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Exterior power operations on higher K-groups via binary complexes

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We use Grayson's binary multicomplex presentation of algebraic K-theory to give a new construction of exterior power operations on the higher K-groups of a (quasicompact) scheme. We show that these operations satisfy the axioms of a λ-ring, including the product and composition laws. To prove the latter we show that the Grothendieck group of the exact category of integral polynomial functors is the universal λ-ring on one generator.

Introduction

Exterior powers of vector bundles over a scheme X endow its Grothendieck group $K_0(X)$ with a family of operations $\lambda^r : K_0(X) \to K_0(X)$, $r = 0, 1, \ldots$. These λ-operations allow us to define Adams operations and the γ-filtration on $K_0(X)$ and are, more generally, at the heart of Grothendieck's Riemann–Roch theory (see [Fulton and Lang 1985]). This fundamental structure has been extended to the higher K-groups $K_n(X)$, $n \geq 0$, using a variety of sophisticated approaches and in various degrees of generality, by [Kratzer 1980; Hiller 1981; Grayson 1989; Nenashev 1991; Levine 1997], and has been most profoundly studied and applied in Soulé's seminal paper [1985]. Common to all these constructions is that they use homotopy theory.

In this paper we give a purely algebraic construction of the λ-operations on the higher K-groups of any quasicompact scheme X. Our construction is explicit in the following sense: in a surprising paper, Grayson [2012] has given explicit generators and relations for $K_n(X)$, and our construction describes explicit (albeit intricate) images of these generators under the λ-operations. Within the purely algebraic context of this paper, we prove moreover that our λ-operations satisfy the usual axioms, including the product and composition laws. In a forthcoming paper we address the problem of matching up our λ-operations with Hiller's.

To describe our results in more precise terms, we recall the definition of a λ-ring.

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Definition. A *pre-λ-ring* is a commutative unital ring K with maps $\lambda^r : K \to K$, $r \geq 1$, satisfying $\lambda^1(x) = x$ and the following axiom for all $x, y \in K$:

(1) $\lambda^r(x+y) = \lambda^r(x) + \sum_{i=1}^{r-1} \lambda^{r-i}(x)\lambda^i(y) + \lambda^r(y)$.

A *λ-ring* K is a pre-λ-ring satisfying the further axioms

(2) $\lambda^r(xy) = P_r(\lambda^1(x), \dots, \lambda^r(x), \lambda^1(y), \dots, \lambda^r(y))$,

(3) $\lambda^r(\lambda^s(x)) = P_{r,s}(\lambda^1(x), \dots, \lambda^{rs}(x))$,

where P_r and $P_{r,s}$, $r, s > 0$, are certain universal integral polynomials (defined in such a way that the axioms (2) and (3) hold in every pre-λ-ring whose additive group is generated by elements l with $\lambda^r(l) = 0$ for all $r > 1$ and in which products of elements of this type are again of this type; for details see [Fulton and Lang 1985]).

Probably the most prominent example of a λ-ring is $K_0(X)$ (see [loc. cit.]). The object of this paper is to make $K_*(X) = \bigoplus_{n \geq 0} K_n(X)$ into a λ-ring.

For each $n \geq 0$, Grayson [2012] associates to an exact category $\mathcal{P}$ the exact category $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$ of so-called n-dimensional bounded acyclic binary complexes, and proves that $K_n(\mathcal{P})$ is isomorphic to a relatively simple-to-describe quotient of the Grothendieck group $K_0((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P})$ (see Section 1 for a detailed review of Grayson's construction). Using the Dold–Puppe construction [1961], we inductively construct functors

$$\Lambda_n^r : (B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X) \to (B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$$

for all $r, n > 0$ from the usual exterior power functors $\Lambda^r : \mathcal{P}(X) \to \mathcal{P}(X)$, $r \geq 0$, on the category $\mathcal{P}(X)$ of vector bundles on X.

The following theorems are the main results of this paper.

Theorem 6.2. *The functors Λ_n^r induce well-defined homomorphisms*

$$\lambda^r : K_n(X) \to K_n(X)$$

for $r, n > 0$.

The tensor product induces the multiplication in the Grothendieck ring $K_0(X)$ and also an action of $K_0(X)$ on the higher K-groups $K_n(X)$. In particular, $K_*(X) = \bigoplus_{n \geq 0} K_n(X)$ carries the structure of a unital commutative ring in which the product of any two elements in $\bigoplus_{n \geq 1} K_n(X)$ is defined to be zero. Note that, if $n > 0$, axiom (1) for $x, y \in K_n(X)$ then follows from $\lambda^r : K_n(X) \to K_n(X)$ being a homomorphism (Theorem 6.2). Furthermore, the formula in axiom (1) can be used to extend our operations $\lambda^r : K_n(X) \to K_n(X)$, $n \geq 0$, to a pre-λ-ring structure on $K_*(X)$.

Theorems 7.1 and 8.18. *The pre-λ-ring $K_*(X)$ is a λ-ring.*

The first half of the paper is devoted to the construction of the exterior power functors Λ^r_n. Let $C_\mathrm{b}\mathcal{P}(X)$ denote the category of bounded complexes in $\mathcal{P}(X)$. We use the Dold–Kan correspondence (reviewed along with the other necessary homological preliminaries in Section 2) to obtain a chain-homotopy invariant functor $\Lambda^r_1 : C_\mathrm{b}\mathcal{P}(X) \to C_\mathrm{b}\mathcal{P}(X)$ for each $r > 0$; if X is affine, then the bounded acyclic complexes in $\mathcal{P}(X)$ are precisely the contractible ones, so we obtain an endofunctor on the category of bounded acyclic chain complexes in $\mathcal{P}(X)$. By generalising and iterating this procedure over complexes of complexes, we get the desired functors $\Lambda^r_n : (B^\mathrm{q}_\mathrm{b})^n\mathcal{P}(X) \to (B^\mathrm{q}_\mathrm{b})^n\mathcal{P}(X)$. This material is the subject of Subsections 3 and 4.

In the rather long Section 5 we construct a "simplicial tensor product" $\otimes_{\Delta,n}$ on $(B^\mathrm{q}_\mathrm{b})^n\mathcal{P}(X)$. In defining exterior powers on $K_0(X)$ we obtain from a short exact sequence of vector bundles $0 \to V' \to V \to V'' \to 0$ a filtration of $\Lambda^r(V)$ whose successive quotients are $\Lambda^{r-i}(V') \otimes \Lambda^i(V'')$. We use our simplicial tensor product of binary multicomplexes to obtain similar statements for short exact sequences in $(B^\mathrm{q}_\mathrm{b})^n\mathcal{P}(X)$; our tensor product $\otimes_{\Delta,n}$ is to $\otimes$ as the exterior powers Λ^r_n are to Λ^r. The main result of the section (Proposition 5.11) is that the product induced by $\otimes_{\Delta,n}$ on $K_n(X)$ vanishes.

In Section 6 we pass our exterior powers from the affine case to general (quasi-compact) schemes and show they induce well-defined maps $\lambda^r : K_n(X) \to K_n(X)$. As the product on $K_n(X)$ that is compatible with these operations is the zero product (by Proposition 5.11), it follows that the λ^r are group homomorphisms.

In Section 7 we show that the resulting pre-λ-ring $K_*(X)$ satisfies the λ-ring axiom (2) concerning products.

The final λ-ring axiom (3) is proved in Section 8. While the usual geometric splitting principle suffices to prove axiom (2) for $K_*(X)$ (see Section 7) and both axioms (2) and (3) for $K_0(X)$, there seems to be no way of extending that approach to prove axiom (3) for $K_*(X)$. We will rather proceed as follows. As $K_0(X)$ is a λ-ring, there exist short exact sequences in $\mathcal{P}(X)$ that prove the relation $\lambda^r(\lambda^s(x)) = P_{r,s}(\lambda^1(x), \dots, \lambda^{rs}(x))$ in $K_0(X)$ when x is the class of a vector bundle V on X. We will see (in Subsection 8D) that if in fact these short exact sequences exist functorially in V, then we can inductively prove the existence of short exact sequences in $(B^\mathrm{q}_\mathrm{b})^n\mathcal{P}(X)$ that prove the relation above when x is the class of an object in $(B^\mathrm{q}_\mathrm{b})^n\mathcal{P}(X)$; in other words, we have then proved axiom (3) for $K_*(X)$.

We are therefore reduced to showing the existence of such short exact sequences of functors in V. This problem may be seen as a weak variant of the famous plethysm problem (see Remark 8.22). The crucial insight now is that it becomes attackable when we also require these functors to be polynomial (see Definition 8.1). On the one hand, this requirement guarantees the existence of appropriate base change functors and hence reduces the problem to $X = \mathrm{Spec}(\mathbb{Z})$ (see Subsections 8A and 8D). On the other hand, it makes the computation of the corresponding

Grothendieck group of functors feasible; this is the content of the following theorem, which we highlight as it may be of independent interest.

It is well known that there exists a unique λ-ring structure on the ring $\mathbb{Z}[s_1, s_2, \dots]$ of integral polynomials in infinite variables such that $\lambda^r(s_1) = s_r$ for all r. Furthermore, let $\mathrm{Pol}_{<\infty}(\mathbb{Z})$ denote the category of polynomial functors over $\mathbb{Z}$ of bounded degree (whose Grothendieck group is easily seen to be a pre-λ-ring).

Theorem 8.5. *The ring homomorphism*

$$\mathbb{Z}[s_1, s_2, \dots] \to K_0(\mathrm{Pol}_{<\infty}(\mathbb{Z})), \quad s_i \mapsto [\Lambda^i],$$

is an isomorphism of pre-λ-rings.

This obviously implies that the right-hand side is a λ-ring as well and hence that the short exact sequences of functors postulated above indeed exist. After interpreting polynomial functors as modules over certain Schur algebras in Subsection 8B following Krause [2013, Section 8.2], in Subsection 8C we will prove the theorem by following Serre's computation [1968] of the Grothendieck group of representations of the group scheme $\mathrm{GL}_{n,\mathbb{Z}}$. A crucial ingredient here is Green's computation [1980] of the Grothendieck group of polynomial functors over a field.

The fundamental idea of proving λ-ring axioms for Grothendieck groups of complexes via the corresponding axioms for a Grothendieck group of appropriate functors is also sketched in an exchange of letters between Deligne [1967a; 1967b] and Grothendieck [1967].[†] Both their correspondence and the introduction of [Serre 1968] already allude to a role of Serre's result for λ-operations.

In a forthcoming paper we will complement the somewhat intricate constructions of this paper with simpler formulae that (help to) compute our λ-operations in certain cases. For instance, we will give formulae for our λ-operations when applied to K_1-groups of rings or to external products $K_m(X) \times K_n(X) \to K_{m+n}(X)$.

1. Binary multicomplexes and algebraic K-theory

In this section we review the description of algebraic K-groups in terms of binary complexes given in [Grayson 2012]. We also prove a simple lemma about shifted binary complexes to justify a slight modification of Grayson's description. The lemma is also useful for computations.

Recall that an *exact category* in the sense of [Quillen 1973] is an additive category with a distinguished class of "short exact sequences" that behave like the short exact sequences of an abelian category. A small exact category $\mathcal{N}$ may also be thought of as a full subcategory of an ambient abelian category $\mathcal{A}$ such that $\mathcal{N}$

[†]The authors became aware of these unpublished letters only after the present article was posted on arXiv (see Acknowledgements on page 448). After acceptance, at the publisher's request, Deligne kindly supplied scans and his permission to make them public.

is closed under extensions in $\mathcal{A}$.[1] The category of chain complexes in an exact category is again an exact category, with short exact sequences defined to be those sequences of chain maps that are short exact in each degree. In this paper we consider only complexes that are concentrated in nonnegative degrees, those with an underlying $\mathbb{Z}_{\geq 0}$-graded object. We denote this category of chain complexes in $\mathcal{N}$ by $C\mathcal{N}$. A chain complex is bounded if it has only finitely many nonzero objects. The exact subcategory of $C\mathcal{N}$ of bounded chain complexes is denoted by $C_{\mathrm{b}}\mathcal{N}$. An *acyclic* complex in an exact category is a chain complex $N_\bullet$ in $\mathcal{N}$ whose differentials $d_i : N_i \to N_{i-1}$ factor as $N_i \to Z_i \to N_{i-1}$ (with Z_i in $\mathcal{N}$), such that each $0 \to Z_{i+1} \to N_i \to Z_i \to 0$ is a short exact sequence in $\mathcal{N}$.[2] The full subcategories of acyclic complexes in $C\mathcal{N}$ and $C_{\mathrm{b}}\mathcal{N}$ are also exact, and are denoted by $C^{\mathrm{q}}\mathcal{N}$ and $C^{\mathrm{q}}_{\mathrm{b}}\mathcal{N}$.

Since each of these categories of complexes is also an exact category, we can iterate their construction to define *n-dimensional multicomplexes* in $\mathcal{N}$. A 1-dimensional multicomplex in $\mathcal{N}$ is simply a chain complex, an object of $C\mathcal{N}$. An $n+1$-dimensional multicomplex in $\mathcal{N}$ is a chain complex in the exact category $C^n\mathcal{N}$ of n-dimensional multicomplexes in $\mathcal{N}$. We define categories of bounded and/or acyclic multicomplexes, $(C_{\mathrm{b}})^n\mathcal{N}$, $(C^{\mathrm{q}})^n\mathcal{N}$ and $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$, analogously. With these notions in place, we can define binary complexes and multicomplexes.

Definition 1.1. (1) A *binary complex* in an exact category $\mathcal{N}$ is a triple $(N_\bullet, d, \tilde{d})$ consisting of a $\mathbb{Z}_{\geq 0}$-graded collection of objects of $\mathcal{N}$ together with two differentials d and d' such that $(N_\bullet, d)$ and $(N_\bullet, \tilde{d})$ are chain complexes in $\mathcal{N}$. A binary complex can be regarded as pair of objects of $C\mathcal{N}$ that have the same underlying graded object. A morphism of binary complexes is a degree 0 map between these underlying objects that commutes with both differentials. The category of binary complexes in $\mathcal{N}$ is denoted by $B\mathcal{N}$. This is an exact category in the same way that $C\mathcal{N}$ is.

(2) A *bounded acyclic* binary complex in $\mathcal{N}$ is a binary complex such that the chain complexes $(N_\bullet, d)$ and $(N_\bullet, \tilde{d})$ are bounded and acyclic. The category of bounded acyclic binary complexes in $\mathcal{N}$ is denoted by $B^{\mathrm{q}}_{\mathrm{b}}\mathcal{N}$. It is an exact subcategory of $B\mathcal{N}$.

(3) An *n-dimensional binary multicomplex* is an object of the exact category $B^n\mathcal{N} = B\cdots B\mathcal{N}$ (defined in the same way as $C^n\mathcal{N}$). An n-dimensional bounded acyclic binary multicomplex is an object of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$.

[1] This is the *Gabriel–Quillen embedding theorem* [Thomason and Trobaugh 1990, Theorem A.7.1 and Proposition A.7.16].

[2] This is not in general the same thing as being a long exact sequence in the ambient abelian category $\mathcal{A}$. However in this paper we work only with *idempotent complete* exact categories, in which case the two notions coincide. See [Grayson 2012; Thomason and Trobaugh 1990, Section A.9.2].

Remark 1.2. A (bounded acyclic) binary multicomplex $N_\bullet$ of dimension n is equivalent to the following data: a (bounded) $\mathbb{Z}^n_{\geq 0}$-graded collection of objects of $\mathcal{N}$ equipped with *two* (acyclic) differentials, denoted by d^i and $\tilde{d}^i$, in each direction $1 \leq i \leq n$, subject to the commutativity requirements

(1) $d^i d^j = d^j d^i$,
(2) $d^i \tilde{d}^j = \tilde{d}^j d^i$,
(3) $\tilde{d}^i d^j = d^j \tilde{d}^i$,
(4) $\tilde{d}^i \tilde{d}^j = \tilde{d}^j \tilde{d}^i$,

whenever $i \neq j$.

Another way to look at these commutativity restraints is that the various subsets of the differentials form (nonbinary) multicomplexes: for each $i = 1, \ldots, n$, choose d^i or $\tilde{d}^i$, and consider the object that has the same underlying $\mathbb{Z}^n$-graded object as $N_\bullet$, but now has *one* acyclic differential in each direction i, given by d^i or $\tilde{d}^i$, depending on our choice. For each of the 2^n choices of differentials, the resulting object is a bounded acyclic multicomplex, i.e., an object of $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$; conversely, given a pair of differentials d^i and $\tilde{d}^i$ in each direction, if the 2^n choices all form objects of $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$, then the whole assembly is an object of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$.

Since this category of bounded acyclic binary complexes in $\mathcal{N}$ is itself an exact category, we can form its Grothendieck group $K_0(B^{\mathrm{q}}_{\mathrm{b}}\mathcal{N})$. The main theorem of [Grayson 2012] is a surprising connection between this group and the n-th higher K-group of $\mathcal{N}$. We call an n-dimensional binary multicomplex *diagonal* if the pair of differentials in some direction are equal, i.e, if $d^i = \tilde{d}^i$ for some $1 \leq i \leq n$. Grayson's theorem, which we shall hereafter use as our definition of the K-groups, says that $K_n(\mathcal{N})$ is isomorphic to the quotient of the Grothendieck group of $B^{\mathrm{q}}_{\mathrm{b}}\mathcal{N}$ by the subgroup generated by the classes of the diagonal bounded acyclic binary multicomplexes. More formally:

Theorem/Definition 1.3 [Grayson 2012, Corollary 7.4]. *For $\mathcal{N}$ an exact category and $n \geq 0$, the abelian group $K_n(\mathcal{N})$ is presented as follows. There is one generator for each bounded acyclic binary multicomplex of dimension n, and there are two families of relations*:

(1) $[N'] + [N''] = [N]$ *if there is a short exact sequence*

$$0 \to N' \to N \to N'' \to 0$$

in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$, and

(2) $[D] = 0$ *if D is a diagonal bounded acyclic binary multicomplex.*

We remark that our statement of Theorem/Definition 1.3 is subtly different than the one originally given by Grayson. Our bounded acyclic binary multicomplexes are first-quadrant multicomplexes, those that are supported in $\mathbb{Z}^n_{\geq 0}$, whereas

Grayson's do not have to satisfy this condition. The absolute lower bound for complexes is a technical constraint that we need in order to use the Dold–Kan correspondence. Our additional condition is harmless, as the following proposition shows. For this, let $K_n^{\mathrm{Gr}}(\mathcal{N})$ temporarily denote the n-th K-group of $\mathcal{N}$ as defined in [Grayson 2012].

Proposition 1.4. *For every exact category $\mathcal{N}$ and every $n \geq 0$, the canonical homomorphism $K_n(\mathcal{N}) \to K_n^{\mathrm{Gr}}(\mathcal{N})$ is bijective.*

Proof. For ease of presentation we shall prove this for $n = 1$ only: there is no additional difficulty for $n > 1$. Let $B^{\mathrm{q}}_{\infty}\mathcal{N}$ denote the category of bounded acyclic binary complexes in $\mathcal{N}$ that may be supported anywhere on $\mathbb{Z}$. For $i \geq 0$, let $B^{\mathrm{q}}_{\geq -i}\mathcal{N}$ denote the full subcategory of $B^{\mathrm{q}}_{\infty}\mathcal{N}$ consisting of complexes that are supported on $[-i, \infty]$. We then have $\bigcup_i B^{\mathrm{q}}_{\geq -i}\mathcal{N} = B^{\mathrm{q}}_{\infty}\mathcal{N}$ and hence $\lim_i K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N}) = K_0(B^{\mathrm{q}}_{\infty}\mathcal{N})$. Let T_i denote the subgroup of $K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N})$ generated by diagonal complexes and let T denote the similarly defined subgroup of $K_0(B^{\mathrm{q}}_{\infty}\mathcal{N})$. The resulting injective homomorphism $\lim_i T_i \to T$ is also surjective because all complexes are assumed to be bounded. We therefore obtain an isomorphism

$$\lim_i (K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N})/T_i) \cong \lim_i K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N})/\lim_i T_i \cong K_0(B^{\mathrm{q}}_{\infty}\mathcal{N})/T = K_1^{\mathrm{Gr}}(\mathcal{N}).$$

The following lemma (after generalising it from $B^{\mathrm{q}}_{\geq 0}\mathcal{N}$ to $B^{\mathrm{q}}_{\geq -i}\mathcal{N}$) shows that, for every $i \geq 0$, "shifting" induces a two-sided inverse to the negative of the canonical homomorphism $K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N})/T_i \to K_0(B^{\mathrm{q}}_{\geq -i-1}\mathcal{N})/T_{i+1}$. Hence the canonical map

$$K_1(\mathcal{N}) = K_0(B^{\mathrm{q}}_{\geq 0}\mathcal{N})/T_0 \to \lim_i (K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N})/T_i) \cong K_1^{\mathrm{Gr}}(\mathcal{N})$$

is an isomorphism, as was to be shown. □

Definition 1.5. Let $N_\bullet$ be an acyclic binary complex with differentials d and $\tilde{d}$. The k-th *shift* of N, denoted by $N[k]$, is the acyclic binary complex that has the same collection of objects as N but "shifted" k places, i.e., $(N[k])_i = N_{i-k}$, and differentials given by $(-1)^k d$ and $(-1)^k \tilde{d}$.

Lemma 1.6. *For any bounded acyclic binary complex $N_\bullet$ and $k \in \mathbb{Z}_{\geq 0}$, we have $[N[k]] = (-1)^k[N]$ in $K_1\mathcal{N}$.*

Proof. It is enough to show that $[N[1]] = -[N]$. There is a short exact sequence

$$0 \to N_\bullet \to \mathrm{cone}(N_\bullet) \to N_\bullet[1] \to 0,$$

where $\mathrm{cone}(N_\bullet)$ denotes the mapping cone of the identity map $N_\bullet \xrightarrow{1} N_\bullet$ ($\mathrm{cone}(N_\bullet)$ is a binary complex in the obvious way). So it suffices to show that $\mathrm{cone}(N_\bullet)$ vanishes in $K_1\mathcal{N}$. Let N_n be the left-most nonzero object of $N_\bullet$, and let $\mathrm{trun}(N_\bullet)$ be

the (not necessarily acyclic) binary complex formed by truncating $N_\bullet$ to forget N_n; that is, $\mathrm{trun}(N_\bullet)$ has a 0 in place of N_n. Then there is a short exact sequence

$$0 \to \mathrm{cone}(\mathrm{trun}(N_\bullet)) \to \mathrm{cone}(N_\bullet) \to \Delta(N_n \xrightarrow{1} N_n) \to 0,$$

where $\Delta(N_n \xrightarrow{1} N_n)$ is the diagonal binary complex

$$0 \rightrightarrows N_n \underset{1}{\overset{1}{\rightrightarrows}} N_n \rightrightarrows 0,$$

which is supported in degrees $n+1$ and n. Mapping cones of identities are always acyclic, so $\mathrm{cone}(\mathrm{trun}(N_\bullet))$ is acyclic even when $\mathrm{trun}(N_\bullet)$ is not. Since $\Delta(N_n \xrightarrow{1} N_n)$ is diagonal its class vanishes in $K_1\mathcal{N}$, so the above short exact sequence yields the relation $[\mathrm{cone}(N_\bullet)] = [\mathrm{cone}(\mathrm{trun}(N_\bullet))]$. We iterate this procedure by repeatedly truncating $\mathrm{trun}(N_\bullet)$ to show that $[\mathrm{cone}(N_\bullet)]$ is zero. □

The same proof gives the analogous result for binary multicomplexes: for N in $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{N}$, the class of N shifted one place in any of the n possible directions in $K_n(\mathcal{N})$ is $-[N]$. From this the actions of more general shifts (in multiple directions) follow immediately.

2. Preliminaries from homological algebra

In this section we recall some preliminaries from the homological algebra of exact categories. We say what it means for an exact category to be idempotent complete or split, and show that the notions of acyclicity and contractibility of complexes coincide in exact categories that have both of these properties. We then review simplicial objects and the Dold–Kan correspondence. Finally we discuss functors of finite degree, a weakening of the concept of additive functors. These three topics may seem rather disjoint here, but we bring them together in the next section to produce functors between categories of chain complexes that preserve boundedness and acyclicity, paving the way for a functor on binary multicomplexes that induces a map on K-theory.

Definition 2.1. An exact category $\mathcal{N}$ is *idempotent complete* if every idempotent endomorphism in $\mathcal{N}$ has a kernel in $\mathcal{N}$.

This does not hold, for example, for the category of free modules over a ring when there exists a nonfree projective module. All of the exact categories we use in this paper are idempotent complete. This is an assumption on the "base level" exact categories we introduce, but will need to be proven for categories of multicomplexes (Lemma 3.4). Idempotent complete exact categories come with an embedding into an abelian category $\mathcal{N} \hookrightarrow \mathcal{A}$ that *supports long exact sequences*: a chain complex is acyclic in $\mathcal{N}$ if and only if it is exact when considered as a chain

complex of $\mathcal{A}$ (see [Grayson 2012, §1]). Homological algebra is therefore quite straightforward in idempotent complete exact categories.

Contractible complexes in idempotent complete exact categories are always acyclic; indeed this is an equivalent characterisation of idempotent completeness [Bühler 2010, Proposition 10.9]. Acyclic complexes in exact categories (even idempotent complete ones) are not usually contractible. There is a useful criterion for contractibility, however. Recall that a chain complex $(C_\bullet, d)$ is called *split* if there exist maps $s_n : C_{n-1} \to C_n$ such that $d_n s_n d_n = d_n$.

Lemma 2.2. *A chain complex in an idempotent complete exact category is contractible if and only if it is acyclic and split.*

Proof. It follows the definition of a chain homotopy that contractible complexes in idempotent complete exact categories are also split. Conversely, an elementary argument shows that if a complex in an exact category is acyclic and split, then the collection of splitting maps $\{s_n\}$ describes a homotopy from its identity map to its zero map. □

If an acyclic complex is split, each of the constituent short exact sequences that it factors into is split: that is, isomorphic to a canonical direct sum sequence (the converse is obviously true as well). Recall that an exact category is called *split exact* if all of its declared short exact sequences are split. In such an exact category, all acyclic complexes are split. Hence the notions of contractibility and acyclicity coincide for complexes in a split exact category that is also idempotent complete. An example of such an exact category is the category $\mathcal{P}(R)$ of (finitely generated) projective modules over a ring R. That acyclic complexes are contractible in this category is key to the results of this paper.

We now turn to the Dold–Kan correspondence. To give its statement we need the language of simplicial objects. Recall that Δ denotes the *simplex category*: the category whose objects are the finite nonempty ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and whose morphisms are the order-preserving maps. A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$, and the natural transformations between such functors make $\mathcal{C}^{\Delta^{\mathrm{op}}}$ into a category. Equivalently, a simplicial object C in $\mathcal{C}$ can be specified to be a collection of objects C_n, $n \in \mathbb{N}$, of $\mathcal{C}$ together with face maps $\delta_i : C_n \to C_{n-1}$ and degeneracy maps $\sigma_j : C_n \to C_{n+1}$, $i, j = 0, \ldots, n$, satisfying various combinatorial identities. A morphism between simplicial objects C and D is a collection of morphisms $C_n \to D_n$ that commutes with the faces and degeneracies. A homotopy $h : f \simeq g$ between simplicial maps $f, g : C \to D$ is a simplicial morphism $h : C \times \Delta^1 \to D$ (where Δ^1 denotes the simplicial set corresponding to the ordered set $\{0 < 1\}$, as usual) such that $h|_{C\times\{0\}} = f$ and $h|_{D\times\{1\}} = g$; it can also be described as collection of morphisms $h_i : C_n \to D_{n+1}$,

$i = 0, \dots, n$, which satisfy further combinatorial identities determined by compositions relating f, g, the h_i, and the faces and degeneracies of C and D. See, for example, Chapter 8 of [Weibel 1994] for full definitions of simplicial objects and homotopies.

If $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor, then postcomposition with F induces a functor between categories of simplicial objects $\mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{D}^{\Delta^{\mathrm{op}}}$. Abusing notation, we shall also call this functor F. Importantly, if $h : f \simeq g$ is a simplicial homotopy between $f, g : C \to D$, then $F(h) : F(f) \simeq F(g)$ is a simplicial homotopy between $F(f), F(g) : F(C) \to F(D)$. The analogous statement for chain homotopies between chain maps is not true if F is not additive. The Dold–Kan correspondence shows that chain complexes and simplicial objects are equivalent in a nonobvious way, and allows us to induce homotopy-preserving functors between categories of chain complexes, even when the original functors are not additive.

Definition 2.3. Let $\mathcal{P}$ be an additive category. Given a chain complex $C_\bullet \in C\mathcal{P}$, we define a simplicial object $\Gamma(C_\bullet) \in \mathcal{P}^{\Delta^{\mathrm{op}}}$ as follows:

(1) *Objects*: Given $p \le n$, let η range over all surjections $[n] \twoheadrightarrow [p]$ in Δ, and let $C_p\langle\eta\rangle$ denote a copy of C_p that is labelled by η. For each n, set

$$\Gamma(C)_n := \bigoplus_{p \le n} \bigoplus_{\eta} C_p\langle\eta\rangle.$$

(2) *Maps*: If $\alpha : [m] \to [n]$ is a morphism in Δ, we describe $\Gamma(\alpha)$ by describing each $\Gamma(\alpha, \eta)$, the restriction of $\Gamma(\alpha)$ to the summand $C_p\langle\eta\rangle$ of $\Gamma(C)_n$. Let

$$[m] \xrightarrow{\eta'} [q] \xhookrightarrow{\varepsilon} [p]$$

be the unique epi-monic factorisation of $\eta\alpha$. Then

$$\Gamma(\alpha, \eta) := \begin{cases} 1 : C_p\langle\eta\rangle \to C_p\langle\eta'\rangle & \text{if } q = p, \\ d_p : C_p\langle\eta\rangle \to C_{p-1}\langle\eta'\rangle & \text{if } q = p-1 \text{ and } \varepsilon = \varepsilon_p, \\ 0 & \text{otherwise.} \end{cases}$$

This construction extends to a functor[3] $\Gamma : C\mathcal{P} \to \mathcal{P}^{\Delta^{\mathrm{op}}}$.

Theorem (Dold–Kan correspondence). *If $\mathcal{P}$ is idempotent complete, then the functor $\Gamma : C\mathcal{P} \to \mathcal{P}^{\Delta^{\mathrm{op}}}$ is an equivalence of categories. Furthermore, Γ is exact and preserves homotopies.*

Proof. Chapter 8 of [Weibel 1994] proves this when $\mathcal{P}$ is an abelian category. The general case is [Lurie 2014, §1.2.3]. □

The inverse functor to Γ is most simply described for an abelian category.

[3] Other authors (e.g., Weibel [1994]) use K in place of Γ; we avoid this notation for obvious reasons.

Definition 2.4. Let A be a simplicial object in an abelian category $\mathcal{A}$.

(1) The *associated chain complex* $C(A)$ has objects $C(A)_n = A_n$ and differential

$$d_n = \sum_{i=0}^{n} (-1)^i \delta_i : C(A)_n \to C(A)_{n-1}.$$

(2) The subcomplex

$$D(A)_n = \sum_{i=0}^{n} \mathrm{Im}(\sigma_i : A_{n-1} \to A_n)$$

is called the *degenerate subcomplex* of $C(A)$.

(3) The *normalised Moore complex* $N(A)$ has objects

$$N_n(A) = A_n / D(A)_n$$

with the induced differential $\bar{d}_n$.

The associated chain complex splits globally as $C(A) = N(A) \oplus D(A)$.

The normalised Moore complex defines a functor $N : \mathcal{A}^{\Delta^{\mathrm{op}}} \to C\mathcal{A}$. It is exact and preserves homotopies, and is inverse to Γ (up to natural isomorphism). Now if $\mathcal{P}$ is an idempotent complete exact category, then there is an embedding $\mathcal{P} \subseteq \mathcal{A}$ into an abelian category such that $\mathcal{P}$ is closed under taking direct summands in $\mathcal{A}$. If P is an object of $\mathcal{P}^{\Delta^{\mathrm{op}}} \subseteq \mathcal{A}^{\Delta^{\mathrm{op}}}$, then the associated chain complex $C(P)$ is a chain complex $\mathcal{A}$ with objects in $\mathcal{P}$. But $N(P)$ is a direct summand of $C(P)$, which has objects in $\mathcal{P}$, so $N(P)$ has objects in $\mathcal{P}$. Therefore N restricts to a functor $\mathcal{P}^{\Delta^{\mathrm{op}}} \to C\mathcal{P}$. Furthermore the functor N is exact and preserves homotopies. See [Lurie 2014] for further details.

We conclude our preliminaries by discussing functors of finite degree.

Definition 2.5. Let $F : \mathcal{C} \to \mathcal{D}$ be any functor between additive categories that satisfies $F(0) = 0$. Then there is a functorial decomposition

$$F(X \oplus Y) = F(X) \oplus \mathrm{cr}_2(F)(X, Y) \oplus F(Y),$$

where $\mathrm{cr}_2(F) : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ is the second *cross-effect functor* (see [Eilenberg and Mac Lane 1954]), which is defined to be the kernel of the natural projection $F(X \oplus Y) \to F(X) \oplus F(Y)$. The functor F is said to have *degree* ≤ 1 if it is additive (i.e., if $\mathrm{cr}_2(F)$ vanishes), and we say that F has degree $\leq d$ if $\mathrm{cr}_2(F)(X, Y)$ is of degree $\leq d - 1$ in each argument. If F is of degree $\leq d$, then F is of degree $\leq d'$ for all $d' \geq d$. We say that F *has degree* d if it has degree $\leq d$ but does not have degree $\leq d - 1$.

Example 2.6. For R a nonzero commutative ring, the exterior power $\Lambda^r : \mathcal{P}(R) \to \mathcal{P}(R)$ has degree r for each $r > 0$. This follows from the canonical decomposition

$$\Lambda^r(X \oplus Y) \cong \Lambda^r(X) \oplus \left(\bigoplus_{i=1}^{r-1} \Lambda^{r-i}(X) \otimes \Lambda^i(Y) \right) \oplus \Lambda^r(Y).$$

If $F : \mathcal{P} \to \mathcal{Q}$ is an *additive* functor between exact categories, and if $P_\bullet$ is a bounded complex, then $NF\Gamma(P_\bullet)$ is certainly bounded again. This also holds true for functors of finite degree, as the following lemma shows:

Lemma 2.7 [Satkurunath and Köck 2010, Corollary 4.6]. *Let $P_\bullet$ be a chain complex in $C\mathcal{P}$ of length ℓ, and let $F : \mathcal{P} \to \mathcal{Q}$ be a functor of degree d between exact categories. Then $NF\Gamma(P_\bullet)$ has length less than or equal to $d\ell$.* □

3. Operations on acyclic complexes

In this rather abstract section we describe how to use the Dold–Kan correspondence to extend a functor $F : \mathcal{P} \to \mathcal{P}$ on an idempotent complete exact category to a functor on each category of multicomplexes $F_n : C^n\mathcal{P} \to C^n\mathcal{P}$, $n \geq 1$. We show that if $\mathcal{P}$ is split exact, then the extended functors F_n send acyclic multicomplexes to acyclic multicomplexes. We also show that if F is of finite degree, then each F_n preserves bounded multicomplexes and is also of finite degree.

Proposition 3.1. *Let $F : \mathcal{P} \to \mathcal{P}$ be a covariant functor on an idempotent complete exact category, with $F(0) = 0$. Let $F_1 := NF\Gamma : C\mathcal{P} \to C\mathcal{P}$ denote the induced functor. Then*:

(1) $F_1(0) = 0$.

(2) *F_1 sends contractible complexes to contractible complexes.*

(3) *If $\mathcal{P}$ is split exact, then F_1 sends acyclic complexes to acyclic complexes.*

(4) *If F is of degree at most d, then F_1 sends bounded complexes to bounded complexes and F_1 is again of degree at most d.*

Proof. Part (1) is trivial.

For (2), the functors $\Gamma : C\mathcal{P} \to \mathcal{P}^{\Delta^{\mathrm{op}}}$ and $N : \mathcal{P}^{\Delta^{\mathrm{op}}} \to C\mathcal{P}$ preserve homotopies and send 0 to 0, so they both send contractible objects to contractible objects. Furthermore, F sends homotopies in $\mathcal{P}^{\Delta^{\mathrm{op}}}$ to homotopies in $\mathcal{P}^{\Delta^{\mathrm{op}}}$ — if $h : f \sim g$ is a homotopy, then $F(h) : F(f) \sim F(g)$ is a homotopy. Since F also has the property that $F(0) = 0$, we see that if $A \simeq 0$ in $\mathcal{P}^{\Delta^{\mathrm{op}}}$, then $F(A) \simeq F(0) = 0$. Therefore $NF\Gamma(P_\bullet)$ is contractible in $C\mathcal{P}$.

Following Lemma 2.2, the acyclic complexes in a split exact idempotent complete exact category coincide with the contractible ones, so (3) follows from (2).

Finally we consider (4). The first part of this statement is Lemma 2.7. For the second part we note that, since N and Γ are additive, it is enough to show that $F : \mathcal{A}^{\Delta^{\mathrm{op}}} \to \mathcal{B}^{\Delta^{\mathrm{op}}}$ is of degree $\leq d$. This is proven by induction on d. □

Proposition 3.1(3) may not hold in an exact category that is not split exact, as is shown in the following example:

Example 3.2. Let F be the degree 2 endofunctor $A \mapsto A^{\otimes 2}$ on the abelian category of abelian groups, and let $C_\bullet$ be the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

viewed as an acyclic complex concentrated in degrees 0, 1 and 2. Then $NF\Gamma(C_\bullet) = N\operatorname{diag}(\Gamma(C_\bullet) \otimes \Gamma(C_\bullet))$ is homotopy equivalent to $\operatorname{Tot}(C_\bullet \otimes C_\bullet)$ by the Eilenberg–Zilber theorem [May 1967, §29].[4] But the homology group $H_2(\operatorname{Tot}(C_\bullet \otimes C_\bullet))$ is $\mathbb{Z}/2\mathbb{Z}$, so $NF\Gamma(C_\bullet)$ is not exact. Furthermore, the short exact sequence of functors

$$0 \to N\Lambda^2\Gamma \to NF\Gamma \to N\mathrm{Sym}^2\Gamma \to 0$$

shows that at least one of $N\Lambda^2\Gamma(C_\bullet)$ or $N\mathrm{Sym}^2\Gamma(C_\bullet)$ is not exact either.

We now iterate the Dold–Kan correspondence to describe induced functors on categories of acyclic multicomplexes.

Definition 3.3. Let $F : \mathcal{P} \to \mathcal{P}$ be a covariant functor on an idempotent complete exact category. We define functors

$$F_n : C^n\mathcal{P} \to C^n\mathcal{P}$$

for all $n \geq 0$ recursively, as follows:

(1) $F_0 := F : \mathcal{P} \to \mathcal{P}$.

(2) By regarding an object of $C^{n+1}\mathcal{P}$ as a chain complex in the exact category $C^n\mathcal{P}$, we define $F_{n+1} := NF_n\Gamma$.

To show that F_n sends acyclic multicomplexes to acyclic multicomplexes in a nice exact category, we need to know that $(C^{\mathrm{q}})^n\mathcal{P}$ satisfies the same hypotheses as $\mathcal{P}$. This is the content of the following technical lemma. The proof is not enlightening for the rest of the paper, so we relegate it to the Appendix.

Lemma 3.4. *Let $\mathcal{P}$ be an exact category. For all $n > 0$ we have the following*:

(1) *If $\mathcal{P}$ is idempotent complete, then $C^n\mathcal{P}$ and $(C^{\mathrm{q}})^n\mathcal{P}$ are also idempotent complete.*

(2) *If $\mathcal{P}$ is split exact, then $(C^{\mathrm{q}})^n\mathcal{P}$ is also split exact.*

The analogous results for the categories $C^n_{\mathrm{b}}\mathcal{P}$ and $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$ of bounded multicomplexes also hold.

Corollary 3.5. *Let $\mathcal{P}$ be a split exact idempotent complete exact category, and $F : \mathcal{P} \to \mathcal{P}$ a covariant functor such that $F(0) = 0$. Then for $n \geq 0$ the functors of Definition 3.3 restrict to functors*

$$F_n : (C^{\mathrm{q}})^n\mathcal{P} \to (C^{\mathrm{q}})^n\mathcal{P}.$$

[4] See also Definition 5.4 and Lemma 5.5 here.

Furthermore, if F is of finite degree, then F_n sends bounded multicomplexes to bounded multicomplexes. That is, each F_n restricts to a functor

$$F_n : (C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P} \to (C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}.$$

Proof. We consider the unbounded case first. By Proposition 3.1(1), we easily see that $F_n(0) = 0$ for all n. Assume that F_n restricts to a functor on the idempotent complete split exact category $(C^{\mathrm{q}})^n\mathcal{P}$. Regarding objects of $(C^{\mathrm{q}})^{n+1}\mathcal{P}$ as acyclic complexes in $(C^{\mathrm{q}})^n\mathcal{P}$, the functor $F_{n+1} = NF_n\Gamma$ restricts to a functor on $(C^{\mathrm{q}})^{n+1}\mathcal{P} = C^{\mathrm{q}}((C^{\mathrm{q}})^n\mathcal{P})$, by Proposition 3.1(3) and Lemma 3.4(2). The first part of the result follows by induction.

For the second part, if $F_0 = F$ is of finite degree, then the same induction over n shows that F_n is of finite degree for every n, by Proposition 3.1(4). In particular, for each $n \geq 1$, the functor $F_n = (F_{n-1})_1$ sends bounded complexes to bounded complexes, that is, it restricts to a functor

$$F_n : C^{\mathrm{q}}_{\mathrm{b}}((C^{\mathrm{q}})^{n-1}\mathcal{P}) \to C^{\mathrm{q}}_{\mathrm{b}}((C^{\mathrm{q}})^{n-1}\mathcal{P}).$$

But we can say more: considering $P_\bullet$ in $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$ as a chain complex, each of its objects is in $(C^{\mathrm{q}}_{\mathrm{b}})^{n-1}\mathcal{P}$, i.e., they are bounded. We claim that the objects of $F_n(P_\bullet) = NF_{n-1}\Gamma(P_\bullet)$ are also bounded. The objects of $\Gamma(P_\bullet)$ are finite direct sums of the objects of $P_\bullet$. Finite sums of bounded objects are bounded, so the objects of $\Gamma(P_\bullet)$ are bounded. Therefore, by the inductive hypothesis, the objects of $F_{n-1}\Gamma(P_\bullet)$ are also bounded. Finally, the objects of $NF_{n-1}\Gamma(P_\bullet)$ are direct summands of the objects of $F_{n-1}\Gamma(P_\bullet)$ (from Definition 2.4, after embedding into an abelian category), so they are bounded as well. Therefore F_n sends bounded chain complexes of bounded objects in $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$ to bounded chain complexes of bounded objects in $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$. This is exactly the statement that F_n restricts to a functor

$$F_n : (C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P} \to (C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P},$$

which was to be proved. □

Remark 3.6. Throughout this section we work with the inductive definition of $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$, that is $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P} := C^{\mathrm{q}}_{\mathrm{b}}((C^{\mathrm{q}}_{\mathrm{b}})^{n-1}\mathcal{P})$ for $n > 1$. As explained in Remark 1.2, one can instead think of objects in $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$ as $\mathbb{Z}^n_{\geq 0}$-graded objects of $\mathcal{N}$ (together with certain differentials) without specifying the order of directions in which the objects have been obtained in the inductive definition. The purpose of this remark is to convince the reader that our construction of the functors F_n given in this section (and hence our construction of exterior powers in the sequel) does not depend on the order of directions either. Rather than including a complete proof, we sketch the idea in the case $n = 2$. Let $F_0 = F$ be as before. The functor F_2 is defined as

$$N_h F_1 \Gamma_h = N_h N_v F_0 \Gamma_v \Gamma_h,$$

where the indices h and v indicated the horizontal and vertical directions respectively. It is quite straightforward to see that the composition $N_h N_v$ sends a bisimplicial object C to the double complex whose objects are obtained from the corresponding objects of C by dividing out the images of all of the horizontal and vertical degeneracy maps. This latter description of course does not depend on the order of N_h and N_v. One can show that the same holds for Γ_h and Γ_v by a similar argument, or just by recalling that Γ_h and Γ_v are adjoint to N_h and N_v, respectively.

We can now describe the exterior power functors that we will use to induce operations on higher K-groups. The following example is the motivation for our work so far.

Main Example 3.7. Let $\mathcal{P}(R)$ be the category of finitely generated projective modules over a commutative ring R. This category is both idempotent complete and split exact. For each $r > 0$, the usual exterior power functor $\Lambda^r : \mathcal{P}(R) \to \mathcal{P}(R)$ satisfies the hypotheses of Corollary 3.5 (Λ^r has degree r). We therefore have induced functors

$$\Lambda^r_n : (C^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(R) \to (C^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(R)$$

for all $n \geq 0$.

In general, the complex $N\Lambda^r\Gamma(P_\bullet)$ is difficult to write down explicitly. Satkurunath and Köck [2010] give an algorithm that addresses this problem. We conclude this section by computing $N\Lambda^r\Gamma(P_\bullet)$ for a very simple choice of $P_\bullet$.

Example 3.8. Let $\varphi : P \to Q$ be an isomorphism of invertible modules over some commutative ring R, considered as an acyclic complex concentrated in degrees 0 and 1:

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{\varphi} & Q & \longrightarrow & 0 \\ 2 & & 1 & & 0 & & -1 \end{array}$$

or $P \xrightarrow{\varphi} Q$ for short. Köck [2001, Lemma 2.2] gives an explicit calculation of $N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)$ in terms of higher cross-effect functors (in fact, he does this for more general P, Q and φ). Specifically, in degree n we have

$$N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)_n = \mathrm{cr}_n(\Lambda^r)(P, \ldots, P) \oplus \mathrm{cr}_{n+1}(\Lambda^r)(Q, P, \ldots, P).$$

We do not wish to expound on the theory of cross-effect functors here; the interested reader can see [Eilenberg and Mac Lane 1954] or [Köck 2001, Section 1]. Instead we merely quote the properties of $\mathrm{cr}_n(\Lambda^r)$ that we need. Firstly, $\mathrm{cr}_n(\Lambda^r) = 0$ for $n > r$, as Λ^r is of degree r; secondly, $\mathrm{cr}_r(\Lambda^r)(P_1, \ldots, P_r) = P_1 \otimes \cdots \otimes P_r$; thirdly, if $n < r$ and if $P_1, \ldots, P_n$ are all invertible, then $\mathrm{cr}_n(\Lambda^r)(P_1, \ldots, P_n) = 0$. From

these we see that

$$N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)_n = \begin{cases} P^{\otimes r} & \text{if } n = r, \\ Q \otimes P^{\otimes(r-1)} & \text{if } n = r-1, \\ 0 & \text{otherwise.} \end{cases}$$

We can also read off the differential $P^{\otimes r} \to Q \otimes P^{\otimes(r-1)}$ from [ibid., Lemma 2.2]: it is $\mathrm{cr}_r(\Lambda^r)(\varphi, 1, \dots, 1) = \varphi \otimes 1 \otimes \cdots \otimes 1$. So $N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)$ is the acyclic complex

$$\underset{r+1}{0} \longrightarrow \underset{r}{P \otimes P^{\otimes(r-1)}} \xrightarrow{\varphi\otimes 1} \underset{r-1}{Q \otimes P^{\otimes(r-1)}} \longrightarrow \underset{r-2}{0}$$

Of particular note is the special case in which P and Q are equal to R considered as a module over itself, and φ is given by multiplication by some $x \in R^\times$. Then $N\Lambda^r\Gamma(R \xrightarrow{x} R)$ is equal to the complex $(R \xrightarrow{x} R)$, shifted so that it is concentrated in degrees r and $r-1$.

4. Operations on binary multicomplexes

The goal of this section is to extend the functors F_n between multicomplexes of the previous section to functors of *binary* multicomplexes. Together with the results of the previous section, this shows that if $P_\bullet$ is a bounded acyclic binary multicomplex, then so is $\Lambda^r_n(P_\bullet)$.

Categories of binary complexes are not so well behaved as categories of complexes. In particular, the category of bounded acyclic binary complexes in a split exact category is not split exact.

Example 4.1. Let P be an object in a split exact category $\mathcal{P}$. The following diagram is an admissible epimorphism in the category of bounded acyclic binary complexes in $\mathcal{P}$:

$$\begin{array}{ccccc} P & \underset{i_2}{\overset{i_1}{\rightrightarrows}} & P \oplus P & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & P \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \Sigma} & & \downarrow \\ P & \underset{1}{\overset{1}{\rightrightarrows}} & P & \rightrightarrows & 0 \end{array}$$

(where i_1 and i_2 are the inclusions into the first and second summands, p_1 and p_2 are the corresponding projections and $\Sigma = p_1 + p_2$). But there is no splitting $P \to P \oplus P$ that commutes with both the top and bottom differentials, so $B^{\mathrm{q}}_{\mathrm{b}}\mathcal{P}$ is not split exact.

This difficulty means that we cannot define exterior powers of binary multicomplexes recursively in exactly the way we have for multicomplexes. This problem is resolvable: we shall show that if $P_\bullet$ is an object of $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(R)$, then the objects of

$\Lambda^r_n(P_\bullet)$ are independent of the differentials of $P_\bullet$. Therefore it will make sense to define the exterior power of a binary complex by applying the exterior powers we developed above individually to the two differentials of the binary complex. The resulting pair of complexes will have the same objects, so we consider them as a binary complex.

Lemma 4.2. *Let $F : \mathcal{P} \to \mathcal{P}$ be a covariant functor on an idempotent complete exact category. If $P_\bullet$ and $Q_\bullet$ are chain complexes with the same underlying graded object, then $NF\Gamma(P_\bullet)$ and $NF\Gamma(Q_\bullet)$ have the same underlying graded object.*

Proof. Let $B \in \mathcal{P}^{\Delta^{\mathrm{op}}}$ be a simplicial object. The objects of the complex $N(B)$ are given by

$$N(B)_n := B_n \Big/ \left(\sum_{i=0}^{n} \mathrm{Im}(\sigma_i : B_{n-1} \to B_n) \right)$$

(after embedding $\mathcal{P}$ in a suitable abelian category), where the σ_i are the degeneracies of B. It is enough therefore to show that the objects and degeneracy maps of $F\Gamma(P_\bullet)$ do not depend upon the differential of $P_\bullet$. The objects of $\Gamma(P_\bullet)$ are direct sums of the objects of $P_\bullet$, indexed by the surjections out of $[n]$ in Δ, and do not depend on the differential. The degeneracy operator $\sigma_i : \Gamma(P_\bullet)_{n-1} \to \Gamma(P_\bullet)_n$ is the image of the degeneracy map $\eta_i : [n] \to [n-1]$ in Δ. For any surjection $\eta : [n-1] \twoheadrightarrow [p]$, the composition $\eta\eta_i$ is also a surjection, so the monomorphism in the epi-monic factorisation of $\eta\eta_i$ is just the identity on $[p]$. Therefore, the degeneracy operator σ_i acts on $\Gamma(P_\bullet)_{n-1}$ by sending the summand corresponding to the surjection η by the identity to the summand of $\Gamma(P_\bullet)_n$ corresponding to the surjection $\eta_i\eta$. Thus σ_i does not depend on the differential of $P_\bullet$. Since the objects and degeneracies of $\Gamma(P_\bullet)$ only depend on the underlying graded object of $P_\bullet$, the same is true of $F\Gamma(P_\bullet)$. Therefore the objects of $NF\Gamma(P_\bullet)$ only depend on the underlying graded object as well. □

Corollary 4.3. *Let $n \geq 1$, and let $P_\bullet$ and $Q_\bullet$ be objects of $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$. If $P_\bullet$ and $Q_\bullet$ have the same underlying $\mathbb{Z}^n$-graded object, then $F_n(P_\bullet)$ and $F_n(Q_\bullet)$ have the same underlying $\mathbb{Z}^n$-graded object.*

Proof. This is a straightforward induction on n. □

We are now ready at last to define exterior powers of acyclic binary multicomplexes. Let $P_\bullet$ be an n-dimensional, bounded, acyclic binary multicomplex in $\mathcal{P}$, i.e., an object of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$. We view the commutativity constraints on the differentials of $P_\bullet$ in the same way as described in Remark 1.2: as a collection of 2^n objects of $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$.

Definition 4.4. For a functor F that satisfies the hypotheses of Corollary 3.5, we define induced functors

$$F_n : (B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P} \to (B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$$

by the following procedure: Let $P_\bullet$ be an object of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$, viewed as a collection of 2^n (nonbinary) multicomplexes in the manner described above. Since these multicomplexes all have the same underlying $\mathbb{Z}^n$-graded object, by Corollary 4.3 the same is true of the 2^n multicomplexes obtained by applying F_n (the functor defined on $(C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$ in Corollary 3.5) to the multicomplexes describing $P_\bullet$. We define $F_n(P_\bullet)$ to be the binary multicomplex described by the resulting collection of multicomplexes.

We now return to our main example of interest: the exterior power functors. Let R be a commutative ring. We have seen in Example 3.7 that the usual exterior power operations Λ^r satisfy the hypotheses of Corollary 3.5, so the exterior powers

$$\Lambda_n^r : (C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(R) \to (C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(R)$$

lift to exterior powers of binary multicomplexes

$$\Lambda_n^r : (B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(R) \to (B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(R)$$

for all $n \geq 0$ and $r \geq 1$.

5. Simplicial tensor products

In this section we develop a tensor product for multicomplexes that is compatible with the exterior powers we have defined in the previous sections. We show that the class of this product vanishes in the appropriate K-group, which will eventually be the key to showing that exterior power operations provide homomorphisms on higher K-groups.

5A. ***Constructing simplicial tensor products.*** In this subsection, using the Dold–Kan correspondence again, we construct the so-called simplicial tensor product of multicomplexes and prove it preserves acyclicity and boundedness of complexes.

Although we are ultimately interested in the products induced from the usual tensor products of modules (or sheaves), it is convenient in this section to work in the rather more abstract setting of a generic idempotent complete exact category with some form of well-behaved tensor product.

Definition 5.1. Let $\mathcal{P}$ be an idempotent complete exact category. We say that a biadditive bifunctor $\otimes : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is a *tensor product* if $P \otimes -$ and $- \otimes P$ are exact functors on $\mathcal{P}$ for each object P of $\mathcal{P}$.

For the rest of this section, we fix such a category $\mathcal{P}$ with a tensor product $\otimes$. The reader may wish to keep in mind the example $\mathcal{P} = \mathcal{P}(R)$, with the usual tensor product of R-modules.

Definition 5.2. Let P be an object of $\mathcal{P}$, and let $(Q_\bullet, d_Q)$ and $(R_\bullet, d_R)$ be chain complexes in $\mathcal{P}$.

(1) By $P \otimes Q_\bullet$ we mean the chain complex whose i-th object is $P \otimes Q_i$, with differential $1 \otimes d_Q$. The complex $Q_\bullet \otimes P$ is defined analogously.

(2) By $\mathrm{Tot}(Q_\bullet \otimes R_\bullet)$ we mean the chain complex formed by taking the total complex of the bicomplex whose (i, j)-th object is $Q_i \otimes R_j$, and whose differentials are $d^{\mathrm{ver}} = d_Q \otimes (-1)^j$ and $d^{\mathrm{hor}} = 1 \otimes d_R$. This bicomplex's i-th row is $Q_i \otimes R_\bullet$ and its j-th column is $Q_\bullet \otimes R_j$.

It is clear that if $Q_\bullet$ and $R_\bullet$ are bounded complexes, then the products $P \otimes Q_\bullet$ and $\mathrm{Tot}(Q_\bullet \otimes R_\bullet)$ are bounded as well. We'll need a couple of properties of these products.

Lemma 5.3. *Let $P_\bullet$ be a chain complex in $\mathcal{P}$.*

(1) *The functor*

$$P_\bullet \otimes - : \mathcal{P} \to C\mathcal{P}, \quad Q \mapsto P_\bullet \otimes Q,$$

is exact.

(2) *If $Q_\bullet$ is an acyclic complex in $\mathcal{P}$, then the complex $\mathrm{Tot}(P_\bullet \otimes Q_\bullet)$ is acyclic.*

Proof. The first part is straightforward, as each $P_i \otimes -$ is an exact functor. For the second part, if $Q_\bullet$ is acyclic, then, since acyclic complexes are spliced together from short exact sequences, each of the complexes $P_n \otimes Q_\bullet$ is acyclic. Therefore the rows of the bicomplex $P_\bullet \otimes Q_\bullet$ are acyclic. Our complexes are nonnegative, so the total complex of this bicomplex is exact in an ambient abelian category by the acyclic assembly lemma [Weibel 1994, Lemma 2.7.3]. Since $\mathcal{P}$ is idempotent complete, it supports long exact sequences, so $\mathrm{Tot}(P_\bullet \otimes Q_\bullet)$ is acyclic in $\mathcal{P}$. □

To define the simplicial tensor product of complexes we need to go beyond regular simplicial objects. A *bisimplicial object* B in $\mathcal{P}$ is a functor $B : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathcal{P}$. The *diagonal* of B is the simplicial object defined by precomposition with the usual diagonal functor $\mathrm{diag} : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$,

$$\mathrm{diag}(B) := B \circ \mathrm{diag} : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathcal{P}.$$

If C and D are simplicial objects in $\mathcal{P}$, then we define $C \otimes D$ to be the bisimplicial object given by $(C \otimes D)([m], [n]) = C_m \otimes D_n$ and $(C \otimes D)(\alpha, \beta) = C(\alpha) \otimes D(\beta)$ for $\alpha : [m] \to [m']$, $\beta : [n] \to [n']$. We can now push the tensor product around the Dold–Kan correspondence.

Definition 5.4. The *simplicial tensor product* of chain complexes $P_\bullet$ and $Q_\bullet$ in $\mathcal{P}$ is defined to be

$$P_\bullet \otimes_\Delta Q_\bullet := N\big(\mathrm{diag}(\Gamma(P_\bullet) \otimes \Gamma(Q_\bullet))\big).$$

A word of warning here: although the tensor product is an additive functor in each variable, the complex $P_\bullet \otimes_\Delta Q_\bullet$ is not equal to the product complex $\mathrm{Tot}(P_\bullet \otimes Q_\bullet)$

discussed above. They are related by the Eilenberg–Zilber theorem, which we shall use in the proof of the following lemma:

Lemma 5.5. *Let $P_\bullet$ and $Q_\bullet$ be chain complexes in $\mathcal{P}$, and suppose that at least one of them is acyclic. Then $P_\bullet \otimes_\Delta Q_\bullet$ is acyclic in $\mathcal{P}$.*

Proof. We suppose, without loss of generality, that $Q_\bullet$ is acyclic. By the Eilenberg–Zilber theorem [May 1967, Section 29], the simplicial tensor product $P_\bullet \otimes_\Delta Q_\bullet = N\,\mathrm{diag}(\Gamma(P_\bullet) \otimes \Gamma(Q_\bullet))$ is homotopy equivalent to $\mathrm{Tot}(P_\bullet \otimes Q_\bullet)$, and is therefore acyclic by Lemmas 5.3(2) and 2.2. □

The following is an analogue of Lemma 2.7 for the simplicial tensor product:

Lemma 5.6. *If $P_\bullet$ and $Q_\bullet$ are both bounded chain complexes in $\mathcal{P}$, of lengths k and l, respectively, then $P_\bullet \otimes_\Delta Q_\bullet$ is of length at most kl and so is bounded as well.*

Proof. Examining the Dold–Kan functors applied to a tensor product, one sees that the object $(P_\bullet \otimes_\Delta Q_\bullet)_n$ is equal to

$$N\big(\mathrm{diag}(\Gamma(P_\bullet) \otimes \Gamma(Q_\bullet))\big)_n = \bigoplus_{\varphi} P_i \otimes Q_j,$$

where φ runs over all injections $[n] \hookrightarrow [i] \times [j]$ whose composition with the projections onto $[i]$ and $[j]$ gives surjections $[n] \twoheadrightarrow [i]$ and $[n] \twoheadrightarrow [j]$ (this is derived in [Lawson 2012]). The complexes $P_\bullet$ and $Q_\bullet$ are of length k and l, so $P_i = 0$ and $Q_j = 0$ for all $i > k$ and $j > l$. But for $n > kl$ there is no injection $[n] \hookrightarrow [i] \times [j]$, with $i \leq k$ and $j \leq l$, such that $[n] \twoheadrightarrow [i]$ and $[n] \twoheadrightarrow [j]$ are order-preserving surjections. So $(P_\bullet \otimes_\Delta Q_\bullet)_n = 0$ for $n > kl$. □

We now verify that $\otimes_\Delta$ is a tensor product in the sense of Definition 5.1.

Proposition 5.7. *The simplicial tensor product $\otimes_\Delta$ is a tensor product on the idempotent complete exact category $C\mathcal{P}$ and restricts to a tensor product on the full subcategory $C_\mathrm{b}^\mathrm{q}\mathcal{P}$.*

Proof. If $P_\bullet$ and $Q_\bullet$ are in $C_\mathrm{b}^\mathrm{q}\mathcal{P}$, then so is $P_\bullet \otimes_{\Delta,n} Q_\bullet$, by Lemmas 5.5 and 5.6. So it remains to show that $- \otimes_\Delta -$ is biadditive, and that the functors $P_\bullet \otimes_\Delta -$ and $- \otimes_\Delta P_\bullet$ are exact when $P_\bullet$ is in $C\mathcal{P}$.

The functors N and Γ are both additive and exact, so we only need to inspect $\mathrm{diag}(- \otimes -)$. This is easily seen to be biadditive, as $- \otimes -$ is biadditive. Therefore $- \otimes_\Delta -$ is biadditive as well.

Let B be a simplicial object in $\mathcal{P}$. For a short exact sequence of simplicial objects $0 \to A' \to A \to A'' \to 0$, the sequence

$$0 \to \mathrm{diag}(B \otimes A')_n \to \mathrm{diag}(B \otimes A)_n \to \mathrm{diag}(B \otimes A'')_n \to 0$$

is equal to

$$0 \to B_n \otimes A'_n \to B_n \otimes A_n \to B_n \otimes A''_n \to 0,$$

which is short exact since each $0 \to A'_n \to A_n \to A''_n \to 0$ is short exact and $B_n \otimes -$ is exact. So the sequence

$$0 \to \mathrm{diag}(B \otimes A') \to \mathrm{diag}(B \otimes A) \to \mathrm{diag}(B \otimes A'') \to 0$$

is short exact in every degree for any simplicial object B in $\mathcal{P}$. Therefore the functor $\mathrm{diag}(\Gamma(P_\bullet) \otimes -) : \mathcal{P}^{\Delta^{\mathrm{op}}} \to \mathcal{P}^{\Delta^{\mathrm{op}}}$ is exact. The same is true for $\mathrm{diag}(- \otimes \Gamma(P_\bullet))$. It follows that $P_\bullet \otimes_\Delta -$ and $- \otimes_\Delta P_\bullet$ are exact functors. □

We are now ready to iteratively define simplicial tensor products on categories of multicomplexes.

Definition 5.8. We define *simplicial tensor products*

$$\otimes_{\Delta,n} : C^n\mathcal{P} \times C^n\mathcal{P} \to C^n\mathcal{P}$$

for all $n \geq 0$ recursively:

(1) $\otimes_{\Delta,0} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is the usual tensor product $\otimes$, and

(2) by regarding objects $P_\bullet$ and $Q_\bullet$ of $C^{n+1}\mathcal{P}$ as chain complexes in the idempotent complete exact category $C^n\mathcal{P}$ with the tensor product $\otimes_{\Delta,n}$, we define $P_\bullet \otimes_{\Delta,n+1} Q_\bullet := N\big(\mathrm{diag}(\Gamma(P_\bullet) \otimes_{\Delta,n} \Gamma(Q_\bullet))\big)$.

The following iteration of Proposition 5.7 is now straightforward. The case $n = 0$ is an assumption of this section, and we iterate using $(C^{\mathrm{q}}_{\mathrm{b}})^{n+1}\mathcal{P} = C^{\mathrm{q}}_{\mathrm{b}}((C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P})$.

Corollary 5.9. *For all $n \geq 0$, the simplicial tensor product $\otimes_{\Delta,n}$ is a tensor product in the sense of Definition 5.1 on $C^n\mathcal{P}$ and on $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$.* □

In fact we can say a little more than this. The following lemma is crucial to the proof of the main result of this section:

Lemma 5.10. *Let $P_\bullet$ be an object of $C_{\mathrm{b}}((C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P})$ and let $Q_\bullet$ be an object of $(C^{\mathrm{q}}_{\mathrm{b}})^{n+1}\mathcal{P}$. Then $P_\bullet \otimes_{\Delta,n+1} Q_\bullet$ is an object of $(C^{\mathrm{q}}_{\mathrm{b}})^{n+1}\mathcal{P}$.*

Proof. Noting that $P_\bullet$ and $Q_\bullet$ both have their objects in $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$, and that $Q_\bullet$ is an acyclic complex of objects in that category, this follows immediately from Lemmas 5.5 and 5.6 applied to the tensor product $\otimes_{\Delta,n}$ on the category $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$. □

We can extend the simplicial tensor products to categories of binary complexes in the same way that we did for exterior powers in Section 4. The simplicial tensor product of a pair of binary complexes $(P_\bullet, d_P, \tilde{d}_P)$ and $(Q_\bullet, d_Q, \tilde{d}_Q)$ is obtained by considering the pair of chain complexes $(P_\bullet, d_C) \otimes_\Delta (Q_\bullet, d_Q)$ and $(P_\bullet, \tilde{d}_P) \otimes_\Delta (Q_\bullet, \tilde{d}_Q)$ as a binary complex (it is straightforward to prove that they have the same underlying graded object, in the same manner as Lemma 4.2). The analogue of Corollary 4.3 then follows, and we define the simplicial tensor product of binary multicomplexes just as we did for a functor of one variable in Definition 4.4.

5B. ***Vanishing of products.*** In this subsection we prove that the class of any simplicial tensor product vanishes in the corresponding K-group. Our proof resembles Grayson's procedure [1992, p. 103] of verifying that the second Euler characteristic of a doubly acyclic bicomplex vanishes.

Let $n > 0$, and let $P_\bullet$ and $Q_\bullet$ be n-dimensional bounded acyclic binary complexes of objects of $\mathcal{P}$. That is, $P_\bullet$ and $Q_\bullet$ are objects of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$. Then the simplicial tensor product $P_\bullet \otimes_{\Delta,n} Q_\bullet$ is in $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$ as well by Corollary 5.9. Since the objects of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$ are the generators of $K_n(\mathcal{P})$, one would like to use $\otimes_{\Delta,n}$ to induce a product $K_n(\mathcal{P}) \times K_n(\mathcal{P}) \to K_n(\mathcal{P})$. On first inspection this appears not to work, because the product $P_\bullet \otimes_{\Delta,n} Q_\bullet$ is not diagonal if only one of $P_\bullet$ or $Q_\bullet$ is diagonal. This is not a problem in the end though, since the whole product vanishes on $K_n(\mathcal{P})$.

Proposition 5.11. *Let $n > 0$. For any pair of n-dimensional bounded acyclic multicomplexes $P_\bullet$ and $Q_\bullet$ in $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$, the class $[P_\bullet \otimes_{\Delta,n} Q_\bullet]$ vanishes in $K_n(\mathcal{P})$.*

Proof. First we filter $P_\bullet$ by degree. Regard $P_\bullet$ as an acyclic binary complex of objects of $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}$. For $i \geq 0$, let $P|_{[0,i]}$ be the binary complex obtained by "restricting" $P_\bullet$ to be supported on $[0, i]$. That is, $(P|_{[0,i]})_j$ is equal to P_j if $0 \leq j \leq i$, and $(P|_{[0,i]})_j = 0$ otherwise. The differentials on $P|_{[0,i]}$ are inherited from $P_\bullet$. We write $P_j[0]$ for P_j considered as a binary complex concentrated in degree 0. Then $P_j[j]$, which denotes P_j considered as a binary complex concentrated in degree j, is the quotient of the inclusion $P|_{[0,j-1]} \hookrightarrow P|_{[0,j]}$ (if $j \geq 1$). If $P_\bullet$ is supported on $[0, n]$, so that $P_j = 0$ for $j > n$, we therefore have an n-stage filtration

$$P_0[0] = P|_{[0,0]} \hookrightarrow P|_{[0,1]} \hookrightarrow \cdots \hookrightarrow P|_{[0,n-1]} \hookrightarrow P|_{[0,n]} = P_\bullet$$

whose successive quotients determine short exact sequences

$$0 \to P|_{[0,j-1]} \to P|_{[0,j]} \to P_j[j] \to 0.$$

We take the simplicial tensor product with $Q_\bullet$ of this whole filtration, obtaining sequences

$$0 \to P|_{[0,j-1]} \otimes_{\Delta,n} Q_\bullet \to P|_{[0,j]} \otimes_{\Delta,n} Q_\bullet \to P_j[j] \otimes_{\Delta,n} Q_\bullet \to 0 \qquad (5.12)$$

for $j = 1, \ldots, n$, which are short exact by Corollary 5.9.

By Lemma 5.10, all of the objects are in the right category, so each of the short exact sequences of (5.12) yields an equation

$$[P|_{[0,j]} \otimes_{\Delta,n} Q_\bullet] = [P|_{[0,j-1]} \otimes_{\Delta,n} Q_\bullet] + [P_j[j] \otimes_{\Delta,n} Q_\bullet]$$

in $K_n(\mathcal{P})$. Putting these together gives

$$[P_\bullet \otimes_{\Delta,n} Q_\bullet] = \sum_{j=0}^{n} [P_j[j] \otimes_{\Delta,n} Q_\bullet].$$

To proceed we need to assume a small lemma, for which the second type of relation in $K_n(\mathcal{P})$ (diagonal binary multicomplexes vanish) is crucial.

Lemma 5.13. *The following equality holds in* $K_n(\mathcal{P})$:

$$[P_j[j] \otimes_{\Delta,n} Q_\bullet] = (-1)^j [P_j[0] \otimes_{\Delta,n} Q_\bullet].$$

Continuing with the main proof, our equation now reads

$$[P_\bullet \otimes_{\Delta,n} Q_\bullet] = \sum_{j=0}^{n} (-1)^j [P_j[0] \otimes_{\Delta,n} Q_\bullet].$$

By inspection we see that $\Gamma(P_j[0])$ is the constant simplicial object which has P_j in each degree. The functor

$$\operatorname{diag}\big(\Gamma(P_j[0]) \otimes_{\Delta,n-1} -\big) : ((C_{\mathrm{b}})^{n-1}\mathcal{A})^{\Delta^{\mathrm{op}}} \to ((C_{\mathrm{b}})^{n-1}\mathcal{A})^{\Delta^{\mathrm{op}}}$$

is therefore isomorphic to the functor

$$P_j \otimes_{\Delta,n-1} - : ((C_{\mathrm{b}})^{n-1}\mathcal{A})^{\Delta^{\mathrm{op}}} \to ((C_{\mathrm{b}})^{n-1}\mathcal{A})^{\Delta^{\mathrm{op}}},$$

since they both have the same effect of "tensoring everywhere by P_j". This functor is additive, so we have an isomorphism of functors

$$N(P_j \otimes_{\Delta,n-1} \Gamma(-)) \cong P_j \otimes_{\Delta,n-1} -.$$

Hence,

$$P_j[0] \otimes_{\Delta,n} Q_\bullet = N \operatorname{diag}\big(\Gamma(P_j[0]) \otimes_{\Delta,n-1} \Gamma(Q_\bullet)\big) \cong P_j \otimes_{\Delta,n-1} Q_\bullet,$$

so we have

$$[P_\bullet \otimes_{\Delta,n} Q_\bullet] = \sum_{j=0}^{n} (-1)^j [P_j \otimes_{\Delta,n-1} Q_\bullet].$$

There is an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0,$$

since $P_\bullet$ is acyclic. The objects of $Q_\bullet$ are in $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}$, so $- \otimes_{\Delta,n-1} Q_\bullet$ is an exact functor by Lemma 5.3(1), and so the following sequence is exact:

$$0 \to P_n \otimes_{\Delta,n-1} Q_\bullet \to P_{n-1} \otimes_{\Delta,n-1} Q_\bullet \to \cdots \to P_1 \otimes_{\Delta,n-1} Q_\bullet \to P_0 \otimes_{\Delta,n-1} Q_\bullet \to 0.$$

Exact sequences translate into alternating sums in the Grothendieck group, so this exact sequence gives exactly the identity

$$\sum_{j=0}^{n} (-1)^j [P_j \otimes_{\Delta,n-1} Q_\bullet] = 0$$

in $K_0((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P})$, thus the same relation holds in $K_n(\mathcal{P})$. Therefore $[P_\bullet \otimes_{\Delta,n} Q_\bullet] = 0$, as required. □

It remains to prove Lemma 5.13.

Proof of Lemma 5.13. Consider the following diagram as a short exact sequence of binary complexes concentrated in degrees j and $j-1$:

$$\begin{array}{ccc} 0 & \rightarrow & P_j \\ \downarrow & & \downarrow \\ P_j & \xrightarrow[1]{1} & P_j \\ \downarrow & & \downarrow \\ P_j & \rightarrow & 0 \end{array}$$

We will use this diagram to show that $[P_j[j]\otimes_{\Delta,n} Q_\bullet] = -[P_j[j-1]\otimes_{\Delta,n} Q_\bullet]$. The argument can be iterated $j-1$ times to yield $[P_j[j]\otimes_{\Delta,n} Q_\bullet] = (-1)^j[P_j[0]\otimes_{\Delta,n} Q_\bullet]$ in $K_n(\mathcal{P})$, as required. For lack of a better notation, we will denote the middle row of the diagram by $(P_j = P_j)$. Then the diagram represents a short exact sequence of binary complexes

$$0 \to P_j[j-1] \to (P_j = P_j) \to P_j[j] \to 0,$$

which upon tensoring with $Q_\bullet$ becomes the short exact sequence

$$0 \to P_j[j-1]\otimes_{\Delta,n} Q_\bullet \to (P_j = P_j)\otimes_{\Delta,n} Q_\bullet \to P_j[j]\otimes_{\Delta,n} Q_\bullet \to 0$$

by Lemma 5.10. Since $Q_\bullet$ is acyclic and has objects in $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}$, each of the terms of this short exact sequence is an object of $(B_{\mathrm{b}}^{\mathrm{q}})^{n}\mathcal{P}$ by Lemma 5.10, so we have a relation

$$[(P_j = P_j)\otimes_{\Delta,n} Q_\bullet] = [P_j[j-1]\otimes_{\Delta,n} Q_\bullet] + [P_j[j]\otimes_{\Delta,n} Q_\bullet]$$

in $K_0((B_{\mathrm{b}}^{\mathrm{q}})^{n}\mathcal{P})$, and hence in $K_n(\mathcal{P})$. We claim that

$$[(P_j = P_j)\otimes_{\Delta,n} Q_\bullet] = 0$$

in $K_n(\mathcal{P})$, so that $[P_j[j]\otimes_{\Delta,n} Q_\bullet] = -[P_j[j-1]\otimes_{\Delta,n} Q_\bullet]$. We can filter $Q_\bullet$ in the same manner that we have filtered $P_\bullet$ in the main proof above:

$$Q_0[0] = Q|_{[0,0]} \hookrightarrow Q|_{[0,1]} \hookrightarrow \cdots \hookrightarrow Q|_{[0,n-1]} \hookrightarrow Q|_{[0,n]} = Q_\bullet,$$

giving short exact sequences

$$0 \to Q|_{[0,i-1]} \to Q|_{[0,i]} \to Q_i[i] \to 0.$$

Upon tensoring with $(P_j = P_j)$, we have short exact sequences

$$0 \to (P_j = P_j)\otimes_{\Delta,n} Q|_{[0,i-1]} \to (P_j = P_j)\otimes_{\Delta,n} Q|_{[0,i]} \to (P_j = P_j)\otimes_{\Delta,n} Q_i[i] \to 0$$

(by Lemma 5.10). Furthermore, since $(P_j = P_j)$ is an acyclic binary complex of objects of $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}$, each of the terms of these short exact sequences is an object

of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$, by Lemma 5.10. We therefore have the equation

$$[(P_j = P_j) \otimes_{\Delta,n} Q_\bullet] = \sum_i [(P_j = P_j) \otimes_{\Delta,n} Q_i[i]]$$

in $K_0((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P})$, and hence in $K_n(\mathcal{P})$. But $(P_j = P_j)$ is a diagonal binary complex, as is each $Q_i[i]$ (trivially). The simplicial tensor product of a pair of diagonal complexes is again diagonal, so each of the acyclic binary complexes $(P_j = P_j) \otimes_{\Delta,n} Q_i[i]$ is diagonal and hence vanishes in $K_n(\mathcal{P})$. Therefore $[(P_j = P_j) \otimes_{\Delta,n} Q_\bullet] = 0$, so the desired relation holds. □

This finally completes the proof of Proposition 5.11. Having taken the trouble to set up an alternative product of bounded acyclic binary multicomplexes, one that is compatible with the exterior powers, we've now shown that (like the usual tensor product) it is always zero! It was not all for naught though: at least we know now that the induced operation $\otimes_{\Delta,n} : K_n(\mathcal{P}) \times K_n(\mathcal{P}) \to K_n(\mathcal{P})$ is well-defined. Furthermore, the vanishing of this product proves that the exterior power operations induce homomorphisms on $K_n(R)$ (and, more generally, on the higher K-groups of schemes). This is shown in the next section.

6. Exterior power operations on K-groups of schemes

The goal of this section is to extend the endofunctor Λ_n^r defined in Section 4 to bounded acyclic multicomplexes of locally free modules of finite rank on a scheme X, and to prove that it induces a well-defined operation λ^r on the higher K-group $K_n(X)$. We will see that, for $n > 0$, this operation λ^r is not just a map but in fact a homomorphism.

Let X be a quasicompact scheme, and let $\mathcal{P}(X)$ be the category of locally free $\mathcal{O}_X$-modules of finite rank. Then $\mathcal{P}(X)$ is an exact category in the usual sense. It is idempotent complete but not split exact in general. We write $K_n(X)$ for the K-group $K_n(\mathcal{P}(X))$.

As in Section 3, we inductively define an endofunctor Λ_n^r on $C^n\mathcal{P}(X)$ for $r \geq 1$ and $n \geq 0$ as follows: the functor Λ_0^r is the usual r-th exterior power functor on $C^0\mathcal{P}(X) = \mathcal{P}(X)$, and Λ_n^r is defined as $N\Lambda_{n-1}^r\Gamma$, with N and Γ as introduced in Section 2.

Proposition 6.1. *For all $r, n > 0$, the functor Λ_n^r restricts to an endofunctor on the subcategory $(C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$ of $C^n\mathcal{P}(X)$.*

Proof. Given any open affine subscheme $U = \mathrm{Spec}(R)$ of X, a straightforward inductive argument shows that the following diagram commutes:

$$\begin{array}{ccc} C^n\mathcal{P}(X) & \xrightarrow{\Lambda_n^r} & C^n\mathcal{P}(X) \\ \downarrow & & \downarrow \\ C^n\mathcal{P}(U) & \xrightarrow{\Lambda_n^r} & C^n\mathcal{P}(U) \\ \| & & \| \\ C^n\mathcal{P}(R) & \xrightarrow{\Lambda_n^r} & C^n\mathcal{P}(R) \end{array}$$

The vertical arrows are induced by the restriction functor $\mathcal{P}(X) \to \mathcal{P}(U)$, $P \mapsto P|_U$, and the lower horizontal arrow is the functor Λ_n^r introduced in Section 3. A complex in $C^n\mathcal{P}(X)$ is acyclic, or bounded, if and only if its restriction to every open affine subscheme has the respective property, so Proposition 6.1 follows from the results of Section 3. □

As in Section 4, one easily deduces that, for any complex $P_\bullet$ in $C^n\mathcal{P}(X)$, the objects in $\Lambda_n^r(P_\bullet)$ do not depend on the differentials in $P_\bullet$. We can therefore extend the endofunctor Λ_n^r to an endofunctor of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$, which we denote by Λ_n^r again. The goal of the rest of this section is to prove the following theorem:

Theorem 6.2. *Let $n > 0$ and $r > 0$. The endofunctor Λ_n^r of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$ induces a well-defined homomorphism $\lambda^r : K_n(X) \to K_n(X)$.*

Definition 6.3. The homomorphism λ^r in the previous theorem is called the *r-th exterior power operation* on $K_n(X)$.

Proof of Theorem 6.2. If $P_\bullet$ is a diagonal multicomplex in $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$, then the multicomplex $\Lambda_n^r(P_\bullet)$ is diagonal as well, by definition of Λ_n^r. It therefore suffices to show that the association $[P_\bullet] \mapsto [\Lambda_n^r(P_\bullet)]$ induces a well-defined homomorphism of groups

$$\lambda^r : K_0((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)) \to K_n(X).$$

Thus we need to show that the equality

$$[\Lambda_n^r(P_\bullet)] = [\Lambda_n^r(P'_\bullet)] + [\Lambda_n^r(P''_\bullet)]$$

holds in $K_n(X)$ for every short exact sequence $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ in $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$. The classes $[\Lambda_n^{r-i}(P'_\bullet) \otimes_{\Delta,n} \Lambda_n^i(P''_\bullet)]$ for $i = 1, \ldots, r-1$ vanish in $K_n(X)$ by Proposition 5.11 applied to the category $\mathcal{P} = \mathcal{P}(X)$, where the simplicial tensor product has been constructed inductively from the usual tensor product of quasicoherent $\mathcal{O}_X$-modules. So the desired equality is equivalent in $K_n(X)$ to the more familiar-looking identity

$$[\Lambda_n^r(P_\bullet)] = [\Lambda_n^r(P'_\bullet)] + \sum_{i=1}^{r-1} [\Lambda_n^{r-i}(P'_\bullet) \otimes_{\Delta,n} \Lambda_n^i(P''_\bullet)] + [\Lambda_n^r(P''_\bullet)].$$

In order to prove this latter formula, we cannot just apply the usual formula for the r-th exterior power of a direct sum because the given short exact sequence of binary complexes, $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$, does not split in general, even if X is affine (see Example 4.1). Instead, by induction on n, we construct for every sequence $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ in $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$ a natural induced filtration

$$\Lambda_n^r(P'_\bullet) \hookrightarrow \Lambda_n^{r-1}(P'_\bullet) \wedge_n \Lambda_n^1(P_\bullet) \hookrightarrow \cdots \hookrightarrow \Lambda_n^1(P'_\bullet) \wedge_n \Lambda_n^{r-1}(P_\bullet) \hookrightarrow \Lambda_n^r(P_\bullet)$$

of $\Lambda_n^r(P_\bullet)$ by certain subobjects $\Lambda_n^{r-i}(P'_\bullet) \wedge_n \Lambda_n^i(P_\bullet)$, $i = 0, \ldots, r$, of $\Lambda_n^r(P_\bullet)$, also belonging to $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$, together with short exact sequences

$$0 \to \Lambda_n^{r-i+1}(P'_\bullet) \wedge_n \Lambda_n^{i-1}(P_\bullet) \to \Lambda_n^{r-i}(P'_\bullet) \wedge_n \Lambda_n^i(P_\bullet) \to \Lambda_n^{r-i}(P'_\bullet) \otimes_{\Delta,n} \Lambda_n^i(P''_\bullet) \to 0 \quad (6.4)$$

for $i = 1, \ldots, n$.

For $n = 0$ and $i \in \{0, \ldots, r\}$, the object $\Lambda_0^{r-i}(P') \wedge_0 \Lambda_0^i(P)$ is defined to be what is usually meant by $\Lambda^{r-i}(P') \wedge \Lambda^i(P)$: the image of the canonical homomorphism $\Lambda^{r-i}(P') \otimes \Lambda^i(P) \to \Lambda^r(P)$. It is well known that these objects come with the required short exact sequences (6.4).

If $n > 0$ and if, for a moment, the sequence $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ is given in $C_{\mathrm{b}}^{\mathrm{q}}(C_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}(X)$ rather than in $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$, we first note that applying the exact functor Γ to the sequence, we get the short exact sequence

$$0 \to \Gamma(P'_\bullet) \to \Gamma(P_\bullet) \to \Gamma(P''_\bullet) \to 0$$

of simplicial objects in $(C_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}(X)$. By the inductive hypothesis, the complexes $\Lambda_{n-1}^{r-i}(\Gamma(P')_m) \wedge_{n-1} \Lambda_{n-1}^i(\Gamma(P)_m)$ for $i = 0, \ldots, n$ and $m \geq 0$ are in $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}(X)$ and we have short exact sequences

$$0 \to \Lambda_{n-1}^{r-i+1}(\Gamma(P')_m) \wedge_{n-1} \Lambda_{n-1}^{i-1}(\Gamma(P)_m) \to \Lambda_{n-1}^{r-i}(\Gamma(P')_m) \wedge_{n-1} \Lambda_{n-1}^i(\Gamma(P)_m) \to \Lambda_{n-1}^{r-i}(\Gamma(P')_m) \otimes_{\Delta,n-1} \Lambda_{n-1}^i(\Gamma(P'')_m) \to 0$$

for $i = 1, \ldots, r$ and $m \geq 0$. These short exact sequences assemble to short exact sequences of simplicial objects in $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}(X)$. By applying the exact functor N, we finally obtain the required objects

$$\Lambda_n^{r-i}(P') \wedge_n \Lambda_n^i(P) := N\big(\Lambda_{n-1}^{r-i}(\Gamma(P')) \wedge_{n-1} \Lambda_{n-1}^i(\Gamma(P))\big)$$

for $i = 0, \ldots, r$ and the required short exact sequences (6.4). As the objects of the multicomplex $\Lambda_n^{r-i}(P') \wedge_n \Lambda_n^i(P)$ are independent of the differentials in the multicomplexes $P'_\bullet$ and $P_\bullet$, this construction of $\wedge_n$ passes to the category $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$ as in Section 4.

From Proposition 6.1 and Section 5 we know that the complex $\Lambda_n^r(P_\bullet)$ and the complexes $\Lambda_n^{r-i}(P'_\bullet) \otimes_{\Delta,n} \Lambda_n^i(P''_\bullet)$ for $i = 0, \ldots, r$ belong to $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$. Now a

straightforward downwards induction on i based on the short exact sequences (6.4) shows that the complexes $\Lambda_n^{r-i}(P'_\bullet) \wedge_n \Lambda_n^i(P_\bullet)$ for $i = 0, \dots, r$ are bounded and acyclic, so they belong to $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$, as was to be shown. □

7. The second λ-ring axiom

Given a scheme X, there is a "trivial" way to equip the graded abelian group $K_*(X) := \bigoplus_{n\geq 0} K_n(X)$ with a multiplication, and to extend the exterior power operations defined in the previous section to $K_*(X)$ so that they are compatible with addition in $K_*(X)$ in the usual sense. The main result of this section is that they are also compatible with multiplication in the expected way — that is to say, they satisfy the λ-ring axiom (2).

Let X be a quasicompact scheme. We recall that $K_0(X)$ together with the usual exterior power operations $\lambda^r : K_0(X) \to K_0(X)$, $r \geq 0$, is a λ-ring as defined in the introduction (see Chapter V of [Fulton and Lang 1985]). Furthermore, $K_n(X)$ is a $K_0(X)$-module via $[P] \cdot [Q_\bullet] := [P \otimes Q_\bullet]$ for P in $\mathcal{P}(X)$ and $Q_\bullet$ in $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$; see also Definition 5.2(1).

We define a multiplication on $K_*(X) := \bigoplus_{n\geq 0} K_n(X)$ by

$$(a_0, a_1, a_2, \dots)\bullet(b_0, b_1, b_2, \dots) = (a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_2b_0, \dots);$$

in particular, the product of any two elements in $\bigoplus_{n\geq 1} K_n(X)$ is defined to be zero. With this multiplication, $K_*(X)$ is a commutative ring. Furthermore, we define exterior power operations $\lambda^r : K_*(X) \to K_*(X)$, $r \geq 0$, by the formula

$$\lambda^r((a_0, a_1, a_2, \dots)) = \left(\lambda^r(a_0), \sum_{i=0}^{r-1} \lambda^i(a_0)\lambda^{r-i}(a_1), \sum_{i=0}^{r-1} \lambda^i(a_0)\lambda^{r-i}(a_2), \dots\right).$$

By definition, we then have $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for all $x \in K_*(X)$. A straightforward calculation using Theorem 6.2 and the fact that $K_0(X)$ satisfies axiom (1) of a λ-ring shows that $K_*(X)$ also satisfies axiom (1). The next theorem addresses axiom (2).

Theorem 7.1. *The pre-λ-ring $K_*(X)$ defined above satisfies axiom (2) of a λ-ring.*

Proof. Axiom (2) holds for elements of the form $x = (a_0, 0, 0, \dots)$ and $y = (b_0, 0, 0, \dots)$ in $K_*(X)$ because it holds for $K_0(X)$. It also holds for elements of the form $x = (0, a_1, a_2, \dots)$ and $y = (0, b_1, b_2, \dots)$ because $\lambda^r(0) = 0$ for all $r \geq 1$ and because every monomial in the ring $\mathbb{Z}[X_1, \dots, X_r, Y_1, \dots, Y_r]$ whose coefficient in $P_r(X_1, \dots, X_r, Y_1, \dots, Y_r)$ is nonzero is divisible by some product X_iY_j. Furthermore, it suffices to check axiom (2) for x and y belonging to a set of additive generators of $K_*(X)$ because $K_*(X)$ satisfies axiom (1) and because axiom (2) is

equivalent to the multiplicativity of the homomorphism

$$\lambda_t : K_*(X) \to 1 + t \cdot K_*(X)[\![t]\!], \quad x \mapsto \sum_{r \geq 0} \lambda^r(x) t^r.$$

We are therefore reduced to showing that the equality

$$\lambda^r(xy) = P_r(\lambda^1(x), \ldots, \lambda^r(x), \lambda^1(y), \ldots, \lambda^r(y)) \tag{7.2}$$

holds in $K_n(X)$ for elements $y \in K_n(X)$ and $x \in K_0(X)$ of the form $x = [\mathcal{E}]$ for some locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank.

We now invoke the projective bundle theorem [Quillen 1973, §8, Theorem 2.1]. We remark that its proof in [loc. cit.] only relies on the additivity and resolution theorems, and not, for instance, on the dévissage theorem or localisation sequence. The additivity and resolution theorems have been proved in [Harris 2015] within the context of Grayson's definition of higher K-groups, so the projective bundle theorem also has a proof within that context, without resorting to topological methods.

It is well known that an iterated application of the projective bundle theorem yields the following splitting principle: there exists a projective morphism $f : Y \to X$ such that $f^*[\mathcal{E}]$ is the sum of invertible $\mathcal{O}_Y$-modules in $K_0(Y)$ and such that $f^* : K_*(X) \to K_*(Y)$ is injective. It is straightforward to check that $f^* : K_*(X) \to K_*(Y)$ is a homomorphism of (pre-)λ-rings. Using the above argument about additive generators again, we are therefore reduced to showing the equality (7.2) only when x is the class $[\mathcal{L}]$ of an invertible $\mathcal{O}_X$-module $\mathcal{L}$. In that case, (7.2) becomes the much simpler formula

$$\lambda^r([\mathcal{L}] \cdot y) = [\mathcal{L}^{\otimes r}] \cdot \lambda^r(y),$$

because $\lambda^2[\mathcal{L}] = \cdots = \lambda^r[\mathcal{L}] = 0$, and because P_r satisfies the identity

$$P_r(1, 0, \ldots, 0, Y_1, \ldots, Y_r) = Y_r$$

and has X-degree r (where X_i is defined to be of degree i for $i = 1, \ldots, r$). Using the argument about additive generators again, it suffices to show that for any object $P_\bullet$ of $(B_{\mathrm{b}}^{\mathrm{q}})^n \mathcal{P}(X)$, the object $\Lambda_n^r(\mathcal{L} \otimes P_\bullet)$ is isomorphic to $\mathcal{L}^{\otimes r} \otimes \Lambda_n^r(P_\bullet)$. This is well known if $n = 0$, and follows by induction on n from the following chain of isomorphisms applied to each of the 2^n multicomplexes associated with the binary multicomplex $P_\bullet$ (which we again denote by $P_\bullet$):

$$\begin{aligned}
\Lambda_n^r(\mathcal{L} \otimes P_\bullet) &= N\Lambda_{n-1}^r \Gamma(\mathcal{L} \otimes P_\bullet) \\
&\cong N(\mathcal{L}^{\otimes r} \otimes \Lambda_{n-1}^r \Gamma(P_\bullet)) \\
&\cong \mathcal{L}^{\otimes r} \otimes N\Lambda_{n-1}^r \Gamma(P_\bullet) \\
&\cong \mathcal{L}^{\otimes r} \otimes \Lambda_n^r(P_\bullet).
\end{aligned}$$

□

8. The final λ-ring axiom

The goal of this section is to prove that the pre-λ-ring $K_*(X)$ (introduced and proven to satisfy λ-ring axiom (2) in the previous section) also satisfies the final λ-ring axiom (3) and is therefore a λ-ring. The main ingredients are the language of polynomial functors, the identification of polynomial functors with modules over the Schur algebra, and Serre's method of computing the Grothendieck group of representations of the group scheme $\mathrm{GL}_{n,\mathbb{Z}}$.

8A. ***Polynomial functors.*** In this subsection we introduce the notion of polynomial functors and state that the Grothendieck group of the category of polynomial functors over $\mathbb{Z}$ is isomorphic to the universal λ-ring in one variable; see Theorem 8.5 below. This theorem will allow us to prove the final λ-ring axiom for $K_*(X)$ in Subsection 8D. The proof of Theorem 8.5 occupies Subsections 8B and 8C.

We recall $\mathcal{P}(S)$ denotes the category of $\mathcal{O}_S$-modules that are locally free of finite rank on a scheme S. We define a category $\underline{\mathcal{P}}(S)$ "enriched in schemes over S" as follows. The objects are the same as the objects of $\mathcal{P}(S)$, and for every $V, W \in \mathcal{P}(S)$ we have an S-scheme

$$\underline{\mathrm{Hom}}(V, W) := \underline{\mathrm{Spec}}_S \, \mathrm{Sym}^\bullet(\mathcal{H}\mathrm{om}(V, W)^\vee).$$

This is the "physical vector bundle" corresponding to the locally free $\mathcal{O}_S$-module $\mathcal{H}\mathrm{om}(V, W)$ and we have

$$\underline{\mathrm{Hom}}(V, W)(T) = \mathrm{Hom}_{\mathcal{O}_T}(V_T, W_T)$$

for every S-scheme T. In fact, by Yoneda's lemma, we may think of $\underline{\mathrm{Hom}}(V, W)$ as the functor which associates $\mathrm{Hom}_{\mathcal{O}_T}(V_T, W_T)$ with every S-scheme T. The latter viewpoint is used in a lot of literature about polynomial functors. Composition in $\underline{\mathcal{P}}(S)$ is given by the natural maps

$$\underline{\mathrm{Hom}}(U, V) \times_S \underline{\mathrm{Hom}}(V, W) \to \underline{\mathrm{Hom}}(U, W)$$

of schemes over S, and the identities are given by the obvious sections id_V in $\underline{\mathrm{Hom}}(V, V)(S)$.

Definition 8.1. A *polynomial functor over S* is an enriched functor $F : \underline{\mathcal{P}}(S) \to \underline{\mathcal{P}}(S)$. A *morphism of polynomial functors* is a natural transformation. We denote the category of polynomial functors over S by $\mathrm{Pol}(S)$.

In other words, a polynomial functor consists of objects $FV \in \mathcal{P}(S)$, $V \in \mathcal{P}(S)$, and of morphisms of S-schemes

$$F : \underline{\mathrm{Hom}}(V, W) \to \underline{\mathrm{Hom}}(FV, FW) \quad \text{for } V, W \in \mathcal{P}(S),$$

which satisfy the usual functor axioms. In less precise terms, F being a morphism of S-schemes means that if, for instance, $S = \mathrm{Spec}(k)$ with k a field, the map $F : \mathrm{Hom}(V, W) \to \mathrm{Hom}(FV, FW)$ is given by polynomials in coordinates of V and W. Note that we do not ask F to be additive. Every polynomial functor F induces an "ordinary" endofunctor of $\mathcal{P}(S)$, denoted by F again. A morphism $\eta : F \to G$ consists of a morphism of $\mathcal{O}_S$-modules

$$\eta_V : FV \to GV$$

for every $V \in \mathcal{P}(S)$, satisfying the usual conditions for a natural transformation.

Example 8.2 (exterior powers). Functoriality of Λ^d implies that for all $V, W \in \mathcal{P}(S)$ we have a map

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(\Lambda^d V, \Lambda^d W).$$

This is a priori a map of sets, but its formation commutes with base change $T \to S$, and hence by Yoneda it defines a map of S-schemes

$$\underline{\mathrm{Hom}}(V, W) \to \underline{\mathrm{Hom}}(\Lambda^d V, \Lambda^d W).$$

We obtain a polynomial functor $\Lambda^d : \underline{\mathcal{P}}(S) \to \underline{\mathcal{P}}(S)$.

The category $\mathrm{Pol}(S)$ is a $\Gamma(S, \mathcal{O}_S)$-linear category. We declare a sequence

$$0 \to F \to G \to H \to 0$$

in $\mathrm{Pol}(S)$ to be exact if the sequence

$$0 \to FV \to GV \to HV \to 0$$

is exact for every V; this way $\mathrm{Pol}(S)$ becomes an exact category [Touzé 2013, Section 2.1.1]. It carries a tensor product

$$\otimes : \mathrm{Pol}(S) \times \mathrm{Pol}(S) \to \mathrm{Pol}(S)$$

as well as exterior power operators

$$\Lambda^n : \mathrm{Pol}(S) \to \mathrm{Pol}(S), \qquad F \mapsto \Lambda^n F := \Lambda^n \circ F.$$

These data turn $K_0(\mathrm{Pol}(S))$ into a pre-λ-ring. To prove this, one proceeds as in the proof of Theorem 6.2. As there, the category $\mathrm{Pol}(S)$ is in general not split exact, but for every short exact sequence as above, one can construct a natural filtration

$$0 \subset \Lambda^n F \subset F \wedge \cdots \wedge F \wedge G \subset \cdots \subset F \wedge G \wedge \cdots \wedge G \subset \Lambda^n G$$

of $\Lambda^n G$ whose successive quotients are isomorphic to $\Lambda^{n-k} F \otimes \Lambda^k H$, $k = 0, \ldots, n$.

Less evident is that for every morphism $f : T \to S$ there is a natural *base change functor* $f^* : \mathrm{Pol}(S) \to \mathrm{Pol}(T)$. This can be constructed as follows. Let $F : \underline{\mathcal{P}}(S) \to \underline{\mathcal{P}}(S)$ be a polynomial functor. Given $V \in \mathcal{P}(T)$ one chooses an open

cover (U_i) of T, vector bundles $V_i \in \underline{\mathcal{P}}(S)$ and isomorphisms $\alpha_i : (f^*V_i)|_{U_i} \to V|_{U_i}$. These define gluing data $\alpha_{ij} := \alpha_i^{-1}\alpha_j$ and one constructs the desired $(f^*F)V$ by gluing the bundles $f^*(FV_i)$ over the U_{ij} using the maps $F(\alpha_{ij})$. Note that the expression $F(\alpha_{ij})$ makes sense as F is a polynomial functor. For an alternative description of f^*, see Remark 8.9.

Thus, every polynomial functor $F \in \mathrm{Pol}(S)$ induces a family of functors

$$F_T : \mathcal{P}(T) \to \mathcal{P}(T),$$

indexed by $T \to S$, and that the F_T commute with base change.

The functor f^* is exact, and commutes with the operations $\otimes$ and Λ^n, so that f^* induces a morphism

$$f^* : K_0(\mathrm{Pol}(S)) \to K_0(\mathrm{Pol}(T))$$

of pre-λ-rings.

Definition 8.3. A polynomial functor $F \in \mathrm{Pol}(S)$ is said to be *homogeneous of degree d* if, for every $V \in \mathcal{P}(S)$, the diagram

$$\begin{array}{ccc} \mathbb{G}_{m,S} & \xrightarrow{x \mapsto x^d} & \mathbb{G}_{m,S} \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(V, V) & \xrightarrow{F} & \underline{\mathrm{Hom}}(FV, FV) \end{array}$$

commutes; here, the vertical morphisms are given by scalar multiplication. We denote by $\mathrm{Pol}_d(S)$ the category of polynomial functors homogeneous of degree d, and by $\mathrm{Pol}_{<\infty}(S)$ the category of polynomial functors that are finite direct sums of homogeneous polynomial functors.

Example 8.4. The polynomial functor Λ^d is homogeneous of degree d. The infinite direct sum $\bigoplus_{d\geq 0} \Lambda^d$ is well-defined as it becomes finite when applied to any V; it is a polynomial functor, but not in $\mathrm{Pol}_{<\infty}(S)$.

Let $\mathbb{Z}[s_1, s_2, \dots]$ denote the ring of symmetric functions, with s_i the i-th elementary symmetric function. This is a λ-ring, with $\lambda^i(s_1) = s_i$, also called the universal λ-ring in one variable; see [Yau 2010, §1.3]. It is also a graded ring with $\deg s_d = d$.

Theorem 8.5. *The ring homomorphism*

$$\mathbb{Z}[s_1, s_2, \dots] \to K_0(\mathrm{Pol}_{<\infty}(\mathbb{Z})), \quad s_i \mapsto [\Lambda^i],$$

is an isomorphism of pre-λ-rings.

The proof of this theorem will be given at the end of Subsection 8C.

Corollary 8.6. *$K_0(\mathrm{Pol}_{<\infty}(\mathbb{Z}))$ is a λ-ring.* □

8B. ***The Schur algebra.*** The object of this subsection is to relate polynomial functors to the Schur algebra; see [Roby 1963, Chapters I & IV; Krause 2013, §2] for details.

Throughout this subsection, R is a commutative ring. If M is a locally free R-module and d a nonnegative integer, then the R-module of degree d divided powers is the module of symmetric degree d tensors:

$$\Gamma^d M = \Gamma^d_R M = (M^{\otimes d})^{S_d}.$$

If A is an associative and locally free R-algebra and M is moreover an A-module, then $\Gamma^d_R A$ is a sub-R-algebra of $A^{\otimes d}$ and the obvious multiplication of $\Gamma^d_R A$ on $\Gamma^d_R M$ turns $\Gamma^d_R M$ into a $\Gamma^d_R A$-module.

Let n be a positive integer. Consider the *Schur algebra* $\Gamma^d \operatorname{Mat}(n, R)$ of R associated with n and d. It is free as an R-module. For every R-module V, the module $V^n = \operatorname{Hom}(R^n, V)$ is a right $\operatorname{Mat}(n, R)$-module, hence $\Gamma^d(V^n)$ is a right $\Gamma^d \operatorname{Mat}(n, R)$-module.

Lemma 8.7. *If V is a projective R-module, then $\Gamma^d(V^n)$ is a projective right $\Gamma^d \operatorname{Mat}(n, R)$-module.*

Proof. If V is a direct summand of W, then $\Gamma^d(V^n)$ is a direct summand of $\Gamma^d(W^n)$, so without loss of generality we may assume that V is a free R-module. Then $\Gamma^d(V^n)$ is a direct sum of $\Gamma^d \operatorname{Mat}(n, R)$-modules of the form

$$\Gamma^{d_1}(R^n) \otimes_R \cdots \otimes_R \Gamma^{d_i}(R^n)$$

with $\sum d_i = d$. By [Akin and Buchsbaum 1988, Proposition 2.1] these are projective over the Schur algebra $\Gamma^d \operatorname{Mat}(n, R)$, and the lemma follows. □

We denote by $\mathcal{M}(R, n, d)$ the category of finitely generated left modules over the Schur algebra $\Gamma^d \operatorname{Mat}(n, R)$, and by $\mathcal{M}_{\mathrm{p}}(R, n, d)$ the full subcategory consisting of those modules whose underlying R-module is projective.

We have a "truncation" functor,

$$\operatorname{Pol}_d(R) \to \mathcal{M}_{\mathrm{p}}(R, n, d), \quad F \mapsto F(R^n),$$

where the structure of left $\Gamma^d \operatorname{Mat}(n, R)$-module on $F(R^n)$ is defined as follows. We have a map

$$\underline{\operatorname{End}}(R^n) \xrightarrow{F} \underline{\operatorname{End}}(F(R^n))$$

which is homogeneous of degree d. By the universal property of divided powers (see [Roby 1963, Proposition IV.1; Ferrand 1998, Proposition 2.5.1]), this map is induced by an R-module homomorphism

$$\Gamma^d \operatorname{End}(R^n) \to \operatorname{End}(F(R^n)),$$

which is moreover multiplicative, hence giving $F(R^n)$ the structure of a $\Gamma^d \operatorname{End}(R^n)$-module.

Theorem 8.8. *If $n \geq d$, the functor $\mathrm{Pol}_d(R) \to \mathcal{M}_{\mathrm{p}}(R, n, d)$ is an equivalence of categories.*

Proof. See [Krause 2013], where the same result is shown for polynomial functors taking values in arbitrary R-modules, and arbitrary $\Gamma^d \mathrm{Mat}(n, R)$-modules. The same argument works in our context; we only need to check that the inverse functor maps $\mathcal{M}_{\mathrm{p}}(R, n, d)$ to $\mathrm{Pol}_d(R)$ (that is, that the inverse functor preserves "finite type and projective").

The inverse functor is defined as follows. Let M be a $\Gamma^d \mathrm{Mat}(n, R)$-module. Then we define a functor

$$F_M : \mathrm{Mod}(R) \to \mathrm{Mod}(R), \quad V \mapsto \Gamma^d(V^n) \otimes_{\Gamma^d \mathrm{Mat}(n,R)} M,$$

where the right $\Gamma^d \mathrm{Mat}(n, R)$-module structure on $\Gamma^d(V^n)$ is inherited from the structure of right $\mathrm{Mat}(n, R)$-module on $V^n = \mathrm{Hom}(R^n, V)$. Formation of F_M commutes with base change.

Now assume that both M and V are finitely generated and projective R-modules. Then the module $F_M(V)$ is also finitely generated. We claim that $F_M(V)$ is also projective. By Lemma 8.7 the module $\Gamma^d(V^n)$ is projective, hence a direct summand of a free $\Gamma^d \mathrm{Mat}(n, R)$-module $\bigoplus_I \Gamma^d \mathrm{Mat}(n, R)$, and hence $F_M(V)$ is a direct summand of a projective R-module $\bigoplus_I M$. □

Remark 8.9. Theorem 8.8 gives an alternative way for producing the base change of a polynomial functor. If $R \to S$ is a map of commutative rings, and if M is a $\Gamma^d \mathrm{Mat}(n, R)$-module, then the base change $M \otimes_R S$ is a $\Gamma^d \mathrm{Mat}(n, S)$-module, since formation of $\Gamma^d \mathrm{Mat}(n, -)$ commutes with base change.

8C. ***The Grothendieck group of polynomial functors over $\mathbb{Z}$.*** We fix n and d satisfying $n \geq d$. For brevity we write $\mathcal{M}(R) := \mathcal{M}(R, n, d)$ and $\mathcal{M}_{\mathrm{p}}(R) := \mathcal{M}_{\mathrm{p}}(R, n, d)$. Furthermore we write $\mathbb{Z}[s_1, s_2, \dots]_d$ for the weighted degree d part of the polynomial ring $\mathbb{Z}[s_1, s_2, \dots]$. It is equal to $\mathbb{Z}[s_1, \dots, s_n]_d$.

In this subsection, following [Serre 1968], we compute the Grothendieck group $K_0(\mathcal{M}_{\mathrm{p}}(\mathbb{Z}))$. Together with Theorem 8.8 this then implies Theorem 8.5.

If R is an integral domain, there is a natural homomorphism

$$K_0(\mathcal{M}_{\mathrm{p}}(R)) = K_0(\mathrm{Pol}_d(R)) \to \mathbb{Z}[s_1, s_2, \dots]_d \quad \text{(by Theorem 8.8)}$$

that sends a polynomial functor F to the weights of the action of $\mathbb{G}_m^n$ on $F(R^n)$.

Theorem 8.10. *For every field K the map $K_0(\mathcal{M}(K)) \to \mathbb{Z}[s_1, s_2, \dots]_d$ is an isomorphism.*

Proof. See [Green 1980, Sections 2.2 and 3.5, especially Remark 3.5(ii)]. Green assumes the field K to be infinite, but this assumption is only used in relating modules over $\Gamma^d \mathrm{Mat}(n, K)$ to representations of the monoid $\mathrm{Mat}(n, K)$, as opposed to

representations of the *monoid scheme* $\mathrm{Mat}_{n,K}$, which would also work over a finite field K. See also [Jantzen 2003, Section II.A]. □

We will deduce from the cases $K = \mathbb{Q}$ and $K = \mathbb{F}_p$ in this theorem that the map

$$K_0(\mathrm{Pol}_d(\mathbb{Z})) \to \mathbb{Z}[s_1, s_2, \dots]_d$$

is an isomorphism. The proof is essentially identical to Serre's proof [1968] that

$$K_0(\mathrm{GL}_{n,\mathbb{Z}}) \to K_0(\mathrm{GL}_{n,\mathbb{Q}})$$

is an isomorphism.

Lemma 8.11 (projective resolutions). *The canonical map*

$$K_0(\mathcal{M}_{\mathrm{p}}(\mathbb{Z})) \to K_0(\mathcal{M}(\mathbb{Z}))$$

is an isomorphism.

Proof. (Compare [Serre 1968, §§2.2–2.3].) Let M be a finitely generated module over $\Gamma^d \mathrm{Mat}(n, \mathbb{Z})$. If M can be generated by m elements, we obtain a presentation

$$0 \to P_0 \to P_1 \to M \to 0$$

with $P_1 = (\Gamma^d \mathrm{Mat}(n, \mathbb{Z}))^m$. Both P_1 and P_0 are torsion-free, hence projective as $\mathbb{Z}$-modules. The same argument as [Serre 1968, Proposition 4] shows that $[P_0] - [P_1] \in K_0(\mathcal{M}_{\mathrm{p}}(\mathbb{Z}))$ is independent of the choice of presentation, and that $M \mapsto [P_0] - [P_1]$ defines a two-sided inverse to the map of the proposition. □

Lemma 8.12 (localisation sequence). *The obvious sequence*

$$\bigoplus_{\ell \text{ prime}} K_0(\mathcal{M}(\mathbb{F}_\ell)) \to K_0(\mathcal{M}(\mathbb{Z})) \to K_0(\mathcal{M}(\mathbb{Q})) \to 0$$

is exact.

Proof. The argument is identical to [Serre 1968, Théorème 1]. The main point is to verify that every $\Gamma^d \mathrm{Mat}(n, \mathbb{Q})$-module V of finite $\mathbb{Q}$-dimension contains a $\Gamma^d \mathrm{Mat}(n, \mathbb{Z})$-submodule Λ with $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda = V$. To construct such Λ, take an arbitrary sub-$\mathbb{Z}$-module Λ_0 with $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_0 = V$, and take $\Lambda := \Gamma^d \mathrm{Mat}(n, \mathbb{Z})\Lambda_0$. □

Lemma 8.13 (decomposition maps). *For every prime ℓ there is a unique homomorphism d_ℓ making the triangle*

$$\begin{array}{ccc} K_0(\mathcal{M}_p(\mathbb{Z})) & \longrightarrow & K_0(\mathcal{M}(\mathbb{Q})) \\ \downarrow & \swarrow_{d_\ell} & \\ K_0(\mathcal{M}(\mathbb{F}_\ell)) & & \end{array}$$

commute.

Proof. The argument is identical to [Serre 1968, Théorème 2]. □

Lemma 8.14. *The composition*

$$K_0(\mathcal{M}(\mathbb{Q})) \xrightarrow{d_\ell} K_0(\mathcal{M}(\mathbb{F}_\ell)) \to K_0(\mathcal{M}(\mathbb{Z}))$$

is the zero map.

Proof. See [Serre 1968, Lemme 4]. Similarly to there, one uses that for every $\mathbb{Z}$-torsion-free $\Gamma^d \operatorname{Mat}(n, \mathbb{Z})$-module Λ the map $\Lambda \to \ell\Lambda,\ x \mapsto \ell x$, is an isomorphism of $\Gamma^d \operatorname{Mat}(n, \mathbb{Z})$-modules. □

Proposition 8.15. *For every prime ℓ, the map d_ℓ is an isomorphism.*

Proof. By Theorem 8.10 the functors

$$\Lambda^{d_1} \otimes \Lambda^{d_2} \otimes \cdots \otimes \Lambda^{d_m}$$

with $\sum d_i = d$ define a basis of $K_0(\mathcal{M}(\mathbb{Q}))$ and of $K_0(\mathcal{M}(\mathbb{F}_\ell))$. Since the map d_ℓ preserves this basis, it is an isomorphism. □

Corollary 8.16. *The canonical map*

$$K_0(\mathcal{M}(\mathbb{Z})) \to K_0(\mathcal{M}(\mathbb{Q}))$$

is an isomorphism.

Proof. By Proposition 8.15 and Lemma 8.14 the maps

$$K_0(\mathcal{M}(\mathbb{F}_\ell)) \to K_0(\mathcal{M}(\mathbb{Z}))$$

are the zero maps. But then the localisation sequence of Lemma 8.12 shows that the map $K_0(\mathcal{M}(\mathbb{Z})) \to K_0(\mathcal{M}(\mathbb{Q}))$ is an isomorphism. □

Proof of Theorem 8.5. The degree d part of the homomorphism

$$\mathbb{Z}[s_1, s_2, \dots] \to K_0(\mathrm{Pol}_{<\infty}(\mathbb{Z})), \quad s_i \mapsto [\Lambda^i],$$

is obviously inverse to the composition of the isomorphisms

$$K_0(\mathrm{Pol}_d(\mathbb{Z})) \cong K_0(\mathcal{M}_{\mathrm{p}}(\mathbb{Z})) \cong K_0(\mathcal{M}(\mathbb{Z})) \cong K_0(\mathcal{M}(\mathbb{Q})) \cong \mathbb{Z}[s_1, s_2, \dots]_d$$

given by Theorem 8.8, Lemma 8.11, Corollary 8.16 and Theorem 8.10, respectively, and is hence bijective and compatible with exterior power operations. □

Remark 8.17. The category $\mathrm{Pol}_d(R)$ is equivalent with the category of weight d representations of the monoid $\mathrm{Mat}_{n,R}$, which forms a full subcategory of the category of representations of $\mathrm{GL}_{n,R}$. Rather than translating Serre's argument from the GL_n to the Mat_n context, one could also deduce our result from Serre's. However, some care has to be taken because the right adjoint to the inclusion, mapping a $\mathrm{GL}_{n,\mathbb{Z}}$-representation V to the largest subrepresentation that extends to $\mathrm{Mat}_{n,\mathbb{Z}}$, is not exact; see [Jantzen 2003, Section II.A].

8D. ***Proof of the final λ-ring axiom.*** In this subsection, we derive from Corollary 8.6 that, for every quasicompact scheme X, the pre-λ-ring $K_*(X)$ satisfies the final λ-ring axiom. Together with Theorem 7.1, this implies that $K_*(X)$ is a λ-ring and finishes the proof of this paper's main result.

Theorem 8.18. *The ring $K_*(X)$ equipped with the exterior power operations defined in Section 7 satisfies axiom (3) of a λ-ring.*

Proof. Let $r, s \geq 1$ and $n \geq 0$. For every $x \in K_n(X)$ we want to show that the identity

$$\lambda^r(\lambda^s(x)) = P_{r,s}(\lambda^1(x), \ldots, \lambda^{rs}(x)) \tag{8.19}$$

holds in $K_n(X)$. We recall that, if $n \geq 1$, all products occurring on the right-hand side of (8.19) are trivial (and hence that the right-hand side of (8.19) happens to be just a multiple of $\lambda^{rs}(x)$). We will show the stronger statement that the identity (8.19) in fact holds in $K_0((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X))$ for all $x \in K_0((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X))$. Now the products occurring on the right-hand side of (8.19) are induced by the simplicial tensor product introduced in Section 5; these products become trivial in $K_n(X)$ by Proposition 5.11. By a standard argument (see the proof of Theorem 7.1) we may assume that x is the class of an object $P_\bullet$ of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$. One easily checks, for instance using the Gabriel–Quillen embedding theorem [Thomason and Trobaugh 1990, Theorem A.7.1 and Proposition A.7.16], that for every exact category $\mathcal{P}$ and any skeletally small category $\mathcal{I}$, the category of functors from $\mathcal{I}$ to $\mathcal{P}$ is again an exact category in the obvious way. In particular, the category $\mathrm{End}((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X))$ of endo-functors of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$ is an exact category. Furthermore it carries a tensor product and exterior power operations (given by $F \mapsto \Lambda_n^d \circ F$). Via the homomorphism $K_0\big(\mathrm{End}((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X))\big) \to K_0((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X))$ given by $F \mapsto F(P_\bullet)$, the desired identity now follows from the even stronger identity

$$[\Lambda_n^r \circ \Lambda_n^s] = P_{r,s}([\Lambda_n^1], \ldots, [\Lambda_n^{rs}]) \tag{8.20}$$

in $K_0(\mathrm{End}((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)))$, which we now prove. We remember that the identity (8.20) (with the subscripts n omitted) holds in the Grothendieck group $K_0(\mathrm{Pol}_{<\infty}(\mathbb{Z}))$ by Corollary 8.6. Then it also holds in $K_0(\mathrm{Pol}^0_{<\infty}(\mathbb{Z}))$, where $\mathrm{Pol}^0_{<\infty}(\mathbb{Z})$ denotes the full subcategory of $\mathrm{Pol}_{<\infty}(\mathbb{Z})$ consisting of functors F satisfying $F(0) = 0$; this follows from the fact that the canonical inclusion $\mathrm{Pol}^0_{<\infty}(\mathbb{Z}) \hookrightarrow \mathrm{Pol}_{<\infty}(\mathbb{Z})$ is split by $F \mapsto \big(V \mapsto \ker(F(V) \to F(0))\big)$. The identity (8.20) therefore follows from Corollary 8.6 once we have shown that we have a pre-λ-ring homomorphism

$$K_0(\mathrm{Pol}^0_{<\infty}(\mathbb{Z})) \to K_0\big(\mathrm{End}((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X))\big) \tag{8.21}$$

that sends the class of the identity functor to the class of the identity functor. By base change (see Subsection 8A), every functor in $\mathrm{Pol}(\mathbb{Z})$ induces a functor in $\mathrm{Pol}(U)$ for every open subset U of X and this construction is compatible with

restriction with respect to any inclusion of open subsets of X. The constructions of Sections 3, 4 and 6 therefore inductively induce a functor

$$\mathrm{Pol}^0_{<\infty}(\mathbb{Z}) \to \mathrm{End}((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X));$$

this functor is exact and compatible with tensor products and exterior power operations, as one easily verifies by induction on n. Thus it induces the desired homomorphism (8.21) and the proof of Theorem 8.18 is complete. □

Remark 8.22. We have seen in the previous proof that the λ-ring axiom (3) already holds in $K_0((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X))$, i.e., before dividing out the subgroup generated by classes of diagonal multicomplexes. The same holds true for the λ-ring axiom (2). This can be shown similarly by using Corollary 8.6 or by using the characteristic-free Cauchy decomposition as constructed in [Akin et al. 1982]. Whereas Corollary 8.6 only proves the existence of short exact sequences, Akin, Buchsbaum and Weyman [Akin et al. 1982] *explicitly* construct short exact sequences that prove axiom (2) of a λ-ring. The problem of explicitly describing short exact sequences of polynomial functors that prove axiom (3) seems however to be even harder than the famous and related plethysm problem in representation theory. Such explicit short exact sequences for the plethysm $\Lambda^2 \circ \Lambda^2$ can be found in [Akin and Buchsbaum 1985, page 175]. Although there also exist solutions of the classical plethysm problem for $\Lambda^r \circ \Lambda^2$ and $\Lambda^2 \circ \Lambda^s$, we are not aware of any corresponding characteristic-free short exact sequences.

Appendix: Proof of Lemma 3.4

In this appendix we prove Lemma 3.4, which states that:

(1) If $\mathcal{P}$ is an idempotent complete exact category, then so are $C^n\mathcal{P}$ and $(C^{\mathrm{q}})^n\mathcal{P}$.

(2) If $\mathcal{P}$ is a split exact category, then so is $(C^{\mathrm{q}})^n\mathcal{P}$.

Note that to prove each of these statements it is enough to prove the case $n = 1$.

Proof of Lemma 3.4(1). Let $e : P_\bullet \to P_\bullet$ be an idempotent map of chain complexes. Then each map $e_n : P_n \to P_n$ is an idempotent of $\mathcal{P}$ and so has a kernel $\ker(e_n)$ which is an object of $\mathcal{P}$. By the universal property of kernels, the chain map on $P_\bullet$ induces a map $\ker(e_n) \to \ker(e_{n-1})$ for each n, and these assemble to form a chain complex of kernels. Thus every idempotent in $C\mathcal{P}$ has a kernel in $C\mathcal{P}$, so $C\mathcal{P}$ is idempotent complete. To show that $C^{\mathrm{q}}\mathcal{P}$ is idempotent complete as well, we must show that this kernel chain complex is acyclic in $\mathcal{P}$ if $P_\bullet$ is. To do this, it suffices to consider the case when the complex is a short exact sequence; the general case then follows because $\mathcal{P}$ supports long exact sequences.

If $P_\bullet = (0 \to P_2 \to P_1 \to P_0 \to 0)$ and if $e : P_\bullet \to P_\bullet$ is an idempotent chain map, then $P_\bullet$ is isomorphic to a sequence of the form

$$0 \to \ker(e_2) \oplus \operatorname{im}(e_2) \to \ker(e_1) \oplus \operatorname{im}(e_1) \to \ker(e_0) \oplus \operatorname{im}(e_0) \to 0.$$

Furthermore, as the morphisms in this short exact sequence commute with the idempotents $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, they split as direct sums of induced morphisms. Hence the sequence

$$0 \to \ker(e_2) \to \ker(e_1) \to \ker(e_0) \to 0$$

is exact as well. □

Proof of Lemma 3.4(2). We wish to show that every admissible monomorphism $i : P_\bullet \rightarrowtail Q_\bullet$ in $C^{\mathrm{q}}\mathcal{P}$ is split; that is, that there exists a chain map $s : Q_\bullet \to P_\bullet$ such that each $s_n i_n : P_n \to Q_n$ is the identity. Let us restrict to the case in which $P_\bullet$ and $Q_\bullet$ are short exact sequences of $\mathcal{P}$. Consider the diagram

$$\begin{array}{ccccc} P' & \overset{j_P}{\rightarrowtail} & P & \overset{q_P}{\twoheadrightarrow} & P'' \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i''} \\ Q' & \overset{j_Q}{\rightarrowtail} & Q & \overset{q_Q}{\twoheadrightarrow} & Q'' \end{array}$$

and fix a splitting s'' for i''. We claim that there exist splittings s' and s of i' and i such that the resulting $s_\bullet : Q_\bullet \to P_\bullet$ is a chain map (and hence a splitting of $i_\bullet$). The general case follows from this claim. Indeed, since acyclic complexes are spliced together from short exact sequences, we construct a splitting for a monomorphism of acyclic complexes $i : P_\bullet \rightarrowtail Q_\bullet$ by splitting each monomorphism of short exact sequences separately. The part of the claim concerning a fixed splitting s'' allows us to choose these splittings of short exact sequences in a compatible manner (beginning in degree 0). So it is enough to prove the claim.

We choose compatible splittings h_P and t_P of j_P and q_P, respectively, i.e., $j_P h_P + t_P q_P = 1$. We also choose a splitting s_0 for i and now set $s = j_P h_P s_0 + t_P s'' q_Q$. Then we compute

(1) $si = j_P h_P s_0 i + t_P s'' q_Q i = j_P h_P + t_P s'' i'' q_P = j_P h_P + t_P q_P = 1$,

(2) $q_P s = q_P j_P h_P s_0 + q_P t_P s'' q_Q = s'' q_Q$,

so s is a splitting for i, and s and s'' commute with q_P, q_Q. We therefore get an induced map of kernels $s' : Q' \to P'$ satisfying $j_P s' = s j_Q$. Moreover, $j_P s' i' = s j_Q i' = s i j_P = j_P$, and j_P is monic, so $s' i' = 1$. □

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