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We describe an arithmetic K_2-valued invariant for longitudes of a link $L \subset \mathbb{R}^3$, obtained from an SL_2 representation of the link group. Furthermore, we show a nontriviality on the elements, and compute the elements for some links. As an application, we develop a method for computing longitudes in $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ representations for link groups, where $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ is the universal covering group of $\mathrm{SL}_2(\mathbb{R})$.

1. Introduction

Algebraic K-groups provide a uniform language for the study of many mathematical phenomena. When it comes to knot theory in topology, the Chern–Simons invariant (i.e., complex volume) and the twisted Alexander polynomial have been extensively studied as important invariants of 3-manifolds, and appear as elements in the K_1- and K_3-groups as follows:[1]

K-group	link invariant
K_1 [Bass 1968]	twisted Alexander polynomial
K_2 [Milnor 1971]	*unknown*
K_3 [Quillen 1973]	Chern–Simons invariant

In contrast, there are few such studies on the second K-group in low-dimensional topology. Although the paper [Cooper et al. 1994, §4] in topology introduced the A-polynomial and a Steinberg symbol "$\{\mathfrak{m}, \mathfrak{l}\} \in K_2^M(F)$", the symbol was defined only for "the tautological representation" (However, this $\{\mathfrak{m}, \mathfrak{l}\}$ has a relation to the study of incompressible surfaces in Culler–Shalen theory; see [Cooper et al. 1994, Introduction].) Moreover, we should emphasize that fields F in most papers on the Culler–Shalen theory are assumed to be (over) the complex field $\mathbb{C}$, which is local from the viewpoint of number theory. Nevertheless, the Milnor–Witt K_2-group

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[1]For the reader interested in the link invariants with relation to the K_1- and K_3-groups, see [Milnor 1966; Friedl and Kim 2008; Zickert 2009] and references therein.

$K_2^{\mathrm{MW}}(F)$ is defined, from any field F, to be the kernel of the universal central extension $\mathcal{E}$ (which exists because $\mathrm{SL}_2(F)$ is perfect),

$$K_2^{\mathrm{MW}}(F) := \mathrm{Ker}\big(\mathcal{E} : \widetilde{\mathrm{SL}}_2(F) \longrightarrow \mathrm{SL}_2(F)\big);$$

this $K_2^{\mathrm{MW}}(F)$ has been extensively studied in relation to, e.g., metaplectic groups, stability problems and $\mathbb{A}^1$-homotopy theory (see [Matsumoto 1969; Morel 2012; Hutchinson and Tao 2008; Suslin 1987]). Actually, $K_2^{\mathrm{MW}}(F)$ contains some obstructions, as in the class number formula, the Beilinson conjecture and so on (see [Weibel 2013]).

In this paper, we propose a natural construction of an element in $K_2^{\mathrm{MW}}(F)$ from any parabolic[2] representation $f : \pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$, where F is any infinite field and L is an arbitrary link in $\mathbb{R}^3$. The construction is done in a simple way, wherein the longitudes of L play a key role: First, we show (Proposition 3.1) that f can be algebraically lifted to $\tilde{f} : \pi_1(\mathbb{R}^3 \setminus L) \to \widetilde{\mathrm{SL}}_2(F)$; it follows from the parabolicity that, for each (preferred) longitude $\mathfrak{l}_i \in \pi_1(\mathbb{R}^3 \setminus L)$, $f(\mathfrak{l}_i)$ lies in the unipotent subgroup U_F of $\mathrm{SL}_2(F)$ up to conjugacy. Therefore, the lifted $\tilde{f}(\mathfrak{l}_i)$ lies in the preimage $\mathcal{E}^{-1}(U_F)$, which will be shown to be isomorphic to the product $F \times \widetilde{K}_2^{\mathrm{MW}}(F)$ as abelian groups (Lemma 2.4). Here $\widetilde{K}_2^{\mathrm{MW}}(F)$ is a $\mathbb{Z}/2$-extension of $K_2^{\mathrm{MW}}(F)$; see (2.3). Further, we will show that the value $\tilde{f}(\mathfrak{l}_i)$ is independent of the choice of the lift $\tilde{f}$, and call it *the K_2 invariant of f* (Definition 3.3). Here is a summary:

$$\begin{tikzcd}
 & \tilde{f}(\mathfrak{l}_i) \in F \times \widetilde{K}_2^{\mathrm{MW}}(F) \cong \mathcal{E}^{-1}(U_F) \arrow[r, hook] & \widetilde{\mathrm{SL}}_2(F) \arrow[d, "\mathcal{E}"] \\
\mathfrak{l}_i \in \pi_1(\mathbb{R}^3 \setminus L) \arrow[ur, mapsto] \arrow[urr, dashed, "\tilde{f}"'] \arrow[rr, "f"'] & & \mathrm{SL}_2(F)
\end{tikzcd}$$

In addition, using a homotopical result in [Nosaka 2015], we will show that any (algebraic) 2-cycle in $\widetilde{K}_2^{\mathrm{MW}}(F)$ can be represented as the K_2 invariant of some parabolic representation of some link (Theorem 3.4). Consequently, this theorem ensures that many links give nontrivial examples of the K_2 invariants. Furthermore, the K_2 invariants are partially computable for some links, by the help of arithmetic studies on the K_2-groups. Here, the Matsumoto–Moore 2-cocycle [Matsumoto 1969; Moore 1968] is useful, to formulate $\tilde{f}(\mathfrak{l}_i)$ explicitly in $K_2^{\mathrm{MW}}(F)$ (see Section 2), and $K_2^{\mathrm{MW}}(F)$ in some cases is computable (see Section 4); Thus, we

[2]Notation in topology: Here, a *link* L is a C^∞-embedding of solid tori into the 3-space $\mathbb{R}^3$. Namely, $L : \bigsqcup(D^2 \times S^1) \hookrightarrow \mathbb{R}^3$. By $\#L$ we mean the number of tori, and $\pi_1(\mathbb{R}^3 \setminus L)$ is called *the link group of* L. Furthermore, with a choice $x_0 \in S^1$, a *meridian* is one component in the image of $\bigsqcup(\partial D^2 \times \{x_0\})$ and a *longitude* is that of $\bigsqcup(\{x_0\} \times S^1)$. A homomorphism $f : \pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$ is *parabolic* if every meridian $\mathfrak{m}$ in $\pi_1(\mathbb{R}^3 \setminus L)$ satisfies $\mathrm{Tr}\, f(\mathfrak{m}) = \pm 2$.

will explicitly determine the K_2 invariants of some small knots (see Section 5), although geometric and arithmetic features appearing in the K_2 invariants have many unknown aspects (see the A-polynomial [Cooper et al. 1994]).

Furthermore, in Section 6B, we will give two applications from the K_2 studies to low-dimensional topology. The first is with respect to the unlifted object $f(\mathfrak{l}_i) = \mathcal{E}(\tilde{f}(\mathfrak{l}_i)) \in \mathrm{SL}_2(F)$, which is commonly called a *cusp shape* in hyperbolic geometry (see [Maclachlan and Reid 2003; Zickert 2009]). While the cusp shape seems, by definition, to be a noncommutative object arising from the link groups $\pi_1(\mathbb{R}^3 \setminus L)$, we explicitly introduce an additive sum formula for $f(\mathfrak{l}_i)$, as in the abelian group $K_2^{\mathrm{MW}}(F)$; see Theorem 6.1. The second is an application to the method of Boyer, Gordon and Watson [Boyer et al. 2013] for finding new 3-dimensional manifolds, $M_r(K)$, obtained by r-surgery on a knot K such that $\pi_1(M_r(K))$ is "left-orderable". This result (Theorem 6.7) gives evidence supporting to a conjecture in [Boyer et al. 2013] that relates L-spaces to left-orderability. The key here is Proposition 6.3, which closely relates the real $K_2^{\mathrm{MW}}(\mathbb{R})$ to $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$, where $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ is the universal cover group of the Lie group $\mathrm{SL}_2(\mathbb{R})$. See Section 6B for the details.

This paper is organized as follows: We first review the K_2-groups in Section 2, and define the K_2 invariants in Section 3. After explaining computation on K_2 in Section 4, we quantitatively compute some K_2 invariants in Section 5. Furthermore, we describe the two applications in Section 6. Finally, Section 7 discusses parabolic representations by means of quandle theory, and proves the theorems.

Notational conventions. Throughout this paper, F is a commutative field of infinite order, and $\mathrm{Char}(F)$ is the characteristic (possibly $\mathrm{Char}(F) = 0, 2$).

2. Review: the Milnor–Witt K_2-group

Before stating the results, we should briefly review the Matsumoto–Moore theorem [Matsumoto 1969; Moore 1968], which provides a presentation of the second group homology $H_2^{\mathrm{gr}}(\mathrm{SL}_2(F))$.

Define $K_2^{\mathrm{MW}}(F)$ to be the abelian group[3] generated by the symbols $[a, b]$ with $a, b \in F$ subject to the relations

(i) $[a, bc] + [b, c] = [ab, c] + [a, b]$ and $[a, 1] = [1, b] = 0$,

(ii) $[a, b] = [b^{-1}, a]$ and $[a, b] = [a, -ab]$ for $a, b, c \in F^\times$,

(iii) $[d, e] = [d, (1-d)e]$,

(iv) $[d, 0] = [d, 0] = 0$ for $d, e \in F$.

[3] The original presentation did not contain the generators $[0, d]$, $[d, 0]$ or the relation (iv). In order to simplify our statements we employ this presentation, although we can easily see that it coincides with the original presentation through the relation (iv).

Noting that the group $\mathrm{SL}_2(F)$ is perfect, i.e., $\mathrm{SL}_2(F)_{\mathrm{ab}} = 0$, we now describe the theorem:

Theorem 2.1 [Moore 1968; Matsumoto 1969, Corollaire 5.12]. *Let F be an infinite field. There is an isomorphism $H_2^{\mathrm{gr}}(\mathrm{SL}_2(F)) \cong K_2^{\mathrm{MW}}(F)$. Moreover, the universal group 2-cocycle is represented as a map $\theta_{\mathrm{uni}} : \mathrm{SL}_2(F) \times \mathrm{SL}_2(F) \to K_2^{\mathrm{MW}}(F)$ defined by*

$$\theta_{\mathrm{uni}}(g, g') := [\chi(gg'), -\chi(g)^{-1}\chi(g')] - [\chi(g), \chi(g')] \in K_2^{\mathrm{MW}}(F). \tag{2.2}$$

(Here, for $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(F)$, we define $\chi(g) := \gamma$ if $\gamma \neq 0$ and $\chi(g) := \delta \in F^\times$ if $\gamma = 0$.) *In particular, the set $K_2^{\mathrm{MW}}(F) \times \mathrm{SL}_2(F)$ with the group operation $(\alpha, g) \cdot (\beta, h) = (\alpha + \beta + \theta_{\mathrm{uni}}(g, h), gh)$ is isomorphic to the universal extension $\widetilde{\mathrm{SL}}_2(F)$.*

Here we note two facts: First, the inclusion $\mathrm{SL}_2(F)$ into the symplectic group $\mathrm{Sp}_{2n}(F)$ induces an isomorphism $H_2^{\mathrm{gr}}(\mathrm{SL}_2(F)) \cong H_2^{\mathrm{gr}}(\mathrm{Sp}_{2n}(F))$ for any $n \in \mathbb{N}$ (see [Hutchinson and Tao 2008; Suslin 1987]). Next, for any finite field $\mathbb{F}_q$ with $q > 10$, the $H_2^{\mathrm{gr}}(\mathrm{SL}_2(\mathbb{F}_q))$ vanishes. Therefore, in this paper, we restrict ourselves to SL_2 and infinite fields.

To end the section, we will introduce some terminology and Lemma 2.4. We first observe the preimage of $\{\pm \mathrm{id}_{F^2}\}$ via the extension $\mathcal{E} : \widetilde{\mathrm{SL}}_2(F) \to \mathrm{SL}_2(F)$. Let $\widetilde{K}_2^{\mathrm{MW}}(F)$ denote the preimage. Since $\theta_{\mathrm{uni}}(a, b) = \theta_{\mathrm{uni}}(b, a)$ for any $a, b \in \{\pm \mathrm{id}_{F^2}\}$ by the definitions, $\widetilde{K}_2^{\mathrm{MW}}(F)$ is abelian. To summarize, if $\mathrm{Char}(F) \neq 2$, we have

$$0 \to K_2^{\mathrm{MW}}(F) \to \widetilde{K}_2^{\mathrm{MW}}(F) \to \mathbb{Z}/2 \to 0 \quad \text{(exact)}. \tag{2.3}$$

If $\mathrm{Char}(F) = 2$, we have $\widetilde{K}_2^{\mathrm{MW}}(F) = K_2^{\mathrm{MW}}(F)$. Furthermore, consider the preimage of the unipotent subgroup U_F, where U_F is of the form $\left\{\left(\begin{smallmatrix} \pm 1 & a \\ 0 & \pm 1 \end{smallmatrix}\right) \mid a \in F\right\}$ as usual. Notice the group isomorphism $U_F \cong \mathbb{Z}/2 \times F$ or $\cong F$ and that the restriction of θ_{uni} on this summand F is zero. Hence, we can readily see the following:

Lemma 2.4. *The preimage $\mathcal{E}^{-1}(U_F)$ is isomorphic to $\widetilde{K}_2^{\mathrm{MW}}(F) \times F$ as an abelian group.*

3. Definition: K_2 invariants

In this section, as a topological part, we introduce the K_2 invariant with respect to SL_2-parabolic representations of link groups (Definition 3.3), and state a theorem. The knot-theoretic notation that we will use is mentioned in the introduction (see the textbook [Lickorish 1997] for more details).

The key in the construction is the following proposition:

Proposition 3.1. *Let F be an infinite field, and $L \subset \mathbb{R}^3$ be a link. Every parabolic representation $f : \pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$ admits the lift $\tilde{f} : \pi_1(\mathbb{R}^3 \setminus L) \to \widetilde{\mathrm{SL}}_2(F)$*

such that any meridian $\mathfrak{m} \in \pi_1(\mathbb{R}^3 \setminus L)$ *satisfies*

$$\tilde{f}(\mathfrak{m}) = (0, f(\mathfrak{m})) \in K_2^{\mathrm{MW}}(F) \times \mathrm{SL}_2(F) = \widetilde{\mathrm{SL}}_2(F).$$

Remark 3.2. The proof appears in Section 7B, not as standard discussions on second homology. Actually, for example, if $\#L > 1$, then $\mathbb{R}^3 \setminus L$ and $S^3 \setminus L$ are not always $K(\pi, 1)$-spaces and $H_2(\mathbb{R}^3 \setminus L; \mathbb{Z}) \neq 0$. To summarize, the lifting is guaranteed from special properties of K_2^{MW} and parabolicity.

Next, we will see that the lifted longitude lies in the preimage $\mathcal{E}^{-1}(U_F) \cong \widetilde{K}_2^{\mathrm{MW}}(F) \times F$ in Lemma 2.4. For this, choose a meridian–longitude pair $(\mathfrak{m}_j, \mathfrak{l}_j)$ with respect to each link-component of L, where $1 \leq j \leq \#L$. Notice that the centralizer of the unipotent subgroup U_F is U_F itself in $\mathrm{SL}_2(F)$. Therefore, since f is parabolic and each $\mathfrak{m}_j$ commutes with $\mathfrak{l}_j$, the image $f(\mathfrak{l}_j) \in \mathrm{SL}_2(F)$ is contained in U_F. Hence, the lifted object $\tilde{f}(\mathfrak{l}_j)$ lies in the product $\widetilde{K}_2^{\mathrm{MW}}(F) \times F \subset \widetilde{\mathrm{SL}}_2(F)$ as required. Furthermore, this $\tilde{f}(\mathfrak{l}_j)$ up to conjugacy of $\mathrm{SL}_2(F)$ is independent of the choice of the lifting $\tilde{f}$, because $\widetilde{K}_2^{\mathrm{MW}}(F)$ is the center in $\widetilde{\mathrm{SL}}_2(F)$.

Definition 3.3. Let f be a parabolic representation $\pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$. For a link-component j of L, fix a meridian–longitude pair $(\mathfrak{m}_j, \mathfrak{l}_j)$. We define the K_2 *invariant* of f to be the value of $\tilde{f}(\mathfrak{l}_j)$ after projecting it onto $\widetilde{K}_2^{\mathrm{MW}}(F)$.

In Section 5, we will compute concretely the K_2 invariants of some links.

Speaking of invariants, we shall observe the nontriviality of the invariant (we prove this theorem from a homotopical viewpoint in Section 7D).

Theorem 3.4. *Let F be an infinite field. For any element* $(\alpha, \beta) \in \widetilde{K}_2^{\mathrm{MW}}(F) \times F$, *there are a link L and a parabolic representation* $f : \pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$ *such that the sum* $\tilde{f}(\mathfrak{l}_1) + \cdots + \tilde{f}(\mathfrak{l}_{\#L})$ *is equal to* $(\alpha, \beta) \in \widetilde{K}_2^{\mathrm{MW}}(F) \times F$.

In summary, this theorem implies that any (algebraic) cycle in $K_2^{\mathrm{MW}}(F)$ may be represented as some parabolic representation of a link via longitudes, and it ensures many links which have the nontriviality of the K_2 invariants.

Incidentally, from the viewpoint of $\mathbb{A}^1$-homotopy theory, we note a homotopical interpretation of the invariant $\tilde{f}(\mathfrak{l})$ for perfect fields F. The following isomorphisms of $\mathbb{A}^1$-fundamental groups are known (see [Morel 2012, §7]):

$$\pi_1^{\mathbb{A}^1}(\mathrm{SL}_2(F)) \cong \pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus \{0\}) \cong K_2^{\mathrm{MW}}(F).$$

Moreover, via the $\mathbb{A}^1$-Galois correspondence, the extension $\mathcal{E} : \widetilde{\mathrm{SL}}_2(F) \to \mathrm{SL}_2(F)$ is the universal covering constructed from a simplicial scheme. Accordingly, the value $\tilde{f}(\mathfrak{l}_i) \in K_2^{\mathrm{MW}}(F)$ can be interpreted as a lift of the covering. We refer the reader to [Morel 2012] for more properties of Milnor–Witt K-theory.

4. Some computations of the Milnor–Witt K_2-group

In preparation for computing the K_2 invariants, this section analyses $K_2^{\mathrm{MW}}(F)$ quantitatively. The key here is a result of Suslin [1987]. To explain this, we will review the two groups $K_2^M(F)$ and $I^2(F)$.

First, let us review the Matsumoto theorem on the Milnor K_2-group $K_2^M(F)$. It says that this $K_2^M(F)$ is the quotient group generated by (Steinberg) symbols $\{x, y\}$ with $x, y \in F^\times$ subject to the relations

$$\{a, bc\} = \{a, b\} + \{a, c\}, \quad \{ab, c\} = \{a, c\} + \{b, c\} \quad \text{for all } a, b, c \in F^\times,$$
$$\{a, 1-a\} = 0 \quad \text{for all } a \in F^\times \setminus \{1\}.$$

Formally, $K_2^M(F)$ can be also presented as the multiplicative group

$$F^\times \otimes_{\mathbb{Z}} F^\times / \big\langle a \otimes (1-a) \mid a \in F^\times \setminus \{1\} \big\rangle.$$

Furthermore, as is known, the correspondence $[a, b] \mapsto \{a, b\}$ defines an epimorphism $\mu : K_2^{\mathrm{MW}}(F) \to K_2^M(F)$. Hence, any element of the form $\{x, -1\} \in K_2^M(F)$ is annihilated by 2. Actually, $2\{x, -1\} = \{x, 1\}$ comes from $[x, 1] = 0 \in K_2^{\mathrm{MW}}(F)$.

Next, let $\mathrm{WG}(F)$ be the Witt–Grothendieck ring of F, that is, the Grothendieck ring of isometric classes of all quadratic forms of finite dimension (see, e.g., [Lam 2005, Chapter II] for the definition). For $a \in F^\times$, let us denote by the symbol $\langle a \rangle$ the quadratic form ax^2 on F. Furthermore, let $I(F) \subset \mathrm{WG}(F)$ denote the augmentation ideal, i.e., $I(F) := \mathrm{Ker}(\mathrm{WG}(F) \to \mathbb{Z})$. Note (see [Suslin 1987, §6]) that the homomorphism

$$\nu : K_2^{\mathrm{MW}}(F) \to I^2(F), \quad [a, b] \mapsto (\langle 1 \rangle - \langle a \rangle)(\langle 1 \rangle - \langle b \rangle),$$

induces the homomorphism $\xi : K_2^M(F) \to I^2(F)/I^3(F)$, called *the Milnor map*.

Suslin [1987] showed that the above homomorphisms provide a pullback diagram

$$\begin{tikzcd} K_2^{\mathrm{MW}}(F) \arrow[r, "\mu"] \arrow[d, "\nu"'] & K_2^M(F) \arrow[d, "\text{Milnor map } \xi"] \\ I^2(F) \arrow[r, "\text{projection}"] & I^2/I^3(F) \end{tikzcd} \tag{4.1}$$

See [Hutchinson and Tao 2008] for another proof. We should make some remarks on this diagram. It is known (see [Lam 2005, §V.6; Weibel 2013, Theorem 7.9]) that the Milnor map induces an isomorphism $K_2^M(F)/2 \cong I^2/I^3(F)$. In particular, the quotient $I^2/I^3(F)$ is an elementary abelian 2-group. Hence, for any prime $l \neq 2$, the pullback localized at l means a direct product. Furthermore, it is known (see [Kramer and Tent 2010] and references therein) that the composite 2-cocycle $\nu \circ \theta_{\mathrm{uni}} : \mathrm{SL}_2(F)^2 \to K_2^{\mathrm{MW}}(F) \to I^2(F)$ coincides with “a Maslov 2-cocycle”.

Next, we mention the Merkujev–Suslin theorem, which deals with torsion parts of the Milnor groups of F; see, e.g., [Weibel 2013]. It says that if F contains a primitive m-th root of unity then "the Galois symbol" gives isomorphisms

$$K_2^M(F)/m \cong H^2_{\text{ét}}(\mathrm{Spec}(F); \mu_m^{\otimes 2}) \cong {}_m\mathrm{Br}(F).$$

Here, the last term ${}_m\mathrm{Br}(F)$ is the set of elements in the Brauer group $\mathrm{Br}(F)$ that are of order m. The original proof of the theorem can be outlined as a reduction to a discussion of the algebraic closure in F of some algebraic subfields. Furthermore, we should remark that the K_2-group of the algebraic closure $\overline{\mathbb{Q}}$ is known to be zero, i.e., $K_2^M(\overline{\mathbb{Q}}) \cong 0$. In particular, the map $K_2^M(F) \to K_2^M(\mathbb{C})$ induced from any complex embedding $F \to \mathbb{C}$ of a number field is zero.[4] In summary, to study the torsion $K_2^M(F)/m$, it is natural to assume that F is a number field, i.e., a finite extension field of $\mathbb{Q}$.

Accordingly, we will restrict ourselves to discussing number fields F. Let r_1 be the number of real embeddings of F and let $\mathrm{Spm}(\mathcal{O}_F)$ be the set of finite primes in the algebraic integer $\mathcal{O}_F$. We first write the localization sequence of the Milnor groups (see [Weibel 2013, §III.6]):

$$0 \to K_2^M(\mathcal{O}_F) \xrightarrow{i_*} K_2^M(F) \xrightarrow{\partial} \bigoplus_{\mathfrak{p} \in \mathrm{Spm}(\mathcal{O}_F)} k(\mathfrak{p})^\times \to 0 \quad \text{(exact)}. \tag{4.2}$$

Here, the symbol i denotes the inclusion $\mathcal{O}_F \hookrightarrow F$ and ∂ is the sum of tame symbols associated with all primes $\mathfrak{p} \in \mathrm{Spm}(\mathcal{O}_F)$. Note further that the tame kernel $K_2^M(\mathcal{O}_F)$ is known to be of finite order. Hence, any element of $K_2^M(F)$ is of finite order.

On the other hand, for the study of the squared ideal $I^2(F)$ in (4.1), consider the sum of all completions $\Upsilon : F \to \mathbb{R}^{r_1} \oplus \big(\bigoplus_{\mathfrak{p} \in \mathrm{Spm}(\mathcal{O}_F)} \mathbb{Q}_\mathfrak{p}\big)$. The induced map on $I^2(\bullet)$ is known to be injective because of the Hasse–Minkowski principle [Lam 2005, §VI.3]. Furthermore, concerning the quotient $I^2/I^3(\bullet)$, the sum Υ yields an exact sequence

$$0 \to I^2/I^3(F) \xrightarrow{\Upsilon_*} (I^2/I^3(\mathbb{R}))^{r_1} \oplus \bigoplus_{\mathfrak{p} \in \mathrm{Spm}(\mathcal{O}_F)} I^2(\mathbb{Q}_\mathfrak{p}) \longrightarrow \mathbb{Z}/2 \to 0 \tag{4.3}$$

which is known as uniqueness of the Hilbert reciprocity. Here, we should note (see [Lam 2005, §VI.2]) that each $I^2(\mathbb{Q}_\mathfrak{p})$ is annihilated by 2, that $I^3(\mathbb{Q}_\mathfrak{p}) = 0$, and that $I^2(\mathbb{R}) \cong 4\mathbb{Z}$. Hence, $I^2(F)$ turns out to be a sum of $\mathbb{Z}^{r_1}$ and some 2-elementary abelian groups. In particular, the pullback (4.1) above immediately leads to a lemma:

Lemma 4.4. *The kernel of the map $\mu : K_2^{\mathrm{MW}}(F) \to K_2^M(F)$ is isomorphic to $\mathbb{Z}^{r_1}$. As a special case, if $r_1 = 0$, then the isomorphism $K_2^{\mathrm{MW}}(F) \cong K_2^M(F)$ holds.*

[4] In contrast to K_1 and K_3, the maps $K_i(F) \to K_i(\mathbb{C})^{r_2}$ induced by the complex embeddings are injective for $i = 1, 3$ (see [Zickert 2009] for details).

Incidentally, the sequence (4.3) implies that the group $K_2^{\mathrm{MW}}(F)$ includes the main information about the metaplectic group defined to be a double cover of $\mathrm{SL}_2(F)$.

Example 4.5 ($F=\mathbb{Q}$). Finally, let us compute $K_2^{\mathrm{MW}}(\mathbb{Q})$ as an application of the above results. Note from a result of Tate (see [Weibel 2013, §III.6.3] or [Milnor 1971]) that the sequence (4.2) splits and there is an isomorphism $K_2^M(\mathbb{Z})\cong\mathbb{Z}/2$. Since $r_1=1$, a careful observation of the pullback diagram (4.1) leads to $K_2^{\mathrm{MW}}(\mathbb{Q})\cong\mathbb{Z}\oplus\bigoplus_p\mathbb{Z}/(p-1)$, where p ranges over all odd primes.

5. Computation of the K_2 invariants; hyperbolic links

We will compute the K_2 invariants of some links. This section assumes that the characteristics of fields are zero, for simplicity.

5A. ***Example*: *the figure-eight knot.*** Consider the figure-eight knot K_{4_1} of Figure 1. By the Wirtinger presentation of $\pi_1(\mathbb{R}^3\setminus K_{4_1})$, the group is formally generated by the arcs α_i. Precisely, by definition,

$$\pi_1(\mathbb{R}^3\setminus K_{4_1})\cong\big\langle \mathfrak{m}_{\alpha_1},\mathfrak{m}_{\alpha_2},\mathfrak{m}_{\alpha_3},\mathfrak{m}_{\alpha_4}\mid \mathfrak{m}_{\alpha_3}=\mathfrak{m}_{\alpha_2}^{-1}\mathfrak{m}_{\alpha_1}\mathfrak{m}_{\alpha_2}=\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_4}\mathfrak{m}_{\alpha_1},\ \mathfrak{m}_{\alpha_2}=\mathfrak{m}_{\alpha_4}^{-1}\mathfrak{m}_{\alpha_1}\mathfrak{m}_{\alpha_4}=\mathfrak{m}_{\alpha_3}^{-1}\mathfrak{m}_{\alpha_4}\mathfrak{m}_{\alpha_3}\big\rangle.$$

Then, we can easily see that the following assignment yields an SL_2 representation $f:\pi_1(\mathbb{R}^3\setminus K_{4_1})\to\mathrm{SL}_2(F)$ if and only if $x^2\pm x+1=0$:

$$f(\mathfrak{m}_{\alpha_1}):=\begin{pmatrix}1&0\\-1&1\end{pmatrix},\quad f(\mathfrak{m}_{\alpha_3}):=\begin{pmatrix}x&(x-1)^2\\-1&2-x\end{pmatrix},$$
$$f(\mathfrak{m}_{\alpha_2}):=\begin{pmatrix}1&x^2\\0&1\end{pmatrix},\quad f(\mathfrak{m}_{\alpha_4}):=\begin{pmatrix}1-x+x^2&(x-1)^2\\-x^2&1+x-x^2\end{pmatrix}.$$

Moreover, according to Proposition 7.3, it can be seen that every parabolic representation turns out to be this f, up to conjugacy. Thus, it is sensible to consider the quadratic field $\mathbb{Q}(\sqrt{-3})=\mathbb{Q}[x]/(x^2\pm x+1)$.

Thus, we set $F=\mathbb{Q}(\sqrt{-3})$, and compute the K_2 invariant of f. Note that the preferred longitude $\mathfrak{l}$ forms

$$\mathfrak{l}=\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_2}\mathfrak{m}_{\alpha_3}\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_4}^{-1}\mathfrak{m}_{\alpha_3}\mathfrak{m}_{\alpha_2}\mathfrak{m}_{\alpha_4}^{-1}=\mathfrak{m}_{\alpha_2}\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_3}\mathfrak{m}_{\alpha_4}^{-1}\in\pi_1(\mathbb{R}^3\setminus K_{4_1}).$$

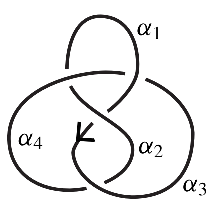

Figure 1. The figure-eight knot K_{4_1} with four arcs.

Here, $K_2^{\mathrm{MW}}(F) = K_2^M(F)$ by Lemma 4.4 with $r_1 = 0$. Hence, from the definitions of K_2^{MW} and the 2-cocycle θ_{uni}, we can compute the K_2 invariant as

$$\begin{aligned}
P_{K_2} \circ \tilde{f}(\mathfrak{l}) &= \theta_{\mathrm{uni}}\big(\big(\tilde{f}(\mathfrak{m}_{\alpha_2}),\, \tilde{f}(\mathfrak{m}_{\alpha_1}^{-1})\big) + \big(\tilde{f}(\mathfrak{m}_{\alpha_2}\mathfrak{m}_{\alpha_1}^{-1}),\, \tilde{f}(\mathfrak{m}_{\alpha_3})\big) \\
&\qquad\qquad + \big(\tilde{f}(\mathfrak{m}_{\alpha_2}\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_3}),\, \tilde{f}(\mathfrak{m}_{\alpha_4}^{-1})\big)\big) \\
&= (\{1,-1\} - \{1,1\}) + (\{x^2,1\} - \{1,-1\}) + (\{2+4x^2,-1\} - \{x^2,x^2\}) \\
&= \{2+4x^2,-1\} - \{x^2,-1\} \\
&= \{(2+4x^2)x^2,-1\} = \{-2-x^2,-1\} \in K_2^M(\mathbb{Q}(\sqrt{-3})).
\end{aligned}$$

Further, let us analyse this $\{-2-x^2,-1\}$ in $K_2^M(\mathbb{Q}(\sqrt{-3}))$. Since the tame kernel $K_2^M(\mathcal{O}_{\mathbb{Q}(\sqrt{-3})})$ is known to be zero (Tate), the sequence (4.2) means that the sum ∂ is an isomorphism. Furthermore, for any prime $\mathfrak{p} \in \mathrm{Spm}(\mathcal{O}_F)$, the tame symbol $\partial_{\mathfrak{p}}(-2-x^2,-1)$ equals $(-1)^{v_{\mathfrak{p}}(-2-x^2)} \in k(\mathfrak{p})$ by definition. Since $(2+x^2)(x^2-1) = -3$ and $2+x^2$ and x^2-1 are prime elements over 3, we can conclude the following:

Proposition 5.1. *Let $F = \mathbb{Q}(\sqrt{-3})$. Then $\partial_{(x^2-1)} \oplus \partial_{(2+x^2)}(\{-2-x^2,-1\}) = (-1,-1) \in (\mathbb{F}_3^\times)^2$, and for any other prime $\mathfrak{p}$ we have $\partial_{\mathfrak{p}}(\{-2-x^2,-1\}) = 1$.*

In summary, the K_2 invariant $\tilde{f}(\mathfrak{l})$ in $K_2^M(\mathbb{Q}(\sqrt{-3}))$ turns out to be nontrivial by means of the tame symbols, whereas the representation f factors through the algebraic integer $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$.

5B. ***Other links.*** Next, let us discuss other links. Here, notice from Lemma 4.4 that it is relatively easy to compute the kernel (which is isomorphic to $\mathbb{Z}^{r_1}$) of $\mu : K_2^{\mathrm{MW}}(F) \to K_2^M(F)$. Thus, this subsection is specialized to some parabolic representations and gives in Table 1 a list of these values $\mu(\tilde{f}(\mathfrak{l}))$ without performing detailed computations (see [Maclachlan and Reid 2003, Appendix 13.3] for the defining polynomials).

In each case, by $F/\mathbb{Q}$ we mean the minimal field extension that splits the defining polynomial. Furthermore, we can see that the class numbers of the splitting fields vanish; we can easily study prime ideals in $\mathcal{O}_F$ and compute the associated valuations $\partial_{\mathfrak{p}}$. For example, the tame symbols at the primes (x^2+1) and (x^2+2) distinguish the K_2-values of the 6_1-knot from one of the 7_7-knot, whereas the defining polynomials are equal. In doing so, we can find further examples of nontrivial K_2 invariants of other links; however, it remains a problem for the future to clarify the topological and arithmetic features reflected in the K_2 invariants.

Finally, let us briefly comment on the K_2 invariants of hyperbolic small links with $\#L > 1$. As seen in [Baker 2001], we find many holonomies contained in $\mathrm{SL}_2(F)$ with some quadratic fields F; we can easily compute the longitudes of such holonomies, since the finite primes of F and the tame kernel $\mathcal{O}_F$ have been

knot	defining polynomial	r_1	$\mu(\tilde{f}(\mathfrak{l})) \in K_2^M(F)$
3_1	x^2-1	1	$\{3,-1\}$
5_1	x^4+3x^2+1	2	$\{x^2+2,-1\}+\left\{\frac{1}{10}(9-8x^2),\,-21+10x^2\right\}$
5_2	$1-2x^2+x^4-x^6$	1	$\{2(1+x^4)(5+3x^4),-1\}$
6_1	$1+x^2+3x^4+2x^6+x^8$	0	$\{x^2+2,-1\}$
7_7	$1+x^2+3x^4+2x^6+x^8$	0	$\left\{\frac{2(2+6x^2+4x^4+x^6)}{3+2x^2+3x^4+x^6},\,1+2x^2+x^4\right\}$ $+\left\{-1+4x^2+7x^4+4x^6,\,\frac{-2+8x^2+4x^4+4x^6}{-2-x^2-x^4}\right\}$ $+\{-3-2x^2-3x^4-x^6,\,2x^2+2x^4+x^6\}$

Table 1. Values of $\mu(\tilde{f}(\mathfrak{l}))$ for some defining polynomials of knots.

well studied (see [Keune 1989; Weibel 2013], for example). However, we remark that, concerning the Whitehead and the Borromean links as the simplest examples, these $\tilde{f}(\mathfrak{l}_i)$ are trivial, unfortunately.

6. Two applications

This paper aims to applications of K_2-groups to low-dimensional topology. This section furthermore gives two applications, although these results are a bit tangential. In this section, although we roughly review some notions in knot theory, we refer the reader to [Lickorish 1997, §1 and §11] or [Maclachlan and Reid 2003] for detailed definitions.

6A. ***On the cusp shape.*** While we discuss the K_2invariant in Sections 3–5, we will focus on another summand F in $F \times \widetilde{K}_2^{\mathrm{MW}}(F)$. The value $\tilde{f}(\mathfrak{l}_i)$ restricted on this F is called *the cusp shape* as an important concept in hyperbolic geometry; see, e.g., [Maclachlan and Reid 2003]. We give a sum formula of the cusp shape.

To state Theorem 6.1, we introduce some terminology. Fix a parabolic representation $f : \pi_1(\mathbb{R}^3 \setminus L) \to \mathrm{SL}_2(F)$, and a link diagram D of L. Roughly speaking, as seen in Figure 2, D is the image $p(L) \subset \mathbb{R}^2$ with over–under information, where p is a "generic" projection $p : \mathbb{R}^3 \to \mathbb{R}^2$. Then, we can consider the over-arcs $\alpha_1, \alpha_2, \ldots, \alpha_{N_j}$ along the orientation of the longitude $\mathfrak{l}_j$ as illustrated in Figure 2. Let β_i be the arc which divides α_{i-1} and α_i, and $\epsilon_i \in \{\pm 1\}$ be the sign of the crossing between α_i and β_i, according to Figure 4 (see Section 7A). We denote a loop circling around an arc α by $\mathfrak{m}_\alpha$. As is known from the Wirtinger presentation, every $\mathfrak{m}_\alpha$ is conjugate to some meridian in $\pi_1(\mathbb{R}^3 \setminus L)$. Hence, by parabolicity of f, it can seen, as in (7.1), that any arc α uniquely, up to sign,

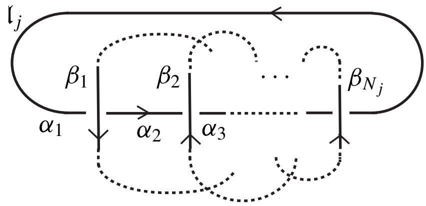

Figure 2. The longitude $\mathfrak{l}_j$ and arcs α_i and β_i in the diagram D.

admits $(c_\alpha, d_\alpha) \in F \times F \setminus \{(0,0)\}$ such that

$$f(\mathfrak{m}_\alpha) = \begin{pmatrix} 1 + c_\alpha d & d_\alpha^2 \\ -c_\alpha^2 & 1 - c_\alpha d_\alpha \end{pmatrix}.$$

Furthermore, we define a map $\mathcal{S} : (F \times F \setminus \{(0,0)\})^2 \to F$ by setting

$$\mathcal{S}((a,b),(c,d)) := \begin{cases} -1 + c^2/(a^2 - abc^2 + a^2cd) & \text{if } a(bc^2 - a - acd) \neq 0, \\ -1 + (-c^2 + c^3 d)/a^2, & \text{if } a \neq 0,\ bc^2 - a - acd = 0, \\ -1 + (-1 - cd)/b^2c^2, & \text{if } a = 0,\ c \neq 0, \\ -1 + d^2/b^2, & \text{if } a = c = 0. \end{cases}$$

We now analyse the sum $\sum_{i=1}^{N_j} \epsilon_i \cdot \mathcal{S}\big((c_{\alpha_i}, d_{\alpha_i}), (c_{\beta_i}, d_{\beta_i})\big) \in F$, as follows:

Theorem 6.1. *The sum coincides with the cusp shape $P_F \circ \tilde{f}(\mathfrak{l}_j)$ in F, where P_F is the projection $\widetilde{K}_2^{\mathrm{MW}}(F) \times F \to F$.*

The proof will appear in Section 7D; the point here is that the sum formula is independent of the order of the crossings, while the longitudes seem to be noncommutative. Moreover, it is interesting and applicable to computations that we need not describe the longitude $\mathfrak{l}_i$ in the formula, with $\mathfrak{l}_i$ complicated, as in (6.5).

6B. ***Another application*: *the real $K_2(\mathbb{R})$ and left-orderable 3-manifold groups.*** This section focuses on the real case $F = \mathbb{R}$ and compares the K_2-group $K_2^{\mathrm{MW}}(\mathbb{R})$ with $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$, where $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ is the topological universal cover of $\mathrm{SL}_2(\mathbb{R})$ associated with $\pi_1(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{Z}$. As an application, we give a formula to compute longitudes lifted to $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$. We hope that this computation will be useful for studying the left-orderability of 3-manifold groups (see [Boyer et al. 2013], for example). In fact, we give new 3-manifold groups which are left-orderable.

We now explain Proposition 6.3, which strictly describes $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$. Consider the map $\mathrm{Sign} : \mathbb{R}^2 \to \mathbb{Z}$ defined by $\mathrm{Sign}(a,b) = 1$ if $a < 0$ and $b < 0$, and $\mathrm{Sign}(a,b) = 0$ otherwise. Recalling the 2-cocycle θ_{uni} in (2.2), we equip $\mathbb{Z} \times \mathrm{SL}_2(\mathbb{R})$ with the group operation

$$(n,g) \cdot (m,h) := (n + m + \mathrm{Sign} \circ \theta_{\mathrm{uni}}(g,h), gh). \tag{6.2}$$

Proposition 6.3. *This group structure on* $\mathbb{Z} \times \mathrm{SL}_2(\mathbb{R})$ *is isomorphic to the universal cover* $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ (forgetting the topology, of course).

Here, we should emphasize that this result is simpler than the known formula for the group operation on $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$, because it was formulated using logarithms (see [Bargmann 1947] for details).

Proof. First, let us compute $K_2^{\mathrm{MW}}(\mathbb{R})$. Since $I(F)$ for any algebraically closed F is known to be zero, we obtain $K_2^{\mathrm{MW}}(F) \cong K_2^M(F)$ from the pullback diagram (4.1). Moreover, it is known (see [Weibel 2013, Theorem III.6.4 and Application III.6.8.3]) that $K_2^M(F)$ is of uncountable cardinality and is a uniquely divisible group, i.e., a $\mathbb{Q}$-vector space, and that an isomorphism $K_2^M(\mathbb{R}) \cong \mathbb{Z}/2 \oplus K_2^M(\mathbb{C})^+$ is obtained as a corollary of Hilbert's Theorem 90. Here the first summand $\mathbb{Z}/2$ is widely known to be generated by the (Steinberg) symbol $\{-1, -1\}$ and the second one is the invariant subspace by complex conjugation. Recalling from Section 4 that $I^2(\mathbb{R}) \cong 4\mathbb{Z}$ is generated by $(\langle 1\rangle - \langle -a^2\rangle)^2$ with $a \in \mathbb{R}$, the pullback diagram (4.1) implies that

$$K_2^{\mathrm{MW}}(\mathbb{R}) \cong \mathbb{Z} \oplus K_2^M(\mathbb{C})^+. \tag{6.4}$$

Here, notice that the induced homomorphism $\mathrm{Sign}_* : K_2^{\mathrm{MW}}(\mathbb{R}) \to \mathbb{Z}$ from $\mathrm{Sign} : \mathbb{R}^2 \to \mathbb{Z}$ coincides, by construction, with the projection in the decomposition (6.4).

Finally, we complete the proof. Since the cover $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ is a central extension of $\mathrm{SL}_2(\mathbb{R})$ with fiber $\mathbb{Z}$, the universal extension $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ surjects onto $\widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$. By noticing the isomorphism (6.4) and that every quotient of a divisible group is divisible, the central kernel is $K_2^M(\mathbb{C})^+$. Hence, the surjection to the group (6.2) induces the desired isomorphism. □

Thanks to Proposition 6.3, given an $\tilde{f} : \pi_1(\mathbb{R}^3 \setminus K) \to \widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ we can compute the value $P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l}))$ of the longitude $\mathfrak{l}$. This section will give an application (Proposition 6.6 and Theorem 6.7). Throughout this subsection, we will denote by $P_{\mathbb{Z}}$ the set-theoretic projection $\mathbb{Z} \times \mathrm{SL}_2(\mathbb{R}) \to \mathbb{Z}$.

First, let us comment on some known results. Note that the connection between the summand $\mathbb{Z}$ in (6.4) and the Euler classes of $U(1)$-bundles over surfaces is well-understood (see [Wood 1971]). For example, the Milnor–Wood inequality gives an estimate of the value $P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l})) \in \mathbb{Z}$ bounded by the Seifert genus $g(K)$ of a knot K. Precisely, since the longitude forms a product of $g(K)$ elements in the commutator subgroup $\pi_1(\mathbb{R}^3 \setminus K)'$, we have $|P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l}))| \leq g(K) - \frac{1}{4}$; see [Wood 1971, (5.5)]. As a corollary, for any knot K of Seifert genus one, the value $P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l}))$ is zero (this result was crucial in [Boyer et al. 2013; Hakamata and Teragaito 2014; Tran 2015]). However, no value $P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l}))$ with respect to knots K of Seifert genus > 1 has been computed so far.

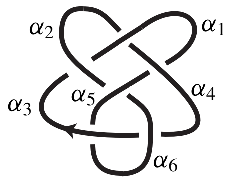

Figure 3. The diagram D of the knot 6_2.

As the nontorus knot of Seifert genus > 1 and of the minimal crossing number, we will focus on the 6_2-knot K. The diagram D with arcs $\alpha_1, \ldots, \alpha_6$ is illustrated in Figure 3.

Inspired by a method in [Hakamata and Teragaito 2014; Tran 2015], we will find elliptic homomorphisms $f : \pi_1(\mathbb{R}^3 \setminus K) \to \mathrm{SL}_2(\mathbb{R})$ such that

$$f(\mathfrak{m}_{\alpha_1}) = \begin{pmatrix} \sqrt{t} & \sqrt{t} \\ 0 & \sqrt{t}^{-1} \end{pmatrix}, \quad f(\mathfrak{m}_{\alpha_2}) = \begin{pmatrix} \sqrt{t} & 0 \\ -s\sqrt{t}^{-1} & \sqrt{t}^{-1} \end{pmatrix} \quad \text{for some } s, t \in \mathbb{R}.$$

Moreover, we set $T = t + t^{-1}$. Then, from the Wirtinger presentation, we can easily see that s and t must satisfy the equation $R_{6_2}(s, T) = 0$, where $R_{6_2}(s, T)$ is the polynomial

$$1 + 3s + s^2 + 2s^3 + 3s^4 + s^5 - (3 + 2s + 4s^2 + 9s^3 + 4s^4)T \\ + (1 + 2s + 9s^2 + 6s^3)T^2 - (3s + 4s^2)T^3 + sT^4.$$

Owing to this quartic equation with respect to $T = t + t^{-1}$, this t can be formulated as an algebraic function of s. Here, suppose a (unique) positive solution $s_0 = 1.48288\ldots$ for which the discriminant $\Delta(s)$ of $R_{6_2}(s, t)$ with respect to t is zero. Then, following the quartic formula for T, if $0 < s < s_0$ (resp. $s_0 < s < 200$), there are two (resp. four) real solutions $t \in \mathbb{R}_{>0}$ of the equation $R_{6_2}(s, t + t^{-1}) = 0$.[5] Choose the two solutions which are smallest and denote them by $t_{\min}$ and $t_{\sec}$. We denote by $f_{s,t}$ the resulting homomorphism $\pi_1(\mathbb{R}^3 \setminus K) \to \mathrm{SL}_2(\mathbb{R})$, and denote by $\tilde{f}_{s,t} : \pi_1(\mathbb{R}^3 \setminus K) \to \mathrm{SL}_2^{\mathrm{top}}(\mathbb{R})$ the lift of $f_{s,t}$.

We will compute the resulting value $P_{\mathbb{Z}}(\tilde{f}_{s,t}(\mathfrak{l}))$, where we will use a longitude $\mathfrak{l}$ of the form

$$\mathfrak{l} = \mathfrak{m}_{\alpha_1}\mathfrak{m}_{\alpha_4}^{-1}\mathfrak{m}_{\alpha_3}\mathfrak{m}_{\alpha_5}^{-1}(\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_6}\mathfrak{m}_{\alpha_1})(\mathfrak{m}_{\alpha_1}^{-1}\mathfrak{m}_{\alpha_2}^{-1}\mathfrak{m}_{\alpha_1}) \in \pi_1(\mathbb{R}^3 \setminus K_{6_2}). \tag{6.5}$$

Hence, according to (6.2), we can formulate the value $P_{\mathbb{Z}}(\tilde{f}_{s,t}(\mathfrak{l}))$ as a function of s. By definition, the function is upper semicontinuous with respect to s. Furthermore, it is possible to list all the (finitely many) noncontinuous points of $P_{\mathbb{Z}}(\tilde{f}_{s,t}(\mathfrak{l}))$ for a given interval in $\mathbb{R}$. Here, we focus on the interval $[0, 200]$. Then, with the help of

[5] Incidentally, if $3000 < s$, we have eight real solutions $t \in \mathbb{R}_{>0}$ of the equation $R_{6_2}(s, T) = 0$.

a computer, we can investigate noncontinuous points in the interval (here we use the above quartic formula), and hence get the following conclusion:

Proposition 6.6. *For $s > 0$, let $t_{\min}$ and $t_{\sec}$ be the above solutions of $R_{6_2}(s,t) = 0$. Then the value $P_{\mathbb{Z}}(\tilde{f}_{s,t_{\min}}(\mathfrak{l}))$ is 0 if $0 < s < 200$, while $P_{\mathbb{Z}}(\tilde{f}_{s,t_{\sec}}(\mathfrak{l}))$ is 1 if $s_0 < s < 200$.*

It is worth noting that, by a computer program, if $2700 < s < 2900$, the value $P_{\mathbb{Z}}(\tilde{f}_{s,t_{\min}}(\mathfrak{l}))$ is 1; hence Proposition 6.6 does not hold for any $s > s_0$. However, we emphasize that it is the first to discover infinitely many homomorphisms $\tilde{f}_{s,t}$ such that the values $P_{\mathbb{Z}}(\tilde{f}_{s,t}(\mathfrak{l}))$ are not zero, and that it seems to be hard to compute the value $P_{\mathbb{Z}}(\tilde{f}(\mathfrak{l}))$ for general knots K.

Finally, we give an application using the ideas of [Boyer et al. 2013; Hakamata and Teragaito 2014]. It is known [Boyer et al. 2013] that, if an irreducible closed 3-manifold M admits a nontrivial homomorphism $\pi_1(M) \to \widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$, then M has left-orderable fundamental group. Here, a group G is *left-orderable* if it has a total order $\le$ such that $g, x, y \in G$ with $x \le y$ implies $gx \le gy$. Based on their ideas, we will show the following:

Theorem 6.7. *Let $r = p/q \in \mathbb{Q}$. Let $M_r(K)$ be the closed 3-manifold obtained by r-Dehn surgery along the 6_2-knot K. If $0.1 < r < 7.99$, then the fundamental group $\pi_1(M_r(K))$ is left-orderable.*

Proof. We will construct a nontrivial homomorphism $f : \pi_1(\mathbb{R}^3 \setminus K) \to \widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ which sends $\mathfrak{m}_{\alpha_1}^p \mathfrak{l}^q$ to the identity. Here note that the 3-manifold $M_r(K)$ obtained from the 2-bridge knot 6_2 is known to be irreducible. If we have such a map, the van Kampen theorem admits the induced map $\pi_1(M_r(K)) \to \widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$ and, hence, gives the desired left-orderability.

The construction of f is as follows: First notice that the commutator subgroup of $f_{s,t}(\mathfrak{m}_{\alpha_1})$ forms

$$\left\{ \begin{pmatrix} u & (u-u^{-1})/(1-t^{-2}) \\ 0 & u^{-1} \end{pmatrix} \,\middle|\, u \in \mathbb{R}^\times \right\}$$

without $t^2 = 1$. Hence, by the definition of $\mathfrak{l}$, we can see that $f_{s,t}(\mathfrak{l}) \in \mathrm{SL}_2(\mathbb{R})$ is of the form

$$\begin{pmatrix} g(s,t) & * \\ 0 & g(s,t)^{-1} \end{pmatrix}$$

for some $* \in \mathbb{R}$, where $g(s,t)$ is a polynomial in s of the form

$$\big(1 - 2t + t^2 - 2t^4 + t^5 + s(4t^2 - 3t - t^3 - 2t^4 + 3t^5 - 2t^6 + t^7) \\ + s^2(3t^2 - 2t^3 - t^4 + 2t^5 - 2t^6) + s^3(t^5 - t^3)\big)/t^2.$$

Since the commutator subgroup is isomorphic to $\mathbb{R}^\times$, the equality $f(\mathfrak{m}_{\alpha_1})^p f(\mathfrak{l})^q = \mathrm{id}_{\mathbb{R}^2} \in \mathrm{SL}_2(\mathbb{R})$ holds if and only if $t^{-p/2} = g(s,t)^q$. To solve this, we consider the function $\mathcal{R}: [0, 100] \to [0, \infty)$ defined by $\mathcal{R}(s) := 2\log(g(s, t_{\min}))/\log(t_{\min})$. Here we note the estimate $\mathcal{R}(10^{-4}) < 10^{-1}$ and $\mathcal{R}(10^2) > 7.99$, which are obtained from a computer program. Since this $\mathcal{R}$ is continuous by construction, the image of $\mathcal{R}$ includes the interval $[0.1, 7.99]$. To summarize, for $10^{-1} < r < 7.99$ there are s and $t_{\min}$ with $0 < s < 100$ which ensure a homomorphism $f_{s,t_{\min}}$ that sends $\mathfrak{m}_{\alpha_1}^p \mathfrak{l}^q$ to the identity in $\mathrm{SL}_2(\mathbb{R})$.

Moreover, we consider a lifted $\tilde{f}_{s,t_{\min}} : \pi_1(\mathbb{R}^3 \setminus K) \to \widetilde{\mathrm{SL}}_2^{\mathrm{top}}(\mathbb{R})$. By Proposition 6.6, we have $P_{\mathbb{Z}}(\tilde{f}_{s,t_{\min}}(\mathfrak{m}_{\alpha_1})) = P_{\mathbb{Z}}(\tilde{f}_{s,t_{\min}}(\mathfrak{l})) = 0$. Hence this lift is one of the required maps. $\square$

It is well known (see [Boyer et al. 2013], for example) that the resulting 3-manifold, $M_r(K)$, of r-surgery on any 2-bridge knot K is not an L-space, i.e., the Heegaard Floer homology of $M_r(K)$ is not isomorphic to that of the lens space $L(p, q)$ for any $(p, q) \in \mathbb{Z}^2$. Theorem 6.7 is supporting evidence for a conjecture in [Boyer et al. 2013], which predicts an equivalence between L-spaces and the left-orderability. As seen in the proof above, we hope that our computation will be applicable to other knots of genus > 1.

7. Proofs of the theorems

We will prove the theorems from Sections 2 and 6. For this, this section employs an approach to obtaining parabolic representations by means of quandles. This approach, using quandle, has some benefits: first, while $\mathrm{SL}_2(F)$ is of dimension 3 over F, the approach can deal with parabolic representations from a certain 2-dimensional object $(\mathbb{A}^2 \setminus 0)/\{\pm\}$; see Proposition 7.3 (in contrast to [Riley 1972] in a group-theoretic approach). Furthermore, the results of [Carter et al. 2005; Eisermann 2014; Nosaka 2015] in quandle theory gave some topological applications; here the point is that quandle theory sometimes ensures nontriviality of some knot invariants and makes a reduction to knot diagrams without 3-dimensional discussion of $\mathbb{R}^3 \setminus L$. Correspondingly, we will see that our setting of $\mathrm{SL}_2(F)$ satisfies conditions necessary to the results, and will give the proofs of Theorems 3.4 and 6.1.

7A. ***Parabolic representations in terms of quandles.*** Let us begin by reviewing quandles. A *quandle* [Joyce 1982] is a set, X, with a binary operation $\lhd : X \times X \to X$ such that

(I) $a \lhd a = a$ for any $a \in X$;

(II) the map $(\bullet \lhd a) : X \to X$ defined by $x \mapsto x \lhd a$ is bijective for any $a \in X$;

(III) $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$ for any $a, b, c \in X$.

A map $f : X \to Y$ between quandles is a (*quandle*) *homomorphism* if $f(a \lhd b) = f(a) \lhd f(b)$ for any $a, b \in X$. For example, any group G is a quandle with the

Figure 4. Positive and negative crossings.

conjugacy operation $x \lhd y := y^{-1}xy$ for any $x, y \in G$, and is called *the conjugacy quandle* in G and denoted by $\mathrm{Conj}(G)$. Furthermore, given an infinite field F, consider the quotient set $F^2 \setminus \{(0,0)\}/\sim$ subject to the relation $(a,b) \sim (-a,-b)$, and equip this set with the quandle operation

$$(a \;\; b) \lhd (c \;\; d) = (a \;\; b) \begin{pmatrix} 1+cd & d^2 \\ -c^2 & 1-cd \end{pmatrix}.$$

This quandle in the case $F = \mathbb{C}$ was introduced in [Inoue and Kabaya 2014, §5], which refers to it as a *parabolic quandle* (over F) and denotes it by X_F. Furthermore, consider the map

$$\begin{aligned} &\iota : X_F \to \mathrm{SL}_2(F), \\ &(c,d) \mapsto \begin{pmatrix} 1+cd & d^2 \\ -c^2 & 1-cd \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned} \tag{7.1}$$

We can easily see that this ι is injective and a quandle homomorphism, and the image is the conjugacy class of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Hence, the quandle X_F is a subquandle composed of parabolic elements of the conjugacy quandle in $\mathrm{SL}_2(F)$ (furthermore, it is a subquandle in $\mathrm{PSL}_2(F)$).

Next, we will review X-colorings. Let X be a quandle and D be an oriented link diagram of a link $L \subset S^3$. An *X-coloring* of D is a map $\mathcal{C} : \{\text{arcs of } D\} \to X$ such that $\mathcal{C}(\gamma_k) = \mathcal{C}(\gamma_i) \lhd \mathcal{C}(\gamma_j)$ at each crossing of D as in Figure 4.

For example, when X is the conjugacy quandle of a group G, the coloring condition coincides with the relations in the Wirtinger presentation of a link L. Hence, we have a bijection

$$\mathrm{Col}_{\mathrm{Conj}(G)}(D) \xleftrightarrow{1:1} \mathrm{Hom}_{\mathrm{gr}}(\pi_1(\mathbb{R}^3 \setminus L), G). \tag{7.2}$$

Next, let us focus on colorings with respect to the parabolic quandles X_F over fields F. Since X_F is a conjugacy class of $\mathrm{SL}_2(F)$ via (7.1), we can easily prove:

Proposition 7.3 (a special case of [Nosaka 2015, Corollary B.1]). *Let D be a diagram of a link L. Fix meridians $\mathfrak{m}_1, \dots, \mathfrak{m}_{\#L} \in \pi_1(\mathbb{R}^3 \setminus L)$ in each link-component which is compatible with the orientation of D. Then the restriction of (7.2) gives a bijection from the set $\mathrm{Col}_{X_F}(D)$ to the following set, composed of parabolic* *representations from* $\pi_1(\mathbb{R}^3 \setminus L)$:

$$\{f \in \mathrm{Hom}(\pi_1(\mathbb{R}^3 \setminus L), \mathrm{SL}_2(F)) \mid f(\mathfrak{m}_i) = \iota(x_i) \text{ for some } x_i \in X_F\}.$$

In particular, if L is a hyperbolic link and $F = \mathbb{C}$, the holonomy is regarded as a nontrivial $X_{\mathbb{C}}$-coloring in $\mathrm{Col}_{X_{\mathbb{C}}}(D)$ (see Appendix 13.3 of [Maclachlan and Reid 2003] for the hyperbolic knots of crossing number < 9).

We remark that it is very often (but not always) the case that the quotient set of $\mathrm{Col}_{X_F}(D)$ modulo conjugation in $\mathrm{SL}_2(F)$ is of finite order. In a special case, we will see that small knots satisfy finiteness (Proposition 7.4). Here, a knot K is said to be *small* if there is no incompressible surface except for a boundary-parallel torus in the knot exterior. For example, the 2-bridge knots and torus knots are known to be small.

Proposition 7.4. *Let F be a field embedded in the complex field $\mathbb{C}$. If D is a diagram of a small knot K, then the quotient set of* $\mathrm{Col}_{X_F}(D)$ *subject to the conjugacy operation of* $\mathrm{SL}_2(F)$ *is of finite order.*

We will omit the proof, since it follows from standard arguments in Culler–Shalen theory similar to those in [Culler and Shalen 1983] or [Cooper et al. 1994, Proposition 2.4].

Example 7.5. It is known that every knot of crossing number < 9 is small. Furthermore, we can see that the quotient set is bijective to $\{x \in F^\times/\{\pm 1\} \mid f(x)f(-x) = 0\}$ for some polynomial $f(x)$. Without proof, we list the defining polynomials of some knots for the case $\mathrm{Char}(F) = 0$ in Table 2.

7B. ***Proof of Proposition 3.1.*** From Proposition 7.3 and the definition of $K_2^{\mathrm{MW}}(F)$, we will prove Proposition 3.1.

Proof of Proposition 3.1. By definition of parabolicity, $f(\mathfrak{m})$ for every meridian $\mathfrak{m}$ is contained in the image of ι (recall Proposition 7.3), where ι is the map in (7.1).

knot	the defining polynomial $f(x)$
3_1	$x-1$
4_1	x^2-x+1
5_1	x^2+x-1
5_2	x^3-x^2+1
6_1	x^4+x^2-x+1
7_4	$(x^3+2x-1)(x^4-x^3+2x^2-2x+1)$
7_7	$(x^4+x^2-x+1)(x^6+x^5+2x^4+2x^3+2x^2+2x+1)$

Table 2. The defining polynomials for some knots.

Hence, from the Wirtinger presentation and Lemma 7.6 below, we can canonically obtain a lift $\tilde{f}:\pi_1(\mathbb{R}^3\setminus L)\to\widetilde{\mathrm{SL}}_2(F)$, defined by setting

$$\tilde{f}(\mathfrak{m})=(0,\,f(\mathfrak{m}))\in K_2^{\mathrm{MW}}(F)\times\mathrm{SL}_2(F).\qquad\square$$

Lemma 7.6. *Consider the composite $\theta_{\mathrm{uni}}\circ(\iota\times\iota):(X_F)^2\to K_2^{\mathrm{MW}}(F)$ of the universal 2-cocycle θ_{uni}. Then, for any $(a,b),(c,d)\in X_F$, the composite satisfies the equality*

$$\theta_{\mathrm{uni}}\circ(\iota\times\iota)\big((a,b),(c,d)\big)=\theta_{\mathrm{uni}}\circ(\iota\times\iota)\big((c,d),(a,b)\lhd(c,d)\big).$$

We will prove Lemma 7.6 by a tedious computation. To this end, denote the restriction $\theta_{\mathrm{uni}}\circ(\iota\times\iota)$ by Θ. Then a direct calculation shows an easy formula for this Θ: precisely, for any $(a,b),(c,d)\in X_F$, the map $\Theta:(X_F)^2\to K_2^{\mathrm{MW}}(F)$ satisfies the equality

$$\Theta\big((a,b),(c,d)\big)=\begin{cases}[(ab-1)c^2-(1+cd)a^2,\,-c^2/a^2] & \\ \qquad\qquad\qquad -[-a^2,-c^2] & \text{if } ac\neq 0,\\ \qquad\qquad 0 & \text{if } ac=0.\end{cases}\tag{7.7}$$

Proof of Lemma 7.6. When $ac=0$, we can easily obtain the desired equality in Lemma 7.6 by a direct calculation, although we omit the details.

Thus, we will assume $ac\neq 0$, and compute $\Theta\big((a,b),(c,d)\big)$ in some detail. Denote $(a,b)\lhd(c,d)$ by $(H,I)\in X_F$ for short. Then a direct calculation can show the identity

$$(1-cd)H^2+(1+HI)c^2=(1-ab)c^2+(1+cd)a^2.\tag{7.8}$$

Let $\mathcal{B}$ be the right-hand side in (7.8). Noting that $[-a^2,-c^2]=[-a^2,-c^2/a^2]$ by axiom (ii), the $\Theta\big((a,b),(c,d)\big)$ in (7.7) becomes $[-\mathcal{B},-c^2/a^2]-[-a^2,-c^2/a^2]$. Further, this is equal to $[\mathcal{B}/a^2,-c^2/a^2]-2[-1,-\mathcal{B}]$ by Lemma 7.9(2) below.

Hence it is enough to show $[\mathcal{B}/c^2,-H^2/c^2]=[\mathcal{B}/a^2,-c^2/a^2]$ for the proof. For this purpose, note $[\mathcal{B}/a^2,-c^2/a^2]=[\mathcal{B}/c^2,-c^2/a^2]$ by Lemma 7.9(1). Therefore, from the identity $\mathcal{B}=aH+c^2$ by definition and the axiom (iii), we deduce that

$$\begin{aligned}[\mathcal{B}/c^2,-c^2/a^2]&=[\mathcal{B}/c^2,-(\mathcal{B}/c^2-1)^2(c^2/a^2)]\\&=[\mathcal{B}/c^2,-a^2H^2/(c^2a^2)]=[\mathcal{B}/c^2,-H^2/c^2].\end{aligned}$$

In summary, we have the desired equality $[\mathcal{B}/c^2,-H^2/c^2]=[\mathcal{B}/a^2,-c^2/a^2]$. $\square$

Lemma 7.9. (1) $[x,y]=[x^{-1},y^{-1}]=[-xy,y]$ *for any* $x,y\in F^\times$.

(2) $[x,-z^2]+[-y^2,-z^2]=[-xy^2,-z^2]+2[-1,x]$ *for any* $x,y,z\in F^\times$.

Proof. First, (1) is directly obtained from the axiom (ii) of $K_2^{\mathrm{MW}}(F)$.

Next we will prove (2). Following [Suslin 1987], we use the notation $[a,b,c]:=[a,b]+[a,c]-[a,bc]$. Since $[A,-z^2]=[-z^{-2},A]$, the goal is equivalent to the equality $[-z^{-2}, x, -y^2] = [x, -1, -1]$. To show this, we set up two identities proven in [Suslin 1987, Lemma 6.1] of the forms

$$[ab, x, c] = [a, bx, c] + [b, x, c] - [a, b, c], \quad [d, e, f] = [d^{-1}, e, f] \tag{7.10}$$

for any $x, a, b, c, d, e, f \in F^\times$. By applying $a = -z$, $b = z$ and $c = -y^2$ to these identities, we have

$$\begin{aligned}[-z^{-2}, x, -y^2] = [-z^2, x, -y^2] &= [-z, zx, -y^2] + [z, x, -y^2] - [-z, z, x]\\ &= [-z^{-1}, -zx, -y^2] + [z, x, -y^2] - [-z^{-1}, z, x]\\ &= [-1, x, -y^2].\end{aligned}$$

Lastly, since the equalities $[x, b, c] = [x, c, b] = [b, c, x]$ are known [Suslin 1987, Lemma 6.1], repeating the computation leads to $[-z^{-2}, x, -y^2] = [-1, x, -y^2] = [-y^2, x, -1] = [-1, x, -1] = [x, -1, -1]$, as desired. □

7C. *Preliminaries.* In the next subsection, we will prove Theorems 3.4 and 6.1, which remain to be proved. For this purpose, this subsection reviews some results [Carter et al. 2005; Eisermann 2014] of quandle theory, which explain a relation between quandles and longitudes.

To this end, we begin by setting up some terminology. Consider the group defined by generators e_x labeled by $x \in X$ modulo the relations $e_x \cdot e_y = e_y \cdot e_{x \lhd y}$ for $x, y \in X$. This group is called *the associated group* and denoted by $\mathrm{As}(X)$, and has a right action on X defined by $x \cdot e_y := x \lhd y$. Letting $\mathrm{O}(X)$ be the set of the orbits, we consider the orbit decomposition of X, i.e., $X = \bigsqcup_{\lambda \in \mathrm{O}(X)} X_\lambda$. In addition, fix a quotient group G of $\mathrm{As}(X)$ subject to a central subgroup. Denote the quotient map $\mathrm{As}(X) \to G$ by p_G.

Switching to topology, given an X-coloring $\mathcal{C} \in \mathrm{Col}_X(D)$ of a link L, let us correspond each arc γ to $p_G(e_{\mathcal{C}(\gamma)}) \in G$. Regarding the arcs as generators of $\pi_1(\mathbb{R}^3 \setminus L)$ by the Wirtinger presentation (see Figure 5), the correspondence defines a group homomorphism $\Gamma_{\mathcal{C}} : \pi_1(\mathbb{R}^3 \setminus L) \to G$.

Furthermore, with respect to link-components of L, we fix an arc γ_j on D with $1 \le j \le \#L$. Let $x_j := \mathcal{C}(\gamma_j) \in X_j$, and fix a preferred longitude $\mathfrak{l}_j$ obtained from D. Noticing that each $\mathfrak{l}_j$ commutes with the meridian γ_j, we have $\Gamma_{\mathcal{C}}(\mathfrak{l}_j) \in \mathrm{Stab}(x_j)$.

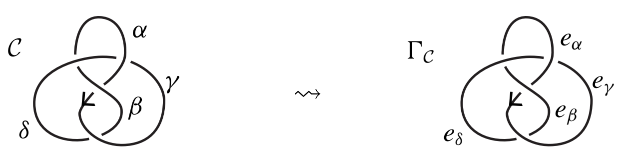

Figure 5. The correspondence $\Gamma_{\mathcal{C}}$.

We will give a computation for the value $\Gamma_{\mathcal{C}}(\mathfrak{l}_j)$ as follows. Fix $x_\lambda \in X_\lambda$ for any $\lambda \in \mathrm{O}(X)$, Since the action of G on X_λ is transitive, we can choose a section $\mathfrak{s}_\lambda : X_\lambda \to G$ such that $x_\lambda \cdot \mathfrak{s}_\lambda(y) = y$ for any $y \in X_\lambda$. Then we define a map $\phi : X^2 \to G$ by

$$\phi(g,h) = \mathfrak{s}_\lambda(g) p_G(e_g^{-1} e_h) \mathfrak{s}_\lambda(g \lhd h)^{-1} \quad \text{for } g \in X_\lambda,\ h \in X. \tag{7.11}$$

By definition, we see that $\phi(g,h)$ lies in the stabilizer $\mathrm{Stab}(x_\lambda) \subset G$ of x_λ if $g \in X_\lambda$.

With respect to the coloring $\mathcal{C}$, similar to in Section 6A, we define a product of the form

$$S_{\mathcal{C},j} := \phi(\mathcal{C}(\alpha_1), \mathcal{C}(\beta_1))^{\epsilon_1} \phi(\mathcal{C}(\alpha_2), \mathcal{C}(\beta_2))^{\epsilon_2} \cdots \phi(\mathcal{C}(\alpha_{N_j}), \mathcal{C}(\beta_{N_j}))^{\epsilon_{N_j}} \in \mathrm{Stab}(x_j),$$

where the terminology of arcs α_i and β_i and of signs ϵ_i are as in Section 6A (see also Figure 2). Although this construction depends on the choice of the x_λ and the sections $\mathfrak{s}_\lambda$, the following is known:

Proposition 7.12 [Carter et al. 2005, Lemma 5.8]. *The product $S_{\mathcal{C},j}$ equals*

$$\mathfrak{s}_\lambda(\mathcal{C}(\gamma_1))^{-1} \Gamma_{\mathcal{C}}(\mathfrak{l}_j) \mathfrak{s}_\lambda(\mathcal{C}(\gamma_1))$$

in $\mathrm{Stab}(x_j)$. *In particular*, *if* $\mathrm{Stab}(x_j)$ *is abelian*, *the equality $S_{\mathcal{C},j} = \Gamma_{\mathcal{C}}(\mathfrak{l}_j)$ holds in* $\mathrm{Stab}(x_j)$.

The proof immediately follows from the definitions of ϕ and of the preferred longitude $\mathfrak{l}_i$.

We next review a computation, shown by Eisermann [2014], of the second quandle homology $H_2^Q(X)$ (see [Carter et al. 2003] for the original definition).

Theorem 7.13 [Eisermann 2014, Theorem 9.9]. *Let X be a quandle with $|O(X)| = 1$. Fix $x_\lambda \in X$. Let* $\mathrm{Stab}(x_\lambda) \subset \mathrm{As}(X)$ *denote the stabilizer of x_λ. Then the abelianization* $\mathrm{Stab}(x_\lambda)_{\mathrm{ab}}$ *is isomorphic to $\mathbb{Z} \oplus H_2^Q(X)$.*

In particular, the class $[\Gamma_{\mathcal{C}}(\mathfrak{l}_j)]$ in the abelianization is contained in $\mathbb{Z} \oplus H_2^Q(X)$ by Theorem 7.13. Then, as a corollary of a homotopical study of the homology $H_2^Q(X)$, we can state a sufficient condition to ensure the nontriviality of the classes in the $\mathbb{Z} \oplus H_2^Q(X)$ as follows:

Proposition 7.14 [Nosaka 2015, Remark 6.4]. *Let X be a quandle such that the orbit $O(X)$ is single. If the group homology $H_2^{\mathrm{gr}}(\mathrm{As}(X); \mathbb{Z})$ is zero, then any element $\Upsilon \in H_2^Q(X)$ admits some X-coloring $\mathcal{C}$ of a link such that the equality $\Upsilon = [\Gamma_{\mathcal{C}}(\mathfrak{l}_1)] + \cdots + [\Gamma_{\mathcal{C}}(\mathfrak{l}_{\#L})]$ holds in $\mathbb{Z} \oplus H_2^Q(X)$.*

7D. *Proofs of Theorems 3.4 and 6.1.* First, we aim to prove Theorem 3.4. Inspired by Theorem 7.13, we first determine the associated groups $\mathrm{As}(X_F)$ of the parabolic quandles over F.

Theorem 7.15. *Take the map $\iota : X_F \to \mathrm{SL}_2(F)$ given in (7.1). Then the map*

$$X_F \to \mathbb{Z} \times K_2^{\mathrm{MW}}(K) \times \mathrm{SL}_2(F), \quad x \mapsto (1, 0, \iota(x)).$$

gives rise to a group homomorphism $\mathrm{As}(X_F) \to \mathbb{Z} \times \widetilde{\mathrm{SL}}_2(F)$, which is an isomorphism.

Proof. We can first verify that the map ι in (7.1) yields a group epimorphism $\mathrm{As}(X_F) \to \mathrm{SL}_2(F)$, which is a central extension. It then follows from Lemma 7.6 that the above map yields a group homomorphism $\mathrm{As}(X_F) \to \mathbb{Z} \times \widetilde{\mathrm{SL}}_2(F)$. Since $H_1(\mathrm{As}(X_F)) \cong \mathbb{Z}$, the universality of central extensions implies that the homomorphism must be an isomorphism. $\square$

Corollary 7.16. *The second quandle homology $H_2^Q(X_F; \mathbb{Z})$ is isomorphic to the group $F \oplus \widetilde{K}_2^{\mathrm{MW}}(F)$.*

Proof. We will compute $H_2^Q(X_F)$ using Theorem 7.13. Fix $x_0 = (0, 1) \in X_F$, and the universal extension $\mathcal{E} : \widetilde{\mathrm{SL}}_2(F) \to \mathrm{SL}_2(F)$. Noticing that the SL_2 standard representation $X_F \curvearrowleft \mathrm{As}(X_F)$ is transitive, i.e., $|O(X)| = 1$, we will calculate the abelianization of the stabilizer $\mathrm{Stab}(x_0) \subset \mathrm{As}(X_F)$. We easily check that $\mathcal{E}(\mathrm{Stab}(x_0)) \subset \mathrm{SL}_2(F)$ is the subgroup U_F. Hence, $\mathrm{Stab}(x_0) \cong \mathbb{Z} \times \mathcal{E}^{-1}(U_F) \cong \mathbb{Z} \times \widetilde{K}_2^{\mathrm{MW}}(F) \times F$ by Lemma 2.4. Since this is abelian, Theorem 7.13 readily implies the conclusion $\mathbb{Z} \oplus H_2^Q(X_F) \cong H_1^{\mathrm{gr}}(\mathrm{Stab}(x_0)) \cong \mathbb{Z} \times \widetilde{K}_2^{\mathrm{MW}}(F) \times F$. $\square$

Proof of Theorem 3.4. Theorem 7.15 says that the quandle X_F satisfies the assumption of Proposition 7.14. Moreover, $\mathrm{Stab}(x_0) \cong \mathbb{Z} \times \widetilde{K}_2^{\mathrm{MW}}(F) \times F \cong \mathbb{Z} \oplus H_2^Q(X_F; \mathbb{Z})$ is abelian by Corollary 7.16. As a consequence, Proposition 7.14 implies the conclusion. $\square$

Next we will turn to proving Theorem 6.1.

Proof of Theorem 6.1. Let G be $\mathrm{PSL}_2(F)$, and let p_G be the composite of projections $\mathrm{As}(X_F) \to \widetilde{\mathrm{SL}}_2(F) \to \mathrm{SL}_2(F) \xrightarrow{\pi} \mathrm{PSL}_2(F)$. Let x_λ be $(0, 1) \in X_F$. Then we easily see that the stabilizer $\mathrm{Stab}(x_\lambda) \subset G$ is an abelian group $\pi(U_F) \cong F$.

Furthermore, we define a section $\mathfrak{s}_F : X_F \to \mathrm{PSL}_2(F)$ by setting $\mathfrak{s}_F(0, b) := \mathrm{diag}(b^{-1}, b)$ and $\mathfrak{s}_F(a, b) := \begin{pmatrix} 0 & -a^{-1} \\ a & b \end{pmatrix}$ if $a \neq 0$. Then, according to (7.11), we have the resulting map $\phi : (X_F)^2 \to \pi(U_F) \cong F$. By an elementary computation, the map ϕ agrees with the map $\mathcal{S}$. Hence, Proposition 7.12 immediately implies the equality claimed in Theorem 6.1. $\square$

Remark 7.17. Similar to the previous proof, by considering the case $(X, G) = (X_F, \widetilde{\mathrm{SL}}_2(F))$, we can give a sum formula for the K_2 invariant. However, as the author can not formulate a section $X_F \to \widetilde{\mathrm{SL}}_2(F)$ in a simple way, the resulting formula is a little complicated and is far from applications. The desired formula would be simple; So this paper omits describing formulae for the K_2 invariants.

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