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Motivic complexes over nonperfect fields

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We show that the theory of motivic complexes developed by Voevodsky over perfect fields works over nonperfect fields as well provided that we work with sheaves with transfers of $\mathbb{Z}[1/p]$-modules ($p = \operatorname{char} F$). In particular we show that every homotopy invariant sheaf with transfers of $\mathbb{Z}[1/p]$-modules is strictly homotopy invariant.

0. Introduction

Voevodsky defined the category of motives over an arbitrary perfect field. The main reason why the same construction does not work over arbitrary fields is that we do not know whether the theorem of Voevodsky, which states that every homotopy invariant Nisnevich sheaf with transfers is strictly homotopy invariant, holds over nonperfect fields. This makes life fairly inconvenient because when we start with a field F of characteristic $p > 0$ and take a look on the function field $F(S)$ of a smooth scheme of finite type over F we never get a perfect field unless $\dim S = 0$.

The main purpose of this paper is to show that the above theorem holds over a nonperfect field of characteristic p provided that we work with sheaves with transfers of $\mathbb{Z}[1/p]$-modules. In Section 1 we use the theory of Frobenius twists to show that extension of scalars from F to its perfect closure F^{1/p^∞} defines an equivalence on the category PT_p of presheaves with transfers of $\mathbb{Z}[1/p]$-modules. In Section 2 we show that the above functor takes sheaves to sheaves, and the resulting functor on the category NST_p of Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules is an equivalence as well. Finally we show that this functor preserves cohomology. In Section 3 we show that the extension of scalars functor takes homotopy invariant presheaves to homotopy invariant presheaves and hence every homotopy invariant Nisnevich sheaf with transfers of $\mathbb{Z}[1/p]$-modules over F is strictly homotopy invariant. This fact readily implies that all results concerning homotopy invariant presheaves with transfers proved by Voevodsky for perfect fields are true over arbitrary fields once we work with sheaves of $\mathbb{Z}[1/p]$-modules. In particular we may define the motivic category $\mathrm{DM}_p^-(F)$ of effective motives in the standard way as

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the category of bounded above complexes of Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules with homotopy invariant cohomology sheaves.

In Section 4 we discuss in some detail the extension of scalars for presheaves with transfers of $\mathbb{Z}[1/p]$-modules in the case of an arbitrary field extension $F \subset E$. We show that this functor takes sheaves to sheaves, takes homotopy invariant sheaves to homotopy invariant sheaves and is exact. This allows us to define the extension of scalars functor $\phi^* : \mathrm{DM}_p^-(F) \to \mathrm{DM}_p^-(E)$ in the most straightforward way — just applying the extension of scalars functor pointwise. We show also that this functor takes tensor products to tensor products and happens to be an equivalence of categories if $E = F^{1/p^\infty}$.

In Section 5 we show that extension of scalars commutes with the internal Hom functor.

1. Presheaves with transfers over nonperfect fields

For a field F we denote by Cor_F the category of finite correspondences over F and by $\mathrm{PT}(F)$ the category of presheaves with transfers over F, i.e., additive contravariant functors $\mathrm{Cor}_F \to \mathrm{Ab}$; see [Mazza et al. 2006] for definitions. We use the notation NST for the category of Nisnevich sheaves with transfers. Let E/F be a field extension. In this case we have an obvious extension of scalars functor $\phi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ taking X to X_E and $Z \in \mathrm{Cor}(X, Y)$ to $Z_E \in \mathrm{Cor}(X_E, Y_E)$. Taking the composition with ϕ we get a direct image functor $\phi_* : \mathrm{PT}(E) \to \mathrm{PT}(F)$. The functor ϕ_* is obviously exact and preserves direct sums and direct products. In particular, ϕ_* is continuous and hence has a left adjoint $\phi^\#$; see [MacLane 1971].

Proposition 1.1. (1) *The functor $\phi^\#$ is uniquely characterized by the following properties*:

(a) *$\phi^\#$ is right exact.*
(b) *$\phi^\#$ preserves arbitrary direct sums.*
(c) $\phi^\#(\mathbb{Z}_{\mathrm{tr}}(X)) = \mathbb{Z}_{\mathrm{tr}}(X_E)$.

(2) *The functor $\phi^\#$ is given by the formula*

$$\phi^\#(\mathcal{F}) = \mathrm{Coker}\Bigg(\bigoplus_{s \in \mathrm{Cor}_F(X, X_1)} \mathbb{Z}_{\mathrm{tr}}(X_E) \otimes \mathcal{F}(X_1) \to \bigoplus_{X \in \mathrm{Sm}_F} \mathbb{Z}_{\mathrm{tr}}(X_E) \otimes \mathcal{F}(X) \Bigg).$$

Proof. Note that every left adjoint functor is right exact and preserves direct sums. For any presheaf with transfers $\mathcal{M} \in \mathrm{PT}(E)$ we have

$$\begin{aligned}\mathrm{Hom}(\phi^\#(\mathbb{Z}_{\mathrm{tr}}(X), \mathcal{M})) &= \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(X), \phi_*(\mathcal{M})) \\ &= \phi_*(\mathcal{M})(X) = \mathcal{M}(X_E) = \mathrm{Hom}(\mathbb{Z}_{\mathrm{tr}}(X_E), \mathcal{M}).\end{aligned}$$

This proves the last formula. Assume now that $a : \mathrm{PT}(F) \to \mathrm{PT}(E)$ is a functor having the above three properties. Note that in this case we also have the following property: for any $\mathcal{F} \in \mathrm{PT}(F)$ and any abelian group A we have a natural isomorphism,

$$a(\mathcal{F} \otimes A) = a(\mathcal{F}) \otimes A.$$

In fact the case when A is a free abelian group is clear since a preserves direct sums. The general case follows in view of right exactness of a. Note now that any presheaf with transfers $\mathcal{F} \in \mathrm{PT}(F)$ has a canonical resolution

$$\bigoplus_{s \in \mathrm{Cor}_F(X, X_1)} \mathbb{Z}_{\mathrm{tr}}(X) \otimes \mathcal{F}(X_1) \to \bigoplus_{X \in \mathrm{Sm}_F} \mathbb{Z}_{\mathrm{tr}}(X) \otimes \mathcal{F}(X) \to \mathcal{F} \to 0.$$

Applying the functor a to this presentation and using the right exactness of a we get the presentation for $a(\mathcal{F})$. This proves that

$$a(\mathcal{F}) = \mathrm{Coker}\Bigg(\bigoplus_{s \in \mathrm{Cor}_F(X, X_1)} \mathbb{Z}_{\mathrm{tr}}(X_E) \otimes \mathcal{F}(X_1) \to \bigoplus_{X \in \mathrm{Sm}_F} \mathbb{Z}_{\mathrm{tr}}(X_E) \otimes \mathcal{F}(X) \Bigg). \quad \square$$

Recall that a presheaf with transfers is called free in case it is a direct sum of presheaves of the form $\mathbb{Z}_{\mathrm{tr}}(X)$. Since all schemes in question are Noetherian it follows that every such presheaf is actually a sheaf in the Nisnevich topology. Later we'll need the following additional properties of the functor $\phi^{\#}$:

Lemma 1.2. (1) *The functor $\phi^{\#}$ takes free presheaves to free presheaves.*

(2) *For any presheaves with transfers $\mathcal{F}$, $\mathcal{G} \in \mathrm{PT}(F)$ we have a natural isomorphism $\phi^{\#}(\mathcal{F} \otimes_{\mathrm{tr}}^{\mathrm{pr}} \mathcal{G}) = \phi^{\#}(\mathcal{F}) \otimes_{\mathrm{tr}}^{\mathrm{pr}} \phi^{\#}(\mathcal{G})$* (here $\otimes_{\mathrm{tr}}^{\mathrm{pr}}$ stands for the tensor product operation in the category of presheaves with transfers; see [Suslin and Voevodsky 2000]).

Proof. The first claim is obvious. To prove the second one we note that, according to the defining properties of the tensor product operation, to compute $\mathcal{F} \otimes_{\mathrm{tr}}^{\mathrm{pr}} \mathcal{G}$ we may start with arbitrary free presentations

$$\bigoplus_i \mathbb{Z}_{\mathrm{tr}}(X_i) \xrightarrow{p} \bigoplus_j \mathbb{Z}_{\mathrm{tr}}(Y_j) \to \mathcal{F} \to 0 \quad \text{and} \quad \bigoplus_s \mathbb{Z}_{\mathrm{tr}}(X_s) \xrightarrow{q} \bigoplus_t \mathbb{Z}_{\mathrm{tr}}(Y_t) \to \mathcal{G} \to 0,$$

in which case $\mathcal{F} \otimes_{\mathrm{tr}}^{\mathrm{pr}} \mathcal{G}$ coincides with the cokernel of the resulting map

$$\bigoplus_{i,t} \mathbb{Z}_{\mathrm{tr}}(X_i \times Y_t) \oplus \bigoplus_{j,s} \mathbb{Z}_{\mathrm{tr}}(Y_j \times X_s) \xrightarrow{p \otimes 1 - 1 \otimes q} \bigoplus_{j,t} \mathbb{Z}_{\mathrm{tr}}(Y_j \times Y_t).$$

Applying the functor $\phi^{\#}$ to the above presentations we get free presentations of $\phi^{\#}(\mathcal{F})$ and $\phi^{\#}(\mathcal{G})$, and the same computation as above yields our claim. $\square$

Note that ϕ_* takes Zariski (resp. Nisnevich) sheaves with transfer over E to Zariski (resp. Nisnevich) sheaves with transfers over F. However it's not clear whether the functor $\phi^\#$ takes sheaves to sheaves, so working with Nisnevich sheaves we need to sheafify $\phi^\#(\mathcal{F})$ in Nisnevich topology. The resulting sheaf with transfers will be denoted $\phi^*(\mathcal{F})$. Sometimes we denote $\phi^*(\mathcal{F})$ by $\mathcal{F}\otimes_F E$ or $\mathcal{F}_E$ and call it the extension of scalars in $\mathcal{F}$.

Corollary 1.3. (1) *For any presheaf with transfers $\mathcal{F}$ we have a natural isomorphism $\phi^\#(\mathcal{F})_{\mathrm{Nis}} = \phi^*(\mathcal{F}_{\mathrm{Nis}})$.*

(2) *For any sheaves with transfers $\mathcal{F}, \mathcal{G} \in \mathrm{NST}(F)$ we have a natural isomorphism $\phi^*(\mathcal{F}\otimes_{\mathrm{tr}}\mathcal{G}) = \phi^*(\mathcal{F})\otimes_{\mathrm{tr}}\phi^*(\mathcal{G})$.*

Proof. The first claim is clear since composition of left adjoints coincides with the left adjoint to the composition. The second claim follows from the first one and the previous lemma. □

Assume now that F is a field of positive characteristic p and take $E = F^{1/p}$. The field E may be identified with F via the Frobenius isomorphism, so that the category Cor_E identifies with Cor_F and the resulting functor ϕ identifies with the Frobenius twist functor $X \mapsto X^{(1)} = X \otimes_f F$, where the subscript f indicates that we view F as F-algebra via the Frobenius embedding $f : F \to F$, $x \mapsto x^p$. The important property of the Frobenius twist functor is the presence, for any $X \in \mathrm{Sch}/F$, of the canonical Frobenius morphism $\Phi_X : X \to X^{(1)}$ (see, for example, [Friedlander and Suslin 1997]), which in the affine case corresponds to the F-algebra homomorphism $A\otimes_f F \to A$, $a\otimes\lambda \mapsto \lambda\cdot a^p$. The following elementary lemma sums up some of the properties of the Frobenius map:

Lemma 1.4. (1) *$\Phi_X : X \to X^{(1)}$ is a natural transformation of functors from Sch/F to itself, i.e., for a morphism $f : X \to Y$ we get a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\Phi_X}\downarrow & & \downarrow{\scriptstyle\Phi_Y} \\ X^{(1)} & \xrightarrow{f^{(1)}} & Y^{(1)} \end{array}$$

(2) $\Phi_{X\times Y} = \Phi_X \times \Phi_Y$.

(3) *Φ_X is a finite surjective morphism for any X. If X/F is a smooth irreducible scheme of dimension d, the morphism $\Phi_X : X \to X^{(1)}$ is a finite flat purely inseparable morphism of degree p^d.*

Let $f : X \to Y$ be a finite surjective morphism of irreducible schemes. In this case the graph $\Gamma_f \subset X\times Y$ is finite over Y and so its transpose $\Gamma_f^T \subset Y\times X$ defines a finite correspondence from Y to X (which we'll call f^T).

Theorem 1.5. *The family of maps* $\Phi_X : X \to X^{(1)}$ *defines a natural transformation of functors from* Cor_F *to itself*, $\Phi : \mathrm{Id}_{\mathrm{Cor}_F} \to \phi$. *Moreover, after inverting* p *this natural transformation becomes an isomorphism of functors with inverse given by the family of maps*

$$\frac{1}{p^{\dim X}} \Phi_X^T : X^{(1)} \to X.$$

The proof is based on several easy but useful lemmas.

Lemma 1.6. *Let* $f : X \to Y$ *be a finite flat purely inseparable morphism of smooth irreducible schemes of the same dimension. Then the maps* $f_* : Z^i(X) \to Z^i(Y)$ *and* $f^* : Z^i(Y) \to Z^i(X)$ *are defined on all cycles and both compositions coincide with multiplication by* $\deg f$.

Proof. Note that the operation f_* is defined on all cycles for any proper morphism and the operation f^* is defined on all cycles for any flat morphism. Furthermore, the composition $f_* \circ f^*$ coincides with multiplication by $\deg f$ in view of the projection formula. Finally, since f is finite and purely inseparable, it is injective. Hence f_* is equally injective and, since $f_* \circ f^* \circ f_* = \deg f \cdot f_*$, we conclude that $f^* \circ f_* = \deg f$ □

We will also need the following well-known and elementary fact:

Lemma 1.7. *Let* $Z \in \mathrm{Cor}(X, Y)$ *be a finite correspondence from a* (*smooth*) *scheme* X *to a* (*smooth*) *scheme* Y. *Let further* $f : Y \to Y'$ *and* $g : X' \to X$ *be morphisms of* (*smooth*) *schemes. Then*:

(1) $f \circ Z \in \mathrm{Cor}(X, Y')$ *coincides with* $(1_X \times f)_*(Z)$.

(2) $Z \circ g \in \mathrm{Cor}(X', Y)$ *coincides with* $(g \times 1_Y)^*(Z)$.

Corollary 1.8. (a) *Let* $f : X \to Y$ *be a finite surjective morphism of smooth irreducible schemes. Then* $f \circ f^T = \deg f$.

(b) *Assume in addition that* f *is purely inseparable. Then* $f^T \circ f = \deg f$.

Proof. Note that for any morphism $f : X \to Y$ the cycle $\Gamma_f \subset X \times Y$ coincides with $(1_X, f)_*(X) = (1_X \times f)_*(\Delta_X)$ and hence $\Gamma_f^T = (f, 1_X)_*(X) = (f \times 1_X)_*(\Delta_X)$. Lemma 1.7 shows now that $f \circ f^T = (1_Y \times f)_*(\Gamma_f^T) = (1_Y \times f)_*(f, 1_X)_*(X) = (f, f)_*(X) = (\Delta_Y \circ f)_*(X) = (\Delta_Y)_*(\deg f \cdot Y) = \deg f \cdot \Delta_Y$ To prove the second claim we note first that f is flat, so that we may apply Lemma 1.6. Applying Lemma 1.7 once again we get

$$f^T \circ f = (f \times 1_X)^*(\Gamma_f^T) = (f \times 1_X)^*(f \times 1_X)_*(\Delta_X) = \deg f \cdot \Delta_X. \qquad \square$$

Proposition 1.9. *Let* X *be a smooth equidimensional scheme and let* $Z \subset X$ *be an equidimensional cycle of dimension* d. *Then* $(\Phi_X)_*(Z) = p^d \cdot Z^{(1)}$.

Proof. It suffices, clearly, to treat the case when Z is irreducible, i.e., is represented by a closed integral subscheme $Z \subset X$. In this case the cycle $Z^{(1)}$ is defined by the closed subscheme $Z^{(1)} \subset X^{(1)}$, which is irreducible but which however need not be reduced. Thus, denoting the integral scheme $Z^{(1)}_{\text{red}}$ by Z' we see that the cycle $Z^{(1)}$ equals $l \cdot Z'$, where l is the length of the local Artinian ring $F(Z) \otimes_F F^{1/p} = F(Z) \otimes_f F$. Furthermore, Lemma 1.4 shows that $\Phi_X(Z) = Z'$ and hence $(\Phi_X)_*(Z) = [F(Z) : F(Z')] \cdot Z'$. Thus we only need to check the formula

$$[F(Z) : F(Z')] = l \cdot p^d.$$

Let $x_1, \dots, x_d$ be the transcendence basis of $F(Z)$ over F. Note that dimension of $F(Z) \otimes_f F$ over $F(x_1, \dots, x_d) \otimes_f F = F(x_1, \dots, x_d)$ equals $[F(Z) : F(x_1, \dots, x_d)]$ and hence $[F(Z) : F(x_1, \dots, x_d)] = l \cdot [F(Z') : F(x_1, \dots, x_d)]$. The field embedding $F(Z') \hookrightarrow F(Z)$ takes x_i to x_i^p. Thus identifying $F(Z')$ with a subfield of $F(Z)$ we identify its subfield $F(x_1, \dots, x_d)$ with the subfield $F(x_1^p, \dots, x_d^p) \subset F(Z)$. So we get

$$\begin{aligned} l \cdot p^d &= \frac{[F(Z) : F(x_1, \dots, x_d)]}{[F(Z') : F(x_1^p, \dots, x_d^p)]} \cdot [F(x_1, \dots, x_d) : F(x_1^p, \dots, x_d^p)] \\ &= \frac{[F(Z) : F(x_1^p, \dots, x_d^p)]}{[F(Z') : F(x_1^p, \dots, x_d^p)]} = [F(Z) : F(Z')]. \end{aligned}$$ □

Proof of Theorem 1.5. We first show that Φ is a natural transformation of functors from Cor_F to itself. Let $Z \in \mathrm{Cor}(X, Y)$ be a finite correspondence from a smooth, irreducible scheme X to a (smooth, irreducible) Y. We have to verify the commutativity of the diagram

$$\begin{tikzcd} X \arrow[r, "Z"] \arrow[d, "\Phi_X"'] & Y \arrow[d, "\Phi_Y"] \\ X^{(1)} \arrow[r, "Z^{(1)}"] & Y^{(1)} \end{tikzcd}$$

By Lemma 1.7, $\Phi_Y \circ Z = (1_X \times \Phi_Y)_*(Z)$ and $Z^{(1)} \circ \Phi_X = (\Phi_X \times 1_{Y^{(1)}})^*(Z^{(1)})$. Thus we need to establish the relation $(1_X \times \Phi_Y)_*(Z) = (\Phi_X \times 1_{Y^{(1)}})^*(Z^{(1)})$. Applying to both sides the injective map $(\Phi_X \times 1_{Y^{(1)}})_*$ and using Lemma 1.6 we see that the formula in question takes the form $(\Phi_X \times \Phi_Y)_*(Z) = p^{\dim X} \cdot Z^{(1)}$. Since $\Phi_X \times \Phi_Y = \Phi_{X \times Y}$ we conclude from Proposition 1.9 that the left-hand side equals $p^{\dim Z} \cdot Z^{(1)}$. It suffices to note now that $\dim X = \dim Z$.

To finish the proof it suffices to establish that, for any smooth irreducible X,

$$\Phi_X \circ \Phi_X^T = p^{\dim X} \cdot \Delta_{X^{(1)}} \quad \text{and} \quad \Phi_X^T \circ \Phi_X = p^{\dim X} \cdot \Delta_X.$$

However, these formulae follow from Corollary 1.8. □

Denote by $\mathrm{Cor}_F[1/p]$ the category with same objects as Cor_F (i.e., all smooth schemes of finite type over F) but whose morphisms are obtained from those of Cor_F by inverting p.

Corollary 1.10. *The Frobenius twist functor induces an equivalence*

$$\mathrm{Cor}_F[1/p] \to \mathrm{Cor}_F[1/p], \quad X \mapsto X^{(1)}.$$

Proof. According to Theorem 1.5, the Frobenius twist functor is isomorphic to the identity functor and hence is an equivalence. □

Set $F_n = F^{1/p^n}$ and let $F_\infty = F^{1/p^\infty} = \varinjlim F_n$ be the perfect closure of F.

Theorem 1.11. *Extension of scalars defines an equivalence of categories*

$$\phi : \mathrm{Cor}_F[1/p] \to \mathrm{Cor}_{F_\infty}[1/p].$$

Proof. Corollary 1.10 shows that extension of scalars from F to F_1 gives an equivalence of categories on $\mathrm{Cor}[1/p]$. Induction on n implies that the same is true for the extension of scalars from F to F_n. Note further that, for $X \in \mathrm{Sm}_F$, every closed subscheme in X_{F_∞} is defined over F_n for appropriate n. Hence, for any $X, Y \in \mathrm{Sm}_F$ we have

$$\mathrm{Cor}(X_{F_\infty}, Y_{F_\infty}) = \varinjlim_n \mathrm{Cor}(X_{F_n}, Y_{F_n}).$$

Thus the extension of scalars map

$$\mathrm{Cor}(X, Y)[1/p] \to \mathrm{Cor}(X_{F_\infty}, Y_{F_\infty})[1/p]$$

is an isomorphism. It remains to show that the functor ϕ is essentially surjective, i.e., every object of $\mathrm{Cor}_{F_\infty}[1/p]$ is isomorphic to X_{F_∞} for appropriate X. However, this follows from Corollary 1.8 and the following result:

Lemma 1.12. *Let $Z \in \mathrm{Sm}_{F_\infty}$ be a smooth irreducible scheme. Then there exists a smooth irreducible scheme $X \in \mathrm{Sm}_F$ and a finite, surjective, purely inseparable morphism $Z \to X_{F_\infty}$.*

Proof. Since every scheme of finite type over F_∞ is defined over F_n for sufficiently large n, we easily conclude that there exists a smooth irreducible scheme $Y \in \mathrm{Sm}_{F_n}$ such that Z is isomorphic to Y_{F_∞}. In the case Y that is defined over F, the scheme Z is also defined over F and we have nothing to prove. In the general case we may identify F_n with F via the n-th power of the Frobenius isomorphism. In this way, Y defines a smooth irreducible scheme X over F. As we pointed out before, the scheme X_{F_n}, viewed as a scheme over F, coincides with the n-th Frobenius twist $X^{(n)}$. The n-th power of the Frobenius morphism

$$\Phi_X^n : X \to X^{(n)}$$

is a finite, surjective, purely inseparable morphism of degree $p^{n\cdot\dim X}$. When we return from F to F_n, the morphism Φ_X^n defines a finite, surjective, purely inseparable morphism $Y \to X_{F_n}$. Extending scalars from F_n to F_∞, we get the required morphism $Z \to X_{F_\infty}$. □

We are going to use the notation $\mathrm{PT}_p(F)$ for the category of the presheaves with transfers of $\mathbb{Z}[1/p]$-modules (i.e., additive functors from Cor_F to $\mathbb{Z}[1/p]$-mod). Note that presheaves with transfers of $\mathbb{Z}[1/p]$-modules may be identified with presheaves with transfers, all of whose groups of sections are uniquely p-divisible, so that $\mathrm{PT}_p(F) \subset \mathrm{PT}(F)$.

Theorem 1.13. *The direct image functor*

$$\phi_* : \mathrm{PT}_p(F_\infty) \to \mathrm{PT}_p(F)$$

is an equivalence of categories with quasi-inverse $\phi^\#$.

Proof. Note first that, for any additive category $\mathbb{C}$, every additive functor $\mathcal{M} : \mathbb{C} \to \mathbb{Z}[1/p]$-mod extends uniquely to an additive functor $\mathbb{C}[1/p] \to \mathbb{Z}[1/p]$-mod. Thus the category $\mathrm{PT}_p(F)$ may be identified with the category of additive functors from $\mathrm{Cor}_F[1/p]$ to $\mathbb{Z}[1/p]$-mod. Our first claim follows now immediately from Theorem 1.11. Since $\phi^\#$ is left adjoint to an equivalence ϕ_*, we conclude that it coincides with the quasi-inverse equivalence. □

Corollary 1.14. *Let E/F be any purely inseparable field extension. Denote by $\psi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. Then $\psi_* : \mathrm{PT}_p(E) \to \mathrm{PT}_p(F)$ is an equivalence of categories with quasi-inverse $\psi^\#$.*

Proof. Note that $E \subset F_\infty$ and, moreover, $F_\infty = E_\infty$. Denote by ϕ' the extension of scalars functor corresponding to the field extension $E \subset E_\infty$. Our claim follows from Theorem 1.13 and the commutative diagram

$$\begin{tikzcd} \mathrm{PT}_p(E_\infty) \arrow[r, "="] \arrow[d, "\phi'"'] & \mathrm{PT}_p(F_\infty) \arrow[d, "\phi_*"'] \\ \mathrm{PT}_p(E) \arrow[r, "\psi_*"] & \mathrm{PT}_p(F) \end{tikzcd}$$

□

In the next section we'll need the following result:

Lemma 1.15. *Let $\mathcal{F} \in \mathrm{PT}_p(F)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules. Let, further, $f : X \to Y$ be a finite, surjective, purely inseparable morphism of irreducible smooth schemes. Then the homomorphism $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ is an isomorphism.*

Proof. It suffices to note that in Cor_F we have a morphism $f^T : Y \to X$, which yields a homomorphism $(f^T)^* : \mathcal{F}(X) \to \mathcal{F}(Y)$ and both compositions of f^* and $(f^T)^*$ are equal (according to Corollary 1.8) to multiplication by $\deg f$. □

2. Sheaves with transfers over nonperfect fields

We keep the notations introduced in the previous section. In particular we denote by $E = F_\infty$ the perfect closure of F. We denote by $\phi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. Finally we use the notation

$$\phi_* : \mathrm{PT}_p(F_\infty) \to \mathrm{PT}_p(F) \quad \text{and} \quad \phi^\# : \mathrm{PT}_p(F) \to \mathrm{PT}_p(F_\infty)$$

for the corresponding functors on presheaves with transfers of $\mathbb{Z}[1/p]$-modules.

Theorem 2.1. *Let $\mathcal{F} \in \mathrm{PT}_p(F_\infty)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules. Assume that $\phi_*(\mathcal{F})$ is a Zariski (resp. Nisnevich) sheaf. Then $\mathcal{F}$ itself is also a Zariski (resp. Nisnevich) sheaf.*

We start with the case of Zariski sheaves, which is somewhat more transparent.

Lemma 2.2. *Let $\pi : Y \to X$ be an integral surjective morphism of integral schemes. Assume further that the scheme X is normal and the extension of rational function fields is purely inseparable. Then π is a homeomorphism.*

Proof. Since π is integral we conclude that it is a closed map. Thus it suffices to establish that π is bijective. Our conditions show that π is surjective and we only need to verify the injectivity of π. Replacing X by an open affine scheme and Y by its inverse image, we see that it suffices to consider the case when $X = \mathrm{Spec}\, A$ (A is an integrally closed domain) and $Y = \mathrm{Spec}\, B$ is an affine integral scheme. Surjectivity of π readily implies that $\pi^* : A \to B$ is injective. Denote by F (resp. E) the field of fractions of A (resp. B). Since E/F is purely inseparable and A is integrally closed there is exactly one prime ideal in B over each prime ideal of A — see [Bourbaki 1972] — which means that π is bijective. □

Corollary 2.3. *Under the conditions and notation of Lemma 2.2, let $\mathcal{F}$ be a presheaf on the small Zariski site of Y. If $\pi_*(\mathcal{F})$ is a sheaf, $\mathcal{F}$ itself is a sheaf as well.*

Let $\mathcal{F}$ be a presheaf on Sm_F. For any $X \in \mathrm{Sm}_F$, restricting $\mathcal{F}$ to the small Zariski (resp. Nisnevich) site of X we get a presheaf $\mathcal{F}_X$ on X_{Zar} (resp. X_{Nis}). Moreover, for any morphism $f : X \to Y$ we get a canonical homomorphism $\alpha_f : \mathcal{F}_Y \to f_*(\mathcal{F}_X)$. Finally, for a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$,

$$\alpha_{gf} = g_*(\alpha_f) \circ \alpha_g.$$

Lemma 2.4. *Let $\mathcal{F} \in \mathrm{PT}_p(F)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules. Let, further, $f : X \to Y$ be a finite surjective purely inseparable morphism of smooth schemes. Then the associated homomorphism*

$$\alpha_f : \mathcal{F}_Y \to f_*(\mathcal{F}_X)$$

is an isomorphism (for both the Nisnevich and Zariski topologies).

Proof. We need to show that for any $U \in Y_{\mathrm{Nis}}$ (resp. $U \in Y_{\mathrm{Zar}}$) the canonical map

$$\mathcal{F}(U) \xrightarrow{p_2^*} \mathcal{F}(X \times_Y U)$$

is an isomorphism. It suffices to treat the case when U is irreducible, in which case (since f is purely inseparable) $X \times_Y U$ is also irreducible. Now our claim follows from Lemma 1.15, since p_2 is a finite, surjective, purely inseparable morphism. □

Proof of Theorem 2.1 for the Zariski topology. To show that $\mathcal{F}$ is a Zariski sheaf we have to verify that for any $Y \in \mathrm{Sm}_E$ the restriction $\mathcal{F}_Y$ is a sheaf. It suffices, clearly, to treat the case when Y is irreducible. Assume first that $Y = X_E$ for an appropriate $X \in \mathrm{Sm}_F$. Let $\pi : Y \to X$ be the structure morphism. Note that $\pi_*(\mathcal{F}_Y) = (\phi_*(\mathcal{F}))_X$ and hence $\pi_*(\mathcal{F}_Y)$ is a sheaf. Since $\pi : Y \to X$ is an integral, surjective, purely inseparable morphism of integral normal schemes we conclude by Corollary 2.3 that $\mathcal{F}_Y$ is a sheaf. In the general case, we use Lemma 1.12 and find a finite surjective purely inseparable morphism $Y \xrightarrow{f} X_E$. Lemma 2.4 shows that $\alpha_f : \mathcal{F}_{X_E} \to f_*(\mathcal{F}_Y)$ is an isomorphism. Thus $f_*(\mathcal{F}_Y)$ is a sheaf. Applying Corollary 2.3 again, we conclude that $\mathcal{F}_Y$ is a sheaf. □

Lemma 2.5. *Let $\mathcal{O}$ be a henselian local ring. Let, further, A be an integral $\mathcal{O}$-algebra. If A is local it is a local henselian ring.*

Proof. Assume first that A is finite over $\mathcal{O}$. Every finite A-algebra happens to be a finite $\mathcal{O}$-algebra and hence is a finite product of local rings. However, this property characterizes henselian local rings — see [Milne 1980] — so we conclude that A is henselian. In the general case, let B be any finitely generated subalgebra of A. Since A is integral over B, we conclude that there is a maximal ideal of A over each maximal ideal of B. Thus B has to be local and hence has to be a henselian local ring. The same reasoning shows that whenever $B \subset B'$ the inclusion $B \hookrightarrow B'$ is a local homomorphism. Now it suffices to use the following lemma. □

Lemma 2.6. *Let $\{B_i\}_{i \in I}$ be a filtering direct system of local rings and local homomorphisms. If all B_i are henselian then $A = \varinjlim_{i \in I} B_i$ is also a henselian local ring.*

Proof. Denote by $\mathfrak{M}_i$ and k_i the maximal ideal and the residue field of B_i. It's perfectly trivial to verify that A is a local ring with maximal ideal $\mathfrak{M} = \varinjlim \mathfrak{M}_i$ and residue field $k = \varinjlim k_i$. Let $f \in A[T]$ be a monic polynomial and assume that $\bar{f} = g_0 \cdot h_0$, where $g_0, h_0 \in k[T]$ are coprime monic polynomials. Clearly there exists $i \in I$ such that f comes from the monic polynomial $f(i) \in B_i[T]$ and $g_0, h_0 \in k[T]$ come from monic polynomials $g_0(i), h_0(i) \in k_i[T]$. Moreover, increasing i if required, we may assume that the identity $\overline{f(i)} = g_0(i) \cdot h_0(i)$ holds in $k_i[T]$. Clearly $g_0(i)$ and $h_0(i)$ are coprime in $k_i[T]$. Since B_i is henselian we conclude that $f(i) = g(i) \cdot h(i)$, where $g(i), h(i) \in B_i[T]$ are monic polynomials with $\overline{g(i)} = g_0(i)$ and $\overline{h(i)} = h_0(i)$. Taking the images of $g(i), h(i) \in B_i[T]$ in $A[T]$, we get the required factorization for $f \in A[T]$. $\square$

Corollary 2.7. *Let $\pi : Y \to X$ be an integral surjective morphism of integral schemes. Assume further that the scheme X is normal and the extension of fields of rational functions is purely inseparable. Let $x \in X$ be a point and $y \in Y$ be the unique point over x. In this case,*

$$Y \times_X \operatorname{Spec} \mathcal{O}_x^h = \operatorname{Spec} \mathcal{O}_y^h.$$

Proof. Obviously $Y \times_X \operatorname{Spec} \mathcal{O}_x^h = \operatorname{Spec} A$, where A is integral over $\mathcal{O}_x^h$. To give a maximal ideal in A is the same as to give a point in the fiber of $\operatorname{Spec} A \to \operatorname{Spec} \mathcal{O}_x^h$ over $\mathfrak{M}_x^h$. Since this fiber coincides with the fiber of $Y \to X$ over x, we conclude that A is local and its unique maximal ideal lies over $\mathfrak{M}_y$. Lemma 2.5 shows that A is a henselian local ring. On the other hand A is ind-étale over Y and hence coincides with $\mathcal{O}_y^h$. $\square$

Recall that whenever x is a point on the scheme X an étale neighborhood of x on X is an étale morphism $f : U \to X$ together with a point $u \in U$ such that $f(u) = x$ and the embedding of residue fields $k(x) \xrightarrow{f^*} k(u)$ is an isomorphism. Given two étale neighborhoods (U, u) and (V, v) of x on X, we say that U is finer than V if there exists a morphism $g : U \to V$ over X that takes u to v. Assume that $\pi : Y \to X$ is a morphism of schemes. Let, further, x be a point on X and $y \in Y$ be a point over x. Finally, let (V, v) be an étale neighborhood of x on X. In this case we define the induced étale neighborhood $\pi^{-1}(V)$ of y as follows: we take $U = V \times_X Y$ and define a morphism $\operatorname{Spec} k(y) \to U$ using the canonical morphisms (over X) $\operatorname{Spec} k(y) \to \operatorname{Spec} k(x) = \operatorname{Spec} k(v) \to V$ and $\operatorname{Spec} k(y) \to Y$. Straightforward verification shows that the resulting point $u \in U$ lies over y and its residue field equals $k(y)$.

Corollary 2.8. *Under the conditions and notation of Corollary 2.7, let (U, u) be an étale neighborhood of (Y, y). Then there exists an étale neighborhood (V, v) of (X, x) such that $\pi^{-1}(V)$ is finer than U.*

Proof. Obviously it suffices to treat the case when $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $U = \operatorname{Spec} R$ are affine. Since $\mathcal{O}_y^h$ may be identified with the direct limit of coordinate algebras of affine étale neighborhoods of (Y, y), we get a canonical B-algebra homomorphism $R \xrightarrow{\phi} \mathcal{O}_y^h = \mathcal{O}_x^h \otimes_A B$. Since R is a finitely presented B-algebra and $\mathcal{O}_x^h \otimes_A B = \varinjlim C \otimes_A B$, where C runs through coordinate algebras of affine étale neighborhoods of (X, x), we conclude that ϕ factors through $C \otimes_A B$ for appropriate C. Thus $V = \operatorname{Spec} C$ is an étale neighborhood of (X, x), whose inverse image to Y is finer than U. $\square$

Proposition 2.9. *Under the conditions and notation of Corollary 2.7, let $\mathcal{F}$ be a presheaf on the small Nisnevich site of Y. Then*

$$\check{H}^0_{\mathrm{Nis}}(Y, \mathcal{F}) = \check{H}^0_{\mathrm{Nis}}(X, \pi_*(\mathcal{F})).$$

Proof. Given a Nisnevich covering $\mathcal{V}$ of X we get the induced Nisnevich covering $\pi^{-1}(\mathcal{V})$ of Y. For these two coverings we have the obvious relation

$$\check{H}^0(\pi^{-1}(\mathcal{V}), \mathcal{F}) = \check{H}^0(\mathcal{V}, \pi_*(\mathcal{F})).$$

Since Čech cohomology may be computed using any cofinal family of coverings, it suffices to check that any Nisnevich covering of Y admits a refinement of the form $\pi^{-1}(\mathcal{V})$. However this readily follows from Corollary 2.8. □

Proof of Theorem 2.1 for the Nisnevich topology. We start with a presheaf $\mathcal{F} \in \mathrm{PT}_p(E)$ such that $\phi_*(\mathcal{F})$ is a Nisnevich sheaf. Since applying the functor $\check{H}^0_{\mathrm{Nis}}$ to a presheaf twice we get the associated sheaf, we see that to show that $\mathcal{F}$ is a sheaf it suffices to verify that, for any irreducible $Y \in \mathrm{Sm}_E$, the natural map $\mathcal{F}(Y) \to \check{H}^0_{\mathrm{Nis}}(Y, \mathcal{F}_Y)$ is an isomorphism. Assume first that $Y = X_E$ for an appropriate $X \in \mathrm{Sm}_F$. Apply Proposition 2.9 to the structure morphism $\pi : Y \to X$. In this way we get

$$\check{H}^0_{\mathrm{Nis}}(Y, \mathcal{F}) = \check{H}^0_{\mathrm{Nis}}(X, p_*(\mathcal{F})) = p_*(\mathcal{F})(X) = \mathcal{F}(Y).$$

In the general case we apply Lemma 1.12 and find a finite, surjective, purely inseparable morphism $f : Y \to X_E$. Lemma 2.4 shows that the natural map $\alpha_f : \mathcal{F}_{X_E} \to f_*(\mathcal{F}_Y)$ is an isomorphism and in particular $f^* : \mathcal{F}(X_E) \to \mathcal{F}(Y)$ is an isomorphism. Proposition 2.9 shows that the pull-back map

$$f^* : \check{H}^0_{\mathrm{Nis}}(X_E, \mathcal{F}_{X_E}) \to \check{H}^0_{\mathrm{Nis}}(X_E, f_*(\mathcal{F}_Y)) \to \check{H}^0_{\mathrm{Nis}}(Y, \mathcal{F}_Y)$$

is an isomorphism. Our claim follows now from the commutative diagram

$$\begin{tikzcd}
\mathcal{F}(X_E) \arrow[r, "\sim", "f^*"'] \arrow[d, "\cong"'] & \mathcal{F}(Y) \arrow[d] \\
\check{H}^0_{\mathrm{Nis}}(X_E, \mathcal{F}_{X_E}) \arrow[r, "\sim", "f^*"'] & \check{H}^0_{\mathrm{Nis}}(Y, \mathcal{F}_Y)
\end{tikzcd}$$

□

Corollary 2.10. *Let E/F be any purely inseparable field extension. Denote by $\psi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. Let, further, $\mathcal{F} \in \mathrm{PT}_p(E)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules over E. Assume that $\psi_*(\mathcal{F})$ is a Zariski (resp. Nisnevich) sheaf. Then $\mathcal{F}$ itself is also a Zariski (resp. Nisnevich) sheaf.*

Proof. Denote by ϕ (resp. ϕ') the extension of scalars functor corresponding to the field extension $F \subset F_\infty$ (resp. $E \subset E_\infty = F_\infty$), and set $\mathcal{G} = (\phi')^\#(\mathcal{F})$. Corollary 1.14

shows that $\mathcal{F} = \phi'_*(\mathcal{G})$. Thus $\phi_*(\mathcal{G}) = \psi_*(\phi'_*(\mathcal{G})) = \psi_*(\mathcal{F})$ is a sheaf. Theorem 2.1 shows that $\mathcal{G}$ is a sheaf and hence $\mathcal{F} = \phi'_*(\mathcal{G})$ is a sheaf as well. $\square$

Corollary 2.11. *Under the conditions and notation of Corollary 2.10 the functor $\psi^\# : \mathrm{PT}_p(F) \to \mathrm{PT}_p(E)$ takes Zariski (resp. Nisnevich) sheaves to Zariski (resp. Nisnevich) sheaves. More precisely, $\mathcal{F} \in \mathrm{PT}_p(F)$ is a sheaf if and only if $\psi^\#(\mathcal{F})$ is a sheaf. In particular for Nisnevich sheaves we have an identification $\psi^*(\mathcal{G}) = \psi^\#(\mathcal{G})$.*

Proof. This follows immediately from Corollary 2.10, since $\psi_*(\psi^\#(\mathcal{G}) = \mathcal{G}$ for any $\mathcal{G} \in \mathrm{PT}_p(F)$ according to Corollary 1.14. $\square$

Denote by $\mathrm{NST}_p(F)$ the category of Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules.

Corollary 2.12. *Let E/F be any purely inseparable field extension. Denote by $\psi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. Then $\psi_* : \mathrm{NST}_p(E) \to \mathrm{NST}_p(F)$ is an equivalence of categories with quasi-inverse $\psi^* = \psi^\#$.*

Theorem 2.13. *Let $\mathcal{F} \in \mathrm{NST}_p(F)$ be a Nisnevich sheaf with transfers of $\mathbb{Z}[1/p]$-modules. Then, for any scheme $X \in \mathrm{Sm}_F$, we have a natural identification of cohomology groups*

$$H^i_{\mathrm{Nis}}(X, \mathcal{F}) = H^i_{\mathrm{Nis}}(X_{F_\infty}, \mathcal{F}_{F_\infty}).$$

Proof. By Corollary 2.12, $\phi_*(\mathcal{F}_{F_\infty}) = \mathcal{F}$. Denote by $\pi : Y = X_{F_\infty} \to X$ the structure morphism and by $\mathcal{G}$ the sheaf $(\mathcal{F}_{F_\infty})_Y$. Thus $\pi_*(\mathcal{G}) = \mathcal{F}_X$. Applying the Leray spectral sequence to π we see that it suffices to establish that $R^i\pi_*\mathcal{G} = 0$ for $i > 0$. However $(R^i\pi_*\mathcal{G})_x = H^i_{\mathrm{Nis}}(Y \times_X \operatorname{Spec} \mathcal{O}^h_x, \mathcal{G}) = H^i(\operatorname{Spec} \mathcal{O}^h_y, \mathcal{G}) = 0$. $\square$

3. Homotopy invariant sheaves with transfers over nonperfect fields

Proposition 3.1. *Let $\mathcal{F} \in \mathrm{PT}_p(F_\infty)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules. Assume that $\phi_*(\mathcal{F})$ is homotopy invariant, where ϕ_* is as in Theorem 1.13. Then $\mathcal{F}$ itself is also homotopy invariant.*

Proof. We need to verify that for any irreducible $Y \in \mathrm{Sm}_{F_\infty}$ the pull-back map $\mathcal{F}(Y) \xrightarrow{p_1^*} F(Y \times \mathbb{A}^1)$ is an isomorphism. To do so we use Lemma 1.12 and find a finite, surjective, purely inseparable morphism $f : Y \to X_{F_\infty}$. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{F}(X_{F_\infty}) = (\phi)_*(\mathcal{F})(X) \arrow[r, "f^*"] \arrow[d, "p_1^*"] & \mathcal{F}(Y) \arrow[d, "p_1^*"] \\
\mathcal{F}(X_{F_\infty} \times \mathbb{A}^1) = \phi_*(\mathcal{F})(X \times \mathbb{A}^1) \arrow[r, "(f \times 1_{\mathbb{A}^1})^*"] & \mathcal{F}(Y \times \mathbb{A}^1)
\end{tikzcd}$$

Lemma 1.15 shows that both horizontal arrows are isomorphisms and the left vertical arrow is an isomorphism by assumption. Thus the right vertical arrow is an isomorphism as well. □

Using the same machinery as in the proof of Corollary 2.10 we readily verify:

Corollary 3.2. *Let E/F be any purely inseparable field extension. Denote by $\psi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. Let, further, $\mathcal{F} \in \mathrm{PT}_p(E)$ be a presheaf with transfers of $\mathbb{Z}[1/p]$-modules over E. Assume that the presheaf $\psi_*(\mathcal{F})$ is homotopy invariant. Then $\mathcal{F}$ itself is also homotopy invariant.*

Corollary 3.3. *Under the conditions and notation of Corollary 3.2 let $\mathcal{F} \in \mathrm{PT}_p(F)$ be a homotopy invariant presheaf with transfers of $\mathbb{Z}[1/p]$-modules. Then $\mathcal{F}_E$ is also homotopy invariant. Conversely, if $\mathcal{F}_E$ is homotopy invariant then $\mathcal{F}$ is also.*

Proof. This follows immediately from Proposition 3.1, since $\psi_*(\mathcal{F}_E) = \mathcal{F}$ according to Corollary 2.12. □

Theorem 3.4. *Let $\mathcal{F} \in \mathrm{NST}_p(F)$ be a homotopy invariant sheaf with transfers of $\mathbb{Z}[1/p]$-modules. Then $\mathcal{F}$ is strictly homotopy invariant, i.e., $H^i_{\mathrm{Nis}}(X \times \mathbb{A}^1, \mathcal{F}) = H^i_{\mathrm{Nis}}(X, \mathcal{F})$ for any i and any $X \in \mathrm{Sm}_F$.*

Proof. Corollary 3.3 shows that $\mathcal{F}_{F_\infty}$ is a homotopy invariant sheaf with transfers over a perfect field F_∞. Thus Voevodsky's Theorem 13.8 [Mazza et al. 2006] shows that $\mathcal{F}_{F_\infty}$ is strictly homotopy invariant. Finally, using Theorem 2.13 we get

$$H^i_{\mathrm{Nis}}(X \times \mathbb{A}^1, \mathcal{F}) = H^i_{\mathrm{Nis}}(X_{F_\infty} \times \mathbb{A}^1, \mathcal{F}_{F_\infty})) = H^i_{\mathrm{Nis}}(X_{F_\infty}, \mathcal{F}_{F_\infty}) = H^i_{\mathrm{Nis}}(X, \mathcal{F}). \quad \square$$

Theorem 3.4 shows that all results proved in [Mazza et al. 2006] over perfect fields hold over all fields once we work with presheaves with transfers of $\mathbb{Z}[1/p]$-modules. In particular we have the following result:

Theorem 3.5. *Let $\mathcal{F} \in \mathrm{PT}_p(F)$ be a homotopy invariant presheaf with transfers of $\mathbb{Z}[1/p]$-modules; then:*

(1) *The sheaf $\mathcal{F}_{\mathrm{Zar}}$ coincides with $\mathcal{F}_{\mathrm{Nis}}$ and has a natural structure of a homotopy invariant sheaf with transfers.*
(2) *$H^i_{\mathrm{Zar}}(X, \mathcal{F}_{\mathrm{Zar}}) = H^i_{\mathrm{Nis}}(X, \mathcal{F}_{\mathrm{Nis}})$ for any $X \in \mathrm{Sm}_F$ and any $i \geq 0$.*
(3) *The presheaves $X \mapsto H^i_{\mathrm{Zar}}(X, \mathcal{F}_{\mathrm{Zar}}) = H^i_{\mathrm{Nis}}(X, \mathcal{F}_{\mathrm{Nis}})$ are homotopy invariant presheaves with transfers.*

Proof. The first claim is proved in [Mazza et al. 2006, Theorems 22.1 and 22.2] for arbitrary homotopy invariant presheaves with transfers. In view of this fact, in the sequel we may assume that $\mathcal{F} \in \mathrm{NST}_p(F)$ is a homotopy invariant Nisnevich sheaf with transfers of $\mathbb{Z}[1/p]$-modules. Homotopy invariance of the presheaf $X \mapsto H^i_{\mathrm{Nis}}(X, \mathcal{F})$ is proved above in Theorem 3.4. The fact that $X \mapsto H^i_{\mathrm{Nis}}(X, \mathcal{F})$

has a natural structure of a presheaf with transfers is proved in [Mazza et al. 2006, Lemma 13.4] for arbitrary Nisnevich sheaves with transfers. Finally, the coincidence of Zariski and Nisnevich cohomology follows easily from the homotopy invariance of Nisnevich cohomology; see the proof of [Mazza et al. 2006, Proposition 13.9]. □

Once Theorem 3.5 is proved we may proceed the same way as in [Suslin and Voevodsky 2000] and define the category of effective motives $\mathrm{DM}_p^-(F)$. Specifically we define the category $\mathrm{DM}_p^-(F)$ as a full subcategory of the derived category $D^-(\mathrm{NST}_p(F))$ of bounded above complexes of Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules comprising the complexes with homotopy invariant cohomology sheaves.

For any $X \in \mathrm{Sm}_F$ we define its motive $M_p(X)$ as the complex $C^\bullet(\mathbb{Z}[1/p]_{\mathrm{tr}}(X))$ in $\mathrm{DM}_p^-(F)$; cf. [Suslin and Voevodsky 2000, §1]. Nisnevich cohomology may be recovered in terms of $\mathrm{DM}_p^-(F)$. The proof of the following result is identical to the proof of Theorem 1.5 in [Suslin and Voevodsky 2000].

Theorem 3.6. *For any complex $A^\bullet \in \mathrm{DM}_p^-(F)$ and any $X \in \mathrm{Sm}_F$ we have natural isomorphisms*

$$H^i_{\mathrm{Nis}}(X, A^\bullet) = \mathrm{Hom}_{\mathrm{DM}_p^-(F)}(M_p(X), A^\bullet[i]).$$

The category $\mathrm{DM}_p^-(F)$ may be also viewed as a localization of the category $D^-(\mathrm{NST}_p(F))$ with respect to a thick triangulated subcategory $\mathcal{A}$. Recall that $A^\bullet \mapsto C^\bullet(A^\bullet)$ defines a functor $C^\bullet : D^-(\mathrm{NST}_p(F)) \to \mathrm{DM}_p^-(F)$ and take $\mathcal{A} \subset D^-(\mathrm{NST}_p(F))$ to be the full triangulated subcategory consisting of those complexes $A^\bullet$ for which the complex $C^\bullet(A^\bullet)$ is acyclic. The proof of the following result is identical to the proof of Theorem 1.12 in [Suslin and Voevodsky 2000].

Theorem 3.7. (1) *The functor $C^\bullet$ is left adjoint to the embedding functor*

$$\mathrm{DM}_p^-(F) \subset D^-(\mathrm{NST}_p(F))$$

and shows the equivalence of $\mathrm{DM}_p^-(F)$ with the localization of $D^-(\mathrm{NST}_p(F))$ with respect to the thick triangulated subcategory $\mathcal{A}$.

(2) *A complex $A^\bullet \in D^-(\mathrm{NST}_p(F))$ is in $\mathcal{A}$ if and only if it is quasi-isomorphic to a bounded above complex of contractible Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules.*

Finally recall (see [Suslin and Voevodsky 2000, §2]) that the category $\mathrm{DM}_p^-(F)$ has a natural tensor structure given by the formula $A^\bullet \otimes B^\bullet = C^\bullet(A^\bullet \otimes^L B^\bullet)$. The main properties of this operation listed in Proposition 2.8 of [Suslin and Voevodsky 2000] remain true over nonperfect fields provided that we work with complexes of Nisnevich sheaves with transfers of $\mathbb{Z}[1/p]$-modules.

For future use we recall briefly the explicit definition of $A^\bullet \otimes^L B^\bullet$. See [Mazza et al. 2006, Definition 8.2]. Given two complexes $A^\bullet$ and $B^\bullet$ in $\mathrm{DM}_p^-(F)$ we pick quasi-isomorphisms $A_1^\bullet \to A^\bullet$ and $B_1^\bullet \to B^\bullet$ with free complexes $A_1^\bullet$, $B_1^\bullet$ (i.e., consisting of direct sums of sheaves $\mathbb{Z}[1/p]_{\mathrm{tr}}(X)$) and set $A^\bullet \otimes^L B^\bullet = \mathrm{Tot}(A_1^\bullet \otimes B_1^\bullet)$, where $A_1^\bullet \otimes B_1^\bullet$ is a bicomplex consisting of sheaves $(A_1)^i \otimes_{\mathrm{tr}} (B_1)^j$. The same reasoning as in [Suslin and Voevodsky 2000] show that the resulting complex is independent of the choice of free resolutions $A_1^\bullet$ and $B_1^\bullet$ up to a natural quasi-isomorphism.

4. Extension of scalars for the category $\mathrm{DM}_p^-(F)$

Let E/F be a field extension. Denote by $\phi : \mathrm{Cor}_F \to \mathrm{Cor}_E$ the corresponding extension of scalars functor. In the sequel we denote by p the exponential characteristic of F (i.e., $p = 1$ for fields of characteristic zero). We work with the category $\mathrm{PT}_p(F)$ of presheaves with transfers of $\mathbb{Z}[1/p]$-modules.

Theorem 4.1. *The functor $\phi^\# : \mathrm{PT}_p(F) \to \mathrm{PT}_p(E)$ takes Nisnevich (resp. Zariski) sheaves to Nisnevich (resp. Zariski) sheaves, so that $\phi^* = \phi^\#$. Furthermore, the functor $\phi^\# : \mathrm{PT}_p(F) \to \mathrm{PT}_p(E)$ is exact.*

Proof. Consider first the special case when the field F is perfect and E is finitely generated over F. In this case E may be written in the form $E = F(S)$ for an appropriate smooth, irreducible scheme of finite type $S \in \mathrm{Sm}_F$. In this case Spec E may be further identified with the inverse limit $\mathrm{Spec}\, E = \varprojlim U$, where U runs through a directed inverse system of open affine neighborhoods of the generic point $\eta \in S$ and we may apply the following classical result; see [EGA IV$_1$ 1964]:

Theorem 4.2. *Let I be a directed partially ordered set. Let, further, $\{S_i\}_{i\in I}$ be an inverse system of schemes over I with affine transition morphisms. Assume that all the S_i are quasicompact and quasiseparated and set $S = \varprojlim S_i$.*

(1) *For any morphism of finite presentation $X \to S$ there exists an index $i \in I$ and a morphism of finite presentation $X_i \to S_i$ such that $X \cong X_i \times_{S_i} S$ (as schemes over S).*

(2) *Given $i \in I$, schemes of finite presentation X_i and Y_i over S_i and a morphism $\phi : X_i \times_{S_i} S \to Y_i \times_{S_i} S$ over S, there exists an index $i' \geq i$ and a morphism $\phi_{i'} : X_i \times_{S_i} S_{i'} \to Y_i \times_{S_i} S_{i'}$ over $S_{i'}$ whose base change to S is ϕ.*

(3) *Given $i \in I$, X_i and Y_i of finite presentation over S_i and a pair of morphisms $\phi_i, \psi_i : X_i \to Y_i$ (over S_i) whose base changes to S are equal, there exists $i' \geq i$ such that the base changes of ϕ_i and ψ_i to $S_{i'}$ are equal.*

(4) *Assume that X_i and Y_i are schemes of finite presentation over S_i and $\phi_i : X_i \to Y_i$ is a morphism over S_i. Denote by X (resp. Y) the scheme obtained*

from X_i (resp. Y_i) by base change from S_i to S. In a similar way denote by $\phi : X \to Y$ the morphism obtained from ϕ_i by base change from S_i to S. Finally, for any $j \geq i$ let X_j, Y_j and ϕ_j be schemes and morphisms obtained from X_i, Y_i and ϕ_i by base change from S_i to S_j. Assume that the morphism ϕ has one of the following properties:

(a) *ϕ is an isomorphism.*
(b) *ϕ is an open (resp closed) embedding.*
(c) *ϕ is surjective.*
(d) *ϕ admits a section.*
(e) *ϕ is finite.*
(f) *ϕ is étale.*
(g) *ϕ is smooth.*

Then there exists an index $j \geq i$ such that ϕ_j has the same property.

Corollary 4.3. *Under the conditions and notation of Theorem 4.2 assume that the scheme S is Noetherian. Let, further, X/S be a reduced scheme of finite type and let $Y \xrightarrow{\phi} X$ be a (singleton) Nisnevich covering of X. Then there exists $i \in I$, schemes of finite presentation X_i and Y_i over S_i, and a Nisnevich covering $Y_i \xrightarrow{\phi_i} X_i$ such that $X = X_i \times_{S_i} S$, $Y = Y_i \times_{S_i} S$ and $\phi = \phi_i \times_{S_i} S$.*

Proof. Theorem 4.2 shows that we may assume that $X = X_i \times_{S_i} S$, $Y = Y_i \times_{S_i} S$ and $\phi = \phi_i \times_{S_i} S$ for appropriate X_i, Y_i and ϕ_i. Since ϕ is étale we conclude from Theorem 4.2 that, increasing i, we may assume that ϕ_i is étale as well. Finally, since ϕ is a Nisnevich covering of a Noetherian scheme we conclude from [Hoyois 2012, Proposition 1.6] that there exists a chain of closed subschemes $\varnothing = X_0 \subset X_1 \subset \cdots \subset X_n = X$ such that ϕ admits a section over $X_i \setminus X_{i-1}$ for $1 \leq i \leq n$. Theorem 4.2 shows that we may assume that $X_k = X_{ik} \times_{S_i} S$, where $\varnothing = X_{i0} \subset X_{i1} \subset \cdots \subset X_{in} = X_i$ is a chain of closed subschemes of X_i and ϕ_i admits a section over $X_{ik} \setminus X_{i,k-1}$. In this case $\phi_i : Y_i \to X_i$ is obviously a Nisnevich covering of X_i. □

We still assume that F is perfect and E is finitely generated over F. We fix a smooth scheme of finite type $S \in \mathrm{Sm}_F$ such that $E = F(S)$. Let $\mathcal{F} : \mathrm{Sm}_F \to \mathrm{Ab}$ be a presheaf of abelian groups on the category Sm_F. In this case we define a new presheaf $a\mathcal{F}$ on Sm_E using the following construction. Let $X \in \mathrm{Sm}_E$ be a smooth scheme of finite type over $\operatorname{Spec} E$. According to Theorem 4.2 we may find a nonempty open $U \subset S$ and a smooth scheme of finite type $\widetilde{X} \to U$ whose generic fiber $\widetilde{X}_\eta$ coincides with X. In this case we'll be saying that $\widetilde{X}$ is a model of X defined over U. In this situation we set

$$a\mathcal{F}(X) = \varinjlim_{V \subset U} \mathcal{F}(\widetilde{X}_V).$$

Let $Y \in \mathrm{Sm}_E$ be another smooth scheme of finite type over Spec E and let $f : X \to Y$ be a morphism over Spec E. Let, further, $\widetilde{Y} \to V$ be a model of Y defined over V. Theorem 4.2 shows that, shrinking V, we may find a morphism $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ over V whose generic fiber equals f. Moreover, given two such extensions $\tilde{f}_1$ and $\tilde{f}_2$ of f, we may find a nonempty open set $W \subset V$ such that $\tilde{f}_1$ and $\tilde{f}_2$ agree on $\widetilde{X}_W$. These considerations readily imply that the definition of $a\mathcal{F}(X)$ is independent of the particular choice of $\widetilde{X}$ and moreover that $a\mathcal{F}$ has a natural structure of an abelian presheaf on Sm_E.

Proposition 4.4. *Assume that $\mathcal{F}$ is a Zariski* (*resp. Nisnevich*) *sheaf. Then $a\mathcal{F}$ is also a Zariski* (*resp. Nisnevich*) *sheaf.*

Proof. Consider first the case of the Zariski topology. Since all schemes in question are quasicompact it suffices to consider finite coverings. Let $X = \bigcup_{i=1}^n X_i$ be an open covering of $X \in \mathrm{Sm}_E$. Let, further, $\widetilde{X}$ be the model of X defined over $U \subset S$. Theorem 4.2 readily implies that, shrinking U, we may find an open covering $\widetilde{X} = \bigcup_{i=1}^n \widetilde{X}_i$ such that $\widetilde{X}_i$ is a model of X_i. For any $V \subset U$ we have an open covering $\widetilde{X}_V = \bigcup_{i=1}^n (\widetilde{X}_i)_V$ and hence an exact sequence

$$0 \to \mathcal{F}(\widetilde{X}_V) \to \prod_{i=1}^n \mathcal{F}((\widetilde{X}_i)_V) \to \prod_{i,j=1}^n \mathcal{F}((\widetilde{X}_i \cap \widetilde{X}_j)_V).$$

Passing to the direct limit over V we get the required exact sequence

$$0 \to a\mathcal{F}(X) \to \prod_{i=1}^n a\mathcal{F}(X_i) \to \prod_{i,j=1}^n a\mathcal{F}(X_i \cap X_j).$$

Next we consider the case of the Nisnevich topology. Let X be a smooth scheme of finite type over Spec E and let $\widetilde{X} \to U$ be its model defined over an open $U \subset S$.

To finish the proof of Proposition 4.4 we start with a singleton Nisnevich covering $f : Y \to X$. According to Corollary 4.3 we may assume that $f = \tilde{f}_\eta$, where $\tilde{f} : \widetilde{Y} \to \widetilde{X}_V$ is a Nisnevich covering of $\widetilde{X}_V$. For any open set $W \subset V$ the induced map $\tilde{f}_W : \widetilde{Y}_W \to \widetilde{X}_W$ is also a Nisnevich covering and, since $\mathcal{F}$ is a Nisnevich sheaf, we conclude that the sequence

$$0 \to \mathcal{F}(\widetilde{X}_W) \to \mathcal{F}(\widetilde{Y}_W) \to \mathcal{F}(\widetilde{Y}_W \times_{\widetilde{X}_W} \widetilde{Y}_W)$$

is exact. Passing to the direct limit over W we get the exactness of the sequence

$$0 \to a\mathcal{F}(X) \to a\mathcal{F}(Y) \to a\mathcal{F}(Y \times_X Y).$$

The case of an arbitrary Nisnevich covering readily follows since we already know that $a\mathcal{F}$ is a Zariski sheaf. □

Proposition 4.5. *Assume that $\mathcal{F}$ is a presheaf with transfers. Then $a\mathcal{F}$ also has a natural structure of a presheaf with transfers.*

Proof. For the proof we need the following elementary definition of finite correspondences in a relative situation. The definition we give is a very special and absolutely elementary case of the construction discussed in Section 1A of [Mazza et al. 2006].

Let U be a smooth irreducible scheme of finite type over F. Let, further, X/U and Y/U be smooth schemes of finite type over U (and a fortiori over F). Denote by $\mathrm{Cor}_U(X, Y)$ the free abelian group generated by closed integral subschemes (called elementary finite correspondences) $T \subset X \times_U Y$ that are finite and surjective over a component of X. Note that $\mathrm{Cor}_U(X, Y)$ is a subgroup in $\mathrm{Cor}_F(X, Y)$. If Z/U is another smooth scheme of finite type over U, we have the usual composition map $\mathrm{Cor}_F(Y, Z)\times\mathrm{Cor}_F(X, Y)\to\mathrm{Cor}_F(X, Z)$, and a straightforward verification shows that this composition map takes $\mathrm{Cor}_U(Y, Z)\times\mathrm{Cor}_U(X, Y)$ to $\mathrm{Cor}_U(X, Z)$. In this way we get the category of finite correspondences over U, whose objects are smooth schemes of finite type over U and morphisms from X/U to Y/U are represented by finite correspondences over U. We denote the corresponding category by Cor_U. Note further that if $V \subset U$ is a nonempty open set we get a canonical functor res^U_V from Cor_U to Cor_V, which takes X/U to X_V/V and takes the relative finite correspondence $T \subset X \times_U Y$ to $T_V \subset X_V \times_V Y_V$.

Let η be the generic point of U. Denote by E' the residue field at η (i.e., the field of rational functions $E' = F(U)$). In this case we get a canonical functor from Cor_U to $\mathrm{Cor}_{E'}$, which takes X/U to its generic fiber $X_\eta = X \times_U \operatorname{Spec} E'$ and takes an elementary correspondence $T \subset X \times_U Y$ to the elementary correspondence represented by the generic fiber $T_\eta \subset X_\eta \times_{\operatorname{Spec} E'} Y_\eta$. Whenever V is a nonempty open subscheme of U the following diagram obviously commutes:

$$\begin{array}{ccc} \mathrm{Cor}_U & \longrightarrow & \mathrm{Cor}_V \\ \downarrow & & \downarrow \\ \mathrm{Cor}_{E'} & \xrightarrow{=} & \mathrm{Cor}_{E'} \end{array}$$

Hence, for any $X, Y \in \mathrm{Sm}_U$ we get the induced map

$$\varinjlim_{V\subset U} \mathrm{Cor}_V(X_V, Y_V) \to \mathrm{Cor}_{E'}(X_\eta, Y_\eta).$$

Lemma 4.6. *For any $X, Y \in \mathrm{Sm}_U$ the natural map*

$$\varinjlim_{V\subset U} \mathrm{Cor}_V(X_V, Y_V) \to \mathrm{Cor}_{E'}(X_\eta, Y_\eta)$$

is an isomorphism.

Proof. To prove the injectivity of the map in question it suffices to establish that the canonical map $\mathrm{Cor}_U(X, Y) \to \mathrm{Cor}_{E'}(X_\eta, Y_\eta)$ is injective. Since this map takes the generators of the free abelian group $\mathrm{Cor}_U(X, Y)$ to the generators of the free abelian group $\mathrm{Cor}_{E'}(X_\eta, Y_\eta)$, we only need to verify that different generators go to different generators, and this follows from the fact that the generic fiber T_η is dense in T.

To prove surjectivity, start with a closed integral subscheme $T_0 \subset X_\eta \times_{\mathrm{Spec}\, E'} Y_\eta = (X \times_U Y)_\eta$ that is finite and surjective over X_η. Theorem 4.2 shows that after shrinking U we can find a model T for T_0 defined over U. The E'-morphism $i : T_\eta \to (X \times_U Y)_\eta$ may be extended (after diminishing U) to a morphism $\tilde{i} : T \to X \times_U Y$. Since i is a closed embedding we see that, diminishing U, we may assume that $\tilde{i}$ is a closed embedding as well, i.e., we may assume that T is a closed subscheme of $X \times_U Y$. Furthermore, since the projection $p_1 : T_\eta = T_0 \to X_\eta$ is finite we conclude that we may assume that T is finite over X. Finally, since T_η is integral it follows easily — cf. [EGA IV$_3$ 1966, Corollaire (8.7.3)] — that we may assume that T is integral. In this way we get an elementary correspondence $T \in \mathrm{Cor}_U(X, Y)$ whose image in $\mathrm{Cor}_{E'}(X_\eta, Y_\eta)$ equals T_0. □

Corollary 4.7. *Assume that $X \in \mathrm{Sm}_U$ and $Y \in \mathrm{Sm}_F$. Then*

$$\mathrm{Cor}_{E'}(X_\eta, Y_{E'}) = \varinjlim_{V \subset U} \mathrm{Cor}_F(X_V, Y).$$

Proof. This follows immediately from the previous proposition in view of the obvious identification (valid for any $V \subset U$)

$$\mathrm{Cor}_V(X_V, Y \times_{\mathrm{Spec}\, F} V) = \mathrm{Cor}_F(X_V, Y)$$ □

To finish the proof of Proposition 4.5 we note that given a section $s_0 \in a\mathcal{F}(Y)$ and a finite correspondence $T_0 \in \mathrm{Cor}_E(X, Y)$ we may pick an open set $U \subset S$, models $\widetilde{X}/U$ and $\widetilde{Y}/U$ for X and Y, and representatives $s \in \mathcal{F}(\widetilde{Y})$ and $T \in \mathrm{Cor}_U(\widetilde{X}, \widetilde{Y})$ for s_0 and T_0, respectively, and take $T_0^*(s_0) \in a\mathcal{F}(X)$ to be the canonical image of $T^*(s) \in \mathcal{F}(\widetilde{X})$. A standard verification based on the repeated use of Theorem 4.2 shows that the resulting section is independent of all choices made and we really get a presheaf with transfers structure on $a\mathcal{F}$. □

In a similar way we show that whenever $f : \mathcal{F}_1 \to \mathcal{F}_2$ is a homomorphism of presheaves with transfers the resulting map $af : a\mathcal{F}_1 \to a\mathcal{F}_2$ is also compatible with transfers. In other words we get a functor $a : \mathrm{PT}(F) \to \mathrm{PT}(E)$. This functor is obviously exact and commutes with arbitrary direct sums. Moreover, Corollary 4.7 shows that when we apply this functor to $\mathbb{Z}_{\mathrm{tr}}(Y)$ we get $\mathbb{Z}_{\mathrm{tr}}(Y_E)$. Proposition 1.1 readily implies that the functor $\phi^\#$ coincides with a and hence takes sheaves to sheaves and is exact.

Still assuming that F is perfect, consider the case of an arbitrary extension E/F. Let $\{E_i\}_{i\in I}$ be the direct system of finitely generated subextensions of E (ordered by inclusion). Set $\mathcal{F}_i = \mathcal{F}_{E_i}$. Assume that $i \le j$. In this case denote by $\phi_i^j : \mathrm{Cor}_{E_i} \to \mathrm{Cor}_{E_j}$ the extension of scalars functor from E_i to E_j. Since $(\phi_i^j)^{\#}(\mathcal{F}_i) = \mathcal{F}_j$ we get a canonical homomorphism $\mathcal{F}_i \to (\phi_i^j)_*(\mathcal{F}_j)$ and so, for any $X_i \in \mathrm{Sm}_{E_i}$, setting $X_j = (X_i)_{E_j}$, we have a canonical map $\mathcal{F}_i(X_i) \to \mathcal{F}_j(X_j)$. A straightforward verification shows that $\{\mathcal{F}_j(X_j)\}_{j\ge i}$ is a direct system of abelian groups.

We use the same approach as before to construct an appropriate model for the functor $\phi^{\#}$. Let X be a smooth scheme of finite type over E. Theorem 4.2 shows that we can find $i \in I$ and a smooth scheme of finite type X_i over E_i such that $X = (X_i)_E$. Defining X_j for $j \ge i$ in the same way as before we set $a\mathcal{F}(X) = \varinjlim_{j\ge i} \mathcal{F}_j(X_j)$. The same reasoning as before shows that the resulting group is independent of the particular choice of i and X_i; moreover, $a\mathcal{F}$ is a presheaf with transfers whenever $\mathcal{F}$ is and $a\mathcal{F}$ is a sheaf in Zariski or Nisnevich topology whenever $\mathcal{F}$ is. The resulting functor a is clearly exact and commutes with arbitrary direct sums. Since $a\mathbb{Z}_{\mathrm{tr}}(X) = \mathbb{Z}_{\mathrm{tr}}(X_E)$, we conclude from Proposition 1.1 that $a = \phi^{\#}$.

Now we are ready to finish the proof of Theorem 4.1. Denote by F_∞ and E_∞ the perfect closures of F and E, respectively. We get a commutative diagram of fields

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d, "\psi_F"'] & E \arrow[d, "\psi_E"] \\ F_\infty \arrow[r, "\phi_\infty"] & E_\infty \end{tikzcd}$$

which yields the associated commutative diagram of functors

$$\begin{tikzcd} \mathrm{PT}(F) \arrow[r, "\phi^{\#}"] \arrow[d, "\psi_F^{\#}"'] & \mathrm{PT}(E) \arrow[d, "\psi_E^{\#}"] \\ \mathrm{PT}(F_\infty) \arrow[r, "\phi_\infty^{\#}"] & \mathrm{PT}(E_\infty). \end{tikzcd}$$

Since both vertical arrows are equivalences and the bottom horizontal arrow is exact we conclude that the top horizontal arrow is exact as well. The claim concerning sheaves is proved in the same way, using Corollary 2.11. $\square$

Corollary 4.8. *Under the conditions and notations of Theorem 4.1 the base change functor* $\phi^* : \mathrm{NST}_p(F) \to \mathrm{NST}_p(E)$ *is exact.*

Proof. This functor is right exact as a left adjoint to the functor ϕ_* and is left exact because left exactness in the category of sheaves is equivalent to the left exactness in the category of presheaves. $\square$

Proposition 4.9. *Under the conditions and notations of Theorem 4.1, the functor $\phi^\#$ takes homotopy invariant presheaves with transfers to homotopy invariant presheaves with transfers.*

Proof. We follow the same steps as in the proof of the Theorem 4.1. Assume first that F is perfect and E is finitely generated over F. Let S/F be a smooth irreducible scheme of finite type for which $E = F(S)$. Let X be a scheme of finite type over E. Find a model $\widetilde{X}/U$ for X defined over an appropriate $U \subset S$. In the course of the proof of Theorem 4.1 we established that $\phi^\#(\mathcal{F})(X) = \varinjlim_{V\subset U} \mathcal{F}(\widetilde{X}_V)$. At the same time $\mathbb{A}^1 \times \widetilde{X}$ is a model for $\mathbb{A}^1 \times X$ and hence

$$(\phi^\#\mathcal{F})(\mathbb{A}^1 \times X) = \varinjlim_{V\subset U} \mathcal{F}(\mathbb{A}^1 \times \widetilde{X}_V) = \varinjlim_{V\subset U} \mathcal{F}(\widetilde{X}_V) = \phi^\#\mathcal{F}(X)$$

In a similar way we consider the case of an arbitrary extension of a perfect field and finally use Corollary 3.3. □

The same reasoning establishes the validity of the following claim:

Lemma 4.10. *Under the conditions and notations of Theorem 4.1, for any presheaf $\mathcal{F} \in \mathrm{PT}_p(F)$ we have the formula*

$$\phi^\#(C_n(\mathcal{F})) = C_n(\phi^\#(\mathcal{F})).$$

The same arguments may be used to prove the following result, which concerns categories of all Nisnevich sheaves with transfers.

Corollary 4.11. *Let F be a perfect field and let E/F be an arbitrary field extension. The functor $\phi^\# : \mathrm{PT}(F) \to \mathrm{PT}(E)$ takes Nisnevich (resp. Zariski) sheaves to Nisnevich (resp. Zariski) sheaves, so that $\phi^* = \phi^\#$. Furthermore, the functor $\phi^\# : \mathrm{PT}_p(F) \to \mathrm{PT}_p(E)$ is exact, takes homotopy invariant presheaves to homotopy invariant presheaves and commutes with C_n.*

Theorem 4.1 together with Proposition 4.9 and Lemma 4.10 immediately imply that, associating to a complex $A^\bullet \in \mathrm{DM}_p^-(F)$ the complex $A^\bullet_E$, we get a well-defined triangulated functor $\phi^* : \mathrm{DM}_p^-(F) \to \mathrm{DM}_p^-(E)$. The following result summarizes the properties of this functor:

Theorem 4.12. (1) *The functor ϕ^* takes exact triangles to exact triangles and commutes with shifts.*

(2) *The functor ϕ^* takes tensor products to tensor products.*

(3) *$\phi^*(M_p(X)) = M_p(X_E)$ for any smooth scheme of finite type X over F.*

Proof. The first claim is obvious; the third one follows from Proposition 1.1 and Lemma 4.10. To prove the second claim we start with arbitrary complexes $A^\bullet, B^\bullet \in \mathrm{DM}_p^-(F)$ and pick their free resolutions $A^\bullet_1 \to A^\bullet$ and $B^\bullet_1 \to B^\bullet$. In this case $A^\bullet \otimes B^\bullet$ is quasi-isomorphic to $C^\bullet(A_1^\bullet \otimes_{\mathrm{tr}} B_1^\bullet)$ and hence $\phi^*(A^\bullet \otimes B^\bullet)$ is quasi-isomorphic to $\phi^*(C^\bullet(A_1^\bullet \otimes_{\mathrm{tr}} B_1^\bullet) = C^\bullet(\phi^*(A_1^\bullet \otimes_{\mathrm{tr}} B_1^\bullet)) = C^\bullet(\phi^*(A_1^\bullet) \otimes_{\mathrm{tr}} \phi^*(B_1^\bullet))$. Since $\phi^*(A_1^\bullet)$ and $\phi^*(B_1^\bullet)$ are free resolutions of $\phi^*(A^\bullet)$ and $\phi^*(B^\bullet)$, respectively, we conclude that the last complex is quasi-isomorphic to $\phi^*(A^\bullet) \otimes \phi^*(B^\bullet)$. □

The following result, which is an immediate consequence of the above discussion and Corollary 2.12, shows that there is essentially no difference between a nonperfect field F and its perfect closure F_∞.

Corollary 4.13. *Let E/F be a purely inseparable field extension. Then the corresponding functor $\phi^* : \mathrm{DM}_p^-(F) \to \mathrm{DM}_p^-(E)$ is an equivalence of categories.*

5. Extension of scalars and internal $\mathcal{H}om$-objects

In this section we'll often write $A_E^\bullet$ instead of $\phi^*(A^\bullet)$. Recall that, for a perfect field F, the category $\mathrm{DM}^-(F)$ has internal $\mathcal{H}om$-objects, i.e., for any $A^\bullet \in \mathrm{DM}^-(F)$ and any $X \in \mathrm{Sm}/F$ we have a new object $\mathcal{H}om(M(X), A^\bullet)$ and a universal morphism $\mathcal{H}om(M(X), A^\bullet) \otimes M(X) \to A^\bullet$ such that the resulting map

$$\mathrm{Hom}_{\mathrm{DM}^-(F)}(M, \mathcal{H}om(\mathcal{M}(X), A^\bullet)) \to \mathrm{Hom}_{\mathrm{DM}^-(F)}(M \otimes M(X), A^\bullet)$$

is an isomorphism for any $M \in \mathrm{DM}^-(F)$. Corollary 4.13 immediately implies that the same result is valid for the category $\mathrm{DM}_p^-(F)$ for arbitrary F. The purpose of this section is to show that extension of scalars functor preserves internal $\mathcal{H}om$-objects.

Theorem 5.1. *Let E/F be any field extension. Let, further, $A^\bullet \in \mathrm{DM}_p^-(F)$ be any motivic complex and let $X \in \mathrm{Sm}/F$ be any smooth scheme. In this case we have a natural isomorphism*

$$\mathcal{H}om(M_p(X), A^\bullet)_E = \mathcal{H}om(M_p(X_E), A_E^\bullet).$$

Proof. Applying the extension of scalars functor to the canonical homomorphism $\mathcal{H}om(M_p(X), A^\bullet) \otimes M_p(X) \to A^\bullet$ and using Theorem 4.12, we get a homomorphism $\mathcal{H}om(M_p(X), A^\bullet)_E \otimes M_p(X_E)) \to A_E^\bullet$ and hence the induced map

$$\mathcal{H}om(M_p(X), A^\bullet)_E \to \mathcal{H}om(M_p(X_E), A_E^\bullet).$$

We claim that this map is a quasi-isomorphism. If E/F is purely inseparable our claim readily follows from Corollary 4.13. Using the same machinery as in the proof of Theorem 4.1 we easily reduce the general case to the special one, when F is perfect. We start with the following observation:

Lemma 5.2. *Let $I \in \mathrm{NST}_p(F)$ be an injective Nisnevich sheaf with transfers. Then, for any smooth $X/\operatorname{Spec} E$, the cohomology groups $H^*_{\mathrm{Nis}}(X, I_E)$ are trivial in positive dimensions.*

Proof. We first consider the Čech cohomology. It clearly suffices to treat the case of a singleton Nisnevich covering $Y \to X$. Assume first that E/F is finitely generated, pick a smooth irreducible scheme of finite type S/F such that $E = F(S)$, and denote by η the generic point of S. According to Theorem 4.2 and Corollary 4.3 we may assume that $X = \widetilde{X}_\eta$, $Y = \widetilde{Y}_\eta$ and $\phi = \tilde{\phi}_\eta$, where $\widetilde{X}$ and $\widetilde{Y}$ are smooth schemes over an open $U \subset S$ and $\tilde{\phi} : \widetilde{Y} \to \widetilde{X}$ is a Nisnevich covering. For any open $V \subset U$ the morphism $\tilde{\phi}_V : \widetilde{Y}_V \to \widetilde{X}_V$ is still a Nisnevich covering and hence $\check{H}^i(\widetilde{Y}_V/\widetilde{X}_V, I) = 0$ for $i > 0$; see [Suslin and Voevodsky 2000, Lemma 1.6]. Passing to the direct limit over $V \subset U$ we see that $\check{H}^i(Y/X, I_E) = 0$ for $i > 0$. In the general case we may write E as a direct limit of finitely generated subextensions, $E = \varinjlim E_i$, and represent Y, X and ϕ in the form $Y = Y_{iE}$, $X = X_{iE}$ and $\phi = \phi_{iE}$, where Y_i and X_i are smooth schemes of finite type over E_i and $\phi_i : Y_i \to X_i$ is a Nisnevich covering. For any $j \geq i$, $\phi_j = (\phi_i)_{E_j} : Y_j \to X_j$ is still a Nisnevich covering. Thus, by what was proved above, $\check{H}^*(Y_j/X_j, I_{E_J}) = 0$ for $i > 0$. Passing to the direct limit over $j \geq i$ we conclude that $\check{H}^i(Y/X, I_E) = 0$ for $i > 0$.

Now the standard argument involving the Cartan–Leray spectral sequence completes the proof. □

Proposition 5.3. *Let $A^\bullet$ be an arbitrary complex of Nisnevich sheaves with transfers.*

(1) *Assume first that E/F is finitely generated, pick a smooth irreducible S/F such that $E = F(S)$ and denote by η the generic point of S. Then, for any smooth scheme of finite type $\widetilde{X}/S$, we have a natural identification*
$$H^*_{\mathrm{Nis}}(\widetilde{X}_\eta, A^\bullet_E) = \varinjlim_{U \subset S} H^*_{\mathrm{Nis}}(\widetilde{X}_U, A^\bullet).$$

(2) *For arbitrary E/F write $E = \varinjlim E_i$, where E_i/F are finitely generated subextensions. Then for any smooth scheme of finite type X_i/E_i we have a natural identification $H^*_{\mathrm{Nis}}((X_i)_E, A^\bullet_E) = \varinjlim_{j \geq i} H^*_{\mathrm{Nis}}((X_i)_{E_j}, A^\bullet_{E_j})$.*

Proof. (1) Recall (see [Suslin and Voevodsky 2000, §0]) that we define hypercohomology with coefficients in nonbounded below complexes via Cartan–Eilenberg resolutions. This approach gives correct answers since all schemes involved have finite cohomological dimension in Nisnevich topology. Thus let $A^\bullet \to I^{\bullet\bullet}$ be a Cartan–Eilenberg resolution of $A^\bullet$ and let $A^\bullet_E \to J^{\bullet\bullet}$ be a Cartan–Eilenberg resolution of $A^\bullet_E$. Note that, according to the definitions and results of Section 4,
$$H^*(I^{\bullet\bullet}_E(\widetilde{X}_\eta)) = \varinjlim_{U \subset S} H^*(I^{\bullet\bullet}(\widetilde{X}_U)) = \varinjlim_{U \subset S} H^*(\widetilde{X}_U, A^\bullet),$$
$$H^*(J^{\bullet\bullet}(\widetilde{X}_\eta)) = H^*(\widetilde{X}_\eta, A^\bullet_E).$$

Since the functor $M \mapsto M_E$ is exact one checks immediately that $I^{\bullet\bullet}_E$ is an admissible resolution of $A^\bullet_E$, i.e., cycles, boundaries and cohomology of rows of this bicomplex give resolutions of cycles etc. of $A^\bullet_E$. The universal property of Cartan–Eilenberg resolutions shows that there exists a unique up to homotopy homomorphism $I^{\bullet\bullet}_E \to J^{\bullet\bullet}$ of bicomplexes under $A^\bullet_E$. Finally the induced homomorphism $I^{\bullet\bullet}_E(\widetilde{X}_\eta) \to J^{\bullet\bullet}(\widetilde{X}_\eta)$ is a quasi-isomorphism since the columns of $I^{\bullet\bullet}_E$ are acyclic resolutions of A^i_E (according to Lemma 5.2) while the columns of $J^{\bullet\bullet}$ are injective resolutions of A^i_E, and sheaf cohomology may be computed using any acyclic resolutions.

(2) The proof of this is essentially the same; we skip the trivial details. □

The end of the proof of Theorem 5.1. Consider first the case when $E = F(S)$ for a smooth irreducible scheme of finite type S/F. To show that the canonical map $\mathcal{H}om(M_p(X), A^\bullet)_E \to \mathcal{H}om(M_p(X_E), A^\bullet_E)$ is a quasi-isomorphism we have to check that its cone is trivial in $\mathrm{DM}^-_p(E)$. Since the category $\mathrm{DM}^-_p(E)$ is weakly generated by objects of the form $M_p(\widetilde{Y}_\eta)[i]$ (with $\widetilde{Y}/S$ smooth of finite type) it suffices to verify that $\mathrm{Hom}_{\mathrm{DM}^-_p(E)}(M_p(\widetilde{Y}_\eta)[i], \text{cone}) = 0$. In other words we have to verify that the induced map

$$\mathrm{Hom}_{\mathrm{DM}^-_p(E)}(M_p(\widetilde{Y}_\eta)[i], \mathcal{H}om(M_p(X), A^\bullet)_E) \\ \to \mathrm{Hom}_{\mathrm{DM}^-_p(E)}\big(M_p(\widetilde{Y}_\eta)[i], \mathcal{H}om(M_p(X_E, A^\bullet_E))\big)$$

is an isomorphism. The previous results show that we may compute the above Hom-groups as follows:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}^-_p(E)}&\big(M_p(\widetilde{Y}_\eta)[i], \mathcal{H}om(M_p(X), A^\bullet)_E\big) \\ &= H^{-i}(\widetilde{Y}_\eta, \mathcal{H}om(M_p(X), A^\bullet)_E) \\ &= \varinjlim_{U\subset S} H^{-i}(\widetilde{Y}_U, \mathcal{H}om(M_p(X), A^\bullet)) \\ &= \varinjlim_{U\subset S} \mathrm{Hom}_{\mathrm{DM}^-_p(F)}(M_p(\widetilde{Y}_U)[i], \mathcal{H}om(M_p(X), A^\bullet)) \\ &= \varinjlim_{U\subset S} \mathrm{Hom}_{\mathrm{DM}^-_p(F)}(M_p(\widetilde{Y}_U \times_F X, A^\bullet)) \\ &= \varinjlim_{U\subset S} H^{-i}(\widetilde{Y}_U \times_F X, A^\bullet).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{DM}^-_p(E)}&(M_p(\widetilde{Y}_\eta)[i], \mathcal{H}om(M_p(X_E), A^\bullet_E)) \\ &= \mathrm{Hom}_{\mathrm{DM}^-_p(E)}(M_p(\widetilde{Y}_\eta \times_E X_E)[i], A^\bullet_E) \\ &= H^{-i}((\widetilde{Y} \times_F X)_\eta, A^\bullet_E) \\ &= \varinjlim_{U\subset S} H^{-i}(\widetilde{Y}_U \times_F X, A^\bullet).\end{aligned}$$

Thus the above Hom-groups identify canonically and it's not hard to trace through the above computations to see that this identification coincides with the canonical homomorphism defined before.

The general case is treated once again by passing to a direct limit over finite subextensions in E and using Proposition 5.3(2). □

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