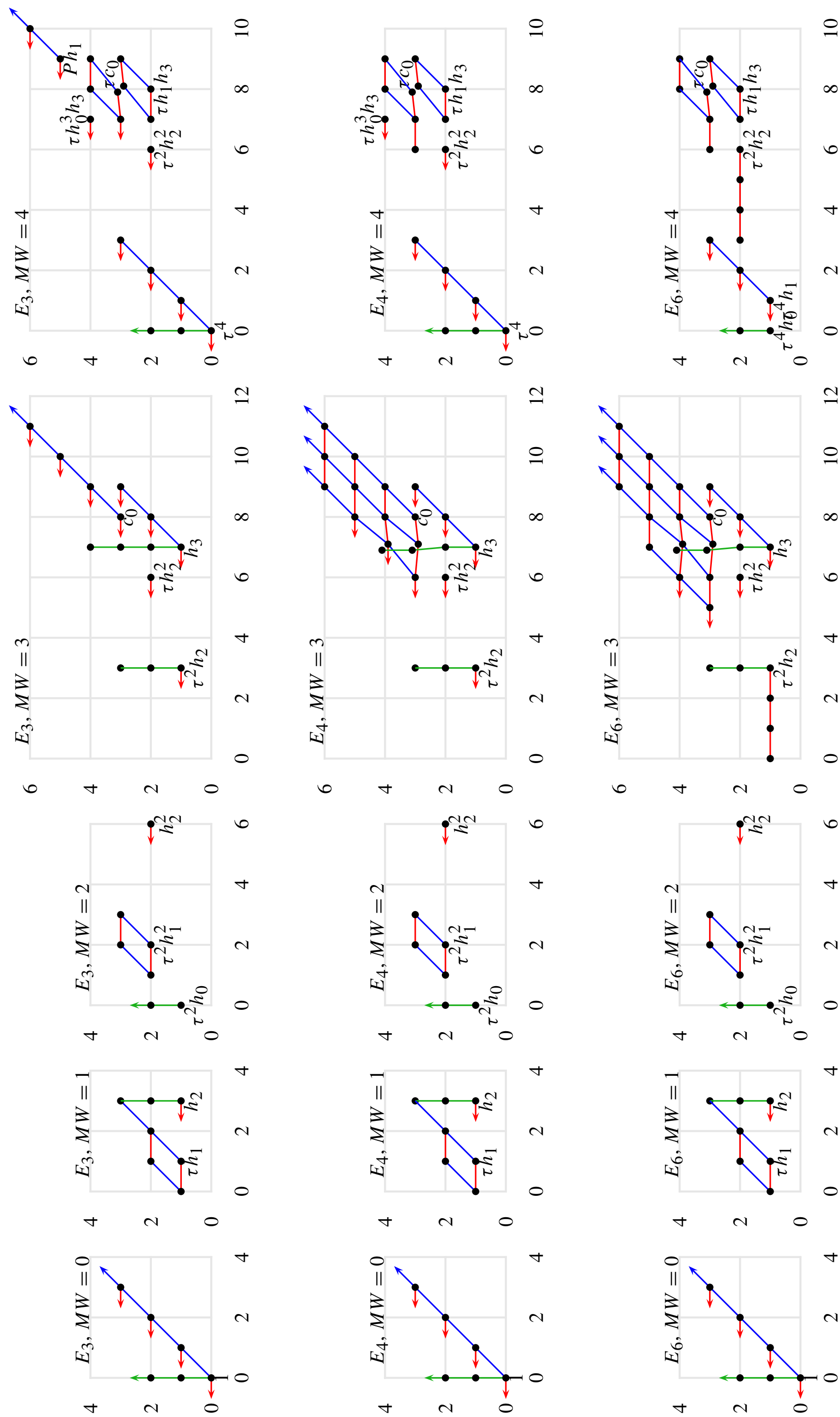



**Figure 4**. $\rho$-Bockstein spectral sequence (page 1 of 3). See p. 208 for conventions and Section 5.3 for discussion.


E_3, MW = 0
E_3, MW = 1
E_3, MW = 2
E_3, MW = 3
E_3, MW = 4
E_4, MW = 0
E_4, MW = 1
E_4, MW = 2
E_4, MW = 3
E_4, MW = 4
E_6, MW = 0
E_6, MW = 1
E_6, MW = 2
E_6, MW = 3
E_6, MW = 4

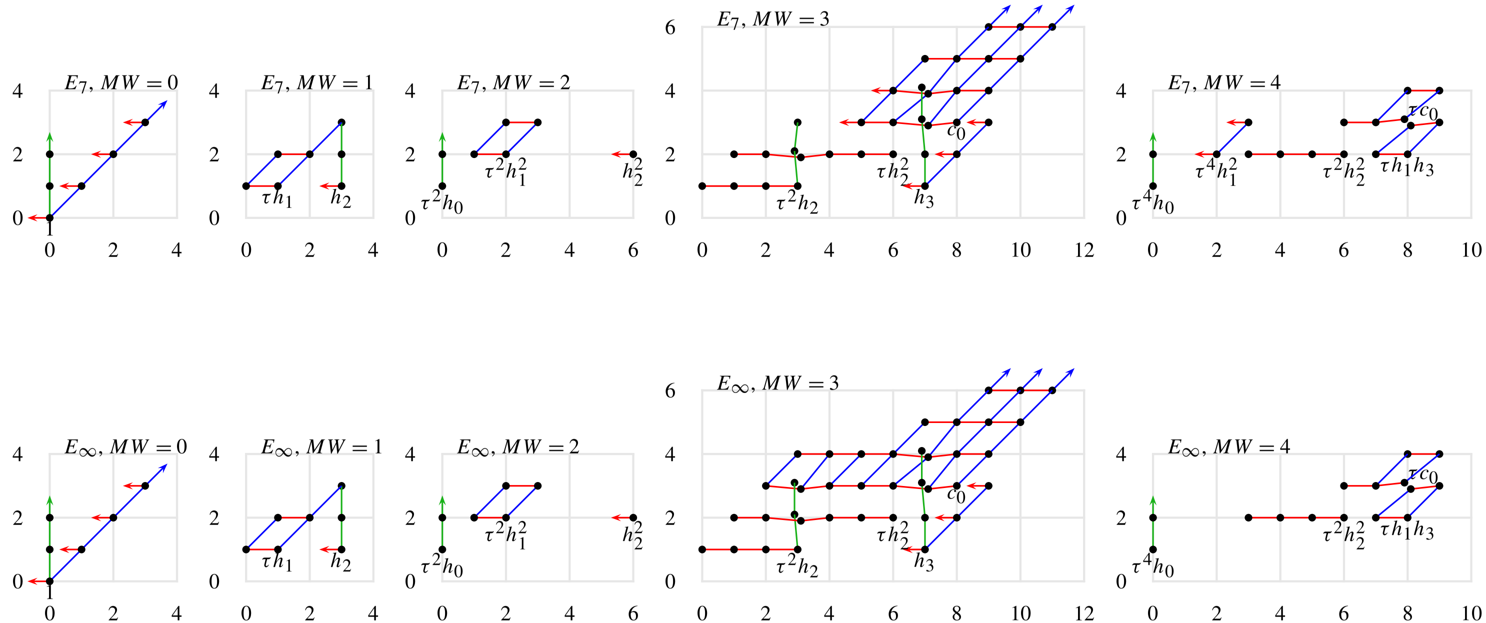



**Figure 4**. $\rho$-Bockstein spectral sequence (page 3 of 3). Black dots: copies of $\mathbb{F}_2$. Red lines: multiplications by $\rho$. Green lines: multiplications by $h_0$. Blue lines: multiplications by $h_1$. Red (or blue) arrows indicate infinitely many copies of $\mathbb{F}_2$ connected by $\rho$ (or $\eta$) multiplications.

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# Longitudes in $\mathrm{SL}_2$ representations of link groups and Milnor–Witt $K_2$-groups of fields

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We describe an arithmetic $K_2$-valued invariant for longitudes of a link $L \subset \mathbb{R}^3$, obtained from an $\mathrm{SL}_2$ representation of the link group. Furthermore, we show a nontriviality on the elements, and compute the elements for some links. As an application, we develop a method for computing longitudes in $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ representations for link groups, where $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ is the universal covering group of $\mathrm{SL}_2(\mathbb{R})$.

## 1. Introduction

Algebraic $K$-groups provide a uniform language for the study of many mathematical phenomena. When it comes to knot theory in topology, the Chern–Simons invariant (i.e., complex volume) and the twisted Alexander polynomial have been extensively studied as important invariants of 3-manifolds, and appear as elements in the $K_1$- and $K_3$-groups as follows:[1]

| $K$-group | link invariant |
|---|---|
| $K_1$ [Bass 1968] | twisted Alexander polynomial |
| $K_2$ [Milnor 1971] | *unknown* |
| $K_3$ [Quillen 1973] | Chern–Simons invariant |

In contrast, there are few such studies on the second $K$-group in low-dimensional topology. Although the paper [Cooper et al. 1994, §4] in topology introduced the $A$-polynomial and a Steinberg symbol "$\{\mathfrak{m}, \mathfrak{l}\} \in K_2^M(F)$", the symbol was defined only for "the tautological representation" (However, this $\{\mathfrak{m}, \mathfrak{l}\}$ has a relation to the study of incompressible surfaces in Culler–Shalen theory; see [Cooper et al. 1994, Introduction].) Moreover, we should emphasize that fields $F$ in most papers on the Culler–Shalen theory are assumed to be (over) the complex field $\mathbb{C}$, which is local from the viewpoint of number theory. Nevertheless, the Milnor–Witt $K_2$-group

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[1]For the reader interested in the link invariants with relation to the $K_1$- and $K_3$-groups, see [Milnor 1966; Friedl and Kim 2008; Zickert 2009] and references therein.

$K_2^{\mathrm{MW}}(F)$ is defined, from any field $F$, to be the kernel of the universal central extension $\mathcal{E}$ (which exists because $\mathrm{SL}_2(F)$ is perfect),

$$K_2^{\mathrm{MW}}(F) := \mathrm{Ker}\big(\mathcal{E} : \widetilde{\mathrm{SL}}_2(F) \longrightarrow \mathrm{SL}_2(F)\big);$$

this $K_2^{\mathrm{MW}}(F)$ has been extensively studied in relation to, e.g., metaplectic groups, stability problems and $\mathbb{A}^1$-homotopy theory (see [Matsumoto 1969; Morel 2012; Hutchinson and Tao 2008; Suslin 1987]). Actually, $K_2^{\mathrm{MW}}(F)$ contains some obstructions, as in the class number formula, the Beilinson conjecture and so on (see [Weibel 2013]).

In this paper, we propose a natural construction of an element in $K_2^{\mathrm{MW}}(F)$ from any parabolic[2] representation $f : \pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$, where $F$ is any infinite field and $L$ is an arbitrary link in $\mathbb{R}^3$. The construction is done in a simple way, wherein the longitudes of $L$ play a key role: First, we show (Proposition 3.1) that $f$ can be algebraically lifted to $\tilde{f} : \pi_1(\mathbb{R}^3 \setminus L) \to \widetilde{\mathrm{SL}}_2(F)$; it follows from the parabolicity that, for each (preferred) longitude $\mathfrak{l}_i \in \pi_1(\mathbb{R}^3 \setminus L)$, $f(\mathfrak{l}_i)$ lies in the unipotent subgroup $U_F$ of $\mathrm{SL}_2(F)$ up to conjugacy. Therefore, the lifted $\tilde{f}(\mathfrak{l}_i)$ lies in the preimage $\mathcal{E}^{-1}(U_F)$, which will be shown to be isomorphic to the product $F \times \widetilde{K}_2^{\mathrm{MW}}(F)$ as abelian groups (Lemma 2.4). Here $\widetilde{K}_2^{\mathrm{MW}}(F)$ is a $\mathbb{Z}/2$-extension of $K_2^{\mathrm{MW}}(F)$; see (2.3). Further, we will show that the value $\tilde{f}(\mathfrak{l}_i)$ is independent of the choice of the lift $\tilde{f}$, and call it *the $K_2$ invariant of $f$* (Definition 3.3). Here is a summary:

$$\begin{tikzcd}
 & \tilde{f}(\mathfrak{l}_i) \in F \times \widetilde{K}_2^{\mathrm{MW}}(F) \cong \mathcal{E}^{-1}(U_F) \arrow[r, hook] & \widetilde{\mathrm{SL}}_2(F) \arrow[d, "\mathcal{E}"] \\
\mathfrak{l}_i \in \pi_1(\mathbb{R}^3 \setminus L) \arrow[ur, mapsto] \arrow[urr, dashed, "\tilde{f}"'] \arrow[rr, "f"'] & & \mathrm{SL}_2(F)
\end{tikzcd}$$

In addition, using a homotopical result in [Nosaka 2015], we will show that any (algebraic) 2-cycle in $\widetilde{K}_2^{\mathrm{MW}}(F)$ can be represented as the $K_2$ invariant of some parabolic representation of some link (Theorem 3.4). Consequently, this theorem ensures that many links give nontrivial examples of the $K_2$ invariants. Furthermore, the $K_2$ invariants are partially computable for some links, by the help of arithmetic studies on the $K_2$-groups. Here, the Matsumoto–Moore 2-cocycle [Matsumoto 1969; Moore 1968] is useful, to formulate $\tilde{f}(\mathfrak{l}_i)$ explicitly in $K_2^{\mathrm{MW}}(F)$ (see Section 2), and $K_2^{\mathrm{MW}}(F)$ in some cases is computable (see Section 4); Thus, we

[2]Notation in topology: Here, a *link* $L$ is a $C^\infty$-embedding of solid tori into the 3-space $\mathbb{R}^3$. Namely, $L : \bigsqcup (D^2 \times S^1) \hookrightarrow \mathbb{R}^3$. By $\#L$ we mean the number of tori, and $\pi_1(\mathbb{R}^3 \setminus L)$ is called *the link group of* $L$. Furthermore, with a choice $x_0 \in S^1$, a *meridian* is one component in the image of $\bigsqcup(\partial D^2 \times \{x_0\})$ and a *longitude* is that of $\bigsqcup(\{x_0\} \times S^1)$. A homomorphism $f : \pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$ is *parabolic* if every meridian $\mathfrak{m}$ in $\pi_1(\mathbb{R}^3 \setminus L)$ satisfies $\mathrm{Tr}\, f(\mathfrak{m}) = \pm 2$.

will explicitly determine the $K_2$ invariants of some small knots (see Section 5), although geometric and arithmetic features appearing in the $K_2$ invariants have many unknown aspects (see the $A$-polynomial [Cooper et al. 1994]).

Furthermore, in Section 6B, we will give two applications from the $K_2$ studies to low-dimensional topology. The first is with respect to the unlifted object $f(\mathfrak{l}_i) = \mathcal{E}(\tilde{f}(\mathfrak{l}_i)) \in \mathrm{SL}_2(F)$, which is commonly called a *cusp shape* in hyperbolic geometry (see [Maclachlan and Reid 2003; Zickert 2009]). While the cusp shape seems, by definition, to be a noncommutative object arising from the link groups $\pi_1(\mathbb{R}^3 \setminus L)$, we explicitly introduce an additive sum formula for $f(\mathfrak{l}_i)$, as in the abelian group $K_2^{\mathrm{MW}}(F)$; see Theorem 6.1. The second is an application to the method of Boyer, Gordon and Watson [Boyer et al. 2013] for finding new 3-dimensional manifolds, $M_r(K)$, obtained by $r$-surgery on a knot $K$ such that $\pi_1(M_r(K))$ is "left-orderable". This result (Theorem 6.7) gives evidence supporting to a conjecture in [Boyer et al. 2013] that relates $L$-spaces to left-orderability. The key here is Proposition 6.3, which closely relates the real $K_2^{\mathrm{MW}}(\mathbb{R})$ to $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$, where $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ is the universal cover group of the Lie group $\mathrm{SL}_2(\mathbb{R})$. See Section 6B for the details.

This paper is organized as follows: We first review the $K_2$-groups in Section 2, and define the $K_2$ invariants in Section 3. After explaining computation on $K_2$ in Section 4, we quantitatively compute some $K_2$ invariants in Section 5. Furthermore, we describe the two applications in Section 6. Finally, Section 7 discusses parabolic representations by means of quandle theory, and proves the theorems.

*Notational conventions.* Throughout this paper, $F$ is a commutative field of infinite order, and $\mathrm{Char}(F)$ is the characteristic (possibly $\mathrm{Char}(F) = 0, 2$).

## 2. Review: the Milnor–Witt $K_2$-group

Before stating the results, we should briefly review the Matsumoto–Moore theorem [Matsumoto 1969; Moore 1968], which provides a presentation of the second group homology $H_2^{\mathrm{gr}}(\mathrm{SL}_2(F))$.

Define $K_2^{\mathrm{MW}}(F)$ to be the abelian group[3] generated by the symbols $[a, b]$ with $a, b \in F$ subject to the relations

(i) $[a, bc] + [b, c] = [ab, c] + [a, b]$ and $[a, 1] = [1, b] = 0$,

(ii) $[a, b] = [b^{-1}, a]$ and $[a, b] = [a, -ab]$ for $a, b, c \in F^\times$,

(iii) $[d, e] = [d, (1-d)e]$,

(iv) $[d, 0] = [d, 0] = 0$ for $d, e \in F$.

[3]The original presentation did not contain the generators $[0, d]$, $[d, 0]$ or the relation (iv). In order to simplify our statements we employ this presentation, although we can easily see that it coincides with the original presentation through the relation (iv).

Noting that the group $\mathrm{SL}_2(F)$ is perfect, i.e., $\mathrm{SL}_2(F)_{\mathrm{ab}} = 0$, we now describe the theorem:

**Theorem 2.1** [Moore 1968; Matsumoto 1969, Corollaire 5.12]. *Let $F$ be an infinite field. There is an isomorphism $H_2^{\mathrm{gr}}(\mathrm{SL}_2(F)) \cong K_2^{\mathrm{MW}}(F)$. Moreover, the universal group 2-cocycle is represented as a map $\theta_{\mathrm{uni}} : \mathrm{SL}_2(F) \times \mathrm{SL}_2(F) \to K_2^{\mathrm{MW}}(F)$ defined by*

$$\theta_{\mathrm{uni}}(g, g') := [\chi(gg'), -\chi(g)^{-1}\chi(g')] - [\chi(g), \chi(g')] \in K_2^{\mathrm{MW}}(F). \tag{2.2}$$

(Here, for $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(F)$, we define $\chi(g) := \gamma$ if $\gamma \neq 0$ and $\chi(g) := \delta \in F^\times$ if $\gamma = 0$.) *In particular, the set $K_2^{\mathrm{MW}}(F) \times \mathrm{SL}_2(F)$ with the group operation $(\alpha, g) \cdot (\beta, h) = (\alpha + \beta + \theta_{\mathrm{uni}}(g, h), gh)$ is isomorphic to the universal extension $\widetilde{\mathrm{SL}}_2(F)$.*

Here we note two facts: First, the inclusion $\mathrm{SL}_2(F)$ into the symplectic group $\mathrm{Sp}_{2n}(F)$ induces an isomorphism $H_2^{\mathrm{gr}}(\mathrm{SL}_2(F)) \cong H_2^{\mathrm{gr}}(\mathrm{Sp}_{2n}(F))$ for any $n \in \mathbb{N}$ (see [Hutchinson and Tao 2008; Suslin 1987]). Next, for any finite field $\mathbb{F}_q$ with $q > 10$, the $H_2^{\mathrm{gr}}(\mathrm{SL}_2(\mathbb{F}_q))$ vanishes. Therefore, in this paper, we restrict ourselves to $\mathrm{SL}_2$ and infinite fields.

To end the section, we will introduce some terminology and Lemma 2.4. We first observe the preimage of $\{\pm \mathrm{id}_{F^2}\}$ via the extension $\mathcal{E} : \widetilde{\mathrm{SL}}_2(F) \to \mathrm{SL}_2(F)$. Let $\widetilde{K}_2^{\mathrm{MW}}(F)$ denote the preimage. Since $\theta_{\mathrm{uni}}(a, b) = \theta_{\mathrm{uni}}(b, a)$ for any $a, b \in \{\pm \mathrm{id}_{F^2}\}$ by the definitions, $\widetilde{K}_2^{\mathrm{MW}}(F)$ is abelian. To summarize, if $\mathrm{Char}(F) \neq 2$, we have

$$0 \to K_2^{\mathrm{MW}}(F) \to \widetilde{K}_2^{\mathrm{MW}}(F) \to \mathbb{Z}/2 \to 0 \quad \text{(exact)}. \tag{2.3}$$

If $\mathrm{Char}(F) = 2$, we have $\widetilde{K}_2^{\mathrm{MW}}(F) = K_2^{\mathrm{MW}}(F)$. Furthermore, consider the preimage of the unipotent subgroup $U_F$, where $U_F$ is of the form $\left\{\left(\begin{smallmatrix} \pm 1 & a \\ 0 & \pm 1 \end{smallmatrix}\right) \mid a \in F\right\}$ as usual. Notice the group isomorphism $U_F \cong \mathbb{Z}/2 \times F$ or $\cong F$ and that the restriction of $\theta_{\mathrm{uni}}$ on this summand $F$ is zero. Hence, we can readily see the following:

**Lemma 2.4.** *The preimage $\mathcal{E}^{-1}(U_F)$ is isomorphic to $\widetilde{K}_2^{\mathrm{MW}}(F) \times F$ as an abelian group.*

## 3. Definition: $K_2$ invariants

In this section, as a topological part, we introduce the $K_2$ invariant with respect to $\mathrm{SL}_2$-parabolic representations of link groups (Definition 3.3), and state a theorem. The knot-theoretic notation that we will use is mentioned in the introduction (see the textbook [Lickorish 1997] for more details).

The key in the construction is the following proposition:

**Proposition 3.1.** *Let $F$ be an infinite field, and $L \subset \mathbb{R}^3$ be a link. Every parabolic representation $f : \pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$ admits the lift $\tilde{f} : \pi_1(\mathbb{R}^3 \setminus L) \to \widetilde{\mathrm{SL}}_2(F)$*

*such that any meridian* $\mathfrak{m} \in \pi_1(\mathbb{R}^3 \setminus L)$ *satisfies*

$$\tilde{f}(\mathfrak{m}) = (0, f(\mathfrak{m})) \in K_2^{\mathrm{MW}}(F) \times \mathrm{SL}_2(F) = \widetilde{\mathrm{SL}}_2(F).$$

**Remark 3.2.** The proof appears in Section 7B, not as standard discussions on second homology. Actually, for example, if $\#L > 1$, then $\mathbb{R}^3 \setminus L$ and $S^3 \setminus L$ are not always $K(\pi, 1)$-spaces and $H_2(\mathbb{R}^3 \setminus L; \mathbb{Z}) \neq 0$. To summarize, the lifting is guaranteed from special properties of $K_2^{\mathrm{MW}}$ and parabolicity.

Next, we will see that the lifted longitude lies in the preimage $\mathcal{E}^{-1}(U_F) \cong \widetilde{K}_2^{\mathrm{MW}}(F) \times F$ in Lemma 2.4. For this, choose a meridian–longitude pair $(\mathfrak{m}_j, \mathfrak{l}_j)$ with respect to each link-component of $L$, where $1 \leq j \leq \#L$. Notice that the centralizer of the unipotent subgroup $U_F$ is $U_F$ itself in $\mathrm{SL}_2(F)$. Therefore, since $f$ is parabolic and each $\mathfrak{m}_j$ commutes with $\mathfrak{l}_j$, the image $f(\mathfrak{l}_j) \in \mathrm{SL}_2(F)$ is contained in $U_F$. Hence, the lifted object $\tilde{f}(\mathfrak{l}_j)$ lies in the product $\widetilde{K}_2^{\mathrm{MW}}(F) \times F \subset \widetilde{\mathrm{SL}}_2(F)$ as required. Furthermore, this $\tilde{f}(\mathfrak{l}_j)$ up to conjugacy of $\mathrm{SL}_2(F)$ is independent of the choice of the lifting $\tilde{f}$, because $\widetilde{K}_2^{\mathrm{MW}}(F)$ is the center in $\widetilde{\mathrm{SL}}_2(F)$.

**Definition 3.3.** Let $f$ be a parabolic representation $\pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$. For a link-component $j$ of $L$, fix a meridian–longitude pair $(\mathfrak{m}_j, \mathfrak{l}_j)$. We define the $K_2$ *invariant* of $f$ to be the value of $\tilde{f}(\mathfrak{l}_j)$ after projecting it onto $\widetilde{K}_2^{\mathrm{MW}}(F)$.

In Section 5, we will compute concretely the $K_2$ invariants of some links.

Speaking of invariants, we shall observe the nontriviality of the invariant (we prove this theorem from a homotopical viewpoint in Section 7D).

**Theorem 3.4.** *Let $F$ be an infinite field. For any element* $(\alpha, \beta) \in \widetilde{K}_2^{\mathrm{MW}}(F) \times F$, *there are a link $L$ and a parabolic representation* $f : \pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$ *such that the sum* $\tilde{f}(\mathfrak{l}_1) + \cdots + \tilde{f}(\mathfrak{l}_{\#L})$ *is equal to* $(\alpha, \beta) \in \widetilde{K}_2^{\mathrm{MW}}(F) \times F$.

In summary, this theorem implies that any (algebraic) cycle in $K_2^{\mathrm{MW}}(F)$ may be represented as some parabolic representation of a link via longitudes, and it ensures many links which have the nontriviality of the $K_2$ invariants.

Incidentally, from the viewpoint of $\mathbb{A}^1$-homotopy theory, we note a homotopical interpretation of the invariant $\tilde{f}(\mathfrak{l})$ for perfect fields $F$. The following isomorphisms of $\mathbb{A}^1$-fundamental groups are known (see [Morel 2012, §7]):

$$\pi_1^{\mathbb{A}^1}(\mathrm{SL}_2(F)) \cong \pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus \{0\}) \cong K_2^{\mathrm{MW}}(F).$$

Moreover, via the $\mathbb{A}^1$-Galois correspondence, the extension $\mathcal{E} : \widetilde{\mathrm{SL}}_2(F) \to \mathrm{SL}_2(F)$ is the universal covering constructed from a simplicial scheme. Accordingly, the value $\tilde{f}(\mathfrak{l}_i) \in K_2^{\mathrm{MW}}(F)$ can be interpreted as a lift of the covering. We refer the reader to [Morel 2012] for more properties of Milnor–Witt $K$-theory.

## 4. Some computations of the Milnor–Witt $K_2$-group

In preparation for computing the $K_2$ invariants, this section analyses $K_2^{\mathrm{MW}}(F)$ quantitatively. The key here is a result of Suslin [1987]. To explain this, we will review the two groups $K_2^M(F)$ and $I^2(F)$.

First, let us review the Matsumoto theorem on the Milnor $K_2$-group $K_2^M(F)$. It says that this $K_2^M(F)$ is the quotient group generated by (Steinberg) symbols $\{x, y\}$ with $x, y \in F^\times$ subject to the relations

$$\{a, bc\} = \{a, b\} + \{a, c\}, \quad \{ab, c\} = \{a, c\} + \{b, c\} \quad \text{for all } a, b, c \in F^\times,$$
$$\{a, 1-a\} = 0 \quad \text{for all } a \in F^\times \setminus \{1\}.$$

Formally, $K_2^M(F)$ can be also presented as the multiplicative group

$$F^\times \otimes_{\mathbb{Z}} F^\times / \big\langle a \otimes (1-a) \mid a \in F^\times \setminus \{1\} \big\rangle.$$

Furthermore, as is known, the correspondence $[a, b] \mapsto \{a, b\}$ defines an epimorphism $\mu : K_2^{\mathrm{MW}}(F) \to K_2^M(F)$. Hence, any element of the form $\{x, -1\} \in K_2^M(F)$ is annihilated by 2. Actually, $2\{x, -1\} = \{x, 1\}$ comes from $[x, 1] = 0 \in K_2^{\mathrm{MW}}(F)$.

Next, let $\mathrm{WG}(F)$ be the Witt–Grothendieck ring of $F$, that is, the Grothendieck ring of isometric classes of all quadratic forms of finite dimension (see, e.g., [Lam 2005, Chapter II] for the definition). For $a \in F^\times$, let us denote by the symbol $\langle a \rangle$ the quadratic form $ax^2$ on $F$. Furthermore, let $I(F) \subset \mathrm{WG}(F)$ denote the augmentation ideal, i.e., $I(F) := \mathrm{Ker}(\mathrm{WG}(F) \to \mathbb{Z})$. Note (see [Suslin 1987, §6]) that the homomorphism

$$\nu : K_2^{\mathrm{MW}}(F) \to I^2(F), \quad [a, b] \mapsto (\langle 1 \rangle - \langle a \rangle)(\langle 1 \rangle - \langle b \rangle),$$

induces the homomorphism $\xi : K_2^M(F) \to I^2(F)/I^3(F)$, called *the Milnor map*.

Suslin [1987] showed that the above homomorphisms provide a pullback diagram

$$\begin{tikzcd} K_2^{\mathrm{MW}}(F) \arrow[r, "\mu"] \arrow[d, "\nu"'] & K_2^M(F) \arrow[d, "\text{Milnor map } \xi"] \\ I^2(F) \arrow[r, "\text{projection}"] & I^2/I^3(F) \end{tikzcd} \tag{4.1}$$

See [Hutchinson and Tao 2008] for another proof. We should make some remarks on this diagram. It is known (see [Lam 2005, §V.6; Weibel 2013, Theorem 7.9]) that the Milnor map induces an isomorphism $K_2^M(F)/2 \cong I^2/I^3(F)$. In particular, the quotient $I^2/I^3(F)$ is an elementary abelian 2-group. Hence, for any prime $l \neq 2$, the pullback localized at $l$ means a direct product. Furthermore, it is known (see [Kramer and Tent 2010] and references therein) that the composite 2-cocycle $\nu \circ \theta_{\mathrm{uni}} : \mathrm{SL}_2(F)^2 \to K_2^{\mathrm{MW}}(F) \to I^2(F)$ coincides with "a Maslov 2-cocycle".

Next, we mention the Merkujev–Suslin theorem, which deals with torsion parts of the Milnor groups of $F$; see, e.g., [Weibel 2013]. It says that if $F$ contains a primitive $m$-th root of unity then "the Galois symbol" gives isomorphisms

$$K_2^M(F)/m \cong H^2_{\text{ét}}(\mathrm{Spec}(F); \mu_m^{\otimes 2}) \cong {}_m\mathrm{Br}(F).$$

Here, the last term ${}_m\mathrm{Br}(F)$ is the set of elements in the Brauer group $\mathrm{Br}(F)$ that are of order $m$. The original proof of the theorem can be outlined as a reduction to a discussion of the algebraic closure in $F$ of some algebraic subfields. Furthermore, we should remark that the $K_2$-group of the algebraic closure $\overline{\mathbb{Q}}$ is known to be zero, i.e., $K_2^M(\overline{\mathbb{Q}}) \cong 0$. In particular, the map $K_2^M(F) \to K_2^M(\mathbb{C})$ induced from any complex embedding $F \to \mathbb{C}$ of a number field is zero.[4] In summary, to study the torsion $K_2^M(F)/m$, it is natural to assume that $F$ is a number field, i.e., a finite extension field of $\mathbb{Q}$.

Accordingly, we will restrict ourselves to discussing number fields $F$. Let $r_1$ be the number of real embeddings of $F$ and let $\mathrm{Spm}(\mathcal{O}_F)$ be the set of finite primes in the algebraic integer $\mathcal{O}_F$. We first write the localization sequence of the Milnor groups (see [Weibel 2013, §III.6]):

$$0 \to K_2^M(\mathcal{O}_F) \xrightarrow{i_*} K_2^M(F) \xrightarrow{\partial} \bigoplus_{\mathfrak{p}\in \mathrm{Spm}(\mathcal{O}_F)} k(\mathfrak{p})^\times \to 0 \quad \text{(exact)}. \tag{4.2}$$

Here, the symbol $i$ denotes the inclusion $\mathcal{O}_F \hookrightarrow F$ and $\partial$ is the sum of tame symbols associated with all primes $\mathfrak{p} \in \mathrm{Spm}(\mathcal{O}_F)$. Note further that the tame kernel $K_2^M(\mathcal{O}_F)$ is known to be of finite order. Hence, any element of $K_2^M(F)$ is of finite order.

On the other hand, for the study of the squared ideal $I^2(F)$ in (4.1), consider the sum of all completions $\Upsilon : F \to \mathbb{R}^{r_1} \oplus \big(\bigoplus_{\mathfrak{p}\in\mathrm{Spm}(\mathcal{O}_F)} \mathbb{Q}_\mathfrak{p}\big)$. The induced map on $I^2(\bullet)$ is known to be injective because of the Hasse–Minkowski principle [Lam 2005, §VI.3]. Furthermore, concerning the quotient $I^2/I^3(\bullet)$, the sum $\Upsilon$ yields an exact sequence

$$0 \to I^2/I^3(F) \xrightarrow{\Upsilon_*} (I^2/I^3(\mathbb{R}))^{r_1} \oplus \bigoplus_{\mathfrak{p}\in \mathrm{Spm}(\mathcal{O}_F)} I^2(\mathbb{Q}_\mathfrak{p}) \longrightarrow \mathbb{Z}/2 \to 0 \tag{4.3}$$

which is known as uniqueness of the Hilbert reciprocity. Here, we should note (see [Lam 2005, §VI.2]) that each $I^2(\mathbb{Q}_\mathfrak{p})$ is annihilated by 2, that $I^3(\mathbb{Q}_\mathfrak{p}) = 0$, and that $I^2(\mathbb{R}) \cong 4\mathbb{Z}$. Hence, $I^2(F)$ turns out to be a sum of $\mathbb{Z}^{r_1}$ and some 2-elementary abelian groups. In particular, the pullback (4.1) above immediately leads to a lemma:

**Lemma 4.4.** *The kernel of the map* $\mu : K_2^{\mathrm{MW}}(F) \to K_2^M(F)$ *is isomorphic to* $\mathbb{Z}^{r_1}$. *As a special case, if* $r_1 = 0$, *then the isomorphism* $K_2^{\mathrm{MW}}(F) \cong K_2^M(F)$ *holds.*

[4] In contrast to $K_1$ and $K_3$, the maps $K_i(F) \to K_i(\mathbb{C})^{r_2}$ induced by the complex embeddings are injective for $i = 1, 3$ (see [Zickert 2009] for details).

Incidentally, the sequence (4.3) implies that the group $K_2^{\mathrm{MW}}(F)$ includes the main information about the metaplectic group defined to be a double cover of $\mathrm{SL}_2(F)$.

**Example 4.5** ($F=\mathbb{Q}$). Finally, let us compute $K_2^{\mathrm{MW}}(\mathbb{Q})$ as an application of the above results. Note from a result of Tate (see [Weibel 2013, §III.6.3] or [Milnor 1971]) that the sequence (4.2) splits and there is an isomorphism $K_2^M(\mathbb{Z})\cong\mathbb{Z}/2$. Since $r_1=1$, a careful observation of the pullback diagram (4.1) leads to $K_2^{\mathrm{MW}}(\mathbb{Q})\cong\mathbb{Z}\oplus\bigoplus_p\mathbb{Z}/(p-1)$, where $p$ ranges over all odd primes.

## 5. Computation of the $K_2$ invariants; hyperbolic links

We will compute the $K_2$ invariants of some links. This section assumes that the characteristics of fields are zero, for simplicity.

**5A.** ***Example*: *the figure-eight knot.*** Consider the figure-eight knot $K_{4_1}$ of Figure 1. By the Wirtinger presentation of $\pi_1(\mathbb{R}^3\setminus K_{4_1})$, the group is formally generated by the arcs $\alpha_i$. Precisely, by definition,

$$\pi_1(\mathbb{R}^3\setminus K_{4_1})\cong\big\langle\mathfrak{m}_{\alpha_1},\mathfrak{m}_{\alpha_2},\mathfrak{m}_{\alpha_3},\mathfrak{m}_{\alpha_4}\mid\mathfrak{m}_{\alpha_3}=\mathfrak{m}_{\alpha_2}^{-1}\mathfrak{m}_{\alpha_1}\mathfrak{m}_{\alpha_2}=\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_4}\mathfrak{m}_{\alpha_1},\\ \mathfrak{m}_{\alpha_2}=\mathfrak{m}_{\alpha_4}^{-1}\mathfrak{m}_{\alpha_1}\mathfrak{m}_{\alpha_4}=\mathfrak{m}_{\alpha_3}^{-1}\mathfrak{m}_{\alpha_4}\mathfrak{m}_{\alpha_3}\big\rangle.$$

Then, we can easily see that the following assignment yields an $\mathrm{SL}_2$ representation $f:\pi_1(\mathbb{R}^3\setminus K_{4_1})\to\mathrm{SL}_2(F)$ if and only if $x^2\pm x+1=0$:

$$f(\mathfrak{m}_{\alpha_1}):=\begin{pmatrix}1&0\\-1&1\end{pmatrix},\quad f(\mathfrak{m}_{\alpha_3}):=\begin{pmatrix}x&(x-1)^2\\-1&2-x\end{pmatrix},$$
$$f(\mathfrak{m}_{\alpha_2}):=\begin{pmatrix}1&x^2\\0&1\end{pmatrix},\quad f(\mathfrak{m}_{\alpha_4}):=\begin{pmatrix}1-x+x^2&(x-1)^2\\-x^2&1+x-x^2\end{pmatrix}.$$

Moreover, according to Proposition 7.3, it can be seen that every parabolic representation turns out to be this $f$, up to conjugacy. Thus, it is sensible to consider the quadratic field $\mathbb{Q}(\sqrt{-3})=\mathbb{Q}[x]/(x^2\pm x+1)$.

Thus, we set $F=\mathbb{Q}(\sqrt{-3})$, and compute the $K_2$ invariant of $f$. Note that the preferred longitude $\mathfrak{l}$ forms

$$\mathfrak{l}=\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_2}\mathfrak{m}_{\alpha_3}\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_4}^{-1}\mathfrak{m}_{\alpha_3}\mathfrak{m}_{\alpha_2}\mathfrak{m}_{\alpha_4}^{-1}=\mathfrak{m}_{\alpha_2}\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_3}\mathfrak{m}_{\alpha_4}^{-1}\in\pi_1(\mathbb{R}^3\setminus K_{4_1}).$$

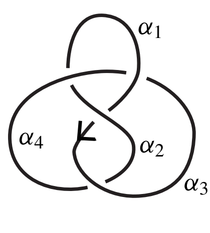


**Figure 1.** The figure-eight knot $K_{4_1}$ with four arcs.

Here, $K_2^{\mathrm{MW}}(F) = K_2^M(F)$ by Lemma 4.4 with $r_1 = 0$. Hence, from the definitions of $K_2^{\mathrm{MW}}$ and the 2-cocycle $\theta_{\mathrm{uni}}$, we can compute the $K_2$ invariant as

$$\begin{aligned} P_{K_2} \circ \tilde{f}(\mathfrak{l}) &= \theta_{\mathrm{uni}}\big(\big(\tilde{f}(\mathfrak{m}_{\alpha_2}), \tilde{f}(\mathfrak{m}_{\alpha_1}^{-1})\big) + \big(\tilde{f}(\mathfrak{m}_{\alpha_2}\mathfrak{m}_{\alpha_1}^{-1}), \tilde{f}(\mathfrak{m}_{\alpha_3})\big) \\ &\qquad\qquad + \big(\tilde{f}(\mathfrak{m}_{\alpha_2}\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_3}), \tilde{f}(\mathfrak{m}_{\alpha_4}^{-1})\big)\big) \\ &= (\{1,-1\} - \{1,1\}) + (\{x^2,1\} - \{1,-1\}) + (\{2+4x^2,-1\} - \{x^2,x^2\}) \\ &= \{2+4x^2,-1\} - \{x^2,-1\} \\ &= \{(2+4x^2)x^2,-1\} = \{-2-x^2,-1\} \in K_2^M(\mathbb{Q}(\sqrt{-3})). \end{aligned}$$

Further, let us analyse this $\{-2-x^2,-1\}$ in $K_2^M(\mathbb{Q}(\sqrt{-3}))$. Since the tame kernel $K_2^M(\mathcal{O}_{\mathbb{Q}(\sqrt{-3})})$ is known to be zero (Tate), the sequence (4.2) means that the sum $\partial$ is an isomorphism. Furthermore, for any prime $\mathfrak{p} \in \mathrm{Spm}(\mathcal{O}_F)$, the tame symbol $\partial_{\mathfrak{p}}(-2-x^2,-1)$ equals $(-1)^{v_{\mathfrak{p}}(-2-x^2)} \in k(\mathfrak{p})$ by definition. Since $(2+x^2)(x^2-1) = -3$ and $2+x^2$ and $x^2-1$ are prime elements over 3, we can conclude the following:

**Proposition 5.1.** *Let $F = \mathbb{Q}(\sqrt{-3})$. Then $\partial_{(x^2-1)} \oplus \partial_{(2+x^2)}(\{-2-x^2,-1\}) = (-1,-1) \in (\mathbb{F}_3^{\times})^2$, and for any other prime $\mathfrak{p}$ we have $\partial_{\mathfrak{p}}(\{-2-x^2,-1\}) = 1$.*

In summary, the $K_2$ invariant $\tilde{f}(\mathfrak{l})$ in $K_2^M(\mathbb{Q}(\sqrt{-3}))$ turns out to be nontrivial by means of the tame symbols, whereas the representation $f$ factors through the algebraic integer $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$.

**5B. *Other links.*** Next, let us discuss other links. Here, notice from Lemma 4.4 that it is relatively easy to compute the kernel (which is isomorphic to $\mathbb{Z}^{r_1}$) of $\mu : K_2^{\mathrm{MW}}(F) \to K_2^M(F)$. Thus, this subsection is specialized to some parabolic representations and gives in Table 1 a list of these values $\mu(\tilde{f}(\mathfrak{l}))$ without performing detailed computations (see [Maclachlan and Reid 2003, Appendix 13.3] for the defining polynomials).

In each case, by $F/\mathbb{Q}$ we mean the minimal field extension that splits the defining polynomial. Furthermore, we can see that the class numbers of the splitting fields vanish; we can easily study prime ideals in $\mathcal{O}_F$ and compute the associated valuations $\partial_{\mathfrak{p}}$. For example, the tame symbols at the primes $(x^2+1)$ and $(x^2+2)$ distinguish the $K_2$-values of the $6_1$-knot from one of the $7_7$-knot, whereas the defining polynomials are equal. In doing so, we can find further examples of nontrivial $K_2$ invariants of other links; however, it remains a problem for the future to clarify the topological and arithmetic features reflected in the $K_2$ invariants.

Finally, let us briefly comment on the $K_2$ invariants of hyperbolic small links with $\#L > 1$. As seen in [Baker 2001], we find many holonomies contained in $\mathrm{SL}_2(F)$ with some quadratic fields $F$; we can easily compute the longitudes of such holonomies, since the finite primes of $F$ and the tame kernel $\mathcal{O}_F$ have been

| knot | defining polynomial | $r_1$ | $\mu(\tilde{f}(\mathfrak{l})) \in K_2^M(F)$ |
|---|---|---|---|
| $3_1$ | $x^2-1$ | 1 | $\{3, -1\}$ |
| $5_1$ | $x^4+3x^2+1$ | 2 | $\{x^2+2, -1\}+\left\{\frac{1}{10}(9-8x^2), -21+10x^2\right\}$ |
| $5_2$ | $1-2x^2+x^4-x^6$ | 1 | $\{2(1+x^4)(5+3x^4), -1\}$ |
| $6_1$ | $1+x^2+3x^4+2x^6+x^8$ | 0 | $\{x^2+2, -1\}$ |
| $7_7$ | $1+x^2+3x^4+2x^6+x^8$ | 0 | $\left\{\frac{2(2+6x^2+4x^4+x^6)}{3+2x^2+3x^4+x^6}, 1+2x^2+x^4\right\}$ $+\left\{-1+4x^2+7x^4+4x^6, \frac{-2+8x^2+4x^4+4x^6}{-2-x^2-x^4}\right\}$ $+\{-3-2x^2-3x^4-x^6, 2x^2+2x^4+x^6\}$ |

**Table 1.** Values of $\mu(\tilde{f}(\mathfrak{l}))$ for some defining polynomials of knots.

well studied (see [Keune 1989; Weibel 2013], for example). However, we remark that, concerning the Whitehead and the Borromean links as the simplest examples, these $\tilde{f}(\mathfrak{l}_i)$ are trivial, unfortunately.

## 6. Two applications

This paper aims to applications of $K_2$-groups to low-dimensional topology. This section furthermore gives two applications, although these results are a bit tangential. In this section, although we roughly review some notions in knot theory, we refer the reader to [Lickorish 1997, §1 and §11] or [Maclachlan and Reid 2003] for detailed definitions.

**6A.** ***On the cusp shape.*** While we discuss the $K_2$invariant in Sections 3–5, we will focus on another summand $F$ in $F \times \widetilde{K}_2^{\mathrm{MW}}(F)$. The value $\tilde{f}(\mathfrak{l}_i)$ restricted on this $F$ is called *the cusp shape* as an important concept in hyperbolic geometry; see, e.g., [Maclachlan and Reid 2003]. We give a sum formula of the cusp shape.

To state Theorem 6.1, we introduce some terminology. Fix a parabolic representation $f : \pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$, and a link diagram $D$ of $L$. Roughly speaking, as seen in Figure 2, $D$ is the image $p(L) \subset \mathbb{R}^2$ with over–under information, where $p$ is a "generic" projection $p : \mathbb{R}^3 \to \mathbb{R}^2$. Then, we can consider the over-arcs $\alpha_1, \alpha_2, \ldots, \alpha_{N_j}$ along the orientation of the longitude $\mathfrak{l}_j$ as illustrated in Figure 2. Let $\beta_i$ be the arc which divides $\alpha_{i-1}$ and $\alpha_i$, and $\epsilon_i \in \{\pm 1\}$ be the sign of the crossing between $\alpha_i$ and $\beta_i$, according to Figure 4 (see Section 7A). We denote a loop circling around an arc $\alpha$ by $\mathfrak{m}_\alpha$. As is known from the Wirtinger presentation, every $\mathfrak{m}_\alpha$ is conjugate to some meridian in $\pi_1(\mathbb{R}^3 \setminus L)$. Hence, by parabolicity of $f$, it can seen, as in (7.1), that any arc $\alpha$ uniquely, up to sign,

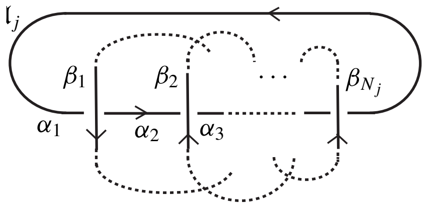


**Figure 2.** The longitude $\mathfrak{l}_j$ and arcs $\alpha_i$ and $\beta_i$ in the diagram $D$.

admits $(c_\alpha, d_\alpha) \in F \times F \setminus \{(0,0)\}$ such that

$$f(\mathfrak{m}_\alpha) = \begin{pmatrix} 1 + c_\alpha d & d_\alpha^2 \\ -c_\alpha^2 & 1 - c_\alpha d_\alpha \end{pmatrix}.$$

Furthermore, we define a map $\mathcal{S} : (F \times F \setminus \{(0,0)\})^2 \to F$ by setting

$$\mathcal{S}((a,b),(c,d)) := \begin{cases} -1 + c^2/(a^2 - abc^2 + a^2cd) & \text{if } a(bc^2 - a - acd) \neq 0, \\ -1 + (-c^2 + c^3d)/a^2, & \text{if } a \neq 0,\ bc^2 - a - acd = 0, \\ -1 + (-1 - cd)/b^2c^2, & \text{if } a = 0,\ c \neq 0, \\ -1 + d^2/b^2, & \text{if } a = c = 0. \end{cases}$$

We now analyse the sum $\sum_{i=1}^{N_j} \epsilon_i \cdot \mathcal{S}\big((c_{\alpha_i}, d_{\alpha_i}), (c_{\beta_i}, d_{\beta_i})\big) \in F$, as follows:

**Theorem 6.1.** *The sum coincides with the cusp shape $P_F \circ \tilde{f}(\mathfrak{l}_j)$ in $F$, where $P_F$ is the projection $\widetilde{K}_2^{\mathrm{MW}}(F) \times F \to F$.*

The proof will appear in Section 7D; the point here is that the sum formula is independent of the order of the crossings, while the longitudes seem to be non-commutative. Moreover, it is interesting and applicable to computations that we need not describe the longitude $\mathfrak{l}_i$ in the formula, with $\mathfrak{l}_i$ complicated, as in (6.5).

**6B.** ***Another application: the real $K_2(\mathbb{R})$ and left-orderable 3-manifold groups.*** This section focuses on the real case $F = \mathbb{R}$ and compares the $K_2$-group $K_2^{\mathrm{MW}}(\mathbb{R})$ with $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$, where $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ is the topological universal cover of $\mathrm{SL}_2(\mathbb{R})$ associated with $\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z}$. As an application, we give a formula to compute longitudes lifted to $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$. We hope that this computation will be useful for studying the left-orderability of 3-manifold groups (see [Boyer et al. 2013], for example). In fact, we give new 3-manifold groups which are left-orderable.

We now explain Proposition 6.3, which strictly describes $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$. Consider the map $\mathrm{Sign} : \mathbb{R}^2 \to \mathbb{Z}$ defined by $\mathrm{Sign}(a,b) = 1$ if $a < 0$ and $b < 0$, and $\mathrm{Sign}(a,b) = 0$ otherwise. Recalling the 2-cocycle $\theta_{\mathrm{uni}}$ in (2.2), we equip $\mathbb{Z} \times \mathrm{SL}_2(\mathbb{R})$ with the group operation

$$(n,g) \cdot (m,h) := (n + m + \mathrm{Sign} \circ \theta_{\mathrm{uni}}(g,h), gh). \tag{6.2}$$

**Proposition 6.3.** *This group structure on $\mathbb{Z} \times \mathrm{SL}_2(\mathbb{R})$ is isomorphic to the universal cover* $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ (forgetting the topology, of course).

Here, we should emphasize that this result is simpler than the known formula for the group operation on $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$, because it was formulated using logarithms (see [Bargmann 1947] for details).

*Proof.* First, let us compute $K_2^{\mathrm{MW}}(\mathbb{R})$. Since $I(F)$ for any algebraically closed $F$ is known to be zero, we obtain $K_2^{\mathrm{MW}}(F) \cong K_2^M(F)$ from the pullback diagram (4.1). Moreover, it is known (see [Weibel 2013, Theorem III.6.4 and Application III.6.8.3]) that $K_2^M(F)$ is of uncountable cardinality and is a uniquely divisible group, i.e., a $\mathbb{Q}$-vector space, and that an isomorphism $K_2^M(\mathbb{R}) \cong \mathbb{Z}/2 \oplus K_2^M(\mathbb{C})^+$ is obtained as a corollary of Hilbert's Theorem 90. Here the first summand $\mathbb{Z}/2$ is widely known to be generated by the (Steinberg) symbol $\{-1, -1\}$ and the second one is the invariant subspace by complex conjugation. Recalling from Section 4 that $I^2(\mathbb{R}) \cong 4\mathbb{Z}$ is generated by $(\langle 1\rangle - \langle -a^2\rangle)^2$ with $a \in \mathbb{R}$, the pullback diagram (4.1) implies that

$$K_2^{\mathrm{MW}}(\mathbb{R}) \cong \mathbb{Z} \oplus K_2^M(\mathbb{C})^+. \tag{6.4}$$

Here, notice that the induced homomorphism $\mathrm{Sign}_* : K_2^{\mathrm{MW}}(\mathbb{R}) \to \mathbb{Z}$ from $\mathrm{Sign} : \mathbb{R}^2 \to \mathbb{Z}$ coincides, by construction, with the projection in the decomposition (6.4).

Finally, we complete the proof. Since the cover $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ is a central extension of $\mathrm{SL}_2(\mathbb{R})$ with fiber $\mathbb{Z}$, the universal extension $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ surjects onto $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$. By noticing the isomorphism (6.4) and that every quotient of a divisible group is divisible, the central kernel is $K_2^M(\mathbb{C})^+$. Hence, the surjection to the group (6.2) induces the desired isomorphism. □

Thanks to Proposition 6.3, given an $\tilde{f} : \pi_1(\mathbb{R}^3 \setminus K) \to \widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ we can compute the value $P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l}))$ of the longitude $\mathfrak{l}$. This section will give an application (Proposition 6.6 and Theorem 6.7). Throughout this subsection, we will denote by $P_{\mathbb{Z}}$ the set-theoretic projection $\mathbb{Z} \times \mathrm{SL}_2(\mathbb{R}) \to \mathbb{Z}$.

First, let us comment on some known results. Note that the connection between the summand $\mathbb{Z}$ in (6.4) and the Euler classes of $U(1)$-bundles over surfaces is well-understood (see [Wood 1971]). For example, the Milnor–Wood inequality gives an estimate of the value $P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l})) \in \mathbb{Z}$ bounded by the Seifert genus $g(K)$ of a knot $K$. Precisely, since the longitude forms a product of $g(K)$ elements in the commutator subgroup $\pi_1(\mathbb{R}^3 \setminus K)'$, we have $|P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l}))| \leq g(K) - \frac{1}{4}$; see [Wood 1971, (5.5)]. As a corollary, for any knot $K$ of Seifert genus one, the value $P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l}))$ is zero (this result was crucial in [Boyer et al. 2013; Hakamata and Teragaito 2014; Tran 2015]). However, no value $P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l}))$ with respect to knots $K$ of Seifert genus $> 1$ has been computed so far.

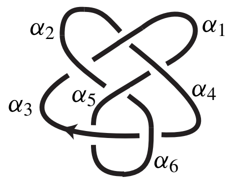


**Figure 3.** The diagram $D$ of the knot $6_2$.

As the nontorus knot of Seifert genus $> 1$ and of the minimal crossing number, we will focus on the $6_2$-knot $K$. The diagram $D$ with arcs $\alpha_1, \dots, \alpha_6$ is illustrated in Figure 3.

Inspired by a method in [Hakamata and Teragaito 2014; Tran 2015], we will find elliptic homomorphisms $f : \pi_1(\mathbb{R}^3 \setminus K) \to \mathrm{SL}_2(\mathbb{R})$ such that

$$f(\mathfrak{m}_{\alpha_1}) = \begin{pmatrix} \sqrt{t} & \sqrt{t} \\ 0 & \sqrt{t}^{-1} \end{pmatrix}, \quad f(\mathfrak{m}_{\alpha_2}) = \begin{pmatrix} \sqrt{t} & 0 \\ -s\sqrt{t}^{-1} & \sqrt{t}^{-1} \end{pmatrix} \quad \text{for some } s, t \in \mathbb{R}.$$

Moreover, we set $T = t + t^{-1}$. Then, from the Wirtinger presentation, we can easily see that $s$ and $t$ must satisfy the equation $R_{6_2}(s, T) = 0$, where $R_{6_2}(s, T)$ is the polynomial

$$1 + 3s + s^2 + 2s^3 + 3s^4 + s^5 - (3 + 2s + 4s^2 + 9s^3 + 4s^4)T \\ + (1 + 2s + 9s^2 + 6s^3)T^2 - (3s + 4s^2)T^3 + sT^4.$$

Owing to this quartic equation with respect to $T = t + t^{-1}$, this $t$ can be formulated as an algebraic function of $s$. Here, suppose a (unique) positive solution $s_0 = 1.48288\dots$ for which the discriminant $\Delta(s)$ of $R_{6_2}(s, t)$ with respect to $t$ is zero. Then, following the quartic formula for $T$, if $0 < s < s_0$ (resp. $s_0 < s < 200$), there are two (resp. four) real solutions $t \in \mathbb{R}_{>0}$ of the equation $R_{6_2}(s, t + t^{-1}) = 0$.[5] Choose the two solutions which are smallest and denote them by $t_{\min}$ and $t_{\sec}$. We denote by $f_{s,t}$ the resulting homomorphism $\pi_1(\mathbb{R}^3 \setminus K) \to \mathrm{SL}_2(\mathbb{R})$, and denote by $\tilde{f}_{s,t} : \pi_1(\mathbb{R}^3 \setminus K) \to \mathrm{SL}_2^{\mathrm{top}}(\mathbb{R})$ the lift of $f_{s,t}$.

We will compute the resulting value $P_{\mathbb{Z}}(\tilde{f}_{s,t}(\mathfrak{l}))$, where we will use a longitude $\mathfrak{l}$ of the form

$$\mathfrak{l} = \mathfrak{m}_{\alpha_1} \mathfrak{m}_{\alpha_4}^{-1} \mathfrak{m}_{\alpha_3} \mathfrak{m}_{\alpha_5}^{-1} (\mathfrak{m}_{\alpha_1}^{-1} \mathfrak{m}_{\alpha_6} \mathfrak{m}_{\alpha_1})(\mathfrak{m}_{\alpha_1}^{-1} \mathfrak{m}_{\alpha_2}^{-1} \mathfrak{m}_{\alpha_1}) \in \pi_1(\mathbb{R}^3 \setminus K_{6_2}). \tag{6.5}$$

Hence, according to (6.2), we can formulate the value $P_{\mathbb{Z}}(\tilde{f}_{s,t}(\mathfrak{l}))$ as a function of $s$. By definition, the function is upper semicontinuous with respect to $s$. Furthermore, it is possible to list all the (finitely many) noncontinuous points of $P_{\mathbb{Z}}(\tilde{f}_{s,t}(\mathfrak{l}))$ for a given interval in $\mathbb{R}$. Here, we focus on the interval $[0, 200]$. Then, with the help of

[5] Incidentally, if $3000 < s$, we have eight real solutions $t \in \mathbb{R}_{>0}$ of the equation $R_{6_2}(s, T) = 0$.

a computer, we can investigate noncontinuous points in the interval (here we use the above quartic formula), and hence get the following conclusion:

**Proposition 6.6.** *For $s > 0$, let $t_{\min}$ and $t_{\sec}$ be the above solutions of $R_{6_2}(s,t) = 0$. Then the value $P_{\mathbb{Z}}(\tilde{f}_{s,t_{\min}}(\mathfrak{l}))$ is $0$ if $0 < s < 200$, while $P_{\mathbb{Z}}(\tilde{f}_{s,t_{\sec}}(\mathfrak{l}))$ is $1$ if $s_0 < s < 200$.*

It is worth noting that, by a computer program, if $2700 < s < 2900$, the value $P_{\mathbb{Z}}(\tilde{f}_{s,t_{\min}}(\mathfrak{l}))$ is 1; hence Proposition 6.6 does not hold for any $s > s_0$. However, we emphasize that it is the first to discover infinitely many homomorphisms $\tilde{f}_{s,t}$ such that the values $P_{\mathbb{Z}}(\tilde{f}_{s,t}(\mathfrak{l}))$ are not zero, and that it seems to be hard to compute the value $P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l}))$ for general knots $K$.

Finally, we give an application using the ideas of [Boyer et al. 2013; Hakamata and Teragaito 2014]. It is known [Boyer et al. 2013] that, if an irreducible closed 3-manifold $M$ admits a nontrivial homomorphism $\pi_1(M) \to \widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$, then $M$ has left-orderable fundamental group. Here, a group $G$ is *left-orderable* if it has a total order $\leq$ such that $g, x, y \in G$ with $x \leq y$ implies $gx \leq gy$. Based on their ideas, we will show the following:

**Theorem 6.7.** *Let $r = p/q \in \mathbb{Q}$. Let $M_r(K)$ be the closed 3-manifold obtained by $r$-Dehn surgery along the $6_2$-knot $K$. If $0.1 < r < 7.99$, then the fundamental group $\pi_1(M_r(K))$ is left-orderable.*

*Proof.* We will construct a nontrivial homomorphism $f : \pi_1(\mathbb{R}^3 \setminus K) \to \widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ which sends $\mathfrak{m}_{\alpha_1}^p \mathfrak{l}^q$ to the identity. Here note that the 3-manifold $M_r(K)$ obtained from the 2-bridge knot $6_2$ is known to be irreducible. If we have such a map, the van Kampen theorem admits the induced map $\pi_1(M_r(K)) \to \widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ and, hence, gives the desired left-orderability.

The construction of $f$ is as follows: First notice that the commutator subgroup of $f_{s,t}(\mathfrak{m}_{\alpha_1})$ forms

$$\left\{ \begin{pmatrix} u & (u - u^{-1})/(1 - t^{-2}) \\ 0 & u^{-1} \end{pmatrix} \,\middle|\, u \in \mathbb{R}^{\times} \right\}$$

without $t^2 = 1$. Hence, by the definition of $\mathfrak{l}$, we can see that $f_{s,t}(\mathfrak{l}) \in \mathrm{SL}_2(\mathbb{R})$ is of the form

$$\begin{pmatrix} g(s,t) & * \\ 0 & g(s,t)^{-1} \end{pmatrix}$$

for some $* \in \mathbb{R}$, where $g(s,t)$ is a polynomial in $s$ of the form

$$\begin{aligned}\big(1 - 2t + t^2 - 2t^4 + t^5 + s(4t^2 - 3t - t^3 - 2t^4 + 3t^5 - 2t^6 + t^7)& \\ + s^2(3t^2 - 2t^3 - t^4 + 2t^5 - 2t^6) + s^3(t^5 - t^3)\big)/t^2&.\end{aligned}$$

Since the commutator subgroup is isomorphic to $\mathbb{R}^{\times}$, the equality $f(\mathfrak{m}_{\alpha_1})^p f(\mathfrak{l})^q = \mathrm{id}_{\mathbb{R}^2} \in \mathrm{SL}_2(\mathbb{R})$ holds if and only if $t^{-p/2} = g(s,t)^q$. To solve this, we consider the function $\mathcal{R}:[0, 100]\to[0, \infty)$ defined by $\mathcal{R}(s):=2\log(g(s, t_{\min}))/\log(t_{\min})$. Here we note the estimate $\mathcal{R}(10^{-4})<10^{-1}$ and $\mathcal{R}(10^2)>7.99$, which are obtained from a computer program. Since this $\mathcal{R}$ is continuous by construction, the image of $\mathcal{R}$ includes the interval $[0.1, 7.99]$. To summarize, for $10^{-1}<r<7.99$ there are $s$ and $t_{\min}$ with $0<s<100$ which ensure a homomorphism $f_{s,t_{\min}}$ that sends $\mathfrak{m}_{\alpha_1}^p\mathfrak{l}^q$ to the identity in $\mathrm{SL}_2(\mathbb{R})$.

Moreover, we consider a lifted $\tilde{f}_{s,t_{\min}}:\pi_1(\mathbb{R}^3\setminus K)\to\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$. By Proposition 6.6, we have $P_{\mathbb{Z}}(\tilde{f}_{s,t_{\min}}(\mathfrak{m}_{\alpha_1}))=P_{\mathbb{Z}}(\tilde{f}_{s,t_{\min}}(\mathfrak{l}))=0$. Hence this lift is one of the required maps. □

It is well known (see [Boyer et al. 2013], for example) that the resulting 3-manifold, $M_r(K)$, of $r$-surgery on any 2-bridge knot $K$ is not an $L$-space, i.e., the Heegaard Floer homology of $M_r(K)$ is not isomorphic to that of the lens space $L(p,q)$ for any $(p,q)\in\mathbb{Z}^2$. Theorem 6.7 is supporting evidence for a conjecture in [Boyer et al. 2013], which predicts an equivalence between $L$-spaces and the left-orderability. As seen in the proof above, we hope that our computation will be applicable to other knots of genus $>1$.

## 7. Proofs of the theorems

We will prove the theorems from Sections 2 and 6. For this, this section employs an approach to obtaining parabolic representations by means of quandles. This approach, using quandle, has some benefits: first, while $\mathrm{SL}_2(F)$ is of dimension 3 over $F$, the approach can deal with parabolic representations from a certain 2-dimensional object $(\mathbb{A}^2\setminus 0)/\{\pm\}$; see Proposition 7.3 (in contrast to [Riley 1972] in a group-theoretic approach). Furthermore, the results of [Carter et al. 2005; Eisermann 2014; Nosaka 2015] in quandle theory gave some topological applications; here the point is that quandle theory sometimes ensures nontriviality of some knot invariants and makes a reduction to knot diagrams without 3-dimensional discussion of $\mathbb{R}^3\setminus L$. Correspondingly, we will see that our setting of $\mathrm{SL}_2(F)$ satisfies conditions necessary to the results, and will give the proofs of Theorems 3.4 and 6.1.

**7A.** ***Parabolic representations in terms of quandles.*** Let us begin by reviewing quandles. A *quandle* [Joyce 1982] is a set, $X$, with a binary operation $\lhd: X\times X\to X$ such that

(I) $a\lhd a=a$ for any $a\in X$;

(II) the map $(\bullet\lhd a): X\to X$ defined by $x\mapsto x\lhd a$ is bijective for any $a\in X$;

(III) $(a\lhd b)\lhd c=(a\lhd c)\lhd(b\lhd c)$ for any $a,b,c\in X$.

A map $f: X\to Y$ between quandles is a (*quandle*) *homomorphism* if $f(a\lhd b)=f(a)\lhd f(b)$ for any $a,b\in X$. For example, any group $G$ is a quandle with the


**Figure 4.** Positive and negative crossings.

conjugacy operation $x \lhd y := y^{-1}xy$ for any $x, y \in G$, and is called *the conjugacy quandle* in $G$ and denoted by $\mathrm{Conj}(G)$. Furthermore, given an infinite field $F$, consider the quotient set $F^2 \setminus \{(0,0)\}/\!\sim$ subject to the relation $(a,b) \sim (-a,-b)$, and equip this set with the quandle operation

$$(\,a \;\; b\,) \lhd (\,c \;\; d\,) = (\,a \;\; b\,)\begin{pmatrix} 1+cd & d^2 \\ -c^2 & 1-cd \end{pmatrix}.$$

This quandle in the case $F = \mathbb{C}$ was introduced in [Inoue and Kabaya 2014, §5], which refers to it as a *parabolic quandle* (over $F$) and denotes it by $X_F$. Furthermore, consider the map

$$\begin{aligned} &\iota : X_F \to \mathrm{SL}_2(F), \\ &(c,d) \mapsto \begin{pmatrix} 1+cd & d^2 \\ -c^2 & 1-cd \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned} \tag{7.1}$$

We can easily see that this $\iota$ is injective and a quandle homomorphism, and the image is the conjugacy class of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Hence, the quandle $X_F$ is a subquandle composed of parabolic elements of the conjugacy quandle in $\mathrm{SL}_2(F)$ (furthermore, it is a subquandle in $\mathrm{PSL}_2(F)$).

Next, we will review $X$-colorings. Let $X$ be a quandle and $D$ be an oriented link diagram of a link $L \subset S^3$. An *$X$-coloring* of $D$ is a map $\mathcal{C} : \{\text{arcs of } D\} \to X$ such that $\mathcal{C}(\gamma_k) = \mathcal{C}(\gamma_i) \lhd \mathcal{C}(\gamma_j)$ at each crossing of $D$ as in Figure 4.

For example, when $X$ is the conjugacy quandle of a group $G$, the coloring condition coincides with the relations in the Wirtinger presentation of a link $L$. Hence, we have a bijection

$$\mathrm{Col}_{\mathrm{Conj}(G)}(D) \xleftrightarrow{1:1} \mathrm{Hom}_{\mathrm{gr}}(\pi_1(\mathbb{R}^3 \setminus L), G). \tag{7.2}$$

Next, let us focus on colorings with respect to the parabolic quandles $X_F$ over fields $F$. Since $X_F$ is a conjugacy class of $\mathrm{SL}_2(F)$ via (7.1), we can easily prove:

**Proposition 7.3** (a special case of [Nosaka 2015, Corollary B.1]). *Let $D$ be a diagram of a link $L$. Fix meridians $\mathfrak{m}_1, \dots, \mathfrak{m}_{\#L} \in \pi_1(\mathbb{R}^3 \setminus L)$ in each link-component which is compatible with the orientation of $D$. Then the restriction of* (7.2) *gives a bijection from the set $\mathrm{Col}_{X_F}(D)$ to the following set, composed of parabolic* *representations from* $\pi_1(\mathbb{R}^3 \setminus L)$:

$$\{ f \in \mathrm{Hom}(\pi_1(\mathbb{R}^3 \setminus L), \mathrm{SL}_2(F)) \mid f(\mathfrak{m}_i) = \iota(x_i) \text{ for some } x_i \in X_F \}.$$

In particular, if $L$ is a hyperbolic link and $F = \mathbb{C}$, the holonomy is regarded as a nontrivial $X_{\mathbb{C}}$-coloring in $\mathrm{Col}_{X_{\mathbb{C}}}(D)$ (see Appendix 13.3 of [Maclachlan and Reid 2003] for the hyperbolic knots of crossing number $< 9$).

We remark that it is very often (but not always) the case that the quotient set of $\mathrm{Col}_{X_F}(D)$ modulo conjugation in $\mathrm{SL}_2(F)$ is of finite order. In a special case, we will see that small knots satisfy finiteness (Proposition 7.4). Here, a knot $K$ is said to be *small* if there is no incompressible surface except for a boundary-parallel torus in the knot exterior. For example, the 2-bridge knots and torus knots are known to be small.

**Proposition 7.4.** *Let $F$ be a field embedded in the complex field $\mathbb{C}$. If $D$ is a diagram of a small knot $K$, then the quotient set of* $\mathrm{Col}_{X_F}(D)$ *subject to the conjugacy operation of* $\mathrm{SL}_2(F)$ *is of finite order.*

We will omit the proof, since it follows from standard arguments in Culler–Shalen theory similar to those in [Culler and Shalen 1983] or [Cooper et al. 1994, Proposition 2.4].

**Example 7.5.** It is known that every knot of crossing number $< 9$ is small. Furthermore, we can see that the quotient set is bijective to $\{x \in F^\times/\{\pm 1\} \mid f(x)f(-x) = 0\}$ for some polynomial $f(x)$. Without proof, we list the defining polynomials of some knots for the case $\mathrm{Char}(F) = 0$ in Table 2.

**7B. *Proof of Proposition 3.1.*** From Proposition 7.3 and the definition of $K_2^{\mathrm{MW}}(F)$, we will prove Proposition 3.1.

*Proof of Proposition 3.1.* By definition of parabolicity, $f(\mathfrak{m})$ for every meridian $\mathfrak{m}$ is contained in the image of $\iota$ (recall Proposition 7.3), where $\iota$ is the map in (7.1).

| knot | the defining polynomial $f(x)$ |
|---|---|
| $3_1$ | $x-1$ |
| $4_1$ | $x^2-x+1$ |
| $5_1$ | $x^2+x-1$ |
| $5_2$ | $x^3-x^2+1$ |
| $6_1$ | $x^4+x^2-x+1$ |
| $7_4$ | $(x^3+2x-1)(x^4-x^3+2x^2-2x+1)$ |
| $7_7$ | $(x^4+x^2-x+1)(x^6+x^5+2x^4+2x^3+2x^2+2x+1)$ |

**Table 2.** The defining polynomials for some knots.

Hence, from the Wirtinger presentation and Lemma 7.6 below, we can canonically obtain a lift $\tilde{f}:\pi_1(\mathbb{R}^3\setminus L)\to\widetilde{\mathrm{SL}}_2(F)$, defined by setting

$$\tilde{f}(\mathfrak{m})=(0,\,f(\mathfrak{m}))\in K_2^{\mathrm{MW}}(F)\times\mathrm{SL}_2(F).\qquad\square$$

**Lemma 7.6.** *Consider the composite $\theta_{\mathrm{uni}}\circ(\iota\times\iota):(X_F)^2\to K_2^{\mathrm{MW}}(F)$ of the universal 2-cocycle $\theta_{\mathrm{uni}}$. Then, for any $(a,b),(c,d)\in X_F$, the composite satisfies the equality*

$$\theta_{\mathrm{uni}}\circ(\iota\times\iota)\big((a,b),(c,d)\big)=\theta_{\mathrm{uni}}\circ(\iota\times\iota)\big((c,d),(a,b)\lhd(c,d)\big).$$

We will prove Lemma 7.6 by a tedious computation. To this end, denote the restriction $\theta_{\mathrm{uni}}\circ(\iota\times\iota)$ by $\Theta$. Then a direct calculation shows an easy formula for this $\Theta$: precisely, for any $(a,b),(c,d)\in X_F$, the map $\Theta:(X_F)^2\to K_2^{\mathrm{MW}}(F)$ satisfies the equality

$$\Theta\big((a,b),(c,d)\big)=\begin{cases}[(ab-1)c^2-(1+cd)a^2,\,-c^2/a^2]\\ \qquad\qquad\qquad\qquad -[-a^2,\,-c^2] & \text{if } ac\neq 0,\\ \qquad\qquad 0 & \text{if } ac=0.\end{cases}\tag{7.7}$$

*Proof of Lemma 7.6.* When $ac=0$, we can easily obtain the desired equality in Lemma 7.6 by a direct calculation, although we omit the details.

Thus, we will assume $ac\neq 0$, and compute $\Theta\big((a,b),(c,d)\big)$ in some detail. Denote $(a,b)\lhd(c,d)$ by $(H,I)\in X_F$ for short. Then a direct calculation can show the identity

$$(1-cd)H^2+(1+HI)c^2=(1-ab)c^2+(1+cd)a^2.\tag{7.8}$$

Let $\mathcal{B}$ be the right-hand side in (7.8). Noting that $[-a^2,-c^2]=[-a^2,-c^2/a^2]$ by axiom (ii), the $\Theta\big((a,b),(c,d)\big)$ in (7.7) becomes $[-\mathcal{B},-c^2/a^2]-[-a^2,-c^2/a^2]$. Further, this is equal to $[\mathcal{B}/a^2,-c^2/a^2]-2[-1,-\mathcal{B}]$ by Lemma 7.9(2) below.

Hence it is enough to show $[\mathcal{B}/c^2,-H^2/c^2]=[\mathcal{B}/a^2,-c^2/a^2]$ for the proof. For this purpose, note $[\mathcal{B}/a^2,-c^2/a^2]=[\mathcal{B}/c^2,-c^2/a^2]$ by Lemma 7.9(1). Therefore, from the identity $\mathcal{B}=aH+c^2$ by definition and the axiom (iii), we deduce that

$$\begin{aligned}[\mathcal{B}/c^2,-c^2/a^2]&=[\mathcal{B}/c^2,-(\mathcal{B}/c^2-1)^2(c^2/a^2)]\\&=[\mathcal{B}/c^2,-a^2H^2/(c^2a^2)]=[\mathcal{B}/c^2,-H^2/c^2].\end{aligned}$$

In summary, we have the desired equality $[\mathcal{B}/c^2,-H^2/c^2]=[\mathcal{B}/a^2,-c^2/a^2]$. $\square$

**Lemma 7.9.** (1) $[x,y]=[x^{-1},y^{-1}]=[-xy,y]$ *for any* $x,y\in F^\times$.

(2) $[x,-z^2]+[-y^2,-z^2]=[-xy^2,-z^2]+2[-1,x]$ *for any* $x,y,z\in F^\times$.

*Proof.* First, (1) is directly obtained from the axiom (ii) of $K_2^{\mathrm{MW}}(F)$.

Next we will prove (2). Following [Suslin 1987], we use the notation $[a,b,c]:=[a,b]+[a,c]-[a,bc]$. Since $[A,-z^2]=[-z^{-2},A]$, the goal is equivalent to the equality $[-z^{-2}, x, -y^2] = [x, -1, -1]$. To show this, we set up two identities proven in [Suslin 1987, Lemma 6.1] of the forms

$$[ab, x, c] = [a, bx, c] + [b, x, c] - [a, b, c], \quad [d, e, f] = [d^{-1}, e, f] \tag{7.10}$$

for any $x, a, b, c, d, e, f \in F^\times$. By applying $a = -z$, $b = z$ and $c = -y^2$ to these identities, we have

$$\begin{aligned}[-z^{-2}, x, -y^2] = [-z^2, x, -y^2] &= [-z, zx, -y^2] + [z, x, -y^2] - [-z, z, x] \\ &= [-z^{-1}, -zx, -y^2] + [z, x, -y^2] - [-z^{-1}, z, x] \\ &= [-1, x, -y^2].\end{aligned}$$

Lastly, since the equalities $[x, b, c] = [x, c, b] = [b, c, x]$ are known [Suslin 1987, Lemma 6.1], repeating the computation leads to $[-z^{-2}, x, -y^2] = [-1, x, -y^2] = [-y^2, x, -1] = [-1, x, -1] = [x, -1, -1]$, as desired. □

**7C.** ***Preliminaries.*** In the next subsection, we will prove Theorems 3.4 and 6.1, which remain to be proved. For this purpose, this subsection reviews some results [Carter et al. 2005; Eisermann 2014] of quandle theory, which explain a relation between quandles and longitudes.

To this end, we begin by setting up some terminology. Consider the group defined by generators $e_x$ labeled by $x \in X$ modulo the relations $e_x \cdot e_y = e_y \cdot e_{x \lhd y}$ for $x, y \in X$. This group is called *the associated group* and denoted by $\mathrm{As}(X)$, and has a right action on $X$ defined by $x \cdot e_y := x \lhd y$. Letting $\mathrm{O}(X)$ be the set of the orbits, we consider the orbit decomposition of $X$, i.e., $X = \bigsqcup_{\lambda \in \mathrm{O}(X)} X_\lambda$. In addition, fix a quotient group $G$ of $\mathrm{As}(X)$ subject to a central subgroup. Denote the quotient map $\mathrm{As}(X) \to G$ by $p_G$.

Switching to topology, given an $X$-coloring $\mathcal{C} \in \mathrm{Col}_X(D)$ of a link $L$, let us correspond each arc $\gamma$ to $p_G(e_{\mathcal{C}(\gamma)}) \in G$. Regarding the arcs as generators of $\pi_1(\mathbb{R}^3 \setminus L)$ by the Wirtinger presentation (see Figure 5), the correspondence defines a group homomorphism $\Gamma_{\mathcal{C}} : \pi_1(\mathbb{R}^3 \setminus L) \to G$.

Furthermore, with respect to link-components of $L$, we fix an arc $\gamma_j$ on $D$ with $1 \le j \le \#L$. Let $x_j := \mathcal{C}(\gamma_j) \in X_j$, and fix a preferred longitude $\mathfrak{l}_j$ obtained from $D$. Noticing that each $\mathfrak{l}_j$ commutes with the meridian $\gamma_j$, we have $\Gamma_{\mathcal{C}}(\mathfrak{l}_j) \in \mathrm{Stab}(x_j)$.

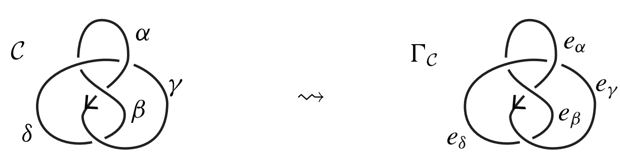


**Figure 5.** The correspondence $\Gamma_{\mathcal{C}}$.

We will give a computation for the value $\Gamma_{\mathcal{C}}(\mathfrak{l}_j)$ as follows. Fix $x_\lambda \in X_\lambda$ for any $\lambda \in \mathrm{O}(X)$, Since the action of $G$ on $X_\lambda$ is transitive, we can choose a section $\mathfrak{s}_\lambda : X_\lambda \to G$ such that $x_\lambda \cdot \mathfrak{s}_\lambda(y) = y$ for any $y \in X_\lambda$. Then we define a map $\phi : X^2 \to G$ by

$$\phi(g,h) = \mathfrak{s}_\lambda(g) p_G(e_g^{-1} e_h) \mathfrak{s}_\lambda(g \lhd h)^{-1} \quad \text{for } g \in X_\lambda,\ h \in X. \tag{7.11}$$

By definition, we see that $\phi(g,h)$ lies in the stabilizer $\mathrm{Stab}(x_\lambda) \subset G$ of $x_\lambda$ if $g \in X_\lambda$.

With respect to the coloring $\mathcal{C}$, similar to in Section 6A, we define a product of the form

$$S_{\mathcal{C},j} := \phi(\mathcal{C}(\alpha_1), \mathcal{C}(\beta_1))^{\epsilon_1} \phi(\mathcal{C}(\alpha_2), \mathcal{C}(\beta_2))^{\epsilon_2} \cdots \phi(\mathcal{C}(\alpha_{N_j}), \mathcal{C}(\beta_{N_j}))^{\epsilon_{N_j}} \in \mathrm{Stab}(x_j),$$

where the terminology of arcs $\alpha_i$ and $\beta_i$ and of signs $\epsilon_i$ are as in Section 6A (see also Figure 2). Although this construction depends on the choice of the $x_\lambda$ and the sections $\mathfrak{s}_\lambda$, the following is known:

**Proposition 7.12** [Carter et al. 2005, Lemma 5.8]. *The product $S_{\mathcal{C},j}$ equals*

$$\mathfrak{s}_\lambda(\mathcal{C}(\gamma_1))^{-1} \Gamma_{\mathcal{C}}(\mathfrak{l}_j) \mathfrak{s}_\lambda(\mathcal{C}(\gamma_1))$$

*in* $\mathrm{Stab}(x_j)$. *In particular, if* $\mathrm{Stab}(x_j)$ *is abelian, the equality $S_{\mathcal{C},j} = \Gamma_{\mathcal{C}}(\mathfrak{l}_j)$ holds in* $\mathrm{Stab}(x_j)$.

The proof immediately follows from the definitions of $\phi$ and of the preferred longitude $\mathfrak{l}_i$.

We next review a computation, shown by Eisermann [2014], of the second quandle homology $H_2^Q(X)$ (see [Carter et al. 2003] for the original definition).

**Theorem 7.13** [Eisermann 2014, Theorem 9.9]. *Let $X$ be a quandle with $|O(X)| = 1$. Fix $x_\lambda \in X$. Let* $\mathrm{Stab}(x_\lambda) \subset \mathrm{As}(X)$ *denote the stabilizer of $x_\lambda$. Then the abelianization* $\mathrm{Stab}(x_\lambda)_{\mathrm{ab}}$ *is isomorphic to $\mathbb{Z} \oplus H_2^Q(X)$.*

In particular, the class $[\Gamma_{\mathcal{C}}(\mathfrak{l}_j)]$ in the abelianization is contained in $\mathbb{Z} \oplus H_2^Q(X)$ by Theorem 7.13. Then, as a corollary of a homotopical study of the homology $H_2^Q(X)$, we can state a sufficient condition to ensure the nontriviality of the classes in the $\mathbb{Z} \oplus H_2^Q(X)$ as follows:

**Proposition 7.14** [Nosaka 2015, Remark 6.4]. *Let $X$ be a quandle such that the orbit $O(X)$ is single. If the group homology $H_2^{\mathrm{gr}}(\mathrm{As}(X);\mathbb{Z})$ is zero, then any element $\Upsilon \in H_2^Q(X)$ admits some $X$-coloring $\mathcal{C}$ of a link such that the equality $\Upsilon = [\Gamma_{\mathcal{C}}(\mathfrak{l}_1)] + \cdots + [\Gamma_{\mathcal{C}}(\mathfrak{l}_{\#L})]$ holds in $\mathbb{Z} \oplus H_2^Q(X)$.*

**7D.** ***Proofs of Theorems 3.4 and 6.1.*** First, we aim to prove Theorem 3.4. Inspired by Theorem 7.13, we first determine the associated groups $\mathrm{As}(X_F)$ of the parabolic quandles over $F$.

**Theorem 7.15.** *Take the map* $\iota : X_F \to \mathrm{SL}_2(F)$ *given in* (7.1). *Then the map*

$$X_F \to \mathbb{Z} \times K_2^{\mathrm{MW}}(K) \times \mathrm{SL}_2(F), \quad x \mapsto (1, 0, \iota(x)).$$

*gives rise to a group homomorphism* $\mathrm{As}(X_F) \to \mathbb{Z} \times \widetilde{\mathrm{SL}}_2(F)$, *which is an isomorphism.*

*Proof.* We can first verify that the map $\iota$ in (7.1) yields a group epimorphism $\mathrm{As}(X_F) \to \mathrm{SL}_2(F)$, which is a central extension. It then follows from Lemma 7.6 that the above map yields a group homomorphism $\mathrm{As}(X_F) \to \mathbb{Z} \times \widetilde{\mathrm{SL}}_2(F)$. Since $H_1(\mathrm{As}(X_F)) \cong \mathbb{Z}$, the universality of central extensions implies that the homomorphism must be an isomorphism. □

**Corollary 7.16.** *The second quandle homology* $H_2^Q(X_F; \mathbb{Z})$ *is isomorphic to the group* $F \oplus \widetilde{K}_2^{\mathrm{MW}}(F)$.

*Proof.* We will compute $H_2^Q(X_F)$ using Theorem 7.13. Fix $x_0 = (0, 1) \in X_F$, and the universal extension $\mathcal{E} : \widetilde{\mathrm{SL}}_2(F) \to \mathrm{SL}_2(F)$. Noticing that the $\mathrm{SL}_2$ standard representation $X_F \curvearrowleft \mathrm{As}(X_F)$ is transitive, i.e., $|O(X)| = 1$, we will calculate the abelianization of the stabilizer $\mathrm{Stab}(x_0) \subset \mathrm{As}(X_F)$. We easily check that $\mathcal{E}(\mathrm{Stab}(x_0)) \subset \mathrm{SL}_2(F)$ is the subgroup $U_F$. Hence, $\mathrm{Stab}(x_0) \cong \mathbb{Z} \times \mathcal{E}^{-1}(U_F) \cong \mathbb{Z} \times \widetilde{K}_2^{\mathrm{MW}}(F) \times F$ by Lemma 2.4. Since this is abelian, Theorem 7.13 readily implies the conclusion $\mathbb{Z} \oplus H_2^Q(X_F) \cong H_1^{\mathrm{gr}}(\mathrm{Stab}(x_0)) \cong \mathbb{Z} \times \widetilde{K}_2^{\mathrm{MW}}(F) \times F$. □

*Proof of Theorem 3.4.* Theorem 7.15 says that the quandle $X_F$ satisfies the assumption of Proposition 7.14. Moreover, $\mathrm{Stab}(x_0) \cong \mathbb{Z} \times \widetilde{K}_2^{\mathrm{MW}}(F) \times F \cong \mathbb{Z} \oplus H_2^Q(X_F; \mathbb{Z})$ is abelian by Corollary 7.16. As a consequence, Proposition 7.14 implies the conclusion. □

Next we will turn to proving Theorem 6.1.

*Proof of Theorem 6.1.* Let $G$ be $\mathrm{PSL}_2(F)$, and let $p_G$ be the composite of projections $\mathrm{As}(X_F) \to \widetilde{\mathrm{SL}}_2(F) \to \mathrm{SL}_2(F) \xrightarrow{\pi} \mathrm{PSL}_2(F)$. Let $x_\lambda$ be $(0, 1) \in X_F$. Then we easily see that the stabilizer $\mathrm{Stab}(x_\lambda) \subset G$ is an abelian group $\pi(U_F) \cong F$.

Furthermore, we define a section $\mathfrak{s}_F : X_F \to \mathrm{PSL}_2(F)$ by setting $\mathfrak{s}_F(0, b) := \mathrm{diag}(b^{-1}, b)$ and $\mathfrak{s}_F(a, b) := \left(\begin{smallmatrix} 0 & -a^{-1} \\ a & b \end{smallmatrix}\right)$ if $a \neq 0$. Then, according to (7.11), we have the resulting map $\phi : (X_F)^2 \to \pi(U_F) \cong F$. By an elementary computation, the map $\phi$ agrees with the map $\mathcal{S}$. Hence, Proposition 7.12 immediately implies the equality claimed in Theorem 6.1. □

**Remark 7.17.** Similar to the previous proof, by considering the case $(X, G) = (X_F, \widetilde{\mathrm{SL}}_2(F))$, we can give a sum formula for the $K_2$ invariant. However, as the author can not formulate a section $X_F \to \widetilde{\mathrm{SL}}_2(F)$ in a simple way, the resulting formula is a little complicated and is far from applications. The desired formula would be simple; So this paper omits describing formulae for the $K_2$ invariants.

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# Equivariant vector bundles, their derived category and $K$-theory on affine schemes

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Let $G$ be an affine group scheme over a noetherian commutative ring $R$. We show that every $G$-equivariant vector bundle on an affine toric scheme over $R$ with $G$-action is equivariantly extended from $\mathrm{Spec}(R)$ for several cases of $R$ and $G$.

We show that, given two affine schemes with group scheme actions, an equivalence of the equivariant derived categories implies isomorphism of the equivariant $K$-theories as well as equivariant $K'$-theories.

## 1. Introduction

The goal of this paper is to answer some well-known questions related to group scheme actions on affine schemes over a fixed affine base scheme. Our particular interest is to explore when are the equivariant vector bundles on such schemes equivariantly trivial and when does an equivalence of their derived categories imply homotopy equivalence of the equivariant $K$-theory. Both questions have been extensively studied and are now satisfactorily answered in the nonequivariant case (see [Lindel 1981; Rickard 1989; Dugger and Shipley 2004]).

**1A.** ***Equivariant Bass–Quillen question.*** The starting point for the first question is the following classical problem from [Bass 1973, Problem IX]:

**Conjecture 1.1** (Bass–Quillen). *Let $R$ be a regular commutative noetherian ring of finite Krull dimension. Then every finitely generated projective module over the polynomial ring $R[x_1, \ldots, x_n]$ is extended from $R$.*

The most complete answer to this conjecture was given by Lindel [1981], who showed (based on the earlier solutions by Quillen and Suslin when $R$ is a field) that the above conjecture has an affirmative solution when $R$ is essentially of finite type over a field. For regular rings which are not of this type, some cases have been solved (see [Rao 1988], for example), but the complete answer is still unknown.

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In this paper, we are interested in the equivariant version of this conjecture, which can be loosely phrased as follows.

Let $R$ be a noetherian regular ring and let $G$ be a flat affine group scheme over $R$. Let $A = R[x_1, \dots, x_n]$ be a polynomial $R$-algebra with a linear $G$-action and let $P$ be a finitely generated $G$-equivariant projective $A$-module. The equivariant version of the above conjecture asks:

**Question 1.2.** *Is $P$ an equivariant extension of a $G$-equivariant projective module over $R$?*

The equivariant Bass–Quillen question was studied, for example, in [Knop 1991; Kraft and Schwarz 1992; 1995; Masuda et al. 1996] when $R = \mathbb{C}$ is the field of complex numbers. This question is known to be very closely related to the linearization problem for reductive group action on affine spaces.

The first breakthrough was achieved by Knop [1991], who found counterexamples to this question when $G$ is a nonabelian reductive group over $\mathbb{C}$. In fact, he showed that every connected reductive nonabelian group over $\mathbb{C}$ admits a linear action on a polynomial ring for which the equivariant Bass–Quillen conjecture fails. Later, such counterexamples were found by Masuda and Petrie [1995] when $G$ is a finite nonabelian group. Thus the only hope to prove this conjecture is when $G$ is diagonalizable. It was subsequently shown by Masuda, Moser-Jauslin and Petrie [Masuda et al. 1996] that the equivariant Bass–Quillen conjecture indeed has a positive solution when $R = \mathbb{C}$ and $G$ is diagonalizable. This was independently shown also by Kraft and Schwarz [1995].

It is not yet known if the equivariant Bass–Quillen conjecture has a positive solution over any field other than $\mathbb{C}$. One of the two goals of this paper is to solve the general case of the equivariant Bass–Quillen question for diagonalizable group schemes over an arbitrary ring or field. Our approach to solving this problem in fact allows us to prove the stronger assertion that such a phenomenon holds over all affine toric schemes over an affine base. This approach was motivated by a similar result of Masuda [1998] over the field of complex numbers.

Let $R$ be a commutative noetherian ring. Recall from [SGA $3_{\text{II}}$ 1970, Exposé VIII] that an affine group scheme $G$ over $R$ is called *diagonalizable* if there is a finitely generated abelian group $P$ such that $G = \operatorname{Spec}(R[P])$, where $R[P]$ is the group algebra of $P$ over $R$.

Let $L$ be a lattice and let $\sigma \subseteq L_{\mathbb{Q}}$ be a strongly convex, polyhedral, rational cone. Let $\Delta$ denote the set of all faces of $\sigma$. Let $A = R[\sigma \cap L]$ be the monoid algebra over $R$. Let $\psi : L \to P$ be a homomorphism which makes $\operatorname{Spec}(A)$ a scheme with $G$-action. Let $A^G$ denote the subring of $G$-invariant elements in $A$. Let us assume that every finitely generated projective module over $R[Q]$ is extended from $R$ if $Q$ is torsion-free (see Theorem 5.2).

Our main result can now be stated as follows (see Theorem 7.8). The underlying terms and notations can be found in the body of this text.

**Theorem 1.3.** *Let $R$ and $A$ be as above. Assume that all finitely generated projective modules over $A_\tau$ and $(A_\tau)^G$ are extended from $R$ for every $\tau \in \Delta$. Then every finitely generated $G$-equivariant projective $A$-module is equivariantly extended from $R$.*

For examples of rings satisfying the hypothesis of the theorem, see Sections 5, 6 and 7.

Let us now assume that $R$ is either a PID, or a regular local ring of dimension at most 2, or a regular local ring containing a field. As a consequence of the above theorem, we obtain the following solution to the equivariant Bass–Quillen question:

**Theorem 1.4.** *Let $R$ be as above and let $G$ be a diagonalizable group scheme over $R$ acting linearly on a polynomial algebra $R[x_1, \dots, x_n, y_1, \dots, y_r]$. Then the following hold*:

(1) *If $A = R[x_1, \dots, x_n]$, then every finitely generated equivariant projective $A$-module is equivariantly extended from $R$.*
(2) *If $R$ is a PID and $A = R[x_1, \dots, x_n, y_1^{\pm 1}, \dots, y_r^{\pm 1}]$, then every finitely generated equivariant projective $A$-module is equivariantly extended from $R$.*

This theorem is generalized to the case of nonlocal regular rings in Theorem 8.4. We note here that, previously, it was not even known whether every $G$-equivariant bundle on a polynomial ring over $R$ is "stably" extended from $R$.

The above results were motivated in part by the following important classification problem for equivariant vector bundles over smooth affine schemes. One of the most notable (among many) recent applications of the nonequivariant Bass–Quillen conjecture is Morel's classification [2012, Theorem 8.1] of vector bundles over smooth affine schemes. He showed, using Lindel's theorem [1981], that all isomorphism classes of rank-$n$ vector bundles on a smooth affine scheme $X$ over a field $k$ are in bijection with the set of $\mathbb{A}^1$-homotopy classes of maps from $X$ to the classifying space of $\mathrm{GL}_{n,k}$. It is important to note here that, even though Morel's final result is over a field, its proof crucially depends on Lindel's theorem for geometric regular local rings.

The equivariant version of the Morel–Voevodsky $\mathbb{A}^1$-homotopy category was constructed in [Heller et al. 2015]. One can make sense of the equivariant classifying space in this category, analogous to the one in the topological setting [May 1996]. The equivariant analogue (Theorem 1.4) of Lindel's theorem now completes one very important step in solving the classification problem for equivariant vector bundles. It remains to see how one can use Theorem 1.4 to complete the proof of the equivariant version of Morel's classification theorem. This will be taken up elsewhere.

**1B.** ***Equivariant derived category and K-theory.*** We now turn to the second question. To motivate this, recall that it is a classical question in algebraic $K$-theory to determine if it is possible that two schemes with equivalent derived categories of quasicoherent sheaves (or vector bundles) have (homotopy) equivalent algebraic $K$-theories. This question gained prominence when Thomason and Trobaugh [1990] showed that the equivalence of $K$-theories is true if the given equivalence of derived categories is induced by a morphism between the underlying schemes. There has been no improvement of this result for the general case of schemes to date.

However, Dugger and Shipley [2004] (see also [Rickard 1989]) showed a remarkable improvement over the result of Thomason and Trobaugh for affine schemes. They showed more generally that any two (possibly noncommutative) noetherian rings with equivalent derived categories (which may not be induced by a map of rings!) have equivalent $K$-theories.

Parallel to the equivariant analogue of the Bass–Quillen question, one can now ask if it is true that two affine schemes with group scheme actions have equivalent equivariant $K$-theories if their equivariant derived categories are equivalent. No case of this problem has been known yet.

In this paper, we show that the general results of Dugger and Shipley [2004] apply in the equivariant setup too, which allows us to solve the above question. More precisely, we combine Dugger and Shipley's results and Proposition 4.6 to prove the following theorem.

Let $R$ be a commutative noetherian ring and let $G$ be an affine group scheme over $R$. Assume that either $G$ is diagonalizable or $R$ contains a field of characteristic zero and $G$ is a split reductive group scheme over $R$. Given a finitely generated $R$-algebra $A$ with $G$-action, let us denote this datum by $(R, G, A)$. Let $D^G(A)$ and $D^G(\mathrm{proj}/A)$ denote the derived categories of $G$-equivariant $A$-modules and $G$-equivariant (finitely generated) projective $A$-modules, respectively. Let $K^G(A)$ and $K'_G(A)$ denote the $K$-theory spectra of $G$-equivariant (finitely generated) projective $A$-modules and $G$-equivariant $A$-modules, respectively.

**Theorem 1.5.** *Let $(R_1, G_1, A_1)$ and $(R_2, G_2, A_2)$ be two data of the above type. Then $D^{G_1}(A_1)$ and $D^{G_2}(A_2)$ are equivalent as triangulated categories if and only if $D^{G_1}(\mathrm{proj}/A_1)$ and $D^{G_2}(\mathrm{proj}/A_2)$ are equivalent as triangulated categories.*

*In either case, there are homotopy equivalences of spectra $K^{G_1}(A_1) \simeq K^{G_2}(A_2)$ and $K'_{G_1}(A_1) \simeq K'_{G_2}(A_2)$.*

In other words, this theorem says that the equivariant $K$-theory as well as the $K'$-theory of affine schemes with group action can be completely determined by the equivariant derived category, which is much simpler to study than the full equivariant geometry of the scheme.

*Brief outline of the proofs.* We end this section with an outline of our methods. Our proof of Theorem 1.3 is based on the techniques used in [Kraft and Schwarz 1995] to solve the equivariant Bass–Quillen question over $\mathbb{C}$. As in [loc. cit.], we show that all equivariant vector bundles actually descend to bundles on the quotient scheme for the group action. This allows us then to use the solution to the nonequivariant Bass–Quillen question to conclude the final proof.

In order to do this, one runs into several technical ring-theoretic issues and one has to find algebraic replacements for the geometric techniques available only over $\mathbb{C}$. Another problem is that the approach of [Masuda et al. 1996] to solve Question 1.2 for $R = \mathbb{C}$ crucially uses the result of [Bass and Haboush 1985] that every equivariant vector bundle over $\mathbb{C}[x_1, \dots, x_n]$ is stably extended from $\mathbb{C}$. But we do not know this over other rings.

Our effort is to resolve these issues by a careful analysis of group scheme actions on affine schemes. Instead of working with schemes, we translate the problem into studying comodules over some Hopf algebras. Sections 2 and 3 are meant to do this. In Section 4, we prove some crucial properties of equivariant vector bundles on affine schemes, which play a very important role in proving Theorem 1.5. These sections generalize several results of [Bass and Haboush 1985] to more general rings.

In Section 5, we prove some properties of equivariant projective modules over monoid algebras, which are the main object of study. In Section 6, we show how to descend an equivariant vector bundle to the quotient scheme and then we use the solution to the Bass–Quillen conjecture in the nonequivariant case to complete the proof of Theorem 1.3 in Section 7. Theorem 1.4 and its generalization are proven in Section 8.

We prove Theorem 1.5 in Section 9 by combining the results of Section 4, [Dugger and Shipley 2004] and a generalization of a theorem of Rickard [1989]. This generalization is shown in the Appendix.

## 2. Recollection of group scheme action and invariants

In this section, we recall some aspects of group schemes and their actions over a given affine scheme from [SGA $3_{\mathrm{I}}$ 1970, Exposé III; SGA $3_{\mathrm{II}}$ 1970, Exposé VIII]. We prove some elementary results about these actions which are of relevance to the proofs of our main results. In this text, a *ring* will always mean a commutative noetherian ring with unit.

Let $S = \mathrm{Spec}(R)$ be a noetherian affine scheme and let $\mathbf{Sch}_S$ denote the category of schemes which are separated and of finite type over $S$. Let $\mathbf{Alg}_R$ denote the category of finite-type $R$-algebras. We shall assume throughout this text that $S$ is connected. If $R$ and $S$ are clear in a context, the fiber product $X \times_S Y$ and tensor product $A \otimes_R B$ will be simply written as $X \times Y$ and $A \otimes B$, respectively. For an $R$-module $M$ and an $R$-algebra $A$, the base extension $M \otimes_R A$ will be denoted by $M_A$.

**2A. *Group schemes and Hopf algebras.*** Recall that a group scheme $G$ over $S$ (equivalently, over $R$) is an object of $\mathbf{Sch}_S$ which is equipped with morphisms $\mu_G : G \times G \to G$ (multiplication), $\eta : S \to G$ (unit) and $\tau : G \to G$ (inverse) that satisfy the known associativity, unit and symmetry axioms. These axioms are equivalent to saying that the presheaf $X \mapsto h_G(X) := \mathrm{Hom}_{\mathbf{Sch}_S}(X, G)$ on $\mathbf{Sch}_S$ is a group-valued (contravariant) functor.

If $G$ is an affine group scheme over $S$, one can represent it algebraically in terms of Hopf algebras over $R$. As this Hopf algebra representation will be a crucial part of our proofs, we recall it briefly.

Let us assume that $G$ is an affine group scheme with coordinate ring $R[G]$. Then the multiplication, unit section and inverse maps above are equivalent to having the morphisms $\Delta : R[G] \to R[G] \otimes R[G]$, $\epsilon : R[G] \to R$ and $\sigma : R[G] \to R[G]$ in $\mathbf{Alg}_R$ such that $\mu_G = \mathrm{Spec}(\Delta)$, $\eta = \mathrm{Spec}(\epsilon)$ and $\tau = \mathrm{Spec}(\sigma)$. The associativity, unit and symmetry axioms are equivalent to the commutative diagrams

$$\begin{tikzcd} R[G] \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & R[G] \otimes R[G] \arrow[r, "\Delta\otimes\mathrm{Id}"] & (R[G] \otimes R[G]) \otimes R[G] \\ & R[G] \otimes R[G] \arrow[r, "\mathrm{Id}\otimes\Delta"] & R[G] \otimes (R[G] \otimes R[G]) \arrow[u, "\text{can. iso.}"'] \end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd} R[G] & R[G] \otimes R[G] \arrow[l, "\mathrm{Id}\otimes\epsilon"'] \arrow[r, "\epsilon\otimes\mathrm{Id}"] & R[G] \\ & R[G] \arrow[ul, "\mathrm{Id}"] \arrow[u, "\Delta"] \arrow[ur, "\mathrm{Id}"'] & \end{tikzcd} \qquad \begin{tikzcd} R[G] \otimes R[G] \arrow[r, "\sigma\cdot\mathrm{Id}"] & R[G] \\ R[G] \arrow[u, "\Delta"] \arrow[r, "\epsilon"] & R \arrow[u] \end{tikzcd} \tag{2.2}$$

In other words, $(R[G], \Delta, \epsilon, \sigma)$ is a Hopf algebra over $R$ and it is well known that the transformation $(G, \mu_G, \eta, \tau) \mapsto (R[G], \Delta, \epsilon, \sigma)$ gives an equivalence between the categories of affine group schemes over $S$ and finite-type Hopf algebras over $R$ (see [Waterhouse 1979, Chapter 1]).

**2A1.** *$R$-$G$-modules.* Let $G$ be an affine group scheme over $R$. An $R$-$G$-module is an $R$-module $M$ equipped with a natural transformation $h_G(\mathrm{Spec}(A)) \to \mathrm{GL}(M)(A)$ of group functors, where the functor $\mathrm{GL}(M)$ associates the group $\mathrm{Aut}_A(A \otimes_R M)$ to an $R$-algebra $A$.

Equivalently, an $R$-$G$-module is an $R$-module $M$ which is also a comodule over the Hopf algebra $R[G]$, in the sense that there is an $R$-linear map $\rho : M \to R[G] \otimes_R M$

such that the following diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & R[G]\otimes M \arrow[d, "\Delta\otimes\mathrm{Id}_M"] \\ R[G]\otimes M \arrow[r, "\mathrm{Id}_{R[G]}\otimes\rho"] & R[G]\otimes R[G]\otimes M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\mathrm{Id}_M"'] & R[G]\otimes M \arrow[d, "\epsilon\otimes\mathrm{Id}_M"] \\ M \arrow[r, "\simeq"] & R\otimes M \end{tikzcd} \tag{2.3}$$

The reader can check that the comodule structure on $M$ associated to a natural transformation of functors $h_G \to \mathrm{GL}(M)$ is given by the map $\rho : M \to R[G]\otimes M$ with $\rho(m) = h_G(R[G])(\mathrm{Id}_{R[G]})(1\otimes m)$. We shall denote an $R$-$G$-module $M$ in the sequel in terms of an $R[G]$-comodule by $(M, \rho)$.

A morphism $f : (M, \rho) \to (M', \rho')$ between $R$-$G$-modules is an $R$-linear map $f : M \to M'$ such that $\rho' \circ f = (\mathrm{Id}_{R[G]}\otimes f)\circ\rho$. We say that $M$ is an $R$-$G$-submodule of $M'$ if $f$ is injective. The set of all $R$-$G$-module homomorphisms from $M$ to $M'$ will be denoted by $\mathrm{Hom}_{RG}(M, M')$.

We shall say that an $R$-$G$-module $M$ is finitely generated (resp. projective) if it is finitely generated (resp. projective) as an $R$-module. The categories of $R$-$G$-modules will be denoted by $(R\text{-}G)$-Mod. The category of finitely generated projective $R$-$G$-modules will be denoted by $(R\text{-}G)$-proj. The category of not necessarily finitely generated projective $R$-$G$-modules will be denoted by $(R\text{-}G)$-Proj.

If $G$ is an affine group scheme which is flat over $R$, then it is easy to check that $(R\text{-}G)$-Mod is an abelian category and $(R\text{-}G)$-proj is an exact category. The flatness is essential here because in its absence the kernel of an $R$-$G$-module map $f : M \to M'$ may not acquire a $G$-action as $R[G]\otimes_R \mathrm{Ker}(f)$ may fail to be a submodule of $R[G]\otimes_R M$.

**2A2.** *Submodule of invariants.* Let $G$ be an affine group scheme over $R$ and let $(M, \rho)$ be an $R$-$G$-module. An element $m \in M$ is said to be *$G$-invariant* under the action of $G$ if $\rho(m) = 1\otimes m$. The $R$-submodule of $G$-invariant elements of $M$ will be denoted by $M^G$.

Given an element $\lambda \in R[G]$, we say that $m \in M$ is *semi-invariant* of weight $\lambda$ under the $G$-action if $\rho(m) = \lambda\otimes m$. The following is a straightforward consequence of the definitions and $R$-linearity of $\rho$.

The group scheme $G$ is called *linearly reductive* if $\mathrm{Inv} : (R\text{-}G)\text{-Mod} \to R\text{-Mod}$ sending $M$ to $M^G$ is an exact functor.

**Lemma 2.4.** *Given an $R$-$G$-module $(M, \rho)$ and character $\lambda \in R[G]$, the set*

$$M_\lambda := \{m \in M \mid \rho(m) = \lambda\otimes m\}$$

*is an $R$-$G$-submodule of $M$. In particular, $M^G$ is an $R$-$G$-submodule of $M$. Every $R$-submodule of $M_\lambda$ is an $R$-$G$-submodule of $M_\lambda$.*

**Example 2.5.** Let $k$ be an algebraically closed field and let $G$ be a linear algebraic group over $k$. In this case, a (finite) $k$-$G$-module is the same as a finite-dimensional representation $V$ of $G$. We can now check that the above notion of $G$-invariants is same as the classical definition of $V^G$, given by $V^G = \{v \in V \mid g \cdot v = v \text{ for all } g \in G\}$. Choose a $k$-basis $\{v_1, \ldots, v_n\}$ for $V$ and suppose that

$$\rho(v_i) = \sum_{j=1}^{n} a_{ij} \otimes v_j. \tag{2.6}$$

One can use (2.3) to see that $V$ becomes a $G$-representation via the homomorphism $\rho' : G \to \mathrm{GL}(V)$ given by $\rho'(g) = (a_{ij}(g))$. Recall here that an element of $k[G]$ is the same as a morphism $G \to \mathbb{A}^1_k$. If we write an element of $V$ in terms of a row vector $x = (x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i v_i$, then it follows easily from (2.6) that $\rho(x) = 1 \otimes x$ if and only if $(a_{ij}(g))x = x$ for $g \in G$. But this is the same as saying that $\rho'(g)(x) = x$ for all $g \in G$.

**2A3.** *Group scheme action.* Let $G$ be a group scheme over $S = \mathrm{Spec}(R)$ and let $X \in \mathbf{Sch}_S$. Recall that a $G$-action on $X$ is a morphism $\mu_X : G \times_S X \to X$ which satisfies the usual associative and unital identities for an action.

If $G$ is an affine group scheme over $S$ and $X = \mathrm{Spec}(A)$ is an affine $S$-scheme, then a $G$-action on $X$ as above is equivalent to a map $\phi : A \to R[G] \otimes_R A$ in $\mathbf{Alg}_R$ such that $\phi$ defines an $R[G]$-comodule structure on $A$. In this case, one has $\mu_X = \mathrm{Spec}(\phi)$. We shall denote this $G$-action on $X$ by the pair $(A, \phi)$ and call $A$ an $R$-$G$-algebra. Note that this notion of $R$-$G$-algebra makes sense for any (possibly noncommutative) $R$-algebra $R \to A$ such that the image of $R$ is contained in the center of $A$. We shall use this $R$-$G$-algebra structure on the endomorphism rings (see Lemma 3.8).

We also recall, in the language of Hopf algebras, the $G$-action on an $R$-$G$-algebra $A$ is *free* if the map $\Phi : A \otimes_R A \to R[G] \otimes_R A$ given by $\Phi(a_1 \otimes a_2) = \phi(a_1)(1 \otimes a_2)$ is surjective.

## 3. Equivariant quasicoherent sheaves on affine schemes

Recall from [Thomason 1987, §1.2] that if $X \in \mathbf{Sch}_S$ has a $G$-action $\mu_X : G \times_S X \to X$ then a $G$-equivariant quasicoherent sheaf on $X$ is a quasicoherent sheaf $\mathcal{F}$ on $X$ together with an isomorphism of sheaves of $\mathcal{O}_{G \times_S X}$-modules on $G \times_S X$

$$\theta : p^*(\mathcal{F}) \xrightarrow{\simeq} \mu_X^*(\mathcal{F}), \tag{3.1}$$

where $p : G \times_S X \to X$ is the projection map. This isomorphism satisfies the cocycle condition on $G \times_S G \times_S X$

$$(1 \times \mu_X)^*(\theta) \circ p_{23}^*(\theta) = (\mu_G \times 1)^*(\theta), \tag{3.2}$$

where $p_{23} : G \times_S G \times_S X \to G \times_S X$ is the projection to the last two factors.

A morphism of $G$-equivariant sheaves $f : (\mathcal{F}_1, \theta_1) \to (\mathcal{F}_2, \theta_2)$ is a map of sheaves $f : \mathcal{F}_1 \to \mathcal{F}_2$ such that $\mu_X^*(f) \circ \theta_1 = \theta_2 \circ p^*(f)$.

**3A. *A-G-modules.*** Let us now assume that $G$ is an affine group scheme over $S = \mathrm{Spec}(R)$ which acts on an affine $S$-scheme $X = \mathrm{Spec}(A)$ with $A \in \mathbf{Alg}_R$. Let $\phi : A \to R[G] \otimes_R A$ be the action map such that $\mu_X = \mathrm{Spec}(\phi)$.

**Definition 3.3.** An $A$-module $M$ is an $A$-$G$-module if $(M, \rho)$ is an $R$-$G$-module such that

$$\rho(a.m) = \phi(a).\rho(m) \quad \text{for all } a \in A \text{ and } m \in M. \tag{3.4}$$

An $A$-$G$-module homomorphism is an $A$-module homomorphism which is also an $R$-$G$-module homomorphism. Given a pair of $A$-$G$-modules, the set of $A$-$G$-module homomorphisms will be denoted by $\mathrm{Hom}_{AG}(\_, \_)$.

We shall denote the category of $A$-$G$-modules by $(A$-$G)$-Mod. An $A$-$G$-module $M$ will be called projective, if it is projective as an $A$-module. We shall denote the category of finitely generated projective $A$-$G$-modules by $(A$-$G)$-proj. The category of (not necessarily finitely generated) projective $A$-$G$-modules will be denoted by $(A$-$G)$-Proj. Notice that, given a morphism of $R$-$G$ algebras $f : (A, \phi_A) \to (B, \phi_B)$, there is a pull-back map $f^* : (A$-$G)$-Mod $\to (B$-$G)$-Mod which preserves projective modules. It is easy to check that, given an $R$-$G$-module $M$ and an $A$-$G$-module $N$, the extension of scalars gives an isomorphism

$$\mathrm{Hom}_{RG}(M, N) \xrightarrow{\simeq} \mathrm{Hom}_{AG}(M_A, N). \tag{3.5}$$

**Proposition 3.6.** *There is an equivalence between the category of $G$-equivariant quasicoherent $\mathcal{O}_X$-modules and the category of $A$-$G$-modules.*

*Proof.* Let $M$ be an $A$-module which defines a $G$-equivariant quasicoherent sheaf on $X$ and let $\theta : R[G] \otimes_R M \xrightarrow{\simeq} R[G] \otimes_R M$ be an isomorphism of $R[G] \otimes_R A$-modules as in (3.1) satisfying (3.2).

We define an $A$-$G$-module structure on $M$ by setting $\rho : M \to R[G] \otimes_R M$ to be the map $\rho(m) = \theta(1 \otimes m)$. The map $\rho$ is clearly $R$-linear and one checks that

$$\rho(a \cdot m) = \theta(1 \otimes a \cdot m) = \theta(a \cdot (1 \otimes m)) = \phi(a) \cdot \theta(1 \otimes m) = \phi(a) \cdot \rho(m).$$

Since the map $\phi : A \to R[G] \otimes_R A$ is just the inclusion map $a \mapsto 1 \otimes a$ when restricted to $R$, one checks easily from (3.2) that

$$\begin{aligned}(1 \times \mu_X)^*(\theta) \circ p_{23}^*(\theta)(1 \otimes 1 \otimes m) &= (1 \times \mu_S)^*(\theta) \circ p_{23}^*(\theta)(1 \otimes 1 \otimes m) \\ &= (\mathrm{Id}_{R[G]} \otimes \rho) \circ \rho(m)\end{aligned}$$

and it is also immediate that $(\mu_G \times 1)^*(\theta)(1 \otimes 1 \otimes m) = (\Delta \otimes \mathrm{Id}_{R[G]}) \circ \rho(m)$. This is the first square of (2.3). The second square of (2.3) is obtained at once by applying the map $(\eta \times \eta \times 1)^*$ to (3.2), where $\eta : S \to G$ is the unit map. We have thus shown that $M$ is an $A$-$G$-module.

Conversely, suppose that $M$ is an $A$-$G$-module. We define $\theta : R[G] \otimes_R M \to R[G] \otimes_R M$ by setting $\theta(x \otimes m) = x \cdot \rho(m)$. In other words, we have

$$\theta = (\alpha \otimes \mathrm{Id}_M) \circ (\mathrm{Id}_{R[G]} \otimes \rho), \tag{3.7}$$

where $\alpha : R[G] \otimes_R R[G] \to R[G]$ is the multiplication of the ring $R[G]$.

Since $\rho$ is $R$-linear, we see that $\theta$ is $R[G]$-linear. To show that $\theta$ is $(R[G] \otimes_R A)$-linear, it is thus enough to show that it is $A$-linear. This is standard and can be checked as follows: For any $a \in A$, $x \in R[G]$ and $m \in M$, we get, inside $R[G] \otimes_R M = R[G] \otimes_R A \otimes_A M$,

$$\begin{aligned}\theta(a \cdot (x \otimes m)) &= \theta(x \otimes a \otimes m)\\ &= \theta(x \otimes 1 \otimes a \cdot m)\\ &= (x \otimes 1) \cdot \rho(a \cdot m)\\ &= (x \otimes 1) \cdot (\phi(a) \cdot \rho(m))\\ &= (x \otimes 1) \cdot \phi(a) \cdot \rho(m).\end{aligned}$$

The fourth equality above follows from (3.4). On the other hand, we have

$$\begin{aligned}a \cdot \theta(x \otimes m) &= \phi(a) \cdot \theta(x \otimes 1 \otimes m)\\ &= \phi(a) \cdot (x \otimes 1) \cdot \theta(1 \otimes m)\\ &= \phi(a) \cdot (x \otimes 1) \cdot \rho(m).\end{aligned}$$

The two sets of identities above show that $\theta$ is $(R[G] \otimes_R A)$-linear. To show that $\theta$ is an isomorphism, we define $\theta^{-1} : R[G] \otimes_R M \to R[G] \otimes_R M$ by

$$\theta^{-1} = (\alpha \otimes \mathrm{Id}_M) \circ (\mathrm{Id}_{R[G]} \otimes \sigma \otimes \mathrm{Id}_M) \circ (\mathrm{Id}_{R[G]} \otimes \rho),$$

where $\sigma : R[G] \to R[G]$ is the inverse map of its Hopf algebra structure.

It is easy to check using (2.2) and (2.3) that $\theta \circ \theta^{-1} = \theta^{-1} \circ \theta = \mathrm{Id}_{R[G] \otimes_R M}$. The cocycle condition (3.2) is a formal consequence of the left square in (2.3). It is also straightforward to check that the two constructions given above yield the desired equivalence between the categories of $G$-equivariant quasicoherent sheaves on $X$ and $A$-$G$-modules on $A$. We leave these verifications as an exercise. $\square$

**Lemma 3.8.** *Assume that $G$ is flat over $R$ and let $(A, \phi)$ be an $R$-$G$-algebra. Let $(L, \rho_L)$, $(M, \rho_M)$ and $(N, \rho_N)$ be $A$-$G$-modules and let $p : (M, \rho_M) \to (N, \rho_N)$ be an $A$-$G$-linear map. Assume that $(L, \rho_L)$ is finitely generated. Then* $\mathrm{Hom}_A(L, N)$ *has a natural $A$-$G$-module structure and* $\mathrm{Hom}_A(L, L)$ *has a natural $A$-$G$-algebra structure such that the following hold*:

(1) *The induced map* $\mathrm{Hom}_A(L, M) \xrightarrow{p\circ\_} \mathrm{Hom}_A(L, N)$ *is $A$-$G$-linear.*

(2) $\mathrm{Hom}_{AG}(L, N) = \mathrm{Hom}_A(L, N)^G$.

(3) *If* $(M, \rho_M)$ *and* $(N, \rho_N)$ *are finitely generated, then*

$$\mathrm{Hom}_A(N, L) \xrightarrow{-\circ p} \mathrm{Hom}_A(M, L)$$

*is* $A$-$G$-*linear.*

*Proof.* To define an $A$-$G$-module structure on $\mathrm{Hom}_A(L, N)$, we need to define an $R$-linear map $\psi_{LN} : \mathrm{Hom}_A(L, N) \to R[G] \otimes_R \mathrm{Hom}_A(L, N)$ satisfying (2.3) and (3.4).

Since $R[G]$ is flat over $R$ and $L$ is a finitely generated $A$-module, it is well known (see [Eisenbud 1995, Proposition 2.10], for example) that there is a canonical isomorphism of $(R[G]\otimes_R A)$-modules:

$$\beta : R[G] \otimes_R \mathrm{Hom}_A(L, N) \to \mathrm{Hom}_{R[G]\otimes_R A}(R[G] \otimes_R L, R[G] \otimes_R N).$$

Using $\beta$, we can define $\psi_{LN}(f)$ for any $f \in \mathrm{Hom}_A(L, N)$ to be the composition

$$R[G] \otimes_R L \xrightarrow{\theta_L^{-1}} R[G] \otimes_R L \xrightarrow{\mathrm{Id}\otimes f} R[G] \otimes_R N \xrightarrow{\theta_N} R[G] \otimes_R N, \tag{3.9}$$

where $\theta_L$ and $\theta_N$ are as in (3.7). One checks using (3.2), (3.4) and (3.7) that $\psi_{LN}$ defines an $A$-$G$-module structure on $\mathrm{Hom}_A(L, N)$. To show that $\mathrm{Hom}_A(L, L)$ has an $A$-$G$-algebra structure, we need to show that $\psi_{LL}(f \circ g) = \psi_{LL}(f) \circ \psi_{LL}(g)$. But this is immediate from (3.9).

The map $\mathrm{Hom}_A(L, M) \xrightarrow{p\circ -} \mathrm{Hom}_A(L, N)$ is known to be $A$-linear. Thus we only need to show that it is $R$-$G$-linear in order to prove (1). Using (3.9), this is equivalent to showing that, for any $f \in \mathrm{Hom}_A(L, M)$, the identity

$$(\mathrm{Id}_{R[G]} \otimes p) \circ \theta_M \circ (\mathrm{Id}_{R[G]} \otimes f) \circ \theta_L^{-1} = \theta_N \circ (\mathrm{Id}_{R[G]} \otimes (p \circ f)) \circ \theta_L^{-1} \tag{3.10}$$

holds in $\mathrm{Hom}_{R[G]\otimes_R A}(R[G] \otimes_R L, R[G] \otimes_R N)$. In order to prove this identity, it suffices to show that $(\mathrm{Id}_{R[G]} \otimes p) \circ \theta_M = \theta_N \circ (\mathrm{Id}_{R[G]} \otimes p)$. But this is equivalent to saying that $p$ is $R$-$G$-linear (see the definition of morphism of $G$-equivariant sheaves below (3.2)). This proves (1), and the proof of (3) is similar.

To prove (2), recall that $f \in \mathrm{Hom}_A(L, N)^G$ if and only if

$$\psi_{LN}(f) = \theta_N \circ (\mathrm{Id} \otimes f) \circ \theta_L^{-1} = \mathrm{Id} \otimes f$$

(see Section 2A2), or equivalently if $\theta_N \circ (\mathrm{Id} \otimes f) = (\mathrm{Id} \otimes f) \circ \theta_L$. We are thus left with showing that $\theta_N \circ (\mathrm{Id}\otimes f) = (\mathrm{Id}\otimes f) \circ \theta_L$ if and only if $\rho_N \circ f = (\mathrm{Id}\otimes f) \circ \rho_L$. But the "if" part follows directly from (3.7) and the "only if" part follows by evaluating $\theta_L$ on $1 \otimes L \hookrightarrow R[G] \otimes_R L$. □

**3B.** ***Diagonalizable group schemes.*** Recall from [SGA $3_{\text{II}}$ 1970, Exposé VIII] that an affine group scheme $G$ over $R$ is called *diagonalizable* if there is a finitely generated abelian group $P$ such that $G = \operatorname{Spec}(R[P])$, where $R[P]$ is the group algebra of $P$ over $R$. Recall that there is a group homomorphism (the exponential map) $e : P \to (R[P])^{\times}$ and the $R$-algebra $R[P]$ carries the following Hopf algebra structure: $\Delta(e_a) = e_a \otimes e_a$, $\sigma(e_a) = e_{-a}$ and $\epsilon(e_a) = 1$ for $a \in P$, where we write $e_a$ for $e(a)$. As $R[P]$ is a free $R$-module with basis $P$, we see that $G$ is a commutative group scheme which is flat over $R$. It is smooth over $R$ if and only if the order of the finite part of $P$ is prime to all residue characteristics of $R$.

Taking $P = \mathbb{Z}$, we get the group scheme $\mathbb{G}_m = \operatorname{Spec}(R[\mathbb{Z}]) = \operatorname{Spec}(R[t^{\pm 1}])$. For an affine group scheme $G$ over $R$, its group of characters is the set $X(G) := \operatorname{Hom}(G, \mathbb{G}_m)$, whose elements are the morphisms $f : G \to \mathbb{G}_m$ in the category of affine group schemes over $R$. Every element of $P$ defines a unique homomorphism of abelian groups $\mathbb{Z} \to P$ and defines a unique morphism of group schemes $\operatorname{Spec}(R[P]) \to \mathbb{G}_m$. One checks that this defines an isomorphism $P \xrightarrow{\simeq} X(G)$ and yields an antiequivalence of categories from finitely generated abelian groups to diagonalizable group schemes over $R$. In particular, the category $\mathbf{Diag}_R$ of diagonalizable group schemes over $R$ is abelian. We shall use the following known facts about the diagonalizable group schemes and quasicoherent sheaves for the action of such group schemes.

**Proposition 3.11** [SGA $3_{\text{II}}$ 1970, Exposé VIII, §3]. *Let $\phi : G \to G'$ be a morphism of diagonalizable group schemes. Then there are diagonalizable group schemes $H$, $G/H$ and $G'/G$ together with exact sequences in $\mathbf{Diag}_R$*

$$0 \to H \to G \xrightarrow{\phi} G/H \to 0 \quad \textit{and} \quad 0 \to G/H \to G' \to G'/G \to 0.$$

**Proposition 3.12** [SGA $3_{\text{I}}$ 1970, Exposé I, Proposition 4.7.3]. *Let $G = \operatorname{Spec}(R[P])$ be a diagonalizable group scheme. Then the category of $R$-$G$-modules is equivalent to the category of $P$-graded $R$-modules. The equivalence is given by associating to every $R$-$G$-module $(M, \rho)$ the $P$-graded $R$-module $M = \bigoplus_{a \in P} M_a$, where $M_a := \{m \in M \mid \rho(m) = e_a \otimes m\}$ is the subspace of $M$ containing elements of weight $e_a$* (see Section 2A2). *To every $P$-graded $R$-module $M = \bigoplus_{a \in P} M_a$, we associate the $R$-$G$-module $(M, \rho)$, where $\rho(m) := (e_a \otimes m)$ for all $m \in M_a$ and $a \in P$.*

**Corollary 3.13.** *Let $G = \operatorname{Spec}(R[P])$ be a diagonalizable group scheme and let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

*be an exact sequence of $R$-$G$-modules. Then the following hold*:

(1) *For each $a \in P$, there is an exact sequence $0 \to (M_1)_a \to (M_2)_a \to (M_3)_a \to 0$ of $R$-$G$-modules.*

(2) $0 \to M_1^G \to M_2^G \to M_3^G \to 0$ *is an exact sequence of $R$-$G$-modules.*

(3) *The sequence* $0 \to M_1 \to M_2 \to M_3 \to 0$ *splits as a sequence of $R$-$G$-modules if and only if it splits as a sequence of $R$-modules.*

*Proof.* Assertions (1) and (2) follow directly from Lemma 2.4 and Proposition 3.12. The "only if" part of (3) is immediate and, to prove the "if" part, it is enough, using (1) and Proposition 3.12, to give a splitting of the $R$-$G$-linear map $t_a : (M_2)_a \to (M_3)_a$ for $a \in P$.

Let $s : M_3 \to M_2$ be an $R$-linear splitting of $t : M_2 \to M_3$. For $a \in P$, consider the composite map $u_a : (M_3)_a \xrightarrow{i_a} M_3 \xrightarrow{s} M_2 \xrightarrow{p_a} (M_2)_a$, where $i_a$ and $p_a$ are the inclusion and the projection maps, respectively. As $t = \bigoplus_{a \in P} t_a$ and hence $t_a \circ p_a = p_a \circ t$, one checks at once that $t_a \circ u_a$ is the identity on $(M_3)_a$. Moreover, for each $m \in (M_3)_a$, one has

$$(\mathrm{Id}_{R[G]} \otimes u_a) \circ \rho_3(m) = e_a \otimes u_a(m) = \rho_2 \circ u_a(m)$$

and this shows that $u_a : (M_3)_a \to (M_2)_a$ is an $R$-$G$-linear splitting of $t_a$. □

Given any $v \in P$, we shall denote the free $R$-$G$-module of rank one with constant weight $e_v$ by $R_v$ (see Section 2A2).

**Lemma 3.14.** *Let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme and let $(A, \phi)$ be an $R$-$G$-algebra. Given two free $R$-$G$-modules $(V, \rho_V)$ and $(W, \rho_W)$ of rank one and respective constant weights $e_v$ and $e_w$, the $A$-$G$-module structure on $\mathrm{Hom}_A(V_A, W_A)$ is given by*

$$\mathrm{Hom}_A(V_A, W_A) \simeq (R_{w-v}) \otimes_R A.$$

*In particular*, $\mathrm{Hom}_{AG}(V_A, W_A) \simeq A_{v-w}$ *and* $\mathrm{End}_{AG}((V, \rho_V)) \simeq A^G$.

*Proof.* This follows directly from Lemma 3.8 by unraveling the $A$-$G$-module structure defined on $\mathrm{Hom}_A(V_A, W_A)$. □

**Lemma 3.15.** *Let*

$$0 \to P_1 \xrightarrow{\phi_1} P_2 \xrightarrow{\phi_2} P_3 \to 0$$

*be an exact sequence of finitely generated abelian groups and set $G_i = \mathrm{Spec}(R[P_i])$. Let $\phi_i^* : R[P_i] \to R[P_{i+1}]$ denote the corresponding map of group algebras. Let $(A, \theta)$ be an $R$-$G_1$-algebra.*

(1) $(A, (\phi_1^* \otimes \mathrm{Id}_A) \circ \theta)$ *is an $R$-$G_2$-algebra.*

(2) *If* $(E, \rho) \in (A\text{-}G_2)\text{-Mod}$, *then* $E_b := \{\lambda \in E \mid (\phi_2^* \otimes \mathrm{Id}_E) \circ \rho(\lambda) = e_b \otimes \lambda\} \subseteq E$ *is an $A$-$G_2$-submodule for each $b \in P_3$.*

(3) *If* $E \in (A\text{-}G_2)\text{-proj}$, *then so does $E_b$.*

*Proof.* The item (1) is clear. For (2), we can write $E = \bigoplus_{a \in P_2} E_a$, where each $E_a$ is an $R$-$G_2$-submodule. In particular, each $E_b$ is an $R$-$G_2$-submodule. To see that it is an $A$-$G_2$-submodule, it suffices to know that $E_b$ is an $A$-submodule of $E$. Setting $A = \bigoplus_{c \in P_1} A_c$, it suffices to check that $x\lambda \in E_b$ for $x \in A_c$ and $\lambda \in E_b$. But this is a straightforward verification using the fact that $(\phi_2^* \circ \phi_1^*)(e_c) = 1$ and we skip it. The item (3) is clear as each $E_b$ is a direct factor of $E$ as an $A$-module. $\square$

**Corollary 3.16.** *With the assumptions of Lemma 3.15, assume furthermore that the action of $G_1$ on $A$ is free and that every finitely generated projective module over $A^{G_1}$ is extended from $R$. Given any finitely generated projective $A$-$G_2$-module $E$, we have $E \simeq F_A$ for some $R$-$G_2$-module $F$.*

*Proof.* We can use Lemma 3.15 to assume that $E = E_b$ for some $b \in P_3$. For any $a \in \phi_2^{-1}(b)$, it is easy to check that the evaluation map

$$\mathrm{Hom}_{AG_3}(R_a \otimes_R A, E) \otimes_A (R_a \otimes_R A) \to E \tag{3.17}$$

is an isomorphism of $A$-$G_3$-modules. Lemma 3.15, however, says that $E' := \mathrm{Hom}_{AG_3}(R_a \otimes_R A, E) = (\mathrm{Hom}_A(R_a \otimes_R A, E))^{G_3}$ is an $A$-$G_2$-module. It follows that (3.17) is an $A$-$G_2$-linear isomorphism.

As $E'$ has trivial $G_3$-action, it can be viewed as a projective $A$-$G_1$-module. It follows from our assumption and [Vistoli 2005, Theorem 4.46] that this is the pull-back of a finitely generated projective module over $A^{G_1}$. Since every such module over $A^{G_1}$ is extended from $R$, we conclude that $E' \simeq F' \otimes_R A$ as an $A$-$G_2$-module for some finitely generated projective $R$-module $F'$. Taking $F = F' \otimes_R R_a$, we get $E \simeq F_A$. $\square$

## 4. Structure of ringoid modules on $(A\text{-}G)$-Mod

Let $R$ be a commutative noetherian ring and let $G$ be a flat affine group scheme over $R$. Let $(A, \phi)$ be an $R$-$G$-algebra. We have observed in Section 2A1 that the flatness of $G$ ensures that $(A$-$G)$-Mod is an abelian category. In this section, we show that $A$-$G$-modules have the structure of modules over a ringoid (defined below) for various cases of $G$. We shall say that an $A$-$G$-module is *$A$-$G$-projective* if it is a projective object of the abelian category $(A$-$G)$-Mod.

**Lemma 4.1** (resolution property). *Let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme over $R$. Then every finitely generated $A$-$G$-module is a quotient of a finitely generated, free $A$-$G$-module in the category* $(A$-$G)$-Mod.

*Proof.* Let $M$ be a finitely generated $A$-$G$-module. As an $R$-$G$-module, we can write $M = \bigoplus_{a \in P} M_a$, where each $M_a$ is an $R$-module and has constant weight "$e_a$". We can find a finite set of elements $S = \{m_{a_1}^1, \dots, m_{a_1}^{k_1}, \dots, m_{a_m}^1, \dots, m_{a_m}^{k_m}\} \subset M$ which generates $M$ as an $A$-module, with $a_1, \dots, a_m \in P$, $k_i \in \mathbb{N}$ and $m_{a_i}^j \in M_{a_i}$.

Consider the free $R$-$G$-module $F = \bigoplus_{i=1}^{m} R_{a_i}^{k_i}$, where $R_{a_i}$ denotes the free rank-1 $R$-$G$-module with constant weight $e_{a_i}$. Then we have an $R$-$G$-module map $F \to M$ such that the set $S$ lies in its image. Therefore, (3.5) yields a unique $A$-$G$-module surjection $F_A \twoheadrightarrow M$, where $F_A$ is a free $A$-$G$-module of finite rank. □

**Remark 4.2.** A similar argument shows that every $A$-$G$-module (not necessarily finitely generated) has an $A$-$G$-linear epimorphism from a direct sum of (possibly infinite) rank-1 free $A$-$G$-modules.

**Lemma 4.3.** *Let $G$ be as above. Then a finitely generated $A$-$G$-module is $A$-$G$-projective if and only if it is projective as an $A$-module. In particular, the category* $(A$-$G)$-Mod *has enough projectives.*

*Proof.* Suppose $L$ is a finitely generated projective $A$-$G$-module. Let $M \xrightarrow{\phi} N$ be a surjective $A$-$G$-module homomorphism. Then $\mathrm{Hom}_A(L, M) \xrightarrow{\phi\circ-} \mathrm{Hom}_A(L, N)$ is an $A$-$G$-linear map by Lemma 3.8 and is surjective as $L$ is a projective $A$-module. By Corollary 3.13(2), the map $\mathrm{Hom}_A(L, M)^G \xrightarrow{\phi\circ-} \mathrm{Hom}_A(L, N)^G$ is also surjective and, therefore, $\mathrm{Hom}_{AG}(L, M) \xrightarrow{\phi\circ-} \mathrm{Hom}_{AG}(L, N)$ is surjective by Lemma 3.8. Hence, $L$ is $A$-$G$-projective.

Conversely, suppose $L$ is $A$-$G$-projective. By Lemma 4.1, there exists a finitely generated free $A$-$G$-module $F$ and an $A$-$G$-module surjection $F \twoheadrightarrow L$. Since $L$ is $A$-$G$-projective, there is a splitting and hence it is a direct summand of $F$. Since $F$ is a projective $A$-module, $L$ is $A$-projective as well. The existence of enough projectives in $(A$-$G)$-Mod now follows from this, Lemma 4.1 and Remark 4.2 since any direct sum of $A$-$G$-projectives is also $A$-$G$-projective. □

Let us now consider more general situations. Recall from [SGA 3$_{\mathrm{III}}$ 1970, Exposé XIX] that an affine group scheme $G$ over $R$ is called *reductive* if it is smooth over $R$ and, for every point $x \in S = \mathrm{Spec}(R)$, the geometric fiber $G \times_S \mathrm{Spec}(\overline{k(x)})$ is a reductive linear algebraic group over $\mathrm{Spec}(\overline{k(x)})$. We say that $G$ is split reductive if it is a connected and reductive group scheme over $R$ and it admits a maximal torus $T \simeq \mathbb{G}_{m,R}^r$ such that the pair $(G, T)$ corresponds to a (reduced) root system $(A, \mathcal{R}, A^\vee, \mathcal{R}^\vee)$ defined over $\mathbb{Z}$ (see [SGA 3$_{\mathrm{III}}$ 1970, Exposé XXII]). It is known that all Chevalley groups, such as $\mathrm{GL}_n$, $\mathrm{SL}_n$, $\mathrm{PGL}_n$, $\mathrm{Sp}_{2n}$ and $\mathrm{SO}_n$, are split reductive group schemes over $R$.

Using similar techniques, we can now extend Lemmas 4.1 and 4.3 to the class of split reductive group schemes over $R$, as follows:

**Lemma 4.4.** *Let $R$ be a unique factorization domain containing a field of characteristic zero. Let $G$ be a connected reductive group scheme over $R$ which contains a split maximal torus $\mathbb{G}_{m,R}^r$. Let $(A, \phi)$ be an $R$-$G$-algebra. Then*:

(1) *Every finitely generated $A$-$G$-module is a quotient of a finitely generated*, *free $A$-$G$-module in the category* $(A$-$G)$-Mod.

(2) *A finitely generated $A$-$G$-module is $A$-$G$-projective if and only if it is projective as an $A$-module.*

*Proof.* Let $k \hookrightarrow R$ be a field of characteristic zero. Since $R$ is a UFD and $G$ contains a split maximal torus, it is known in this case (see [SGA $3_{\text{III}}$ 1970, Exposé XXII, Proposition 2.2], for example) that $G$ is in fact a split reductive group scheme over $R$. In particular, it is defined over the ring $\mathbb{Z}$ and hence over $k$. Let $G_0$ be a $k$-form for $G$. In other words, $G_0$ is a connected reductive group over $k$ such that $k[G_0] \otimes_k R \simeq R[G]$.

Let $M$ be a finitely generated $A$-$G$-module. Since $G_0$ is reductive and $\operatorname{char}(k) = 0$, we see that it is linearly reductive (see Section 2A2). Since $R[G] = k[G_0] \otimes_k R$, we see that the $R$-$G$-module structure on $M$ given by $(M, \rho)$ is same thing as the $k$-$G_0$-module structure $(M, \rho)$ (see Section 2A1). With this $k$-$G_0$-module structure, we can write $M$ as a (possibly infinite) direct sum of irreducible $k$-$G_0$-modules. Let $S = \{m_1, \dots, m_s\}$ be a generating set of $M$ as an $A$-module. Then we can find finitely many irreducible $k$-$G_0$-submodules of $M$ whose direct sum contains $S$. Letting $F$ denote this direct sum, we get a $k$-$G_0$-linear map $F \to M$ whose image contains $S$. This map uniquely defines an $R$-$G$-linear map $F_R \to M$. Extending this further to $A$ using (3.5), we get a unique $A$-$G$-linear map $F_A \to M$, which is clearly surjective. This proves (1).

Suppose $L$ is a finitely generated projective $A$-$G$-module. Let $M \xrightarrow{\phi} N$ be a surjective $A$-$G$-module homomorphism. Then $\operatorname{Hom}_A(L, M) \xrightarrow{\phi\circ-} \operatorname{Hom}_A(L, N)$ is an $A$-$G$-linear map by Lemma 3.8 and is surjective as $L$ is a projective $A$-module. Using the linear reductivity of $G_0$ and arguing as in the proof of Lemma 4.3, we see that the map $\operatorname{Hom}_A(L, M)^{G_0} \xrightarrow{\phi\circ-} \operatorname{Hom}_A(L, N)^{G_0}$ is surjective. As argued in the proof of (1) above, it is easy to see from the identification of $(M, \rho_R)$ with $(M, \rho_k)$ and Section 2A2 that $E^G = E^{G_0}$ for any $R$-$G$-module $E$. We conclude that the map $\operatorname{Hom}_A(L, M)^G \xrightarrow{\phi\circ-} \operatorname{Hom}_A(L, N)^G$ is surjective. Therefore, $\operatorname{Hom}_{AG}(L, M) \xrightarrow{\phi\circ-} \operatorname{Hom}_{AG}(L, N)$ is surjective. Hence $L$ is $A$-$G$-projective. The converse follows exactly as in the diagonalizable group case using (1). □

We recall a few definitions in category theory:

**Definition 4.5.** Let $\mathcal{A}$ be a cocomplete abelian category. We say that a set of objects $\{P_\alpha\}_\alpha$ is a *set of strong generators* for $\mathcal{A}$ if for every object $X$ in $\mathcal{A}$ we have $X = 0$ whenever $\operatorname{Hom}_{\mathcal{A}}(P_\alpha, X) = 0$ for all $\alpha$.

An object $P$ is called *small* if $\bigoplus_\lambda \operatorname{Hom}_{\mathcal{A}}(P, X_\lambda) \to \operatorname{Hom}_{\mathcal{A}}(P, \bigoplus_\lambda X_\lambda)$ is a bijection for every set of objects $\{X_\lambda\}_\lambda$.

Recall that a *ringoid* $\mathcal{R}$ is a small category which is enriched over the category **Ab** of abelian groups. This means that the hom-sets in $\mathcal{R}$ are abelian groups and the compositions of morphisms are bilinear maps of abelian groups. A ringoid with only one object can be easily seen to be equivalent to a (possibly noncommutative) ring $R$.

A (right) $\mathcal{R}$-module is a contravariant functor $M : (\mathcal{R})^{\mathrm{op}} \to \mathbf{Ab}$. It is known that the category $\mathcal{R}$-Mod of (right) $\mathcal{R}$-modules is a complete and cocomplete abelian category, where the limits and colimits are defined objectwise. An $\mathcal{R}$-module is called free of rank one if it is of the form $B \mapsto \mathrm{Hom}_{\mathcal{R}}(B, A)$ for some $A \in \mathcal{R}$. Such modules are denoted by $H_A$. We say that an $\mathcal{R}$-module is finitely generated if it is a quotient of a finite coproduct of rank-one free $\mathcal{R}$-modules. It is known that $\mathcal{R}$-Mod is a Grothendieck category which has a set of small and projective strong generators. This set is given by the collection $\{H_A \mid A \in \mathrm{Obj}(\mathcal{R})\}$. We refer to [Mitchell 1972] for more details about ringoids.

A combination of the previous few results gives us the following conclusion:

**Proposition 4.6.** *Given a commutative noetherian ring $R$, an affine group scheme $G$ over $R$ and an $R$-$G$-algebra $(A, \phi)$, the following hold:*

(1) *If $G = \mathrm{Spec}(R[P])$ is a diagonalizable group scheme, then the category $(A\text{-}G)$-Mod has a set of small and projective strong generators.*

(2) *If $R$ is a UFD containing a field of characteristic zero and $G$ is a split reductive group scheme, then the category $(A\text{-}G)$-Mod has a set of small and projective strong generators.*

*In either case, the category $(A\text{-}G)$-Mod is equivalent to the category $\mathcal{R}$-mod for some ringoid $\mathcal{R}$ and this equivalence preserves finitely generated projective objects.*

*Proof.* If $G = \mathrm{Spec}(R[P])$ is diagonalizable, we set $S = \{A \otimes_R R_a \mid a \in P\}$, and if $G$ is split reductive, we set $S = \{A \otimes_k V_a\}_a$, where $\{V_a\}_a$ is the set of isomorphism classes of all irreducible $k$-$G_0$-modules. The proposition now follows from Lemmas 4.1, 4.3 and 4.4 and Remark 4.2. It is shown as part of the proofs of these lemmas that $S$ is a set of strong generators for $(A\text{-}G)$-Mod.

The last part follows from (1) and (2) and [Freyd 1964, Exercise 5.3H], which says that the functor

$$\mathrm{Hom}(S, \_) : (A\text{-}G)\text{-Mod} \to \mathrm{End}(S)\text{-Mod}$$

is an equivalence of categories, where $\mathrm{End}(S)$ is the full subcategory of $(A\text{-}G)$-Mod consisting of objects in $S$. To show that this equivalence preserves finitely generated projective objects, we only need to show that it preserves finitely generated objects, since any equivalence of abelian categories preserves projective objects. Suppose now that $M$ is a finitely generated $A$-$G$-module in case (1).

It was shown in the proof of Lemma 4.1 that there is a finite set $\{a_1, \dots, a_m\} \subseteq P$ and a surjective $A$-$G$-linear map $\bigoplus_{i=1}^m (A \otimes_R R_{a_i}) \twoheadrightarrow M$. But this precisely means that $\bigoplus_{i=1}^{m} H_{a_i}(A \otimes_R R_a) \twoheadrightarrow \mathrm{Hom}(S, M)(A \otimes_R R_a)$ for all $a \in P$ and this means $\mathrm{Hom}(S, M)$ is a finitely generated object of $\mathrm{End}(S)$-Mod. The case (2) follows similarly. □

**Remark 4.7.** If $G$ is a finite constant group scheme over $R$ whose order is invertible in $R$, then one can show using the same argument as above that the category $(A\text{-}G)$-Mod has a single small and projective generator given by $A \otimes_R R[G]$. In particular, a variant of Freyd's theorem implies that $(A\text{-}G)$-Mod is equivalent to the category of right $S$-modules, where $S$ is the endomorphism ring of $A \otimes_R R[G]$.

## 5. Group action on monoid algebras

In this section, we prove some properties of projective modules over the ring of invariants when a diagonalizable group acts on a monoid algebra. We fix a commutative noetherian ring $R$ and a diagonalizable group scheme $G = \mathrm{Spec}(R[P])$ over $R$.

Let $Q$ be a monoid, i.e., a commutative semigroup with unit. Let $G(Q)$ be the Grothendieck group associated to $Q$.

**Definition 5.1.** We say that $Q$ is

- *cancellative* if $ax = ay$ implies $x = y$ in $Q$;
- *seminormal* if $x \in G(Q)$ and $x^2, x^3 \in Q$ implies $x \in Q$;
- *normal* if $x \in G(Q)$ and $x^n \in Q$ for any $n > 0$ implies $x \in Q$;
- *torsion-free* if $x^n = y^n$ for some $n > 0$ implies $x = y$;
- *having no nontrivial unit* if $x, y \in Q$ and $xy = 1$ imply that $x$ is the unit of $Q$.

Given a monoid $Q$, we can form the monoid algebra $R[Q]$. As an $R$-module, $R[Q]$ is free with a basis consisting of the symbols $\{e_a \mid a \in Q\}$, and the multiplication on $R[Q]$ is defined by the $R$-bilinear extension of $e_a \cdot e_b = e_{ab}$. The elements $e_a$ are called the monomials of $R[Q]$. For example, polynomial ring $R[x_1, \dots, x_n]$ is a monoid algebra defined by the monoid $\mathbb{Z}_+^n$, and the monomials of $R[\mathbb{Z}_+^n]$ are exactly the monomials of the polynomial ring.

**5A.** ***Projective modules over monoid algebras.*** For $R$ as above, consider the following conditions.

(†) Every (not necessarily finitely generated) projective $R$-module is free and every finitely generated projective $R[Q]$-module is extended from $R$ if $Q$ is a torsion-free abelian group.

(††) Every (not necessarily finitely generated) projective $R$-module is free and every (finitely generated) projective module over $R[Q \times \mathbb{Z}^n]$ is extended from $R$ if $Q$ is a torsion-free, seminormal and cancellative monoid which has no nontrivial unit and $n \geq 0$ is an integer.

**Theorem 5.2.** *Let $R$ be a commutative noetherian ring that is any of the following*:

(1) *A principal ideal domain.*

(2) *A regular local ring of dimension $\leq 2$.*

(3) *A regular local ring containing a field.*

*Then $R$ satisfies* (†) *and* (††).

*Proof.* The first part of (†) holds more generally for any commutative noetherian ring $R$ which is either local or a principal ideal domain. This follows from [Kaplansky 1958, Theorem 2; Bass 1973].

That the principal ideal domains satisfy (†) and (††) follows from [Bruns and Gubeladze 2009, Theorem 8.4]. These conditions for (2) follow from [Swan 1992, Theorem 1.2, Corollary 3.5]. To show (†) and (††) for (3), we first reduce to the case when $R$ is essentially of finite type over a field, using the methods of [Swan 1998, Theorem 2.1] and Neron–Popescu desingularization. In the special case when $R$ is essentially of finite type over a field, (3) follows from [Swan 1992, Theorem 1.2, Corollary 3.5]. $\square$

**5B.** ***Projective modules over the ring of invariants.*** Let $Q$ be a monoid and let $u : Q \to P$ be a homomorphism of monoids. Consider the graph homomorphism $\gamma_u : Q \to P \times Q$ given by $\gamma_u(a) = (u(a), a)$. This defines a unique morphism $\phi : R[Q] \to R[P \times Q] \simeq R[P] \otimes_R R[Q]$ of monoid $R$-algebras, given by $\phi(f_a) = g_{\gamma_u(a)} = e_{u(a)} \otimes f_a$, where $e : P \to (R[P])^\times$, $f : Q \to (R[Q])^\times$ and $g : P \times Q \to (R[P \times Q])^\times$ are the exponential maps (see Section 3B). Notice that these exponential maps are injective. Setting $A = R[Q]$, we thus get a canonical map of $R$-algebras

$$\phi : A \to R[P] \otimes_R A. \tag{5.3}$$

One checks at once that this makes $(A, \phi)$ into an $R$-$G$-algebra.

**Proposition 5.4.** *Let $Q' = \mathrm{Ker}(u)$ be the submonoid of $Q$. Assume that $Q$ satisfies any of the properties listed in Definition 5.1. Then $Q'$ also satisfies the same property. In each case*, *there is an isomorphism of $R$-algebras $R[Q'] \xrightarrow{\simeq} A^G$.*

*Proof.* Since we work with (commutative) monoids, we shall write their elements additively. It is immediate from the definition that the properties of being cancellative, torsion-free and having no nontrivial units are shared by all submonoids of $Q$. The only issue is to show that $Q'$ is seminormal (resp. normal) if $Q$ is so.

So let us assume that $Q$ is seminormal and let $x \in G(Q')$ be such that $2x, 3x \in Q'$. Since $G(Q') \subseteq G(Q)$, we see that $x \in Q$. Setting $y = u(x)$, we get $2y = u(2x) = 0 = u(3x) = 3y$. Since $P = G(P)$, we get $y = 3y - 2y = 0$ and this means $x \in Q'$.

Suppose now that $Q$ is normal and $x \in G(Q')$ is such that $nx \in Q'$ for some $n > 0$. As $G(Q') \subseteq G(Q)$ and $Q$ is normal, we get $x \in Q$. The commutative diagram

$$\begin{tikzcd} Q' \arrow[r] \arrow[d] & Q \arrow[d] \arrow[dr, "u"] & \\ G(Q') \arrow[r] & G(Q) \arrow[r, "G(u)"'] & P \end{tikzcd}$$

now shows that $u(x) = G(u)(x) = 0$ and hence $x \in Q'$.

It is clear from the definition that $R[Q'] \subseteq A^G$ and so we only need to show the reverse inclusion to prove the second part of the proposition. Let $p = \sum_a r_a f_a \in A^G$ with $0 \neq r_a \in R$. This means that $\phi(p) = 1 \otimes p = e_0 \otimes p$. Equivalently, we get

$$\begin{aligned} \sum_a r_a(e_{u(a)} \otimes f_a) = \sum_a r_a(e_0 \otimes f_a) &\iff \sum_a r_a(e_{u(a)} - e_0) \otimes f_a = 0 \\ &\iff r_a(e_{u(a)} - e_0) = 0 \quad \text{for all } a \\ &\iff e_{u(a)} = e_0 \quad \text{for all } a \\ &\iff u(a) = 0 \quad \text{for all } a \\ &\iff a \in Q' \quad \text{for all } a. \end{aligned}$$

The second equivalence follows from the fact that $R[P] \otimes_R R[Q]$ is a free $R[P]$-module with basis $\{f_a \mid a \in Q\}$ and the third follows from the fact that $R[P]$ is a free $R$-module with basis $\{e_b \mid b \in P\}$ and $r_a \neq 0$. The last statement implies that each summand of $p$ belongs to $R[Q']$ and so does $p$. This proves the proposition. □

**Corollary 5.5.** *Assume that $R$ satisfies* (††). *Let $Q$ be a monoid which is cancellative, torsion-free, seminormal and has no nontrivial unit. Let $A = R[Q]$ be the monoid algebra having the $R$-$G$-algebra structure given by* (5.3). *Then finitely generated projective modules over $A$ and $A^G$ are free.*

**Corollary 5.6.** *Let $R$ be a principal ideal domain and let $Q$ be a monoid which is cancellative, torsion-free and seminormal* (*possibly having nontrivial units*). *Let $A = R[Q]$ be the monoid algebra having the $R$-$G$-algebra structure given by* (5.3). *Then finitely generated projective modules over $A$ and $A^G$ are free.*

*Proof.* This follows from Proposition 5.4 and the main result of [Gubeladze 1988]. □

We end this section with the following description of finitely generated free $R$-$G$-modules when $R$ satisfies (†) and its consequence:

**Lemma 5.7.** *Assume that $R$ satisfies* (†). *Then every finitely generated free $R$-$G$-module is a direct sum of free $R$-$G$-modules of rank one. Every free $R$-$G$-module of rank one has constant weight of the form $e_a$ for some $a \in P$.*

*Proof.* Let $M$ be a finitely generated free $R$-$G$-module. By Proposition 3.12, we can write $M = \bigoplus_{a \in P} M_a$. Lemma 2.4 says that this is a direct sum decomposition as $R$-$G$-modules. Moreover, each $M_a$ is a direct factor of the free $R$-module $M$ and hence is projective and thus free as $R$ satisfies (†).

Therefore, it is enough to show that, if $M$ is a free $R$-$G$-module of constant weight $e_a$, then every $R$-submodule of $M$ is an $R$-$G$-submodule. But this follows directly from Lemma 2.4. The decomposition $M = \bigoplus_{a \in P} M_a$ also shows that a free rank-one $R$-$G$=module must have a constant weight of the form $e_a$ with $a \in P$. □

**Corollary 5.8.** *Assume that $R$ satisfies* (†). *Under the assumptions of Corollary 3.16 suppose that $F, F' \in (R\text{-}G_2)$*-proj *are isomorphic as $R$-$G_3$-modules. Then $F_A \simeq F'_A$ as $A$-$G_2$-modules.*

*Proof.* By Lemma 5.7 and Proposition 3.12, it is enough to prove that, if $F$ and $F'$ are one-dimensional free $R$-$G_2$-modules of constant weights $e_a$ and $e_{a'}$, where $a, a' \in P_2$ with $\phi_2(a) = \phi_2(a')$, then $F_A \simeq F'_A$ as $A$-$G_2$-modules.

As $G_3$ acts trivially on $A$ and $\phi_2(a) = \phi_2(a')$, we have $\mathrm{Hom}_{AG_3}(F_A, F'_A) = \mathrm{Hom}_A(F_A, F'_A)$. By Lemma 3.15, $\mathrm{Hom}_{AG_3}(F_A, F'_A) = \mathrm{Hom}_A(F_A, F'_A)$ as $A$-$G_2$-modules and $\mathrm{Hom}_A(F_A, F'_A) \simeq R_{a'-a} \otimes A$ as an $A$-$G_2$-module by Lemma 3.14. The argument of Corollary 3.16 shows that $\mathrm{Hom}_{AG_3}(F_A, F'_A) \simeq A$ as an $A$-$G_2$-module. Therefore, $R_{a'-a} \otimes A \simeq A$ and hence $R_a \otimes A \simeq R_{a'} \otimes A$ as $A$-$G_2$-modules. □

## 6. Toric schemes and their quotients

Let $R$ be a commutative noetherian ring and let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme over $R$. In this section, we recall the notion of affine $G$-toric schemes and study their quotients for the $G$-action.

**6A.** ***Toric schemes.*** Let $L$ be a lattice (a free abelian group of finite rank). A subset of $L_{\mathbb{Q}}$ of the form $l^{-1}(\mathbb{Q}_+)$, where $l : L_{\mathbb{Q}} \to \mathbb{Q}$ is a nonzero linear functional and $\mathbb{Q}_+ = \{r \in \mathbb{Q} \mid r \geq 0\}$, is called a half-space of $L_{\mathbb{Q}}$. A *cone* of $L_{\mathbb{Q}}$ is an intersection of a finite number of half-spaces. A cone is always assumed to be convex, polyhedral and rational ("rational" means that it is generated by vectors in the lattice). The *dimension* of a cone $\sigma$ is defined to be the dimension of the smallest subspace of $L_{\mathbb{Q}}$ containing $\sigma$. We say that $\sigma$ is *strongly convex* in $L_{\mathbb{Q}}$ if it spans $L_{\mathbb{Q}}$. By replacing $L_{\mathbb{Q}}$ by its subspace $\sigma + (-1)\sigma$, there is no loss of generality in assuming that $\sigma$ is a strongly convex cone in $L_{\mathbb{Q}}$.

The intersection $\sigma \cap L$ is clearly a cancellative, torsion-free monoid. Moreover, $L_\sigma = \sigma \cap L$ is known to be finitely generated and normal (see [Danilov 1978, Lemma 1.3; Bruns and Gubeladze 2009, Corollary 2.24]). It follows from [Bruns and Gubeladze 2009, Theorem 4.40] that the monoidal $R$-algebra $A = R[L_\sigma]$ is a normal integral domain if $R$ is so. The scheme $X_\sigma = \mathrm{Spec}(R[L_\sigma])$ is called an *affine toric scheme* over $R$. The inclusion $\iota_\sigma : L_\sigma \hookrightarrow L$ defines a Hopf algebra map $\phi_\sigma : A \to R[L] \otimes_R A$ (the graph of $\iota_\sigma$), which is equivalent to giving an action of the "big torus" $T_\sigma = \mathrm{Spec}(R[L])$ on $X_\sigma$. The inclusion $R[L_\sigma] \hookrightarrow R[L]$ embeds $T_\sigma$ as a $T_\sigma$-invariant affine open subset of $X_\sigma$, where $T_\sigma$ acts on itself by multiplication.

A *face* of $\sigma$ is its subset of the form $\sigma \cap l^{-1}(0)$, where $l : L_{\mathbb{Q}} \to \mathbb{Q}$ is a linear functional that is positive on $\sigma$. A face of a cone is again a cone, so for each face $\tau$ of $\sigma$, we have a toric scheme $X_\tau$ which has an action of $T_\sigma$ given by the inclusion $L_\tau \hookrightarrow L$ and this action factors through the action of the big torus $T_\tau = \mathrm{Spec}(R[M])$ of $X_\tau$ (where $M$ is the smallest sublattice of $L$ such that $M_{\mathbb{Q}}$ is a subspace containing $\tau$) on $X_\tau$. Let $\chi$ be the characteristic function of the face $\tau$, i.e., the function which is 1 on $\tau$ and 0 outside $\tau$. The assignment $e_m \mapsto \chi(m)e_m$ (for $m \in L_\sigma$) extends to a surjective homomorphism of $R$-algebras $i_\tau : R[L_\sigma] \twoheadrightarrow R[L_\tau]$, which defines a closed embedding of $X_\tau$ in $X_\sigma$. The natural inclusion $L_\tau \hookrightarrow L_\sigma$ defines a retraction morphism $\pi_\tau : R[L_\tau] \to R[L_\sigma]$. Both $i_\tau$ and $\pi_\tau$ are $R$-$T_\sigma$-algebra morphisms such that the composition $i_\tau \circ \pi_\tau$ is the identity.

If $\tau' \subseteq \sigma$ is another face different from $\tau$ and $\eta$ is their intersection, then we get a commutative diagram

$$\begin{tikzcd} R[L_\tau] \arrow[r, "\pi_\tau"] \arrow[d, "\iota_\eta"'] & R[L_\sigma] \arrow[r, "\iota_\tau"] \arrow[d, "\iota_{\tau'}"] & R[L_\tau] \arrow[d, "\iota_\eta"] \\ R[L_\eta] \arrow[r, "\pi_\eta"'] & R[L_{\tau'}] \arrow[r, "\iota_\eta"'] & R[L_\eta] \end{tikzcd} \tag{6.1}$$

in which the composite horizontal maps are the identity.

Let $J$ denote the ideal of $R[L_\sigma]$ generated by all the monomials $e_m$ with $m$ strictly inside $\sigma$. Then $J$ is a $T_\sigma$-invariant ideal of $R[L_\sigma]$ such that $X_\sigma \setminus Y = T_\sigma$, where $Y = \mathrm{Spec}(R[L_\sigma]/J)$ (see [Danilov 1978, Section 2.6.1], for example).

**Lemma 6.2.** *Let $\Delta^1$ denote the set of codimension 1 faces of $X_\sigma$. Then the ideal $J$ is the ideal defining the closed subscheme $\bigcup_{\tau\in\Delta^1} X_\tau$ of $X_\sigma$, i.e., $Y = \bigcup_{\tau\in\Delta^1} X_\tau$.*

*Proof.* The ideal $\mathcal{I}(X_\tau)$ that defines $X_\tau$ is generated by all monomials $e_m$ with $m \in (\sigma \setminus \tau) \cap L$. Since $\mathcal{I}\big(\bigcup_{\tau\in\Delta^1} X_\tau\big) = \bigcap_{\tau\in\Delta^1} \mathcal{I}(X_\tau)$, the lemma follows. □

**Lemma 6.3.** *For any $m \in L$, there is a sufficiently large integer $N$ such that $f/e_m \in R[L_\sigma]$ for any $f \in J^N$.*

*Proof.* It is enough to prove the lemma when $f = \prod_{k=1}^N e_{m_k}$ with $m_k$ strictly inside $\sigma$. Let $v_1, \dots, v_p$ be generators of $L_\sigma$ and let $l_1, \dots, l_q$ be linear functionals defining $\sigma$. Set $s = \min_{i,j}\{l_i(v_j) > 0\}$. Since $m_k$ lies strictly inside $\sigma$, $l_i(m_k) > 0$ for any $i$. Since $m_k$ is a linear combination of the $v_j$ with nonnegative integer coefficients, we get $l_i(m_k) \geq s$ for any $i$. Therefore, $l_i\big(\sum_{k=1}^N m_k - m\big) \geq Ns - l_i(m)$ for any $i$. Since $s$ is positive, we must have $l_i\big(\sum_{k=1}^N m_k - m\big) \geq 0$ for any $i$ if $N$ is sufficiently large. That is, $\sum_{k=1}^{N} m_k - m \in L_\sigma$ independent of the choice of the $m_k$. □

**6B.** ***G-toric schemes and their quotients.*** Let $\sigma$ be a strongly convex, rational, polyhedral cone in $L_{\mathbb{Q}}$, where $L$ is a lattice of finite rank. Let $A = R[L_\sigma]$ and $X = X_\sigma = \mathrm{Spec}(A)$. Let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme over $R$.

**Definition 6.4.** An *affine $G$-toric scheme* is an affine toric scheme $X_\sigma$ as above with a $G$-action such that the action of $G$ on $X_\sigma$ factors through the action of $T_\sigma$.

Since $\mathrm{Spec}(R)$ is connected, a $G$-toric scheme structure on $X_\sigma$ is equivalent to having a map of monoids $\psi : L \to P$ such that the $R$-$G$-algebra structure on $A = R[L_\sigma]$ is defined by the composite action map

$$\phi_P : A \to R[L] \otimes_R A \xrightarrow{\psi \otimes \mathrm{Id}} R[P] \otimes_R A. \tag{6.5}$$

**Examples 6.6.** We shall say that $G$ acts linearly on a polynomial algebra $A = R[t_1, \dots, t_n]$ if there is a free $R$-$G$-module $(V, \rho)$ of rank $n$ such that $A = \mathrm{Sym}_R(V)$. In this case, we also say that $G$ acts linearly on $\mathrm{Spec}(A) = \mathbb{A}^n_R$.

Assume that $R$ satisfies (†). Let $A = R[x_1, \dots, x_n, y_1, \dots, y_r]$ be a polynomial $R$-algebra with a linear $G$-action, with $n, r \geq 0$. Using Lemma 5.7, we can assume that the $G$-action on $A$ is given by $\phi(x_i) = e_{\lambda_i} \otimes x_i$ for $1 \leq i \leq n$ and $\phi(y_j) = e_{\gamma_j} \otimes y_j$ for $1 \leq j \leq r$.

(1) Let $A = R[x_1, \dots, x_n]$. Consider the cone $\sigma = \mathbb{Q}^n_+$ of $L_{\mathbb{Q}}$, where $L$ is the lattice $\mathbb{Z}^n$. Then $A = R[\sigma \cap L]$ and $\mathrm{Spec}(A)$ is an affine $G$-toric scheme via the morphism $\psi : \mathbb{Z}^n_+ \to P$ given by $\psi(\alpha_i) = \lambda_i$, where $\{\alpha_1, \dots, \alpha_n\}$ is the standard basis of $\mathbb{Z}^n_+$.

(2) Let $A = R[x_1, \dots, x_n, y_1^{\pm 1}, \dots, y_r^{\pm 1}]$. Then it can be seen, as in (1) above, that $\mathrm{Spec}(A)$ is an affine $G$-toric scheme by considering the lattice $L = \mathbb{Z}^{n+r}$ and the cone $\sigma = \mathbb{Q}^n_+ \oplus \mathbb{Q}^r$ in $L_{\mathbb{Q}}$.

**Lemma 6.7.** *Let $\theta : L \to P$ be a homomorphism from $L$ to a finitely generated abelian group and let $M = \mathrm{Ker}(\theta)$. Then $R[\sigma \cap M]$ is a finitely generated $R$-algebra.*

*Proof.* By replacing $P$ by the image of $\theta$, we can assume that $\theta$ is an epimorphism. This yields an exact sequence

$$0 \to M_{\mathbb{Q}} \xrightarrow{i_M} L_{\mathbb{Q}} \to P_{\mathbb{Q}} \to 0. \tag{6.8}$$

We write $\sigma = \bigcap_{i=1}^{r} \sigma_i$, where $\sigma_i = l_i^{-1}(\mathbb{Q}_+)$ is a half-space. By taking repeated intersections of $M$ with these $\sigma_i$ and using induction, we easily reduce to the case when $r = 1$. We set $\tau = \sigma \cap M_{\mathbb{Q}}$. Then $\tau = l^{-1}(\mathbb{Q}_+) \cap M_{\mathbb{Q}} = m^{-1}(\mathbb{Q}_+)$, where $m = l \circ i_M$. In particular, $\tau$ is a cone in $M_{\mathbb{Q}}$. Furthermore, as $M \hookrightarrow L$, it is a free abelian group and hence a lattice in $M_{\mathbb{Q}}$. It follows from Gordon's lemma (see [Fulton 1993, Proposition 1.2.1], for instance) that $\tau \cap M$ is a finitely generated monoid. Therefore, $\sigma \cap M = \sigma \cap M_{\mathbb{Q}} \cap M = \tau \cap M$ is a finitely generated monoid. Since any generating set of $\sigma \cap M$ generates $R[\sigma \cap M]$ as an $R$-algebra, the lemma follows. □

Combining Lemma 6.7 and Proposition 5.4, we get:

**Corollary 6.9.** *Let $A^G$ denote the ring of $G$-invariants of $A$ with respect to $\phi_P$. Then $A^G$ is a finitely generated $R$-algebra.*

**Lemma 6.10.** *Let $B$ be any flat $A^G$-algebra. Then $B = (A \otimes_{A^G} B)^G$.*

*Proof.* Set $B' = A \otimes_{A^G} B$. To prove this lemma, we need to recall how $G$ acts on $B'$. The map $\phi_P : A \to R[P] \otimes_R A$ induces a $B'$-algebra map

$$B' = A \otimes_{A^G} B \xrightarrow{\phi_P \otimes 1_B} (R[P] \otimes_R A) \otimes_{A^G} B.$$

This can also be written as $\phi_{P,B} : B' \to R[P] \otimes_R B'$ with $\phi_{P,B} = \phi_P \otimes 1_B$, which gives a $G$-action on $\operatorname{Spec}(B')$.

Let $\gamma_P : A \to R[P] \otimes_R A$ be the ring homomorphism $\gamma_P(a) = 1 \otimes a$, which gives the projection map $G \times X \to X$. Set $\gamma_{P,B} = \gamma_P \otimes 1_B : B' \to R[P] \otimes_R B'$. It is clear that

$$\gamma_{P,B}(a \otimes b) = \gamma_P(a) \otimes b = 1 \otimes a \otimes b = 1 \otimes (a \otimes b).$$

Since $B'$ is generated by elements of the form $a \otimes b$ with $a \in A$ and $b \in B$, we see that $\gamma_{P,B}(\alpha) = 1 \otimes \alpha$ for all $\alpha \in B'$.

Since $A = R[L_\sigma]$ is flat (in fact free) over $R$ (see Lemma 6.7), the map $\gamma_P : A \to R[P] \otimes_R A$ is injective. Furthermore, there is an exact sequence (by definition of $A^G$)

$$0 \to A^G \to A \xrightarrow{\phi_P - \gamma_P} R[P] \otimes_R A. \tag{6.11}$$

As $B$ is flat over $A^G$, the tensor product with $B$ over $A^G$ yields an exact sequence

$$0 \to B \to B' \xrightarrow{(\phi_P \otimes 1_B) - (\gamma_P \otimes 1_B)} R[P] \otimes_R B'. \tag{6.12}$$

Since $\phi_P \otimes 1_B = \phi_{P,B}$ and $\gamma_P \otimes 1_B = \gamma_{P,B}$, we get an exact sequence

$$0 \to B \to B' \xrightarrow{\phi_{P,B} - \gamma_{P,B}} R[P] \otimes_R B'. \tag{6.13}$$

But this is equivalent to saying that $B = (B')^G$. □

**Lemma 6.14.** *Let $I$, $I' \subseteq A$ be inclusions of $A$-$G$-modules such that $I + I' = A$. Then the sequence*

$$0 \to I^G \to A^G \to (A/I)^G \to 0$$

*is exact and* $I^G + I'^G = A^G$. *In particular*, *the map* $\mathrm{Spec}((A/I)^G) \hookrightarrow \mathrm{Spec}(A^G)$ *is a closed immersion and* $\mathrm{Spec}((A/I)^G) \cap \mathrm{Spec}((A/I')^G) = \varnothing$ *in* $\mathrm{Spec}(A^G)$.

*Proof.* The assumption $I + I' = A$ is equivalent to saying that the map $I \oplus I' \to A$ is surjective. The lemma is now an immediate consequence of Corollary 3.13(2). □

Combining the above lemmas, we obtain the following. We refer to [Mumford et al. 1994, §0.1] for the terms used in this result.

**Proposition 6.15.** *Let* $X = X_\sigma$ *be a* $G$*-toric scheme over* $R$ *as above. Then a categorical quotient in* $\mathbf{Sch}_S$, $p : X \to X'$, *for* $G$*-action* (in the sense of [Mumford et al. 1994, Definition 0.5]) *exists. Moreover*, *the following hold*:

(1) *If* $Z \subseteq X$ *is a* $G$*-invariant closed subscheme*, *then* $p(Z)$ *is a closed subscheme of* $Y$.

(2) *If* $Z_1, Z_2 \subseteq X$ *are* $G$*-invariant closed subschemes with* $Z_1 \cap Z_2 = \varnothing$, *then* $p(Z_1) \cap p(Z_2) = \varnothing$.

(3) *The map* $p : X \to X'$ *is a uniform categorical quotient in* $\mathbf{Sch}_S$.

(4) *The quotient map* $p$ *is submersive.*

*Proof.* We take $X' = \mathrm{Spec}(A^G)$. It follows from Lemma 6.7 that $X'$ is an affine scheme of finite type over $R$. The fact that $p : X \to X'$, given by the inclusion $A^G \hookrightarrow A$, is a categorical quotient follows at once from the exact sequence (6.11). The universality of $p$ with respect to $G$-invariant maps $p' : Y' \to X'$ of affine $G$-schemes with trivial $G$-action on $Y'$ also follows immediately from (6.11). The properties (1) and (2) are direct consequences of Lemma 6.14. To prove (3), let $Y' \to X'$ be a flat morphism between finite type $R$-schemes. To show that $p' : Y' \times_{X'} X \to Y'$ is a categorical quotient, we can use the descent argument of [Mumford et al. 1994, §0.2, Remark 8] to reduce to the case when $Y'$ is affine. In this case, the desired property follows at once from Lemma 6.10. Item (4) follows from (1)–(3) and [Mumford et al. 1994, §0.2, Remark 6]. □

**Corollary 6.16.** *Let* $X = \mathrm{Spec}(A)$ *be a* $G$*-toric scheme as above and let* $p : X \to X'$ *be the quotient map. Let* $Y \subsetneq X$ *be a closed subscheme defined by a* $G$*-invariant ideal* $J$. *Let* $h \in A^G$ *be a nonunit such that* $h \equiv 1 \pmod J$ *and set* $V' = \mathrm{Spec}(A^G[h^{-1}])$. *Then we can find an open subscheme* $U'$ *of* $X'$ *such that* $X' = U' \cup V'$ *and* $p^{-1}(U') \cap Y = \varnothing$.

*Proof.* Our assumption says that $V' \subsetneq X'$ is a proper open subset of $X'$ and $Y \subset V = p^{-1}(V')$ is a $G$-invariant closed subset. Setting $Y' = p(Y)$, it follows from Proposition 6.15 that $Y' \subsetneq X'$ is a closed subset contained in $V'$. In particular, $Y_1 = p^{-1}(Y')$ is a $G$-invariant closed subscheme of $X$ such that $Y \subseteq Y_1 \subsetneq V \subsetneq X$. The open subset $U' = X' \setminus Y'$ now satisfies our requirements. □

## 7. Equivariant vector bundles on $G$-toric schemes

In this section, we prove our main result about equivariant vector bundles on affine $G$-toric schemes.

**7A.** ***The setup.*** We shall prove Theorem 7.8 under the following setup. Let $R$ be a commutative noetherian ring and let $S = \mathrm{Spec}(R)$. Let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme over $R$. Let $L$ be a lattice of finite rank and let $\sigma$ be a strongly convex, polyhedral, rational cone in $L_{\mathbb{Q}}$. Let $\Delta$ denote the set of all faces of $\sigma$.

Let $A = R[L_\sigma]$ be such that $X = \mathrm{Spec}(A)$ is a $G$-toric scheme via a homomorphism $\psi : L \to P$ (see (6.5)). Set $Y = \bigcup_{\tau \in \Delta^1} X_\tau$. Let $X' = \mathrm{Spec}(A^G)$ and let $p : X \to X'$ denote the uniform categorical quotient in $\mathbf{Sch}_S$ defined by the inclusion $A^G \hookrightarrow A$.

**7B.** ***Reduction to faithful action.*** We set $Q = \psi(L)$ and $H = \mathrm{Spec}(R[Q])$. Then $H$ is a diagonalizable closed subgroup of $T_\sigma$ which acts faithfully on $X$ and $G$ acts on $X$ via the quotient $G \twoheadrightarrow H$ (see Proposition 3.11). The following lemma reduces the proof of the main theorem of this section to the case of faithful action of $G$ on $X$.

We shall say that a finitely generated projective $A$-$G$-module $M$ over an $R$-$G$-algebra $A$ is *trivial* if it can be equivariantly extended from $R$, that is, there is a finitely generated projective $R$-$G$-module $F$ such that $M \simeq F_A$.

**Lemma 7.1.** *If every finitely generated projective $A$-$H$-module is trivial, then so is every finitely generated projective $A$-$G$-module.*

*Proof.* Given any $E \in (A\text{-}G)\text{-proj}$, we can write $E = \bigoplus_{b \in P/Q} E_b$ with $E_b = \bigoplus_{\{a \mid b = a \bmod Q\}} E_a$. Lemma 3.15 says that each $E_b \in (A\text{-}G)\text{-proj}$. It suffices to show that each $E_b$ is trivial.

Now, $E_b$ is trivial if and only if $E_b \otimes_R R_{-a}$ is trivial for any $a$ with $b = a \bmod Q$. But $E_b \otimes_R R_{-a}$ is a projective $A$-$H$-module and so we can find an $A$-$H$-module isomorphism $\phi : E_b \otimes_R R_{-a} \xrightarrow{\simeq} F_A$ for some $F \in (R\text{-}H)\text{-proj}$. This is then an $A$-$G$-module isomorphism as well. □

**7C.** ***Trivialization in a neighborhood of $Y$.*** Note that, if $X = \mathrm{Spec}(A)$ is an affine G-toric scheme and $\tau$ is any face of the cone $\sigma$, then $X_\tau$ is a $G$-invariant closed subscheme of $X$. Moreover, the map $\pi_\tau : R[L_\tau] \to A = R[L_\sigma]$ defined before is $G$-equivariant (because it is $T_\sigma$-equivariant).

**Lemma 7.2.** *Let $\tau_1, \dots, \tau_k$ denote the codimension-1 faces of $\sigma$ and let $I_j$ denote the ideal of $A$ defining the closed subscheme $X_{\tau_j}$ associated to the face $\tau_j$. Let $E$ be an $A$-$G$-module and $F$ be an $R$-$G$-module such that $E/I_j \simeq F_{A/I_j}$ for all $1 \le j \le k$. Then $E/J \simeq F_{A/J}$, where $J$ denotes the ideal defining $Y = \bigcup_{i=1}^k X_{\tau_i}$.*

*Proof.* Let $J_r$ be the ideal defining the $G$-invariant closed subscheme $Y_r = \bigcup_{i=1}^{r} X_{\tau_i}$ for $1 \le r \le k$. We prove by induction on $r$ that $E/J_r \simeq F_{A/J_r}$. Assume that $\phi : E/J_r \simeq F_{A/J_r}$ and $\eta : E/I_{r+1} \simeq F_{A/I_{r+1}}$ are given isomorphisms. This gives us a $G$-equivariant automorphism $\eta \circ \phi^{-1}$ of $F_{A/(J_r+I_{r+1})}$. Under the $G$-equivariant retraction $\Pi_{r+1} : X_\sigma \to X_{\tau_{r+1}} = \mathrm{Spec}(A/I_{r+1})$ (where $\Pi_i = \mathrm{Spec}(\pi_i)$), we have $\Pi_{r+1}(Y_r) \subset Y_r \cap X_{\tau_{r+1}}$ (see (6.1)).

Therefore $\phi' = (\Pi_{r+1}|_{Y_r})^*(\eta \circ \phi^{-1})$ defines an $A/(J_r)$-$G$-linear automorphism of $F_{A/J_r}$. Replacing $\phi$ by the isomorphism $\phi' \circ \phi$, we can arrange that $\phi$ and $\eta$ agree modulo $(J_r + I_{r+1})$. So they define a unique isomorphism $E/J_{r+1} \to F_{A/J_{r+1}}$. To see this, use the exact sequence

$$0 \to E/J_{r+1} \to E/J_r \times E/I_{r+1} \to E/(J_r + I_{r+1}) \to 0. \qquad \square$$

**Lemma 7.3.** *Let $P \in M_m(A^G)$ be a rank-$m$ matrix with entries in $A^G$ such that $P$ is invertible modulo $I_j$ for all $1 \le j \le k$, where $I_j$ and $J$ are as in Lemma 7.2. Then, for any positive integer $N$, there is $\widetilde{P}_N \in \mathrm{GL}_m(A^G)$ such that $(P\widetilde{P}_N)_{ij} \in J^N$ for all $i \ne j$.*

*Proof.* For $1 \le i \le k$, we consider the commutative diagram of retractions

$$\begin{array}{ccc} (A/I_i)^G & \longrightarrow & A/I_i \\ \Big\downarrow{\scriptstyle \pi^G_{\tau_i}} & & \Big\downarrow{\scriptstyle \pi_{\tau_i}} \\ A^G & \longrightarrow & A \end{array} \tag{7.4}$$

Since $P \bmod I_1$ is invertible, $P_1 := \pi_{\tau_1}(P \bmod I_1) \in \mathrm{GL}_m(A^G)$ and hence $PP_1^{-1} \equiv \mathrm{Id}_m \pmod{I_1}$. We now let $P_2$ denote the image of $PP_1^{-1} \bmod I_2$ under the $G$-equivariant retraction $\pi_{\tau_2}$. This yields $P_2 \equiv \mathrm{Id}_m \pmod{I_1}$ (see (6.1)) and so $PP_1^{-1}P_2^{-1} \equiv \mathrm{Id}_m \pmod{I_1 \cap I_2}$. Repeating this procedure and using Lemma 6.2, we can find $\widetilde{P}_1 \in \mathrm{GL}_m(A^G)$ such that $P\widetilde{P}_1 \equiv \mathrm{Id}_m \pmod{J}$, which proves the lemma for $N = 1$.

Assume now that there exists $\widetilde{P}_N \in \mathrm{GL}_m(A^G)$ such that $(P\widetilde{P}_N)_{ij} \equiv 0 \pmod{J^N}$ for $i \ne j$ and $(P\widetilde{P}_N)_{ii} \equiv 1 \pmod{J}$. By elementary column operations

$$C_i \mapsto C_i - (P\widetilde{P}_N)_{ji} C_j \quad \text{for } i > j = 1, \dots, m-1$$

and

$$C_i \mapsto C_i - (P\widetilde{P}_N)_{ji} C_j \quad \text{for } i < j = 2, \dots, m$$

on $P\widetilde{P}_N$, we get a matrix whose off-diagonal elements are $0 \pmod{J^{N+1}}$ and diagonal elements are $1 \pmod{J}$. These operations correspond to right multiplication by some $P' \in \mathrm{GL}_m(A^G)$. Taking $\widetilde{P}_{N+1} = \widetilde{P}_N P'$ completes the induction step. $\square$

**Lemma 7.5.** *Assume that $R$ satisfies (†) and let $I$ be a $G$-invariant ideal of $A$. Let $F$ and $E$ be finitely generated free $R$-$G$- and $A$-$G$-modules, respectively. Given* *any $(A/I)$-$G$-module isomorphism $\phi : E/I \xrightarrow{\simeq} F_{A/I}$, there exists $h \in A^G$ such that $h \equiv 1$ modulo $I$ and $\phi$ extends to an $A_h$-$G$-module isomorphism $\phi_h : E_h \xrightarrow{\simeq} F_{A_h}$.*

*Proof.* Let $\phi'$ denote the inverse of $\phi$. Since $E$ and $F_A$ are projective $A$-$G$-modules, $\phi$ and $\phi'$ extend to $A$-$G$-module homomorphisms $T : E \to F_A$ and $T' : F_A \to E$ by Lemma 4.3. As $R$ satisfies (†), $F$ is a direct sum of rank-1 free $R$-$G$-modules by Lemma 5.7. Since $E$ and $F_A$ are isomorphic modulo $I$, they have the same rank, say $m$. Fix an $R$-basis $\{v_1, \dots, v_m\}$ of $F$ consisting of elements of constant weights $e_{w_1}, \dots, e_{w_m}$ ($w_i \in P$) and fix any $A$-basis of $E$.

With respect to the chosen bases, $T$ and $T'$ define matrices in $M_m(A)$ which are invertible modulo $I$. Moreover, as $TT' = (a_{ij})$ defines an $A$-$G$-module endomorphism of $F_A$, it can be easily checked using Lemma 3.14 that $a_{ij} \in A_{w_i - w_j}$ and, using the Leibniz formula for the determinant, one checks that $\det(TT') \in A^G$. We take $h = \det(TT')$ to finish the proof. □

**7D.** ***Descent to the quotient scheme.*** The following unique "descent to the quotient" property of the $G$-equivariant maps will be crucial for proving our main results on equivariant vector bundles:

**Lemma 7.6.** *Assume that $R$ satisfies (†). Let $q : W \to W'$ be a uniform categorical quotient in $\mathbf{Sch}_S$ for a $G$-action on $W$, where $w : W \to S$ and $w' : W' \to S$ are structure maps. Assume that $q$ is an affine morphism. Let $F$ be a finitely generated projective $R$-module. Given any $G$-equivariant endomorphism $f$ of $w^*(F)$, there exists a unique endomorphism $\tilde{f}$ of $w'^*(F)$ such that $f = q^*(\tilde{f})$. In particular, $\tilde{f}$ is an automorphism if $f$ is so.*

*Proof.* The second part follows from the uniqueness assertion in the first part, so we only have to prove the existence of a unique $\tilde{f}$. Since $W'$ is noetherian, we can write $W' = \bigcup_{i=1}^r U_i'$, where each $U_i'$ is affine open. We prove the lemma by induction on $r$. If $r = 1$, then $W'$ is affine and hence so is $W$. We can write $W = \mathrm{Spec}(B)$ and $W' = \mathrm{Spec}(B^G)$ for some finite-type $R$-$G$-algebra $B$ (see Proposition 6.15). As $F$ is a free $R$-$G$-module of constant weight $e_0$, it follows from Lemma 3.14 that $f \in M_n(B^G)$ with $n = \mathrm{rank}(F)$. In particular, it defines a unique endomorphism $\tilde{f}$ of $w'^*(F)$ such that $f = q^*(\tilde{f})$.

We now assume $r \geq 2$ and set $U' = \bigcup_{i=2}^r U_i'$. Then $q : U_1 := q^{-1}(U_1') \to U_1'$ and $q : U := q^{-1}(U') \to U'$ are uniform categorical quotients. As $U_1'$ is affine, there exists a unique $\tilde{f}_{U_1'} : F_{U_1'} \to F_{U_1'}$ such that $q^*(\tilde{f}_{U_1'}) = f|_{U_1}$. By the induction hypothesis, there exists a unique $\tilde{f}_{U'} : F_{U'} \to F_{U'}$ such that $q^*(\tilde{f}_{U'}) = f|_U$. As $V' := U_1' \cap U'$ has a cover by $r - 1$ affine opens, the induction hypothesis and uniqueness imply that $\tilde{f}_{U_1'}|_V = \tilde{f}_{U'}|_V$. The reader can check that $\tilde{f}_{U_1'}$ and $\tilde{f}_{U'}$ glue together to define the desired unique endomorphism $\tilde{f} : w'^*(F) \to w'^*(F)$. □

**7E. *The main theorem.*** We now use the above reduction steps to prove our main result of this section. We first consider the case of faithful action.

**Lemma 7.7.** *Suppose $\psi : L \twoheadrightarrow P$. Assume that $R$ satisfies* (†) *and that every finitely generated projective $A^G$-module is extended from $R$. Let $E \in (A\text{-}G)$*-proj *and $F \in (R\text{-}G)$*-proj. *Suppose there exist $G$-equivariant isomorphisms $\eta : E|_U \xrightarrow{\simeq} F_A|_U$ and $\phi : E|_V \xrightarrow{\simeq} F_A|_V$, where $U = X \setminus Y$ is the big torus of $X$ and $V = \mathrm{Spec}(A[h^{-1}])$ for some $h \in A^G$ such that $h \equiv 1 \pmod J$, where $J$ is the defining ideal of the inclusion $Y \hookrightarrow X$. Then $E \simeq F_A$ as $A$-$G$-modules.*

*Proof.* If $h$ is a unit in $A^G$, we have $V = X$ and we are done. So assume that $h$ is not a unit in $A^G$. Let $p : X \to X'$ denote the quotient map as in Proposition 6.15. Set $V' = \mathrm{Spec}(A^G[h^{-1}])$ so that $V = p^{-1}(V')$ and let $U' \subseteq X'$ be as obtained in Corollary 6.16 so that $U_1 := p^{-1}(U') \subseteq U$. Set $W' = U' \cap V'$ and $W = p^{-1}(W')$. Then $\eta : E|_{U_1} \to F_A|_{U_1}$ is a $G$-equivariant isomorphism. Let $\Phi = \phi \circ \eta^{-1}$ denote the $G$-equivariant automorphism of $F_A|_W$.

By Lemma 5.7, we can write $F = \bigoplus_{i=1}^m \widetilde{F}_{\lambda_i}$, where $\lambda_i \in P$ are not necessarily distinct and $\widetilde{F}_{\lambda_i}$ are free $R$-$G$-modules of rank 1 and constant weight $e_{\lambda_i}$. Since $L \twoheadrightarrow P$, there exist monomials in $R[L]$ of any given weight. Suppose $d_i \in R[L]$ is a monomial having weight $e_{\lambda_i}$. Let $D$ be the diagonal matrix with diagonal entries $d_1, \dots, d_m$. Then $D \in \mathrm{Hom}_{R[L]G}(F_{R[L]}, F'_{R[L]})$ is an isomorphism of $R[L]$-$G$-modules, where $F'$ is a free $R$-$G$-module of rank $m$ and constant weight $e_0$. Thus $\widetilde{\Phi} := D\Phi D^{-1}$ is a $G$-equivariant automorphism of $F'_A|_W$.

Since $p : W \to W'$ is a uniform categorical quotient which is an affine morphism, we can apply Lemma 7.6 to find a unique automorphism $f$ of $F'_{A^G}|_{W'}$ such that $\widetilde{\Phi} = p^*(f)$. As $X' = U' \cup V'$, such an automorphism defines a locally free sheaf on $X'$ by gluing of sheaves [Hartshorne 1977, Exercise II.1.22]. Since every such locally free sheaf on $X'$ is free by assumption, we have [loc. cit.] $f = f_2 \circ f_1$ for some automorphisms $f_1$ and $f_2$ of $F'_{A^G}|_{U'}$ and $F'_{A^G}|_{V'}$, respectively. Then $\widetilde{\Phi} = p^*(f) = p^*(f_2) \circ p^*(f_1)$ and hence we get $\Phi = (D^{-1} p^*(f_2) D)(D^{-1} p^*(f_1) D)$. As $p^*(f_2)$ defines a matrix $P_1$ in $\mathrm{GL}_m(A^G[h^{-1}])$, by an appropriate choice of basis we can find $s \geq 0$ such that $P := h^s P_1 \in M_m(A^G)$.

By Lemma 7.3, we can find $\widetilde{P}_N \in \mathrm{GL}_m(A^G)$ such that $(P\widetilde{P}_N)_{ij} \in J^N$ for $i \neq j$. The $(ij)$-th entry of $D^{-1} P \widetilde{P}_N D$ is $d_i^{-1} d_j (P\widetilde{P}_N)_{ij}$. Taking $N$ sufficiently large, we may assume that $d_i^{-1} d_j (P\widetilde{P}_N)_{ij} \in A$ by Lemma 6.3.

Setting $\theta_1 = (D^{-1} \widetilde{P}_N^{-1} p^*(f_1) D)$ and $\theta_2 = (D^{-1} h^{-s} P \widetilde{P}_N D)$, we see that $\theta_1$ and $\theta_2$ define G-equivariant automorphisms of $F_A|_{U_1}$ and $F_A|_V$, respectively, such that $\theta_2 \circ \theta_1 = \Phi = \phi \circ \eta^{-1}$.

If we set $\eta' = \theta_1 \circ \eta$ and $\phi' = \theta_2^{-1} \circ \phi$, we see that $\eta' : E|_{U_1} \to F_A|_{U_1}$ and $\phi' : E|_V \to F_A|_V$ are $G$-equivariant isomorphisms such that $\eta'|_W = \phi'|_W$. By gluing therefore, we get a $G$-equivariant isomorphism $E \to F_A$ on $X$. □

**Theorem 7.8.** *Consider the setup of Section 7A. Assume that $R$ satisfies* (†) *and that finitely generated projective modules over $A_\tau$ and $(A_\tau)^G$ are extended from $R$ for every $\tau \in \Delta$. Then every finitely generated projective $A$-$G$-module is trivial.*

*Proof.* We can assume that the map $\psi : L \to P$ is surjective by Lemma 7.1. Let $E \in (A\text{-}G)$-proj. Since $R$ satisfies (†) and every finitely generated projective $A$-module is extended from $R$, we see that $E$ is a free $A$-module of finite rank. In particular, Lemma 7.5 applies.

Let $\tilde{\tau}$ denote the face of $\sigma$ of smallest dimension. Then $X_{\tilde{\tau}}$ is a torus whose dimension is that of the largest subspace of $L_{\mathbb{Q}}$ contained in $\sigma$. Let $M$ denote the smallest sublattice of $L$ such that $\tilde{\tau} = M_{\mathbb{Q}}$. Let $\phi : M \hookrightarrow L \to P$ denote the composite map. Consider the abelian groups $Q_1 := \mathrm{Im}(\phi)$ and $Q_2 := P/Q_1$. Fix a finitely generated projective $R$-$G$-module $F$ such that $E|_{X_{\tilde{\tau}}} \simeq F \otimes_R R[L_{\tilde{\tau}}]$. This exists by Corollary 3.16, applied to the sequence

$$0 \to Q_1 \to P \to P/Q_1 \to 0.$$

We prove by induction on the dimension of the cone $\sigma$ that $E \simeq F_{R[L_\sigma]}$. Assume that $E|_{X_\tau} \simeq F_{R[L_\tau]}$ for all codimension-1 faces $\tau$ of $\sigma$. Let $Y = \bigcup_{\tau \in \Delta^1} X_\tau$ be as before. We first apply Lemma 7.2 to get an isomorphism $\tilde{\phi} : E/J \simeq F_{A/J}$. We next apply Lemma 7.5 to find $h \in A^G$ such that $\tilde{\phi}$ extends to an isomorphism $\phi$ on $V = \mathrm{Spec}(A_h) \supseteq Y$.

Applying Corollaries 3.16 and 5.8 to the torus $T_\sigma = \mathrm{Spec}(R[L])$, there exists an $R[L]$-$G$-module isomorphism $\eta : E|_{T_\sigma} \xrightarrow{\simeq} F_{R[L]} = F_A|_{T_\sigma}$ (consider the exact sequence $0 \to P \to P \to 0 \to 0$ and note that the action of $G$ on $T_\sigma$ is free). We now apply Lemma 7.7 to conclude that $E \simeq F_A$. This completes the induction step and proves the theorem. □

As an easy consequence of Corollary 5.6 and Theorem 7.8, we obtain:

**Corollary 7.9.** *Consider the setup of Section 7A and assume that $R$ is a principal ideal domain. Then every finitely generated projective $A$-$G$-module is trivial.*

## 8. Vector bundles over $\mathbb{A}^n_R \times \mathbb{G}^r_{m,R}$

In this section, we apply Theorem 7.8 to prove triviality of $G$-equivariant projective modules over polynomial and Laurent polynomial rings. When $R$ satisfies (††), we have the following answer to the equivariant Bass–Quillen question:

**Theorem 8.1.** *Let $R$ be a regular ring and let $R[x_1, \dots, x_n, y_1, \dots, y_r]$ be a polynomial $R$-algebra with a linear $G$-action with $n, r \geq 0$. Then the following hold*:

(1) *If $R$ satisfies* (††) *and $A = R[x_1, \dots, x_n]$, then every finitely generated projective $A$-$G$-module is trivial.*

(2) *If $R$ is a PID and $A = R[x_1, \dots, x_n, y_1^{\pm 1}, \dots, y_r^{\pm 1}]$, then every finitely generated projective $A$-$G$-module is trivial.*

*Proof.* As shown in Examples 6.6, $\mathrm{Spec}(A)$ is an affine toric $G$-scheme in both cases. To prove (1), note that $R$ satisfies the hypotheses of Theorem 7.8, by Corollary 5.5. Therefore, (1) follows from Theorem 7.8. Similarly, (2) is a special case of Corollary 7.9. □

**8A. *Vector bundles over $\mathbb{A}^n_R$ without condition* (††).** Let $R$ be a noetherian ring and let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme over $R$. We now show that if the localizations of $R$ satisfy (††) then the equivariant vector bundles over $\mathbb{A}^n_R$ can be extended from $\mathrm{Spec}(R)$. In order to show this, we shall need the following equivariant version of Quillen's patching lemma [1976, Lemma 1]. In this section, we shall allow our $R$-$G$-algebras to be noncommutative (see Section 2A3).

Given a (possibly noncommutative) $R$-$G$-algebra $A$, a polynomial $A$-$G$-algebra is an $R$-$G$-algebra $A[t]$ which is a polynomial algebra over $A$ with indeterminate $t$ such that the inclusion $A \hookrightarrow A[t]$ is a morphism of $R$-$G$-algebras and $t \in A[t]$ is semi-invariant (see Section 2A2). For a polynomial $A$-$G$-algebra $A[t]$, let $(1 + tA[t])^\times$ denote the (possibly noncommutative) group of units $\phi(t) \in A[t]$ such that $\phi(0) = 1$.

Given an $A[t]$-$G$-module $M$ (with $A$ commutative), we shall say that $M$ is extended from $A$ if there is an $A$-$G$-module $N$ and an $A[t]$-$G$-linear isomorphism $\theta : N \otimes_A A[t] \xrightarrow{\simeq} M$. It is easy to check that this condition is equivalent to saying that there is an $A[t]$-$G$-linear isomorphism $\theta : (M/tM) \otimes_A A[t] \xrightarrow{\simeq} M$.

**Lemma 8.2** (equivariant patching lemma). *Let $(A, \phi)$ be an $R$-$G$-algebra and let $(A[t], \tilde{\phi})$ be a polynomial $A$-$G$-algebra as above. Let $0 \neq f \in R$ and let $\theta(t) \in (1 + tA_f[t])^\times$ be a $G$-invariant polynomial. Then there exists $k \geq 0$ such that, for any $a$, $b \in R$ with $a - b \in f^k R$, we can find a $G$-invariant element $\psi(t) \in (1 + tA[t])^\times$ with $\psi_f(t) = \theta(at)\theta(bt)^{-1}$.*

*Proof.* This is a straightforward generalization of [Quillen 1976, Lemma 1] with the same proof almost verbatim. The only extra thing we need to check is that if $\theta(t) \in (1 + tA_f[t])^\times \cap (A[t])^G$ then $\psi(t)$ (as constructed in [loc. cit.]) is also $G$-invariant. But this can be checked directly, using the fact that $t$ is semi-invariant. We leave the details to the reader. □

**Lemma 8.3.** *Let $(A, \phi)$ and $(A[t], \tilde{\phi})$ be as in Lemma 8.2. Assume that $A$ is commutative. Let $M$ be a finitely generated $A[t]$-$G$-module and let $Q(M) = \{f \in R \mid M_f$ is an extended $A_f[t]$-$G$-module$\}$. Then $Q(M) \cup \{0\}$ is an ideal of $R$.*

*Proof.* We only need to check that if $f_0$, $f_1 \in Q(M)$ then $f_0 + f_1 \in Q(M)$. We can assume that $f_0 + f_1$ is invertible in $R$. In particular, $(f_0, f_1) = R$. Set

$$A_i = A_{f_i},\ M_i = M_{f_i} \quad \text{for } i = 0, 1, \quad N = M/tM \quad \text{and} \quad E = \mathrm{Hom}_A(N, N).$$

Given isomorphisms $u_i : N \otimes_A A_i[t] \xrightarrow{\simeq} M_i$, Quillen [1976, Theorem 1] constructs automorphisms $\psi_i(t) \in \mathrm{Hom}_{A_i[t]}(N \otimes_A A_i[t], N \otimes_A A_i[t]) = E_i[t]$ for $i = 0, 1$ with the following properties:

$$u'_i := u_i \cdot \psi_i(t) : N \otimes_A A_i[t] \xrightarrow{\simeq} M_i \quad \text{and} \quad (u'_0)_{f_1} = (u'_1)_{f_0}.$$

One should observe here that the isomorphism

$$E_i[t] \xrightarrow{\simeq} \mathrm{Hom}_{A_i[t]}(N \otimes_A A_i[t], N \otimes_A A_i[t]), \qquad f \otimes t^i \mapsto (n \otimes a \mapsto f(n) \otimes at^i),$$

is $R$-$G$-linear (see Lemma 3.8).

To prove the lemma, we only need to show that each $\psi_i(t)$ is $G$-equivariant. By Lemma 3.8, this is equivalent to showing that $\psi_i(t) \in (E_i[t])^G$ for $i = 0, 1$. But this follows at once (as the reader can check by hand) by observing that each $u_i$ is $G$-invariant and subsequently applying Lemma 8.2 to $E_{f_0}$ and $E_{f_1}$, which are (possibly noncommutative) $R$-$G$-algebras by Lemma 3.8. □

The following result generalizes Theorem 8.1 to the case when the base ring $R$ does not necessarily satisfy (††), but whose local rings satisfy (††). For examples of local rings satisfying (††), see Theorem 5.2.

**Theorem 8.4.** *Let $R$ be a noetherian integral domain such that its localizations at all maximal ideals satisfy* (††). *Let $G = \mathrm{Spec}(R[P])$ be a diagonalizable group scheme over $R$. Let $V = \bigoplus_{i=1}^n Rx_i$ be a direct sum of one-dimensional free $R$-$G$-modules and let $A = R[x_1, \dots, x_n] = \mathrm{Sym}_R(V)$. Then every finitely generated projective $A$-$G$-module is extended from $R$.*

*Proof.* We prove the theorem by induction on $n$. There is nothing to prove when $n = 0$ and the case $n = 1$ is an easy consequence of Theorem 8.1 and Lemma 8.3. Suppose now that $n \geq 2$ and every projective $R[x_1, \dots, x_{n-1}]$-$G$-module is extended from $R$.

Let $M$ be a finitely generated projective $A$-$G$-module and set $A_i = R[x_1, \dots, x_i]$. It follows from Theorem 8.1 that $M_{\mathfrak{m}}$ is extended from $(A_{n-1})_m$ for every maximal ideal $\mathfrak{m}$ of $R$. We now apply Lemma 8.3 to $(A_{n-1}, \phi_{n-1})$ and $(A_{n-1}[x_n], \tilde{\phi}_{n-1}) = (A, \phi)$ to conclude that $M$ is extended from $A_{n-1}$. It follows by induction that $M$ is extended from $R$. □

## 9. Derived equivalence and equivariant *K*-theory

In this section, we shall apply the results of Section 4 to show that the derived equivalence of equivariant quasicoherent sheaves on affine schemes with group action implies the equivalence of the equivariant *K*-theory of these schemes. When the underlying group is trivial, this was shown by Dugger and Shipley [2004]. In the equivariant setup too, we make essential use of some general results of Dugger and Shipley, which we now recall.

**9A.** ***Some results of Dugger and Shipley.*** Recall that an object $X$ in a cocomplete triangulated category $\mathcal{T}$ is called *compact* if the natural map $\varinjlim_\alpha \mathrm{Hom}_{\mathcal{T}}(X, Z_\alpha) \to \mathrm{Hom}_{\mathcal{T}}(X, \varinjlim_\alpha Z_\alpha)$ is a bijection for every direct system $\{Z_\alpha\}$ of objects in $\mathcal{T}$.

If $\mathcal{A}$ is an abelian category, then an object of the category $\mathrm{Ch}_{\mathcal{A}}$ of chain complexes over $\mathcal{A}$ is called *compact* if its image in the derived category $D(\mathcal{A})$ is compact in the above sense.

The key steps in the proof of our main theorem of this section are Propositions 4.6 and A.1 and the following general results of [Dugger and Shipley 2004]:

**Theorem 9.1** [Dugger and Shipley 2004, Theorem D]. *Let $\mathcal{A}$ and $\mathcal{B}$ be cocomplete abelian categories which have sets of small*, *projective*, *strong generators. Let $K_c(\mathcal{A})$* (*resp. $K_c(\mathcal{B})$*) *denote the Waldhausen K-theory of the compact objects in* $\mathrm{Ch}(\mathcal{A})$ (*resp.* $\mathrm{Ch}(\mathcal{B})$). *Then*:

(1) *$\mathcal{A}$ and $\mathcal{B}$ are derived equivalent if and only if* $\mathrm{Ch}(\mathcal{A})$ *and* $\mathrm{Ch}(\mathcal{B})$ *are equivalent as pointed model categories.*

(2) *If $\mathcal{A}$ and $\mathcal{B}$ are derived equivalent*, *then $K_c(\mathcal{A}) \simeq K_c(\mathcal{B})$*.

**Theorem 9.2** [Dugger and Shipley 2004, Corollary 3.9]. *Let $\mathcal{M}$ and $\mathcal{N}$ be pointed model categories connected by a zigzag of Quillen equivalences. Let $\mathcal{U}$ be a complete Waldhausen subcategory of $\mathcal{M}$*, *and let $\mathcal{V}$ consist of all cofibrant objects in $\mathcal{N}$ which are carried into $\mathcal{U}$ by the composite of the derived functors of the Quillen equivalences. Then $\mathcal{V}$ is a complete Waldhausen subcategory of $\mathcal{N}$*, *and there is an induced zigzag of weak equivalences between $K(\mathcal{U})$ and $K(\mathcal{V})$*.

**Theorem 9.3** [Dugger and Shipley 2004, Theorems 4.2 and 7.5]. *Let $\mathcal{R}$ and $\mathcal{S}$ be two ringoids* (see Section 4). *Then the following conditions are equivalent*:

(1) *There is a zigzag of Quillen equivalences between* $\mathrm{Ch}(\mathrm{Mod}\text{-}\mathcal{R})$ *and* $\mathrm{Ch}(\mathrm{Mod}\text{-}\mathcal{S})$.

(2) *$D(\mathcal{R}) \simeq D(\mathcal{S})$ are triangulated equivalent.*

(3) *The bounded derived categories of finitely generated projective $\mathcal{R}$- and $\mathcal{S}$-modules are triangulated equivalent.*

**9B.** ***Derived equivalence and K-theory under group action.*** Let $R$ be a commutative noetherian ring and let $G$ be an affine group scheme over $R$. Let $(A, \phi)$ be an $R$-$G$-algebra and let $X = \mathrm{Spec}(A)$ be the associated affine $S$-scheme with $G$-action, where $S = \mathrm{Spec}(R)$. We shall denote this datum in this section by $(R, G, A)$. Let $\mathrm{Ch}^G(A)$ denote the abelian category of unbounded chain complexes of $A$-$G$-modules and let $D^G(A)$ denote the associated derived category. One knows that $D^G(A)$ is a cocomplete triangulated category.

We shall say that $(R, G, A)$ has the *resolution property* if for every finitely generated $A$-$G$-module $M$ there is a finitely generated projective $A$-$G$-module $E$ and a $G$-equivariant epimorphism $E \twoheadrightarrow M$.

Recall that a bounded chain complex of finitely generated, projective $A$-$G$-modules is called a *strict perfect complex*. A (possibly unbounded) chain complex of $A$-$G$-modules is called a *perfect complex* if it is isomorphic to a strict perfect complex in $D^G(A)$. We shall denote the categories of strict perfect and perfect complexes of $A$-$G$-modules by $\mathrm{Sperf}^G(A)$ and $\mathrm{Perf}^G(A)$, respectively. It is known that $\mathrm{Sperf}^G(A)$ and $\mathrm{Perf}^G(A)$ are both complicial bi-Waldhausen categories in the sense of [Thomason and Trobaugh 1990] and there is a natural inclusion $\mathrm{Sperf}^G(A) \hookrightarrow \mathrm{Perf}^G(A)$ of complicial bi-Waldhausen categories. As this inclusion induces an equivalence of the associated derived categories, it follows from [Thomason and Trobaugh 1990, Theorem 1.9.8] that this induces a homotopy equivalence of the associated Waldhausen $K$-theory spectra. We shall denote the common derived category by $D^G(\mathrm{Perf}/A)$ and the common $K$-theory spectrum by $K^G(A)$. It follows from [Thomason and Trobaugh 1990, Theorem 1.11.7] that $K^G(A)$ is homotopy equivalent to the $K$-theory spectrum of the exact category of finitely generated projective $A$-$G$-modules.

Let $\mathrm{Ch}^G_b(A)$ denote the category of bounded chain complexes of finitely generated $A$-$G$-modules and let $D^G_b(A)$ denote its derived category. The Waldhausen $K$-theory spectrum of $\mathrm{Ch}^G_b(A)$ will be denoted by $K'_G(A)$. Let $\mathrm{Ch}^{\mathrm{hb},-}(A\text{-}G\text{-proj})$ be the category of chain complexes of finitely generated projective $A$-$G$-modules which are bounded above and cohomologically bounded. Let $D^{\mathrm{hb},-}(A\text{-}G\text{-proj})$ denote the associated derived category.

If $(R, G, A)$ has the resolution property, then every complex of $\mathrm{Ch}^G_b(A)$ is quasi-isomorphic to a complex of $\mathrm{Ch}^{\mathrm{hb},-}(A\text{-}G\text{-proj})$ and vice versa. It follows from [Thomason and Trobaugh 1990, Theorem 1.9.8] that they have homotopy equivalent Waldhausen $K$-theory spectra:

$$K'_G(A) \simeq K(\mathrm{Ch}^{\mathrm{hb},-}(A\text{-}G\text{-proj})). \tag{9.4}$$

**Lemma 9.5.** *Assume that $(R, G, A)$ has the resolution property. Given any complex $K \in \mathrm{Ch}^G(A)$, there exists a direct system of strict perfect complexes $F_\alpha$, and a quasi-isomorphism*

$$\varinjlim_{\alpha} F_\alpha \xrightarrow{\sim} K.$$

*Proof.* The nonequivariant case of this result was proven in [Thomason and Trobaugh 1990, Proposition 2.3.2] and a similar proof applies here as well once we verify that [ibid., Hypothesis 1.9.5.1] holds for $\mathcal{A} = (A\text{-}G)\text{-Mod}$, $\mathcal{D}$ the category of (possibly infinite) direct sums of finitely generated projective A-G-modules and $\mathcal{C}$ the category of cohomologically bounded above complexes in $\mathrm{Ch}^G(A)$. For this, it is enough to show that if $M \to N$ is a surjective map of $A$-$G$-modules then there is a (possibly infinite) direct sum $F$ of finitely generated projective $A$-$G$ modules and an $A$-$G$-linear map $F \to M$ such that the composite $F \to M \to N$ is surjective. But $M$

is the direct limit of its finitely generated $A$-$G$-submodules, as shown in [Laumon and Moret-Bailly 2000, Proposition 15.4] (see also [Thomason 1987, Lemma 2.1] when $G$ is faithfully flat over $S$). Therefore, it follows from the resolution property that $M$ is a quotient of a direct sum of finitely generated projective $A$-$G$-modules. $\square$

In order to lift the derived equivalence to an equivalence of Waldhausen categories, we need to use model structures on the category of chain complexes of $A$-$G$-modules. We refer to [Hovey 1999] for model structures and various related terms that we shall use here. Let $\mathcal{A}$ be a Grothendieck abelian category with enough projective objects and let $\mathrm{Ch}_{\mathcal{A}}$ denote the category of unbounded chain complexes over $\mathcal{A}$. Recall from [Hovey 2007, Proposition 7.4] that $\mathrm{Ch}_{\mathcal{A}}$ has the *projective model structure*, in which the weak equivalences are the quasi-isomorphisms, fibrations are termwise surjections and the cofibrations are the maps having the left lifting property with respect to fibrations which are also weak equivalences.

**Lemma 9.6.** *Let $E$ be a bounded above complex of projective objects in a Grothendieck abelian category $\mathcal{A}$ with enough projective objects. Then $E$ is cofibrant in the projective model structure on* $\mathrm{Ch}_{\mathcal{A}}$.

*Proof.* This is proved in [Hovey 1999, Lemma 2.3.6] in the case when $\mathcal{A}$ is the category of modules over a ring. The same proof goes through for any abelian category for which the projective model structure exists. $\square$

Given a datum $(R, G, A)$ as above, let $\mathrm{Ch}^G_{\mathrm{cc}}(A)$ denote the full subcategory of $\mathrm{Ch}^G(A)$ consisting of chain complexes which are compact and cofibrant (in projective model structure). For the notion of Waldhausen subcategories of a model category, see [Dugger and Shipley 2004, §3].

**Proposition 9.7.** *Let $(R, G, A)$ be as in Proposition 4.6. Then there is an inclusion* $\mathrm{Sperf}^G(A) \hookrightarrow \mathrm{Ch}^G_{\mathrm{cc}}(A)$ *of Waldhausen subcategories of* $\mathrm{Ch}^G(A)$ *such that the induced map on the $K$-theory spectra is a homotopy equivalence.*

*Proof.* It follows from the results of Section 4 that $\mathrm{Sperf}^G(A)$ is same as the category of bounded chain complexes of finitely generated projective objects of $(A$-$G)$-Mod. To check now that $\mathrm{Sperf}^G(A)$ and $\mathrm{Ch}^G_{\mathrm{cc}}(A)$ are Waldhausen subcategories of $\mathrm{Ch}^G(A)$, we only need to check that they are closed under taking push-outs. But this is true for the first category because every cofibration in $\mathrm{Ch}^G(A)$ is a termwise split injection with projective cokernels (see [Hovey 1999, Theorem 2.3.11]) and this is true for the second category because of the well-known fact that the cofibrations are closed under push-out and, if two vertices of a triangle in a triangulated category are compact, then so is the third.

To show that $\mathrm{Sperf}^G(A)$ is a subcategory of $\mathrm{Ch}^G_{\mathrm{cc}}(A)$, we have to show that every object of $\mathrm{Sperf}^G(A)$ is cofibrant and compact. The first property follows from Lemmas 4.3, 4.4 and 9.6. To prove compactness, we can use Proposition 4.6 to replace $(A\text{-}G)$-Mod by $\mathcal{R}$-Mod, where $\mathcal{R}$ is a ringoid. But, in this case, it is shown in [Keller 1994, §4.2] that a bounded complex of finitely generated projective objects of $\mathcal{R}$-Mod is compact.

To show that the inclusion $\mathrm{Sperf}^G(A) \hookrightarrow \mathrm{Ch}^G_{\mathrm{cc}}(A)$ induces a homotopy equivalence of $K$-theory spectra, we can use [Blumberg and Mandell 2011, Theorem 1.3] to reduce to showing that this inclusion induces an equivalence of the associated derived subcategories of $D^G(A)$. To do this, all we need to show is that every compact object of $D^G(A)$ is isomorphic to an object of $\mathrm{Sperf}^G(A)$. We have just shown above that every object of $\mathrm{Sperf}^G(A)$ is compact. It follows now from Lemma 9.5 and [Neeman 1996, Theorem 2.1] that every compact object of $D^G(A)$ comes from $\mathrm{Sperf}^G(A)$. Notice that we have shown in Lemmas 4.1 and 4.4 that the hypothesis of Lemma 9.5 is satisfied in our case. The proof of the proposition is now complete. □

For $i = 1, 2$, let $R_i$ be a commutative noetherian ring, $G_i$ an affine group scheme over $R_i$ and $A_i$ an $R_i$-$G_i$-algebra such that one of the following holds:

(1) $G_i$ is a diagonalizable group scheme over $R_i$.

(2) $R_i$ is a UFD containing a field of characteristic zero and $G_i$ is a split reductive group scheme over $R_i$.

We are now ready to prove the main result of this section.

**Theorem 9.8.** *Let $(R_1, G_1, A_1)$ and $(R_2, G_2, A_2)$ be as above. Then $D^{G_1}(A_1)$ and $D^{G_2}(A_2)$ are equivalent as triangulated categories if and only if $D^{G_1}(\mathrm{Perf}/A_1)$ and $D^{G_2}(\mathrm{Perf}/A_2)$ are equivalent as triangulated categories. In either case, the following hold*:

(1) *There is a homotopy equivalence of spectra $K^{G_1}(A_1) \simeq K^{G_2}(A_2)$.*

(2) *There is a homotopy equivalence of spectra $K'_{G_1}(A_1) \simeq K'_{G_2}(A_2)$.*

*Proof.* It follows from Lemmas 4.3 and 4.4 that the derived categories of perfect complexes are the same as the bounded derived categories of finitely generated projective objects. The first assertion of the theorem is now an immediate consequence of Proposition 4.6 and Theorem 9.3.

If $D^{G_1}(A_1)$ and $D^{G_2}(A_2)$ are equivalent as triangulated categories, it follows from Theorem 9.1 and Propositions 4.6 and 9.7 that there is a homotopy equivalence of spectra $K^{G_1}(A_1) \simeq K^{G_2}(A_2)$.

To prove (2), we first conclude from Proposition 4.6 and Theorem 9.3 that the equivalence of the derived categories is induced by a zigzag of Quillen equivalences between $\mathrm{Ch}^{G_1}(A_1)$ and $\mathrm{Ch}^{G_2}(A_2)$. It follows from Propositions 4.6 and A.1 that this derived equivalence induces an equivalence between the triangulated subcategories $D^{\mathrm{hb},-}(A_1\text{-}G_1\text{-proj})$ and $D^{\mathrm{hb},-}(A_2\text{-}G_2\text{-proj})$ of the corresponding derived categories. It follows that this zigzag of Quillen equivalences carries the Waldhausen subcategory $\mathrm{Ch}^{\mathrm{hb},-}(A_1\text{-}G_1\text{-proj})$ of $\mathrm{Ch}^{G_1}(A_1)$ onto the Waldhausen subcategory $\mathrm{Ch}^{\mathrm{hb},-}(A_2\text{-}G_2\text{-proj})$ of $\mathrm{Ch}^{G_2}(A_2)$. Furthermore, it follows from Proposition 4.6 and Lemma 9.6 that the objects of $\mathrm{Ch}^{\mathrm{hb},-}(A_1\text{-}G_1\text{-proj})$ and $\mathrm{Ch}^{\mathrm{hb},-}(A_2\text{-}G_2\text{-proj})$ are cofibrant objects for the projective model structure on the chain complexes. We can therefore apply Theorem 9.2 and (9.4) to conclude that there is a homotopy equivalence of spectra $K'_{G_1}(A_1)$ and $K'_{G_2}(A_2)$. This finishes the proof. □

**Remark 9.9.** If $G$ is a finite constant group scheme whose order is invertible in the base ring $R$, then one can check that the analogue of Theorem 9.8 is a direct consequence of Remark 4.7 and the main results of [Dugger and Shipley 2004].

## Appendix: Ringoid version of Rickard's theorem

In the proof of Theorem 9.8, we used the following ringoid (see Section 4) version of a theorem of Rickard [1989, Proposition 8.1] for rings. We shall say that a ringoid $\mathcal{R}$ is (right) coherent if every submodule of a finitely generated (right) $\mathcal{R}$-module is finitely generated. We say that $\mathcal{R}$ is complete if every $\mathcal{R}$-module is a filtered direct limit of its finitely generated submodules. We shall assume in our discussion that the ringoids are complete and right coherent. Given a ringoid $\mathcal{R}$, we have the following categories: Mod-$\mathcal{R}$ is the category of $\mathcal{R}$-modules; mod-$\mathcal{R}$ is the category of finitely generated $\mathcal{R}$-modules; Free-$\mathcal{R}$ (resp. free-$\mathcal{R}$) is the category of free (resp. finitely generated free) $\mathcal{R}$-modules; Proj-$\mathcal{R}$ (resp. proj-$\mathcal{R}$) is the category of projective (resp. finitely generated projective) $\mathcal{R}$-modules. Let $\mathrm{Ch}(-)$ denote the category of chain complexes and $D(-)$ denote the derived category of unbounded chain complexes. The superscripts $-$, b and hb denote the full subcategories of bounded above, bounded and cohomologically bounded chain complexes, respectively. $D(\text{Mod-}\mathcal{R})$ is denoted by $D(\mathcal{R})$.

Since every bounded above complex of finitely generated projective $\mathcal{R}$-modules has a resolution by a bounded above complex of finitely generated free modules, we see that there are equivalences of subcategories $D^-(\text{free-}\mathcal{R}) \simeq D^-(\text{proj-}\mathcal{R})$ and $D^{\mathrm{b}}(\text{mod-}\mathcal{R}) \simeq D^{\mathrm{hb},-}(\text{proj-}\mathcal{R})$. We shall say that two ringoids $\mathcal{R}$ and $\mathcal{S}$ are derived equivalent if there is an equivalence $D(\mathcal{R}) \simeq D(\mathcal{S})$ of triangulated categories. We shall say that a set $\mathbb{T}$ of objects in $D^{\mathrm{b}}(\text{proj-}\mathcal{R})$ is a set of *tiltors* if it generates $D(\mathcal{R})$ and $\mathrm{Hom}_{D(\mathcal{R})}(T, T'[n]) = 0$ unless $n = 0$ for any $T, T' \in \mathbb{T}$.

**Proposition A.1.** *Let $\mathcal{R}$ and $\mathcal{S}$ be ringoids which are derived equivalent. Then $D^{\mathrm{hb},-}(\text{proj-}\mathcal{R})$ and $D^{\mathrm{hb},-}(\text{proj-}\mathcal{S})$ are equivalent as triangulated categories.*

*Proof.* Any equivalence of triangulated categories $D(\mathcal{R})$ and $D(\mathcal{S})$ induces an equivalence of its compact objects and hence induces an equivalence between $D^{\mathrm{hb}}(\text{Mod-}\mathcal{R})$ and $D^{\mathrm{hb}}(\text{Mod-}\mathcal{S})$, because an object $X$ of $D(\mathcal{R})$ is in $D^{\mathrm{hb}}(\text{Mod-}\mathcal{R})$ if and only if, for every compact object $A$, one has $\mathrm{Hom}_{D(\mathcal{R})}(A, X[n]) = 0$ for all but finitely many $n$. Since $D^{\mathrm{hb},-}(\text{proj-}\mathcal{R}) = D^{-}(\text{proj-}\mathcal{R}) \cap D^{\mathrm{hb}}(\text{Mod-}\mathcal{R})$, the proposition is about showing that the triangulated categories $D^{-}(\text{proj-}\mathcal{R})$ and $D^{-}(\text{proj-}\mathcal{S})$ are equivalent.

This result was proven by Rickard [1989, Proposition 8.1] when $\mathcal{R}$ and $\mathcal{S}$ are both rings. We only explain here how Rickard's proof goes through even for ringoids without further changes. The completeness assumption and our hypotheses together imply that the triangulated categories $D^{-}(\text{Proj-}\mathcal{R})$ and $D^{-}(\text{Proj-}\mathcal{S})$ are equivalent. It follows from [Dugger and Shipley 2004, Theorem 7.5] that this induces an equivalence of the triangulated subcategories $D^{\mathrm{b}}(\text{proj-}\mathcal{R})$ and $D^{\mathrm{b}}(\text{proj-}\mathcal{S})$. Let $S$ denote the set of images of the objects of $\mathcal{S}$ (the representable objects of $\mathcal{S}$-Mod) under this equivalence and let $\mathcal{T} := \mathrm{End}(S)$ denote the full subcategory of $D^{\mathrm{b}}(\text{proj-}\mathcal{R})$ consisting of objects in $S$. One easily checks that $S$ is a set of tiltors such that $\mathrm{End}(S) \simeq \mathcal{S}$ as ringoids (see [Dugger and Shipley 2004, Theorem 7.5]).

Rickard constructs (in the case of rings) a functor $F : D^{-}(\text{Proj-}\mathcal{T}) \to D^{-}(\text{Proj-}\mathcal{R})$ of triangulated categories which is an equivalence and shows that it induces an equivalence between $D^{-}(\text{proj-}\mathcal{T})$ and $D^{-}(\text{proj-}\mathcal{R})$. We recall his construction, which works for ringoids as well. The functor $\mathrm{Hom}_{D(\mathcal{R})}(\mathcal{T}, -)$ from $D^{-}(\text{Proj-}\mathcal{R})$ to $\mathcal{T}$-Mod induces an equivalence between the direct sums of objects of $\mathcal{T}$ and free objects of $\mathcal{T}$-Mod. Moreover, the completeness assumption on $\mathcal{S}$ implies that the inclusion $\mathrm{Ch}^{-}(\text{Free-}\mathcal{T}) \to \mathrm{Ch}^{-}(\text{Proj-}\mathcal{T})$ induces an equivalence of their homotopy categories. One is thus reduced to constructing a functor from the category $D^{-}(\text{Free-}\mathcal{T})$ of bounded above chain complexes of direct sums of copies of objects in $S$ to $D^{-}(\text{Proj-}\mathcal{R})$ with the requisite properties.

An object $X$ of $D^{-}(\text{Free-}\mathcal{T})$ consists of a bigraded object $X = (X^{**}, d, \delta)$ of projective $\mathcal{R}$-modules such that each row is a chain complex of objects which are direct sums of objects in $S$ but the columns are not necessarily chain complexes. The goal is then to modify the differentials of $X^{**}$ so that it becomes a double complex and then one defines $F(X)$ to be the total complex of $X^{**}$, which is an object of $D^{-}(\text{Proj-}\mathcal{R})$.

In order to modify the differentials of $X^{**}$, Rickard uses his Lemma 2.3, whose proof works in the ringoid case if we know that $\mathrm{Hom}_{D(\mathcal{R})}(T, T'[n]) = 0$ unless $n = 0$ for any $T, T' \in S$. But this is true in our case as $S$ is a set of tiltors. The rest of [Rickard 1989, §2] shows how one can indeed modify $X^{**}$ to get a double complex under this assumption. The point of the other sections is to show how this yields an equivalence of triangulated categories, which only uses the requirement that $S$ is a set of tiltors and, in particular, it generates $D^{\mathrm{b}}(proj{-}\mathcal{R})$ and hence $D(\mathcal{R})$.

Finally, the functor $F$ will take $D^{-}(\text{proj-}\mathcal{T})$ to $D^{-}(\text{proj-}\mathcal{R})$ if $F(\mathrm{Tot}(X^{**}))$ is a bounded above complex of finitely generated projective $\mathcal{R}$-modules whenever each row of $X^{**}$ is a finite direct sum of objects in $S$. But this is obvious because each object of $S$ is a bounded complex of finitely generated projective $\mathcal{R}$-modules. □

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# Motivic complexes over nonperfect fields

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We show that the theory of motivic complexes developed by Voevodsky over perfect fields works over nonperfect fields as well provided that we work with sheaves with transfers of $\mathbb{Z}[1/p]$-modules ($p = \operatorname{char} F$). In particular we show that every homotopy invariant sheaf with transfers of $\mathbb{Z}[1/p]$-modules is strictly homotopy invariant.

## 0. Introduction

Voevodsky defined the category of motives over an arbitrary perfect field. The main reason why the same construction does not work over arbitrary fields is that we do not know whether the theorem of Voevodsky, which states that every homotopy invariant Nisnevich sheaf with transfers is strictly homotopy invariant, holds over nonperfect fields. This makes life fairly inconvenient because when we start with a field $F$ of characteristic $p > 0$ and take a look on the function field $F(S)$ of a smooth scheme of finite type over $F$ we never get a perfect field unless $\dim S = 0$.

The main purpose of this paper is to show that the above theorem holds over a nonperfect field of characteristic $p$ provided that we work with sheaves with transfers of $\mathbb{Z}[1/p]$-modules. In Section 1 we use the theory of Frobenius twists to show that extension of scalars from $F$ to its perfect closure $F^{1/p^\infty}$ defines an equivalence on the category $\mathrm{PT}_p$ of presheaves with transfers of $\mathbb{Z}[1/p]$-modules. In Section 2 we show that the above functor takes sheaves to sheaves, and the resulting functor on the category $\mathrm{NST}_p$ of Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules is an equivalence as well. Finally we show that this functor preserves cohomology. In Section 3 we show that the extension of scalars functor takes homotopy invariant presheaves to homotopy invariant presheaves and hence every homotopy invariant Nisnevich sheaf with transfers of $\mathbb{Z}[1/p]$-modules over $F$ is strictly homotopy invariant. This fact readily implies that all results concerning homotopy invariant presheaves with transfers proved by Voevodsky for perfect fields are true over arbitrary fields once we work with sheaves of $\mathbb{Z}[1/p]$-modules. In particular we may define the motivic category $\mathrm{DM}_p^-(F)$ of effective motives in the standard way as

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the category of bounded above complexes of Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules with homotopy invariant cohomology sheaves.

In Section 4 we discuss in some detail the extension of scalars for presheaves with transfers of $\mathbb{Z}[1/p]$-modules in the case of an arbitrary field extension $F \subset E$. We show that this functor takes sheaves to sheaves, takes homotopy invariant sheaves to homotopy invariant sheaves and is exact. This allows us to define the extension of scalars functor $\phi^* : \mathrm{DM}_p^-(F) \to \mathrm{DM}_p^-(E)$ in the most straightforward way — just applying the extension of scalars functor pointwise. We show also that this functor takes tensor products to tensor products and happens to be an equivalence of categories if $E = F^{1/p^\infty}$.

In Section 5 we show that extension of scalars commutes with the internal Hom functor.

## 1. Presheaves with transfers over nonperfect fields

For a field $F$ we denote by $\mathrm{Cor}_F$ the category of finite correspondences over $F$ and by $\mathrm{PT}(F)$ the category of presheaves with transfers over $F$, i.e., additive contravariant functors $\mathrm{Cor}_F \to \mathrm{Ab}$; see [Mazza et al. 2006] for definitions. We use the notation NST for the category of Nisnevich sheaves with transfers. Let $E/F$ be a field extension. In this case we have an obvious extension of scalars functor $\phi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ taking $X$ to $X_E$ and $Z \in \mathrm{Cor}(X, Y)$ to $Z_E \in \mathrm{Cor}(X_E, Y_E)$. Taking the composition with $\phi$ we get a direct image functor $\phi_* : \mathrm{PT}(E) \to \mathrm{PT}(F)$. The functor $\phi_*$ is obviously exact and preserves direct sums and direct products. In particular, $\phi_*$ is continuous and hence has a left adjoint $\phi^\#$; see [MacLane 1971].

**Proposition 1.1.** (1) *The functor $\phi^\#$ is uniquely characterized by the following properties*:

(a) *$\phi^\#$ is right exact.*
(b) *$\phi^\#$ preserves arbitrary direct sums.*
(c) *$\phi^\#(\mathbb{Z}_{\mathrm{tr}}(X)) = \mathbb{Z}_{\mathrm{tr}}(X_E)$.*

(2) *The functor $\phi^\#$ is given by the formula*

$$\phi^\#(\mathcal{F}) = \mathrm{Coker}\Bigg( \bigoplus_{s \in \mathrm{Cor}_F(X, X_1)} \mathbb{Z}_{\mathrm{tr}}(X_E) \otimes \mathcal{F}(X_1) \to \bigoplus_{X \in \mathrm{Sm}_F} \mathbb{Z}_{\mathrm{tr}}(X_E) \otimes \mathcal{F}(X) \Bigg).$$

*Proof.* Note that every left adjoint functor is right exact and preserves direct sums. For any presheaf with transfers $\mathcal{M} \in \mathrm{PT}(E)$ we have

$$\begin{aligned}\mathrm{Hom}(\phi^\#(\mathbb{Z}_{\mathrm{tr}}(X), \mathcal{M})) &= \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(X), \phi_*(\mathcal{M})) \\ &= \phi_*(\mathcal{M})(X) = \mathcal{M}(X_E) = \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(X_E), \mathcal{M}).\end{aligned}$$

This proves the last formula. Assume now that $a : \mathrm{PT}(F) \to \mathrm{PT}(E)$ is a functor having the above three properties. Note that in this case we also have the following property: for any $\mathcal{F} \in \mathrm{PT}(F)$ and any abelian group $A$ we have a natural isomorphism,

$$a(\mathcal{F} \otimes A) = a(\mathcal{F}) \otimes A.$$

In fact the case when $A$ is a free abelian group is clear since $a$ preserves direct sums. The general case follows in view of right exactness of $a$. Note now that any presheaf with transfers $\mathcal{F} \in \mathrm{PT}(F)$ has a canonical resolution

$$\bigoplus_{s \in \mathrm{Cor}_F(X,X_1)} \mathbb{Z}_{\mathrm{tr}}(X) \otimes \mathcal{F}(X_1) \to \bigoplus_{X \in \mathrm{Sm}_F} \mathbb{Z}_{\mathrm{tr}}(X) \otimes \mathcal{F}(X) \to \mathcal{F} \to 0.$$

Applying the functor $a$ to this presentation and using the right exactness of $a$ we get the presentation for $a(\mathcal{F})$. This proves that

$$a(\mathcal{F}) = \mathrm{Coker}\Bigl( \bigoplus_{s \in \mathrm{Cor}_F(X,X_1)} \mathbb{Z}_{\mathrm{tr}}(X_E) \otimes \mathcal{F}(X_1) \to \bigoplus_{X \in \mathrm{Sm}_F} \mathbb{Z}_{\mathrm{tr}}(X_E) \otimes \mathcal{F}(X) \Bigr). \quad \square$$

Recall that a presheaf with transfers is called free in case it is a direct sum of presheaves of the form $\mathbb{Z}_{\mathrm{tr}}(X)$. Since all schemes in question are Noetherian it follows that every such presheaf is actually a sheaf in the Nisnevich topology. Later we'll need the following additional properties of the functor $\phi^{\#}$:

**Lemma 1.2.** (1) *The functor $\phi^{\#}$ takes free presheaves to free presheaves.*

(2) *For any presheaves with transfers $\mathcal{F}$, $\mathcal{G} \in \mathrm{PT}(F)$ we have a natural isomorphism $\phi^{\#}(\mathcal{F} \otimes_{\mathrm{tr}}^{\mathrm{pr}} \mathcal{G}) = \phi^{\#}(\mathcal{F}) \otimes_{\mathrm{tr}}^{\mathrm{pr}} \phi^{\#}(\mathcal{G})$* (here $\otimes_{\mathrm{tr}}^{\mathrm{pr}}$ stands for the tensor product operation in the category of presheaves with transfers; see [Suslin and Voevodsky 2000]).

*Proof.* The first claim is obvious. To prove the second one we note that, according to the defining properties of the tensor product operation, to compute $\mathcal{F} \otimes_{\mathrm{tr}}^{\mathrm{pr}} \mathcal{G}$ we may start with arbitrary free presentations

$$\bigoplus_i \mathbb{Z}_{\mathrm{tr}}(X_i) \xrightarrow{p} \bigoplus_j \mathbb{Z}_{\mathrm{tr}}(Y_j) \to \mathcal{F} \to 0 \quad \text{and} \quad \bigoplus_s \mathbb{Z}_{\mathrm{tr}}(X_s) \xrightarrow{q} \bigoplus_t \mathbb{Z}_{\mathrm{tr}}(Y_t) \to \mathcal{G} \to 0,$$

in which case $\mathcal{F} \otimes_{\mathrm{tr}}^{\mathrm{pr}} \mathcal{G}$ coincides with the cokernel of the resulting map

$$\bigoplus_{i,t} \mathbb{Z}_{\mathrm{tr}}(X_i \times Y_t) \oplus \bigoplus_{j,s} \mathbb{Z}_{\mathrm{tr}}(Y_j \times X_s) \xrightarrow{p \otimes 1 - 1 \otimes q} \bigoplus_{j,t} \mathbb{Z}_{\mathrm{tr}}(Y_j \times Y_t).$$

Applying the functor $\phi^{\#}$ to the above presentations we get free presentations of $\phi^{\#}(\mathcal{F})$ and $\phi^{\#}(\mathcal{G})$, and the same computation as above yields our claim. $\square$

Note that $\phi_*$ takes Zariski (resp. Nisnevich) sheaves with transfer over $E$ to Zariski (resp. Nisnevich) sheaves with transfers over $F$. However it's not clear whether the functor $\phi^\#$ takes sheaves to sheaves, so working with Nisnevich sheaves we need to sheafify $\phi^\#(\mathcal{F})$ in Nisnevich topology. The resulting sheaf with transfers will be denoted $\phi^*(\mathcal{F})$. Sometimes we denote $\phi^*(\mathcal{F})$ by $\mathcal{F} \otimes_F E$ or $\mathcal{F}_E$ and call it the extension of scalars in $\mathcal{F}$.

**Corollary 1.3.** (1) *For any presheaf with transfers $\mathcal{F}$ we have a natural isomorphism $\phi^\#(\mathcal{F})_{\mathrm{Nis}} = \phi^*(\mathcal{F}_{\mathrm{Nis}})$.*

(2) *For any sheaves with transfers $\mathcal{F}, \mathcal{G} \in \mathrm{NST}(F)$ we have a natural isomorphism $\phi^*(\mathcal{F} \otimes_{\mathrm{tr}} \mathcal{G}) = \phi^*(\mathcal{F}) \otimes_{\mathrm{tr}} \phi^*(\mathcal{G})$.*

*Proof.* The first claim is clear since composition of left adjoints coincides with the left adjoint to the composition. The second claim follows from the first one and the previous lemma. □

Assume now that $F$ is a field of positive characteristic $p$ and take $E = F^{1/p}$. The field $E$ may be identified with $F$ via the Frobenius isomorphism, so that the category $\mathrm{Cor}_E$ identifies with $\mathrm{Cor}_F$ and the resulting functor $\phi$ identifies with the Frobenius twist functor $X \mapsto X^{(1)} = X \otimes_f F$, where the subscript $f$ indicates that we view $F$ as $F$-algebra via the Frobenius embedding $f : F \to F,\ x \mapsto x^p$. The important property of the Frobenius twist functor is the presence, for any $X \in \mathrm{Sch}/F$, of the canonical Frobenius morphism $\Phi_X : X \to X^{(1)}$ (see, for example, [Friedlander and Suslin 1997]), which in the affine case corresponds to the $F$-algebra homomorphism $A \otimes_f F \to A,\ a \otimes \lambda \mapsto \lambda \cdot a^p$. The following elementary lemma sums up some of the properties of the Frobenius map:

**Lemma 1.4.** (1) *$\Phi_X : X \to X^{(1)}$ is a natural transformation of functors from $\mathrm{Sch}/F$ to itself, i.e., for a morphism $f : X \to Y$ we get a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \Phi_X}\downarrow & & \downarrow{\scriptstyle \Phi_Y} \\ X^{(1)} & \xrightarrow{f^{(1)}} & Y^{(1)} \end{array}$$

(2) $\Phi_{X \times Y} = \Phi_X \times \Phi_Y$.

(3) *$\Phi_X$ is a finite surjective morphism for any $X$. If $X/F$ is a smooth irreducible scheme of dimension $d$, the morphism $\Phi_X : X \to X^{(1)}$ is a finite flat purely inseparable morphism of degree $p^d$.*

Let $f : X \to Y$ be a finite surjective morphism of irreducible schemes. In this case the graph $\Gamma_f \subset X \times Y$ is finite over $Y$ and so its transpose $\Gamma_f^T \subset Y \times X$ defines a finite correspondence from $Y$ to $X$ (which we'll call $f^T$).

**Theorem 1.5.** *The family of maps $\Phi_X : X \to X^{(1)}$ defines a natural transformation of functors from* $\mathrm{Cor}_F$ *to itself*, $\Phi : \mathrm{Id}_{\mathrm{Cor}_F} \to \phi$. *Moreover, after inverting $p$ this natural transformation becomes an isomorphism of functors with inverse given by the family of maps*

$$\frac{1}{p^{\dim X}} \Phi_X^T : X^{(1)} \to X.$$

The proof is based on several easy but useful lemmas.

**Lemma 1.6.** *Let $f : X \to Y$ be a finite flat purely inseparable morphism of smooth irreducible schemes of the same dimension. Then the maps $f_* : Z^i(X) \to Z^i(Y)$ and $f^* : Z^i(Y) \to Z^i(X)$ are defined on all cycles and both compositions coincide with multiplication by* $\deg f$.

*Proof.* Note that the operation $f_*$ is defined on all cycles for any proper morphism and the operation $f^*$ is defined on all cycles for any flat morphism. Furthermore, the composition $f_* \circ f^*$ coincides with multiplication by $\deg f$ in view of the projection formula. Finally, since $f$ is finite and purely inseparable, it is injective. Hence $f_*$ is equally injective and, since $f_* \circ f^* \circ f_* = \deg f \cdot f_*$, we conclude that $f^* \circ f_* = \deg f$ □

We will also need the following well-known and elementary fact:

**Lemma 1.7.** *Let $Z \in \mathrm{Cor}(X, Y)$ be a finite correspondence from a* (*smooth*) *scheme $X$ to a* (*smooth*) *scheme $Y$. Let further $f : Y \to Y'$ and $g : X' \to X$ be morphisms of* (*smooth*) *schemes. Then*:

(1) $f \circ Z \in \mathrm{Cor}(X, Y')$ *coincides with* $(1_X \times f)_*(Z)$.

(2) $Z \circ g \in \mathrm{Cor}(X', Y)$ *coincides with* $(g \times 1_Y)^*(Z)$.

**Corollary 1.8.** (a) *Let $f : X \to Y$ be a finite surjective morphism of smooth irreducible schemes. Then $f \circ f^T = \deg f$.*

(b) *Assume in addition that $f$ is purely inseparable. Then $f^T \circ f = \deg f$.*

*Proof.* Note that for any morphism $f : X \to Y$ the cycle $\Gamma_f \subset X \times Y$ coincides with $(1_X, f)_*(X) = (1_X \times f)_*(\Delta_X)$ and hence $\Gamma_f^T = (f, 1_X)_*(X) = (f \times 1_X)_*(\Delta_X)$. Lemma 1.7 shows now that $f \circ f^T = (1_Y \times f)_*(\Gamma_f^T) = (1_Y \times f)_*(f, 1_X)_*(X) = (f, f)_*(X) = (\Delta_Y \circ f)_*(X) = (\Delta_Y)_*(\deg f \cdot Y) = \deg f \cdot \Delta_Y$ To prove the second claim we note first that $f$ is flat, so that we may apply Lemma 1.6. Applying Lemma 1.7 once again we get

$$f^T \circ f = (f \times 1_X)^*(\Gamma_f^T) = (f \times 1_X)^*(f \times 1_X)_*(\Delta_X) = \deg f \cdot \Delta_X. \qquad \square$$

**Proposition 1.9.** *Let $X$ be a smooth equidimensional scheme and let $Z \subset X$ be an equidimensional cycle of dimension $d$. Then $(\Phi_X)_*(Z) = p^d \cdot Z^{(1)}$.*

*Proof.* It suffices, clearly, to treat the case when $Z$ is irreducible, i.e., is represented by a closed integral subscheme $Z \subset X$. In this case the cycle $Z^{(1)}$ is defined by the closed subscheme $Z^{(1)} \subset X^{(1)}$, which is irreducible but which however need not be reduced. Thus, denoting the integral scheme $Z^{(1)}_{\mathrm{red}}$ by $Z'$ we see that the cycle $Z^{(1)}$ equals $l \cdot Z'$, where $l$ is the length of the local Artinian ring $F(Z) \otimes_F F^{1/p} = F(Z) \otimes_f F$. Furthermore, Lemma 1.4 shows that $\Phi_X(Z) = Z'$ and hence $(\Phi_X)_*(Z) = [F(Z) : F(Z')] \cdot Z'$. Thus we only need to check the formula

$$[F(Z) : F(Z')] = l \cdot p^d.$$

Let $x_1, \dots, x_d$ be the transcendence basis of $F(Z)$ over $F$. Note that dimension of $F(Z) \otimes_f F$ over $F(x_1, \dots, x_d) \otimes_f F = F(x_1, \dots, x_d)$ equals $[F(Z) : F(x_1, \dots, x_d)]$ and hence $[F(Z) : F(x_1, \dots, x_d)] = l \cdot [F(Z') : F(x_1, \dots, x_d)]$. The field embedding $F(Z') \hookrightarrow F(Z)$ takes $x_i$ to $x_i^p$. Thus identifying $F(Z')$ with a subfield of $F(Z)$ we identify its subfield $F(x_1, \dots, x_d)$ with the subfield $F(x_1^p, \dots, x_d^p) \subset F(Z)$. So we get

$$\begin{aligned} l \cdot p^d &= \frac{[F(Z) : F(x_1, \dots, x_d)]}{[F(Z') : F(x_1^p, \dots, x_d^p)]} \cdot [F(x_1, \dots, x_d) : F(x_1^p, \dots, x_d^p)] \\ &= \frac{[F(Z) : F(x_1^p, \dots, x_d^p)]}{[F(Z') : F(x_1^p, \dots, x_d^p)]} = [F(Z) : F(Z')]. \end{aligned}$$ □

*Proof of Theorem 1.5.* We first show that $\Phi$ is a natural transformation of functors from $\mathrm{Cor}_F$ to itself. Let $Z \in \mathrm{Cor}(X, Y)$ be a finite correspondence from a smooth, irreducible scheme $X$ to a (smooth, irreducible) $Y$. We have to verify the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "Z"] \arrow[d, "\Phi_X"'] & Y \arrow[d, "\Phi_Y"] \\ X^{(1)} \arrow[r, "Z^{(1)}"] & Y^{(1)} \end{tikzcd}$$

By Lemma 1.7, $\Phi_Y \circ Z = (1_X \times \Phi_Y)_*(Z)$ and $Z^{(1)} \circ \Phi_X = (\Phi_X \times 1_{Y^{(1)}})^*(Z^{(1)})$. Thus we need to establish the relation $(1_X \times \Phi_Y)_*(Z) = (\Phi_X \times 1_{Y^{(1)}})^*(Z^{(1)})$. Applying to both sides the injective map $(\Phi_X \times 1_{Y^{(1)}})_*$ and using Lemma 1.6 we see that the formula in question takes the form $(\Phi_X \times \Phi_Y)_*(Z) = p^{\dim X} \cdot Z^{(1)}$. Since $\Phi_X \times \Phi_Y = \Phi_{X \times Y}$ we conclude from Proposition 1.9 that the left-hand side equals $p^{\dim Z} \cdot Z^{(1)}$. It suffices to note now that $\dim X = \dim Z$.

To finish the proof it suffices to establish that, for any smooth irreducible $X$,

$$\Phi_X \circ \Phi_X^T = p^{\dim X} \cdot \Delta_{X^{(1)}} \quad \text{and} \quad \Phi_X^T \circ \Phi_X = p^{\dim X} \cdot \Delta_X.$$

However, these formulae follow from Corollary 1.8. □

Denote by $\mathrm{Cor}_F[1/p]$ the category with same objects as $\mathrm{Cor}_F$ (i.e., all smooth schemes of finite type over $F$) but whose morphisms are obtained from those of $\mathrm{Cor}_F$ by inverting $p$.

**Corollary 1.10.** *The Frobenius twist functor induces an equivalence*

$$\mathrm{Cor}_F[1/p] \to \mathrm{Cor}_F[1/p], \quad X \mapsto X^{(1)}.$$

*Proof.* According to Theorem 1.5, the Frobenius twist functor is isomorphic to the identity functor and hence is an equivalence. □

Set $F_n = F^{1/p^n}$ and let $F_\infty = F^{1/p^\infty} = \varinjlim F_n$ be the perfect closure of $F$.

**Theorem 1.11.** *Extension of scalars defines an equivalence of categories*

$$\phi : \mathrm{Cor}_F[1/p] \to \mathrm{Cor}_{F_\infty}[1/p].$$

*Proof.* Corollary 1.10 shows that extension of scalars from $F$ to $F_1$ gives an equivalence of categories on $\mathrm{Cor}[1/p]$. Induction on $n$ implies that the same is true for the extension of scalars from $F$ to $F_n$. Note further that, for $X \in \mathrm{Sm}_F$, every closed subscheme in $X_{F_\infty}$ is defined over $F_n$ for appropriate $n$. Hence, for any $X, Y \in \mathrm{Sm}_F$ we have

$$\mathrm{Cor}(X_{F_\infty}, Y_{F_\infty}) = \varinjlim_n \mathrm{Cor}(X_{F_n}, Y_{F_n}).$$

Thus the extension of scalars map

$$\mathrm{Cor}(X, Y)[1/p] \to \mathrm{Cor}(X_{F_\infty}, Y_{F_\infty})[1/p]$$

is an isomorphism. It remains to show that the functor $\phi$ is essentially surjective, i.e., every object of $\mathrm{Cor}_{F_\infty}[1/p]$ is isomorphic to $X_{F_\infty}$ for appropriate $X$. However, this follows from Corollary 1.8 and the following result:

**Lemma 1.12.** *Let $Z \in \mathrm{Sm}_{F_\infty}$ be a smooth irreducible scheme. Then there exists a smooth irreducible scheme $X \in \mathrm{Sm}_F$ and a finite, surjective, purely inseparable morphism $Z \to X_{F_\infty}$.*

*Proof.* Since every scheme of finite type over $F_\infty$ is defined over $F_n$ for sufficiently large $n$, we easily conclude that there exists a smooth irreducible scheme $Y \in \mathrm{Sm}_{F_n}$ such that $Z$ is isomorphic to $Y_{F_\infty}$. In the case $Y$ that is defined over $F$, the scheme $Z$ is also defined over $F$ and we have nothing to prove. In the general case we may identify $F_n$ with $F$ via the $n$-th power of the Frobenius isomorphism. In this way, $Y$ defines a smooth irreducible scheme $X$ over $F$. As we pointed out before, the scheme $X_{F_n}$, viewed as a scheme over $F$, coincides with the $n$-th Frobenius twist $X^{(n)}$. The $n$-th power of the Frobenius morphism

$$\Phi_X^n : X \to X^{(n)}$$

is a finite, surjective, purely inseparable morphism of degree $p^{n\cdot\dim X}$. When we return from $F$ to $F_n$, the morphism $\Phi_X^n$ defines a finite, surjective, purely inseparable morphism $Y \to X_{F_n}$. Extending scalars from $F_n$ to $F_\infty$, we get the required morphism $Z \to X_{F_\infty}$. $\square$

We are going to use the notation $\mathrm{PT}_p(F)$ for the category of the presheaves with transfers of $\mathbb{Z}[1/p]$-modules (i.e., additive functors from $\mathrm{Cor}_F$ to $\mathbb{Z}[1/p]$-mod). Note that presheaves with transfers of $\mathbb{Z}[1/p]$-modules may be identified with presheaves with transfers, all of whose groups of sections are uniquely $p$-divisible, so that $\mathrm{PT}_p(F) \subset \mathrm{PT}(F)$.

**Theorem 1.13.** *The direct image functor*

$$\phi_* : \mathrm{PT}_p(F_\infty) \to \mathrm{PT}_p(F)$$

*is an equivalence of categories with quasi-inverse $\phi^\#$.*

*Proof.* Note first that, for any additive category $\mathbb{C}$, every additive functor $\mathcal{M}$ : $\mathbb{C} \to \mathbb{Z}[1/p]$-mod extends uniquely to an additive functor $\mathbb{C}[1/p] \to \mathbb{Z}[1/p]$-mod. Thus the category $\mathrm{PT}_p(F)$ may be identified with the category of additive functors from $\mathrm{Cor}_F[1/p]$ to $\mathbb{Z}[1/p]$-mod. Our first claim follows now immediately from Theorem 1.11. Since $\phi^\#$ is left adjoint to an equivalence $\phi_*$, we conclude that it coincides with the quasi-inverse equivalence. $\square$

**Corollary 1.14.** *Let $E/F$ be any purely inseparable field extension. Denote by $\psi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. Then $\psi_* : \mathrm{PT}_p(E) \to \mathrm{PT}_p(F)$ is an equivalence of categories with quasi-inverse $\psi^\#$.*

*Proof.* Note that $E \subset F_\infty$ and, moreover, $F_\infty = E_\infty$. Denote by $\phi'$ the extension of scalars functor corresponding to the field extension $E \subset E_\infty$. Our claim follows from Theorem 1.13 and the commutative diagram

$$\begin{array}{ccc} \mathrm{PT}_p(E_\infty) & \xrightarrow{=} & \mathrm{PT}_p(F_\infty) \\ {\scriptstyle \phi'}\big\downarrow & & \big\downarrow{\scriptstyle \phi_*} \\ \mathrm{PT}_p(E) & \xrightarrow{\psi_*} & \mathrm{PT}_p(F) \end{array}$$

$\square$

In the next section we'll need the following result:

**Lemma 1.15.** *Let $\mathcal{F} \in \mathrm{PT}_p(F)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules. Let, further, $f : X \to Y$ be a finite, surjective, purely inseparable morphism of irreducible smooth schemes. Then the homomorphism $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ is an isomorphism.*

*Proof.* It suffices to note that in $\mathrm{Cor}_F$ we have a morphism $f^T : Y \to X$, which yields a homomorphism $(f^T)^* : \mathcal{F}(X) \to \mathcal{F}(Y)$ and both compositions of $f^*$ and $(f^T)^*$ are equal (according to Corollary 1.8) to multiplication by $\deg f$. $\square$

## 2. Sheaves with transfers over nonperfect fields

We keep the notations introduced in the previous section. In particular we denote by $E = F_\infty$ the perfect closure of $F$. We denote by $\phi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. Finally we use the notation

$$\phi_* : \mathrm{PT}_p(F_\infty) \to \mathrm{PT}_p(F) \quad \text{and} \quad \phi^\# : \mathrm{PT}_p(F) \to \mathrm{PT}_p(F_\infty)$$

for the corresponding functors on presheaves with transfers of $\mathbb{Z}[1/p]$-modules.

**Theorem 2.1.** *Let $\mathcal{F} \in \mathrm{PT}_p(F_\infty)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules. Assume that $\phi_*(\mathcal{F})$ is a Zariski (resp. Nisnevich) sheaf. Then $\mathcal{F}$ itself is also a Zariski (resp. Nisnevich) sheaf.*

We start with the case of Zariski sheaves, which is somewhat more transparent.

**Lemma 2.2.** *Let $\pi : Y \to X$ be an integral surjective morphism of integral schemes. Assume further that the scheme $X$ is normal and the extension of rational function fields is purely inseparable. Then $\pi$ is a homeomorphism.*

*Proof.* Since $\pi$ is integral we conclude that it is a closed map. Thus it suffices to establish that $\pi$ is bijective. Our conditions show that $\pi$ is surjective and we only need to verify the injectivity of $\pi$. Replacing $X$ by an open affine scheme and $Y$ by its inverse image, we see that it suffices to consider the case when $X = \mathrm{Spec}\, A$ ($A$ is an integrally closed domain) and $Y = \mathrm{Spec}\, B$ is an affine integral scheme. Surjectivity of $\pi$ readily implies that $\pi^* : A \to B$ is injective. Denote by $F$ (resp. $E$) the field of fractions of $A$ (resp. $B$). Since $E/F$ is purely inseparable and $A$ is integrally closed there is exactly one prime ideal in $B$ over each prime ideal of $A$ — see [Bourbaki 1972] — which means that $\pi$ is bijective. □

**Corollary 2.3.** *Under the conditions and notation of Lemma 2.2, let $\mathcal{F}$ be a presheaf on the small Zariski site of $Y$. If $\pi_*(\mathcal{F})$ is a sheaf, $\mathcal{F}$ itself is a sheaf as well.*

Let $\mathcal{F}$ be a presheaf on $\mathrm{Sm}_F$. For any $X \in \mathrm{Sm}_F$, restricting $\mathcal{F}$ to the small Zariski (resp. Nisnevich) site of $X$ we get a presheaf $\mathcal{F}_X$ on $X_{\mathrm{Zar}}$ (resp. $X_{\mathrm{Nis}}$). Moreover, for any morphism $f : X \to Y$ we get a canonical homomorphism $\alpha_f : \mathcal{F}_Y \to f_*(\mathcal{F}_X)$. Finally, for a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$,

$$\alpha_{gf} = g_*(\alpha_f) \circ \alpha_g.$$

**Lemma 2.4.** *Let $\mathcal{F} \in \mathrm{PT}_p(F)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules. Let, further, $f : X \to Y$ be a finite surjective purely inseparable morphism of smooth schemes. Then the associated homomorphism*

$$\alpha_f : \mathcal{F}_Y \to f_*(\mathcal{F}_X)$$

*is an isomorphism (for both the Nisnevich and Zariski topologies).*

*Proof.* We need to show that for any $U \in Y_{\mathrm{Nis}}$ (resp. $U \in Y_{\mathrm{Zar}}$) the canonical map

$$\mathcal{F}(U) \xrightarrow{p_2^*} \mathcal{F}(X \times_Y U)$$

is an isomorphism. It suffices to treat the case when $U$ is irreducible, in which case (since $f$ is purely inseparable ) $X \times_Y U$ is also irreducible. Now our claim follows from Lemma 1.15, since $p_2$ is a finite, surjective, purely inseparable morphism. □

*Proof of Theorem 2.1 for the Zariski topology.* To show that $\mathcal{F}$ is a Zariski sheaf we have to verify that for any $Y \in \mathrm{Sm}_E$ the restriction $\mathcal{F}_Y$ is a sheaf. It suffices, clearly, to treat the case when $Y$ is irreducible. Assume first that $Y = X_E$ for an appropriate $X \in \mathrm{Sm}_F$. Let $\pi : Y \to X$ be the structure morphism. Note that $\pi_*(\mathcal{F}_Y) = (\phi_*(\mathcal{F}))_X$ and hence $\pi_*(\mathcal{F}_Y)$ is a sheaf. Since $\pi : Y \to X$ is an integral, surjective, purely inseparable morphism of integral normal schemes we conclude by Corollary 2.3 that $\mathcal{F}_Y$ is a sheaf. In the general case, we use Lemma 1.12 and find a finite surjective purely inseparable morphism $Y \xrightarrow{f} X_E$. Lemma 2.4 shows that $\alpha_f : \mathcal{F}_{X_E} \to f_*(\mathcal{F}_Y)$ is an isomorphism. Thus $f_*(\mathcal{F}_Y)$ is a sheaf. Applying Corollary 2.3 again, we conclude that $\mathcal{F}_Y$ is a sheaf. □

**Lemma 2.5.** *Let $\mathcal{O}$ be a henselian local ring. Let, further, $A$ be an integral $\mathcal{O}$-algebra. If $A$ is local it is a local henselian ring.*

*Proof.* Assume first that $A$ is finite over $\mathcal{O}$. Every finite $A$-algebra happens to be a finite $\mathcal{O}$-algebra and hence is a finite product of local rings. However, this property characterizes henselian local rings — see [Milne 1980] — so we conclude that $A$ is henselian. In the general case, let $B$ be any finitely generated subalgebra of $A$. Since $A$ is integral over $B$, we conclude that there is a maximal ideal of $A$ over each maximal ideal of $B$. Thus $B$ has to be local and hence has to be a henselian local ring. The same reasoning shows that whenever $B \subset B'$ the inclusion $B \hookrightarrow B'$ is a local homomorphism. Now it suffices to use the following lemma. □

**Lemma 2.6.** *Let $\{B_i\}_{i \in I}$ be a filtering direct system of local rings and local homomorphisms. If all $B_i$ are henselian then $A = \varinjlim_{i \in I} B_i$ is also a henselian local ring.*

*Proof.* Denote by $\mathfrak{M}_i$ and $k_i$ the maximal ideal and the residue field of $B_i$. It's perfectly trivial to verify that $A$ is a local ring with maximal ideal $\mathfrak{M} = \varinjlim \mathfrak{M}_i$ and residue field $k = \varinjlim k_i$. Let $f \in A[T]$ be a monic polynomial and assume that $\bar{f} = g_0 \cdot h_0$, where $g_0, h_0 \in k[T]$ are coprime monic polynomials. Clearly there exists $i \in I$ such that $f$ comes from the monic polynomial $f(i) \in B_i[T]$ and $g_0, h_0 \in k[T]$ come from monic polynomials $g_0(i), h_0(i) \in k_i[T]$. Moreover, increasing $i$ if required, we may assume that the identity $\overline{f(i)} = g_0(i) \cdot h_0(i)$ holds in $k_i[T]$. Clearly $g_0(i)$ and $h_0(i)$ are coprime in $k_i[T]$. Since $B_i$ is henselian we conclude that $f(i) = g(i) \cdot h(i)$, where $g(i), h(i) \in B_i[T]$ are monic polynomials with $\overline{g(i)} = g_0(i)$ and $\overline{h(i)} = h_0(i)$. Taking the images of $g(i), h(i) \in B_i[T]$ in $A[T]$, we get the required factorization for $f \in A[T]$. □

**Corollary 2.7.** *Let $\pi : Y \to X$ be an integral surjective morphism of integral schemes. Assume further that the scheme $X$ is normal and the extension of fields of rational functions is purely inseparable. Let $x \in X$ be a point and $y \in Y$ be the unique point over $x$. In this case,*

$$Y \times_X \operatorname{Spec} \mathcal{O}_x^h = \operatorname{Spec} \mathcal{O}_y^h.$$

*Proof.* Obviously $Y \times_X \operatorname{Spec} \mathcal{O}_x^h = \operatorname{Spec} A$, where $A$ is integral over $\mathcal{O}_x^h$. To give a maximal ideal in $A$ is the same as to give a point in the fiber of $\operatorname{Spec} A \to \operatorname{Spec} \mathcal{O}_x^h$ over $\mathfrak{M}_x^h$. Since this fiber coincides with the fiber of $Y \to X$ over $x$, we conclude that $A$ is local and its unique maximal ideal lies over $\mathfrak{M}_y$. Lemma 2.5 shows that $A$ is a henselian local ring. On the other hand $A$ is ind-étale over $Y$ and hence coincides with $\mathcal{O}_y^h$. □

Recall that whenever $x$ is a point on the scheme $X$ an étale neighborhood of $x$ on $X$ is an étale morphism $f : U \to X$ together with a point $u \in U$ such that $f(u) = x$ and the embedding of residue fields $k(x) \xrightarrow{f^*} k(u)$ is an isomorphism. Given two étale neighborhoods $(U, u)$ and $(V, v)$ of $x$ on $X$, we say that $U$ is finer than $V$ if there exists a morphism $g : U \to V$ over $X$ that takes $u$ to $v$. Assume that $\pi : Y \to X$ is a morphism of schemes. Let, further, $x$ be a point on $X$ and $y \in Y$ be a point over $x$. Finally, let $(V, v)$ be an étale neighborhood of $x$ on $X$. In this case we define the induced étale neighborhood $\pi^{-1}(V)$ of $y$ as follows: we take $U = V \times_X Y$ and define a morphism $\operatorname{Spec} k(y) \to U$ using the canonical morphisms (over $X$) $\operatorname{Spec} k(y) \to \operatorname{Spec} k(x) = \operatorname{Spec} k(v) \to V$ and $\operatorname{Spec} k(y) \to Y$. Straightforward verification shows that the resulting point $u \in U$ lies over $y$ and its residue field equals $k(y)$.

**Corollary 2.8.** *Under the conditions and notation of Corollary 2.7, let $(U, u)$ be an étale neighborhood of $(Y, y)$. Then there exists an étale neighborhood $(V, v)$ of $(X, x)$ such that $\pi^{-1}(V)$ is finer than $U$.*

*Proof.* Obviously it suffices to treat the case when $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $U = \operatorname{Spec} R$ are affine. Since $\mathcal{O}_y^h$ may be identified with the direct limit of coordinate algebras of affine étale neighborhoods of $(Y, y)$, we get a canonical $B$-algebra homomorphism $R \xrightarrow{\phi} \mathcal{O}_y^h = \mathcal{O}_x^h \otimes_A B$. Since $R$ is a finitely presented $B$-algebra and $\mathcal{O}_x^h \otimes_A B = \varinjlim C \otimes_A B$, where $C$ runs through coordinate algebras of affine étale neighborhoods of $(X, x)$, we conclude that $\phi$ factors through $C \otimes_A B$ for appropriate $C$. Thus $V = \operatorname{Spec} C$ is an étale neighborhood of $(X, x)$, whose inverse image to $Y$ is finer than $U$. □

**Proposition 2.9.** *Under the conditions and notation of Corollary 2.7, let $\mathcal{F}$ be a presheaf on the small Nisnevich site of $Y$. Then*

$$\check{H}^0_{\mathrm{Nis}}(Y, \mathcal{F}) = \check{H}^0_{\mathrm{Nis}}(X, \pi_*(\mathcal{F})).$$

*Proof.* Given a Nisnevich covering $\mathcal{V}$ of $X$ we get the induced Nisnevich covering $\pi^{-1}(\mathcal{V})$ of $Y$. For these two coverings we have the obvious relation

$$\check{H}^0(\pi^{-1}(\mathcal{V}), \mathcal{F}) = \check{H}^0(\mathcal{V}, \pi_*(\mathcal{F})).$$

Since Čech cohomology may be computed using any cofinal family of coverings, it suffices to check that any Nisnevich covering of $Y$ admits a refinement of the form $\pi^{-1}(\mathcal{V})$. However this readily follows from Corollary 2.8. $\square$

*Proof of Theorem 2.1 for the Nisnevich topology.* We start with a presheaf $\mathcal{F} \in \mathrm{PT}_p(E)$ such that $\phi_*(\mathcal{F})$ is a Nisnevich sheaf. Since applying the functor $\check{H}^0_{\mathrm{Nis}}$ to a presheaf twice we get the associated sheaf, we see that to show that $\mathcal{F}$ is a sheaf it suffices to verify that, for any irreducible $Y \in \mathrm{Sm}_E$, the natural map $\mathcal{F}(Y) \to \check{H}^0_{\mathrm{Nis}}(Y, \mathcal{F}_Y)$ is an isomorphism. Assume first that $Y = X_E$ for an appropriate $X \in \mathrm{Sm}_F$. Apply Proposition 2.9 to the structure morphism $\pi : Y \to X$. In this way we get

$$\check{H}^0_{\mathrm{Nis}}(Y, \mathcal{F}) = \check{H}^0_{\mathrm{Nis}}(X, p_*(\mathcal{F})) = p_*(\mathcal{F})(X) = \mathcal{F}(Y).$$

In the general case we apply Lemma 1.12 and find a finite, surjective, purely inseparable morphism $f : Y \to X_E$. Lemma 2.4 shows that the natural map $\alpha_f : \mathcal{F}_{X_E} \to f_*(\mathcal{F}_Y)$ is an isomorphism and in particular $f^* : \mathcal{F}(X_E) \to \mathcal{F}(Y)$ is an isomorphism. Proposition 2.9 shows that the pull-back map

$$f^* : \check{H}^0_{\mathrm{Nis}}(X_E, \mathcal{F}_{X_E}) \to \check{H}^0_{\mathrm{Nis}}(X_E, f_*(\mathcal{F}_Y)) \to \check{H}^0_{\mathrm{Nis}}(Y, \mathcal{F}_Y)$$

is an isomorphism. Our claim follows now from the commutative diagram

$$\begin{array}{ccc}
\mathcal{F}(X_E) & \xrightarrow[f^*]{\sim} & \mathcal{F}(Y) \\
\cong\downarrow & & \downarrow \\
\check{H}^0_{\mathrm{Nis}}(X_E, \mathcal{F}_{X_E}) & \xrightarrow[f^*]{\sim} & \check{H}^0_{\mathrm{Nis}}(Y, \mathcal{F}_Y)
\end{array}$$

$\square$

**Corollary 2.10.** *Let $E/F$ be any purely inseparable field extension. Denote by $\psi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. Let, further, $\mathcal{F} \in \mathrm{PT}_p(E)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules over $E$. Assume that $\psi_*(\mathcal{F})$ is a Zariski* (*resp. Nisnevich*) *sheaf. Then $\mathcal{F}$ itself is also a Zariski* (*resp. Nisnevich*) *sheaf.*

*Proof.* Denote by $\phi$ (resp. $\phi'$) the extension of scalars functor corresponding to the field extension $F \subset F_\infty$ (resp. $E \subset E_\infty = F_\infty$), and set $\mathcal{G} = (\phi')^{\#}(\mathcal{F})$. Corollary 1.14

shows that $\mathcal{F} = \phi'_*(\mathcal{G})$. Thus $\phi_*(\mathcal{G}) = \psi_*(\phi'_*(\mathcal{G})) = \psi_*(\mathcal{F})$ is a sheaf. Theorem 2.1 shows that $\mathcal{G}$ is a sheaf and hence $\mathcal{F} = \phi'_*(\mathcal{G})$ is a sheaf as well. $\square$

**Corollary 2.11.** *Under the conditions and notation of Corollary 2.10 the functor $\psi^\# : \mathrm{PT}_p(F) \to \mathrm{PT}_p(E)$ takes Zariski (resp. Nisnevich) sheaves to Zariski (resp. Nisnevich) sheaves. More precisely, $\mathcal{F} \in \mathrm{PT}_p(F)$ is a sheaf if and only if $\psi^\#(\mathcal{F})$ is a sheaf. In particular for Nisnevich sheaves we have an identification $\psi^*(\mathcal{G}) = \psi^\#(\mathcal{G})$.*

*Proof.* This follows immediately from Corollary 2.10, since $\psi_*(\psi^\#(\mathcal{G}) = \mathcal{G}$ for any $\mathcal{G} \in \mathrm{PT}_p(F)$ according to Corollary 1.14. $\square$

Denote by $\mathrm{NST}_p(F)$ the category of Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules.

**Corollary 2.12.** *Let $E/F$ be any purely inseparable field extension. Denote by $\psi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. Then $\psi_* : \mathrm{NST}_p(E) \to \mathrm{NST}_p(F)$ is an equivalence of categories with quasi-inverse $\psi^* = \psi^\#$.*

**Theorem 2.13.** *Let $\mathcal{F} \in \mathrm{NST}_p(F)$ be a Nisnevich sheaf with transfers of $\mathbb{Z}[1/p]$-modules. Then, for any scheme $X \in \mathrm{Sm}_F$, we have a natural identification of cohomology groups*

$$H^i_{\mathrm{Nis}}(X, \mathcal{F}) = H^i_{\mathrm{Nis}}(X_{F_\infty}, \mathcal{F}_{F_\infty}).$$

*Proof.* By Corollary 2.12, $\phi_*(\mathcal{F}_{F_\infty}) = \mathcal{F}$. Denote by $\pi : Y = X_{F_\infty} \to X$ the structure morphism and by $\mathcal{G}$ the sheaf $(\mathcal{F}_{F_\infty})_Y$. Thus $\pi_*(\mathcal{G}) = \mathcal{F}_X$. Applying the Leray spectral sequence to $\pi$ we see that it suffices to establish that $R^i\pi_*\mathcal{G} = 0$ for $i > 0$. However $(R^i\pi_*\mathcal{G})_x = H^i_{\mathrm{Nis}}(Y \times_X \operatorname{Spec} \mathcal{O}^h_x, \mathcal{G}) = H^i(\operatorname{Spec} \mathcal{O}^h_y, \mathcal{G}) = 0$. $\square$

## 3. Homotopy invariant sheaves with transfers over nonperfect fields

**Proposition 3.1.** *Let $\mathcal{F} \in \mathrm{PT}_p(F_\infty)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules. Assume that $\phi_*(\mathcal{F})$ is homotopy invariant, where $\phi_*$ is as in Theorem 1.13. Then $\mathcal{F}$ itself is also homotopy invariant.*

*Proof.* We need to verify that for any irreducible $Y \in \mathrm{Sm}_{F_\infty}$ the pull-back map $\mathcal{F}(Y) \xrightarrow{p_1^*} F(Y \times \mathbb{A}^1)$ is an isomorphism. To do so we use Lemma 1.12 and find a finite, surjective, purely inseparable morphism $f : Y \to X_{F_\infty}$. Consider the commutative diagram

$$\begin{array}{ccc}
\mathcal{F}(X_{F_\infty}) = (\phi)_*(\mathcal{F})(X) & \xrightarrow{\quad f^* \quad} & \mathcal{F}(Y) \\
\Big\downarrow p_1^* & & \Big\downarrow p_1^* \\
\mathcal{F}(X_{F_\infty} \times \mathbb{A}^1) = \phi_*(\mathcal{F})(X \times \mathbb{A}^1) & \xrightarrow{(f \times 1_{\mathbb{A}^1})^*} & \mathcal{F}(Y \times \mathbb{A}^1)
\end{array}$$

Lemma 1.15 shows that both horizontal arrows are isomorphisms and the left vertical arrow is an isomorphism by assumption. Thus the right vertical arrow is an isomorphism as well. □

Using the same machinery as in the proof of Corollary 2.10 we readily verify:

**Corollary 3.2.** *Let $E/F$ be any purely inseparable field extension. Denote by $\psi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. Let, further, $\mathcal{F} \in \mathrm{PT}_p(E)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules over $E$. Assume that the presheaf $\psi_*(\mathcal{F})$ is homotopy invariant. Then $\mathcal{F}$ itself is also homotopy invariant.*

**Corollary 3.3.** *Under the conditions and notation of Corollary 3.2 let $\mathcal{F} \in \mathrm{PT}_p(F)$ be a homotopy invariant presheaf with transfers of $\mathbb{Z}[1/p]$-modules. Then $\mathcal{F}_E$ is also homotopy invariant. Conversely, if $\mathcal{F}_E$ is homotopy invariant then $\mathcal{F}$ is also.*

*Proof.* This follows immediately from Proposition 3.1, since $\psi_*(\mathcal{F}_E) = \mathcal{F}$ according to Corollary 2.12. □

**Theorem 3.4.** *Let $\mathcal{F} \in \mathrm{NST}_p(F)$ be a homotopy invariant sheaf with transfers of $\mathbb{Z}[1/p]$-modules. Then $\mathcal{F}$ is strictly homotopy invariant, i.e., $H^i_{\mathrm{Nis}}(X \times \mathbb{A}^1, \mathcal{F}) = H^i_{\mathrm{Nis}}(X, \mathcal{F})$ for any $i$ and any $X \in \mathrm{Sm}_F$.*

*Proof.* Corollary 3.3 shows that $\mathcal{F}_{F_\infty}$ is a homotopy invariant sheaf with transfers over a perfect field $F_\infty$. Thus Voevodsky's Theorem 13.8 [Mazza et al. 2006] shows that $\mathcal{F}_{F_\infty}$ is strictly homotopy invariant. Finally, using Theorem 2.13 we get

$$H^i_{\mathrm{Nis}}(X \times \mathbb{A}^1, \mathcal{F}) = H^i_{\mathrm{Nis}}(X_{F_\infty} \times \mathbb{A}^1, \mathcal{F}_{F_\infty})) = H^i_{\mathrm{Nis}}(X_{F_\infty}, \mathcal{F}_{F_\infty}) = H^i_{\mathrm{Nis}}(X, \mathcal{F}). \ \square$$

Theorem 3.4 shows that all results proved in [Mazza et al. 2006] over perfect fields hold over all fields once we work with presheaves with transfers of $\mathbb{Z}[1/p]$-modules. In particular we have the following result:

**Theorem 3.5.** *Let $\mathcal{F} \in \mathrm{PT}_p(F)$ be a homotopy invariant presheaf with transfers of $\mathbb{Z}[1/p]$-modules; then:*

(1) *The sheaf $\mathcal{F}_{\mathrm{Zar}}$ coincides with $\mathcal{F}_{\mathrm{Nis}}$ and has a natural structure of a homotopy invariant sheaf with transfers.*

(2) $H^i_{\mathrm{Zar}}(X, \mathcal{F}_{\mathrm{Zar}}) = H^i_{\mathrm{Nis}}(X, \mathcal{F}_{\mathrm{Nis}})$ *for any $X \in \mathrm{Sm}_F$ and any $i \geq 0$.*

(3) *The presheaves $X \mapsto H^i_{\mathrm{Zar}}(X, \mathcal{F}_{\mathrm{Zar}}) = H^i_{\mathrm{Nis}}(X, \mathcal{F}_{\mathrm{Nis}})$ are homotopy invariant presheaves with transfers.*

*Proof.* The first claim is proved in [Mazza et al. 2006, Theorems 22.1 and 22.2] for arbitrary homotopy invariant presheaves with transfers. In view of this fact, in the sequel we may assume that $\mathcal{F} \in \mathrm{NST}_p(F)$ is a homotopy invariant Nisnevich sheaf with transfers of $\mathbb{Z}[1/p]$-modules. Homotopy invariance of the presheaf $X \mapsto H^i_{\mathrm{Nis}}(X, \mathcal{F})$ is proved above in Theorem 3.4. The fact that $X \mapsto H^i_{\mathrm{Nis}}(X, \mathcal{F})$

has a natural structure of a presheaf with transfers is proved in [Mazza et al. 2006, Lemma 13.4] for arbitrary Nisnevich sheaves with transfers. Finally, the coincidence of Zariski and Nisnevich cohomology follows easily from the homotopy invariance of Nisnevich cohomology; see the proof of [Mazza et al. 2006, Proposition 13.9]. □

Once Theorem 3.5 is proved we may proceed the same way as in [Suslin and Voevodsky 2000] and define the category of effective motives $\mathrm{DM}_p^-(F)$. Specifically we define the category $\mathrm{DM}_p^-(F)$ as a full subcategory of the derived category $D^-(\mathrm{NST}_p(F))$ of bounded above complexes of Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules comprising the complexes with homotopy invariant cohomology sheaves.

For any $X \in \mathrm{Sm}_F$ we define its motive $M_p(X)$ as the complex $C^\bullet(\mathbb{Z}[1/p]_{\mathrm{tr}}(X))$ in $\mathrm{DM}_p^-(F)$; cf. [Suslin and Voevodsky 2000, §1]. Nisnevich cohomology may be recovered in terms of $\mathrm{DM}_p^-(F)$. The proof of the following result is identical to the proof of Theorem 1.5 in [Suslin and Voevodsky 2000].

**Theorem 3.6.** *For any complex $A^\bullet \in \mathrm{DM}_p^-(F)$ and any $X \in \mathrm{Sm}_F$ we have natural isomorphisms*

$$H^i_{\mathrm{Nis}}(X, A^\bullet) = \mathrm{Hom}_{\mathrm{DM}_p^-(F)}(M_p(X), A^\bullet[i]).$$

The category $\mathrm{DM}_p^-(F)$ may be also viewed as a localization of the category $D^-(\mathrm{NST}_p(F))$ with respect to a thick triangulated subcategory $\mathcal{A}$. Recall that $A^\bullet \mapsto C^\bullet(A^\bullet)$ defines a functor $C^\bullet : D^-(\mathrm{NST}_p(F)) \to \mathrm{DM}_p^-(F)$ and take $\mathcal{A} \subset D^-(\mathrm{NST}_p(F))$ to be the full triangulated subcategory consisting of those complexes $A^\bullet$ for which the complex $C^\bullet(A^\bullet)$ is acyclic. The proof of the following result is identical to the proof of Theorem 1.12 in [Suslin and Voevodsky 2000].

**Theorem 3.7.** (1) *The functor $C^\bullet$ is left adjoint to the embedding functor*

$$\mathrm{DM}_p^-(F) \subset D^-(\mathrm{NST}_p(F))$$

*and shows the equivalence of $\mathrm{DM}_p^-(F)$ with the localization of $D^-(\mathrm{NST}_p(F))$ with respect to the thick triangulated subcategory $\mathcal{A}$.*

(2) *A complex $A^\bullet \in D^-(\mathrm{NST}_p(F))$ is in $\mathcal{A}$ if and only if it is quasi-isomorphic to a bounded above complex of contractible Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules.*

Finally recall (see [Suslin and Voevodsky 2000, §2]) that the category $\mathrm{DM}_p^-(F)$ has a natural tensor structure given by the formula $A^\bullet \otimes B^\bullet = C^\bullet(A^\bullet \otimes^L B^\bullet)$. The main properties of this operation listed in Proposition 2.8 of [Suslin and Voevodsky 2000] remain true over nonperfect fields provided that we work with complexes of Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules.

For future use we recall briefly the explicit definition of $A^\bullet \otimes^L B^\bullet$. See [Mazza et al. 2006, Definition 8.2]. Given two complexes $A^\bullet$ and $B^\bullet$ in $\mathrm{DM}_p^-(F)$ we pick quasi-isomorphisms $A_1^\bullet \to A^\bullet$ and $B_1^\bullet \to B^\bullet$ with free complexes $A_1^\bullet$, $B_1^\bullet$ (i.e., consisting of direct sums of sheaves $\mathbb{Z}[1/p]_{\mathrm{tr}}(X)$) and set $A^\bullet \otimes^L B^\bullet = \mathrm{Tot}(A_1^\bullet \otimes B_1^\bullet)$, where $A_1^\bullet \otimes B_1^\bullet$ is a bicomplex consisting of sheaves $(A_1)^i \otimes_{\mathrm{tr}} (B_1)^j$. The same reasoning as in [Suslin and Voevodsky 2000] show that the resulting complex is independent of the choice of free resolutions $A_1^\bullet$ and $B_1^\bullet$ up to a natural quasi-isomorphism.

## 4. Extension of scalars for the category $\mathrm{DM}_p^-(F)$

Let $E/F$ be a field extension. Denote by $\phi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. In the sequel we denote by $p$ the exponential characteristic of $F$ (i.e., $p = 1$ for fields of characteristic zero). We work with the category $\mathrm{PT}_p(F)$ of presheaves with transfers of $\mathbb{Z}[1/p]$-modules.

**Theorem 4.1.** *The functor $\phi^\# : \mathrm{PT}_p(F) \to \mathrm{PT}_p(E)$ takes Nisnevich (resp. Zariski) sheaves to Nisnevich (resp. Zariski) sheaves, so that $\phi^* = \phi^\#$. Furthermore, the functor $\phi^\# : \mathrm{PT}_p(F) \to \mathrm{PT}_p(E)$ is exact.*

*Proof.* Consider first the special case when the field $F$ is perfect and $E$ is finitely generated over $F$. In this case $E$ may be written in the form $E = F(S)$ for an appropriate smooth, irreducible scheme of finite type $S \in \mathrm{Sm}_F$. In this case $\operatorname{Spec} E$ may be further identified with the inverse limit $\operatorname{Spec} E = \varprojlim U$, where $U$ runs through a directed inverse system of open affine neighborhoods of the generic point $\eta \in S$ and we may apply the following classical result; see [EGA $\mathrm{IV}_1$ 1964]:

**Theorem 4.2.** *Let $I$ be a directed partially ordered set. Let, further, $\{S_i\}_{i \in I}$ be an inverse system of schemes over $I$ with affine transition morphisms. Assume that all the $S_i$ are quasicompact and quasiseparated and set $S = \varprojlim S_i$.*

(1) *For any morphism of finite presentation $X \to S$ there exists an index $i \in I$ and a morphism of finite presentation $X_i \to S_i$ such that $X \cong X_i \times_{S_i} S$ (as schemes over $S$).*

(2) *Given $i \in I$, schemes of finite presentation $X_i$ and $Y_i$ over $S_i$ and a morphism $\phi : X_i \times_{S_i} S \to Y_i \times_{S_i} S$ over $S$, there exists an index $i' \geq i$ and a morphism $\phi_{i'} : X_i \times_{S_i} S_{i'} \to Y_i \times_{S_i} S_{i'}$ over $S_{i'}$ whose base change to $S$ is $\phi$.*

(3) *Given $i \in I$, $X_i$ and $Y_i$ of finite presentation over $S_i$ and a pair of morphisms $\phi_i, \psi_i : X_i \to Y_i$ (over $S_i$) whose base changes to $S$ are equal, there exists $i' \geq i$ such that the base changes of $\phi_i$ and $\psi_i$ to $S_{i'}$ are equal.*

(4) *Assume that $X_i$ and $Y_i$ are schemes of finite presentation over $S_i$ and $\phi_i : X_i \to Y_i$ is a morphism over $S_i$. Denote by $X$ (resp. $Y$) the scheme obtained*

*from $X_i$ (resp. $Y_i$) by base change from $S_i$ to $S$. In a similar way denote by $\phi : X \to Y$ the morphism obtained from $\phi_i$ by base change from $S_i$ to $S$. Finally, for any $j \geq i$ let $X_j$, $Y_j$ and $\phi_j$ be schemes and morphisms obtained from $X_i$, $Y_i$ and $\phi_i$ by base change from $S_i$ to $S_j$. Assume that the morphism $\phi$ has one of the following properties*:

(a) *$\phi$ is an isomorphism.*
(b) *$\phi$ is an open (resp closed) embedding.*
(c) *$\phi$ is surjective.*
(d) *$\phi$ admits a section.*
(e) *$\phi$ is finite.*
(f) *$\phi$ is étale.*
(g) *$\phi$ is smooth.*

*Then there exists an index $j \geq i$ such that $\phi_j$ has the same property.*

**Corollary 4.3.** *Under the conditions and notation of Theorem 4.2 assume that the scheme $S$ is Noetherian. Let, further, $X/S$ be a reduced scheme of finite type and let $Y \xrightarrow{\phi} X$ be a (singleton) Nisnevich covering of $X$. Then there exists $i \in I$, schemes of finite presentation $X_i$ and $Y_i$ over $S_i$, and a Nisnevich covering $Y_i \xrightarrow{\phi_i} X_i$ such that $X = X_i \times_{S_i} S$, $Y = Y_i \times_{S_i} S$ and $\phi = \phi_i \times_{S_i} S$.*

*Proof.* Theorem 4.2 shows that we may assume that $X = X_i \times_{S_i} S$, $Y = Y_i \times_{S_i} S$ and $\phi = \phi_i \times_{S_i} S$ for appropriate $X_i$, $Y_i$ and $\phi_i$. Since $\phi$ is étale we conclude from Theorem 4.2 that, increasing $i$, we may assume that $\phi_i$ is étale as well. Finally, since $\phi$ is a Nisnevich covering of a Noetherian scheme we conclude from [Hoyois 2012, Proposition 1.6] that there exists a chain of closed subschemes $\varnothing = X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that $\phi$ admits a section over $X_i \setminus X_{i-1}$ for $1 \leq i \leq n$. Theorem 4.2 shows that we may assume that $X_k = X_{ik} \times_{S_i} S$, where $\varnothing = X_{i0} \subset X_{i1} \subset \cdots \subset X_{in} = X_i$ is a chain of closed subschemes of $X_i$ and $\phi_i$ admits a section over $X_{ik} \setminus X_{i,k-1}$. In this case $\phi_i : Y_i \to X_i$ is obviously a Nisnevich covering of $X_i$. □

We still assume that $F$ is perfect and $E$ is finitely generated over $F$. We fix a smooth scheme of finite type $S \in \mathrm{Sm}_F$ such that $E = F(S)$. Let $\mathcal{F} : \mathrm{Sm}_F \to \mathrm{Ab}$ be a presheaf of abelian groups on the category $\mathrm{Sm}_F$. In this case we define a new presheaf $a\mathcal{F}$ on $\mathrm{Sm}_E$ using the following construction. Let $X \in \mathrm{Sm}_E$ be a smooth scheme of finite type over Spec $E$. According to Theorem 4.2 we may find a nonempty open $U \subset S$ and a smooth scheme of finite type $\widetilde{X} \to U$ whose generic fiber $\widetilde{X}_\eta$ coincides with $X$. In this case we'll be saying that $\widetilde{X}$ is a model of $X$ defined over $U$. In this situation we set

$$a\mathcal{F}(X) = \varinjlim_{V \subset U} \mathcal{F}(\widetilde{X}_V).$$

Let $Y \in \mathrm{Sm}_E$ be another smooth scheme of finite type over Spec $E$ and let $f : X \to Y$ be a morphism over Spec $E$. Let, further, $\widetilde{Y} \to V$ be a model of $Y$ defined over $V$. Theorem 4.2 shows that, shrinking $V$, we may find a morphism $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ over $V$ whose generic fiber equals $f$. Moreover, given two such extensions $\tilde{f}_1$ and $\tilde{f}_2$ of $f$, we may find a nonempty open set $W \subset V$ such that $\tilde{f}_1$ and $\tilde{f}_2$ agree on $\widetilde{X}_W$. These considerations readily imply that the definition of $a\mathcal{F}(X)$ is independent of the particular choice of $\widetilde{X}$ and moreover that $a\mathcal{F}$ has a natural structure of an abelian presheaf on $\mathrm{Sm}_E$.

**Proposition 4.4.** *Assume that $\mathcal{F}$ is a Zariski* (*resp. Nisnevich*) *sheaf. Then $a\mathcal{F}$ is also a Zariski* (*resp. Nisnevich*) *sheaf.*

*Proof.* Consider first the case of the Zariski topology. Since all schemes in question are quasicompact it suffices to consider finite coverings. Let $X = \bigcup_{i=1}^n X_i$ be an open covering of $X \in \mathrm{Sm}_E$. Let, further, $\widetilde{X}$ be the model of $X$ defined over $U \subset S$. Theorem 4.2 readily implies that, shrinking $U$, we may find an open covering $\widetilde{X} = \bigcup_{i=1}^n \widetilde{X}_i$ such that $\widetilde{X}_i$ is a model of $X_i$. For any $V \subset U$ we have an open covering $\widetilde{X}_V = \bigcup_{i=1}^n (\widetilde{X}_i)_V$ and hence an exact sequence

$$0 \to \mathcal{F}(\widetilde{X}_V) \to \prod_{i=1}^n \mathcal{F}((\widetilde{X}_i)_V) \to \prod_{i,j=1}^n \mathcal{F}((\widetilde{X}_i \cap \widetilde{X}_j)_V).$$

Passing to the direct limit over $V$ we get the required exact sequence

$$0 \to a\mathcal{F}(X) \to \prod_{i=1}^n a\mathcal{F}(X_i) \to \prod_{i,j=1}^n a\mathcal{F}(X_i \cap X_j).$$

Next we consider the case of the Nisnevich topology. Let $X$ be a smooth scheme of finite type over Spec $E$ and let $\widetilde{X} \to U$ be its model defined over an open $U \subset S$.

To finish the proof of Proposition 4.4 we start with a singleton Nisnevich covering $f : Y \to X$. According to Corollary 4.3 we may assume that $f = \tilde{f}_\eta$, where $\tilde{f} : \widetilde{Y} \to \widetilde{X}_V$ is a Nisnevich covering of $\widetilde{X}_V$. For any open set $W \subset V$ the induced map $\tilde{f}_W : \widetilde{Y}_W \to \widetilde{X}_W$ is also a Nisnevich covering and, since $\mathcal{F}$ is a Nisnevich sheaf, we conclude that the sequence

$$0 \to \mathcal{F}(\widetilde{X}_W) \to \mathcal{F}(\widetilde{Y}_W) \to \mathcal{F}(\widetilde{Y}_W \times_{\widetilde{X}_W} \widetilde{Y}_W)$$

is exact. Passing to the direct limit over $W$ we get the exactness of the sequence

$$0 \to a\mathcal{F}(X) \to a\mathcal{F}(Y) \to a\mathcal{F}(Y \times_X Y).$$

The case of an arbitrary Nisnevich covering readily follows since we already know that $a\mathcal{F}$ is a Zariski sheaf. $\square$

**Proposition 4.5.** *Assume that $\mathcal{F}$ is a presheaf with transfers. Then $a\mathcal{F}$ also has a natural structure of a presheaf with transfers.*

*Proof.* For the proof we need the following elementary definition of finite correspondences in a relative situation. The definition we give is a very special and absolutely elementary case of the construction discussed in Section 1A of [Mazza et al. 2006].

Let $U$ be a smooth irreducible scheme of finite type over $F$. Let, further, $X/U$ and $Y/U$ be smooth schemes of finite type over $U$ (and a fortiori over $F$). Denote by $\mathrm{Cor}_U(X, Y)$ the free abelian group generated by closed integral subschemes (called elementary finite correspondences) $T \subset X \times_U Y$ that are finite and surjective over a component of $X$. Note that $\mathrm{Cor}_U(X, Y)$ is a subgroup in $\mathrm{Cor}_F(X, Y)$. If $Z/U$ is another smooth scheme of finite type over $U$, we have the usual composition map $\mathrm{Cor}_F(Y, Z)\times\mathrm{Cor}_F(X, Y) \to \mathrm{Cor}_F(X, Z)$, and a straightforward verification shows that this composition map takes $\mathrm{Cor}_U(Y, Z) \times \mathrm{Cor}_U(X, Y)$ to $\mathrm{Cor}_U(X, Z)$. In this way we get the category of finite correspondences over $U$, whose objects are smooth schemes of finite type over $U$ and morphisms from $X/U$ to $Y/U$ are represented by finite correspondences over $U$. We denote the corresponding category by $\mathrm{Cor}_U$. Note further that if $V \subset U$ is a nonempty open set we get a canonical functor $\mathrm{res}^U_V$ from $\mathrm{Cor}_U$ to $\mathrm{Cor}_V$, which takes $X/U$ to $X_V/V$ and takes the relative finite correspondence $T \subset X \times_U Y$ to $T_V \subset X_V \times_V Y_V$.

Let $\eta$ be the generic point of $U$. Denote by $E'$ the residue field at $\eta$ (i.e., the field of rational functions $E' = F(U)$). In this case we get a canonical functor from $\mathrm{Cor}_U$ to $\mathrm{Cor}_{E'}$, which takes $X/U$ to its generic fiber $X_\eta = X \times_U \mathrm{Spec}\, E'$ and takes an elementary correspondence $T \subset X \times_U Y$ to the elementary correspondence represented by the generic fiber $T_\eta \subset X_\eta \times_{\mathrm{Spec}\, E'} Y_\eta$. Whenever $V$ is a nonempty open subscheme of $U$ the following diagram obviously commutes:

$$\begin{tikzcd}
\mathrm{Cor}_U \arrow[r] \arrow[d] & \mathrm{Cor}_V \arrow[d] \\
\mathrm{Cor}_{E'} \arrow[r, "="] & \mathrm{Cor}_{E'}
\end{tikzcd}$$

Hence, for any $X, Y \in \mathrm{Sm}_U$ we get the induced map

$$\varinjlim_{V\subset U} \mathrm{Cor}_V(X_V, Y_V) \to \mathrm{Cor}_{E'}(X_\eta, Y_\eta).$$

**Lemma 4.6.** *For any $X, Y \in \mathrm{Sm}_U$ the natural map*

$$\varinjlim_{V\subset U} \mathrm{Cor}_V(X_V, Y_V) \to \mathrm{Cor}_{E'}(X_\eta, Y_\eta)$$

*is an isomorphism.*

*Proof.* To prove the injectivity of the map in question it suffices to establish that the canonical map $\mathrm{Cor}_U(X, Y) \to \mathrm{Cor}_{E'}(X_\eta, Y_\eta)$ is injective. Since this map takes the generators of the free abelian group $\mathrm{Cor}_U(X, Y)$ to the generators of the free abelian group $\mathrm{Cor}_{E'}(X_\eta, Y_\eta)$, we only need to verify that different generators go to different generators, and this follows from the fact that the generic fiber $T_\eta$ is dense in $T$.

To prove surjectivity, start with a closed integral subscheme $T_0 \subset X_\eta \times_{\mathrm{Spec}\, E'} Y_\eta = (X \times_U Y)_\eta$ that is finite and surjective over $X_\eta$. Theorem 4.2 shows that after shrinking $U$ we can find a model $T$ for $T_0$ defined over $U$. The $E'$-morphism $i : T_\eta \to (X \times_U Y)_\eta$ may be extended (after diminishing $U$) to a morphism $\tilde{i} : T \to X \times_U Y$. Since $i$ is a closed embedding we see that, diminishing $U$, we may assume that $\tilde{i}$ is a closed embedding as well, i.e., we may assume that $T$ is a closed subscheme of $X \times_U Y$. Furthermore, since the projection $p_1 : T_\eta = T_0 \to X_\eta$ is finite we conclude that we may assume that $T$ is finite over $X$. Finally, since $T_\eta$ is integral it follows easily — cf. [EGA IV$_3$ 1966, Corollaire (8.7.3)] — that we may assume that $T$ is integral. In this way we get an elementary correspondence $T \in \mathrm{Cor}_U(X, Y)$ whose image in $\mathrm{Cor}_{E'}(X_\eta, Y_\eta)$ equals $T_0$. □

**Corollary 4.7.** *Assume that $X \in \mathrm{Sm}_U$ and $Y \in \mathrm{Sm}_F$. Then*

$$\mathrm{Cor}_{E'}(X_\eta, Y_{E'}) = \varinjlim_{V \subset U} \mathrm{Cor}_F(X_V, Y).$$

*Proof.* This follows immediately from the previous proposition in view of the obvious identification (valid for any $V \subset U$)

$$\mathrm{Cor}_V(X_V, Y \times_{\mathrm{Spec}\, F} V) = \mathrm{Cor}_F(X_V, Y) \qquad \square$$

To finish the proof of Proposition 4.5 we note that given a section $s_0 \in a\mathcal{F}(Y)$ and a finite correspondence $T_0 \in \mathrm{Cor}_E(X, Y)$ we may pick an open set $U \subset S$, models $\widetilde{X}/U$ and $\widetilde{Y}/U$ for $X$ and $Y$, and representatives $s \in \mathcal{F}(\widetilde{Y})$ and $T \in \mathrm{Cor}_U(\widetilde{X}, \widetilde{Y})$ for $s_0$ and $T_0$, respectively, and take $T_0^*(s_0) \in a\mathcal{F}(X)$ to be the canonical image of $T^*(s) \in \mathcal{F}(\widetilde{X})$. A standard verification based on the repeated use of Theorem 4.2 shows that the resulting section is independent of all choices made and we really get a presheaf with transfers structure on $a\mathcal{F}$. □

In a similar way we show that whenever $f : \mathcal{F}_1 \to \mathcal{F}_2$ is a homomorphism of presheaves with transfers the resulting map $af : a\mathcal{F}_1 \to a\mathcal{F}_2$ is also compatible with transfers. In other words we get a functor $a : \mathrm{PT}(F) \to \mathrm{PT}(E)$. This functor is obviously exact and commutes with arbitrary direct sums. Moreover, Corollary 4.7 shows that when we apply this functor to $\mathbb{Z}_{\mathrm{tr}}(Y)$ we get $\mathbb{Z}_{\mathrm{tr}}(Y_E)$. Proposition 1.1 readily implies that the functor $\phi^\#$ coincides with $a$ and hence takes sheaves to sheaves and is exact.

Still assuming that $F$ is perfect, consider the case of an arbitrary extension $E/F$. Let $\{E_i\}_{i\in I}$ be the direct system of finitely generated subextensions of $E$ (ordered by inclusion). Set $\mathcal{F}_i = \mathcal{F}_{E_i}$. Assume that $i \leq j$. In this case denote by $\phi_i^j : \mathrm{Cor}_{E_i} \to \mathrm{Cor}_{E_j}$ the extension of scalars functor from $E_i$ to $E_j$. Since $(\phi_i^j)^{\#}(\mathcal{F}_i) = \mathcal{F}_j$ we get a canonical homomorphism $\mathcal{F}_i \to (\phi_i^j)_*(\mathcal{F}_j)$ and so, for any $X_i \in \mathrm{Sm}_{E_i}$, setting $X_j = (X_i)_{E_j}$, we have a canonical map $\mathcal{F}_i(X_i) \to \mathcal{F}_j(X_j)$. A straightforward verification shows that $\{\mathcal{F}_j(X_j)\}_{j\geq i}$ is a direct system of abelian groups.

We use the same approach as before to construct an appropriate model for the functor $\phi^{\#}$. Let $X$ be a smooth scheme of finite type over $E$. Theorem 4.2 shows that we can find $i \in I$ and a smooth scheme of finite type $X_i$ over $E_i$ such that $X = (X_i)_E$. Defining $X_j$ for $j \geq i$ in the same way as before we set $a\mathcal{F}(X) = \varinjlim_{j\geq i} \mathcal{F}_j(X_j)$. The same reasoning as before shows that the resulting group is independent of the particular choice of $i$ and $X_i$; moreover, $a\mathcal{F}$ is a presheaf with transfers whenever $\mathcal{F}$ is and $a\mathcal{F}$ is a sheaf in Zariski or Nisnevich topology whenever $\mathcal{F}$ is. The resulting functor $a$ is clearly exact and commutes with arbitrary direct sums. Since $a\mathbb{Z}_{\mathrm{tr}}(X) = \mathbb{Z}_{\mathrm{tr}}(X_E)$, we conclude from Proposition 1.1 that $a = \phi^{\#}$.

Now we are ready to finish the proof of Theorem 4.1. Denote by $F_\infty$ and $E_\infty$ the perfect closures of $F$ and $E$, respectively. We get a commutative diagram of fields

$$\begin{array}{ccc} F & \xrightarrow{\phi} & E \\ {\scriptstyle\psi_F}\downarrow & & \downarrow{\scriptstyle\psi_E} \\ F_\infty & \xrightarrow{\phi_\infty} & E_\infty \end{array}$$

which yields the associated commutative diagram of functors

$$\begin{array}{ccc} \mathrm{PT}(F) & \xrightarrow{\phi^{\#}} & \mathrm{PT}(E) \\ {\scriptstyle\psi_F^{\#}}\downarrow & & \downarrow{\scriptstyle\psi_E^{\#}} \\ \mathrm{PT}(F_\infty) & \xrightarrow{\phi_\infty^{\#}} & \mathrm{PT}(E_\infty). \end{array}$$

Since both vertical arrows are equivalences and the bottom horizontal arrow is exact we conclude that the top horizontal arrow is exact as well. The claim concerning sheaves is proved in the same way, using Corollary 2.11. □

**Corollary 4.8.** *Under the conditions and notations of Theorem 4.1 the base change functor $\phi^* : \mathrm{NST}_p(F) \to \mathrm{NST}_p(E)$ is exact.*

*Proof.* This functor is right exact as a left adjoint to the functor $\phi_*$ and is left exact because left exactness in the category of sheaves is equivalent to the left exactness in the category of presheaves. □

**Proposition 4.9.** *Under the conditions and notations of Theorem 4.1, the functor $\phi^{\#}$ takes homotopy invariant presheaves with transfers to homotopy invariant presheaves with transfers.*

*Proof.* We follow the same steps as in the proof of the Theorem 4.1. Assume first that $F$ is perfect and $E$ is finitely generated over $F$. Let $S/F$ be a smooth irreducible scheme of finite type for which $E = F(S)$. Let $X$ be a scheme of finite type over $E$. Find a model $\widetilde{X}/U$ for $X$ defined over an appropriate $U \subset S$. In the course of the proof of Theorem 4.1 we established that $\phi^{\#}(\mathcal{F})(X) = \varinjlim_{V \subset U} \mathcal{F}(\widetilde{X}_V)$. At the same time $\mathbb{A}^1 \times \widetilde{X}$ is a model for $\mathbb{A}^1 \times X$ and hence

$$(\phi^{\#}\mathcal{F})(\mathbb{A}^1 \times X) = \varinjlim_{V \subset U} \mathcal{F}(\mathbb{A}^1 \times \widetilde{X}_V) = \varinjlim_{V \subset U} \mathcal{F}(\widetilde{X}_V) = \phi^{\#}\mathcal{F}(X)$$

In a similar way we consider the case of an arbitrary extension of a perfect field and finally use Corollary 3.3. □

The same reasoning establishes the validity of the following claim:

**Lemma 4.10.** *Under the conditions and notations of Theorem 4.1, for any presheaf $\mathcal{F} \in \mathrm{PT}_p(F)$ we have the formula*

$$\phi^{\#}(C_n(\mathcal{F})) = C_n(\phi^{\#}(\mathcal{F})).$$

The same arguments may be used to prove the following result, which concerns categories of all Nisnevich sheaves with transfers.

**Corollary 4.11.** *Let $F$ be a perfect field and let $E/F$ be an arbitrary field extension. The functor $\phi^{\#} : \mathrm{PT}(F) \to \mathrm{PT}(E)$ takes Nisnevich (resp. Zariski) sheaves to Nisnevich (resp. Zariski) sheaves, so that $\phi^* = \phi^{\#}$. Furthermore, the functor $\phi^{\#} : \mathrm{PT}_p(F) \to \mathrm{PT}_p(E)$ is exact, takes homotopy invariant presheaves to homotopy invariant presheaves and commutes with $C_n$.*

Theorem 4.1 together with Proposition 4.9 and Lemma 4.10 immediately imply that, associating to a complex $A^\bullet \in \mathrm{DM}_p^-(F)$ the complex $A_E^\bullet$, we get a well-defined triangulated functor $\phi^* : \mathrm{DM}_p^-(F) \to \mathrm{DM}_p^-(E)$. The following result summarizes the properties of this functor:

**Theorem 4.12.** (1) *The functor $\phi^*$ takes exact triangles to exact triangles and commutes with shifts.*

(2) *The functor $\phi^*$ takes tensor products to tensor products.*

(3) $\phi^*(M_p(X)) = M_p(X_E)$ *for any smooth scheme of finite type $X$ over $F$.*

*Proof.* The first claim is obvious; the third one follows from Proposition 1.1 and Lemma 4.10. To prove the second claim we start with arbitrary complexes $A^\bullet, B^\bullet \in \mathrm{DM}_p^-(F)$ and pick their free resolutions $A_1^\bullet \to A^\bullet$ and $B_1^\bullet \to B^\bullet$. In this case $A^\bullet \otimes B^\bullet$ is quasi-isomorphic to $C^\bullet(A_1^\bullet \otimes_{\mathrm{tr}} B_1^\bullet)$ and hence $\phi^*(A^\bullet \otimes B^\bullet)$ is quasi-isomorphic to $\phi^*(C^\bullet(A_1^\bullet \otimes_{\mathrm{tr}} B_1^\bullet) = C^\bullet(\phi^*(A_1^\bullet \otimes_{\mathrm{tr}} B_1^\bullet)) = C^\bullet(\phi^*(A_1^\bullet) \otimes_{\mathrm{tr}} \phi^*(B_1^\bullet))$. Since $\phi^*(A_1^\bullet)$ and $\phi^*(B_1^\bullet)$ are free resolutions of $\phi^*(A^\bullet)$ and $\phi^*(B^\bullet)$, respectively, we conclude that the last complex is quasi-isomorphic to $\phi^*(A^\bullet) \otimes \phi^*(B^\bullet)$. □

The following result, which is an immediate consequence of the above discussion and Corollary 2.12, shows that there is essentially no difference between a nonperfect field $F$ and its perfect closure $F_\infty$.

**Corollary 4.13.** *Let $E/F$ be a purely inseparable field extension. Then the corresponding functor $\phi^* : \mathrm{DM}_p^-(F) \to \mathrm{DM}_p^-(E)$ is an equivalence of categories.*

## 5. Extension of scalars and internal $\mathcal{H}om$-objects

In this section we'll often write $A_E^\bullet$ instead of $\phi^*(A^\bullet)$. Recall that, for a perfect field $F$, the category $\mathrm{DM}^-(F)$ has internal $\mathcal{H}om$-objects, i.e., for any $A^\bullet \in \mathrm{DM}^-(F)$ and any $X \in \mathrm{Sm}/F$ we have a new object $\mathcal{H}om(M(X), A^\bullet)$ and a universal morphism $\mathcal{H}om(M(X), A^\bullet) \otimes M(X) \to A^\bullet$ such that the resulting map

$$\mathrm{Hom}_{\mathrm{DM}^-(F)}(M, \mathcal{H}om(\mathcal{M}(X), A^\bullet)) \to \mathrm{Hom}_{\mathrm{DM}^-(F)}(M \otimes M(X), A^\bullet)$$

is an isomorphism for any $M \in \mathrm{DM}^-(F)$. Corollary 4.13 immediately implies that the same result is valid for the category $\mathrm{DM}_p^-(F)$ for arbitrary $F$. The purpose of this section is to show that extension of scalars functor preserves internal $\mathcal{H}om$-objects.

**Theorem 5.1.** *Let $E/F$ be any field extension. Let, further, $A^\bullet \in \mathrm{DM}_p^-(F)$ be any motivic complex and let $X \in \mathrm{Sm}/F$ be any smooth scheme. In this case we have a natural isomorphism*

$$\mathcal{H}om(M_p(X), A^\bullet)_E = \mathcal{H}om(M_p(X_E), A_E^\bullet).$$

*Proof.* Applying the extension of scalars functor to the canonical homomorphism $\mathcal{H}om(M_p(X), A^\bullet) \otimes M_p(X) \to A^\bullet$ and using Theorem 4.12, we get a homomorphism $\mathcal{H}om(M_p(X), A^\bullet)_E \otimes M_p(X_E)) \to A_E^\bullet$ and hence the induced map

$$\mathcal{H}om(M_p(X), A^\bullet)_E \to \mathcal{H}om(M_p(X_E), A_E^\bullet).$$

We claim that this map is a quasi-isomorphism. If $E/F$ is purely inseparable our claim readily follows from Corollary 4.13. Using the same machinery as in the proof of Theorem 4.1 we easily reduce the general case to the special one, when $F$ is perfect. We start with the following observation:

**Lemma 5.2.** *Let $I \in \mathrm{NST}_p(F)$ be an injective Nisnevich sheaf with transfers. Then, for any smooth $X/\operatorname{Spec} E$, the cohomology groups $H^*_{\mathrm{Nis}}(X, I_E)$ are trivial in positive dimensions.*

*Proof.* We first consider the Čech cohomology. It clearly suffices to treat the case of a singleton Nisnevich covering $Y \to X$. Assume first that $E/F$ is finitely generated, pick a smooth irreducible scheme of finite type $S/F$ such that $E = F(S)$, and denote by $\eta$ the generic point of $S$. According to Theorem 4.2 and Corollary 4.3 we may assume that $X = \widetilde{X}_\eta$, $Y = \widetilde{Y}_\eta$ and $\phi = \tilde{\phi}_\eta$, where $\widetilde{X}$ and $\widetilde{Y}$ are smooth schemes over an open $U \subset S$ and $\tilde{\phi} : \widetilde{Y} \to \widetilde{X}$ is a Nisnevich covering. For any open $V \subset U$ the morphism $\tilde{\phi}_V : \widetilde{Y}_V \to \widetilde{X}_V$ is still a Nisnevich covering and hence $\check{H}^i(\widetilde{Y}_V/\widetilde{X}_V, I) = 0$ for $i > 0$; see [Suslin and Voevodsky 2000, Lemma 1.6]. Passing to the direct limit over $V \subset U$ we see that $\check{H}^i(Y/X, I_E) = 0$ for $i > 0$. In the general case we may write $E$ as a direct limit of finitely generated subextensions, $E = \varinjlim E_i$, and represent $Y$, $X$ and $\phi$ in the form $Y = Y_{i\,E}$, $X = X_{i\,E}$ and $\phi = \phi_{i\,E}$, where $Y_i$ and $X_i$ are smooth schemes of finite type over $E_i$ and $\phi_i : Y_i \to X_i$ is a Nisnevich covering. For any $j \geq i$, $\phi_j = (\phi_i)_{E_j} : Y_j \to X_j$ is still a Nisnevich covering. Thus, by what was proved above, $\check{H}^*(Y_j/X_j, I_{E_j}) = 0$ for $i > 0$. Passing to the direct limit over $j \geq i$ we conclude that $\check{H}^i(Y/X, I_E) = 0$ for $i > 0$.

Now the standard argument involving the Cartan–Leray spectral sequence completes the proof. □

**Proposition 5.3.** *Let $A^\bullet$ be an arbitrary complex of Nisnevich sheaves with transfers.*

(1) *Assume first that $E/F$ is finitely generated, pick a smooth irreducible $S/F$ such that $E = F(S)$ and denote by $\eta$ the generic point of $S$. Then, for any smooth scheme of finite type $\widetilde{X}/S$, we have a natural identification*

$$H^*_{\mathrm{Nis}}(\widetilde{X}_\eta, A^\bullet_E) = \varinjlim_{U \subset S} H^*_{\mathrm{Nis}}(\widetilde{X}_U, A^\bullet).$$

(2) *For arbitrary $E/F$ write $E = \varinjlim E_i$, where $E_i/F$ are finitely generated subextensions. Then for any smooth scheme of finite type $X_i/E_i$ we have a natural identification $H^*_{\mathrm{Nis}}((X_i)_E, A^\bullet_E) = \varinjlim_{j \geq i} H^*_{\mathrm{Nis}}((X_i)_{E_j}, A^\bullet_{E_j})$.*

*Proof.* (1) Recall (see [Suslin and Voevodsky 2000, §0]) that we define hypercohomology with coefficients in nonbounded below complexes via Cartan–Eilenberg resolutions. This approach gives correct answers since all schemes involved have finite cohomological dimension in Nisnevich topology. Thus let $A^\bullet \to I^{\bullet\bullet}$ be a Cartan–Eilenberg resolution of $A^\bullet$ and let $A^\bullet_E \to J^{\bullet\bullet}$ be a Cartan–Eilenberg resolution of $A^\bullet_E$. Note that, according to the definitions and results of Section 4,

$$H^*(I^{\bullet\bullet}_E(\widetilde{X}_\eta)) = \varinjlim_{U \subset S} H^*(I^{\bullet\bullet}(\widetilde{X}_U)) = \varinjlim_{U \subset S} H^*(\widetilde{X}_U, A^\bullet),$$
$$H^*(J^{\bullet\bullet}(\widetilde{X}_\eta)) = H^*(\widetilde{X}_\eta, A^\bullet_E).$$

Since the functor $M \mapsto M_E$ is exact one checks immediately that $I^{\bullet\bullet}_E$ is an admissible resolution of $A^\bullet_E$, i.e., cycles, boundaries and cohomology of rows of this bicomplex give resolutions of cycles etc. of $A^\bullet_E$. The universal property of Cartan–Eilenberg resolutions shows that there exists a unique up to homotopy homomorphism $I^{\bullet\bullet}_E \to J^{\bullet\bullet}$ of bicomplexes under $A^\bullet_E$. Finally the induced homomorphism $I^{\bullet\bullet}_E(\widetilde{X}_\eta) \to J^{\bullet\bullet}(\widetilde{X}_\eta)$ is a quasi-isomorphism since the columns of $I^{\bullet\bullet}_E$ are acyclic resolutions of $A^i_E$ (according to Lemma 5.2 ) while the columns of $J^{\bullet\bullet}$ are injective resolutions of $A^i_E$, and sheaf cohomology may be computed using any acyclic resolutions.

(2) The proof of this is essentially the same; we skip the trivial details. □

*The end of the proof of Theorem 5.1.* Consider first the case when $E = F(S)$ for a smooth irreducible scheme of finite type $S/F$. To show that the canonical map $\mathcal{H}om(M_p(X), A^\bullet)_E \to \mathcal{H}om(M_p(X_E), A^\bullet_E)$ is a quasi-isomorphism we have to check that its cone is trivial in $\mathrm{DM}^-_p(E)$. Since the category $\mathrm{DM}^-_p(E)$ is weakly generated by objects of the form $M_p(\widetilde{Y}_\eta)[i]$ (with $\widetilde{Y}/S$ smooth of finite type) it suffices to verify that $\mathrm{Hom}_{\mathrm{DM}^-_p(E)}(M_p(\widetilde{Y}_\eta)[i], \text{cone}) = 0$. In other words we have to verify that the induced map

$$\mathrm{Hom}_{\mathrm{DM}^-_p(E)}(M_p(\widetilde{Y}_\eta)[i], \mathcal{H}om(M_p(X), A^\bullet)_E) \\ \to \mathrm{Hom}_{\mathrm{DM}^-_p(E)}\big(M_p(\widetilde{Y}_\eta)[i], \mathcal{H}om(M_p(X_E, A^\bullet_E))\big)$$

is an isomorphism. The previous results show that we may compute the above Hom-groups as follows:

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{DM}^-_p(E)}\big(M_p(\widetilde{Y}_\eta)[i], \mathcal{H}om(M_p(X), A^\bullet)_E\big) \\
&\qquad = H^{-i}(\widetilde{Y}_\eta, \mathcal{H}om(M_p(X), A^\bullet)_E) \\
&\qquad = \varinjlim_{U\subset S} H^{-i}(\widetilde{Y}_U, \mathcal{H}om(M_p(X), A^\bullet)) \\
&\qquad = \varinjlim_{U\subset S} \mathrm{Hom}_{\mathrm{DM}^-_p(F)}(M_p(\widetilde{Y}_U)[i], \mathcal{H}om(M_p(X), A^\bullet)) \\
&\qquad = \varinjlim_{U\subset S} \mathrm{Hom}_{\mathrm{DM}^-_p(F)}(M_p(\widetilde{Y}_U \times_F X, A^\bullet)) \\
&\qquad = \varinjlim_{U\subset S} H^{-i}(\widetilde{Y}_U \times_F X, A^\bullet).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{DM}^-_p(E)}(M_p(\widetilde{Y}_\eta)[i], \mathcal{H}om(M_p(X_E), A^\bullet_E)) \\
&\qquad = \mathrm{Hom}_{\mathrm{DM}^-_p(E)}(M_p(\widetilde{Y}_\eta \times_E X_E)[i], A^\bullet_E) \\
&\qquad = H^{-i}((\widetilde{Y} \times_F X)_\eta, A^\bullet_E) \\
&\qquad = \varinjlim_{U\subset S} H^{-i}(\widetilde{Y}_U \times_F X, A^\bullet).
\end{aligned}$$

Thus the above Hom-groups identify canonically and it's not hard to trace through the above computations to see that this identification coincides with the canonical homomorphism defined before.

The general case is treated once again by passing to a direct limit over finite subextensions in $E$ and using Proposition 5.3(2). □

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# $K$-theory of derivators revisited

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We define a $K$-theory for pointed right derivators and show that it agrees with Waldhausen $K$-theory in the case where the derivator arises from a good Waldhausen category. This $K$-theory is not invariant under general equivalences of derivators, but only under a stronger notion of equivalence that is defined by considering a simplicial enrichment of the category of derivators. We show that derivator $K$-theory, as originally defined, is the best approximation to Waldhausen $K$-theory by a functor that is invariant under equivalences of derivators.



## 1. Introduction

Recent developments in the theory of derivators have shown both that the theory is sufficiently rich to contend for an independent approach to homotopical algebra and that its language is very useful in formulating precisely universal properties in homotopy theory. Since models for homotopical algebra typically give rise to derivators, the theory reflects a minimalist approach employing basically only purely (2-)categorical arguments, albeit technically quite complex at times, to address problems of abstract homotopy theory.

Derivators codify structure lying somewhere between the model and its associated homotopy category, but fairly closer to the model than the homotopy category,

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and restructure the presentation of the homotopy theory defined by the model in a surprisingly efficient way. This intermediate structure involves the collection of all homotopy categories of diagrams of various shapes in the model together with the network of restriction functors between them and their adjoint homotopy Kan extensions. The theory of derivators is based on an abstract axiomatization of collections of such (homotopy) Kan extensions (and their adjoints), which does not involve any underlying model. On the one hand, it is often the case that questions about the model are really questions about the associated derivator and thus they can instructively be handled more abstractly at this level of generality. On the other hand, for the theory to be successful, one is normally required to supply a large amount of data in order to compensate for the lack of an underlying homotopy theory and, consequently, working with these objects can be cumbersome.

The main problem is to understand how close this passage from the model to its associated derivator actually is to being faithful. This paper is a contribution to this problem in connection with Waldhausen $K$-theory regarded as an invariant of homotopy theories. The proven failure to reconstruct $K$-theory from the triangulated structure of the homotopy category in such a way that it satisfies certain desirable properties [Schlichting 2002] suggested turning to the more highly structured world of derivators for such a reconstruction. Indeed, it turned out that the structure of a derivator is rich enough to allow for a natural definition of $K$-theory. This was introduced by Maltsiniotis [2007], who also conjectured that it satisfies an *additivity* property, *agreement* with Quillen $K$-theory and a *localization* property. In previous work [Muro and Raptis 2011], we showed that *agreement* fails for Waldhausen $K$-theory and moreover, that derivator $K$-theory cannot satisfy both *agreement* with Quillen $K$-theory and the *localization* property. On the other hand, Cisinski and Neeman [2008] showed that *additivity* for derivator $K$-theory holds for triangulated derivators.

The purpose of this paper is twofold. First, we define a new $K$-theory of derivators, which we also call Waldhausen $K$-theory, and show in Theorem 4.3.1 that it agrees with the usual Waldhausen $K$-theory for all well-behaved Waldhausen categories. The proof rests crucially on the homotopically flexible versions of the $\mathcal{S}_\bullet$-construction due to Blumberg and Mandell [2011] and Cisinski [2010b].

The price to be paid for such a strong version of agreement is that this new definition is *provably* not invariant under equivalences of derivators. Here it is worthwhile to recall that Toën and Vezzosi [2004] showed that Waldhausen $K$-theory cannot factor through the 2-category of derivators. However, we consider here a simplicial enrichment of the category of derivators which enhances the 2-categorical structure. This leads to a stronger and more refined notion of equivalence of derivators which basically encodes higher coherence and is closer to an equivalence of homotopy theories. We show that Waldhausen $K$-theory of derivators is invariant under this stronger notion of equivalence. We think that the simplicial enrichment of the category of derivators and the accompanying stronger notion of equivalence have independent interest and may prove useful also in other applications of the theory.

There is a natural comparison transformation from Waldhausen $K$-theory to derivator $K$-theory. The second main result of the paper (Theorem 5.2.2) says that this comparison transformation is homotopically initial among all natural transformations from Waldhausen $K$-theory to a functor which is invariant under equivalences of derivators.

The paper is organized as follows. In Section 2, we review some background material from the (2-categorical) theory of derivators and fix some notational conventions. In Section 3, we discuss the simplicial enrichment of the category of derivators, the associated notion of strong equivalence, and the comparison with the 2-categorical viewpoint.

Section 4 is concerned with the definition of Waldhausen $K$-theory for derivators and some of its general properties. We present two canonically homotopy equivalent models for Waldhausen $K$-theory, both of which we use in the paper. Then we show that the Waldhausen $K$-theory of derivators is invariant under strong equivalences of derivators and agrees with the Waldhausen $K$-theory of derivable strongly saturated Waldhausen categories. In Section 5, we recall the definition of derivator $K$-theory and discuss its dependence on the 2-categorical theory of derivators. Then we recall the definition of the comparison map from Waldhausen $K$-theory to derivator $K$-theory and show that derivator $K$-theory is the best approximation to Waldhausen $K$-theory by a functor that is invariant under equivalences of derivators.

There are several remaining open questions, regarding either the notion of strong equivalence or the $K$-theory of derivators, some of which are briefly mentioned in Section 6. The paper ends with two appendices on topics of related interest but which are, strictly speaking, independent of the rest of the paper. In Appendix A, we recall the results from the comparison between combinatorial model categories and the 2-category of derivators due to Renaudin [2009] and discuss some slight improvements with an eye towards understanding the comparison with the simplicial category of derivators. Appendix B is concerned with the approximation theorem in $K$-theory, which in a version due to Cisinski [2010b] shows that $K$-theory is invariant under derived equivalences, and a partial converse which shows that derived equivalences are detected by the homotopy type of the $\mathcal{S}_\bullet$-construction.

## 2. Preliminaries on (pre)derivators

**2.1.** ***Prederivators.*** Let $\mathcal{C}at$ denote the 2-category of small categories. We fix a 1- and 2-full sub-2-category of diagrams $\mathcal{D}ia \subset \mathcal{C}at$ which is closed under all required constructions appearing below (e.g., taking opposite categories or finite (co)products, passing to comma categories, etc.); see [Maltsiniotis 2007] for the precise list of axioms. We think of the collection of categories in $\mathcal{D}ia$ as possible shapes for indexing diagrams in other categories.

The smallest option for $\mathcal{D}ia$ is $\mathcal{P}os_f$, spanned by the finite posets, and the largest option is, of course, $\mathcal{C}at$ itself. An intermediate option, which appears prominently in connection with $K$-theory, is the 2-category of diagrams $\mathcal{D}ir_f$ spanned by the finite direct categories. Recall that a *finite direct category* is a small category whose nerve has finitely many nondegenerate simplices. This is equivalent to saying that the underlying graph spanned by the nonidentity arrows of the category has no cycles. Every finite poset is a finite direct category.

A *prederivator* (with domain $\mathcal{D}ia$) is[1] a strict 2-functor $\mathbb{D} : \mathcal{D}ia^{\mathrm{op}} \to \mathcal{C}at$. More explicitly, for every category $X$ in $\mathcal{D}ia$ there is a small category $\mathbb{D}(X)$, for every functor $f : X \to Y$ in $\mathcal{D}ia$ there is an *inverse image* functor

$$f^* = \mathbb{D}(f) : \mathbb{D}(Y) \to \mathbb{D}(X),$$

for every natural transformation $\alpha : f \Rightarrow g$ in $\mathcal{D}ia$ there is a natural transformation $\alpha^* = \mathbb{D}(\alpha) : f^* \Rightarrow g^*$,

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y, \end{tikzcd} \qquad \begin{tikzcd} \mathbb{D}(X) \arrow[r, phantom, "\alpha^*\Downarrow"] & \mathbb{D}(Y), \arrow[l, bend right, "f^*"'] \arrow[l, bend left, "g^*"] \end{tikzcd}$$

and all these are required to satisfy the obvious strict 2-functoriality properties.

A (1-)*morphism of prederivators* $\phi : \mathbb{D} \to \mathbb{D}'$ is a pseudonatural transformation of contravariant 2-functors, i.e., for every $X$ in $\mathcal{D}ia$ there is a functor

$$\phi(X) : \mathbb{D}(X) \to \mathbb{D}'(X),$$

and for every $f : X \to Y$ in $\mathcal{D}ia$ there is a natural isomorphism $\phi(f)$,

$$\begin{tikzcd} \mathbb{D}(Y) \arrow[r, "\phi(Y)"] \arrow[d, "f^*"'] & \mathbb{D}'(Y) \arrow[d, "f^*"] \arrow[dl, phantom, "\swarrow\phi(f)"] \\ \mathbb{D}(X) \arrow[r, "\phi(X)"'] & \mathbb{D}'(X) \end{tikzcd} \tag{2.1.1}$$

such that certain coherence laws are satisfied. The morphism $\phi$ is called *strict* if $\phi(X) f^* = f^* \phi(Y)$ and $\phi(f)$ is the identity natural transformation for every $f$.

[1] The reader should be warned about the slight variations of this definition that appear in the literature. These pertain to the choice of domain and the different ways of forming the opposite of a 2-category.

A 2-*morphism* $\tau : \phi \Rightarrow \phi'$ between 1-morphisms of prederivators is a *modification* of pseudonatural transformations. This is defined by a collection of suitably compatible natural transformations in $\mathcal{C}at$,

$$\begin{tikzcd} \mathbb{D}(X) \arrow[r, bend left, "\phi(X)"{name=U}] \arrow[r, bend right, "\phi'(X)"'{name=D}] & \mathbb{D}'(X) \arrow[Rightarrow, from=U, to=D, "\tau(X)"] \end{tikzcd}$$

for every $X$ in $\mathcal{D}ia$ (see, e.g., [Borceux 1994, Section 7.5] for the precise definitions).

Let $\mathtt{PDer}$ (resp. $\mathtt{PDer}^{\mathrm{str}}$) denote the resulting 2-category of prederivators, morphisms (resp. strict morphisms) and 2-morphisms. This is an example of a 2-category formed by 2-functors, pseudonatural (or 2-natural) transformations and modifications (see [Borceux 1994, Propositions 7.3.3 and 7.5.4]). The notion of *equivalence* of prederivators is defined in the usual way in terms of the 2-categorical structure of $\mathtt{PDer}$. Equivalently, a morphism $\phi : \mathbb{D} \to \mathbb{D}'$ is an equivalence if and only if $\phi(X) : \mathbb{D}(X) \to \mathbb{D}'(X)$ is an equivalence of categories for every $X$ in $\mathcal{D}ia$. We also consider the 1-full sub-2-categories $\mathtt{PDer}_{\mathrm{eq}}$ and $\mathtt{PDer}^{\mathrm{str}}_{\mathrm{eq}}$ of $\mathtt{PDer}$ and $\mathtt{PDer}^{\mathrm{str}}$ respectively, which have the same objects and 1-morphisms but whose 2-morphisms are the invertible modifications. These are categories enriched in groupoids.

**Remark 2.1.2.** A basic example of a prederivator is the representable prederivator defined by a small category $X$:

$$\mathcal{C}at(-, X) : \mathcal{D}ia^{\mathrm{op}} \to \mathcal{C}at.$$

This construction yields a 2-categorical Yoneda functor

$$\mathcal{C}at \to \mathtt{PDer}^{\mathrm{str}},$$

which is 1- and 2-fully faithful when restricted to $\mathcal{D}ia$. If we restrict to the 1-full sub-2-category whose 2-morphisms are the invertible natural transformations, we obtain a 2-functor to $\mathtt{PDer}^{\mathrm{str}}_{\mathrm{eq}}$.

Let $e$ denote the final category with one object $e$ and one morphism $\mathrm{id}_e$. Given a small category $X$, there is a canonical isomorphism of categories $i_{X,-} : X \cong \mathcal{C}at(e, X)$ defined as follows. An object $x \in \mathrm{Ob}\, X$ defines a functor $i_{X,x} : e \to X$ with $i_{X,x}(e) = x$, and a morphism $g : x \to x'$ in $X$ induces a natural transformation $i_{X,g} : i_{X,x} \Rightarrow i_{X,x'}$ with $i_{X,g}(e) = g$.

Let $X$ be a category in $\mathcal{D}ia$. For every prederivator $\mathbb{D}$ there is a functor

$$\mathrm{dia}_{X,e} : \mathbb{D}(X) \to \mathcal{C}at(X, \mathbb{D}(e)) \tag{2.1.3}$$

which sends an object $F$ in $\mathbb{D}(X)$ to the functor $\mathrm{dia}_{X,e}(F) : X \to \mathbb{D}(e)$ defined by

$$\mathrm{dia}_{X,e}(F)(x) = i_{X,x}^* F, \qquad \mathrm{dia}_{X,e}(F)(g : x \to x') = i_{X,g}^* F,$$

and a morphism $\varphi : F \to G$ in $\mathbb{D}(X)$ to $\mathrm{dia}_{X,e}(\varphi) : \mathrm{dia}_{X,e}(F) \Rightarrow \mathrm{dia}_{X,e}(G)$, the natural transformation given by

$$\mathrm{dia}_{X,e}(\varphi)(x) = i_{X,x}^* \varphi.$$

This suggests a useful analogy, namely, to regard $\mathbb{D}(e)$ as the underlying category of $\mathbb{D}$ and the elements of $\mathbb{D}(X)$ as $X$-indexed diagrams in $\mathbb{D}$. We will often write $F_x$ for $i_{X,x}^*(F)$.

**Remark 2.1.4.** The functors (2.1.3) assemble to a morphism of prederivators

$$\mathrm{dia}_{-,e} : \mathbb{D} \to \mathcal{C}at(-, \mathbb{D}(e))$$

which is the unit of the 2-adjoint pair

$$\mathtt{PDer}^{\mathrm{str}} \underset{\text{2-Yoneda}}{\overset{\text{evaluation at } e}{\rightleftarrows}} \mathcal{C}at.$$

The counit is the natural isomorphism $\mathcal{C}at(e, X) \cong X$ described above.

The product of 2-categories is 2-functorial, hence for any $Y$ in $\mathcal{D}ia$ and any prederivator $\mathbb{D}$, we obtain a new prederivator

$$\mathbb{D}_Y := \mathbb{D}(- \times Y) : \mathcal{D}ia^{\mathrm{op}} \to \mathcal{C}at.$$

The morphism $\mathrm{dia}_{X,e}$ for this new prederivator will be denoted by

$$\mathrm{dia}_{X,Y} : \mathbb{D}(X \times Y) \to \mathcal{C}at(X, \mathbb{D}(Y)). \tag{2.1.5}$$

Here we use the obvious isomorphism $e \times Y \cong Y$ as an identification. This functor sends an object $F$ in $\mathbb{D}(X \times Y)$ to the functor $\mathrm{dia}_{X,Y} : X \to \mathbb{D}(Y)$ defined by

$$\mathrm{dia}_{X,Y}(F)(x) = (i_{X,x} \times Y)^* F, \qquad \mathrm{dia}_{X,Y}(F)(g : x \to x') = (i_{X,g} \times Y)^* F,$$

and a morphism $\varphi : F \to G$ in $\mathbb{D}(X \times Y)$ to $\mathrm{dia}_{X,Y}(\varphi) : \mathrm{dia}_{X,Y}(F) \Rightarrow \mathrm{dia}_{X,Y}(G)$, the natural transformation given by

$$\mathrm{dia}_{X,Y}(\varphi)(x) = (i_{X,x} \times Y)^* \varphi.$$

The functor $\mathrm{dia}_{X,Y}$ may be viewed as taking an $(X \times Y)$-indexed diagram to the underlying $X$-diagram of $Y$-indexed diagrams in $\mathbb{D}$.

**2.2. *Derivators.*** A (*right* or *left, pointed, stable/triangulated*) *derivator* is a prederivator that satisfies certain additional properties. We only briefly review the definitions here. With the exception of Appendix A, we are mainly concerned with the case of pointed right derivators.

A *right derivator* is a prederivator $\mathbb{D}$ satisfying the following properties:

(Der1) For every pair of small categories $X$ and $Y$ in $\mathcal{D}ia$, the functor induced by the inclusions of the factors to the coproduct $X \sqcup Y$,

$$\mathbb{D}(X \sqcup Y) \to \mathbb{D}(X) \times \mathbb{D}(Y),$$

is an equivalence of categories. Moreover, $\mathbb{D}(\varnothing)$ is the final category $e$.

(Der2) For every small category $X$ in $\mathcal{D}ia$, the functor

$$(i^*_{X,x})_{x\in \mathrm{Ob}\, X} : \mathbb{D}(X) \to \prod_{x\in \mathrm{Ob}\, X} \mathbb{D}(e)$$

reflects isomorphisms.

(Der3) For every morphism $f : X \to Y$ in $\mathcal{D}ia$, the inverse image $f^* : \mathbb{D}(Y) \to \mathbb{D}(X)$ admits a left adjoint $f_! : \mathbb{D}(X) \to \mathbb{D}(Y)$.

(Der4) Given $f : X \to Y$ in $\mathcal{D}ia$ and $y$ an object of $Y$, consider the following diagram in $\mathcal{D}ia$:

$$\begin{tikzcd} f\downarrow y \arrow[r, "j_{f,y}"] \arrow[d, "p_{f\downarrow y}"'] & X \arrow[d, "f"] \arrow[dl, phantom, "\swArrow\,\alpha_{f,y}"] \\ e \arrow[r, "i_{Y,y}"'] & Y \end{tikzcd}$$

Here $f \downarrow y$ is the comma category whose objects $(x, f(x) \to y)$ are pairs given by an object $x$ in $X$ and a map $f(x) \to y$ in $Y$, $j_{f,y}$ is the functor $j_{f,y}(x, f(x) \to y) = x$, and $\alpha_{f,y}(x, f(x) \to y) = (f(x) \to y)$. Then the diagram obtained by applying $\mathbb{D}$ satisfies the Beck–Chevalley condition, i.e., the mate natural transformation

$$c_{f,y} : (p_{f\downarrow y})_! j^*_{f,y} \Longrightarrow i^*_{Y,y} f_!,$$

which is the adjoint of

$$j^*_{f,y} \xRightarrow{\substack{\text{unit of}\\ f_!\dashv f^*}} j^*_{f,y} f^* f_! = (f j_{f,y})^* f_! \xRightarrow{\alpha^*_{f,y} f_!} (i_{Y,y} p_{f\downarrow y})^* f_! = p^*_{f\downarrow y} i^*_{Y,y} f_!,$$

is a natural isomorphism.

A *left derivator* $\mathbb{D}$ is a prederivator whose *opposite prederivator* $\mathbb{D}^{\mathrm{op}}$, defined by $\mathbb{D}^{\mathrm{op}}(X) = \mathbb{D}(X^{\mathrm{op}})^{\mathrm{op}}$, is a right derivator. A prederivator which is both a left and a right derivator is simply called a *derivator*. There is yet another axiom that a prederivator may satisfy:

(Der5) For every pair of small categories $X$ and $\mathcal{I}$ in $\mathcal{D}ia$ where $\mathcal{I}$ is a free finite category, the canonical functor

$$\mathrm{dia}_{\mathcal{I},X} : \mathbb{D}(\mathcal{I} \times X) \to \mathcal{C}at(\mathcal{I}, \mathbb{D}(X))$$

is full and essentially surjective.

The inclusion of this axiom in the definition of derivator is somewhat controversial in the literature. Heller [1988] includes (Der5) as part of the definition. Other authors prefer either to omit it and reserve it for an additional "strongness" property of a derivator, or to replace it with the seemingly weaker version in which $\mathcal{I} = [1]$; see, e.g., [Maltsiniotis 2001; 2007; Franke 1996; Groth 2013]. The inclusion of (Der5) matters very little for our purposes here, but we choose to exclude it from the basic definition.

We recall that a small category is called *pointed* if it has a zero object. A functor between pointed categories is called pointed if it preserves zero objects. A prederivator $\mathbb{D}$ is called *pointed* if $\mathbb{D}(X)$ is a pointed category and $f^* : \mathbb{D}(Y) \to \mathbb{D}(X)$ is a pointed functor for all $X$ and $f : X \to Y$ in $\mathcal{D}ia$. This definition follows Groth [2013], who showed that it is equivalent for derivators to the original definition; see, e.g., [Maltsiniotis 2007].

We recall the definition of cocartesian squares for right derivators. Consider the "commutative square" category $\square = [1] \times [1]$. A *commutative square* in a prederivator $\mathbb{D}$ is an object $F$ of $\mathbb{D}(\square)$. There is a subcategory $\ulcorner \subseteq \square$ which can be depicted as follows:

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (1,0) \arrow[d] \\ (0,1) \arrow[r] & (1,1) \end{tikzcd} \qquad \begin{tikzcd} (0,0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & (1,0) \\ (0,1) & \end{tikzcd}$$

Denote the inclusion functor by

$$i_\ulcorner : \ulcorner \to \square.$$

If $\mathbb{D}$ is a right derivator, a commutative square $F$ in $\mathbb{D}$ is called *cocartesian* if the counit

$$(i_\ulcorner)_! i_\ulcorner^* F \to F$$

is an isomorphism. If $\mathbb{D}$ is a left derivator, a commutative square $F$ in $\mathbb{D}$ is called *cartesian* if it is cocartesian in $\mathbb{D}^{\mathrm{op}}$. A pointed derivator which satisfies (Der5) is called *stable* (or *triangulated*) if cocartesian and cartesian squares coincide.

A crucial point to note regarding all of the above definitions is that only the notion of a prederivator constitutes *structure*, while the additional axioms assert *properties*. We also emphasize that the property of being a right (or left, pointed, stable/triangulated) derivator is invariant under equivalences of prederivators. Moreover, if $\mathbb{D}$ is a pointed right derivator, then so is $\mathbb{D}_Y$ for every $Y$ in $\mathcal{D}ia$.

A morphism of right derivators $\phi : \mathbb{D} \to \mathbb{D}'$ is called *cocontinuous* if for every $f : X \to Y$ in $\mathcal{D}ia$, the canonical natural transformation

$$f_! \phi(X) \Longrightarrow \phi(Y) f_!,$$

adjoint to

$$\phi(X) \xRightarrow[\;]{\text{unit of } f_! \dashv f^*} \phi(X) f^* f_! \xRightarrow{\phi(f)^{-1} f_!} f^* \phi(Y) f_!,$$

is an isomorphism.[2] An easy application of (Der1) shows that the components of a cocontinuous morphism between pointed right derivators are automatically pointed functors.

Let $\mathtt{Der}$ and $\mathtt{Der}^{\text{str}}$ denote the 2-full sub-2-categories of $\mathtt{PDer}$ and $\mathtt{PDer}^{\text{str}}$, respectively, given by the pointed right derivators, cocontinuous (strict) morphisms and 2-morphisms. Let $\mathtt{Der}_{\text{eq}}$ and $\mathtt{Der}^{\text{str}}_{\text{eq}}$ denote the 1-full sub-2-categories of $\mathtt{Der}$ and $\mathtt{Der}^{\text{str}}$ whose 2-morphisms are the invertible modifications.

**2.3.** ***Examples.*** The examples of prederivators that we are interested in arise from categories with weak equivalences as follows. Let $(\mathcal{C}, \mathcal{W})$ be a pair consisting of a small category $\mathcal{C}$ together with a subcategory $\mathcal{W}$ which contains the isomorphisms. The morphisms of $\mathcal{W}$ are called *weak equivalences*. The *homotopy category* of $(\mathcal{C}, \mathcal{W})$ is the localization

$$\operatorname{Ho}\mathcal{C} := \mathcal{C}[\mathcal{W}^{-1}].$$

For every object $X$ in $\mathcal{D}ia$, the diagram category $\mathcal{C}^X$ together with the subcategory of objectwise weak equivalences of functors is again a category with weak equivalences $(\mathcal{C}^X, \mathcal{W}^X)$. The choice of objectwise weak equivalences is natural in $X$, so there is a prederivator $\mathbb{D}(\mathcal{C}, \mathcal{W}) : \mathcal{D}ia^{\text{op}} \to \mathcal{C}at$ given by the homotopy categories of all relevant diagram categories, i.e., it is defined on objects by

$$\mathbb{D}(\mathcal{C}, \mathcal{W})(X) := \operatorname{Ho}(\mathcal{C}^X)$$

and on 1- and 2-morphisms in the canonically induced way.

A functor $F : \mathcal{C} \to \mathcal{C}'$ that preserves the weak equivalences $F(\mathcal{W}) \subset \mathcal{W}'$ induces a (strict) morphism of prederivators $\mathbb{D}(F) : \mathbb{D}(\mathcal{C}, \mathcal{W}) \to \mathbb{D}(\mathcal{C}', \mathcal{W}')$. Such functors are called *homotopical*. A homotopical functor $F : \mathcal{C} \to \mathcal{C}'$ is a *derived equivalence* if

[2]This condition is comparable to *right* exactness of a functor. This justifies the term *right* derivator.

it induces an equivalence of homotopy categories $\operatorname{Ho} F : \operatorname{Ho} \mathcal{C} \xrightarrow{\simeq} \operatorname{Ho} \mathcal{C}'$. A natural transformation $\alpha : F \Rightarrow F'$ of homotopical functors $F, F' : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ defines a 2-morphism $\mathbb{D}(\alpha) : \mathbb{D}(F) \Rightarrow \mathbb{D}(F')$ in $\mathtt{PDer}^{\mathrm{str}}$. If the components of the natural transformation $\alpha$ are given by weak equivalences, then $\mathbb{D}(\alpha)$ is in $\mathtt{PDer}^{\mathrm{str}}_{\mathrm{eq}}$.

We note that if $\mathcal{W}$ is the subcategory of isomorphisms, $\mathbb{D}(\mathcal{C}, \mathcal{W}) = \mathcal{C}at(-, \mathcal{C})$ is the representable prederivator of Remark 2.1.2. We will normally write $\mathbb{D}(\mathcal{C})$ when the choice of $\mathcal{W}$ is clear from the context.

For well-behaved categories with weak equivalences $(\mathcal{C}, \mathcal{W})$, the associated prederivator $\mathbb{D}(\mathcal{C})$ is a (right or left, pointed, stable/triangulated) derivator. We refer the reader to [Cisinski 2010a] for a systematic treatment of the results in this direction. Here we will be particularly concerned with categories with weak equivalences that arise from Waldhausen categories [Waldhausen 1985]. Following [Cisinski 2010b], we say that a Waldhausen category $(\mathcal{C}, co\mathcal{C}, w\mathcal{C})$ is *derivable* if it satisfies the "2-out-of-3" axiom and every morphism in $\mathcal{C}$ can be written as the composition of a cofibration followed by a weak equivalence. The following theorem is a special case of results proved in [Cisinski 2010a; 2010b].

**Theorem 2.3.1** (Cisinski). (a) *Let $(\mathcal{C}, co\mathcal{C}, w\mathcal{C})$ be a derivable Waldhausen category. Then the associated prederivator $\mathbb{D}(\mathcal{C}) : \mathcal{D}ir_f^{\mathrm{op}} \to \mathcal{C}at$ is a pointed right derivator which also satisfies* (Der5).

(b) *An exact functor of derivable Waldhausen categories*

$$F : (\mathcal{C}, co\mathcal{C}, w\mathcal{C}) \to (\mathcal{C}', co\mathcal{C}', w\mathcal{C}')$$

*induces a cocontinuous morphism $\mathbb{D}(F) : \mathbb{D}(\mathcal{C}) \to \mathbb{D}(\mathcal{C}')$ in* $\mathtt{Der}^{\mathrm{str}}$.

(c) *Moreover, the morphism $\mathbb{D}(F) : \mathbb{D}(\mathcal{C}) \to \mathbb{D}(\mathcal{C}')$ is an equivalence in* $\mathtt{Der}^{\mathrm{str}}$ *if and only if* $\operatorname{Ho} F : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{C}'$ *is an equivalence of categories.*

A derivable Waldhausen category $(\mathcal{C}, co\mathcal{C}, w\mathcal{C})$ is called *strongly saturated* if it satisfies the property that a morphism in $\mathcal{C}$ is a weak equivalence if and only if it becomes an isomorphism in the homotopy category. A derivable Waldhausen category with functorial factorizations is strongly saturated if and only if the weak equivalences are closed under retracts; see [Blumberg and Mandell 2011, Theorems 5.5 and 6.4].

## 3. Simplicial enrichments of (pre)derivators

We recall that the *simplex category* $\Delta$ consists of the finite ordinals

$$[n] = \{0 < \cdots < n\}$$

for $n \geq 0$, and the nondecreasing maps between them. Thus, it is contained in $\mathcal{D}ir_f$, and in fact also in any other possible category of diagrams $\mathcal{D}ia$. The naturality of the construction $Y \mapsto \mathbb{D}_Y$ shows that there is a 2-functor

$$\mathcal{D}ia^{\mathrm{op}} \times \mathtt{PDer}^{\mathrm{str}} \to \mathtt{PDer}^{\mathrm{str}}$$

which may be regarded as a "cotensor 2-structure" of $\mathtt{PDer}^{\mathrm{str}}$ over $\mathcal{D}ia$. Using this, we can associate to every prederivator $\mathbb{D}$ a simplicial object $\mathbb{D}_\bullet$ in $\mathtt{PDer}^{\mathrm{str}}$ with

$$\mathbb{D}_n = \mathbb{D}([n] \times -).$$

In particular, we have $\mathbb{D}_0 = \mathbb{D}$. Faces and degeneracies are morphisms of prederivators since both $\mathbb{D}$ and the product of 2-categories are 2-functorial. This natural simplicial object will be used to define an enrichment of the underlying category of $\mathtt{PDer}^{\mathrm{str}}$ over simplicial sets.

**3.1.** ***Definition of*** $\underline{\mathtt{PDer}}^{\mathbf{str}}$**.** We define a simplicially enriched category $\underline{\mathtt{PDer}}^{\mathrm{str}}$ with prederivators as objects and morphism simplicial sets

$$\underline{\mathtt{PDer}}^{\mathrm{str}}(\mathbb{D}, \mathbb{D}')_\bullet = \operatorname{Ob} \mathtt{PDer}^{\mathrm{str}}(\mathbb{D}, \mathbb{D}'_\bullet).$$

The composition is defined by simplicial maps

$$\underline{\mathtt{PDer}}^{\mathrm{str}}(\mathbb{D}', \mathbb{D}'')_\bullet \times \underline{\mathtt{PDer}}^{\mathrm{str}}(\mathbb{D}, \mathbb{D}')_\bullet \to \underline{\mathtt{PDer}}^{\mathrm{str}}(\mathbb{D}, \mathbb{D}'')_\bullet$$

which send pairs of strict morphisms

$$\begin{aligned} \phi &: \mathbb{D} \to \mathbb{D}'([n] \times -), \\ \psi &: \mathbb{D}' \to \mathbb{D}''([n] \times -) \end{aligned}$$

to the composite

$$\mathbb{D} \xrightarrow{\phi} \mathbb{D}'([n] \times -) \xrightarrow{\psi([n]\times -)} \mathbb{D}''([n] \times [n] \times -) \xrightarrow{\mathbb{D}''(\triangle \times -)} \mathbb{D}''([n] \times -),$$

where $\triangle : [n] \to [n] \times [n]$ is the diagonal functor.

To see that the composition is associative, consider strict morphisms

$$\begin{aligned} \phi &: \mathbb{D} \to \mathbb{D}'([n] \times -), \\ \psi &: \mathbb{D}' \to \mathbb{D}''([n] \times -), \\ \xi &: \mathbb{D}'' \to \mathbb{D}'''([n] \times -). \end{aligned}$$

Then it suffices to show that the leftmost and rightmost morphisms in the following diagram coincide.

$$
\begin{tikzcd}
 & \mathbb{D} \arrow[d, "\phi"'] \arrow[ddr, bend left, "\psi\phi"] \arrow[ddddd, bend right=80, "(\xi\psi)\phi"'] \arrow[ddddd, bend left=80, "\xi(\psi\phi)"] & \\
 & \mathbb{D}'([n]\times -) \arrow[d, "\psi([n]\times -)"'] \arrow[ddd, bend right=70, "(\xi\psi)([n]\times -)"'] & \\
 & \mathbb{D}''([n]\times[n]\times -) \arrow[dl, "\xi([n]\times[n]\times -)"'] \arrow[dr, "\mathbb{D}''(\triangle\times -)"] & \\
\mathbb{D}'''([n]\times[n]\times[n]\times -) \arrow[dr, "\mathbb{D}'''(\triangle\times[n]\times -)"'] & & \mathbb{D}''([n]\times -) \arrow[dl, "\xi([n]\times -)"] \\
 & \mathbb{D}'''([n]\times[n]\times -) \arrow[d, "\mathbb{D}'''(\triangle\times -)"'] & \\
 & \mathbb{D}'''([n]\times -) &
\end{tikzcd}
$$

All cells in this diagram commute by definition, except for the inner square. If the inner square were commutative, the result would follow immediately. However, the post-composition of the square with $\mathbb{D}'''(\triangle \times -)$ yields a commutative square, and this suffices. Indeed, since the diagonal functor is coassociative, i.e., $(\triangle \times [n])\triangle = ([n] \times \triangle)\triangle$, it is enough to show that the slightly different square

$$
\begin{tikzcd}
 & \mathbb{D}''([n]\times[n]\times -) \arrow[dl, "\xi([n]\times[n]\times -)"'] \arrow[dr, "\mathbb{D}''(\triangle\times -)"] & \\
\mathbb{D}'''([n]\times[n]\times[n]\times -) \arrow[dr, "\mathbb{D}'''([n]\times\triangle\times -)"'] & & \mathbb{D}''([n]\times -) \arrow[dl, "\xi([n]\times -)"] \\
 & \mathbb{D}'''([n]\times[n]\times -) &
\end{tikzcd}
$$

commutes. Note that the only difference between this last square and the inner square in the previous diagram is in the lower left arrow. The latter square commutes because $\xi$ is a strict morphism.

This simplicial enrichment can be used to introduce homotopy theoretic notions into the world of prederivators, but these will be too coarse for our purposes here. For a more appropriate notion of homotopy, we consider the subobject of $\mathbb{D}_\bullet$ defined by "simplicially constant" diagrams.

**3.2.** ***Definition of*** $\underline{\mathtt{PDer}}^{\mathbf{str}}_{\mathbf{eq}}$**.** Given a prederivator $\mathbb{D}$ and $Y$ in $\mathcal{D}ia$, there is a prederivator $\mathbb{D}(Y\times -)_{\mathrm{eq}}$ equipped with a strict morphism $i_{\mathrm{eq}} : \mathbb{D}(Y\times -)_{\mathrm{eq}} \to \mathbb{D}(Y\times -)$ such that for all $X$ in $\mathcal{D}ia$,

$$i_{\mathrm{eq}}(Y) : \mathbb{D}(Y\times X)_{\mathrm{eq}} \to \mathbb{D}(Y\times X)$$

is the inclusion of the full subcategory spanned by the objects $F$ such that the underlying $Y$-diagram

$$\mathrm{dia}_{Y,X}(F) : Y \longrightarrow \mathbb{D}(X)$$

sends each morphism of $Y$ to an isomorphism in $\mathbb{D}(X)$.

To show that this is well-defined, it is enough to check that given $f : X \to Z$ in $\mathcal{D}ia$ and $F$ in $\mathbb{D}(Y\times Z)_{\mathrm{eq}}$, the object $(Y\times f)^*(F)$ in $\mathbb{D}(Y\times X)$ is actually in $\mathbb{D}(Y\times X)_{\mathrm{eq}}$. Let $g : y \to y'$ be a morphism in $Y$. We have

$$\begin{aligned}(i_{Y,g}\times X)^*(Y\times f)^*(F) &= ((Y\times f)(i_{Y,g}\times X))^*(F)\\ &= (i_{Y,g}\times f)^*(F)\\ &= ((i_{Y,g}\times Z)(Y\times f))^*(F)\\ &= (e\times f)^*(i_{Y,g}\times Z)^*(F).\end{aligned}$$

Since $F$ is in $\mathbb{D}(Y\times Z)_{\mathrm{eq}}$, it follows that $(i_{Y,g}\times Z)^*(F)$ is an isomorphism. Hence, so is $(i_{Y,g}\times X)^*(Y\times f)^*(F)$ for any morphism $g$ in $Y$, and therefore $(Y\times f)^*(F)$ is in $\mathbb{D}(Y\times X)_{\mathrm{eq}}$. (See also Remark 2.1.4.)

Hence, for any prederivator $\mathbb{D}$, there is a simplicial prederivator $\mathbb{D}_{\mathrm{eq},\bullet}$ with $\mathbb{D}_{\mathrm{eq},n} = \mathbb{D}([n]\times -)_{\mathrm{eq}}$ equipped with a morphism $i_{\mathrm{eq}} : \mathbb{D}_{\mathrm{eq},\bullet} \to \mathbb{D}_\bullet$ of simplicial prederivators. Note that $\mathbb{D}_{\mathrm{eq},0} = \mathbb{D}$.

We define a simplicially enriched category $\underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}}$ with prederivators as objects and morphism simplicial sets

$$\underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}}(\mathbb{D},\mathbb{D}')_\bullet = \mathrm{Ob}\,\mathtt{PDer}^{\mathrm{str}}(\mathbb{D},\mathbb{D}'_{\mathrm{eq},\bullet})$$

such that the morphisms of simplicial prederivators $i_{\mathrm{eq}} : \mathbb{D}'_{\mathrm{eq},\bullet} \to \mathbb{D}'_\bullet$ induce a simplicial functor $i_{\mathrm{eq}} : \underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}} \to \underline{\mathtt{PDer}}^{\mathrm{str}}$.

To show that this is a well-defined simplicial subcategory, we check that the composition in $\underline{\mathtt{PDer}}^{\mathrm{str}}$ of two composable $n$-simplices in $\underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}}$ is again in $\underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}}$, i.e., given strict morphisms $\phi : \mathbb{D} \to \mathbb{D}'([n]\times -)_{\mathrm{eq}}$ and $\psi : \mathbb{D}' \to \mathbb{D}''([n]\times -)_{\mathrm{eq}}$, we show that the composite

$$\mathbb{D} \xrightarrow{\phi} \mathbb{D}'([n]\times -) \xrightarrow{\psi([n]\times -)} \mathbb{D}''([n]\times[n]\times -) \xrightarrow{(\triangle\times -)^*} \mathbb{D}''([n]\times -)$$

takes values in $\mathbb{D}''([n]\times -)_{\mathrm{eq}}$. Given an object $F$ in $\mathbb{D}(X)$ and a morphism $g : x \to x'$ in $[n]$, consider the following diagram:

$$\begin{tikzcd}
\mathbb{D}(X) \arrow[r, "\phi(X)"] & \mathbb{D}'([n]\times X) \arrow[r, "\psi([n]\times X)"] \arrow[d, bend right=50, "(i_{[n],x}\times X)^*"'] \arrow[d, bend left=50, "(i_{[n],x'}\times X)^*"] \arrow[d, phantom, "\overset{(i_{[n],g}\times X)^*}{\Longrightarrow}"] & \mathbb{D}''([n]\times[n]\times X) \arrow[r, "(\triangle\times X)^*"] \arrow[d, bend right=50, "([n]\times i_{[n],x}\times X)^*"'] \arrow[d, bend left=50, "([n]\times i_{[n],x'}\times X)^*"] \arrow[d, phantom, "\overset{([n]\times i_{[n],g}\times X)^*}{\Longrightarrow}"] & \mathbb{D}''([n]\times X) \arrow[dd, bend right=40, "(i_{[n],x}\times X)^*"'] \arrow[dd, bend left=40, "(i_{[n],x'}\times X)^*"] \arrow[dd, phantom, "\overset{(i_{[n],g}\times X)^*}{\Longrightarrow}"] \\
& \mathbb{D}'(X) \arrow[r, "\psi(X)"'] & \mathbb{D}''([n]\times X) \arrow[d, bend right=50, "(i_{[n],x}\times X)^*"'] \arrow[d, bend left=50, "(i_{[n],x'}\times X)^*"] \arrow[d, phantom, "\overset{(i_{[n],g}\times X)^*}{\Longrightarrow}"] & \\
& & \mathbb{D}''(X) \arrow[r, equal] & \mathbb{D}''(X)
\end{tikzcd}$$

This diagram satisfies several commutativity properties. The subdiagram of functors formed by the straight arrows and the arrows which are curved to the left is commutative, and likewise the subdiagram of straight arrows and arrows curved to the right. In the middle square, the natural transformations

$$\psi(X)(i_{[n],g}\times X)^* = ([n]\times i_{[n],g}\times X)^*\psi([n]\times X)$$

coincide, since $\psi$ is a strict morphism. In the rightmost region, the two horizontally composable natural transformations compose to $(i_{[n],g}\times X)^*(\triangle\times X)^*$, since $\mathbb{D}''$ is a 2-functor.

Since $\phi$ takes values in $\mathbb{D}'([n]\times -)_{\mathrm{eq}}$, we have that $(i_{[n],g}\times X)^*\phi(X)$ is a natural isomorphism. Moreover, $\psi$ takes values in $\mathbb{D}''([n]\times -)_{\mathrm{eq}}$, and therefore $(i_{[n],g}\times X)^*\psi(X)$ is also a natural isomorphism. This, together with the aforementioned commutativity properties, shows that

$$(i_{[n],g}\times X)^*(\triangle\times X)^*\psi([n]\times X)\phi(X)(F)$$

is an isomorphism.

The passage from $\underline{\mathtt{PDer}}^{\mathrm{str}}$ to $\underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}}$ is reminiscent of the passage from the category of $\infty$-categories, regarded as an $(\infty, 2)$-category, to the associated $\infty$-category defined by restriction to the maximal $\infty$-groupoids of the morphism $\infty$-categories. More on the viewpoint that regards well-behaved types of prederivators as models for homotopy theories will be discussed in Appendix A; see also [Renaudin 2009].

**3.3.** ***Strong equivalences.*** The prederivator $\mathbb{D}([1]\times -)_{\mathrm{eq}}$ together with the factorization

$$\mathbb{D} \xrightarrow{s_0} \mathbb{D}([1]\times -)_{\mathrm{eq}} \xrightarrow{(d_1,d_0)} \mathbb{D}\times\mathbb{D}$$

of the diagonal natural transformation will be regarded as a path object associated with $\mathbb{D}$. We can now introduce some basic homotopical notions in the context of prederivators.

**Definition 3.3.1.** Let $\phi_0, \phi_1 : \mathbb{D} \to \mathbb{D}'$ be two strict morphisms of prederivators. A *strong isomorphism from* $\phi_0$ *to* $\phi_1$ is a 1-simplex of $\underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}}(\mathbb{D}, \mathbb{D}')$,

$$\Psi : \mathbb{D} \to \mathbb{D}'([1]\times -)_{\mathrm{eq}},$$

such that $d_1(\Psi) = \phi_0$ and $d_0(\Psi) = \phi_1$. We say that $\phi_0$ is *strongly isomorphic* to $\phi_1$, written $\phi_0 \simeq \phi_1$, if there is a zigzag of strong isomorphisms from $\phi_0$ to $\phi_1$.

Obviously the relation $\simeq$ is exactly the relation that two vertices of $\underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}}(\mathbb{D}, \mathbb{D}')$ lie on the same component.

**Definition 3.3.2.** Let $\mathbb{D}$ and $\mathbb{D}'$ be prederivators.

(a) A strict morphism $\phi : \mathbb{D} \to \mathbb{D}'$ is called a *strong* (or *coherent*) *equivalence* if there is a strict morphism $\psi : \mathbb{D}' \to \mathbb{D}$ such that $\mathrm{id}_{\mathbb{D}} \simeq \psi\phi$ and $\phi\psi \simeq \mathrm{id}_{\mathbb{D}'}$.

(b) $\mathbb{D}$ and $\mathbb{D}'$ are called *strongly* (or *coherently*) *equivalent* if there is a strong equivalence $\phi : \mathbb{D} \to \mathbb{D}'$.

**Remark 3.3.3.** A strong isomorphism $\Phi$ from $\phi_0$ to $\phi_1$ induces a natural isomorphism

$$\mathrm{dia}_{[1],-}(0 \to 1) : \phi_0 \Longrightarrow \phi_1.$$

From this follows that strong equivalences of prederivators are also equivalences in the 2-categorical sense of the previous section.

**Example 3.3.4.** For every prederivator $\mathbb{D}$ and any $X$ in $\mathcal{D}ia$ with an initial object $x_0 \in \mathrm{Ob}\, X$, the prederivator $\mathbb{D}(X\times -)_{\mathrm{eq}}$ is strongly equivalent to $\mathbb{D}$. Indeed, consider the morphisms

$$(p\times -)^* : \mathbb{D}\mathbb{D}(X\times -)_{\mathrm{eq}}, \qquad (i_{X,x_0}\times -)^* : \mathbb{D}(X\times -)_{\mathrm{eq}} \to \mathbb{D},$$

where $p : X \to e$ is the unique functor in this direction. Note that the underlying $X$-diagrams of elements in the image of $(p\times -)^*$ are constant functors (cf. Remark 2.1.4). Since $pi_{X,x_0}$ is the identity functor on $e$, we have

$$(i_{X,x_0}\times -)^*(p\times -)^* = \mathrm{id}_{\mathbb{D}}\,.$$

Moreover, $i_{X,x_0}p : X \to X$ is the constant functor $x \mapsto x_0$, and since $x_0$ is initial, there is a unique functor $H : [1]\times X \to X$ with $H(0,-) = i_{X,x_0}p$ and $H(1,-) = \mathrm{id}_X$.

The induced functor

$$(H \times -)^* : \mathbb{D}(X \times -)_{\mathrm{eq}} \to \mathbb{D}(([1] \times X) \times -)_{\mathrm{eq}} = \mathbb{D}(X \times -)_{\mathrm{eq},1}$$

is a strong isomorphism from $(p \times -)^*(i_{X,x_0} \times -)^*$ to $\mathrm{id}_{\mathbb{D}(X\times -)_{\mathrm{eq}}}$. One can argue similarly if $X$ has a final object. This shows, in particular, that the face and degeneracy operators in $\mathbb{D}_{\mathrm{eq},\bullet}$ are strong equivalences.

The notion of strong equivalence differs from the standard notion of equivalence defined in terms of the 2-categorical structure of $\mathtt{PDer}$. This observation will be crucial in connection with the definitions of $K$-theory that follow in the next sections.

**Example 3.3.5.** Let $\mathbb{D}$ be a prederivator and $\mathrm{iso}_n\,\mathbb{D}$ denote the prederivator for which $(\mathrm{iso}_n\,\mathbb{D})(X)$ is the full subcategory spanned by the strings of $n$ composable isomorphisms in the diagram category of $\mathcal{C}at([n], \mathbb{D}(X))$. Then the canonical "inclusion of identities" morphism $\mathbb{D} \to \mathrm{iso}_n\,\mathbb{D}$ is clearly an equivalence of prederivators, but not a strong equivalence in general. This assertion is a consequence of the invariance properties of Waldhausen $K$-theory and will be justified in Remark 5.1.5 below.

**Remark 3.3.6.** In connection with the examples of Section 2.3, a natural transformation $\alpha : F \Rightarrow F'$ between homotopical functors $F, F' : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ induces a 1-simplex in $\underline{\mathtt{PDer}}^{\mathrm{str}}$,

$$\alpha_* : \mathbb{D}(\mathcal{C}', \mathcal{W}') \to \mathbb{D}(\mathcal{C}, \mathcal{W})([1] \times -),$$

which upgrades the 2-morphism $\mathbb{D}(\alpha) : \mathbb{D}(F) \Rightarrow \mathbb{D}(F')$. If $\alpha$ takes values in $\mathcal{W}'$ then $\alpha_*$ is a 1-simplex in $\underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}}$. This implies that for every homotopical functor $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ which admits a "homotopy inverse" (i.e., there is a homotopical functor $G : \mathcal{C}' \to \mathcal{C}$ such that the composites $FG$ and $GF$ can be connected to the respective identity functors via zigzags of natural weak equivalences), the associated morphism $\mathbb{D}(F)$ is a strong equivalence.

A 2-category $\mathcal{C}$ will be regarded as a simplicial category $N_\bullet\mathcal{C}$ via the nerve functor $N_\bullet : \mathcal{C}at \to \mathcal{SS}et$ from small categories to simplicial sets, which preserves products. We have a simplicial functor

$$\underline{\mathtt{PDer}}^{\mathrm{str}} \to N_\bullet\mathtt{PDer}^{\mathrm{str}},$$

which is the identity on objects, and is given on morphisms by the simplicial maps

$$\underline{\mathtt{PDer}}^{\mathrm{str}}(\mathbb{D}, \mathbb{D}') \to N_\bullet\mathtt{PDer}^{\mathrm{str}}(\mathbb{D}, \mathbb{D}') \tag{3.3.7}$$

defined using the functors $\mathrm{dia}_{[\bullet],-}$. These simplicial maps also restrict to simplicial maps

$$\underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}}(\mathbb{D}, \mathbb{D}') \to N_\bullet\mathtt{PDer}^{\mathrm{str}}_{\mathrm{eq}}(\mathbb{D}, \mathbb{D}')$$

which assemble to a simplicial functor

$$\rho : \underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}} \to N_\bullet \mathtt{PDer}^{\mathrm{str}}_{\mathrm{eq}}$$

given by the identity on objects.

Consider the following adjoint pairs

$$\mathcal{S}\mathcal{S}et \underset{N_\bullet}{\overset{\tau_1}{\rightleftarrows}} \mathcal{C}at \underset{\text{incl.}}{\rightleftarrows} \mathcal{G}rd.$$

Here $\mathcal{G}rd$ is the category of groupoids, $\mathcal{G}rd \to \mathcal{C}at$ is the inclusion, the lower arrows are the right adjoints, $\tau_1$ is the fundamental category functor, and the composite $\mathcal{S}\mathcal{S}et \to \mathcal{G}rd$ is the fundamental groupoid functor, denoted by $\Pi_1$. All these functors preserve products, hence, for example, we can apply them to a simplicial category $\mathcal{S}$ to obtain a 2-category $\tau_1\mathcal{S}$, or a category enriched in groupoids $\Pi_1\mathcal{S}$. In particular, the simplicial functors (3.3.7) and $\rho$ above also define 2-functors

$$\tau_1\underline{\mathtt{PDer}}^{\mathrm{str}} \to \mathtt{PDer}^{\mathrm{str}}, \qquad \Pi_1\underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}} \to \mathtt{PDer}^{\mathrm{str}}_{\mathrm{eq}},$$

by adjunction. These functors are not 2-equivalences of 2-categories. This means that the simplicial enrichment of the category of prederivators encodes more structure than the 2-category of prederivators.

Similarly let $\underline{\mathtt{Der}}^{\mathrm{str}}$ and $\underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}}$ denote the corresponding simplicial subcategories of $\underline{\mathtt{PDer}}^{\mathrm{str}}$ and $\underline{\mathtt{PDer}}^{\mathrm{str}}_{\mathrm{eq}}$, respectively. In both cases the objects are pointed right derivators, and for a pair of pointed right derivators $\mathbb{D}$ and $\mathbb{D}'$ we have

$$\underline{\mathtt{Der}}^{\mathrm{str}}(\mathbb{D}, \mathbb{D}')_\bullet = \operatorname{Ob} \mathtt{Der}^{\mathrm{str}}(\mathbb{D}, \mathbb{D}'_\bullet),$$
$$\underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}}(\mathbb{D}, \mathbb{D}')_\bullet = \operatorname{Ob} \mathtt{Der}^{\mathrm{str}}(\mathbb{D}, \mathbb{D}'_{\mathrm{eq},\bullet}).$$

This is well-defined because if $X$ is in $\mathcal{D}ia$ and $\mathbb{D}$ is a right (or left, pointed, stable/triangulated) derivator, then so are $\mathbb{D}(X \times -)$ and $\mathbb{D}(X \times -)_{\mathrm{eq}}$.

Specializing the discussion above to pointed right derivators, we define similarly a simplicial functor

$$\rho : \underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}} \to N_\bullet \mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}}.$$

Again, the associated 2-functor

$$\Pi_1\underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}} \to \mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}}$$

is not a 2-equivalence.

## 4. Waldhausen *K*-theory of derivators

In this section, we define the Waldhausen $K$-theory of a pointed right derivator and show that it agrees with the Waldhausen $K$-theory of a strongly saturated derivable Waldhausen category.

**4.1.** ***The $\mathcal{S}_{\bullet\bullet}$-construction.*** First we recall the analogue of Waldhausen's $\mathcal{S}_\bullet$-construction in the setting of derivators due to Garkusha [2005; 2006]. Let $\mathbb{D}$ be a pointed right derivator. We denote by $\mathrm{Ar}[n]$ the category (finite poset) of arrows of the poset $[n]$. Let $\mathcal{S}_n\mathbb{D}$ denote the full subcategory of $\mathbb{D}(\mathrm{Ar}[n])$ spanned by objects $F$ that satisfy the following conditions:

(i) for every $0 \le i \le n$, the object $i^*_{\mathrm{Ar}[n], i \to i} F \in \mathrm{Ob}\, \mathbb{D}(e)$ is a zero object;

(ii) for every $0 \le i \le j \le k \le n$, the restriction of $F$ along the inclusion of the subcategory

$$\begin{tikzcd} (i \to j) \arrow[r] \arrow[d] & (i \to k) \arrow[d] \\ (j \to j) \arrow[r] & (j \to k) \end{tikzcd}$$

of $\mathrm{Ar}[n]$, isomorphic to $\square$, is a cocartesian object of $\mathbb{D}(\square)$.

This defines a simplicial category $\mathcal{S}_\bullet\mathbb{D}$ where the simplicial operators are defined by the structure of $\mathbb{D}$ as a prederivator. Since morphisms in $\mathtt{Der}^{\mathrm{str}}$ preserve cocartesian squares, it follows easily that the correspondence $\mathbb{D} \mapsto \mathcal{S}_\bullet\mathbb{D}$ defines a functor from the underlying 1-category $\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0}$ of $\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}}$ (or $\mathtt{Der}^{\mathrm{str}}$) — which can also be obtained by forgetting the simplices of positive dimension in $\underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}}$ — to the (1-)category of simplicial categories.

For the definition of Waldhausen $K$-theory, we need to consider a more refined version of this construction. Let $\mathcal{S}_{\bullet\bullet}\mathbb{D}$ be the bisimplicial set whose set of $(n, m)$-simplices $\mathcal{S}_{n,m}\mathbb{D}$ is the set of objects

$$F \in \mathrm{Ob}\, \mathbb{D}([m] \times \mathrm{Ar}[n])_{\mathrm{eq}}$$

such that

(∗) for every $j : [0] \to [m]$ the object $(j \times \mathrm{Ar}[n])^*(F) \in \mathrm{Ob}\, \mathbb{D}(\mathrm{Ar}[n])$ is in $\mathcal{S}_n\mathbb{D}$.

Note that if this condition holds for *some* $j : [0] \to [m]$ then it holds for all $j$. The bisimplicial operators are again defined using the structure of the underlying prederivator. Moreover, it is easy to see that the construction is natural in $\mathbb{D}$; that is, we obtain a functor $\mathbb{D} \mapsto \mathcal{S}_{\bullet\bullet}\mathbb{D}$ from the underlying 1-category $\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0}$ of $\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}}$ to the category of bisimplicial sets.

**Definition 4.1.1.** The Waldhausen $K$-theory of a pointed right derivator $\mathbb{D}$ is defined to be the space $K^W(\mathbb{D}) := \Omega|\mathcal{S}_{\bullet\bullet}\mathbb{D}|$.

Our next goal is to show that the functor $K^W$ can be extended to a simplicial functor from $\underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}}$ to the simplicially enriched category of topological spaces $\underline{\mathtt{Top}}$. Here the $n$-simplices of the simplicial mapping space $\underline{\mathtt{Top}}(X, Y)$ between topological spaces $X$ and $Y$ are the continuous maps $X \times \Delta^n \to Y$, where $\Delta^n$ denotes the geometric $n$-simplex. Since both the geometric realization functor and the loop space functor are simplicial, it is enough to show that the functor

$$\mathbb{D} \mapsto \operatorname{diag} \mathcal{S}_{\bullet\bullet}\mathbb{D}$$

can be extended to a simplicial functor from $\underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}}$ to the standard simplicially enriched category of simplicial sets $\underline{\mathtt{SSet}}$. We recall that for simplicial sets $X$ and $Y$, the $n$-simplices of $\underline{\mathtt{SSet}}(X, Y)$ are the simplicial maps $X \times \Delta[n] \to Y$, where $\Delta[n]$ denotes the $n$-simplex and $|\Delta[n]| \cong \Delta^n$. A useful way of describing an $n$-simplex of $\underline{\mathtt{SSet}}(X, Y)$ is by giving a natural transformation as follows (cf. [Waldhausen 1985, Section 1.4]):

$$\begin{tikzcd}
 & \Delta^{\mathrm{op}} \arrow[dd, Rightarrow, "\alpha"'] \arrow[dr, bend left, "X"] & \\
(\Delta \downarrow [n])^{\mathrm{op}} \arrow[ur, bend left, "\text{source}"] \arrow[dr, bend right, "\text{source}"'] & & \mathtt{Set} \\
 & \Delta^{\mathrm{op}} \arrow[ur, bend right, "Y"'] &
\end{tikzcd}$$

Such a natural transformation $\alpha$ produces a simplicial map $\phi : X \times \Delta[n] \to Y$ which is defined by

$$\phi(x, [k] \xrightarrow{\sigma} [n]) = \alpha(\sigma)(x).$$

Conversely, a simplicial map $\phi : X \times \Delta[n] \to Y$ defines the components of such a natural transformation by setting $\alpha([k] \xrightarrow{\sigma} [n]) = \phi(-, \sigma) : X_k \to Y_k$.

**Proposition 4.1.2.** *Waldhausen K-theory extends to a simplicial functor*

$$K^W : \underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}} \to \underline{\mathtt{Top}}.$$

*Proof.* As remarked above, it suffices to show that the (1-)functor

$$\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0} \to \mathtt{SSet},$$
$$\mathbb{D} \mapsto \operatorname{diag} \mathcal{S}_{\bullet\bullet}\mathbb{D},$$

extends to a simplicial functor from $\underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}}$ to $\underline{\mathtt{SSet}}$. This extension is defined as follows: given an $n$-simplex $\phi : \mathbb{D} \to \mathbb{D}'([n] \times -)_{\mathrm{eq}}$ in $\underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}}$, its image in $\underline{\mathtt{SSet}}$ is an $n$-simplex, which we specify by giving the associated natural transformation

$$\begin{tikzcd}
 & \Delta^{\mathrm{op}} \arrow[dd, Rightarrow, "\phi_*"'] \arrow[dr, bend left, "\operatorname{diag} \mathcal{S}_{\bullet\bullet}\mathbb{D}"] & \\
(\Delta \downarrow [n])^{\mathrm{op}} \arrow[ur, bend left, "\text{source}"] \arrow[dr, bend right, "\text{source}"'] & & \mathtt{Set} \\
 & \Delta^{\mathrm{op}} \arrow[ur, bend right, "\operatorname{diag} \mathcal{S}_{\bullet\bullet}\mathbb{D}'"'] &
\end{tikzcd}$$

The component of $\phi_*$ at an object $\sigma : [k] \to [n]$ in $\Delta \downarrow [n]$ is the map

$$\phi_*(\sigma) : \mathcal{S}_{k,k}\mathbb{D} \to \mathcal{S}_{k,k}\mathbb{D}'$$

defined as the (co)restriction of the map on objects that comes from the following functor:

$$
\begin{array}{c}
\mathbb{D}([k]\times \mathrm{Ar}[k])_{\mathrm{eq}} \\
\Big\downarrow {\scriptstyle \phi([k]\times\mathrm{Ar}[k])} \\
\mathbb{D}'(([n]\times[k])\times \mathrm{Ar}[k])_{\mathrm{eq}} \\
\Big\downarrow {\scriptstyle \mathbb{D}'(\sigma\times[k]\times\mathrm{Ar}[k])} \\
\mathbb{D}'(([k]\times[k])\times \mathrm{Ar}[k])_{\mathrm{eq}} \\
\Big\downarrow {\scriptstyle \mathbb{D}'(\triangle\times\mathrm{Ar}[k])} \\
\mathbb{D}'([k]\times \mathrm{Ar}[k])_{\mathrm{eq}}
\end{array}
$$

It is straightforward to check that $\phi_*$ is a natural transformation. Moreover, it is easy to check that the correspondence $\phi \mapsto \phi_*$ respects the composition (by arguments analogous to those in 3.1). □

As an immediate consequence, we have the following invariance property of Waldhausen $K$-theory.

**Corollary 4.1.3.** *Let $\phi : \mathbb{D} \to \mathbb{D}'$ be a strong equivalence of pointed right derivators. Then the induced map $K^W(\phi) : K^W(\mathbb{D}) \to K^W(\mathbb{D}')$ is a homotopy equivalence.*

**4.2.** ***The $\mathsf{s}_\bullet$-construction.*** We mention a variant of the $\mathcal{S}_{\bullet\bullet}$-construction, which is actually the analogue of Waldhausen's $\mathsf{s}_\bullet$-construction in this context (cf. [Waldhausen 1985, Section 1.4]). Let $\mathsf{s}_\bullet\mathbb{D}$ denote the simplicial set with $n$-simplices

$$\mathsf{s}_n\mathbb{D} := \mathrm{Ob}\,\mathcal{S}_n\mathbb{D} = \mathcal{S}_{n,0}\mathbb{D},$$

and define

$$K^{W,\mathrm{ob}}(\mathbb{D}) := \Omega|\mathsf{s}_\bullet\mathbb{D}|.$$

The inclusion of the 0-simplices defines a canonical comparison map

$$\iota : K^{W,\mathrm{ob}}(\mathbb{D}) \to K^W(\mathbb{D}).$$

**Proposition 4.2.1.** *The comparison map $\iota$ is a weak equivalence.*

Proposition 4.2.1 is a consequence of the following lemma (cf. [Waldhausen 1985, Lemma 1.4.1]).

**Lemma 4.2.2.** *Let $\phi, \phi' : \mathbb{D} \to \mathbb{D}'$ be two cocontinuous strict morphisms of pointed right derivators. Then a* 1*-simplex $\Psi$ in $\underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}}$ with $d_1\Psi = \phi$ and $d_0\Psi = \phi'$ induces a simplicial homotopy $\mathsf{s}_\bullet\phi \simeq \mathsf{s}_\bullet\phi' : \mathsf{s}_\bullet\mathbb{D} \to \mathsf{s}_\bullet\mathbb{D}'$.*

*Proof.* The idea is analogous to the definition of the simplicial enhancement in Proposition 4.1.2. The required homotopy $\mathsf{s}_\bullet\phi \simeq \mathsf{s}_\bullet\phi'$ is a map $\mathsf{s}_\bullet\mathbb{D} \times \Delta[1] \to \mathsf{s}_\bullet\mathbb{D}'$, which we will specify by defining a natural transformation as follows:

$$\begin{tikzcd} & \Delta^{\mathrm{op}} \arrow[dr, bend left, "\mathsf{s}_\bullet\mathbb{D}"] \arrow[d, phantom, "\alpha\Downarrow"] & \\ (\Delta \downarrow [1])^{\mathrm{op}} \arrow[ur, bend left, "\text{source}"] \arrow[dr, bend right, "\text{source}"'] & & \mathtt{Set} \\ & \Delta^{\mathrm{op}} \arrow[ur, bend right, "\mathsf{s}_\bullet\mathbb{D}'"'] & \end{tikzcd}$$

Recall that $\Psi$ is a strict cocontinuous morphism $\Psi : \mathbb{D} \to \mathbb{D}'([1] \times -)_{\mathrm{eq}}$. Given an object $\sigma : [k] \to [1]$ in $\Delta \downarrow [1]$, we define $\alpha(\sigma)$ to be the (co)restriction of the map on objects that comes from the following functor:

$$\begin{tikzcd} \mathbb{D}(\mathrm{Ar}[k]) \arrow[d, "\Psi(\mathrm{Ar}[k])"] \\ \mathbb{D}'([1] \times \mathrm{Ar}[k])_{\mathrm{eq}} \arrow[d, "(p\times\mathrm{Ar}[k])^*"] \\ \mathbb{D}'(\mathrm{Ar}[1] \times \mathrm{Ar}[k])_{\mathrm{eq}} \arrow[d, "(\mathrm{Ar}(\sigma),\mathrm{id}_{\mathrm{Ar}[k]})^*"] \\ \mathbb{D}'(\mathrm{Ar}[k]) \end{tikzcd}$$

Here $p : \mathrm{Ar}[1] \to [1]$ is the functor defined by $p(0, 0) = 0$, $p(0, 1) = 1$, $p(1, 1) = 1$. The restriction of the composite functor to the $\mathsf{s}_\bullet$-construction is well-defined because the 1-simplex $\Psi$ is in $\underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}}$ (and not merely in $\underline{\mathtt{Der}}^{\mathrm{str}}$). The naturality of $\alpha$ is straightforward to check. $\square$

**Remark 4.2.3.** Moreover, the functor $\mathbb{D} \mapsto \mathsf{s}_\bullet\mathbb{D}$ extends to a simplicial functor from $\underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}}$ to $\underline{\mathtt{SSet}}$. The same argument works with [1] replaced more generally by $[n]$ and $p$ by the functor $\mathrm{Ar}[n] \to [n]$, $(i \to j) \mapsto j$.

An immediate consequence is the following invariance under strong equivalences.

**Corollary 4.2.4.** *Let $\phi : \mathbb{D} \to \mathbb{D}'$ be a strong equivalence of pointed right derivators. Then the induced maps $|\mathsf{s}_\bullet\phi| : |\mathsf{s}_\bullet\mathbb{D}| \to |\mathsf{s}_\bullet\mathbb{D}'|$ and $K^{W,\mathrm{ob}}(\phi) : K^{W,\mathrm{ob}}(\mathbb{D}) \to K^{W,\mathrm{ob}}(\mathbb{D}')$ are homotopy equivalences.*

We can now return to the proof of Proposition 4.2.1.

*Proof of Proposition 4.2.1.* Since

$$\big|([n], [m]) \mapsto \mathcal{S}_{n,m}\mathbb{D}\big| \cong \big|[m] \mapsto |[n] \mapsto \mathsf{s}_n\mathbb{D}([m] \times -)_{\mathrm{eq}}|\big|,$$

it suffices to show that every simplicial operator in the $m$-direction is a weak equivalence of simplicial sets after realizing in the $n$-direction. This follows from Corollary 4.2.4 and Example 3.3.4. $\square$

**4.3.** ***Agreement with Waldhausen K-theory.*** The agreement of $K^W$ with the Waldhausen $K$-theory of well-behaved Waldhausen categories is based on results about the homotopically flexible variations of the $\mathcal{S}_\bullet$-construction by Blumberg and Mandell [2008] and Cisinski [2010b]. We recall that Waldhausen's original $\mathcal{S}_\bullet$-construction [1985] of a Waldhausen category $\mathcal{C}$ is a simplicial Waldhausen category $[n] \mapsto \mathcal{S}_n\mathcal{C}$, where the objects of $\mathcal{S}_n\mathcal{C}$ are given by diagrams $F : \mathrm{Ar}[n] \to \mathcal{C}$ such that $F(i \to i)$ is the zero object for all $i \in [n]$, and for every $i \le j \le k$, the square

$$\begin{tikzcd} F(i \to j) \arrow[r] \arrow[d] & F(i \to k) \arrow[d] \\ F(j \to j) \arrow[r] & F(j \to k) \end{tikzcd}$$

has cofibrations as horizontal maps and is required to be a pushout. Restricting degreewise to the subcategory of (pointwise) weak equivalences gives a simplicial category $[n] \mapsto w\mathcal{S}_n\mathcal{C}$. We denote by $N_\bullet w\mathcal{S}_n\mathcal{C}$ the nerve of $w\mathcal{S}_n\mathcal{C}$. Then the Waldhausen $K$-theory of $\mathcal{C}$ is defined to be the space $K(\mathcal{C}) := \Omega|N_\bullet wS_\bullet\mathcal{C}|$.

**Theorem 4.3.1.** *Let $\mathcal{C}$ be a strongly saturated derivable Waldhausen category. Then there is a natural weak equivalence*

$$K(\mathcal{C}) \xrightarrow{\sim} K^W(\mathbb{D}(\mathcal{C})).$$

*Proof.* The map is induced by a bisimplicial map $N_m w\mathcal{S}_n\mathcal{C} \to \mathcal{S}_{n,m}\mathbb{D}(\mathcal{C})$ which sends an element $[m] \times \mathrm{Ar}[n] \to \mathcal{C}$ in $N_m w\mathcal{S}_n\mathcal{C}$ to the corresponding object of $\mathcal{S}_{n,m}\mathbb{D}$. Since $\mathcal{C}$ is strongly saturated, so too are the Waldhausen categories $\mathcal{S}_n\mathcal{C}$ for every $n$. It follows that the bisimplicial set $\mathcal{S}_{*,\bullet}\mathbb{D}(\mathcal{C})$ is isomorphic to the bisimplicial set $N_\bullet wS^h_*\mathcal{C}$ of [Cisinski 2010b] (and also to the bisimplicial set $N_\bullet wS'_*\mathcal{C}$ of [Blumberg and Mandell 2008], since every map can be replaced by a cofibration). Then the result follows from the agreement of the $\mathcal{S}^h_\bullet$-construction with the $\mathcal{S}_\bullet$-construction; see [Cisinski 2010b, Proposition 4.3] (cf. [Blumberg and Mandell 2008, Theorem 2.9] under the assumption that factorizations are functorial). $\square$

## 5. Derivator *K*-theory

**5.1.** ***Recollections and 2-categorical properties.*** Derivator $K$-theory was first defined for triangulated derivators by Maltsiniotis [2007]. The definition, however, applies similarly to all pointed right derivators. Here we consider the explicit model defined in terms of the $\mathcal{S}_\bullet$-construction which was introduced by Garkusha [2005;

2006], who also showed that it is equivalent to Maltsiniotis's in the triangulated setting.

**Definition 5.1.1.** The *derivator $K$-theory* of a pointed right derivator $\mathbb{D}$ is defined to be the space $K(\mathbb{D}) := \Omega|N_\bullet \operatorname{iso} \mathcal{S}_\bullet \mathbb{D}|$.

Since a cocontinuous strict morphism $\phi : \mathbb{D} \to \mathbb{D}'$ preserves cocartesian squares and zero objects, it can be easily checked that derivator $K$-theory defines a functor from $\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0}$ to the category of topological spaces. Moreover, it is invariant under equivalences of derivators.

**Proposition 5.1.2.** *If the strict morphism $\phi : \mathbb{D} \to \mathbb{D}'$ is an equivalence of pointed right derivators, then the induced map $K(\phi) : K(\mathbb{D}) \to K(\mathbb{D}')$ is a weak equivalence.*

*Proof.* This is an immediate consequence of the fact that the geometric realization of simplicial categories sends pointwise (weak) equivalences to weak equivalences of spaces. □

We emphasize that an equivalence of right pointed derivators does not necessarily admit a *strict* inverse. This means that an equivalence in $\mathtt{Der}$ is not in general a (2-categorical) equivalence in $\mathtt{Der}^{\mathrm{str}}$. However, the two concepts are closely related as morphisms of prederivators can be made strict up to a strict equivalence in $\mathtt{PDer}$; see [Cisinski and Neeman 2008, Proposition 10.14].

Derivator $K$-theory is compatible with the 2-categorical structure of $\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}}$. We show next how to enhance derivator $K$-theory to a simplicial functor from $N_\bullet\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}}$ to $\underline{\mathtt{Top}}$.

**Proposition 5.1.3.** *Derivator $K$-theory extends to a simplicial functor*

$$K : N_\bullet \mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}} \to \underline{\mathtt{Top}}.$$

*Proof.* It suffices to construct a simplicial enhancement for the (1-)functor

$$\begin{aligned} \mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0} &\to \mathtt{SSet}, \\ \mathbb{D} &\mapsto \operatorname{diag} N_\bullet \operatorname{iso} \mathcal{S}_\bullet \mathbb{D}. \end{aligned}$$

Suppose we are given an $n$-simplex

$$\alpha = (\phi_0 \xRightarrow{\alpha_1} \phi_1 \xRightarrow{\alpha_2} \cdots \xRightarrow{\alpha_n} \phi_n)$$

in $\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}}(\mathbb{D}, \mathbb{D}')$, where

$$\begin{tikzcd} \mathbb{D} \arrow[r, bend left, "\phi_{k-1}"] \arrow[r, bend right, "\phi_k"'] \arrow[r, phantom, "\alpha_k \Downarrow"] & \mathbb{D}' \end{tikzcd}$$

are invertible modifications. We construct a simplicial map

$$(\operatorname{diag} N_\bullet \operatorname{iso} \mathcal{S}_\bullet)(\alpha) : (\operatorname{diag} N_\bullet \operatorname{iso} \mathcal{S}_\bullet \mathbb{D}) \times \Delta[n] \to (\operatorname{diag} N_\bullet \operatorname{iso} \mathcal{S}_\bullet \mathbb{D}')$$

by defining a natural transformation as follows:

$$\begin{tikzcd}
 & \Delta^{\mathrm{op}} \arrow[dd, Rightarrow, "\alpha_*"'] \arrow[dr, bend left, "\operatorname{diag} N_\bullet \operatorname{iso} \mathcal{S}_\bullet \mathbb{D}"] & \\
(\Delta \downarrow [n])^{\mathrm{op}} \arrow[ur, bend left, "\text{source}"] \arrow[dr, bend right, "\text{source}"'] & & \mathrm{Set} \\
 & \Delta^{\mathrm{op}} \arrow[ur, bend right, "\operatorname{diag} N_\bullet \operatorname{iso} \mathcal{S}_\bullet \mathbb{D}'"'] &
\end{tikzcd}$$

Given $\sigma : [k] \to [n]$ in $\Delta$, the map

$$\alpha_*(\sigma) : N_k \operatorname{iso} \mathcal{S}_k \mathbb{D} \to N_k \operatorname{iso} \mathcal{S}_k \mathbb{D}'$$

is defined as follows. Let

$$\beta = \sigma^*(\alpha) = (\psi_0 \xRightarrow{\beta_1} \psi_1 \xRightarrow{\beta_2} \cdots \xRightarrow{\beta_k} \psi_k).$$

Consider an element in the domain of $\alpha_*(\sigma)$, denoted by $(f_1, \ldots, f_k)$, which is a chain of $k$ composable isomorphisms in $\mathcal{S}_k\mathbb{D} \subset \mathbb{D}(\mathrm{Ar}[k])$:

$$X_0 \to \cdots \to X_{r-1} \xrightarrow{f_r} X_r \to \cdots \to X_k.$$

The $k$-simplex $\beta$ gives rise to a $k \times k$ grid of commutative squares of solid arrows in $\mathcal{S}_k\mathbb{D}' \subset \mathbb{D}'(\mathrm{Ar}[k])$:

$$\begin{tikzcd}
\psi_0(X_0) \arrow[r] \arrow[d] \arrow[dr, dashed] & \cdots \arrow[r] & \psi_0(X_{r-1}) \arrow[r, "\psi_0(f_r)"] \arrow[d] & \psi_0(X_r) \arrow[r] \arrow[d] & \cdots \arrow[r] & \psi_0(X_k) \arrow[d] \\
\vdots \arrow[d] & \ddots \arrow[dr, dashed] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] \\
\psi_{r-1}(X_0) \arrow[r] \arrow[d, "\beta_r(X_0)"'] & \cdots \arrow[r] & \psi_{r-1}(X_{r-1}) \arrow[r, "\psi_{r-1}(f_r)"] \arrow[d, "\beta_r(X_{r-1})"'] \arrow[dr, dashed] & \psi_{r-1}(X_r) \arrow[r] \arrow[d, "\beta_r(X_r)"] & \cdots \arrow[r] & \psi_{r-1}(X_k) \arrow[d, "\beta_r(X_k)"] \\
\psi_r(X_0) \arrow[r] \arrow[d] & \cdots \arrow[r] & \psi_r(X_{r-1}) \arrow[r, "\psi_r(f_r)"'] \arrow[d] & \psi_r(X_r) \arrow[r] \arrow[d] \arrow[dr, dashed] & \cdots \arrow[r] & \psi_r(X_k) \arrow[d] \\
\vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \ddots \arrow[dr, dashed] & \vdots \arrow[d] \\
\psi_k(X_0) \arrow[r] & \cdots \arrow[r] & \psi_k(X_{r-1}) \arrow[r, "\psi_k(f_r)"] & \psi_k(X_r) \arrow[r] & \cdots \arrow[r] & \psi_k(X_k)
\end{tikzcd}$$

We set $\alpha_*(\sigma)(f_1, \dots, f_k)$ to be the sequence of $k$ diagonal morphisms, depicted above as dashed arrows,

$$\psi_0(X_0) \longrightarrow \cdots \longrightarrow \psi_{r-1}(X_{r-1}) \xrightarrow[\psi_r(f_r)\beta_r(X_{r-1})]{\beta_r(X_r)\psi_{r-1}(f_r)} \psi_r(X_r) \longrightarrow \cdots \longrightarrow \psi_k(X_k).$$

The naturality of $\alpha_*$ in $\sigma$ is straightforward. For the compatibility with composition, we consider an $n$-simplex in $N_\bullet\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}}(\mathbb{D}', \mathbb{D}'')$,

$$\alpha' = (\phi'_0 \overset{\alpha'_1}{\Longrightarrow} \phi'_1 \overset{\alpha'_2}{\Longrightarrow} \cdots \overset{\alpha'_n}{\Longrightarrow} \phi'_n),$$

and then it suffices to check that for all $\sigma : [k] \to [n]$ in $\Delta \downarrow [n]$, we have

$$(\alpha'\alpha)_*(\sigma) = \alpha'_*(\sigma)\alpha_*(\sigma).$$

Indeed, if

$$\beta' = \sigma^*(\alpha') = (\psi'_0 \overset{\beta'_1}{\Longrightarrow} \psi'_1 \overset{\beta'_2}{\Longrightarrow} \cdots \overset{\beta'_k}{\Longrightarrow} \psi'_k)$$

then each of the maps above, when applied to an element $(f_1, \dots, f_k) \in N_k \operatorname{iso} \mathcal{S}_k\mathbb{D}$, gives

$$\begin{array}{l} \psi'_0\psi_0(X_0) \longrightarrow \cdots \longrightarrow \psi'_{r-1}\psi_{r-1}(X_{r-1}) \\ \qquad\qquad\qquad {\scriptstyle (\beta'_r\beta_r)(X_r)(\psi'_r\psi_{r-1})(f_r)} \Big\downarrow {\scriptstyle \beta'_r(\psi_r(X_r))\psi'_{r-1}(\psi_r(f_r)\beta_r(X_{r-1}))} \\ \qquad\qquad\qquad\qquad\qquad \psi'_r\psi_r(X_r) \longrightarrow \cdots \longrightarrow \psi'_k\psi_k(X_k). \end{array}$$

This $k$-simplex can be obtained as a diagonal in a 3-dimensional cube, in the same way that $\alpha_*(\sigma)(f_1, \dots, f_k)$ is a diagonal in a square. Therefore, the vertical map can be written in six different ways. We have just chosen two of them. □

**Remark 5.1.4.** The proposition shows that the homotopy class of the morphism $K(\phi) : K(\mathbb{D}) \to K(\mathbb{D}')$ depends only on the isomorphism class of $\phi : \mathbb{D} \to \mathbb{D}'$ in the 2-category $\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}}$. This together with the invariance of derivator $K$-theory under equivalences implies that derivator $K$-theory is in fact functorial in the homotopy category of spaces with respect to *all* morphisms of derivators. More precisely, if for a category $\mathcal{G}$ which is enriched in groupoids, we denote by $\pi_0\mathcal{G}$ the 1-category obtained by identifying isomorphic morphisms, then there exists a unique factorization

$$\begin{tikzcd} \pi_0\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}} \arrow[r, "K"] \arrow[d, "\pi_0\rho"'] & \mathtt{Top}/\simeq \\ \pi_0\mathtt{Der}_{\mathrm{eq}} \arrow[ur] & \end{tikzcd}$$

Here $\mathtt{Top}/\simeq$ is the homotopy category of topological spaces. Compare [Cisinski and Neeman 2008, Corollary 10.19].

**Remark 5.1.5.** Derivator $K$-theory $K(\mathbb{D})$ is weakly equivalent to the geometric realization of Waldhausen $K$-theories

$$|\,[n] \mapsto K^{W,\mathrm{ob}}(\mathrm{iso}_n\,\mathbb{D})\,|.$$

Note that the derivators $\{\mathrm{iso}_n\,\mathbb{D}\}_{n\geq 0}$ are equivalent in $\mathtt{Der}$ and the simplicial operators are equivalences of derivators. Therefore, given that Waldhausen and derivator $K$-theory are different in general [Muro and Raptis 2011], it follows that Waldhausen $K$-theory is not invariant under equivalences of derivators. In particular, it follows that there are equivalences of (pre)derivators which are not strong, and more specifically, there are derivators $\mathbb{D}$ such that the canonical "degeneracy" equivalence $\mathbb{D} \to \mathrm{iso}_n\,\mathbb{D}$ is not a strong equivalence.

In the case where $\mathbb{D} = \mathbb{D}(\mathcal{C})$ for some derivable Waldhausen category $(\mathcal{C}, co\mathcal{C}, w\mathcal{C})$, the following variant of derivator $K$-theory is available. Passing to the homotopy categories of the $\mathcal{S}_\bullet$-construction, we obtain a new simplicial category $[n] \mapsto \mathrm{Ho}\,\mathcal{S}_n\mathcal{C}$, and a canonical morphism of simplicial categories $\mathrm{Ho}\,\mathcal{S}_\bullet\mathcal{C} \to \mathcal{S}_\bullet\mathbb{D}(\mathcal{C})$. This is degreewise an equivalence of categories, and therefore the induced map

$$\Omega|N_\bullet\,\mathrm{iso}\,\mathrm{Ho}\,\mathcal{S}_\bullet\mathcal{C}| \xrightarrow{\sim} K(\mathbb{D}(\mathcal{C}))$$

is a weak equivalence.

**5.2.** ***Comparison with Waldhausen K-theory.*** There is a natural comparison map from Waldhausen to derivator $K$-theory. For $n, m \geq 0$, the functors

$$\mathrm{dia}_{[m],\mathrm{Ar}[n]} : \mathbb{D}([m] \times \mathrm{Ar}[n]) \to \mathcal{C}at([m], \mathbb{D}(\mathrm{Ar}[n]))$$

assemble to define a bisimplicial map

$$\mathcal{S}_{\bullet\bullet}\mathbb{D} \to N_\bullet\,\mathrm{iso}\,\mathcal{S}_\bullet\mathbb{D}$$

which then induces the comparison map

$$\mu : K^W(\mathbb{D}) \to K(\mathbb{D})$$

from Waldhausen $K$-theory to derivator $K$-theory (cf. [Maltsiniotis 2007; Garkusha 2005; Muro and Raptis 2011]). We note that composing with the weak equivalence $\iota : K^{W,\mathrm{ob}}(\mathbb{D}) \to K^W(\mathbb{D})$, we obtain

$$\begin{tikzcd} K^{W,\mathrm{ob}}(\mathbb{D}) \arrow[r, "\iota", "\sim"'] \arrow[dr, "\mu^{\mathrm{ob}}"'] & K^W(\mathbb{D}) \arrow[d, "\mu"] \\ & K(\mathbb{D}) \end{tikzcd}$$

where the map $\mu^{\mathrm{ob}}$ is given degreewise simply by the inclusion of objects. The comparison maps $\mu$ and $\mu^{\mathrm{ob}}$ define natural transformations. Moreover, $\mu$ defines a natural transformation of simplicially enriched functors

$$\begin{tikzcd} \underline{\mathtt{Der}}^{\mathrm{str}}_{\mathrm{eq}} \arrow[d, "\rho"'] \arrow[dr, "K^W"] \arrow[d, phantom, shift left=6ex, "\mu\Downarrow"] & \\ N_\bullet \mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}} \arrow[r, "K"'] & \underline{\mathtt{Top}} \end{tikzcd}$$

and the same holds for $\mu^{\mathrm{ob}}$ (cf. Lemma 4.2.2 and Remark 4.2.3). However, making use of these simplicial enrichments will not be required in what follows since it is possible to think of them, in a homotopical fashion, only as asserting certain invariance properties. We will concentrate instead on the natural transformation

$$\begin{tikzcd} \mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0} \arrow[r, bend left, "K^{W,\mathrm{ob}}"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\mu^{\mathrm{ob}}\Downarrow"] & \mathtt{Top} \end{tikzcd}$$

because this is technically a more convenient model of the comparison map for the statement of our results. Here $\mathtt{Top}$ is the ordinary category of topological spaces.

In connection with the diagram above, it is interesting to mention that Toën and Vezzosi [2004] gave a neat abstract argument, based only on functoriality, to show that Waldhausen $K$-theory cannot factor through $N_\bullet \mathtt{Der}^{\mathrm{str}}_{\mathrm{eq}}$ by a functor which is invariant under equivalences of derivators.

Maltsiniotis [2007] conjectured that $\mu$ is a weak equivalence when $\mathbb{D}$ is the triangulated derivator associated with an exact category [Keller 2007]. This conjecture remains open, but several relevant results are known. Garkusha [2006], based on previous results by Neeman on the $K$-theory of triangulated categories, showed that $\mu$ admits a retraction when $\mathbb{D}$ arises from an abelian category. Maltsiniotis [2007] and Muro [2008] showed that $\mu$ induces an isomorphism on $\pi_0$ and $\pi_1$, respectively, for any $\mathbb{D}$ that arises from a strongly saturated derivable Waldhausen category. In [Muro and Raptis 2011], we showed that $\mu$ fails to be a weak equivalence in general for triangulated derivators that arise from differential graded algebras (or stable module categories). Moreover, we showed that the conjecture fails if derivator $K$-theory satisfies *localization*, a property also conjectured by Maltsiniotis [2007].

However, the pair $(K, \mu)$ turns out to be the best approximation to Waldhausen $K$-theory by a functor which sends equivalences of derivators to weak equivalences. We choose a rather ad hoc but direct way of formulating this property precisely as follows.

First, in order to ensure that our categories remain locally small and so to avoid set-theoretical troubles, we fix a (small) set $S$ of pointed right derivators $\mathbb{D}$ closed under taking iso $\mathbb{D}$, and restrict to the full subcategory of $\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0}$ spanned by $S$, which we still denote by $\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0}$. Second, it will be more convenient to work here with simplicial techniques and the delooped versions of Waldhausen and derivator $K$-theory. Thus we set

$$\begin{aligned}\Omega^{-1}K^{W}(\mathbb{D}) &:= \operatorname{diag}\mathcal{S}_{\bullet\bullet}\mathbb{D},\\ \Omega^{-1}K^{W,\mathrm{ob}}(\mathbb{D}) &:= \mathsf{s}_{\bullet}\mathbb{D},\\ \Omega^{-1}K(\mathbb{D}) &:= \operatorname{diag}\operatorname{iso}_{\bullet}\mathcal{S}_{\bullet}\mathbb{D},\end{aligned}$$

and we have natural transformations

$$\begin{aligned}\mu &: \Omega^{-1}K^{W} \Rightarrow \Omega^{-1}K,\\ \mu^{\mathrm{ob}} &: \Omega^{-1}K^{W,\mathrm{ob}} \Rightarrow \Omega^{-1}K,\end{aligned}$$

and a natural weak equivalence $\iota : \Omega^{-1}K^{W,\mathrm{ob}} \Rightarrow \Omega^{-1}K^{W}$ with $\mu^{\mathrm{ob}} = \mu\iota$.

**Definition 5.2.1.** Let $\mathcal{S}\mathcal{S}et^{\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0}}$ be the functor category. The *category* $\mathtt{App}$ *of invariant approximations* to Waldhausen $K$-theory $\Omega^{-1}K^{W,\mathrm{ob}}$ is the full subcategory of the comma category $\Omega^{-1}K^{W,\mathrm{ob}} \downarrow \mathcal{S}\mathcal{S}et^{\mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0}}$ spanned by the objects $\eta : \Omega^{-1}K^{W,\mathrm{ob}} \Rightarrow F$ such that $F : \mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0} \to \mathcal{S}\mathcal{S}et$ sends equivalences of derivators to weak equivalences. A morphism

$$\begin{tikzcd} & \Omega^{-1}K^{W,\mathrm{ob}} \arrow[dl, Rightarrow, "\eta"'] \arrow[dr, Rightarrow, "\eta'"] & \\ F \arrow[rr, Rightarrow, "u"'] & & F' \end{tikzcd}$$

in $\mathtt{App}$ is a *weak equivalence* if the components of $u$ are weak equivalences of simplicial sets.

Note that $\mu^{\mathrm{ob}} : \Omega^{-1}K^{W,\mathrm{ob}} \Rightarrow \Omega^{-1}K$ is an object of $\mathtt{App}$. Following [Dwyer et al. 2004], we say that an object $X$ of a category with weak equivalences $(\mathcal{C}, \mathcal{W})$ (satisfying in addition the "2-out-of-6" property) is *homotopically initial* if there are homotopical functors $F_0, F_1 : \mathcal{C} \to \mathcal{C}$ and a natural transformation $f : F_0 \Rightarrow F_1$ such that

(i) $F_0$ is naturally weakly equivalent to the constant functor at $X$,

(ii) $F_1$ is naturally weakly equivalent to the identity functor on $\mathcal{C}$, and

(iii) $f_X : F_0(X) \to F_1(X)$ is a weak equivalence.

If $X$ is initial in $\mathcal{C}$, then it is also homotopically initial in this sense. If $X$ is homotopically initial in $\mathcal{C}$, then $X$ is initial in $\operatorname{Ho}\mathcal{C}$. Finally, the category of homotopically initial objects in $(\mathcal{C}, \mathcal{W})$ is either empty or homotopically contractible. We refer the reader to [Dwyer et al. 2004] for more details.

**Theorem 5.2.2.** *The object $\mu^{\mathrm{ob}} : \Omega^{-1}K^{W,\mathrm{ob}} \Rightarrow \Omega^{-1}K$ is homotopically initial in the category with weak equivalences* $\mathtt{App}$.

*Proof.* Let $F : \mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0} \to \mathcal{S}Set$ be a functor. Then there is a canonical way of associating to $F$ a new functor

$$\begin{aligned}\mathbb{H}F : \mathtt{Der}^{\mathrm{str}}_{\mathrm{eq},0} &\to \mathcal{S}Set,\\ \mathbb{D} &\mapsto \operatorname{diag} F(\mathrm{iso}_\bullet\, \mathbb{D}).\end{aligned}$$

The inclusion of 0-simplices defines a natural transformation

$$\iota_F : F \Longrightarrow \mathbb{H}F.$$

By definition, we have

$$\iota_{\Omega^{-1}K^{W,\mathrm{ob}}} = \mu^{\mathrm{ob}} : \Omega^{-1}K^{W,\mathrm{ob}} \Longrightarrow \mathbb{H}\Omega^{-1}K^{W,\mathrm{ob}} = \Omega^{-1}K.$$

If $F$ sends equivalences of derivators to weak equivalences then the simplicial operators of $F(\mathrm{iso}_\bullet\, \mathbb{D})$ in the $\mathrm{iso}_\bullet$-direction are weak equivalences, so $\iota_F$ is a natural weak equivalence. In this case, it follows that $\mathbb{H}F$ also sends equivalences of derivators to weak equivalences. Using this fact, we can view the $\mathbb{H}$-construction as an endofunctor, denoted $\widetilde{\mathbb{H}} : \mathtt{App} \to \mathtt{App}$, which sends an object $\eta : \Omega^{-1}K^{W,\mathrm{ob}} \Rightarrow F$ in $\mathtt{App}$ to the natural transformation $\widetilde{\mathbb{H}}(\eta)$ given by the diagonal in the following commutative square:

$$\begin{tikzcd} \Omega^{-1}K^{W,\mathrm{ob}} \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\mu^{\mathrm{ob}}"'] & F \arrow[d, Rightarrow, "\iota_F", "\sim"'] \\ \Omega^{-1}K \arrow[r, Rightarrow, "\mathbb{H}\eta"] & \mathbb{H}F \end{tikzcd}$$

The natural transformation $\iota$ induces a natural weak equivalence $\iota' : \mathrm{id}_{\mathtt{App}} \Rightarrow \widetilde{\mathbb{H}}$ given by the right vertical arrow, and the bottom horizontal arrow defines a natural transformation from the constant functor at $\mu^{\mathrm{ob}} : \Omega^{-1}K^{W,\mathrm{ob}} \Rightarrow \Omega^{-1}K$ to $\widetilde{\mathbb{H}}$. Hence, the result follows. $\square$

We wish to remark that we could have worked entirely with simplicially enriched categories and functors in this section. More specifically, the construction $\mathbb{H}F$ in the proof of the last theorem has a simplicial enhancement which can be constructed as in the proof of Proposition 5.1.3. We decided to work with 1-categories in order to avoid the ensuing technicalities.

## 6. Some open questions

**6.1.** ***Derivators and the homotopy theory of homotopy theories.*** The simplicial enrichment of the category of derivators leads to a homotopy theory of derivators which is more discerning than the 2-categorical one and is closer to the homotopy theory of categories with weak equivalences. An interesting problem is to understand exactly how close this relationship is, and find out whether this homotopy theory of derivators is rich and structured enough to be (or contain a part of) a model for the homotopy theory of homotopy theories. In the case of the 2-category of derivators, a theorem of Renaudin [2009] specified the relationship between combinatorial model categories and their associated derivators (see also Appendix A for a review). In this context, the question would be whether this result can be improved in view of the simplicial enrichment of derivators. The results of Appendix A may be a first step in this direction.

**6.2.** ***Derived equivalences vs. strong equivalences.*** We do not know whether an exact functor of well-behaved Waldhausen categories which is a derived equivalence also induces a strong equivalence between the associated pointed right derivators. This is clear in the case where the derived equivalence admits a homotopy inverse (cf. Remark 3.3.6), but such an inverse may not exist strictly at the level of models in general. If the statement is true, then we will be able to deduce the invariance of Waldhausen $K$-theory (of Waldhausen categories) under derived equivalences also from the invariance of Waldhausen $K$-theory of pointed right derivators under strong equivalences.

**6.3.** ***Additivity for derivator K-theory.*** The additivity of derivator $K$-theory was proved by Cisinski and Neeman [2008] for triangulated derivators. However, the more general case of additivity for derivator $K$-theory of pointed right derivators seems to remain an open problem. We emphasize that this seems to be so also in the case where the derivator admits a model. In this case, we tried to apply Waldhausen's original proof and generalize the approach in [Garkusha 2005], but we discovered a gap in the proof of [Garkusha 2005, Theorem 3.1] which we could not fix. (Namely, in diagram (7), at the bottom of page 655, the arrow $\varphi^*_{X_i} v_i c_i : V''_i \to \overline{X}_i$ need not be a weak equivalence.) In particular, we do not know whether derivator $K$-theory of pointed right derivators is invariant under an appropriately defined notion of stabilization which would produce a triangulated derivator.

A related problem is to show that additivity holds for the Waldhausen $K$-theory of pointed right derivators. Of course, this is true if the derivator admits a model. However it would still be interesting to establish the general case as it is through this generality that the concept of derivator can also be tested.

## Appendix A. Combinatorial model categories and derivators

The purpose of this appendix is to highlight some results on the connections between combinatorial model categories and derivators. Since the discussion is heavily based on [Renaudin 2009], we will give a very concise review of his results while providing precise references where necessary. Then we will record some minor strengthenings of Renaudin's main theorem with a view to addressing the questions of Section 6.1.

Let $\mathscr{MOD}$ denote the 2-category of left proper combinatorial model categories, Quillen adjunctions and natural transformations between left Quillen functors. Following [Renaudin 2009], we view the morphism categories as categories with weak equivalences where the weak equivalences are given by *Quillen homotopies*. We recall that a natural transformation of left Quillen functors is a Quillen homotopy if it is pointwise a weak equivalence at the cofibrant objects; see [Renaudin 2009, Définition 2.1.2]. Passing to the homotopy categories of all morphism categories yields a new 2-category $\underline{\mathscr{MOD}}$. We note that $\underline{\mathscr{MOD}}$ is enriched in the category of all categories $\mathscr{CAT}$.

The class of Quillen equivalences in $\underline{\mathscr{MOD}}$ admits a calculus of right fractions [Renaudin 2009, Proposition 2.3.2]. Thus the bilocalization of $\underline{\mathscr{MOD}}$ at the class of Quillen equivalences exists, denoted here by $\underline{\mathscr{MOD}}[\mathcal{Q}^{-1}]$, and is actually equivalent to the bilocalization of $\mathscr{MOD}$ at the class of Quillen equivalences [Renaudin 2009, Théorème 2.3.3].

Let $\mathscr{MOD}^p$ be the 1- and 2-full subcategory of presentable model categories — that is, combinatorial model categories that arise from a left Bousfield localization of the projective model category of $C$-diagrams in $\mathcal{SSet}$, for some small category $C$, at a set of morphisms $S$. Every combinatorial model category is equivalent to a presentable one [Dugger 2001]. Presentable model categories have certain nice "cofibrancy" properties which can in particular be used to show that the 2-functor

$$\underline{\mathscr{MOD}}^p \to \underline{\mathscr{MOD}}[\mathcal{Q}^{-1}]$$

is a biequivalence [Renaudin 2009, Proposition 2.3.4]. Here $\underline{\mathscr{MOD}}^p$ denotes the corresponding 1- and 2-full subcategory of $\underline{\mathscr{MOD}}$. The restriction to presentable model categories in what follows is mainly a technical matter and is due essentially to the rigidity of $\mathscr{MOD}$ compared, say, to the essentially equivalent context of presentable $\infty$-categories.

Let $\mathscr{DER}$ (resp. $\mathscr{DER}_!$, $\mathscr{DER}_{ad}$) denote the 2-category of derivators with domain $\mathcal{D}ia = \mathcal{C}at$ and values in the 2-category $\mathscr{CAT}$ together with pseudonatural transformations (resp. cocontinuous morphisms, adjunctions between derivators) as 1-morphisms, and modifications as 2-morphisms. Cisinski [2003] constructed a pseudofunctor

$$\mathbb{D}(-) : \mathscr{MOD} \to \mathscr{DER}_{ad}$$

which is defined on objects by $\mathcal{M} \mapsto \mathbb{D}(\mathcal{M})$ (cf. Section 2.3) and sends Quillen equivalences to equivalences of derivators. We note that $\mathbb{D}(\mathcal{M})$ takes values in locally small categories. There is an induced pseudofunctor of 2-categories

$$\underline{\mathbb{D}}(-) : \underline{\mathscr{MOD}}[\mathcal{Q}^{-1}] \to \mathscr{DER}_{ad}.$$

Renaudin [2009, Théorème 3.3.2] showed that this functor is a *local equivalence*, i.e., it induces equivalences between the morphism categories. This could be interpreted as identifying a part of $\mathscr{DER}_{ad}$ with a truncation of the homotopy theory of homotopy theories as modeled by $\mathscr{MOD}$. For our purposes, it will be necessary to reformulate this result in terms of the larger 2-category $\mathscr{DER}_!$ (cf. [Renaudin 2009, Remarque 3.3.3]).

**Theorem A.1.** *The canonical pseudofunctor*

$$\underline{\mathbb{D}}(-) : \underline{\mathscr{MOD}}^p \to \mathscr{DER}_!$$

*is a local equivalence.*

*Proof.* Since the composition $\underline{\mathscr{MOD}}^p \to \underline{\mathscr{MOD}}[\mathcal{Q}^{-1}] \to \mathscr{DER}_{ad}$ is a local equivalence, it suffices to show that for all $\mathcal{M}$ and $\mathcal{N}$ in $\mathscr{MOD}^p$, the fully faithful inclusion functor

$$\mathscr{DER}_{ad}(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})) \hookrightarrow \mathscr{DER}_!(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})) \tag{A.2}$$

is also essentially surjective. Let $F : \mathbb{D}(\mathcal{M}) \to \mathbb{D}(\mathcal{N})$ be a cocontinuous morphism. Suppose that $\mathcal{M} = L_S\mathcal{S}Set^C$. The Quillen adjunction $\mathrm{Id} : \mathcal{S}Set^C \rightleftarrows \mathcal{M} : \mathrm{Id}$ induces a morphism in $\mathscr{DER}_{ad}$, denoted as

$$\mathbb{L}_S(\mathrm{Id}) : \mathbb{D}(\mathcal{S}Set^C) \rightleftarrows \mathbb{D}(\mathcal{M}) : \mathbb{R}_S(\mathrm{Id}).$$

The composite $F' = F \circ \mathbb{L}_S(\mathrm{Id}) : \mathbb{D}(\mathcal{S}Set^C) \to \mathbb{D}(\mathcal{N})$ is a cocontinuous morphism. By [Renaudin 2009, Remarque 3.3.3] (or more directly, by using the universal property of $\mathcal{S}Set^C$ due to Dugger [Renaudin 2009, Proposition 2.2.7], and that of $\mathbb{D}(\mathcal{S}Set^C)$ due to Cisinski [Renaudin 2009, Théorème 3.3.1]), there is a Quillen adjunction

$$\widetilde{F}' : \mathcal{S}Set^C \rightleftarrows \mathcal{N} : \widetilde{G}'$$

such that $\mathbb{D}(\widetilde{F}')$ is isomorphic to $F'$ in $\mathscr{DER}_!(\mathbb{D}(\mathcal{S}Set^C), \mathbb{D}(\mathcal{N}))$. Then the universal property of Bousfield localization shows that $(\widetilde{F}', \widetilde{G}')$ descends to a Quillen adjunction

$$\widetilde{F}'' : L_S\mathcal{S}Set^C \rightleftarrows \mathcal{N} : \widetilde{G}''$$

such that $\mathbb{D}(\widetilde{F}'') \circ \mathbb{L}_S(\mathrm{Id})$ is isomorphic to $F \circ \mathbb{L}_S(\mathrm{Id})$. Then

$$\mathbb{D}(\widetilde{F}'') : \mathbb{D}(\mathcal{M}) \rightleftarrows \mathbb{D}(\mathcal{N}) : \mathbb{D}(\widetilde{G}'')$$

is an adjunction of derivators and the left adjoint $\mathbb{D}(\widetilde{F}'')$ is isomorphic to $F$ since the functor

$$\mathbb{L}_S(\mathrm{Id})^* : \mathscr{DER}_!(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})) \to \mathscr{DER}_!(\mathbb{D}(\mathcal{S}Set^C), \mathbb{D}(\mathcal{N}))$$

is fully faithful [Tabuada 2008, Definition 4.2, Lemma 4.3 and Theorem 4.4]. □

We would like to emphasize that the equivalence of categories (A.2) in the last proof can be regarded as an adjoint functor theorem for derivators that arise from combinatorial model categories.

We recall from [Groth 2012] the construction of internal hom-objects in the 2-category of derivators. Given prederivators $\mathbb{D}, \mathbb{D}' : \mathcal{C}at \to \mathscr{CAT}$ there is a prederivator $\mathrm{HOM}(\mathbb{D}, \mathbb{D}') : \mathcal{C}at^{\mathrm{op}} \to \mathscr{CAT}$ which is defined explicitly by

$$\mathrm{HOM}(\mathbb{D}, \mathbb{D}')(X) = \mathscr{DER}(\mathbb{D}, \mathbb{D}'_X).$$

Moreover, if $\mathbb{D}'$ is a derivator, then so is $\mathrm{HOM}(\mathbb{D}, \mathbb{D}')$; see [Groth 2012, Proposition 1.20]. The simplicial enrichments of the previous sections are obtained from this by setting $X = [n]$ and restricting to the objects. If $\mathbb{D}$ and $\mathbb{D}'$ are derivators we also consider the following closely related prederivator

$$\begin{aligned}\mathrm{HOM}_!(\mathbb{D}, \mathbb{D}') : \mathcal{C}at^{\mathrm{op}} &\to \mathscr{CAT}, \\ X &\mapsto \mathscr{DER}_!(\mathbb{D}, \mathbb{D}'_X).\end{aligned}$$

To see that this is again a prederivator, it suffices to consider $u : X \to Y$ in $\mathcal{C}at$ and a cocontinuous morphism $\phi : \mathbb{D} \to \mathbb{D}'_Y$, and then note that the morphism

$$\mathrm{HOM}(\mathbb{D}, \mathbb{D}')(u)(\phi) := u^*\phi : \mathbb{D} \to \mathbb{D}'_Y \to \mathbb{D}'_X$$

is again cocontinuous because $u^* : \mathbb{D}'_Y \to \mathbb{D}'_X$ is cocontinuous (in fact, it admits a right adjoint $u_* : \mathbb{D}'_X \to \mathbb{D}'_Y$).

Similarly, it is easy to check that $\mathrm{HOM}_!(\mathbb{D}, \mathbb{D}')$ is in fact a right derivator. For every $u : X \to Y$, the pullback functor

$$u^* : \mathscr{DER}_!(\mathbb{D}, \mathbb{D}'_Y) \to \mathscr{DER}_!(\mathbb{D}, \mathbb{D}'_X)$$

defined above admits a left adjoint

$$u_! : \mathscr{DER}_!(\mathbb{D}, \mathbb{D}'_X) \to \mathscr{DER}_!(\mathbb{D}, \mathbb{D}'_Y)$$

which is defined as for the derivator $\mathrm{HOM}(\mathbb{D}, \mathbb{D}')$: given a cocontinuous morphism $\phi : \mathbb{D} \to \mathbb{D}'_X$, set

$$u_!(\phi) := u_!\phi : \mathbb{D} \to \mathbb{D}'_X \to \mathbb{D}'_Y.$$

We refer the reader to [Groth 2013, Propositions 2.5 and 2.9, Example 2.10] for more details about adjunctions.

The purpose of this appendix is to show that the functor $\mathbb{D}(-)$ also preserves hom-objects in the sense of the following theorem. For $\mathcal{M}$ and $\mathcal{N}$ in $\mathscr{MOD}$, let $\mathscr{MOD}_l(\mathcal{M}, \mathcal{N})$ denote the category of left Quillen functors $\mathcal{M} \to \mathcal{N}$ and natural transformations. This is again a category with weak equivalences, the Quillen homotopies, and the forgetful functor $\mathscr{MOD}(\mathcal{M}, \mathcal{N}) \to \mathscr{MOD}_l(\mathcal{M}, \mathcal{N})$ is an equivalence.

**Theorem A.3.** *Let $\mathcal{M}$ and $\mathcal{N}$ be presentable model categories. Then there is an equivalence of prederivators*

$$\Phi(\mathcal{M},\mathcal{N}) : \mathbb{D}(\mathscr{MOD}_l(\mathcal{M},\mathcal{N})) \simeq \mathrm{HOM}_!(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})).$$

*Proof.* For every small category $X$, there is a natural equivalence of categories

$$\mathbb{D}(\mathscr{MOD}_l(\mathcal{M},\mathcal{N}))(X) \simeq \mathrm{Ho}(\mathscr{MOD}_l(\mathcal{M}, (\mathcal{N}^X)_{\mathrm{inj}})),$$

since an $X$-diagram of left Quillen functors $\mathcal{M} \to \mathcal{N}$ is the same as a left Quillen functor $\mathcal{M} \to (\mathcal{N}^X)_{\mathrm{inj}}$ where the target is given the injective model structure. The latter model category is strictly speaking no longer presentable, but we can find a natural replacement for it by a presentable one $\mathcal{N}^X$ simply by a change to the projective (co)fibrations. Then by [Renaudin 2009, Proposition 2.2.9], we have an equivalence of categories

$$\mathrm{Ho}(\mathscr{MOD}_l(\mathcal{M}, (\mathcal{N}^X)_{\mathrm{inj}})) \xleftarrow{\simeq} \mathrm{Ho}(\mathscr{MOD}_l(\mathcal{M},\mathcal{N}^X)).$$

There is a morphism of prederivators

$$\Phi(\mathcal{M},\mathcal{N}) : \mathrm{Ho}(\mathscr{MOD}_l(\mathcal{M}, (\mathcal{N}^?)_{\mathrm{inj}}) \to \mathrm{HOM}_!(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N})),$$

induced by $\mathbb{D}(-)$, whose components are equivalences of categories, because we have commutative diagrams

$$\begin{tikzcd}
\mathrm{Ho}(\mathscr{MOD}_l(\mathcal{M}, (\mathcal{N}^X)_{\mathrm{inj}})) \arrow[rr, "\Phi(\mathcal{M},\mathcal{N})_X"] & & \mathscr{DER}_!(\mathbb{D}(\mathcal{M}), \mathbb{D}(\mathcal{N}^X)) \\
& \mathrm{Ho}(\mathscr{MOD}_l(\mathcal{M},\mathcal{N}^X)) \arrow[ul, "\simeq"] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

where the indicated equivalence on the right is a consequence of Theorem A.1. □

## Appendix B. A remark on the approximation theorem

The original approximation theorem of Waldhausen [1985] states sufficient conditions for an exact functor of Waldhausen categories to induce an equivalence in $K$-theory. Although Waldhausen did not analyze the meaning of these conditions from the viewpoint of homotopical algebra, various authors have later studied connections between abstract homotopy theory and Waldhausen $K$-theory and have shown more general and refined versions of the approximation theorem; see [Thomason and Trobaugh 1990; Sagave 2004; Dugger and Shipley 2004; Cisinski 2010b; Blumberg and Mandell 2011]. These results ultimately say that Waldhausen $K$-theory is an invariant of homotopy theories and allow definitions of the theory via $\infty$-categories or simplicial categories; see also [Toën and Vezzosi 2004].

**Theorem B.1** [Cisinski 2010b; Blumberg and Mandell 2011]. *Let $F : \mathcal{C} \to \mathcal{C}'$ be an exact functor of strongly saturated derivable Waldhausen categories. If the induced functor* $\operatorname{Ho} F : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}'$ *is an equivalence of categories, then the map* $w\mathcal{S}_n F : w\mathcal{S}_n\mathcal{C} \to w\mathcal{S}_n\mathcal{C}'$ *is a weak equivalence for all* $n \geq 0$. *In particular, the map* $K(F) : K(\mathcal{C}) \to K(\mathcal{C}')$ *is also a weak equivalence.*

The purpose of this appendix is to note the following result, which may be regarded as a partial converse to the approximation theorem. The proof is based on ideas of Dwyer and Kan for modeling mapping spaces in homotopical algebra via zigzag diagrams (see, e.g., [Dwyer and Kan 1980]) and related results from [Blumberg and Mandell 2011].

**Theorem B.2.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be an exact functor of derivable Waldhausen categories. Suppose that*

(i) *$wF : w\mathcal{C} \to w\mathcal{C}'$ induces isomorphisms on $\pi_0$ and $\pi_1$ for all basepoints,*

(ii) *$w\mathcal{S}_2F : w\mathcal{S}_2\mathcal{C} \to w\mathcal{S}_2\mathcal{C}'$ is* 1*-connected* (*i.e., it induces an isomorphism on $\pi_0$ and an epimorphism on $\pi_1$ for all basepoints*).

*Then* $\operatorname{Ho} F : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}'$ *is an equivalence of categories.*

*Proof.* Consider the commutative square

$$\begin{tikzcd} w\mathcal{S}_2\mathcal{C} \arrow[r, "w\mathcal{S}_2F"] \arrow[d, "(d_1,d_2)"] & w\mathcal{S}_2\mathcal{C}' \arrow[d, "(d_1,d_2)"] \\ w\mathcal{C} \times w\mathcal{C} \arrow[r, "wF\times wF"] & w\mathcal{C}' \times w\mathcal{C}' \end{tikzcd}$$

Using the properties of the long exact sequence of homotopy groups and assumptions (i) and (ii), it follows that the induced map between the homotopy fibers of the vertical maps (at any basepoint) induces an isomorphism on $\pi_0$; see, e.g., [May and Ponto 2012, Lemma 1.4.7]. Applying [Blumberg and Mandell 2011, Theorem 1.2], the map between the homotopy fibers at the points defined by $(X_1, X_2) \in \operatorname{Ob}\mathcal{C} \times \operatorname{Ob}\mathcal{C}$ and $(F(X_1), F(X_2)) \in \operatorname{Ob}\mathcal{C}' \times \operatorname{Ob}\mathcal{C}'$ can be identified with the map induced by $F$ between the corresponding mapping spaces in the respective hammock localizations

$$L^H(\mathcal{C})(X_1, X_2) \to L^H(\mathcal{C}')(F(X_1), F(X_2)).$$

Thus, applying $\pi_0$ to this map gives an isomorphism

$$\operatorname{Ho} F : \operatorname{Ho}\mathcal{C}(X_1, X_2) \cong \operatorname{Ho}\mathcal{C}'(F(X_1), F(X_2)),$$

and therefore $\operatorname{Ho} F$ is fully faithful. It is also essentially surjective because $wF$ is an epimorphism on $\pi_0$. □

To sum up, we have the following corollary.

**Corollary B.3.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be an exact functor of strongly saturated derivable Waldhausen categories. If*

(i) *$wF : w\mathcal{C} \to w\mathcal{C}'$ induces isomorphisms on $\pi_0$ and $\pi_1$, and*

(ii) *$w\mathcal{S}_2F : w\mathcal{S}_2\mathcal{C} \to w\mathcal{S}_2\mathcal{C}'$ is 1-connected,*

*then $w\mathcal{S}_nF : w\mathcal{S}_n\mathcal{C} \to w\mathcal{S}_n\mathcal{C}'$ is a weak equivalence for all $n \geq 0$. In particular, the induced map $K(F) : K(\mathcal{C}) \to K(\mathcal{C}')$ is also a weak equivalence.*

These results show that being a derived equivalence is much stronger than being a $K$-equivalence. More specifically, the property of being a derived equivalence does not take into account the "group completion" process that takes place in the definition of $K$-theory. To obtain an ideal approximation theorem that encodes this group completion process, one would need to "localize $\mathcal{C}$ and $\mathcal{C}'$" at all the relations which are derived from the additivity property, and then ask for the weaker property that these localized objects are equivalent. This localization is accomplished using $\infty$-categories with the construction of the universal additive invariant in [Blumberg et al. 2013], and it is essentially shown that it is equivalent to Waldhausen $K$-theory.

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# Chow groups of some generically twisted flag varieties

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We classify the split simple affine algebraic groups $G$ of types A and C over a field with the property that the Chow group of the quotient variety $E/P$ is torsion-free, where $P \subset G$ is a special parabolic subgroup (e.g., a Borel subgroup) and $E$ is a generic $G$-torsor (over a field extension of the base field). Examples of $G$ include the adjoint groups of type A. Examples of $E/P$ include the Severi–Brauer varieties of generic central simple algebras.

## 1. Introduction

Let $G$ be a split semisimple affine algebraic group over a field $k$ and let $P$ be a parabolic subgroup of $G$. The quotient $G/P$ is a smooth projective algebraic $k$-variety sometimes called a *flag variety* of $G$. The variety $G/P$ is (absolutely) *cellular* (in the sense of [Elman et al. 2008, §66]). In particular, its Chow group $\mathrm{CH}(G/P)$ is torsion-free.

Given a $G$-torsor $E$ over $k$, the quotient variety $E/P$ is a *twisted flag variety*, a twisted form of $G/P$. The Chow group $\mathrm{CH}(E/P)$ may have a large torsion subgroup and is far from being understood. The situation is still the same when we restrict our attention to the case of a *special* parabolic subgroup $P$. Recall that $P$ is *special* if any $P$-torsor over any field extension of $k$ is trivial. (For instance, any Borel subgroup of $G$ is special parabolic.) For any special parabolic $P$, every $G$-torsor $E$ over $k$ splits over the function field $k(E/P)$ (see [Karpenko and Merkurjev 2006, Lemma 6.5]), showing that $E/P$ is a generically cellular variety, i.e., becomes cellular over its own function field.

Let, now, $E$ be a *generic* $G$-torsor. By this we mean a $G$-torsor over a certain field extension $F/k$, obtained by the following construction (see Remark 2.3): We fix an imbedding of $G$ into the general linear group $\mathrm{GL}_N$ for some $N$. This makes $\mathrm{GL}_N$ a $G$-torsor over the quotient variety $S := \mathrm{GL}_N/G$. We define $F$ to be the

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function field $k(S)$ and we define the generic $G$-torsor $E$ to be the $G$-torsor over $F$ given by the generic fiber of $\mathrm{GL}_N \to S$.

For any other $G$-torsor $E'$ over any field extension $k'/k$, there exists a $k'$-point of $S$ such that $E'$ is isomorphic to the fiber of $\mathrm{GL}_N \to S$ over the point. Moreover, for infinite $k'$, the set of such $k'$-points is dense in $S$ [Serre 2003, §5.3]. This suggests that $E$, being the generic fiber of $\mathrm{GL}_N \to S$, is the most complicated $G$-torsor and that the variety $E/P$, which we call a *generically twisted flag variety*, is the most complicated twisted flag variety (for given $G$ and $P$). Nevertheless, the Chow group $\mathrm{CH}(E/P)$ for a generic $E$ turns out to be more accessible than in general.

In this paper, we classify the split simple affine algebraic groups $G$ of types A and C over a field with the property that the Chow group $\mathrm{CH}(E/P)$ of $E/P$ is torsion-free — see Theorems 3.1 and 4.1. Examples of $G$ include adjoint groups of type A (Theorem 3.7). Examples of $E/P$ include the Severi–Brauer varieties of generic central simple algebras.

An application to computation of the *topological* (also called *geometrical*) filtration on the Grothendieck ring of twisted flag varieties is provided as well as some other applications — see Corollaries 3.9, 3.10 and 3.14.

For $G$ of type $\mathrm{B}_n$, an analogue of Theorems 3.1 and 4.1 is known. Note that $G$ is isomorphic to $\mathrm{Spin}_{2n+1}$ (the simply connected case) or to $\mathrm{O}^+_{2n+1}$ (the adjoint case). Since $\mathrm{B}_n = \mathrm{C}_n$ for $n = 1, 2$, let us assume that $n \geq 3$. By [Petrov 2007] (see also [Smirnov and Vishik 2014]), $\mathrm{CH}(E/P)$ is torsion-free for $G = \mathrm{O}^+_{2n+1}$. And it is easy to see that $\mathrm{CH}^2(E/P)$ contains an element of order 2 for $G = \mathrm{Spin}_{2n+1}$.

For the type $\mathrm{D}_n$ (with $n \geq 4$), $\mathrm{CH}(E/P)$ is torsion-free if $G = \mathrm{O}^+_{2n}$ (see [Petrov 2007] or [Smirnov and Vishik 2014]) and $\mathrm{CH}^2(E/P)$ has an element of order 2 for $G = \mathrm{Spin}_{2n}$. However, the analysis of the remaining projective orthogonal and semispinor groups has not been completed so far.

For $G$ of type $\mathrm{G}_2$ and *any* nonsplit $G$-torsor $E$ over a field, $\mathrm{CH}^2(E/P)$ has an element of order 2; see [Yagita 2016], for example, which also has computations concerning Chow groups of some other twisted flag varieties.

## 2. Generic torsors

For $G$ as in the introduction and $P$ a parabolic subgroup of $G$, we consider a generically twisted flag variety $E/P$, where $E$ is the generic $G$-torsor over $F$ obtained from an imbedding $G \hookrightarrow \mathrm{GL}_N$ for some $N$. Here $F$ is the function field $k(S)$ of the $k$-variety $S := \mathrm{GL}_N / G$.

We consider the pull-back homomorphism of *$P$-equivariant Chow groups* (see [Edidin and Graham 1998])

$$\mathrm{CH}_P \operatorname{Spec} F \to \mathrm{CH}_P E$$

with respect to the ($P$-equivariant) structure morphism $E \to \operatorname{Spec} F$ of the $F$-variety $E$ (where $P$ acts on $\operatorname{Spec} F$ trivially). Note that the $P$-equivariant Chow group $\operatorname{CH}_P E$ coincides with the ordinary Chow group of $E/P$. The following statement is proved but not formulated in [Karpenko and Merkurjev 2006]:

**Lemma 2.1.** *The homomorphism* $\operatorname{CH}_P \operatorname{Spec} F \to \operatorname{CH}(E/P)$ *is surjective.*

*Proof.* The variety $\mathrm{GL}_N$ is a $\mathrm{GL}_N$-equivariant open subvariety of the affine space $\operatorname{End} k^N$. It is enough to prove that the composition

$$\operatorname{CH}_P \operatorname{Spec} k \to \operatorname{CH}_P \operatorname{Spec} F \to \operatorname{CH}(E/P) = \operatorname{CH}_P E$$

with the change of field homomorphism $\operatorname{CH}_P \operatorname{Spec} k \to \operatorname{CH}_P \operatorname{Spec} F$ is surjective. The homomorphism $\operatorname{CH}_P \operatorname{Spec} k \to \operatorname{CH}_P E$ decomposes as

$$\operatorname{CH}_P \operatorname{Spec} k \to \operatorname{CH}_P \operatorname{End} k^N \to \operatorname{CH}_P \mathrm{GL}_N \to \operatorname{CH}_P E.$$

The first homomorphism here is the pull-back with respect to the structure morphism of the $k$-variety $\operatorname{End} k^N$; it is an isomorphism by homotopy invariance of equivariant Chow groups. The second and the third homomorphisms are pull-backs with respect to the open imbedding $\mathrm{GL}_N \hookrightarrow \operatorname{End} k^N$ and the localization morphism $E \to \mathrm{GL}_N$; they are surjective by the localization property of equivariant Chow groups. □

**Example 2.2.** For the quotient $G := \mathrm{SL}_n / \mu_m$ of the special linear group $\mathrm{SL}_n$ by the central subgroup $\mu_m$ of the $m$-th roots of unity, where $m \geq 1$ is a divisor of $n \geq 2$, any $G$-torsor over $k$ gives rise to a central simple $k$-algebra $A$ of degree $n$ and exponent $m$. We refer to an algebra $A$ corresponding to a generic $G$-torsor as a *generic central simple algebra of degree n and exponent m*. In the decomposition $n = n_1 n_2$ with $n_1 \geq 1$ having the same prime divisors as $m$ and with $n_2$ relatively prime to $m$, the factor $n_1$ is the index of $A$. Let $P$ be a parabolic subgroup in $G$ with conjugacy class corresponding to the subset of the Dynkin diagram of $G$ obtained by removing the first vertex. The variety $E/P$ is the Severi–Brauer variety $X$ of $A$. It is shown in [Karpenko and Merkurjev 2006, §8.1] that the graded ring $\operatorname{CH}_P \operatorname{Spec} F$ is generated by some homogeneous elements with at most one element in every codimension. Therefore, by Lemma 2.1, the Chow ring $\operatorname{CH} X$ is generated by some homogeneous elements with at most one element in every codimension.

In the particular case of $G := \mathrm{PGL}_n = \mathrm{SL}_n / \mu_n$, we refer to $A$ as a *generic central simple algebra of degree n*. The index and exponent of such $A$ are equal to $n$ as well.

**Remark 2.3.** The construction of a generic $G$-torsor we use in this paper is a particular case of the construction of [Serre 2003, Example 5.4], which nowadays is more common. For two generic $G$-torsors $E$ and $E'$ over fields $F/k$ and $F'/k$

produced by this more general construction, there is a canonical construction of a field $L/k$, containing both $F/k$ and $F'/k$, and of an isomorphism $E_L \simeq E'_L$ such that the extensions $L/F$ and $L/F'$ are purely transcendental. Since Chow groups do not change under purely transcendental base field extensions, we get a canonical isomorphism $\mathrm{CH}(E/P) \simeq \mathrm{CH}(E'/P)$ for any $P$. Thanks to A. Merkurjev for pointing this out.

The relationship between $\mathrm{CH}(E/P)$ and $\mathrm{CH}(E/P')$ for different special parabolic subgroups $P, P' \subset G$ is explained in the proof of Lemma 3.6.

**Example 2.4.** For any split semisimple $G$, a generic $G$-torsor $E$, and a Borel subgroup $B \subset G$, the topological filtration on the Grothendieck ring $K(E/B)$ coincides with the gamma filtration. Indeed, by [Edidin and Graham 1998, Proposition 6], the graded ring $\mathrm{CH}_B \operatorname{Spec} F$ is identified with the symmetric algebra $S(\hat{T})$ of the character group $\hat{T}$ of a maximal split torus $T \subset B$. It follows that the ring $\mathrm{CH}_B \operatorname{Spec} F$ is generated by elements of codimension 1. By Lemma 2.1, this implies that the ring $\mathrm{CH}(E/B)$ is generated by elements of codimension 1. Therefore the ring $\mathrm{CH}(E/B)$ is generated by Chern classes. In particular, the associated graded ring of the topological filtration on $K(E/B)$ is generated by Chern classes, which precisely means that the topological filtration coincides with the gamma filtration; see [Karpenko 1998, Remark 2.17].

The above considerations also show that the ring $\mathrm{CH}(E/B)$ is finitely generated. In particular, its torsion subgroup $\operatorname{Tors}\mathrm{CH}(E/B)$ is finite.

## 3. Type $\mathrm{A}_{n-1}$

Let $n \geq 2$. Any split simple affine algebraic group $G$ of type $\mathrm{A}_{n-1}$ over any field $k$ is isomorphic to the quotient $\mathrm{SL}_n/\mu_m$, where $m \geq 1$ is a divisor of $n$. Here is the main result of this section:

**Theorem 3.1.** *For $G := \mathrm{SL}_n/\mu_m$ (with $n$ and $m$ as above) over any field $k$, let $P \subset G$ be a special parabolic subgroup and let $E$ be a generic $G$-torsor over a field extension $F/k$. The group $\mathrm{CH}(E/P)$ is torsion-free if and only if the g.c.d. $(m, n/m)$ is bounded by 2. Moreover, for every odd prime divisor $p$ of $(m, n/m)$, as well as $p = 2$ if 4 divides $(m, n/m)$, the group $\mathrm{CH}^2(E/P)$ contains an element of order $p$.*

We will prove Theorem 3.1 after some preparation. The most significant cases of torsion-free $\mathrm{CH}(E/P)$ are the cases $G = \mathrm{PGL}_n = \mathrm{SL}_n/\mu_n$ and $G = \mathrm{SL}_{2^r}/\mu_{2^{r-1}}$ (for any $r \geq 1$). Since $\mathrm{SL}_n$ is special, the case $G = \mathrm{SL}_n$ is trivial. We start with a result covering the case $G = \mathrm{PGL}_n$:

**Proposition 3.2.** *Let $F$ be a field and $A$ a central simple $F$-algebra. Assume that the Chow ring $\mathrm{CH}\,X$ of the Severi–Brauer variety $X$ of $A$ is generated (as a ring)* *by some homogeneous elements with at most one element in every codimension. Then the group* $\mathrm{CH}\,X$ *is* $p$*-torsion-free for every prime number* $p$ *such that the* $p$*-primary parts of the exponent and the index of* $A$ *coincide.*

**Remark 3.3.** According to Example 2.2, Proposition 3.2 applies to any generic central simple algebra $A$ of any given degree (without restriction on its exponent), implying that the Chow ring of the Severi–Brauer variety of $A$ is torsion-free.

**Remark 3.4.** In the case where $\exp A = \operatorname{ind} A$, Proposition 3.2 provides a complete description of the ring $\mathrm{CH}\,X$. Indeed, for any $n \geq 1$ and any central simple $F$-algebra $A$ of degree $n$, the kernel of the change of field homomorphism

$$\mathrm{CH}\,X \to \mathrm{CH}\,X_L = \mathrm{CH}\,\mathbb{P}^{n-1} = \mathbb{Z}[H]/(H^n),$$

given by any splitting field $L/F$ of the algebra, where $H$ corresponds to the hyperplane class in $\mathrm{CH}\,\mathbb{P}^{n-1}$, is the torsion subgroup of $\mathrm{CH}\,X$. Moreover, by [Karpenko 1995b, Theorem 1], if $\exp A = \operatorname{ind} A =: d$, then for any $0 \leq j \leq n-1 = \dim X$ the image of $\mathrm{CH}^j X$ in $\mathrm{CH}^j\,\mathbb{P}^{n-1} = \mathbb{Z}$ is generated by $d/(j,d)$.

*Proof of Proposition 3.2.* Let $n$ be the degree of $A$. Let $x_i \in \mathrm{CH}^i X$, $i = 0, 1, \dots, n-1$, be elements generating the ring $\mathrm{CH}\,X$.

We fix an arbitrary prime number $p$ such that the $p$-primary parts of the exponent and the index of $A$ coincide. For the remainder of the proof, we switch to the Chow groups $\mathrm{CH} \otimes \mathbb{Z}_{(p)}$ with coefficients in $\mathbb{Z}_{(p)}$ — the localization of $\mathbb{Z}$ at the prime ideal $(p)$ generated by $p$. To prove Proposition 3.2 it suffices to show that the group $\mathrm{CH}\,X \otimes \mathbb{Z}_{(p)}$ is torsion-free.

Let $p^r$ be the $p$-primary part of $\operatorname{ind} A$. By Lemma 3.5, we only need to check that $\mathrm{CH}^j X \otimes \mathbb{Z}_{(p)}$ is torsion-free for $j < p^r$.

Let $L/F$ be a finite Galois field extension splitting $A$. Let $L_r$ be the intermediate field corresponding to a $p$-Sylow subgroup of $\mathrm{Gal}(L/F)$, so that $[L_r : F]$ is prime to $p$ and $[L : L_r]$ is a $p$-power. Let $L_0$ be a minimal subfield of $L$ containing $L_r$ and splitting $A$. We have $[L_0 : L_r] = p^r$. By [Hall 1959, Theorem 4.2.1], there is a chain of subfields

$$L_r \subset L_{r-1} \subset \cdots \subset L_0$$

with $[L_{i-1} : L_i] = p$ for every $i = r, \dots, 1$. Note that $\operatorname{ind} A_{L_i} = p^i$ for $i = 0, 1, \dots, r$.

We claim that, for any $j = 1, \dots, p^r - 1$, the norm map

$$N_i^j : \mathrm{CH}^j X_{L_i} \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}^j X \otimes \mathbb{Z}_{(p)}$$

is surjective, where $i = v_p(j)$ and $v_p$ is the $p$-adic valuation. Since $\operatorname{ind} A_{L_i} = p^i$ divides $j$, we have $\mathrm{CH}^j X_{L_i} = \mathbb{Z}$ (by [Karpenko 1995a, Corollary 1.3.2]). More precisely, $\mathrm{CH}^j X_L = \mathrm{CH}^j\,\mathbb{P}^{n-1} = \mathbb{Z}$, where $1 \in \mathbb{Z}$ corresponds to the class in $\mathrm{CH}^j\,\mathbb{P}^{n-1}$

of a linear subspace in $\mathbb{P}^{n-1}$ of codimension $j$, and the change of field homomorphism $\mathrm{CH}^j X_{L_i} \to \mathrm{CH}^j X_L$ is an isomorphism. Therefore the claim implies that $\mathrm{CH}^j X \otimes \mathbb{Z}_{(p)}$ is torsion-free.

We prove the claim by induction on $j$. Given an arbitrary positive $j \le p^r - 1$, we assume that the claim holds in positive codimensions $< j$. We first check that every element of $\mathrm{CH}^j X \otimes \mathbb{Z}_{(p)}$ that is a polynomial in $x_1, \dots, x_{j-1}$ (without $x_j$) is in the image of the norm map $N_i^j$. It suffices to consider the case where the polynomial is a monomial. Since the degree $j$ of the monomial is not divisible by $p^{i+1}$, the monomial contains a factor $x_k$ for some $k \in \{1, \dots, j-1\}$ not divisible by $p^{i+1}$. Since $v_p(k) \le i$, it follows by the induction hypothesis that $x_k$ is in the image of $N_i^k$. Therefore, by the projection formula [Elman et al. 2008, Proposition 56.8], the monomial is in the image of $N_i^j$.

To finish the proof of the claim (and therefore the proof of Proposition 3.2), it suffices to check that $x_j$ is also in the image of $N_i^j$. For this we decompose the element $N_i^j(1) \in \mathrm{CH}^j X \otimes \mathbb{Z}_{(p)}$, where 1 is the generator of $\mathrm{CH}^j X_{L_i} \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}$, into a linear combination of the degree-$j$ monomials in $x_1, x_2, \dots, x_j$ and check that the coefficient $\lambda \in \mathbb{Z}_{(p)}$ at the monomial $x_j$ is invertible.

Let us observe that $v_p(N_i^j(1)_L) = v_p([L_i : F]) = r - i$. On the other hand, if $\lambda$ is not invertible, then $(\lambda x_j)_L$ is divisible by $p^{r-i+1}$ because $x_L$ is divisible by $p^{r-i}$ for any element $x \in \mathrm{CH}^j X$; see Remark 3.4.[1] Also $M_L$ is divisible by $p^{r-i+1}$ for any monomial $M \in \mathrm{CH}^j X$ in $x_1, \dots, x_{j-1}$ because $M$ contains $x_k$ with some $k$ not divisible by $p^{i+1}$: $x_{kL}$ is then divisible by $p^{r-i}$; at the same time, $M$ necessarily contains another factor $x_l$ with some $l = 1, \dots, j-1$ ($l = k$ is also possible). Our assumption that $j < p^r$ ensures that $l$ is not divisible by $p^r$ so that $x_{lL}$ is divisible by $p$. □

Here is the lemma used in the proof of Proposition 3.2:

**Lemma 3.5.** *Let $A$ be a central simple algebra over a field $F$ of degree $n \ge 1$. Let $p$ be a prime number and $p^r$ the $p$-primary part of* $\mathrm{ind}\, A$. *Let $X$ be the Severi–Brauer variety of $A$. For any integer $0 \le j \le \dim X = n-1$, the group $\mathrm{CH}^j X \otimes \mathbb{Z}_{(p)}$ is isomorphic to the group $\mathrm{CH}^{j'} X \otimes \mathbb{Z}_{(p)}$, where $0 \le j' \le p^r - 1$ is the remainder after division of $j$ by $p^r$.*

*Proof.* Let $A_p$ be the $p$-primary part of the underlying division algebra of $A$ (so that $\mathrm{ind}\, A_p = p^r$). Let $X_p$ be the Severi–Brauer variety of $A_p$.

Let $L/F$ be a finite Galois field extension splitting the algebra $A$. Let $K/F$ be the subextension corresponding to a $p$-Sylow subgroup of $\mathrm{Gal}(L/F)$. Therefore the degree of $K/F$ is prime to $p$, the degree of $L/K$ is a $p$-power, and the algebra $A_K$ is isomorphic to a matrix algebra over $A_{pK}$.

[1] This is the only place in the proof where we use the fact that the $p$-primary part of the exponent of $A$ coincides with the $p$-primary part of its index.

Below we work in the category of Chow motives [Elman et al. 2008, §64], first with integral coefficients, then with coefficients in $\mathbb{Z}_{(p)}$. The integral Chow motive $M(X_K)$ of the $K$-variety $X_K$ is isomorphic to the direct sum of shifts of the Chow motive of $X_{pK}$ with the shifting numbers of the summands being the multiples of $p^r$ from 0 to $n - p^r$ [Karpenko 1995a, Corollary 1.3.2]:

$$M(X_K) \simeq \bigoplus_{i=0}^{n/p^r-1} M(X_{pK})\{ip^r\}.$$

We switch to the Chow motives with coefficients in $\mathbb{Z}_{(p)}$. Let $f$ be the above isomorphism after the switch. We apply the norm $N_{K/F}$ to $f$ and divide the result by $[K : F] \in \mathbb{Z}_{(p)}^{\times}$. This way we get a morphism $g : M(X) \to \bigoplus_{i=0}^{n/p^r-1} M(X)\{ip^r\}$ with the property that $g_L = f_L$. In particular, $g_L$ is an isomorphism. It follows by [Elman et al. 2008, Corollary 92.7 with Remark 92.3], a consequence of the nilpotence theorem for projective homogeneous varieties, that $g$ is an isomorphism. Thus $\mathrm{CH}^j X \otimes \mathbb{Z}_{(p)} \simeq \mathrm{CH}^{j'} X_p \otimes \mathbb{Z}_{(p)} \simeq \mathrm{CH}^{j'} X \otimes \mathbb{Z}_p$. □

**Lemma 3.6.** *Let $G$ be a split semisimple linear algebraic group over a field $k$ and let $E$ be a $G$-torsor over $k$. If the Chow group* $\mathrm{CH}(E/P)$ *is torsion-free for at least one special parabolic subgroup $P$ of $G$, then it is torsion-free for every special parabolic subgroup. The same holds with* $\mathrm{CH}^2(E/P)$ *in place of* $\mathrm{CH}(E/P)$.

*Proof.* Let $P$ and $P'$ be special parabolic subgroups of $G$ with torsion-free $\mathrm{CH}(E/P)$. Since $E$ splits over $F(E/P)$ (see [Karpenko and Merkurjev 2006, Lemma 6.5]), the Chow motive of the variety $E/P \times E/P'$ is a direct sum of shifts of the motive of $E/P$ [Petrov et al. 2008, Corollary 3.4]. Therefore $\mathrm{CH}(E/P \times E/P')$ is torsion-free. At the same time, the Chow motive of $E/P \times E/P'$ is a direct sum of shifts of the motive of $E/P'$, so that $\mathrm{CH}(E/P')$ is torsion-free as well.

The same chain of conclusions goes through for $\mathrm{CH}^2(E/P)$ in place of $\mathrm{CH}(E/P)$, because one shifting number is 0 and the remaining shifting numbers are positive in both motivic decompositions mentioned. (Recall that, for any projective homogeneous variety, the groups $\mathrm{CH}^0$ and $\mathrm{CH}^1$ are torsion-free.) □

At this point we have already proved Theorem 3.1 for $m = n$, i.e., for $G = \mathrm{PGL}_n$:

**Theorem 3.7.** *For any field $k$ and any $n \geq 2$, let $G$ be the projective linear group* $\mathrm{PGL}_n$ *over $k$, let $P$ be a special parabolic subgroup of $G$, and let $E$ be a generic $G$-torsor* (*over a field extension of $k$*). *Then the Chow group of the generically twisted flag variety $E/P$ is torsion-free.* □

The Severi–Brauer variety $X$ of a degree-$n$ central simple algebra $A$ is, by definition, a closed subvariety of the Grassmannian of $n$-dimensional subspaces in the $n^2$-dimensional vector space $A$. The tautological bundle on $X$ has rank $n$ and is the restriction of the tautological bundle on the Grassmannian.

**Corollary 3.8.** *For any $n$, let $X$ be the Severi–Brauer variety of a generic central simple algebra of degree $n$. Then the Chow ring* $\mathrm{CH}\,X$ *is generated by the Chern classes of the tautological vector bundle on $X$.*

*Proof.* Let $\overline{X}$ be $X$ over a splitting field of the algebra. As shown in [Karpenko and Merkurjev 2006], the image of the change of field homomorphism $\mathrm{CH}\,X \to \mathrm{CH}\,\overline{X}$ is generated by the Chern classes of the tautological vector bundle. Since $\mathrm{CH}\,X$ is torsion-free, the change of field homomorphism $\mathrm{CH}\,X \to \mathrm{CH}\,\overline{X}$ is injective and it follows that $\mathrm{CH}\,X$ itself is generated by the Chern classes of the tautological vector bundle. □

Here are a couple of applications:

**Corollary 3.9.** *Let $X$ be the Severi–Brauer variety of a central simple algebra $A$ over a field $k$ satisfying* $\operatorname{ind} A = \exp A$. *Then the torsion subgroup* $\operatorname{Tors}\mathrm{CH}\,X$ *of* $\mathrm{CH}\,X$ *splits off canonically as a direct summand of* $\mathrm{CH}\,X$.

*Proof.* By [Karpenko 1995a, Corollary 1.3.2], we may assume that $A$ is a division algebra. By specialization, all relations between the Chern classes of the tautological vector bundle on the Severi–Brauer variety of a generic central simple algebra of degree $\deg A$ hold for the Chern classes of the tautological vector bundle on our $X$. It follows that the subring $C \subset \mathrm{CH}\,X$ generated by these Chern classes is mapped under the quotient map $\mathrm{CH}\,X \to \mathrm{CH}\,X/\operatorname{Tors}\mathrm{CH}\,X$ isomorphically onto the quotient (see Remark 3.4), whence the statement. □

The following result has been proved in [Karpenko 1998] for division algebras of $p$-primary index. Those assumptions can be dropped:

**Corollary 3.10.** *Let $X$ be the Severi–Brauer variety of a central simple algebra $A$ over a field $k$ satisfying* $\operatorname{ind} A = \exp A$. *Then the topological filtration on the Grothendieck ring $K(X)$ coincides with the gamma filtration. Moreover, for any finite product $Y$ of any generalized Severi–Brauer varieties of any tensor powers of $A$, the topological filtration on the Grothendieck ring $K(X_{k(Y)})$ coincides with the gamma filtration.*

*Proof.* Let $\tilde{X}$ be the Severi–Brauer variety of a generic central simple algebra $\tilde{A}$ of degree $\deg A$ over a field $F$. Note that $\exp \tilde{A} = \operatorname{ind} \tilde{A} = \deg \tilde{A}$. By Corollary 3.8, the ring $\mathrm{CH}\,\tilde{X}$ is generated by Chern classes. Therefore, the topological filtration on the Grothendieck ring $K(\tilde{X})$ coincides with the gamma filtration. Let $T$ be the generalized Severi–Brauer variety $\mathrm{SB}_{\operatorname{ind} A}(\tilde{A})$ (of right ideals in $\tilde{A}$ of reduced dimension $\operatorname{ind} A$; the usual Severi–Brauer variety $\mathrm{SB}(\tilde{A})$ is $\mathrm{SB}_1(\tilde{A})$ in this notation). By the index reduction formula [Merkurjev et al. 1996, (5.11)], the index and the exponent of the central simple $F(T)$-algebra $\tilde{A}_{F(T)}$ are equal to $\operatorname{ind} A$. Since the projection $T \times \tilde{X} \to \tilde{X}$ is a Grassmann bundle, the topological filtration on the Grothendieck ring $K(\tilde{X}_{F(T)})$ coincides with the gamma filtration; see

[Karpenko 1998]. Moreover, by [Karpenko 1995b], since $\operatorname{ind} \tilde{A}_{F(T)} = \exp \tilde{A}_{F(T)}$ the topological filtration on $K(\tilde{X}_{F(T)})$ coincides with the filtration induced by the topological filtration on the Grothendieck ring of $\tilde{X}$ considered over an algebraic closure of $F(T)$.

Turning back to $A$ and $X$ over $k$, we have three embedded filtrations on $K(X)$: the gamma filtration, which is contained in the topological filtration, which in turn is contained in the filtration induced by the topological filtration over an algebraic closure of $k$. By [Quillen 1973], since for any $i \geq 1$ the indexes of the $i$-th tensor powers of the algebras $A$ and $\tilde{A}_{F(T)}$ coincide (see [Karpenko 1998, Example 3.9]), the rings $K(X)$ and $K(\tilde{X}_{F(T)})$ are identified. Under this identification, both gamma filtrations and both filtrations induced from the respective algebraic closures are identified as well. It follows that all three filtrations on $K(X)$ coincide. In particular, the topological filtration on the Grothendieck ring $K(X)$ coincides with the gamma filtration.

From this point, the deduction of the statement on $K(X_{k(Y)})$ is standard; see [Karpenko 1998]. □

The following statement will be of help in the proof of Proposition 3.12:

**Corollary 3.11.** *Let $A$ be an arbitrary central simple algebra over a field $F$ and let $L$ be a maximal subfield of the underlying division algebra. Let $p$ be a prime integer. For $i > 0$, let $c_i \in \mathrm{CH}^i X \otimes \mathbb{Z}_{(p)}$ be the $i$-th Chern class of the tautological vector bundle on the Severi–Brauer $X$ variety of $A$, considered in the Chow group with coefficients in $\mathbb{Z}_{(p)}$. For any $i > 0$ coprime with $p$, the class $c_i$ is in the image of the norm map $N_{L/F}$.*

*Proof.* We fix some $i > 0$ coprime with $p$ and set $n := \deg A$. The image of $1 \in \mathbb{Z} = \mathrm{CH}^i X_L$ under $N_{L/F} : \mathrm{CH}^i X_L \to \mathrm{CH}^i X$ equals $h^i_*(e)$, where $e \in \mathrm{CH}_0 X$ is the class of a closed point of degree $\operatorname{ind} A$ (the canonical generator of the torsion-free group $\mathrm{CH}_0 X$; see [Panin 1984] or [Chernousov and Merkurjev 2006]) and $h \in \mathrm{CH}^1(X \times X)$ is the first Chern class of the canonical line bundle on $X \times X$. (In particular, $N_{L/F}(1)$ does not depends on the choice of $L$.) We need to show that $c_i$ is a multiple of $h^i_*(e)$ (in the Chow group with coefficients in $\mathbb{Z}_{(p)}$).

By Theorem 3.7, $c_i$ is a multiple of $h^i_*(e)$ provided that $A$ is replaced by a generic central simple algebra of degree $n$ (over a field extension of $F$). Indeed, for generic $A$, the Chow group with integer coefficients is torsion-free (by Theorem 3.7) and, by Remark 3.4, the image of $\mathrm{CH}^i X \otimes \mathbb{Z}_{(p)}$ in $\mathrm{CH}^i X_L \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}$ is generated by the image $[L : F] = \operatorname{ind} A$ of $h^i_*(e)$.

It follows by specialization that $c_i$ is a multiple of $h^i_*(e)$ for our initial $A$ as well. □

Here is the result serving the case of $G = \mathrm{SL}_n / \mu_{n/2}$:

**Proposition 3.12.** *Let $F$ be a field and let $A$ be a central simple $F$-algebra such that the 2-primary part of its exponent is equal to the half of the 2-primary part of its index $d$ (this implied that $d$ is divisible by 4) and the index of the tensor power $A^{\otimes(d/4)}$ is divisible by 4. Assume that the Chow ring $\mathrm{CH}\,X$ of the Severi–Brauer variety $X$ of $A$ is generated (as a ring) by some element of codimension 1 and the Chern classes of the tautological vector bundle. Then the group $\mathrm{CH}\,X$ is 2-torsion-free.*

**Remark 3.13.** In the case $d := \mathrm{ind}\,A = 2\exp A$ and $4 \mid \mathrm{ind}\,A^{\otimes(d/4)}$, Proposition 3.12 provides a complete description of the ring $\mathrm{CH}\,X$. Indeed, for any $n \geq 1$ and any central simple $F$-algebra $A$ of degree $n$, the kernel of the change of field homomorphism $\mathrm{CH}\,X \to \mathrm{CH}\,\mathbb{P}^{n-1} = \mathbb{Z}[H]/(H^n)$, given by any splitting field of the algebra, is the torsion subgroup of $\mathrm{CH}\,X$. Moreover, if $\exp A = \frac{1}{2}d$, where $d := \mathrm{ind}\,A$, and $4 \mid \mathrm{ind}\,A^{\otimes(d/4)}$, then, for any $0 \leq j \leq n-1$ and any prime integer $p$, the $p$-adic valuation of a generator of the image of $\mathrm{CH}^j X$ in $\mathrm{CH}^j \mathbb{P}^{n-1} = \mathbb{Z}$ is determined as follows: for odd $p$ it is $v_p(d/(j,d))$; for $p = 2$ it is $v_2(d/(j,d))$ provided that $v_2(j-1) < v_2(d)$ and it is $v_2(d)-1$ otherwise. This is a consequence of Remark 3.4 (for odd $p$) and of [Karpenko 1998, proof of Proposition 4.9] (for $p = 2$), since by the proof of Lemma 3.5 we only need to consider the case where $d$ is a $p$-power.

*Proof of Proposition 3.12.* We obtain a proof of Proposition 3.12, appropriately modifying the proof of Proposition 3.2. Let $n$ be the degree of $A$. For $i \geq 2$, let $x_i \in \mathrm{CH}^i X$ be the $i$-th Chern class of the tautological vector bundle on $X$. As a ring, $\mathrm{CH}\,X$ is generated by some element $x_1 \in \mathrm{CH}^1 X$ and the elements $x_i \in \mathrm{CH}^i X$, $i = 1, \dots, \dim X = n-1$.

For the remainder of the proof, we switch to the Chow groups with coefficients in $\mathbb{Z}_{(2)}$ — the localization of $\mathbb{Z}$ in the prime ideal generated by 2. To prove Proposition 3.12, it suffices to show that the group $\mathrm{CH}\,X \otimes \mathbb{Z}_{(2)}$ is torsion-free.

Let $2^r$ be the 2-primary part of $d = \mathrm{ind}\,A$. Recall that $d$ is divisible by 4, that is to say, $r \geq 2$. By Lemma 3.5, we only need to check that $\mathrm{CH}^j X \otimes \mathbb{Z}_{(2)}$ is torsion-free for $j < 2^r$.

Let $L/F$ be a finite Galois field extension splitting $A$. Let $L_r$ be the intermediate field corresponding to a 2-Sylow subgroup of $\mathrm{Gal}(L/F)$, so that $[L_r : F]$ is odd and $[L : L_r]$ is a 2-power. Let $L_0$ be a minimal subfield of $L$ containing $L_r$ and splitting $A$. We have $[L_0 : L_r] = 2^r$. By [Hall 1959, Theorem 4.2.1], there is a chain of subfields

$$L_r \subset L_{r-1} \subset \cdots \subset L_0$$

with $[L_{i-1} : L_i] = 2$ for every $i = r, \dots, 1$. Note that $\mathrm{ind}\,A_{L_i} = 2^i$ for $i = 0, 1, \dots, r$.

We claim that, for any $j = 2, \dots, 2^r - 1$, the norm map

$$N_i^j : \mathrm{CH}^j X_{L_i} \otimes \mathbb{Z}_{(2)} \to \mathrm{CH}^j X \otimes \mathbb{Z}_{(2)}$$

is surjective, where $i = v_2(j)$ and $v_2$ is the 2-adic valuation. In contrast with the proof of Proposition 3.2, where the exponent of $A$ was equal to the index of $A$, not to half that, the norm map $N_0^1$ is not surjective; moreover, none of the maps $N_1^1, \dots, N_{r-1}^1$ is surjective. However, and this will be used in the proof below, the image of the change of field homomorphism $\mathrm{CH}^1 X \otimes \mathbb{Z}_{(2)} \to \mathrm{CH}^1 X_{L_{r-1}} \otimes \mathbb{Z}_{(2)}$ coincides with the image of the norm map

$$N_{L_0/L_{r-1}} : \mathrm{CH}^1 X_{L_0} \otimes \mathbb{Z}_{(2)} \to \mathrm{CH}^1 X_{L_{r-1}} \otimes \mathbb{Z}_{(2)}.$$

This is so because the change of field homomorphism $\mathrm{CH}^1 X \to \mathrm{CH}^1 X_L = \mathbb{Z}$ is injective and its image is generated by the integer $\exp A$ [Artin 1982, §2].

Since $\operatorname{ind} A_{L_i} = 2^i$ divides $j$, we have $\mathrm{CH}^j X_{L_i} = \mathbb{Z}$ (by [Karpenko 1995a, Corollary 1.3.2]). More precisely, $\mathrm{CH}^j X_L = \mathrm{CH}^j \mathbb{P}^{n-1} = \mathbb{Z}$, where $1 \in \mathbb{Z}$ corresponds to the class in $\mathrm{CH}^j \mathbb{P}^{n-1}$ of a linear subspace in $\mathbb{P}^{n-1}$ of codimension $j$, and the change of field homomorphism $\mathrm{CH}^j X_{L_i} \to \mathrm{CH}^j X_L$ is an isomorphism. Therefore the claim implies that $\mathrm{CH}^j X \otimes \mathbb{Z}_{(2)}$ is torsion-free.

We prove the claim by induction on $j$. Given an arbitrary $j$ with $2 \le j \le 2^r - 1$, we assume that the claim holds in codimensions $2, \dots, j-1$. We first check that every element of $\mathrm{CH}^j X \otimes \mathbb{Z}_{(2)}$ that is a polynomial in $x_1, \dots, x_{j-1}$ (without $x_j$) is in the image of the norm map $N_i^j$. It suffices to consider the case where the polynomial is a monomial. Since the degree $j$ of the monomial is not divisible by $2^{i+1}$, the monomial contains the factor $x_k$ for some $k \in \{1, \dots, j-1\}$ not divisible by $2^{i+1}$. If $k \neq 1$, then it follows by the induction hypothesis that $x_k$ is in the image of $N_i^k$; therefore, by the projection formula, the monomial is in the image of $N_i^j$.

Now assume $k = 1$. There is at least one more factor $x_l$, for some $l \in \{1, \dots, j-1\}$. If $l \neq 1$, it follows by the induction hypothesis that $x_l$ is in the image of $N_{r-1}^l$ (our assumption that $j < 2^r$ ensures that $l$ is not divisible by $2^r$), so that $x_1 x_l = N_{r-1}^l(x_{1L_{r-1}} y)$ for some $y \in \mathrm{CH}^l X_{L_{r-1}}$. Since $x_{1L_{r-1}}$ is in the image of the norm map $N_{L_0/L_{r-1}}$, the product $x_1 x_l$ is in the image of $N_0^{l+1}$ (and therefore in the image of $N_i^{l+1}$ for any $i$).

It remains to consider the case $l = 1$. We show that $x_1^2$ is in the image of $N_0^2$. The Chow group $\mathrm{CH}^2 X$ coincides with the quotient $K(X)^{(2)}/K(X)^{(3)}$ of the second term of the topological filtration on the Grothendieck ring $K(X)$ by the third term. The second term of the topological filtration coincides with the second term of the gamma filtration. The third topological term contains the third gamma term and the quotient consists of torsion elements; see [Karpenko 1998, Proposition 2.14]. Since $4 \mid \operatorname{ind} A^{\otimes(d/4)}$, the quotient of the second gamma term by the third gamma term is torsion-free by [Karpenko 1998, Proposition 4.9 with Lemma 3.10] and the proof of Lemma 3.5. It follows that the third gamma term coincides with the third topological term. In particular, the quotient of the topological terms is torsion-free.

Therefore the group $\mathrm{CH}^2 X$ is torsion-free as well. So, by Remark 3.13, it is identified with $2^{r-1}\mathbb{Z} \subset \mathbb{Z} = \mathrm{CH}^2 X_{L_0}$. The image of the norm map $N_0^2$ is $2^r \mathbb{Z}_{(2)}$, and $x_1^2 = 2^{2r-2}$. Since $r \geq 2$, we have $2r - 2 \geq r$, showing that $x_1^2$ is indeed in the image of $N_0^2$.

To finish the proof of the claim (and therefore the proof of Proposition 3.12), it suffices to check that $x_j$ is also in the image of $N_i^j$. For odd $j$, this holds by Corollary 3.11 (we recall that $x_j$ is the $j$-th Chern class of the tautological vector bundle). For even $j$, we decompose the element $N_i^j(1) \in \mathrm{CH}^j X$ into a linear combination of the degree-$j$ monomials in $x_1, x_2, \dots, x_j$ and check that the coefficient $\lambda \in \mathbb{Z}_{(2)}$ at the monomial $x_j$ is invertible.

Let us observe that $v_2(N_i^j(1)_L) = v_2([L_i : F]) = r - i$. On the other hand, if $\lambda$ is not invertible, then $(\lambda x_j)_L$ is divisible by $2^{r-i+1}$ because $x_L$ is divisible by $2^{r-i}$ for any element $x \in \mathrm{CH}^j X$; see Remark 3.13. Also $M_L$ is divisible by $2^{r-i+1}$ for any monomial $M \in \mathrm{CH}^j X$ in $x_1, \dots, x_{j-1}$, because $M$ contains $x_k$ for some $k$ not divisible by $2^{i+1}$; $x_{kL}$ is then divisible by $2^{r-i}$ (even if $k = 1$, because $i \geq 1$ since $j$ is even); at the same time $M$ necessarily contains another factor $x_l$ for some $l = 1, \dots, j-1$ ($l = k$ is also possible). Our assumption that $j < 2^r$ ensures that $l$ is not divisible by $2^r$, so that $x_{lL}$ is divisible by 2. □

*Proof of Theorem 3.1.* Let $A$ be the central simple $F$-algebra corresponding to the generic $G$-torsor $E$. By Lemma 3.6, we may assume that $E/P$ is the Severi–Brauer variety $X$ of $A$. By [Karpenko 2016, proof of Theorem 1.1], the ring $\mathrm{CH}\, X$ is generated by $\mathrm{CH}^1 X$ and the Chern classes of the tautological vector bundle. This, in particular, implies that the topological filtration on $K(X)$ coincides with the gamma filtration.

We start by assuming that the condition $(m, n/m) \leq 2$ fails. Then the integer $(m, n/m)$ is divisible by an odd prime number $p$ or by 4. In the first case, let us show that the group $\mathrm{CH}^2(E/P)$ has an element of order $p$. The group $\mathrm{CH}^2 X$ is isomorphic to the quotient $K(X)^{(2)}/K(X)^{(3)}$ of the topological filtration on the Grothendieck group $K(X)$. Let $L/F$ be a finite extension of degree prime to $p$ such that the index of the $L$-algebra $A_L$ is a $p$-power. Note that $\operatorname{ind} A_L = p^{v_p(n)}$ and $\exp A_L = p^{v_p(m)}$, so that $\exp A_L < \operatorname{ind} A_L$. The change of field homomorphism $K(X) \otimes \mathbb{Z}_{(p)} \to K(X_L) \otimes \mathbb{Z}_{(p)}$ is an isomorphism of rings with filtrations. The topological filtration on $K(X_L) \otimes \mathbb{Z}_{(p)}$ coincides with the gamma filtration. By [Karpenko 1998, Proposition 4.7], the 2nd quotient of the gamma filtration on $K(X_L)$ has an element of order $p$. So, we get an element of order $p$ in $\mathrm{CH}^2 X$.

Let now assume that 4 divides $(m, n/m)$ and prove that $\mathrm{CH}^2(E/P)$ has an element of order 2. We proceed as above and come to a 2-primary algebra $A_L$ with $\exp A_L < \frac{1}{2} \operatorname{ind} A_L$. By [Karpenko 1998, Proposition 4.9], the 2nd quotient of the gamma filtration on $K(X_L)$ has an element of order 2. So, we get an element of order 2 in $\mathrm{CH}^2 X$.

Finally, let us assume that $(m, n/m) \leq 2$. For an arbitrary prime number $p$ we claim that the $p$-torsion of $\mathrm{CH}\,X$ is trivial. If $v_p(m) = 0$, then $p$ does not divide the index of $A$, so that the claim is obvious. Below we assume that $v_p(m) > 0$, in which case $v_p(m) = v_p(n)$ or $p = 2$ and $v_2(m) = v_2(n) - 1$.

If $v_p(m) = v_p(n)$, Proposition 3.2 does the job.

If $p = 2$ and $v_2(m) = v_2(n) - 1$, we are done by Proposition 3.12. Indeed, by [Karpenko 1998, Lemma 3.10], there exists a central simple algebra $A$ (over a field extension of $k$) of degree $n$ and exponent $m$, satisfying the condition $4 \mid \operatorname{ind} A^{\otimes(d/4)}$ of Proposition 3.12, where $d := \operatorname{ind} A$. Therefore any generic algebra of degree $n$ and exponent $m$ satisfies this condition. □

The following statement is an application proved similarly to Corollaries 3.9 and 3.10:

**Corollary 3.14.** *Let $X$ be the Severi–Brauer variety of a central simple $k$-algebra $A$ such that $d := \operatorname{ind} A = 2\exp A$ and $4 \mid \operatorname{ind} A^{\otimes(d/4)}$. Then the torsion subgroup* $\mathrm{Tors}\,\mathrm{CH}\,X$ *splits off canonically as a direct summand of* $\mathrm{CH}\,X$. *Furthermore, the topological filtration on the Grothendieck ring $K(X)$ coincides with the gamma filtration. Moreover, for any finite product $Y$ of any generalized Severi–Brauer varieties of any tensor powers of $A$, the topological filtration on the Grothendieck ring $K(X_{k(Y)})$ coincides with the gamma filtration.* □

## 4. Type $\mathrm{C}_n$

A split simple group $G$ over $k$ of type $\mathrm{C}_n$ ($n \geq 1$) is isomorphic to $\mathrm{Sp}_{2n}$ (the simply connected case) or $\mathrm{PGSp}_{2n}$ (the adjoint case). The group $\mathrm{Sp}_{2n}$ is special. For this reason, we only treat the adjoint case $G = \mathrm{PGSp}_{2n}$ here.

The set of isomorphism classes of $G$-torsors over $k$ is identified with the set of isomorphism classes of central simple $k$-algebras of degree $2n$ endowed with a symplectic involution. Let $E$ be a $G$-torsor over $k$ and let $A$ be a corresponding $k$-algebra. Since $A$ possesses a $k$-linear involution, the exponent of $A$ is 2 or $A$ is split. The index of $A$ is a 2-power, a divisor of the 2-primary part of $2n$. If $E$ is a generic $G$-torsor (over $F \supset k$), then $\exp A = 2$ and $\operatorname{ind} A$ is the 2-primary part of $2n$.

Let $P \subset G$ be a parabolic subgroup of type $\mathrm{C}_{n-1}$. Then $P$ is special and the variety $E/P$ can be viewed as the variety of isotropic right ideals in $A$ of reduced dimension 1. But every right ideal of reduced dimension 1 is isotropic with respect to any symplectic involution on $A$, therefore $E/P$, which is a priori a closed subvariety in the Severi–Brauer variety $\mathrm{SB}(A)$, coincides with $\mathrm{SB}(A)$.

If $n$ is not divisible by 4, then $\operatorname{ind} A$ divides 4 and it follows that the group $\mathrm{CH}\,X$ of $X := \mathrm{SB}(A)$ is torsion-free. In more detail, $\mathrm{CH}\,X$ is a direct sum of shifted copies of $\mathrm{CH}\,X'$, where $X'$ is the Severi–Brauer variety of a degree-4 central simple algebra $A'$ Brauer-equivalent to $A$. For $i \leq 2$ the group $\mathrm{CH}^i X'$ coincides with the $i$-th quotient of the topological filtration on $K(X')$, which is torsion-free (for $i = 2$, see [Karpenko 1998, Proposition 4.9], for example). The group $\mathrm{CH}^3 X' = \mathrm{CH}_0 X'$ is torsion-free by [Chernousov and Merkurjev 2006] (originally proved in [Panin 1984]).

For any $n$ and generic $E$ (over $F \supset k$), it follows by Corollary 3.10 and specialization that the topological filtration on $K(X)$ coincides with the gamma filtration. Indeed, over a suitable field extension $k''/k$, there exists a central division algebra $A''$ with $2n = \deg A'' = \operatorname{ind} A'' = \exp A''$. Taking for $Y$ in Corollary 3.10 the Severi–Brauer variety of the tensor square of $A''$ and setting $k' := k''(Y)$ and $A' := A''_{k'}$, we get that, for $X' := \mathrm{SB}(A')$, the topological filtration on $K(X')$ coincides with the gamma filtration. By the index reduction formula for Severi–Brauer varieties [Schofield and Van den Bergh 1992] (see also [Merkurjev et al. 1996, (5.11)]), the index of the algebra $A'$ is the 2-primary part of $2n$ and its exponent is 2. In particular, $A'$ admits a symplectic involution [Knus et al. 1998, Theorem 3.1(1) and Corollary 2.8(2)]. The pair, consisting of the algebra with a fixed symplectic involution on it, is given by a $G$-torsor $E'$ over $k'$. Using specialization, we identify $K(X)$ with $K(X')$. Under this identification, the gamma filtration on $K(X)$ is identified with the gamma filtration on $K(X')$, while each term of the topological filtration on $K(X)$ is identified with a subgroup of the corresponding term of the topological filtration on $K(X')$. Since each term of the topological filtration on $K(X)$ contains the corresponding term of the gamma filtration, both filtrations on $K(X)$ coincide.

By [Karpenko 1998, Proposition 4.9], if $n$ is divisible by 4, the second quotient of the gamma filtration contains an element of order 2. We have proven:

**Theorem 4.1.** *For $G := \mathrm{PGSp}_{2n}$ ($n \geq 1$) over any field $k$, let $P \subset G$ be a special parabolic subgroup and let $E$ be a generic $G$-torsor over a field extension $F/k$. The group $\mathrm{CH}(E/P)$ is torsion-free if and only if $n$ is not divisible by 4. Moreover, if $n$ is divisible by 4, the group $\mathrm{CH}^2(E/P)$ contains an element of order 2.* □

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