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Real cohomology and the powers of the fundamental ideal in the Witt ring

Jeremy A. Jacobson

Let A be a local ring in which 2 is invertible. It is known that the localization of the cohomology ring $H^*_{\acute{e}t}(A, \mathbb{Z}/2)$ with respect to the class $(-1) \in H^1_{\acute{e}t}(A, \mathbb{Z}/2)$ is isomorphic to the ring $\mathrm{C}(\operatorname{sper} A, \mathbb{Z}/2)$ of continuous $\mathbb{Z}/2$-valued functions on the real spectrum of A. Let $I^n(A)$ denote the powers of the fundamental ideal in the Witt ring of symmetric bilinear forms over A. The starting point of this article is the "integral" version: the localization of the graded ring $\bigoplus_{n\geq 0} I^n(A)$ with respect to the class $\langle\langle -1\rangle\rangle := \langle 1, 1\rangle \in I(A)$ is isomorphic to the ring $\mathrm{C}(\operatorname{sper} A, \mathbb{Z})$ of continuous $\mathbb{Z}$-valued functions on the real spectrum of A.

This has interesting applications to schemes. For instance, for any algebraic variety X over the field of real numbers $\mathbb{R}$ and any integer n strictly greater than the Krull dimension of X, we obtain a bijection between the Zariski cohomology groups $H^*_{\mathrm{Zar}}(X, \mathcal{I}^n)$ with coefficients in the sheaf $\mathcal{I}^n$ associated to the n-th power of the fundamental ideal in the Witt ring $W(X)$ and the singular cohomology groups $H^*_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z})$.

1. Introduction

Let X be an algebraic variety over the field of real numbers and let d denote the Krull dimension of X. Let $\mathcal{H}^n$ denote the Zariski sheaf associated to the presheaf $U \mapsto H^n_{\acute{e}t}(U, \mathbb{Z}/2)$, where $H^n_{\acute{e}t}(U, \mathbb{Z}/2)$ denotes the étale cohomology of U with $\mathbb{Z}/2\mathbb{Z}$-coefficients. Under the hypotheses that X is smooth, integral, and quasiprojective, a classic theorem of Jean-Louis Colliot-Thélène and Raman Parimala [1990, Theorem 2.3.1] states that the sections of $\mathcal{H}^n$ are in bijection with

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$H^0_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}/2)$ when $n \geq d+1$; it follows from this that there is a bijection of cohomology groups

$$H^*_{\mathrm{Zar}}(X, \mathcal{H}^n) \simeq H^*_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}/2) \tag{1.1}$$

when $n \geq d+1$, where $X(\mathbb{R})$ denotes the real points of X equipped with the Euclidean topology (defined in Remark 4.4) and $H^*_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}/2)$ denotes the singular cohomology groups of the real points with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

Let $W(X)$ denote the Witt ring of symmetric bilinear forms over X and $I^n(X)$ the powers of the fundamental ideal; see [Knebusch 1977]. Let $\mathcal{I}^n$ denote the Zariski sheaf associated to the presheaf $U \mapsto I^n(U)$. Let $\overline{\mathcal{I}^n}$ denote the sheaf associated to the presheaf $U \mapsto \mathcal{I}^n(U)/\mathcal{I}^{n+1}(U)$. The short exact sequence of sheaves

$$0 \to \mathcal{I}^{n+1} \to \mathcal{I}^n \to \overline{\mathcal{I}^n} \to 0$$

induces a long exact sequence in Zariski cohomology

$$\begin{aligned}\cdots \to H^m_{\mathrm{Zar}}(X, \mathcal{I}^{n+1}) \to H^m_{\mathrm{Zar}}(X, \mathcal{I}^n)& \\ \to H^m_{\mathrm{Zar}}(X, \overline{\mathcal{I}^n}) \xrightarrow{\partial} H^{m+1}_{\mathrm{Zar}}(X, \mathcal{I}^{n+1}) \to \cdots .&\end{aligned} \tag{1.2}$$

The introduction to [Fasel 2013] made the following assertions:

- the Zariski cohomology groups $H^*_{\mathrm{Zar}}(X, \mathcal{I}^n)$ are the analogue of the singular cohomology groups $H^*_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z})$, while $H^*_{\mathrm{Zar}}(X, \overline{\mathcal{I}^n})$ are the analogue of $H^*_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}/2)$;
- the map $H^*_{\mathrm{Zar}}(X, \mathcal{I}^{n+1}) \to H^*_{\mathrm{Zar}}(X, \mathcal{I}^n)$ corresponds to the homomorphism
$$H^*_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}) \xrightarrow{2} H^*_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z})$$
induced by the multiplication by 2 on the coefficients;
- the connecting homomorphism $H^*_{\mathrm{Zar}}(X, \overline{\mathcal{I}^n}) \xrightarrow{\partial} H^*_{\mathrm{Zar}}(X, \mathcal{I}^{n+1})$ is analogous to the Bockstein homomorphism
$$H^*_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\beta} H^{*+1}_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}).$$

Under the additional hypothesis that X is affine, smooth, and has trivial canonical sheaf, Fasel [2011, Proposition 5.1] proved that $H^d_{\mathrm{Zar}}(X, \mathcal{I}^n) \simeq H^d_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z})$ for all $n \geq d$.

We prove these assertions as a consequence of our more general results on real cohomology and the powers of the fundamental ideal. Precisely, we show in Corollary 8.11 that when $n \geq d+1$, the global signature induces an isomorphism $H^m_{\mathrm{Zar}}(X, \mathcal{I}^n) \simeq^{\mathrm{sign}} H^m_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z})$ for all $m \geq 0$, which in turn induces an isomorphism of long exact sequences from (1.2) to

$$\cdots \to H^m_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}) \xrightarrow{2} H^m_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}) \\ \to H^m_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}/2) \xrightarrow{\beta} H^{m+1}_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z}) \to \cdots .$$

Real cohomology is a cohomology theory for schemes that globalizes singular cohomology to any scheme X in the sense that when X is a real variety, the real cohomology groups $H^m(X_r, \mathbb{Z})$ may be identified with the singular cohomology groups $H^m_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z})$. For details, see Remark 4.4. The foundations and fundamental results on real cohomology are due to Claus Scheiderer [1994]. There is a close relationship between real and étale cohomology: the étale cohomology of X with 2-primary coefficients stabilizes in high degrees against the real cohomology of X with 2-primary coefficients [Scheiderer 1994, Corollary 7.19, Proposition 19.8]. Scheiderer also obtained a generalization to schemes of the bijection (1.1). To introduce it, first recall that for any scheme X, multiplication by cup product with $(-1) \in H^1(X_{\text{ét}}, \mathbb{Z}/2)$ induces a morphism of sheaves $\mathcal{H}^n \to \mathcal{H}^{n+1}$. Consequently, one may consider the colimit $\varinjlim \mathcal{H}^n$ over the system

$$\mathcal{H}^0 \xrightarrow{(-1)} \mathcal{H}^1 \xrightarrow{(-1)} \mathcal{H}^2 \xrightarrow{(-1)} \cdots .$$

The signature modulo 2 induces an isomorphism of sheaves $\varinjlim \mathcal{H}^n \to \operatorname{supp}_* \mathbb{Z}/2$ which induces an isomorphism of cohomology groups

$$H^m_{\mathrm{Zar}}(X, \varinjlim \mathcal{H}^n) \simeq H^m(X_r, \mathbb{Z}/2) \tag{1.3}$$

for all $m \geq 0$, where $H^m(X_r, \mathbb{Z}/2)$ denotes the real cohomology of X with coefficients in the constant sheaf $\mathbb{Z}/2$ [Scheiderer 1994, Corollary 19.5.1].

Note that one cannot obtain integral coefficient versions of the isomorphisms (1.1) and (1.3) by simply replacing everywhere $\mathbb{Z}/2$ with $\mathbb{Z}$, because when $n > d$ the étale cohomology groups $H^n_{\text{ét}}(U, \mathbb{Z})$ are always torsion for any open subscheme U of X [Scheiderer 1994, Corollary 7.23.3].

Here, we obtain integral versions by demonstrating in Theorem 8.6 that for any scheme X with 2 invertible in its global sections, the signature induces an isomorphism of sheaves $\varinjlim \mathcal{I}^n \to \operatorname{supp}_* \mathbb{Z}$ which induces an isomorphism of cohomology groups

$$H^m_{\mathrm{Zar}}(X, \varinjlim \mathcal{I}^n) \overset{\text{sign}}{\simeq} H^m(X_r, \mathbb{Z})$$

for all $m \geq 0$, where $\varinjlim \mathcal{I}^n$ denotes the Zariski sheaf on X obtained by taking the colimit of the system of sheaves

$$\mathcal{W} \xrightarrow{\langle\langle -1\rangle\rangle} \mathcal{I} \xrightarrow{\langle\langle -1\rangle\rangle} \mathcal{I}^2 \xrightarrow{\langle\langle -1\rangle\rangle} \cdots$$

and $\mathcal{I}^n \xrightarrow{\langle\langle -1\rangle\rangle} \mathcal{I}^{n+1}$ denotes the map induced by tensor product with the Pfister form $\langle\langle -1\rangle\rangle := \langle 1, 1\rangle$.

These global results follow from the local case, that is, the statement on the localization of the graded ring $I^*(A)$ from the abstract. Another way of stating this is to say that

$$\operatorname{sign} : \varinjlim I^n(A) \to \mathrm{C}(\operatorname{sper} A, \mathbb{Z}) \tag{1.4}$$

is bijective for any local ring A with 2 invertible. Injectivity of (1.4) is well-known and follows from the local ring version of Pfister's local-global principal (for instance [Knebusch 1977, Chapter II, §5], or directly in terms of the signature used in this article [Mahé 1982, Théorème 2.1 and Corollaire]). The statement that (1.4) is surjective is stronger than Mahé's theorem, which states that the cokernel of $\operatorname{sign} : W(A) \to \mathrm{C}(\operatorname{sper} A, \mathbb{Z})$ is 2-primary torsion for any commutative ring with 2 invertible. We believe that surjectivity of (1.4) when A is local is known as well, but we don't know of a reference in the literature. We give a proof of bijectivity of (1.4) in Proposition 7.2 in a much different way using cohomological methods. For instance, in Theorem 5.3 we prove the Gersten conjecture for the Witt groups with 2 inverted of any regular excellent local ring. From this we deduce injectivity of (1.4) for any local ring with 2 invertible using "Hoobler's trick". Similarly, in Proposition 6.3 we prove a purity result for $\varinjlim I^n(A)$ in "geometric" cases and deduce surjectivity in general from this.

2. Total signature

Throughout this section, let F be a field of characteristic different from 2, though the hypothesis on the characteristic is not necessary for the definitions.

Definition 2.1. An *ordering* on F is a subset $P \subset F$ satisfying

(1) $P + P \subset P$, $PP \subset P$;

(2) $P \cap (-P) = 0$;

(3) $P \cup -P = F$.

If $b - a \in P$, then we write $a \leq_P b$. If $a \in P$ and $a \neq 0$, then $a >_P 0$. It follows from the axioms that if F is nontrivial, then $1 >_P 0$. Also, for any $a \neq 0$ we write $\operatorname{sgn}_P(a) = 1$ if $a \in P$ and $\operatorname{sgn}_P(a) = -1$ if $a \in -P$. From the axioms one has that $\operatorname{sgn}_P(ab) = \operatorname{sgn}_P(a)\operatorname{sgn}_P(b)$ for any $a, b \in F^\times$; consequently, assigning any $a \in F^\times$ to $\operatorname{sgn}_P(a)$ determines a homomorphism $\operatorname{sgn}_P : F^\times \to \{\pm 1\}$ of groups. The pair (F, P) is called an *ordered field* [Knebusch and Scheiderer 1989, Kapitel I, Definition 1 and Bemerkungen].

Definition 2.2. The *real spectrum of* F, denoted $\operatorname{sper} F$, is the topological space formed by equipping the set of all orderings on F with the topology generated by the subbasis consisting of subsets $H(a) \subset \operatorname{sper} F$, $a \in F$, where $H(a)$ denotes the set of all orderings P satisfying $a >_P 0$.

Definition 2.3. Let P be an ordering on F. Any nondegenerate quadratic form ϕ over F splits as an orthogonal sum $\phi \simeq \phi_+ \perp \phi_-$, where the form ϕ_+ is positive definite with respect to the ordering (for all $0 \neq v$, $q(v) > 0$ with respect to P) and the form ϕ_- is negative definite with respect to the ordering (i.e., $-\phi_-$ is positive definite). The numbers $n_+ := \dim \phi_+$ and $n_- := \dim \phi_-$ do not change under an isometry of ϕ [Knebusch and Scheiderer 1989, Kapitel I, §2, Satz 2]. The integer $\operatorname{sign}_P([\phi]) := n_+ - n_-$ is called the *signature of* $[\phi]$ *with respect to* P. As the signature of the hyperbolic form is trivial, assigning to an isometry class $[\phi]$ its signature $\operatorname{sign}_P([\phi])$ defines a map

$$\operatorname{sign}_P : W(F) \to \mathbb{Z}$$

which is a homomorphism of rings [loc. cit.]. Let $\mathrm{C}(\operatorname{sper} F, \mathbb{Z})$ denote the set of continuous integer-valued functions on the real spectrum of F. The *total signature* is the ring homomorphism

$$\operatorname{sign} : W(F) \to \mathrm{C}(\operatorname{sper} F, \mathbb{Z})$$

which assigns to an isometry class $[\phi]$ the continuous function $P \mapsto \operatorname{sign}_P([\phi])$ [Knebusch and Scheiderer 1989, Kapitel III, §8, Satz 1]. If F has no ordering, then sign is trivial.

The following lemma is obtained directly from the definition of the signature and the fact that the signature is a ring homomorphism.

Lemma 2.4. *Let P be an ordering on F.*

(1) *If ϕ is a diagonalizable form, $\phi \simeq \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ for some $a_1, \ldots, a_n \in F^\times$, then*

$$\operatorname{sign}_P([\phi]) := \sum_{i=1}^{n} \operatorname{sgn}_P(a_i).$$

(2) *Let $a \in F^\times$. The Pfister form $\langle\langle a \rangle\rangle := \langle 1, -a \rangle$ has total signature*

$$\operatorname{sign}(\langle\langle a \rangle\rangle) = 2\mathbb{1}_{\{a<0\}}.$$

(3) *Let $a_1, a_2, \ldots, a_n \in F^\times$. The n-fold Pfister form*

$$\langle\langle a_1, \ldots, a_n \rangle\rangle := \langle\langle a_1 \rangle\rangle \otimes \cdots \otimes \langle\langle a_n \rangle\rangle$$

has total signature

$$\operatorname{sign}(\langle\langle a_1, \ldots, a_n \rangle\rangle) = 2^n \mathbb{1}_{\{a_1<0, \ldots, a_n<0\}}.$$

Definition 2.5. As hyperbolic forms have even rank, assigning a quadratic form to its rank modulo 2 determines a ring homomorphism $W(F) \to \mathbb{Z}/2\mathbb{Z}$. The kernel is denoted $I(F)$ and is called the fundamental ideal of F. The powers of the fundamental ideal $I^j(F)$ are additively generated by Pfister forms $\langle\langle a_1, \dots, a_j \rangle\rangle$, so it follows from Lemma 2.4 that the signature induces a group homomorphism

$$\mathrm{sign} : I^j(F) \to \mathrm{C}(\mathrm{sper}\, F, 2^j\mathbb{Z})$$

and the diagram

$$\begin{array}{ccc} I^j(F) & \xrightarrow{\mathrm{sign}} & \mathrm{C}(\mathrm{sper}\, F, 2^j\mathbb{Z}) \\ \Big\downarrow \langle\langle -1\rangle\rangle & & \Big\downarrow 2 \\ I^{j+1}(F) & \xrightarrow{\mathrm{sign}} & \mathrm{C}(\mathrm{sper}\, F, 2^{j+1}\mathbb{Z}) \end{array}$$

commutes. So after identifying

$$\varinjlim\bigl(\mathrm{C}(\mathrm{sper}\, F, \mathbb{Z}) \xrightarrow{2} \mathrm{C}(\mathrm{sper}\, F, 2\mathbb{Z}) \xrightarrow{2} \mathrm{C}(\mathrm{sper}\, F, 2^2\mathbb{Z}) \xrightarrow{2} \cdots\bigr) \simeq \mathrm{C}(\mathrm{sper}\, F, \mathbb{Z}),$$

one obtains the map

$$\varinjlim\bigl(W(F) \xrightarrow{\langle\langle -1\rangle\rangle} I(F) \xrightarrow{\langle\langle -1\rangle\rangle} I^2(F) \xrightarrow{\langle\langle -1\rangle\rangle} \cdots\bigr) \xrightarrow{\mathrm{sign}} \mathrm{C}(\mathrm{sper}\, F, \mathbb{Z}), \qquad (2.6)$$

where $\varinjlim$ denotes the colimit of the directed system of groups.

The following result first appeared in a paper of J. Arason and M. Knebusch. Injectivity follows from A. Pfister's local-global principal [Pfister 1966, Satz 22], and surjectivity follows immediately from the "normality theorem" of R. Elman and T.Y. Lam [1972, 3.2].

Proposition 2.7 [Arason and Knebusch 1978, Satz 2a]. *The morphism (2.6) is a bijection.*

3. Residues

Throughout this section A denotes a discrete valuation ring with fraction field K and residue field $k = A/\boldsymbol{m}$ of characteristic different from 2. Let π be a uniformizing parameter for A. The following lemma restates well-known facts on the second residue for Witt groups; see [Milnor and Husemoller 1973, Chapter IV (1.2)–(1.3)].

Lemma 3.1. (1) *Every rank one quadratic form over K is isometric to some $\langle c\rangle$, where $c = b\pi^n$, b is a unit in A, and either $n = 0$ or $n = 1$.*

(2) *The second residue $\partial_\pi : W(K) \to W(k)$ has the description*

$$\partial_\pi(\langle c\rangle) = \begin{cases} \langle \bar{b}\rangle & \text{if } n = 1, \\ 0 & \text{if } n = 0, \end{cases}$$

on rank one forms $\langle c\rangle$ as in (1).

(3) *The second residue respects the powers of the fundamental ideal, that is, for any integer* $n \geq 1$, *it induces a homomorphism of groups*

$$\partial_\pi : I^n(K) \to I^{n-1}(k),$$

where $I^0(k) := W(k)$.

Definition 3.2. Let P be an ordering on the fraction field K. One says that A is *convex in* K (with respect to P) when for all $x, y, z \in K$,

$$\{x \leq_P z \leq_P y \text{ and } x, y \in A\} \Rightarrow z \in A;$$

see [Knebusch and Scheiderer 1989, Kapitel II, §1, Definition 1 and §2, Satz 3; Bochnak et al. 1998, Definition 10.1.3(ii), Proposition 10.1.4]. If A is convex in K, then the subset $\overline{P} := \sigma(P \cap A) \subset k$, where $\sigma : A \to k$ is the surjection onto the residue field, defines an ordering on k called the *induced ordering* [Knebusch and Scheiderer 1989, Kapitel II, §2, Bemerkungen]. For any ordering $\xi \in \operatorname{sper} k$, let $Y_\xi \subset \operatorname{sper} K$ denote the subset consisting of orderings such that A is convex in K and $\xi = \overline{P}$ is the induced ordering. The assignment

$$P \mapsto \operatorname{sgn}_P(\pi)$$

defines a bijection from Y_ξ to the set $\{\pm 1\}$ [Knebusch and Scheiderer 1989, Kapitel II, §7, Theorem (Baer–Krull)], cf. [Bochnak et al. 1998, Theorem 10.1.10 and its proof]. That is to say, there are exactly two orderings in Y_ξ, say η_+ and η_-, where $\operatorname{sgn}_{\eta_+}(\pi) = 1$ and $\operatorname{sgn}_{\eta_-}(\pi) = -1$. The group homomorphism

$$\beta_\pi : \mathrm{C}(\operatorname{sper} K, \mathbb{Z}) \to \mathrm{C}(\operatorname{sper} A/\boldsymbol{m}, \mathbb{Z})$$

is defined by assigning $s \in \mathrm{C}(\operatorname{sper} K, \mathbb{Z})$ to the map $\xi \mapsto \beta_\pi(s)(\xi)$, where

$$\beta_\pi(s)(\xi) := s(\eta_+) - s(\eta_-).$$

If $\operatorname{sper} A/\boldsymbol{m} = \varnothing$, then it is defined to be zero.

Lemma 3.3. *Let* π *be a uniformizing parameter for* A. *The morphism* β_π *of Definition 3.2 has the following description on elements* $\operatorname{sign}(\langle c\rangle)$, *where* $c = b\pi^n$, b *is a unit in* A, *and either* $n = 0$ *or* $n = 1$:

$$\beta_\pi(\operatorname{sign}(\langle c\rangle)) = \begin{cases} 2\operatorname{sign}(\langle \bar{b}\rangle) & \text{if } n \text{ is } 1, \\ 0 & \text{if } n \text{ is } 0. \end{cases}$$

Proof. Let $c = b\pi^n$, where b is a unit in A, and either $n = 0$ or $n = 1$. For any $\xi \in \operatorname{sper} A/\boldsymbol{m}$,

$$
\begin{aligned}
&\beta_\pi(\mathrm{sign}(\langle c\rangle))(\xi)\\
&\qquad = \mathrm{sign}_{\eta_+}(\langle c\rangle) - \mathrm{sign}_{\eta_-}(\langle c\rangle)\\
&\qquad = \mathrm{sgn}_{\eta_+}(c) - \mathrm{sgn}_{\eta_-}(c)\\
&\qquad = \begin{cases} \mathrm{sgn}_\xi(\bar c) - \mathrm{sgn}_\xi(\bar c) & \text{if } n=0 \text{ (both orderings induce } \xi\text{)},\\ \mathrm{sgn}_{\eta_+}(b\pi) - \mathrm{sgn}_{\eta_-}(b\pi) & \text{if } n=1,\end{cases}\\
&\qquad = \begin{cases} 0 & \text{if } n=0,\\ \mathrm{sgn}_{\eta_+}(b)\,\mathrm{sgn}_{\eta_+}(\pi) \\ \quad - \mathrm{sgn}_{\eta_-}(b)\,\mathrm{sgn}_{\eta_-}(\pi) & \text{if } n=1,\end{cases}\\
&\qquad = \begin{cases} 0 & \text{if } n=0,\\ \mathrm{sgn}_{\eta_+}(b) + \mathrm{sgn}_{\eta_-}(b) & \text{if } n=1 \text{ (by definition of } \eta_+ \text{ and } \eta_-\text{)},\end{cases}\\
&\qquad = \begin{cases} 0 & \text{if } n=0,\\ \mathrm{sgn}_\xi(\bar b) + \mathrm{sgn}_\xi(\bar b) & \text{if } n=1 \text{ (both orderings induce } \xi\text{)},\end{cases}\\
&\qquad = \begin{cases} 0 & \text{if } n=0,\\ 2\,\mathrm{sgn}_\xi(\bar b) & \text{if } n=1.\end{cases}
\end{aligned}
$$

The above equalities prove the lemma. □

The next lemma follows from Lemmas 3.1 and 3.3.

Lemma 3.4. *The diagram of abelian groups below is commutative*:

$$
\begin{array}{ccc}
\varinjlim I^n(K) & \xrightarrow{\partial_\pi} & \varinjlim_{n\geq -1} I^n(k)\\
\downarrow{\scriptstyle \mathrm{sign}} & & \downarrow{\scriptstyle 2\,\mathrm{sign}}\\
\mathrm{C}(\mathrm{sper}\,K,\mathbb{Z}) & \xrightarrow{\beta_\pi} & \mathrm{C}(\mathrm{sper}\,k,\mathbb{Z})
\end{array}
$$

where $\varinjlim_{n\geq -1} I^n(k)$ *denotes the colimit over*

$$
W(k) \xrightarrow{\langle\langle -1\rangle\rangle} W(k) \xrightarrow{\langle\langle -1\rangle\rangle} I(k) \xrightarrow{\langle\langle -1\rangle\rangle} I^2(k) \xrightarrow{\langle\langle -1\rangle\rangle} \cdots .
$$

4. Real cohomology

C. Scheiderer [1994] developed a theory of *real cohomology* for schemes. It “globalizes” to schemes the singular cohomology of the real points of a real variety in the same way that étale cohomology globalizes the singular cohomology of the complex points of a complex variety. Following [Scheiderer 1994], we recall the definition and some needed properties.

Definition 4.1. The *real spectrum of a ring* A is a topological space denoted by $\mathrm{sper}\,A$. As a set it consists of all pairs $\xi = (\boldsymbol{p}, P)$ with $\boldsymbol{p} \in \mathrm{spec}\,A$ and P an ordering of the residue field $k(\boldsymbol{p})$. For any point $\xi \in \mathrm{sper}\,A$, let $k(\xi)$ denote the real closure of the ordered field $k(\boldsymbol{p})$ with respect to P. For $a \in A$, write

$a(\xi) > 0$ to indicate that the image of a in $k(\xi)$ is positive. The sets of the form $D(a) := \{\xi \in \operatorname{sper} A : a(\xi) > 0\}$ for $a \in A$ are a subbasis for the topology on $\operatorname{sper} A$. The *real spectrum of a scheme* X is the topological space X_r formed by glueing the real spectra of its open affine subschemes. This does not depend on the open cover of X that was chosen. Furthermore, any map of schemes $f : Y \to X$ induces a continuous map of real spectra $f_r : Y_r \to X_r$. The assignment $(\boldsymbol{p}, P) \mapsto \boldsymbol{p}$ defines a continuous map of topological spaces $\operatorname{sper} A \to \operatorname{spec} A$, and similarly one has a continuous map $\operatorname{supp} : X_r \to X$ called the *support map*.

Definition 4.2. Let X be a scheme. First we recall the definition of the *real site* of X, which we also denote by X_r. It is the category $\mathrm{O}(X_r)$ of open subsets of X_r equipped with the "usual" coverings, i.e., a family of open subspaces $\{U_\lambda \to U\}$ is a covering of $U \in \mathrm{O}(X_r)$ if $U = \bigcup U_\lambda$.[1] The category of sheaves of abelian groups on X_r is denoted $\mathrm{Ab}(X_r)$ and the category of abelian groups by Ab. For any $\mathcal{F} \in \mathrm{Ab}(X_r)$, the *real cohomology groups of X with coefficients in $\mathcal{F}$* are the right derived functors of the global sections functor $\Gamma : \mathrm{Ab}(X_r) \to \mathrm{Ab}$. They are denoted by

$$H^p(X_r, \mathcal{F}) := R^p \Gamma \mathcal{F},$$

where $R^p\Gamma$ is the p-th derived functor of Γ. When $X = \operatorname{spec} A$ is affine, we may write $H^p(\operatorname{sper} A, \mathcal{F})$ instead of $H^p(X_r, \mathcal{F})$. For any abelian group M, we also denote by M the sheaf on X_r associated to the presheaf $U \mapsto M$ for U any open set in X_r. Such a sheaf is called a *constant sheaf*. Moreover, when the group M is equipped with the discrete topology we may write $\mathrm{C}(\operatorname{sper} A, M)$ instead of $H^0(\operatorname{sper} A, M)$. If $i : S \to X_r$ is a closed subspace, then for any abelian sheaf F on X_r, define

$$H^0_S(X_r, F) := \ker(F(X_r) \to F(X_r \setminus S)).$$

The functor $F \mapsto H^0_S(X_r, F)$ is left exact and its right derived functors

$$H^q_S(X_r, F) := R^q H^0_S(X_r, F)$$

are called the *relative cohomology of F with support in S* [Scheiderer 1995, Notations] see [SGA 4_3 1973, Exposé V, 6.3] or [SGA 2 2005, Exposé I, §2, Defini-

[1]The *real étale site*, denoted $X_{\text{rét}}$, is obtained by equipping the category of étale X-schemes with coverings given by the *real surjective families*, that is, $\{f_\lambda : U_\lambda \to U\}$ is a covering if the real spectrum U_r equals the union of the images $(f_\lambda)_r((U_\lambda)_r)$. For any sheaf F on X_r,

$$\{X' \xrightarrow{f} X\} \mapsto H^0(X'_r, f_r^* F)$$

defines a sheaf on $X_{\text{rét}}$ denoted $F^\flat$. This determines a functor from the category $\widetilde{X_r}$ of sheaves on X_r to the category $\widetilde{X_{\text{rét}}}$ of sheaves on $X_{\text{rét}}$, which is an equivalence of categories compatible with morphisms $Y \to X$ of schemes [Scheiderer 1994, Theorem 1.3, Theorem 1.14, and Remark 1.16]. We follow [Scheiderer 1995, Notation] in defining real cohomology and cohomology with supports as sheaf cohomology on the topological space X_r.

tion 2.1]. Additionally, $i^! F$ is defined to be the sheaf

$$S \cap U \mapsto \ker(F(U) \to F(U \setminus (S \cap U)))$$

on S (U open in X_r) and one has that

$$H^0_S(X_r, F) = H^0(X_r, i_* i^! F)$$

using the exact sequence

$$0 \to i_* i^! F \to F \to j^* j_* F \to i_* R^1 i^! F \to 0; \tag{4.3}$$

see [SGA 4_3 1973, Exposé V, Proposition 6.5] or [SGA 2 2005, Exposé I, Corollaire 2.11], noting that $R^1 i_* i^! F \simeq i_* R^1 i^! F$ since i_* is exact [Scheiderer 1994, Corollary 3.11.1].

Remark 4.4. Let X be an algebraic variety over $\mathbb{R}$, by which we mean an $\mathbb{R}$-scheme that is separated and of finite type. We explain in this remark how to equip $X(\mathbb{R})$ with a topology and identify its singular cohomology with the real cohomology of X_r. For any affine scheme $U = \operatorname{spec} \mathbb{R}[T_1, T_2, \ldots, T_n]/I$, we consider the $\mathbb{R}$-points $U(\mathbb{R})$ as a topological space by equipping $U(\mathbb{R}) \subset \mathbb{R}^n$ with the subspace topology, where $\mathbb{R}^n$ has the Euclidean topology. The *Euclidean topology* on the set of $\mathbb{R}$-points $X(\mathbb{R})$ is the topological space formed by glueing the $U(\mathbb{R})$ of the open affine subschemes U taken from an open cover of X. This does not depend on the open cover of X that was chosen. The inclusion map $i : X(\mathbb{R}) \to X_r$, sending an $\mathbb{R}$-point x to the pair $(x, \mathbb{R}_{\geq 0})$, is continuous and i^{-1} induces a bijection from connected components of X_r to connected components of $X(\mathbb{R})$ and from connected components of any basic open $D(a_1, a_2, \ldots, a_n)$ in X_r to connected components of $i^{-1}(D(a_1, a_2, \ldots, a_n))$ [Coste and Roy 1982, Corollaire 3.7]. Hence, the functor i_* determines an equivalence from the category of constant sheaves of abelian groups on $X(\mathbb{R})$ to the category of constant sheaves of abelian groups on X_r. Consequently, for any abelian group M, the sheaf cohomology $H^*(X(\mathbb{R}), M)$ coincides with the real cohomology groups $H^*(X_r, i_* M)$ and $H^*(X_r, M)$. Also, singular cohomology $H^*_{\text{sing}}(X(\mathbb{R}), M)$ is canonically isomorphic to sheaf cohomology $H^*(X(\mathbb{R}), M)$; see [Scheiderer 1994, Remark 13.6]. In particular, the real cohomology groups $H^*(X_r, \mathbb{Z})$ are finitely generated groups, isomorphic to $H^*_{\text{sing}}(X(\mathbb{R}), \mathbb{Z})$.

Definition 4.5. Let $\mathrm{Ab}(X_{\mathrm{Zar}})$ denote the category of sheaves of abelian groups on the Zariski site X_{Zar}. Since the support map is a continuous map of topological spaces, it induces the direct image functor

$$\mathrm{supp}_* : \mathrm{Ab}(X_r) \to \mathrm{Ab}(X_{\mathrm{Zar}}),$$

and this functor is faithful and exact [Scheiderer 1994, Theorem 19.2].

Lemma 4.6. *Let X be a scheme. For any sheaf $\mathcal{F} \in \mathrm{Ab}(X_r)$,*

$$H^p(X_r, \mathcal{F}) \simeq H^p_{\mathrm{Zar}}(X, \mathrm{supp}_* \mathcal{F}).$$

Proof. Using the Grothendieck spectral sequence for the composition of the functors supp_* and the global sections functor Γ, we obtain a spectral sequence with $\mathrm{E}_2^{p,q} = H^p_{\mathrm{Zar}}(X, R^q \,\mathrm{supp}_* \mathcal{F})$ that abuts to $H^{p+q}(X_r, \mathcal{F})$. For $q > 0$, the sheaves $R^q \,\mathrm{supp}_* \mathcal{F}$ vanish [Scheiderer 1994, Theorem 19.2]. Therefore the edge maps in this spectral sequence determine isomorphisms $H^p(X_r, \mathcal{F}) \xrightarrow{\simeq} H^p_{\mathrm{Zar}}(X, \mathrm{supp}_* \mathcal{F})$ for $p \geq 0$. □

Next we recall the work of C. Scheiderer [1995], in which he constructs a "Bloch–Ogus" style complex that computes real cohomology. The codimension of support filtration on X determines a spectral sequence abutting to real cohomology. Scheiderer shows that for regular excellent schemes the E_1-page is zero except for the complex $\mathrm{E}_1^{*,0}$, and hence obtains the result below. Recall that a locally noetherian scheme is called excellent if X can be covered by open affine subschemes $\operatorname{spec} A_\alpha$, where the A_α are excellent rings [EGA IV$_2$ 1965, 7.8.5]. For a point $x \in X$ of a scheme, we denote $\operatorname{sper} k(x)$ by x_r.

Proposition 4.7 [Scheiderer 1995, Theorem 2.1]. *Let X be a noetherian regular excellent scheme. Let W be an open constructible subset of X_r, and let $\mathcal{F}$ be a locally constant sheaf on W. Then there is a complex of abelian groups*

$$\bigoplus_{x \in X^{(0)}} H^0_x(W, \mathcal{F}) \to \bigoplus_{x \in X^{(1)}} H^1_x(W, \mathcal{F}) \to \bigoplus_{x \in X^{(2)}} H^2_x(W, \mathcal{F}) \to \cdots \tag{4.8}$$

natural in W and $\mathcal{F}$, whose q-th cohomology group is canonically isomorphic to $H^q(W, \mathcal{F})$, $q \geq 0$. Here $H^q_x(W, \mathcal{F}) := H^q_{x_r \cap W}(\operatorname{sper} \mathcal{O}_{X,x} \cap W, \mathcal{F})$ are the relative cohomology groups of $\operatorname{sper} \mathcal{O}_{X,x}$ *with support in $x_r \cap W$* (Definition 4.2) *and $X^{(i)}$ denotes, for $i \geq 0$, the set of codimension i points* ($\dim \mathcal{O}_{X,x} = i$) *of X. This complex is contravariantly functorial for flat morphisms of schemes.*

The following lemma is based on the proof of [Scheiderer 1995, Proposition 2.6], where $M = \mathbb{Z}/2\mathbb{Z}$.

Lemma 4.9. *Let X be a noetherian regular excellent scheme which is integral with function field K. Let $x \in X^{(1)}$ and let π denote a choice of uniformizing parameter for $\mathcal{O}_{X,x}$. Fix an integer $n \geq 0$ and let M denote the constant sheaf $\mathbb{Z}$. Denote by ∂ the map*

$$H^0(\operatorname{sper} K, M) \to H^1_{x_r}(\operatorname{sper} \mathcal{O}_{X,x}, M)$$

induced by first differential of the complex (4.8) from Proposition 4.7. Then there is an isomorphism $\iota_\pi : H^1_{x_r}(\operatorname{sper} \mathcal{O}_{X,x}, M) \to H^0(x_r, M)$ for which $\iota_\pi \circ \partial = \beta_\pi$, where β_π is the map of Definition 3.2.

Proof. Let $X' = \operatorname{sper} \mathcal{O}_{X,x}$ and $Z' = x_r$. Let $i : Z' \to X'$ denote the inclusion, and let $j : \operatorname{sper} K \to X'$ denote the inclusion of the complement to Z'. For any abelian sheaf M on X' the sequence

$$M \to j_* j^* M \to i_* R^1 i^!(M) \to 0$$

is exact (Definition 4.2, (4.3)). By [Scheiderer 1995, Lemma 1.3], for any locally constant sheaf M on X' the sequence

$$M \to j_* j^* M \xrightarrow{\beta} i_* i^* M \to 0$$

is exact, where β is defined on stalks as $(\beta s)_\zeta = s(\eta_+) - s(\eta_-) \in M$. Hence we get an isomorphism ι_π of cokernels and a commutative diagram

$$\begin{tikzcd} j_* j^* M(X') \arrow[r, "\partial"] \arrow[dr, "\beta"'] & i_* R^1 i^! M(X') \arrow[d, "\iota_\pi"] \\ & i_* i^* M(X') \end{tikzcd} \tag{4.10}$$

Tracking down all the definitions, one finds that (4.10) is equal to the diagram

$$\begin{tikzcd} H^0(X' - Z', M) \arrow[r, "\partial"] \arrow[dr, "\beta_\pi"'] & H^1_{Z'}(X', M) \arrow[d, "\iota_\pi"] \\ & H^0(Z', M) \end{tikzcd}$$

where the vertical map is the isomorphism ι_π, the diagonal map is the map β_π of Definition 3.2, and $\operatorname{sper} K$ equals $X' - Z'$. This finishes the proof of the lemma. $\square$

Lemma 4.11. *Let A be a regular excellent local ring with fraction field K. Let $X = \operatorname{spec} A$, and for any $x \in X^{(1)}$, let π_x be a choice of uniformizing parameter for $\mathcal{O}_{X,x}$. Then the sequence*

$$0 \longrightarrow C(\operatorname{sper} A, \mathbb{Z}) \longrightarrow C(\operatorname{sper} K, \mathbb{Z}) \xrightarrow{\oplus \beta_\pi} \bigoplus_{x \in X^{(1)}} C(\operatorname{sper} k(x), \mathbb{Z})$$

is exact, where β_π is the map of Definition 3.2.

Proof. To prove the lemma, choose isomorphisms ι_π for each $x \in X^{(1)}$ as in Lemma 4.9, and then use Proposition 4.7. $\square$

5. On the Gersten conjecture with 2 inverted

Definition 5.1. Let A be a regular local ring with 2 invertible and let $X = \operatorname{spec} A$. Let d denote the Krull dimension of A and K the fraction field of A. We work with the Gersten complex for the Witt groups of X as found for instance in [Balmer et al. 2002, Definition 3.1], which we denote by $C^\bullet(A, W)$. Recall that for any integer $p \geq 0$, after choosing local parameters for $\mathcal{O}_{X,x}$ for each $x \in X^{(p)}$ one may write down isomorphisms $\iota_p : C^p(A, W) \xrightarrow{\simeq} \bigoplus_{x \in X^{(p)}} W(k(x))$. Then $C^\bullet(A, W)$ is isomorphic to the complex

$$C^\bullet(A, W, \iota) := W(K) \xrightarrow{\partial_\iota} \bigoplus_{x \in X^{(1)}} W(k(x)) \xrightarrow{\partial_\iota} \cdots \xrightarrow{\partial_\iota} \bigoplus_{x \in X^{(d)}} W(k(x)),$$

where the differentials are $\partial_\iota := \iota_{p+1} \circ \partial \circ \iota_p^{-1}$ and ∂ is the differential leaving $C^p(A, W)$. The differentials ∂_ι may differ for different choices of isomorphisms ι_p but the resulting complexes will all be isomorphic. For all $x \in X^{(1)}$ we may choose parameters $\pi \in \mathcal{O}_{X,x}$ so that $\partial_\iota : W(K) \to W(k(x))$ equals the second residue ∂_π of Lemma 3.1; see [Balmer and Walter 2002, Lemma 8.4], cf. [Gille 2007, Proposition 6.5]. It was proved by J. Arason that the second residue ∂_π respects the filtration by powers of the fundamental ideal, that is, $\partial_\pi(I^n(K)) \subset I^{n-1}(k(x))$ [Arason 1975] and similarly one may show that all the differentials ∂_ι respect this filtration; for instance, this was shown by S. Gille [2007, Corollary 7.3] for coherent Witt groups, which gives the same complex since A is regular [Balmer et al. 2002, Section 3]. So one may obtain a subcomplex

$$C^\bullet(A, I^n, \iota) := \bigoplus_{x \in X^{(0)}} I^n(k(x)) \xrightarrow{\partial_\iota} \bigoplus_{x \in X^{(1)}} I^{n-1}(k(x)) \xrightarrow{\partial_\iota} \cdots \xrightarrow{\partial_\iota} \bigoplus_{x \in X^{(d)}} I^{n-d}(k(x)),$$

where we set $I^m(k(x)) = W(k(x))$ when $m \leq 0$. Define

$$C^\bullet(A, W/I^n) := C^\bullet(A, W)/C^\bullet(A, I^n, \iota)$$

to be the quotient complex. The exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet(A, I^n, \iota) \arrow[r] \arrow[d, "2"] & C^\bullet(A, W) \arrow[r] \arrow[d, "2"] & C^\bullet(A, W/I^n) \arrow[r] \arrow[d, "2"] & 0 \\
0 \arrow[r] & C^\bullet(A, I^{n+1}, \iota) \arrow[r] & C^\bullet(A, W) \arrow[r] & C^\bullet(A, W/I^{n+1}) \arrow[r] & 0
\end{tikzcd}$$

determines an exact sequence of colimits

$$0 \to C^\bullet(A, \varinjlim I^n) \to C^\bullet(A, \varinjlim W) \to C^\bullet(A, \varinjlim W/I^n) \to 0, \tag{5.2}$$

where we define

$$\begin{aligned}
C^\bullet(A, \varinjlim I^n) &:= \varinjlim C^\bullet(A, I^n, \iota), \\
C^\bullet(A, \varinjlim W/I^n) &:= \varinjlim C^\bullet(A, W/I^n), \\
C^\bullet\big(A, W\big[\tfrac{1}{2}\big]\big) &:= \varinjlim\big(C^\bullet(A, W) \xrightarrow{2} C^\bullet(A, W) \xrightarrow{2} C^\bullet(A, W) \xrightarrow{2} \cdots\big).
\end{aligned}$$

Theorem 5.3. *If A is a regular excellent local ring with 2 invertible, then the Gersten complex $C^\bullet(A, W[\frac{1}{2}])$ is exact and $H^0(C^\bullet(A, W[\frac{1}{2}])) = W(A)[\frac{1}{2}]$.*

Proof. We proceed by induction on the Krull dimension of A. The Gersten complex without inverting 2 is exact already in low dimensions for any regular local ring [Balmer et al. 2002, Lemma 3.2]. Fix A and assume that the statement of the proposition is known for regular excellent local rings of Krull dimension less than that of A. It is sufficient to show that the cohomology of $C^\bullet(A, W[\frac{1}{2}])$ vanishes in degrees 2 and higher: one may use the Balmer–Walter spectral sequence with 2 inverted for Witt groups to show that this implies $H^*(C^\bullet(A, W[\frac{1}{2}])) = 0$ in positive degree and $H^0(C^\bullet(A, W[\frac{1}{2}])) = W(A)[\frac{1}{2}]$; see, e.g., [Balmer et al. 2002, Lemma 3.2]. For any regular parameter $f \in A$, there is a short exact sequence of complexes

$$0 \to C^\bullet(A, W) \to C^\bullet(A_f, W) \to C^\bullet(A/f, W)[-1] \to 0;$$

see, for instance, [Balmer et al. 2002, Lemma 3.3 and proof of Theorem 4.4]. Taking colimits it remains exact. As $\dim A/f$ is strictly less than $\dim A$ and A/f is again regular and excellent we have that $C^\bullet(A/f, W[\frac{1}{2}])[-1]$ is exact.

Hence it remains to see that $C^\bullet(A_f, W)$ is exact in degrees 2 and higher. Note that for any $\boldsymbol{p} \in \operatorname{spec} A_f$, $\dim(A_f)_{\boldsymbol{p}}$ is strictly less than $\dim A$ and $(A_f)_{\boldsymbol{p}}$ is again regular and excellent, hence the cohomology of $C^\bullet(A_f, W[\frac{1}{2}])$ agrees with $H^*_{\mathrm{Zar}}(\operatorname{spec} A_f, \varinjlim \mathcal{W})$, where $\varinjlim \mathcal{W}$ denotes the colimit over the sheaves

$$\mathcal{W} \xrightarrow{\langle\langle -1\rangle\rangle} \mathcal{W} \xrightarrow{\langle\langle -1\rangle\rangle} \mathcal{W} \xrightarrow{\langle\langle -1\rangle\rangle} \cdots .$$

For any point $\boldsymbol{p}$ in $\operatorname{spec} A_p$, using the induction hypothesis we have that the top row in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim W((A_f)_{\boldsymbol{p}}) \arrow[r] \arrow[d] & \varinjlim W(K) \arrow[r, "\oplus\partial_\pi"] \arrow[d, "\mathrm{sign}"] & \bigoplus_{x \in Y^{(1)}} \varinjlim W(k(x)) \arrow[d, "2\,\mathrm{sign}"] \\
0 \arrow[r] & \mathrm{C}(\operatorname{sper}(A_f)_{\boldsymbol{p}}, \mathbb{Z}[\frac{1}{2}]) \arrow[r] & \mathrm{C}(\operatorname{sper} K, \mathbb{Z}[\frac{1}{2}]) \arrow[r, "\oplus\beta_\pi"] & \bigoplus_{x \in Y^{(1)}} \mathrm{C}(\operatorname{sper} k(x), \mathbb{Z}[\frac{1}{2}])
\end{tikzcd}$$

is exact, and using Lemma 4.11 we have that the bottom row is exact, where $Y := \operatorname{spec}(A_f)_{\boldsymbol{p}}$. Proposition 2.7 implies the middle vertical map is a bijection and the rightmost vertical map is an injection, from which it follows that the leftmost vertical map is bijective. Thus we get an isomorphism $\varinjlim \mathcal{W} \xrightarrow{\simeq} \operatorname{supp}_* \mathbb{Z}[\frac{1}{2}]$ of sheaves on A_f as it is an isomorphism on stalks, where we use Lemma 4.6 to identify the sheaf $\operatorname{supp}_* \mathbb{Z}[\frac{1}{2}]$ as the sheaf $U \mapsto \mathrm{C}(U_r, \mathbb{Z}[\frac{1}{2}])$. Then the real cohomology groups $H^*(\operatorname{sper} A_f, \mathbb{Z}[\frac{1}{2}])$ are isomorphic to $H^*_{\mathrm{Zar}}(\operatorname{spec} A_f, \varinjlim \mathcal{W})$, so it remains to prove their vanishing in degree 2 and higher. This is true since the real cohomology of local rings vanish in positive degree (in fact, semilocal too)

[Scheiderer 1994, Proposition 19.2.1] and the real cohomology of $\operatorname{sper} A_f$ sits in a long exact sequence with that of $\operatorname{sper} A/f$ and $\operatorname{sper} A$ whenever A is regular excellent [Scheiderer 1995, Corollary (1.10)]. This finishes the proof. □

Since the diagram

$$\begin{array}{ccc} \varinjlim I^n(A) & \longrightarrow & W(A)\left[\frac{1}{2}\right] \\ \downarrow & & \downarrow \\ \varinjlim I^n(K) & \longrightarrow & W(K)\left[\frac{1}{2}\right] \end{array}$$

is commutative and the horizontal maps in the diagram are injective, we have the following corollary to Theorem 5.3.

Corollary 5.4. *Let A be a regular excellent local ring with 2 invertible. The map*

$$\varinjlim I^n(A) \to \varinjlim I^n(K)$$

is injective.

We will also need the following result later.

Lemma 5.5. *Let A be a regular excellent local ring with 2 invertible. The cohomology groups $H^m(C^\bullet(A, \varinjlim I^n))$ vanish when $m \geq 2$.*

Proof. Consider the long exact sequence in cohomology

$$\cdots \to H^m(C^\bullet(A, \varinjlim I^n)) \to H^m(C^\bullet(A, \varinjlim W)) \to H^m(C^\bullet(A, \varinjlim W/I^n)) \to \cdots$$

associated to the short exact sequence of complexes (5.2). The cohomology groups $H^m(C^\bullet(A, \varinjlim W))$ vanish when $m > 0$ by Theorem 5.3. Then $H^m(C^\bullet(A, \varinjlim I^n))$ is isomorphic to $H^{m-1}(C^\bullet(A, \varinjlim W/I^n))$ for all $m \geq 2$. The cohomology groups $H^m(C^\bullet(A, \varinjlim W/I^n))$ are 2-primary torsion since the complex $C^\bullet(A, \varinjlim W/I^n)$ is, while the groups $H^m(C^\bullet(A, \varinjlim I^n))$ have no 2-primary torsion since multiplication by 2

$$C^\bullet(A, \varinjlim I^n) \xrightarrow{2} C^\bullet(A, \varinjlim I^n)$$

is an isomorphism of complexes. Thus both groups vanish proving, the lemma. □

6. Purity of the limit in the local "geometric" case

For any prime p, we use $\mathbb{Z}_{\langle p\rangle}$ to denote the localization of $\mathbb{Z}$ at the prime ideal $\langle p\rangle \in \operatorname{spec} \mathbb{Z}$. In this section we prove purity of $\varinjlim I^n(A)$ in the case that A is essentially smooth over either $\mathbb{Q}$ or $\mathbb{Z}_{\langle p\rangle}$ (Proposition 6.3). When A is a local ring of mixed characteristic $(0, p)$ with $p \neq 2$ (that is to say, the characteristic of the fraction field K is 0 and the characteristic of the residue field is p) we say that A is essentially smooth over $\mathbb{Z}_{\langle p\rangle}$ if $A = R_{\boldsymbol{p}}$ is the localization at a prime $\boldsymbol{p} \in \operatorname{spec} R$ of a smooth and finite type $\mathbb{Z}_{\langle p\rangle}$-algebra $R = \mathbb{Z}_{\langle p\rangle}[T_1, T_2, \ldots, T_n]/I$.

Lemma 6.1. *If A is essentially smooth over $\mathbb{Z}_{\langle p\rangle}$ for some prime $p \neq 2$ or over $\mathbb{Q}$, then the sequence*

$$I^n(A)/I^{n+1}(A) \longrightarrow I^n(K)/I^{n+1}(K) \xrightarrow{\oplus\partial_\pi} \bigoplus_{x\in X^{(1)}} I^{n-1}(k(x))/I^n(k(x))$$

is exact, where $X = \operatorname{spec} A$ and K is the fraction field of A.

Proof. Let $K_n^M(A)/2$ denote the "naive" Milnor K-theory defined exactly as for a field. Kummer theory gives a "symbol map" $K_n^M(A)/2 \to H^n_{\text{ét}}(A, \mathbb{Z}/2)$, and in the commutative diagram

$$\begin{array}{ccccccc}
 & & K_n^M(A)/2 & \longrightarrow & K_n^M(K)/2 & \longrightarrow & \bigoplus_{x\in X^{(1)}} K_{n-1}^M(k(x))/2 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^n_{\text{ét}}(A,\mathbb{Z}/2) & \longrightarrow & H^n_{\text{ét}}(K,\mathbb{Z}/2) & \longrightarrow & \bigoplus_{x\in X^{(1)}} H^{n-1}_{\text{ét}}(k(x),\mathbb{Z}/2)
\end{array}$$

where $X = \operatorname{spec} A$ and K is the fraction field of A, the lower row is exact as a consequence of Gillet's Gersten conjecture for étale cohomology in the $\mathbb{Z}_{\langle p\rangle}$ case,[2] and Bloch–Ogus in the $\mathbb{Q}$ case. Furthermore, the Galois symbol

$$K_n^M(A)/2 \to H^n_{\text{ét}}(A, \mathbb{Z}/2)$$

is surjective when A is essentially smooth over $\mathbb{Q}$ [Kerz 2009; 2010] and when A is essentially smooth over a discrete valuation ring[3]; see [Kahn 2002, p. 114, surjectivity of the Galois symbol]. Applying the Milnor conjecture as proved by V. Voevodsky, we have that the vertical maps in the middle and on the right are bijections. It follows that the upper row is exact in the middle. Since $\langle\langle a, 1-a\rangle\rangle = 0$ in $W(A)$ for $a \in A^\times$ such that $1-a \in A^\times$, there is a well-defined homomorphism $K_n^M(A)/2 \to I^n(A)/I^{n+1}(A)$. Hence, in the commutative diagram

$$\begin{array}{ccccc}
K_n^M(A)/2 & \longrightarrow & K_n^M(K)/2 & \longrightarrow & \bigoplus_{x\in X^{(1)}} K_{n-1}^M(k(x))/2 \\
\downarrow & & \downarrow & & \downarrow \\
I^n(A)/I^{n+1}(A) & \longrightarrow & I^n(K)/I^{n+1}(K) & \longrightarrow & \bigoplus_{x\in X^{(1)}} I^{n-1}(k(x))/I^n(k(x))
\end{array}$$

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[2]Manuscript notes titled "Bloch–Ogus for the étale cohomology of certain arithmetic schemes" distributed at the 1997 Seattle algebraic K-theory conference. Also, this follows from Thomas Geisser's proof of the Gersten conjecture for motivic cohomology [Geisser 2004]. This is explicitly stated in the sentence after Geisser's Theorem 1.2, because $R^n\epsilon_*\mu_2$ is the Zariski sheaf associated to the presheaf $U \mapsto H^n_{\text{ét}}(U, \mu_2)$, and the affirmation of the Milnor conjecture allows one to identify the Gersten complex for motivic cohomology with the Gersten complex for étale cohomology.

[3]In a correspondence with the author, B. Kahn explained that the passage from surjectivity in the essentially smooth over a field case to this case is easy and goes back to Lichtenbaum, if you grant Gillet's Gersten conjecture for étale cohomology.

after using again the Milnor conjecture, by which the vertical maps in the middle and on the right are bijections, we have that the lower row is exact in the middle. □

Lemma 6.2. *Let A be essentially smooth over $\mathbb{Z}_{\langle p\rangle}$ $(p \neq 2)$ or $\mathbb{Q}$.*

(1) *There exists an integer N such that $C^\bullet(A, I^s, \iota) \xrightarrow{2} C^\bullet(A, I^{s+1}, \iota)$ is an isomorphism of complexes for all $s \geq N$.*

(2) *The groups $H^m(C^\bullet(A, W))$ are 2^N-torsion for all $m \geq 2$.*

(3) *There exists an integer $B \geq 0$ such that $2^B H^0(C^\bullet(A, W)) \subset i_*(W(A))$, where $i^* : W(A) \to W(K)$ denotes the map induced by $i : \operatorname{spec} K \to \operatorname{spec} A$.*

(4) $2^{B+N} H^0(C^\bullet(A, W)) \subset i_*(I^N(A))$.

Proof. To prove (1), note that the cohomological 2-dimension of $k(x)[\sqrt{-1}]$ is finite and, for all points x, bounded strictly less than some integer n. Using the Arason–Pfister Hauptsatz and the Milnor conjecture for fields it follows that $I^n(k(x)[\sqrt{-1}])$ vanishes for all x, and from this it follows that, for all x, we have an isomorphism $I^s(k(x)) \xrightarrow{2} I^{s+1}(k(x))$ for all $s \geq n$ [Elman et al. 2008, Corollary 35.27]. Hence $C^\bullet(A, I^s, \iota) \xrightarrow{2} C^\bullet(A, I^{s+1}, \iota)$ is an isomorphism of complexes for all $s \geq N$, where $N := n + \dim X$. Then $C^\bullet(A, \varinjlim I^n)$ and $C^\bullet(A, I^N, \iota)$ are isomorphic complexes, so the cohomology group $H^m(C^\bullet(A, I^N, \iota))$ vanishes when $m \geq 2$ by Lemma 5.5. It follows that the groups $H^m(C^\bullet(A, W))$ are 2^N-torsion when $m \geq 2$ since $H^m(C^\bullet(A, W)) \xrightarrow{2^N} H^m(C^\bullet(A, W))$ factors

$$\begin{tikzcd}
H^m(C^\bullet(A, W)) \arrow[rr, "2^N"] \arrow[dr, "2^N"'] & & H^m(C^\bullet(A, W)) \\
& H^m(C^\bullet(A, I^N, \iota)) \arrow[ur] &
\end{tikzcd}$$

proving (2).

Now to prove (3), let $q \in H^0(C^\bullet(A, W))$. From the Balmer–Walter spectral sequence for Witt groups [Balmer and Walter 2002] we have that $W(A)$ surjects onto $E_\infty^{0,0}$, which consists of the elements in $H^0(C^\bullet(A, W))$ mapped to zero under all the differentials in the spectral sequence leaving $H^0(C^\bullet(A, W))$. So it suffices to show that some 2-power of q maps to zero under all of these finitely many nontrivial differentials. The first nontrivial differential is $d : H^0(C^\bullet(A, W)) \to H^5(C^\bullet(A, W))$. Since $2^N H^5(C^\bullet(A, W)) = 0$, we have that $d(2^N q) = 0$. Repeating this argument for each nontrivial differential $d : H^0(C^\bullet(A, W)) \to H^{4*+1}(C^\bullet(A, W))$ we eventually find some 2-power 2^B, which does not depend on q, such that $2^B q$ is in the kernel of all differentials, hence is in $E_\infty^{0,0}$. Finally, to prove (4), let $q \in 2^{B+N} H^0(C^\bullet(A, W))$. Write it as $q = 2^{B+N} q_{unr}$ for some $q_{unr} \in H^0(C^\bullet(A, W))$. By (3), we have that $2^B q_{unr} = i_*(Q)$ for some $Q \in W(A)$. So $i_*(2^N Q) = q$ and $2^N Q \in I^N(A)$. This proves $2^{B+N} H^0(C^\bullet(A, W)) \subset i_*(I^N(A))$, finishing the proof of the lemma. □

Proposition 6.3. *Let A be essentially smooth over either $\mathbb{Z}_{\langle p\rangle}$ $(p \neq 2)$ or $\mathbb{Q}$. The sequence*

$$\varinjlim I^n(A) \longrightarrow \varinjlim I^n(K) \xrightarrow{\oplus\partial_\pi} \bigoplus_{x\in X^{(1)}} \varinjlim_{n\geq -1} I^n(k(x))$$

is exact, where $\varinjlim_{n\geq -1} I^n(k(x))$ denotes the colimit over

$$W(k(x)) \xrightarrow{\langle\langle -1\rangle\rangle} W(k(x)) \xrightarrow{\langle\langle -1\rangle\rangle} I(k(x)) \xrightarrow{\langle\langle -1\rangle\rangle} I^2(k(x)) \xrightarrow{\langle\langle -1\rangle\rangle} \cdots .$$

Proof. Let q be in the kernel of the residue, hence $q \in H^0(C^\bullet(A, I^N, \iota))$ for some $N \geq 0$. We may assume that N is the integer N from Lemma 6.2(1) by either multiplying by 2 or dividing by 2 as needed. Using Lemma 6.1 we find $\overline{Q_N} \in I^N(A)/I^{N+1}(A)$, which we may then lift to obtain $Q_N \in I^N(A)$ satisfying $q - i_*(Q_N) \in H^0(C^\bullet(A, I^{N+1}, \iota))$. By repeating this argument we find that $q - i_*(Q_N + Q_{N+1} + \cdots + Q_{B+2N-1}) \in H^0(C^\bullet(A, I^{B+2N}, \iota))$, where B is the integer from Lemma 6.2(3). Since we are in the "stable" range we have that $H^0(C^\bullet(A, I^{B+2N}, \iota)) = 2^{B+N} H^0(C^\bullet(A, I^N, \iota)) \subset 2^{B+N} H^0(C^\bullet(A, W)) \subset i_*(I^N(A))$, where we used Lemma 6.2(4) to obtain the rightmost inclusion. Hence we have $Q_N^{'} \in I^N(A)$ such that

$$q = i_*(Q_N + Q_{N+1} + \cdots + Q_{B+2N-1} + Q_N^{'}),$$

where $Q_N + Q_{N+1} + \cdots + Q_{B+2N-1} + Q_N^{'} \in I^N(A)$. This finishes the proof. $\square$

7. On the signature: local case

In this section we use "Hoobler's trick", which is a method due to R. Hoobler [2006] for passing from the smooth geometric case to the geometric case for many questions involving cohomological invariants satisfying "rigidity" in the sense of the following lemma.

Lemma 7.1. *If B is a local ring and (B, I) a henselian pair such that 2 is invertible in both B and B/I, then for all integers $n \geq 0$, the homomorphisms of groups*

$$I^n(B) \to I^n(B/I),$$
$$I^n(B)/I^{n+1}(B) \to I^n(B/I)/I^{n+1}(B/I)$$

induced by the surjection $B \to B/I$ are bijections.

Proof. Let B be a local ring and (B, I) a henselian pair such that 2 is invertible in both B and B/I. Considering the diagram

$$\begin{tikzcd}
0 \arrow[r] & I^{n+1}(B) \arrow[r] \arrow[d] & I^n(B) \arrow[r] \arrow[d] & I^n(B)/I^{n+1}(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^{n+1}(B/I) \arrow[r] & I^n(B/I) \arrow[r] & I^n(B/I)/I^{n+1}(B/I) \arrow[r] & 0
\end{tikzcd}$$

we see, by the two out of three lemma, that it suffices to prove $I^n(B) \to I^n(B/I)$ is a bijection for all $n \geq 0$. To prove injectivity for all $n \geq 0$, note that as $I^n(B)$ is contained in $W(B)$, it suffices to prove that $W(B) \to W(B/I)$ is injective.

We now claim that the assignment $b + I \mapsto b$ determines a well-defined map $(B/I)^\times/(B/I)^{\times 2} \to B^\times/B^{\times 2}$. This claim follows from rigidity for étale cohomology due to Strano [1984] and Gabber [1994] (independently), but one may also prove it directly from the definition of Henselian pair[4]: let $b_1, b_2 \in B^\times$ be such that $b_1 + I = b_2 + I$; the polynomial $T^2 - b_1/b_2$ has image $T^2 - 1$ in $B/I[T]$; as (B, I) is a henselian pair, from the factorization $T^2 - 1 = (T-1)(T+1)$ in $B/I[T]$ we obtain a factorization $T^2 - b_1/b_2 = (T-a)(T+a)$ in $B[T]$, for some $a \in B$; hence $b_1 = a^2 b_2$ for some $a \in B^\times$, that is, $b_1 = b_2$ in $B^\times/(B^\times)^2$. The claim follows.

Next recall that for any semilocal ring A, the Witt group $W(A)$ is a quotient of the group ring $\mathbb{Z}[A^\times/A^{\times 2}]$ modulo the set of relations R additively generated by $[1] + [-1]$ and all elements

$$\sum_{i=1}^{h} [a_i] - \sum_{i=1}^{h} [b_i]$$

satisfying

$$\perp_{i=1}^{h} \langle a_i \rangle \simeq \perp_{i=1}^{h} \langle b_i \rangle$$

with $h = 4$ [Knebusch 1977, Chapter 2, §4, Theorem 2]. Hence, the rows are exact in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] & \mathbb{Z}[B^\times/B^{\times 2}] \arrow[r] & W(B) \arrow[r] & 0 \\
0 \arrow[r] & R \arrow[r] \arrow[u] & \mathbb{Z}[(B/I)^\times/(B/I)^{\times 2}] \arrow[r] \arrow[u] & W(B/I) \arrow[r] & 0
\end{tikzcd}$$

Thus we obtain a well-defined map of cokernels $W(B/I) \to W(B)$ such that the composition $W(B) \to W(B/I) \to W(B)$ is the identity. This proves the desired injectivity. The composition $W(B/I) \to W(B) \to W(B/I)$ is the identity, hence $W(B) \to W(B/I)$ is surjective. To prove surjectivity of $I^n(B) \to I^n(B/I)$ for all $n \geq 0$, recall that $I^n(B/I)$ is additively generated by Pfister forms $\langle\langle \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n \rangle\rangle$, where $\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n$ are units in B/I [Baeza 1978, Chapter V, Section 1, Remark 1.3]. For any Pfister form $\langle\langle \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n \rangle\rangle$ we may lift the $\bar{b}_i$ to units b_i of B to obtain an element $\langle\langle b_1, b_2, \ldots, b_n \rangle\rangle \in I^n(B)$ mapping to it, proving surjectivity of $I^n(B) \to I^n(B/I)$ and finishing the proof of the lemma. □

Proposition 7.2. *If A is a local ring with $2 \in A^\times$, then the signature map*

$$\varinjlim I^n(A) \to \mathrm{C}(\operatorname{sper} A, \mathbb{Z})$$

is a bijection.

[4] The author learned this from a recent preprint of Stefan Gille [2015].

Proof. As both groups respect filtered colimits, it suffices to consider the case where A is a localization of a finite type $\mathbb{Z}$-algebra: any local ring may be written as a union of its finitely generated subrings A_α; pulling back the maximal ideal of A over $A_\alpha \to A$ yields a prime ideal $\boldsymbol{p}_\alpha \in \operatorname{spec} A_\alpha$; localizing the A_α with respect to these primes yields a directed system of local rings $A_{\boldsymbol{p}_\alpha}$, and taking the direct limit yields A.

From now on we assume $A = R_{\boldsymbol{p}}$, where $\boldsymbol{p} \in \operatorname{spec} R$ and $R = \mathbb{Z}[T_1, T_2, \ldots, T_n]/I$ for some ideal I. We obtain a *henselian pair* (B, I) *for* A as follows: let s denote the quotient map $\mathbb{Z}[T_1, T_2, \ldots, T_n] \to R$, and let $B_0 := \mathbb{Z}[T_1, T_2, \ldots, T_n]_{s^{-1}(\boldsymbol{p})}$ and similarly $I_0 := I_{s^{-1}(\boldsymbol{p})}$; also let B denote the henselization of B_0 along I_0 and $I := I_0 B$. Recall that the henselization along I_0 is obtained by taking the colimit over the directed category consisting of those étale B_0-algebras C having the property that $B_0/I_0 \to C/I_0C$ is an isomorphism. The map $B_0 \to B$ induces on quotients $A = B_0/I_0 \to B/I$ an isomorphism of local rings. In the commutative diagram

$$\begin{array}{ccc} \varinjlim I^n(B) & \longrightarrow & \varinjlim I^n(A) \\ \Big\downarrow{\scriptstyle \text{sign}} & & \Big\downarrow{\scriptstyle \text{sign}} \\ \mathrm{C}(\operatorname{sper} B, \mathbb{Z}) & \longrightarrow & \mathrm{C}(\operatorname{sper} A, \mathbb{Z}) \end{array}$$

the horizontal maps induced by the surjection $B \to B/I \simeq A$ are isomorphisms for the powers of the fundamental ideal (Lemma 7.1) and for real cohomology.[5] Therefore it suffices to prove bijectivity for B.

We claim that the local ring B is a filtered colimit of local rings which are essentially smooth over either $\mathbb{Z}_{\langle p\rangle}$ ($p \neq 2$) or over $\mathbb{Q}$. To prove the claim, first note that the pullback of $s^{-1}(\boldsymbol{p}) \in \operatorname{spec} \mathbb{Z}[T_1, T_2, \ldots, T_n]$ over $\mathbb{Z} \to \mathbb{Z}[T_1, T_2, \ldots, T_n]$ yields a prime $\langle p\rangle \in \operatorname{spec} \mathbb{Z}$, and localizing with respect to this prime induces $\mathbb{Z}_{\langle p\rangle} \hookrightarrow B_0$. When $\langle p\rangle = 0$ it follows that B_0 contains $\mathbb{Q}$, otherwise B_0 contains $\mathbb{Z}_{\langle p\rangle}$, $p \neq 2$. The morphisms $\mathbb{Z}_{\langle p\rangle} \to B_0$ and $B_0 \to B$ are both flat with geometrically regular fibers, hence the composition $\mathbb{Z}_{\langle p\rangle} \to B$ has these properties. Then it follows from Popescu's theorem that B is a filtered colimit of either smooth $\mathbb{Z}_{\langle p\rangle}$-algebras or $\mathbb{Q}$-algebras A_α. Pulling back the maximal ideal over $A_\alpha \to B$ and localizing, one obtains the statement of the claim. Thus, we may assume that B is essentially smooth over $\mathbb{Q}$ or $\mathbb{Z}_{\langle p\rangle}$. Then we may apply Lemma 4.11 to get exactness of the lower row in the commutative diagram

[5] The following proof was communicated to the author by C. Scheiderer: every point in $\operatorname{sper} B$ specializes to a point in $\operatorname{sper} B/I$ by the henselian property; since any real spectrum is a "normal" spectral space, meaning that every point of X_r specializes to a unique closed point, the restriction map in sheaf cohomology $H^*(X_r, \mathcal{F}) \to H^*((X_r)_{\max}, \mathcal{F})$ is an isomorphism for every sheaf $\mathcal{F}$ on any scheme X; thus restriction gives isomorphisms

$$H^*(\operatorname{sper}(B), \mathcal{F}) \to H^*(\operatorname{sper}(B)_{\max}, \mathcal{F}) \xleftarrow{\simeq} H^*(\operatorname{sper}(B/I), \mathcal{F}).$$

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim I^n(B) \arrow[r] \arrow[d] & \varinjlim I^n(K) \arrow[r, "\oplus\partial_\pi"] \arrow[d, "\mathrm{sign}"] & \bigoplus_{x\in Y^{(1)}} \varinjlim_{n\geq -1} I^n(k(x)) \arrow[d, "2\,\mathrm{sign}"] \\
0 \arrow[r] & \mathrm{C}(\mathrm{sper}\, B, \mathbb{Z}) \arrow[r] & \mathrm{C}(\mathrm{sper}\, K, \mathbb{Z}) \arrow[r, "\oplus\beta_\pi"] & \bigoplus_{x\in Y^{(1)}} \mathrm{C}(\mathrm{sper}\, k(x), \mathbb{Z})
\end{tikzcd}$$

where $Y = \mathrm{spec}\, B$. We have exactness of the upper row by Proposition 6.3 and Corollary 5.4. Using the bijection of Proposition 2.7 we get that the middle vertical map in the diagram above is bijective and the rightmost vertical map is injective. The square on the right commutes by Lemma 3.4. Hence $\varinjlim I^n(B) \to \mathrm{C}(\mathrm{sper}\, B, \mathbb{Z})$ is bijective, finishing the proof of the theorem. □

The following corollary is well-known, as mentioned in the introduction.

Corollary 7.3. *Let A be a local ring with $2 \in A^\times$. Then the signature induces an isomorphism*

$$W(A)\left[\tfrac{1}{2}\right] \to \mathrm{C}(\mathrm{sper}\, A, \mathbb{Z})\left[\tfrac{1}{2}\right].$$

Proof. From the preceding theorem, any $f \in \mathrm{C}(\mathrm{sper}\, A, \mathbb{Z})$ has $2^n f = \mathrm{sign}(Q)$ for some $Q \in I^n(A) \subset W(A)$, proving surjectivity, and for any $Q' \in W(A)$, if $\mathrm{sign}(Q') = 0$ then $2^n Q' = 0$ for some n, proving injectivity. □

Remark 7.4. Let $A = \bigoplus_{n\geq 0} A_n$ be a $\mathbb{Z}_+$-graded ring and let $s \in A_1$ be a homogeneous element of degree 1. Recall that the homogeneous localization $A_{(s)}$ is the subring of degree zero elements in the localization of A with respect to $\{1, s, s^2, \ldots\}$, and that $A_{(s)} \simeq A/(s-1)A$ as rings. Furthermore, $A_{(s)}$ may be obtained by taking the direct limit of the sequence $A_0 \xrightarrow{s} A_1 \xrightarrow{s} A_2 \xrightarrow{s} \cdots$.

Corollary 7.5. *Let A be a local ring with 2 invertible.*

(1) *Let $I^*(A)_{\langle\langle -1\rangle\rangle}$ be the homogeneous localization of the graded ring $\bigoplus_{n\geq 0} I^n(A)$ with respect to the element $\langle\langle -1\rangle\rangle = \langle 1, 1\rangle \in I(A)$. The signature defines an isomorphism of rings*

$$I^*(A)_{\langle\langle -1\rangle\rangle} \simeq \mathrm{C}(\mathrm{sper}\, A, \mathbb{Z}).$$

(2) *Let $\overline{I^*}(A)_{\langle\langle -1\rangle\rangle}$ be the homogeneous localization of the graded ring $\bigoplus_{n\geq 0} \overline{I^n}(A)$ with respect to $\langle\langle -1\rangle\rangle = \langle 1, 1\rangle \in \overline{I^1}(A)$, where $\overline{I^n}(A) := I^n(A)/I^{n+1}(A)$. The signature defines an isomorphism of rings*

$$\overline{I^*}(A)_{\langle\langle -1\rangle\rangle} \simeq \mathrm{C}(\mathrm{sper}\, A, \mathbb{Z}/2).$$

Proof. Recall (Remark 7.4) that one may identify $\varinjlim I^n(A)$ with $I^*(A)_{\langle\langle -1\rangle\rangle}$: using the direct sum construction of the direct limit $\varinjlim I^n(A)$, the relations one finds are the same as the relations defining the localization $I^*(A)_{\langle\langle -1\rangle\rangle}$; explicitly, the isomorphism $\varphi : \varinjlim I^n(A) \to I^*(A)_{\langle\langle -1\rangle\rangle}$ is given by $\varphi_n : I^n(A) \to I^*(A)_{\langle\langle -1\rangle\rangle}$ defined by

$$q \mapsto \frac{q}{\langle\langle -1\rangle\rangle^n},$$

and consequently we obtain using the preceding proposition that the assignment

$$\frac{q}{\langle\langle -1\rangle\rangle^n} \mapsto \frac{\operatorname{sign}(q)}{2^n},$$

for $q \in I^n(A)$, defines an isomorphism from $I^*(A)_{\langle\langle -1\rangle\rangle}$ to $\mathrm{C}(\operatorname{sper} A, \mathbb{Z})$. To prove (2), we obtain the desired isomorphism as an isomorphism of cokernels in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim_{n\geq 1} I^n(A) \arrow[r] \arrow[d] & \varinjlim I^n(A) \arrow[r] \arrow[d] & \varinjlim \overline{I^n}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{C}(\operatorname{sper} A, 2\mathbb{Z}) \arrow[r] & \mathrm{C}(\operatorname{sper} A, \mathbb{Z}) \arrow[r] & \mathrm{C}(\operatorname{sper} A, \mathbb{Z}/2) \arrow[r] & 0
\end{tikzcd}$$

where $\varinjlim_{n\geq 1} \to \mathrm{C}(\operatorname{sper} A, 2\mathbb{Z})$ is an isomorphism since in the commutative diagram

$$\begin{tikzcd}
\varinjlim_{n\geq 1} I^n(A) \arrow[r] & \mathrm{C}(\operatorname{sper} A, 2\mathbb{Z}) \\
\varinjlim I^n(A) \arrow[r] \arrow[u, "\langle\langle -1\rangle\rangle"] & \mathrm{C}(\operatorname{sper} A, \mathbb{Z}) \arrow[u, "2"']
\end{tikzcd}$$

the vertical maps and the lower horizontal map are isomorphisms. □

Corollary 7.6. *Let A be a local ring with 2 invertible. Let $H^*_{\text{ét}}(A, \mathbb{Z}/2)_{(-1)}$ denote the homogeneous localization of the cohomology ring $\bigoplus_{n\geq 0} H^n_{\text{ét}}(A, \mathbb{Z}/2\mathbb{Z})$ with respect to $(-1) \in H^1_{\text{ét}}(A, \mathbb{Z}/2\mathbb{Z})$. Then the n-th cohomological invariant*

$$\bar{e}_n : \overline{I^n} \to H^n_{\text{ét}}(A, \mathbb{Z}/2),$$

which assigns the class of a Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ to the cup product $(a_1) \cup \cdots \cup (a_n)$, determines a well-defined homomorphism

$$\bar{e}_* : \overline{I^*}(A)_{\langle\langle -1\rangle\rangle} \simeq H^*_{\text{ét}}(A, \mathbb{Z}/2)_{(-1)}$$

which is an isomorphism of rings.

Proof. For any local ring A essentially smooth over $\mathbb{Z}_{\langle p\rangle}$ or $\mathbb{Q}$, the diagram

$$\begin{tikzcd}
 & I^n(A) \arrow[r] \arrow[d] & I^n(K) \arrow[r] \arrow[d] & \bigoplus_{x\in X^{(1)}} I^{n-1}(k(x))/2 \arrow[d] \\
0 \arrow[r] & H^n_{\text{ét}}(A, \mathbb{Z}/2) \arrow[r] & H^n_{\text{ét}}(K) \arrow[r] & \bigoplus_{x\in X^{(1)}} H^{n-1}_{\text{ét}}(k(x))
\end{tikzcd}$$

commutes and the lower row is exact as the Gersten conjecture is known for étale cohomology in this case.

As the diagram commutes it follows that $I^n(A)/I^{n+1}(A)$ maps into $H^n(A, \mathbb{Z}/2)$. Let $\bar{e}_n$ denote this map. As the lower row is exact, it has the description asserted on Pfister forms. Using rigidity and the fact that both groups respect filtered colimits as in the proof of Theorem 8.6, we obtain the map $\bar{e}_n$ for any local ring, and after localizing, we obtain the map in the commutative diagram

$$\begin{tikzcd} \overline{I^*}(A)_{\langle\langle -1\rangle\rangle} \arrow[r, "\bar{e}_*"] \arrow[dr, "\simeq"] & H^*_{\text{ét}}(A, \mathbb{Z}/2)_{(-1)} \arrow[d, "\simeq"] \\ & \mathrm{C}(\operatorname{sper} A, \mathbb{Z}/2) \end{tikzcd}$$

where we use the fact that for any semilocal ring A with 2 invertible, the signature modulo 2 defines an isomorphism

$$H^*_{\text{ét}}(A, \mathbb{Z}/2)_{(-1)} \xrightarrow{\simeq} \mathrm{C}(\operatorname{sper} A, \mathbb{Z}/2) \tag{7.7}$$

of rings. This is due to J. Burési and L. Mahé in the semilocal case [Burési 1995; Mahé 1995] and C. Scheiderer in general [Scheiderer 1994, Corollaries 7.10.3 and 7.19]. From the isomorphisms in the diagram, the desired isomorphism follows. □

8. Globalization

In this section X always denotes a scheme. Let $W(X)$ denote the Witt ring of symmetric bilinear forms over X; see [Knebusch 1977].

Definition 8.1. Recall that the *global signature* is the ring homomorphism

$$\operatorname{sign} : W(X) \to H^0(X_r, \mathbb{Z})$$

that assigns an isometry class $[\phi]$ of a symmetric bilinear form ϕ over X to the function on X_r defined by

$$\operatorname{sign}([\phi])(x, P) := \operatorname{sign}_P([i_x^*\phi]),$$

where $i_x : x \to X$ is any point and P is any ordering on $k(x)$; see [Mahé 1982].

Definition 8.2. There exists a well-defined ring homomorphism on the Witt ring $W(X) \to H^0_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$, called the *rank*, which assigns an isometry class of a symmetric bilinear form $[\mathcal{E}, \phi]$ over X to the rank of its underlying vector bundle $\mathcal{E}$ modulo 2; see [Knebusch 1977, Chapter 1, §7]. The kernel of the rank map is called the *fundamental ideal* and is denoted by $I(X)$.

It follows from the definitions that the diagram

$$\begin{tikzcd} W(X) \arrow[r, "\mathrm{sign}"] \arrow[d, "\mathrm{rank\ mod\ 2}"] & H^0(X_r, \mathbb{Z}) \arrow[d] \\ H^0_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \arrow[r, "h_0"] & H^0(X_r, \mathbb{Z}/2\mathbb{Z}) \end{tikzcd} \tag{8.3}$$

commutes, where h_0 denotes the *signature modulo 2* defined as follows: given $\alpha \in H^0_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$, if $\xi : x \to X$ is the inclusion of a "real" point (that is, for some $(x, P) \in X_r$), then $h_0(\alpha)$ evaluated at ξ is $\xi^*\alpha \in H^0(x_{\text{ét}}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$; write $\alpha(\xi)$ for this element of $\mathbb{Z}/2\mathbb{Z}$, so $h_0(\alpha)$ is the locally constant map $X_r \to \mathbb{Z}/2\mathbb{Z}$, $\xi \mapsto \alpha(\xi)$; see [Scheiderer 1994, (7.19.1)].

Definition 8.4. As there is an exact sequence

$$0 \to H^0(X_r, 2\mathbb{Z}) \to H^0(X_r, \mathbb{Z}) \to H^0(X_r, \mathbb{Z}/2\mathbb{Z})$$

one finds using commutativity of (8.3) that the restriction of the signature to $I(X)$ defines the homomorphism of groups

$$I(X) \to H^0(X_r, 2\mathbb{Z}).$$

For $n \geq 0$, let $I^n(X)$ denote the powers of the fundamental ideal and $I^0(X) = W(X)$. Since the global signature is a ring homomorphism that maps elements of $I(X)$ into $H^0(X_r, 2\mathbb{Z})$, it follows that for any $n \geq 0$ it induces a homomorphism

$$I^n(X) \to H^0(X_r, 2^n\mathbb{Z})$$

of groups. Moreover, multiplication by $2 = \langle\langle -1 \rangle\rangle \in I(X)$ induces a homomorphism $I^n(X) \xrightarrow{\langle\langle -1 \rangle\rangle} I^{n+1}(X)$ such that the diagram

$$\begin{tikzcd} I^j(X) \arrow[r, "\mathrm{sign}"] \arrow[d, "\langle\langle -1 \rangle\rangle"] & H^0(X_r, 2^j\mathbb{Z}) \arrow[d, "2"] \\ I^{j+1}(X) \arrow[r, "\mathrm{sign}"] & H^0(X_r, 2^{j+1}\mathbb{Z}) \end{tikzcd}$$

commutes. Hence, we obtain a homomorphism

$$\varinjlim I^n(X) \to H^0(X_r, \mathbb{Z}),$$

where $\varinjlim I^n(X)$ denotes the direct limit of the sequence of groups

$$W(X) \xrightarrow{\langle\langle -1 \rangle\rangle} I(X) \xrightarrow{\langle\langle -1 \rangle\rangle} I^2(X) \xrightarrow{\langle\langle -1 \rangle\rangle} \cdots .$$

Definition 8.5. It follows from Lemma 4.6 that $\mathrm{supp}_* \mathbb{Z}$ is the Zariski sheaf $U \mapsto H^0(U_r, \mathbb{Z})$ on X. Recall that $\mathcal{I}^n$ denotes the Zariski sheaf on X associated to the presheaf $U \mapsto I^n(U)$. For any integer $n \geq 0$, the restriction of the global signature to the powers of the fundamental ideal of Definition 8.4 induces a homomorphism

$$\mathcal{I}^n \to \operatorname{supp}_* 2^n\mathbb{Z}$$

of Zariski sheaves on X. Similarly, $I^n(X) \xrightarrow{\langle\langle -1\rangle\rangle} I^{n+1}(X)$ induces a homomorphism $\mathcal{I}^n \xrightarrow{\langle\langle -1\rangle\rangle} \mathcal{I}^{n+1}$ of sheaves for any $n \geq 0$, and a homomorphism of sheaves

$$\varinjlim I^n \to \operatorname{supp}_* \mathbb{Z},$$

where $\varinjlim I^n$ denotes the direct limit of the sequence of sheaves

$$\mathcal{W} \xrightarrow{\langle\langle -1\rangle\rangle} \mathcal{I} \xrightarrow{\langle\langle -1\rangle\rangle} \mathcal{I}^2 \xrightarrow{\langle\langle -1\rangle\rangle} \cdots .$$

Similarly, the signature induces a morphism of sheaves

$$\mathcal{W}\big[\tfrac{1}{2}\big] \to \operatorname{supp}_* \mathbb{Z}\big[\tfrac{1}{2}\big],$$

where $\mathcal{W}\big[\tfrac{1}{2}\big]$ is the sheaf associated to the presheaf $U \mapsto W(U)\big[\tfrac{1}{2}\big]$ and $\operatorname{supp}_* \mathbb{Z}\big[\tfrac{1}{2}\big]$ is the sheaf $U \mapsto H^0\big(U_r, \mathbb{Z}\big[\tfrac{1}{2}\big]\big)$.

Theorem 8.6. *Let X be a scheme with 2 invertible in its global sections.*

(1) *The signature morphism of sheaves*

$$\varinjlim \mathcal{I}^n \to \operatorname{supp}_* \mathbb{Z} \tag{8.7}$$

of Definition 8.5 is an isomorphism.

(2) *The signature morphism of sheaves*

$$\mathcal{W}\big[\tfrac{1}{2}\big] \to \operatorname{supp}_* \mathbb{Z}\big[\tfrac{1}{2}\big] \tag{8.8}$$

of Definition 8.5 is an isomorphism.

(3) *The signature induces an isomorphism of short exact sequence of sheaves on X,*

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim \mathcal{I}^n \arrow[r] \arrow[d] & \mathcal{W}\big[\tfrac{1}{2}\big] \arrow[r] \arrow[d] & \varinjlim \mathcal{W}/\mathcal{I}^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{supp}_* \mathbb{Z} \arrow[r] & \operatorname{supp}_* \mathbb{Z}\big[\tfrac{1}{2}\big] \arrow[r] & \operatorname{supp}_*\big(\mathbb{Z}\big[\tfrac{1}{2}\big]/\mathbb{Z}\big) \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{W}/\mathcal{I}^n$ denotes the sheaf associated to the presheaf $U \mapsto \mathcal{W}(U)/\mathcal{I}^n(U)$.

(4) *The signature induces an isomorphism of short exact sequence of sheaves on* X,

$$\begin{tikzcd} 0 \arrow[r] & \varinjlim \mathcal{I}^n \arrow[r, "\langle\langle -1 \rangle\rangle"] \arrow[d] & \varinjlim \mathcal{I}^n \arrow[r] \arrow[d] & \varinjlim \overline{\mathcal{I}^n} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{supp}_* \mathbb{Z} \arrow[r, "2"] & \mathrm{supp}_* \mathbb{Z} \arrow[r] & \mathrm{supp}_* \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

where $\overline{\mathcal{I}^n}$ *denotes the sheaf associated to the presheaf* $U \mapsto \mathcal{I}^n(U)/\mathcal{I}^{n+1}(U)$.

Proof. Statements (1) and (2) follow immediately from the local case, Proposition 7.2 and Corollary 7.3 respectively, as it is sufficient to prove that they induce an isomorphism on stalks. As supp_* is exact, statements (3) and (4) may be obtained by applying supp_* to the analogous short exact sequences of groups and then using the two out of three lemma to conclude. For (4), one should also note that

$$\varinjlim \mathcal{I}^n \xrightarrow{\langle\langle -1 \rangle\rangle} \varinjlim_{n \geq 1} \mathcal{I}^n$$

is an isomorphism to obtain exactness of the top row of the diagram in (4). □

The next corollary is an immediate consequence of the previous theorem and Lemma 4.6.

Corollary 8.9. *Let* X *be a scheme with* 2 *invertible.*

(1) *For any* $m \geq 0$, *the morphism* (8.7) *induces an isomorphism of cohomology groups*
$$H^m_{\mathrm{Zar}}(X, \varinjlim \mathcal{I}^n) \to H^m(X_r, \mathbb{Z}).$$

(2) *For any* $m \geq 0$, *the morphism* (8.8) *induces an isomorphism of cohomology groups*
$$H^m_{\mathrm{Zar}}\big(X, \mathcal{W}[\tfrac{1}{2}]\big) \to H^m\big(X_r, \mathbb{Z}[\tfrac{1}{2}]\big).$$

Corollary 8.10. *Let* X *be a scheme with* 2 *invertible which is quasiseparated and quasicompact. Then there is an isomorphism of cohomology groups for all* $m \geq 0$,

$$\bigoplus_{m \geq 0} H^m_{\mathrm{Zar}}(X, \varinjlim \overline{\mathcal{I}^n}) \simeq \varinjlim H^n_{\text{ét}}(X, \mathbb{Z}/2).$$

Proof. Under the hypotheses stated C. Scheiderer [1994, Corollary 7.19] has proved that there is an isomorphism

$$\varinjlim H^n_{\text{ét}}(X, \mathbb{Z}/2) \xrightarrow{\simeq} \bigoplus_{m \geq 0} H^m(X_r, \mathbb{Z}/2),$$

and from Theorem 8.6 one has an isomorphism $H^m_{\mathrm{Zar}}(X, \overline{\mathcal{I}^n}) \xrightarrow{\simeq} H^m(X_r, \mathbb{Z}/2)$ for all $m \geq 0$. Thus one obtains the isomorphism stated. □

Corollary 8.11. *If X is a real variety* (*by which we mean a scheme which is separated and of finite type over* $\mathbb{R}$) *and the Krull dimension of X is d, then whenever $n \geq d+1$, the signature induces an isomorphism in cohomology*

$$H^m_{\mathrm{Zar}}(X, \mathcal{I}^n) \overset{\mathrm{sign}}{\simeq} H^m_{\mathrm{sing}}(X(\mathbb{R}), \mathbb{Z})$$

for all integers $m \geq 0$ and an isomorphism of long exact sequences as stated in the introduction.

Proof. It suffices to see that the morphism of sheaves $\mathcal{I}^n \xrightarrow{\langle\langle -1\rangle\rangle} \mathcal{I}^{n+1}$ is an isomorphism for $n \geq d+1$, for then multiplication by 2^{d+1} defines an isomorphism of sheaves $\mathcal{I}^{d+1} \simeq \varinjlim \mathcal{I}^n$ and hence we obtain the statement of the corollary using Theorem 8.6 in view of Remark 4.4. When $n \geq d+1$, for any U open in X we have an isomorphism of kernels in the diagram of residues

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}^n(U) \arrow[r] \arrow[d, "2"] & I^n(K) \arrow[r] \arrow[d, "2"] & \bigoplus_{x\in X^{(1)}} I^{n-1}(k(x)) \arrow[d, "2"] \\
0 \arrow[r] & \mathcal{I}^{n+1}(U) \arrow[r] & I^{n+1}(K) \arrow[r] & \bigoplus_{x\in X^{(1)}} I^{n}(k(x))
\end{tikzcd}$$

since the two rightmost vertical maps are isomorphisms for $n \geq d+1$, which proves the desired isomorphism. $\square$

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Colocalising subcategories of modules over finite group schemes

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Dedicated to Amnon Neeman on the occasion of his 60th birthday

The Hom closed colocalising subcategories of the stable module category of a finite group scheme are classified. This complements the classification of the tensor closed localising subcategories in our previous work. Both classifications involve π-points in the sense of Friedlander and Pevtsova. We identify for each π-point an endofinite module which both generates the corresponding minimal localising subcategory and cogenerates the corresponding minimal colocalising subcategory.

1. Introduction

Let G be a finite group scheme over a field k of positive characteristic. There is a notion of π-cosupport [Benson et al. 2017] for any G-module M, based on the notion of π-points of G introduced by Friedlander and Pevtsova [2005]. The π-cosupport of M, denoted by $\pi\text{-cosupp}_G(M)$, is a subset of $\operatorname{Proj} H^*(G,k)$. The main result in this work is a classification of the colocalising subcategories of $\mathsf{StMod}\, G$, the stable module category of possibly infinite-dimensional G-modules, in terms of π-cosupport.

Theorem 1.1. *Let G be a finite group scheme over a field k. Then the assignment*

$$\mathsf{C} \mapsto \bigcup_{M\in\mathsf{C}} \pi\text{-cosupp}_G(M)$$

induces a bijection between the colocalising subcategories of $\mathsf{StMod}\, G$ *that are closed under tensor product with simple G-modules and the subsets of* $\operatorname{Proj} H^*(G,k)$.

This is proved after Corollary 4.9. Recall that a *colocalising subcategory* C is a full triangulated subcategory that is closed under set-indexed products. Such a C is closed under tensor product with simple G-modules if and only if it is *Hom closed*: if M is in C, so is $\operatorname{Hom}_k(L,M)$ for any G-module L. Theorem 1.1 complements

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the classification of the tensor closed localising subcategories of $\mathsf{StMod}\, G$ from [Benson et al. 2016]. Combining them gives a remarkable bijection:

Corollary 1.2. *The map sending a localising subcategory* C *of* $\mathsf{StMod}\, G$ *to* $\mathsf{C}^{\perp}$ *induces a bijection*

$$\left\{ \begin{matrix} \textit{tensor closed localising} \\ \textit{subcategories of } \mathsf{StMod}\, G \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \textit{Hom closed colocalising} \\ \textit{subcategories of } \mathsf{StMod}\, G \end{matrix} \right\}.$$

The inverse map sends a colocalising subcategory C *to* ${}^{\perp}\mathsf{C}$. □

Predecessors of these results are the analogues for the derived category of a commutative noetherian ring by Neeman [2011], and the stable module category of a finite group [Benson et al. 2012]. Any finite group gives rise to a finite group scheme, and we obtain an entirely new proof in that case.

Products of modules tend to be more complicated than coproducts. This is reflected by the fact that the classification of colocalising subcategories formally implies the classification of localising subcategories in terms of π-supports of G-modules; see [Benson et al. 2012, Theorem 9.7]. So Theorem 1.1 implies the classification result in our work presented in [Benson et al. 2016]. However, the arguments in the present work rely heavily on the tools developed in [loc. cit.], which, in turn, depend on the fundamental results and geometric techniques for the representation theory and cohomology of finite group schemes from [Suslin 2006; Suslin et al. 1997].

An essential ingredient in the proof of Theorem 1.1 is a family of G-modules, one arising from each π-point of G. We call them *point modules* and write $\Delta_G(\alpha)$, where $\alpha : K[t]/(t^p) \to KG$ is the corresponding π-point. They appear already in [Benson et al. 2016, Section 9] and play the role of residue fields in commutative algebra. Indeed, while they are not usually finite-dimensional, they are always endofinite in the sense of Crawley-Boevey [1991], as we prove in Proposition 3.8. It follows from results in [Benson et al. 2016] that the π-support of $\Delta_G(\alpha)$ is equal to the prime ideal $\mathfrak{p}$ corresponding to α, and that the localising subcategory generated by $\Delta_G(\alpha)$ is $\varGamma_{\mathfrak{p}}\, \mathsf{StMod}\, G$, the full subcategory of $\mathfrak{p}$-local and $\mathfrak{p}$-torsion objects.

In Theorem 4.4, we prove that $\Delta_G(\alpha)$ also *cogenerates* $\varLambda^{\mathfrak{p}}\, \mathsf{StMod}\, G$, the full subcategory of $\mathfrak{p}$-local and $\mathfrak{p}$-complete G-modules, in the sense of [Benson et al. 2012]. This result is an important step in the proof of Theorem 1.1, because the subcategories $\varLambda^{\mathfrak{p}}\, \mathsf{StMod}\, G$, as $\mathfrak{p}$ varies over $\operatorname{Proj} H^*(G, k)$, cogenerate $\mathsf{StMod}\, G$. From this it follows that *every* Hom closed colocalising subcategories of $\mathsf{StMod}\, G$ is cogenerated by point modules, which again highlights the special role played by them.

There is a parallel between point modules and standard objects in highest weight categories studied by Cline, Parshall and Scott [Cline et al. 1988]. This is explained towards the end of this article. The notation $\Delta_G(\alpha)$ reflects this connection.

2. Recollections

In this section we recall basic notions and results we will need about modules over finite group schemes. Our standard references are [Jantzen 2003; Waterhouse 1979]. For the later parts, and for the notation, we follow [Benson et al. 2016].

Let G be a finite group scheme over a field k. Thus G is an affine group scheme such that its coordinate algebra $k[G]$ is finite-dimensional as a k-vector space. The k-linear dual of $k[G]$ is a cocommutative Hopf algebra, called the group algebra of G, and denoted by kG. We identify G-modules with modules over the group algebra kG. The category of all (left) G-modules is denoted by $\mathsf{Mod}\, G$.

The stable module category $\mathsf{StMod}\, G$ is obtained from $\mathsf{Mod}\, G$ by identifying two morphisms between G-modules when they factor through a projective G-module. The tensor product of G-modules passes to $\mathsf{StMod}\, G$ and we obtain a compactly generated tensor triangulated category with suspension Ω^{-1}, the inverse of the syzygy functor. We use the notation $\underline{\mathrm{Hom}}_G(M, N)$ for the Hom-sets in $\mathsf{StMod}\, G$. For details, readers might consult [Carlson 1996, §5; Happel 1988, Chapter 1].

In the context of finite groups there is a duality theorem due to Tate [Cartan and Eilenberg 1956, Chapter XII, Theorem 6.4] that is helpful in computing morphisms in the stable category. In the proof of Lemma 4.1 we need an extension of this to finite group schemes, which is recalled below.

Duality. Given a k-vector space V, we set $V^\vee := \mathrm{Hom}_k(V, k)$ to be the dual vector space. If V is a G-module, then $V^\vee$ can also be endowed with a structure of a G-module using the Hopf algebra structure of kG.

Let G^{op} denote the opposite group scheme that is given by the group algebra $(kG)^{\mathrm{op}}$. Given a G^{op}-module M, we write $DM := \mathrm{Hom}_k(M, k)$ for the dual vector space considered as a G-module. Let

$$\tau = D \circ \mathrm{Tr} : \mathsf{stmod}\, G \xrightarrow{\sim} \mathsf{stmod}\, G$$

be the composition of the duality functor D and the transpose

$$\mathrm{Tr} : \mathsf{stmod}\, G \to \mathsf{stmod}\, G^{\mathrm{op}};$$

see [Skowroński and Yamagata 2011, Section III.4] for the definition. For any G-module M and finite-dimensional G-module N, there is a natural isomorphism of vector spaces

$$\underline{\mathrm{Hom}}_G(N, M)^\vee \cong \underline{\mathrm{Hom}}_G(M, \Omega^{-1}\tau N). \tag{2.1}$$

This isomorphism can be deduced from a formula of Auslander and Reiten [Auslander 1978, Proposition I.3.4] — see also [Krause 2003, Corollary, p. 269] — which yields the first isomorphism below:

$$\underline{\mathrm{Hom}}_G(N, M)^\vee \cong \mathrm{Ext}^1_G(M, \tau N) \cong \underline{\mathrm{Hom}}_G(M, \Omega^{-1}\tau N).$$

When kG is symmetric (in particular, whenever G is a finite group), we have $\tau N \cong \Omega^2 N$. This follows from [Skowroński and Yamagata 2011, Section IV.8] and reduces (2.1) to Tate duality.

Extending the base field. Let G be a finite group scheme over a field k. If K is a field extension of k, we write $K[G]$ for $K \otimes_k k[G]$, which is a commutative Hopf algebra over K. This defines a finite group scheme over K, denoted by G_K. We have a natural isomorphism $KG_K \cong K \otimes_k kG$ and we simplify notation by writing KG. The restriction functor

$$\operatorname{res}_k^K : \operatorname{Mod} G_K \to \operatorname{Mod} G$$

admits a left adjoint that sends a G-module M to

$$M_K := K \otimes_k M,$$

and a right adjoint sending M to

$$M^K := \operatorname{Hom}_k(K, M).$$

The next result tracks how these functors interact with taking tensors and modules of homomorphisms. We give proofs for lack of an adequate reference.

Lemma 2.2. *Let K be a field extension of k. For a G_K-module M and a G-module N, there are natural isomorphisms of G-modules*

$$\operatorname{res}_k^K (M \otimes_K N_K) \cong (\operatorname{res}_k^K M) \otimes_k N,$$
$$\operatorname{res}_k^K \operatorname{Hom}_K(M, N^K) \cong \operatorname{Hom}_k(\operatorname{res}_k^K M, N).$$

Proof. The first isomorphism is clear since the k-linear isomorphism

$$M \otimes_K (K \otimes_k N) \cong (M \otimes_K K) \otimes_k N \cong M \otimes_k N$$

is compatible with the diagonal G-actions.

The second isomorphism follows from the first one, because the functor

$$\operatorname{res}_k^K \operatorname{Hom}_K(M, (-)^K) \text{ is right adjoint to } \operatorname{res}_k^K (M \otimes_K (-)_K),$$

while the functor

$$\operatorname{Hom}_k(\operatorname{res}_k^K M, -) \text{ is right adjoint to } (\operatorname{res}_k^K M) \otimes_k -. \qquad \square$$

Subgroup schemes. For each subgroup scheme H of G restriction is a functor

$$\operatorname{res}_H^G : \operatorname{Mod} G \to \operatorname{Mod} H.$$

This has a right adjoint, called *induction*,

$$\operatorname{ind}_H^G : \operatorname{Mod} H \to \operatorname{Mod} G,$$

as described in [Jantzen 2003, Section I.3.3], and a left adjoint, called *coinduction*,

$$\operatorname{coind}_H^G : \mathsf{Mod}\, H \to \mathsf{Mod}\, G,$$

as described in [Jantzen 2003, Section I.8.14].

Lemma 2.3. *Let H be a subgroup scheme of G. For any H-module M and G-module N there are natural isomorphisms*

$$\begin{aligned}\operatorname{coind}_H^G(M \otimes_k \operatorname{res}_H^G N) &\cong (\operatorname{coind}_H^G M) \otimes_k N,\\ \operatorname{ind}_H^G \operatorname{Hom}_k(M, \operatorname{res}_H^G N) &\cong \operatorname{Hom}_k(\operatorname{coind}_H^G M, N).\end{aligned}$$

In particular, for $M = k$ these give isomorphisms

$$\begin{aligned}\operatorname{coind}_H^G \operatorname{res}_H^G N &\cong (\operatorname{coind}_H^G k) \otimes_k N,\\ \operatorname{ind}_H^G \operatorname{res}_H^G N &\cong \operatorname{Hom}_k(\operatorname{coind}_H^G k, N).\end{aligned}$$

Proof. Recalling that $\operatorname{coind}_H^G = kG \otimes_{kH} -$, the first isomorphism follows from associativity of tensor products:

$$\begin{aligned}\operatorname{coind}_H^G(M \otimes_k \operatorname{res}_H^G N) &\cong kG \otimes_{kH} (M \otimes_k \operatorname{res}_H^G N)\\ &\cong (kG \otimes_{kH} M) \otimes_k N\\ &\cong (\operatorname{coind}_H^G M) \otimes_k N.\end{aligned}$$

The second isomorphism follows from the first one, because the functor

$$\operatorname{ind}_H^G \operatorname{Hom}_k(M, -) \operatorname{res}_H^G \quad \text{is right adjoint to} \quad \operatorname{coind}_H^G(M \otimes_k -) \operatorname{res}_H^G,$$

while the functor

$$\operatorname{Hom}_k(\operatorname{coind}_H^G M, -) \;\text{ is right adjoint to }\; (\operatorname{coind}_H^G M) \otimes_k -. \qquad \square$$

Cohomology and π-points. Let k be a field of positive characteristic p and G a finite group scheme over k. We write $H^*(G, k)$ for the cohomology algebra of G and $\operatorname{Proj} H^*(G, k)$ for the set of its homogeneous prime ideals not containing $H^{\geqslant 1}(G, k)$, the elements of positive degree.

A *π-point* of G, defined over a field extension K of k, is a morphism of K-algebras

$$\alpha : K[t]/(t^p) \to KG$$

that factors through the group algebra of a unipotent abelian subgroup scheme of G_K, and such that KG is flat when viewed as a left (equivalently, as a right) module over $K[t]/(t^p)$ via α. Given such an α, restriction yields a functor

$$\alpha^* : \mathsf{Mod}\, KG \to \mathsf{Mod}\, K[t]/(t^p).$$

We write $H^*(\alpha)$ for the composition of homomorphisms of k-algebras

$$H^*(G,k)=\operatorname{Ext}^*_G(k,k)\xrightarrow{K\otimes_k -}\operatorname{Ext}^*_{G_K}(K,K)\xrightarrow{\alpha^*}\operatorname{Ext}^*_{K[t]/(t^p)}(K,K).$$

The radical of the ideal $\operatorname{Ker} H^*(\alpha)$ is a prime ideal in $H^*(G,k)$, and the assignment $\alpha\mapsto\sqrt{\operatorname{Ker} H^*(\alpha)}$ yields a bijection between the equivalence classes of π-points and $\operatorname{Proj} H^*(G,k)$; see [Friedlander and Pevtsova 2007, Theorem 3.6]. Recall that π-points $\alpha: K[t]/(t^p)\to KG$ and $\beta: L[t]/(t^p)\to LG$ are *equivalent* if for every G-module M the module $\alpha^*(M_K)$ is projective if and only if $\beta^*(M_L)$ is projective. In the sequel, we identify a prime in $\operatorname{Proj} H^*(G,k)$ and the corresponding equivalence class of π-points.

Given a point in $\operatorname{Proj} H^*(G,k)$, there is some flexibility in choosing a π-point representing it. This will be important in the sequel.

Remark 2.4. We call a group scheme $\mathscr{E}$ quasielementary if there is an isomorphism $\mathscr{E}\cong\mathbb{G}_{a(r)}\times E$, where $\mathbb{G}_{a(r)}$ is the r-th Frobenius kernel of the additive group $\mathbb{G}_a$ and E is an elementary abelian p-group.

By Proposition 4.2 of [Friedlander and Pevtsova 2005], given a π-point α : $K[t]/(t^p)\to KG$, there exists an equivalent π-point $\beta: K[t]/(t^p)\to KG$ that factors through a quasielementary subgroup scheme of G_K.

A point $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$ is *closed* if there is no point in $\operatorname{Proj} H^*(G,k)$ properly containing it as a prime ideal. Then there exists a π-point $\alpha: K[t]/(t^p)\to KG$ such that K is finite-dimensional over k; see Theorem 4.2 of [Friedlander and Pevtsova 2007]. In view of the preceding paragraph, one may choose an α that factors through a quasielementary subgroup scheme of G_K.

Local cohomology and completions. We recall from [Benson et al. 2008; 2012] the definition of local cohomology and completion for G-modules.

The algebra $H^*(G,k)$ acts on $\mathsf{StMod}\, G$. This means that for G-modules M and N there is a natural action of $H^*(G,k)$ on

$$\underline{\operatorname{Hom}}^*_G(M,N)=\bigoplus_{i\in\mathbb{Z}}\underline{\operatorname{Hom}}_G(\Omega^i M,N)$$

via the homomorphism of k-algebras

$$-\otimes_k M: H^*(G,k)=\operatorname{Ext}^*_G(k,k)\to\underline{\operatorname{Hom}}^*_G(M,M).$$

Fix $\mathfrak{p}\in\operatorname{Proj} H^*(G,k)$. There is a localisation functor $\mathsf{StMod}\, G\to\mathsf{StMod}\, G$ sending M to $M_\mathfrak{p}$ such that the natural morphism $M\to M_\mathfrak{p}$ induces an isomorphism

$$\underline{\operatorname{Hom}}^*_G(-,M)_\mathfrak{p}\xrightarrow{\sim}\underline{\operatorname{Hom}}^*_G(-,M_\mathfrak{p})$$

when restricted to finite-dimensional G-modules. A G-module M is called $\mathfrak{p}$-*local* if $M\xrightarrow{\sim}M_\mathfrak{p}$, and we write $(\mathsf{StMod}\, G)_\mathfrak{p}$ for the full subcategory of $\mathfrak{p}$-local G-modules. The module M is $\mathfrak{p}$-*torsion* if $M_\mathfrak{q}=0$ for all $\mathfrak{q}\in\operatorname{Spec} H^*(G,k)$ that do not contain $\mathfrak{p}$. There is a colocalisation functor $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{p})} : \mathsf{StMod}\, G \to \mathsf{StMod}\, G$ such that the natural morphism $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{p})}(M) \to M$ is an isomorphism if and only M is $\mathfrak{p}$-torsion. The functor $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{p})}$ admits a right adjoint, denoted by $\Lambda^{\mathcal{V}(\mathfrak{p})}$ and called $\mathfrak{p}$-*completion*. We say that M is $\mathfrak{p}$-*complete* if the natural map $M \to \Lambda^{\mathcal{V}(\mathfrak{p})} M$ is an isomorphism.

The functor $\mathit{\Gamma}_{\mathfrak{p}} : \mathsf{StMod}\, G \to \mathsf{StMod}\, G$ sending M to $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{p})}(M_{\mathfrak{p}})$ gives *local cohomology at* $\mathfrak{p}$. It has a right adjoint $\Lambda^{\mathfrak{p}} : \mathsf{StMod}\, G \to \mathsf{StMod}\, G$ that plays the role of *completion at* $\mathfrak{p}$ for modules over commutative rings.

Koszul objects and reduction to closed points. Given a cohomology class ζ in $H^*(G, k)$, let $k/\!\!/\zeta$ be a mapping cone of the morphism $k \to \Omega^{-d} k$ in $\mathsf{StMod}\, G$ defined by ζ. Note that $k/\!\!/\zeta \cong \Omega^{-d-1} L_\zeta$, where L_ζ is the Carlson module [1983] defined by ζ. For a homogeneous ideal $\mathfrak{a}$ in $H^*(G, k)$, we pick a system of homogeneous generators $\zeta_1, \dots, \zeta_n$, and define a Koszul object $k/\!\!/\mathfrak{a}$ to be

$$k/\!\!/\mathfrak{a} := k/\!\!/\zeta_1 \otimes_k \cdots \otimes_k k/\!\!/\zeta_n.$$

Observe that the map $k \to \Omega^{-d} k$ defined by ζ becomes an isomorphism when localised at any prime ideal $\mathfrak{p}$ of $H^*(G, k)$ not containing ζ. Given this, the next result is [Benson et al. 2016, Theorem 8.8].

Theorem 2.5. *Let* $\mathfrak{p}$ *be a point in* Proj $H^*(G, k)$. *There exists a field extension* L/k *and an ideal* $\mathfrak{q}$ *of* $H^*(G_L, L)$ *with radical* $\sqrt{\mathfrak{q}}$ *a closed point in* Proj $H^*(G_L, L)$ *lying over* $\mathfrak{p}$ *such that there is an isomorphism*

$$\operatorname{res}^L_k(L/\!\!/\mathfrak{q}) \cong (k/\!\!/\mathfrak{p})_{\mathfrak{p}}.$$

The construction of $L/\!\!/\mathfrak{q}$ involves a choice of generators for $\mathfrak{q}$, so the theorem effectively states that there exist an ideal $\mathfrak{q}$ *and* a choice of generators that produces the Koszul object with required properties. For details, see [Benson et al. 2016, Section 8].

Brown representability. Let C be a finite-dimensional G-module and I an injective $H^*(G, k)$-module. Recall that $H^*(G, k)$ acts on $\underline{\operatorname{Hom}}^*_G(C, M)$ for any M in $\mathsf{StMod}\, G$, and consider the contravariant functor

$$\operatorname{Hom}_{H^*(G,k)}(\underline{\operatorname{Hom}}^*_G(C, -), I) : \mathsf{StMod}\, G \to \mathsf{Ab}$$

This functor takes triangles to exact sequences and coproducts to products. Hence, by the contravariant version of Brown representability (see [Brown 1965] or [Neeman 1996]), there exists a G-module $T_C(I)$ such that

$$\operatorname{Hom}_{H^*(G,k)}(\underline{\operatorname{Hom}}^*_G(C, -), I) \cong \underline{\operatorname{Hom}}_G(-, T_C(I)). \tag{2.6}$$

We refer to [Benson et al. 2012; Benson and Krause 2002] for details about these modules.

Support and cosupport. The following definitions of π-support and π-cosupport of a G-module M are from [Friedlander and Pevtsova 2007] and [Benson et al. 2017], respectively. We set

$$\pi\text{-supp}_G(M) := \{\mathfrak{p} \in \operatorname{Proj} H^*(G,k) \mid \alpha_{\mathfrak{p}}^*(M_K) \text{ is not projective}\},$$
$$\pi\text{-cosupp}_G(M) := \{\mathfrak{p} \in \operatorname{Proj} H^*(G,k) \mid \alpha_{\mathfrak{p}}^*(M^K) \text{ is not projective}\}.$$

Here $\alpha_{\mathfrak{p}} : K[t]/(t^p) \to KG$ denotes a π-point corresponding to $\mathfrak{p}$. Both π-supp and π-cosupp are well-defined on the equivalence classes of π-points [Benson et al. 2017, Theorem 3.1].

The local cohomology functors $\Gamma_{\mathfrak{p}}$ and their right adjoints $\Lambda^{\mathfrak{p}}$ yield alternative notions of support and cosupport for a G-module M; see [Benson et al. 2008; Benson et al. 2012]. We set

$$\operatorname{supp}_G(M) := \{\mathfrak{p} \in \operatorname{Proj} H^*(G,k) \mid \Gamma_{\mathfrak{p}} M \neq 0\},$$
$$\operatorname{cosupp}_G(M) := \{\mathfrak{p} \in \operatorname{Proj} H^*(G,k) \mid \Lambda^{\mathfrak{p}} M \neq 0\}.$$

It is an important fact that these notions agree with the ones defined via π-points. This will be used freely throughout this work.

Theorem 2.7 [Benson et al. 2016, Theorems 6.1 and 9.3]. *For every G-module M there are equalities*

$$\pi\text{-supp}_G(M) = \operatorname{supp}_G(M) \quad \textit{and} \quad \pi\text{-cosupp}_G(M) = \operatorname{cosupp}_G(M).$$

For ease of reference we recall basic facts concerning support and cosupport.

Remark 2.8. Let M and N be G-modules.

(1) M is projective if and only if $\operatorname{supp}_G(M) = \varnothing$ if and only if $\operatorname{cosupp}_G(M) = \varnothing$.

(2) $\operatorname{supp}_G(M)$ and $\operatorname{cosupp}_G(M)$ have the same maximal elements with respect to inclusion.

(3) $\operatorname{supp}_G(M \otimes_k N) = \operatorname{supp}_G(M) \cap \operatorname{supp}_G(N)$.

(4) $\operatorname{cosupp}_G \operatorname{Hom}_k(M, N) = \operatorname{supp}_G(M) \cap \operatorname{cosupp}_G(N)$.

(5) $\operatorname{supp}_G(k) = \operatorname{Proj} H^*(G,k) = \operatorname{cosupp}_G(k)$.

Keeping in mind Theorem 2.7, parts (1) and (2) are recombinations of [Benson et al. 2016, Theorem 5.3 and Corollary 9.4]. Parts (3) and (4) are from [Benson et al. 2017, Theorem 4.4], while (5) is contained in [Benson et al. 2017, Lemma 4.5].

Remark 2.9. For an ideal $\mathfrak{a}$ in $H^*(G,k)$ we write $\mathcal{V}(\mathfrak{a})$ for the closed subset of those points in $\operatorname{Proj} H^*(G,k)$ corresponding to homogeneous prime ideals containing $\mathfrak{a}$.

Let $\zeta_1, \ldots, \zeta_n$ be a system of homogeneous generators of an ideal $\mathfrak{a} \subset H^*(G,k)$. By a theorem of Carlson [1983], one has $\operatorname{supp}_G(k/\!\!/\zeta) = \mathcal{V}(\zeta)$ for any $\zeta \in H^d(G,k)$. The tensor product property, recalled in Remark 2.8, now implies that

$$\operatorname{supp}_G(k/\!\!/\mathfrak{a}) = \mathcal{V}(\zeta_1) \cap \cdots \cap \mathcal{V}(\zeta_n) = \mathcal{V}(\mathfrak{a}).$$

In particular, for L and $\mathfrak{q}$ as in Theorem 2.5, one gets

$$\operatorname{supp}_{G_L}(L/\!\!/\mathfrak{q}) = \mathcal{V}(\mathfrak{q}) = \{\sqrt{\mathfrak{q}}\} \subset \operatorname{Proj} H^*(G_L, L),$$

since $\sqrt{\mathfrak{q}}$ is a closed point in $\operatorname{Proj} H^*(G_L, L)$.

3. Point modules

In this section we discuss a distinguished class of G-modules that correspond to a π-point. Later on we will see that these modules serve as cogenerators of colocalising subcategories.

Point modules. Fix a π-point $\alpha : k[t]/(t^p) \to kG$. The restriction functor

$$\alpha^* : \operatorname{Mod} G \to \operatorname{Mod} k[t]/(t^p)$$

admits a left adjoint and a right adjoint,

$$\alpha_* := kG \otimes_{k[t]/(t^p)} - \quad \text{and} \quad \alpha_! := \operatorname{Hom}_{k[t]/(t^p)}(kG, -).$$

These functors are isomorphic, as the next result asserts.

Theorem 3.1. *For any π-point $\alpha : k[t]/(t^p) \to kG$ and $k[t]/(t^p)$-module M, there is a natural isomorphism of G-modules*

$$\alpha_*(M) \cong \alpha_!(M).$$

Proof. It is convenient to set $R := k[t]/(t^p)$. It is easy to verify that the R-module $\operatorname{Hom}_k(R,k)$ is isomorphic to R. This will be used further below. We will also use the fact that kG is a Frobenius algebra, that is to say that there is an isomorphism of G-modules

$$kG \cong \operatorname{Hom}_k(kG, k).$$

See [Jantzen 2003, Lemma I.8.7; Skowroński and Yamagata 2011, Chapter VI, Theorem 3.6]. This justifies the third step in the following chain of isomorphisms of G-modules:

$$\operatorname{Hom}_R(kG, R) \cong \operatorname{Hom}_R(kG, \operatorname{Hom}_k(R,k)) \cong \operatorname{Hom}_k(kG,k) \cong kG. \tag{3.2}$$

The second is standard adjunction.

We are now ready to justify the stated result. Consider first the case when G is abelian. Then kG and $\operatorname{Hom}_R(kG,R)$ also have G^{op}-actions. As G is abelian, the isomorphism (3.2) is compatible with these structures. It follows that it is also compatible with the induced R^{op}-actions on kG and $\operatorname{Hom}_R(kG, R)$. This justifies the second isomorphism below:

$$\alpha_!(M) = \operatorname{Hom}_R(kG, M) \cong \operatorname{Hom}_R(kG, R) \otimes_R M \cong kG \otimes_R M = \alpha_*(M).$$

The first isomorphism holds because kG is a finitely generated projective R-module. The composition of the maps is the desired isomorphism.

Now let G be an arbitrary finite group scheme. By definition, the π-point α factors as $R \xrightarrow{\beta} kU \hookrightarrow kG$, where U is an unipotent abelian subgroup scheme of G. Note that $\beta_* = \beta_!$ by what we have already verified, since U is abelian. Observing that $\alpha_* = \operatorname{coind}_U^G \beta_*$ and $\alpha_! = \operatorname{ind}_U^G \beta_!$, it thus remains to show that $\operatorname{coind}_U^G \cong \operatorname{ind}_U^G$. By [Jantzen 2003, Proposition I.8.17], there is an isomorphism

$$\operatorname{coind}_U^G(M) \cong \operatorname{ind}_U^G(M \otimes_k (\delta_G){\downarrow_U} \delta_U^{-1}),$$

where δ_G and δ_U are certain characters of G and U, respectively. Since U is a unipotent group scheme, it has no nontrivial characters; see [Waterhouse 1979, Section 8.3], for example. This yields the last claim and therefore the proof is complete. □

Definition 3.3. Let K be a field extension of k and $\alpha : K[t]/(t^p) \to KG$ a π-point. We call the G-module

$$\Delta_G(\alpha) := \operatorname{res}_k^K \alpha_*(K) \cong \operatorname{res}_k^K \alpha_!(K)$$

the *point module* corresponding to α.

As an example, we describe the point modules for the Klein four group, following the description of the π-points in [Friedlander and Pevtsova 2007, Example 2.3]; see also [Benson et al. 2017, Example 3.6].

Example 3.4. Let $V = \mathbb{Z}/2 \times \mathbb{Z}/2$ and k a field of characteristic two. The group algebra kV is isomorphic to $k[x, y]/(x^2, y^2)$, where $x + 1$ and $y + 1$ correspond to the generators of V, and $\operatorname{Proj} H^*(V, k) \cong \mathbb{P}_k^1$. A kV-module M is given by a k-vector space together with two k-linear endomorphisms x_M and y_M, representing the action of x and y, respectively.

For each closed point $\mathfrak{p} \in \mathbb{P}_k^1$ there is some finite field extension K of k such that $\mathbb{P}_K^1$ contains a rational point $[a, b]$ over $\mathfrak{p}$ (using homogeneous coordinates). The π-point corresponding to $\mathfrak{p}$ is represented by the map of K-algebras

$$K[t]/(t^p) \to K[x, y]/(x^2, y^2), \quad \text{where } t \mapsto ax + by,$$

and the corresponding point module is given by $\Delta = K \oplus K$ together with

$$x_\Delta = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} \quad \text{and} \quad y_\Delta = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}.$$

Now let K denote the field of rational functions in a variable s. The generic point of $\mathbb{P}^1_k$ then corresponds to the map of K-algebras

$$K[t]/(t^p) \to K[x, y]/(x^2, y^2), \quad \text{where } t \mapsto x + sy,$$

and the corresponding point module is given by $\Delta = K \oplus K$ together with

$$x_\Delta = \begin{bmatrix} 0 & 0 \\ s & 0 \end{bmatrix} \quad \text{and} \quad y_\Delta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The next example illustrates that the G-module $\Delta_G(\alpha)$ depends on α and not only on the point in $\operatorname{Proj} H^*(G, k)$ that it represents.

Example 3.5. Let k be a field of characteristic $p \geq 3$ and set $G := \mathbb{Z}/p \times \mathbb{Z}/p$. Thus, $kG = k[x, y]/(x^p, y^p)$ and $\operatorname{Proj} H^*(G, k) = \mathbb{P}^1_k$. The homomorphism

$$\alpha_\lambda : k[t]/(t^p) \to kG, \quad \text{where } t \mapsto x - \lambda y^2,$$

defines a π-point for any $\lambda \in k$, corresponding to the same point in $\mathbb{P}^1_k$, namely $[1, 0]$. On the other hand, the point modules

$$\Delta_G(\alpha_\lambda) \cong k[x, y]/(x - \lambda y^2, y^p)$$

are pairwise nonisomorphic; for example, their annihilators differ. They are also indecomposable, because they are cyclic and kG is a local ring.

The next example shows that point modules need not be indecomposable.

Example 3.6. Let k be a field of characteristic 3 and set $G := \Sigma_3 \times \mathbb{Z}/3$. The π-point $\alpha : k[t]/(t^3) \to kG$ given by the inclusion $\mathbb{Z}/3 \hookrightarrow G$ as a direct factor yields a point module $\Delta_G(\alpha)$ that decomposes into two nonisomorphic indecomposable G-modules, because it is isomorphic to $k\Sigma_3$.

Endofinite modules. Let G be a group scheme defined over k. A point module defined over a field extension K is finite-dimensional, as a G-module, if and only if K is finite-dimensional over k. Nonetheless, point modules always enjoy a strong finiteness property because they arise as restrictions of finite-dimensional modules.

Let A be any ring. Following Crawley-Boevey [1991; 1992], we say that an A-module M is *endofinite* if it has finite composition length when viewed as a module over its endomorphism ring $\operatorname{End}_A(M)$. The following result, due to Crawley-Boevey, collects some of the basic properties of endofinite modules. The proof employs the fact that endofinite modules are Σ-pure-injective.

Theorem 3.7. *An indecomposable endofinite module has a local endomorphism ring and any endofinite module can be written essentially uniquely as a direct sum of indecomposable endofinite modules. Conversely, a direct sum of endofinite* *modules is endofinite if and only if there are only finitely many isomorphism classes of indecomposables involved.*

The class of endofinite modules is closed under finite direct sums, direct summands, and arbitrary products or direct sum of copies of one module.

Proof. See [Crawley-Boevey 1991, Section 1.1; 1992, Section 4]. □

For an A-module M, we write $\operatorname{Add}(M)$ for the full subcategory of A-modules that are direct summands of direct sums of copies of M. Analogously, $\operatorname{Prod}(M)$ denotes the subcategory of all direct summands of products of copies of M. For an endofinite module M it follows from Theorem 3.7 that $\operatorname{Add}(M)$ and $\operatorname{Prod}(M)$ coincide: they consist of all direct sums of indecomposable direct summands of M. This observation explains the formal part of the following proposition:

Proposition 3.8. *For any π-point α of G, the G-module $\Delta_G(\alpha)$ is endofinite and there is an equality*

$$\operatorname{Add}(\Delta_G(\alpha)) = \operatorname{Prod}(\Delta_G(\alpha)).$$

Proof. Let $\alpha : K[t]/(t^p) \to KG$ be the given π-point. Then $\Delta_G(\alpha)$ is a kG-K-bimodule and there is a homomorphism of rings $K \to \operatorname{End}_G(\Delta_G(\alpha))$. In particular, $\dim_K(\Delta_G(\alpha))$ is an upper bound for the length of $\Delta_G(\alpha)$ as a module over $\operatorname{End}_G(\Delta_G(\alpha))$. Since one has inequalities

$$\dim_K(\Delta_G(\alpha)) = (1/p)\dim_K(KG) \le \dim_K(KG) < \infty,$$

it follows that $\Delta_G(\alpha)$ is endofinite. The remaining assertion is by Theorem 3.7. □

Support and cosupport. Next we explain how point modules can be used to compute support and cosupport; this is partly why we are interested in them.

Proposition 3.9. *Let α be a π-point corresponding to $\mathfrak{p} \in \operatorname{Proj} H^*(G, k)$ and M a G-module. The following statements are equivalent*:

(1) $\mathfrak{p} \notin \operatorname{cosupp}_G(M)$.

(2) $\operatorname{Hom}_k(\Delta_G(\alpha), M)$ *is projective.*

(3) $\underline{\operatorname{Hom}}_G(\Delta_G(\alpha), M) = 0$.

(4) $\underline{\operatorname{Hom}}^*_G(\Delta_G(\alpha), M) = 0$.

Proof. The equivalences (1) $\iff$ (2) $\iff$ (3) are [Benson et al. 2016, Lemma 9.2].

(1) $\iff$ (4): With α the map $K[t]/(t^p) \to KG$, adjunctions yield isomorphisms

$$\underline{\operatorname{Hom}}^*_G(\operatorname{res}^K_k \alpha_*(K), M) \cong \underline{\operatorname{Hom}}^*_{G_K}(\alpha_*(K), M^K) \cong \underline{\operatorname{Hom}}^*_{K[t]/(t^p)}(K, \alpha^*(M^K)).$$

Clearly, the right-hand term vanishes if and only if $\alpha^*(M^K)$ is projective. □

Here is the analogous statement for supports. As in the context of commutative rings, one can use also tensor products with the point modules to detect support.

Proposition 3.10. *Let α be a π-point corresponding to $\mathfrak{p} \in \operatorname{Proj} H^*(G, k)$ and M a G-module. The following statements are equivalent*:

(1) $\mathfrak{p} \notin \operatorname{supp}_G(M)$.

(2) $\Delta_G(\alpha) \otimes_k M$ *is projective.*

(3) $\operatorname{Hom}_k(M, \Delta_G(\alpha))$ *is projective.*

(4) $\underline{\operatorname{Hom}}_G(M, \Delta_G(\alpha)) = 0$.

(5) $\underline{\operatorname{Hom}}^*_G(M, \Delta_G(\alpha)) = 0$.

Proof. (1) $\iff$ (2): Since $\operatorname{supp}_G(\Delta_G(\alpha)) = \{\mathfrak{p}\}$ by [Benson et al. 2016, Lemma 9.1], Remark 2.8(3) yields the first equivalence below:

$$\mathfrak{p} \notin \operatorname{supp}_G(M) \iff \operatorname{supp}_G(\Delta_G(\alpha) \otimes_k M) = \varnothing \iff \Delta_G(\alpha) \otimes_k M \text{ is projective.}$$

The second one holds because support detects projectivity, by Remark 2.8(1).

(1) $\iff$ (4): With α the map $K[t]/(t^p) \to KG$, adjunctions yield isomorphisms

$$\underline{\operatorname{Hom}}_G(M, \operatorname{res}^K_k \alpha_!(K)) \cong \underline{\operatorname{Hom}}_{G_K}(M_K, \alpha_!(K)) \cong \underline{\operatorname{Hom}}_{K[t]/(t^p)}(\alpha^*(M_K), K).$$

Clearly, the right-hand term vanishes if and only if $\alpha^*(M_K)$ is projective.

(1) $\iff$ (5): This is analogous to (1) $\iff$ (4).

(1) $\Longrightarrow$ (3): When $\mathfrak{p}$ is not in $\operatorname{supp}_G(M)$, it is not in $\operatorname{supp}_G(C \otimes_k M)$ for any finite-dimensional G-module C, by Remark 2.8(3). Thus, the already-established equivalence of conditions (1) and (4) yields that

$$\underline{\operatorname{Hom}}_G(C, \operatorname{Hom}_k(M, \Delta_G(\alpha))) \cong \underline{\operatorname{Hom}}_G(C \otimes_k M, \Delta_G(\alpha)) = 0.$$

Therefore $\operatorname{Hom}_k(M, \Delta_G(\alpha))$ is projective.

(3) $\Longrightarrow$ (4): This is clear. ∎

In the next result, the claim about the support of $\Delta_G(\alpha)$ is from [Benson et al. 2016, Lemma 9.1], and has been used in the proofs of the Propositions 3.9 and 3.10.

Corollary 3.11. *Let α be a π-point of G. A π-point β of G is equivalent to α if and only if $\underline{\operatorname{Hom}}^*_G(\Delta_G(\beta), \Delta_G(\alpha)) \neq 0$. In particular, there are equalities*

$$\operatorname{supp}_G(\Delta_G(\alpha)) = \{\mathfrak{p}\} = \operatorname{cosupp}_G(\Delta_G(\alpha)),$$

where $\mathfrak{p}$ is the point in $\operatorname{Proj} H^*(G, k)$ *corresponding to α.*

Proof. If β corresponds to a point $\mathfrak{q}$ in $\operatorname{Proj} H^*(G, k)$, then $\operatorname{supp}_G(\Delta_G(\beta)) = \{\mathfrak{q}\}$ by [loc. cit.], so Proposition 3.10 yields that $\underline{\operatorname{Hom}}^*_G(\Delta_G(\beta), \Delta_G(\alpha))$ is nonzero precisely when $\mathfrak{q} = \mathfrak{p}$. Given this, it follows from Proposition 3.9 that the cosupport of $\Delta_G(\alpha)$ is $\{\mathfrak{p}\}$. ∎

4. $\mathfrak{p}$-local and $\mathfrak{p}$-complete objects

The proof of Theorem 1.1 amounts to showing that for any homogeneous prime ideal $\mathfrak{p}$ of $H^*(G,k)$ the $\mathfrak{p}$-local and $\mathfrak{p}$-complete objects in $\mathsf{StMod}\,G$ form a minimal Hom closed colocalising subcategory. Here, a Hom closed colocalising subcategory $\mathsf{C} \subseteq \mathsf{StMod}\,G$ is *minimal* if $\mathsf{C}' \subseteq \mathsf{C}$ implies $\mathsf{C}' = 0$ or $\mathsf{C}' = \mathsf{C}$ for any Hom closed colocalising subcategory $\mathsf{C}' \subseteq \mathsf{StMod}\,G$.

$\mathfrak{p}$-local and $\mathfrak{p}$-complete objects. We recall from [Benson et al. 2008; 2012] the definitions and basic facts about $\mathfrak{p}$-local and $\mathfrak{p}$-complete objects in $\mathsf{StMod}\,G$.

Fix $\mathfrak{p} \in \operatorname{Proj} H^*(G,k)$. We write $\varGamma_{\mathfrak{p}}\,\mathsf{StMod}\,G$ for the full subcategory of G-modules M such that $\varGamma_{\mathfrak{p}}(M) \cong M$ and have, from Corollary 5.9 of [Benson et al. 2008],

$$\varGamma_{\mathfrak{p}}\,\mathsf{StMod}\,G = \big\{M \in \mathsf{StMod}\,G \mid \operatorname{supp}_G(M) \subseteq \{\mathfrak{p}\}\big\}.$$

From [Benson et al. 2012, Corollaries 4.8 and 4.9], it follows that a G-module M satisfies $\varLambda^{\mathfrak{p}}(M) \cong M$ if and only if M is $\mathfrak{p}$-local and $\mathfrak{p}$-complete, and that

$$\varLambda^{\mathfrak{p}}\,\mathsf{StMod}\,G = \big\{M \in \mathsf{StMod}\,G \mid \operatorname{cosupp}_G(M) \subseteq \{\mathfrak{p}\}\big\}.$$

Note that the adjoint pair $(\varGamma_{\mathfrak{p}}, \varLambda^{\mathfrak{p}})$ restricts by [Benson et al. 2012, Proposition 5.1] to an equivalence

$$\varGamma_{\mathfrak{p}}\,\mathsf{StMod}\,G \xrightarrow{\sim} \varLambda^{\mathfrak{p}}\,\mathsf{StMod}\,G.$$

Cogenerators for $\mathfrak{p}$-local and $\mathfrak{p}$-complete objects. Given a set T of G-modules, let $\mathsf{Coloc}(T)$ be the smallest colocalising subcategory of $\mathsf{StMod}\,G$ that contains T. We say that T *cogenerates* a class C of G-modules if $\mathsf{C} \subseteq \mathsf{Coloc}(T)$. The class C is *Hom closed* if, for every pair of G-modules M and N with $N \in \mathsf{C}$, we have $\operatorname{Hom}_k(M,N) \in \mathsf{C}$. We write $\mathsf{Coloc}^{\mathrm{Hom}}(T)$ for the smallest Hom closed colocalising subcategory that contains T.

An object T is a *perfect cogenerator* of a colocalising subcategory $\mathsf{C} \subseteq \mathsf{StMod}\,G$ if the following holds:

(1) If M is an object in C and $\underline{\operatorname{Hom}}_G(M,T) = 0$ then $M = 0$.

(2) If a countable family of morphisms $M_i \to N_i$ in C is such that, for all i,
$$\underline{\operatorname{Hom}}_G(N_i,T) \to \underline{\operatorname{Hom}}_G(M_i,T)$$
is surjective, then so is the induced map
$$\underline{\operatorname{Hom}}_G\Big(\prod_i N_i, T\Big) \to \underline{\operatorname{Hom}}_G\Big(\prod_i M_i, T\Big).$$

Any perfect cogenerator is a cogenerator; see [Benson et al. 2012, Section 5].

Recall from Remark 2.4 that any closed point of $\operatorname{Proj} H^*(G,k)$ is represented by a π-point $\alpha : K[t]/t^p \to KG$ defined over a finite field extension K/k.

Lemma 4.1. *Let $\alpha : K[t]/(t^p) \to KG$ be a π-point representing $\mathfrak{p} \in \operatorname{Proj} H^*(G, k)$. If K is finite-dimensional over k, then $\Delta_G(\alpha)$ perfectly cogenerates $\Lambda^{\mathfrak{p}} \operatorname{StMod} G$.*

Proof. We check the conditions (1) and (2) for $\Delta_G(\alpha)$.

(1) If $M \in \Lambda^{\mathfrak{p}} \operatorname{StMod} G$ is nonzero, then $\operatorname{cosupp}_G(M) = \{\mathfrak{p}\}$ and hence $\mathfrak{p}$ is in $\operatorname{supp}_G(M)$ by Remark 2.8(2). Thus, $\underline{\operatorname{Hom}}_G(M, \Delta_G(\alpha)) \neq 0$ by Proposition 3.10.

(2) Since extension of scalars is left adjoint to restriction of scalars, we have

$$\underline{\operatorname{Hom}}_G(M, \Delta_G(\alpha)) \cong \underline{\operatorname{Hom}}_{G_K}(M_K, \alpha_*(K)).$$

As $\alpha_*(K)$ is finite-dimensional as a G_K-module, using the duality isomorphism (2.1) we may rewrite the right-hand term as

$$\underline{\operatorname{Hom}}_{G_K}(\tau^{-1}\Omega(\alpha_*(K)), M_K)^\vee.$$

So $\underline{\operatorname{Hom}}_G(N, \Delta_G(\alpha)) \to \underline{\operatorname{Hom}}_G(M, \Delta_G(\alpha))$ is surjective if and only if

$$\underline{\operatorname{Hom}}_{G_K}(\tau^{-1}\Omega(\alpha_*(K)), M_K) \to \underline{\operatorname{Hom}}_{G_K}(\tau^{-1}\Omega(\alpha_*(K)), N_K)$$

is injective. It remains to observe that $M \mapsto M_K$ preserves products, as K is finite-dimensional over k. $\square$

Let I be an injective $H^*(G, k)$-module and C a finite-dimensional G-module. In what follows, we use the representing objects $T_C(I)$ and the Koszul objects $k/\!\!/\mathfrak{p}$ defined in Section 1.

Lemma 4.2. *Fix a point $\mathfrak{p}$ in $\operatorname{Proj} H^*(G, k)$ and I an injective $H^*(G, k)$-module.*

(1) *For any finite-dimensional G-modules C and M, there is a natural isomorphism*

$$\operatorname{Hom}_k(M, T_C(I)) \cong T_{\operatorname{Hom}_k(M,C)}(I).$$

(2) *With I the injective envelope of $H^*(G, k)/\mathfrak{p}$, the modules $\operatorname{Hom}_k(k/\!\!/\mathfrak{p}, T_C(I))$, as C varies over the simple G-modules, perfectly cogenerate $\Lambda^{\mathfrak{p}} \operatorname{StMod} G$.*

Proof. Recall that $(-)^\vee$ denotes the functor $\operatorname{Hom}_k(-, k)$. For a G-module M, we consider $M^\vee$ with the diagonal G-action, and we have

$$\operatorname{Hom}_k(M, -) \cong - \otimes_k M^\vee$$

when M is finite-dimensional. Combining this with standard adjunctions and the definition of $T_C(I)$ gives the following isomorphisms, which justify (1):

$$\begin{aligned}
\underline{\operatorname{Hom}}_G(-, \operatorname{Hom}_k(M, T_C(I))) &\cong \underline{\operatorname{Hom}}_G(- \otimes_k M, T_C(I)) \\
&\cong \operatorname{Hom}_{H^*(G,k)}(\underline{\operatorname{Hom}}^*_G(C, - \otimes_k M), I) \\
&\cong \operatorname{Hom}_{H^*(G,k)}(\underline{\operatorname{Hom}}^*_G(C \otimes_k M^\vee, -), I) \\
&\cong \operatorname{Hom}_{H^*(G,k)}(\underline{\operatorname{Hom}}^*_G(\operatorname{Hom}_k(M, C), -), I) \\
&\cong \underline{\operatorname{Hom}}_G(-, T_{\operatorname{Hom}_k(M,C)}(I)).
\end{aligned}$$

As to (2), given the isomorphism in (1) applied to $M = k/\!\!/\mathfrak{p}$, one can deduce the desired result by mimicking the proof of [Benson et al. 2012, Proposition 5.4]. □

For the next result we employ the reduction to closed points technique from Section 1.

Proposition 4.3. *Let $\mathfrak{p}$ be a point in $\operatorname{Proj} H^*(G,k)$ and M a $\mathfrak{p}$-local G-module. There exists a field extension L/k and an ideal $\mathfrak{q}$ in $H^*(G_L, L)$ with radical a closed point in $\operatorname{Proj} H^*(G_L, L)$ lying over $\mathfrak{p}$ such that $\operatorname{res}^L_k \operatorname{Hom}_L(L/\!\!/\mathfrak{q}, M^L)$ and $\operatorname{Hom}_k(k/\!\!/\mathfrak{p}, M)$ are isomorphic as G-modules.*

Proof. By Theorem 2.5, we can find L and $\mathfrak{q}$ such that there is an isomorphism $\operatorname{res}^L_k(L/\!\!/\mathfrak{q}) \cong (k/\!\!/\mathfrak{p})_{\mathfrak{p}}$. Thus there are isomorphisms

$$\begin{aligned}\operatorname{res}^L_k \operatorname{Hom}_L(L/\!\!/\mathfrak{q}, M^L) &\cong \operatorname{Hom}_k(\operatorname{res}^L_k(L/\!\!/\mathfrak{q}), M)\\ &\cong \operatorname{Hom}_k((k/\!\!/\mathfrak{p})_{\mathfrak{p}}, M)\\ &\cong \operatorname{Hom}_k(k/\!\!/\mathfrak{p}, M).\end{aligned}$$

The first one follows from Lemma 2.2 and the last one holds as M is $\mathfrak{p}$-local. □

In what follows, $\mathsf{Thick}(M)$ denotes the thick subcategory of $\mathsf{StMod}\, G$ generated by a G-module M.

Theorem 4.4. *Given $\mathfrak{p} \in \operatorname{Proj} H^*(G,k)$, there exists a π-point $\alpha : K[t]/(t^p) \to KG$ corresponding to $\mathfrak{p}$ that factors through a quasielementary subgroup scheme of G_K and has the following properties*:

(1) $\Delta_G(\alpha)$ *is a compact object in* $(\mathsf{StMod}\, G)_{\mathfrak{p}}$.

(2) $\mathsf{Coloc}(\Delta_G(\alpha)) = \varLambda^{\mathfrak{p}} \mathsf{StMod}\, G$.

Proof. Let L and $\mathfrak{q}$ be as in Proposition 4.3, and let $\mathfrak{m} = \sqrt{\mathfrak{q}}$. Since $\mathfrak{m}$ is a closed point in $\operatorname{Proj} H^*(G_L, L)$, there exists a finite extension K of L and a π-point $\alpha : K[t]/(t^p) \to KG$ of G_L corresponding to $\mathfrak{m}$, and factoring through a quasielementary subgroup scheme of G_K; see Remark 2.4. It then follows directly from the definitions that α corresponds to $\mathfrak{p}$, when viewed as a π-point of G.

(1) Set $M := L/\!\!/\mathfrak{q}$. This is a finite-dimensional G_L-module with support $\{\mathfrak{m}\}$; see Remark 2.9. From the construction it is clear that the G_L-module $\operatorname{res}^K_L \alpha_*(K)$ is also finite-dimensional and has support $\{\mathfrak{m}\}$. Thus the classification [Benson et al. 2016, Corollary 10.2] of tensor closed thick subcategories of $\mathsf{stmod}\, G_L$ yields that $\operatorname{res}^K_L \alpha_*(K)$ is in $\mathsf{Thick}^{\otimes}(M)$. Any simple G_L-module is a direct summand of S_L, where S is the sum of representatives of isomorphism classes of simple G-modules, so one gets

$$\operatorname{res}^K_L \alpha_*(K) \in \mathsf{Thick}(M \otimes_L S_L).$$

Applying res_k^L and using Lemma 2.2, one then gets that

$$\Delta_G(\alpha) = \operatorname{res}_k^L \operatorname{res}_L^K \alpha_*(K) \in \mathsf{Thick}((\operatorname{res}_k^L M) \otimes_k S).$$

It remains to verify that $(\operatorname{res}_k^L M) \otimes_k S$ is a compact object in $(\mathsf{StMod}\, G)_{\mathfrak{p}}$. To this end, note that there are isomorphisms

$$(\operatorname{res}_k^L M) \otimes_k S \cong (k/\!\!/\mathfrak{p})_{\mathfrak{p}} \otimes_k S \cong (k/\!\!/\mathfrak{p} \otimes_k S)_{\mathfrak{p}},$$

where the first one is by Theorem 2.5 and the second is by [Benson et al. 2008, Theorem 8.2], for example. It remains to note that $k/\!\!/\mathfrak{p} \otimes_k S$ is a finite-dimensional G-module and hence compact in $\mathsf{StMod}\, G$, so that its localisation at $\mathfrak{p}$ is compact in $(\mathsf{StMod}\, G)_{\mathfrak{p}}$.

(2) Let I denote the injective envelope of the $H^*(G,k)$-module $H^*(G,k)/\mathfrak{p}$. Since $\operatorname{supp}_{G_L}(L/\!\!/\mathfrak{q}) = \{\mathfrak{m}\}$, Remark 2.8(4) implies that for any finite-dimensional G-module C the module $\operatorname{Hom}_L(L/\!\!/\mathfrak{q}, T_C(I)^L)$ belongs to $\Lambda^{\mathfrak{m}}\, \mathsf{StMod}\, G_L$. Given the choice of α, Lemma 4.1 thus implies that this module is cogenerated by $\Delta_{G_L}(\alpha)$. So, by Proposition 4.3, the G-module $\operatorname{res}_k^L \Delta_{G_L}(\alpha)$, that is to say $\Delta_G(\alpha)$, cogenerates $\operatorname{Hom}_k(k/\!\!/\mathfrak{p}, T_C(I))$. It remains to apply Lemma 4.2(2). □

Minimality. Next we prove that $\Lambda^{\mathfrak{p}}\, \mathsf{StMod}\, G$ is a minimal Hom closed colocalising subcategory. This requires further preparation.

Lemma 4.5. *Let K be a field extension of k and H a subgroup scheme of G_K. Set $F = \operatorname{res}_k^K \operatorname{coind}_H^{G_K}(K)$. If M is a G-module then*

$$\operatorname{res}_k^K \operatorname{ind}_H^{G_K} \operatorname{res}_H^{G_K}(M^K) = \operatorname{Hom}_k(F, M).$$

When K is a finite extension of k, the G-module F is finite-dimensional over k.

Proof. The desired result is a consequence of the isomorphisms

$$\begin{aligned}\operatorname{res}_k^K \operatorname{ind}_H^{G_K} \operatorname{res}_H^{G_K}(M^K) &\cong \operatorname{res}_k^K \operatorname{Hom}_K(\operatorname{coind}_H^{G_K}(K), M^K)\\ &\cong \operatorname{Hom}_k(\operatorname{res}_k^K \operatorname{coind}_H^{G_K}(K), M).\end{aligned}$$

The first one follows from Lemma 2.3 and the second from Lemma 2.2. The last assertion follows from the fact that, in general, there are inequalities

$$\dim_K \operatorname{coind}_H^{G_K}(K) = \frac{\dim_K(KG)}{\dim_K(KH)} \le \dim_K(KG),$$

and hence the number on the left is finite. □

Lemma 4.6. *Given a quasielementary group scheme $\mathcal{E}$ over K and a π-point $\beta : K[t]/(t^p) \to K\mathcal{E}$, for any $\mathcal{E}$-module M the $\mathcal{E}$-module $\beta_!\beta^*(M)$ is in* $\mathsf{Thick}(M)$.

Proof. Note that neither β^* nor $\beta_!$ involve the coproduct on $\mathscr{E}$, so we may change that and assume that KU is the group algebra of an elementary abelian p-group and that β is the inclusion $KH \to K\mathscr{E}$, where H is a cyclic subgroup $\mathscr{E}$. Lemma 4.5 then yields that $\operatorname{ind}_H^{\mathscr{E}} \operatorname{res}_H^{\mathscr{E}}(M)$, that is to say $\beta_!\beta^*(M)$, equals $\operatorname{Hom}_k(F, M)$ for some finite-dimensional $\mathscr{E}$-module F. Since k is the only simple $\mathscr{E}$-module, F is in $\mathsf{Thick}(k)$, and hence $\operatorname{Hom}_k(F, M)$ is in $\mathsf{Thick}(\operatorname{Hom}_k(k, M))$. It remains to recall that $\operatorname{Hom}_k(k, M) \cong M$ as $\mathscr{E}$-modules. □

Combining the preceding results one obtains the following:

Proposition 4.7. *Let $\alpha : K[t]/(t^p) \to KG$ be a π-point of G that factors through a quasielementary subgroup scheme of G_K. Then $\operatorname{res}_k^K \alpha_!\alpha^*(M^K)$ is in $\mathsf{Coloc}^{\mathrm{Hom}}(M)$ for any G-module M.*

Proof. By hypothesis, there exists a quasielementary subgroup scheme U of G_K such that $\alpha = \gamma \circ \beta$, where $\beta : K[t]/(t^p) \to KU$ and $\gamma : KU \to KG$. Then

$$\operatorname{res}_k^K \alpha_!\alpha^*(M^K) = \operatorname{res}_k^K \gamma_!\beta_!\beta^*\gamma^*(M^K).$$

Since $\beta_!\beta^*\gamma^*(M^K)$ is in $\mathsf{Thick}(\gamma^*(M^K))$ by Lemma 4.6, one has that

$$\operatorname{res}_k^K \alpha_!\alpha^*(M^K) \in \mathsf{Thick}(\operatorname{res}_k^K \gamma_!\gamma^*(M^K)).$$

Since $\operatorname{res}_k^K \gamma_!\gamma^*(M^K)$ is in $\mathsf{Coloc}^{\mathrm{Hom}}(M)$ by Lemma 4.5, it follows that

$$\operatorname{res}_k^K \alpha_!\alpha^*(M^K) \in \mathsf{Coloc}^{\mathrm{Hom}}(M).$$ □

The next result complements Theorem 4.4.

Theorem 4.8. *Let M be a G-module and $\mathfrak{p} \in \operatorname{cosupp}_G(M)$. If $\alpha : K[t]/(t^p) \to KG$ is a π-point that factors through a quasielementary subgroup scheme of G_K and represents $\mathfrak{p}$, then $\Delta_G(\alpha)$ is in $\mathsf{Coloc}^{\mathrm{Hom}}(M)$.*

Proof. By hypothesis on $\mathfrak{p}$, the $k[t]/(t^p)$-module $\alpha^*(M^K)$ is not projective, and hence K is in $\mathsf{Coloc}(\alpha^*(M^K))$. This implies that $\alpha_!(K)$ is in $\mathsf{Coloc}(\alpha_!\alpha^*(M^K))$, and hence, by restriction of scalars, that

$$\Delta_G(\alpha) \in \mathsf{Coloc}(\operatorname{res}_k^K \alpha_!\alpha^*(M^K)).$$

Finally, by Proposition 4.7, the module on the right is in $\mathsf{Coloc}^{\mathrm{Hom}}(M)$. □

Corollary 4.9. *For $\mathfrak{p} \in \operatorname{Proj} H^*(G, k)$, the colocalising subcategory $\varLambda^{\mathfrak{p}} \mathsf{StMod}\, G$ of $\mathsf{StMod}\, G$ contains no proper nonzero Hom closed colocalising subcategories.*

Proof. Fix a π-point α as in Theorem 4.4, factoring through a quasielementary subgroup scheme. Since $\mathfrak{p}$ is in the π-cosupport of any nonzero module M in $\varLambda^{\mathfrak{p}} \mathsf{StMod}\, G$, Theorem 4.8 yields the inclusion below:

$$\varLambda^{\mathfrak{p}} \mathsf{StMod}\, G = \mathsf{Coloc}(\Delta_G(\alpha)) \subseteq \mathsf{Coloc}^{\mathrm{Hom}}(M).$$

The equality is from Theorem 4.4. This is the desired result. □

Proof of Theorem 1.1. In the terminology of [Benson et al. 2012], Corollary 4.9 means that $\mathsf{StMod}\, G$ is *costratified* by the action of $H^*(G, k)$. Given this, [Benson et al. 2012, Corollary 9.2] yields the desired bijection between Hom closed colocalising subcategories of $\mathsf{StMod}\, G$ and subsets of $\operatorname{Proj} H^*(G, k)$. □

Colocalising and localising subcategories. A key step in the proof of the classification theorem above is that, given a point $\mathfrak{p}$ in $\operatorname{Proj} H^*(G, k)$, the point module associated to a certain type of π-point representing $\mathfrak{p}$ cogenerates $\Lambda^{\mathfrak{p}}\, \mathsf{StMod}\, G$; see Theorem 4.4. As a corollary of the classification result, it follows that any π-point may be used, as long as we also allow tensor products with simple modules.

Corollary 4.10. *For any point $\mathfrak{p}$ in $\operatorname{Proj} H^*(G, k)$ and any π-point representing $\mathfrak{p}$,*

$$\mathsf{Loc}^{\otimes}(\Delta_G(\alpha)) = \Gamma_{\mathfrak{p}}\, \mathsf{StMod}\, G \quad \textit{and} \quad \mathsf{Coloc}^{\mathrm{Hom}}(\Delta_G(\alpha)) = \Lambda^{\mathfrak{p}}\, \mathsf{StMod}\, G.$$

Proof. Since $\operatorname{supp}_G(\Delta_G(\alpha)) = \{\mathfrak{p}\}$, the first equality is a direct consequence of the bijection between tensor closed localising subcategories of $\mathsf{StMod}\, G$ and subsets of $\operatorname{Proj} H^*(G, k)$ established in [Benson et al. 2016, Theorem 8.1]. In the same vein, the second equality follows from Theorem 1.1, since $\operatorname{cosupp}_G(\Delta_G(\alpha)) = \{\mathfrak{p}\}$. □

Given a subcategory C of $\mathsf{StMod}\, G$ we set

$$\operatorname{supp}_G(\mathsf{C}) := \bigcup_{M \in \mathsf{C}} \operatorname{supp}_G(M) \quad \text{and} \quad \operatorname{cosupp}_G(\mathsf{C}) := \bigcup_{M \in \mathsf{C}} \operatorname{cosupp}_G(M).$$

For any subset $\mathcal{U} \subseteq \operatorname{Proj} H^*(G, k)$ set

$$\operatorname{cl}(\mathcal{U}) := \{\mathfrak{p} \in \operatorname{Proj} H^*(G, k) \mid \mathfrak{p} \subseteq \mathfrak{q} \text{ for some } \mathfrak{q} \in \mathcal{U}\}.$$

This is the closure of $\mathcal{U}$ with respect to the Hochster dual of the Zariski topology [Hochster 1969], and we call $\mathcal{U}$ *generalisation closed* if $\operatorname{cl}(\mathcal{U}) = \mathcal{U}$.

Corollary 4.11. *For a subcategory $\mathsf{C} \subseteq \mathsf{StMod}\, G$ the following are equivalent*:

(1) C *is a tensor closed localising subcategory and closed under all products.*

(2) C *is a Hom closed colocalising subcategory and closed under all coproducts.*

In that case we have $\operatorname{supp}_G(\mathsf{C}) = \operatorname{cosupp}_G(\mathsf{C})$ and this set is generalisation closed. Moreover, any generalisation closed subset of $\operatorname{Proj} H^*(G, k)$ arises in that way.

Proof. Benson et al. [2016] prove that, as a tensor triangulated category, $\mathsf{StMod}\, G$ is stratified by $H^*(G, k)$. It follows that the assignment $\mathsf{C} \mapsto \operatorname{supp}_G(\mathsf{C})$ yields a bijection between the tensor closed localising subcategories of $\mathsf{StMod}\, G$ that are closed under all products and the generalisation closed subsets of $\operatorname{Proj} H^*(G, k)$. This can be verified by mimicking the argument used to prove the implication (a) $\Longleftrightarrow$ (c) of [Benson et al. 2011, Theorem 11.8]; see also [Benson et al. 2011, Theorem 6.3]. The desired assertion now follows from the bijection between localising and colocalising subcategories (Corollary 1.2), noticing that for any tensor ideal localising subcategory C we have

$$\operatorname{supp}_G(\mathsf{C}) \sqcup \operatorname{cosupp}_G(\mathsf{C}^{\perp}) = \operatorname{Proj} H^*(G,k). \qquad \square$$

For any generalisation closed subset $\mathcal{U} \subseteq \operatorname{Proj} H^*(G,k)$ we set

$$(\mathsf{StMod}\, G)_{\mathcal{U}} := \{M \in \mathsf{StMod}\, G \mid \operatorname{supp}_G(M) \subseteq \mathcal{U}\}.$$

We collect some basic properties of this category.

Remark 4.12. There is an equality

$$(\mathsf{StMod}\, G)_{\mathcal{U}} = \{M \in \mathsf{StMod}\, G \mid \operatorname{cosupp}_G(M) \subseteq \mathcal{U}\}$$

and this is compactly generated as a triangulated category. The first assertion is justified by Remark 2.8(2), and compact generation follows from the fact that

$$(\mathsf{StMod}\, G)_{\mathcal{U}} = \varGamma_{\mathcal{U}^{\mathsf{c}}}(\mathsf{StMod}_G)^{\perp},$$

where $\mathcal{U}^{\mathsf{c}} := \operatorname{Proj} H^*(G,k) \setminus \mathcal{U}$. Indeed, the subset $\mathcal{U}^{\mathsf{c}}$ is specialisation closed, so $\varGamma_{\mathcal{U}^{\mathsf{c}}}(\mathsf{StMod}_G)$ is compactly generated (see [Benson et al. 2011, Proposition 2.7], for example). Now the assertion is a formal consequence of [Neeman 1992, Theorem 2.1; 2001, Theorem 9.1.16].

Given generalisation closed subsets $\mathcal{V} \subseteq \mathcal{U} \subseteq \operatorname{Proj} H^*(G,k)$, it follows from Brown representability [Neeman 2001] that the inclusion

$$(\mathsf{StMod}\, G)_{\mathcal{V}} \to (\mathsf{StMod}\, G)_{\mathcal{U}}$$

admits a left adjoint and a right adjoint, because the functor preserves products and coproducts.

Now fix a point $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$ and consider the generalisation closure of $\mathfrak{p}$. Then $(\mathsf{StMod}\, G)_{\leq\mathfrak{p}}$ equals the full subcategory of $\mathfrak{p}$-local G-modules and we obtain the following pair of equivalent recollements:

$$\begin{tikzcd}[column sep=large]
(\mathsf{StMod}\, G)_{<\mathfrak{p}} \arrow[r, tail] & (\mathsf{StMod}\, G)_{\leq\mathfrak{p}} \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \arrow[r, two heads, "\varGamma_{\mathfrak{p}}"] & \varGamma_{\mathfrak{p}}(\mathsf{StMod}\, G), \arrow[l, tail, shift right=2, "\mathrm{incl}"'] \arrow[l, tail, shift left=2, "\varLambda^{\mathfrak{p}}"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large]
(\mathsf{StMod}\, G)_{<\mathfrak{p}} \arrow[r, tail] & (\mathsf{StMod}\, G)_{\leq\mathfrak{p}} \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \arrow[r, two heads, "\varLambda^{\mathfrak{p}}"] & \varLambda^{\mathfrak{p}}(\mathsf{StMod}\, G) \arrow[l, tail, shift right=2, "\varGamma_{\mathfrak{p}}"'] \arrow[l, tail, shift left=2, "\mathrm{incl}"]
\end{tikzcd}$$

Note that for a π-point α representing $\mathfrak{p}$ we have, in $(\mathsf{StMod}\, G)_{\leq\mathfrak{p}}$,

$$\Delta_G(\alpha)^{\perp} = (\mathsf{StMod}\, G)_{<\mathfrak{p}} = {}^{\perp}\Delta_G(\alpha).$$

There is an analogy between point modules over finite group schemes and standard objects of highest weight categories. In fact, the analogy includes costandard objects, depending on whether one thinks of a point module as induced or coinduced from a trivial representation; see Theorem 3.1.

Remark 4.13. Let A be a highest weight category [Cline et al. 1988] with partially ordered set of weights Λ, which is assumed to be finite for simplicity. Thus A is an abelian length category with simple objects $\{L(\lambda)\}_{\lambda\in\Lambda}$. Now fix $\lambda\in\Lambda$ and consider the full subcategory $\mathsf{A}_{\le\lambda}$ of objects in A that have composition factors $L(\mu)$ with $\mu\le\lambda$. The standard object $\Delta(\lambda)$ is a projective cover of $L(\lambda)$ in $\mathsf{A}_{\le\lambda}$ and its endomorphism ring is a division ring, which we denote by K_λ. This situation gives rise to the following recollement [Cline et al. 1988, Theorem 3.9]:

$$\begin{tikzcd}[column sep=huge] \mathsf{A}_{<\lambda} \arrow[r, tail] & \mathsf{A}_{\le\lambda} \arrow[l, two heads, shift right=2ex] \arrow[l, two heads, shift left=2ex] \arrow[r, two heads, "\mathrm{Hom}(\Delta(\lambda),-)"] & \operatorname{mod} K_\lambda \arrow[l, hook', shift right=2ex] \arrow[l, hook, shift left=2ex] \end{tikzcd}$$

Note that $\Delta(\lambda)^\perp=\mathsf{A}_{<\lambda}={}^\perp\nabla(\lambda)$, where $\nabla(\lambda)$ denotes the costandard object corresponding to λ, namely the injective envelope of $L(\lambda)$ in $\mathsf{A}_{\le\lambda}$.

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Exterior power operations on higher K-groups via binary complexes

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We use Grayson's binary multicomplex presentation of algebraic K-theory to give a new construction of exterior power operations on the higher K-groups of a (quasicompact) scheme. We show that these operations satisfy the axioms of a λ-ring, including the product and composition laws. To prove the latter we show that the Grothendieck group of the exact category of integral polynomial functors is the universal λ-ring on one generator.

Introduction

Exterior powers of vector bundles over a scheme X endow its Grothendieck group $K_0(X)$ with a family of operations $\lambda^r : K_0(X) \to K_0(X)$, $r = 0, 1, \dots$. These λ-operations allow us to define Adams operations and the γ-filtration on $K_0(X)$ and are, more generally, at the heart of Grothendieck's Riemann–Roch theory (see [Fulton and Lang 1985]). This fundamental structure has been extended to the higher K-groups $K_n(X)$, $n \geq 0$, using a variety of sophisticated approaches and in various degrees of generality, by [Kratzer 1980; Hiller 1981; Grayson 1989; Nenashev 1991; Levine 1997], and has been most profoundly studied and applied in Soulé's seminal paper [1985]. Common to all these constructions is that they use homotopy theory.

In this paper we give a purely algebraic construction of the λ-operations on the higher K-groups of any quasicompact scheme X. Our construction is explicit in the following sense: in a surprising paper, Grayson [2012] has given explicit generators and relations for $K_n(X)$, and our construction describes explicit (albeit intricate) images of these generators under the λ-operations. Within the purely algebraic context of this paper, we prove moreover that our λ-operations satisfy the usual axioms, including the product and composition laws. In a forthcoming paper we address the problem of matching up our λ-operations with Hiller's.

To describe our results in more precise terms, we recall the definition of a λ-ring.

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Definition. A *pre-λ-ring* is a commutative unital ring K with maps $\lambda^r : K \to K$, $r \geq 1$, satisfying $\lambda^1(x) = x$ and the following axiom for all $x, y \in K$:

(1) $\lambda^r(x+y) = \lambda^r(x) + \sum_{i=1}^{r-1} \lambda^{r-i}(x)\lambda^i(y) + \lambda^r(y)$.

A *λ-ring* K is a pre-λ-ring satisfying the further axioms

(2) $\lambda^r(xy) = P_r(\lambda^1(x), \dots, \lambda^r(x), \lambda^1(y), \dots, \lambda^r(y))$,

(3) $\lambda^r(\lambda^s(x)) = P_{r,s}(\lambda^1(x), \dots, \lambda^{rs}(x))$,

where P_r and $P_{r,s}$, $r, s > 0$, are certain universal integral polynomials (defined in such a way that the axioms (2) and (3) hold in every pre-λ-ring whose additive group is generated by elements l with $\lambda^r(l) = 0$ for all $r > 1$ and in which products of elements of this type are again of this type; for details see [Fulton and Lang 1985]).

Probably the most prominent example of a λ-ring is $K_0(X)$ (see [loc. cit.]). The object of this paper is to make $K_*(X) = \bigoplus_{n\geq 0} K_n(X)$ into a λ-ring.

For each $n \geq 0$, Grayson [2012] associates to an exact category $\mathcal{P}$ the exact category $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$ of so-called n-dimensional bounded acyclic binary complexes, and proves that $K_n(\mathcal{P})$ is isomorphic to a relatively simple-to-describe quotient of the Grothendieck group $K_0((B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P})$ (see Section 1 for a detailed review of Grayson's construction). Using the Dold–Puppe construction [1961], we inductively construct functors

$$\Lambda^r_n : (B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X) \to (B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$$

for all $r, n > 0$ from the usual exterior power functors $\Lambda^r : \mathcal{P}(X) \to \mathcal{P}(X)$, $r \geq 0$, on the category $\mathcal{P}(X)$ of vector bundles on X.

The following theorems are the main results of this paper.

Theorem 6.2. *The functors Λ^r_n induce well-defined homomorphisms*

$$\lambda^r : K_n(X) \to K_n(X)$$

for $r, n > 0$.

The tensor product induces the multiplication in the Grothendieck ring $K_0(X)$ and also an action of $K_0(X)$ on the higher K-groups $K_n(X)$. In particular, $K_*(X) = \bigoplus_{n\geq 0} K_n(X)$ carries the structure of a unital commutative ring in which the product of any two elements in $\bigoplus_{n\geq 1} K_n(X)$ is defined to be zero. Note that, if $n > 0$, axiom (1) for $x, y \in K_n(X)$ then follows from $\lambda^r : K_n(X) \to K_n(X)$ being a homomorphism (Theorem 6.2). Furthermore, the formula in axiom (1) can be used to extend our operations $\lambda^r : K_n(X) \to K_n(X)$, $n \geq 0$, to a pre-λ-ring structure on $K_*(X)$.

Theorems 7.1 and 8.18. *The pre-λ-ring $K_*(X)$ is a λ-ring.*

The first half of the paper is devoted to the construction of the exterior power functors Λ^r_n. Let $C_{\mathrm{b}}\mathcal{P}(X)$ denote the category of bounded complexes in $\mathcal{P}(X)$. We use the Dold–Kan correspondence (reviewed along with the other necessary homological preliminaries in Section 2) to obtain a chain-homotopy invariant functor $\Lambda^r_1 : C_{\mathrm{b}}\mathcal{P}(X) \to C_{\mathrm{b}}\mathcal{P}(X)$ for each $r > 0$; if X is affine, then the bounded acyclic complexes in $\mathcal{P}(X)$ are precisely the contractible ones, so we obtain an endofunctor on the category of bounded acyclic chain complexes in $\mathcal{P}(X)$. By generalising and iterating this procedure over complexes of complexes, we get the desired functors $\Lambda^r_n : (B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X) \to (B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$. This material is the subject of Subsections 3 and 4.

In the rather long Section 5 we construct a "simplicial tensor product" $\otimes_{\Delta,n}$ on $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$. In defining exterior powers on $K_0(X)$ we obtain from a short exact sequence of vector bundles $0 \to V' \to V \to V'' \to 0$ a filtration of $\Lambda^r(V)$ whose successive quotients are $\Lambda^{r-i}(V') \otimes \Lambda^i(V'')$. We use our simplicial tensor product of binary multicomplexes to obtain similar statements for short exact sequences in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$; our tensor product $\otimes_{\Delta,n}$ is to $\otimes$ as the exterior powers Λ^r_n are to Λ^r. The main result of the section (Proposition 5.11) is that the product induced by $\otimes_{\Delta,n}$ on $K_n(X)$ vanishes.

In Section 6 we pass our exterior powers from the affine case to general (quasi-compact) schemes and show they induce well-defined maps $\lambda^r : K_n(X) \to K_n(X)$. As the product on $K_n(X)$ that is compatible with these operations is the zero product (by Proposition 5.11), it follows that the λ^r are group homomorphisms.

In Section 7 we show that the resulting pre-λ-ring $K_*(X)$ satisfies the λ-ring axiom (2) concerning products.

The final λ-ring axiom (3) is proved in Section 8. While the usual geometric splitting principle suffices to prove axiom (2) for $K_*(X)$ (see Section 7) and both axioms (2) and (3) for $K_0(X)$, there seems to be no way of extending that approach to prove axiom (3) for $K_*(X)$. We will rather proceed as follows. As $K_0(X)$ is a λ-ring, there exist short exact sequences in $\mathcal{P}(X)$ that prove the relation $\lambda^r(\lambda^s(x)) = P_{r,s}(\lambda^1(x), \dots, \lambda^{rs}(x))$ in $K_0(X)$ when x is the class of a vector bundle V on X. We will see (in Subsection 8D) that if in fact these short exact sequences exist functorially in V, then we can inductively prove the existence of short exact sequences in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$ that prove the relation above when x is the class of an object in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$; in other words, we have then proved axiom (3) for $K_*(X)$.

We are therefore reduced to showing the existence of such short exact sequences of functors in V. This problem may be seen as a weak variant of the famous plethysm problem (see Remark 8.22). The crucial insight now is that it becomes attackable when we also require these functors to be polynomial (see Definition 8.1). On the one hand, this requirement guarantees the existence of appropriate base change functors and hence reduces the problem to $X = \mathrm{Spec}(\mathbb{Z})$ (see Subsections 8A and 8D). On the other hand, it makes the computation of the corresponding

Grothendieck group of functors feasible; this is the content of the following theorem, which we highlight as it may be of independent interest.

It is well known that there exists a unique λ-ring structure on the ring $\mathbb{Z}[s_1, s_2, \dots]$ of integral polynomials in infinite variables such that $\lambda^r(s_1) = s_r$ for all r. Furthermore, let $\mathrm{Pol}_{<\infty}(\mathbb{Z})$ denote the category of polynomial functors over $\mathbb{Z}$ of bounded degree (whose Grothendieck group is easily seen to be a pre-λ-ring).

Theorem 8.5. *The ring homomorphism*

$$\mathbb{Z}[s_1, s_2, \dots] \to K_0(\mathrm{Pol}_{<\infty}(\mathbb{Z})), \quad s_i \mapsto [\Lambda^i],$$

is an isomorphism of pre-λ-rings.

This obviously implies that the right-hand side is a λ-ring as well and hence that the short exact sequences of functors postulated above indeed exist. After interpreting polynomial functors as modules over certain Schur algebras in Subsection 8B following Krause [2013, Section 8.2], in Subsection 8C we will prove the theorem by following Serre's computation [1968] of the Grothendieck group of representations of the group scheme $\mathrm{GL}_{n,\mathbb{Z}}$. A crucial ingredient here is Green's computation [1980] of the Grothendieck group of polynomial functors over a field.

The fundamental idea of proving λ-ring axioms for Grothendieck groups of complexes via the corresponding axioms for a Grothendieck group of appropriate functors is also sketched in an exchange of letters between Deligne [1967a; 1967b] and Grothendieck [1967].[†] Both their correspondence and the introduction of [Serre 1968] already allude to a role of Serre's result for λ-operations.

In a forthcoming paper we will complement the somewhat intricate constructions of this paper with simpler formulae that (help to) compute our λ-operations in certain cases. For instance, we will give formulae for our λ-operations when applied to K_1-groups of rings or to external products $K_m(X) \times K_n(X) \to K_{m+n}(X)$.

1. Binary multicomplexes and algebraic K-theory

In this section we review the description of algebraic K-groups in terms of binary complexes given in [Grayson 2012]. We also prove a simple lemma about shifted binary complexes to justify a slight modification of Grayson's description. The lemma is also useful for computations.

Recall that an *exact category* in the sense of [Quillen 1973] is an additive category with a distinguished class of "short exact sequences" that behave like the short exact sequences of an abelian category. A small exact category $\mathcal{N}$ may also be thought of as a full subcategory of an ambient abelian category $\mathcal{A}$ such that $\mathcal{N}$

[†]The authors became aware of these unpublished letters only after the present article was posted on arXiv (see Acknowledgements on page 448). After acceptance, at the publisher's request, Deligne kindly supplied scans and his permission to make them public.

is closed under extensions in $\mathcal{A}$.[1] The category of chain complexes in an exact category is again an exact category, with short exact sequences defined to be those sequences of chain maps that are short exact in each degree. In this paper we consider only complexes that are concentrated in nonnegative degrees, those with an underlying $\mathbb{Z}_{\geq 0}$-graded object. We denote this category of chain complexes in $\mathcal{N}$ by $C\mathcal{N}$. A chain complex is bounded if it has only finitely many nonzero objects. The exact subcategory of $C\mathcal{N}$ of bounded chain complexes is denoted by $C_{\mathrm{b}}\mathcal{N}$. An *acyclic* complex in an exact category is a chain complex $N_\bullet$ in $\mathcal{N}$ whose differentials $d_i : N_i \to N_{i-1}$ factor as $N_i \to Z_i \to N_{i-1}$ (with Z_i in $\mathcal{N}$), such that each $0 \to Z_{i+1} \to N_i \to Z_i \to 0$ is a short exact sequence in $\mathcal{N}$.[2] The full subcategories of acyclic complexes in $C\mathcal{N}$ and $C_{\mathrm{b}}\mathcal{N}$ are also exact, and are denoted by $C^{\mathrm{q}}\mathcal{N}$ and $C^{\mathrm{q}}_{\mathrm{b}}\mathcal{N}$.

Since each of these categories of complexes is also an exact category, we can iterate their construction to define *n-dimensional multicomplexes* in $\mathcal{N}$. A 1-dimensional multicomplex in $\mathcal{N}$ is simply a chain complex, an object of $C\mathcal{N}$. An $n+1$-dimensional multicomplex in $\mathcal{N}$ is a chain complex in the exact category $C^n\mathcal{N}$ of n-dimensional multicomplexes in $\mathcal{N}$. We define categories of bounded and/or acyclic multicomplexes, $(C_{\mathrm{b}})^n\mathcal{N}$, $(C^{\mathrm{q}})^n\mathcal{N}$ and $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$, analogously. With these notions in place, we can define binary complexes and multicomplexes.

Definition 1.1. (1) A *binary complex* in an exact category $\mathcal{N}$ is a triple $(N_\bullet, d, \tilde{d})$ consisting of a $\mathbb{Z}_{\geq 0}$-graded collection of objects of $\mathcal{N}$ together with two differentials d and d' such that $(N_\bullet, d)$ and $(N_\bullet, \tilde{d})$ are chain complexes in $\mathcal{N}$. A binary complex can be regarded as pair of objects of $C\mathcal{N}$ that have the same underlying graded object. A morphism of binary complexes is a degree 0 map between these underlying objects that commutes with both differentials. The category of binary complexes in $\mathcal{N}$ is denoted by $B\mathcal{N}$. This is an exact category in the same way that $C\mathcal{N}$ is.

(2) A *bounded acyclic* binary complex in $\mathcal{N}$ is a binary complex such that the chain complexes $(N_\bullet, d)$ and $(N_\bullet, \tilde{d})$ are bounded and acyclic. The category of bounded acyclic binary complexes in $\mathcal{N}$ is denoted by $B^{\mathrm{q}}_{\mathrm{b}}\mathcal{N}$. It is an exact subcategory of $B\mathcal{N}$.

(3) An *n-dimensional binary multicomplex* is an object of the exact category $B^n\mathcal{N} = B \cdots B\mathcal{N}$ (defined in the same way as $C^n\mathcal{N}$). An n-dimensional bounded acyclic binary multicomplex is an object of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$.

[1] This is the *Gabriel–Quillen embedding theorem* [Thomason and Trobaugh 1990, Theorem A.7.1 and Proposition A.7.16].

[2] This is not in general the same thing as being a long exact sequence in the ambient abelian category $\mathcal{A}$. However in this paper we work only with *idempotent complete* exact categories, in which case the two notions coincide. See [Grayson 2012; Thomason and Trobaugh 1990, Section A.9.2].

Remark 1.2. A (bounded acyclic) binary multicomplex $N_\bullet$ of dimension n is equivalent to the following data: a (bounded) $\mathbb{Z}^n_{\geq 0}$-graded collection of objects of $\mathcal{N}$ equipped with *two* (acyclic) differentials, denoted by d^i and $\tilde{d}^i$, in each direction $1 \leq i \leq n$, subject to the commutativity requirements

(1) $d^i d^j = d^j d^i$,

(2) $d^i \tilde{d}^j = \tilde{d}^j d^i$,

(3) $\tilde{d}^i d^j = d^j \tilde{d}^i$,

(4) $\tilde{d}^i \tilde{d}^j = \tilde{d}^j \tilde{d}^i$,

whenever $i \neq j$.

Another way to look at these commutativity restraints is that the various subsets of the differentials form (nonbinary) multicomplexes: for each $i = 1, \dots, n$, choose d^i or $\tilde{d}^i$, and consider the object that has the same underlying $\mathbb{Z}^n$-graded object as $N_\bullet$, but now has *one* acyclic differential in each direction i, given by d^i or $\tilde{d}^i$, depending on our choice. For each of the 2^n choices of differentials, the resulting object is a bounded acyclic multicomplex, i.e., an object of $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$; conversely, given a pair of differentials d^i and $\tilde{d}^i$ in each direction, if the 2^n choices all form objects of $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$, then the whole assembly is an object of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$.

Since this category of bounded acyclic binary complexes in $\mathcal{N}$ is itself an exact category, we can form its Grothendieck group $K_0(B^{\mathrm{q}}_{\mathrm{b}}\mathcal{N})$. The main theorem of [Grayson 2012] is a surprising connection between this group and the n-th higher K-group of $\mathcal{N}$. We call an n-dimensional binary multicomplex *diagonal* if the pair of differentials in some direction are equal, i.e, if $d^i = \tilde{d}^i$ for some $1 \leq i \leq n$. Grayson's theorem, which we shall hereafter use as our definition of the K-groups, says that $K_n(\mathcal{N})$ is isomorphic to the quotient of the Grothendieck group of $B^{\mathrm{q}}_{\mathrm{b}}\mathcal{N}$ by the subgroup generated by the classes of the diagonal bounded acyclic binary multicomplexes. More formally:

Theorem/Definition 1.3 [Grayson 2012, Corollary 7.4]. *For $\mathcal{N}$ an exact category and $n \geq 0$, the abelian group $K_n(\mathcal{N})$ is presented as follows. There is one generator for each bounded acyclic binary multicomplex of dimension n, and there are two families of relations*:

(1) $[N'] + [N''] = [N]$ *if there is a short exact sequence*

$$0 \to N' \to N \to N'' \to 0$$

in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$, and

(2) $[D] = 0$ *if D is a diagonal bounded acyclic binary multicomplex.*

We remark that our statement of Theorem/Definition 1.3 is subtly different than the one originally given by Grayson. Our bounded acyclic binary multicomplexes are first-quadrant multicomplexes, those that are supported in $\mathbb{Z}^n_{\geq 0}$, whereas

Grayson's do not have to satisfy this condition. The absolute lower bound for complexes is a technical constraint that we need in order to use the Dold–Kan correspondence. Our additional condition is harmless, as the following proposition shows. For this, let $K_n^{\mathrm{Gr}}(\mathcal{N})$ temporarily denote the n-th K-group of $\mathcal{N}$ as defined in [Grayson 2012].

Proposition 1.4. *For every exact category $\mathcal{N}$ and every $n \geq 0$, the canonical homomorphism $K_n(\mathcal{N}) \to K_n^{\mathrm{Gr}}(\mathcal{N})$ is bijective.*

Proof. For ease of presentation we shall prove this for $n = 1$ only: there is no additional difficulty for $n > 1$. Let $B^{\mathrm{q}}_{\infty}\mathcal{N}$ denote the category of bounded acyclic binary complexes in $\mathcal{N}$ that may be supported anywhere on $\mathbb{Z}$. For $i \geq 0$, let $B^{\mathrm{q}}_{\geq -i}\mathcal{N}$ denote the full subcategory of $B^{\mathrm{q}}_{\infty}\mathcal{N}$ consisting of complexes that are supported on $[-i, \infty]$. We then have $\bigcup_i B^{\mathrm{q}}_{\geq -i}\mathcal{N} = B^{\mathrm{q}}_{\infty}\mathcal{N}$ and hence $\lim_i K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N}) = K_0(B^{\mathrm{q}}_{\infty}\mathcal{N})$. Let T_i denote the subgroup of $K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N})$ generated by diagonal complexes and let T denote the similarly defined subgroup of $K_0(B^{\mathrm{q}}_{\infty}\mathcal{N})$. The resulting injective homomorphism $\lim_i T_i \to T$ is also surjective because all complexes are assumed to be bounded. We therefore obtain an isomorphism

$$\lim_i (K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N})/T_i) \cong \lim_i K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N})/\lim_i T_i \cong K_0(B^{\mathrm{q}}_{\infty}\mathcal{N})/T = K_1^{\mathrm{Gr}}(\mathcal{N}).$$

The following lemma (after generalising it from $B^{\mathrm{q}}_{\geq 0}\mathcal{N}$ to $B^{\mathrm{q}}_{\geq -i}\mathcal{N}$) shows that, for every $i \geq 0$, "shifting" induces a two-sided inverse to the negative of the canonical homomorphism $K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N})/T_i \to K_0(B^{\mathrm{q}}_{\geq -i-1}\mathcal{N})/T_{i+1}$. Hence the canonical map

$$K_1(\mathcal{N}) = K_0(B^{\mathrm{q}}_{\geq 0}\mathcal{N})/T_0 \to \lim_i (K_0(B^{\mathrm{q}}_{\geq -i}\mathcal{N})/T_i) \cong K_1^{\mathrm{Gr}}(\mathcal{N})$$

is an isomorphism, as was to be shown. □

Definition 1.5. Let $N_\bullet$ be an acyclic binary complex with differentials d and $\tilde{d}$. The k-th *shift* of N, denoted by $N[k]$, is the acyclic binary complex that has the same collection of objects as N but "shifted" k places, i.e., $(N[k])_i = N_{i-k}$, and differentials given by $(-1)^k d$ and $(-1)^k \tilde{d}$.

Lemma 1.6. *For any bounded acyclic binary complex $N_\bullet$ and $k \in \mathbb{Z}_{\geq 0}$, we have $[N[k]] = (-1)^k [N]$ in $K_1\mathcal{N}$.*

Proof. It is enough to show that $[N[1]] = -[N]$. There is a short exact sequence

$$0 \to N_\bullet \to \operatorname{cone}(N_\bullet) \to N_\bullet[1] \to 0,$$

where $\operatorname{cone}(N_\bullet)$ denotes the mapping cone of the identity map $N_\bullet \xrightarrow{1} N_\bullet$ ($\operatorname{cone}(N_\bullet)$ is a binary complex in the obvious way). So it suffices to show that $\operatorname{cone}(N_\bullet)$ vanishes in $K_1\mathcal{N}$. Let N_n be the left-most nonzero object of $N_\bullet$, and let $\operatorname{trun}(N_\bullet)$ be the (not necessarily acyclic) binary complex formed by truncating $N_\bullet$ to forget N_n; that is, $\mathrm{trun}(N_\bullet)$ has a 0 in place of N_n. Then there is a short exact sequence

$$0 \to \mathrm{cone}(\mathrm{trun}(N_\bullet)) \to \mathrm{cone}(N_\bullet) \to \Delta(N_n \xrightarrow{1} N_n) \to 0,$$

where $\Delta(N_n \xrightarrow{1} N_n)$ is the diagonal binary complex

$$0 \rightrightarrows N_n \overset{1}{\underset{1}{\rightrightarrows}} N_n \rightrightarrows 0,$$

which is supported in degrees $n+1$ and n. Mapping cones of identities are always acyclic, so $\mathrm{cone}(\mathrm{trun}(N_\bullet))$ is acyclic even when $\mathrm{trun}(N_\bullet)$ is not. Since $\Delta(N_n \xrightarrow{1} N_n)$ is diagonal its class vanishes in $K_1\mathcal{N}$, so the above short exact sequence yields the relation $[\mathrm{cone}(N_\bullet)] = [\mathrm{cone}(\mathrm{trun}(N_\bullet))]$. We iterate this procedure by repeatedly truncating $\mathrm{trun}(N_\bullet)$ to show that $[\mathrm{cone}(N_\bullet)]$ is zero. □

The same proof gives the analogous result for binary multicomplexes: for N in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{N}$, the class of N shifted one place in any of the n possible directions in $K_n(\mathcal{N})$ is $-[N]$. From this the actions of more general shifts (in multiple directions) follow immediately.

2. Preliminaries from homological algebra

In this section we recall some preliminaries from the homological algebra of exact categories. We say what it means for an exact category to be idempotent complete or split, and show that the notions of acyclicity and contractibility of complexes coincide in exact categories that have both of these properties. We then review simplicial objects and the Dold–Kan correspondence. Finally we discuss functors of finite degree, a weakening of the concept of additive functors. These three topics may seem rather disjoint here, but we bring them together in the next section to produce functors between categories of chain complexes that preserve boundedness and acyclicity, paving the way for a functor on binary multicomplexes that induces a map on K-theory.

Definition 2.1. An exact category $\mathcal{N}$ is *idempotent complete* if every idempotent endomorphism in $\mathcal{N}$ has a kernel in $\mathcal{N}$.

This does not hold, for example, for the category of free modules over a ring when there exists a nonfree projective module. All of the exact categories we use in this paper are idempotent complete. This is an assumption on the "base level" exact categories we introduce, but will need to be proven for categories of multicomplexes (Lemma 3.4). Idempotent complete exact categories come with an embedding into an abelian category $\mathcal{N} \hookrightarrow \mathcal{A}$ that *supports long exact sequences*: a chain complex is acyclic in $\mathcal{N}$ if and only if it is exact when considered as a chain complex of $\mathcal{A}$ (see [Grayson 2012, §1]). Homological algebra is therefore quite straightforward in idempotent complete exact categories.

Contractible complexes in idempotent complete exact categories are always acyclic; indeed this is an equivalent characterisation of idempotent completeness [Bühler 2010, Proposition 10.9]. Acyclic complexes in exact categories (even idempotent complete ones) are not usually contractible. There is a useful criterion for contractibility, however. Recall that a chain complex $(C_\bullet, d)$ is called *split* if there exist maps $s_n : C_{n-1} \to C_n$ such that $d_n s_n d_n = d_n$.

Lemma 2.2. *A chain complex in an idempotent complete exact category is contractible if and only if it is acyclic and split.*

Proof. It follows the definition of a chain homotopy that contractible complexes in idempotent complete exact categories are also split. Conversely, an elementary argument shows that if a complex in an exact category is acyclic and split, then the collection of splitting maps $\{s_n\}$ describes a homotopy from its identity map to its zero map. $\square$

If an acyclic complex is split, each of the constituent short exact sequences that it factors into is split: that is, isomorphic to a canonical direct sum sequence (the converse is obviously true as well). Recall that an exact category is called *split exact* if all of its declared short exact sequences are split. In such an exact category, all acyclic complexes are split. Hence the notions of contractibility and acyclicity coincide for complexes in a split exact category that is also idempotent complete. An example of such an exact category is the category $\mathcal{P}(R)$ of (finitely generated) projective modules over a ring R. That acyclic complexes are contractible in this category is key to the results of this paper.

We now turn to the Dold–Kan correspondence. To give its statement we need the language of simplicial objects. Recall that Δ denotes the *simplex category*: the category whose objects are the finite nonempty ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and whose morphisms are the order-preserving maps. A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$, and the natural transformations between such functors make $\mathcal{C}^{\Delta^{\mathrm{op}}}$ into a category. Equivalently, a simplicial object C in $\mathcal{C}$ can be specified to be a collection of objects C_n, $n \in \mathbb{N}$, of $\mathcal{C}$ together with face maps $\delta_i : C_n \to C_{n-1}$ and degeneracy maps $\sigma_j : C_n \to C_{n+1}$, $i, j = 0, \ldots, n$, satisfying various combinatorial identities. A morphism between simplicial objects C and D is a collection of morphisms $C_n \to D_n$ that commutes with the faces and degeneracies. A homotopy $h : f \simeq g$ between simplicial maps $f, g : C \to D$ is a simplicial morphism $h : C \times \Delta^1 \to D$ (where Δ^1 denotes the simplicial set corresponding to the ordered set $\{0 < 1\}$, as usual) such that $h|_{C\times\{0\}} = f$ and $h|_{D\times\{1\}} = g$; it can also be described as collection of morphisms $h_i : C_n \to D_{n+1}$,

$i = 0, \ldots, n$, which satisfy further combinatorial identities determined by compositions relating f, g, the h_i, and the faces and degeneracies of C and D. See, for example, Chapter 8 of [Weibel 1994] for full definitions of simplicial objects and homotopies.

If $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor, then postcomposition with F induces a functor between categories of simplicial objects $\mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{D}^{\Delta^{\mathrm{op}}}$. Abusing notation, we shall also call this functor F. Importantly, if $h : f \simeq g$ is a simplicial homotopy between $f, g : C \to D$, then $F(h) : F(f) \simeq F(g)$ is a simplicial homotopy between $F(f), F(g) : F(C) \to F(D)$. The analogous statement for chain homotopies between chain maps is not true if F is not additive. The Dold–Kan correspondence shows that chain complexes and simplicial objects are equivalent in a nonobvious way, and allows us to induce homotopy-preserving functors between categories of chain complexes, even when the original functors are not additive.

Definition 2.3. Let $\mathcal{P}$ be an additive category. Given a chain complex $C_. \in C\mathcal{P}$, we define a simplicial object $\Gamma(C_\bullet) \in \mathcal{P}^{\Delta^{\mathrm{op}}}$ as follows:

(1) *Objects*: Given $p \leq n$, let η range over all surjections $[n] \twoheadrightarrow [p]$ in Δ, and let $C_p\langle\eta\rangle$ denote a copy of C_p that is labelled by η. For each n, set

$$\Gamma(C)_n := \bigoplus_{p \leq n} \bigoplus_{\eta} C_p\langle\eta\rangle.$$

(2) *Maps*: If $\alpha : [m] \to [n]$ is a morphism in Δ, we describe $\Gamma(\alpha)$ by describing each $\Gamma(\alpha, \eta)$, the restriction of $\Gamma(\alpha)$ to the summand $C_p\langle\eta\rangle$ of $\Gamma(C)_n$. Let

$$[m] \xrightarrow{\eta'} [q] \xhookrightarrow{\varepsilon} [p]$$

be the unique epi-monic factorisation of $\eta\alpha$. Then

$$\Gamma(\alpha, \eta) := \begin{cases} 1 : C_p\langle\eta\rangle \to C_p\langle\eta'\rangle & \text{if } q = p, \\ d_p : C_p\langle\eta\rangle \to C_{p-1}\langle\eta'\rangle & \text{if } q = p-1 \text{ and } \varepsilon = \varepsilon_p, \\ 0 & \text{otherwise.} \end{cases}$$

This construction extends to a functor[3] $\Gamma : C\mathcal{P} \to \mathcal{P}^{\Delta^{\mathrm{op}}}$.

Theorem (Dold–Kan correspondence). *If $\mathcal{P}$ is idempotent complete, then the functor $\Gamma : C\mathcal{P} \to \mathcal{P}^{\Delta^{\mathrm{op}}}$ is an equivalence of categories. Furthermore, Γ is exact and preserves homotopies.*

Proof. Chapter 8 of [Weibel 1994] proves this when $\mathcal{P}$ is an abelian category. The general case is [Lurie 2014, §1.2.3]. □

The inverse functor to Γ is most simply described for an abelian category.

[3] Other authors (e.g., Weibel [1994]) use K in place of Γ; we avoid this notation for obvious reasons.

Definition 2.4. Let A be a simplicial object in an abelian category $\mathcal{A}$.

(1) The *associated chain complex* $C(A)$ has objects $C(A)_n = A_n$ and differential
$$d_n = \sum_{i=0}^{n} (-1)^i \delta_i : C(A)_n \to C(A)_{n-1}.$$

(2) The subcomplex
$$D(A)_n = \sum_{i=0}^{n} \mathrm{Im}(\sigma_i : A_{n-1} \to A_n)$$
is called the *degenerate subcomplex* of $C(A)$.

(3) The *normalised Moore complex* $N(A)$ has objects
$$N_n(A) = A_n / D(A)_n$$
with the induced differential $\bar{d}_n$.

The associated chain complex splits globally as $C(A) = N(A) \oplus D(A)$.

The normalised Moore complex defines a functor $N : \mathcal{A}^{\Delta^{\mathrm{op}}} \to C\mathcal{A}$. It is exact and preserves homotopies, and is inverse to Γ (up to natural isomorphism). Now if $\mathcal{P}$ is an idempotent complete exact category, then there is an embedding $\mathcal{P} \subseteq \mathcal{A}$ into an abelian category such that $\mathcal{P}$ is closed under taking direct summands in $\mathcal{A}$. If P is an object of $\mathcal{P}^{\Delta^{\mathrm{op}}} \subseteq \mathcal{A}^{\Delta^{\mathrm{op}}}$, then the associated chain complex $C(P)$ is a chain complex $\mathcal{A}$ with objects in $\mathcal{P}$. But $N(P)$ is a direct summand of $C(P)$, which has objects in $\mathcal{P}$, so $N(P)$ has objects in $\mathcal{P}$. Therefore N restricts to a functor $\mathcal{P}^{\Delta^{\mathrm{op}}} \to C\mathcal{P}$. Furthermore the functor N is exact and preserves homotopies. See [Lurie 2014] for further details.

We conclude our preliminaries by discussing functors of finite degree.

Definition 2.5. Let $F : \mathcal{C} \to \mathcal{D}$ be any functor between additive categories that satisfies $F(0) = 0$. Then there is a functorial decomposition
$$F(X \oplus Y) = F(X) \oplus \mathrm{cr}_2(F)(X, Y) \oplus F(Y),$$
where $\mathrm{cr}_2(F) : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$ is the second *cross-effect functor* (see [Eilenberg and Mac Lane 1954]), which is defined to be the kernel of the natural projection $F(X \oplus Y) \to F(X) \oplus F(Y)$. The functor F is said to have *degree* ≤ 1 if it is additive (i.e., if $\mathrm{cr}_2(F)$ vanishes), and we say that F has degree $\leq d$ if $\mathrm{cr}_2(F)(X, Y)$ is of degree $\leq d-1$ in each argument. If F is of degree $\leq d$, then F is of degree $\leq d'$ for all $d' \geq d$. We say that F *has degree* d if it has degree $\leq d$ but does not have degree $\leq d-1$.

Example 2.6. For R a nonzero commutative ring, the exterior power $\Lambda^r : \mathcal{P}(R) \to \mathcal{P}(R)$ has degree r for each $r > 0$. This follows from the canonical decomposition
$$\Lambda^r(X \oplus Y) \cong \Lambda^r(X) \oplus \left(\bigoplus_{i=1}^{r-1} \Lambda^{r-i}(X) \otimes \Lambda^i(Y) \right) \oplus \Lambda^r(Y).$$

If $F : \mathcal{P} \to \mathcal{Q}$ is an *additive* functor between exact categories, and if $P_\bullet$ is a bounded complex, then $NF\Gamma(P_\bullet)$ is certainly bounded again. This also holds true for functors of finite degree, as the following lemma shows:

Lemma 2.7 [Satkurunath and Köck 2010, Corollary 4.6]. *Let $P_\bullet$ be a chain complex in $C\mathcal{P}$ of length ℓ, and let $F : \mathcal{P} \to \mathcal{Q}$ be a functor of degree d between exact categories. Then $NF\Gamma(P_\bullet)$ has length less than or equal to $d\ell$.* □

3. Operations on acyclic complexes

In this rather abstract section we describe how to use the Dold–Kan correspondence to extend a functor $F : \mathcal{P} \to \mathcal{P}$ on an idempotent complete exact category to a functor on each category of multicomplexes $F_n : C^n\mathcal{P} \to C^n\mathcal{P}$, $n \geq 1$. We show that if $\mathcal{P}$ is split exact, then the extended functors F_n send acyclic multicomplexes to acyclic multicomplexes. We also show that if F is of finite degree, then each F_n preserves bounded multicomplexes and is also of finite degree.

Proposition 3.1. *Let $F : \mathcal{P} \to \mathcal{P}$ be a covariant functor on an idempotent complete exact category, with $F(0) = 0$. Let $F_1 := NF\Gamma : C\mathcal{P} \to C\mathcal{P}$ denote the induced functor. Then*:

(1) $F_1(0) = 0$.

(2) *F_1 sends contractible complexes to contractible complexes.*

(3) *If $\mathcal{P}$ is split exact, then F_1 sends acyclic complexes to acyclic complexes.*

(4) *If F is of degree at most d, then F_1 sends bounded complexes to bounded complexes and F_1 is again of degree at most d.*

Proof. Part (1) is trivial.

For (2), the functors $\Gamma : C\mathcal{P} \to \mathcal{P}^{\Delta^{\mathrm{op}}}$ and $N : \mathcal{P}^{\Delta^{\mathrm{op}}} \to C\mathcal{P}$ preserve homotopies and send 0 to 0, so they both send contractible objects to contractible objects. Furthermore, F sends homotopies in $\mathcal{P}^{\Delta^{\mathrm{op}}}$ to homotopies in $\mathcal{P}^{\Delta^{\mathrm{op}}}$ — if $h : f \sim g$ is a homotopy, then $F(h) : F(f) \sim F(g)$ is a homotopy. Since F also has the property that $F(0) = 0$, we see that if $A \simeq 0$ in $\mathcal{P}^{\Delta^{\mathrm{op}}}$, then $F(A) \simeq F(0) = 0$. Therefore $NF\Gamma(P_\bullet)$ is contractible in $C\mathcal{P}$.

Following Lemma 2.2, the acyclic complexes in a split exact idempotent complete exact category coincide with the contractible ones, so (3) follows from (2).

Finally we consider (4). The first part of this statement is Lemma 2.7. For the second part we note that, since N and Γ are additive, it is enough to show that $F : \mathcal{A}^{\Delta^{\mathrm{op}}} \to \mathcal{B}^{\Delta^{\mathrm{op}}}$ is of degree $\leq d$. This is proven by induction on d. □

Proposition 3.1(3) may not hold in an exact category that is not split exact, as is shown in the following example:

Example 3.2. Let F be the degree 2 endofunctor $A \mapsto A^{\otimes 2}$ on the abelian category of abelian groups, and let $C_\bullet$ be the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

viewed as an acyclic complex concentrated in degrees 0, 1 and 2. Then $NF\Gamma(C_\bullet) = N\,\mathrm{diag}(\Gamma(C_\bullet) \otimes \Gamma(C_\bullet))$ is homotopy equivalent to $\mathrm{Tot}(C_\bullet \otimes C_\bullet)$ by the Eilenberg–Zilber theorem [May 1967, §29].[4] But the homology group $H_2(\mathrm{Tot}(C_\bullet \otimes C_\bullet))$ is $\mathbb{Z}/2\mathbb{Z}$, so $NF\Gamma(C_\bullet)$ is not exact. Furthermore, the short exact sequence of functors

$$0 \to N\Lambda^2\Gamma \to NF\Gamma \to N\mathrm{Sym}^2\Gamma \to 0$$

shows that at least one of $N\Lambda^2\Gamma(C_\bullet)$ or $N\mathrm{Sym}^2\Gamma(C_\bullet)$ is not exact either.

We now iterate the Dold–Kan correspondence to describe induced functors on categories of acyclic multicomplexes.

Definition 3.3. Let $F : \mathcal{P} \to \mathcal{P}$ be a covariant functor on an idempotent complete exact category. We define functors

$$F_n : C^n\mathcal{P} \to C^n\mathcal{P}$$

for all $n \geq 0$ recursively, as follows:

(1) $F_0 := F : \mathcal{P} \to \mathcal{P}$.

(2) By regarding an object of $C^{n+1}\mathcal{P}$ as a chain complex in the exact category $C^n\mathcal{P}$, we define $F_{n+1} := NF_n\Gamma$.

To show that F_n sends acyclic multicomplexes to acyclic multicomplexes in a nice exact category, we need to know that $(C^{\mathrm{q}})^n\mathcal{P}$ satisfies the same hypotheses as $\mathcal{P}$. This is the content of the following technical lemma. The proof is not enlightening for the rest of the paper, so we relegate it to the Appendix.

Lemma 3.4. *Let $\mathcal{P}$ be an exact category. For all $n > 0$ we have the following*:

(1) *If $\mathcal{P}$ is idempotent complete, then $C^n\mathcal{P}$ and $(C^{\mathrm{q}})^n\mathcal{P}$ are also idempotent complete.*

(2) *If $\mathcal{P}$ is split exact, then $(C^{\mathrm{q}})^n\mathcal{P}$ is also split exact.*

The analogous results for the categories $C^n_{\mathrm{b}}\mathcal{P}$ and $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$ of bounded multicomplexes also hold.

Corollary 3.5. *Let $\mathcal{P}$ be a split exact idempotent complete exact category, and $F : \mathcal{P} \to \mathcal{P}$ a covariant functor such that $F(0) = 0$. Then for $n \geq 0$ the functors of Definition 3.3 restrict to functors*

$$F_n : (C^{\mathrm{q}})^n\mathcal{P} \to (C^{\mathrm{q}})^n\mathcal{P}.$$

[4] See also Definition 5.4 and Lemma 5.5 here.

Furthermore, if F is of finite degree, then F_n sends bounded multicomplexes to bounded multicomplexes. That is, each F_n restricts to a functor

$$F_n : (C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P} \to (C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}.$$

Proof. We consider the unbounded case first. By Proposition 3.1(1), we easily see that $F_n(0) = 0$ for all n. Assume that F_n restricts to a functor on the idempotent complete split exact category $(C^{\mathrm{q}})^n\mathcal{P}$. Regarding objects of $(C^{\mathrm{q}})^{n+1}\mathcal{P}$ as acyclic complexes in $(C^{\mathrm{q}})^n\mathcal{P}$, the functor $F_{n+1} = NF_n\Gamma$ restricts to a functor on $(C^{\mathrm{q}})^{n+1}\mathcal{P} = C^{\mathrm{q}}((C^{\mathrm{q}})^n\mathcal{P})$, by Proposition 3.1(3) and Lemma 3.4(2). The first part of the result follows by induction.

For the second part, if $F_0 = F$ is of finite degree, then the same induction over n shows that F_n is of finite degree for every n, by Proposition 3.1(4). In particular, for each $n \geq 1$, the functor $F_n = (F_{n-1})_1$ sends bounded complexes to bounded complexes, that is, it restricts to a functor

$$F_n : C_{\mathrm{b}}^{\mathrm{q}}((C^{\mathrm{q}})^{n-1}\mathcal{P}) \to C_{\mathrm{b}}^{\mathrm{q}}((C^{\mathrm{q}})^{n-1}\mathcal{P}).$$

But we can say more: considering $P_\bullet$ in $(C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$ as a chain complex, each of its objects is in $(C_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}$, i.e., they are bounded. We claim that the objects of $F_n(P_\bullet) = NF_{n-1}\Gamma(P_\bullet)$ are also bounded. The objects of $\Gamma(P_\bullet)$ are finite direct sums of the objects of $P_\bullet$. Finite sums of bounded objects are bounded, so the objects of $\Gamma(P_\bullet)$ are bounded. Therefore, by the inductive hypothesis, the objects of $F_{n-1}\Gamma(P_\bullet)$ are also bounded. Finally, the objects of $NF_{n-1}\Gamma(P_\bullet)$ are direct summands of the objects of $F_{n-1}\Gamma(P_\bullet)$ (from Definition 2.4, after embedding into an abelian category), so they are bounded as well. Therefore F_n sends bounded chain complexes of bounded objects in $(C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$ to bounded chain complexes of bounded objects in $(C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$. This is exactly the statement that F_n restricts to a functor

$$F_n : (C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P} \to (C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P},$$

which was to be proved. □

Remark 3.6. Throughout this section we work with the inductive definition of $(C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$, that is $(C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P} := C_{\mathrm{b}}^{\mathrm{q}}((C_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P})$ for $n > 1$. As explained in Remark 1.2, one can instead think of objects in $(C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$ as $\mathbb{Z}^n_{\geq 0}$-graded objects of $\mathcal{N}$ (together with certain differentials) without specifying the order of directions in which the objects have been obtained in the inductive definition. The purpose of this remark is to convince the reader that our construction of the functors F_n given in this section (and hence our construction of exterior powers in the sequel) does not depend on the order of directions either. Rather than including a complete proof, we sketch the idea in the case $n = 2$. Let $F_0 = F$ be as before. The functor F_2 is defined as

$$N_h F_1 \Gamma_h = N_h N_v F_0 \Gamma_v \Gamma_h,$$

where the indices h and v indicated the horizontal and vertical directions respectively. It is quite straightforward to see that the composition $N_h N_v$ sends a bisimplicial object C to the double complex whose objects are obtained from the corresponding objects of C by dividing out the images of all of the horizontal and vertical degeneracy maps. This latter description of course does not depend on the order of N_h and N_v. One can show that the same holds for Γ_h and Γ_v by a similar argument, or just by recalling that Γ_h and Γ_v are adjoint to N_h and N_v, respectively.

We can now describe the exterior power functors that we will use to induce operations on higher K-groups. The following example is the motivation for our work so far.

Main Example 3.7. Let $\mathcal{P}(R)$ be the category of finitely generated projective modules over a commutative ring R. This category is both idempotent complete and split exact. For each $r > 0$, the usual exterior power functor $\Lambda^r : \mathcal{P}(R) \to \mathcal{P}(R)$ satisfies the hypotheses of Corollary 3.5 (Λ^r has degree r). We therefore have induced functors

$$\Lambda^r_n : (C^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(R) \to (C^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(R)$$

for all $n \geq 0$.

In general, the complex $N\Lambda^r\Gamma(P_\bullet)$ is difficult to write down explicitly. Satkurunath and Köck [2010] give an algorithm that addresses this problem. We conclude this section by computing $N\Lambda^r\Gamma(P_\bullet)$ for a very simple choice of $P_\bullet$.

Example 3.8. Let $\varphi : P \to Q$ be an isomorphism of invertible modules over some commutative ring R, considered as an acyclic complex concentrated in degrees 0 and 1:

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{\varphi} & Q & \longrightarrow & 0 \\ 2 & & 1 & & 0 & & -1 \end{array}$$

or $P \xrightarrow{\varphi} Q$ for short. Köck [2001, Lemma 2.2] gives an explicit calculation of $N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)$ in terms of higher cross-effect functors (in fact, he does this for more general P, Q and φ). Specifically, in degree n we have

$$N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)_n = \mathrm{cr}_n(\Lambda^r)(P, \ldots, P) \oplus \mathrm{cr}_{n+1}(\Lambda^r)(Q, P, \ldots, P).$$

We do not wish to expound on the theory of cross-effect functors here; the interested reader can see [Eilenberg and Mac Lane 1954] or [Köck 2001, Section 1]. Instead we merely quote the properties of $\mathrm{cr}_n(\Lambda^r)$ that we need. Firstly, $\mathrm{cr}_n(\Lambda^r) = 0$ for $n > r$, as Λ^r is of degree r; secondly, $\mathrm{cr}_r(\Lambda^r)(P_1, \ldots, P_r) = P_1 \otimes \cdots \otimes P_r$; thirdly, if $n < r$ and if $P_1, \ldots, P_n$ are all invertible, then $\mathrm{cr}_n(\Lambda^r)(P_1, \ldots, P_n) = 0$. From these we see that

$$N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)_n = \begin{cases} P^{\otimes r} & \text{if } n = r, \\ Q \otimes P^{\otimes(r-1)} & \text{if } n = r-1, \\ 0 & \text{otherwise.} \end{cases}$$

We can also read off the differential $P^{\otimes r} \to Q \otimes P^{\otimes(r-1)}$ from [ibid., Lemma 2.2]: it is $\mathrm{cr}_r(\Lambda^r)(\varphi, 1, \dots, 1) = \varphi \otimes 1 \otimes \cdots \otimes 1$. So $N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)$ is the acyclic complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & P \otimes P^{\otimes(r-1)} & \xrightarrow{\varphi\otimes 1} & Q \otimes P^{\otimes(r-1)} & \longrightarrow & 0 \\ r+1 & & r & & r-1 & & r-2 \end{array}$$

Of particular note is the special case in which P and Q are equal to R considered as a module over itself, and φ is given by multiplication by some $x \in R^\times$. Then $N\Lambda^r\Gamma(R \xrightarrow{x} R)$ is equal to the complex $(R \xrightarrow{x} R)$, shifted so that it is concentrated in degrees r and $r-1$.

4. Operations on binary multicomplexes

The goal of this section is to extend the functors F_n between multicomplexes of the previous section to functors of *binary* multicomplexes. Together with the results of the previous section, this shows that if $P_\bullet$ is a bounded acyclic binary multicomplex, then so is $\Lambda^r_n(P_\bullet)$.

Categories of binary complexes are not so well behaved as categories of complexes. In particular, the category of bounded acyclic binary complexes in a split exact category is not split exact.

Example 4.1. Let P be an object in a split exact category $\mathcal{P}$. The following diagram is an admissible epimorphism in the category of bounded acyclic binary complexes in $\mathcal{P}$:

$$\begin{tikzcd} P \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] \arrow[d, "1"] & P \oplus P \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d, "\Sigma"] & P \arrow[d] \\ P \arrow[r, shift left, "1"] \arrow[r, shift right, "1"'] & P \arrow[r, shift left] \arrow[r, shift right] & 0 \end{tikzcd}$$

(where i_1 and i_2 are the inclusions into the first and second summands, p_1 and p_2 are the corresponding projections and $\Sigma = p_1 + p_2$). But there is no splitting $P \to P \oplus P$ that commutes with both the top and bottom differentials, so $B^{\mathrm{q}}_{\mathrm{b}}\mathcal{P}$ is not split exact.

This difficulty means that we cannot define exterior powers of binary multicomplexes recursively in exactly the way we have for multicomplexes. This problem is resolvable: we shall show that if $P_\bullet$ is an object of $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(R)$, then the objects of

$\Lambda^r_n(P_\bullet)$ are independent of the differentials of $P_\bullet$. Therefore it will make sense to define the exterior power of a binary complex by applying the exterior powers we developed above individually to the two differentials of the binary complex. The resulting pair of complexes will have the same objects, so we consider them as a binary complex.

Lemma 4.2. *Let $F : \mathcal{P} \to \mathcal{P}$ be a covariant functor on an idempotent complete exact category. If $P_\bullet$ and $Q_\bullet$ are chain complexes with the same underlying graded object, then $NF\Gamma(P_\bullet)$ and $NF\Gamma(Q_\bullet)$ have the same underlying graded object.*

Proof. Let $B \in \mathcal{P}^{\Delta^{\mathrm{op}}}$ be a simplicial object. The objects of the complex $N(B)$ are given by

$$N(B)_n := B_n \Big/ \left(\sum_{i=0}^{n} \mathrm{Im}(\sigma_i : B_{n-1} \to B_n) \right)$$

(after embedding $\mathcal{P}$ in a suitable abelian category), where the σ_i are the degeneracies of B. It is enough therefore to show that the objects and degeneracy maps of $F\Gamma(P_\bullet)$ do not depend upon the differential of $P_\bullet$. The objects of $\Gamma(P_\bullet)$ are direct sums of the objects of $P_\bullet$, indexed by the surjections out of $[n]$ in Δ, and do not depend on the differential. The degeneracy operator $\sigma_i : \Gamma(P_\bullet)_{n-1} \to \Gamma(P_\bullet)_n$ is the image of the degeneracy map $\eta_i : [n] \to [n-1]$ in Δ. For any surjection $\eta : [n-1] \twoheadrightarrow [p]$, the composition $\eta\eta_i$ is also a surjection, so the monomorphism in the epi-monic factorisation of $\eta\eta_i$ is just the identity on $[p]$. Therefore, the degeneracy operator σ_i acts on $\Gamma(P_\bullet)_{n-1}$ by sending the summand corresponding to the surjection η by the identity to the summand of $\Gamma(P_\bullet)_n$ corresponding to the surjection $\eta_i\eta$. Thus σ_i does not depend on the differential of $P_\bullet$. Since the objects and degeneracies of $\Gamma(P_\bullet)$ only depend on the underlying graded object of $P_\bullet$, the same is true of $F\Gamma(P_\bullet)$. Therefore the objects of $NF\Gamma(P_\bullet)$ only depend on the underlying graded object as well. □

Corollary 4.3. *Let $n \geq 1$, and let $P_\bullet$ and $Q_\bullet$ be objects of $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$. If $P_\bullet$ and $Q_\bullet$ have the same underlying $\mathbb{Z}^n$-graded object, then $F_n(P_\bullet)$ and $F_n(Q_\bullet)$ have the same underlying $\mathbb{Z}^n$-graded object.*

Proof. This is a straightforward induction on n. □

We are now ready at last to define exterior powers of acyclic binary multicomplexes. Let $P_\bullet$ be an n-dimensional, bounded, acyclic binary multicomplex in $\mathcal{P}$, i.e., an object of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$. We view the commutativity constraints on the differentials of $P_\bullet$ in the same way as described in Remark 1.2: as a collection of 2^n objects of $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$.

Definition 4.4. For a functor F that satisfies the hypotheses of Corollary 3.5, we define induced functors

$$F_n : (B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P} \to (B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$$

by the following procedure: Let $P_\bullet$ be an object of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$, viewed as a collection of 2^n (nonbinary) multicomplexes in the manner described above. Since these multicomplexes all have the same underlying $\mathbb{Z}^n$-graded object, by Corollary 4.3 the same is true of the 2^n multicomplexes obtained by applying F_n (the functor defined on $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$ in Corollary 3.5) to the multicomplexes describing $P_\bullet$. We define $F_n(P_\bullet)$ to be the binary multicomplex described by the resulting collection of multicomplexes.

We now return to our main example of interest: the exterior power functors. Let R be a commutative ring. We have seen in Example 3.7 that the usual exterior power operations Λ^r satisfy the hypotheses of Corollary 3.5, so the exterior powers

$$\Lambda^r_n : (C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(R) \to (C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(R)$$

lift to exterior powers of binary multicomplexes

$$\Lambda^r_n : (B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(R) \to (B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(R)$$

for all $n \geq 0$ and $r \geq 1$.

5. Simplicial tensor products

In this section we develop a tensor product for multicomplexes that is compatible with the exterior powers we have defined in the previous sections. We show that the class of this product vanishes in the appropriate K-group, which will eventually be the key to showing that exterior power operations provide homomorphisms on higher K-groups.

5A. *Constructing simplicial tensor products.* In this subsection, using the Dold–Kan correspondence again, we construct the so-called simplicial tensor product of multicomplexes and prove it preserves acyclicity and boundedness of complexes.

Although we are ultimately interested in the products induced from the usual tensor products of modules (or sheaves), it is convenient in this section to work in the rather more abstract setting of a generic idempotent complete exact category with some form of well-behaved tensor product.

Definition 5.1. Let $\mathcal{P}$ be an idempotent complete exact category. We say that a biadditive bifunctor $\otimes : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is a *tensor product* if $P \otimes -$ and $- \otimes P$ are exact functors on $\mathcal{P}$ for each object P of $\mathcal{P}$.

For the rest of this section, we fix such a category $\mathcal{P}$ with a tensor product $\otimes$. The reader may wish to keep in mind the example $\mathcal{P} = \mathcal{P}(R)$, with the usual tensor product of R-modules.

Definition 5.2. Let P be an object of $\mathcal{P}$, and let $(Q_\bullet, d_Q)$ and $(R_\bullet, d_R)$ be chain complexes in $\mathcal{P}$.

(1) By $P \otimes Q_\bullet$ we mean the chain complex whose i-th object is $P \otimes Q_i$, with differential $1 \otimes d_Q$. The complex $Q_\bullet \otimes P$ is defined analogously.

(2) By $\mathrm{Tot}(Q_\bullet \otimes R_\bullet)$ we mean the chain complex formed by taking the total complex of the bicomplex whose (i, j)-th object is $Q_i \otimes R_j$, and whose differentials are $d^{\mathrm{ver}} = d_Q \otimes (-1)^j$ and $d^{\mathrm{hor}} = 1 \otimes d_R$. This bicomplex's i-th row is $Q_i \otimes R_\bullet$ and its j-th column is $Q_\bullet \otimes R_j$.

It is clear that if $Q_\bullet$ and $R_\bullet$ are bounded complexes, then the products $P \otimes Q_\bullet$ and $\mathrm{Tot}(Q_\bullet \otimes R_\bullet)$ are bounded as well. We'll need a couple of properties of these products.

Lemma 5.3. *Let $P_\bullet$ be a chain complex in $\mathcal{P}$.*

(1) *The functor*

$$P_\bullet \otimes - : \mathcal{P} \to C\mathcal{P}, \quad Q \mapsto P_\bullet \otimes Q,$$

is exact.

(2) *If $Q_\bullet$ is an acyclic complex in $\mathcal{P}$, then the complex* $\mathrm{Tot}(P_\bullet \otimes Q_\bullet)$ *is acyclic.*

Proof. The first part is straightforward, as each $P_i \otimes -$ is an exact functor. For the second part, if $Q_\bullet$ is acyclic, then, since acyclic complexes are spliced together from short exact sequences, each of the complexes $P_n \otimes Q_\bullet$ is acyclic. Therefore the rows of the bicomplex $P_\bullet \otimes Q_\bullet$ are acyclic. Our complexes are nonnegative, so the total complex of this bicomplex is exact in an ambient abelian category by the acyclic assembly lemma [Weibel 1994, Lemma 2.7.3]. Since $\mathcal{P}$ is idempotent complete, it supports long exact sequences, so $\mathrm{Tot}(P_\bullet \otimes Q_\bullet)$ is acyclic in $\mathcal{P}$. □

To define the simplicial tensor product of complexes we need to go beyond regular simplicial objects. A *bisimplicial object* B in $\mathcal{P}$ is a functor $B : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathcal{P}$. The *diagonal* of B is the simplicial object defined by precomposition with the usual diagonal functor $\mathrm{diag} : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$,

$$\mathrm{diag}(B) := B \circ \mathrm{diag} : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathcal{P}.$$

If C and D are simplicial objects in $\mathcal{P}$, then we define $C \otimes D$ to be the bisimplicial object given by $(C \otimes D)([m], [n]) = C_m \otimes D_n$ and $(C \otimes D)(\alpha, \beta) = C(\alpha) \otimes D(\beta)$ for $\alpha : [m] \to [m']$, $\beta : [n] \to [n']$. We can now push the tensor product around the Dold–Kan correspondence.

Definition 5.4. The *simplicial tensor product* of chain complexes $P_\bullet$ and $Q_\bullet$ in $\mathcal{P}$ is defined to be

$$P_\bullet \otimes_\Delta Q_\bullet := N\big(\mathrm{diag}(\Gamma(P_\bullet) \otimes \Gamma(Q_\bullet))\big).$$

A word of warning here: although the tensor product is an additive functor in each variable, the complex $P_\bullet \otimes_\Delta Q_\bullet$ is not equal to the product complex $\mathrm{Tot}(P_\bullet \otimes Q_\bullet)$

discussed above. They are related by the Eilenberg–Zilber theorem, which we shall use in the proof of the following lemma:

Lemma 5.5. *Let $P_\bullet$ and $Q_\bullet$ be chain complexes in $\mathcal{P}$, and suppose that at least one of them is acyclic. Then $P_\bullet \otimes_\Delta Q_\bullet$ is acyclic in $\mathcal{P}$.*

Proof. We suppose, without loss of generality, that $Q_\bullet$ is acyclic. By the Eilenberg–Zilber theorem [May 1967, Section 29], the simplicial tensor product $P_\bullet \otimes_\Delta Q_\bullet = N\,\mathrm{diag}(\Gamma(P_\bullet) \otimes \Gamma(Q_\bullet))$ is homotopy equivalent to $\mathrm{Tot}(P_\bullet \otimes Q_\bullet)$, and is therefore acyclic by Lemmas 5.3(2) and 2.2. □

The following is an analogue of Lemma 2.7 for the simplicial tensor product:

Lemma 5.6. *If $P_\bullet$ and $Q_\bullet$ are both bounded chain complexes in $\mathcal{P}$, of lengths k and l, respectively, then $P_\bullet \otimes_\Delta Q_\bullet$ is of length at most kl and so is bounded as well.*

Proof. Examining the Dold–Kan functors applied to a tensor product, one sees that the object $(P_\bullet \otimes_\Delta Q_\bullet)_n$ is equal to

$$N\big(\mathrm{diag}(\Gamma(P_\bullet) \otimes \Gamma(Q_\bullet))\big)_n = \bigoplus_{\varphi} P_i \otimes Q_j,$$

where φ runs over all injections $[n] \hookrightarrow [i] \times [j]$ whose composition with the projections onto $[i]$ and $[j]$ gives surjections $[n] \twoheadrightarrow [i]$ and $[n] \twoheadrightarrow [j]$ (this is derived in [Lawson 2012]). The complexes $P_\bullet$ and $Q_\bullet$ are of length k and l, so $P_i = 0$ and $Q_j = 0$ for all $i > k$ and $j > l$. But for $n > kl$ there is no injection $[n] \hookrightarrow [i] \times [j]$, with $i \le k$ and $j \le l$, such that $[n] \twoheadrightarrow [i]$ and $[n] \twoheadrightarrow [j]$ are order-preserving surjections. So $(P_\bullet \otimes_\Delta Q_\bullet)_n = 0$ for $n > kl$. □

We now verify that $\otimes_\Delta$ is a tensor product in the sense of Definition 5.1.

Proposition 5.7. *The simplicial tensor product $\otimes_\Delta$ is a tensor product on the idempotent complete exact category $C\mathcal{P}$ and restricts to a tensor product on the full subcategory $C_b^q\mathcal{P}$.*

Proof. If $P_\bullet$ and $Q_\bullet$ are in $C_b^q\mathcal{P}$, then so is $P_\bullet \otimes_{\Delta,n} Q_\bullet$, by Lemmas 5.5 and 5.6. So it remains to show that $- \otimes_\Delta -$ is biadditive, and that the functors $P_\bullet \otimes_\Delta -$ and $- \otimes_\Delta P_\bullet$ are exact when $P_\bullet$ is in $C\mathcal{P}$.

The functors N and Γ are both additive and exact, so we only need to inspect $\mathrm{diag}(- \otimes -)$. This is easily seen to be biadditive, as $- \otimes -$ is biadditive. Therefore $- \otimes_\Delta -$ is biadditive as well.

Let B be a simplicial object in $\mathcal{P}$. For a short exact sequence of simplicial objects $0 \to A' \to A \to A'' \to 0$, the sequence

$$0 \to \mathrm{diag}(B \otimes A')_n \to \mathrm{diag}(B \otimes A)_n \to \mathrm{diag}(B \otimes A'')_n \to 0$$

is equal to

$$0 \to B_n \otimes A'_n \to B_n \otimes A_n \to B_n \otimes A''_n \to 0,$$

which is short exact since each $0 \to A'_n \to A_n \to A''_n \to 0$ is short exact and $B_n \otimes -$ is exact. So the sequence

$$0 \to \mathrm{diag}(B \otimes A') \to \mathrm{diag}(B \otimes A) \to \mathrm{diag}(B \otimes A'') \to 0$$

is short exact in every degree for any simplicial object B in $\mathcal{P}$. Therefore the functor $\mathrm{diag}(\Gamma(P_\bullet) \otimes -) : \mathcal{P}^{\Delta^{\mathrm{op}}} \to \mathcal{P}^{\Delta^{\mathrm{op}}}$ is exact. The same is true for $\mathrm{diag}(- \otimes \Gamma(P_\bullet))$. It follows that $P_\bullet \otimes_\Delta -$ and $- \otimes_\Delta P_\bullet$ are exact functors. $\square$

We are now ready to iteratively define simplicial tensor products on categories of multicomplexes.

Definition 5.8. We define *simplicial tensor products*

$$\otimes_{\Delta,n} : C^n\mathcal{P} \times C^n\mathcal{P} \to C^n\mathcal{P}$$

for all $n \geq 0$ recursively:

(1) $\otimes_{\Delta,0} : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is the usual tensor product $\otimes$, and

(2) by regarding objects $P_\bullet$ and $Q_\bullet$ of $C^{n+1}\mathcal{P}$ as chain complexes in the idempotent complete exact category $C^n\mathcal{P}$ with the tensor product $\otimes_{\Delta,n}$, we define $P_\bullet \otimes_{\Delta,n+1} Q_\bullet := N\big(\mathrm{diag}(\Gamma(P_\bullet) \otimes_{\Delta,n} \Gamma(Q_\bullet))\big)$.

The following iteration of Proposition 5.7 is now straightforward. The case $n = 0$ is an assumption of this section, and we iterate using $(C^{\mathrm{q}}_{\mathrm{b}})^{n+1}\mathcal{P} = C^{\mathrm{q}}_{\mathrm{b}}((C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P})$.

Corollary 5.9. *For all $n \geq 0$, the simplicial tensor product $\otimes_{\Delta,n}$ is a tensor product in the sense of Definition 5.1 on $C^n\mathcal{P}$ and on $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$.* $\square$

In fact we can say a little more than this. The following lemma is crucial to the proof of the main result of this section:

Lemma 5.10. *Let $P_\bullet$ be an object of $C_{\mathrm{b}}((C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P})$ and let $Q_\bullet$ be an object of $(C^{\mathrm{q}}_{\mathrm{b}})^{n+1}\mathcal{P}$. Then $P_\bullet \otimes_{\Delta,n+1} Q_\bullet$ is an object of $(C^{\mathrm{q}}_{\mathrm{b}})^{n+1}\mathcal{P}$.*

Proof. Noting that $P_\bullet$ and $Q_\bullet$ both have their objects in $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$, and that $Q_\bullet$ is an acyclic complex of objects in that category, this follows immediately from Lemmas 5.5 and 5.6 applied to the tensor product $\otimes_{\Delta,n}$ on the category $(C^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}$. $\square$

We can extend the simplicial tensor products to categories of binary complexes in the same way that we did for exterior powers in Section 4. The simplicial tensor product of a pair of binary complexes $(P_\bullet, d_P, \tilde{d}_P)$ and $(Q_\bullet, d_Q, \tilde{d}_Q)$ is obtained by considering the pair of chain complexes $(P_\bullet, d_C) \otimes_\Delta (Q_\bullet, d_Q)$ and $(P_\bullet, \tilde{d}_P) \otimes_\Delta (Q_\bullet, \tilde{d}_Q)$ as a binary complex (it is straightforward to prove that they have the same underlying graded object, in the same manner as Lemma 4.2). The analogue of Corollary 4.3 then follows, and we define the simplicial tensor product of binary multicomplexes just as we did for a functor of one variable in Definition 4.4.

5B. ***Vanishing of products.*** In this subsection we prove that the class of any simplicial tensor product vanishes in the corresponding K-group. Our proof resembles Grayson's procedure [1992, p. 103] of verifying that the second Euler characteristic of a doubly acyclic bicomplex vanishes.

Let $n > 0$, and let $P_\bullet$ and $Q_\bullet$ be n-dimensional bounded acyclic binary complexes of objects of $\mathcal{P}$. That is, $P_\bullet$ and $Q_\bullet$ are objects of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$. Then the simplicial tensor product $P_\bullet \otimes_{\Delta,n} Q_\bullet$ is in $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$ as well by Corollary 5.9. Since the objects of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$ are the generators of $K_n(\mathcal{P})$, one would like to use $\otimes_{\Delta,n}$ to induce a product $K_n(\mathcal{P}) \times K_n(\mathcal{P}) \to K_n(\mathcal{P})$. On first inspection this appears not to work, because the product $P_\bullet \otimes_{\Delta,n} Q_\bullet$ is not diagonal if only one of $P_\bullet$ or $Q_\bullet$ is diagonal. This is not a problem in the end though, since the whole product vanishes on $K_n(\mathcal{P})$.

Proposition 5.11. *Let $n > 0$. For any pair of n-dimensional bounded acyclic multicomplexes $P_\bullet$ and $Q_\bullet$ in $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$, the class $[P_\bullet \otimes_{\Delta,n} Q_\bullet]$ vanishes in $K_n(\mathcal{P})$.*

Proof. First we filter $P_\bullet$ by degree. Regard $P_\bullet$ as an acyclic binary complex of objects of $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}$. For $i \geq 0$, let $P|_{[0,i]}$ be the binary complex obtained by "restricting" $P_\bullet$ to be supported on $[0, i]$. That is, $(P|_{[0,i]})_j$ is equal to P_j if $0 \leq j \leq i$, and $(P|_{[0,i]})_j = 0$ otherwise. The differentials on $P|_{[0,i]}$ are inherited from $P_\bullet$. We write $P_j[0]$ for P_j considered as a binary complex concentrated in degree 0. Then $P_j[j]$, which denotes P_j considered as a binary complex concentrated in degree j, is the quotient of the inclusion $P|_{[0,j-1]} \hookrightarrow P|_{[0,j]}$ (if $j \geq 1$). If $P_\bullet$ is supported on $[0, n]$, so that $P_j = 0$ for $j > n$, we therefore have an n-stage filtration

$$P_0[0] = P|_{[0,0]} \hookrightarrow P|_{[0,1]} \hookrightarrow \cdots \hookrightarrow P|_{[0,n-1]} \hookrightarrow P|_{[0,n]} = P_\bullet$$

whose successive quotients determine short exact sequences

$$0 \to P|_{[0,j-1]} \to P|_{[0,j]} \to P_j[j] \to 0.$$

We take the simplicial tensor product with $Q_\bullet$ of this whole filtration, obtaining sequences

$$0 \to P|_{[0,j-1]} \otimes_{\Delta,n} Q_\bullet \to P|_{[0,j]} \otimes_{\Delta,n} Q_\bullet \to P_j[j] \otimes_{\Delta,n} Q_\bullet \to 0 \tag{5.12}$$

for $j = 1, \ldots, n$, which are short exact by Corollary 5.9.

By Lemma 5.10, all of the objects are in the right category, so each of the short exact sequences of (5.12) yields an equation

$$[P|_{[0,j]} \otimes_{\Delta,n} Q_\bullet] = [P|_{[0,j-1]} \otimes_{\Delta,n} Q_\bullet] + [P_j[j] \otimes_{\Delta,n} Q_\bullet]$$

in $K_n(\mathcal{P})$. Putting these together gives

$$[P_\bullet \otimes_{\Delta,n} Q_\bullet] = \sum_{j=0}^{n} [P_j[j] \otimes_{\Delta,n} Q_\bullet].$$

To proceed we need to assume a small lemma, for which the second type of relation in $K_n(\mathcal{P})$ (diagonal binary multicomplexes vanish) is crucial.

Lemma 5.13. *The following equality holds in $K_n(\mathcal{P})$:*

$$[P_j[j] \otimes_{\Delta,n} Q_\bullet] = (-1)^j [P_j[0] \otimes_{\Delta,n} Q_\bullet].$$

Continuing with the main proof, our equation now reads

$$[P_\bullet \otimes_{\Delta,n} Q_\bullet] = \sum_{j=0}^{n} (-1)^j [P_j[0] \otimes_{\Delta,n} Q_\bullet].$$

By inspection we see that $\Gamma(P_j[0])$ is the constant simplicial object which has P_j in each degree. The functor

$$\operatorname{diag}\bigl(\Gamma(P_j[0]) \otimes_{\Delta,n-1} -\bigr) : ((C_{\mathrm{b}})^{n-1}\mathcal{A})^{\Delta^{\mathrm{op}}} \to ((C_{\mathrm{b}})^{n-1}\mathcal{A})^{\Delta^{\mathrm{op}}}$$

is therefore isomorphic to the functor

$$P_j \otimes_{\Delta,n-1} - : ((C_{\mathrm{b}})^{n-1}\mathcal{A})^{\Delta^{\mathrm{op}}} \to ((C_{\mathrm{b}})^{n-1}\mathcal{A})^{\Delta^{\mathrm{op}}},$$

since they both have the same effect of "tensoring everywhere by P_j". This functor is additive, so we have an isomorphism of functors

$$N(P_j \otimes_{\Delta,n-1} \Gamma(-)) \cong P_j \otimes_{\Delta,n-1} -.$$

Hence,

$$P_j[0] \otimes_{\Delta,n} Q_\bullet = N \operatorname{diag}\bigl(\Gamma(P_j[0]) \otimes_{\Delta,n-1} \Gamma(Q_\bullet)\bigr) \cong P_j \otimes_{\Delta,n-1} Q_\bullet,$$

so we have

$$[P_\bullet \otimes_{\Delta,n} Q_\bullet] = \sum_{j=0}^{n} (-1)^j [P_j \otimes_{\Delta,n-1} Q_\bullet].$$

There is an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0,$$

since $P_\bullet$ is acyclic. The objects of $Q_\bullet$ are in $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}$, so $- \otimes_{\Delta,n-1} Q_\bullet$ is an exact functor by Lemma 5.3(1), and so the following sequence is exact:

$$0 \to P_n \otimes_{\Delta,n-1} Q_\bullet \to P_{n-1} \otimes_{\Delta,n-1} Q_\bullet \to \cdots \to P_1 \otimes_{\Delta,n-1} Q_\bullet \to P_0 \otimes_{\Delta,n-1} Q_\bullet \to 0.$$

Exact sequences translate into alternating sums in the Grothendieck group, so this exact sequence gives exactly the identity

$$\sum_{j=0}^{n} (-1)^j [P_j \otimes_{\Delta,n-1} Q_\bullet] = 0$$

in $K_0((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P})$, thus the same relation holds in $K_n(\mathcal{P})$. Therefore $[P_\bullet \otimes_{\Delta,n} Q_\bullet] = 0$, as required. □

It remains to prove Lemma 5.13.

Proof of Lemma 5.13. Consider the following diagram as a short exact sequence of binary complexes concentrated in degrees j and $j-1$:

$$\begin{tikzcd}
0 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, tail] & P_j \arrow[d, tail] \\
P_j \arrow[r, shift left, "1"] \arrow[r, shift right, "1"'] \arrow[d, two heads] & P_j \arrow[d, two heads] \\
P_j \arrow[r, shift left] \arrow[r, shift right] & 0
\end{tikzcd}$$

We will use this diagram to show that $[P_j[j] \otimes_{\Delta,n} Q_\bullet] = -[P_j[j-1] \otimes_{\Delta,n} Q_\bullet]$. The argument can be iterated $j-1$ times to yield $[P_j[j] \otimes_{\Delta,n} Q_\bullet] = (-1)^j [P_j[0] \otimes_{\Delta,n} Q_\bullet]$ in $K_n(\mathcal{P})$, as required. For lack of a better notation, we will denote the middle row of the diagram by $(P_j = P_j)$. Then the diagram represents a short exact sequence of binary complexes

$$0 \to P_j[j-1] \to (P_j = P_j) \to P_j[j] \to 0,$$

which upon tensoring with $Q_\bullet$ becomes the short exact sequence

$$0 \to P_j[j-1] \otimes_{\Delta,n} Q_\bullet \to (P_j = P_j) \otimes_{\Delta,n} Q_\bullet \to P_j[j] \otimes_{\Delta,n} Q_\bullet \to 0$$

by Lemma 5.10. Since $Q_\bullet$ is acyclic and has objects in $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}$, each of the terms of this short exact sequence is an object of $(B_{\mathrm{b}}^{\mathrm{q}})^{n}\mathcal{P}$ by Lemma 5.10, so we have a relation

$$[(P_j = P_j) \otimes_{\Delta,n} Q_\bullet] = [P_j[j-1] \otimes_{\Delta,n} Q_\bullet] + [P_j[j] \otimes_{\Delta,n} Q_\bullet]$$

in $K_0((B_{\mathrm{b}}^{\mathrm{q}})^{n}\mathcal{P})$, and hence in $K_n(\mathcal{P})$. We claim that

$$[(P_j = P_j) \otimes_{\Delta,n} Q_\bullet] = 0$$

in $K_n(\mathcal{P})$, so that $[P_j[j] \otimes_{\Delta,n} Q_\bullet] = -[P_j[j-1] \otimes_{\Delta,n} Q_\bullet]$. We can filter $Q_\bullet$ in the same manner that we have filtered $P_\bullet$ in the main proof above:

$$Q_0[0] = Q|_{[0,0]} \hookrightarrow Q|_{[0,1]} \hookrightarrow \cdots \hookrightarrow Q|_{[0,n-1]} \hookrightarrow Q|_{[0,n]} = Q_\bullet,$$

giving short exact sequences

$$0 \to Q|_{[0,i-1]} \to Q|_{[0,i]} \to Q_i[i] \to 0.$$

Upon tensoring with $(P_j = P_j)$, we have short exact sequences

$$0 \to (P_j = P_j) \otimes_{\Delta,n} Q|_{[0,i-1]} \to (P_j = P_j) \otimes_{\Delta,n} Q|_{[0,i]} \to (P_j = P_j) \otimes_{\Delta,n} Q_i[i] \to 0$$

(by Lemma 5.10). Furthermore, since $(P_j = P_j)$ is an acyclic binary complex of objects of $(B_{\mathrm{b}}^{\mathrm{q}})^{n-1}\mathcal{P}$, each of the terms of these short exact sequences is an object of $(B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}$, by Lemma 5.10. We therefore have the equation

$$[(P_j = P_j) \otimes_{\Delta,n} Q_\bullet] = \sum_i [(P_j = P_j) \otimes_{\Delta,n} Q_i[i]]$$

in $K_0((B_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P})$, and hence in $K_n(\mathcal{P})$. But $(P_j = P_j)$ is a diagonal binary complex, as is each $Q_i[i]$ (trivially). The simplicial tensor product of a pair of diagonal complexes is again diagonal, so each of the acyclic binary complexes $(P_j = P_j) \otimes_{\Delta,n} Q_i[i]$ is diagonal and hence vanishes in $K_n(\mathcal{P})$. Therefore $[(P_j = P_j) \otimes_{\Delta,n} Q_\bullet] = 0$, so the desired relation holds. □

This finally completes the proof of Proposition 5.11. Having taken the trouble to set up an alternative product of bounded acyclic binary multicomplexes, one that is compatible with the exterior powers, we've now shown that (like the usual tensor product) it is always zero! It was not all for naught though: at least we know now that the induced operation $\otimes_{\Delta,n} : K_n(\mathcal{P}) \times K_n(\mathcal{P}) \to K_n(\mathcal{P})$ is well-defined. Furthermore, the vanishing of this product proves that the exterior power operations induce homomorphisms on $K_n(R)$ (and, more generally, on the higher K-groups of schemes). This is shown in the next section.

6. Exterior power operations on K-groups of schemes

The goal of this section is to extend the endofunctor Λ_n^r defined in Section 4 to bounded acyclic multicomplexes of locally free modules of finite rank on a scheme X, and to prove that it induces a well-defined operation λ^r on the higher K-group $K_n(X)$. We will see that, for $n > 0$, this operation λ^r is not just a map but in fact a homomorphism.

Let X be a quasicompact scheme, and let $\mathcal{P}(X)$ be the category of locally free $\mathcal{O}_X$-modules of finite rank. Then $\mathcal{P}(X)$ is an exact category in the usual sense. It is idempotent complete but not split exact in general. We write $K_n(X)$ for the K-group $K_n(\mathcal{P}(X))$.

As in Section 3, we inductively define an endofunctor Λ_n^r on $C^n\mathcal{P}(X)$ for $r \geq 1$ and $n \geq 0$ as follows: the functor Λ_0^r is the usual r-th exterior power functor on $C^0\mathcal{P}(X) = \mathcal{P}(X)$, and Λ_n^r is defined as $N\Lambda_{n-1}^r\Gamma$, with N and Γ as introduced in Section 2.

Proposition 6.1. *For all $r, n > 0$, the functor Λ_n^r restricts to an endofunctor on the subcategory $(C_{\mathrm{b}}^{\mathrm{q}})^n\mathcal{P}(X)$ of $C^n\mathcal{P}(X)$.*

Proof. Given any open affine subscheme $U = \mathrm{Spec}(R)$ of X, a straightforward inductive argument shows that the following diagram commutes:

$$\begin{array}{ccc} C^n\mathcal{P}(X) & \xrightarrow{\Lambda^r_n} & C^n\mathcal{P}(X) \\ \downarrow & & \downarrow \\ C^n\mathcal{P}(U) & \xrightarrow{\Lambda^r_n} & C^n\mathcal{P}(U) \\ \| & & \| \\ C^n\mathcal{P}(R) & \xrightarrow{\Lambda^r_n} & C^n\mathcal{P}(R) \end{array}$$

The vertical arrows are induced by the restriction functor $\mathcal{P}(X) \to \mathcal{P}(U)$, $P \mapsto P|_U$, and the lower horizontal arrow is the functor Λ^r_n introduced in Section 3. A complex in $C^n\mathcal{P}(X)$ is acyclic, or bounded, if and only if its restriction to every open affine subscheme has the respective property, so Proposition 6.1 follows from the results of Section 3. □

As in Section 4, one easily deduces that, for any complex $P_\bullet$ in $C^n\mathcal{P}(X)$, the objects in $\Lambda^r_n(P_\bullet)$ do not depend on the differentials in $P_\bullet$. We can therefore extend the endofunctor Λ^r_n to an endofunctor of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$, which we denote by Λ^r_n again. The goal of the rest of this section is to prove the following theorem:

Theorem 6.2. *Let $n > 0$ and $r > 0$. The endofunctor Λ^r_n of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$ induces a well-defined homomorphism $\lambda^r : K_n(X) \to K_n(X)$.*

Definition 6.3. The homomorphism λ^r in the previous theorem is called the *r-th exterior power operation* on $K_n(X)$.

Proof of Theorem 6.2. If $P_\bullet$ is a diagonal multicomplex in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$, then the multicomplex $\Lambda^r_n(P_\bullet)$ is diagonal as well, by definition of Λ^r_n. It therefore suffices to show that the association $[P_\bullet] \mapsto [\Lambda^r_n(P_\bullet)]$ induces a well-defined homomorphism of groups

$$\lambda^r : K_0((B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)) \to K_n(X).$$

Thus we need to show that the equality

$$[\Lambda^r_n(P_\bullet)] = [\Lambda^r_n(P'_\bullet)] + [\Lambda^r_n(P''_\bullet)]$$

holds in $K_n(X)$ for every short exact sequence $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ in $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$. The classes $[\Lambda^{r-i}_n(P'_\bullet) \otimes_{\Delta,n} \Lambda^i_n(P''_\bullet)]$ for $i = 1, \dots, r-1$ vanish in $K_n(X)$ by Proposition 5.11 applied to the category $\mathcal{P} = \mathcal{P}(X)$, where the simplicial tensor product has been constructed inductively from the usual tensor product of quasicoherent $\mathcal{O}_X$-modules. So the desired equality is equivalent in $K_n(X)$ to the more familiar-looking identity

$$[\Lambda^r_n(P_\bullet)] = [\Lambda^r_n(P'_\bullet)] + \sum_{i=1}^{r-1}[\Lambda^{r-i}_n(P'_\bullet) \otimes_{\Delta,n} \Lambda^i_n(P''_\bullet)] + [\Lambda^r_n(P''_\bullet)].$$

In order to prove this latter formula, we cannot just apply the usual formula for the r-th exterior power of a direct sum because the given short exact sequence of binary complexes, $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$, does not split in general, even if X is affine (see Example 4.1). Instead, by induction on n, we construct for every sequence $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ in $(B^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(X)$ a natural induced filtration

$$\Lambda^r_n(P'_\bullet) \hookrightarrow \Lambda^{r-1}_n(P'_\bullet) \wedge_n \Lambda^1_n(P_\bullet) \hookrightarrow \cdots \hookrightarrow \Lambda^1_n(P'_\bullet) \wedge_n \Lambda^{r-1}_n(P_\bullet) \hookrightarrow \Lambda^r_n(P_\bullet)$$

of $\Lambda^r_n(P_\bullet)$ by certain subobjects $\Lambda^{r-i}_n(P'_\bullet) \wedge_n \Lambda^i_n(P_\bullet)$, $i = 0, \dots, r$, of $\Lambda^r_n(P_\bullet)$, also belonging to $(B^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(X)$, together with short exact sequences

$$\begin{aligned} 0 \to \Lambda^{r-i+1}_n(P'_\bullet) \wedge_n \Lambda^{i-1}_n(P_\bullet) \to \Lambda^{r-i}_n(P'_\bullet) \wedge_n \Lambda^i_n(P_\bullet)& \\ \to \Lambda^{r-i}_n(P'_\bullet) \otimes_{\Delta,n} \Lambda^i_n(P''_\bullet) \to 0& \end{aligned} \tag{6.4}$$

for $i = 1, \dots, n$.

For $n = 0$ and $i \in \{0, \dots, r\}$, the object $\Lambda^{r-i}_0(P') \wedge_0 \Lambda^i_0(P)$ is defined to be what is usually meant by $\Lambda^{r-i}(P') \wedge \Lambda^i(P)$: the image of the canonical homomorphism $\Lambda^{r-i}(P') \otimes \Lambda^i(P) \to \Lambda^r(P)$. It is well known that these objects come with the required short exact sequences (6.4).

If $n > 0$ and if, for a moment, the sequence $0 \to P'_\bullet \to P_\bullet \to P''_\bullet \to 0$ is given in $C^{\mathrm{q}}_{\mathrm{b}}(C^{\mathrm{q}}_{\mathrm{b}})^{n-1} \mathcal{P}(X)$ rather than in $(B^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(X)$, we first note that applying the exact functor Γ to the sequence, we get the short exact sequence

$$0 \to \Gamma(P'_\bullet) \to \Gamma(P_\bullet) \to \Gamma(P''_\bullet) \to 0$$

of simplicial objects in $(C^{\mathrm{q}}_{\mathrm{b}})^{n-1} \mathcal{P}(X)$. By the inductive hypothesis, the complexes $\Lambda^{r-i}_{n-1}(\Gamma(P')_m) \wedge_{n-1} \Lambda^i_{n-1}(\Gamma(P)_m)$ for $i = 0, \dots, n$ and $m \geq 0$ are in $(B^{\mathrm{q}}_{\mathrm{b}})^{n-1} \mathcal{P}(X)$ and we have short exact sequences

$$\begin{aligned} 0 \to \Lambda^{r-i+1}_{n-1}(\Gamma(P')_m) \wedge_{n-1} \Lambda^{i-1}_{n-1}(\Gamma(P)_m) \to \Lambda^{r-i}_{n-1}(\Gamma(P')_m) \wedge_{n-1} \Lambda^i_{n-1}(\Gamma(P)_m)& \\ \to \Lambda^{r-i}_{n-1}(\Gamma(P')_m) \otimes_{\Delta,n-1} \Lambda^i_{n-1}(\Gamma(P'')_m) \to 0& \end{aligned}$$

for $i = 1, \dots, r$ and $m \geq 0$. These short exact sequences assemble to short exact sequences of simplicial objects in $(B^{\mathrm{q}}_{\mathrm{b}})^{n-1} \mathcal{P}(X)$. By applying the exact functor N, we finally obtain the required objects

$$\Lambda^{r-i}_n(P') \wedge_n \Lambda^i_n(P) := N\big(\Lambda^{r-i}_{n-1}(\Gamma(P')) \wedge_{n-1} \Lambda^i_{n-1}(\Gamma(P))\big)$$

for $i = 0, \dots, r$ and the required short exact sequences (6.4). As the objects of the multicomplex $\Lambda^{r-i}_n(P') \wedge_n \Lambda^i_n(P)$ are independent of the differentials in the multicomplexes $P'_\bullet$ and $P_\bullet$, this construction of $\wedge_n$ passes to the category $(B^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(X)$ as in Section 4.

From Proposition 6.1 and Section 5 we know that the complex $\Lambda^r_n(P_\bullet)$ and the complexes $\Lambda^{r-i}_n(P'_\bullet) \otimes_{\Delta,n} \Lambda^i_n(P''_\bullet)$ for $i = 0, \dots, r$ belong to $(B^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(X)$. Now a straightforward downwards induction on i based on the short exact sequences (6.4) shows that the complexes $\Lambda_n^{r-i}(P'_\bullet) \wedge_n \Lambda_n^i(P_\bullet)$ for $i = 0, \ldots, r$ are bounded and acyclic, so they belong to $(B_\mathrm{b}^\mathrm{q})^n \mathcal{P}(X)$, as was to be shown. □

7. The second λ-ring axiom

Given a scheme X, there is a "trivial" way to equip the graded abelian group $K_*(X) := \bigoplus_{n\geq 0} K_n(X)$ with a multiplication, and to extend the exterior power operations defined in the previous section to $K_*(X)$ so that they are compatible with addition in $K_*(X)$ in the usual sense. The main result of this section is that they are also compatible with multiplication in the expected way — that is to say, they satisfy the λ-ring axiom (2).

Let X be a quasicompact scheme. We recall that $K_0(X)$ together with the usual exterior power operations $\lambda^r : K_0(X) \to K_0(X)$, $r \geq 0$, is a λ-ring as defined in the introduction (see Chapter V of [Fulton and Lang 1985]). Furthermore, $K_n(X)$ is a $K_0(X)$-module via $[P] \cdot [Q_\bullet] := [P \otimes Q_\bullet]$ for P in $\mathcal{P}(X)$ and $Q_\bullet$ in $(B_\mathrm{b}^\mathrm{q})^n \mathcal{P}(X)$; see also Definition 5.2(1).

We define a multiplication on $K_*(X) := \bigoplus_{n\geq 0} K_n(X)$ by

$$(a_0, a_1, a_2, \ldots)\bullet(b_0, b_1, b_2, \ldots) = (a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_2 b_0, \ldots);$$

in particular, the product of any two elements in $\bigoplus_{n\geq 1} K_n(X)$ is defined to be zero. With this multiplication, $K_*(X)$ is a commutative ring. Furthermore, we define exterior power operations $\lambda^r : K_*(X) \to K_*(X)$, $r \geq 0$, by the formula

$$\lambda^r((a_0, a_1, a_2, \ldots)) = \left(\lambda^r(a_0), \sum_{i=0}^{r-1} \lambda^i(a_0)\lambda^{r-i}(a_1), \sum_{i=0}^{r-1} \lambda^i(a_0)\lambda^{r-i}(a_2), \ldots\right).$$

By definition, we then have $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for all $x \in K_*(X)$. A straightforward calculation using Theorem 6.2 and the fact that $K_0(X)$ satisfies axiom (1) of a λ-ring shows that $K_*(X)$ also satisfies axiom (1). The next theorem addresses axiom (2).

Theorem 7.1. *The pre-λ-ring $K_*(X)$ defined above satisfies axiom* (2) *of a λ-ring.*

Proof. Axiom (2) holds for elements of the form $x = (a_0, 0, 0, \ldots)$ and $y = (b_0, 0, 0, \ldots)$ in $K_*(X)$ because it holds for $K_0(X)$. It also holds for elements of the form $x = (0, a_1, a_2, \ldots)$ and $y = (0, b_1, b_2, \ldots)$ because $\lambda^r(0) = 0$ for all $r \geq 1$ and because every monomial in the ring $\mathbb{Z}[X_1, \ldots, X_r, Y_1, \ldots, Y_r]$ whose coefficient in $P_r(X_1, \ldots, X_r, Y_1, \ldots, Y_r)$ is nonzero is divisible by some product $X_i Y_j$. Furthermore, it suffices to check axiom (2) for x and y belonging to a set of additive generators of $K_*(X)$ because $K_*(X)$ satisfies axiom (1) and because axiom (2) is equivalent to the multiplicativity of the homomorphism

$$\lambda_t : K_*(X) \to 1 + t \cdot K_*(X)[\![t]\!], \quad x \mapsto \sum_{r \geq 0} \lambda^r(x) t^r.$$

We are therefore reduced to showing that the equality

$$\lambda^r(xy) = P_r(\lambda^1(x), \ldots, \lambda^r(x), \lambda^1(y), \ldots, \lambda^r(y)) \tag{7.2}$$

holds in $K_n(X)$ for elements $y \in K_n(X)$ and $x \in K_0(X)$ of the form $x = [\mathcal{E}]$ for some locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank.

We now invoke the projective bundle theorem [Quillen 1973, §8, Theorem 2.1]. We remark that its proof in [loc. cit.] only relies on the additivity and resolution theorems, and not, for instance, on the dévissage theorem or localisation sequence. The additivity and resolution theorems have been proved in [Harris 2015] within the context of Grayson's definition of higher K-groups, so the projective bundle theorem also has a proof within that context, without resorting to topological methods.

It is well known that an iterated application of the projective bundle theorem yields the following splitting principle: there exists a projective morphism $f : Y \to X$ such that $f^*[\mathcal{E}]$ is the sum of invertible $\mathcal{O}_Y$-modules in $K_0(Y)$ and such that $f^* : K_*(X) \to K_*(Y)$ is injective. It is straightforward to check that $f^* : K_*(X) \to K_*(Y)$ is a homomorphism of (pre-)λ-rings. Using the above argument about additive generators again, we are therefore reduced to showing the equality (7.2) only when x is the class $[\mathcal{L}]$ of an invertible $\mathcal{O}_X$-module $\mathcal{L}$. In that case, (7.2) becomes the much simpler formula

$$\lambda^r([\mathcal{L}] \cdot y) = [\mathcal{L}^{\otimes r}] \cdot \lambda^r(y),$$

because $\lambda^2[\mathcal{L}] = \cdots = \lambda^r[\mathcal{L}] = 0$, and because P_r satisfies the identity

$$P_r(1, 0, \ldots, 0, Y_1, \ldots, Y_r) = Y_r$$

and has X-degree r (where X_i is defined to be of degree i for $i = 1, \ldots, r$). Using the argument about additive generators again, it suffices to show that for any object $P_\bullet$ of $(B^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(X)$, the object $\Lambda^r_n(\mathcal{L} \otimes P_\bullet)$ is isomorphic to $\mathcal{L}^{\otimes r} \otimes \Lambda^r_n(P_\bullet)$. This is well known if $n = 0$, and follows by induction on n from the following chain of isomorphisms applied to each of the 2^n multicomplexes associated with the binary multicomplex $P_\bullet$ (which we again denote by $P_\bullet$):

$$\begin{aligned}
\Lambda^r_n(\mathcal{L} \otimes P_\bullet) &= N\Lambda^r_{n-1}\Gamma(\mathcal{L} \otimes P_\bullet) \\
&\cong N(\mathcal{L}^{\otimes r} \otimes \Lambda^r_{n-1}\Gamma(P_\bullet)) \\
&\cong \mathcal{L}^{\otimes r} \otimes N\Lambda^r_{n-1}\Gamma(P_\bullet) \\
&\cong \mathcal{L}^{\otimes r} \otimes \Lambda^r_n(P_\bullet).
\end{aligned}$$

□

8. The final λ-ring axiom

The goal of this section is to prove that the pre-λ-ring $K_*(X)$ (introduced and proven to satisfy λ-ring axiom (2) in the previous section) also satisfies the final λ-ring axiom (3) and is therefore a λ-ring. The main ingredients are the language of polynomial functors, the identification of polynomial functors with modules over the Schur algebra, and Serre's method of computing the Grothendieck group of representations of the group scheme $\mathrm{GL}_{n,\mathbb{Z}}$.

8A. *Polynomial functors.* In this subsection we introduce the notion of polynomial functors and state that the Grothendieck group of the category of polynomial functors over $\mathbb{Z}$ is isomorphic to the universal λ-ring in one variable; see Theorem 8.5 below. This theorem will allow us to prove the final λ-ring axiom for $K_*(X)$ in Subsection 8D. The proof of Theorem 8.5 occupies Subsections 8B and 8C.

We recall $\mathcal{P}(S)$ denotes the category of $\mathcal{O}_S$-modules that are locally free of finite rank on a scheme S. We define a category $\underline{\mathcal{P}}(S)$ "enriched in schemes over S" as follows. The objects are the same as the objects of $\mathcal{P}(S)$, and for every $V, W \in \mathcal{P}(S)$ we have an S-scheme

$$\underline{\mathrm{Hom}}(V, W) := \underline{\mathrm{Spec}}_S \,\mathrm{Sym}^\bullet(\mathcal{H}\mathrm{om}(V, W)^\vee).$$

This is the "physical vector bundle" corresponding to the locally free $\mathcal{O}_S$-module $\mathcal{H}\mathrm{om}(V, W)$ and we have

$$\underline{\mathrm{Hom}}(V, W)(T) = \mathrm{Hom}_{\mathcal{O}_T}(V_T, W_T)$$

for every S-scheme T. In fact, by Yoneda's lemma, we may think of $\underline{\mathrm{Hom}}(V, W)$ as the functor which associates $\mathrm{Hom}_{\mathcal{O}_T}(V_T, W_T)$ with every S-scheme T. The latter viewpoint is used in a lot of literature about polynomial functors. Composition in $\underline{\mathcal{P}}(S)$ is given by the natural maps

$$\underline{\mathrm{Hom}}(U, V) \times_S \underline{\mathrm{Hom}}(V, W) \to \underline{\mathrm{Hom}}(U, W)$$

of schemes over S, and the identities are given by the obvious sections id_V in $\underline{\mathrm{Hom}}(V, V)(S)$.

Definition 8.1. A *polynomial functor over S* is an enriched functor $F : \underline{\mathcal{P}}(S) \to \underline{\mathcal{P}}(S)$. A *morphism of polynomial functors* is a natural transformation. We denote the category of polynomial functors over S by $\mathrm{Pol}(S)$.

In other words, a polynomial functor consists of objects $FV \in \mathcal{P}(S)$, $V \in \mathcal{P}(S)$, and of morphisms of S-schemes

$$F : \underline{\mathrm{Hom}}(V, W) \to \underline{\mathrm{Hom}}(FV, FW) \quad \text{for } V, W \in \mathcal{P}(S),$$

which satisfy the usual functor axioms. In less precise terms, F being a morphism of S-schemes means that if, for instance, $S = \mathrm{Spec}(k)$ with k a field, the map $F : \mathrm{Hom}(V, W) \to \mathrm{Hom}(FV, FW)$ is given by polynomials in coordinates of V and W. Note that we do not ask F to be additive. Every polynomial functor F induces an "ordinary" endofunctor of $\mathcal{P}(S)$, denoted by F again. A morphism $\eta : F \to G$ consists of a morphism of $\mathcal{O}_S$-modules

$$\eta_V : FV \to GV$$

for every $V \in \mathcal{P}(S)$, satisfying the usual conditions for a natural transformation.

Example 8.2 (exterior powers). Functoriality of Λ^d implies that for all $V, W \in \mathcal{P}(S)$ we have a map

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(\Lambda^d V, \Lambda^d W).$$

This is a priori a map of sets, but its formation commutes with base change $T \to S$, and hence by Yoneda it defines a map of S-schemes

$$\underline{\mathrm{Hom}}(V, W) \to \underline{\mathrm{Hom}}(\Lambda^d V, \Lambda^d W).$$

We obtain a polynomial functor $\Lambda^d : \underline{\mathcal{P}}(S) \to \underline{\mathcal{P}}(S)$.

The category $\mathrm{Pol}(S)$ is a $\Gamma(S, \mathcal{O}_S)$-linear category. We declare a sequence

$$0 \to F \to G \to H \to 0$$

in $\mathrm{Pol}(S)$ to be exact if the sequence

$$0 \to FV \to GV \to HV \to 0$$

is exact for every V; this way $\mathrm{Pol}(S)$ becomes an exact category [Touzé 2013, Section 2.1.1]. It carries a tensor product

$$\otimes : \mathrm{Pol}(S) \times \mathrm{Pol}(S) \to \mathrm{Pol}(S)$$

as well as exterior power operators

$$\Lambda^n : \mathrm{Pol}(S) \to \mathrm{Pol}(S), \qquad F \mapsto \Lambda^n F := \Lambda^n \circ F.$$

These data turn $K_0(\mathrm{Pol}(S))$ into a pre-λ-ring. To prove this, one proceeds as in the proof of Theorem 6.2. As there, the category $\mathrm{Pol}(S)$ is in general not split exact, but for every short exact sequence as above, one can construct a natural filtration

$$0 \subset \Lambda^n F \subset F \wedge \cdots \wedge F \wedge G \subset \cdots \subset F \wedge G \wedge \cdots \wedge G \subset \Lambda^n G$$

of $\Lambda^n G$ whose successive quotients are isomorphic to $\Lambda^{n-k} F \otimes \Lambda^k H$, $k = 0, \ldots, n$.

Less evident is that for every morphism $f : T \to S$ there is a natural *base change functor* $f^* : \mathrm{Pol}(S) \to \mathrm{Pol}(T)$. This can be constructed as follows. Let $F : \underline{\mathcal{P}}(S) \to \underline{\mathcal{P}}(S)$ be a polynomial functor. Given $V \in \mathcal{P}(T)$ one chooses an open cover (U_i) of T, vector bundles $V_i \in \underline{P}(S)$ and isomorphisms $\alpha_i : (f^*V_i)|_{U_i} \to V|_{U_i}$. These define gluing data $\alpha_{ij} := \alpha_i^{-1}\alpha_j$ and one constructs the desired $(f^*F)V$ by gluing the bundles $f^*(FV_i)$ over the U_{ij} using the maps $F(\alpha_{ij})$. Note that the expression $F(\alpha_{ij})$ makes sense as F is a polynomial functor. For an alternative description of f^*, see Remark 8.9.

Thus, every polynomial functor $F \in \mathrm{Pol}(S)$ induces a family of functors

$$F_T : \mathcal{P}(T) \to \mathcal{P}(T),$$

indexed by $T \to S$, and that the F_T commute with base change.

The functor f^* is exact, and commutes with the operations $\otimes$ and Λ^n, so that f^* induces a morphism

$$f^* : K_0(\mathrm{Pol}(S)) \to K_0(\mathrm{Pol}(T))$$

of pre-λ-rings.

Definition 8.3. A polynomial functor $F \in \mathrm{Pol}(S)$ is said to be *homogeneous of degree d* if, for every $V \in \mathcal{P}(S)$, the diagram

$$\begin{array}{ccc} \mathbb{G}_{m,S} & \xrightarrow{x \mapsto x^d} & \mathbb{G}_{m,S} \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(V, V) & \xrightarrow{F} & \underline{\mathrm{Hom}}(FV, FV) \end{array}$$

commutes; here, the vertical morphisms are given by scalar multiplication. We denote by $\mathrm{Pol}_d(S)$ the category of polynomial functors homogeneous of degree d, and by $\mathrm{Pol}_{<\infty}(S)$ the category of polynomial functors that are finite direct sums of homogeneous polynomial functors.

Example 8.4. The polynomial functor Λ^d is homogeneous of degree d. The infinite direct sum $\bigoplus_{d\geq 0} \Lambda^d$ is well-defined as it becomes finite when applied to any V; it is a polynomial functor, but not in $\mathrm{Pol}_{<\infty}(S)$.

Let $\mathbb{Z}[s_1, s_2, \dots]$ denote the ring of symmetric functions, with s_i the i-th elementary symmetric function. This is a λ-ring, with $\lambda^i(s_1) = s_i$, also called the universal λ-ring in one variable; see [Yau 2010, §1.3]. It is also a graded ring with $\deg s_d = d$.

Theorem 8.5. *The ring homomorphism*

$$\mathbb{Z}[s_1, s_2, \dots] \to K_0(\mathrm{Pol}_{<\infty}(\mathbb{Z})), \quad s_i \mapsto [\Lambda^i],$$

is an isomorphism of pre-λ-rings.

The proof of this theorem will be given at the end of Subsection 8C.

Corollary 8.6. $K_0(\mathrm{Pol}_{<\infty}(\mathbb{Z}))$ *is a λ-ring.* □

8B. ***The Schur algebra.*** The object of this subsection is to relate polynomial functors to the Schur algebra; see [Roby 1963, Chapters I & IV; Krause 2013, §2] for details.

Throughout this subsection, R is a commutative ring. If M is a locally free R-module and d a nonnegative integer, then the R-module of degree d divided powers is the module of symmetric degree d tensors:

$$\Gamma^d M = \Gamma^d_R M = (M^{\otimes d})^{S_d}.$$

If A is an associative and locally free R-algebra and M is moreover an A-module, then $\Gamma^d_R A$ is a sub-R-algebra of $A^{\otimes d}$ and the obvious multiplication of $\Gamma^d_R A$ on $\Gamma^d_R M$ turns $\Gamma^d_R M$ into a $\Gamma^d_R A$-module.

Let n be a positive integer. Consider the *Schur algebra* $\Gamma^d \operatorname{Mat}(n, R)$ of R associated with n and d. It is free as an R-module. For every R-module V, the module $V^n = \operatorname{Hom}(R^n, V)$ is a right $\operatorname{Mat}(n, R)$-module, hence $\Gamma^d(V^n)$ is a right $\Gamma^d \operatorname{Mat}(n, R)$-module.

Lemma 8.7. *If V is a projective R-module, then $\Gamma^d(V^n)$ is a projective right $\Gamma^d \operatorname{Mat}(n, R)$-module.*

Proof. If V is a direct summand of W, then $\Gamma^d(V^n)$ is a direct summand of $\Gamma^d(W^n)$, so without loss of generality we may assume that V is a free R-module. Then $\Gamma^d(V^n)$ is a direct sum of $\Gamma^d \operatorname{Mat}(n, R)$-modules of the form

$$\Gamma^{d_1}(R^n) \otimes_R \cdots \otimes_R \Gamma^{d_i}(R^n)$$

with $\sum d_i = d$. By [Akin and Buchsbaum 1988, Proposition 2.1] these are projective over the Schur algebra $\Gamma^d \operatorname{Mat}(n, R)$, and the lemma follows. $\square$

We denote by $\mathcal{M}(R, n, d)$ the category of finitely generated left modules over the Schur algebra $\Gamma^d \operatorname{Mat}(n, R)$, and by $\mathcal{M}_{\mathrm{p}}(R, n, d)$ the full subcategory consisting of those modules whose underlying R-module is projective.

We have a "truncation" functor,

$$\operatorname{Pol}_d(R) \to \mathcal{M}_{\mathrm{p}}(R, n, d), \quad F \mapsto F(R^n),$$

where the structure of left $\Gamma^d \operatorname{Mat}(n, R)$-module on $F(R^n)$ is defined as follows. We have a map

$$\underline{\operatorname{End}}(R^n) \xrightarrow{F} \underline{\operatorname{End}}(F(R^n))$$

which is homogeneous of degree d. By the universal property of divided powers (see [Roby 1963, Proposition IV.1; Ferrand 1998, Proposition 2.5.1]), this map is induced by an R-module homomorphism

$$\Gamma^d \operatorname{End}(R^n) \to \operatorname{End}(F(R^n)),$$

which is moreover multiplicative, hence giving $F(R^n)$ the structure of a $\Gamma^d \operatorname{End}(R^n)$-module.

Theorem 8.8. *If $n \geq d$, the functor $\mathrm{Pol}_d(R) \to \mathcal{M}_{\mathrm{p}}(R, n, d)$ is an equivalence of categories.*

Proof. See [Krause 2013], where the same result is shown for polynomial functors taking values in arbitrary R-modules, and arbitrary $\Gamma^d \operatorname{Mat}(n, R)$-modules. The same argument works in our context; we only need to check that the inverse functor maps $\mathcal{M}_{\mathrm{p}}(R, n, d)$ to $\mathrm{Pol}_d(R)$ (that is, that the inverse functor preserves "finite type and projective").

The inverse functor is defined as follows. Let M be a $\Gamma^d \operatorname{Mat}(n, R)$-module. Then we define a functor

$$F_M : \mathrm{Mod}(R) \to \mathrm{Mod}(R), \quad V \mapsto \Gamma^d(V^n) \otimes_{\Gamma^d \operatorname{Mat}(n,R)} M,$$

where the right $\Gamma^d \operatorname{Mat}(n, R)$-module structure on $\Gamma^d(V^n)$ is inherited from the structure of right $\operatorname{Mat}(n, R)$-module on $V^n = \operatorname{Hom}(R^n, V)$. Formation of F_M commutes with base change.

Now assume that both M and V are finitely generated and projective R-modules. Then the module $F_M(V)$ is also finitely generated. We claim that $F_M(V)$ is also projective. By Lemma 8.7 the module $\Gamma^d(V^n)$ is projective, hence a direct summand of a free $\Gamma^d \operatorname{Mat}(n, R)$-module $\bigoplus_I \Gamma^d \operatorname{Mat}(n, R)$, and hence $F_M(V)$ is a direct summand of a projective R-module $\bigoplus_I M$. $\square$

Remark 8.9. Theorem 8.8 gives an alternative way for producing the base change of a polynomial functor. If $R \to S$ is a map of commutative rings, and if M is a $\Gamma^d \operatorname{Mat}(n, R)$-module, then the base change $M \otimes_R S$ is a $\Gamma^d \operatorname{Mat}(n, S)$-module, since formation of $\Gamma^d \operatorname{Mat}(n, -)$ commutes with base change.

8C. ***The Grothendieck group of polynomial functors over $\mathbb{Z}$.*** We fix n and d satisfying $n \geq d$. For brevity we write $\mathcal{M}(R) := \mathcal{M}(R, n, d)$ and $\mathcal{M}_{\mathrm{p}}(R) := \mathcal{M}_{\mathrm{p}}(R, n, d)$. Furthermore we write $\mathbb{Z}[s_1, s_2, \dots]_d$ for the weighted degree d part of the polynomial ring $\mathbb{Z}[s_1, s_2, \dots]$. It is equal to $\mathbb{Z}[s_1, \dots, s_n]_d$.

In this subsection, following [Serre 1968], we compute the Grothendieck group $K_0(\mathcal{M}_{\mathrm{p}}(\mathbb{Z}))$. Together with Theorem 8.8 this then implies Theorem 8.5.

If R is an integral domain, there is a natural homomorphism

$$K_0(\mathcal{M}_{\mathrm{p}}(R)) = K_0(\mathrm{Pol}_d(R)) \to \mathbb{Z}[s_1, s_2, \dots]_d \quad \text{(by Theorem 8.8)}$$

that sends a polynomial functor F to the weights of the action of $\mathbb{G}_m^n$ on $F(R^n)$.

Theorem 8.10. *For every field K the map $K_0(\mathcal{M}(K)) \to \mathbb{Z}[s_1, s_2, \dots]_d$ is an isomorphism.*

Proof. See [Green 1980, Sections 2.2 and 3.5, especially Remark 3.5(ii)]. Green assumes the field K to be infinite, but this assumption is only used in relating modules over $\Gamma^d \operatorname{Mat}(n, K)$ to representations of the monoid $\operatorname{Mat}(n, K)$, as opposed to representations of the *monoid scheme* $\mathrm{Mat}_{n,K}$, which would also work over a finite field K. See also [Jantzen 2003, Section II.A]. $\square$

We will deduce from the cases $K=\mathbb{Q}$ and $K=\mathbb{F}_p$ in this theorem that the map

$$K_0(\mathrm{Pol}_d(\mathbb{Z})) \to \mathbb{Z}[s_1, s_2, \dots]_d$$

is an isomorphism. The proof is essentially identical to Serre's proof [1968] that

$$K_0(\mathrm{GL}_{n,\mathbb{Z}}) \to K_0(\mathrm{GL}_{n,\mathbb{Q}})$$

is an isomorphism.

Lemma 8.11 (projective resolutions). *The canonical map*

$$K_0(\mathcal{M}_{\mathrm{p}}(\mathbb{Z})) \to K_0(\mathcal{M}(\mathbb{Z}))$$

is an isomorphism.

Proof. (Compare [Serre 1968, §§2.2–2.3].) Let M be a finitely generated module over $\Gamma^d\,\mathrm{Mat}(n,\mathbb{Z})$. If M can be generated by m elements, we obtain a presentation

$$0 \to P_0 \to P_1 \to M \to 0$$

with $P_1 = (\Gamma^d\,\mathrm{Mat}(n,\mathbb{Z}))^m$. Both P_1 and P_0 are torsion-free, hence projective as $\mathbb{Z}$-modules. The same argument as [Serre 1968, Proposition 4] shows that $[P_0]-[P_1] \in K_0(\mathcal{M}_{\mathrm{p}}(\mathbb{Z}))$ is independent of the choice of presentation, and that $M \mapsto [P_0]-[P_1]$ defines a two-sided inverse to the map of the proposition. $\square$

Lemma 8.12 (localisation sequence). *The obvious sequence*

$$\bigoplus_{\ell \text{ prime}} K_0(\mathcal{M}(\mathbb{F}_\ell)) \to K_0(\mathcal{M}(\mathbb{Z})) \to K_0(\mathcal{M}(\mathbb{Q})) \to 0$$

is exact.

Proof. The argument is identical to [Serre 1968, Théorème 1]. The main point is to verify that every $\Gamma^d\,\mathrm{Mat}(n,\mathbb{Q})$-module V of finite $\mathbb{Q}$-dimension contains a $\Gamma^d\,\mathrm{Mat}(n,\mathbb{Z})$-submodule Λ with $\mathbb{Q}\otimes_{\mathbb{Z}}\Lambda = V$. To construct such Λ, take an arbitrary sub-$\mathbb{Z}$-module Λ_0 with $\mathbb{Q}\otimes_{\mathbb{Z}}\Lambda_0 = V$, and take $\Lambda := \Gamma^d\,\mathrm{Mat}(n,\mathbb{Z})\Lambda_0$. $\square$

Lemma 8.13 (decomposition maps). *For every prime ℓ there is a unique homomorphism d_ℓ making the triangle*

$$\begin{tikzcd} K_0(\mathcal{M}_p(\mathbb{Z})) \arrow[r] \arrow[d] & K_0(\mathcal{M}(\mathbb{Q})) \arrow[dl, "d_\ell"] \\ K_0(\mathcal{M}(\mathbb{F}_\ell)) & \end{tikzcd}$$

commute.

Proof. The argument is identical to [Serre 1968, Théorème 2]. $\square$

Lemma 8.14. *The composition*

$$K_0(\mathcal{M}(\mathbb{Q})) \xrightarrow{d_\ell} K_0(\mathcal{M}(\mathbb{F}_\ell)) \to K_0(\mathcal{M}(\mathbb{Z}))$$

is the zero map.

Proof. See [Serre 1968, Lemme 4]. Similarly to there, one uses that for every $\mathbb{Z}$-torsion-free $\Gamma^d \operatorname{Mat}(n, \mathbb{Z})$-module Λ the map $\Lambda \to \ell\Lambda,\ x \mapsto \ell x$, is an isomorphism of $\Gamma^d \operatorname{Mat}(n, \mathbb{Z})$-modules. □

Proposition 8.15. *For every prime ℓ, the map d_ℓ is an isomorphism.*

Proof. By Theorem 8.10 the functors

$$\Lambda^{d_1} \otimes \Lambda^{d_2} \otimes \cdots \otimes \Lambda^{d_m}$$

with $\sum d_i = d$ define a basis of $K_0(\mathcal{M}(\mathbb{Q}))$ and of $K_0(\mathcal{M}(\mathbb{F}_\ell))$. Since the map d_ℓ preserves this basis, it is an isomorphism. □

Corollary 8.16. *The canonical map*

$$K_0(\mathcal{M}(\mathbb{Z})) \to K_0(\mathcal{M}(\mathbb{Q}))$$

is an isomorphism.

Proof. By Proposition 8.15 and Lemma 8.14 the maps

$$K_0(\mathcal{M}(\mathbb{F}_\ell)) \to K_0(\mathcal{M}(\mathbb{Z}))$$

are the zero maps. But then the localisation sequence of Lemma 8.12 shows that the map $K_0(\mathcal{M}(\mathbb{Z})) \to K_0(\mathcal{M}(\mathbb{Q}))$ is an isomorphism. □

Proof of Theorem 8.5. The degree d part of the homomorphism

$$\mathbb{Z}[s_1, s_2, \dots] \to K_0(\operatorname{Pol}_{<\infty}(\mathbb{Z})), \quad s_i \mapsto [\Lambda^i],$$

is obviously inverse to the composition of the isomorphisms

$$K_0(\operatorname{Pol}_d(\mathbb{Z})) \cong K_0(\mathcal{M}_{\mathrm{p}}(\mathbb{Z})) \cong K_0(\mathcal{M}(\mathbb{Z})) \cong K_0(\mathcal{M}(\mathbb{Q})) \cong \mathbb{Z}[s_1, s_2, \dots]_d$$

given by Theorem 8.8, Lemma 8.11, Corollary 8.16 and Theorem 8.10, respectively, and is hence bijective and compatible with exterior power operations. □

Remark 8.17. The category $\operatorname{Pol}_d(R)$ is equivalent with the category of weight d representations of the monoid $\operatorname{Mat}_{n,R}$, which forms a full subcategory of the category of representations of $\mathrm{GL}_{n,R}$. Rather than translating Serre's argument from the GL_n to the Mat_n context, one could also deduce our result from Serre's. However, some care has to be taken because the right adjoint to the inclusion, mapping a $\mathrm{GL}_{n,\mathbb{Z}}$-representation V to the largest subrepresentation that extends to $\operatorname{Mat}_{n,\mathbb{Z}}$, is not exact; see [Jantzen 2003, Section II.A].

8D. ***Proof of the final λ-ring axiom.*** In this subsection, we derive from Corollary 8.6 that, for every quasicompact scheme X, the pre-λ-ring $K_*(X)$ satisfies the final λ-ring axiom. Together with Theorem 7.1, this implies that $K_*(X)$ is a λ-ring and finishes the proof of this paper's main result.

Theorem 8.18. *The ring $K_*(X)$ equipped with the exterior power operations defined in Section 7 satisfies axiom (3) of a λ-ring.*

Proof. Let $r, s \geq 1$ and $n \geq 0$. For every $x \in K_n(X)$ we want to show that the identity

$$\lambda^r(\lambda^s(x)) = P_{r,s}(\lambda^1(x), \dots, \lambda^{rs}(x)) \tag{8.19}$$

holds in $K_n(X)$. We recall that, if $n \geq 1$, all products occurring on the right-hand side of (8.19) are trivial (and hence that the right-hand side of (8.19) happens to be just a multiple of $\lambda^{rs}(x)$). We will show the stronger statement that the identity (8.19) in fact holds in $K_0((B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X))$ for all $x \in K_0((B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X))$. Now the products occurring on the right-hand side of (8.19) are induced by the simplicial tensor product introduced in Section 5; these products become trivial in $K_n(X)$ by Proposition 5.11. By a standard argument (see the proof of Theorem 7.1) we may assume that x is the class of an object $P_\bullet$ of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$. One easily checks, for instance using the Gabriel–Quillen embedding theorem [Thomason and Trobaugh 1990, Theorem A.7.1 and Proposition A.7.16], that for every exact category $\mathcal{P}$ and any skeletally small category $\mathcal{I}$, the category of functors from $\mathcal{I}$ to $\mathcal{P}$ is again an exact category in the obvious way. In particular, the category $\mathrm{End}((B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X))$ of endo-functors of $(B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)$ is an exact category. Furthermore it carries a tensor product and exterior power operations (given by $F \mapsto \Lambda^d_n \circ F$). Via the homomorphism $K_0\big(\mathrm{End}((B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X))\big) \to K_0((B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X))$ given by $F \mapsto F(P_\bullet)$, the desired identity now follows from the even stronger identity

$$[\Lambda^r_n \circ \Lambda^s_n] = P_{r,s}([\Lambda^1_n], \dots, [\Lambda^{rs}_n]) \tag{8.20}$$

in $K_0(\mathrm{End}((B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X)))$, which we now prove. We remember that the identity (8.20) (with the subscripts n omitted) holds in the Grothendieck group $K_0(\mathrm{Pol}_{<\infty}(\mathbb{Z}))$ by Corollary 8.6. Then it also holds in $K_0(\mathrm{Pol}^0_{<\infty}(\mathbb{Z}))$, where $\mathrm{Pol}^0_{<\infty}(\mathbb{Z})$ denotes the full subcategory of $\mathrm{Pol}_{<\infty}(\mathbb{Z})$ consisting of functors F satisfying $F(0) = 0$; this follows from the fact that the canonical inclusion $\mathrm{Pol}^0_{<\infty}(\mathbb{Z}) \hookrightarrow \mathrm{Pol}_{<\infty}(\mathbb{Z})$ is split by $F \mapsto \big(V \mapsto \ker(F(V) \to F(0))\big)$. The identity (8.20) therefore follows from Corollary 8.6 once we have shown that we have a pre-λ-ring homomorphism

$$K_0(\mathrm{Pol}^0_{<\infty}(\mathbb{Z})) \to K_0\big(\mathrm{End}((B^{\mathrm{q}}_{\mathrm{b}})^n\mathcal{P}(X))\big) \tag{8.21}$$

that sends the class of the identity functor to the class of the identity functor. By base change (see Subsection 8A), every functor in $\mathrm{Pol}(\mathbb{Z})$ induces a functor in $\mathrm{Pol}(U)$ for every open subset U of X and this construction is compatible with restriction with respect to any inclusion of open subsets of X. The constructions of Sections 3, 4 and 6 therefore inductively induce a functor

$$\mathrm{Pol}^0_{<\infty}(\mathbb{Z}) \to \mathrm{End}((B^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(X));$$

this functor is exact and compatible with tensor products and exterior power operations, as one easily verifies by induction on n. Thus it induces the desired homomorphism (8.21) and the proof of Theorem 8.18 is complete. □

Remark 8.22. We have seen in the previous proof that the λ-ring axiom (3) already holds in $K_0((B^{\mathrm{q}}_{\mathrm{b}})^n \mathcal{P}(X))$, i.e., before dividing out the subgroup generated by classes of diagonal multicomplexes. The same holds true for the λ-ring axiom (2). This can be shown similarly by using Corollary 8.6 or by using the characteristic-free Cauchy decomposition as constructed in [Akin et al. 1982]. Whereas Corollary 8.6 only proves the existence of short exact sequences, Akin, Buchsbaum and Weyman [Akin et al. 1982] *explicitly* construct short exact sequences that prove axiom (2) of a λ-ring. The problem of explicitly describing short exact sequences of polynomial functors that prove axiom (3) seems however to be even harder than the famous and related plethysm problem in representation theory. Such explicit short exact sequences for the plethysm $\Lambda^2 \circ \Lambda^2$ can be found in [Akin and Buchsbaum 1985, page 175]. Although there also exist solutions of the classical plethysm problem for $\Lambda^r \circ \Lambda^2$ and $\Lambda^2 \circ \Lambda^s$, we are not aware of any corresponding characteristic-free short exact sequences.

Appendix: Proof of Lemma 3.4

In this appendix we prove Lemma 3.4, which states that:

(1) If $\mathcal{P}$ is an idempotent complete exact category, then so are $C^n\mathcal{P}$ and $(C^{\mathrm{q}})^n\mathcal{P}$.

(2) If $\mathcal{P}$ is a split exact category, then so is $(C^{\mathrm{q}})^n\mathcal{P}$.

Note that to prove each of these statements it is enough to prove the case $n = 1$.

Proof of Lemma 3.4(1). Let $e : P_\bullet \to P_\bullet$ be an idempotent map of chain complexes. Then each map $e_n : P_n \to P_n$ is an idempotent of $\mathcal{P}$ and so has a kernel $\ker(e_n)$ which is an object of $\mathcal{P}$. By the universal property of kernels, the chain map on $P_\bullet$ induces a map $\ker(e_n) \to \ker(e_{n-1})$ for each n, and these assemble to form a chain complex of kernels. Thus every idempotent in $C\mathcal{P}$ has a kernel in $C\mathcal{P}$, so $C\mathcal{P}$ is idempotent complete. To show that $C^{\mathrm{q}}\mathcal{P}$ is idempotent complete as well, we must show that this kernel chain complex is acyclic in $\mathcal{P}$ if $P_\bullet$ is. To do this, it suffices to consider the case when the complex is a short exact sequence; the general case then follows because $\mathcal{P}$ supports long exact sequences.

If $P_\bullet = (0 \to P_2 \to P_1 \to P_0 \to 0)$ and if $e : P_\bullet \to P_\bullet$ is an idempotent chain map, then $P_\bullet$ is isomorphic to a sequence of the form

$$0 \to \ker(e_2) \oplus \operatorname{im}(e_2) \to \ker(e_1) \oplus \operatorname{im}(e_1) \to \ker(e_0) \oplus \operatorname{im}(e_0) \to 0.$$

Furthermore, as the morphisms in this short exact sequence commute with the idempotents $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, they split as direct sums of induced morphisms. Hence the sequence

$$0 \to \ker(e_2) \to \ker(e_1) \to \ker(e_0) \to 0$$

is exact as well. □

Proof of Lemma 3.4(2). We wish to show that every admissible monomorphism $i : P_\bullet \rightarrowtail Q_\bullet$ in $C^{\mathrm{q}}\mathcal{P}$ is split; that is, that there exists a chain map $s : Q_\bullet \to P_\bullet$ such that each $s_n i_n : P_n \to Q_n$ is the identity. Let us restrict to the case in which $P_\bullet$ and $Q_\bullet$ are short exact sequences of $\mathcal{P}$. Consider the diagram

$$\begin{tikzcd} P' \arrow[r, tail, "j_P"] \arrow[d, tail, "i'"] & P \arrow[r, two heads, "q_P"] \arrow[d, tail, "i"] & P'' \arrow[d, tail, "i''"] \\ Q' \arrow[r, tail, "j_Q"] & Q \arrow[r, two heads, "q_Q"] & Q'' \end{tikzcd}$$

and fix a splitting s'' for i''. We claim that there exist splittings s' and s of i' and i such that the resulting $s_\bullet : Q_\bullet \to P_\bullet$ is a chain map (and hence a splitting of $i_\bullet$). The general case follows from this claim. Indeed, since acyclic complexes are spliced together from short exact sequences, we construct a splitting for a monomorphism of acyclic complexes $i : P_\bullet \rightarrowtail Q_\bullet$ by splitting each monomorphism of short exact sequences separately. The part of the claim concerning a fixed splitting s'' allows us to choose these splittings of short exact sequences in a compatible manner (beginning in degree 0). So it is enough to prove the claim.

We choose compatible splittings h_P and t_P of j_P and q_P, respectively, i.e., $j_P h_P + t_P q_P = 1$. We also choose a splitting s_0 for i and now set $s = j_P h_P s_0 + t_P s'' q_Q$. Then we compute

(1) $si = j_P h_P s_0 i + t_P s'' q_Q i = j_P h_P + t_P s'' i'' q_P = j_P h_P + t_P q_P = 1$,

(2) $q_P s = q_P j_P h_P s_0 + q_P t_P s'' q_Q = s'' q_Q$,

so s is a splitting for i, and s and s'' commute with q_P, q_Q. We therefore get an induced map of kernels $s' : Q' \to P'$ satisfying $j_P s' = s j_Q$. Moreover, $j_P s' i' = s j_Q i' = s i j_P = j_P$, and j_P is monic, so $s' i' = 1$. □

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