ANNALS OF
K-THEORY

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Hochster's theta invariant is defined for a pair of finitely generated modules on a hypersurface ring having only an isolated singularity. Up to a sign, it agrees with the Euler invariant of a pair of matrix factorizations.

Working over the complex numbers, Buchweitz and van Straten have established an interesting connection between Hochster's theta invariant and the classical linking form on the link of the singularity. In particular, they establish the vanishing of the theta invariant if the hypersurface is even-dimensional by exploiting the fact that the (reduced) cohomology of the Milnor fiber is concentrated in odd degrees in this situation.

We give purely algebraic versions of some of these results. In particular, we establish the vanishing of the theta invariant for isolated hypersurface singularities of even dimension in characteristic $p > 0$ under some mild extra assumptions. This confirms, in a large number of cases, a conjecture of Hailong Dao.

1. Introduction

In this paper, a *hypersurface* will refer to a ring R that can be expressed as a quotient of a regular (Noetherian) ring Q by a non-zero-divisor f; i.e., $R = Q/f$.

(We do not require Q to be local.) If M and N are finitely generated R-modules then, using the standard long exact sequence

$$\cdots \to \operatorname{Tor}_{i-1}^Q(M, N) \to \operatorname{Tor}_i^R(M, N) \to \operatorname{Tor}_{i-2}^R(M, N) \to \operatorname{Tor}_{i-2}^Q(M, N) \to \cdots$$

and the fact that $\operatorname{Tor}_j^Q(M, N) = 0$ for $j \gg 0$ since Q is regular, we conclude that $\operatorname{Tor}_*^R(M, N)$ is eventually two-periodic: there is an isomorphism

$$\operatorname{Tor}_i^R(M, N) \xrightarrow{\;\cong\;} \operatorname{Tor}_{i+2}^R(M, N) \quad \text{for } i \gg 0.$$

If we assume, in addition, that $\operatorname{Tor}_i^R(M, N)$ has finite length for all $i \gg 0$, then Hochster's *theta invariant* [1981] of the pair (M, N) is defined to be the integer

$$\theta^R(M, N) = \operatorname{length} \operatorname{Tor}_{2i}^R(M, N) - \operatorname{length} \operatorname{Tor}_{2i+1}^R(M, N) \quad \text{for } i \gg 0.$$

The modules $\operatorname{Tor}_i^R(M, N)$ will have finite length for $i \gg 0$ if, for example, there exists a maximal ideal $\mathfrak{m}$ of R such that one of the modules, say N, is locally of finite projective dimension on the punctured spectrum $\operatorname{Spec}(R) \setminus \mathfrak{m}$. In this situation, N determines a coherent sheaf on the quasiaffine scheme $\operatorname{Spec}(R) \setminus \mathfrak{m}$ that admits a finite resolution by locally free coherent sheaves, and hence N determines a class in $[N] \in K_0(\operatorname{Spec}(R) \setminus \mathfrak{m})$. The module M, of course, determines a class $[M] \in G_0(R)$. Hochster [1981, Theorem 1.2] proves that θ is biadditive for such pairs of modules and hence determines a pairing

$$G_0(R) \times K_0(\operatorname{Spec} R \setminus \mathfrak{m}) \to \mathbb{Z}.$$

Moreover, $\theta(R/\mathfrak{m}, -)$ is identically zero and, since $G_0(R) \to G_0(\operatorname{Spec}(R) \setminus \mathfrak{m})$ is surjective with kernel generated by $[R/\mathfrak{m}]$, we obtain an induced pairing

$$\theta = \theta_{(Q, \mathfrak{m}, f)} : G_0(\operatorname{Spec}(R) \setminus \mathfrak{m}) \times K_0(\operatorname{Spec}(R) \setminus \mathfrak{m}) \to \mathbb{Z}. \qquad (1.1)$$

We will refer to this as *Hochster's θ pairing associated to the data* $(Q, f, \mathfrak{m})$.

Buchweitz and van Straten [2012] relate Hochster's θ pairing for isolated hypersurface singularities of the form $R = \mathbb{C}\{x_0, \ldots, x_n\}/(f)$, where $\mathbb{C}\{x_0, \ldots, x_n\}$ is the ring of convergent power series, to the linking form on the link of the singularity. Using this relationship, they also prove the θ pairing vanishes when $\dim(R)$ is even.

The goal of this paper is to give a purely algebraic interpretation of some of the results of Buchweitz and van Straten, ones which are also valid in characteristic $p > 0$. In particular, we obtain the vanishing of θ for isolated hypersurface singularities of even dimension in all characteristics for a large number of rings, confirming in many cases a conjecture of H. Dao [2013, Conjecture 3.15]. We refer the reader to Corollaries 7.16 and 7.17 for the most general statements, but an important special case of our results is given by the following theorem:

Theorem 1.2. *Let k be a perfect field, Q a finitely generated and regular k-algebra, and $f \in Q$ a non-zero-divisor. Assume the associated morphism of affine varieties*

$$f : \text{Spec}(Q) \to \mathbb{A}_k^1$$

has only isolated singularities. Set $R = Q/f$.

If $\dim(R)$ is even, then $\theta^R(M, N) = 0$ for all finitely generated R-modules M and N.

The theorem may be extended easily to allow for localizations of smooth algebras, and thus justifies examples such as the following:

Example 1.3. Let k be a perfect field, let $f \in k[x_0, \dots, x_n]$ be a polynomial in $n+1$ variables contained in $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$, and assume $\langle \partial f/\partial x_0, \dots, \partial f/\partial x_n \rangle_{\mathfrak{m}}$ is an $\mathfrak{m}$-primary ideal of $k[x_0, \dots, x_n]_{\mathfrak{m}}$. Set $R = k[x_0, \dots, x_n]_{\mathfrak{m}}/f$.

If n is even, then $\theta^R(M, N) = 0$ for all pairs of finitely generated R-modules M and N.

One technical aspect of this paper may be of independent interest: in Section 3, we prove a slightly weakened form of a conjecture of C. Sherman [2004, §2] concerning his "star pairing" in algebraic K-theory.

2. Overview

We give an overview of this paper by comparing the details we use with those occurring in the work of Buchweitz and van Straten.

The Milnor fibration. Given a power series $f \in \mathbb{C}\{x_0, \dots, x_n\}$ with positive radius of convergence, we interpret f as defining a holomorphic function defined on an open neighborhood of 0 in $\mathbb{C}^{n+1}$. Let $\bar{B}$, the *Milnor ball*, be a closed ball of radius ϵ centered at the origin of $\mathbb{C}^{n+1}$ and let D be an open disk of radius δ centered at the origin in the complex plane $\mathbb{C}$. Here, ϵ is chosen first and to be sufficiently small, and δ is chosen to be sufficiently smaller than ϵ: $0 < \delta \ll \epsilon \ll 1$. Set $X = f^{-1}(D) \cap \bar{B}$ and also write f for the restricted function $f : X \to D$.

Let us assume that the fiber X_0 of f over $0 \in D$ is an isolated singularity at the origin. It follows that X_0 is homeomorphic to the cone over the *link* L of the singularity, defined as $L := X_0 \cap S$, where $S := \partial \bar{B} \cong S^{2n+1}$ is the *Milnor sphere*. The link L is a smooth orientable manifold of real dimension $2n - 1$.

The map away from the singular locus,

$$X^* := (X \setminus X_0) \to (D \setminus 0) =: D^*,$$

induced by f is a fibration, called the *Milnor fibration*. For any $t \neq 0$ in D, the fiber X_t of f over t is a smooth manifold with boundary of (real) dimension $2n$. Up to diffeomorphism, X_t is independent of t, and it is called the *Milnor fiber* of

the singularity. A key result of Milnor [1968] gives that the Milnor fiber has the
homotopy type of a bouquet of n-dimensional spheres; the number of spheres is the
Milnor number, often written as μ. In particular, the reduced singular cohomology
of the Milnor fiber, $\widetilde{H}^*(X_t)$, is a free abelian group of rank μ concentrated in
degree n.

The restriction of f to $S \cap X$ is a fibration and thus, up to diffeomorphism, we
may identify the boundary of the Milnor fiber, $\partial X_t = (f|_{X \cap S})^{-1}(t)$, with the link
$L = (f|_{X \cap S})^{-1}(0)$. In particular, for each $t \in D^*$ there is a continuous map

$$\rho_t : L \to X^*$$

given by composing the diffeomorphism $L \cong \partial X_t$ with the inclusion $\partial X_t \subseteq X^*$.

The vanishing of the pairing when n is even. With the above notation set, we
sketch the proof of Buchweitz and van Straten for the vanishing of θ when n is
even.

Associated to maximal Cohen–Macaulay (MCM) $\mathbb{C}\{x_0, \ldots, x_n\}/f$-modules M
and N, Buchweitz and van Straten associate classes in topological K-theory:

$$\alpha(M) \in K_{\text{top}}^1(X^*) \quad \text{and} \quad [N]_{\text{top}} \in K_{\text{top}}^0(L).$$

The class $[N]_{\text{top}}$ is given by the evident restriction to L of the coherent sheaf asso-
ciated to N the class $\alpha(M)$ is built from a *matrix factorization* representation
of M. They prove [Buchweitz and van Straten 2012, Theorem 4.2] that

$$\theta(M, N) = \chi_{\text{top}}\big(\rho_t^*(\alpha(M)) \cup [N]_{\text{top}}\big).$$

Here, $\rho_t^* : K_{\text{top}}^1(X^*) \to K_{\text{top}}^1(L)$ is the map induced by the map $\rho_t : L \rightarrowtail X^*$ defined
above, $\cup$ is the product operation for the ring K_{top}^*, and

$$\chi_{\text{top}} : K_{\text{top}}^1(L) \to K_{\text{top}}^0(\text{pt}) \cong \mathbb{Z}$$

is induced by push-forward. (Recall L is odd-dimensional and so χ_{top} switches the
parity of degrees.)

Notice that ρ_t^* factors through $K_{\text{top}}^1(X_t)$, since ρ_t factors through X_t by its very
construction. When n is even, we have $K_{\text{top}}^1(X_t) = 0$, since X_t is a bouquet of
n-dimensional spheres by Milnor's theorem [1968]. It follows that $\rho_t^*(\alpha(M)) = 0$,
and hence

$$\theta(M, N) = \chi_{\text{top}}\big(\rho_t^*(\alpha(M)) \cup [N]_{\text{top}}\big) = 0.$$

The algebraic analogue of the Milnor fibration. The algebraic analogue of the
Milnor fibration is constructed as follows: Assume V is a Henselian DVR with
algebraically closed residue field k and field of fractions F. For example, V could
be $k[\![t]\!]$ with $k = \bar{k}$. The affine scheme $\text{Spec}(V)$ is the analogue of D, a small open

disk in the complex plane. Let Q be a flat V-algebra of finite type and assume the associated morphism of affine schemes

$$\mathrm{Spec}(Q) \to \mathrm{Spec}(V)$$

is smooth except at a single point $\mathfrak{m} \in \mathrm{Spec}(Q)$, necessarily belonging to the closed fiber. Let $Q_{\mathfrak{m}}^{\mathrm{hen}}$ denote the Henselization of Q at $\mathfrak{m}$. The affine scheme $\mathrm{Spec}(Q_{\mathfrak{m}}^{\mathrm{hen}})$ is the analogue of $X = f^{-1}(D) \cap \bar{B}$ in the notation above. Thus the morphism

$$X^{\mathrm{alg}} := \mathrm{Spec}(Q_{\mathfrak{m}}^{\mathrm{hen}}) \to \mathrm{Spec}(V) =: D^{\mathrm{alg}}$$

is the algebraic analogue of the analytic map $f : X \to D$ considered above. The generic fiber of $X^{\mathrm{alg}} \to D^{\mathrm{alg}}$,

$$X_{\mathrm{alg}}^{*} := \mathrm{Spec}(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V F) \to \mathrm{Spec}(F) =: D_{\mathrm{alg}}^{*},$$

is the analogue of the Milnor fibration, and the geometric generic fiber,

$$X_t^{\mathrm{alg}} := \mathrm{Spec}(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \bar{F}),$$

where $\bar{F}$ is the algebraic closure of F, is the analogue of the Milnor fiber.

Remark 2.1. It is important to be aware that $Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \bar{F}$ need not be a Noetherian ring.

The singularity is the closed fiber of $X^{\mathrm{alg}} \to D^{\mathrm{alg}}$,

$$X_0^{\mathrm{alg}} := \mathrm{Spec}(R_{\mathfrak{m}}^{\mathrm{hen}}) = \mathrm{Spec}(Q_{\mathfrak{m}}^{\mathrm{hen}}/f),$$

where f denotes the image in Q of a specified uniformizing local parameter $t \in V$ and we set $R = Q/f$. The algebraic analogue of the link is the punctured spectrum of the singularity,

$$L^{\mathrm{alg}} := \mathrm{Spec}(R_{\mathfrak{m}}^{\mathrm{hen}}) \setminus \mathfrak{m}.$$

We summarize these constructions and introduce a few more analogies in Table 1.

Remark 2.2. The algebraic analogue of the Milnor ball and the algebraic analogue of X coincide, in contrast with what occurs in the analytic setting. The analogy could be slightly improved if one does not assume from the start that V is Henselian. Then the algebraic analogue the Milnor ball remains $\mathrm{Spec}(Q_{\mathfrak{m}}^{\mathrm{hen}})$ but the algebraic analogue of $X \to D$ would become $\mathrm{Spec}(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V V^{\mathrm{hen}}) \to \mathrm{Spec}(V^{\mathrm{hen}})$, where V^{hen} denotes the Henselization of V at its unique maximal ideal. For this paper, however, this more general construction has no advantage.

With this notation fixed, let us sketch our proof of the vanishing of the θ pairing for $R = Q/f$.

Given finitely generated MCM R-modules M and N, we define classes

$$\alpha^{\mathrm{alg}}(M) \in K_1(X_{\mathrm{alg}}^{*}) = K_1(Q[1/f])$$

geometric notion	algebraic analogue
D, a small open disc in complex plane	$D^{alg} := \mathrm{Spec}(V)$, where V is a Henselian DVR with uniformizing parameter t, algebraically closed residue field k, and field of fractions F
convergent power series $f \in \mathbb{C}\{x_0, \ldots, x_n\}$ representing an isolated singularity	a flat, finite-type ring map $V \to Q$ sending $t \in V$ to $f \in Q$ that is smooth away from some point $\mathfrak{m}$ in the closed fiber
X, the intersection of a small closed ball in $\mathbb{C}^{n+1}$ with $f^{-1}(D)$	$X^{alg} = \mathrm{Spec}(Q_{\mathfrak{m}}^{hen})$
Milnor fibration $X^* \to D^*$	the generic fiber of $X^{alg} \to D^{alg}$; namely, $\mathrm{Spec}(Q^{hen} \otimes_V F) \to \mathrm{Spec}(F)$
the Milnor fiber $X_t := f^{-1}(t), t \neq 0$	the geometric generic fiber of $X^{alg} \to D^{alg}$; namely, $X_t^{alg} := \mathrm{Spec}(Q_{\mathfrak{m}}^{hen} \otimes_V \bar{F})$
the singularity $X_0 = f^{-1}(0)$	$X_0^{alg} := \mathrm{Spec}(R_{\mathfrak{m}}^{hen})$, where $R = Q/f$ (i.e., the closed fiber of $X^{alg} \to D^{alg}$)
the link L	the punctured spectrum $\mathrm{Spec}(R_{\mathfrak{m}}^{hen}) \setminus \mathfrak{m} =: L^{alg}$
Milnor ball $\bar{B}$	$\bar{B}^{alg} = \mathrm{Spec}(Q_{\mathfrak{m}}^{hen})$
Milnor sphere $S = \partial \bar{B}$	$S^{alg} = \mathrm{Spec}(Q_{\mathfrak{m}}^{hen}) \setminus \mathfrak{m}$
$\rho_t^* : K_{top}^1(X^*) \to K_{top}^1(L)$	the "specialization map" in K-theory with finite coefficients.

Table 1. Summary of geometric notions and their algebraic analogues.

and
$$[N] \in K_0(L^{alg}) = K_0(\mathrm{Spec}(R_{\mathfrak{m}}) \setminus \mathfrak{m}),$$

where $[N]$ is defined as the image of the class of N in $G_0(R)$ under the restriction map $G_0(R) \to G_0(\mathrm{Spec}(R) \setminus \mathfrak{m}) \cong K_0(\mathrm{Spec}(R) \setminus \mathfrak{m})$ and $\alpha^{alg}(M) = [A] \in K_1(Q[1/f])$ for any matrix factorization (A, B) representation of M. Our first key result (see Corollary 4.10) is the equation

$$\theta^R(M, N) = \chi\big(\partial(\alpha(M) \cup [f]) \cup [N]\big). \tag{2.3}$$

Let us explain the components of this formula. Since f is a unit of $Q[1/f]$, it determines a class $[f] \in K_1(Q[1/f])$. The symbol $\cup$ denotes the product operation in K-theory. The map $\partial : K_2(Q[1/f]) \to K_1(\mathrm{Spec}(R) \setminus \mathfrak{m})$ is the boundary map in the long exact localization sequence associated to the closed subscheme $L^{alg} =$

$\operatorname{Spec}(R) \setminus \mathfrak{m}$ of $S^{\mathrm{alg}} = \operatorname{Spec}(Q) \setminus \mathfrak{m}$. (Since $Q[1/f]$ and $\operatorname{Spec}(R) \setminus \mathfrak{m}$ are regular, we use, as we may, K-theory in place of G-theory.) Finally,

$$\chi : K_1(\operatorname{Spec}(R) \setminus \mathfrak{m}) \to \mathbb{Z}$$

is the composition of the boundary map

$$K_1(\operatorname{Spec}(R) \setminus \mathfrak{m}) \to K_0(\operatorname{Spec}(R/\mathfrak{m}))$$

with the canonical isomorphism $K_0(\operatorname{Spec}(R/\mathfrak{m})) \cong \mathbb{Z}$, sending $[R/\mathfrak{m}]$ to 1.

All of these facts remain true if we use K-theory with $\mathbb{Z}/l$ coefficients, where l is a prime distinct from the characteristic of $Q/\mathfrak{m}$ (and, for technical reasons, $l \geq 5$). Moreover, we may replace algebraic K-theory with (a version of) étale K-theory with $\mathbb{Z}/l$ coefficients, which we write as $K_*^{\mathrm{top}}(-, \mathbb{Z}/l)$.

A portion of the formula (2.3), using $K_*^{\mathrm{top}}(-, \mathbb{Z}/l)$ instead of $K_*(-)$, is given by the mapping

$$K_1^{\mathrm{top}}(X_{\mathrm{top}}^*, \mathbb{Z}/l) = K_1^{\mathrm{top}}(Q[1/f], \mathbb{Z}/l) \to K_1^{\mathrm{top}}(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) = K_1^{\mathrm{top}}(L^{\mathrm{alg}}, \mathbb{Z}/l)$$

that sends α to $\partial(\alpha \cup [f])$. We interpret this map, under some additional hypotheses, as a "specialization map" in K-theory, and it is the analogue of the map ρ_t^* occurring in the work of Buchweitz and van Straten. The next key result is that this specialization map factors through

$$K_1^{\mathrm{top}}(X_t^{\mathrm{alg}}, \mathbb{Z}/l) = K_1^{\mathrm{top}}(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \bar{F}, \mathbb{Z}/l).$$

This factorization is the analogue of the factorization of ρ_t^* through the K-theory of the Milnor fiber used in the work of Buchweitz and van Straten.

Finally, we combine a theorem of Rosenschon and Østvær [2006, Theorem 4.3] (which is a generalization of a celebrated theorem of Thomason [1985, Theorem 4.1]) and a theorem of Illusie [2003, Corollaire 2.10] to prove

$$K_1^{\mathrm{top}}(X_t^{\mathrm{alg}}, \mathbb{Z}/l) = 0$$

if n is even (and some additional mild hypotheses hold). Illusie's theorem is in fact a good analogue of Milnor's theorem that the Milnor fiber of an isolated singularity is homotopy equivalent to a bouquet of n-spheres.

These results combine to prove the vanishing of θ for n even. See Theorem 7.15 and its corollaries for the precise statement of our vanishing result.

Finally, when n is odd, Buchweitz and van Straten prove that the θ pairing is induced by the linking form on the homology of the link of the singularity. Our Corollary 4.10 may be interpreted as analogue of this result in algebraic K-theory; see the discussion at the end of Section 4.

3. On Sherman's star pairing

Suppose Q is a regular ring, $f \in Q$ is a non-zero-divisor, and (A, B) is a *matrix factorization* of f. Recall that this means A and B are $m \times m$ matrices with entries in Q such that $AB = f I_m = BA$. Note that A may be regarded as an element of $\mathrm{GL}_m(Q[1/f])$ and hence it determines a class $[A] \in K_1(Q[1/f])$. The main result of this section, Corollary 3.11, gives an explicit description of the image of $[A]$ under the boundary map

$$K_1(Q[1/f]) \xrightarrow{\partial} G_0(Q/f)$$

in the long exact localization sequence in G-theory. (Since $Q[1/f]$ is regular, $K_1(Q[1/f]) \cong G_1(Q[1/f])$.) Our description of this image, and its proof, builds on work of C. Sherman, which we review here.

Pairings in K-theory. We will need some results about various pairings in algebraic K-theory, including Sherman's *star pairing*.

For an exact category $\mathcal{P}$, an object M of it, and a pair of commuting automorphisms α and β of M, Sherman [2004] defines an element

$$\alpha \star_{\mathrm{Sh}} \beta \in K_2(\mathcal{P}).$$

In this definition, the group $K_2(\mathcal{P})$ is taken to be $\pi_2|G(\mathcal{P})|$ where $G(\mathcal{P})$ is a simplicial set, the "G-construction", defined by Gillet and Grayson [1987]. The reader is referred to their paper for the full definition, but let us recall the definition of zero, one and two simplices. A zero simplex in $G(\mathcal{P})$ is an ordered pair (X, Y) of objects of $\mathcal{P}$. An edge (i.e., a one simplex) connecting (X, Y) to (X', Y') is given by a pair of short exact sequences of the form

$$0 \to X \xrightarrow{i} X' \xrightarrow{p} Z \to 0 \quad \text{and} \quad 0 \to Y \xrightarrow{j} Y' \xrightarrow{q} Z \to 0.$$

(Notice the right-most nonzero object is the same in both sequences.) Equivalently, an edge is a pair of monomorphisms, together with a compatible collection of isomorphisms on the various representations of their cokernels. We typically write an edge as $(X, Y) \xrightarrow{(i,j)} (X', Y')$, leaving Z, p, and q implicit.

A two simplex is represented by a pair of commutative diagrams
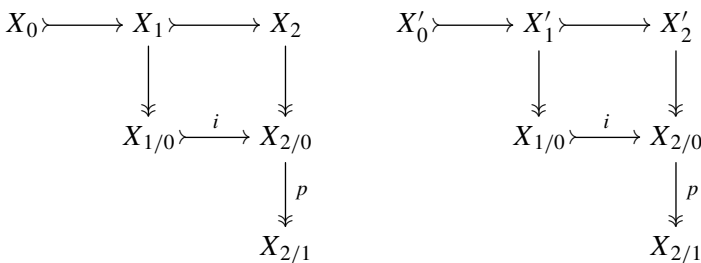
such that $0 \to X_i \to X_j \to X_{i/j} \to 0$ and $0 \to X_i' \to X_j' \to X_{i/j} \to 0$ for all $0 \leq i < j \leq 2$, and $0 \to X_{1/0} \to X_{2/0} \to X_{2/1} \to 0$ are short exact sequences.

The element $\alpha \star_{\mathrm{Sh}} \beta$ of $K_2(\mathcal{P})$ is represented by the following diagram of simplices in $G(\mathcal{P})$:

$$
\begin{array}{ccc}
& (M, M) \xrightarrow{(1,\beta)} (M, M) & \\
(0,0) \quad (\alpha,\alpha) & (\alpha,\alpha\beta) & (\alpha,\alpha) \quad (0,0) \\
& (M, M) \xrightarrow{(1,\beta)} (M, M) &
\end{array}
\tag{3.1}
$$

In this diagram, each triangle is a two simplex. The lower-middle triangle is the two simplex

$$
\begin{array}{ccc}
M \xrightarrow{\alpha} M \xrightarrow{=} M & \qquad & M \xrightarrow{\alpha} M \xrightarrow{\beta} M \\
\downarrow \qquad \downarrow & & \downarrow \qquad \downarrow \\
0 \longrightarrow 0 & & 0 \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

and the others are defined similarly. Since the top and bottom paths in (3.1) represent the same loop in $|G(\mathcal{P})|$, this diagram represents a map from the two-sphere to $|G(\mathcal{P})|$ and hence an element of $K_2(\mathcal{P})$.

Sherman's pairing is functorial in the following sense: if $F : \mathcal{P} \to \mathcal{P}'$ is an exact functor between exact categories, $M \in \mathrm{ob}\,\mathcal{P}$, and α and β are commuting automorphisms of M, then $F_* : K_2(\mathcal{P}) \to K_2(\mathcal{P}')$ sends $\alpha \star_{\mathrm{Sh}} \beta$ to $F(\alpha) \star_{\mathrm{Sh}} F(\beta)$.

If $\mathcal{P} = \mathcal{P}(S)$, the category of projective right S-modules for some (not necessarily commutative) ring S, and $A, B \in \mathrm{GL}_n(S)$ are matrices that commute, we define

$$A \star_{\mathrm{Sh}} B \in K_2(S)$$

by viewing A and B as commuting automorphisms of the right S-module S^n, with S^n thought of as column vectors and the action of A and B given by multiplication on the left of S^n. If $g : S \to S'$ is a ring map and A and B are commuting elements of $\mathrm{GL}_n(S)$, then $g_*(A \star_{\mathrm{Sh}} B) = g(A) \star_{\mathrm{Sh}} g(B)$ holds in $K_2(S')$.

Grayson [1979] has also defined a "star" pairing, defined for pairs of commuting $n \times n$ invertible matrices (A, B) in a (not necessarily commutative) ring S. Grayson's definition amounts to

$$A \star_{\mathrm{Gr}} B := D(A) \star_{\mathrm{Mi}} D'(B),$$

where $D(A)$ and $D'(B)$ are the $3n \times 3n$ elementary matrices

$$D(A) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^{-1} & 0 \\ 0 & 0 & I_n \end{bmatrix} \quad \text{and} \quad D'(B) = \begin{bmatrix} B & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & B^{-1} \end{bmatrix},$$

and $\star_{\mathrm{Mi}}$ denotes *Milnor's star pairing*. The latter pairing was defined by Milnor [1971] for pairs of commuting *elementary* matrices E_1 and E_2 (in $E(R) \subset \mathrm{GL}(R) := \mathrm{GL}_\infty(R)$) and is given by

$$E_1 \star_{\mathrm{Mi}} E_2 = [\widetilde{E}_1, \widetilde{E}_2],$$

where $\widetilde{E}_1$ and $\widetilde{E}_2$ are lifts of E_1 and E_2 to elements of the Steinberg group, $\mathrm{St}(R)$, under the canonical surjection $\mathrm{St}(R) \twoheadrightarrow E(R)$, and $[-, -]$ denotes the commutator of a pair of elements of $\mathrm{St}(R)$.

We also have the multiplication rule for $K_*(S)$ when S is a commutative ring, which gives a pairing we will write as a cup product:

$$- \cup - : K_1(S) \otimes_{\mathbb{Z}} K_1(S) \to K_2(S).$$

There are several equivalent methods of describing this multiplication rule [Weibel 1981]; we use Milnor's [1971] original description of it: For $A \in \mathrm{GL}_m(S)$ and $B \in \mathrm{GL}_n(S)$, we write $[A], [B] \in K_1(R)$ for the associated K-theory classes. One then defines

$$[A] \cup [B] = D(A \otimes I_n) \star_{\mathrm{Mi}} D'(I_m \otimes B),$$

where $A \otimes I_n$ and $I_m \otimes B$ are identified with elements of $\mathrm{GL}_{mn}(R)$ by viewing each as an automorphism of $S^m \otimes_S S^n$ and choosing, arbitrarily, a basis of $S^m \otimes_S S^n$.

It is important to realize that $A \star_{\mathrm{Gr}} B$ and $[A] \cup [B]$ are not the same element of $K_2(S)$ in general. This is clear even for diagonal matrices: Say A and B are 2×2 diagonal matrices with diagonal entries a_1, a_2 and b_1, b_2 in a commutative ring S. Then $A \star_{\mathrm{Gr}} B = [a_1] \cup [b_1] + [a_2] \cup [b_2] \in K_2(R)$ but $[A] \cup [B] = [a_1] \cup [b_1] + [a_1] \cup [b_2] + [a_2] \cup [b_1] + [a_2] \cup [b_2]$. (We have written the group law for $K_2(S)$ additively here.)

We do have, however, the identity

$$A \star_{\mathrm{Gr}} u I_n = [A] \cup [u]$$

for any commutative ring S, unit $u \in S$ and matrix $A \in \mathrm{GL}_n(S)$. In particular,

$$u \star_{\mathrm{Gr}} v = [u] \cup [v] \tag{3.2}$$

for any pair of units u and v in a commutative ring.

It is clear that Grayson's star pairing is also functorial for ring maps: if $g : S \to S'$ is any ring homomorphism and A and B are elements of $\mathrm{GL}_n(S)$ that commute,

then

$$g_*(A \star_{\mathrm{Gr}}^S B) = g(A) \star_{\mathrm{Gr}}^{S'} g(B),$$

where $g_* : K_2(S) \to K_2(S')$ is the induced map on K_2 and the superscript indicates in which ring the star operation is being performed.

Both Grayson's and Sherman's operations are also preserved by Morita equivalence, in the following sense: if we identify an $n \times n$ invertible matrix with entries in $\mathrm{Mat}_m(S)$ with an $nm \times nm$ invertible matrix with entries in S in the usual manner, then

$$\psi(A \star_{\mathrm{Sh}}^{\mathrm{Mat}_m(S)} B) = A \star_{\mathrm{Sh}}^S B$$
$$\psi(A \star_{\mathrm{Gr}}^{\mathrm{Mat}_m(S)} B) = A \star_{\mathrm{Gr}}^S B,$$

where $\psi : K_2(\mathrm{Mat}_m(S)) \xrightarrow{\cong} K_2(S)$ is the canonical isomorphism induced from the isomorphisms $\psi : \mathrm{St}(\mathrm{Mat}_m(S)) \xrightarrow{\cong} \mathrm{St}(S)$ and $\psi : \mathrm{GL}(\mathrm{Mat}_m(S)) \xrightarrow{\cong} \mathrm{GL}(S)$ (see [Grayson 1979, Lemma 4.5]).

Sherman's theorem on the connecting homomorphism. Sherman's theorem [2004, Theorem 3.6] describes the image of certain elements of the form $\alpha \star_{\mathrm{Sh}} \beta \in K_1(\mathcal{P})$ under the boundary map in a long exact localization sequence. To state it precisely, and for use again in Section 4, we review some technical details of Sherman's work.

In an exact category $\mathcal{A}$, a *mirror image sequence* is a pair (E, F) of short exact sequences on the same three objects, but in the opposite order,

$$E = \left(0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0\right) \quad \text{and} \quad F = \left(0 \to Z \xrightarrow{j} Y \xrightarrow{q} X \to 0\right). \quad (3.3)$$

Sherman [1998, p. 18] associates to a mirror image sequence (E, F) a loop in $|G(\mathcal{A})|$ given by the diagram of one simplices in $G(\mathcal{A})$

$$(0, 0) \to (X, X) \xrightarrow{(\iota_1, i)} (X \oplus Z, Y) \xleftarrow{(\iota_2, j)} (Z, Z) \leftarrow (0, 0),$$

where ι_1 and ι_2 denote inclusions into the first and second summands, and he writes $G(E, F)$ for the associated element of $K_1(\mathcal{A}) = \pi_1|G(\mathcal{A})|$.

Mainly for use in Section 4, we recall here an alternative description of the class $G(E, F)$. A *double short exact sequence* in an exact category $\mathcal{A}$, defined originally by Nenashev [1996], is a pair of short exact sequences involving the same three objects, in the same order:

$$l = \left(0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0, \; 0 \to A \xrightarrow{j} B \xrightarrow{q} C \to 0\right).$$

Nenashev associates to a double short exact sequence l the class $m(l) \in K_1(\mathcal{A})$ represented by the loop

$$(0, 0) \to (A, A) \xrightarrow{(i, j)} (B, B) \leftarrow (0, 0).$$

(Moreover, Nenashev [1996; 1998] proves that $K_1(\mathcal{A})$ is generated by these loops and gives explicit generators for all the relations.)

Associated to a mirror image sequence (3.3), we have the double short exact sequence $l_{(E,F)}$ given by

$$
\begin{aligned}
0 \to X \oplus Z \xrightarrow{\begin{bmatrix} 0 & j \\ 0 & 0 \\ 1 & 0 \end{bmatrix}} Y \oplus Z \oplus X \xrightarrow{\begin{bmatrix} 0 & 1 & 0 \\ q & 0 & 0 \end{bmatrix}} Z \oplus X \to 0, \\
0 \to X \oplus Z \xrightarrow{\begin{bmatrix} i & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} Y \oplus Z \oplus X \xrightarrow{\begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} Z \oplus X \to 0.
\end{aligned}
\tag{3.4}
$$

The following result connects explicitly the two types of elements of $K_1(\mathcal{A})$:

Proposition 3.5 [Sherman 2013, p. 164]. *For any mirror image sequence (3.3) in an exact category $\mathcal{A}$, we have*

$$
G(E, F) = m(l_{E,F}) - G(Y, -1) \in K_1(\mathcal{A}),
$$

where $(Y, -1)$ denotes the mirror image sequence

$$
\left(0 \to Y \xrightarrow{-1} Y \to 0 \to 0, \; 0 \to 0 \to Y \xrightarrow{1} Y \to 0\right).
$$

Suppose now that $\mathcal{A}$ is an abelian category, $\mathcal{S} \subset \mathcal{A}$ is a Serre subcategory, and $\mathcal{P} = \mathcal{A}/\mathcal{S}$, the associated quotient abelian category. Recall that Quillen's localization sequence is a long exact sequence of the form

$$
\cdots \to K_i(\mathcal{S}) \to K_i(\mathcal{A}) \to K_i(\mathcal{P}) \xrightarrow{\partial} K_{i-1}(\mathcal{S}) \to \cdots .
$$

Suppose also that α and β are commuting injective endomorphisms of an object M of $\mathcal{A}$ whose cokernels lie in $\mathcal{S}$. These form a commutative diagram
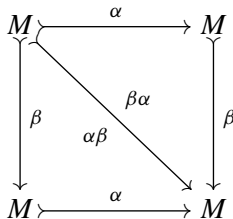
and by the "snake chasing its tail lemma", this diagram determines the mirror image sequence (E, F):

$$
\begin{aligned}
E &:= \left(0 \to \operatorname{coker}(\alpha) \xrightarrow{\beta} \operatorname{coker}(\beta\alpha) \longrightarrow \operatorname{coker}(\beta) \to 0\right), \\
F &:= \left(0 \to \operatorname{coker}(\beta) \xrightarrow{\alpha} \operatorname{coker}(\alpha\beta) \longrightarrow \operatorname{coker}(\alpha) \to 0\right).
\end{aligned}
\tag{3.6}
$$

Since we are assuming that coker(α) and coker(β) belong to $\mathcal{S}$, the mirror image sequence (E, F) determines the class $G(E, F) \in K_1(\mathcal{S})$. Also, the morphisms in $\mathcal{P} = \mathcal{A}/\mathcal{S}$ induced by α and β, which we will write as $\bar{\alpha}$ and $\bar{\beta}$, are commuting automorphisms of $M \in \mathcal{P}$, so that we have the element $\bar{\alpha} \star_{\mathrm{Sh}} \bar{\beta} \in K_2(\mathcal{P})$.

Theorem 3.7 [Sherman 2004, Theorem 3.6]. *Under these hypotheses,*

$$\partial(\bar{\alpha} \star_{\mathrm{Sh}} \bar{\beta}) = \pm G(E, F) \in K_1(\mathcal{S}), \tag{3.8}$$

where (E, F) is the mirror image sequence (3.6).

A proof of (a weakened form of) a conjecture of Sherman. Sherman has conjectured that his pairing $\star_{\mathrm{Sh}}$ and Grayson's pairing $\star_{\mathrm{Gr}}$ coincide up to a sign. We prove here a slightly weaker form of this conjecture:

Theorem 3.9. *If S is a (not necessarily commutative) ring and A and B are $n \times n$ invertible matrices with entries in S that commute, then $A \star_{\mathrm{Sh}} B$ and $A \star_{\mathrm{Gr}} B$ agree up to a sign and two-torsion — i.e.,*

$$2A \star_{\mathrm{Sh}} B = \pm 2A \star_{\mathrm{Gr}} B \in K_2(S).$$

Proof. We first prove this when $S = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ and A and B are the 1×1 matrices x and y. Since

$$K_2(S) \cong K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z})^{\oplus 2} \oplus K_0(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus (\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z},$$

to prove the theorem in this case it suffices to show the images of $x \star_{\mathrm{Sh}} y$ and $x \star_{\mathrm{Gr}} y$ under the canonical map $K_2(S) \twoheadrightarrow \mathbb{Z}$, given by modding out torsion, are both generators of $\mathbb{Z}$.

We have the localization long exact sequence

$$\cdots \to K_2(\mathbb{Z}[x^{\pm 1}, y]) \to K_2(S) \xrightarrow{\partial} G_1(\mathbb{Z}[x^{\pm 1}, y]/y)$$
$$\xrightarrow{0} K_1(\mathbb{Z}[x^{\pm 1}, y]) \rightarrowtail K_1(S) \to \cdots.$$

The map labeled as vanishing does so because the next map is injective, and this injectivity holds since

$$K_1(\mathbb{Z}[x^{\pm 1}]) \xrightarrow{\cong} K_1(\mathbb{Z}[x^{\pm 1}, y])$$

is an isomorphism and the map $\mathbb{Z}[x^{\pm 1}] \to S$ splits.

Observe $\mathbb{Z}[x^{\pm 1}, y]/y \cong \mathbb{Z}[x^{\pm 1}]$ and hence

$$G_1(\mathbb{Z}[x^{\pm 1}, y]/y) \cong K_1(\mathbb{Z}[x^{\pm 1}]) \cong K_1(\mathbb{Z}) \oplus K_0(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}.$$

Also,

$$K_2(\mathbb{Z}[x^{\pm 1}, y]) \cong K_2(\mathbb{Z}[x^{\pm 1}]) \cong K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

From these calculations it follows that, in order to prove the theorem in this special case, it suffices to prove $\partial(x \star_{\text{Sh}} y)$ and $\partial(x \star_{\text{Gr}} y)$ each map to a generator of $\mathbb{Z}$ under the canonical surjection $G_1(\mathbb{Z}[x^{\pm 1}, y]/y) \twoheadrightarrow \mathbb{Z}$ given by modding out torsion.

To prove $\partial(x \star_{\text{Sh}} y)$ maps to a generator, we apply Sherman's equation (3.8) with $\mathcal{A} = \mathcal{M}(\mathbb{Z}[x^{\pm 1}, y])$ and $\mathcal{S}$ the Serre subcategory of y-torsion modules, so that $\mathcal{A}/\mathcal{S} = \mathcal{M}(\mathbb{Z}[x^{\pm 1}, y^{\pm 1}])$. (In general, $\mathcal{M}(A)$ denotes the abelian category of finitely generated A-modules.) This gives $\partial(x \star_{\text{Sh}} y) = \pm G(E, F)$, where E and F are the short exact sequences, in $\mathcal{S}$,

$$E = \big(0 \to 0 \to \mathbb{Z}[x^{\pm 1}] \xrightarrow{=} \mathbb{Z}[x^{\pm 1}] \to 0\big),$$
$$F = \big(0 \to \mathbb{Z}[x^{\pm 1}] \xrightarrow{x} \mathbb{Z}[x^{\pm 1}] \to 0 \to 0\big).$$

Under the isomorphism $K_1(\mathbb{Z}[x^{\pm 1}]) \cong K_1(\mathcal{S})$ that is induced by the inclusion of $\mathcal{P}(\mathbb{Z}[x^{\pm 1}, y]/y)$ into $\mathcal{S}$, the element $G(E, F)$ corresponds to $[x] \in K_1(\mathbb{Z}[x^{\pm 1}])$, which maps to a generator under $K_1(\mathbb{Z}[x^{\pm 1}]) \twoheadrightarrow \mathbb{Z}$.

To prove $\partial(x \star_{\text{Gr}} y)$ also maps to a generator of $\mathbb{Z}$, we first observe that $x \star_{\text{Gr}} y = [x] \cup [y] \in K_2(S)$ by (3.2). Note that $[x]$ lifts to an element of $K_1(\mathbb{Z}[x^{\pm 1}, y])$ and hence, by [Suslin 1983, (1.1)], we have

$$\partial([x] \cup [y]) = \overline{[x]} \cup \partial'([y]),$$

where $\overline{[x]}$ is the image of $[x]$ under the map

$$K_1(\mathbb{Z}[x^{\pm 1}, y]) \to K_1(\mathbb{Z}[x^{\pm 1}, y]/y) \cong K_1(\mathbb{Z}[x^{\pm 1}])$$

and

$$\partial' : K_1(S) \to G_0(\mathbb{Z}[x^{\pm 1}, y]/y) \cong G_0(\mathbb{Z}[x^{\pm 1}]) \cong K_0(\mathbb{Z}[x^{\pm 1}]) \cong \mathbb{Z}$$

is the boundary map in the localization long exact sequence. Now $\partial'([y]) = \text{coker}(\mathbb{Z}[x^{\pm 1}, y] \xrightarrow{y} \mathbb{Z}[x^{\pm 1}, y]) = 1$ in $K_0(\mathbb{Z}[x^{\pm 1}])$ and thus

$$\partial([x] \cup [y]) = \overline{[x]} \cup \partial'([y]) = \overline{[x]},$$

whose image under

$$K_1(\mathbb{Z}[x^{\pm 1}]) \twoheadrightarrow \mathbb{Z}$$

is a generator. This completes the proof of the theorem in this special case.

The general form of the theorem now follows readily by naturality. Let $\text{Mat}_n(S)$ be the ring of $n \times n$ matrices with entries in S and define $g : \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \to \text{Mat}_n(S)$ by $g(x) = A$ and $g(y) = B$. This is well-defined since A and B are assumed to commute. We have proven $2x \star_{\text{Sh}} y = \pm 2x \star_{\text{Gr}} y$. Using the naturality of $\star_{\text{Sh}}$ and $\star_{\text{Gr}}$ for ring maps and Morita equivalence, it follows that

$$2A \star_{\text{Sh}} B = 2\psi(A \star_{\text{Sh}}^{\text{Mat}_n(S)} B) = 2\psi g_*(x \star_{\text{Sh}} y) = \pm 2\psi g_*(x \star_{\text{Gr}} y)$$
$$= \pm 2\psi(A \star_{\text{Gr}}^{\text{Mat}_n(S)} B) = \pm 2A \star_{\text{Gr}} B. \qquad \square$$

Corollary 3.10. *Suppose S is a commutative ring, A and B are $n \times n$ invertible matrices with entries in S, and $AB = BA = fI_n$ for some (unit) $f \in S$. Then $2A \star_{\mathrm{Gr}} B = 2[A] \cup [f]$ and $2A \star_{\mathrm{Sh}} B = \pm 2[A] \cup [f]$.*

Proof. By Theorem 3.9, it suffices to prove the first equation.

Since $B = fA^{-1}$, we have $D'(B) = D'(fI_n)D'(A^{-1}) = D'(fI_n)D'(A)^{-1}$ and hence the bimultiplicativity of $\star_{\mathrm{Mi}}$ (see [Milnor 1971, Lemma 8.1]) gives

$$A \star_{\mathrm{Gr}} B = D(A) \star_{\mathrm{Mi}} D'(B) = D(A) \star_{\mathrm{Mi}} D'(fI_n) + D(A) \star_{\mathrm{Mi}} D'(A^{-1})$$
$$= [A] \cup [f] - D(A) \star_{\mathrm{Mi}} D'(A)$$

(where, as before, the group law for K_2 is written additively).

We now extend slightly an argument of Milnor [1971, Lemma 8.2]. Let

$$P = \begin{bmatrix} -I_n & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix}.$$

Then $PD(A)P^{-1} = D'(A)$ and $PD(B)P^{-1} = D'(A)$. So, using the invariance of $\star_{\mathrm{Mi}}$ under inner automorphisms [Milnor 1971, Lemma 8.1], we have

$$D(A) \star_{\mathrm{Mi}} D'(A) = D'(A) \star_{\mathrm{Mi}} D(A) = -D(A) \star_{\mathrm{Mi}} D'(A).$$

It follows that $2D(A) \star_{\mathrm{Mi}} D'(A) = 0$ and hence

$$2A \star_{\mathrm{Gr}} B = 2[A] \cup [f]. \qquad \square$$

The following result will be used to reinterpret Hochster's theta invariant in the next section. We will apply it in the case when Q is assumed to be regular, but state it here in its natural level of generality.

Corollary 3.11. *Let Q be a commutative ring and $f \in Q$ a non-zero-divisor, and set $R = Q/f$. Assume (A, B) is a matrix factorization of f in Q; i.e., A and B are $m \times m$ matrices with entries in Q such that $AB = BA = fI_m$.*

Then, under the composition of

$$K_2(Q[1/f]) \xrightarrow{\mathrm{can}} G_2(Q[1/f]) \xrightarrow{\partial} G_1(R),$$

where ∂ is the boundary map in the localization long exact sequence, the image of the element $[A] \cup [f] \in K_2(Q[1/f])$ is, up to a sign and two-torsion, equal to $G(E, F) \in G_1(R)$, where (E, F) is the mirror image sequence

$$E := (0 \to \mathrm{coker}(A) \xrightarrow{\beta} R^m \to \mathrm{coker}(B) \to 0),$$
$$F := (0 \to \mathrm{coker}(B) \xrightarrow{\alpha} R^m \to \mathrm{coker}(A) \to 0).$$

Proof. This follows immediately from Theorem 3.7 and Corollary 3.10. $\qquad \square$

4. Reinterpreting theta

In this section, we use the results of the previous section to give a reformulation of the θ pairing in terms of more familiar K-theoretic constructions. Our results here are valid for a general hypersurface ring.

Let Q be a regular (Noetherian) ring, $\mathfrak{m}$ a maximal ideal of Q, and $f \in \mathfrak{m}$ a non-zero-divisor. Set $R = Q/f$ and also write $\mathfrak{m}$ for the image of $\mathfrak{m}$ in R. Recall that Hochster's theta invariant may be viewed as a bilinear pairing of the form

$$\theta = \theta_{(Q,\mathfrak{m},f)} : G_0(\operatorname{Spec} R \setminus \mathfrak{m}) \times K_0(\operatorname{Spec} R \setminus \mathfrak{m}) \longrightarrow \mathbb{Z},$$

and is determined by the formula

$$(M, N) \mapsto \operatorname{length} \operatorname{Tor}_{2j}^R(M, N) - \operatorname{length} \operatorname{Tor}_{2j+1}^R(M, N), \quad j \gg 0,$$

where M and N are finitely generated R-modules such that $\operatorname{pd}_{R_\mathfrak{p}} N_\mathfrak{p} < \infty$ for all $\mathfrak{p} \neq \mathfrak{m}$. We will need to know the θ pairing factors through localization at $\mathfrak{m}$; in fact, more is true:

Proposition 4.1. *Let* Q, $\mathfrak{m}$, f, *and* R *be as above. Suppose* $Q \to Q'$ *is a flat ring map such that* $\mathfrak{m}' := \mathfrak{m}Q'$ *is a maximal ideal of* Q' *and* $Q/\mathfrak{m} \xrightarrow{\cong} Q'/\mathfrak{m}'$ *is an isomorphism. Let* $f' \in Q'$ *be the image of* f *in* Q' *and set* $R' = Q'/f' = Q' \otimes_Q R$. *The triangle*

commutes, where $\phi : \operatorname{Spec}(R') \setminus \mathfrak{m}' \to \operatorname{Spec}(R) \setminus \mathfrak{m}$ *is the induced flat morphism.*
In particular, the θ *pairing factors through the localization at* $\mathfrak{m}$; *that is,*

commutes.

Proof. The hypothesis $\mathfrak{m}Q' = \mathfrak{m}'$ implies that $Q \to Q'$ does indeed determine a flat morphism $\phi : \mathrm{Spec}(R) \setminus \mathfrak{m} \to \mathrm{Spec}(R') \setminus \mathfrak{m}'$ of schemes. Since $R \to R'$ is flat, we have an isomorphism

$$\mathrm{Tor}_j^R(M, N) \otimes_R R' \cong \mathrm{Tor}_j^{R'}(M \otimes_R R', N \otimes_R R').$$

Finally, for any finitely generated R-module T supported at $\mathfrak{m}$, we have

$$\mathrm{length}_R(T) = \mathrm{length}_{R'}(T \otimes_R R'),$$

since $R' \otimes_R R/\mathfrak{m} \cong R'/\mathfrak{m}R' = R'/\mathfrak{m}'$ by assumption. $\qquad\square$

For any Noetherian scheme U, we write $K_*(U)$ for the (Quillen) K-groups of the exact category of locally free coherent sheaves on U and $G_*(U)$ for the K-groups of the abelian category of all coherent sheaves on U. The groups $K_*(U)$ form a graded ring with the multiplication rule, which we write as a cup product $- \cup -$, induced by the tensor product. The tensor product also defines the cap product pairing

$$- \cap - : K_i(U) \times G_j(U) \to G_{i+j}(U),$$

making $G_*(U)$ into a graded $K_*(U)$-module.

For any integer $l \geq 0$, let $K_*(U, \mathbb{Z}/l)$ and $G_*(U, \mathbb{Z}/l)$ denote K-theory and G-theory with $\mathbb{Z}/l$ coefficients. These are defined as the homotopy groups of spectra obtains from the $\mathcal{K}$- and $\mathcal{G}$-theory spectra by smashing them with the mod-l Moore space. There are long exact coefficient sequences

$$\cdots \to K_m(U) \xrightarrow{\cdot l} K_m(U) \longrightarrow K_m(U, \mathbb{Z}/l) \longrightarrow K_{m-1}(U) \to \cdots$$

and similarly for G-theory.

We also will need the mod-l versions of the cup and cap product pairings. To avoid complications in the multiplication rule for the Moore spaces for small primes, we assume for simplicity that $\mathbb{Z}/l$ has no 2- or 3-torsion; i.e., either $l = 0$ or it is not divisible by 2 or 3. In this case, $K_*(U, \mathbb{Z}/l)$ is a graded ring under cup product and $G_*(U, \mathbb{Z}/l)$ is a module over this ring under cap product. Moreover, $K_*(U) \to K_*(U, \mathbb{Z}/l)$ is a ring map and $G_*(U) \to G_*(U, \mathbb{Z}/l)$ is $K_*(U)$-linear. See [Thomason 1985, Section A.6] for more information.

The open complement of the closed subscheme $\mathrm{Spec}(R) \setminus \mathfrak{m}$ of $\mathrm{Spec}(Q) \setminus \mathfrak{m}$ is $\mathrm{Spec}\, Q[1/f]$ and (using that Q is regular) we have a long exact localization sequence in K-theory

$$\cdots \to K_{i+1}(\mathrm{Spec}(Q) \setminus \mathfrak{m}, \mathbb{Z}/l) \longrightarrow K_{i+1}(\mathrm{Spec}\, Q[1/f], \mathbb{Z}/l)$$
$$\xrightarrow{\partial} G_i(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \longrightarrow K_i(\mathrm{Spec}(Q) \setminus \mathfrak{m}, \mathbb{Z}/l) \to \cdots. \quad (4.2)$$

We also have the long exact localization sequence in G-theory associated to the closed subscheme $\mathrm{Spec}(R/\mathfrak{m})$ of $\mathrm{Spec}(R)$,

$$\cdots \to G_{i+1}(\mathrm{Spec}\, R, \mathbb{Z}/l) \longrightarrow G_{i+1}(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l)$$
$$\xrightarrow{\partial'} K_i(R/\mathfrak{m}, \mathbb{Z}/l) \longrightarrow G_i(R, \mathbb{Z}/l) \to \cdots . \quad (4.3)$$

Definition 4.4. Assume Q is a (not necessarily regular) Noetherian ring, $\mathfrak{m}$ is a maximal ideal, $f \in \mathfrak{m}$ is any element, and l is an integer such that $\mathbb{Z}/l$ has no 2- or 3-torsion. Set $R = Q/f$. Let ∂ and ∂' be the boundary maps in the G-theory localization sequences (4.2) and (4.3).

Define $\tilde{\theta} = \tilde{\theta}_{(Q,f,\mathfrak{m}),l,(i,j)}$ to be the pairing

$$\tilde{\theta} : G_i(Q[1/f], \mathbb{Z}/l) \times K_j(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \to K_{i+j-1}(R/\mathfrak{m}, \mathbb{Z}/l)$$

defined by the formula

$$\tilde{\theta}(\alpha, \gamma) = \partial'\big(\partial(\alpha \cap [f]) \cap \gamma\big),$$

where $[f] \in K_1(Q[1/f])$ denotes the class of the unit f of $Q[1/f]$.

Taking $i = 1$ and $j = 0$, we have in particular the pairing

$$\tilde{\theta} : G_1(Q[1/f], \mathbb{Z}/l) \times K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \to K_0(R/\mathfrak{m}, \mathbb{Z}/l) \cong \mathbb{Z}/l, \quad (4.5)$$

where the isomorphism sends $[R/\mathfrak{m}]$ to $1 \in \mathbb{Z}/l$. The goal of this section is to relate this pairing to Hochster's θ pairing.

We shall need the following analogue of Proposition 4.1 for $\tilde{\theta}$:

Proposition 4.6. *Suppose* $(Q, \mathfrak{m}, f, l, R)$ *are as in Definition 4.4 and that* $Q \to Q'$ *is a flat ring map such that* $\mathfrak{m}' := \mathfrak{m}Q'$ *is a maximal ideal of* Q' *and* $Q/\mathfrak{m} \xrightarrow{\cong} Q'/\mathfrak{m}'$ *is an isomorphism. Let* $f' \in Q'$ *be the image of* f *in* Q' *and set* $R' = Q'/f' = Q' \otimes_Q R$. *Then the square*

$$
\begin{array}{ccc}
G_i(Q[1/f], \mathbb{Z}/l) \times K_j(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) & \xrightarrow{\tilde{\theta}} & K_{i+1-1}(R/\mathfrak{m}, \mathbb{Z}/l) \\
\downarrow & & \downarrow{\scriptstyle\cong} \\
G_i(Q'[1/f'], \mathbb{Z}/l) \times K_j(\mathrm{Spec}(R') \setminus \mathfrak{m}', \mathbb{Z}/l) & \xrightarrow{\tilde{\theta}} & K_{i+1-1}(R'/\mathfrak{m}', \mathbb{Z}/l)
\end{array}
$$

commutes.

Proof. Write $\phi : \mathrm{Spec}(Q') \to \mathrm{Spec}(Q)$ for the associated map of affine schemes. The hypotheses ensure that each square in

$$
\begin{array}{ccccccc}
\mathrm{Spec}(R') \setminus \mathfrak{m}' & \longrightarrow & \mathrm{Spec}(Q') \setminus \mathfrak{m}' & \longrightarrow & \mathrm{Spec}(Q') & \longleftarrow & \mathrm{Spec}(Q'[1/f']) \\
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\
\mathrm{Spec}(R) \setminus \mathfrak{m} & \longrightarrow & \mathrm{Spec}(Q) \setminus \mathfrak{m} & \longrightarrow & \mathrm{Spec}(Q) & \longleftarrow & \mathrm{Spec}(Q[1/f])
\end{array}
$$

is a pull-back. It follows that the diagram

$$\begin{array}{ccccc}
\mathcal{G}(\operatorname{Spec}(R) \setminus \mathfrak{m}) & \longrightarrow & \mathcal{G}(\operatorname{Spec}(Q) \setminus \mathfrak{m}) & \longrightarrow & \mathcal{G}(\operatorname{Spec}(Q[1/f])) \\
\downarrow \phi^* & & \downarrow \phi^* & & \downarrow \phi^* \\
\mathcal{G}(\operatorname{Spec}(R') \setminus \mathfrak{m}) & \longrightarrow & \mathcal{G}(\operatorname{Spec}(Q') \setminus \mathfrak{m}') & \longrightarrow & \mathcal{G}(\operatorname{Spec}(Q'[1/f']))
\end{array}$$

is homotopy commutative. This gives us that ϕ^* commutes with the boundary maps ∂ in the associated long exact sequence. Whenever ϕ is flat, ϕ^* commutes with the cap product. This proves that

$$\phi^*\big(\partial(\alpha \cap [f]) \cap \gamma\big) = \partial(\phi^*(\alpha) \cap [f']) \cap \phi^*(\gamma) \in G_{i+j}(\operatorname{Spec}(R') \setminus \mathfrak{m}').$$

Similarly, the diagram

$$\begin{array}{ccccc}
\mathcal{G}(\operatorname{Spec}(R/m)) & \longrightarrow & \mathcal{G}(R) & \longrightarrow & \mathcal{G}(\operatorname{Spec}(R) \setminus \mathfrak{m}) \\
\downarrow \phi^* & & \downarrow \phi^* & & \downarrow \phi^* \\
\mathcal{G}(\operatorname{Spec}(R'/m')) & \longrightarrow & \mathcal{G}(R') & \longrightarrow & \mathcal{G}(\operatorname{Spec}(R') \setminus \mathfrak{m}')
\end{array}$$

is homotopy commutative, so that ϕ^* commutes with the boundary maps ∂'. $\qquad\square$

Example 4.7. We will use the previous result, in particular, when $Q' = Q_{\mathfrak{m}}^{\mathrm{hen}}$, the Henselization of Q at the maximal ideal $\mathfrak{m}$. That is, the $\tilde{\theta}$ pairing for $R = Q/f$ factors through its Henselization at $\mathfrak{m}$.

Assume Q is a regular ring and $f \in Q$ is a non-zero-divisor. Given a matrix factorization (A, B) of f in Q, the module $\operatorname{coker}(A)$ is annihilated by f and thus may be regarded as an R-module, where $R := Q/f$. It is necessarily a MCM R-module, as seen by the depth formula and the fact that $\operatorname{pd}_Q(M) = 1$. Moreover, if Q is local, every MCM R-module is the cokernel of some matrix factorization. Recall that A is an invertible matrix when regarded as a matrix with entries in $Q[1/f]$ and we write $[A]$ for the class in $K_1(Q[1/f])$ it determines. We write $[A]_l$ for the image of this class in $K_1(Q[1/f], \mathbb{Z}/l)$. (Note that $[A]_0 = [A]$.)

For finitely generated R-modules M and N such that $N_{\mathfrak{p}}$ has finite projective dimension for all $\mathfrak{p} \neq \mathfrak{m}$, let $\theta_l(M, N) \in \mathbb{Z}/l$ be the value of $\theta(M, N)$ modulo l.

Recall that such a module N determines a coherent sheaf on $\operatorname{Spec}(R) \setminus \mathfrak{m}$ that admits a finite resolution by locally free coherent sheaves, and hence N determines a class $[N'] \in K_0(\operatorname{Spec}(R) \setminus \mathfrak{m})$. We write $[N']_l$ for the image of $[N']$ in $K_0(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l)$. (As before, $[N]_0 = [N]$.)

Theorem 4.8. *Assume Q is a regular ring, $\mathfrak{m}$ is a maximal ideal of Q and $f \in \mathfrak{m}$ is a non-zero-divisor. Let $R = Q/f$ and let $\mathfrak{m}$ also denote the image of $\mathfrak{m}$ in R. Let (A, B) be a matrix factorization of f in Q and let $M = \operatorname{coker}(A)$ be the associated*

MCM R-module. Let N be a finitely generated R-module such that $\mathrm{pd}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}) < \infty$
for all $\mathfrak{p} \neq \mathfrak{m}$, *and let* $[N']$ *be the class it determines in* $K_0(\mathrm{Spec}(R) \setminus \mathfrak{m})$.

For any integer l such that $\mathbb{Z}/l$ *has no 2- or 3-torsion,*

$$\theta_l(M, N) = \pm\tilde{\theta}_l([A]_l, [N']_l)$$

Proof. The compatibility of the cup and cap products with the canonical maps
$K_*(U) \to K_*(U, \mathbb{Z}/l)$ implies that the diagram

$$
\begin{array}{ccc}
K_1(Q[1/f]) \times K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}) & \xrightarrow{\ \tilde{\theta}\ } & G_0(R/\mathfrak{m}) \\
\downarrow & & \downarrow \\
K_1(Q[1/f], \mathbb{Z}/l) \times K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) & \xrightarrow{\ \tilde{\theta}_l\ } & G_0(R/\mathfrak{m}, \mathbb{Z}/l)
\end{array}
$$

commutes. It therefore suffices to prove the theorem with integral coefficients.

To shorten notation, set $U = \mathrm{Spec}(Q)$, $U' = U \setminus \mathfrak{m}$, $V = \mathrm{Spec}(R)$ and $V' = V \setminus \mathfrak{m}$.
Note that U' is open in U, V is closed in U, and $V' = V \cap U$.

Let N' denote the coherent sheaf on V' given by the restriction of N, viewed as
a coherent sheaf on V, to V'. Since a high enough syzygy for an R-free resolution
of N will be locally free on V' and since the equation in question depends only
on the class of N' in $K_0(V')$, we may assume N' is a locally free coherent sheaf
on V'.

From [Corollary 3.11](#) we have

$$2\partial([A] \cup [f]) = \pm 2G(E, F) \in G_1(\mathrm{Spec}\, R) = G_1(V),$$

where E and F are the short exact sequences of coherent sheaves on $V = \mathrm{Spec}(R)$

$$E := \big(0 \to \mathrm{coker}(A) \to \mathcal{O}_V^m \to \mathrm{coker}(B) \to 0\big),$$
$$F := \big(0 \to \mathrm{coker}(B) \to \mathcal{O}_V^m \to \mathrm{coker}(A) \to 0\big).$$

(We are assuming A and B are $m \times m$ matrices.) The image of $G(E, F)$ under
$G_1(V) \to G_1(V')$ is $G(E', F')$, where

$$E' := \big(0 \to \mathrm{coker}(A') \to \mathrm{coker}\, \mathcal{O}_{V'}^m \to \mathrm{coker}(B') \to 0\big),$$
$$F' := \big(0 \to \mathrm{coker}(B') \to \mathrm{coker}\, \mathcal{O}_{V'}^m \to \mathrm{coker}(A') \to 0\big),$$

and A' and B' are the injective maps $\mathcal{O}_U^m \rightarrowtail \mathcal{O}_U^m$ induced by A and B. Thus

$$2\partial([A] \cup [f]) = \pm 2G(E', F') \in G_1(V').$$

We may describe the mirror image sequence (E, F) in the following equivalent
way: Let $\bar{A}$ and $\bar{B}$ be the matrices in $R = Q/f$ determined by A and B. We have

an unbounded, two-periodic exact sequence of locally free $\mathcal{O}_V$-modules

$$\cdots \to \mathcal{O}_V^r \xrightarrow{\bar{A}} \mathcal{O}_V^r \xrightarrow{\bar{B}} \mathcal{O}_V^r \xrightarrow{\bar{A}} \mathcal{O}_V^r \xrightarrow{\bar{B}} \cdots . \tag{4.9}$$

Then E and F are the two canonical short exact sequences involving syzygies coming from this two-periodic exact sequence.

The key result we need is that

$$\partial'(G(E', F') \cap [N']) = \theta(M, N),$$

where ∂' is the boundary map $G_1(V') \to G_0(R/\mathfrak{m}) \cong \mathbb{Z}$. This is essentially proven by Buchweitz and van Straten [2012, Theorem 3.4], but they use the notation of double short exact sequences, and not that of mirror image sequences. We proceed to summarize their proof in the notation we need.

Since N' is locally free on V', the exact sequences E' and F' remain exact upon tensoring with N' and it follows that

$$G(E', F') \cap [N'] = G(E' \otimes_{\mathcal{O}_{V'}} N', F' \otimes_{\mathcal{O}_{V'}} N') \in G_1(V').$$

Tensoring the sequence (4.9) with N yields the two-periodic complex of coherent sheaves on V

$$\cdots \xrightarrow{\bar{B} \otimes \mathrm{id}_N} \mathcal{O}_V^r \otimes_{\mathcal{O}_V} N \xrightarrow{\bar{A} \otimes \mathrm{id}_N} \mathcal{O}_V^r \otimes_{\mathcal{O}_V} N \xrightarrow{\bar{B} \otimes \mathrm{id}_N} \mathcal{O}_V^r \otimes_{\mathcal{O}_V} N \xrightarrow{\bar{A} \otimes \mathrm{id}_N} \cdots$$

and, again using that N' is locally free on V', this complex is exact on the open subset V' of V. From this complex, one builds what Buchweitz and van Straten call a "cyclic diagram":

$$\zeta := \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\bar{A} \otimes \mathrm{id}_N) & \longrightarrow & \mathcal{O}_V^r & \longrightarrow & \mathrm{im}(\bar{A} \otimes \mathrm{id}_N) & \longrightarrow & 0 \\ & & \subseteq \uparrow & & \| & & \downarrow \subseteq & & \\ 0 & \longleftarrow & \mathrm{im}(\bar{B} \otimes \mathrm{id}_N) & \longleftarrow & \mathcal{O}_V^r & \longleftarrow & \ker(\bar{B} \otimes \mathrm{id}_N) & \longleftarrow & 0 \end{array}$$

(Note that the diagram does *not* commute.) The left and right vertical arrows in this are the canonical inclusions. Buchweitz and van Straten [2012, p. 247] define $\{\zeta\} \in G_1(R) = G_1(V)$ to be $m(l_\zeta)$, where l_ζ is a certain double short exact sequence associated to a cyclic diagram ζ; see their paper for the precise formula.

The image of $\{\zeta\}$ under $G_1(V) \to G_1(V')$ is $\{\zeta'\}$, where

$$\zeta' := \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\bar{A}' \otimes \mathrm{id}_{N'}) & \longrightarrow & \mathcal{O}_{V'}^r & \longrightarrow & \mathrm{im}(\bar{A}' \otimes \mathrm{id}_{N'}) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longleftarrow & \mathrm{im}(\bar{B}' \otimes \mathrm{id}_{N'}) & \longleftarrow & \mathcal{O}_{V'}^r & \longleftarrow & \ker(\bar{B}' \otimes \mathrm{id}_{N'}) & \longleftarrow & 0 \end{array}$$

Observe that the vertical arrows are now all identity maps, and thus we may interpret ζ' as determining the mirror image sequence (E'', F''), where

$$E'' := \big(0 \to \ker(\bar{A}' \otimes \mathrm{id}_{N'}) \overset{\subseteq}{\longrightarrow} \mathcal{O}_{V'}^r \xrightarrow{\bar{A}' \otimes \mathrm{id}_{N'}} \mathrm{im}(\bar{A}' \otimes \mathrm{id}_{N'}) \to 0\big),$$

$$F'' := \big(0 \to \mathrm{im}(\bar{A}' \otimes \mathrm{id}_{N'}) \overset{\subseteq}{\longrightarrow} \mathcal{O}_{V'}^r \xrightarrow{\bar{B}' \otimes \mathrm{id}_{N'}} \ker(\bar{A}' \otimes \mathrm{id}_{N'}) \to 0\big).$$

Since $\ker(\bar{A}') = \mathrm{im}(\bar{B}') \cong \mathrm{coker}(\bar{A}')$ and $\ker(\bar{B}') = \mathrm{im}(\bar{A}') \cong \mathrm{coker}(\bar{B}')$ we have an isomorphism of mirror image sequences

$$(E'', F'') \cong (E' \otimes_{\mathcal{O}_{V'}} N', F' \otimes_{\mathcal{O}_{V'}} N').$$

Moreover, by comparing the definitions $l_{\zeta'}$ and $l_{(E'', F'')}$ (see [loc. cit.] and (3.4)), it is clear that

$$l_{\zeta'} = -l_{(E'', F'')}.$$

Using also that $m(l_{(E'', F'')}) = G(E'', F'') + G(\mathcal{O}_{V'}^r, -1)$ [Sherman 2013, p. 164], we obtain

$$-\{\zeta'\} = (G(E', F') \cap [N']) + G(\mathcal{O}_{V'}^r, -1) \in G_1(V').$$

The class $G(\mathcal{O}_{V'}^r, -1)$ clearly lifts along $G_1(V) \to G_1(V')$ to the class $G(\mathcal{O}_V^r, -1)$ and hence $\partial'(G(\mathcal{O}_{V'}^r, -1)) = 0$ and

$$\partial'(G(E', F') \cap [N']) = \partial'(\zeta').$$

Finally, Buchweitz and van Straten show [2012, proof of Theorem 3.4] that

$$\partial'(\{\zeta'\}) = \theta(M, N).$$

We have proven that

$$2\theta(M, N) = \pm 2\partial'\big(\partial([A] \cup [f]) \cup [N']\big) = \pm 2\tilde{\theta}_l(M, N)$$

and, since this is an equation in $\mathbb{Z}$, we may divide by 2. $\square$

Corollary 4.10. *Let Q be a regular ring, $\mathfrak{m}$ a maximal ideal of Q, $f \in \mathfrak{m}$ a non-zero-divisor, $R = Q/f$, and $l \geq 0$ an integer such that $\mathbb{Z}/l$ is without 2- and 3-torsion. The diagram*

$$
\begin{array}{c}
K_1(Q[1/f], \mathbb{Z}/l) \times K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \\
\Big\downarrow{\scriptstyle \partial \times \mathrm{id}} \qquad \qquad \searrow^{\tilde{\theta}} \\
G_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \times K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \xrightarrow[\theta]{} \mathbb{Z}/l
\end{array}
$$

commutes up to a sign.

Proof. By Propositions 4.1 and 4.6, θ and $\tilde{\theta}$ factor through the localization of R at $\mathfrak{m}$, and hence we may assume Q is local.

Theorem 4.8 shows that $\tilde{\theta}_l(A, N) = \theta_l(\operatorname{coker}(A), N)$ for all matrix factorizations (A, B) of f in Q and all N. An arbitrary class of $K_1(Q[1/f])$ is determined by an invertible matrix T with entries in $Q[1/f]$ (not necessarily coming from a matrix factorization). In $G_0(R)$, we may represent $\partial([T])$ as the difference of the classes of MCM modules. Since Q is assumed to be local, every MCM R-module is represented as the cokernel of a matrix factorization. It follows that $K_1(Q[1/f])$ is generated by classes coming from matrix factorizations of f and the image of $K_1(Q) \to K_1(A[1/f])$. It therefore suffices to prove $\tilde{\theta}$ annihilates the image of $K_1(Q)$.

Let, then, T be an invertible matrix with entries in Q. Using [Suslin 1983, (1.1)], we have

$$\tilde{\theta}([T], N) = \partial'\big(\partial([T] \cup [f]) \cap [N]\big) = \partial'\big((\overline{[T]} \cap \partial[f]) \cap [N]\big),$$

where $\overline{[T]}$ is the image of $[T]$ under the canonical map $K_1(Q) \to K_1(\operatorname{Spec}(R) \setminus \mathfrak{m})$. The map ∂ factors as

$$K_1(Q[1/f]) \xrightarrow{\tilde{\partial}} G_0(R) \xrightarrow{\operatorname{can}} G_0(\operatorname{Spec}(R) \setminus \mathfrak{m}).$$

Since $\tilde{\partial}([f]) = [R]$ we have $\overline{[T]} \cap \partial[f] = \overline{[T]}$ and hence

$$\tilde{\theta}([T], N) = \partial'(\overline{[T]} \cap [N]).$$

Finally, since $\overline{[T]}$ and $[N]$ lift to elements of $K_1(R)$ and $G_0(R)$, respectively, $\overline{[T]} \cap [N] \in G_1(\operatorname{Spec}(R) \setminus \mathfrak{m})$ lifts to an element of $G_1(R)$. Thus, $\partial'(\overline{[T]} \cap [N]) = 0$. □

Let us indicate how the previous result relates to the work of Buchweitz and van Straten, at least on the level of analogy. We start with:

Proposition 4.11. *Assume $(Q, \mathfrak{m}, k)$ is a Henselian, regular local ring with algebraically closed residue field k and $0 \neq f \in \mathfrak{m}$. Set $R = Q/f$. Assume l is a positive integer that is relatively prime to $\operatorname{char}(k)$ and is such that $[R/\mathfrak{m}] = 0$ in $G_0(R, \mathbb{Z}/l)$.*

Then the boundary map

$$K_1(Q[1/f], \mathbb{Z}/l) \xrightarrow{\cong} G_0(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \qquad (4.12)$$

is an isomorphism.

Remark 4.13. Before proving this, let us show $[R/\mathfrak{m}] = 0$ in $G_0(R, \mathbb{Z}/l)$, at least when $\dim(R) > 0$, for all but a finite number of primes l, so that the hypotheses of the proposition are often met.

Let $\mathfrak{p}$ be a prime in R of height $\dim(R) - 1$ (recall that we assume $\dim(R) > 0$) and choose $g \in \mathfrak{m} \setminus \mathfrak{p}$. Then the short exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{g} R/\mathfrak{p} \longrightarrow R/(\mathfrak{p}, g) \to 0$$

gives $[R/(\mathfrak{p}, g)] = 0$ in $G_0(R)$. But $R/(\mathfrak{p}, g)$ is a finite length R-module, say of length N, and hence $[R/(\mathfrak{p}, g)] = N[R/\mathfrak{m}]$. This proves $N[R/\mathfrak{m}] = 0$ in $G_0(R)$ and thus, if l is a prime such that $l \nmid N$, then $[R/\mathfrak{m}]$ is divisible by l in $G_0(R)$.

Proof. The localization sequences for G-theory (with $\mathbb{Z}/l$ coefficients), and their naturality, applied to the schemes, closed subschemes, and open complements

$$\operatorname{Spec}(R/\mathfrak{m}) \subseteq \operatorname{Spec}(R) \supseteq \operatorname{Spec}(R) \setminus \mathfrak{m},$$

$$\operatorname{Spec}(Q/\mathfrak{m}) \subseteq \operatorname{Spec}(Q) \supseteq \operatorname{Spec}(Q) \setminus \mathfrak{m},$$

$$\operatorname{Spec}(R) \setminus \mathfrak{m} \subseteq \operatorname{Spec}(Q) \setminus \mathfrak{m} \supseteq \operatorname{Spec}(Q[1/f]),$$

yield the commutative diagram

$$
\begin{array}{ccccccc}
G_1(R) & \longrightarrow & G_1(R \setminus \mathfrak{m}) & \longrightarrow & G_0(R/\mathfrak{m}) & \xrightarrow{0} & G_0(R) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow \\
G_1(Q) & \xrightarrow{0} & G_1(Q \setminus \mathfrak{m}) & \xrightarrow{\cong} & G_0(Q/\mathfrak{m}) & \longrightarrow & G_0(Q) \\
& & \downarrow{\scriptstyle 0} & & & & \\
& & G_1(Q[1/f]) & & & & \\
& & \downarrow & & & & \\
& & G_0(R \setminus \mathfrak{m}) & & & & \\
& & \downarrow{\scriptstyle 0} & & & & \\
& & G_0(Q \setminus \mathfrak{m}) & & & & \\
& & \downarrow{\scriptstyle \cong} & & & & \\
& & G_0(Q[1/f]) & & & &
\end{array}
$$

with exact rows and column. (Here, we have suppressed the coefficients of $\mathbb{Z}/l$ and written $R \setminus \mathfrak{m}$ and $Q \setminus \mathfrak{m}$ for the nonaffine schemes $\operatorname{Spec}(R) \setminus \mathfrak{m}$ and $\operatorname{Spec}(Q) \setminus \mathfrak{m}$.) The proposition follows immediately from this diagram, provided we justify the labels.

The hypothesis that $[R/\mathfrak{m}] = 0$ in $G_0(R, \mathbb{Z}/l)$ gives us that $G_0(R/\mathfrak{m}, \mathbb{Z}/l) \to G_0(R, \mathbb{Z}/l)$ is the zero map, as labeled. The map $G_1(Q, \mathbb{Z}/l) \to G_1(Q \setminus \mathfrak{m}, \mathbb{Z}/l)$ is the zero map, since

$$G_1(Q, \mathbb{Z}/l) \cong K_1(Q, \mathbb{Z}/l) \cong K_1(Q/\mathfrak{m}, \mathbb{Z}/l) = 0$$

by [Gabber 1992; Suslin 1983]. A diagram chase then gives that $G_1(R \setminus \mathfrak{m}, \mathbb{Z}/l) \to G_1(Q \setminus \mathfrak{m}, \mathbb{Z}/l)$ is onto and hence that $G_1(Q \setminus \mathfrak{m}, \mathbb{Z}/l) \to G_1(Q[1/f], \mathbb{Z}/l)$ is the zero map. The group $G_0(Q \setminus \mathfrak{m}, \mathbb{Z}/l) \cong K_0(Q \setminus \mathfrak{m}, \mathbb{Z}/l)$ is isomorphic to $\mathbb{Z}/l$ and is generated by the class of the structure sheaf, since $G_0(Q, \mathbb{Z}/l) \cong K_0(Q, \mathbb{Z}/l) \cong \mathbb{Z}/l$ (using that Q is local) and $K_0(Q, \mathbb{Z}/l) \to K_0(Q \setminus \mathfrak{m}, \mathbb{Z}/l)$ is onto. It follows that $K_0(Q \setminus \mathfrak{m}, \mathbb{Z}/l) \to K_0(Q[1/f], \mathbb{Z}/l)$ is an isomorphism and hence that $G_0(R \setminus \mathfrak{m}, \mathbb{Z}/l) \to G_0(Q \setminus \mathfrak{m}, \mathbb{Z}/l)$ is the zero map. $\qquad\square$

Assuming the hypotheses of the proposition are met and that l is not divisible by 2 or 3, then the mod-l theta pairing

$$\theta_l : G_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \times G_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \to \mathbb{Z}/l$$

may be given by the formula

$$\theta_l(\alpha, \beta) = \langle \gamma(\alpha), \beta \rangle. \tag{4.14}$$

Here, we define

$$\gamma : G_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \to G_1(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \tag{4.15}$$

as the composition of the inverse of the isomorphism (4.12) and the map

$$K_1(Q[1/f], \mathbb{Z}/l) \xrightarrow{\partial(-\cup[t])} G_1(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l). \tag{4.16}$$

(In the next section, we note that the latter map is the "specialization map" in K-theory, and we will explore its properties.) The pairing

$$\langle -, - \rangle : G_1(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \times G_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \to \mathbb{Z}$$

is the composition of the cup product, the boundary map $G_1(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \xrightarrow{\partial'} G_0(R/\mathfrak{m}, \mathbb{Z}/l)$ and the canonical isomorphism $G_0(R/\mathfrak{m}) \cong \mathbb{Z}/l$.

The isomorphism (4.12) is the analogue of the Alexander duality isomorphism $H^{2m+1}(S \setminus L) \xrightarrow{\cong} H^{2m}(L)$ and the map (4.16) is the analogue of the map $\rho_t^* : H^{j+1}(S \setminus L) \to H^{j+1}(L)$ induced by the "push aside" map $\rho_t : L \to \partial_t F \subseteq (S \setminus L)$. Thus the map γ defined in (4.15) is the analogue of the map

$$\gamma^{\mathrm{top}} : H^j(L) \to H^{j+1}(L)$$

occurring in the work of Buchweitz and van Straten. They prove [2012, Proposition 5.1]

$$\mathrm{link}(\alpha, \beta) = \langle \gamma^{\mathrm{top}}(\alpha), \beta \rangle$$

for classes α and β in $H^{n-1}(L) \cong H_n(L)$, where link is the linking form on $H_n(L)$. (The linking form may be defined as the restriction of the Seifert form, defined on the homology of the Milnor fiber $H_n(F_t)$, along the canonical map $H_n(\partial(F_t)) \to H_n(F_t)$, using the diffeomorphism $L \cong \partial F_t$.)

We may thus interpret Corollary 4.10 as saying that the mod-l theta pairing is the analogue for algebraic K-theory with finite coefficients of the linking form in topology.

5. Specialization with finite coefficients (following Suslin)

In the previous section we have given a new interpretation of the θ pairing in terms of the pairing $\tilde{\theta}$ defined as

$$\tilde{\theta}(\alpha, \gamma) = \partial'\big(\partial(\alpha \cap [f]) \cap \gamma\big).$$

Under some additional assumptions, a portion of this formula, namely the map

$$\sigma : G_i(Q[1/f], \mathbb{Z}/l) \to G_i(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l)$$

sending α to $\partial(\alpha \cap [f])$, can be interpreted as a *specialization map*. The goal of this section is to make this precise and to establish properties of this specialization map. When working with the algebraic analogue of the Milnor fiber, the specialization map is the analogue of the map

$$\rho_t^* : K_{\mathrm{top}}^1(X^*) \to K_{\mathrm{top}}^1(L)$$

occurring in the work of Buchweitz and van Straten. As mentioned, ρ_t^* factors through the topological K-theory of the Milnor fiber. We prove the analogous fact here, by showing, under certain hypotheses, that the specialization map σ factors through the G-theory of the algebraic analogue of the Milnor fiber. Our notion of specialization in K- and G-theory is based on work of Andrei Suslin [1983].

Define a *pointed curve* to be a pair (C, c), where C is an affine, Noetherian, integral scheme that is regular of dimension one and $c \in C$ is a closed point such that the associated maximal ideal is principal and the residue field $\kappa(c)$ is algebraically closed. Equivalently, a pointed curve is a pair $(V, \mathfrak{n})$ where V is a Dedekind domain and $\mathfrak{n}$ is a principal maximal ideal of V such that $V/\mathfrak{n}$ is an algebraically closed field. We will use both notations (C, c) and $(V, \mathfrak{n})$ interchangeably. We typically write t for a chosen generator of $\mathfrak{n}$, but such a choice is not part of the defining data.

A typical example occurs when V is a DVR with algebraically closed residue field (and hence t is a uniformizing parameter), but we do not limit ourselves to this case.

We will use the exact sequence

$$K_1(C) \xrightarrow{i^*} K_1(C \setminus c) \xrightarrow{\partial} K_0(c),$$

equivalently, the exact sequence

$$K_1(V) \xrightarrow{i^*} K_1(V[1/t]) \xrightarrow{\partial} K_0(V/\mathfrak{n})$$

coming from a portion of the long exact localization sequence in K-theory. (Since V, $V[1/t]$ and $V/\mathfrak{n}$ are all regular, their K- and G-theories coincide.) The map ∂ sends $[t] \in K_1(V[1/t])$ to $[V/\mathfrak{n}] \in K_0(V/\mathfrak{n})$ and $[V/\mathfrak{n}]$ generates $K_0(V/\mathfrak{n}) \cong \mathbb{Z}$.

Given a pointed curve (C, c) and a morphism of Noetherian schemes $p : X \to C$, write X_c for the fiber over $c \in C$ and $X \setminus X_c$ for its open complement in X. Observe that if V is a DVR then X_c is the closed fiber and $X \setminus X_c$ is the generic fiber. If $X = \mathrm{Spec}(Q)$ is affine, so that p is given by a ring map $V \to Q$, then $X_c = \mathrm{Spec}(Q/f)$ and $X \setminus X_c = \mathrm{Spec}(Q[1/f])$, where $f \in Q$ is the image of a chosen generator t of $\mathfrak{n}$.

Similarly, given a ring map $V \to Q$ and a maximal ideal $\mathfrak{m}$ of Q such that $\mathfrak{m} \cap V = \mathfrak{n}$, then, setting $X = \mathrm{Spec}(Q) \setminus \mathfrak{m}$, we have $X_c = \mathrm{Spec}(Q/f) \setminus \mathfrak{m}$ and $X \setminus X_c = \mathrm{Spec}(Q[1/f])$. This is the main example we have in mind but, for most of this section, we allow X to be an arbitrary Noetherian scheme.

Definition 5.1. Given a pointed curve (C, c) corresponding to $(V, \mathfrak{n})$, a morphism of Noetherian schemes $p : X \to C$, and a positive integer l such that $\mathrm{char}(\kappa(c)) \nmid l$, define the *specialization map in G-theory* (for X with $\mathbb{Z}/l$ coefficients) to be the map

$$\sigma = \sigma_{C,c,p} : G_i(X \setminus X_c, \mathbb{Z}/l) \to G_i(X_c, \mathbb{Z}/l)$$

given as follows: Choose z to be any element of $K_1(V[1/t])$ that maps to $1 \in \mathbb{Z}$ under $\partial : K_1(V[1/t]) \to K_0(V/\mathfrak{n}) = \mathbb{Z}$; for example, z could be $[t]$. Then σ is the composition of

$$G_i(X \setminus X_c, \mathbb{Z}/l) \xrightarrow{-\cap z} G_{i+1}(X \setminus X_c, \mathbb{Z}/l) \xrightarrow{\partial} G_i(X_c, \mathbb{Z}/l),$$

where ∂ is the boundary map in the localization long exact sequence in G-theory associated to the closed subscheme X_c of X.

Lemma 5.2 (Suslin). *The specialization map $\sigma_{C,c,p}$ is independent of the choice of z.*

Proof. Our proof is basically that of Suslin's, with some minor modifications.

Suppose z' is another element of $K_1(V[1/t])$ with $\partial(z') = 1$, and let

$$\sigma' : G_i(X \setminus X_c, \mathbb{Z}/l) \to G_i(X_c, \mathbb{Z}/l)$$

be the map $\sigma'(\alpha) = \partial(\alpha \cap z')$. The difference $\sigma - \sigma'$ sends $\alpha \in G_i(U \setminus U_c, \mathbb{Z}/l)$ to $\partial(\alpha \cap (z - z'))$. Since $\partial(z - z') = 0$, we have $z - z' = i^*(w)$ for some $w \in V$. Using [Suslin 1983, (1.1)], we have

$$\partial(\alpha \cap i^*(w)) = \partial(\alpha) \cap j^*(w) \in G_i(U_c, \mathbb{Z}/l),$$

where $j^* : K_1(V) \to K_1(V/\mathfrak{n})$ is the map induced by modding out by $\mathfrak{n}$. Since we assume $V/\mathfrak{n}$ is algebraically closed and $\mathrm{char}(V/\mathfrak{n}) \nmid l$, $j^*(w)$ is l-divisible, whence $\partial(\alpha) \cap j^*(w) = 0$ since $G_i(U_c, \mathbb{Z}/l)$ is l-torsion. $\square$

Remark 5.3. In fact, the previous result remains true (and the given proof remains valid) if the field $V/\mathfrak{n}$ is merely assumed to be closed under taking l-th roots.

The specialization map is closely related to the map $\tilde{\theta}$. In detail, suppose $(V, \mathfrak{n})$ is a pointed curve with $\mathfrak{n} = (t)$ and $V \to Q$ is a map of rings sending t to f. Let $\mathfrak{m}$ be a maximal ideal of Q with $f \in \mathfrak{m}$ such that $\mathfrak{m} \cap V = \mathfrak{n}$. Set $X = \mathrm{Spec}(Q) \setminus \mathfrak{m}$. Then $X \setminus X_c = \mathrm{Spec}(Q[1/f])$, $X_c = \mathrm{Spec}(Q/f) \setminus \mathfrak{m}$, and the specialization map $\sigma : G_i(Q[1/f], \mathbb{Z}/l) \to G_i(\mathrm{Spec}(Q/f) \setminus \mathfrak{m}, \mathbb{Z}/l)$ (defined by setting $z = [t]$) is given by

$$\sigma(\alpha) = \partial(\alpha \cap [f]).$$

We have proven:

Proposition 5.4. *Suppose Q is a Noetherian ring, $\mathfrak{m}$ is a maximal ideal, $f \in \mathfrak{m}$ is a non-zero-divisor, and l is a positive integer that is not divisible by* $\mathrm{char}(Q/\mathfrak{m})$, 2 *or* 3. *Set $R = Q/f$. If there exists a pointed curve $(V, \mathfrak{n})$ and a ring map $V \to Q$ that sends some generator t of $\mathfrak{n}$ to f, then for all integers i and j the $\tilde{\theta}$ pairing fits into a commutative square*

$$\begin{array}{ccc}
G_i(Q[1/f], \mathbb{Z}/l) \times K_j(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) & \xrightarrow{\tilde{\theta}} & G_{i+j-1}(R/\mathfrak{m}, l) \\
\downarrow{\scriptstyle \sigma \times \mathrm{id}} & & \uparrow{\scriptstyle \partial'} \\
G_i(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \times K_j(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) & \xrightarrow{(-\cap-)} & G_{i+j}(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l)
\end{array} \tag{5.5}$$

Example 5.6. If Q contains a field k, then we may take $(V, \mathfrak{n}) = (k[t], (t))$ and define $V \to Q$ to be the unique k-algebra map sending t to f.

We will need to understand the behavior of the specialization map as we allow the base (C, c) to vary in a suitably nice manner.

Definition 5.7. A *morphism of pointed curves*, say from (C', c') to (C, c), is a flat morphism of schemes $\phi : C' \to C$ such that $\phi^{-1}(c) = \{c'\}$ and the induced map on residue fields is an isomorphism $\kappa(c) \xrightarrow{\cong} \kappa(c')$. Equivalently, if (C, c) corresponds to $(V, \mathfrak{n})$ and (C', c') to $(V', \mathfrak{n}')$, then a morphism is a flat ring map $g : V \to V'$ such that $\mathfrak{n}V'$ is $\mathfrak{n}'$-primary and the induced map $V/\mathfrak{n} \xrightarrow{\cong} V'/\mathfrak{n}'$ is an isomorphism.

Since g is necessarily injective, we often just write a morphism as if it were a ring extension $V \subseteq V'$. If $\mathfrak{n} = (t)$ and $\mathfrak{n} = (t')$ then, since V and V' are Dedekind domains, the condition that $\mathfrak{n}V'$ is $\mathfrak{n}'$-primary is equivalent to the existence of an equation $t = u'(t')^n$ for some integer $n \geq 1$ and unit $u' \in (V')^{\times}$.

Definition 5.8. A morphism of pointed curves is *unramified* if $C' \times_C \mathrm{Spec}(\kappa(c)) \cong \kappa(c')$ (via the canonical map), or, equivalently, if $\mathfrak{n}V = \mathfrak{n}'$. If $\mathfrak{n} = (t)$ and $\mathfrak{n}' = (t')$, being unramified is equivalent to $t = u't'$ for some $u' \in (V')^{\times}$.

A morphism of pointed curves is *finite* if the underlying map of schemes $\phi : C' \to C$ is finite, or, equivalently, if V' is a finitely generated V-module. Since V

and V' are Dedekind domains, $V \subseteq V'$ is finite if and only if the induced map on fields of fractions $E \subseteq E'$ is finite and V' is the integral closure of V in E'.

If $\phi : (C', c') \to (C, c)$ is a morphism of pointed curves and $p : X \to C$ is a morphism of Noetherian schemes, we will write $p' : X' \to C'$ for the pull-back of p along ϕ. Abusing notation, we write $\phi : X' \to X$ for the induced map, and also use ϕ for the induced maps on fibers. Observe that our hypotheses ensure that the induced map on fibers over the marked points is an isomorphism $\phi : X'_{c'} \xrightarrow{\cong} X_c$. Using also that $\phi^{-1}(c) = \{c'\}$, we have that both squares in

$$
\begin{array}{ccccc}
X' \setminus X'_{c'} & \longrightarrow & X' & \longleftarrow & X'_{c'} \\
\phi \downarrow & & \phi \downarrow & & \phi \downarrow \cong \\
X \setminus X_c & \longrightarrow & X & \longleftarrow & X_c
\end{array}
$$

are Cartesian.

Lemma 5.9. *If $\phi : (C', c') \to (C, c)$ is a morphism pointed curves that is either finite or unramified, $p : X \to C$ is any morphism of Noetherian schemes, and l is any integer, the diagram*

$$
\begin{array}{ccc}
G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) & \xrightarrow{\ \partial'\ } & G_{i-1}(X'_{c'}, \mathbb{Z}/l) \\
\phi_* \downarrow & & \phi_* \downarrow \cong \\
G_i(X \setminus X_c, \mathbb{Z}/l) & \xrightarrow{\ \partial\ } & G_{i-1}(X_c, \mathbb{Z}/l)
\end{array}
$$

commutes.

Proof. This follows from the fact that

$$
\begin{array}{ccccc}
\mathcal{G}(X'_{c'}) & \longrightarrow & \mathcal{G}(X') & \longrightarrow & \mathcal{G}(X' \setminus X'_{c'}) \\
\phi_* \downarrow & & \phi_* \downarrow & & \phi_* \downarrow \\
\mathcal{G}(X_c) & \longrightarrow & \mathcal{G}(X) & \longrightarrow & \mathcal{G}(X \setminus X_c)
\end{array}
$$

is a homotopy commutative diagram of spectra. $\qquad\square$

Lemma 5.10. *If $\phi : (C', c') \to (C, c)$ is either a finite or an unramified morphism of pointed curves, $p : X \to C$ is any morphism of Noetherian schemes, and l is a positive integer not divisible by $\mathrm{char}(\kappa(c))$, the diagram*

$$
\begin{array}{ccc}
G_i(X \setminus X_c, \mathbb{Z}/l) & \xrightarrow{\ \sigma\ } & G_i(X_c, \mathbb{Z}/l) \\
\phi^* \downarrow & & \phi^* \downarrow \cong \\
G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) & \xrightarrow{\ \sigma'\ } & G_i(X'_{c'}, \mathbb{Z}/l)
\end{array}
$$

commutes, where $\sigma = \sigma_{C,c,p}$ and $\sigma' = \sigma_{C',c',p'}$ are the specialization maps.

Proof. Assume ϕ is finite. Since $\phi_* : G_i(X_c, \mathbb{Z}/l) \xrightarrow{\cong} G_i(X'_{c'}, \mathbb{Z}/l)$ is the inverse of ϕ^*, it suffices to prove

$$
\begin{array}{ccc}
G_i(X \setminus X_c, \mathbb{Z}/l) & \xrightarrow{\quad\sigma\quad} & G_i(X_c, \mathbb{Z}/l) \\
{\scriptstyle\phi^*}\downarrow & & \uparrow{\scriptstyle\phi_*}\ {\scriptstyle\cong} \\
G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) & \xrightarrow{\quad\sigma'\quad} & G_{i-1}(X'_{c'}, \mathbb{Z}/l)
\end{array}
$$

commutes. Choose any $z' \in K_1(V'[1/t'])$ as in the definition of the specialization map σ'. A special case of Lemma 5.9 gives that $\partial\phi_*(z') = 1$, and hence we may choose $z := \phi_*(z')$ in the definition of σ. Using Lemma 5.9 again gives

$$\phi_* \sigma'(\phi^*\alpha) = \phi_* \partial'(\phi^*(\alpha) \cap z') = \partial\phi_*(\phi^*(\alpha) \cap z').$$

By the projection formula, $\phi_*(\phi^*(\alpha) \cap z') = \alpha \cap \phi_*(z') = \alpha \cap z$, since $z = \phi_*(z')$. The result follows.

Now assume ϕ is unramified. Then

$$
\begin{array}{ccc}
X'_{c'} & \longrightarrow & X' \\
\downarrow{\scriptstyle\cong} & & \downarrow \\
X_c & \longrightarrow & X
\end{array}
$$

is a Cartesian square, and it follows that

$$
\begin{array}{ccccc}
\mathcal{G}(X_c) & \longrightarrow & \mathcal{G}(X) & \longrightarrow & \mathcal{G}(X \setminus X_c) \\
{\scriptstyle\phi^*}\downarrow & & {\scriptstyle\phi^*}\downarrow & & {\scriptstyle\phi^*}\downarrow \\
\mathcal{G}(X'_{c'}) & \longrightarrow & \mathcal{G}(X') & \longrightarrow & \mathcal{G}(X' \setminus X'_{c'})
\end{array}
$$

is a homotopy commutative diagram. The boundary maps ∂ and ∂' in the associated long exact sequences of homotopy groups (with any coefficients) thus commute with ϕ^*.

Let $t \in \mathfrak{n}$ be any generator. Since ϕ is unramified, $t' := \phi(t)$ generates $\mathfrak{n}'$. Using $z = [t]$ and $z' = [t']$ for the definition of σ and σ', we have, for any $\alpha \in G_i(X \setminus X_c, \mathbb{Z}/l)$,

$$\sigma(\alpha) = \phi^* \partial(\alpha \cap [t]) = \partial'\big(\phi^*(\alpha \cap [t])\big) = \partial'\big(\phi^*(a) \cap [t']\big) = \sigma'(\phi^*(\alpha)),$$

since ϕ^* commutes with the cap product and $\phi^*([t]) = [t']$. $\qquad\square$

The notion of the Henselization of a ring at a maximal ideal will be important for the rest of this paper. Let us recall its main properties. Given a Noetherian ring Q and a maximal ideal $\mathfrak{m}$ of it, we write $Q_{\mathfrak{m}}^{\mathrm{hen}}$ for the Henselization of Q at $\mathfrak{m}$. We have:

- $Q_{\mathfrak{m}}^{\text{hen}}$ is a Noetherian local ring with maximal ideal $\mathfrak{m}^{\text{hen}}$, and the pair $(Q^{\text{hen}}, \mathfrak{m}^{\text{hen}})$ satisfies Hensel's lemma: given a monic polynomial $p(x) \in Q_{\mathfrak{m}}^{\text{hen}}[x]$, if its image $\bar{p}(x) \in Q_{\mathfrak{m}}^{\text{hen}}/\mathfrak{m}^{\text{hen}}[t]$ has a simple root a, then $p(x)$ has a root $\alpha \in Q_{\mathfrak{m}}^{\text{hen}}$ whose image in $Q_{\mathfrak{m}}^{\text{hen}}/\mathfrak{m}^{\text{hen}}$ is a.

- There are flat ring maps $Q \to Q_{\mathfrak{m}} \to Q_{\mathfrak{m}}^{\text{hen}}$ and $Q_{\mathfrak{m}} \to Q_{\mathfrak{m}}^{\text{hen}}$ is faithfully flat.

- We have $\mathfrak{m} Q_{\mathfrak{m}}^{\text{hen}} = \mathfrak{m}^{\text{hen}}$, or, in other words, the fiber of $\text{Spec}(Q_{\mathfrak{m}}^{\text{hen}}) \to \text{Spec}(Q)$ over $\mathfrak{m}$ is $\text{Spec}(\kappa(\mathfrak{m}^{\text{hen}})) = \text{Spec}(Q_{\mathfrak{m}}^{\text{hen}}/\mathfrak{m}^{\text{hen}})$.

- $Q_{\mathfrak{m}}$ is regular if and only if $Q_{\mathfrak{m}}^{\text{hen}}$ is regular.

- [EGA IV$_4$ 1967, Théorème (18.6.9)] The fibers of $\text{Spec}(Q_{\mathfrak{m}}^{\text{hen}}) \to \text{Spec}(Q)$ are spectra of finite products of algebraic, separable field extensions. That is, for every $\mathfrak{p} \in \text{Spec}(Q)$, there are a finite number of primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_d$ in $Q_{\mathfrak{m}}^{\text{hen}}$ such that $\mathfrak{q}_i \cap Q = \mathfrak{p}$ and, for each i, the induced field map $\kappa(\mathfrak{p}) \rightarrowtail \kappa(\mathfrak{q}_i)$ is separable algebraic.

For any pointed curve $(V, \mathfrak{n})$, let V^{hen} denote the Henselization of V at $\mathfrak{n}$ and let $\mathfrak{n}^{\text{hen}}$ denote also the maximal ideal of V^{hen}. Then V^{hen} is also a Dedekind domain and $(V, \mathfrak{n}) \to (V^{\text{hen}}, \mathfrak{n}^{\text{hen}})$ is an unramified morphism of pointed curves.

In the following theorem, given a morphism $p : X \to \text{Spec}(V)$ of Noetherian schemes we will consider the G-theory of the geometric generic fiber

$$X_{\bar{F}} := X \times_{\text{Spec}(V)} \text{Spec}(\bar{F}),$$

where F is the field of fractions of a Henselian ring V and $\bar{F}$ is its algebraic closure. Note that $X_{\bar{F}}$ may fail to be Noetherian and G-theory is ordinarily defined for just Noetherian schemes. (For non-Noetherian schemes, the category of coherent sheaves may fail to be an abelian category.) However, since $X_L := X \times_{\text{Spec } V} \text{Spec } L$ is Noetherian for every finite field extension L of F, we have that $X_{\bar{F}}$ is a filtered colimit of Noetherian schemes with flat transition maps. We thus take, as the definition, the G-theory spectrum of $X_{\bar{F}}$ to be the filtered colimit of spectra

$$\mathcal{G}(X_{\bar{F}}) := \underset{L}{\text{colim}}\, \mathcal{G}(X_L)$$

indexed by the finite field extensions of F contained in $\bar{F}$. The associated G-groups are thus also given by colimits:

$$G_n(X_{\bar{F}}) = \underset{L}{\text{colim}}\, G_n(X_L).$$

Remark 5.11. At least when $X = \text{Spec}(A)$ is affine, $G_n(X_L)$ may be interpreted as the K-theory of an abelian category. Indeed, more generally, if A is a filtered colimit $A = \text{colim}_{i \in I} A_i$ of Noetherian rings with flat transition maps, then $G_n(A) := \varinjlim_{i \in I} G_n(A_i)$ is the K-theory of the category of finitely presented A-modules and the latter forms an abelian category under these assumptions.

In later sections we will be mostly interested in the case where $X_F \to \mathrm{Spec}(F)$ is smooth, so that each transition map in the colimit giving $X_{\bar{F}}$ is a flat morphism of regular rings, and we will also be interested only in the case when X is affine. In this situation, G-theory and K-theory coincide, as we now explain.

More generally, suppose $A = \mathrm{colim}_{i \in I} A_i$ is a filtered colimit of regular Noetherian rings with flat transition maps. Even if A is not Noetherian, the correct notion of the K-theory of A is unambiguous: it is the K-theory of the exact category $\mathcal{P}(A)$ of finitely generated projective A-modules. Moreover, we have

$$K_n(A) = \varinjlim_{i \in I} K_n(A_i)$$

for all n. Since we assume each A_i is regular Noetherian, each natural map $K_n(A_i) \to G_n(A_i)$ is an isomorphism and thus the canonical map

$$K_n(A) \xrightarrow{\cong} G_n(A)$$

is an isomorphism. In particular, $K_n(X_{\bar{F}}) \cong G_n(X_{\bar{F}})$ provided $X_F \to \mathrm{Spec}(F)$ is smooth and X is affine.

Theorem 5.12. *Given a pointed curve $(V, \mathfrak{n})$ and a morphism $p : X \to \mathrm{Spec}(V)$ of Noetherian schemes, let $X_{\bar{F}} = X \times_{\mathrm{Spec}(V)} \mathrm{Spec}(\bar{F})$, where $\bar{F}$ is the algebraic closure of the field of fractions of the Henselization V^{hen} of V at $\mathfrak{n}$. For any positive integer l such that $\mathrm{char}(\kappa(c)) \nmid l$, the specialization map $\sigma : G_i(X \setminus X_c, \mathbb{Z}/l) \to G_i(X_c, \mathbb{Z}/l)$ factors through $G_i(X_{\bar{F}})$; i.e., there is a commutative triangle*

$$
\begin{array}{ccc}
G_i(X \setminus X_c, \mathbb{Z}/l) & \xrightarrow{\quad\sigma_{C,c,p}\quad} & G_i(X_c, \mathbb{Z}/l) \\
& \searrow \qquad \nearrow & \\
& G_i(X_{\bar{F}}, \mathbb{Z}/l) &
\end{array}
$$

Moreover, if l is also not divisible by 2 or 3, then the direct sum indexed over $i \geq 0$ of these morphisms are graded $K_*(V, \mathbb{Z}/l)$-module homomorphisms.

Proof. As discussed above, the map $(V, \mathfrak{n}) \to (V^{\mathrm{hen}}, \mathfrak{n})$ is unramified and so, using Lemma 5.10, we may assume without loss of generality that $(V, \mathfrak{n})$ is Henselian with field of fractions F. Note that the integral closure $\bar{V}$ of F is a filtered colimit of rings of the form V', where V' is the integral closure of V in some finite field extension $F \subseteq F'$ of the field of fractions of V. Using Lemma 5.10 again, we have a commutative triangle

$$
\begin{array}{ccc}
G_i(X \setminus X_c, \mathbb{Z}/l) & \xrightarrow{\quad\sigma\quad} & G_i(X_c, \mathbb{Z}/l) \\
\downarrow & \nearrow{\scriptstyle (\phi^*)^{-1} \circ \sigma'} & \\
G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) & &
\end{array}
$$

for each such V'. Moreover, given maps $V \rightarrowtail V' \rightarrowtail V''$ associated to a chain of finite field extensions $F \subseteq F' \subseteq F''$, the diagram

$$
\begin{array}{ccc}
G_i(X \setminus X_c, \mathbb{Z}/l) & \xrightarrow{\ \ \sigma\ \ } & G_i(X_c, \mathbb{Z}/l) \\
\downarrow & \nearrow^{\sigma'} & \nearrow \\
G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) & {\sigma''} & \\
\downarrow & \nearrow & \\
G_i(X'' \setminus X''_{c''}, \mathbb{Z}/l) & &
\end{array}
$$

commutes. Taking colimits thus gives a commutative triangle

$$
\begin{array}{ccc}
G_i(X \setminus X_c, \mathbb{Z}/l) & \xrightarrow{\ \ \sigma\ \ } & G_i(X_c, \mathbb{Z}/l) \\
\downarrow & \nearrow & \\
\operatorname{colim}_{F \subseteq F'} G_i(X' \setminus X'_{c'}, \mathbb{Z}/l) & &
\end{array}
$$

The first result follows, since $\operatorname{colim}_{F \subseteq F'}(X' \setminus X'_{c'}) \cong X_{\bar{F}}$ and

$$
G_i(X_{\bar{F}}, \mathbb{Z}/l) := \operatorname*{colim}_{F \subseteq F'} G_i(X' \setminus X'_{c'}, \mathbb{Z}/l).
$$

For the final claim, note that every morphism occurring in this proof commutes with multiplication by any fixed element of $K_*(V, \mathbb{Z}/l)$. $\qquad\square$

Corollary 5.13. *Suppose Q is a Noetherian ring, $\mathfrak{m}$ is a maximal ideal, $f \in \mathfrak{m}$ is a non-zero-divisor, and l is a positive integer that is not divisible by $\operatorname{char}(Q/\mathfrak{m})$, 2 or 3. Set $R = Q/f$. If there exists a pointed curve $(V, \mathfrak{n})$ and a ring map $V \to Q$ that sends some generator t of $\mathfrak{m}$ to f, then for all i and j the $\tilde{\theta}$ pairing fits into a commutative square*

$$
\begin{array}{ccc}
G_i(Q[1/f], \mathbb{Z}/l) \times K_j(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) & \xrightarrow{\ \tilde{\theta}\ } & K_{i+j-1}(R/\mathfrak{m}, \mathbb{Z}/l) \\
\downarrow & \nearrow & \\
G_i(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) \times K_j(\operatorname{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) & &
\end{array}
$$

where $Q_{\mathfrak{m}}^{\mathrm{hen}}$ is the Henselization of Q at $\mathfrak{m}$ and $\bar{F}$ is the algebraic closure of the field of fractions F of V^{hen}. Moreover, the direct sum indexed by $i, j \geq 0$ of these morphisms are graded $K_*(V, \mathbb{Z}/l)$-module homomorphisms.

If $F \to Q \otimes_V F$ is smooth, then

$$
G_i(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) \cong K_i(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \bar{F}, \mathbb{Z}/l).
$$

Proof. Applying Proposition 4.6 to the flat map $Q \to Q_{\mathfrak{m}}^{\mathrm{hen}}$ allows us to reduce to the case where $Q = Q_{\mathfrak{m}}^{\mathrm{hen}}$. The result then follows from Proposition 5.4 and Theorem 5.12. □

Remark 5.14. In fact, Corollary 5.13 remains valid even if $l = 0$ (i.e., with $\mathbb{Z}$-coefficients). Since we shall not need this fact in this paper, we omit its proof.

6. Étale (Bott inverted) *K*-theory

Recall that the vanishing result of Buchweitz and van Straten uses topological *K*-theory, not algebraic *K*-theory. In a general characteristic setting, the best replacement for topological *K*-theory is étale *K*-theory (with finite coefficients). This leads us to the goal of proving that, under suitable hypotheses, the pairing

$$\tilde{\theta} : G_0(Q[1/f], \mathbb{Z}/l) \times K_1(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \to \mathbb{Z}/l$$

factors through the analogous pairing involving étale *G*- and *K*-theory.

This is roughly what we achieve in this section. But, to avoid some nagging issues in the foundations of étale *K*-theory, it proves simpler to use instead the theory obtained from algebraic *K*-theory (and *G*-theory) with finite coefficients by inverting the so-called "Bott element". Since the resulting theory is closely related to topological *K*-theory, we will write it as K^{top} (and G^{top}); see Definition 6.1 below. Motivation for our approach is provided by Thomason's theorem [1985, Theorem 4.1] (recently extended by Rosenschon and Østvær [2006, Theorem 4.3]), which says, roughly, that "Bott inverted algebraic *K*-theory with $\mathbb{Z}/l$ coefficients and étale *K*-theory with $\mathbb{Z}/l$ coefficients coincide".

Definition 6.1. For any scheme X and prime l, let $\mathcal{K}(X, \mathbb{Z}/l)$ be the result of smashing the algebraic *K*-theory spectrum $\mathcal{K}(X)$ of X with the mod-*l* Moore space and, for any Noetherian scheme X, let $\mathcal{G}(X, \mathbb{Z}/l)$ be the result of smashing $\mathcal{G}(X)$ with the mod-*l* Moore space.

Let $\mathcal{KU}$ denote the spectrum representing (two-periodic) topological *K*-theory and let $L_{\mathcal{KU}}\mathcal{E}$ denote the Bousfield localization of a spectrum $\mathcal{E}$ at $\mathcal{KU}$.

Finally, for any scheme X and prime l, define

$$K_n^{\mathrm{top}}(X, \mathbb{Z}/l) := \pi_n L_{\mathcal{KU}}\mathcal{K}(X, \mathbb{Z}/l), \quad n \in \mathbb{Z},$$

and, for any Noetherian scheme X, define

$$G_n^{\mathrm{top}}(X, \mathbb{Z}/l) := \pi_n L_{\mathcal{KU}}\mathcal{G}(X, \mathbb{Z}/l), \quad n \in \mathbb{Z}.$$

Let us bring these definitions down to Earth a bit. Assume for simplicity that $l \geq 5$ and define $\mu_l = e^{2\pi i/l}$, a primitive *l*-th root of unity in $\mathbb{C}^\times$, and consider the ring $\mathbb{Z}[\mu_l]$. Recall (from [Thomason 1985, Appendix A], for example) that the *Bott element* is a certain canonical element $\beta \in K_2(\mathbb{Z}[\mu_l], \mathbb{Z}/l)$, which maps

to $[\mu_l] \in K_1(\mathbb{Z}[\mu_l])$ under the boundary map. If X is a scheme over $\mathrm{Spec}(\mathbb{Z}[\mu_l])$, then we obtain by pullback a Bott element $\beta \in K_2(X, \mathbb{Z}/l)$.

For example, if $X = \mathrm{Spec}(k)$ is an algebraically closed field with $\mathrm{char}(k) \neq l$, then there is a map $\mathbb{Z}[\mu_l] \to k$, specified by a choice of a primitive l-th root of unity for k. Moreover, the boundary map of the long exact coefficients sequence for the K-theory of k determines (using [Suslin 1983; 1984]) an isomorphism

$$K_2(k, \mathbb{Z}/l) \xrightarrow{\cong} \mu_l(k),$$

and a Bott element β for k maps to the chosen primitive l-th root of unity in k under this map. More generally, if V is a Henselian DVR with algebraically closed residue field k such that $\mathrm{char}(k) \neq l$, then we have

$$K_i(V, \mathbb{Z}/l) \cong K_i(k, \mathbb{Z}/l) \cong \begin{cases} \mathbb{Z}/l & \text{if } i \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

by [Suslin 1983; 1984], and a Bott element for V may also be specified by choosing a primitive l-th root of unity in k.

Given a Bott element $\beta \in K_2(\mathbb{Z}[\mu_l], \mathbb{Z}/l)$ for some prime $l \geq 5$, the element β acts on the K- and G-groups of all schemes over $\mathbb{Z}[\mu_l]$, and this action is compatible with the localization long exact sequences for such schemes and it commutes with the cup and cap products. Thus, formally inverting the action of β preserves all the structure needed in this paper.

Lemma 6.2. *Let $l \geq 5$ be a prime, X a scheme over $\mathbb{Z}[\mu_l]$, and let $\beta \in K_2(X, \mathbb{Z}/l)$ be the associated Bott element for X. There is a natural isomorphism*

$$K_*^{\mathrm{top}}(X, \mathbb{Z}/l) \cong K_*(X, \mathbb{Z}/l)[1/\beta],$$

where $K_*(X, \mathbb{Z}/l)[1/\beta]$ denotes the graded ring obtained by inverting the homogeneous central element β in the graded ring $K_*(X, \mathbb{Z}/l)$.

If X is Noetherian, there is a natural isomorphism

$$G_*^{\mathrm{top}}(X, \mathbb{Z}/l) \cong G_*(X, \mathbb{Z}/l)[1/\beta],$$

where $G_*(X, \mathbb{Z}/l)[1/\beta]$ is the localization by β of the graded $K_*(X, \mathbb{Z}/l)$-module $G_*(X, \mathbb{Z}/l)$.

Proof. See [Thomason 1985, Section A.14]. $\qquad\qquad\square$

Remark 6.3. Since β has degree two, one can view K_*^{top} and G_*^{top} as $\mathbb{Z}/2$-graded abelian groups.

If V is a Henselian DVR with algebraically closed residue field k such that $\mathrm{char}(k) \neq l$, Q is a V-algebra, $\mathfrak{m}$ is a maximal ideal of Q, and $f \in \mathfrak{m}$ is a non-zero-divisor, then the family of pairings

$$\tilde{\theta} : G_i(Q[1/f], \mathbb{Z}/l) \times K_j(\mathrm{Spec}(Q/f) \setminus \mathfrak{m}, \mathbb{Z}/l) \to K_{i+j-1}(Q/\mathfrak{m}, \mathbb{Z}/l)$$

forms a pairing of graded $K_*(V, \mathbb{Z}/l)$-modules, and hence, upon inverting the action of the Bott element $\beta \in K_2(V, \mathbb{Z}/l)$, we obtain the pairing

$$\tilde{\theta}^{\text{top}} : G_i^{\text{top}}(Q[1/f], \mathbb{Z}/l) \times K_j^{\text{top}}(\text{Spec}(Q/f) \setminus \mathfrak{m}, \mathbb{Z}/l) \to K_{i+j-1}^{\text{top}}(Q/\mathfrak{m}, \mathbb{Z}/l).$$

If $Q/\mathfrak{m}$ is algebraically closed (e.g., if Q is a finite-type V-algebra or the localization of such at a maximal ideal), then by [Suslin 1983; 1984] we have

$$K_*(Q/\mathfrak{m}, \mathbb{Z}/l) = \mathbb{Z}/l[\beta]$$

and hence

$$K_*^{\text{top}}(Q/\mathfrak{m}, \mathbb{Z}/l) = \mathbb{Z}/l[\beta, \beta^{-1}].$$

In particular, $K_0(Q/\mathfrak{m}, \mathbb{Z}/l) \cong K_0^{\text{top}}(Q/\mathfrak{m}, \mathbb{Z}/l) \cong \mathbb{Z}/l$.

Proposition 6.4. *Assume V is a DVR with algebraically closed residue field k and field of fractions F, Q is a flat V-algebra, t is a uniformizing parameter of V that maps to an element $f \in Q$, and $\mathfrak{m}$ is a maximal ideal of Q containing f such that $Q/\mathfrak{m}$ is algebraically closed. Set $R = Q/f$. For any prime $l \geq 5$ not divisible by* char(k), *there is a commutative diagram*

$$G_1(Q[1/f], \mathbb{Z}/l) \times K_0(\text{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l)$$

$$\downarrow \qquad\qquad\qquad\qquad \overset{\tilde{\theta}}{\searrow}$$

$$G_1^{\text{top}}(Q[1/f], \mathbb{Z}/l) \times K_0^{\text{top}}(\text{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l) \xrightarrow{\tilde{\theta}^{\text{top}}} \mathbb{Z}/l$$

$$\downarrow \qquad\qquad\qquad\qquad \overset{}{\nearrow}$$

$$G_1^{\text{top}}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) \times K_0^{\text{top}}(\text{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/l)$$

Proof. This diagram is obtained from the commutative diagrams in Corollaries 4.10 and 5.13 by inverting the action of $\beta \in K_2(V, \mathbb{Z}/l)$. $\qquad\square$

Corollary 6.5. *With the assumptions of Proposition 6.4, if $G_1^{\text{top}}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) = 0$ for infinitely many primes $l \geq 5$, then $\theta^R(M, N) = 0$ for all finitely generated R-modules M and N such that $N_{\mathfrak{p}}$ has finite projective dimension for all $\mathfrak{p} \neq \mathfrak{m}$.*

Proof. The proposition implies that $\theta^R(M, N)$ is a multiple of l for infinitely many primes l. $\qquad\square$

7. Vanishing of K_1^{top}

In this section, we combine a theorem of Rosenschon and Østvær [2006, Theorem 4.3], which is an improvement of [Thomason 1985, Theorem 4.1], with a theorem of Illusie [2003, Corollaire 2.10] to establish the vanishing of the odd-degree topological K-groups with $\mathbb{Z}/l$-coefficients of the "algebraic Milnor fiber", $\text{Spec}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F})$, in certain cases.

The Thomason–Rosenschon–Østvær theorem.

Definition 7.1. Fix a prime l. The *mod-l étale cohomological dimension* of a scheme X, written $\mathrm{cd}_l(X) \in \mathbb{N} \cup \{\infty\}$, is defined as

$$\mathrm{cd}_l(X) = \sup\{i \mid H^i_{\text{ét}}(X, \mathcal{F}) \neq 0 \text{ for some } l\text{-power torsion étale sheaf } \mathcal{F}\}.$$

If F is algebraically closed (or even just separably closed), then $\mathrm{cd}_l(F) = 0$ since the étale topology of such a field is trivial. If E is a field extension of F of transcendence degree d, then by [Serre 1994, Proposition II.4.2.11] we have

$$\mathrm{cd}_l(E) \leq \mathrm{cd}_l(F) + d.$$

It follows that if E is a field of transcendence degree M over an algebraically closed field then

$$\mathrm{cd}_l(E) \leq M. \tag{7.2}$$

Definition 7.3. For an odd prime l, a scheme X is *l-good* if X is quasiseparated, quasicompact and of finite Krull dimension, l is a unit of $\Gamma(X, \mathcal{O}_X)$, and there is a uniform finite bound on the mod-l étale cohomological dimensions of all the residue fields of X. A commutative ring A is *l-good* if $\mathrm{Spec}(A)$ is l-good.

Remark 7.4. The definition of l-good is motivated by the hypotheses of the Rosenschon–Østvær theorem. The correct version of l-good for $l = 2$ involves "virtual étale cohomological dimension".

Note that an affine scheme $X = \mathrm{Spec}(A)$ is automatically separated (and hence quasiseparated) and quasicompact. So, A is l-good if and only if A has finite Krull dimension, l is a unit of A, and there is a uniform finite bound on $\mathrm{cd}_l(\kappa(\mathfrak{p}))$ as $\mathfrak{p}$ ranges over all primes ideals of A.

If X is a separated scheme of finite type over an algebraically closed field k, then the transcendence degree of each of its residue fields is at most $\dim(X) < \infty$, and hence we have $\mathrm{cd}_l(\kappa(x)) \leq \dim(X)$ for all $x \in X$. It follows that, for such X, if $l \neq \mathrm{char}(k)$ then X is l-good.

We will need a generalization of this fact that involves Henselizations. Recall that, given a commutative ring V, a V-algebra Q is *essentially of finite type* over V if Q is the localization of a finitely generated V-algebra by some multiplicatively closed subset.

Lemma 7.5. *Suppose V is a Noetherian ring, $\mathfrak{p}$ is any prime of V, Q is a V-algebra essentially of finite type, and $\mathfrak{m}$ is a maximal ideal of Q. Let $Q_{\mathfrak{m}}^{\text{hen}}$ denote the Henselization of Q at $\mathfrak{m}$. If $l \neq \mathrm{char}(\kappa(\mathfrak{p}))$, then $Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \overline{\kappa(\mathfrak{p})}$ is l-good, where $\overline{\kappa(\mathfrak{p})}$ is the algebraic closure of $\kappa(\mathfrak{p})$.*

Remark 7.6. The lemma generalizes to schemes: if S is a Noetherian scheme, $p : X \to S$ is a morphism of schemes essentially of finite type, and $x \in X$ is any

closed point, then each geometric fiber of $X_x^{\mathrm{hen}} \to S$ is l-good for all l not equal to the characteristic of the fiber.

Proof. Since Q is essentially of finite type over V and V is Noetherian, Q is also Noetherian. It follows [EGA IV$_4$ 1967, Théorème (18.6.6)] that $Q_{\mathfrak{m}}^{\mathrm{hen}}$ is Noetherian and, since it is local, $\dim(Q_{\mathfrak{m}}^{\mathrm{hen}}) < \infty$.

Again using that Q is essentially of finite type over V, there is a bound M such that for each prime $\mathfrak{q} \in \mathrm{Spec}(Q)$, the transcendence degree of $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{q} \cap V)$ is at most M. The residue fields of $Q_{\mathfrak{m}}^{\mathrm{hen}}$ are algebraic (and separable) over the corresponding residue fields of Q by [EGA IV$_4$ 1967, Théorème (18.6.9)]. It follows that for each prime $\mathfrak{q} \in \mathrm{Spec}(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \kappa(\mathfrak{p}))$, the transcendence degree of $\kappa(\mathfrak{q})$ over $\kappa(\mathfrak{p})$ is at most M.

To simplify notation, we state and prove a more general assertion: if k is any field, A is a commutative k-algebra of finite Krull dimension, and there exists a finite bound M such that the transcendence degree of $\kappa(\mathfrak{q})$ over k is at most M for all $\mathfrak{q} \in \mathrm{Spec}(A)$, then $A \otimes_k \bar{k}$ is l-good for all $l \neq \mathrm{char}(k)$. The lemma follows from the case $k = \kappa(\mathfrak{p})$ and $A = Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \kappa(\mathfrak{p})$.

The field $\bar{k}$ is the filtered colimit of the finite extensions L of k contained in it, and hence $A \otimes_k \bar{k}$ is the filtered colimit of the collection of rings $\{A \otimes_k L\}$. For each L, each residue field of $A \otimes_k L$ is a finite extension of the corresponding residue field of A. As with any colimit of rings, the residue field of $A \otimes_k \bar{k}$ at a prime $\mathfrak{q}$ is the filtered colimit of the residue fields $\mathfrak{q} \cap (A \otimes_k L)$. We conclude that for each $\mathfrak{q} \in \mathrm{Spec}(A \otimes_k \bar{k})$ the residue field $\kappa(\mathfrak{q})$ is algebraic over $\kappa(\mathfrak{q} \cap A)$ and hence has transcendence degree at most M over k. Since $\kappa(\mathfrak{q})$ contains $\bar{k}$, it has transcendence degree at most M over $\bar{k}$ as well. By (7.2) we have $\mathrm{cd}_l(\kappa(\mathfrak{q})) \leq M$ for all $\mathfrak{q} \in \mathrm{Spec}(A \otimes_k \bar{k})$.

Finally, since $A \subseteq A \otimes_k \bar{k}$ is an integral extension, $\dim(A \otimes_k \bar{k}) = \dim(A) < \infty$.
$\square$

The following result is an improvement of a celebrated theorem of Thomason [1985, Theorem 4.1]:

Theorem 7.7 [Rosenschon and Østvær 2006, Theorem 4.3]. *If X is an l-good scheme for a prime $l \geq 5$, there is a strongly convergent, right half-plane spectral sequence*

$$E_2^{p,q} \Longrightarrow K_{q-p}^{\mathrm{top}}(X, \mathbb{Z}/l),$$

where

$$E_2^{p,q} = \begin{cases} H_{\text{ét}}^p(X, \mu_l^{\otimes i}) & \text{if } q = 2i, \\ 0 & \text{if } q \text{ is odd}, \end{cases}$$

and the differential $d_r : E_r^{p,q} \to E_r^{p+r,q+r-1}$ on the r-th page has bidegree $(r, r-1)$.

Remarks 7.8. (1) In their original paper, the abutment of this spectral sequence is $\pi_{q-p}L_{\mathcal{KU}}\mathcal{K}^B(X, \mathbb{Z}/l)$, where $\mathcal{K}^B$ denotes *Bass's algebraic K-theory spectrum*. Since $\mathcal{K} \to \mathcal{K}^B$ induces an isomorphism on nonnegative homotopy groups, the natural map $L_{\mathcal{KU}}\mathcal{K} \xrightarrow{\sim} L_{\mathcal{KU}}\mathcal{K}^B$ is a weak equivalence.

(2) The integer i in $\mu_l^{\otimes i}$ is allowed to be negative. For $i < 0$, $\mu_l^{\otimes i} = (\mu_l^{-1})^{\otimes |i|}$ where μ_l^{-1} is the $\mathbb{Z}/l$-linear dual of μ_l.

(3) If X is a scheme over an algebraically closed field of characteristic not equal to l, then upon choosing a primitive l-th root of unity, we may identify $\mu_l^{\otimes i}$ with $\mathbb{Z}/l$ for all $i \in \mathbb{Z}$.

(4) We have assumed $l \geq 5$ only to avoid some technical complications and because the cases $l = 2, 3$ will not be important for our purposes. But, appropriately interpreted, this theorem remains valid for $l \in \{2, 3\}$.

The following result gives the special case of the theorem that we will need:

Corollary 7.9. *Assume V is a Noetherian domain with field of fractions F, Q is essentially of finite type over V, the generic fiber of $\mathrm{Spec}(Q) \to \mathrm{Spec}(V)$ (namely, $\mathrm{Spec}(Q \otimes_V F) \to \mathrm{Spec}(F)$) is essentially smooth, $\mathfrak{m}$ is a maximal ideal of Q, and l is a prime such that $l \geq 5$ and $l \neq \mathrm{char}(F)$. Then there is a strongly convergent spectral sequence*

$$E_2^{p,q} \Longrightarrow K_{q-p}^{\mathrm{top}}(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \bar{F}, \mathbb{Z}/l),$$

where

$$E_2^{p,q} = \begin{cases} H_{\text{ét}}^p(Q_{\mathfrak{m}}^{\mathrm{hen}} \otimes_V \bar{F}, \mu_l^{\otimes i}) & \text{if } q = 2i, \\ 0 & \text{if } q \text{ is odd}, \end{cases}$$

and the differential on the r-th page has bidegree $(r, r-1)$.

Proof. This follows from Lemma 7.5 and Theorem 7.7. $\square$

Illusie's theorem. We will need to make the following assumptions:

Assumptions 7.10. Assume $(V, k, F, Q, \mathfrak{m}, f, l)$ satisfy:

(1) V is a Henselian DVR with algebraically closed residue field k and field of fractions F.

(2) Q is a regular ring, $\mathfrak{m}$ is a maximal ideal of Q, and $f \in \mathfrak{m}$.

(3) There is a flat, finite-type map $\mathrm{Spec}(Q) \to \mathrm{Spec}(V)$ of affine schemes of relative dimension n such that the associated map of rings sends some uniformizing parameter $t \in V$ to $f \in Q$.

(4) The morphism $\mathrm{Spec}(Q) \to \mathrm{Spec}(V)$ is smooth at every point except, possibly, $\mathfrak{m} \in \mathrm{Spec}(Q)$. Notice in particular that the generic fiber $\mathrm{Spec}(Q \otimes_V F) \to \mathrm{Spec}(F)$ is smooth.

(5) The morphism $\text{Spec}(Q) \to \text{Spec}(V)$ is a complete intersection near $\mathfrak{m}$ — that is, for some $g \in Q \setminus \mathfrak{m}$, $Q[1/g]$ the quotient of a smooth V-algebra by a regular sequence.

(6) l is prime not equal to 2, 3 or $\text{char}(k)$. Notice this implies $l \neq \text{char}(F)$ too.

Theorem 7.11 [Illusie 2003, Corollaire 2.10]. *Under Assumptions 7.10,*

$$H_{\text{ét}}^j((Q_{\mathfrak{m}}^{\text{hen}}) \otimes_V \bar{F}, \mathbb{Z}/l) = 0$$

if $j \notin \{0, n\}$, where $Q_{\mathfrak{m}}^{\text{hen}}$ is the Henselization of Q at $\mathfrak{m}$ and $\bar{F}$ is the algebraic closure of F.

Remark 7.12. Illusie's theorem is the analogue of Milnor's theorem, stating that the Milnor fiber of an analytic isolated singularity is homotopy equivalent to a bouquet of n-dimensional spheres.

Corollary 7.13. *Under Assumptions 7.10, if n is even then*

$$K_1^{\text{top}}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) = 0.$$

Proof. The assumptions allow us to apply Corollary 7.9, giving a strongly convergent spectral sequence

$$E_2^{p,q} \implies K_{q-p}^{\text{top}}(Q \otimes_V \bar{F}, \mathbb{Z}/l),$$

where

$$E_2^{p,q} = \begin{cases} H_{\text{ét}}^p(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mu_l^{\otimes i}) & \text{if } q = 2i, \\ 0 & \text{if } q \text{ is odd}, \end{cases}$$

and the differential on the r-th page has bidegree $(r, r-1)$. Since $\bar{F}$ is algebraically closed, $\mu_l \cong \mathbb{Z}/l$ (noncanonically) and thus Illusie's theorem applies to give that the only nonzero E_2-terms are $E_2^{n,2i}$ and $E_2^{0,2i}$. Since n is even, these terms only contribute to the even degree part of K_*^{top}. $\qquad\square$

Remark 7.14. The proof also shows that, when n is even, there exists an exact sequence

$$0 \to H_{\text{ét}}^n(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) \to K_0^{\text{top}}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) \to H_{\text{ét}}^0(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) \to 0.$$

Similarly, when n is odd, there exists an exact sequence

$$0 \to K_0^{\text{top}}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) \to H_{\text{ét}}^0(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l)$$
$$\to H_{\text{ét}}^n(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) \to K_1^{\text{top}}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) \to 0.$$

Theorem 7.15. *If conditions (1)–(5) of Assumptions 7.10 hold and n is even, then $\theta^R(M, N) = 0$ for all finitely generated R-modules M and N, where $R = Q/f$.*

Proof. The only singular point of R/f is $\mathfrak{m}$ and hence $\theta^R(M, N)$ is defined for all finitely generated R-modules.

The hypotheses ensure that $Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}$ is a filtered colimit of regular (Noetherian) rings with flat transition maps, and so

$$K_1^{\text{top}}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l) \cong G_1^{\text{top}}(Q_{\mathfrak{m}}^{\text{hen}} \otimes_V \bar{F}, \mathbb{Z}/l).$$

The theorem is thus an immediate consequence of Corollaries 6.5 and 7.13. $\qquad \square$

Corollary 7.16. *Let V be a Dedekind domain, $\mathfrak{n}$ be a maximal ideal of V such that $V/\mathfrak{n}$ is a perfect field, and Q be a regular, flat V-algebra of finite type. Assume the singular locus of the morphism $\mathrm{Spec}(Q) \to \mathrm{Spec}(V)$ is a finite set $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_1\}$ of maximal ideals of Q that lie over $\mathfrak{n}$ and that the morphism $\mathrm{Spec}(Q) \to \mathrm{Spec}(V)$ is a complete intersection in an open neighborhood of each of the $\mathfrak{m}_i$.*

Then $R := Q/f$ is a hypersurface with only isolated singularities and, if $\dim(R)$ is even,

$$\theta^R(M, N) = 0$$

for all finitely generated R-modules M and N.

Proof. We may assume V is local and hence a DVR. Then R is the hypersurface Q/f, where f is the image in Q of a chosen uniformizing parameter t of V. The non-regular locus of R is $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_l\}$ and we have

$$\theta^R(M, N) = \sum_{i=1}^{l} \theta^{R_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}, N_{\mathfrak{m}_i})$$

for all finitely generated R-modules M and N. It suffices to prove $\theta^{R_{\mathfrak{m}_i}} \equiv 0$ for all i, and thus, upon replacing $\mathrm{Spec}(Q)$ by a sufficiently small affine open neighborhood of each $\mathfrak{m}_i$, we may assume that $l = 1$, that $\mathfrak{m} := \mathfrak{m}_1$ is the only singular point of the morphism $\mathrm{Spec}(Q) \to \mathrm{Spec}(V)$, and that this morphism is a complete intersection.

Let $V_{\mathfrak{n}}^{\text{sh}}$ denote the *strict Henselization* of V at its maximal ideal $\mathfrak{n}$. Recall from [EGA IV$_4$ 1967, §18.8] that there is a faithfully flat local ring map $V \to V_{\mathfrak{n}}^{\text{sh}}$, that $\mathfrak{n}^{\text{sh}} := \mathfrak{n} V_{\mathfrak{n}}^{\text{sh}}$ is the maximal ideal of $V_{\mathfrak{n}}^{\text{sh}}$, and that the induced map on residue fields $V/\mathfrak{n} \rightarrowtail V_{\mathfrak{n}}^{\text{sh}}/\mathfrak{n}^{\text{sh}}$ is a (initially chosen) separable closure of $V/\mathfrak{n}$. Since we assume $V/\mathfrak{n}$ is perfect, the residue field of $V_{\mathfrak{n}}^{\text{sh}}$ is, in fact, algebraically closed.

Set $Q' = Q \otimes_V V_{\mathfrak{n}}^{\text{sh}}$ and $R' := R \otimes_V V_{\mathfrak{n}}^{\text{sh}} = Q'/f'$, where f' is the image of f under $Q \to Q'$. The fiber of $\mathrm{Spec}(Q') \to \mathrm{Spec}(Q)$ over $\mathfrak{m}$ is

$$Q/\mathfrak{m} \otimes_V V_{\mathfrak{n}}^{\text{sh}} \cong Q/\mathfrak{m} \otimes_{V/\mathfrak{n}} V^{\text{sh}}/\mathfrak{n}^{\text{sh}} = Q/\mathfrak{m} \otimes_k \bar{k}.$$

Since Q has finite type over V, $k \rightarrowtail Q/\mathfrak{m}$ is a finite field extension. This shows that the fiber of $\mathrm{Spec}(Q') \to \mathrm{Spec}(Q)$ over $\mathfrak{m}$ consists of a finite number of maximal ideals $\mathfrak{m}_1', \ldots, \mathfrak{m}_n'$ of Q'. Since $\mathrm{Spec}(Q) \setminus \mathfrak{m} \to \mathrm{Spec}(V)$ is smooth, so is

$\mathrm{Spec}(Q) \setminus \{\mathfrak{m}_1', \ldots, \mathfrak{m}_n'\} \to \mathrm{Spec}(V^{\mathrm{sh}})$. For each i, upon replacing Q' by a suitably small affine open neighborhood of $\mathfrak{m}_i'$, conditions (1)–(5) of Assumptions 7.10 are met, and thus we have $\theta^{R_{\mathfrak{m}_i}'}(-,-) \equiv 0$ by the theorem.

To prove θ^R vanishes, it suffices to prove the following more general fact: if R is a hypersurface ring having only one singular point, $\mathfrak{m}$, and there is a flat local ring map $(R_{\mathfrak{m}}, \mathfrak{m}) \to (R', \mathfrak{m}')$ such that R' is a local hypersurface with an isolated singularity satisfying $\theta^{R'} \equiv 0$, then $\theta^R \equiv 0$. To prove this, observe that if T is a finitely generated R-module supported on $\mathfrak{m}$ then

$$\mathrm{length}_{R'}(T \otimes_R R') = \mathrm{length}_R(T) \cdot \mathrm{length}_{R'}(R'/\mathfrak{m}_R R').$$

It follows that, for any pair of finitely generated R-modules M and N, we have

$$\mathrm{length}_{R'} \mathrm{Tor}_i^{R'}(M \otimes_R R', N \otimes_R R') = \mathrm{length}_{R'}(\mathrm{Tor}_i^R(M, N) \otimes_R R')$$
$$= \mathrm{length}_R(\mathrm{Tor}_i^R(M, N)) \cdot \mathrm{length}_{R'}(R'/\mathfrak{m}_R R')$$

for $i \gg 0$, and hence

$$\theta^R(M, N) = \frac{\theta^{R'}(M \otimes_R R', N \otimes_R R')}{\mathrm{length}_{R'}(R'/\mathfrak{m}_R R')} = 0. \qquad \square$$

Theorem 1.2 from the introduction follows quickly from the previous corollary by taking $V = k[t]$: Since f is a non-zero-divisor, the map of k-algebras $k[t] \to Q$ sending t to f is flat. Since k is perfect, Q is smooth over k and hence $Q[t]$ is smooth over $k[t]$. The morphism $\mathrm{Spec}(Q) \to \mathrm{Spec}(V) = \mathbb{A}_k^1$ is thus a complete intersection because $Q \cong Q[t]/(f-t)$.

We can extend our main vanishing result slightly by allowing localizations:

Corollary 7.17. *For any ring R as in Corollary 7.16, $\theta^{S^{-1}R}(M, N) = 0$ for any multiplicatively closed set S disjoint from the singular locus of R and any pair of finitely generated $S^{-1}R$-modules M and N.*

Proof. More generally, suppose R is any hypersurface whose nonregular locus is $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\} \subseteq \mathrm{mSpec}(R)$ and $\theta^R \equiv 0$, and let $R' = S^{-1}R$ for any multiplicatively closed set S with $S \cap \mathfrak{m}_i = \varnothing$ for all i. We claim that $\theta^{R'} \equiv 0$ too. It is clear R' is also a hypersurface with isolated singularities. Given finitely generated R'-modules M and N, there exist finitely generated R-modules $\widetilde{M}$ and $\widetilde{N}$ such that $\widetilde{M} \otimes_R R' \cong M$ and $\widetilde{N} \otimes_R R' \cong N$. For all i we have

$$S^{-1} \mathrm{Tor}_i^R(\widetilde{M}, \widetilde{N}) \cong \mathrm{Tor}_i^{R'}(M, N)$$

and, for $i \gg 0$, $\mathrm{Tor}_i^R(\widetilde{M}, \widetilde{N})$ is supported on $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$ so that $S^{-1} \mathrm{Tor}_i^R(\widetilde{M}, \widetilde{N}) \cong \mathrm{Tor}_i^R(\widetilde{M}, \widetilde{N})$. It follows that $\theta^{R'}(M, N) = \theta^R(\widetilde{M}, \widetilde{N}) = 0$. $\qquad \square$

In particular, Corollary 7.17 justifies Example 1.3 in the introduction.

Acknowledgements

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