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Rational mixed Tate motivic graphs

Susama Agarwala and Owen Patashnick

We study the combinatorics of a subcomplex of the Bloch–Kriz cycle complex that was used to construct the category of mixed Tate motives. The algebraic cycles we consider properly contain the subalgebra of cycles that correspond to multiple logarithms (as defined by Gangl, Goncharov and Levin). We associate an algebra of graphs to our subalgebra of algebraic cycles. We give a purely combinatorial criterion for admissibility. We show that sums of bivalent graphs correspond to coboundary elements of the algebraic cycle complex. Finally, we compute the Hodge realization for an infinite family of algebraic cycles represented by sums of graphs that are not describable in the combinatorial language of Gangl et al.

1. Introduction

Let $\mathcal{M}_T$ denote the category of mixed Tate motives and denote its associated Galois group by G_T. This Galois group has been defined in the literature in at least two distinct contexts, first by [Bloch 1991; Bloch and Kriz 1994] but also by [Levine 1993] in what turned out to be Voevodsky's formalism (see [Deligne and Goncharov 2005], for example). Note that Spitzweck [2001; n.d.] and Levine [2005] have shown that the two definitions are equivalent.

We will take the Bloch–Kriz construction as our definition of M_T and G_T.

Although a significant amount of work has gone into understanding G_T, there is still much that is unknown about Tate motives, even over a number field k. In particular, the connection between G_T and the unipotent completions $\pi^1(\mathbb{P}^1_k - n \text{ points})^{\text{unip}}$ of $\pi^1(\mathbb{P}^1_k - n \text{ points})$ is still of current interest.

For $N \geq 1$, let k_N be the cyclotomic field over $\mathbb{Q}$ generated by an N-th root of unity, and $\mathbb{O}_{k_N}$ its ring of integers. Let $M_{T,N}$ denote the full Tannakian subcategory of M_T generated by the motivic fundamental group of $\mathbb{P}^1_{k_N} - \{0, \infty, \mu_N\}$, with associated motivic Galois group $G_{T,N}$ and algebra of periods $\mathcal{P}_{T,N}$. Here μ_N denotes the set of the N-th roots of unity, though geometrically it could just be a set of N distinct points of $\mathbb{C}^*$. A question, probably going back to Grothendieck, is

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how much of the motivic fundamental group G_T is measured by $G_{T,N}$, in particular $G_{T,1}$. This subcategory, and its integral analogues, were studied by Deligne and Goncharov [2005]. They showed that, over a number field, $\mathcal{P}_{T,N}(\mathbb{O}_{k_N})$ is generated as a $\mathbb{Q}$ vector space by values of multiple polylogarithms. There is a natural categorical inclusion $M_{T,N} \hookrightarrow M_T$, which induces surjections $\phi_N : G_T(\mathbb{O}_{k_N}) \twoheadrightarrow G_{T,N}(\mathbb{O}_{k_N})$ (equivalently an injection $\mathcal{P}_{T,N} \hookrightarrow \mathcal{P}_T$). Brown [2012], in the case $N = 1$, and Deligne [2010], in the cases $N \in \{2, 3, 4, 6, 8\}$, showed that ϕ was an isomorphism. Conversely, and more interestingly, Goncharov [2001a] showed that for most N, ϕ has a nontrivial kernel. Little is known about this kernel. Even less is known about this kernel if the ground field is a cyclotomic extension of a general number field (as opposed to a cyclotomic extension of $\mathbb{Q}$). In particular, all known constructions of elements of M_T lie in the subcategory $M_{T,N}$.

What is sorely needed is an approach to construct more general elements of M_T, especially ones that do not come from the motivic fundamental groups of $\mathbb{G}_m - \mu_N$. This paper is motivated in part by the desire to find a suitable framework to study this kernel. We do not claim to have found such a framework here, but are hopeful that we have taken a first step in the right direction.

The Bloch–Kriz definition of M_T relies heavily on the theory of algebraic cycles. While general enough to capture all mixed Tate motives, traditional methods of representing algebraic cycles (such as in terms of formal linear combinations of systems of polynomial equations) are notoriously difficult to work with, so progress in capitalizing on this description of the category to illuminate outstanding conjectures in the field has been slow. Gangl, Goncharov and Levin [Gangl et al. 2009] suggest a simpler way to understand a subcategory of $\mathcal{M}_T$ by relating specific algebraic cycles to rooted, decorated, binary trees. This approach necessarily restricts focus to motives generated by the motivic fundamental groups of $\mathbb{G}_m - \mu_N$. Any attempt to study the kernel of ϕ defined above requires a more general framework.

Soudères [2016a; 2016b] extends the family of algebraic cycles studied by Gangl et al. to include those over a more general base scheme, in particular giving a rigorous construction of unital values of the multiple polylogarithms, i.e., multiple zeta values, as periods (and not just nonunital values of the multiple logarithms). The combinatorial properties of these algebraic cycles, however, are as yet unexplored.

Let $\mathcal{A}$ be the differential graded algebra (DGA) of cycles introduced by Bloch and Kriz [1994]. In this paper we generalize the Gangl–Goncharov–Levin construction as follows: We define a subalgebra of cycles, $\mathcal{A}^{\times}_{1L} \subset \mathcal{A}$, that properly contains the subalgebra associated to multiple logarithms studied in [Gangl et al. 2009], and reinterprets $\mathcal{A}^{\times}_{1L}$ in terms of graphs. By considering graphs, as opposed to trees, and by loosening the valence restriction on the vertices, we enrich the tools available to study algebraic cycles. Therefore, we are able to consider a larger subcomplex of cycles. We hope this will lead to a better understanding of the complexity and richness underlying the Bloch–Kriz cycle complex, even in the restricted subclass we consider. In particular, in Section 4, we describe several examples of classes of algebraic cycles that define motives. Most of these cycles cannot be described by the trees presented in [Gangl et al. 2009]. In Section 5, we compute the Hodge realization of an infinite family of such classes. Furthermore, in Section 3, we present a purely graphical interpretation of admissibility for the family of algebraic cycles we consider. We also give valency requirements for which classes of algebraic cycles will always be coboundaries in $H^0(B(\mathcal{G}_{1L}))$. There is a lot of interesting combinatorial structure in the types of underlying graphs — and their linear combinations — that give rise to allowable classes of motives. We have barely begun to explore this structure and feel strongly that it deserves further study.

The plan for the paper is as follows. In Section 2, we review mixed Tate motives à la [Bloch and Kriz 1994] and introduce the subalgebra, $\mathcal{A}^{\times}_{1L}$, of $\mathbb{P}^1_k$-linear parametrizable cycles of the algebra $\mathcal{A}$ of admissible cycles. This subalgebra is the focus of our attention. We then define a subcomplex $B(\mathcal{A}^{\times}_{1L})$ of the bar construction on admissible cycles, $B(\mathcal{A})$. The category of comodules over $H^0(B(\mathcal{A}^{\times}_{1L}))$ is the (sub)category of motives we wish to study.

Section 3 introduces an algebra of graphs, $\mathcal{G}_{1L}$, that corresponds to the algebra $\mathcal{A}^{\times}_{1L}$. Theorem 3.63 shows that the two algebras are isomorphic as DGAs. Since $\mathcal{A}^{\times}_{1L}$ is a subalgebra of $\mathcal{A}$, this implies that there is an injection from the algebra of graphs developed in this paper to the full Bloch Kriz cycle complex. In the process, we show, in Theorem 3.59, that the conditions for an arbitrary irreducible $\mathbb{P}^1_k$-linear cycle to be admissible, that is, a generator of $\mathcal{A}^{\times}_{1L}$, can be defined and computed completely from the graphical properties of the corresponding graph in $\mathcal{G}_{1L}$.

In Section 4, we give examples of classes in and results about $H^0(B(\mathcal{G}_{1L}))$. In addition we show, in Corollary 4.14, that in any completely decomposable (sum of) graphs either each summand has a valence-two vertex, or none do. We further show, in Theorem 4.16, that if a completely decomposable (sum of) graphs has valence-two vertices, it is a coboundary in $B(\mathcal{G}_{1L})$.

In Section 5, following the algorithm as outlined in [Bloch and Kriz 1994; Gangl et al. 2009] and especially [Kimura 2013], we compute the Hodge realization of a projective system of classes whose defining cycles are not describable by trees. (All previously known explicit computations of the Bloch–Kriz Hodge realization have been of cycles that can be described by trees.)

2. A subcomplex of algebraic cycles

In this section, we define a particular subcomplex of the Bloch–Kriz cycle complex that we develop in this paper. We begin with a review of the general mixed Tate motive construction via algebraic cycles. Then we proceed to describe parametrized cycles, and finally define the subcomplex of $\mathbb{P}_{1L}$-cycles that we use in the remainder of this paper.

2A. ***A review of mixed Tate motives.*** We work with the category of mixed Tate motives over a field k, $\mathcal{M}(T)$, as constructed by [Bloch 1991; Bloch and Kriz 1994]. When k is a number field, this construction does not depend on any conjectures. In [Bloch and Kriz 1994], two conjectures are stated: that $\mathrm{gr}_r K_n(F) \otimes \mathbb{Q} \cong \mathrm{CH}^r(\mathrm{Spec}(F), n) \otimes \mathbb{Q}$, and that a certain algebra is quasiisomorphic to its Sullivan 1-model. The first conjecture was subsequently proved more generally for all varieties X independently by Bloch [1994; 1986], Levine [1994] and Spivakowsky (unpublished). The second conjecture, which is a strengthening of the Beilinson–Soulé conjecture for fields, is known for number fields by the work of Borel and Yang [1994] on the rank conjecture. (The Beilinson–Soulé conjecture was already known to be true for number fields by the work of Borel [1974]).

In the rest of this section we review some details of their construction, following [Bloch and Kriz 1994] closely.

We assume the reader is familiar with the concepts of algebraic cycles, higher Chow groups, minimal models, 1-minimal models and the bar construction for a commutative differential graded algebra (DGA) A. For the reader who wishes to refresh her memory: The concept of a generalized minimal model is due originally to Quillen (see [Quillen 1970], for example). In the form used here (extensions by free one-dimensional models) it is due originally to Sullivan [1977, discussion starting p. 316]. A good reference for the applications of minimal models we have in mind is the treatment in [Kriz and May 1995, Part IV]. The bar construction is due originally to Eilenberg and Mac Lane. Good references for the use of the bar construction in this paper are [Chen 1976; Bloch and Kriz 1994, Section 2].

In order to define the category of mixed Tate motives, $\mathcal{M}(T)$, it suffices to define its motivic Galois group G_T [Bloch and Kriz 1994]. Equivalently, one may work with its dual Hopf algebra, $\mathcal{H}_T$. This is defined from the DGA, $\mathcal{A}$, of admissible algebraic cycles.

Below, following [loc. cit.], we define how to derive a Hopf algebra from a commutative graded DGA, A, which is cohomologically connected. That is, $H^0(A) = \mathbb{Q}$ and $H^{-n}(A) = 0$ for $n > 0$. Our DGA, A, is not a Hopf algebra in general, as the differential does not decompose. The strategy, therefore, is to "linearize" A, i.e., form the minimal model $\mathcal{H}(A)$ of A, which, by construction, is a Hopf algebra which is quasiisomorphic to A. The minimal model can be constructed quite explicitly via the bar construction. We start with a few definitions.

Definition 2.1. (1) Consider the commutative DGA, $A = \bigoplus_i A_i$. Here, we refer to the grading on A by degree: $\deg(a) = i \Leftrightarrow a \in A_i$. The tensor algebra, $T(A) = \bigoplus_n A^{\otimes n}$, is a commutative algebra under the shuffle product, Ш.

(2) Let $D(A)$ be the ideal in $T(A)$ of degenerate tensor products, defined by

$$\{a_1 \otimes \cdots \otimes a_n \mid a_i \in A,\ a_j \in k \ \text{ for some } \ j\}.$$

(3) The bar construction on A is defined as

$$B(A) = T(A)/D(A).$$

It is a bigraded algebra, with grading given by tensor degree and algebraic degree. The total degree of a monomial $a_1 \otimes \cdots \otimes a_n \in B(A)$ is defined by a shift in the degree of the tensor components in A. That is,

$$\operatorname{tot}\deg(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n} (\deg(a_i) - 1).$$

Hence, the total degree of an element of the bar construction is the difference between the algebraic degree and the tensor degree. Write the bar construction as $B(A) = \bigoplus_{i,j} B(A)^i_j$, where

$$B(A)^i_j = \bigoplus_{\sum_1^i (j_k - 1) = j} A_{j_1} \otimes \cdots \otimes A_{j_i}$$

has total degree j.

Since A is a DGA, it is endowed with a differential structure $\partial : A \to A$ and a product structure $\mu : A \otimes A \to A$. These both extend to define differential structures on the bar construction $B(A)$, called the algebraic and multiplicative differentials, respectively. Thus $(B(A), \partial + \mu)$ is the following bicomplex:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r, "\mu"] & B(A)^3_0 \arrow[u, "d"] \arrow[r, "\mu"] & B(A)^2_1 \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^1_2 \arrow[u, "\partial"] \arrow[r, "\varepsilon"] & 0 \\
\cdots \arrow[r, "\mu"] & B(A)^3_{-1} \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^2_0 \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^1_1 \arrow[u, "\partial"] \arrow[r, "\varepsilon"] & 0 \\
\cdots \arrow[r, "\mu"] & B(A)^3_{-2} \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^2_{-1} \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^1_0 \arrow[u, "\partial"] \arrow[r, "\varepsilon"] & \mathbb{Q} \\
\cdots \arrow[r, "\mu"] & B(A)^3_{-3} \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^2_{-2} \arrow[u, "\partial"] \arrow[r, "\mu"] & B(A)^1_{-1} \arrow[u, "\partial"] \arrow[r, "\varepsilon"] & 0 \\
 & \vdots \arrow[u, "\partial"] & \vdots \arrow[u, "\partial"] & \vdots \arrow[u, "\partial"] &
\end{tikzcd} \tag{2.2}$$

Further details and calculations involving the bar complex can be found in Section 4.

When A is connected, cohomologically connected and generated in degree one (a $K(\pi, 1)$ in the sense of Sullivan), then its minimal model is isomorphic to $\mathcal{H}(A) := H^0(B(A))$, where the cohomology is taken under the total derivative $\partial + \mu$. Note that $B(A)$ is a Hopf algebra, with a product structure given by the signed shuffle product and a coproduct structure given by deconcatenation, which satisfy all the axioms for a Hopf algebra. Note that while the product introduces a degree-dependent sign fact, the coproduct has no such sign. This induces a well-defined product, coproduct, and Hopf algebra structure on $\mathcal{H}(A)$.

Bloch and Kriz study a bar construction of a DGA of admissible cycles, $\mathcal{A} = \bigoplus_i \mathcal{A}_i$, defined below. The Hopf algebra H_T dual to the motivic Galois group G_T is exactly the Hopf algebra defined above for the algebra of admissible cycles.

Definition 2.3. (1) Denote $\mathbb{P}^1_k \setminus \{1\}$ by $\square$. Then we may write $\square^n = (\mathbb{P}^1_k \setminus \{1\})^n$. The boundary of this space is defined when one of the coordinates is set to 0 or ∞.

(2) For $I, J \subset \{1, \dots, n\}$ two disjoint subsets, write $F_{I,J}$ to indicate the codimension-$|I \cup J|$ face of $\square^n$ with the coordinates in I set to 0 and the coordinates in J set to ∞. Write $F_{\varnothing,\varnothing} = \square^n$ to indicate the entire space.

(3) As usual, let $\mathcal{Z}^p(\square^n)$ be the free abelian group generated by algebraic cycles of codimension p in $\square^n$. These are the elements of weight p.

(4) Write $\mathcal{Z}^p(\operatorname{Spec} k, n) \subset \mathcal{Z}^p(\square^n)$ for the free abelian subgroup generated by *admissible* algebraic cycles. A cycle $\mathcal{Z} \in \mathcal{Z}^p(\operatorname{Spec} k, n)$ is one that intersects each face $F_{I,J}$ of $\square^n$ in codimension p, or not at all.

(5) Let Alt be the alternating projection with respect to the action of the group $\mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ on $\mathcal{Z}^p(\operatorname{Spec} k, n)$. Here the symmetric group $\mathfrak{S}_n$ acts by permutation of coordinates, and the i-th copy of $(\mathbb{Z}/2\mathbb{Z})^n$ acts by taking a coordinate to its multiplicative inverse.

(6) Write
$$\mathcal{A}^n_i = \operatorname{Alt}(\mathcal{Z}^n(\operatorname{Spec} k, 2n - i) \otimes \mathbb{Q}),$$
where i is the degree of the algebraic cycle and n the codimension. This is a bigraded algebra, by weight and degree. The weight-graded pieces, $\mathcal{A}^n := \bigoplus_i \mathcal{A}^n_i = \bigoplus_i \operatorname{Alt}(\mathcal{Z}^n(\operatorname{Spec} k, 2n - i) \otimes \mathbb{Q})$, define a complex by the differential operator defined in (2.6). Each degree-graded piece is $\mathcal{A}_i := \bigoplus_n \mathcal{A}^n_i = \bigoplus_n \operatorname{Alt}(\mathcal{Z}^n(\operatorname{Spec} k, 2n - i) \otimes \mathbb{Q})$.

Remark 2.4. The main result of Section 3C is to identify which cycles are elements in $\mathcal{A}$. In order to determine *which* algebraic cycles are admissible, we must consider the space of *all* algebraic cycles, including those that are not admissible. Therefore, when we write $\mathcal{Z}^p(\square^n)$, we mean the entire space of algebraic cycles. We denote admissible cycles by the notation $\mathcal{Z}^p(\operatorname{Spec} k, n)$.

We now define the DGA structure of $\mathscr{A}$. Consider two admissible cycles,

$$Z_i \in \mathscr{Z}^n(\operatorname{Spec} k, 2n-i) \quad \text{and} \quad Z_j \in \mathscr{Z}^m(\operatorname{Spec} k, 2m-j).$$

Write $(Z_i, Z_j) \in \mathscr{Z}^{n+m}(\operatorname{Spec} k, 2(n+m)-(i+j))$ to indicate the admissible cycle defined by Z_i on the first $2n-i$ coordinates and Z_j on the last $2m-j$ coordinates. The product on the associated elements in $\mathscr{A}$ is given by

$$\mu(\operatorname{Alt}\mathscr{Z}_i \otimes \operatorname{Alt}\mathscr{Z}_j) = \operatorname{Alt}(\mathscr{Z}_i, \mathscr{Z}_j) = (-1)^{ij} \operatorname{Alt}(\mathscr{Z}_j, \mathscr{Z}_i),$$

where we drop the $\otimes\mathbb{Q}$ notation for simplicity. The last inequality comes from the properties of Alt, and defines a graded commutative structure on $\mathscr{A}$.

Definition 2.5. An element $Z \in \mathscr{A}$ is decomposable if it can be expressed as the product of two nontrivial cycles.

Next, we define the differential structure on $\mathscr{A}$. Consider $Z \in \mathscr{A}$. Let $\partial_{j,0}\mathscr{Z}$ indicate the intersection of $\mathscr{Z}$ with the face $F_{j,\varnothing}$. Similarly, let $\partial_{j,\infty}\mathscr{Z}$ indicate the intersection of $\mathscr{Z}$ with the face $F_{\varnothing,j}$. These two operators define the differential ∂ on $\mathscr{A}$:

$$\partial\mathscr{Z} = \sum_{j=1}^{2n-i} (-1)^{j-1}(\partial_{j,0} - \partial_{j,\infty})\mathscr{Z}. \tag{2.6}$$

Remark 2.7. It is difficult to identify elements of $\mathscr{A}$, that is, to classify the cycles that satisfy the condition of admissibility. One of the achievements of this paper is to give a clear, simple condition for identifying admissible cycles for a large subclass of cycles, called $\mathbb{P}^1_k$-linear cycles. In particular, see Theorem 3.59.

For an element $\varepsilon \in \bigoplus_n B(\mathscr{A})^n_i$ to define a class in $H^i(B(\mathscr{A}))$, each graded component must have decomposable algebraic boundary. This comes from the fact that $(\partial + \mu)(\varepsilon) = 0$. In order to define what it means for a cycle to have decomposable boundary, let π_m be the projection of ε onto the m-th tensor component. That is, $\pi_m(\varepsilon) \in B(\mathscr{A})^m_i$. Then, for each m, $\partial(\pi_m\varepsilon)$ is a decomposable element.

Definition 2.8. Consider an element $\varepsilon \in B(\mathscr{A})$.

(1) The projection, $\pi_i(\varepsilon) \in B(\mathscr{A})^n_i$, is decomposable if it has a decomposable algebraic boundary. That is, if there exists an $\varepsilon' \in B(\mathscr{A})^{n+1}_{i+1}$ such that $\partial(\pi_i(\varepsilon)) = -\mu(\varepsilon')$. That is, the coboundary of the projection $\pi_i(\varepsilon)$ is in the image of the product map μ.

(2) An element $\varepsilon \in B(\mathscr{A})$ is completely decomposable if $\pi_i(\varepsilon)$ is decomposable for all i, with

$$\partial\pi_i(\varepsilon) = -\mu\pi_{i+1}(\varepsilon).$$

Definition 2.9. We say that the element $\varepsilon \in \bigoplus_n B(\mathscr{A})^n_i$ is minimally decomposable if it is completely decomposable, and cannot be written as a sum of two nontrivial completely decomposable elements. That is, one cannot write $\varepsilon = \varepsilon_1 + \varepsilon_2$, where each $\varepsilon_i \neq 0$ and is completely decomposable.

Remark 2.10. Notice that if ε is minimally decomposable, it is determined (up to shuffle products) by $\pi_{n_0}(\epsilon)$, where n_0 is the smallest integer for which $\pi_n(\varepsilon) \neq 0$. Therefore, by abuse of notation, we say that $\pi_{n_0}(\varepsilon)$ defines a class in $H^i(B(\mathscr{A}))$. In all examples in this paper, $n_0 = 1$.

Next we give an example of an admissible cycle that defines a class in $H^0(B(\mathscr{A}))$.

Example 2.11. Consider the cycle $Z_T(a) = \mathrm{Alt}(t, 1-t, 1-a/t) \in \mathscr{A}^2_1$. This is a parametric representation of the algebraic cycle determined by the system of equations $\{x + y = 1,\ xz = x + a\}$. This is the Torato cycle [1992] with codimension 2 in $\square^3$. It is a degree-one element in $\mathscr{A}$, $Z_T(a) \in \mathscr{A}^2_1$,

We check that $Z_T(a)$ has a completely decomposable boundary. Therefore, it defines a class in $H^0(B(\mathscr{A}))$. To see this, compute $\partial Z_T(a)$. The intersections $\partial_{\infty,i} Z_T(a)$ give the empty cycles for $i \in \{1, 2, 3\}$. This is because setting one of the coordinates of $Z_T(a)$ to ∞ sets a different coordinate to 1. The same holds for $\partial_{0,1} Z_T(a)$ and $\partial_{0,2} Z_T(a)$. Therefore,

$$\partial Z_T(a) = \partial_{0,3} Z_T(a) = \mathrm{Alt}(a, 1-a) = \mu[\mathrm{Alt}(a) \mid \mathrm{Alt}(1-a)].$$

The last equality comes from the product structure on $\mathscr{A}$. Since (a) and $(1-a)$ are constant cycles, $\partial[\mathrm{Alt}(a) \mid \mathrm{Alt}(1-a)] = 0$ by the Leibnitz rule. Therefore, $Z_T(a) \oplus -[\mathrm{Alt}(a) \mid \mathrm{Alt}(1-a)] \in \ker(\partial + \mu)$. Since $Z_T(a)$ has total degree 0 in $B(\mathscr{A})$, it defines a class in $H^0(B(\mathscr{A}))$.

The Hodge realization functor associates the period $\mathrm{Li}_2(a)$ to the cycle $Z_T(a)$ [Bloch and Kriz 1994]. To do this, consider the $\mathscr{A}$ module, $\mathscr{T}$, defined by maps from n-simplices, Δ_n, to $\square^n$. There is an element $\zeta(a)$ in the circular bar construction $B(\mathscr{T}, \mathscr{A})$ such that $\zeta(a) + 1 \otimes Z_T(a)$ defines a class in $H^0(B(\mathscr{T}, \mathscr{A}))$. The summands of $\zeta(a)$ that are supported completely on Δ_2 define the integrand of the associated period.

This example hints at another shortcoming of the current state of technology surrounding algebraic cycles. We are interested in defining elements of $B(\mathscr{A})$ that define classes of $H^0(B(\mathscr{A}))$. In particular, we are interested in cycles with boundaries that can be written as products of other cycles, as is the case for the Torato cycle in Example 2.11. In Section 4B1, we provide several examples of such sums of cycles in weight 4. However, we have not yet addressed this issue of how to find such sums in general. We hope that the graphical point of view presented here will shed light on the problem of identifying cycles with completely decomposable boundaries. We leave this for future work.

2B. ***A subalgebra of $\mathcal{A}$.*** Unfortunately, the standard parametric notation for cycles as represented in [Bloch and Kriz 1994; Gangl et al. 2007; 2009; Totaro 1992] is rather misleading. For example, consider the usual form for the Totaro cycle, $Z_T(a) = \mathrm{Alt}(t, 1-t, 1-a/t) \in \mathcal{A}_1^2$, defined in Example 2.11, and in the literature [Totaro 1992; Gangl et al. 2009]. It is technically defined on $\square_k^3 = (\mathbb{P}_k^1 - \{1\})^3$, but is written as if it is defined on $\mathbb{A}_k^3 = (\mathbb{P}_k^1 - \{\infty\})^3$. In actuality, the Totaro cycle (for $a \in k^*$) is an algebraic cycle defined by the system of equations

$$\left\{x + y = 1,\ xz = x - a : (x, y, z) \in (\mathbb{P}_k^1 - \{1\})^3\right\}$$

together with a parametrization map $\mathbb{P}_k^1 \to (\mathbb{P}_k^1 - \{1\})^3$. However, when manipulated in practice, the cycle is understood

- to come equipped with a parametrization map, and
- to be defined at the hyperplanes with one coordinate equal to ∞, and not defined at the hyperplanes with one coordinate equal to 1.

This is unnecessarily obtuse. It can be described as the intersection of the image of

$$\mathbb{P}_k^1 \to (\mathbb{P}_k^1)^3, \quad (T : U) \mapsto \left(\frac{T}{U}, \frac{U-T}{U}, \frac{T-aU}{T}\right),$$

with the complement of the hyperplanes of $(\mathbb{P}_k^1)^3$ defined by setting some coordinate equal to 1.

In light of this example, we work with parametrized cycles.

Definition 2.12. A parametrized cycle is a pair, (Z, ϕ), consisting of an algebraic cycle $Z \in \mathcal{Z}^p(\square^n)$ and a parametrization $\phi : \mathbb{P}_k^{n-p} \to (\mathbb{P}_k^1)^n$ satisfying the following: ϕ induces a map on the group of algebraic cycles,

$$\phi_* : \mathcal{Z}^0(\mathbb{P}_k^{n-p}) \to \mathcal{Z}^p((\mathbb{P}_k^1)^n).$$

Then, given the inclusion $i : \square^n \hookrightarrow (\mathbb{P}_k^1)^n$, we have

$$Z = i^*\phi_*(\mathbb{P}_k^{n-p}),$$

where $\mathbb{P}_k^{n-p}$ is the generator of $\mathcal{Z}^0(\mathbb{P}_k^{n-p})$.

For $Z \in \mathcal{Z}^p(\square^n)$, write the parametrizing map $\phi = (\phi_1, \dots, \phi_n)$, where each ϕ_i corresponds to the image in a coordinate of $\square^n$. There are, of course, multiple possible parametrizations of any cycle $Z \in \mathcal{Z}^p(\square^n)$. Here we are interested in the algebraic cycles themselves, not the particular parametrizations. If the same cycle Z can be represented by two different parametrizations, (Z, ϕ) and (Z, ϕ'), we say that ϕ and ϕ' are equivalent parametrizations. We are interested in cycles that *can* be endowed with a $\mathbb{P}_k^1$-linear parametrization.

Definition 2.13. A cycle $Z \in \mathscr{Z}^p(\square^n)$ is $\mathbb{P}^1_k$-linear if it can be parametrized by a ϕ such that each component can be written as

$$\phi_j \in \left\{ \left(1 - \frac{t_1}{a_j t_2}\right)^{\zeta}, \left(1 - \frac{t_2}{a_j t_1}\right)^{\zeta}, \left(\frac{t_1}{a_j t_2}\right)^{\zeta} \right\},$$

with $a_j \in k^\times$ and $\zeta \in \{\pm 1\}$. In particular, writing the j-th $\mathbb{P}^1_k$ in the image of ϕ as $[U_j : V_j]$, we define $\phi_j = U_j/V_j$, using the standard affine representation. Such a ϕ is called a $\mathbb{P}^1_k$-linear parametrization, and can be written as a map on $\mathbb{P}^1_k$ via the following commutative diagram:

$$\begin{tikzcd} \mathbb{P}^1_k \arrow[r] \arrow[d, "\phi_j"] & \mathbb{P}^{n-p}_k \arrow[d, "\phi"] \\ \mathbb{P}^1_k \arrow[r, hook, "i_j"'] & (\mathbb{P}^1_k)^n \end{tikzcd}$$

The top arrow is given by a map

$$(t_1 : t_2) \mapsto (0 : \cdots : 0 : t_1 : 0 : \cdots : 0 : t_2 : 0 : \cdots : 0),$$

and the bottom arrow is given by inclusion into the j-th coordinate.

Definition 2.14. Denote the free abelian groups of $\mathbb{P}^1_k$-linear cycles by $\mathscr{Z}^n_{1L}(\square^m)$. Write $\mathscr{Z}_{1L}{}^n(\operatorname{Spec} k, m)$ for the free abelian group of $\mathbb{P}^1_k$-linear *admissible* cycles.

The goal of this section is to define a sub-DGA of $\mathscr{A}$, the algebra of admissible cycles, that is generated by $\mathscr{Z}_{1L}{}^n(\operatorname{Spec} k, 2n-i)$. Call it

$$\mathscr{A}_{1L} = \bigoplus_i \mathscr{A}_{1L,i} = \bigoplus_{n,i} \operatorname{Alt} \mathscr{Z}_{1L}{}^n(\operatorname{Spec} k, 2n-i) \otimes \mathbb{Q}.$$

The graded commutative structure on $\mathscr{A}_{1L}$ comes from the product structure on $\mathscr{A}$, along with the fact that the product of two parametrizable cycles is still parametrizable. It remains to check that the differential structure on $\mathscr{A}$ is well-defined on $\mathscr{A}_{1L}$. The differential on $\mathscr{A}$ comes from intersecting each coordinate of an element $\operatorname{Alt} Z \in \mathscr{A}^n_i$ with the appropriate 0 and ∞ face of $\square^{2n-i}_k$. Consider $Z \in \mathscr{Z}_{1L}{}^n(\operatorname{Spec} k, 2n-i)$. Let ϕ be a parametrization on Z. Then the intersection of Z with a particular face corresponds to the pullback of ϕ by the appropriate face map. Therefore, the differential of Z is also a $\mathbb{P}^1_k$-linear parametrizable cycle.

If $\operatorname{Alt} Z \in \mathscr{A}_{1L}$ is a decomposable cycle of codimension i, write

$$\operatorname{Alt} Z = \operatorname{Alt}(Z_1, \ldots, Z_r)$$

as above. The Leibnitz rule and properties of Alt show that $\partial \operatorname{Alt}(Z_1, \ldots, Z_r)$ is also parametrizable.

The algebra $\mathscr{A}_{1L}$ contains all the Totaro cycles. Moreover, it contain a large class of cycles which correspond to the multiple logarithms [Gangl et al. 2009]. Therefore, conjecturally, it contains all the cycles necessary to define the full category of mixed Tate motives. There has been some effort to understand subalgebras of $\mathscr{A}_{1L}$ in terms of polylogarithms and multiple logarithms [Gangl et al. 2009; Soudères 2016a]. Here we study a subalgebra $\mathscr{A}^{\times}_{1L} \subset \mathscr{A}_{1L}$ that specifically excludes the Totaro cycles, but still contains the multiple logarithms.

Definition 2.15. Let $\mathscr{A}^{\times}_{1L}$ be the algebra of $\mathbb{P}^1_k$-linear cycles, where

$$\phi_i \in \left\{ \left(1 - \frac{t_1}{a_i t_2}\right)^{\zeta}, \left(1 - \frac{t_2}{a_i t_1}\right)^{\zeta} \right\},$$

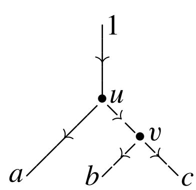
with $a_i \in k^{\times}$ and $\zeta \in \{\pm 1\}$.

The combinatorics of the cycles in $\mathscr{A}^{\times}_{1L}$ are studied in Section 3. The graphs introduced in Section 3 correspond to the subalgebra $\mathscr{A}^{\times}_{1L}$, which excludes cycles with coordinates of the form $a_i t_i / t_j$.

3. Motivic graphs

The first graphical description of some of the algebraic cycles that arise in the category $\mathcal{M}(T)$ of mixed Tate motives was given by [Gangl et al. 2007; 2009] in their description of R-deco trees. These provide a description of a particular proper sub-DGA of $\mathscr{A}^{\times}_{1L}$.

In particular, they represent a subalgebra of cycles by labeled oriented trees. For example,

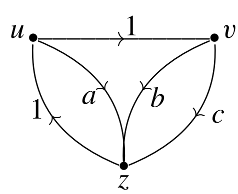


$$\mapsto \left[1 - \frac{1}{u}, 1 - \frac{u}{a}, 1 - \frac{u}{v}, 1 - \frac{v}{b}, 1 - \frac{v}{c}\right].$$

Note that this assignment depends on several choices, such as a choice of root vertex as well as a choice of affine patch.

In this section we give a more general graphical depiction that encapsulates all $\mathscr{A}^{\times}_{1L}$ cycles using decorated, oriented, non-simply connected graphs.

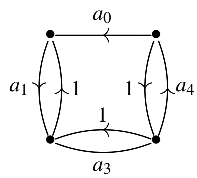
For example, the tree and cycle above come from the labeled oriented graph



$$\mapsto \left[1 - \frac{z}{u}, 1 - \frac{u}{az}, 1 - \frac{u}{v}, 1 - \frac{v}{bz}, 1 - \frac{v}{cz}\right],$$

by taking the affine patch at $z = 1$. Graphically this amounts to removing the vertex labeled z and changing the labels from the edges of the graph to the leaves and root of the tree.

Our approach produces far more algebraic cycles that are not seen via the approach given in [Gangl et al. 2007; 2009]. In particular, we can study cycles represented by graphs that cannot be represented by a tree in any affine patch. For example, the graph



in this paper corresponds to the algebraic cycle

$$\operatorname{Alt}\Big[1-\frac{z}{x},\,1-\frac{x}{a_1 z},\,1-\frac{w}{z},\,1-\frac{z}{a_2 w},\,1-\frac{y}{w},\,1-\frac{w}{a_3 y},\,1-\frac{y}{a_0 x}\Big].$$

Yet there is no affine patch one can take (i.e., a vertex one can remove) that will result in a tree of the form studied in [Gangl et al. 2009].

The aim of Section 3 is to construct an algebra of graphs, $\mathscr{G}_{1L} = \bigoplus_{\bullet,\star} \mathscr{G}_{1L}{}^{\bullet}_{\star}$ that is isomorphic to the algebra of admissible cycles $\mathscr{A}^{\times}_{1L}$ as DGAs. The definition of this algebra is given at the end of Section 3D. Most of Sections 3A–3D are devoted to building up $\mathscr{G}_{1L}$ step by step. We begin with a general set of oriented graphs with labeled and ordered edges, $\mathscr{G}(k^{\times})$. This corresponds to the set of generators of the free abelian group $\mathscr{Z}^{\bullet}_{1L}(\square^{2\bullet-\star})$. We define a monoid structure on the set so that $\mathscr{G}(k^{\times})$ generates an algebra, $\mathbb{Q}[\mathscr{G}]$. Then we consider the alternating representation on the graphs, by imposing an equivalence relation on them by the ordering of their edges. This gives an algebra homomorphism from $\mathbb{Q}[\mathscr{G}]^{\bullet}_{\star}/\sim_{\mathrm{ord}}$ to the algebra of cycles $\operatorname{Alt}\mathscr{Z}^{\star}_{1L}(\square^{2\bullet-\star})$.

However, we wish for a DGA homomorphism to the algebra of admissible, $\mathbb{P}^1_k$-linear cycles, $\mathscr{A}^{\times}_{1L} \subset \mathscr{Z}^{\star}(\operatorname{Spec} k, 2\bullet-\star)$. To do this, we define a subset of $\mathscr{G}_{\mathrm{ad}}(k^{\times}) \subset \mathscr{G}(k^{\times})$, which we show corresponds to admissible graphs in Theorem 3.59. We write $\mathbb{Q}[\mathscr{G}_{\mathrm{ad}}]$ to indicate the algebra generated by $\mathscr{G}_{\mathrm{ad}}(k^{\times})$. In order to establish a DGA isomorphism between $\mathscr{A}^{\times}_{1L}$ and $\mathbb{Q}[\mathscr{G}_{\mathrm{ad}}]$, we must define a differential operator on graphs. To do this, we need two further equivalence relations among graphs, which we call $\sim_{\mathrm{v}}$ and $\sim_{\mathrm{ori}}$. In Section 3C, we show that $\mathbb{Q}[\mathscr{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$ is a DGA of graphs.

In Section 3D, we show one of the main findings of this paper, that *admissibility of $\mathbb{P}^1_k$-linear cycles can be encoded purely by labeled oriented graphs.* In particular, there is no further algebraic input necessary. Imposing the third equivalence relation gives the desired isomorphism

$$\mathscr{G}_{1L} = \mathbb{Q}[\mathscr{G}_{\mathrm{ad}}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}, \sim_{\mathrm{ori}}) \simeq \mathscr{A}^{\times}_{1L}.$$

3A. ***An interesting algebra of graphs.*** In this section, we introduce a general set of biconnected graphs with oriented, labeled, and ordered edges. We impose a product structure on it. This defines an algebra of graphs that corresponds to the algebra of general (not necessarily admissible) algebraic cycles.

We work over a number field k.

Definition 3.1. Let $\mathcal{G}(k^\times)$ be the set of graphs with biconnected connected components, with oriented and ordered edges, each labeled by an element of $k^\times \times \mathbb{Z}/2\mathbb{Z}$.

In practice, we say that the edges of G are labeled by a nonzero number and a sign.

For a graph $G \in \mathcal{G}(k^\times)$, let $V(G)$ be the set of vertices of G, and $E(G)$ be the unordered set of edges of the graph. However, we are working with graphs with ordered edges. Therefore we must consider the ordered set of edges.

Definition 3.2. Let $\omega(G)$ be the ordered set of edges of G, where $\omega(e)$ expresses the ordinality of the edge $e \in E(G)$ in $\omega(G)$. Write $\mathrm{sgn}_{\omega(e)}$ to indicate the sign associated the edge e.

The loop number, or first Betti number, of a graph $G \in \mathcal{G}(k^\times)$ is

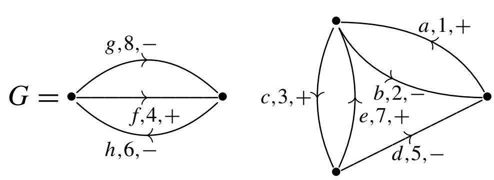
$$h^1(G) = |E(G)| - |V(G)| + h^0(G), \tag{3.3}$$

where $h^0(G)$ counts the number of connected components of the graph. The vector space $H^1(G)$ is spanned by graphical cycles of the unoriented graph underlying G.

Remark 3.4. There are multiple conventions regarding the definition of cycles in graphs in the literature. We take $L \subset E(G)$, together with an orientation (possibly different from the orientation on the individual edges in $E(L)$) is a graphical cycle of the graph G if it defines a path in G that starts and ends at the same vertex. Specifically, the path in G defined by the edges of L does not need to respect the orientation of the edges in L. A graphical *loop* is a graphical cycle that does not intersect itself until the final vertex.

We will concern ourselves only with graphical loops of $G \in \mathcal{G}(k^\times)$.

Example 3.5. Consider the disconnected graph G given by



These are in $\mathcal{G}(k^\times)$, assuming $a, \ldots, g$ are all in $k^\times$. The second labels indicate the ordering of the edges; the final label give the signs.

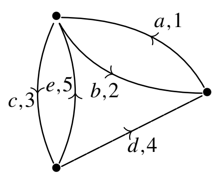
We impose a product structure on the set $\mathcal{G}(k^\times)$. For $G, G' \in \mathcal{G}(k^\times)$, let $G \sqcup G'$ be the disjoint union of the graphs, without an overall ordering imposed on the union of the edges. The product of two graphs $G \cdot G'$ is the graph $G \sqcup G'$, with the edges of G appearing before the edges of G'. In particular, this is a noncommutative product,

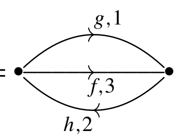
$$G \cdot G' \neq G' \cdot G,$$

as the ordering of the edge set, $E(G \sqcup G')$, in the two cases is not the same.

Example 3.6. In this example, we concern ourselves primarily with the ordering of the edges in the product. Therefore, we write label the edges with elements of $k^\times$ and the ordering, and neglect to indicate the sign. One may assume, without loss of generality, that the signs are all positive in the graphs below.

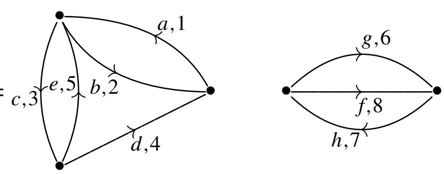
Consider the graphs

$G_1 =$ 

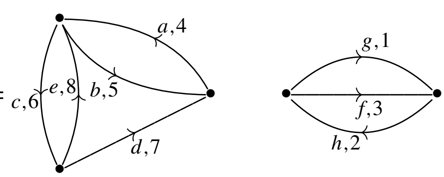
and $G_2 =$ 

First, notice that the graph in Example 3.5 cannot be written as the product of G_1 and G_2, since the edges of one connected component do not precede the edges of the other, as written.

The product in one order is

$G_1 \cdot G_2 =$ 

while the product in the other order is

$G_2 \cdot G_1 =$ 

It is the ordering on the two graphs that distinguishes the two products. Everything else about the labeled oriented graphs $G \cdot G'$ and $G' \cdot G$ is the same.

This noncommutative product gives $(\mathcal{G}(k^\times), \cdot)$ a free monoidal structure. The unit in the monoid is given by the empty graph, which has no loops and no edges, and therefore no labels.

Definition 3.7. Let $\mathbb{Q}[\mathcal{G}]$ be the free algebra generated by the monoid $(\mathcal{G}(k^\times), \cdot)$.

Just as with the cycles, we are not interested in the order of the coordinates, but their image under Alt. Therefore, we are also only interested in an alternating projection on the edges of the graphs. There is a $\mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ action on the edges of a graph $G \in \mathcal{G}(k^\times)$. This action permutes the order of the edges in the graph, and changes the assigned signs. An element $g \in \mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ is of the form $g = (\sigma, \vec{\mathrm{sgn}})$, where $\sigma \in \mathfrak{S}_{|E(G)|}$ and $\vec{\mathrm{sgn}} \in (\mathbb{Z}/2\mathbb{Z})^n$ is an ordered set of signs. Write $\vec{\mathrm{sgn}}_j$ for the j-th entry of the ordered set. Furthermore, write

$$\mathrm{sgn}(g) = \mathrm{sgn}(\sigma) \prod_j \vec{\mathrm{sgn}}_j,$$

where $\mathrm{sgn}(\sigma)$ indicates the sign of the permutation $\sigma \in \mathfrak{S}_{|E(G)|}$.

The action of $\mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ on the algebra of graphs is as follows: $gG = 0$ if $|E(G)| \neq n$, and otherwise gG is given by

$$\begin{cases} \omega(gG) := \sigma(\omega(G)), \\ \vec{\mathrm{sgn}}_i(gG) = \vec{\mathrm{sgn}}_i \vec{\mathrm{sgn}}_i(G). \end{cases}$$

That is, if $|E(G)| = n$, the ordering and signs of the edges in gG for $g = (\sigma, \vec{\mathrm{sgn}})$ are determined by σ and $\vec{\mathrm{sgn}}$, respectively.

The action of $\mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ defines an equivalence relation on $\mathbb{Q}[\mathcal{G}]$.

Lemma 3.8. *Letting n vary, any two monomials G and $G' \in \mathbb{Q}[\mathcal{G}]$ are equivalent if and only if there is an element $g \in \mathfrak{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ relating the two*:

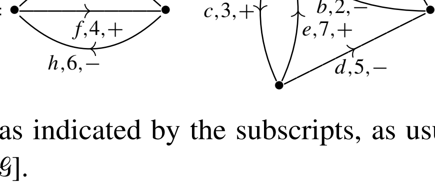
$$G \sim_{\mathrm{ord}} \mathrm{sgn}(g) gG.$$

The proof comes from the identity, inverse and composition laws of the group $\mathfrak{S}_{|E(G)|} \rtimes (\mathbb{Z}/2\mathbb{Z})^n$, and we omit it.

In Lemma 3.14, we show that $\mathbb{Q}[\mathcal{G}]/\sim_{\mathrm{ord}}$ is generated as an algebra by connected graphs. In other words, under the equivalence $\sim_{\mathrm{ord}}$, any disconnected element of $\mathbb{Q}[\mathcal{G}]$ is no longer primitive.

First we give an example.

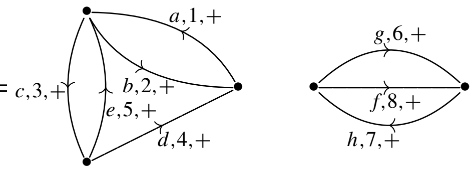
Example 3.9. To illustrate the equivalence relations from Lemma 3.8, consider the graph G in Example 3.5 as a monomial in $\mathbb{Q}[\mathcal{G}]$:

$G =$ 

with the edges ordered as indicated by the subscripts, as usual. This graph is a primitive element of $\mathbb{Q}[\mathcal{G}]$.

However, in the ring quotiented by the equivalence relation, $\mathbb{Q}[\mathcal{G}]/{\sim_{\mathrm{ord}}}$, we see that $G \sim_{\mathrm{ord}} G_1 \cdot G_2$, where G_1 and G_2 are the graphs defined in Example 3.6:

$G \sim_{\mathrm{ord}} G_1 \cdot G_2 =$



which is not primitive. Notice that both signs and orderings have been changed in this example.

As an algebra, $\mathbb{Q}[\mathcal{G}]$ is bigraded by first Betti number, or weight, and degree of the graphs. That is, if $G \in \mathbb{Q}[\mathcal{G}]^{\star}_{\bullet}$, then $h_1(G) = \bullet$, while $\star = h_1(G) - V(G) + h_0(G)$. From the formula for the first Betti number of a graphs in (3.3), if $G \in \mathbb{Q}[\mathcal{G}]^{\star}_{\bullet}$,

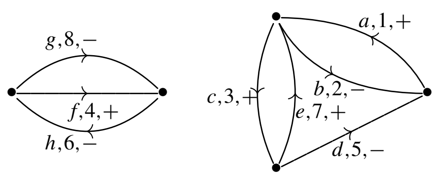
$$|E(G)| = 2\bullet - \star. \tag{3.10}$$

As the equivalence relation $\sim_{\mathrm{ord}}$ does not affect the underlying topology of the graph, $\mathbb{Q}[\mathcal{G}]/{\sim_{\mathrm{ord}}}$ is also bigraded by weight and degree of the graphs.

Remark 3.11. The unit of this algebra is in $\mathbb{Q}[\mathcal{G}]^0_0$. It is represented by the empty graph.

Example 3.12. For instance, consider the graph in Examples 3.5 and 3.9:

$G =$



This graph has five loops, five vertices and two connected components. Therefore, it is in $\mathbb{Q}[\mathcal{G}]^2_5/{\sim_{\mathrm{ord}}}$.

Definition 3.13. Let $\mathcal{G}_0(k^\times) \subset \mathcal{G}(k^\times)$ be the subset of biconnected graphs with ordered, labeled, oriented edges. That is, there are no disconnected graphs in $\mathcal{G}_0(k^\times)$.

Lemma 3.14. *The algebra $\mathbb{Q}[\mathcal{G}]/{\sim_{\mathrm{ord}}}$ is generated by the set $\mathcal{G}_0(k^\times)/{\sim_{\mathrm{ord}}}$ as a skew symmetric bigraded algebra.*

Proof. For any disconnected graph $G \in \mathbb{Q}[\mathcal{G}]^i_n$, there is an element $g = (\sigma, \mathrm{id})$ in $\mathfrak{S}_{2n-i} \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ that rearranges the order of the edges of each connected component consecutively. Since $\mathrm{sgn}(g) = \mathrm{sgn}(\sigma)$, by Lemma 3.8 we obtain

$$G \sim_{\mathrm{ord}} \mathrm{sgn}(g)(gG) = \mathrm{sgn}(g) G_1 \cdot G_2 \cdots G_m,$$

with each $G_i \in \mathcal{G}_0(k^\times)$.

The product preserves the bigrading, as the zeroth and first Betti numbers as well as the sizes of the edge and vertex sets are additive under disjoint union. For $G \in \mathbb{Q}[\mathcal{G}]^i_n/\sim_{\mathrm{ord}}$ and $G' \in \mathbb{Q}[\mathcal{G}]^{i'}_{n'}/\sim_{\mathrm{ord}}$, we have

$$G \cdot G' \in \mathbb{Q}[\mathcal{G}]^{i+i'}_{n+n'}/\sim_{\mathrm{ord}}.$$

To see that this is skew symmetric, as above, write

$$G \cdot G' \sim_{\mathrm{ord}} (-1)^{|E(G)||E(G')|} G' \cdot G = (-1)^{ii'} G' \cdot G.$$

The last equality comes from the fact that $|E(G)| = 2n - i$ and $|E(G')| = 2n' - i'$. $\square$

Since

$$\mathbb{Q}[\mathcal{G}]/\sim_{\mathrm{ord}} = \mathbb{Q}[\mathcal{G}_0]/\sim_{\mathrm{ord}},$$

for the rest of this paper we consider only elements of $\mathcal{G}_0(k^\times)$.

3B. ***A brief interlude on algebraic cycles.*** In this section we introduce the relationship between the graphs defined above and algebraic cycles generating $\mathcal{Z}^p_{1L}(\square^n)$. As of yet, we make no claims on admissibility of cycles.

Definition 3.15. Define $\mathbb{Q}\mathcal{Z}_{1L}$ to be the group ring generated by the free abelian group of $\mathbb{P}^1_k$-linear cycles

$$\mathbb{Q}\mathcal{Z}_{1L} = \bigoplus_{p,i} \mathrm{Alt}(\mathcal{Z}^p_{1L}(\square^{2p-i}) \otimes \mathbb{Q}).$$

This is a skew symmetric algebra. Write $\mathbb{Q}\mathcal{Z}_{1L}{}^i_p = \mathrm{Alt}(\mathcal{Z}^n_{1L}(\square^{2p-i}) \otimes \mathbb{Q})$.

There is a homomorphism, Z, from $\mathbb{Q}[\mathcal{G}]^\star_\bullet/\sim_{\mathrm{ord}}$ to $\mathbb{Q}\mathcal{Z}_{1L}$. Note that $\mathcal{A}^\times_{1L} \subset \mathbb{Q}\mathcal{Z}_{1L}$. In Section 3E, we show that Z is a DGA homomorphism onto $\mathcal{A}^\times_{1L}$ that becomes an isomorphism of DGAs when $\mathbb{Q}[\mathcal{G}]^\star_\bullet$ is subjected to more equivalence relations. That is the isomorphism we seek. In this section, we only show that elements of $\mathbb{Q}[\mathcal{G}]$ correspond to parametrizations of $\mathbb{P}^1_k$-linear algebraic cycles on $\square_k^{|E(G)|}$.

Definition 3.16. Each connected graph $G \in \mathcal{G}(k^\times)$, with loop number p and n edges, defines a parametrization, $\phi : \mathbb{P}_k^{|V(G)|-1} \to (\mathbb{P}^1_k)^n$, of an algebraic cycle $Z(G) \in \mathcal{Z}^p_{1L}(\square^n)$. The $\omega(e)$-th coordinate of the cycle $Z(G)$ is

$$\phi_{\omega(e)} = \left(1 - \frac{x_{s(e)}}{a_e x_{t(e)}}\right)^{\mathrm{sgn}_{\omega(e)}},$$

where $x_{s(e)}$ and $x_{t(e)}$ are variables assigned to the vertices at the source and target of the edge $e \in E(G)$, and a_e is the label of edge e.

Recall from Definition 2.13, each $\phi_{\omega(e)}$ is the ratio of the projective coordinates defining the $\omega(e)$-th copy of $\mathbb{P}^1_k$ in the image.

Thus we have, for $\bullet = h_1(G)$ and $\star = h_1(G) - |V(G)| + h_0(G)$, a set map

$$Z : \mathcal{G}_0(k^\times) \to \bigoplus_{\bullet,\star} \mathcal{Z}^\bullet_{1L}(\square^{2\star-\bullet}), \quad G \mapsto [\phi_1, \ldots, \phi_{|E(G)|}], \tag{3.17}$$

from graphs to parametrized $\mathbb{P}_k$-linear cycles.

To make this map concrete, we explicitly derive the system of polynomials defined by a graph G. First we introduce a function that relates edges of a graph to the loops of G.

Definition 3.18. For $e \in E(G)$, and L a loop of G, define

$$\epsilon(e, L) = \begin{cases} 1 & \text{if } e \in E(L), \text{ oriented as } L \text{ is}, \\ 0 & \text{if } e \notin E(L), \\ -1 & \text{if } e \in E(L), \text{ oriented opposite to } L. \end{cases}$$

Given this notation, we are ready to define the system of polynomials defined by a graph $G \in \mathcal{G}_0(k^\times)$.

Theorem 3.19. *For a graph $G \in \mathcal{G}_0(k^\times)/\sim_{\mathrm{ord}}$, indicate the label of the edge $e \in E(G)$ as $a_e \in k^\times$. Suppose $h_1(G) = p$ and $|E(G)| = n$. Let $\beta = \{L_1, \ldots, L_p\}$ be a loop basis of $H_1(G)$. The algebraic cycle $Z(G)$ is defined by the system of p polynomial equations, each associated to an element of the loop basis, and induced from the rational relations*

$$1 = \prod_{e \in E(G)} (a_e(1 - \phi_{\omega(e)}))^{\epsilon(e, L_i)}. \tag{3.20}$$

Proof. Given a loop basis β for $H_1(G)$, begin with a loop, call it L_1. Subsequent elements of the system of equations are similarly defined.

Consider an edge $e \in E(L_1)$. The $\omega(e)$-th coordinate of the cycle $Z(G)$ is defined by the function $\phi_{\omega(e)}(x, y)$, where x and y are the variables associated to the vertices at the endpoints of $e \in E(G)$. Suppose that, in the orientation inherent in L_1 as an element of a loop basis, L_1 flows from the vertex associated to x directly to the vertex associated to y. This is not necessarily the orientation of the edge connecting the vertices associated to x and y, but the second orientation on the edges induced by the orientation of L_1. Since we are working over $\mathbb{Q}[\mathcal{G}]^\star_\bullet/\sim_{\mathrm{ord}}$, we may choose G such that all the signs on the edges of G are all positive. Then one can associate to the edge $e \in E(G)$ the equation

$$x = y(a_e(1 - \phi_{\omega(e)}))^{\epsilon(e, L_1)}. \tag{3.21}$$

There is a unique edge $e' \neq e$ in L_1 with an endpoint at the vertex associated to the variable y. As above, associate to the edge e' the equation

$$y = z(a_{e'}(1 - \phi_{\omega(e')}))^{\epsilon(e', L_1)}.$$

Substituting this into (3.21) gives

$$x = z(a_e(1-\phi_{\omega(e)}))^{\epsilon(e,L_1)}(a_{e'}(1-\phi_{\omega(e')}))^{\epsilon(e',L_1)}.$$

Continuing along the entire loop in this manner gives

$$x = x\prod_{e\in E(G)}(a_e(1-\phi_{\omega(e)}))^{\epsilon(e,L_1)},$$

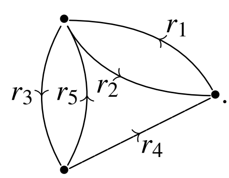
which simplifies to an expression of the form in (3.20):

$$1 = \prod_{e\in E(G)}(a_e(1-\phi_{\omega(e)}))^{\epsilon(e,L_1)}.$$

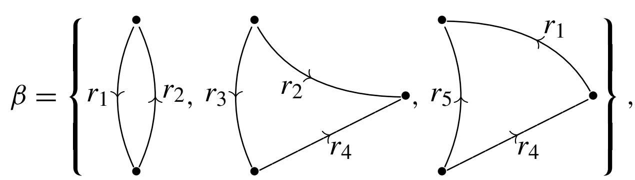
Since β is a loop basis, the function $\phi_{\omega(e)}$, associated to each edge of G is used in the system of equations defined in (3.20), and the functions thus derived are independent of each other. $\square$

Notice that the specific form of this system of equations depends on the loop basis for $H_1(G)$. However, a different loop basis will give an equivalent system of polynomials.

Example 3.22. Recall the graph in Example 3.5:



Define a basis



where all the loops in β are oriented counterclockwise.

A system of equations for this graph is given by the polynomials

$$\begin{aligned}
1 &= r_1r_2(1-\phi_1)(1-\phi_2),\\
1 &= \frac{r_3r_4}{r_2}\frac{(1-\phi_3)(1-\phi_4)}{(1-\phi_2)},\\
1 &= \frac{r_1r_4}{r_5}\frac{(1-\phi_1)(1-\phi_4)}{(1-\phi_5)}.
\end{aligned}$$

This brings us to an important invariant of the graphs in $\mathbb{Q}[\mathcal{G}]^{\star}_{\bullet}/\sim_{\text{ord}}$, the loop coefficient:

Definition 3.23. Given a loop L of G, the loop coefficient of L is defined by

$$\chi_G(L) = \prod_{E(G)} r_e^{\epsilon(e,L)}. \tag{3.24}$$

In this notation, we can restate the image of the map Z. For $G \in \mathbb{Q}[\mathcal{G}]_n^p/\sim_{\mathrm{ord}}$ with $\beta = \{L_1, \dots, L_p\}$ a basis of $H^1(G)$, the cycle $Z(G)$ is defined by the system of polynomial equations

$$\left\{1 = \chi_G(L_i) \prod_{e \in E(L_i)} (1 - \phi_{\omega(e)})^{\epsilon(e,L_i)}\right\}_{L_i \in \beta}. \tag{3.25}$$

We can extend the set map Z thus defined to the algebra $\mathbb{Q}[\mathcal{G}]_\bullet^\star/\sim_{\mathrm{ord}}$, where $Z(G)$ maps a graph to an algebraic cycle under the alternating projection.

Theorem 3.26. *The set map Z in* (3.17) *induces a grading-preserving algebra homomorphism*

$$Z : \mathbb{Q}[\mathcal{G}]/\sim_{\mathrm{ord}} \to \mathbb{Q}\mathcal{Z}_{1L}, \quad G \mapsto \mathrm{Alt}[\phi_1, \dots, \phi_{|E(G)|}].$$

Proof. The equivalence relation $\sim_{\mathrm{ord}}$ equates different orderings of edges of graphs as Alt combines different orderings of coordinates into a single generator of $\mathbb{Q}\mathcal{Z}_{1L}$. Therefore, Z maps generators of $\mathbb{Q}[\mathcal{G}]/\sim_{\mathrm{ord}}$ to generators of $\mathbb{Q}\mathcal{Z}_{1L}$. Lemma 3.14 shows that the algebra structure of $\mathbb{Q}[\mathcal{G}]/\sim_{\mathrm{ord}}$ matches the algebra structure of $\mathbb{Q}\mathcal{Z}_{1L}$.

It remains to check that if $G \in \mathbb{Q}[\mathcal{G}]_p^i/\sim_{\mathrm{ord}}$ then $Z(G) \in \mathbb{Q}\mathcal{Z}_{1L}{}_p^i$. First notice that by the parametrization given in Definition 3.16, Writing $G = G_1 \cdots G_m$ in terms of its connected components, the cycle $Z(G)$ is parametrized by the map

$$\phi : \prod_{i=1}^m \mathbb{P}_k^{|V(G_i)|-1} \to \square_k^{|E(G)|}.$$

Therefore, the cycle $Z(G)$ has codimension

$$E(G) - V(G) + h_0(G) = h_1(G) = p$$

in $\square_k^{|E(G)|}$. By (3.10) this implies that $Z(G) \in \mathcal{Z}_{1L}^p(\square^{2p-i})$. □

Finally, in conjunction with Theorem 3.19, this allows for a statement about irreducible cycles.

Corollary 3.27. *If G is a generator of $\mathbb{Q}[\mathcal{G}]/\sim_{\mathrm{ord}}$, i.e., a disconnected graph, then $Z(G)$ is a reducible cycle.*

Proof. Recall that a reducible cycle is one that arises from a reducible variety. □

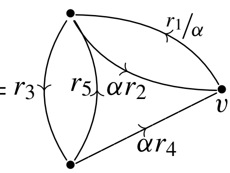
3C. ***The DGA structure on graphs.*** In this section, we define a differential structure on the algebra of graphs. In order to do this, we need to define an additional equivalence relation on $\mathcal{G}_0(k^\times)$.

In particular, we consider graphs that differ only by a rescaling of the labels of the edges attached to a particular vertex.

Definition 3.28. Consider $\alpha \in k^\times$ and $v \in V(G)$. The vertex rescaled graph $v_\alpha(G)$ is the labeled oriented graph G with labels changed as follows: for each edge e of G, if an edge terminates (starts) at v, multiply (divide) its label by α to get the label of the edge in $v_\alpha(G)$; otherwise, keep the same label for e. The signs associated to and the ordering of the edges of G by ω do not change.

Vertex rescaling a graph corresponds to rescaling all instances of a variable in the parametrized $\mathbb{P}_k$-linear cycle $Z(G)$ by a constant multiple. This does not affect the cycle at all. In other words, G and $v_\alpha(G)$ correspond to two different parametrizations of $Z(G)$. We call this procedure label rescaling with respect to a vertex, or label rescaling at v.

Example 3.29. For the graph G in Example 3.5, one can rescale the rightmost vertex by α to obtain the graph

$v_\alpha(G) =$ 

where the ordering of the edges is given by the subscripts.

Remark 3.30. Vertex rescaling is an equivalence relation on the set $\mathcal{G}_0(k^\times)$. We write it as $\sim_{\mathrm{v}}$.

In the sequel, we consider the algebra of graphs up to this equivalence set. We are interested in graphs only as a tool to understand their corresponding algebraic cycles. We work with graphs up to this rescaling since two graphs that differ by a vertex rescaling correspond, under the homomorphism Z defined in Section 3B, to different parametrizations of the same cycle.

To see this, notice that vertex rescaling does not change the loop coefficient of the graph.

Lemma 3.31. *Loop coefficients are invariant under rescaling at vertices.*

Proof. Let L be a loop in G, with $G \in \mathcal{G}_0(k^\times)$. For $v \in V(L)$, a vertex in L, v is attached to exactly two edges e_1 and e_2 of L. We compare $\chi_G(L)$ and $\chi_{v_\alpha(G)}(L)$.

There are three cases to consider. If v is the terminal vertex of e_1 and the source vertex of e_2, then the respective coefficients are r_1 and r_2 in G, and $r_1\alpha$ and r_2/α in $v_\alpha(G)$. Both numbers either appear in the numerator or the denominator of the coefficient of L. Thus the contributions of α cancel in $\chi_{v_\alpha(G)}(L)$.

The other two cases are as follows. The vertex v is either the source or target vertex of both e_1 and e_2. Then the coefficients are r_1/α and r_2/α (or $r_1\alpha$ and $r_2\alpha$). One label appears in the numerator of the loop, the other in the denominator, so the contribution of α cancels $\chi_{v_\alpha(G)}(L)$.

Therefore,

$$\chi_G(L) = \chi_{v_\alpha(G)}(L),$$

as desired. □

Therefore, given the form of the system of polynomials defined by each of these graphs in (3.25), $Z(G) = Z(v_\alpha(G))$.

Theorem 3.32. *The parametrized cycles $Z(G)$ and $Z(v_\alpha(G))$ correspond to the same cycle, under different parametrizations*

$$Z(G) = Z(v_\alpha(G)) \in \mathscr{Z}_{1L}^{h_1(G)}(\square^{|E(G)|}).$$

Proof. Since, by Lemma 3.31, loop coefficients are invariant under vertex rescaling, from the system of equations defined in (3.25), we see that the cycles defined are the same. □

Therefore, the algebra homomorphism, Z, defined in Section 3B passes to an algebra homomorphism under the quotient $\sim_{\mathrm{v}}$

$$Z : \mathbb{Q}[\mathscr{G}]_\bullet^\star/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}) \to \mathrm{Alt}(\mathscr{Z}_{1L}^\bullet(\square^{2\bullet-\star}) \otimes \mathbb{Q}).$$

As we mentioned before, the algebra $\mathrm{Alt}\,\mathscr{Z}_{1L}^\bullet(\square^{2\bullet-\star})$ does not have a DGA structure. However, the algebra $\mathbb{Q}[\mathscr{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$ does. On individual graphs, this is defined by a modified contraction of the edges. We devote the rest of this subsection to developing this differential.

Definition 3.33. Consider $G \in \mathscr{G}_0(k^\times)$. For $e \in E(G)$, define the graph G/e to be that formed by contracting the edge e and identifying the vertices s_e and t_e as a new vertex v. If the edge $e \in E(G)$ has the same source and target vertex, then $G/e = 0$. If contracting the edge e leads to a one connected graph, split the graph into its biconnected components at the articulation vertex.

The above definition is not the standard definition of an edge contraction in graphs. The standard definition has been modified to fit the algebraic properties of the graphs we need, namely the splitting of graphs at the articulation vertex. Furthermore, the ordering of $G/e \in \mathbb{Q}[\mathscr{G}]$ is induced from the ordering of G.

Definition 3.34. Let $\omega(G)$ be the ordering of the edges of the graph G. Then $\hat{\omega}_e(G/e)$ is the ordering of the graph G/e which is the same as $\omega(G)$ with the $\omega(e)$-th element removed.

We are now ready to define a differential operator on $G \in \mathbb{Q}[\mathscr{G}]_\bullet^\star/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$.

Theorem 3.35. *Consider a monomial $G \in \mathbb{Q}[\mathscr{G}]_\bullet^\star/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$. For $e \in E(G)$ an edge, let r_e denote the label of this edge and let s_e denote the source vertex. There* *is a degree*-1 *differential operator*

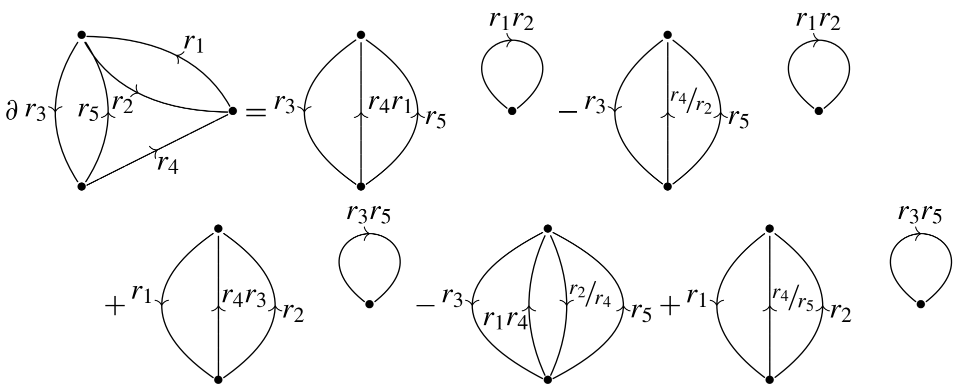
$$\partial : \mathbb{Q}[\mathcal{G}]^{\star}_{\bullet}/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}) \to \mathbb{Q}[\mathcal{G}]^{\star+1}_{\bullet}/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}),$$
$$(\omega, G) \mapsto \sum_{e \in E(G)} (-1)^{\omega(e)-1} (\hat{\omega}_e, ((s_e)_{r_e}(G))/e).$$

By direct calculation, one sees that this operator satisfies the Leibniz rule

$$\partial(G \cdot G') = \partial(G) \cdot G' + (-1)^{\star} G \cdot \partial(G'). \tag{3.36}$$

We prove this theorem in steps. Before starting the proof, we give an example of the action of ∂. Recall that the notation $(s_e)_{r_e}$ in Theorem 3.35 is the vertex rescaling from Definition 3.28.

Example 3.37. For example, for the graph in Example 3.5, with ω ordered according to the numbering of the labels,



First, we define a contraction operator on graphs with labeled edges.

Definition 3.38. For $e \in E(G)$, we write the contraction of an edge as $\partial_e(G) = (s_e)_{r_e}(G)/e$.

In this notation, the operator defined in Theorem 3.35 can be rewritten as

$$\partial(G) = \sum_{e \in E(G)} (-1)^{\omega(e)-1} \partial_e(G).$$

Notice that if $r_e = 1$ then $\partial_e(G) = (G/e)$. This further implies that the loop coefficient is invariant under contraction.

Lemma 3.39. *Consider* $G \in \mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$. *Let* L *be a loop in* G *with more than one edge*, *and* $e \in E(L)$. *Then*

$$\chi_G(L) = \chi_{\partial_e G}(L/e).$$

Proof. It is sufficient to consider G connected. If s is the source vertex of e, and r the label, the equivalent graph $s_r(G)$ is such that the label of e equals 1.

In Lemma 3.43, we show that contraction is well-defined on $\mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}})$. Therefore, $\partial_e(G) \sim_{\text{v}} \partial_e(s_r(G))$. Since the label of e is 1, the contraction $\partial_e(s_r(G))$ equals $s_r(G)/e$, and

$$\chi_G(L) = \chi_{s_r(G)}(L) = \chi_{\partial_e s_r(G)}(L \setminus e) = \chi_{\partial_e G}(L \setminus e).$$

The first equality comes from Lemma 3.31. The second equality comes from the form of $\partial_e(s_r(G))$. Finally, the third equality comes from the equivalence of the two contractions (Lemma 3.43). □

Working under the equivalence relations $\sim_{\text{v}}$ gives an important representation of graphs that simplifies the calculation of the derivatives.

Lemma 3.40. *For any given $G \in \mathcal{G}_0(k^\times)$, and any subtree $T \subset G$, there is a graph G_T such that the labels of the edges in T are* 1 *and $G \sim_{\text{v}} G_T$. In particular, any monomial $G \in \mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}})$ can be rescaled such that any spanning forest of G is labeled by* 1.

Proof. Without loss of generality, assume that the graph $G \in \mathcal{G}_0(k^\times)$ is a connected graph. Otherwise, the following arguments apply to each connected component of G.

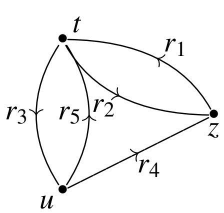
Let T be a spanning tree of G. Label the vertices $\{v_1, \dots, v_{|V(G)|}\} \in V(G)$ such that v_1 has valence 1 in T. Let $\{r_2, \dots, r_{|V(T)|}\}$ be the labels of the edges in $E(T)$, where r_i labels the edge connected to v_i.

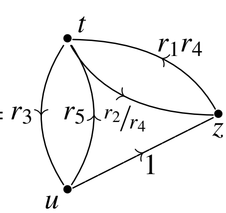
Rescale the graph G at the vertex v_2 by r_2 (resp. $1/r_2$) if v_2 is a source (resp. target) vertex of the edge labeled by r_2. In the rescaled graph $(v_2)_{r_2}(G)$ (resp. $(v_2)_{1/r_2}(G)$) the edge connecting v_1 and v_2 is labeled by 1. By similar logic, there is a series of rescaling coefficients, $\{\alpha_1, \dots, \alpha_{|V(G)|-1}\}$, where each α_i is a rational function of the r_j such that edges of the spanning tree T in

$$(v_{|V(G)|-1})_{\alpha_{|V(G)|-1}}(\cdots((v_1)_{\alpha_1}(G))\cdots)$$

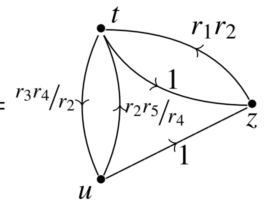
are all labeled by 1. □

Example 3.41. Consider again the graph in Example 3.5. The loop coefficient of the loop defined by the inner triangle of legs, oriented clockwise, is r_2r_5/r_4. The same graph can be relabeled to have a spanning tree labeled with ones as follows:

$G =$ 

; $z_{1/r_4}(G) =$ 

;

$$t_{r_2/r_4}(z_{1/r_4}(G)) =$$ 

Contrary to appearance, we have made no choice in our definition of the derivative ∂_e. We could just as easily have written

$$\partial_e(G) = (-1)^{\omega(e)-1}(t_e)_{1/r_e}(G)/e,$$

where t_e is the target vertex of the edge e. This is because the two graphs are equivalent under vertex rescaling.

Lemma 3.42. *For $G \in \mathbb{Q}[\mathcal{G}]$, let t and s be the target and source vertices, respectively, of the edge $e \in E(G)$. Then*

$$t_{1/a}(G)/e \sim_{\mathrm{v}} s_a(G)/e.$$

Proof. We show that there is a vertex rescaling such that

$$t_{1/a}(G)/e \sim s_a(G)/e.$$

By construction, $e \notin E(G/e)$, and the vertices $t, s \in V(G)$ are replaced by a single vertex $v \in V(G/e)$.

In the graph $t_{1/a}(G)$, the label of e is multiplied by $1/a$, as are all the edges terminating on t. All edges starting at t are multiplied by a. The edges attached to s and not t are unaffected. Similarly, in the graph $s_a(G)$, the label of e is multiplied by $1/a$, as are all the edges starting at s. All edges terminating at s are multiplied by a. The edges attached to t and not s are unaffected.

Therefore, contracting e and identifying s with t at the new vertex in the contracted graph, we get a unique vertex $v = V(G/e) \setminus V(G)$,

$$v_{1/a}(s_a(G)/e) = t_{1/a}(G)/e.$$

Similarly, one may also write

$$s_a(G)/e = v_a(t_{1/a}(G)/e). \qquad \square$$

Choosing $a = r_e$, the label of the edge e, shows that, in $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$, it does not matter if ∂_e is defined according to the source vertex of e or the target vertex.

Next we show that the operator ∂ is well-defined under vertex rescaling.

Lemma 3.43. *The operator ∂ defined above is well-defined on $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$.*

Proof. Since $\partial = \sum_{e \in E(G)} (-1)^{\omega(e)-1} \partial_e$, for any $g \in \mathfrak{S}|E(G)| \rtimes \mathbb{Z}/2\mathbb{Z}^{|E(G)|}$

$$\partial G = \partial g G$$

in the quotient space $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$ for all $G \in \mathcal{G}_0(k^\times)$.

It remains to check that, for $G \in \mathcal{G}_0(k^\times)$,

$$\partial(G) \sim \partial(v_\alpha(G)) \tag{3.44}$$

for any $v \in V(G)$. Before proceeding, we note that vertex rescaling is multiplicative. That is, for $v \in V(G)$,

$$v_\alpha(v_\beta(G)) = v_{\alpha\beta}(G). \tag{3.45}$$

Fix $v \in V(G)$. For any edge e not incident upon v,

$$\partial_e(v_\alpha G) = v_\alpha \partial_e(G) \sim \partial_e(G).$$

Therefore, consider only the edges $e \in E(G)$ that are incident upon v. They are labeled by r_e. By Definition 3.38 and Lemma 3.42,

$$\partial_e(G) \sim_{\mathrm{v}} \begin{cases} v_{r_e}(G)/e, & v \text{ a source of } e, \\ v_{1/r_e}(G)/e, & v \text{ a target of } e. \end{cases}$$

Recall, by the definition of ∂_e, that if v is the source of e, the above equivalence is an exact equality.

Similarly,

$$\partial_e(v_\alpha(G)) \sim_{\mathrm{v}} \begin{cases} v_{r_e/\alpha}(v_\alpha(G))/e, & v \text{ a source of } e, \\ v_{1/r_e\alpha}(v_\alpha(G))/e, & v \text{ a target of } e. \end{cases}$$

By the multiplicativity of vertex rescaling (3.45), we rewrite this

$$\begin{aligned}\partial_e(v_\alpha(G)) &\sim_{\mathrm{v}} \begin{cases} v_{r_e}(G)/e, & v \text{ a source of } e, \\ v_{1/r_e}(G)/e, & v \text{ a target of } e, \end{cases} \\ &\sim_{\mathrm{v}} \partial_e(G).\end{aligned}$$

Therefore, $\partial(G) \sim_{\mathrm{v}} \partial(v_\alpha(G))$ for any $G \in \mathcal{G}_0(k^\times)$ and $v \in V(G)$. □

Thus far, we have shown that the operator ∂ is well-defined on $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$. Next we show that the operators ∂_e commute.

Lemma 3.46. *Contractions along different edges commute in* $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}})$, *that is*, $\partial_e \circ \partial_{e'} = \partial_{e'} \circ \partial_e$.

Proof. There are two cases to consider: when the edges e and e' form a cycle in G, and when they do not.

If $e \cup e'$ is a union of loops in G, then, by Definition 3.33, $\partial_e G = \partial_{e'} G = 0$. If $e \cup e'$ is a loop in G, then e' defines a loop in $\partial_e G$, and e a loop in $\partial_{e'} G$. Therefore, $\partial_e \circ \partial_{e'} G = \partial_{e'} \circ \partial_e G = 0$.

If $e \cup e'$ is not a cycle in G, there is a spanning tree T such that $e, e' \subset E(T)$. By Lemma 3.40, write G such that the edges of T are labeled by 1. In this case, $\partial_e \circ \partial_{e'} G = (G/e')/e = G/\{e' \cup e\} = \partial_{e'} \circ \partial_e G$. □

We are now ready to prove Theorem 3.35.

Proof of Theorem 3.35. Lemma 3.43 shows that the operator

$$\partial : \mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}}) \to \mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}})$$

is well-defined.

To see that $\partial \circ \partial = 0$, write

$$\partial \circ \partial = \sum_{e \in E(G/e)} (-1)^{\omega(e)-1} \partial_e \biggl(\sum_{e' \in E(G)} (-1)^{\omega(e')-1} \partial'_e(G) \biggr).$$

Assume without loss of generality that $\omega(e) < \omega(e')$. Then the term $\partial_e \circ \partial_{e'}$ appears in $\partial \circ \partial$ with sign $(-1)^{\omega(e)}(-1)^{\omega(e')}$, while $\partial_{e'} \circ \partial_e$ appears with sign $(-1)^{\omega(e)-1}(-1)^{\omega(e')}$. By Lemma 3.46, $\partial_e \circ \partial_{e'} = \partial_{e'} \circ \partial_e$. Thus the two contributions cancel.

To see that ∂ is a degree-one operator, note that if G/e is not 0, then

$$h_1(G) = h_1(G/e).$$

However,

$$|V(G/e)| = |V(G)| - 1 + (h_0(G/e) - h_0(G)).$$

Recall from (3.10) that if $G \in \mathcal{G}_{1L}{}^{\bullet}_{i}$, the degree is given by

$$i = h_1(G) - |V(G)| + h_0(G).$$

Similarly, the degree of $(\hat{\omega}_e, G/e)$ is given by

$$\begin{aligned} &h_1(G/e) - |V(G/e)| + h_0(G/e) \\ &\qquad = h_1(G) - \bigl(|V(G)| - 1 + (h_0(G/e) - h_0(G))\bigr) + h_0(G/e) \\ &\qquad = h_1(G) - |V(G)| + h_0(G) + 1 = i + 1. \end{aligned}$$

□

So far, we have shown that $\mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}})$ is a bigraded DGA and that Z is a homomorphism of algebras from $\mathbb{Q}[\mathcal{G}]^{\star}_{\bullet}/(\sim_{\text{ord}}, \sim_{\text{v}})$ to $\operatorname{Alt}\mathcal{Z}^{\bullet}_{1L}(\square^{2\bullet-\star})$. However, we are ultimately interested in graphs $\mathcal{G}_{1L}$ that correspond to $\mathcal{A}^{\times}_{1L}$ under the algebra homomorphism Z defined in Section 3B. In Section 3D, we define the algebra of admissible graphs, and show that $\mathcal{G}_{1L}$ is a DGA under the differential defined in this section. In Section 3E, we show that $\mathcal{G}_{1L}$ is isomorphic to $\mathcal{A}^{\times}_{1L}$ as a DGA.

3D. ***Admissible graphs.*** So far, we have said nothing about admissible cycles. By the arguments presented in Sections 3B and 3C, there is an algebra homomorphism

$$Z : \mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}}) \to \operatorname{Alt}\mathcal{Z}^{\bullet}_{1L}(\square^{2\bullet-\star}).$$

Theorem 3.32 shows that generators of $\mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}})$ map to generic $\mathbb{P}^1_k$-linear cycles under Z, not necessarily to admissible ones. In this section, we define a subalgebra of admissible graphs, which, in Section 3E, we show corresponds to admissible cycles.

There is a compact way of reading off loop coefficients for graphs if the graph is parametrized as in Lemma 3.40, by setting each label of a spanning tree to 1.

Lemma 3.47. *Consider a connected graph $G \in \mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}})$. Each spanning tree T of G defines a loop basis of $H^1(G)$, the loop coefficients of which are the labels of the edges $E(G) \setminus E(T)$.*

Proof. Each spanning tree of a connected graph defines a set of loops in G as follows: For a spanning tree T, each oriented edge $e \in E(G) \setminus E(T)$ defines a graphical loop, L_e, in conjunction with a subset of $E(T)$. The orientation of the graphical loop is determined by the orientation of e. The rank of the loop space of G is $\operatorname{rk} H^1(G) = |E(G)| - |V(G)| + 1$. Since $|E(T)| = |V(G)| - 1$, we see that $\operatorname{rk} H^1(G) = |E(G) \setminus E(T)|$. Furthermore, $\bigcup_{e \in E(G)\setminus E(T)} E(L_e) = E(G)$. Therefore, the set $\{L_e\}_{e \in E(G)\setminus E(T)}$ defines a basis of $H^1(G)$.

By choosing a parametrization where T is labeled by ones, the graphical loop coefficient L_e is exactly the label of e. □

We are now ready to define a class of graphs called admissible graphs. We show in Section 3E that these correspond to admissible cycles under the homomorphism Z defined in Section 3B.

Definition 3.48. A graph $G \in \mathcal{G}_0(k^\times)$ is admissible if:

(1) The connected components of G are strongly connected.

(2) There is no graphical loop in G that has loop coefficient 1.

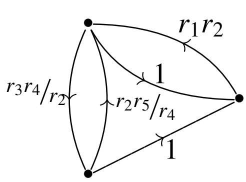
We recall the definition of a strongly connected graph in the first condition.

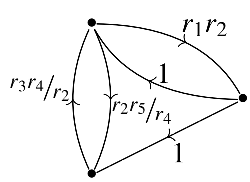
Definition 3.49. An oriented graph is strongly connected if, for any two vertices $v, w \in V(G)$, there is a path from v to w and one from w to v which respect the orientation of the edges of G.

By Lemma 3.47, Definition 3.48 implies that, if a graph $G \in \mathcal{G}_0(k^\times)$ can be parametrized such that there exists a loop with all edged labeled by ones, then G is not admissible.

Finally we add one more equivalence relation among graphs that is useful in Section 3E.

Definition 3.50. For $G \in \mathcal{G}$, let $\overline{G} \in \mathcal{G}$ be the graph with the same underlying labled unoriented graph structure, but with the orientations of every edge switched. Define an equivalence relation $\sim_{\text{ori}}$ that relates graphs with all orientations switched: $G \sim_{\text{ori}} \overline{G}$.

Example 3.51. $G =$ 

$\Longrightarrow$ $\overline{G} =$ 

Switching the orientation of all edges of a graph corresponds to a reparametrization of $Z(G)$. If the $\omega(e)$-th edge of G corresponds to the parametrization $\phi_{\omega(e)} = 1 - t_i/(a_{\omega(e)}t_j)$, then the $\omega(e)$-th edge of $\overline{G}$ corresponds to the parametrization $\bar{\phi}_{\omega(e)} = 1 - t_j/(a_{\omega(e)}t_i)$, which differs from $\phi_{\omega(e)}$ by the change of variables $t_i \to 1/t_i$. We show that these two are both parametrizations of the same cycle in Section 3E, Corollary 3.61.

Definition 3.52. There is a subalgebra

$$\mathcal{G}_{1L} \subset \mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}, \sim_{\mathrm{ori}})$$

generated over $\mathbb{Q}$ by admissible graphs.

$$\mathcal{G}_{1L} = \mathbb{Q}[G \mid G \in \mathcal{G} \text{ admissible}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}, \sim_{\mathrm{ori}}).$$

Lemma 3.53. *The differential operator ∂ restricts to a differential operator on $\mathcal{G}_{1L}$.*

Proof. By the Leibnitz rule, it is sufficient to consider connected graphs. We show that if G is an admissible graph, then so is $\partial_e(G)$ for any $e \in E(G)$.

First, we check that if G is strongly connected, then G/e is as well. If $v, w \in V(G)$ are in the same connected component of G/e, then the paths between v and w are either shortened by the contraction of the edge e, or unaffected. Therefore, the connected components of G/e are strongly connected, as desired.

As taking the derivative along any edge does not affect the loop coefficient of any loop in G, we have $\partial_e(G) \in \mathcal{G}_{1L}{}^{\bullet}_{\star+1}$ for $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$. □

Therefore, $\mathcal{G}_{1L}$ is a sub-DGA of $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}, \sim_{\mathrm{ori}})$. We show that the homomorphism Z defined in Section 3B is well-defined on $\mathbb{Q}[\mathcal{G}]/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}, \sim_{\mathrm{ori}})$.

Theorem 3.54. *Let $\overline{G}$ be as in Definition 3.50. The graphs $G, \overline{G} \in \mathbb{Q}[\mathcal{G}]^{\star}_{\bullet}$ map to the same algebraic cycle in $\mathrm{Alt}\,\mathcal{X}^{\bullet}_{1L}(\square^{2\bullet-\star})$ under Z.*

Proof. Recall from Theorem 3.19 and (3.25) that, given a basis $\beta = \{L_1, \dots, L_\bullet\}$ of $H^1(G)$, the cycle $Z(G)$ is defined by the set of equations

$$\left\{1 = \chi_G(L_i) \prod_{e \in E(L_i)} (1 - \phi_{\omega(e)})^{\epsilon(e, L_i)}\right\}_{L_i \in \beta}.$$

Note that the set β also defines a basis of $H^1(\overline{G})$, and that $\chi_{\overline{G}}(L_i) = (\chi_G(L_i))^{-1}$ for each $L_i \in \beta$, as the only difference between G and $\overline{G}$ is the orientation of the edges. Similarly, the function $\epsilon(e, L_i)$ defined on G is the negative of the same defined on $\overline{G}$. Therefore, the cycle $Z(\overline{G})$ is defined by the set of equations

$$\left\{1 = (\chi_G(L_i))^{-1} \prod_{e \in E(L_i)} (1 - \phi_{\omega(e)})^{-\epsilon(e, L_i)}\right\}_{L_i \in \beta}.$$

That is, $Z(G)$ and $Z(\overline{G})$ are defined by the same algebraic cycles. □

Therefore, $Z : \mathbb{Q}[\mathcal{G}]^{\star}_{\bullet}/(\sim_{\mathrm{ord}}, \sim_{\mathrm{v}}, \sim_{\mathrm{ori}}) \to \mathrm{Alt}\,\mathscr{Z}^{\bullet}_{1L}(\square^{2\bullet-\star})$ is a well-defined algebra homomorphism. In the following section, we show that this sub-DGA is isomorphic to $\mathscr{A}^{\times}_{1L}$.

3E. ***From graphs to admissible cycles.*** We now return to the homomorphism defined in Section 3B. In this section, we show that the map Z defined in (3.17), restricts to an isomorphism of DGAs between $\mathcal{G}_{1L}$ and $(\mathscr{A}^{\times}_{1L})$.

To compare the DGA of admissible cycles to the DGA of admissible graphs, we show that the homomorphism Z, when restricted to $\mathcal{G}_{1L}$ is compatible with both the differential on $(\mathscr{A}^{\times}_{1L})$, defined in (2.6), and the differential on $\mathcal{G}_{1L}{}^{\bullet}_{\star}$, defined in Theorem 3.35.

Recall from Definition 2.3 the faces $F_{I,J}$ of $\square^n$.

Lemma 3.55. *For $G \in \mathcal{G}_{1L}$, the derivative is*

$$Z(\partial_e(G)) = \begin{cases} Z(G) \cap F_{\omega(e),\varnothing} & \text{if } \mathrm{sgn}_{\omega(e)} = +, \\ Z(G) \cap F_{\varnothing,\omega(e)} & \text{if } \mathrm{sgn}_{\omega(e)} = - \,. \end{cases}$$

Proof. Consider G to be a connected graph. We consider two cases, when $\partial_e(G)$ is connected, and when it is a disconnected graph.

The cycle $Z(G)$ is equipped with a parametrization

$$\phi : \mathbb{P}_k^{|V(G)|-1} \to (\mathbb{P}^1_k)^{|E(G)|},$$

where the coordinate of $Z(G)$ corresponding to the $\omega(e)$-th edge is

$$\phi_{\omega(e)} = \Big(1 - x a_{\omega(e)} y\Big)^{\mathrm{sgn}_{\omega(e)}}.$$

Recall from Definition 2.12 that $Z(G)$ is the cycle defined by intersecting the image of ϕ with $\square^{|E(G)|}$. In other words, $Z(G) = i^*\phi_*$, where $i : \square^{|E(G)|} \hookrightarrow (\mathbb{P}^1_k)^{|E(G)|}$.

Let $\iota_{I,J} : F_{I,J} \to \square^n$ be the injection into the appropriate face of codimension $|I \cup J|$. If $\mathrm{sgn}_{\omega(e)} = +$ (resp. $\mathrm{sgn}_{\omega(e)} = -$), the intersection $Z(G) \cap F_{\omega(e),\varnothing}$ (resp. $Z(G) \cap F_{\varnothing,\omega(e)}$) is the further pullback $\iota^*_{\omega(e),\varnothing}(i^*\phi_*)$ (resp. $\iota^*_{\varnothing,\omega(e)}(i^*\phi_*)$).

For the remainder of this proof, we assume that $\mathrm{sgn}_{\omega(e)} = +$. The calculation for $\mathrm{sgn}_{\omega(e)} = -$ is similar, and left to the reader.

The intersection $Z(G) \cap F_{\omega(e),\varnothing}$ imposes the restriction $x = a_e y$. Therefore, it can be parametrized by

$$\phi_{\partial_e} : \mathbb{P}_k^{|V(G)|-2} \to (\mathbb{P}^1_k)^{|E(G)|-1}, \tag{3.56}$$

formed by removing the $\omega(e)$-th coordinate of ϕ and replacing each instance of x with $a_e y$. If $\partial_e(G)$ is connected, this is exactly the parametrization defined by the contracted graph. Therefore, the lemma holds when $\partial_e G$ is connected.

If $\partial_e(G) = \prod_{i=1}^k G_i$ is disconnected, then the parametrization defined by this disconnected graph,

$$\phi' : \prod_{i=1}^{k} \mathbb{P}_k^{|V(G_i)|-1} \to (\mathbb{P}_k^1)^{|E(G)|-1},$$

is different from the parametrization, ϕ_{∂_e}, defined by the contraction ∂_e in (3.56). However, consider the affine space $\mathbb{A}_k^{|V(G)|-2}$ defined by setting $x = a_e y = 1$ in $\mathbb{P}_k^{|V(G)|-1}$. Then there is a product of corresponding affine spaces, $\prod_{i=1}^k \mathbb{A}_k^{|V(G_i)|-1}$, associated to the disconnected parametrization, each formed by setting the variable of the new vertex defined by the contraction to 1. The two parametrizations ϕ_{∂_e} and ϕ' agree on these affine spaces. On the hyperplanes at infinity, at least one of the parametrizing variables is 0. Since G is strongly connected, none of the coordinates correspond to purely sink vertices in either G or $\partial_e(G)$. Therefore, setting a parametrization variable to 0 corresponds to setting a coordinate of the image of ϕ' or ϕ_{∂_e} to 1. However, $\square^{|E(G)|-1}$ omits precisely the points of $\mathbb{P}_k^{|E(G)|-1}$ where one of the coordinates is set to 1. Therefore, the parametrized cycles $Z(\partial_e G) = (i^*\phi'_*)$ and $\partial_e Z(G) = i^*\phi_{\partial_e *}$ agree on the pullback to $\mathbb{P}_k^{|E(G)|-1}$, as desired. $\square$

This is the key step to understanding the relationship between the differential on graphs and the differential on cycles.

Theorem 3.57. *If $G \in \mathcal{G}_{1L}$, then*

$$\partial Z(G) = Z(\partial(G)).$$

Proof. Recall from (2.6) that

$$\partial Z(G) = \sum_{e \in E(G)} (-1)^{\omega(e)-1} (\partial_{\omega(e),\varnothing} - \partial_{\varnothing,\omega(e)}) Z(G).$$

From Lemma 3.55,

$$\partial Z(G) = \sum_{\mathrm{sgn}(e)=+} (-1)^{\omega(e)-1} (Z(\partial_e G) - \partial_{\varnothing,\omega(e)} Z(G)) \\ + \sum_{\mathrm{sgn}(e)=-} (-1)^{\omega(e)-1} (Z(\partial_e G) - \partial_{\omega(e),\varnothing} Z(G)).$$

The theorem follows from the fact that $\partial_{\varnothing,\omega(e)} Z(G)$ is empty if $\mathrm{sgn}(e) = +$ and $\partial_{\omega(e),\varnothing} Z(G) = \varnothing$ if $\mathrm{sgn}(e) = -$.

As above, we only do the calculation for $\mathrm{sgn}(e) = +$, as the calculation for $\mathrm{sgn}(e) = -$ is similar. By definition,

$$\partial_{\varnothing,\omega(e)} Z(G) = Z(G) \cap F_{\varnothing,\omega(e)}.$$

That is, the coordinate $\phi_{\omega(e)} = 1 - x/(a_e y) = \infty$. This implies that $x/y = \infty$. Since G is strongly connected, there is another edge e' such that $t_e = s_{e'}$. Then $\phi_{\omega(e')} = 1 - y/(a_{e'} x) = 1$. Therefore, $\partial_{\varnothing,\omega(e')} Z(G) = \varnothing$. $\square$

For any two edges $e, e' \in E(G)$, with $G \in \mathcal{G}_{1L}$, the derivatives ∂_e and $\partial_{e'}$ commute, by Lemma 3.46. Therefore, we can talk about contracting a subgraph of another graph, without noting the order in which the edges are contracted.

Definition 3.58. Let $G' \subset G$, with $E(G') = \{e_1, \ldots, e_n\}$. We write

$$\partial_{G'}(G) = \partial_{e_n}(\cdots(\partial_{e_1}(G))\cdots),$$

where $e_i \in E(G')$.

Notice that if the contracted graph G' is not a subtree of G, then $\partial_{G'}(G) = 0$.

We use this shorthand to show that the graphs in $\mathcal{G}_{1L}{}^{\bullet}_{\star}$ correspond exactly to admissible cycles in $(\mathcal{A}^{\times}_{1L})^{\bullet}_{\star}$. Recall that an algebraic cycle in $\mathcal{Z}^{\bullet}(\operatorname{Spec} k, \star)$ is admissible if it intersects all faces of $\square^{2\bullet-\star}$ in codimension $\bullet$ or not at all.

Theorem 3.59. *For $G \in \mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}}, \sim_{\text{ori}})$, the cycle $Z(G)$ is admissible if and only if $G \in \mathcal{G}_{1L}$.*

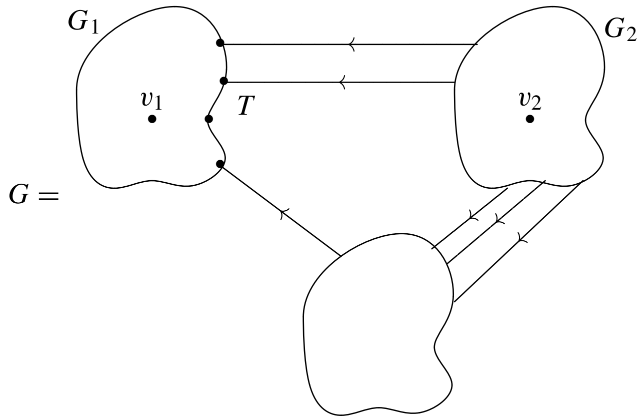
Proof. It is sufficient to look at connected graphs.

Consider a $G \in \mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{v}}, \sim_{\text{ori}})$ such that there exists a loop, L with loop coefficient 1 in G. Specifically, chose a graph $G \notin \mathcal{G}_{1L}{}^{\bullet}_{\star}$. By Lemma 3.40, we can label the edges of any spanning tree of L by ones. Since rescaling does not change the loop coefficient by Lemma 3.31, all edges of L can be labeled by ones. Let $T \subset L$ be a subgraph of the loop L consisting of all but two of the edges of L. Suppose $E(L \setminus T) = \{e_1, e_2\}$. Let $I = \{e \in E(T) \mid \operatorname{sgn}_e = +\}$ and $J = \{e \in E(T) \mid \operatorname{sgn}_e = -\}$. The graph $\partial_T(G)$, formed by taking the derivative of G along the edges in T, corresponds to intersecting $Z(G)$ with the face $F_{I,J}$. The $\omega(e_1)$-th and $\omega(e_2)$-th coordinate of $Z(\partial_T(G))$ are of the form $\operatorname{sgn}(e_i)(1 - x/y)^{\operatorname{sgn}(e_i)}$ for $i \in \{1, 2\}$. This cycle is not admissible.

To see this, notice that the intersection of $Z(\partial_T(G))$ with the face $F_{\omega(e_1),\varnothing}$ (if $\operatorname{sgn}(e_1) = +$) or $F_{\varnothing,\omega(e_1)}$ (if $\operatorname{sgn}(e_1) = -$) also sets the $\omega(e_2)$-th coordinate to 0, giving it the wrong codimension.

Conversely, suppose $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$. Specifically, G is strongly connected. Let G' be a (not necessarily connected) subgraph of G. Let $I = \{e \in E(G') \mid \operatorname{sgn}_e = +\}$ and $J = \{e \in E(G') \mid \operatorname{sgn}_e = -\}$. Consider $D_{G'}(G)$. By Lemma 3.53, $D_{G'}(G)$ is also in $\mathcal{G}_{1L}{}^{\bullet}_{\star}$. If G' is not a forest, then $D_{G'}(G) = 0$. Therefore, we only consider the case when G' is a forest (possibly consisting of a single tree). By Lemma 3.55 $D_{G'}(G)$ amounts to intersecting $Z(G)$ with the face $F_{I,J}$. Since G' is a forest, $h_1(G') = 0$, and $h_1(G) = h_1(D_{G'}(G))$. Therefore $Z(G) \cap F_{I,J}$ has codimension $\star$ in F, making it admissible.

Finally, if G is not strongly connected, then there exists two vertices v_1 and v_2 such that there is not an orientation-respecting path in G from v_1 to v_2. Let G_1 be the largest subgraph of G defined by the vertices that can be reached by orientation-respecting paths from v_1. Let G_2 be the largest subgraph of G defined by the vertices that can reach v_2 by orientation-preserving paths in G. By construction, G_1 and G_2 are disjoint subgraphs:



In particular, the subgraph G_1 has i edges flowing into its vertices from the rest of the graph, $G \setminus G_1$. Let T be a subtree of G_1 connecting all the sink vertices of these incoming edges. The derivative $\partial_T(G)$ has at least two connected components. Write

$$\partial_T(G) = \pm G' \partial_T(G_1),$$

with G' the (possibly disconnected) subgraph of $\partial_T(G)$ that contains G_2 as a subgraph. The graph G' has a sink vertex in the connected component containing (G_2). Therefore, the cycle $Z(G')$ has at least two coordinates of the form $\phi_i = (1 - x/(ay))^{\mathrm{sgn}_i}$ and $\phi_j = (1 - z/(by))^{\mathrm{sgn}_j}$. Setting the coordinate $\phi_i = 0(\infty)$ sets the coordinate $\phi_j = 0(\infty)$, by the arguments above. Since the derivative $\partial_T(G)$ has the wrong codimension intersecting the face $F_{i,\varnothing(\varnothing,i)}$, the cycle $Z(G)$ is not admissible. □

It follows from Theorems 3.26, 3.32, 3.54 and 3.59, that the homomorphism Z is surjective:

Corollary 3.60. *The homomorphism*

$$Z : \mathscr{G}_{1L}{}^{\bullet}_{\star} \to (\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$$

is a surjection of DGAs.

Proof. By Theorems 3.26, 3.32, 3.54 and 3.59, we see that Z is a homomorphism of DGAs with image contained in $(\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$. We check surjection of this map. By definition, if $Z \in (\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$, there is a parametrization $\phi : \mathbb{P}_k^{\bullet-\star} \to (\mathbb{P}^1_k)^{2\bullet-\star}$, with $\phi_i = 1 - x_i/(a_i y_i)$. Assuming that Z is reducible, Corollary 3.27 states that this defines a connected graph G with $2\bullet - \star$ edges and $\bullet - \star + 1$ vertices. Since Z is admissible, by Theorem 3.59, $G \in \mathscr{G}_{1L}$. □

It remains to show that Z is an isomorphism.

Corollary 3.61. *Any cycle in* $(\mathcal{A}^\times_{1L})$ *remains invariant under inverting all the parametrizing variables, or scaling some of them by a constant multiple.*

Proof. This follows from Theorem 3.19.

Let $\mathfrak{X} \in (\mathcal{A}^\times_{1L})$ be the cycle parametrized by the variables $\{v_1, \dots, v_n\}$ such that each coordinate is of the form $\phi_i^{\operatorname{sgn}_i}$ with

$$\phi_i = 1 - \frac{v_{i_s}}{a_i v_{i_t}},$$

and $v_{i_s}, v_{i_t} \in \{v_1, \dots, v_n\}$. Let $\mathfrak{X}' \in (\mathcal{A}^\times_{1L})$ be the cycle with coordinates

$$\phi_i = 1 - \frac{b_{i_s} v_{i_s}}{a_i b_{i_t} v_{i_t}}$$

for $b_{i_j} \in k^\times$, and $\hat{\mathfrak{X}} \in (\mathcal{A}^\times_{1L})$ be the cycle with coordinates

$$\phi_i = 1 - \frac{v_{i_t}}{a_i v_{i_s}}.$$

The claim of this corollary is that

$$\mathfrak{X}' = \mathfrak{X} = \hat{\mathfrak{X}}. \tag{3.62}$$

Algebra and writing the cycles out in the form of (3.20) shows that these equalities hold. □

In terms of graphs, the first equality in (3.62) corresponds to rescaling at vertices to pass from G to $v_{1b_1}(\cdots(v_{nb_n}(G))\cdots)$. The second equality corresponds to changing the orientations of all the edges in the graph.

We are now ready to show that the two algebras $\mathcal{G}_{1L}{}^\bullet_\star$ and $(\mathcal{A}^\times_{1L})^\bullet_\star$ are isomorphic.

Theorem 3.63. *The map* $Z : \mathcal{G}_{1L}{}^\bullet_\star \to (\mathcal{A}^\times_{1L})^\bullet_\star$ *defined in* (3.17) *is an isomorphism of DGAs.*

Proof. Theorem 3.57 shows that Z is a homomorphism of DGAs. Corollary 3.60 shows that this map is surjective.

Rescaling a vertex on a graph G, that is passing from G to $v_\alpha(G)$, corresponds to rescaling the corresponding parametrizing variable in $Z(G)$. Similarly, inverting the orientations of all the edges, passing from G to $\overline{G}$, corresponds to inverting all the parametrizing variables in $Z(G)$. Since, by Corollary 3.61, neither of these reparametrizations changes the underlying cycle, the map Z is one-to-one.

Explicitly, define a map

$$G : (\mathcal{A}^\times_{1L})^\bullet_\star \to \mathcal{G}_{1L}{}^\bullet_\star$$

that is a left inverse of Z. For any cycle parametrized in a $\mathbb{P}^1_k$-linear form,

$$G(\operatorname{Alt}[\phi_1^{\operatorname{sgn}_1}, \dots, \phi_n^{\operatorname{sgn}_n}])$$

is a graph constructed as follows: Write each ϕ_i as $1 - x/(a_i y)$. If ϕ_i is a constant, write it as $1 - 1/(a_i)$. Each independent variable in $\mathrm{Alt}[\phi_1, \dots, \phi_n]$ corresponds to a vertex. For each ϕ_i, draw an oriented edge of G, oriented from the numerator variable to the denominator variable, labeled by a_i. In this scheme, constant coordinates correspond to one edge loops. The term ω is defined by the ordering and signs of the ϕ_i. □

4. Elements of $H^0(B(\mathscr{G}_{1L}))$

In the previous section, we establish an isomorphism between the DGA of $\mathbb{P}^1_k$-linear cycles, $(\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$, and the DGA of admissible graphs $\mathscr{G}_{1L}{}^{\bullet}_{\star}$. We use this to establish that everything that needs to be done for $(\mathscr{A}^{\times}_{1L})^{\bullet}_{\star}$ cycles can be done on the algebra of graphs $\mathscr{G}_{1L}{}^{\bullet}_{\star}$. For the rest of this paper, we restrict our attention to the DGA of graphs.

In particular, to define the category of motives, we are interested in studying the Hopf algebra

$$H_0(B(\mathscr{G}_{1L})) \simeq H_0(B(\mathscr{A}^{\times}_{1L})).$$

We maintain the definition of the bar construction $B(\mathscr{G}_{1L})$ as in Definition 2.1, with $A = \mathscr{G}_{1L}$. Following convention, we indicate the tensor product in the bar construction by $|$.

As in Definition 2.1, write the degree and tensor graded components of $B(\mathscr{G}_{1L})$ as

$$B(\mathscr{G}_{1L})^n_m = \bigoplus_{\sum_1^n (w_i - 1) = m} [\mathscr{G}_{1L}{}^{\bullet}_{w_1} \mid \cdots \mid \mathscr{G}_{1L}{}^{\bullet}_{w_n}]. \tag{4.1}$$

Note that, as in Definition 2.1, the degree of a graph in the bar construction is shifted from the degree of a graph in the algebra. That is, if $G \in \mathscr{G}_{1L}{}^{\bullet}_j$, then $G \in B(\mathscr{G}_{1L})^1_{j-1}$.

Definition 4.2. Due to the multiple degrees assigned to graphs in an algebraic and bar construction context, we write $\deg_B$ (as opposed to simply deg) for the shifted degree of a graph as it contributes to the total degree in the bar construction.

Explicitly, if $G \in \mathscr{G}_{1L}{}^{\bullet}_j$, $\deg(G) = j$ then $\deg_B(G) = j - 1$.

To set notation, we define differentials that make the bar complex $(B(\mathscr{G}_{1L}), \partial + \mu)$ a bicomplex. Write $\partial_{\mathscr{G}}$ and $\mu_{\mathscr{G}}$ for the derivatives and product on the graphs. Then ∂ and μ are the degree-one operators on $B(\mathscr{G}_{1L})$ induced by $\partial_{\mathscr{G}}$ and $\mu_{\mathscr{G}}$, calculated by the degree of graphs in the bar construction under the Leibnitz rule. Let ∂_j be the differential operator that acts by $(-1)^{\deg_B G_i}$ id on the first $j - 1$ tensor components, by $\partial_{\mathscr{G}}$ on the j-th tensor component, and by id on the remaining tensor components. Then for $[G_1 \mid \cdots \mid G_n] \in B(\mathscr{G}_{1L})^n_m$, write

$$\partial[G_1 \mid \cdots \mid G_n] := \sum_{j=1} \partial_j [G_1 \mid \cdots \mid G_n]$$
$$= \sum_{j=1}^{n} (-1)^{\sum_{k=1}^{j-1} \deg_B(G_k)} [G_1 \mid \cdots \mid \partial_{\mathscr{G}}(G_j) \mid \cdots \mid G_n], \tag{4.3}$$

is a degree-one differential operator $\partial : B(\mathscr{G}_{1L})^n_m \to B(\mathscr{G}_{1L})^n_{m+1}$. Similarly, let μ_j be the differential operator that acts by $(-1)^{\deg_B G_i}$ id on the first $j-1$ tensor components, by $(-1)^{\deg_B G_j}\mu$ on the j-th and $(j+1)$-st components, and as id on the remaining components. Then

$$\mu[G_1 \mid \cdots \mid G_n] := \sum_{j=1} \mu_j [G_1 \mid \cdots \mid G_n]$$
$$= \sum_{j=1}^{n-1} (-1)^{\sum_{i=1}^{j} \deg G_i} [G_1 \mid \cdots \mid G_j \cdot G_{j+1} \mid \cdots \mid G_n]. \tag{4.4}$$

This is a degree-one differential operator, as $[G_1 \mid G_2] \in B(\mathscr{G}_{1L})^2_{m_1+m_2-2}$ while $\mu[G_1 \mid G_2] = [G_1 G_2] \in B(\mathscr{G}_{1L})^1_{m_1+m_2-1}$ for $G_i \in \mathscr{G}_{1L}{}^{r_i}_{m_i}$.

In order to study elements of $H^i(B(\mathscr{G}_{1L}))$, identify elements in the kernel of

$$D + \mu : \bigoplus_{n \geq 1} B(\mathscr{G}_{1L})^n_i \to \bigoplus_{n \geq 1} B(\mathscr{G}_{1L})^n_{i+1}.$$

By Definition 2.8, we see that elements of this kernel are exactly the elements with completely decomposable boundaries.

Remark 4.5. Very few generators of $\mathscr{G}_{1L}{}^{\bullet}_{\star}$ as an algebra have a decomposable boundary. The completely decomposable objects in $B(\mathscr{G}_{1L})$ correspond to linear combinations of tensor products of graphs.

In this paper, we wish to study $H^0(B(\mathscr{G}_{1L}))$. Therefore, we study completely decomposable elements of $\bigoplus_{i \geq 1} B(\mathscr{G}_{1L})^i_0$ defined by completely decomposable elements of $B(\mathscr{G}_{1L})^1_0$. From Definition 2.8, a completely decomposable element, ε, of $B(\mathscr{G}_{1L})^1_0$ defines a trivial cycle in $H^0(B(\mathscr{G}_{1L}))$ if it can be written as the coboundary of another sum of graphs $\sum_i G_i \in \mathscr{G}_{1L}{}^{\bullet}_2$,

$$\partial \sum_i G_i = \varepsilon,$$

or if it can be written as the sum of a product of graphs,

$$\mu \sum_i [G_{1,i} \mid G_{2,i}] = \varepsilon.$$

In this section, we first give a result that greatly reduces the number of algebraic cycles in $\mathscr{A}^{\times}_{1L}{}^{\bullet}_{1}$ one needs to construct $H^0(B(\mathscr{G}_{1L}))$.

Theorem 4.6. *If $\varepsilon \in \mathscr{A}^{\times\,\bullet}_{1L\,i}$ is a completely decomposable algebraic cycle which can be written as $Z(\sum G_j)$, where each $G_j \in \mathscr{G}_{1L}{}^{\bullet}_{i}$, and some G_j have valence-two vertices, then ε defines a coboundary element of $B(\mathscr{A}^{\times}_{1L})$.*

In particular, taking $i = 1$, we see that sums of graphs involving valence-two vertices have trivial motivic contributions. This is a major calculational aid in that it identifies a large class of cycles that we need not consider for motivic content. The proof of this theorem is the subject of Section 4A. See Theorem 4.16 for the graphical version of this statement. In Section 4B1 we give examples of some completely decomposable graphs.

Since we are only interested in the zeroth cohomology henceforth, for the remainder of this paper we only consider graphs in $\mathscr{G}_{1L}{}^{\bullet}_{1}$, that is, cycles in $\mathscr{A}^{\times\,\bullet}_{1L\,1}$.

4A. ***Valence-two vertices.*** In this section we show that there is a large class of graphs in $\mathscr{G}_{1L}$ that correspond to the trivial cycles in $H^i(B(\mathscr{G}_{1L}))$. Namely, we show that completely decomposable sums of graphs with two valent vertices can be written as the coboundary of an element of $\mathscr{G}_{1L}{}^{\bullet}_{i-1}$. We start by studying the properties of decomposable graphs in $\mathscr{G}_{1L}$ with two valent vertices.

Definition 4.7. A handle of length $n > 1$ is a linear subgraph $h \in G$ defined by n edges and $n+1$ vertices $\{v_0, \dots, v_n\}$ labeled as follows: The vertex v_i has valence 2 if $1 \leq i < n$. The vertex v_0 and v_n have valence 1 in the handle h, but strictly greater than 2 in G. Write $E(h) = \{e^1, \dots, e^n\}$, with e^i the edge in h connecting v_{i-1} and v_i. Write $H(G)$ to the be set of handles of a graph G. Further, write $H_{\text{odd}}(G)$ for the set of handles of odd lengths and $H_{\text{even}}(G)$ for the set of handles of even length.

Minimally decomposable sums of graphs can be classified by the number of handles they have.

Lemma 4.8. *Consider $G \in \mathscr{G}_{1L}$, a connected graph with handles, $H(G) \neq 0$. Then*

$$\sum_{e \in E(h)} (-1)^{\omega(e)-1} \partial_e G = \begin{cases} 0 & \text{if } h \in H_{\text{even}}(G), \\ (-1)^{\omega(e^1)-1} \partial_{e^1} G & \text{if } h \in H_{\text{odd}}(G). \end{cases}$$

Proof. The essence of this proof comes from showing the relation

$$(-1)^{\omega(e^i)} \partial_{e^i} G = -(-1)^{\omega(e^{i+1})} \partial_{e^{i+1}} G. \tag{4.9}$$

To see this, choose a representation of G such that the edges of h are labeled by ones.

Write $c(e^i, e^{i+1}) \in \mathfrak{S}_{|E(G)|}$ as the cyclic element of order $|\omega(e^{i+1}) - \omega(e^i)|$. Write this as

$$c(e^i, e^{i+1}) := (\omega(e^i))(\omega(e^i)+1)\dots(\omega(e^{i+1})-1)(\omega(e^{i+1})).$$

The sign of this permutation is given by $\operatorname{sgn}(c(e^i, e^{i+1})) = (-1)^{\omega(e^{i+1})-\omega(e^i)+1}$. In this notation, the orderings of the contracted graphs can be related by

$$\hat{\omega}_{e^{i+1}} = \begin{cases} c(e^i, e^{i+1})\hat{\omega}_{e^i} & \text{if } \omega(e^i) < \omega(e^{i+1}), \\ c(e^i, e^{i+1})^{-1}\hat{\omega}_{e^i} & \text{if } \omega(e^i) > \omega(e^{i+1}). \end{cases}$$

Since the underlying contracted graphs, $G/e^i = G/e^{i+1}$, are the same, we have, by Lemma 3.8,

$$(-1)^{\omega(e^{i+1})-\omega(e^i)+1}\partial_{e^{i+1}}(G) = \partial_{e^i} G,$$

which is equivalent to (4.9).

Summing over all edges in a fixed handle h gives

$$\sum_{e\in E(h)} (-1)^{\omega(e)}\partial_e(G) = \begin{cases} 0 & \text{if } n \text{ even}, \\ (-1)^{\omega(e^1)}\partial_{e^1}(G) & \text{if } n \text{ odd}. \end{cases} \qquad \square$$

Call edges of G that are not handles, interior edges of G.

Definition 4.10. By abuse of notation, write $\mathring{G}$ to indicate the interior graph of G. This is the graph G with all its handles removed (not contracted). More precisely,

$$\mathring{G} = G \setminus \{e \mid e \in E(h),\ h \in H(G)\}.$$

In this section, we write

$$\partial|_H(\omega, G) = \sum_{e\in H(G)} (-1)^{\omega(e)-1}\partial_e(G), \tag{4.11}$$

$$\partial|_{\mathring{G}}(G) = \sum_{e\in\mathring{G}} (-1)^{\omega(e)-1}\partial_e(G), \tag{4.12}$$

so that $\partial = \partial|_H + \partial|_{\mathring{G}}$. This allows for a neat reorganizing of the terms in the derivative ∂G by interior edges and edges with valence-two endpoints.

Corollary 4.13. *The derivative is*

$$\partial(G) = \sum_{e\in E(\mathring{G})} (-1)^{\omega(e)-1}\partial_e(G) + \sum_{e\in h,\ h\in H_{\mathrm{odd}}(G)} (-1)^{\omega(e_1(h))-1}\partial_{e_1(h)}(G).$$

As a direct corollary, we see that graphs with valence-two vertices form a separate class of graphs in themselves. If $\varepsilon \in B(\mathcal{G}_{1L})$ is a minimally decomposable sum of graphs, then either all the summands involve a valence-two vertex, or none of them do. In fact, one can be more specific than this.

Corollary 4.14. *Consider a minimally decomposable sum of graphs $\varepsilon = \sum_j G_j$ in $\mathcal{G}_{1L_i^\bullet}$ of fixed degree. The summand graph G_j has a valence-two vertex if and only if the graphs in each of the summand have the same number of handles*:

$$|H(G_j)| = |H(G_{j'})| \quad \textit{for all } j \neq j'.$$

Proof. If $\partial_e(G_j)$ is not decomposable, then it must cancel with a sum of another derivative $\partial_{e'}(G_{j'})$. By Lemma 4.8 and Corollary 4.13, applying ∂ does not change the number of handles on a graph. Since ε has a minimally completely decomposable boundary, there are no summands that do not contribute to the cancellation of the terms in ∂_e. Therefore, G_j and G'_j must have the same number of handles. □

Finally, we show that sums of graphs with decomposable boundaries and valence-two vertices characterize trivial classes in $H^0(B(\mathcal{G}_{1L}))$. In the proof of this theorem, we work up to products of graphs. For this, we establish some notation.

Definition 4.15. For G a connected graph in $\mathcal{G}_{1L}$, if $\partial_e G$ is decomposable, we write

$$\partial_e G \doteq 0.$$

In general, we write $\partial G \doteq G'$, where G' is a linear sum of connected graphs, that is, G' is the sum of graphs corresponding to edge differentials that do not split the graph into two connected components.

Theorem 4.16. *Let $\varepsilon = \sum_j (G_j) \in \mathcal{G}_{1L}{}^{\bullet}_{i}$ be a sum of graphs with minimally completely decomposable boundary, such that each graph has bivalent vertices. Then there exists a sum of graphs $\eta \in \mathcal{G}_{1L}{}^{\bullet}_{i-1}$ such that $[\partial\eta] = [\varepsilon]$. In other words, $[\varepsilon]$ is exact.*

Translated into the language of algebraic cycles, instead of graphs, this theorem gives Theorem 4.6.

Proof. Let ε be a minimal completely decomposable sum of graphs. Write $\varepsilon = \sum_{G\in S} G$, where S is the set of summands (not including multiplicity). By Corollary 4.14, each $G \in S$ has the same number of total handles. That is, $H(G) = m$ for all $G \in S$. It suffices to work with sums of connected graphs.

We can partition the underlying set of graphs S by the number of odd handles they have. Write $\varepsilon_i = \sum_{G\in S_i} G$, where $S_i = \{G \in S \mid H_{\text{odd}}(G) = i\}$. In this way, we may write

$$\varepsilon = \sum_{i=j}^{n} \varepsilon_i .$$

In other words, while every graph in S has m handles, they all have between j and n handles of odd length. From equations (4.11) and (4.12), write the differential operator as $\partial = \partial|_H + \partial|_{\mathring{G}}$. Then the sum $\partial(\varepsilon)$ decomposes into $n - j + 1$ sums that evaluate to 0, up to decomposable elements. By collecting terms according to the number of odd handles are present in the graph:

$$\begin{aligned} \partial|_{\mathring{G}}\varepsilon_n &\doteq 0, \\ \partial|_H\varepsilon_n + \partial|_{\mathring{G}}\varepsilon_{n-1} &\doteq 0, \\ &\vdots \\ \partial|_H\varepsilon_j &\doteq 0, \end{aligned} \tag{4.17}$$

with $j \geq 0$.

In order to better understand the cancellations represented by the equations in (4.17), further classify the handles with odd length of the graphs $G \in S_i$. Write

$$H_R(G) = \{h \in H_{\text{odd}}(G) \mid \partial|_h(G) + \partial|_{h'}(G') = 0 \text{ for some } G' \in S_i,\ h' \in H_{\text{odd}}(G')\}.$$

This is the set of handles that cancel with other handles of other graphs in S_i. Notice that G and G' must be different graphs, otherwise G would be a torsion element and thus 0. Similarly, write

$$H_I(G) = \{h \in H_{\text{odd}}(G) \mid \partial|_h(G) + (-1)^{\omega(e)-1}\partial e(G') = 0 \text{ for some } G' \in S_{i-1},\ e \in \mathring{G}'\}.$$

This is the set of handles that cancel with interior edges of graphs in S_{i-1}. Thus defined, $H_{\text{odd}}(G) = H_R(G) \cup H_I G$.

By construction, no graph in S has fewer than j handles of odd length. Since $\partial|_H \varepsilon_j \doteq 0$, for every $G \in S_j$ we have $H_R(G) = H_{\text{odd}}(G)$.

We define the $\eta = \sum_{i=j}^{n} \eta_{i+1}$ desired in the theorem by extending specific even handles, $h \in H_{\text{even}}(G)$, of summands of ε ($G \in S$). The specifics of which handles are extended is described below.

The construction proceeds by induction on the number of odd handles. For $j > 0$, $H_{\text{odd}} = H_R(G) \neq \varnothing$ for all $G \in S_j$. For every graph handle pair, (G, h) and (G', h'), for $G, G' \in S_j$ and $h \in H_R(G)$, $h' \in H_R(G')$ such that $\partial|_h(G) + \partial|_{h'}(G') = 0$, there is a $\tilde{G} \in \tilde{S}_{j+1}$ with odd handles corresponding both to h and h'. Namely, this is the graph constructed by extending the even handle of G correspond to the odd handle $h' \in H_{\text{odd}}(G')$ to an odd handle. Order the edges of h and $h' \in H_{\text{odd}}(\tilde{G})$ so that $\partial|_h(\tilde{G}) = (-1)^j G$ and $\partial|_{h'}(\tilde{G}) = (-1)^j G'$.

Write

$$\eta_{j+1} = \sum_{\tilde{G} \in \tilde{S}_{j+1}} \tilde{G}$$

with $\tilde{S}_{j+1}$ the set of summands of η_{j+1}. This is a minimal set of the $\tilde{G}$s constructed above to satisfy $\partial|_H \eta_{j+1} = \varepsilon_j$.

If $j = n$, this concludes the proof, as, by construction,

$$\partial|_{\mathring{G}} \eta_{n+1} \doteq 0.$$

Therefore, $\partial \eta_{n+1} = \varepsilon_n$, as desired.

If $j = 0$, then $H_R(G) = \varnothing$ for all $G \in S_0$, and $\partial|_H \varepsilon_0 = 0$. Therefore, we may construct η_1 by extending an arbitrary even handle per graph. This construction is not unique. However, there is a restriction on the choice of edge to extend, as outlined towards the end of this proof. As above, $\partial|_H \eta_1 = \varepsilon_0$.

To understand $\partial|_{\mathring{G}} \eta_{j+1}$ for $n > j \geq 0$, note that

$$\partial|_H \partial|_{\mathring{G}} \eta_{j+1} = -\partial|_{\mathring{G}} \partial|_H \eta_{j+1} = -\partial|_{\mathring{G}} \varepsilon_j \doteq \partial|_H \varepsilon_{j+1}.$$

The first equality comes from the anticommuting of derivatives (as argued in the proof of Theorem 3.35, the second from the construction of $\partial|_H\eta_{j+1}$ and the last from (4.17). This implies that

$$\partial|_H\partial|_{\mathring{G}}\eta_{j+1} = \sum_{\tilde{G}\in\tilde{S}_{j+1}}\sum_{h\in H_I(\tilde{G})}\partial|_h\partial|_{\mathring{\tilde{G}}}\tilde{G} \doteq \sum_{G\in S_{j+1}}\sum_{h\in H_I(G)}\partial|_h G = \partial|_H\varepsilon_{j+1}.$$

By the middle equality, we may divide the summands of $\partial|_{\mathring{G}}\eta_{j+1}$ in two groups: those that correspond to elements of S_{j+1} $(\varepsilon_{j+1}(1))$ (the $G\in S_{j+1}$ such that $H_I(G)\neq\varnothing$), and those that differ from elements of S_{j+1} by the position of one handle of odd length R_{j+1}. Write

$$\partial|_{\mathring{G}}\eta_{j+1} \doteq R_{j+1}+\varepsilon_{j+1}(1),$$

where R_{j+1} is a sum of terms that differ from summands of ε_j by the placement of one odd handle, and $\varepsilon_j(1)$ are the summands of $\partial|_{\mathring{G}}\eta_{j+1}$ that are also summands of ε_j.

We continue constructing η_i by induction on i.

Define $\varepsilon_i(2)=\varepsilon_i-\varepsilon_i(1)$ to be the difference between ε_i and the quantity $\varepsilon_i(1)$ defined in the previous inductive step. Consider the sum of graphs $-R_i+\varepsilon_i(2)$, with R_i as defined in the previous inductive step. Let T_i be the underlying set of graphs in the sum $-R_i+\varepsilon_i(2)$. By construction each summand in R_i differs from a summand of $\varepsilon_i(2)$ by the placement of one odd handle. The remaining summands of $\varepsilon_i(2)$ (those that do not have a corresponding summand in R_i) are precisely the $G\in S_i$ such that $H_I(G)=\varnothing$. Construct η_{i+1} as before, comparing graphs in T_i instead of S_j.

As above, we have

$$\begin{aligned}\partial|_H\partial|_{\mathring{G}}\eta_i &= -\partial|_{\mathring{G}}\partial|_H\eta_i = -\partial|_{\mathring{G}}(-R_{i-1}+\varepsilon_{i-1}(2))\\ &= -\partial|_{\mathring{G}}\big(-(R_{i-1}+\varepsilon_{i-1}(1))+\varepsilon_{i-1}\big) \doteq -\partial|_{\mathring{G}}(-\partial|_{\mathring{G}}\eta_{i-1}+\varepsilon_{i-1}) \doteq \partial|_H\varepsilon_i.\end{aligned}$$

Therefore, we may write

$$\partial|_{\mathring{G}}\eta_i \doteq R_i+\varepsilon_i(1).$$

For $n>i>1$, R_i cannot be 0. If $R_i=0$, then $\varepsilon(1)=\varepsilon_i$, as every graph $G\in S_i$ is such that $H_R(G)\neq\varnothing$. In this case, $\partial\eta_i=\varepsilon_{i+1}+\varepsilon_i(2)-R_i$. Therefore, we may write $\partial\sum_{j+1}^i\eta_k \doteq \sum_j^i\varepsilon_k$, which contradict that ε is a minimal sum. However, for $n>i=1$, one must be careful to extend even handles of ε_0 so that $R_1\neq 0$, otherwise the induction can't continue. This choice can always be made by comparing ε_0 to ε_1.

Finally, if $i=n+1$, note that

$$\partial|_{\mathring{G}}\eta_{n+1}\doteq 0.$$

Therefore, the process terminates. □

So far we have shown a class of minimally decomposable sums of graphs (algebraic cycles) that give rise to trivial motives. We have said nothing about how to find such minimally decomposable sums. There is as yet a short yet significant selection of literature on trying to understand this structure [Gangl et al. 2007; 2009; Soudères 2015; 2016a]. In the next section we give some examples of minimally decomposable sums in degree 4, only one of which has been previously studied [Gangl et al. 2009]. As of yet, we do not claim to add to the existing knowledge about the structure of, and relations between, minimally decomposable sums, other than identifying further examples. In future work, we hope to return to this larger class of example to better understand which sums of graphs define classes in $H^0(B(\mathcal{G}_{1L}))$.

4B. *Examples.* In this section, we give several examples of classes of $H^0(B(\mathcal{G}_{1L}))$. Generally speaking, it is nontrivial to find linear combinations of graphs which define classes in $H^0(B(\mathcal{G}_{1L}))$. Individual graphs do not have decomposable boundaries. It is only when summed with appropriate graphs with whom the boundaries cancel does one find classes in $H^0(B(\mathcal{G}_{1L}))$.

In the following subsection, we give examples of several sums in weight four.

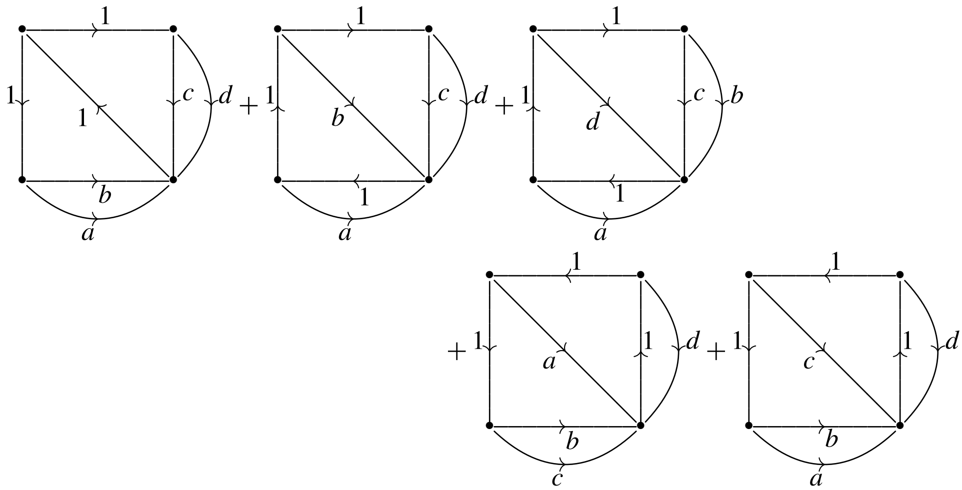
Remark 4.18. In all of these examples in this section, we write only a sum of graphs in $\mathcal{G}_{1L}{}^4_1$, and not the full representative in $B(\mathcal{G}_{1L})$. We can do this since the indecomposable graphs in a completely decomposable sum of graphs determines its class in $B(\mathcal{G}_{1L})$ (see Remark 2.10).

After giving examples in weight 4, we turn our attention to an particularly nice infinite family of graphs for which we compute the Hodge realization functor in Section 5.

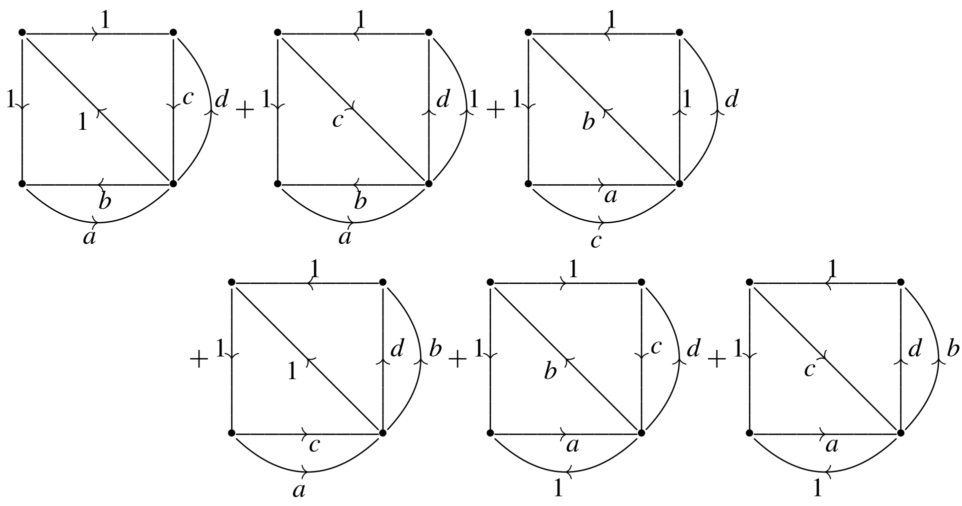
4B1. *Some minimally decomposable examples in degree* 4. In this section we give several examples of minimally decomposable sums of graphs in weight four. One of these, Example 4.19, corresponds exactly to the decomposable cycles identified in [Gangl et al. 2009] that correspond with $\mathrm{Li}_{1,1,1,1}\big(\frac{b}{a}, \frac{c}{b}, \frac{d}{c}, \frac{1}{d}\big)$. We also find a different minimally decomposable sum of graphs that involves the same unoriented graphs, but with different coefficients and orientations on the edges. In Example 4.22 we give two minimally decomposable sums that involve a different underlying graph, though closely related to the underlying graph of the previous example. Example 4.23 gives the degree-four example of the family of graphs studied in detail in Section 4B2. (In Section 5C we calculate the Hodge realization of these graphs.) Finally, Example 4.24 gives a more complicated minimally decomposable sum in degree four involving several distinct underlying graphs.

The reader is encouraged to play with these examples and construct others. There seems to be a lot of variety as to the type and number of underlying graphs in a sum that is decomposable. It would be very interesting to understand this structure better.

Example 4.19. Gangl et al. [2009] define a family of five binary graphs that correspond to $\mathrm{Li}_{1,1,1,1}\left(\frac{b}{a}, \frac{c}{b}, \frac{d}{c}, \frac{1}{d}\right)$. In the notation developed here, we depict this same minimally decomposable sum of trees as



Example 4.20. There is another decomposable sum of graphs involving the same underlying unoriented graphs:



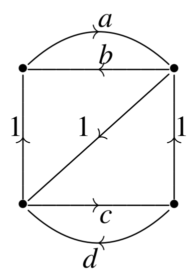
Remark 4.21. For $G \in \mathscr{G}_{1L^\bullet_\star}$ a connected graph, and $\beta = \{L_1, \dots, L_\bullet\}$ a loop basis of $H_1(G)$, let β index the system of polynomial equations f_{L_i} that define the admissible cycle $Z(G)$ in Theorem 3.19. Namely, f_L is the equation

$$1 = \prod_{e \in E(L)} a_e (1 - \phi_{w(e)})^{\epsilon(e,L)}.$$

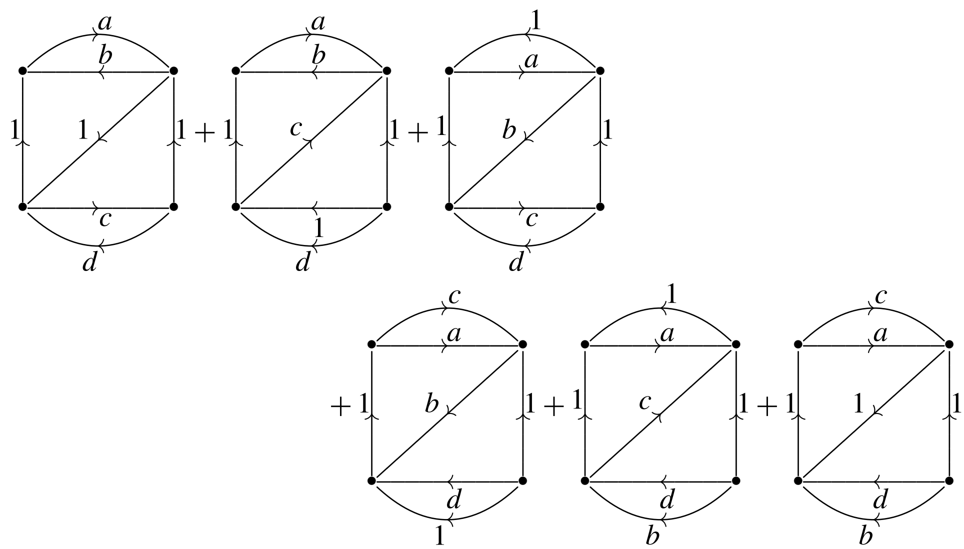
Then reversing the orientation of an edge e in graph G without changing its label replaces every factor of $a_e(1-\phi_{w(e)})$ with $(a_e(1-\phi_{w(e)}))^{-1}$. In other words, such graphs represent closely related algebraic cycles. For instance, in the above example, the first graph in Example 4.19 and the first graph in Example 4.20 differ by changing the orientations of the edge labeled b and the edge labeled d. This is also true of the last graph in the first sum and the second graph in the second sum. The second graph in the first sum and the fifth graph in the second sum differ by the orientation of the edges labeled b and d, along with the orientation of two of the edges labeled 1. Presumably these two sums of graphs give rise to closely related sums of algebraic cycles.

While the motive associated to the first sum has been studied (see [Gangl et al. 2009], for example) the other appears to be new. We suspect that they define dependent classes in $H^0(B(\mathscr{G}_{1L}))$. It would be interesting to use the Hodge realization techniques developed in Section 5 and/or other graphical tools to analyze the motives they represent.

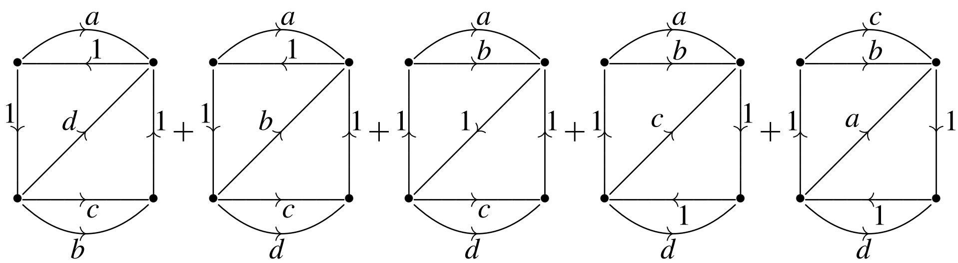
There is a related family of graphs, defined by changing the labelings and orientations of



Example 4.22. The following sum of six diagrams is minimally decomposable:

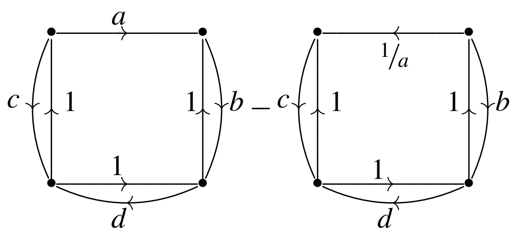


as is this sum of five related diagrams:

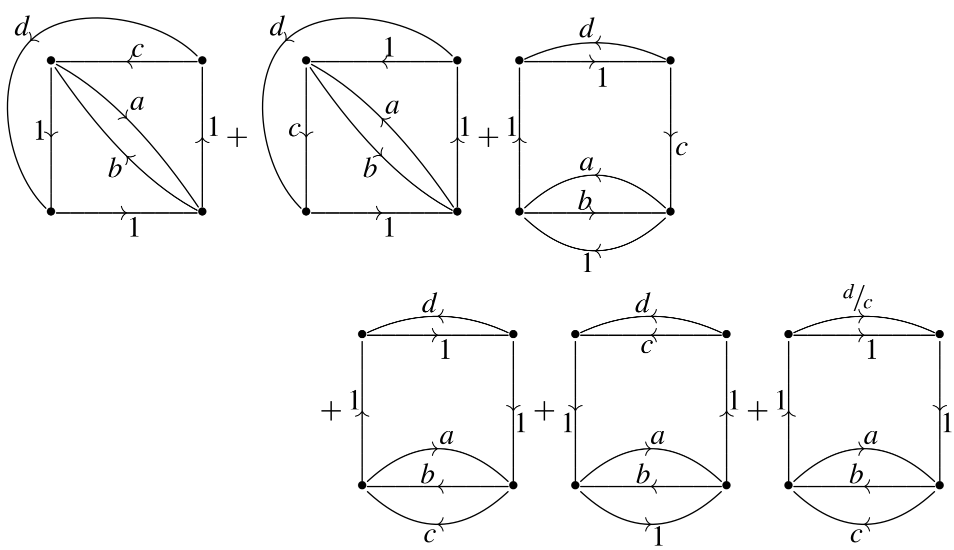


Next we present the weight-four example of the necklace graphs that are the subject of (4.26).

Example 4.23. The following sum of graphs is minimally decomposable:



Example 4.24. We end this section with a complicated minimally decomposable sum that, unlike the previous examples, involves several different types of unoriented graphs:



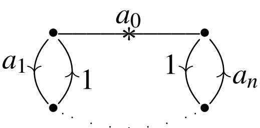
It is highly likely that the classes defined by all of the above examples are related. It would be very interesting to work out the precise dependencies.

These examples illustrate that, even in the vastly simplified case of $\mathscr{A}_{1L}^{\times}$, there is a richness and complexity amongst the minimally decomposable classes of $B(\mathscr{G}_{1L})$. By further studying these minimally decomposable sums of graphs, we hope to gain a better understanding of the structure of (our subcategory of) mixed Tate motives.

4B2. *The n-beaded necklace graph.* In this section, we introduce an infinite family of terms in $H^0(B(\mathscr{G}_{1L}))$, which we refer to as necklace diagrams. In Section 5, we show that these correspond to trivial classes.

Definition 4.25. The necklace graph with n beads is the graph of the form

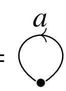
$$G^*(a_0, \ldots, a_n) = \tag{4.26}$$



with $* \in \{L, R\}$ (left, right) to indicate the orientation of the marked edge. The ordering is given as follows: each edge labeled a_i is in the $(2i+1)$-st position; for $i > 0$ the "parallel edge" labeled 1 (which shares vertices with that labeled a_i) is in the $2i$-th position. The signs associated to the edges are all positive.

When $n = 0$, we write

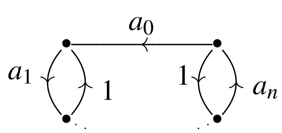
$$G(a) = G^R(a) = G^L(a) = \tag{4.27}$$

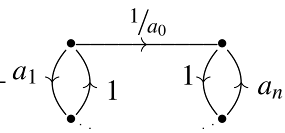


We consider the following linear combination of n-beaded necklace graphs:

$$\varepsilon^n(a_0, \ldots, a_n) = G^L(a_0, a_1, \ldots, a_n) - G^R\Big(\frac{1}{a_0}, a_1, \ldots, a_n\Big)$$

$$= \quad - \tag{4.28}$$





To avoid extreme notational complexity in keeping track of labels of graphs, we introduce some notation.

Definition 4.29. Define a set $\mathbb{n} = \{1, \ldots, n\}$. We define $\mathbb{a}_{\mathbb{n}}$ to be the n-tuple $(a_1, \ldots, a_n)$. For any $S \subset \mathbb{n}$, $\mathbb{a}_{\mathbb{n}\backslash S} = (a_1, \ldots, \hat{a}_S, \ldots, a_n)$ is the $n-|S|$-tuple with the elements labeled by $s \in S$ removed.

Lemma 4.30. *The sum of graphs $\varepsilon^n(a_0, \mathbb{a}_{\mathbb{n}})$ is completely decomposable.*

Proof. By direct calculation,

$$\partial \varepsilon^n(a_0, \mathbb{a}_{\mathbb{n}}) = \sum_{i=1}^{n} \big(\varepsilon^{n-1}(a_0, \mathbb{a}_{\mathbb{n}\backslash i}) - \varepsilon^{n-1}(a_0 a_i, \mathbb{a}_{\mathbb{n}\backslash i})\big) \cdot G(a_i).$$

The proof follows by induction. $\square$

We explicitly write the entire minimally decomposable element of $B(\mathcal{G}_{1L})$ defined by $\varepsilon^n(a_0, \mathbb{a}_{\mathbb{n}})$.

Recall that

$$[a_1 \mid \cdots \mid a_n] \amalg\!\amalg [b_1 \mid \cdots \mid b_m]$$

is the shuffle product of the ordered sets $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_m)$.

In particular, for $a, b \in \mathcal{G}_{1L_1}$,

$$a \amalg\!\amalg b = [a \mid b] + [b \mid a].$$

The shuffle product $a \amalg\!\amalg b$ is in $\ker \mu$. That is

$$\mu(a \amalg\!\amalg b) = 0. \tag{4.31}$$

Lemma 4.32. *The element*

$$\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}}) = \sum_{S \subset \mathbb{n}} (-1)^{|S|} \sum_{J \subseteq S} (-1)^{|J|} \Bigl[\varepsilon^{n-|S|} \Bigl(a_0 \prod_{j \in J} a_j, \mathbb{a}_{\mathbb{n} \setminus S} \Bigr) \Bigm| \coprod_{s \in S} G(a_s) \Bigr]$$

is in $H^0(B(\mathcal{G}_{1L}))$.

Proof. Recall that, since $\varepsilon^n \in {\mathcal{G}_{1L}}_1^{n+1}$, it defines an element of degree 0 in $B(\mathcal{G}_{1L})$.

Consider the component of $\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})$ in $B(\mathcal{G}_{1L})^0_{k+1}$. We compute $\partial + \mu$ on this term. By Lemma 4.30,

$$\begin{aligned} \partial \sum_{\substack{|S|=k \\ J \subseteq S}} (-1)^{|J|} &\Bigl[\varepsilon^{n-k} \Bigl(a_0 \prod_{j \in J} a_j, \mathbb{a}_{\mathbb{n} \setminus S} \Bigr) \Bigm| \coprod_{s \in S} G(a_s) \Bigr] \\ = \sum_{\substack{|S|=k \\ i \in n \setminus S \\ J \subseteq S}} (-1)^{|J|} &\Biggl[\Biggl(\varepsilon^{n-k-1} \Bigl(a_0 \prod_{j \in J} a_j, \mathbb{a}_{\mathbb{n} \setminus (S \cup i)} \Bigr) \\ &\qquad - \varepsilon^{n-k-1} \Bigl(\Bigl(a_0 \prod_{j \in J} a_j \Bigr) a_i, \mathbb{a}_{\mathbb{n} \setminus (S \cup i)} \Bigr) \Biggr) \cdot G(a_i) \Bigm| \coprod_{s \in S} G(a_s) \Biggr]. \end{aligned}$$

Collecting terms, the right-hand side becomes

$$\sum_{\substack{|S|=k \\ i \in n \setminus S \\ J \subseteq S \cup i}} (-1)^{|J|} \Bigl[\varepsilon^{n-k-1} \Bigl(a_0 \prod_{j \in J} a_j, \mathbb{a}_{\mathbb{n} \setminus (S \cup i)} \Bigr) \cdot G(a_i) \Bigm| \coprod_{s \in S} G(a_s) \Bigr].$$

However, by (4.31), this is

$$\mu \sum_{\substack{|S|=k+1 \\ J \subseteq S}} (-1)^{|J|} \Bigl[\varepsilon^{n-k-1} \Bigl(a_0 \prod_{j \in J} a_j, \mathbb{a}_{\mathbb{n} \setminus (S \cup i)} \Bigr) \Bigm| G(a_i) \Bigm| \coprod_{s \in S} G(a_s) \Bigr].$$

Therefore,

$$(\partial + \mu)(\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})) = 0. \qquad \square$$

Therefore, $\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})$ defines a class in $H^0(B(\mathcal{G}_{1L}))$, as stated in Remark 2.10.

Definition 4.33. Let $[\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})] \in H^0(B(\mathcal{G}_{1L}))$ be the class defined by $\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})$.

This choice of notation emphasizes that this is the class in $H^0(B(\mathcal{G}_{1L}))$ associated to an element in ${\mathcal{G}_{1L}}_1^\bullet$ with completely decomposable boundary.

5. Hodge realization

In this section we describe the Hodge realization for a number field k for our category and compute some examples. We follow the approach to constructing a Hodge realization described in [Bloch and Kriz 1994, Sections 7 and 8; Kimura 2013]. Namely, we first note that the Hodge realization as constructed in Section 7 of [Bloch and Kriz 1994] can be defined independently of choice. However, as noted at the beginning of [ibid., Section 8], this construction is not very amenable to computation, and a second description of the Hodge realization functor is given. Here we restrict to this second description of the Hodge realization. Namely, we explicitly construct a comodule J of $\mathcal{H}_T = H^0(B(\mathcal{G}_{1L}))$ and construct a natural mixed Tate Hodge structure on J. This, as in [Gangl et al. 2009], provides the Hodge realization for our graphical structure as J associates a natural mixed Tate Hodge structure on any graded comodule $\mathcal{M}$ of $\mathcal{H}_T$.

In the context of the graphs, the $\mathbb{Q}$ mixed Tate Hodge structure is given by the rational lattice

$$H_{\mathbb{Q}} = H^0(B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})),$$

where $\mathcal{T}_{1L}^{\text{twist}}$ is a right $\mathcal{G}_{1L}$ module, and $B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})$ is the corresponding cyclic bar construction. Both filtrations are induced from the weights of graphs (or the codimension of the corresponding cycles), as defined in Section 3A. These are introduced in detail in Section 5A.

5A. ***Topologically augmented admissible graphs.*** As in [Bloch and Kriz 1994], in order to create the construction outlined above, one must define a set of topologically supported cycles in $\square^n$.

Definition 5.1. Let $\mathcal{X}^\bullet_{\text{top}}(\Delta_\bullet, \square^{2\bullet-\star})$ be the free abelian group (vector space) generated by admissible algebraic cycles supported on the image of a smooth map $\sigma : \Delta_\bullet \to \mathbb{P}_k(\mathbb{C})^{2\bullet-\star}$ of codimension $\bullet$ and algebraic degree $\star$. Then define a vector space $\mathcal{X}_{\text{top}} = \bigoplus_{\bullet,\star} \operatorname{Alt} \mathcal{X}^\bullet_{\text{top}}(\Delta_\bullet, \square^{2\bullet-\star})$.

These topological cycles define a means of passing from the algebraic cycles to integrals by considering the supports. In particular, given a completely decomposable element $\varepsilon \in B(\mathcal{A}_{1L}^\times)$, with $[\varepsilon] \in H^0(B(\mathcal{A}_{1L}^\times))$, one considers the element $1 \otimes \varepsilon$ in the circular bar construction, $B(\mathcal{X}_{\text{top}}, \mathcal{A}_{1L}^\times)$. This does not define a cohomology class. Namely, it is not completely decomposable. The task then is to find an element $\xi \in B(\mathscr{Z}_{\text{top}}, \mathscr{A}_{1L}^{\times})$ such that $1 \otimes \varepsilon + \xi$ is completely decomposable, that is,

$$[1 \otimes \varepsilon + \xi] \in H^0(B(\mathscr{Z}_{\text{top}}, \mathscr{A}_{1L}^{\times})).$$

It is worth noting that, while the cohomology class thus defined is unique, the element ξ need not be. In particular, in the example worked out in Section 5C2, the given ξ is by no means the only possible construction.

In the context of graphs, we parallel this construction by defining topologically augmented admissible graphs, which, under a natural extension of the homomorphism Z defined in Section 3B, correspond to elements of $\mathscr{Z}^{\bullet}_{\text{top}}(\Delta_{\bullet}, \square^{2\bullet-\star})$. These topologically augmented graphs generate a $\mathscr{G}_{1L}$ module, which we develop in this section. First we establish some notation.

Let $\Delta_n \subset \mathbb{R}^n$ be the standard real n-simplex. Let $C^{\infty}(n, m)$ be the set of smooth maps from Δ_n to $(\mathbb{P}_k(\mathbb{C})^1)^N$ of dimension m. Here N is an arbitrary integer $N \geq n$.

Definition 5.2. We say that m is the simplicial dimension of maps in $C^{\infty}(n, m)$.

Note that σ need not be injective, that is, m may be less than n. In particular, $\mathbb{C}^{\infty}(n, 0)$ consists of all constant maps from Δ_n. We view $C^{\infty}(n, m)$ as a chain complex, $C(n)_m$.

We parametrize Δ_n by an ordered set as usual, $0 \leq t_1 \leq \cdots \leq t_n \leq 1$, sometimes writing $0 = t_0$ and $1 = t_{n+1}$. Then any $\sigma \in C(n)_m$ is a continuous function of $\{t_1, \ldots, t_n\}$.

Definition 5.3. Given the standard face maps s_i and degeneracy maps d_i on Δ_n, for any subset $I \in \{t_0, \ldots, t_n\}$ of size $|I| = p$ we write d_I for the standard codimension-p degeneracy map.

Let $\mathbb{n} = \{1, \ldots, n\}$ as before. Any continuous map $\sigma \in C(n)_m$ can be written in terms of codimension-$n-m$ face maps. That is, there is a set $I \in \mathbb{n}$ with $|I| = m$ and $\sigma' \in C(n)_n$ such that

$$\sigma = d_{I*}\sigma'. \tag{5.4}$$

The degeneracy maps define a differential on the chain complex $C(n)_m$. In particular, we write

$$\delta_i : C(m)_m \to C(m)_{m-1}, \quad \sigma \mapsto d_{i*}s_{i*}\sigma,$$

with $\delta = \sum_{i=0}^{m}(-1)^i \delta_i$. More generally, for $\sigma \in C(n)_m$, where $\sigma = d_{I*}\sigma'$, write

$$\delta_i : C(n)_m \to C(n)_{m-1}, \quad \sigma \mapsto \delta_i d_{I*}\sigma'. \tag{5.5}$$

Therefore, we have shown:

Lemma 5.6. *For a fixed n, $(C(n)_*, \delta)$ is a chain complex.*

Remark 5.7. As in the prequel, the symbol $\bullet$ will always correspond to the codimension of a cycle (loop number of a graph). The symbol $\star$ will always correspond to the algebraic degree, and the symbol $*$ always the simplicial dimension of the graph.

Given this notation, we define the right module of topologically augmented admissible graphs. Generators of this algebra are given by the pair $\sigma \in C(n)_*$ and an admissible graph $G \in \mathcal{G}_{1}{L_i^n}$. In particular, the topologically augmented graph (G, σ) has edges labeled, not by elements of $k^\times$ as usual, but by the image of σ. For $t \in \Delta_n$, write $\sigma(t)$ as the $(2n-i)$-tuple $\sigma(t) = (\sigma_1(t), \dots, \sigma_{2n-i}(t))$. The coordinate $\sigma_i(t)$ labels the edge $e \in E(G)$ that is in the i-th position, that is, such that $\omega(e) = i$. There is a natural extension of the vector space homomorphism Z defined in Section 3B to the topologically augmented admissible graphs such that each graph maps to a topologically supported cycle in $\mathcal{X}_{\text{top}}$.

For each $G \in \mathcal{G}_{1}{L_\star^\bullet}$, $\sigma \in C(\bullet)_*$ and $t \in \Delta_\bullet$ such that $\sigma_{\omega(e)}(t) \neq 0, \infty$ for any $e \in E(G)$, the pair $(G, \sigma(t))$ defines a graph in $\mathbb{Q}[\mathcal{G}]/(\sim_{\text{ord}}, \sim_{\text{ori}}, \sim_{\text{v}})$. If $\sigma_{\omega(e)}(t) = 0, \infty$, we say that $(G, \sigma(t))$ is the trivial graph. As we show below, in Lemma 5.15, graphs with such labels correspond to algebraic cycles with 1 in the appropriate coordinate. In particular, for a general σ, the labels $\sigma(t)$ need not correspond to an admissible labeling of the underlying graph G. We wish to consider pairs $(G, \sigma(t))$ which evaluate to admissible graphs almost everywhere on Δ_n. Such $\sigma \in C(\bullet)_*$ are called admissible simplices for G.

Definition 5.8. A map $\sigma \in C(\bullet)_*$ is admissible for a graph G if the following hold:

(1) Let $\delta_J(\sigma)$ indicate the degeneracy map onto the face opposite that defined by J in $\Delta_\bullet$. For all J, each loop of the augmented graph $(G, \delta_J\sigma)$ does not have loop coefficient 1 almost everywhere on $\Delta_\bullet$.

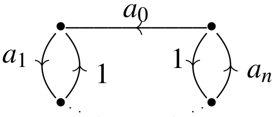
(2) For all $e \in E(G)$, if there exists a $t \in \Delta_\bullet$ such that $\sigma_{\omega(e)}(t) = 0$, there exists an $e' \in E(G)$ such that $\sigma_{\omega(e')}(t) = \infty$. Therefore, the cycle $Z(G, \sigma(t))$ is trivial.

(3) Writing $\delta\sigma = \sum_{i=0}^{\bullet}(-1)^i \delta_i \sigma$, there is some i for which no coordinate of $\delta_i\sigma$ is ∞.

We are now ready to define the vector space of admissible topologically augmented graphs.

Definition 5.9. Let $\mathcal{T}_{1}{L_{2\bullet-*}^\bullet}$ be the vector space of topologically augmented graphs (G, σ), with $h_1(G) = \bullet$ and $\sigma \in C(\bullet)_*$ an admissible labeling.

Example 5.10. Consider the necklace graph $G^L(a_0, \dots, a_n) \in \mathcal{G}_{1}{L_1^{n+1}}$

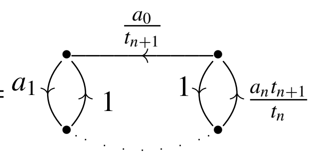
$G^L(a_0, \dots, a_n) =$ 

There is a constant map $\sigma \in C(n+1)_0$ of the form $\sigma(\Delta_{n+1}) = (a_0, 1, a_1, \ldots, 1, a_n)$. As this has 0-dimensional topological support, this is the constant map. The pair $(G^L, \sigma) \in \mathcal{T}_{1L}{}^{n+1}_{2n+2}$ is a trivially topologically augmented graph. That is, $(G^L, \sigma) = G^L \in \mathcal{G}_{1L}{}^{n+1}_{1}$.

Consider a different map, $\sigma' \in C(n+1)_2$, of the form

$$\sigma'(\Delta_{n+1}) = \left(\frac{a_0}{t_{n+1}}, 1, a_1, \ldots, 1, \frac{t_{n+1}a_n}{t_n}\right).$$

Then the pair

$(G^L, \sigma') =$ 

is an element of $\mathcal{T}_{1L}{}^{n+1}_{2n}$.

Note that $\mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}$ is not an algebra. In particular, there is no natural product structure on $C(n)_*$. For general $(G, \sigma) \in \mathcal{T}_{1L}{}^{n}_{2n-*}$ and $(G', \sigma') \in \mathcal{T}_{1L}{}^{n'}_{2n'-*'}$, the product is given by the graph $(GG', \sigma \times \sigma')$. As in (5.4), write σ and σ' as degeneracies $d_{I*}\bar{\sigma}$ and $d_{I*}\bar{\sigma}'$ for some $\bar{\sigma} \in C(n)_n$ and $\bar{\sigma}' \in C(n')_{n'}$. However, $d_{I*}\bar{\sigma} \times d_{I*}\bar{\sigma}'$ does not correspond to a smooth map restricted to some face of $\Delta_{n\times n'}$. Therefore, we consider $\mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}$ as a $\mathcal{G}_{1L}{}^{\bullet}_{\star}$ module.

There is an inclusion of the algebra of admissible nonaugmented graphs into $\mathcal{T}_{1L}$:

Example 5.11. There is an inclusion $\mathcal{G}_{1L}{}^{\bullet}_{\star} \hookrightarrow \mathcal{T}_{1L}{}^{\bullet}_{2\bullet}$. Any graph $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$ can be written as $(G, d_{I*}\sigma_0)$ via the constant map

$$\sigma_0(\Delta_\bullet) = (a_1, \ldots, a_{|E(G)|}),$$

where $a_{\omega(e)}$ is the label of edge $e \in E(G)$.

Proposition 5.12. *The vector space $\mathcal{T}_{1L}$ is a $\mathcal{G}_{1L}$ module.*

Proof. As done in Example 5.11, write $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$, as $(G, \sigma) \in \mathcal{T}_{1L}{}^{\bullet}_{2\bullet}$ with $\sigma \in C(\bullet)_0$. Further consider $(G', \sigma') \in \mathcal{T}_{1L}{}^{\bullet'}_{2\bullet'-m}$ with $\sigma' \in C(\bullet')_m$.

In general, we cannot write $(G, \sigma)(G', \sigma') = (GG', \sigma \times \sigma')$ as an element in $\mathcal{T}_{1L}$. However, since $\sigma \in C(\bullet)_0$, we can rewrite this as $(GG', d_{(I'')*}\sigma_m)$, where I'' is the appropriate face in $\Delta_{\bullet+\bullet'}$.

Therefore, the product of a non-topologically augmented graph $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$ with an augmented one $(G', \sigma') \in \mathcal{T}_{1L}{}^{\bullet'}_{2\bullet'-m}$ is

$$(G, \sigma_0) \cdot (G', \sigma') = (G \cdot G', (\sigma_0, \sigma')) \in \mathcal{T}_{1L}{}^{\bullet+\bullet'}_{2(\bullet+\bullet')-m}.$$

This gives the module structure. □

The vector space $\mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}$ is a bigraded vector space. We may write

$$\mathcal{T}_{1L} = \bigoplus_{0\le\bullet,*} \mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}.$$

Finally, we consider $\mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}$ as a complex. The module has two natural differential structures on it, induced by the topological differential δ on Δ_n and the algebraic differential ∂ on $\mathcal{G}_{1L}$. Before defining these explicitly and the associated bicomplex structure on augmented graphs, it is necessary to introduce a shifted vector space, $\mathcal{T}^{\text{twist}}_{1L}$.

Definition 5.13. For $(G,\sigma) \in \mathcal{T}_{1L}$ we define a twisted module $\mathcal{T}^{\text{twist}}_{1L}$, where the grading of each element is shifted from that of $\mathcal{T}_{1L}$ by the dimension of the range of σ, i.e, the number of edges of the graph G. That is, for $G \in \mathcal{G}_{1L}{}^{\bullet}_{\star}$ and $(G,\sigma) \in \mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*}$, the same element is in $(\mathcal{T}^{\text{twist}}_{1L})^{\bullet}_{\star_t} := \mathcal{T}_{1L}{}^{\bullet}_{2\bullet-*-n}$ for $n = 2\bullet-\star$. Henceforth define a topologically twisted degree $\star_t := \star - *$ to be the difference between the algebraic degree and topological dimension. Write

$$\mathcal{T}^{\text{twist}}_{1L} = \bigoplus_{\bullet,\star_t} (\mathcal{T}^{\text{twist}}_{1L})^{\bullet}_{\star_t}.$$

For $\sigma_m \in C(n)_m$, write $\sigma_m = d_{I*}\sigma'$ for some $\sigma' \in C(m)_m$. The topological differential, δ is induced by the differential on the chain $C(n)_m$ defined in (5.5):

$$\delta : (\mathcal{T}^{\text{twist}}_{1L})^{\bullet}_{\star_t} \to (\mathcal{T}^{\text{twist}}_{1L})^{\bullet}_{\star_t+1}, \quad (G,\sigma_m) \mapsto \sum_{i=0}^{m} (-1)^i (G, \delta_i \sigma_m). \tag{5.14}$$

This is a degree-one differential operator on $(\mathcal{T}^{\text{twist}}_{1L})$.

The algebraic differential ∂ is induced from the differential ∂ on $\mathcal{G}_{1L}$. On $\mathcal{T}^{\text{twist}}_{1L}$, vertex rescaling is a direct generalization of rescaling on $\mathcal{G}_{1L}$, allowing one to rescale by functions $\sigma \in C^\infty(|E(G)|, |E(G)|)$. For s_e and t_e the source and terminal vertices of $e \in G$, write

$$(\partial_e\sigma)_{\omega(e')} = \begin{cases} 1 & \text{if } e = e', \\ \sigma_{\omega(e')} & \text{if } s_e \text{ is not a vertex of } e', \\ \sigma_{\omega(e')}\sigma_{\omega(e)} & \text{if } s_e = t_{e'}, \\ \sigma_{\omega(e')}/\sigma_{\omega(e)} & \text{if } s_e = s_{e'}, \end{cases}$$

as one expects from vertex rescaling and Definition 3.38. Then

$$\partial : (\mathcal{T}^{\text{twist}}_{1L})^{\bullet}_{\star_t} \to (\mathcal{T}^{\text{twist}}_{1L})^{\bullet}_{\star_t+1}, \quad (G,\sigma_m) \mapsto \sum_{e\in E(G)} (-1)^{\omega(e)-1} (\partial_e G, \partial_e \sigma),$$

which is a degree-one differential operator on $\mathcal{T}^{\text{twist}}_{1L}$.

The topologically augmented graphs correspond to the vector space of topologically supported admissible algebraic cycles $\mathcal{Z}^{\bullet}_{\text{top}}(\Delta_*, \square^{2\bullet-\star})$.

Lemma 5.15. *The map Z defined in Section 3B extends to a module homomorphism*

$$Z : (\mathcal{T}^{\text{twist}}_{1L}) \to \mathcal{Z}_{\text{top}},$$

as defined in Definition 5.1.

Proof. Each edge of the augmented graph (G, σ_m) defines a coordinate $\phi_{\omega(e)} = 1 - x_e/(\sigma_{m,\omega(e)} y_e)$, where x_e and y_e are the variables associated to the source and the target vertices of the edge e as usual. Then $\phi = (\phi_1, \dots, \phi_n)$ parametrizes an algebraic cycle supported on an m-simplex in $\square^n$.

It remains to check that $Z(G, \sigma_m)$ is an admissible topologically supported cycle. By Definition 5.8, the loop number of any loop in (G, σ_m) is not 1 almost everywhere in Δ_m or on any of its faces. If $\sigma_{\omega(e)}(t) = 0$ for some $t \in \sigma_m$, then the cycle $Z(G, \sigma_m(t))$ is trivial, as the corresponding coordinate is 1. Therefore, by condition (2) of Definition 5.8, if there is some $t \in \sigma_m$ and an edge $e \in E(G)$, $Z(G, \sigma_m(t))$ is trivial. Therefore, by Theorem 3.59, $Z(G, \sigma_m)$ is admissible almost everywhere on Δ_m. $\square$

The third condition in Definition 5.8 gives rise to the following statement:

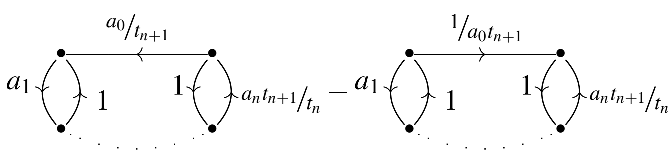
Lemma 5.16. *The image of $\mathcal{T}^{\text{twist}}_{1L}$ under Z is an acyclic chain complex under δ.*

Proof. Equation (5.14) shows that $\mathcal{T}^{\text{twist}}_{1L}$ is a chain complex under δ. In particular,

$$\delta : (\mathcal{T}^{\text{twist}}_{1L})^\bullet_{\star_t} \to (\mathcal{T}^{\text{twist}}_{1L})^\bullet_{\star_t+1}.$$

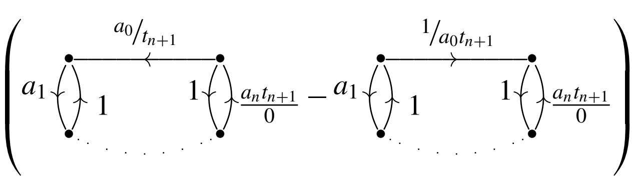
The third condition of Definition 5.8 imposes acyclicity. By Lemma 5.15, if $\delta_i\sigma$ has a coordinate set at ∞, then $Z(G, \delta_i\sigma)$ is a trivial cycle. Requiring that there is some face of $\Delta_\bullet$ such that $(\delta_i\sigma)_{\omega(e)} \neq \infty$ for all $e \in E(G)$ implies that $\delta Z(G, \sigma) \neq 0$. In other words, the image of $Z(\mathcal{T}^{\text{twist}}_{1L})$ is an acyclic chain complex under δ. $\square$

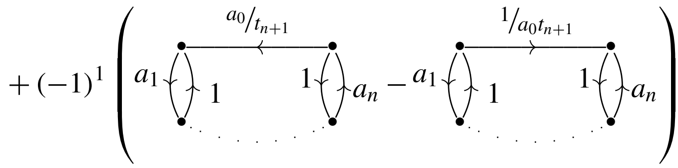
Example 5.17. In this example, we augment the sum of graphs $\varepsilon^n(a_0, \dots, a_n)$ defined in (4.28) by a 2-dimensional support, as in Example 5.10. First, recall notation from Definition 4.29. Writing $\mathbb{n} = \{1, \dots, n\}$, define an n-tuple $\mathbb{a}_\mathbb{n} = (a_1, \dots, a_n)$. Similarly, for any $S \subset \mathbb{n}$, write $\mathbb{a}_{\mathbb{n}\setminus S} = (a_1, \dots, \hat{a}_S, \dots, a_n)$ for the same n-tuple with the elements $\{a_s \mid s \in S\}$ removed. Then write the augmented sum of graphs

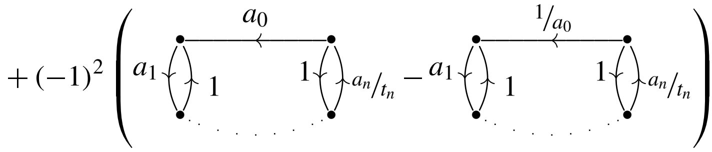
$(\varepsilon^n, \sigma(a_0, \mathbb{a}_\mathbb{n})_2) =$ 

Here, $\sigma(a_0, \mathbb{a}_{\mathbb{n}\setminus\mathbb{S}})_2 \in C(n - |S| + 1)_2$ is a labeling on the decomposable sum of the $n-|S|$-beaded necklace.

Then the topological differential is

$\delta(\varepsilon^n, \sigma(a_0, \mathbb{a}_\mathbb{n})_2) = (-1)^0$ 



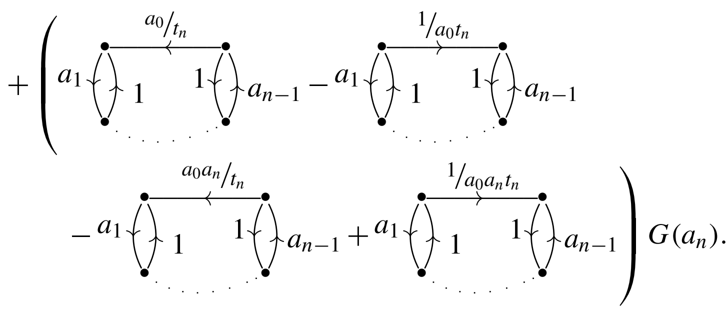


The first two terms in this sum correspond to trivial graphs.

Recall from (4.27) that $G(a)$ is the graph with a single edge and a single loop labeled by a.

The algebraic differential on this graph is

$$\partial(\varepsilon^n, \sigma(a_0, \mathbb{n})_2) = \sum_{i=1}^{n-1} \big((\varepsilon^{n-1}, \sigma(a_0, \mathbb{n}\setminus i)_2) - (\varepsilon^{n-1}, \sigma(a_0 a_i, \mathbb{n}\setminus i)_2)\big)\, G(a_i)$$



Due to the form of the augmentation $\sigma(n+1)_2$ chosen in this example, we may write the second and third lines above as

$$\big((\varepsilon^{n-1}, \delta_{n-1}\sigma(a_0, \mathbb{n-1})_2) - (\varepsilon^{n-1}, \delta_{n-1}(a_0 a_n, \mathbb{n-1})_2)\big) \cdot G(a_n).$$

5B. ***A comodule and Hodge structure.*** We are now ready to define the Hodge comodule J.

First we build the circular bar construction $B(\mathcal{T}^{\mathrm{twist}}_{1L}, \mathcal{G}_{1L}, \mathbb{Q})$. In the sequel, we take the last entry as given, and simply write $B(\mathcal{T}^{\mathrm{twist}}_{1L}, \mathcal{G}_{1L})$. As in [Bloch and Kriz 1994; Kriz and May 1995] and references therein, we define the $B(\mathcal{T}^{\mathrm{twist}}_{1L}, \mathcal{G}_{1L})$ on the tensor algebra $\mathcal{T}^{\mathrm{twist}}_{1L} \otimes T(\mathcal{G}_{1L})/D(\mathcal{G}_{1L})$ as in Definition 2.1.

Consider $(G_0, \sigma)\otimes G_1 \otimes \cdots \otimes G_k \in B(\mathcal{T}^{\mathrm{twist}}_{1L}, \mathcal{G}_{1L})^k_w$ with $G_i \in {\mathcal{G}_{1L}}^{r_i}_{w_i}$ for $0 \le i \le k$, and $\sigma \in C(r_0)_m$. The total degree of this bar element is $w = \sum_{i=0}^k w_i - (k+1) - m$.

We define the bicomplex structure on it by extending the differentials from (4.3) and (4.4) for the bar construction $(B(\mathcal{G}_{1L}), \mu, \partial)$.

As before, for $j > 0$ write ∂_j to indicate the operator on $B(\mathcal{T}^{\mathrm{twist}}_{1L}, \mathcal{G}_{1L})$ that acts as ∂ on the j-th tensor component of $T(\mathcal{G}_{1L})$, as $(-1)^{\deg_B G_i}$ id on $\mathcal{T}^{\mathrm{twist}}_{1L}$ and the first $j-1$ tensor components of $T(\mathcal{G}_{1L})$, and as id on the rest. As before, $\deg_B G_i$ refers to the *graphical* bar degree of the component, excluding any topological considerations. Hence, for $(G_0,\sigma)\in(\mathcal{T}_{1L}^{\text{twist}})^{\bullet}_{\star_t}$ with $\sigma\in C(\bullet)_*$, we have $\deg_B(G_0,\sigma)=\star_t+*-1=\star-1$. Define ∂_0 as $\partial+\delta$ on $\mathcal{T}_{1L}^{\text{twist}}$ and the identity on the other tensor components of the bar element. In this shifted notation, ∂_0 is a degree-one operator on $\mathcal{T}_{1L}^{\text{twist}}$.

For the product, with (G_0,σ) as above, define μ_j as the degree-one operator on $B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})$ that acts as $(-1)^{\deg_B G_0-m}$ id on the zeroth tensor component and as $(-1)^{\deg_B G_i}$id on the next $j-1$ tensor components of $T(\mathcal{G}_{1L})$, as $(-1)^{\deg_B G_j}\mu$ on the j-th and $(j{+}1)$-st components, and as the identity on the remaining elements.

Then, in parallel to (4.4), for $\sigma\in C(\bullet)_m$ write

$$\begin{aligned}\mu[(G_0,\sigma)\mid G_1\mid\cdots\mid G_n]&\\ &:=\sum_{j=0}\mu_j[(G_0,\sigma)\mid G_1\mid\cdots\mid G_n]\\ &=\sum_{j=0}^{n-1}(-1)^{(\sum_{i=0}^{j}\deg_B G_i)-m}[(G_0,\sigma)\mid G_1\mid\cdots\mid G_j\cdot G_{j+1}\mid\cdots\mid G_n].\end{aligned}\tag{5.18}$$

Similarly, in parallel to (4.3), write

$$\begin{aligned}\partial[(G_0,\sigma)\mid G_1\mid\cdots\mid G_n]&\\ &:=\sum_{j=0}\partial_j[(G_0,\sigma)\mid G_1\mid\cdots\mid G_n]\\ &=\sum_{j=0}^{n-1}(-1)^{\sum_{i=0}^{j-1}\deg_B G_i}[(G_0,\sigma)\mid G_1\mid\cdots\mid \partial G_j\mid\cdots\mid G_n].\end{aligned}\tag{5.19}$$

In parallel to (2.2), we explicitly draw a few terms of $(B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L}),\mu,\partial)$ (recall that $\bigoplus_n(\mathcal{T}_{1L}^{\text{twist}})^n_i=B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^0_i$):

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots\\
 & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow\\
\cdots & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^3_0 & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^2_1 & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^1_2 & \xrightarrow{\mu} & \bigoplus_n(\mathcal{T}_{1L}^{\text{twist}})^n_3\\
 & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow\\
\cdots & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^3_{-1} & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^2_0 & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^1_1 & \xrightarrow{\mu} & \bigoplus_n(\mathcal{T}_{1L}^{\text{twist}})^n_2\\
 & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow\\
\cdots & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^3_{-2} & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^2_{-1} & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^1_0 & \xrightarrow{\mu} & \bigoplus_n(\mathcal{T}_{1L}^{\text{twist}})^n_1\\
 & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow\\
\cdots & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^3_{-3} & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^2_{-2} & \xrightarrow{\mu} & B(\mathcal{T}_{1L}^{\text{twist}},\mathcal{G}_{1L})^1_{-1} & \xrightarrow{\mu} & \bigoplus_n(\mathcal{T}_{1L}^{\text{twist}})^n_0\\
 & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow & & \partial\uparrow\\
 & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Definition 5.20. We may now define the comodule $J = H^0(B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L}))$ and $J_{\mathbb{C}} = J \otimes \mathbb{C}$.

Following [Kimura 2013, Proposition 3.3], the weight filtration $W_{2r} = W_{2r-1}$ is induced by the algebraic weight (codimension) filtration on $B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})$. Write

$$B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})(r) = \mathcal{T}_{1L}^{\text{twist}} \otimes B(\mathcal{G}_{1L})(r) = \mathcal{T}_{1L}^{\text{twist}} \otimes \bigoplus_{k\geq 1} \bigoplus_{\sum_{i=1}^{k} r_i = r} {\mathcal{G}_{1L}}_{\star}^{r_1} \otimes \cdots \otimes {\mathcal{G}_{1L}}_{\star}^{r_k}.$$

Here, $B(\mathcal{G}_{1L})(r)$ is the tensor product of unaugmented graphs with total codimension r. That is, we may write

$$W_r(B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})) = \bigoplus_{q\leq r} B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})(q).$$

This induces the weight filtration on J in the usual way, $\text{gr}_{2r}^W J = \text{gr}_{2r-1}^W J = H^0(B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})(r))$. Similarly, $\text{gr}_{2r}^W J_{\mathbb{C}} = \text{gr}_{2r-1}^W J_{\mathbb{C}} = H^0(B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})(r))$.

Definition 5.21. Let

$$\Omega_n = \frac{1}{(2\pi i)^n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$$

be the logarithmic n-form on $\square^n$.

Definition 5.22. For $(G, \sigma) \in (\mathcal{T}_{1L}^{\text{twist}})_{\star_t}^{\bullet}$ and $\sigma \in C(\bullet)_*$, we define an evaluation map

$$\mathcal{I} : \mathcal{T}_{1L} \to \mathbb{C}, \quad (G, \sigma) \mapsto \int_{(G,\sigma)} \Omega_{2\bullet - \star}.$$

This integral is only well-defined if $* = 2\bullet - \star$. That is, $\sigma \in C(\bullet)_{2\bullet-\star}$. However, since $* \leq \bullet \leq 2\bullet - \star$, this implies that $* = \bullet = \star$.

Explicitly,

$$\int_{(G,\sigma)} \Omega_n = \int_{\Delta_n} \sigma_*(\Omega_m) = \frac{1}{(2\pi i)^m} \int_{\Delta_m} \frac{d(1-1/\sigma_1)}{1-1/\sigma_1} \wedge \cdots \wedge \frac{d(1-1/\sigma_m)}{1-1/\sigma_m},$$

where the ordering of the coordinates of σ are given by the ordering of the edges of G.

We call $\mathcal{I}(G, \sigma_m)$ the period associated to (G, σ_m). The evaluation map induces a quasiisomorphism between $B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L}) \otimes \mathbb{C}$ and $B(\mathcal{G}_{1L}) \otimes \mathbb{C}$:

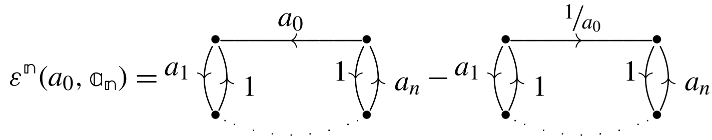
$$\begin{aligned} \mathcal{I} \otimes \text{id} : B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L}) \otimes \mathbb{C} &\to \mathbb{C} \otimes B(\mathcal{G}_{1L}), \\ [(G_0, \sigma) \mid G_1 \mid \cdots \mid G_n] &\mapsto \mathcal{I}(G_0, \sigma)[G_1 \mid \cdots \mid G_n]. \end{aligned} \tag{5.23}$$

Thus (again following [Kimura 2013]) we can define the Hodge filtration by

$$F^k J_{\mathbb{C}} = \bigoplus_{r\geq k} H^0(B(\mathcal{G}_{1L}))(r) \otimes \mathbb{C}.$$

Remark 5.24. The realization functor appears to depend on choices of simplices. However, it is in fact well-defined and independent of choice, as our complex is isomorphic to a subcomplex (via the equivalence with algebraic cycles) of the full realization map on the category of mixed Tate motives as defined in Section 7 of [Bloch and Kriz 1994].

5C. ***Hodge realization for necklace diagrams.*** For the remainder of this paper, we study the Hodge realization of the specific class $[\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})] \in H^0(B(\mathscr{G}_{1L}))$. This is defined in Definition 4.33 by the sum of graphs



As always, $\mathbb{a}_{\mathbb{n}}$ is the n-tuple $(a_1, \dots, a_n)$ that labels the beads of the completely decomposable sum of necklace graphs. Section 5C calculates the period of the class $[\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})] \in H^0(B(\mathscr{G}_{1L}))$ defined by this graph. In Section 5C1, we construct an element $[\xi^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}}) + 1 \otimes \varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})] \in H^0(B(\mathscr{T}_{1L}^{\text{twist}}, \mathscr{G}_{1L}))$ that defines the period. For ease of notation, we drop the arguments $(a_0, \mathbb{a}_{\mathbb{n}})$ whenever possible.

The current state of art for Hodge realization functor calculates the periods associated to elements of $H^0(B(A_{1L}))$ that can be represented by binary trees. See [Bloch and Kriz 1994; Kimura 2013] for cycles that map to classical polylogarithms, and [Gangl et al. 2007; 2009] for cycles that map to multiple polylogarithms. In this section, we compute the period associated to an algebraic cycle that is not in this small family of $\mathbb{P}^1_k$ linear cycles.

5C1. *Corresponding element of* $B(\mathscr{T}_{1L}^{\text{twist}}, \mathscr{G}_{1L})$. By Lemma 4.30, the sum of graphs ε^n is completely decomposable. Therefore, by Lemma 4.32, the sum

$$\varepsilon^{\mathbb{n}} = \sum_{S \subset \mathbb{n}} (-1)^{|S|} \sum_{J \subseteq S} (-1)^{|J|} \left[\varepsilon^{n-|S|} \left(a_0 \prod_{j \in J} a_j, \mathbb{a}_{n \setminus S} \right) \Bigm| \coprod_{s \in S} G(a_s) \right]$$

is a representative element defining the class $[\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})] \in H^0(B(\mathscr{G}_{1L}))$.

In this section, we define an element $\xi^{\mathbb{n}} \in \bigoplus_{i=1}^{n+1} B(\mathscr{T}_{1L}^{\text{twist}}, \mathscr{G}_{1L})_0^i$ such that $\xi^{\mathbb{n}} + 1 \otimes \varepsilon^{\mathbb{n}}$ defines a class in $H^0(B(\mathscr{T}_{1L}^{\text{twist}}, \mathscr{G}_{1L}))$. Since $(\mu + \partial)\varepsilon^{\mathbb{n}} = 0$ in $B(\mathscr{G}_{1L})$, we see that $(\mu + \partial) 1 \otimes \varepsilon^{\mathbb{n}} = \varepsilon^{\mathbb{n}}$, seen as an element in $\bigoplus_{i=1}^{n} B(\mathscr{T}_{1L}^{\text{twist}}, \mathscr{G}_{1L})_1^i$. Here, as in Example 5.11, we write

$$\varepsilon^n = (\varepsilon^n, \sigma(a_0, \mathbb{a}_{\mathbb{n}})_0) \in \mathscr{T}_{1L}^{\text{twist}}(n)^1.$$

It is sufficient to identify an element $\xi^{\mathbb{n}} \in \bigoplus_{i=1}^{n+1} B(\mathscr{T}_{1L}^{\text{twist}}, \mathscr{G}_{1L})_0^i$ such that

$$(\partial + \mu)\xi^{\mathbb{n}} = -\varepsilon^{\mathbb{n}}. \tag{5.25}$$

The remainder of this section is devoted to identifying $\xi^{\mathbb{n}}$, which is a complicated sum of elements in the circular bar construction. We introduce it in stages, starting with the easiest to state, then breaking each sum into component pieces in order to demonstrate the appropriate properties. We state what criteria these summands need to satisfy, and provide proofs along the way.

Write $\xi^{\mathbb{n}} = \sum_{k=0}^{n}(-1)^k \xi^{n-k}$, with $\xi^{n-k} \in B(\mathcal{T}_{1L}^{\mathrm{twist}}, \mathcal{G}_{1L})_0^k$ defined as

$$\xi^{n-k} = \sum_{\substack{S\subset\mathbb{n} \\ |S|=k}} \xi_{\mathrm{top}}^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S}) \otimes \coprod_{i\in S} G(a_i).$$

Here $\xi_{\mathrm{top}}^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S})$ is a topologically augmented graph in $(\mathcal{T}_{1L}^{\mathrm{twist}})_0^{n-k+1}$ such that

$$(\partial+\delta)\partial\xi_{\mathrm{top}}^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S}) + \mu\Bigg(\sum_{i\in\mathbb{n}\setminus S} \xi_{\mathrm{top}}^{n-k-1}(a_0, \mathbb{a}_{\mathbb{n}\setminus S\cup i}) \otimes G(a_i)\Bigg) = -\varepsilon^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S}). \quad (5.26)$$

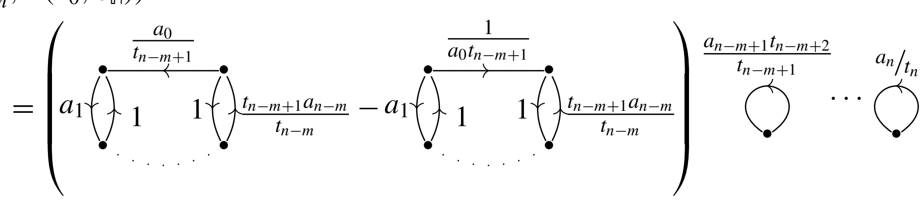
This is the key condition that we prove explicitly in Theorem 5.30.

In order to define ξ_{top}^{n}, we begin with a family of disconnected sums of unaugmented graphs

$$\xi_m^n(a_0, \mathbb{a}_{\mathbb{n}}) = \varepsilon^{n-m}(a_0, \mathbb{a}_{\mathbb{n}-\mathbb{m}})G(a_{n-m+1})\cdots G(a_n).$$

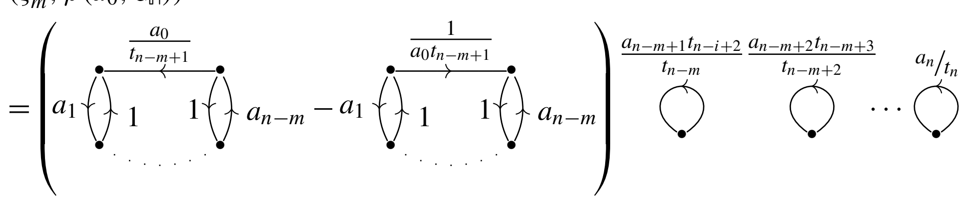
Each graph $\xi_m^n \in \mathcal{G}_{1L}{}_{m+1}^{n+1}$ consists of $m+1$ connected components, with graphical degree $m+1$. We impose upon this family of graphs two topological augmentations $\sigma(a_0, \mathbb{a}_{\mathbb{n}})$ and $\rho(a_0, \mathbb{a}_{\mathbb{n}}) \in C(n+1)_{m+1}$ of the form

$(\xi_m^n, \sigma(a_0, \mathbb{a}_{\mathbb{n}}))$



and

$(\xi_m^n, \rho(a_0, \mathbb{a}_{\mathbb{n}}))$



Note that the only difference between the labeling $\sigma(a_0, \mathbb{a}_{\mathbb{n}})_m$ and $\rho(a_0, \mathbb{a}_{\mathbb{n}})_m$ is the label on the last bead of the first connected component, $\varepsilon^{n-m}(a_0, \mathbb{a}_{\mathbb{n}-\mathbb{m}})$, and that of the second connected component. This distinction is necessary for the appropriate cancellations between algebraic and topological differentials needed to satisfy condition (5.26). Before writing down the expression for ξ_{top}^{n-k}, we introduce some further notation to simplify the expression.

We define two new terms as sums of ξ_m^n with variants of σ and ρ:

$$\lambda_m^n(a_0, \mathbb{a}_{\mathbb{n}}) = \sum_{J \subset \{n-m+1,\dots,n\}} (-1)^{|J|} \left(\xi_m^n, \sigma\left(a_0 \prod_{j \in J} a_j, \mathbb{a}_{\mathbb{n}} \right) \right), \tag{5.27}$$

$$\chi_m^n(a_0, \mathbb{a}_{\mathbb{n}}) = \sum_{I \subset \{n-m+2,\dots,n\}} (-1)^{|I|} \left(\xi_m^n, \rho\left(a_0 \prod_{i \in I} a_i, \mathbb{a}_{\mathbb{n}} \right) \right). \tag{5.28}$$

Under this notation, we write

$$\xi_{\text{top}}^n = \sum_{m=0}^{n} \lambda_m^n - \sum_{m=1}^{n} \chi_m^n.$$

Note that sum for χ_m^n starts at $m = 1$ whilst the sum for λ_m^n starts at $m = 0$. Furthermore, the sets I and J differ. Namely, the first argument for ρ, augmenting χ_m^n, never contains a_{n-m+1}, while this label appears in the first argument of σ summands of λ_m^n. The terms λ_m^n, χ_m^n and ξ_{top}^n are constructed so that the summands of the differentials of λ_m^n cancel with terms in the differentials of χ_{m+1}^n and terms of the form $\sum_{i \in \mathbb{n}} \xi_{\text{top}}^{n-1}(a_0, \mathbb{a}_{\mathbb{n}\setminus i})$ leaving the term ε^n. This is how ξ_{top}^{n-k} satisfies (5.26). We show this cancellation explicitly in Theorem 5.30.

The unaugmented graphs ξ_m^n are in $\mathcal{G}_1{L}_{m+1}^{n+1}$. Therefore the augmented graphs λ_m^n and χ_n^m are in $\mathcal{T}_{1L}^{\text{twist}}(n+1)^0$. Furthermore, ξ_{top}^n is a sum of admissible augmented graphs. If $t_{n-m+k} = 0$, then $t_{n-m+i} = 0$ for all $i < k$. Therefore, the edges labeled a_0/t_{n-m+1} and $1/(a_0 t_{n-m+1})$ are labeled by ∞, making the graphs $(\xi_m^n, \sigma(n+1)_{m+1})(a_0, \dots, a_n)$ and $(\xi_m^n, \rho(n+1)_{m+1})(a_0, \dots, a_n)$ trivial at this point.

Remark 5.29. Recall that, as shown in Lemma 5.15, the labels on the edges of these graphs correspond to the coefficients of the coordinates of the cycles. That is, the augmented cycle is parametrized $\phi_{2n+1} = 1 - x a_0 t_{n-m+1}/y$. Therefore, if $t_{n-m+1} = 0$, then $\phi_{2n+1} = 1$.

It remains to check that $\xi^{\mathbb{n}}$ defined above satisfies the necessary conditions.

Theorem 5.30. *The element $\xi^{\mathbb{n}} + 1 \otimes \varepsilon^{\mathbb{n}} \in \bigoplus_{i=1}^{n+1} B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L})_0^i$ defines a class in $H^0(B(\mathcal{T}_{1L}^{\text{twist}}, \mathcal{G}_{1L}))$.*

Proof. By the arguments presented in this section, it is sufficient to check that ξ_{top}^n satisfies (5.26). It is enough to show this for $k = 0$.

We proceed by computing the four terms of $(\delta+\partial)(\lambda_m^n-\chi_m^n)$ to show that

$$(\delta+\partial)\xi_{\mathrm{top}}^n=-\varepsilon^n-\mu\biggl(\sum_{i\in\mathbb{n}}\xi_{\mathrm{top}}^{n-1}(a_0,\mathbb{a}_{\mathbb{n}\setminus i})\otimes G(a_i)\biggr),$$

as required.

When $m=0$, the graph $(\xi_0^n,\sigma(a_0,\mathbb{a}_{\mathbb{n}}))=\lambda_0^n$ is augmented by a 1-simplex with topological boundary

$$\delta\lambda_0^n=-\delta^1\lambda_0^n=-\varepsilon^n.$$

For more general m, the algebraic boundary of the augmented sum of graphs λ_m^n is

$$\partial\lambda_m^n=-\mu\biggl(\sum_{i=1}^{n-m-1}\lambda_m^{n-1}(a_0,\mathbb{a}_{\mathbb{n}\setminus i})-\lambda_m^{n-1}(a_0a_i,\mathbb{a}_{\mathbb{n}\setminus i})\otimes G(a_i)\biggr)\\ +\delta^2\chi_{m+1}^n(a_0,\mathbb{a}_{\mathbb{n}})-\delta^1\chi_{m+1}^n(a_0a_{n-m},\mathbb{a}_{\mathbb{n}}).\tag{5.31}$$

The algebraic boundary of the augmented sum of graphs χ_m^n is

$$-\partial\chi_m^n=\mu\biggl(\sum_{i=1}^{n-m}\chi_m^{n-1}(a_0,\mathbb{a}_{\mathbb{n}\setminus i})-\chi_m^{n-1}(a_0a_i,\mathbb{a}_{\mathbb{n}\setminus i})\otimes G(a_i)\biggr).\tag{5.32}$$

For $m\geq 1$, the topological boundary of the augmented sum of graphs λ_m^n is

$$\delta\lambda_m^n=-\delta^1\chi_m^n(a_0,\mathbb{a}_{\mathbb{n}})+\delta^1\chi_m^n(a_0a_{n-m+1},\mathbb{a}_{\mathbb{n}})\\ -\mu\biggl(\sum_{i=n-m+1}^{n}\lambda_{m-1}^{n-1}(a_0,\mathbb{a}_{\mathbb{n}\setminus i})\otimes G(a_i)\biggr).\tag{5.33}$$

The topological boundary of the augmented sum of graphs χ_m^n is

$$-\delta\chi_m^n=\delta^1\chi_m^n(a_0,\mathbb{a}_{\mathbb{n}})-\delta^2\chi_{m+1}^n(a_0,\mathbb{a}_{\mathbb{n}})\\ +\mu\biggl(\sum_{i=n-m+1}^{n}\chi_{m-1}^{n-1}(a_0,\mathbb{a}_{\mathbb{n}\setminus i})\otimes G(a_i)\biggr).\tag{5.34}$$

Adding up equations (5.31), (5.32), (5.33) and (5.34), we see that

$$(\delta+\partial)\xi_{\mathrm{top}}^n=-\varepsilon_n-\mu\biggl(\sum_{i\in\mathbb{n}}\xi_{\mathrm{top}}^{n-1}(a_0,\mathbb{a}_{\mathbb{n}\setminus i})\otimes G(a_i)\biggr),$$

which matches (5.26). □

5C2. *Integrals associated to necklace diagrams.* This section is devoted to calculating the period associated to $\varepsilon^{\mathbb{n}}$. We show that this is 0 for $n\geq 1$.

By abuse of notation, in this section we write the augmented graphs λ_m^n and χ_m^n as

$$\lambda_m^n = \varepsilon^{n-m}\left(\frac{a_0}{t_{n-m+1}}, \mathbb{a}_{\mathbb{n}-\mathbb{m}-\mathbb{1}}, a_{n-m}\frac{t_{n-m+1}}{t_{n-m}}\right)G\left(a_{n-m+1}\frac{t_{n-m+2}}{t_{n-m+1}}\right)\cdots G\left(\frac{a_n}{t_n}\right),$$
$$\chi_m^n = \varepsilon^{n-m}\left(\frac{a_0}{t_{n-m+1}}, \mathbb{a}_{\mathbb{n}-\mathbb{m}}\right)G\left(a_{n-m+1}\frac{t_{n-m+2}}{t_{n-m}}\right)G\left(a_{n-m+2}\frac{t_{n-m+3}}{t_{n-m+2}}\right)\cdots G\left(\frac{a_n}{t_n}\right).$$

Theorem 5.35. *The period associated to* $\xi^{\mathbb{n}} + 1 \otimes \varepsilon^{\mathbb{n}}$ *is* 0 *for all* n*. Therefore,* $[\varepsilon^{\mathbb{n}}(a_0, \mathbb{a}_{\mathbb{n}})] \in H^0(B(\mathcal{G}_{1L}))$ *defines a trivial cohomology class.*

Proof. We apply the map $\mathcal{I} \otimes \mathrm{id}$ from (5.23) to the element $\xi^{\mathbb{n}} + 1 \otimes \varepsilon^{\mathbb{n}}$. This integral is only well defined when m, the simplicial dimension of the augmented graph, is equal to n, the loop number of the graph. Therefore,

$$\mathcal{I}\xi^n = \sum_{k=0}^{n}(-1)^k \sum_{\substack{S\subset\mathbb{n}\\|S|=k}} \mathcal{I}\otimes\mathrm{id}\big(\lambda_{n-k}^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S}) - \chi_{n-k}^{n-k}(a_0, \mathbb{a}_{\mathbb{n}\setminus S})\big)\Big[\coprod_{s\in S} G(a_s)\Big].$$

Since $\mathcal{I}(1) = 0$, the evaluation map is $(\mathcal{I}\otimes\mathrm{id})(1\otimes\varepsilon^{\mathbb{n}}) = 0$.

Recall that $\varepsilon_0(a_0) = G(a_0) - G(1/a_0)$. Therefore, from equations (5.27) and (5.28), we have

$$\lambda_n^n(a_0, \mathbb{a}_{\mathbb{n}}) = \sum_{J\subset\{1,\dots,n\}} (-1)^{|J|}\varepsilon_0\left(a_0\prod_{j\in J} a_j\frac{1}{t_0}\right)G\left(a_1\frac{t_2}{t_1}\right)\cdots G\left(a_n\frac{1}{t_n}\right),$$
$$\chi_n^n(a_0, \mathbb{a}_{\mathbb{n}}) = \sum_{I\subset\{2,\dots,n\}} (-1)^{|I|}\varepsilon_0\left(a_0\prod_{i\in I} a_i\frac{1}{t_1}\right)G\left(a_1\frac{t_2}{t_0}\right)\cdots G\left(a_n\frac{1}{t_n}\right).$$

Collecting like terms, we write

$$\lambda_n^n - \chi_n^n = \left(\sum_{J\subset\{1,\dots,n\}} (-1)^{|J|}\varepsilon_0\left(a_0\prod_{j\in J} a_j\frac{1}{t_0}\right)G\left(a_1\frac{t_2}{t_1}\right) - \sum_{I\subset\{2,\dots,n\}} (-1)^{|I|}\varepsilon_0\left(a_0\prod_{i\in I} a_i\frac{1}{t_1}\right)G\left(a_1\frac{t_2}{t_0}\right)\right) \times G\left(a_2\frac{t_3}{t_2}\right)\cdots G\left(a_n\frac{1}{t_n}\right). \quad (5.36)$$

To evaluate this integral, we recall a few facts about the iterated integrals associated to multiple polylogarithms. First of all, for a constant cycle supported on a 1-simplex,

$$\mathcal{I}(G(a/t)) = \int_0^1 \frac{d(1-t/a)}{1-t/a} = -\int_0^1 \frac{dt}{a-t} = -\mathrm{Li}\Big(\frac{1}{a}\Big).$$

Inverting the label of the edge gives

$$\mathcal{I}(G(t/a)) = \int_0^1 \frac{d(1-a/t)}{1-a/t} = \int_0^1 \frac{a\,dt}{t^2-at} = -\int_0^1 \frac{dt}{t} - \int_0^1 \frac{dt}{a-t}.$$

Subtracting the second expression from the first gives $\mathcal{I}(\varepsilon_0(a/t)) = \int_0^1 (1/t)\,dt$. We may write this as $\mathrm{Li}_1(1) = 0$, by standard renormalization of polylogarithms [Goncharov 2001b].

Similarly, for the cycle supported on a two-simplex,

$$\mathcal{I}(G(a/t_0)G(b/t_1)) = -\int_0^1 \frac{1}{b-t_1}\Big(\int_0^{t_1} \frac{dt_0}{t_0}\Big)\,dt_1 = \mathrm{Li}_2\Big(\frac{1}{b}\Big).$$

The last equality in this equation comes from the shuffle product on iterated integrals:

$$\Big(\int_0^z \frac{dt}{b-t}\Big)\Big(\int_0^z \frac{ds}{s}\Big) = \int_0^z \frac{1}{b-t}\Big(\int_0^t \frac{ds}{s}\Big)\,dt + \int_0^z \frac{1}{s}\Big(\int_0^s \frac{dt}{b-t}\Big)\,ds.$$

By the standard regularization arguments above, the left-hand side is 0. Therefore,

$$\int_0^z \frac{1}{t-b}\Big(\int_0^t \frac{ds}{s}\Big)\,dt = \mathrm{Li}_2\Big(\frac{1}{b}\Big). \tag{5.37}$$

This does not depend on the first argument, a. Therefore, the alternating signs in the sums for λ_1^1 and χ_1^1 force $\mathcal{I}(\lambda_1^1(a,b)) = \mathcal{I}(\chi_1^1(a,b)) = 0$.

For cycles supported on a three-simplex, there are two terms to check:

$$\mathcal{I}(\lambda_2^2(a,b,c)) = (-1)^2\int_0^1 \frac{1}{c-t_2}\Big(\int_0^{t_2} \frac{1}{bt_2-t_1}\Big(\int_0^{t_1} \frac{dt_0}{t_0}\Big)\,dt_1\Big)\,dt_2 = \mathrm{Li}_1\Big(\frac{1}{c}\Big)\mathrm{Li}_2\Big(\frac{1}{b}\Big)$$

and

$$\mathcal{I}(\chi_2^2(a,b,c)) = (-1)^2\int_0^1 \frac{1}{c-t_2}\Big(\int_0^{t_2} \frac{1}{t_1}\Big(\int_0^{t_1} \frac{dt_0}{bt_2-t_0}\Big)\,dt_1\Big)\,dt_2 = \mathrm{Li}_1\Big(\frac{1}{c}\Big)\mathrm{Li}_2\Big(\frac{1}{b}\Big).$$

As before, since neither integral depends on a, the alternating signs in the sums for λ_2^2 and χ_2^2 force both $\mathcal{I}(\lambda_2^2(a,b,c)) = \mathcal{I}(\chi_2^2(a,b,c)) = 0$.

For a general $n+1$-simplex, we have

$$\mathcal{I}(\xi_n^n, \sigma(a_0, \mathbb{a}_{\mathbb{n}})_{n+1}) = (-1)^n \prod_{i=2}^n \mathrm{Li}_1\Big(\frac{1}{a_i}\Big)\mathrm{Li}_2\Big(\frac{1}{a_1}\Big).$$

Similarly,

$$\mathcal{I}(\xi_n^n, \rho(a_0, \mathbb{a}_{\mathbb{n}})_{n+1}) = (-1)^n \prod_{i=2}^n \mathrm{Li}_1\Big(\frac{1}{a_i}\Big)\mathrm{Li}_2\Big(\frac{1}{a_1}\Big).$$

Since neither of these expressions depend on a_0 we have that $\mathcal{I}(\lambda_n^n)(a_0, \mathbb{a}_{\mathbb{n}}) = \mathcal{I}(\chi_n^n)(a_0, \mathbb{a}_{\mathbb{n}}) = 0$. Therefore, $\mathcal{I}(\xi_n^{\mathrm{top}}) = 0$ for all n. This is the period associated to ε^n.

This implies that

$$\mathcal{I} \otimes \mathrm{id}(\xi^n + 1 \otimes \varepsilon^n) = 0$$

for all n. Therefore this defines a trivial class in $H^0(B(\mathscr{G}_{1L}))$. $\square$

6. Outlook and future work

This paper is a first step in a program to understand the cohomology of (part of) the Bloch–Kriz cycle complex, and by extension to understand the motives associated to these cycles. By introducing a graphical representation of certain cycles, we pave the way for graph-theoretic methods to be added to the list of tools used to tackle the problem of understanding mixed Tate motives, the algebra of multiple zeta values, and the relations between such values.

Some topics for future study include:

(1) We have not yet dealt systematically with relations between closely related minimally decomposable sums. In particular, we expect the examples listed in Section 4B1 to all be related. A further analysis of these classes, their Hodge realizations, and generalizations of these classes, should give insight into constructing relations among motives and hopefully corresponding relations among the associated periods.

(2) We excluded graphs with edges labeled by 0, i.e., precisely the graphs needed to correspond to the classical polylogarithms. There is an unwritten conjecture of Brown and Gangl that only the multiple logarithms are necessary to generate the entire space of multiple polylogarithms (including the standard polylogarithms). If one assumes this conjecture, then our inability to label our edges with 0 is not a significant setback. However, in future work, we hope to devise a way of encoding edges labeled by 0s, possibly by including colored, unoriented edges, so that all the results of this paper hold in the new general setting.

(3) The graphs we study lend themselves easily to study via the language of matroids. Roughly speaking, a matroid is a combinatorial way of encoding the independence data of a matrix or graph (in this case, the subtrees of a graph). While simple to define, this is a powerful tool when it comes to studying boundaries of geometric objects. We hope that this will lead to some insight for an algorithm for finding, or a classification of, sums of algebraic cycles that lead to elements with completely decomposable boundary.

(4) The Hodge realization functor is admittedly difficult to compute explicitly. The computation of the Hodge realization in the section above, though comparable to previous computations using algebraic cycles, does not really use the graphical machinery developed earlier. We hope in future work to give a simpler and more graphically intuitive description of the Hodge realization.

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SUSAMA AGARWALA: agarwala@usna.edu
Mathematics Department, USNA, Annapolis, MD 21402, United States

OWEN PATASHNICK: o.patashnick@bristol.ac.uk
School of Mathematics, University of Bristol, Bristol, BS8 1TW, United Kingdom

and

Heilbronn Institute for Mathematical Research, Bristol, United Kingdom

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Stable operations and cooperations in derived Witt theory with rational coefficients

Alexey Ananyevskiy

The algebras of stable operations and cooperations in derived Witt theory with rational coefficients are computed and an additive description of cooperations in derived Witt theory is given. The answer is parallel to the well-known case of K-theory of real vector bundles in topology. In particular, we show that stable operations in derived Witt theory with rational coefficients are given by the values on the powers of the Bott element.

1. Introduction

Derived Witt theory, introduced by Balmer [1999] (see also [Balmer 2005] for an extensive survey), immerses Witt groups of (commutative, unital) rings and, more generally, Witt groups of schemes, into the realm of generalized cohomology theories, producing for a smooth variety X a sequence of groups $\mathrm{W}^{[n]}(X)$. This sequence is 4-periodic in n with $\mathrm{W}^{[0]}(X)$ and $\mathrm{W}^{[2]}(X)$ being canonically identified with the Witt group of symmetric vector bundles and the Witt group of symplectic vector bundles, respectively. The latter groups were introduced by Knebusch [1977]. All the groups $\mathrm{W}^{[n]}(X)$ are presented by generators and relations: roughly speaking, one should repeat the classic definition of the Witt group of a field in the setting of derived categories of coherent sheaves (or perfect complexes), carefully treating the notion of metabolic objects. The above-mentioned periodicity yields that in a certain sense we do not have "higher" derived Witt groups, in contrast to the case of algebraic K-theory.

Another approach to derived Witt theory is given by higher Grothendieck–Witt groups $\mathrm{GW}_i^{[n]}(X)$ (also known under the name of hermitian K-theory) defined for schemes by Schlichting [2010b]; see also [Schlichting 2010a; 2017]. For an affine scheme these groups coincide with hermitian K-groups introduced by Karoubi. It turns out [Schlichting 2017, Proposition 6.3] that negative higher Grothendieck–Witt groups coincide with the derived Witt groups defined by Balmer:

$$\mathrm{GW}_i^{[n]}(X) \cong \mathrm{W}^{[n-i]}(X) \quad \text{for } i < 0.$$

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If the characteristic of the base field is not 2, then higher Grothendieck–Witt groups of smooth varieties are representable in the stable motivic homotopy category; see [Hornbostel 2005] or [Schlichting and Tripathi 2015] for a geometric model. It is well-known that derived Witt theory can be obtained from higher Grothendieck–Witt groups inverting the Hopf element η; see, e.g., [Ananyevskiy 2016, Theorem 6.5]. The Hopf element η is the element in the motivic stable homotopy group $\pi_{1,1}(k)$ corresponding to the projection $\mathbb{A}^2-\{0\}\to\mathbb{P}^1$, $(x,y)\mapsto[x:y]$ (see Definition 7.1). Thus derived Witt theory is represented in the stable motivic homotopy category by a spectrum representing higher Grothendieck–Witt groups with η inverted. We denote the latter spectrum KW. This spectrum is not only $(1,1)$-periodic via η but also $(8,4)$-periodic with the periodicity realized by cup product with a Bott element $\beta\in\mathrm{KW}^{-8,-4}(\mathrm{pt})$. In this paper we compute the algebras of operations and cooperations in derived Witt theory with rational coefficients, that is, $\mathrm{KW}_{\mathbb{Q}}^{*,*}(\mathrm{KW}_{\mathbb{Q}})$ and $(\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KW}_{\mathbb{Q}})$, and give an additive description of the cooperations in derived Witt theory, $\mathrm{KW}_{*,*}(\mathrm{KW})$ (see Definition 2.12 for the notation). The answer is as follows (see Theorems 9.4, 10.2 and 10.4).

Theorem 1.1. *Let k be a field of characteristic not 2. Then the homomorphism of left $\mathrm{KW}_{\mathbb{Q}}^{0,0}(\operatorname{Spec}k)\cong\mathrm{W}_{\mathbb{Q}}(k)$-modules*

$$\mathrm{Ev}:\mathrm{KW}_{\mathbb{Q}}^{0,0}(\mathrm{KW}_{\mathbb{Q}})\to\prod_{m\in\mathbb{Z}}\mathrm{W}_{\mathbb{Q}}(k)$$

given by

$$\mathrm{Ev}(\phi)=(\dots,\beta^2\phi(\beta^{-2}),\beta\phi(\beta^{-1}),\phi(1),\beta^{-1}\phi(\beta),\beta^{-2}\phi(\beta^2),\dots)$$

is an isomorphism of algebras. Here the product on the left is given by composition and the product on the right is the componentwise one.

Moreover, $\mathrm{KW}_{\mathbb{Q}}^{p,q}(\mathrm{KW}_{\mathbb{Q}})=0$ *when* $4\nmid p-q$ *and the above isomorphism induces an isomorphism of left* $\mathrm{KW}_{\mathbb{Q}}^{*,*}(\operatorname{Spec}k)\cong\mathrm{W}_{\mathbb{Q}}(k)[\eta^{\pm1},\beta^{\pm1}]$*-modules*

$$\mathrm{KW}_{\mathbb{Q}}^{*,*}(\mathrm{KW}_{\mathbb{Q}})\cong\bigoplus_{r,s\in\mathbb{Z}}\beta^r\eta^s\prod_{m\in\mathbb{Z}}\mathrm{W}_{\mathbb{Q}}(k)$$

with $\deg\beta=(-8,-4)$, $\deg\eta=(-1,-1)$.

Theorem 1.2. *Let k be a field of characteristic not 2. Then the homomorphism of* $\mathrm{W}_{\mathbb{Q}}(k)[\eta^{\pm1}]\cong\bigoplus_{n\in\mathbb{Z}}\mathrm{KW}_{\mathbb{Q}}^{n,n}(\operatorname{Spec}k)$*-algebras*

$$\mathrm{W}_{\mathbb{Q}}(k)[\eta^{\pm1}][\beta_l^{\pm1},\beta_r^{\pm1}]\to(\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KW}_{\mathbb{Q}})$$

given by

$$\beta_l\mapsto\Sigma^{8,4}\beta\wedge u_{\mathrm{KW}_{\mathbb{Q}}},\qquad\beta_r\mapsto u_{\mathrm{KW}_{\mathbb{Q}}}\wedge\Sigma^{8,4}\beta$$

is an isomorphism of rings. Here $u_{\mathrm{KW}_{\mathbb{Q}}} : \mathbb{S} \to \mathrm{KW}_{\mathbb{Q}}$ *is the unit map and*

$$\Sigma^{8,4}\beta \wedge u_{\mathrm{KW}_{\mathbb{Q}}},\ u_{\mathrm{KW}_{\mathbb{Q}}} \wedge \Sigma^{8,4}\beta \\ \in (\mathrm{KW}_{\mathbb{Q}})_{8,4}(\mathrm{KW}_{\mathbb{Q}}) = \mathrm{Hom}_{\mathcal{SH}(k)}(\mathbb{S} \wedge \mathrm{S}^{8,4}, \mathrm{KW}_{\mathbb{Q}} \wedge \mathrm{KW}_{\mathbb{Q}}).$$

Theorem 1.3. *Suppose that k is a field of characteristic not 2 and let* M *be the abelian subgroup of* $\mathbb{Q}[v, v^{-1}]$ *generated by polynomials*

$$f_{j,n} = \frac{v^{-n}\prod_{i=0}^{j-1}(v-(2i+1)^2)}{4^j(2j)!},$$

$j \geq 0$, $n \in \mathbb{Z}$. *Then there are canonical isomorphisms of left* $\mathrm{KW}^{0,0}(\operatorname{Spec} k) \cong \mathrm{W}(k)$-*modules*

$$\mathrm{KW}_{p,q}(\mathrm{KW}) \cong \begin{cases} \mathrm{W}(k) \otimes_{\mathbb{Z}} \mathrm{M}, & 4 \mid p-q, \\ 0, & \text{otherwise}. \end{cases}$$

These theorems show that the algebras of stable operations and cooperations in derived Witt theory with rational coefficients have structure similar to the well-known case of (topological) K-theory of real vector bundles $\mathrm{KO}^{\mathrm{top}}$. This is not an accidental coincidence; these theories have quite a lot in common. $\mathrm{KO}^{\mathrm{top}}$ is built out of real vector bundles and every real vector bundle over a compact space admits a scalar product providing an isomorphism with the dual bundle. Derived Witt theory, roughly speaking, is built out of vector bundles with a fixed isomorphism with the dual bundle. In the motivic setting the element η is invertible in derived Witt theory. Real points of the Hopf map give a double cover of S^1, i.e., real points of η correspond to $2 \in \mathbb{Z} \cong \pi_0^{st}$. Thus $\mathrm{KO}^{\mathrm{top}}_{1/2}$ (K-theory of real vector bundles with inverted 2) should be a nice approximation to derived Witt groups. It is well-known that $(\mathrm{KO}^{\mathrm{top}}_{1/2})^n$ is 4-periodic in n with

$$(\mathrm{KO}^{\mathrm{top}}_{1/2})^0(\mathrm{pt}) = \mathbb{Z}\big[\tfrac{1}{2}\big], \qquad (\mathrm{KO}^{\mathrm{top}}_{1/2})^n(\mathrm{pt}) = 0, \quad n = 1, 2, 3.$$

The same holds for derived Witt theory: $\mathrm{W}^{[n]}$ is 4-periodic in n with

$$\mathrm{W}^{[0]}(\mathrm{pt}) = \mathrm{W}(k), \qquad \mathrm{W}^{[n]}(\mathrm{pt}) = 0, \quad n = 1, 2, 3.$$

In fact, over the real numbers one can show that the (real) realization functor takes the motivic spectrum KW to the spectrum $\mathrm{KO}^{\mathrm{top}}_{1/2}$ and there are deep theorems comparing $\mathrm{W}^{[n]}(X)$ to $(\mathrm{KO}^{\mathrm{top}})^n(X(\mathbb{R}))$ for an algebraic variety X over the field of real numbers; see [Brumfiel 1984; Karoubi et al. 2016]. Moreover, in a private communication Oliver Röndigs outlined to me a strategy for obtaining a description of $\big(\mathrm{KW} \otimes \mathbb{Z}\big[\tfrac{1}{2}\big]\big)_{*,*}\big(\mathrm{KW} \otimes \mathbb{Z}\big[\tfrac{1}{2}\big]\big)$ and $\mathrm{KW}_{\mathbb{Q}}^{*,*}(\mathrm{KW}_{\mathbb{Q}})$ over a base field of characteristic zero applying Brumfiel's theory [1984] to the well-known computation of cooperations and rational operations in topology [Adams et al. 1971].

The algebras of stable operations and cooperations in $\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}}$ can be described as follows. Denote by $\beta^{\mathrm{top}} \in (\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}})^{-4}(\mathrm{pt})$ the element inducing periodicity

$$(\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}})^{n+4} \cong (\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}})^{n}.$$

Every stable operation is uniquely determined by its values on the powers of β^{top}, yielding an isomorphism

$$(\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}})^{*}(\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}}) \cong \bigoplus_{n\in\mathbb{Z}} (\beta^{\mathrm{top}})^{n} \prod_{m\in\mathbb{Z}} \mathbb{Q},$$

while for the cooperations one has

$$(\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}})_{*}(\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}}) \cong \mathbb{Q}[\beta_l^{\pm 1}, \beta_r^{\pm 1}],$$

where β_r and β_l are similar to the ones from Theorem 1.2.

Computations of $(\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}})^{*}(\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}})$ and $(\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}})_{*}(\mathrm{KO}^{\mathrm{top}}_{\mathbb{Q}})$ could be carried out quite easily using Serre's theorem about finiteness of stable homotopy groups of spheres. In the motivic setting the analogous result on stable homotopy groups is not completely settled; moreover, our motivation is just the opposite one. It was pointed out to me by Marc Levine that the above computations of stable operations and cooperations in $\mathrm{KW}_{\mathbb{Q}}$ combined with the technique developed by Cisinski and Déglise [2012] could possibly yield the motivic version of Serre's finiteness. This problem is addressed in a forthcoming paper [Ananyevskiy et al. 2017].

Our approach to the computation of stable operations and cooperations in $\mathrm{KW}_{\mathbb{Q}}$ and cooperations in KW is straightforward. The spectrum KW is obtained by localization from the spectrum KO representing higher Grothendieck–Witt groups, hence

$$\mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{KW}_{\mathbb{Q}}) = \mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{KO}),$$
$$(\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KW}_{\mathbb{Q}}) = (\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KO}),$$
$$\mathrm{KW}_{*,*}(\mathrm{KW}) = \mathrm{KW}_{*,*}(\mathrm{KO}).$$

The odd spaces in the spectrum KO are all the same and coincide with the infinite quaternionic Grassmannian HGr. Derived Witt theory of HGr is known to be given by power series in characteristic classes [Panin and Walter 2010a, Theorem 9.1]. The pullbacks along the structure maps of KO can be described explicitly using the following computation of characteristic classes of triple tensor products of rank 2 symplectic bundles (Lemma 8.2).

Lemma 1.4. *Let E_1, E_2 and E_3 be rank 2 symplectic bundles over a smooth variety X. Put $\xi_i = b_1^{\mathrm{KW}}(E_i) \in \mathrm{KW}^{4,2}(X)$ and denote by $\xi(n_1, n_2, n_3)$ the sum of all the monomials lying in the orbit of $\xi_1^{n_1}\xi_2^{n_2}\xi_3^{n_3}$ under the action of S_3. Then*

$$\begin{aligned}
b_1^{\mathrm{KW}}(E_1 \otimes E_2 \otimes E_3) &= \beta\xi(1, 1, 1),\\
b_2^{\mathrm{KW}}(E_1 \otimes E_2 \otimes E_3) &= \beta\xi(2, 2, 0) - 2\xi(2, 0, 0),\\
b_3^{\mathrm{KW}}(E_1 \otimes E_2 \otimes E_3) &= \beta\xi(3, 1, 1) - 8\xi(1, 1, 1),\\
b_4^{\mathrm{KW}}(E_1 \otimes E_2 \otimes E_3) &= \beta\xi(2, 2, 2) + \xi(4, 0, 0) - 2\xi(2, 2, 0).
\end{aligned}$$

This computation is a derived Witt analogue of the equality

$$c_1^{\mathrm{K}}(L_1 \otimes L_2) = c_1^{\mathrm{K}}(L_1) + c_1^{\mathrm{K}}(L_2) - c_1^{\mathrm{K}}(L_1)c_1^{\mathrm{K}}(L_2)$$

in K-theory, i.e., an analogue of a formal group law. It turns out that the inverse limit $\varprojlim \mathrm{KW}_{\mathbb{Q}}^{*+8n+4,*+4n+2}(\mathrm{HGr})$ can be easily computed yielding the desired answer, while the $\varprojlim^1$ term vanishes. For the cooperations we employ a strategy similar to the one used for operations, the main difference being that in place of the result by Panin and Walter on the derived Witt theory of HGr we use Theorem 5.10, which provides the following description of derived Witt homology of HGr (see Definitions 5.8 and 5.9 for the details).

Theorem 1.5. *Let k be a field of characteristic not 2. Then there is a canonical isomorphism of* $\mathrm{KW}^{*,*}(\operatorname{Spec} k) \cong \mathrm{W}(k)[\eta^{\pm 1}, \beta^{\pm 1}]$*-algebras*

$$\mathrm{KW}_{*,*}(\mathrm{HGr}_+) \cong \mathrm{W}(k)[\eta^{\pm 1}, \beta^{\pm 1}][x_1, x_2, \dots].$$

The paper is organized in the following way. In Section 2 we recall the well-known definitions and constructions in generalized (co)homology theories representable in the stable motivic homotopy category introduced by Morel and Voevodsky. The next section deals with the definitions and basic properties of cup and cap products in the motivic setting. In Section 4 we recall the theory of symplectic orientation in generalized motivic cohomology developed by Panin and Walter [2010a; 2010c]. Section 5 is dual to Section 4 and deals with symplectically oriented homology theories. In Sections 6 and 7 we recall various representability properties of higher Grothendieck–Witt groups and derived Witt theory. In Section 8 we compute characteristic classes of triple tensor products of rank 2 symplectic bundles. In the last two sections we compute the algebras of stable operations and cooperations in $\mathrm{KW}_{\mathbb{Q}}$ and give an additive description of cooperations in KW.

2. Recollection on generalized motivic (co)homology

In this section we recall some basic definitions and constructions in the unstable and stable motivic homotopy categories $\mathcal{H}_{\bullet}(k)$ and $\mathcal{SH}(k)$. We refer the reader to the foundational papers [Morel and Voevodsky 1999; Voevodsky 1998] for an introduction to the subject. We use the version of stable motivic homotopy category based on $\mathcal{HP}^1$-spectra introduced in [Panin and Walter 2010b].

Throughout this paper k is a field of characteristic different from 2.

Definition 2.1. Let Sm/k be the category of smooth varieties over k. A *motivic space* over k is a simplicial presheaf on Sm/k. Each $X \in \mathrm{Sm}/k$ defines an unpointed motivic space $\mathrm{Hom}_{\mathrm{Sm}/k}(-, X)$ constant in the simplicial direction. We often write pt for $\operatorname{Spec} k$ regarded as a motivic space. Inverting all the weak motivic equivalences in the category of the pointed motivic spaces, we obtain the *pointed motivic unstable homotopy category* $\mathcal{H}_\bullet(k)$.

Definition 2.2. Define $\mathrm{S}^{1,1} = (\mathbb{A}^1 - \{0\}, 1)$, $\mathrm{S}^{1,0} = \mathrm{S}^1_s = \Delta^1/\partial(\Delta^1)$ and

$$\mathrm{S}^{p+q,q} = (\mathrm{S}^{1,1})^{\wedge q} \wedge (\mathrm{S}^{1,0})^{\wedge p}$$

for the integers $p, q \geq 0$. Let $\mathrm{T} = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ be the Morel–Voevodsky object, which is canonically isomorphic to $\mathrm{S}^{2,1}$ in $\mathcal{H}_\bullet(k)$ [Morel and Voevodsky 1999, Lemma 3.2.15].

Definition 2.3. Let $V = (k^{\oplus 4}, \phi)$, $\phi(\boldsymbol{x}, \boldsymbol{y}) = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$, be the standard symplectic vector space over k of dimension 4. The *quaternionic projective line* HP^1 is the variety of symplectic planes in V. Alternatively, it can be described as $\mathrm{HP}^1 = \mathrm{Sp}_4/\mathrm{Sp}_2 \times \mathrm{Sp}_2$. Write $* = \langle e_1, e_2\rangle \in \mathrm{HP}^1(k)$ for the standard basis e_1, e_2, e_3, e_4 of V. If not otherwise specified we consider HP^1 as a pointed motivic space $(\mathrm{HP}^1, *)$. Let $\mathcal{HP}^1$ be the pushout of

$$\mathbb{A}^1 \xleftarrow{0} \mathrm{pt} \xrightarrow{*} \mathrm{HP}^1.$$

There is an obvious isomorphism $\mathcal{HP}^1 \xrightarrow{\cong} \mathrm{HP}^1$ in $\mathcal{H}_\bullet(k)$ given by a contraction of $\mathbb{A}^1$ and we usually identify these two objects in $\mathcal{H}_\bullet(k)$.

Remark 2.4. The only reason that we need $\mathcal{HP}^1$ is Definition 6.7, since the morphisms that we use there do no exist for HP^1.

Lemma 2.5 [Panin and Walter 2010b, Theorem 9.8]. *There exists a canonical isomorphism* $\mathrm{HP}^1 \cong \mathrm{T} \wedge \mathrm{T}$ *in* $\mathcal{H}_\bullet(k)$.

Corollary 2.6. *There exists a canonical isomorphism* $\mathrm{HP}^1 \cong \mathrm{S}^{4,2}$ *in* $\mathcal{H}_\bullet(k)$.

Proof. This follows from the lemma by applying the canonical isomorphism $\mathrm{T} \cong \mathrm{S}^{2,1}$ [Morel and Voevodsky 1999, Lemma 3.2.15]. □

Definition 2.7. An $\mathcal{HP}^1$*-spectrum* A is a sequence of pointed motivic spaces

$$(A_0, A_1, A_2, \dots)$$

equipped with structural maps $\sigma_n : \mathcal{HP}^1 \wedge A_n \to A_{n+1}$. A morphism of $\mathcal{HP}^1$-spectra is a sequence of morphisms of pointed motivic spaces compatible with the structural maps. Inverting the stable motivic weak equivalences as in [Jardine 2000], we obtain the motivic stable homotopy category $\mathcal{SH}(k) = \mathcal{SH}_{\mathcal{HP}^1}(k)$. This category has a canonical symmetric monoidal structure. From now on, by a spectrum we mean an $\mathcal{HP}^1$-spectrum.

Lemma 2.8 [Panin and Walter 2010b, Theorem 12.1]. *The stable homotopy categories of* T*-spectra and of* $\mathcal{HP}^1$*-spectra are equivalent.*

Definition 2.9. Every pointed motivic space Y gives rise to a suspension spectrum

$$\Sigma^\infty_{\mathcal{HP}^1} Y = (Y, \mathcal{HP}^1 \wedge Y, (\mathcal{HP}^1)^{\wedge 2} \wedge Y, \dots).$$

Put $\mathbb{S} = \Sigma^\infty_{\mathcal{HP}^1} \mathrm{pt}_+$ for the sphere spectrum.

Definition 2.10. Let $A = (A_0, A_1, \dots)$ be an $\mathcal{HP}^1$-spectrum and m be an integer. Denote by $A\{m\} = (A\{m\}_0, A\{m\}_1, \dots)$ the spectrum given by

$$A\{m\}_n = \begin{cases} A_{m+n}, & m+n \geq 0, \\ \mathrm{pt}, & m+n < 0, \end{cases}$$

with the structure maps induced by the structure maps of A.

Definition 2.11. It follows from Corollary 2.6 that in $\mathcal{SH}(k)$ there is a canonical isomorphism $(A \wedge \mathrm{S}^{4,2})\{-1\} \cong A$. The suspension functors $- \wedge \mathrm{S}^{p+q,q}$, $p, q \geq 0$, become invertible in $\mathcal{SH}(k)$, so we extend the notation to arbitrary integers p, q in an obvious way.

Definition 2.12. For $A, B \in \mathcal{SH}(k)$ put

$$A^{i,j}(B) = \mathrm{Hom}_{\mathcal{SH}(k)}(B, A \wedge \mathrm{S}^{i,j}), \qquad A^{*,*}(B) = \bigoplus_{i,j \in \mathbb{Z}} A^{i,j}(B),$$

$$A_{i,j}(B) = \mathrm{Hom}_{\mathcal{SH}(k)}(\mathbb{S} \wedge \mathrm{S}^{i,j}, A \wedge B), \qquad A_{*,*}(B) = \bigoplus_{i,j \in \mathbb{Z}} A_{i,j}(B).$$

Let $f : B \to B'$ be a morphism in $\mathcal{SH}(k)$. Denote by

$$f^A : A^{*,*}(B') \to A^{*,*}(B), \qquad f_A : A_{*,*}(B) \to A_{*,*}(B')$$

the natural morphisms given by composition with f.

Remark 2.13. Using suspension spectra we may treat every pointed motivic space as a spectrum; in particular, we may treat a smooth variety X as a suspension spectrum $\Sigma^\infty_{\mathcal{HP}^1}(X_+, +)$. Thus all the definitions involving $A^{*,*}(B)$ and $A_{*,*}(B)$ are applicable to the case of B being a pointed motivic space or a smooth variety.

Definition 2.14. For $A, B \in \mathcal{SH}(k)$ we have *suspension isomorphisms*

$$\Sigma^{p,q} : A^{*,*}(B) \xrightarrow{\simeq} A^{*+p,*+q}(B \wedge \mathrm{S}^{p,q}),$$

$$\Sigma^{p,q} : A_{*,*}(B) \xrightarrow{\simeq} A_{*+p,*+q}(B \wedge \mathrm{S}^{p,q}),$$

given by smash product $-\wedge \mathrm{id}_{S^{p,q}}$. The isomorphisms from [Morel and Voevodsky 1999, Lemma 3.2.15] and Corollary 2.6 induce *suspension isomorphisms*

$$\begin{aligned} \Sigma_{\mathrm{T}} &: A^{*,*}(B) \xrightarrow{\simeq} A^{*+2,*+1}(B\wedge \mathrm{T}), & \Sigma_{\mathrm{T}} &: A_{*,*}(B) \xrightarrow{\simeq} A_{*+2,*+1}(B\wedge \mathrm{T}), \\ \Sigma_{\mathrm{HP}^1} &: A^{*,*}(B) \xrightarrow{\simeq} A^{*+4,*+2}(B\wedge \mathrm{HP}^1), & \Sigma_{\mathrm{HP}^1} &: A_{*,*}(B) \xrightarrow{\simeq} A_{*+4,*+2}(B\wedge \mathrm{HP}^1), \\ \Sigma_{\mathcal{HP}^1} &: A^{*,*}(B) \xrightarrow{\simeq} A^{*+4,*+2}(B\wedge \mathcal{HP}^1), & \Sigma_{\mathcal{HP}^1} &: A_{*,*}(B) \xrightarrow{\simeq} A_{*+4,*+2}(B\wedge \mathcal{HP}^1). \end{aligned}$$

We write Σ^n_{T}, $\Sigma^n_{\mathrm{HP}^1}$ and $\Sigma^n_{\mathcal{HP}^1}$ for the n-fold composition of the respective suspension isomorphisms.

Definition 2.15. Let $A = (A_0, A_1, \dots)$ be an $\mathcal{HP}^1$-spectrum. Denote $\mathrm{Tr}_n A$ the spectrum given by

$$(\mathrm{Tr}_n A)_m = \begin{cases} A_m, & m \le n, \\ (\mathcal{HP}^1)^{\wedge m-n} \wedge A_m, & m > n, \end{cases}$$

with the structure maps induced by the structure maps of A.

Remark 2.16. The obvious map $\Sigma^\infty_{\mathcal{HP}^1} A_n\{-n\} \to \mathrm{Tr}_n A$ clearly becomes an isomorphism in $\mathcal{SH}(k)$.

Lemma 2.17. *Consider $A \in \mathcal{SH}(k)$ and let $B = (B_0, B_1, \dots)$ be an $\mathcal{HP}^1$-spectrum with structure maps $\sigma_n : \mathcal{HP}^1 \wedge B_n \to B_{n+1}$. Then*:

(1) *The canonical homomorphism*

$$\varinjlim A_{*+4n,*+2n}(B_n) \to A_{*,*}(B)$$

is an isomorphism, where the limit is taken with respect to the morphisms

$$(\sigma_n)_A \circ \Sigma_{\mathcal{HP}^1} : A_{*+4n,*+2n}(B_n) \to A_{*+4(n+1),*+2(n+1)}(B_{n+1}).$$

(2) *There is a short exact sequence*

$$0 \to \varprojlim{}^1 A^{*+4n-1,*+2n}(B_n) \to A^{*,*}(B) \to \varprojlim A^{*+4n,*+2n}(B_n) \to 0,$$

where the limit is taken with respect to the morphisms

$$\Sigma^{-1}_{\mathcal{HP}^1} \circ \sigma^A_n : A^{*+4(n+1),*+2(n+1)}(B_{n+1}) \to A^{*+4n,*+2n}(B_n).$$

Proof. Straightforward, using $B = \varinjlim \mathrm{Tr}_n B$ and a mapping telescope. In the motivic setting see, for example, [Panin et al. 2009, Lemma A.34]. □

3. Cup and cap product on generalized motivic (co)homology

In this section we recall the well-known constructions of cup and cap product in generalized (co)homology. A classic reference for this theme in (nonmotivic) stable homotopy theory is [Adams 1974, III.9].

Definition 3.1. A *commutative ring spectrum* A is a commutative monoid

$$(A, m_A : A \wedge A \to A, u_A : \mathbb{S} \to A)$$

in $(\mathcal{SH}(k), \wedge, \mathbb{S})$.

Definition 3.2. Let (A, m_A, u_A) be a commutative ring spectrum and $f : B \to C \wedge D$ a morphism in $\mathcal{SH}(k)$. The *cup product*

$$\cup_f : A^{p,q}(C) \times A^{i,j}(D) \to A^{p+i,q+j}(B)$$

is given by $a \cup_f b = (m_A \wedge \sigma) \circ (\mathrm{id}_A \wedge \tau_{\mathrm{S}^{p,q},A} \wedge \mathrm{id}_{\mathrm{S}^{i,j}}) \circ (a \wedge b) \circ f$,

$$a \cup_f b = \left(\begin{array}{c} B \xrightarrow{\quad f \quad} C \wedge D \xrightarrow{a \wedge b} A \wedge \mathrm{S}^{p,q} \wedge A \wedge \mathrm{S}^{i,j} \\ \swarrow\, \mathrm{id}_A \wedge \tau_{\mathrm{S}^{p,q},A} \wedge \mathrm{id}_{\mathrm{S}^{i,j}} \\ A \wedge A \wedge \mathrm{S}^{p,q} \wedge \mathrm{S}^{i,j} \xrightarrow{\quad m_A \wedge \sigma \quad} A \wedge \mathrm{S}^{p+i,q+j} \end{array} \right),$$

where $\tau_{\mathrm{S}^{p,q},A} : \mathrm{S}^{p,q} \wedge A \xrightarrow{\simeq} A \wedge \mathrm{S}^{p,q}$ and $\sigma : \mathrm{S}^{p,q} \wedge \mathrm{S}^{i,j} \xrightarrow{\simeq} \mathrm{S}^{p+i,q+j}$ are permutation isomorphisms. We usually omit the subscript f from the notation when the morphism is clear from the context. The cup product is clearly bilinear and associative. We are going to use this product in the following special cases:

(1) Let $U_1, U_2 \subset X$ be open subsets of a smooth variety X and

$$f : X/(U_1 \cup U_2) \to X/U_1 \wedge X/U_2$$

be the morphism induced by the diagonal embedding. Then the above construction gives a cup product

$$\cup : A^{p,q}(X/U_1) \times A^{i,j}(X/U_2) \to A^{p+i,q+j}(X/(U_1 \cup U_2)).$$

In particular, for $U_1 = U_2 = \varnothing$ we obtain a product

$$\cup : A^{p,q}(X) \times A^{i,j}(X) \to A^{p+i,q+j}(X)$$

endowing $A^{*,*}(X)$ with a ring structure.

(2) Consider $B \in \mathcal{SH}(k)$ and let

$$f_1 = \mathrm{id} : B \to B \wedge \mathbb{S}, \qquad f_2 = \mathrm{id} : B \to \mathbb{S} \wedge B$$

be the identity maps. Then we obtain cup products

$$\cup : A^{p,q}(B) \times A^{i,j}(\mathrm{pt}) \to A^{p+i,q+j}(B),$$
$$\cup : A^{p,q}(\mathrm{pt}) \times A^{i,j}(B) \to A^{p+i,q+j}(B)$$

endowing $A^{*,*}(B)$ with the structure of an $A^{*,*}(\mathrm{pt})$-bimodule.

Definition 3.3. Let $\tau_{S^{2,1},S^{2,1}} : S^{2,1} \wedge S^{2,1} \xrightarrow{\simeq} S^{2,1} \wedge S^{2,1}$ be the permutation isomorphism and let (A, m_A, u_A) be a commutative ring spectrum. Put

$$\varepsilon = \Sigma^{-4,-2}\tau^A_{S^{2,1},S^{2,1}}\Sigma^{4,2}u_A \in A^{0,0}(\mathrm{pt}).$$

Note that $\varepsilon^2 = 1$.

Lemma 3.4. *Let (A, m_A, u_A) be a commutative ring spectrum and $f : B \to C \wedge D$ a morphism in $\mathcal{SH}(k)$. Write $f^\tau = \tau \circ f : B \to D \wedge C$ with $\tau : C \wedge D \xrightarrow{\simeq} D \wedge C$ being the permutation isomorphism. Then*

$$a \cup_f b = (-1)^{pi}\varepsilon^{qj} b \cup_{f^\tau} a \in A^{p+i,q+j}(B)$$

for every $a \in A^{p,q}(C)$ and $b \in A^{i,j}(D)$.

Proof. Examining the definition one notices that

$$a \cup_f b = (\Sigma^{-p-i,-q-j}\tau_{S^{p,q},S^{i,j}}) \cup b \cup_{f^\sigma} a,$$

where $\tau_{S^{p,q},S^{i,j}} : S^{p,q} \wedge S^{i,j} \xrightarrow{\simeq} S^{i,j} \wedge S^{p,q}$ is the permutation isomorphism. By classical homotopy theory one has $\Sigma^{-2,0}(\tau_{S^{1,0},S^{1,0}}) = -1$, so the claim follows. $\square$

Definition 3.5. Let (A, m_A, u_A) be a commutative ring spectrum and $f : B \to C \wedge D$ be a morphism in $\mathcal{SH}(k)$. The *cap product*

$$\cap_f : A^{p,q}(C) \times A_{i,j}(B) \to A_{i-p,j-q}(D)$$

is given by $a \cap_f x = \Sigma^{-p,-q}((m_A \wedge \tau_{S^{p,q},D}) \circ (\mathrm{id}_A \wedge a \wedge \mathrm{id}_D) \circ (\mathrm{id}_A \wedge f) \circ x)$,

$$a \cap_f x = \Sigma^{-p,-q}\left(\begin{tikzcd} & \mathbb{S} \wedge S^{i,j} \arrow[r, "x"] & A \wedge B \arrow[r, "\mathrm{id}_A \wedge f"] & A \wedge C \wedge D \arrow[dll, "\mathrm{id}_A \wedge a \wedge \mathrm{id}_D"'] \\ A \wedge A \wedge S^{p,q} \wedge D \arrow[rrr, "m_A \wedge \tau_{S^{p,q},D}"] & & & A \wedge D \wedge S^{p,q} \end{tikzcd}\right),$$

where $\tau_{S^{p,q},D} : S^{p,q} \wedge D \xrightarrow{\simeq} D \wedge S^{p,q}$ is the permutation isomorphism. The subscript f is usually omitted from the notation when the morphism is clear from the context. We are going to use this product in the following special cases:

(1) Let Y be a pointed motivic space and let $f = \Delta : Y \to Y \wedge Y$ be the diagonal embedding. Then we obtain the cap product

$$\cap : A^{p,q}(Y) \times A_{i,j}(Y) \to A_{i-p,j-q}(Y).$$

One can easily check that $(a \cup a') \cap x = a \cap (a' \cap x)$. This product endows $A_{*,*}(Y)$ with a left $A^{*,*}(Y)$-module structure.

(2) Let U be an open subset of a smooth variety X and $f : X/U \to (X/U) \wedge X_+$ the morphism induced by the diagonal embedding. Then we obtain the cap product

$$\cap : A^{p,q}(X/U) \times A_{i,j}(X/U) \to A_{i-p,j-q}(X).$$

(3) Consider $B \in \mathcal{SH}(k)$ and let $f = \mathrm{id} : B \to B \wedge \mathbb{S}$ be the identity morphism. Then we obtain the *Kronecker pairing*

$$\langle -,- \rangle : A^{p,q}(B) \times A_{i,j}(B) \to A_{i-p,j-q}(\mathrm{pt}) \cong A^{p-i,q-j}(\mathrm{pt}).$$

(4) Consider $B \in \mathcal{SH}(k)$ and let $f = \mathrm{id} : B \to \mathbb{S} \wedge B$ be the identity morphism. Then we obtain a cap product

$$\cap : A^{p,q}(\mathrm{pt}) \times A_{i,j}(B) \to A_{i-p,j-q}(B)$$

endowing $A_{*,*}(B)$ with a left $A^{*,*}(\mathrm{pt})$-module structure.

Lemma 3.6. *Let A be a commutative ring spectrum. Then for a commutative square*

$$\begin{tikzcd} C \wedge D \arrow[r, "r\wedge s"] & C' \wedge D' \\ B \arrow[u, "f"] \arrow[r, "t"] & B' \arrow[u, "f'"'] \end{tikzcd}$$

in $\mathcal{SH}(k)$ and $a \in A^{*,*}(C')$, $x \in A_{*,*}(B)$ we have

$$s_A(r^A(a) \cap x) = a \cap t_A(x).$$

Proof. Straightforward. $\square$

Definition 3.7. Let A be a commutative ring spectrum and let $p : X \to Y$ be a morphism of pointed motivic spaces. Then the pairing

$$p_A \circ (- \cap -) : A^{*,*}(X) \times A_{*,*}(X) \to A_{*,*}(Y)$$

is $A^{*,*}(Y)$-bilinear. Denote by

$$D_p : A_{*,*}(X) \to \mathrm{Hom}_{A^{*,*}(Y)}(A^{*,*}(X), A_{*,*}(Y))$$

the adjoint homomorphism of left $A^{*,*}(X)$-modules.

Definition 3.8. Let (A, m_A, u_A) and (B, m_B, u_B) be commutative ring spectra. The product

$$- \star - : A_{p,q}(B) \times A_{i,j}(B) \to A_{i+p,j+q}(B)$$

is given by $x \star y = (m_A \wedge m_B) \circ (\mathrm{id}_A \wedge \tau_{B,A} \wedge \mathrm{id}_B) \circ (y \wedge x) \circ \sigma$,

$$x \star y = \left(\begin{tikzcd} \mathbb{S} \wedge \mathrm{S}^{i+p,j+q} \arrow[r, "\sigma"] & \mathbb{S} \wedge \mathrm{S}^{i,j} \wedge \mathbb{S} \wedge \mathrm{S}^{p,q} \arrow[r, "y\wedge x"] & A \wedge B \wedge A \wedge B \arrow[dll, "\mathrm{id}_A \wedge \tau_{B,A} \wedge \mathrm{id}_B"'] \\ A \wedge A \wedge B \wedge B \arrow[rr, "m_A \wedge m_B"] & & A \wedge B \end{tikzcd} \right),$$

where $\sigma : \mathrm{S}^{i+p,j+q} \xrightarrow{\simeq} \mathrm{S}^{i,j} \wedge \mathrm{S}^{p,q}$ and $\tau_{B,A} : B \wedge A \xrightarrow{\simeq} A \wedge B$ are the permutation isomorphisms. This product endows $A_{*,*}(B)$ with a ring structure. Moreover, one immediately checks that it agrees with the cap product introduced in the end of Definition 3.5 under the homomorphism

$$A^{-p,-q}(\mathrm{pt}) \simeq A_{p,q}(\mathrm{pt}) \xrightarrow{(u_B)_A} A_{p,q}(B).$$

4. Symplectically oriented cohomology theories

In this section we provide a list of results from the theory of symplectic orientation in generalized motivic cohomology developed in [Panin and Walter 2010c].

Definition 4.1. We adopt the following notation dealing with Grassmannians and flags of symplectic spaces (see [Panin and Walter 2010c]).

- $\mathrm{H}_- = \left(k^{\oplus 2}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)$ is the *standard symplectic plane*.
- $\mathrm{HGr}(2r, 2n) = \mathrm{Sp}_{2n} / \mathrm{Sp}_{2r} \times \mathrm{Sp}_{2n-2r}$ is the *quaternionic Grassmannian*. Alternatively, it can be described as the open subscheme of $\mathrm{Gr}(2r, \mathrm{H}_-^{\oplus n})$ parametrizing subspaces on which the standard symplectic form is nondegenerate.
- $\mathcal{U}^s_{2r,2n}$ is the tautological rank $2r$ symplectic vector bundle over $\mathrm{HGr}(2r, 2n)$.
- $\mathrm{HP}^n = \mathrm{HGr}(2, 2n+2)$ is the *quaternionic projective space*.
- $\mathcal{H}(1) = \mathcal{U}^s_{2,n+2}$ is the tautological rank 2 symplectic vector bundle over HP^n.
- $\mathrm{HFlag}(2^r, 2n) = \mathrm{Sp}_{2n} / (\mathrm{Sp}_2 \times \cdots \times \mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2r})$ is the *quaternionic flag variety*. Alternatively, it can be described as the variety of flags $V_2 \le V_4 \le \cdots \le V_{2r} \le \mathrm{H}_-^{\oplus n}$ such that $\dim V_{2i} = 2i$ and the restriction of the symplectic form is nondegenerate on V_{2i} for every i.
- $\mathrm{HGr}(2r, E)$, $\mathrm{HP}(E)$, $\mathrm{HFlag}(2^r, E)$ are the relative versions of the above varieties defined for a rank $2n$ symplectic bundle E over a smooth variety X.

Definition 4.2 [Panin and Walter 2010c, Definition 14.2; 2010a, Definition 12.1]. A *symplectic orientation* of a commutative ring spectrum A is a rule which assigns to each rank $2n$ symplectic bundle E over a smooth variety X an element

$$\mathrm{th}(E) = \mathrm{th}^A(E) \in A^{4n,2n}(E/(E-X))$$

with the following properties:

(1) For an isomorphism $u : E \xrightarrow{\simeq} E'$, one has $\mathrm{th}(E) = u^A\,\mathrm{th}(E')$.

(2) For a morphism of varieties $f : X \to Y$, symplectic bundle E over Y and pullback morphism $f_E : f^*E \to E$, one has $f_E^A\,\mathrm{th}(E) = \mathrm{th}(f^*E)$.

(3) The homomorphisms $-\cup \mathrm{th}(E) : A^{*,*}(X) \to A^{*+4n,*+2n}(E/(E-X))$ are isomorphisms.

(4) We have $\mathrm{th}(E \oplus E') = q_1^A\,\mathrm{th}(E) \cup q_2^A\,\mathrm{th}(E')$, where q_1, q_2 are the projections from $E \oplus E'$ to its factors.

We refer to the classes $\mathrm{th}(E)$ as *Thom classes*. A commutative ring spectrum A with a chosen symplectic orientation is called a *symplectically oriented spectrum*.

Lemma 4.3. *Let A be a symplectically oriented spectrum, X be a smooth variety and let $p : X \to \mathrm{pt}$ be the projection. Identify $\mathrm{H}_-^{\oplus r}/(\mathrm{H}_-^{\oplus r} - \{0\}) \cong \mathrm{T}^{\wedge 2r}$. Then*

$$\mathrm{th}(p^*\mathrm{H}_-^{\oplus r}) = a\,\Sigma_{\mathrm{T}}^{2r} 1_X$$

for some invertible element $a \in A^{0,0}(\mathrm{pt})$.

Proof. We have the following isomorphisms:

$$A^{0,0}(\mathrm{pt}) \xrightarrow{-\cup \Sigma_{\mathrm{T}}^{2r} 1_{\mathrm{pt}}} A^{4r,2r}(\mathrm{T}^{\wedge 2r}) \cong A^{4r,2r}(\mathrm{H}_-^{\oplus r}/(\mathrm{H}_-^{\oplus r} - \{0\})) \xleftarrow{-\cup \mathrm{th}(\mathrm{H}_-^{\oplus r})} A^{0,0}(\mathrm{pt}),$$

and thus $\mathrm{th}(\mathrm{H}_-^{\oplus r}) = a\,\Sigma_{\mathrm{T}}^{2r} 1_{\mathrm{pt}}$ for some invertible $a \in A^{0,0}(\mathrm{pt})$. The claim follows from the functoriality of Thom classes. □

Remark 4.4. There is a canonical bijection between the sets of symplectic orientations satisfying the additional condition of normalization ($\mathrm{th}(\mathrm{H}_-) = \Sigma_{\mathrm{T}}^2 1$) and homomorphisms of monoids $\mathrm{MSp} \to A$. See [Panin and Walter 2010a, Theorems 12.2 and 13.2] for the details.

Remark 4.5. The main example of a symplectically oriented cohomology theory that we are interested in is that of higher Grothendieck–Witt groups (hermitian K-theory). See Definition 6.7 and Theorems 6.8 and 6.10 for the details.

Definition 4.6 [Panin and Walter 2010c, Definition 14.1]. A theory of *Borel classes* on a commutative ring spectrum A is a rule assigning to every symplectic bundle E over a smooth variety X a sequence of elements $b_i(E) = b_i^A(E) \in A^{4i,2i}(X)$, $i \geq 1$, satisfying:

(1) For $E \cong E'$ we have $b_i(E) = b_i(E')$ for all i.

(2) For a morphism of varieties $f : X \to Y$ and symplectic bundle E over Y we have $f^A b_i(E) = b_i(f^*E)$ for all i.

(3) For every variety X the homomorphism

$$A^{*,*}(X) \oplus A^{*-4,*-2}(X) \to A^{*,*}(\mathrm{HP}^1 \times X)$$

given by $a + a' \mapsto p^A(a) + p^A(a') \cup b_1(\mathcal{H}(1))$ is an isomorphism. Here $p : \mathrm{HP}^1 \times X \to X$ is the canonical projection.

(4) $b_1(\mathrm{H}_-) = 0 \in A^{4,2}(\mathrm{pt})$.

(5) For E of rank $2r$ we have $b_i(E) = 0$ for $i > r$.

(6) For symplectic bundles E, E' over X we have $b_t(E)b_t(E') = b_t(E \oplus E')$, where

$$b_t(E) = 1 + b_1(E)t + b_2(E)t^2 + \cdots \in A^{*,*}(X)[t].$$

We refer to $b_i(E)$ as *Borel classes* of E and $b_t(E)$ is the *total Borel class*.

Remark 4.7. In [Panin and Walter 2010c] the above classes were called *Pontryagin classes*, but as I learned from I. Panin, it was noted by V. Buchstaber that these classes act much more like symplectic Borel classes than Pontryagin classes in topology, so we prefer to adopt this new notation. See also [Ananyevskiy 2015, Definition 7].

Theorem 4.8 [Panin and Walter 2010c, Theorem 14.4]. *Let A be a commutative ring spectrum. Then there is a canonical bijection between the set of symplectic orientations of A and the set of Borel class theories on A.*

Proof. We give a sketch of the definition of a Borel class theory on a symplectically oriented spectrum. First one defines $b_1(E) = z^A\,\mathrm{th}(E)$ for a rank 2 symplectic bundle E over a smooth variety X and morphism $z : X \to E/(E - X)$ induced by the zero section. Then the higher Borel classes are introduced using Theorem 4.9 below. In particular, we have $b_r(E) = z^A\,\mathrm{th}(E)$ for a rank $2r$ symplectic bundle E. See [Panin and Walter 2010c] for the details, but note that we omit the minus sign in front of $b_1(E)$. □

Theorem 4.9 [Panin and Walter 2010c, Theorem 8.2]. *Let A be a symplectically oriented spectrum and E a rank $2r$ symplectic bundle over a smooth variety X. Denote by $\mathrm{HP}(E)$ the relative quaternionic projective space associated to E and put $\xi = b_1(\mathcal{H}(1)) \in A^{4,2}(\mathrm{HP}(E))$. Then the homomorphism of left $A^{*,*}(X)$-modules*

$$\bigoplus_{i=0}^{r-1} A^{*-4i,*-2i}(X) \to A^{*,*}(\mathrm{HP}(E))$$

given by $\sum_{i=0}^{r-1} a_i \mapsto \sum_{i=0}^{r-1} a_i \cup \xi^i$ is an isomorphism.

Corollary 4.10. *Let A be a symplectically oriented spectrum and let E be a rank $2r$ symplectic bundle over a smooth variety X. Denote by $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_s$ the tautological rank 2 symplectic bundles over $\mathrm{HFlag}(2^s, E)$ and put $\xi_i = b_1(\mathcal{U}_i)$. Then the homomorphism of left $A^{*,*}(X)$-modules*

$$\bigoplus_{\substack{0\le n_i\le(r-i)\\ i=1\cdots s}} A^{*-4(n_1+\cdots+n_s),*-2(n_1+\cdots+n_s)}(X) \to A^{*,*}(\mathrm{HFlag}(2^s, E))$$

given by

$$\sum_{\substack{0\le n_i\le(r-i)\\ i=1\cdots s}} a_{n_1n_2\cdots n_s} \mapsto \sum_{\substack{0\le n_i\le(r-i)\\ i=1\cdots s}} a_{n_1n_2\cdots n_s} \cup \xi_1^{n_1}\xi_2^{n_2}\cdots\xi_s^{n_s}$$

is an isomorphism.

Proof. This follows from the theorem, since one can present $\mathrm{HFlag}(2^s, E)$ as an iterated quaternionic projective bundle

$$\mathrm{HFlag}(2^s, E) \to \mathrm{HFlag}(2^{s-1}, E) \to \cdots \to \mathrm{HFlag}(2, E) = \mathrm{HP}(E). \qquad \square$$

Theorem 4.11 [Panin and Walter 2010c, Theorem 10.2]. *Let A be a symplectically oriented spectrum and E a rank $2r$ symplectic bundle over a smooth variety X. Then there exists a canonical morphism of smooth varieties $f : Y \to X$ such that*

(1) $f^A : A^{*,*}(X) \to A^{*,*}(Y)$ *is injective,*

(2) $f^*E \cong E_1 \oplus E_2 \oplus \cdots \oplus E_r$ *for some canonically defined rank 2 symplectic bundles E_i. In particular,*

$$b_i(E) = \sigma_i(b_1(E_1), b_1(E_2), \ldots, b_1(E_r))$$

for the elementary symmetric polynomials σ_i.

Definition 4.12. Let E be a rank $2r$ symplectic bundle over a smooth variety X. In the notation of Theorem 4.11 we refer to $\{b_1(E_1), b_1(E_2), \ldots, b_1(E_r)\}$ as *Borel roots* of E and denote $\xi_i = \xi_i(E) = b_1(E_i)$. Write $s_n(E)$ for the power sums of Borel roots of E,

$$s_n(E) = \xi_1^n + \xi_2^n + \cdots + \xi_r^n \in A^{4n,2n}(X),$$

and let

$$s_t(E) = s_1(E)t + s_2(E)t^2 + \cdots \in A^{*,*}(X)[\![t]\!].$$

It follows from the standard relations between power sums and elementary symmetric polynomials that

$$s_t(E) = -t\frac{d}{dt}\ln b_{-t}(E).$$

Theorem 4.13 [Panin and Walter 2010c, Theorem 11.2]. *Let A be a symplectically oriented spectrum. Then the homomorphism of $A^{*,*}(\mathrm{pt})$-algebras*

$$A^{*,*}(\mathrm{pt})[b_1, b_2, \ldots, b_r]/(h_{n-r+1}, \ldots, h_n) \to A^{*,*}(\mathrm{HGr}(2r, 2n))$$

induced by $b_i \mapsto b_i(\mathcal{U}^s_{2r,2n})$ is an isomorphism. Here $h_j = h_j(b_1, b_2, \ldots, b_r)$ is the polynomial representing the j-th complete symmetric polynomial in r variables via elementary symmetric polynomials.

Definition 4.14. We have the following ind-objects considered as pointed motivic spaces:

- $\mathrm{HGr}(2r) = \varinjlim_n (\mathrm{HGr}(2r, 2n), *)$,
- $\mathrm{HGr} = \varinjlim_{r,n} (\mathrm{HGr}(2r, 2n), *)$,

where $* = \mathrm{HGr}(2, 2) \in \mathrm{HGr}(2r, 2n)$.

Definition 4.15. We have the following classes over the infinite Grassmannians:

- $b_i(\mathcal{U}^s_{2r}) \in A^{4i,2i}(\mathrm{HGr}(2r))$ satisfying $b_i(\mathcal{U}^s_{2r})|_{\mathrm{HGr}(2r,2n)} = b_i(\mathcal{U}^s_{2r,2n})$,
- $b_i(\tau^s) \in A^{4i,2i}(\mathrm{HGr})$ satisfying $b_i(\tau^s)|_{\mathrm{HGr}(2r,2n)} = b_i(\mathcal{U}^s_{2r,2n})$.

The next theorem yields that these elements are uniquely defined by the given restrictions.

Definition 4.16. Let R be a graded ring and let b_i be variables of degree $d_i \in \mathbb{N}$. We denote by $R[\![b_1, b_2, \dots]\!]_h$ the *ring of homogeneous power series*.

Theorem 4.17 [Panin and Walter 2010a, Theorem 9.1]. *Let A be a symplectically oriented spectrum. Then the following homomorphisms of $A^{*,*}(\mathrm{pt})$-algebras are isomorphisms*:

(1) $A^{*,*}(\mathrm{pt})[\![b_1, b_2, \dots, b_r]\!]_h \to A^{*,*}(\mathrm{HGr}(2r)_+)$, *induced by* $b_i \mapsto b_i(\mathcal{U}^s_{2r})$,

(2) $A^{*,*}(\mathrm{pt})[\![b_1, b_2, \dots]\!]_h \to A^{*,*}(\mathrm{HGr}_+)$, *induced by* $b_i \mapsto b_i(\tau^s)$.

5. Symplectically oriented homology theories

The results of this section are dual to the results of the previous one: we compute symplectically oriented homology of quaternionic Grassmannians and flag varieties. Throughout this section A denotes a symplectically oriented commutative ring spectrum in the sense of Definition 4.2.

Lemma 5.1. *Let E be a rank $2r$ symplectic bundle over a smooth variety X. Then*

$$\mathrm{th}(E) \cap - : A_{*,*}(E/(E - X)) \to A_{*-4r,*-2r}(X)$$

is an isomorphism.

Proof. Using a standard Mayer–Vietoris argument we may assume that E is a trivial bundle, i.e., $E = p^*\mathrm{H}_-^{\oplus r}$ for the projection $p : X \to \mathrm{pt}$. By Lemma 4.3 $\mathrm{th}(E) = a\Sigma^{2r}_{\mathrm{T}} 1_X$, and thus $\mathrm{th}(E) \cap -$ coincides up to an invertible scalar with the suspension isomorphism $\Sigma^{-2r}_{\mathrm{T}}$. □

Definition 5.2. Let $i : Y \to X$ be a codimension $2r$ closed embedding of smooth varieties. Suppose that the normal bundle N_i is equipped with a symplectic form. The *transfer map in homology* $i^{!A}$ is given by composition

$$i^{!A} : A_{*,*}(X) \xrightarrow{p_A} A_{*,*}(X/(X-Y)) \xrightarrow{d_A} A_{*,*}(N_i/(N_i-Y)) \xrightarrow{\operatorname{th}(N_i)\cap -} A_{*-4r,*-2r}(Y).$$

Here

- $X \xrightarrow{p} X/(X-Y)$ is the canonical quotient morphism,
- $d : X/(X-Y) \xrightarrow{\simeq} N_i/(N_i-Y)$ is the deformation to the normal bundle isomorphism [Morel and Voevodsky 1999, Theorem 3.2.23].

With this notation the localization sequence in homology could be rewritten as

$$\cdots \xrightarrow{\partial} A_{*,*}(X-Y) \xrightarrow{j_A} A_{*,*}(X) \xrightarrow{i^{!A}} A_{*-4r,*-2r}(Y) \xrightarrow{\partial} \cdots .$$

Lemma 5.3. *Let $i : Y \to X$ be a codimension $2r$ closed embedding of smooth varieties. Suppose that the normal bundle N_i is equipped with a symplectic form. Then the transfer map $i^{!A}$ is a homomorphism of $A^{*,*}(X)$-modules*, *i.e.*,

$$i^{!A}(a \cap x) = i^A(a) \cap i^{!A}(x)$$

for every $x \in A_{*,*}(X)$ and $a \in A^{*,*}(X)$.

Proof. The morphisms p_A and d_A are homomorphisms of $A^{*,*}(X)$-modules by Lemma 3.6, while cap product with the Thom class induces a homomorphism of $A^{*,*}(X)$-modules by Lemma 3.4. □

Lemma 5.4 (cf. [Panin and Walter 2010c, Proposition 7.6]). *Let E be a rank $2r$ symplectic bundle over a smooth variety X and let $s : X \to E$ be a section meeting the zero section $z : X \to E$ transversally in Y. Let $i : Y \to X$ be the closed embedding. Equip the normal bundle N_i with a symplectic form using the canonical isomorphism $i^*E \cong N_i$. Then for every $x \in A_{*,*}(X)$ we have*

$$i_A i^{!A}(x) = b_r(E) \cap x.$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
 & & A_{*,*}(N_i/(N_i-Y)) \arrow[d, "\operatorname{th}(N_i)\cap -"] \arrow[ddl, "j_A"'] \\
A_{*,*}(X) \arrow[r, "p_A"] \arrow[d, shift right, "s_A"'] \arrow[d, shift left, "z_A"] & A_{*,*}(X/(X-Y)) \arrow[ur, "d_A"] \arrow[d, "s_A"] & A_{*-4r,*-2r}(Y) \arrow[d, "i_A"] \\
A_{*,*}(E) \arrow[r, "q_A"] \arrow[u, bend right, "\pi_A"'] & A_{*,*}(E/(E-z(X))) \arrow[r, "\operatorname{th}(E)\cap -"] & A_{*-4r,*-2r}(X)
\end{tikzcd}$$

Here

- the morphisms p_A and q_A are induced by the quotient maps,
- d_A is induced by the deformation to the normal bundle isomorphism,
- π_A is induced by the canonical projection $\pi : E \to X$,
- j_A is induced by the isomorphism $i^*E \cong N_i$.

In the left side of the diagram, s_A and z_A are homomorphisms inverse to an isomorphism π_A, so $s_A = z_A$ and the left square commutes. The middle triangle commutes by the functoriality of the deformation to the normal bundle isomorphism. The right side commutes by the functoriality of Thom classes. Hence

$$i_A i^{!A}(x) = i_A(\mathrm{th}(N_i) \cap d_A p_A(x)) = \mathrm{th}(E) \cap (q_A z_A(x)).$$

By Lemma 3.6 we have

$$\mathrm{th}(E) \cap (q_A z_A(x)) = z^A q^A(\mathrm{th}(E)) \cap x = b_r(E) \cap x. \qquad \square$$

Theorem 5.5. *Let E be a symplectic bundle of rank $2r+2$ over a smooth variety X. Denote by $p : \mathrm{HP}(E) \to X$ the canonical projection and set $\xi = b_1(\mathcal{H}(1))$. Then the homomorphism of left $A^{*,*}(X)$-modules*

$$A_{*,*}(\mathrm{HP}(E)) \to \bigoplus_{n=0}^{r} A_{*-4n,*-2n}(X)$$

given by $x \mapsto p_A(x) + p_A(\xi \cap x) + \cdots + p_A(\xi^r \cap x)$ is an isomorphism.

Proof. A usual Mayer–Vietoris argument yields that it is sufficient to treat the case of a trivial symplectic bundle E, i.e., $\mathrm{HP}(E) = \mathrm{HP}^r \times X$. The proof does not depend on the base X, so we omit it from the notation.

By [Panin and Walter 2010c, Theorems 3.1, 3.2 and 3.4] there is a closed subvariety $Y \subset \mathrm{HP}^r$ satisfying

- Y is a transversal intersection of a section $s : \mathrm{HP}^r \to \mathcal{H}(1)$ and the zero section $z : \mathrm{HP}^r \to \mathcal{H}(1)$,
- $\mathrm{HP}^r - Y$ is $\mathbb{A}^1$-homotopy equivalent to a point,
- there is a morphism $\pi : Y \to \mathrm{HP}^{r-1}$ which is an $\mathbb{A}^2$-bundle such that $\pi^*\mathcal{H}(1) \cong i^*\mathcal{H}(1)$, where $i : Y \to \mathrm{HP}^r$ is the closed embedding.

Equip the normal bundle N_i with the symplectic form induced by the isomorphism $i^*\mathcal{H}(1) \cong N_i$. Identifying $A_{*,*}(\mathrm{HP}^r - Y) \cong A_{*,*}(\mathrm{pt})$ and $A_{*,*}(Y) \cong A_{*,*}(\mathrm{HP}^{r-1})$, we obtain a long exact sequence in homology

$$\cdots \xrightarrow{\partial} A_{*,*}(\mathrm{pt}) \xrightarrow{j_A} A_{*,*}(\mathrm{HP}^r) \xrightarrow{i^{!A}} A_{*-4,*-2}(\mathrm{HP}^{r-1}) \xrightarrow{\partial} \cdots.$$

Here j is the composition $\mathrm{pt} \cong \mathrm{HP}^r - Y \to \mathrm{HP}^r$. The projection $\mathrm{HP}^r \to \mathrm{pt}$ splits the first morphism, thus $i^{!A}$ is surjective. Denote by $q : \mathrm{HP}^{r-1} \to \mathrm{pt}$ the canonical projection and consider the following diagram:

$$\begin{tikzcd}
A_{*,*}(\mathrm{pt}) \arrow[r, "j_A"] \arrow[d, "="] & A_{*,*}(\mathrm{HP}^r) \arrow[r, "i^{!A}"] \arrow[d, "\sum_{n=0}^{r} p_A(\xi^n \cap -)"] & A_{*-4,*-2}(\mathrm{HP}^{r-1}) \arrow[d, "\sum_{n=0}^{r-1} q_A(\xi^n \cap -)"] \\
A_{*,*}(\mathrm{pt}) \arrow[r, "u"] & \bigoplus_{n=0}^{r} A_{*-4n,*-2n}(\mathrm{pt}) \arrow[r, "v"] & \bigoplus_{n=1}^{r-1} A_{*-4n,*-2n}(\mathrm{pt})
\end{tikzcd}$$

Here u is the injection on the zeroth summand and v is the projection forgetting about the zeroth summand. The left square commutes by Lemma 3.6:

$$\xi^n \cap j_A(x) = j_A(j^A(\xi^n) \cap x) = \begin{cases} j_A(x), & n = 0, \\ j_A(0 \cap a) = 0, & n > 0. \end{cases}$$

The right square commutes by Lemmas 3.6 and 5.4:

$$q_A(\xi^n \cap i^{!A} x) = p_A i_A(\xi^n \cap i^{!A} x) = p_A(\xi^n \cap i_A i^{!A} x) = p_A(\xi^{n+1} \cap x).$$

The claim follows by induction. □

Corollary 5.6. *Let E be a symplectic bundle of rank $2r$ over a smooth variety X and $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_s$ the tautological rank 2 symplectic bundles over $\mathrm{HFlag}(2^s, E)$. Set $\xi_i = b_1(\mathcal{U}_i)$ and let $p : \mathrm{HFlag}(2^s, E) \to X$ be the canonical projection. Then the homomorphism of $A^{*,*}(X)$-modules*

$$A_{*,*}(\mathrm{HFlag}(2^s, E)) \to \bigoplus_{\substack{0 \le n_i \le (r-i) \\ i=1 \cdots s}} A_{*-4(n_1+n_2+\cdots+n_s),*-2(n_1+n_2+\cdots+n_s)}(X)$$

given by

$$x \mapsto \sum_{\substack{0 \le n_i \le (r-i) \\ i=1 \cdots s}} p_A((\xi_1^{n_1} \xi_2^{n_2} \cdots \xi_s^{n_s}) \cap x)$$

is an isomorphism.

Proof. This follows from Theorem 5.5, since one can present $\mathrm{HFlag}(2^s, E)$ as an iterated quaternionic projective bundle

$$\mathrm{HFlag}(2^s, E) \to \mathrm{HFlag}(2^{s-1}, E) \to \cdots \to \mathrm{HFlag}(2, E) = \mathcal{HP}(E). \qquad \square$$

Theorem 5.7. *Let E be a symplectic bundle of rank $2r$ over a smooth variety X. Denote by $p : \mathrm{HFlag}(2^s, E) \to X$ and $q : \mathrm{HGr}(2s, E) \to X$ the canonical projections. Then the following duality homomorphisms, given by Definition 3.7, are isomorphisms*:

$$D_p : A_{*,*}(\mathrm{HFlag}(2^s, E)) \to \mathrm{Hom}_{A^{*,*}(X)}(A^{*,*}(\mathrm{HFlag}(2^s, E)), A_{*,*}(X)),$$

$$D_q : A_{*,*}(\mathrm{HGr}(2s, E)) \to \mathrm{Hom}_{A^{*,*}(X)}(A^{*,*}(\mathrm{HGr}(2s, E)), A_{*,*}(X)).$$

Proof. The first morphism is an isomorphism by Corollaries 4.10 and 5.6.

Denote by $p' : \mathrm{HFlag}(2^s, E) \to \mathrm{HGr}(2s, E)$ the canonical projection and abbreviate $\mathrm{HF} = \mathrm{HFlag}(2^s, E)$, $\mathrm{HG} = \mathrm{HGr}(2s, E)$. Recall that HF is a quaternionic flag bundle over HG. Thus,

$$D_{p'} : A_{*,*}(\mathrm{HF}) \to \mathrm{Hom}_{A^{*,*}(\mathrm{HG})}(A^{*,*}(\mathrm{HF}), A_{*,*}(\mathrm{HG}))$$

is an isomorphism by the above. Since $A^{*,*}(\mathrm{HF})$ is a free $A^{*,*}(\mathrm{HG})$-module by Corollary 4.10, it is sufficient to check that the composition

$$\begin{tikzcd}
A_{*,*}(\mathrm{HF}) \arrow[r, "D_{p'}", "\simeq"'] \arrow[dr, "(D_q)_* \circ D_{p'}"'] & \mathrm{Hom}_{A^{*,*}(\mathrm{HG})}(A^{*,*}(\mathrm{HF}), A_{*,*}(\mathrm{HG})) \arrow[d, "(D_q)_*"] \\
 & \mathrm{Hom}_{A^{*,*}(\mathrm{HG})}\big(A^{*,*}(\mathrm{HF}), \mathrm{Hom}_{A^{*,*}(X)}(A^{*,*}(\mathrm{HG}), A_{*,*}(X))\big)
\end{tikzcd}$$

is an isomorphism. The claim follows from the commutativity of the following diagram, which is straightforward.

$$\begin{tikzcd}
 & \mathrm{Hom}_{A^{*,*}(\mathrm{HG})}\big(A^{*,*}(\mathrm{HF}), \mathrm{Hom}_{A^{*,*}(X)}(A^{*,*}(\mathrm{HG}), A_{*,*}(X))\big) \\
A_{*,*}(\mathrm{HF}) \arrow[ur, "(D_q)_* \circ D_{p'}"] \arrow[dr, "\simeq", "D_p"'] & \mathrm{Hom}_{A^{*,*}(X)}(A^{*,*}(\mathrm{HF}) \otimes_{A^{*,*}(\mathrm{HG})} A^{*,*}(\mathrm{HG}), A_{*,*}(X)) \arrow[u, "\cong"'] \\
 & \mathrm{Hom}_{A^{*,*}(X)}(A^{*,*}(\mathrm{HF}), A_{*,*}(X)) \arrow[u, "\cong"']
\end{tikzcd}$$

□

Definition 5.8. The operation of orthogonal sum of symplectic bundles yields a morphism $\mathrm{HGr}_+ \wedge \mathrm{HGr}_+ \to \mathrm{HGr}_+$ endowing $A_{*,*}(\mathrm{HGr}_+)$ with a ring structure

$$A_{*,*}(\mathrm{HGr}_+) \times A_{*,*}(\mathrm{HGr}_+) \to A_{*,*}(\mathrm{HGr}_+).$$

Definition 5.9. For $n \geq 0$ denote by $\chi_n \in A_{4n,2n}(\mathrm{HP}^\infty_+)$ the unique collection of elements satisfying

$$\langle \xi^m, \chi_n \rangle = \begin{cases} 1, & m = n, \\ 0, & m \neq n, \end{cases}$$

for $\xi = b_1(\mathcal{H}(1))$. The existence and uniqueness of these elements is guaranteed by Theorem 5.7 (consider $s = 1$). Also, by the same theorem we know that $A_{*,*}(\mathrm{HP}^\infty_+)$ is a free $A^{*,*}(\mathrm{pt})$-module with a basis given by $\{1, \chi_1, \chi_2, \dots\}$. Abusing the notation, we denote by the same letters the elements $\chi_n = i_A(\chi_n) \in A_{4n,2n}(\mathrm{HGr}_+)$ for the canonical embedding $i : \mathrm{HP}^\infty_+ \to \mathrm{HGr}_+$.

Theorem 5.10. *Identify*

$$A^{*,*}(\mathrm{HGr}_+) \cong A^{*,*}(\mathrm{pt})[\![b_1, b_2, \dots]\!]_h \cong A^{*,*}(\mathrm{pt})[\![\xi_1, \xi_2, \dots]\!]_h^{S_\infty}$$

by Theorems 4.11 and 4.17 via $b_i(\tau^s) \leftrightarrow b_i \leftrightarrow \sigma_i(\xi_1, \xi_2, \dots)$. Given a partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0\}$ denote by $\xi(\lambda) \in A^{*,*}(\mathrm{pt})[\![\xi_1, \xi_2, \dots]\!]_h^{S_\infty}$ the sum of all the elements in the orbit of $\xi_1^{\lambda_1}\xi_2^{\lambda_2}\cdots\xi_k^{\lambda_k}$. Then

(1) $\langle \xi(\lambda), \chi_1^{l_1}\chi_2^{l_2}\cdots\chi_r^{l_r}\rangle = \begin{cases} 1, & l_j = \#\{\lambda_i = j\} \text{ for all } j \geq 1, \\ 0, & \text{otherwise}, \end{cases}$

(2) *the homomorphism of $A^{*,*}(\mathrm{pt})$-algebras*

$$A^{*,*}(\mathrm{pt})[x_1, x_2, \dots] \to A_{*,*}(\mathrm{HGr}_+)$$

induced by $x_i \mapsto \chi_i$ is an isomorphism.

Proof. Put $|l| = l_1 + l_2 + \cdots + l_r$ and consider the canonical embedding

$$i : (\underbrace{\mathrm{HP}^\infty \times \mathrm{HP}^\infty \times \cdots \times \mathrm{HP}^\infty}_{|l|})_+ \to \mathrm{HGr}_+$$

given by orthogonal sum. Identify

$$A^{*,*}((\mathrm{HP}^\infty \times \mathrm{HP}^\infty \times \cdots \times \mathrm{HP}^\infty)_+) = \bigoplus_{i_j \geq 0} A^{*,*}(\mathrm{pt})\xi^{i_1} \otimes \xi^{i_2} \otimes \cdots \otimes \xi^{i_l},$$

$$A_{*,*}((\mathrm{HP}^\infty \times \mathrm{HP}^\infty \times \cdots \times \mathrm{HP}^\infty)_+) = \bigoplus_{i_j \geq 0} A^{*,*}(\mathrm{pt})\chi_{i_1} \otimes \chi_{i_2} \otimes \cdots \otimes \chi_{i_l}.$$

Put

$$\chi^l = \chi_1^{l_1}\chi_2^{l_2}\cdots\chi_r^{l_r}, \quad \chi_\otimes^l = \underbrace{\chi_1 \otimes \dots \otimes \chi_1}_{l_1} \otimes \underbrace{\chi_2 \otimes \cdots \otimes \chi_2}_{l_2} \otimes \cdots \otimes \underbrace{\chi_r \otimes \cdots \otimes \chi_r}_{l_r},$$

and denote by $\xi_\otimes(\lambda)$ the sum of all the elements in the orbit of $\xi^{\lambda_1} \otimes \xi^{\lambda_2} \otimes \cdots \otimes \xi^{\lambda_l}$ under the action of S_l. Here $\lambda_j = 0$ for $j > k$.

We have $i_A(\chi_\otimes^l) = \chi^l$ and

$$i^A(\xi(\lambda)) = \begin{cases} 0, & k > |l|, \\ \xi_\otimes(\lambda), & k \leq |l|. \end{cases}$$

By Lemma 3.6 we have $\langle \xi(\lambda), \chi^l\rangle = \langle i^A(\xi(\lambda)), \chi_\otimes^l\rangle$.

If $k > |l|$ then $i^A(\xi(\lambda)) = 0$ and $\langle \xi(\lambda), \chi^l\rangle = 0$ by the above.

If $k \leq |l|$ then we have

$$\begin{aligned}\langle \xi(\lambda), \chi^l\rangle &= \langle \xi_\otimes(\lambda), \chi_\otimes^l\rangle \\ &= \sum_{\substack{(\lambda'_1,\dots,\lambda'_l) \\ =(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(l)}) \\ \text{for some } \sigma \in S_l}} \langle \xi^{\lambda'_1}, \chi_1\rangle \cdots \langle \xi^{\lambda'_{l_1}}, \chi_1\rangle\langle \xi^{\lambda'_{l_1+1}}, \chi_2\rangle \cdots \langle \xi^{\lambda'_{l_1+l_2}}, \chi_2\rangle \cdots \langle \xi^{\lambda'_l}, \chi_r\rangle.\end{aligned}$$

This expression equals 1 if $l_j = \#\{\lambda_i = j\}$ for every $j \geq 1$ and equals zero otherwise, so the first claim follows.

Lemma 2.17 together with Theorem 5.7 yield

$$\begin{aligned} A_{*,*}(\mathrm{HGr}_+) &= \varinjlim A_{*,*}(\mathrm{HGr}(2r, 2n)) \\ &= \varinjlim \mathrm{Hom}_{A^{*,*}(\mathrm{pt})}(A^{*,*}(\mathrm{HGr}(2r, 2n)), A_{*,*}(\mathrm{pt})). \end{aligned}$$

We have an explicit computation of $A^{*,*}(\mathrm{HGr}(2r, 2n))$ given by Theorem 4.13, so the second claim follows from the first one. □

6. Preliminaries on KO

In this section we gather the representability results for higher Grothendieck–Witt groups (also known as hermitian K-theory) and fix a symplectic orientation on it. Recall that the characteristic of the base field is assumed to be different from 2.

Definition 6.1. Let X be a smooth variety and $U \subset X$ an open subset. For $n, i \in \mathbb{Z}$ denote by $\mathrm{GW}_i^{[n]}(X, U)$ higher Grothendieck–Witt groups defined by Schlichting [2010b, Definition 8]; see also [Schlichting 2010a; 2017]. Recall that by [Schlichting 2017, Proposition 6.3] (cf. [Walter 2003, Theorem 2.4]) for $i < 0$ there is a canonical identification $\mathrm{GW}_i^{[n]}(X, U) \cong \mathrm{W}^{[n-i]}(X, U)$, where the latter groups are derived Witt groups defined by Balmer [1999]. Moreover, $\mathrm{GW}_0^{[0]}(X)$ and $\mathrm{GW}_0^{[2]}(X)$ coincide with the Grothendieck–Witt group of X introduced by Knebusch [1977] and its symplectic version respectively.

For an orthogonal (resp. symplectic) bundle E over a smooth variety X we denote by

- $\langle E\rangle \in \mathrm{GW}_0^{[0]}(X)$ (resp. $\langle E\rangle \in \mathrm{GW}_0^{[2]}(X)$) the corresponding element in the Grothendieck–Witt group,
- $[E] \in \mathrm{W}^{[0]}(X)$ (resp. $[E] \in \mathrm{W}^{[2]}(X)$) the corresponding element in the Witt group.

Definition 6.2. We need the following notation complementary to the one introduced in Definition 4.1 (see [Panin and Walter 2010b]).

- $\mathrm{H}_+ = \left(k^{\oplus 2}, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$ is the *standard hyperbolic plane*.
- $\mathrm{RGr}(2r, 2n) = \mathrm{O}_{2n} / (\mathrm{O}_{2r} \times \mathrm{O}_{2n-2r})$ is the *real Grassmannian*. Here the orthogonal groups are taken with respect to the hyperbolic quadratic form $x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n}$. Similarly to the quaternionic case, the real Grassmannian could be described as the open subscheme of $\mathrm{Gr}(2r, \mathrm{H}_+^{\oplus n})$ parametrizing subspaces on which the standard hyperbolic quadratic form is nondegenerate.
- $\mathcal{U}^o_{2r,2n}$ is the tautological rank $2r$ orthogonal vector bundle over $\mathrm{RGr}(2r, 2n)$.
- $\mathrm{RGr} = \varinjlim_{r,n}(\mathrm{RGr}(2r, 2n), *)$ is the *infinite real Grassmannian* considered as a pointed motivic space. Here $* = \mathrm{RGr}(2, 2) \in \mathrm{RGr}(2r, 2n)$.

Theorem 6.3 ([Schlichting and Tripathi 2015, Theorem 1.1]; see also [Panin and Walter 2010b, Theorem 8.2]). *Let X be a smooth variety and U an open subset of X. Denote by $\underline{\mathbb{Z}}$ the sheaf associated to the presheaf $\mathbb{Z}$. Then there are natural isomorphisms*

$$\mathrm{Hom}_{\mathcal{H}_\bullet(k)}(X/U, \underline{\mathbb{Z}} \times \mathrm{RGr}) \cong \mathrm{GW}_0^{[0]}(X, U),$$
$$\mathrm{Hom}_{\mathcal{H}_\bullet(k)}(X/U, \underline{\mathbb{Z}} \times \mathrm{HGr}) \cong \mathrm{GW}_0^{[2]}(X, U).$$

Under these isomorphisms the tautological morphisms

$$\mathrm{RGr}(2r, 2n) \to \{m\} \times \mathrm{RGr},$$
$$\mathrm{HGr}(2r, 2n) \to \{m\} \times \mathrm{HGr}$$

correspond to

$$\langle \mathcal{U}^o_{2r,2n}\rangle + (m-r)\langle \mathrm{H}_+\rangle \in \mathrm{GW}_0^{[0]}(\mathrm{RGr}(2r, 2n)),$$
$$\langle \mathcal{U}^s_{2r,2n}\rangle + (m-r)\langle \mathrm{H}_-\rangle \in \mathrm{GW}_0^{[2]}(\mathrm{HGr}(2r, 2n)),$$

respectively.

Remark 6.4. Let A be a symplectically oriented spectrum. Then this theorem via the Yoneda lemma allows us to interpret characteristic classes, i.e., elements of $A^{*,*}(\mathrm{HGr})$, as natural transformations $\mathrm{GW}_0^{[2]}(X) \to A^{*,*}(X)$.

Definition 6.5. Let Y be a pointed motivic space. Put

$$\mathrm{GW}_0^{[0]}(Y) = \mathrm{Hom}_{\mathcal{H}_\bullet(k)}(Y, \underline{\mathbb{Z}} \times \mathrm{RGr}),$$
$$\mathrm{GW}_0^{[2]}(Y) = \mathrm{Hom}_{\mathcal{H}_\bullet(k)}(Y, \underline{\mathbb{Z}} \times \mathrm{HGr}).$$

For a family of pointed smooth varieties $(X_1, x_1), (X_2, x_2), \dots, (X_m, x_m)$ and $n = 0$ or 2, we identify $\mathrm{GW}_0^{[n]}((X_1, x_1) \wedge (X_2, x_2) \wedge \cdots \wedge (X_m, x_m))$ with the subgroup of $\mathrm{GW}_0^{[n]}(X_1 \times X_2 \times \cdots \times X_m)$ consisting of all the elements α satisfying

$$\alpha|_{X_1\times\cdots\times X_{j-1}\times\{x_j\}\times X_{j+1}\times\cdots\times X_m} = 0$$

for all j.

Definition 6.6. Let $\tau^s \in \mathrm{GW}_0^{[2]}(\mathrm{HGr})$ and $\tau^o \in \mathrm{GW}_0^{[0]}(\mathrm{RGr})$ be the tautological elements over the infinite Grassmannians represented by identity morphisms $\mathrm{HGr} \to \{0\} \times \mathrm{HGr}$ and $\mathrm{RGr} \to \{0\} \times \mathrm{RGr}$ and satisfying

$$\tau^s|_{\mathrm{HGr}(2r,2n)} = \langle \mathcal{U}^s_{2r,2n}\rangle - r\langle \mathrm{H}_-\rangle, \qquad \tau^o|_{\mathrm{RGr}(2r,2n)} = \langle \mathcal{U}^o_{2r,2n}\rangle - r\langle \mathrm{H}_+\rangle.$$

Definition 6.7. The periodic $\mathcal{HP}^1$-spectrum KO is given by the spaces

$$\mathrm{KO} = (\mathrm{RGr}, \mathrm{HGr}, \mathrm{RGr}, \mathrm{HGr}, \dots)$$

and structure maps

$$\sigma^o_{\mathrm{KO}} : \mathcal{HP}^1 \wedge \mathrm{RGr} \to \mathrm{HGr}, \qquad \sigma^s_{\mathrm{KO}} : \mathcal{HP}^1 \wedge \mathrm{HGr} \to \mathrm{RGr}$$

satisfying

$$(\sigma^o_{\mathrm{KO}})^{\mathrm{GW}}(\tau^s)|_{\mathcal{HP}^1\wedge \mathrm{RGr}(2r,2n)} = (\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle)\boxtimes \tau^o|_{\mathrm{RGr}(2r,2n)},$$
$$(\sigma^s_{\mathrm{KO}})^{\mathrm{GW}}(\tau^o)|_{\mathcal{HP}^1\wedge \mathrm{HGr}(2r,2n)} = (\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle)\boxtimes \tau^s|_{\mathrm{HGr}(2r,2n)}.$$

Here $\boxtimes$ is induced by the external tensor product of vector bundles,

$$E_1\boxtimes E_2 = p_1^*E\otimes p_2^*E_2$$

for vector bundles E_1 over X_1 and E_2 over X_2 with projections $p_i : X_1\times X_2\to X_i$. Note that an (external) tensor product of two symplectic vector bundles has a canonical orthogonal structure, while an (external) tensor product of a symplectic and an orthogonal bundle is symplectic.

The above morphisms σ^o_{KO} and σ^s_{KO} exist as morphisms of pointed sheaves by [Panin and Walter 2010b, Proposition 12.4, Lemmas 12.5 and 12.6]. This defined spectrum is canonically isomorphic in $\mathcal{SH}(k)$ to the spectra $\mathbf{BO}^{\mathrm{geom}}$ and $\mathbf{BO}$ constructed in [Panin and Walter 2010b].

Theorem 6.8 [Panin and Walter 2010b, Theorems 1.3 and 1.5]. *The spectrum* KO *can be endowed with the structure of a commutative ring spectrum* $(\mathrm{KO}, m_{\mathrm{KO}}, u_{\mathrm{KO}})$. *Moreover, this commutative ring spectrum represents higher Grothendieck–Witt groups, i.e., for every smooth variety X and an open subset $U\subset X$ there exist canonical functorial isomorphisms*

$$\Theta : \mathrm{KO}^{i,j}(X/U)\xrightarrow{\cong}\mathrm{GW}^{[j]}_{2j-i}(X,U)$$

satisfying

(1) Θ *commutes with the connecting homomorphisms ∂ in localization sequences*,

(2) *the $\cup$-product on* $\mathrm{KO}^{*,*}(-)$ *induced by the monoid structure of* KO *agrees with the Gille–Nenashev right pairing* (see [Gille and Nenashev 2003, Theorem 2.9]) *lifted to* $\mathrm{GW}^{[*]}_0(-)$ (as in [Panin and Walter 2010b, §4]),

(3) $\Theta(1)=1$, $\Theta(\varepsilon)=\langle -1\rangle$.

Remark 6.9. In view of the above theorem we identify $\mathrm{KO}^{0,0}(X)\cong \mathrm{GW}^{[0]}_0(X)$ and $\mathrm{KO}^{4,2}(X)\cong\mathrm{GW}^{[2]}_0(X)$.

Theorem 6.10. *The rule which assigns to a rank 2 symplectic bundle E over a smooth variety X class $b^{\mathrm{KO}}_1(E)=\langle E\rangle-\langle \mathrm{H}_-\rangle\in \mathrm{KO}^{4,2}(X)$ can be uniquely extended to a Borel class theory and by Theorem 4.8 induces a symplectic orientation of* KO.

Proof. Existence of the Borel class theory follows from [Panin and Walter 2010b, Theorem 5.1], while uniqueness follows from [Panin and Walter 2010c, Theorem 14.4b] □

The next two lemmas follow immediately from the construction of Θ.

Lemma 6.11. *Let X be a smooth variety. Then the following diagram commutes*:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{H}_\bullet(k)}(X_+, \mathrm{HGr}) \arrow[r, "\Sigma^\infty_{\mathcal{HP}^1}"] \arrow[d, "i"] & \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathcal{HP}^1} X_+, \Sigma^\infty_{\mathcal{HP}^1}\mathrm{HGr}) \arrow[d, "\phi", "\cong"'] \\
\mathrm{Hom}_{\mathcal{H}_\bullet(k)}(X_+, \mathbb{Z} \times \mathrm{HGr}) \arrow[dd, "f"', "\cong"] & \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathcal{HP}^1} X_+, \mathrm{Tr}_1\mathrm{KO} \wedge \mathcal{HP}^1) \arrow[d, "j"] \\
& \mathrm{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_{\mathcal{HP}^1} X_+, \mathrm{KO} \wedge \mathcal{HP}^1) \arrow[d, "="] \\
\mathrm{GW}^{[2]}_0(X) & \mathrm{KO}^{4,2}(X) \arrow[l, "\Theta"', "\cong"]
\end{tikzcd}$$

Here

- i *is induced by the identity morphism* $\mathrm{HGr} \to \{0\} \times \mathrm{HGr}$,
- ϕ *is induced by the canonical isomorphisms*

$$\Sigma^\infty_{\mathcal{HP}^1}\mathrm{HGr} \xleftarrow{\simeq} \Sigma^\infty_{\mathcal{HP}^1}\mathrm{HGr}\{-1\} \wedge \mathcal{HP}^1 \xrightarrow{\simeq} \mathrm{Tr}_1\mathrm{KO} \wedge \mathcal{HP}^1,$$

- j *is induced by the canonical morphism* $\mathrm{Tr}_1\mathrm{KO} \to \mathrm{KO}$,
- f *and* Θ *are given by Theorems 6.3 and 6.8*, *respectively.*

Lemma 6.12. *The following diagram commutes*:

$$\begin{tikzcd}
\Sigma^\infty_{\mathcal{HP}^1}\mathcal{HP}^1\{-1\} \arrow[r, "\phi", "\simeq"'] \arrow[dr, "u'_{\mathrm{KO}}"'] & \mathbb{S} \arrow[d, "u_{\mathrm{KO}}"] \\
& \mathrm{KO}
\end{tikzcd}$$

Here

- u_{KO} *is the unit morphism*,
- ϕ *is an isomorphism which is identity starting from the first space*,
- $u'_{\mathrm{KO}} = (f_0, f_1, f_2, \ldots)$ *with* $f_n : (\mathcal{HP}^1)^{\wedge n} \to \mathrm{KO}_n$ *satisfying*

$$f^{\mathrm{GW}}_{2m-1}(\tau^s) = \underbrace{(\langle\mathcal{H}(1)\rangle - \langle\mathrm{H}_-\rangle) \boxtimes \cdots \boxtimes (\langle\mathcal{H}(1)\rangle - \langle\mathrm{H}_-\rangle)}_{2m-1},$$

$$f^{\mathrm{GW}}_{2m}(\tau^o) = \underbrace{(\langle\mathcal{H}(1)\rangle - \langle\mathrm{H}_-\rangle) \boxtimes \cdots \boxtimes (\langle\mathcal{H}(1)\rangle - \langle\mathrm{H}_-\rangle)}_{2m}$$

for $n \geq 1$.

Corollary 6.13. *Let $\mathcal{H}(1)$ be the tautological rank 2 symplectic bundle over* HP^1. *Then*

(1) $\Sigma_{\mathrm{HP}^1} 1 = b_1^{\mathrm{KO}}(\mathcal{H}(1)) \in \mathrm{KO}^{4,2}(\mathrm{HP}^1)$,

(2) $\Sigma_{\mathrm{HP}^1} 1 = \chi_1 \in \mathrm{KO}_{4,2}(\mathrm{HP}^1)$.

Proof. With our definition $b_1^{\mathrm{KO}}(\mathcal{H}(1)) = \langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle$, the first claim is straightforward from the above two lemmas. The second claim follows from the first one since $\langle \Sigma^1_{\mathrm{HP}^1} 1, \Sigma^1_{\mathrm{HP}^1} 1\rangle = 1$ for the Kronecker product. □

Definition 6.14. The cohomology theory $\mathrm{KO}^{*,*}(-)$ is $(8,4)$-periodic with the periodicity isomorphism induced by

$$\mathrm{KO} \wedge \mathrm{S}^{8,4} \cong \mathrm{KO} \wedge (\mathcal{HP}^1)^{\wedge 2} \xrightarrow{\simeq} \mathrm{KO}\{2\} \cong \mathrm{KO}.$$

Here the first isomorphism is given by Corollary 2.6, the second isomorphism is the canonical one identifying double $\mathcal{HP}^1$-suspension with shift by 2 and the third isomorphism is given by the identity map.

One may identify these periodicity isomorphisms with

$$\mathrm{KO} \wedge \mathrm{S}^{8,4} \xrightarrow{-\cup \Sigma^{8,4}\beta} \mathrm{KO},$$

where $\beta \in \mathrm{KO}^{-8,-4}(\mathrm{pt})$ is the element corresponding to $1 \in \mathrm{KO}^{0,0}(\mathrm{pt})$ under the categorical periodicity isomorphism

$$\mathrm{KO}^{0,0}(\mathrm{pt}) \cong \mathrm{GW}_0^{[0]}(\mathrm{pt}) \cong \mathrm{GW}_0^{[-4]}(\mathrm{pt}) \cong \mathrm{KO}^{-8,-4}(\mathrm{pt}),$$

i.e., β is the unique element satisfying

$$\Sigma^2_{\mathrm{HP}^1}\beta = (\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle) \boxtimes (\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle) \in \mathrm{KO}^{0,0}(\mathrm{HP}^1 \wedge \mathrm{HP}^1).$$

We refer to β as the *Bott element*.

Remark 6.15. For a spectrum K representing algebraic K-theory there exists a morphism $\mathrm{KO} \xrightarrow{F} \mathrm{K}$ that induces forgetful maps

$$F : \mathrm{GW}_0^{[0]}(X) \cong \mathrm{KO}^{0,0}(X) \to \mathrm{K}^{0,0}(X) \cong \mathrm{K}_0(X).$$

Recall that K is $(2,1)$-periodic with the periodicity realized by cup product with the element $\beta_{\mathrm{K}} \in \mathrm{K}^{-2,-1}(\mathrm{pt})$ satisfying

$$\Sigma_{\mathbb{P}^1}\beta_{\mathrm{K}} = [\mathcal{O}(-1)] - 1 \in \mathrm{K}^{0,0}(\mathbb{P}^1, \infty).$$

One can show that $F(\beta) = \beta_{\mathrm{K}}^4$.

Remark 6.16. Let E_1, E_2 be symplectic bundles over a smooth variety X. Then

$$\beta \cup \langle E_1\rangle \cup \langle E_2\rangle = \langle E_1 \otimes E_2\rangle.$$

Here, on the left side we consider E_1, E_2 as elements of $\mathrm{KO}^{4,2}(X)$ and on the right side we consider them as symplectic bundles, so $E_1 \otimes E_2$ is an orthogonal bundle which we treat as an element of $\mathrm{KO}^{0,0}(X)$.

7. Hopf element and KW

In this section we recall the definition of the Hopf element and identify $\mathrm{KO}[\eta^{-1}]$ as a spectrum representing derived Witt groups.

Definition 7.1. The *Hopf map* is the projection

$$H : \mathbb{A}^2 - \{0\} \to \mathbb{P}^1$$

given by $H(x, y) = [x, y]$. Pointing $\mathbb{A}^2 - \{0\}$ by $(1, 1)$ and $\mathbb{P}^1$ by $[1 : 1]$ and taking the suspension spectra we obtain a morphism

$$\Sigma^\infty_{\mathcal{HP}^1} H \in \mathrm{Hom}_{\mathcal{SH}(k)}\big(\Sigma^\infty_{\mathcal{HP}^1}(\mathbb{A}^2 - \{0\}, (1, 1)), \Sigma^\infty_{\mathcal{HP}^1}(\mathbb{P}^1, [1 : 1])\big).$$

The *Hopf element* $\eta = \Sigma^{-3,-2}\Sigma^\infty_{\mathcal{HP}^1} H \in \mathbb{S}^{-1,-1}(\mathrm{pt})$ is the element corresponding to $\Sigma^\infty_{\mathcal{HP}^1} H$ under the suspension isomorphism and canonical isomorphisms

$$(\mathbb{P}^1, [1 : 1]) \cong \mathrm{S}^{2,1}, \qquad (\mathbb{A}^2 - \{0\}, (1, 1)) \cong \mathrm{S}^{3,2}$$

given by [Morel and Voevodsky 1999, Lemma 3.2.15, Corollary 3.2.18 and Example 3.2.20].

Definition 7.2. Define

$$\mathbb{S}[\eta^{-1}] = \mathrm{hocolim}\big(\mathbb{S} \xrightarrow{\cup\eta} \mathbb{S} \wedge \mathrm{S}^{-1,-1} \xrightarrow{\cup\eta} \mathbb{S} \wedge \mathrm{S}^{-2,-2} \xrightarrow{\cup\eta} \cdots\big),$$
$$\mathrm{KW} = \mathrm{KO} \wedge \mathbb{S}[\eta^{-1}].$$

This spectrum inherits the structure of an (8, 4)-periodic symplectically oriented commutative ring spectrum from KO.

Remark 7.3. We clearly have

$$\mathrm{KW}^{*,*}(\mathrm{KW}) = \mathrm{KW}^{*,*}(\mathrm{KO}), \qquad \mathrm{KW}_{*,*}(\mathrm{KW}) = \mathrm{KW}_{*,*}(\mathrm{KO}).$$

It is well-known that the spectrum KW represents derived Witt groups defined by Balmer [1999] (see, for example, [Ananyevskiy 2016, Theorem 6.5]).

Theorem 7.4. *For every smooth variety X there exists an isomorphism functorial in X, $\Theta_{\mathrm{W}} : \mathrm{KW}^{i,j}(X) \xrightarrow{\simeq} \mathrm{W}^{[i-j]}(X)$, such that the square*

$$\begin{tikzcd}
\mathrm{KO}^{2n,n}(X) \arrow[r, "\Theta", "\simeq"'] \arrow[d] & \mathrm{GW}^{[n]}_0(X) \arrow[d] \\
\mathrm{KW}^{2n,n}(X) \arrow[r, "\Theta_{\mathrm{W}}", "\simeq"'] & \mathrm{W}^{[n]}(X)
\end{tikzcd}$$

commutes for all n. Here the left vertical morphism is the canonical one arising from localization and the right vertical morphism is given by killing metabolic elements.

Remark 7.5. With the above theorem in mind we identify $\mathrm{KW}^{0,0}(X)$ with $\mathrm{W}^{[0]}(X)$ and $\mathrm{KW}^{4,2}(X)$ with $\mathrm{W}^{[2]}(X)$. In particular, we have $b_1^{\mathrm{KW}}(E) = [E] \in \mathrm{KW}^{4,2}(X)$ for a rank 2 symplectic bundle E over X.

8. Borel classes of triple tensor products in KW

In this section, in Lemma 8.2 we compute characteristic classes of a triple tensor product of rank 2 symplectic bundles. This computation is a derived Witt analogue of the equality

$$c_1^{\mathrm{K}}(L_1 \otimes L_2) = c_1^{\mathrm{K}}(L_1) + c_1^{\mathrm{K}}(L_2) - c_1^{\mathrm{K}}(L_1)c_1^{\mathrm{K}}(L_2)$$

in K-theory, where L_i are line bundles and $c_1^{\mathrm{K}}(L_i) = 1 - [L_i^\vee]$ is the first Chern class in K-theory. As an intermediate step we show how to express Borel classes in derived Witt groups using external powers.

Lemma 8.1. *Let E be a symplectic bundle of rank* 8 *over a smooth variety X. Then*

$$b_1^{\mathrm{KW}}(E) = [E], \qquad \beta b_3^{\mathrm{KW}}(E) = [\Lambda^3 E] - 3[E],$$
$$\beta b_2^{\mathrm{KW}}(E) = [\Lambda^2 E] - 4, \qquad \beta^2 b_4^{\mathrm{KW}}(E) = [\Lambda^4 E] - 2[\Lambda^2 E] + 2.$$

Proof. Using Theorem 4.11 we may assume that $E = E_1 \oplus E_2 \oplus E_3 \oplus E_4$ for rank 2 symplectic bundles E_i. Then $\beta^{[n/2]} b_n^{\mathrm{KW}}(E) = \sigma_n(E_1, E_2, E_3, E_4)$.

Expanding

$$\Lambda^j(E_1 \oplus E_2 \oplus E_3 \oplus E_4) = \bigoplus_{i_1+i_2+i_3+i_4=j} \Lambda^{i_1} E_1 \otimes \Lambda^{i_2} E_2 \otimes \Lambda^{i_3} E_3 \otimes \Lambda^{i_4} E_4$$

and using the given trivializations $\Lambda^2 E_i = \mathbb{1}_X$, we obtain

$$\begin{aligned}
\Lambda^1 E &= \sigma_1(E_1, E_2, E_3, E_4), \\
\Lambda^2 E &= \sigma_2(E_1, E_2, E_3, E_4) + 4, \\
\Lambda^3 E &= \sigma_3(E_1, E_2, E_3, E_4) + 3\sigma_1(E_1, E_2, E_3, E_4), \\
\Lambda^4 E &= \sigma_4(E_1, E_2, E_3, E_4) + 2\sigma_2(E_1, E_2, E_3, E_4) + 6.
\end{aligned}$$

The claim follows. □

Lemma 8.2. *Let E_1, E_2 and E_3 be rank* 2 *symplectic bundles over a smooth variety X. Put $\xi_i = b_1^{\mathrm{KW}}(E_i) \in \mathrm{KW}^{4,2}(X)$ and denote by $\xi(n_1, n_2, n_3)$ the sum of all the monomials lying in the orbit of $\xi_1^{n_1}\xi_2^{n_2}\xi_3^{n_3}$ under the action of S_3. Then*

$$\begin{aligned}
b_1^{\mathrm{KW}}(E_1 \otimes E_2 \otimes E_3) &= \beta\xi(1,1,1),\\
b_2^{\mathrm{KW}}(E_1 \otimes E_2 \otimes E_3) &= \beta\xi(2,2,0) - 2\xi(2,0,0),\\
b_3^{\mathrm{KW}}(E_1 \otimes E_2 \otimes E_3) &= \beta\xi(3,1,1) - 8\xi(1,1,1),\\
b_4^{\mathrm{KW}}(E_1 \otimes E_2 \otimes E_3) &= \beta\xi(2,2,2) + \xi(4,0,0) - 2\xi(2,2,0).
\end{aligned}$$

Proof. Consider the representation ring

$$\begin{aligned}
\mathrm{Rep}(\mathrm{Sp}_2 \times \mathrm{Sp}_2 \times \mathrm{Sp}_2) &\cong \mathbb{Z}[\chi_1^{\pm 1}, \chi_2^{\pm 1}, \chi_3^{\pm 1}]^{\mathbb{Z}/2\times\mathbb{Z}/2\times\mathbb{Z}/2}\\
&= \mathbb{Z}[\chi_1 + \chi_1^{-1}, \chi_2 + \chi_2^{-1}, \chi_3 + \chi_3^{-1}]
\end{aligned}$$

with the action of the i-th copy of $\mathbb{Z}/2$ given by $\chi_i \leftrightarrow \chi_i^{-1}$. The exterior powers of representations give rise to the operations

$$\Lambda^m : \mathbb{Z}[\chi_1+\chi_1^{-1}, \chi_2+\chi_2^{-1}, \chi_3+\chi_3^{-1}] \to \mathbb{Z}[\chi_1+\chi_1^{-1}, \chi_2+\chi_2^{-1}, \chi_3+\chi_3^{-1}], \quad m \in \mathbb{N}_0,$$

which are compatible with the operations

$$\Lambda^m : \mathbb{Z}[\chi_1^{\pm 1}, \chi_2^{\pm 1}, \chi_3^{\pm 1}] \to \mathbb{Z}[\chi_1^{\pm 1}, \chi_2^{\pm 1}, \chi_3^{\pm 1}], \quad m \in \mathbb{N}_0,$$

characterized by the following properties:

(1) $\Lambda^m(0) = 0$,

(2) $\Lambda^m(\chi_1^{i_1}\chi_2^{i_2}\chi_3^{i_3}) = \begin{cases} 1, & m = 0,\\ \chi_1^{i_1}\chi_2^{i_2}\chi_3^{i_3}, & m = 1,\\ 0, & \text{otherwise,}\end{cases}$

(3) $\Lambda^m(f+g) = \bigoplus_{m_1+m_2=m}(\Lambda^{m_1} f)(\Lambda^{m_2} g)$.

Set $e_i = \chi_i + \chi_i^{-1}$. A straightforward computation in $\mathbb{Z}[\chi_1^{\pm 1}, \chi_2^{\pm 1}, \chi_3^{\pm 1}]$ shows that

$$\begin{aligned}
\Lambda^1(e_1e_2e_3) &= e_1e_2e_3,\\
\Lambda^2(e_1e_2e_3) &= e_1^2e_2^2 + e_1^2e_3^2 + e_2^2e_3^2 - 2(e_1^2 + e_2^2 + e_3^2) + 4,\\
\Lambda^3(e_1e_2e_3) &= e_1^3e_2e_3 + e_1e_2^3e_3 + e_1e_2e_3^3 - 5e_1e_2e_3,\\
\Lambda^4(e_1e_2e_3) &= e_1^4 + e_2^4 + e_3^4 + e_1^2e_2^2e_3^2 - 4(e_1^2 + e_2^2 + e_3^2) + 6.
\end{aligned}$$

Thus

$$\begin{aligned}
\Lambda^1(E_1 \otimes E_2 \otimes E_3) &= E_1 \otimes E_2 \otimes E_3,\\
\Lambda^2(E_1 \otimes E_2 \otimes E_3) &= E_1^2 \otimes E_2^2 + E_1^2 \otimes E_3^2 + E_2^2 \otimes E_3^2 - 2(E_1^2 + E_2^2 + E_3^2) + 4,\\
\Lambda^3(E_1 \otimes E_2 \otimes E_3) &= E_1^3 \otimes E_2 \otimes E_3 + E_1 \otimes E_2^3 \otimes E_3\\
&\qquad + E_1 \otimes E_2 \otimes E_3^3 - 5E_1 \otimes E_2 \otimes E_3,\\
\Lambda^4(E_1 \otimes E_2 \otimes E_3) &= E_1^4 + E_2^4 + E_3^4 + E_1^2 \otimes E_2^2 \otimes E_3^2 - 4(E_1^2 + E_2^2 + E_3^2) + 6.
\end{aligned}$$

The claim of the lemma follows by Lemma 8.1. $\square$

9. Stable operations in $\mathrm{KW}_{\mathbb{Q}}$

In this section, we compute the algebra of stable operations in $\mathrm{KW}_{\mathbb{Q}}$, that is, $\mathrm{KW}_{\mathbb{Q}}^{*,*}(\mathrm{KW}_{\mathbb{Q}})$. The computation is straightforward and based on Lemma 2.17 combined with Theorem 4.17.

Lemma 9.1. *Let $B \in \mathrm{Hom}_{\mathcal{H}_\bullet(k)}(\mathrm{HP}^1 \wedge \mathrm{HP}^1 \wedge \mathrm{HGr}, \mathrm{HGr})$ be the morphism characterized by the property*

$$B^{\mathrm{GW}}(\tau^s) = \big(\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) \boxtimes \big(\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) \boxtimes \tau^s.$$

Then

$$B^{\mathrm{KW}}(s_i^{\mathrm{KW}}(\tau^s)) = [\mathcal{H}(1) \boxtimes \mathcal{H}(1)] \cup (a_i s_i^{\mathrm{KW}}(\tau^s) + c_i s_{i-2}^{\mathrm{KW}}(\tau^s))$$

for

$$a_{2j+1} = (2j+1)^2, \qquad c_{2j+1} = -\beta^{-1} 8j(2j+1), \qquad a_{2j} = c_{2j} = 0.$$

Proof. As noted in Remark 6.4, we may interpret s_i^{KW} as a natural transformation $\mathrm{GW}_0^{[2]} \to \mathrm{KW}^{4n,2n}$, whence

$$B^{\mathrm{KW}}(s_i^{\mathrm{KW}}(\tau^s)) = s_i^{\mathrm{KW}}(B^{\mathrm{GW}}(\tau^s)).$$

Thus we need to compute $s_i^{\mathrm{KW}}\big(\big(\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) \boxtimes \big(\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) \boxtimes \tau^s\big)$. The classes s_i^{KW} are additive and $s_i^{\mathrm{KW}}(\langle \mathrm{H}_-\rangle) = 0$, so it is sufficient to show that

$$\begin{aligned} &s_i^{\mathrm{KW}}\big(\big(\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) \boxtimes \big(\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) \boxtimes \big(\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big)\big) \\ &\qquad = [\mathcal{H}(1) \boxtimes \mathcal{H}(1)] \cup \big(a_i s_i^{\mathrm{KW}}(\langle \mathcal{H}(1)\rangle) + c_i s_{i-2}^{\mathrm{KW}}(\langle \mathcal{H}(1)\rangle)\big) \end{aligned}$$

for

$$\begin{aligned} &\big(\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) \boxtimes \big(\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) \boxtimes \big(\langle \mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) \\ &\qquad \in \mathrm{GW}_0^{[2]}(\mathrm{HP}^1 \wedge \mathrm{HP}^1 \wedge \mathrm{HP}^\infty). \end{aligned}$$

Define

$$\begin{gathered} x = b_1^{\mathrm{KW}}(\mathcal{H}(1) \boxtimes \mathbb{1} \boxtimes \mathbb{1}), \quad y = b_1^{\mathrm{KW}}(\mathbb{1} \boxtimes \mathcal{H}(1) \boxtimes \mathbb{1}), \quad \xi = b_1^{\mathrm{KW}}(\mathbb{1} \boxtimes \mathbb{1} \boxtimes \mathcal{H}(1)), \\ b_t(x, y, \xi) = b_t^{\mathrm{KW}}(\mathcal{H}(1) \boxtimes \mathcal{H}(1) \boxtimes \mathcal{H}(1)), \\ s_t(x, y, \xi) = s_t^{\mathrm{KW}}(\mathcal{H}(1) \boxtimes \mathcal{H}(1) \boxtimes \mathcal{H}(1)). \end{gathered}$$

In this notation the claim is equivalent to

$$\begin{aligned} &s_t(x, y, \xi) - s_t(0, y, \xi) - s_t(x, 0, \xi) - s_t(x, y, 0) \\ &\quad + s_t(0, 0, \xi) + s_t(0, y, 0) + s_t(x, 0, 0) - s_t(0, 0, 0) = \beta xy \sum_{i \geq 1} (a_i \xi^i + c_i \xi^{i-2}) t^i. \end{aligned}$$

The main summand on the left side is $s_t(x, y, \xi)$ and the other summands just cancel from $s_t(x, y, \xi)$ all the terms that do not contain $xy\xi$. Since $x^2 = y^2 = 0$, Lemma 8.2 yields

$$b_t(x,y,\xi) = 1 + \beta xy\xi t - 2\xi^2t^2 + (\beta xy\xi^3 - 8xy\xi)t^3 + \xi^4t^4.$$

Thus

$$\begin{aligned} s_t(x,y,\xi) &= -t\frac{d}{dt}\ln b_{-t}(x,y,\xi) \\ &= -t\,\frac{\frac{d}{dt}\big((1-\xi^2t^2)^2 - xy\xi(\beta t + (\beta\xi^2-8)t^3)\big)}{(1-\xi^2t^2)^2 - xy\xi(\beta t + (\beta\xi^2-8)t^3)}. \end{aligned}$$

Put

$$A(\xi,t) = (1-\xi^2t^2)^2, \qquad B(\xi,t) = \xi(\beta t + (\beta\xi^2-8)t^3).$$

Recall that $x^2 = y^2 = 0$, whence $(xy)^2 = 0$ and

$$\begin{aligned} s_t(x,y,\xi) &= -t\,\frac{\frac{d}{dt}(A(\xi,t) - xyB(\xi,t))}{A(\xi,t) - xyB(\xi,t)} \\ &= -t\,\frac{\big(\frac{d}{dt}(A(\xi,t) - xyB(\xi,t))\big)(A(\xi,t) + xyB(\xi,t))}{A(\xi,t)^2}. \end{aligned}$$

Expanding the numerator, applying $x^2y^2 = 0$ and omitting all the terms that do not contain $xy\xi$ we obtain

$$\begin{aligned} s_t(x,y,\xi) &= -t\,\frac{\big(\frac{d}{dt}A(\xi,t)\big)xyB(\xi,t) - A(\xi,t)\frac{d}{dt}(xyB(\xi,t))}{A(\xi,t)^2} \\ &= xyt\frac{d}{dt}\left(\frac{B(\xi,t)}{A(\xi,t)}\right) = \beta xyt\frac{d}{dt}\left(\frac{\xi t + (\xi^3 - 8\beta^{-1}\xi)t^3}{(1-\xi^2t^2)^2}\right) \\ &= \beta xyt\frac{d}{dt}\left((\xi t + (\xi^3 - 8\beta^{-1}\xi)t^3)\left(\sum_{j\ge 0}(j+1)\xi^{2j}t^{2j}\right)\right) \\ &= \beta xyt\frac{d}{dt}\left(\sum_{j\ge 0}((2j+1)\xi^{2j+1} - 8\beta^{-1}j\xi^{2j-1})t^{2j+1}\right) \\ &= \beta xy\sum_{j\ge 0}((2j+1)^2\xi^{2j+1} - 8\beta^{-1}j(2j+1)\xi^{2j-1})t^{2j+1}. \qquad \square \end{aligned}$$

Lemma 9.2. *The following diagram commutes*:

$$\begin{array}{ccc} \mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{KO}) & \longrightarrow & \varprojlim \mathrm{KW}^{*+8n+4,*+4n+2}_{\mathbb{Q}}(\mathrm{HGr}) \\ \Big\downarrow R & & \Big\downarrow T \\ \mathrm{KW}^{*+8,*+4}_{\mathbb{Q}}(\mathrm{KO}) & \longrightarrow & \varprojlim \mathrm{KW}^{*+8(n+1)+4,*+4(n+1)+2}_{\mathbb{Q}}(\mathrm{HGr}) \end{array}$$

Here the horizontal homomorphisms are the canonical ones given by Lemma 2.17,

T is induced by the shift

$$\prod_{n\geq 0}\mathrm{KW_{\mathbb{Q}}}^{*+8n+4,*+4n+2}(\mathrm{HGr})\to\prod_{n\geq 0}\mathrm{KW_{\mathbb{Q}}}^{*+8(n+1)+4,*+4(n+1)+2}(\mathrm{HGr}),$$

$$(t_1,t_3,t_5,\dots)\mapsto(t_3,t_5,\dots)$$

and R is given by $R(\gamma)=(\Sigma^{8,4}\gamma)\circ(-\cup\beta^{-1})$.

Proof. Straightforward from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Tr}_{2n+1}\mathrm{KO} \arrow[rr, "="] \arrow[d, "i"] & & (\mathrm{Tr}_{2(n+1)+1}\mathrm{KO})\{2\} \arrow[d, "i"] \\
\mathrm{KO} \arrow[r, "-\cup\beta^{-1}"] & \mathrm{KO}\wedge \mathrm{S}^{8,4} \arrow[r, "\simeq"] & \mathrm{KO}\{2\}
\end{tikzcd}$$

□

Lemma 9.3. *Let $\gamma\in\mathrm{KW}^{0,0}_{\mathbb{Q}}(\mathrm{KO})$ be a stable operation such that*

$$\gamma\mapsto(\gamma_1,\gamma_3,\dots)\in\varprojlim\mathrm{KW}^{8n+4,4n+2}_{\mathbb{Q}}(\mathrm{HGr})$$

under the canonical morphism given by Lemma 2.17. Let X be a pointed motivic space and let

$$f=(f_0,f_1,f_2,\dots):\Sigma^{\infty}_{\mathcal{HP}^1}X\{-1\}\to\mathrm{KO}$$

be a morphism of spectra. Then

$$\gamma(f)=\Sigma^{-1}_{\mathcal{HP}^1}\gamma_1(f_1),$$

where $f_1\in\mathrm{Hom}_{\mathcal{H}_\bullet(k)}(X,\mathrm{HGr})$ is treated as an element of $\mathrm{GW}^{[2]}_0(X)$ and γ_1 is treated as an operation $\mathrm{GW}^{[2]}_0\to\mathrm{KW}^{4,2}_{\mathbb{Q}}$.

Proof. This follows from Lemma 6.11. □

Theorem 9.4. *The homomorphism of left $\mathrm{KW}^{0,0}_{\mathbb{Q}}(\mathrm{pt})\cong\mathrm{W}_{\mathbb{Q}}(k)$-modules*

$$\mathrm{Ev}:\mathrm{KW}^{0,0}_{\mathbb{Q}}(\mathrm{KW}_{\mathbb{Q}})\to\prod_{m\in\mathbb{Z}}\mathrm{W}_{\mathbb{Q}}(k)$$

given by

$$\mathrm{Ev}(\phi)=\big(\dots,\beta^2\phi(\beta^{-2}),\beta\phi(\beta^{-1}),\phi(1),\beta^{-1}\phi(\beta),\beta^{-2}\phi(\beta^2),\dots\big)$$

is an isomorphism of algebras. Here the product on the left is given by composition and the product on the right is the componentwise one.

Moreover, $\mathrm{KW}^{p,q}_{\mathbb{Q}}(\mathrm{KW}_{\mathbb{Q}})=0$ when $4\nmid p-q$ and the above isomorphism induces an isomorphism of left $\mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{pt})\cong\mathrm{W}_{\mathbb{Q}}(k)[\eta^{\pm1},\beta^{\pm1}]$-modules

$$\mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{KW}_{\mathbb{Q}})\cong\bigoplus_{r,s\in\mathbb{Z}}\beta^r\eta^s\prod_{m\in\mathbb{Z}}\mathrm{W}_{\mathbb{Q}}(k)$$

with $\deg\beta=(-8,-4)$, $\deg\eta=(-1,-1)$.

Proof. Having in mind the canonical identifications

$$\mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{KW}_{\mathbb{Q}}) = \mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{KO}_{\mathbb{Q}}) = \mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{KO}),$$

we focus on the computation of $\mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{KO})$.

Lemma 2.17 yields the short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{KW}^{*+8n+3,*+4n+2}_{\mathbb{Q}}(\mathrm{HGr}) \to \mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{KO}) \to \varprojlim \mathrm{KW}^{*+8n+4,*+4n+2}_{\mathbb{Q}}(\mathrm{HGr}) \to 0$$

with the limit taken with respect to the morphisms

$$\Sigma^{-2}_{\mathcal{HP}^1} \circ B^{\mathrm{KW}} : \mathrm{KW}^{*+8n+12,*+4n+6}_{\mathbb{Q}}(\mathrm{HGr}) \to \mathrm{KW}^{*+8n+4,*+4n+2}_{\mathbb{Q}}(\mathrm{HGr}),$$

where $B = \sigma^o_{\mathrm{KO}} \circ (\mathrm{id}_{\mathcal{HP}^1} \wedge \sigma^s_{\mathrm{KO}})$ is the same morphism as in Lemma 9.1 up to the canonical identification $\mathcal{HP}^1 \cong \mathrm{HP}^1$.

Consider the following diagram:

$$\begin{tikzcd}
\mathrm{KW}^{*+8n+12,*+4n+6}_{\mathbb{Q}}(\mathrm{HGr}) \arrow[r, "\pi"] \arrow[d, "B^{\mathrm{KW}}"] & \mathrm{IQ}^{*+8n+12,*+4n+6}_{\mathbb{Q}}(\mathrm{HGr}) \arrow[dd, "S_{\mathbb{Q}}"] \arrow[ddl, bend right, "S'_{\mathbb{Q}}"'] \\
\mathrm{KW}^{*+8n+12,*+4n+6}_{\mathbb{Q}}(\mathcal{HP}^1 \wedge \mathcal{HP}^1 \wedge \mathrm{HGr}) \arrow[d, "\Sigma^{-2}_{\mathcal{HP}^1}"] & \\
\mathrm{KW}^{*+8n+4,*+4n+2}_{\mathbb{Q}}(\mathrm{HGr}) \arrow[r, "\pi"] & \mathrm{IQ}^{*+8n+4,*+4n+2}_{\mathbb{Q}}(\mathrm{HGr})
\end{tikzcd}$$

Here

- $\mathrm{IQ}^{*,*}_{\mathbb{Q}}(\mathrm{HGr}) = \varprojlim_{m,n} \mathrm{IQ}(\mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{HGr}(2m, 2n), *))$ is the indecomposable quotient (i.e., the ring modulo the reducible elements) of $\mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{HGr})$. The Newton identities yield

  $$(-1)^{i+1} i\overline{b^{\mathrm{KW}}_i(\tau^s)} = \overline{s^{\mathrm{KW}}_i(\tau^s)}$$

  in the indecomposable quotient, and thus Theorem 4.17 allows us to identify

  $$\mathrm{IQ}^{*,*}_{\mathbb{Q}}(\mathrm{HGr}) = \Bigl(\prod_{i\ge 1} \mathrm{KW}^{*-4i,*-2i}_{\mathbb{Q}}(\mathrm{pt})\overline{b^{\mathrm{KW}}_i(\tau^s)}\Bigr)_h = \Bigl(\prod_{i\ge 1} \mathrm{KW}^{*-4i,*-2i}_{\mathbb{Q}}(\mathrm{pt})\bar{s}_i\Bigr)_h$$

  for $s_i = s^{\mathrm{KW}}_i(\tau^s)$.
- π is the canonical projection.
- $S'_{\mathbb{Q}}$ is given by $S'_{\mathbb{Q}}(\bar{s}_i) = \beta a_i s_i + c_i s_{i-2}$ with

  $$a_{2j} = c_{2j} = 0, \qquad a_{2j+1} = (2j+1)^2, \qquad c_{2j+1} = -8j(2j+1).$$
- $S_{\mathbb{Q}} = \pi \circ S'_{\mathbb{Q}}$.

The ring $\mathrm{KW}_{\mathbb{Q}}^{*,*}(\mathcal{HP}^1 \wedge \mathcal{HP}^1 \wedge \mathrm{HGr})$ has trivial multiplication by Theorem 4.13 (since $b_1^{\mathrm{KW}}(\mathcal{H}(1))^2 = 0$ on $\mathcal{HP}^1$). Thus B^{KW} factors through the indecomposable quotient and Lemma 9.1 yields commutativity of the diagram. It follows that the canonical homomorphisms

$$\pi : \varprojlim \mathrm{KW}_{\mathbb{Q}}^{*+8n+4,*+4n+2}(\mathrm{HGr}) \xrightarrow{\simeq} \varprojlim \mathrm{IQ}_{\mathbb{Q}}^{*+8n+4,*+4n+2}(\mathrm{HGr}),$$
$$\pi^1 : \varprojlim{}^1 \mathrm{KW}_{\mathbb{Q}}^{*+8n+3,*+4n+2}(\mathrm{HGr}) \xrightarrow{\simeq} \varprojlim{}^1 \mathrm{IQ}_{\mathbb{Q}}^{*+8n+3,*+4n+2}(\mathrm{HGr})$$

are isomorphisms.

The morphism

$$S_{\mathbb{Q}} : \Big(\prod_{i\geq 1} \mathrm{KW}_{\mathbb{Q}}^{*+8n-4i+12,*+4n-2i+6}(\mathrm{pt})\bar{s}_i\Big)_h \to \Big(\prod_{i\geq 1} \mathrm{KW}_{\mathbb{Q}}^{*+8n-4i+4,*+4n-2i+2}(\mathrm{pt})\bar{s}_i\Big)_h$$

is given by the matrix

$$\begin{pmatrix} \beta a_1 & 0 & c_3 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \beta a_3 & 0 & c_5 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \beta a_5 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where a_{2j+1} and c_{2j+1} are invertible. Clearly we have

$$\mathrm{Im}(S_{\mathbb{Q}} \circ S_{\mathbb{Q}}) = \mathrm{Im}(S_{\mathbb{Q}}) = \Big(\prod_{j\geq 0} \mathrm{KW}_{\mathbb{Q}}^{*+8(n-j),*+4(n-j)}(\mathrm{pt})\bar{s}_{2j+1}\Big)_h,$$

so the $\varprojlim^1$ term vanishes. For $4 \nmid p-q$ we have

$$\mathrm{KW}_{\mathbb{Q}}^{p+8(n-j),q+4(n-j)}(\mathrm{pt}) \cong \mathrm{W}_{\mathbb{Q}}^{[p-q+4(n-j)]}(k) = 0,$$

whence the limit is trivial and $\mathrm{KW}_{\mathbb{Q}}^{p,q}(\mathrm{KW}_{\mathbb{Q}}) = 0$. In view of the periodicities given by η and β, from now on we deal with $\mathrm{KW}_{\mathbb{Q}}^{0,0}(\mathrm{KW}_{\mathbb{Q}})$. Moreover, it is sufficient to show that the homomorphism Ev from the statement of the theorem is an isomorphism, since it clearly agrees with the products.

In order to compute the above limit for $S_{\mathbb{Q}}$ we may drop all the terms involving $\bar{s}_{2j}$ and consider

$$S_{\mathbb{Q}} : \prod_{j\geq 0} \mathrm{KW}_{\mathbb{Q}}^{8(n-j),4(n-j)}(\mathrm{pt})\bar{s}_{2j+1} \to \prod_{j\geq 0} \mathrm{KW}_{\mathbb{Q}}^{8(n-j),4(n-j)}(\mathrm{pt})\bar{s}_{2j+1}.$$

For every $j \geq 0$, choose

$$\bar{\rho}_{2j+1} = \sum_{l \geq j} \alpha_{2j+1,2l+1} \bar{s}_{2l+1} \in \prod_{j \geq 0} \mathrm{KW}_{\mathbb{Q}}^{-8j,-4j}(\mathrm{pt}) \bar{s}_{2j+1}$$

such that

(1) $S_{\mathbb{Q}}(\bar{\rho}_1) = 0$,

(2) $S_{\mathbb{Q}}(\bar{\rho}_{2j+1}) = \beta \bar{\rho}_{2j-1}$,

(3) $\alpha_{1,1} = 1$.

The kernel of $S_{\mathbb{Q}}$ is a free module of rank 1. Thus (1) and (3) uniquely determine $\bar{\rho}_1$. Item (2) together with the condition that the sum for $\bar{\rho}_{2j+1}$ does not contain $\bar{s}_1$ uniquely determines $\bar{\rho}_{2j+1}$. One can easily see that $\alpha_{2j+1,2j+1}$ is invertible for every j, whence

$$\prod_{j \geq 0} \mathrm{KW}_{\mathbb{Q}}^{8(n-j),4(n-j)}(\mathrm{pt}) \bar{s}_{2j+1} = \prod_{j \geq 0} \mathrm{KW}_{\mathbb{Q}}^{*+8n,*+4n}(\mathrm{pt}) \bar{\rho}_{2j+1}.$$

In the new basis consisting of the $\bar{\rho}_{2j+1}$, the morphism $S_{\mathbb{Q}}$ is just a shift multiplied by β. Thus we can easily compute the inverse limit, obtaining

$$\varprojlim \mathrm{KW}_{\mathbb{Q}}^{8n+4,4n+2}(\mathrm{HGr}) = \varprojlim \mathrm{IQ}_{\mathbb{Q}}^{8n+4,4n+2}(\mathrm{HGr}) = \prod_{m \in \mathbb{Z}} \mathrm{KW}_{\mathbb{Q}}^{0,0}(\mathrm{pt}) \rho_m^{st},$$

where $\deg \rho_m^{st} = (0,0)$ and the structure morphisms

$$\prod_{m \in \mathbb{Z}} \mathrm{KW}_{\mathbb{Q}}^{0,0}(\mathrm{pt}) \rho_m^{st} \to \mathrm{KW}_{\mathbb{Q}}^{8n+4,4n+2}(\mathrm{HGr})$$

are given by

$$\rho_m^{st} \mapsto \begin{cases} \beta^{-n} \rho_{2(m+n)+1}, & m+n \geq 0, \\ 0, & m+n < 0, \end{cases}$$

for $\rho_{2(m+n)+1} = \sum_{l \geq m+n} \alpha_{2(m+n)+1,2l+1} s_{2l+1} \in \mathrm{KW}_{\mathbb{Q}}^{4,2}(\mathrm{HGr})$.

In order to obtain the claim of the theorem it is sufficient to check that

$$\beta^{-n} \rho_m^{st}(\beta^n) = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

It follows from Lemma 9.2 that $\rho_m^{st}(\beta^n) = \beta^n \rho_{m-n}^{st}(1)$, so it is sufficient to check that

$$\rho_m^{st}(1) = \begin{cases} 1, & m = 0, \\ 0, & m \neq 0. \end{cases}$$

Lemma 9.3 yields

$$\rho_m^{st}(1) = \Sigma_{\mathcal{HP}^1}^{-1} \rho_{2m+1}\big(\langle \mathcal{H}(1) \rangle - \langle \mathrm{H}_- \rangle\big).$$

By the definition of ρ_{2m+1} we have

$$\rho_{2m+1}\big(\langle\mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) = \begin{cases} \sum_{l\geq m} \alpha_{2m+1,2l+1} s^{\mathrm{KW}}_{2l+1}\big(\langle\mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big), & m \geq 0, \\ 0, & m < 0. \end{cases}$$

All the higher characteristic classes of $\langle\mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle$ vanish while

$$s^{\mathrm{KW}}_1\big(\langle\mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) = [\mathcal{H}(1)].$$

Thus

$$\rho_{2m+1}\big(\langle\mathcal{H}(1)\rangle - \langle \mathrm{H}_-\rangle\big) = \begin{cases} [\mathcal{H}(1)] = \Sigma^1_{\mathcal{HP}^1} 1, & m = 0, \\ 0, & m \neq 0, \end{cases}$$

and the claim follows. □

Remark 9.5. One can restate Theorem 9.4 as follows. Let

$$\mathcal{B} = (\Sigma^{8m,4m}\beta^m)_{m\in\mathbb{Z}} : \bigoplus_{m\in\mathbb{Z}} \mathbb{S}\wedge \mathrm{S}^{8m,4m} \to \mathrm{KW}_{\mathbb{Q}}$$

be the morphism induced by $\Sigma^{8m,4m}\beta^m : \mathbb{S}\wedge \mathrm{S}^{8m,4m} \to \mathrm{KW}_{\mathbb{Q}}$. Then the pullback homomorphism

$$\mathcal{B}^{\mathrm{KW}_{\mathbb{Q}}} : \mathrm{KW}^{*,*}_{\mathbb{Q}}(\mathrm{KW}_{\mathbb{Q}}) \to \mathrm{KW}^{*,*}_{\mathbb{Q}}\Big(\bigoplus_{m\in\mathbb{Z}} \mathbb{S}\wedge \mathrm{S}^{8m,4m}\Big)$$

is an isomorphism.

10. Stable cooperations in $\mathrm{KW}_{\mathbb{Q}}$ and KW

In this section we compute the algebra of cooperations in $\mathrm{KW}_{\mathbb{Q}}$ and give an additive description of the cooperations in KW. The approach is dual to the one used in the proof of Theorem 9.4 and based on Lemma 2.17 and Theorem 5.10.

Lemma 10.1. *The following diagram commutes*:

$$\begin{tikzcd} \varinjlim(\mathrm{KW}_{\mathbb{Q}})_{*+8n+4,*+4n+2}(\mathrm{HGr}) \arrow[r, "\cong"] \arrow[d, "T"] & (\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KO}) \arrow[d, "-\star\beta_r"] \\ \varinjlim(\mathrm{KW}_{\mathbb{Q}})_{*+8(n+1)+4,*+4(n+1)+2}(\mathrm{HGr}) \arrow[r, "\cong"] & (\mathrm{KW}_{\mathbb{Q}})_{*+8,*+4}(\mathrm{KO}) \end{tikzcd}$$

Here the horizontal isomorphisms are the canonical ones given by Lemma 2.17, T *is induced by the shift*

$$\bigoplus_{n\geq 0}(\mathrm{KW}_{\mathbb{Q}})_{*+8n+4,*+4n+2}(\mathrm{HGr}) \to \bigoplus_{n\geq 0}(\mathrm{KW}_{\mathbb{Q}})_{*+8(n+1)+4,*+4(n+1)+2}(\mathrm{HGr}),$$
$$(t_1, t_3, t_5, \ldots) \mapsto (t_3, t_5, \ldots),$$

$\beta_r = u_{\mathrm{KW}_{\mathbb{Q}}} \wedge \Sigma^{8,4}\beta \in (\mathrm{KW}_{\mathbb{Q}})_{8,4}(\mathrm{KO})$ *and* $-\star\beta_r$ *is given by Definition 3.8.*

Proof. This follows from the commutativity of the diagram

$$\begin{tikzcd} \mathrm{Tr}_{2n+1}\mathrm{KO} \arrow[rr, "="] \arrow[d, "i"] & & (\mathrm{Tr}_{2(n+1)+1}\mathrm{KO})\{-2\} \arrow[d, "i"] \\ \mathrm{KO} \arrow[r, "-\cup\beta"] & \mathrm{KO}\wedge \mathrm{S}^{-8,-4} \arrow[r, "\simeq"] & \mathrm{KO}\{-2\} \end{tikzcd}$$

□

Theorem 10.2. *Let* $u_{\mathrm{KW}_{\mathbb{Q}}} : \mathbb{S} \to \mathrm{KW}_{\mathbb{Q}}$ *be the unit map. Then the homomorphism of* $\mathrm{W}_{\mathbb{Q}}(k)[\eta^{\pm 1}] \cong \bigoplus_{n\in\mathbb{Z}} \mathrm{KW}_{\mathbb{Q}}^{n,n}(\mathrm{pt})$*-algebras*

$$\mathrm{W}_{\mathbb{Q}}(k)[\eta^{\pm 1}][\beta_l^{\pm 1}, \beta_r^{\pm 1}] \to (\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KW}_{\mathbb{Q}})$$

given by

$$\beta_l \mapsto \Sigma^{8,4}\beta \wedge u_{\mathrm{KW}_{\mathbb{Q}}}, \qquad \beta_r \mapsto u_{\mathrm{KW}_{\mathbb{Q}}} \wedge \Sigma^{8,4}\beta$$

is an isomorphism. Here the product on the right is given by Definition 3.8.

Proof. Abusing the notation, put

$$\beta_l = \Sigma^{8,4}\beta \wedge u_{\mathrm{KW}_{\mathbb{Q}}}, \qquad \beta_r = u_{\mathrm{KW}_{\mathbb{Q}}} \wedge \Sigma^{8,4}\beta.$$

We need to show that

$$(\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KW}_{\mathbb{Q}}) = \bigoplus_{n,p,q\in\mathbb{Z}} \mathrm{KW}_{\mathbb{Q}}^{n,n}(\mathrm{pt})\beta_l^p \star \beta_r^q.$$

Identifying $(\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KW}_{\mathbb{Q}}) = (\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KO})$ and applying the reasoning dual to the one used in the proof of Theorem 9.4 we obtain that

$$(\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KW}_{\mathbb{Q}}) = \varinjlim (\mathrm{PE}_{\mathbb{Q}})_{*+8n+4,*+4n+2}(\mathrm{HGr}),$$

where

$$(\mathrm{PE}_{\mathbb{Q}})_{*,*}(\mathrm{HGr}) = \bigoplus_{i\geq 1} (\mathrm{KW}_{\mathbb{Q}})_{*-4i,*-2i}(\mathrm{pt})\bar{s}_i^{\vee}$$

is the subspace of $(\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{HGr})$ dual to $\mathrm{IQ}_{\mathbb{Q}}^{*,*}(\mathrm{HGr})$ (see Theorem 5.10). Here $\bar{s}_i^{\vee} \in \mathrm{PE}_{4i,2i}(\mathrm{HGr})$ satisfies $\langle \bar{s}_i, \bar{s}_i^{\vee}\rangle = 1$ and $\langle \bar{s}_l, \bar{s}_i^{\vee}\rangle = 0$ for $l \neq i$. The limit is taken with respect to the morphisms

$$S_{\mathbb{Q}}^{\vee} : \bigoplus_{i\geq 1} (\mathrm{KW}_{\mathbb{Q}})_{*+8n-4i+4,*+4n-2i+2}(\mathrm{pt})\bar{s}_i^{\vee} \to \bigoplus_{i\geq 1} (\mathrm{KW}_{\mathbb{Q}})_{*+8n-4i+12,*+4n-2i+6}(\mathrm{pt})\bar{s}_i^{\vee}$$

given by $S_{\mathbb{Q}}^{\vee}(\bar{s}_i^{\vee}) = \beta a_i \bar{s}_i^{\vee} + c_{i+2}\bar{s}_{i+2}^{\vee}$ for

$$a_{2j} = c_{2j} = 0, \qquad a_{2j+1} = (2j+1)^2, \qquad c_{2j+1} = -8j(2j+1)$$

just as in the proof of Theorem 9.4. The matrix of $S^{\vee}_{\mathbb{Q}}$ is

$$\begin{pmatrix} \beta a_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ c_3 & 0 & \beta a_3 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & c_5 & 0 & \beta a_5 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can drop all the terms involving $\bar{s}^{\vee}_{2j}$, obtaining

$$(\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KW}_{\mathbb{Q}}) = \varinjlim_n \bigoplus_{j\geq 0} (\mathrm{KW}_{\mathbb{Q}})_{*+8(n-j),*+4(n-j)}(\mathrm{pt})\bar{s}^{\vee}_{2j+1}.$$

For $4 \nmid p-q$ the group $(\mathrm{KW}_{\mathbb{Q}})_{p,q}(\mathrm{KW}_{\mathbb{Q}})$ vanishes, and in view of the periodicity realized by cap product with η and cap product with β (that coincides with multiplication by β_l; see Definition 3.8) from now on we deal with $(\mathrm{KW}_{\mathbb{Q}})_{0,0}(\mathrm{KW}_{\mathbb{Q}})$.

Let $\tau_1 = \bar{s}^{\vee}_1$ and $\tau_{2j+1} = \beta^{-1} S^{\vee}_{\mathbb{Q}}(\tau_{2j-1})$. One can easily check that

$$\bigoplus_{j\geq 0} (\mathrm{KW}_{\mathbb{Q}})_{8(n-j),4(n-j)}(\mathrm{pt})\bar{s}^{\vee}_{2j+1} = \bigoplus_{j\geq 0} (\mathrm{KW}_{\mathbb{Q}})_{8n,4n}(\mathrm{pt})\tau_{2j+1}.$$

In this basis $S^{\vee}_{\mathbb{Q}}$ is a shift composed with multiplication by β, so the limit is easily computed:

$$\varinjlim_n \bigoplus_{j\geq 0} (\mathrm{KW}_{\mathbb{Q}})_{8n,4n}(\mathrm{pt})\tau_{2j+1} = \bigoplus_{m\in\mathbb{Z}} (\mathrm{KW}_{\mathbb{Q}})_{0,0}(\mathrm{pt})\tau^{st}_m$$

with the structure morphisms

$$\bigoplus_{j\geq 0} (\mathrm{KW}_{\mathbb{Q}})_{8n,4n}(\mathrm{pt})\tau_{2j+1} \to \bigoplus_{m\in\mathbb{Z}} (\mathrm{KW}_{\mathbb{Q}})_{0,0}(\mathrm{pt})\tau^{st}_m$$

given by $\tau_{2j+1} \mapsto \beta^{-n}\tau^{st}_{j-n}$. Lemma 10.1 yields that

$$\tau^{st}_m = \beta_l^{-1} \star \tau^{st}_{m-1} \star \beta_r,$$

whence $\tau^{st}_m = \beta_l^{-m} \star \tau^{st}_0 \star \beta_r^m$ and

$$(\mathrm{KW}_{\mathbb{Q}})_{0,0}(\mathrm{KW}_{\mathbb{Q}}) = \bigoplus_{m\in\mathbb{Z}} (\mathrm{KW}_{\mathbb{Q}})_{0,0}(\mathrm{pt})\beta_l^{-m} \star \tau^{st}_0 \star \beta_r^m.$$

In order to check that $\tau^{st}_0 = u_{\mathrm{KW}_{\mathbb{Q}}} \wedge u_{\mathrm{KW}_{\mathbb{Q}}}$ (whence $\beta_l^{-m} \star \tau^{st}_0 \star \beta_r^m = \beta_l^{-m} \star \beta_r^m$) recall that $s^{\vee}_1 = \chi_1$ and consider the following diagram:

$$\begin{tikzcd}
 & \mathrm{KW}_{\mathbb{Q}} \wedge (\Sigma^{\infty}_{\mathcal{HP}^1} \mathcal{HP}^1\{-1\}) \arrow[r, "\mathrm{id}_{\mathrm{KW}_{\mathbb{Q}}} \wedge i"] \arrow[d, "\cong"] & \mathrm{KW}_{\mathbb{Q}} \wedge (\Sigma^{\infty}_{\mathcal{HP}^1} \mathrm{HGr}\{-1\}) \arrow[d, "\cong"] \\
\mathbb{S} \arrow[ur, "u_{\mathrm{KW}_{\mathbb{Q}}} \wedge \Sigma^{-1}_{\mathcal{HP}^1} \chi_1"] \arrow[r, "u_{\mathrm{KW}_{\mathbb{Q}}} \wedge \mathrm{id}_{\mathbb{S}}"] \arrow[dr, "\tau_0^{st}"'] & \mathrm{KW}_{\mathbb{Q}} \wedge \mathbb{S} \arrow[d, "\mathrm{id}_{\mathrm{KW}_{\mathbb{Q}}} \wedge u_{\mathrm{KO}}"] & \mathrm{KW}_{\mathbb{Q}} \wedge \mathrm{Tr}_1\mathrm{KO} \arrow[dl, "\mathrm{id}_{\mathrm{KW}_{\mathbb{Q}}} \wedge j"] \\
 & \mathrm{KW}_{\mathbb{Q}} \wedge \mathrm{KO} &
\end{tikzcd}$$

Here

- i is induced by the canonical embedding $\mathrm{HP}^1 \to \mathrm{HGr}$,
- j is the canonical morphism $\mathrm{Tr}_1\mathrm{KO} \to \mathrm{KO}$.

The right half of the diagram commutes by Lemma 6.12, the upper triangle commutes by Corollary 6.13 and the outer contour commutes by the definition of τ_0^{st}. Thus the lower triangle commutes as well and the claim follows. □

Remark 10.3. One can restate Theorem 10.2 as follows. Let

$$\mathcal{B} = (\Sigma^{8m,4m}\beta^m)_{m\in\mathbb{Z}} : \bigoplus_{m\in\mathbb{Z}} \mathbb{S} \wedge \mathrm{S}^{8m,4m} \to \mathrm{KW}_{\mathbb{Q}}$$

be the morphism given by $\Sigma^{8m,4m}\beta^m : \mathbb{S} \wedge \mathrm{S}^{8m,4m} \to \mathrm{KW}_{\mathbb{Q}}$. Then the induced homomorphism in homology

$$\mathcal{B}_{\mathrm{KW}_{\mathbb{Q}}} : (\mathrm{KW}_{\mathbb{Q}})_{*,*}\left(\bigoplus_{m\in\mathbb{Z}} \mathbb{S} \wedge \mathrm{S}^{8m,4m}\right) \to (\mathrm{KW}_{\mathbb{Q}})_{*,*}(\mathrm{KW}_{\mathbb{Q}})$$

is an isomorphism.

Now we turn to the description of integral cooperations.

Theorem 10.4. *Let* M *be the abelian subgroup of* $\mathbb{Q}[v, v^{-1}]$ *generated by polynomials*

$$f_{j,n} = \frac{v^{-n}\prod_{i=0}^{j-1}(v-(2i+1)^2)}{4^j(2j)!}, \quad j \geq 0,\ n \in \mathbb{Z}.$$

Then there are canonical isomorphisms of left $\mathrm{KW}_{0,0}(\mathrm{pt}) \cong \mathrm{W}(k)$*-modules*

$$\mathrm{KW}_{p,q}(\mathrm{KW}) \cong \begin{cases} \mathrm{W}(k) \otimes_{\mathbb{Z}} \mathrm{M}, & 4 \mid p-q, \\ 0, & \textit{otherwise}. \end{cases}$$

Rationally $\mathrm{W}_{\mathbb{Q}}(k) \otimes_{\mathbb{Z}} \mathrm{M} \cong (\mathrm{KW}_{\mathbb{Q}})_{r,r-4t}(\mathrm{KW})$ *is given by*

$$v^m \mapsto \eta^{r-8t}\beta_l^{t-m} \star \beta_r^m$$

in the notation of Theorem 10.2.

Proof. Applying the reasoning dual to the one used in the beginning of the proof of Theorem 9.4, we obtain that

$$\mathrm{KW}_{*,*}(\mathrm{KW}) = \varinjlim \bigoplus_{i\geq 1} \mathrm{KW}_{*+8n-4i+4,*+4n-2i+2}(\mathrm{pt})\bar{b}_i^{\vee}.$$

Here $\bar{b}_i^{\vee}$ belongs to the submodule of $\mathrm{KW}_{*,*}(\mathrm{HGr})$ dual to the indecomposable quotient $\mathrm{IQ}^{*,*}(\mathrm{HGr})$ and satisfies $\langle b_i, \bar{b}_i^{\vee}\rangle = 1$, $\langle b_l, \bar{b}_i^{\vee}\rangle = 0$ for $l \neq i$. The limit is computed along the morphisms $S^{\vee}$ dual to the corresponding morphisms S between indecomposable quotients. Recall that S is induced by a desuspension of an appropriate morphism $\mathcal{HP}^1 \wedge \mathcal{HP}^1 \wedge \mathrm{HGr} \to \mathrm{HGr}$.

It follows from Lemma 8.2 that $S(b_i)$ is a $\mathbb{Z}[\beta, \beta^{-1}]$-linear combination of products of Borel classes b_j (cf. Lemma 9.1), thus there exists a linear map

$$S_{\mathbb{Z}} : \prod_{i\geq 1} \mathbb{Z}[\beta, \beta^{-1}]\bar{b}_i \to \prod_{i\geq 1} \mathbb{Z}[\beta, \beta^{-1}]\bar{b}_i$$

inducing

$$S : \prod_{i\geq 1} \mathrm{KW}^{*+8n-4i+12,*+4n-2i+6}(\mathrm{pt})\bar{b}_i \to \prod_{i\geq 1} \mathrm{KW}^{*+8n-4i+4,*+4n-2i+2}(\mathrm{pt})\bar{b}_i.$$

Moreover, $S_{\mathbb{Z}}$ gives rise to the dual map

$$S_{\mathbb{Z}}^{\vee} : \bigoplus_{i\geq 1} \mathbb{Z}[\beta, \beta^{-1}]\bar{b}_i^{\vee} \to \bigoplus_{i\geq 1} \mathbb{Z}[\beta, \beta^{-1}]\bar{b}_i^{\vee}.$$

and

$$S^{\vee} : \bigoplus_{i\geq 1} \mathrm{KW}_{*+8n-4i+4,*+4n-2i+2}(\mathrm{pt})\bar{b}_i^{\vee} \to \bigoplus_{i\geq 1} \mathrm{KW}_{*+8n-4i+12,*+4n-2i+6}(\mathrm{pt})\bar{b}_i^{\vee}$$

is given by $S^{\vee} = \mathrm{id}_{\mathrm{KW}_{*,*}(\mathrm{pt})} \otimes_{\mathbb{Z}[\beta,\beta^{-1}]} S_{\mathbb{Z}}^{\vee}$.

The proof of Lemma 9.1 yields

$$S_{\mathbb{Z}}(\bar{s}_{2j}) = 0, \qquad S_{\mathbb{Z}}(\bar{s}_{2j+1}) = \beta(2j+1)^2\bar{s}_{2j+1} - 8j(2j+1)\bar{s}_{2j-1}.$$

From the Newton identities we have $\langle \bar{s}_i, \bar{b}_i^{\vee}\rangle = (-1)^{i+1}i$ and $\langle \bar{s}_l, \bar{b}_i^{\vee}\rangle = 0$ for $l \neq i$. Combining this with the above, we obtain

$$\langle \bar{s}_{2j}, S_{\mathbb{Z}}^{\vee}(\bar{b}_i^{\vee})\rangle = \langle S_{\mathbb{Z}}(\bar{s}_{2j}), \bar{b}_i^{\vee}\rangle = 0,$$

$$\langle \bar{s}_{2j+1}, S_{\mathbb{Z}}^{\vee}(\bar{b}_i^{\vee})\rangle = \langle S_{\mathbb{Z}}(\bar{s}_{2j+1}), \bar{b}_i^{\vee}\rangle = \begin{cases} \beta^{-1}(2j+1)^3, & i = 2j+1, \\ -8j(2j-1)(2j+1), & i = 2j-1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence $S_{\mathbb{Z}}^{\vee}(\bar{b}_{2j}^{\vee}) = 0$ and $S_{\mathbb{Z}}^{\vee}(\bar{b}_{2j+1}^{\vee}) = (2j+1)^2\beta\bar{b}_{2j+1}^{\vee} - 8(j+1)(2j+1)\bar{b}_{2j+3}^{\vee}$.

We may therefore drop all of the $\bar{b}_{2j}^\vee$, obtaining

$$\mathrm{KW}_{*,*}(\mathrm{KW}) = \varinjlim \bigoplus_{j\geq 0} \mathrm{KW}_{*+8(n-j),*+4(n-j)}(\mathrm{pt})\bar{b}_{2j+1}^\vee.$$

Specifying to the degree (p, q), $4 \nmid p-q$, we obtain $\mathrm{KW}_{p,q}(\mathrm{KW}) = 0$ since

$$\mathrm{KW}_{p+8(n-j),q+4(n-j)}(\mathrm{pt}) \cong \mathrm{W}^{[q-p-4(n-j)]}(k) = 0.$$

In view of the periodicities given by cap-product with η and β, from now on we deal with $\mathrm{KW}_{0,0}(\mathrm{KW})$.

We have

$$\mathrm{KW}_{0,0}(\mathrm{KW}) = \varinjlim_n \bigoplus_{j\geq 0} \mathrm{KW}_{8(n-j),4(n-j)}(\mathrm{pt})\bar{b}_{2j+1}^\vee = \varinjlim_n \bigoplus_{j\geq 0} \mathrm{W}(k)\beta^{n-j}\bar{b}_{2j+1}^\vee,$$

where the colimit is computed with respect to the morphism

$$S^\vee : \bigoplus_{j\geq 0} \mathrm{W}(k)\beta^{n-j}\bar{b}_{2j+1}^\vee \to \bigoplus_{j\geq 0} \mathrm{W}(k)\beta^{n+1-j}\bar{b}_{2j+1}^\vee$$

given by

$$S^\vee(\beta^{n-j}\bar{b}_{2j+1}^\vee) = (2j+1)^2\beta^{n+1-j}\bar{b}_{2j+1}^\vee - 8(j+1)(2j+1)\beta^{n-j}\bar{b}_{2j+3}^\vee.$$

Colimit commutes with tensor product, so

$$\mathrm{KW}_{0,0}(\mathrm{KW}) = \mathrm{W}(k) \otimes_{\mathbb{Z}} \left(\varinjlim_n \bigoplus_{j\geq 0} \mathbb{Z}\beta^{n-j}\bar{b}_{2j+1}^\vee \right)$$

with the morphisms

$$S_{\mathbb{Z}}^\vee : \bigoplus_{j\geq 0} \mathbb{Z}\beta^{n-j}\bar{b}_{2j+1}^\vee \to \bigoplus_{j\geq 0} \mathbb{Z}\beta^{n+1-j}\bar{b}_{2j+1}^\vee$$

in the bases $\{\beta^{n-j}\bar{b}_{2j+1}^\vee\}_{j\geq 0}$ and $\{\beta^{n+1-j}\bar{b}_{2j+1}^\vee\}_{j\geq 0}$ given by

$$\begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots \\ c_3' & a_3 & 0 & 0 & \cdots \\ 0 & c_5' & a_5 & 0 & \cdots \\ 0 & 0 & c_7' & a_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a_{2j+1} = (2j+1)^2$ and $c_{2j+1}' = -8j(2j-1)$.

The terms in the last colimit are torsion-free, so the canonical morphism

$$\varinjlim_n \bigoplus_{j\geq 0} \mathbb{Z}\beta^{n-j}\bar{b}_{2j+1}^\vee \to \varinjlim_n \bigoplus_{j\geq 0} \mathbb{Q}\beta^{n-j}\bar{b}_{2j+1}^\vee$$

is injective. One computes the right-hand side colimit as in the proof Theorem 10.2. Let

$$T_{\mathbb{Z}}^{\vee} = (\beta^{-1} \cap -) \circ S_{\mathbb{Z}}^{\vee} : \bigoplus_{j \geq 0} \mathbb{Z}\beta^{n-j}\bar{b}_{2j+1}^{\vee} \to \bigoplus_{j \geq 0} \mathbb{Z}\beta^{n-j}\bar{b}_{2j+1}^{\vee}$$

and choose a basis of $\bigoplus_{j \geq 0} \mathbb{Q}\beta^{n-j}\bar{b}_{2j+1}^{\vee}$ to be

$$\{\beta^n \bar{b}_1^{\vee}, T_{\mathbb{Q}}^{\vee}(\beta^n \bar{b}_1^{\vee}), (T_{\mathbb{Q}}^{\vee})^2(\beta^n \bar{b}_1^{\vee}), \dots\}.$$

In these bases $S_{\mathbb{Q}}^{\vee}$ is a shift, so

$$\varinjlim_n \bigoplus_{j \geq 0} \mathbb{Q}\beta^{n-j}\bar{b}_{2j+1}^{\vee} = \bigoplus_{m \in \mathbb{Z}} \mathbb{Q} \cdot [\beta_l^{-m} \star \beta_r^m]$$

with the canonical morphisms

$$\bigoplus_{j \geq 0} \mathbb{Q}\beta^{n-j}\bar{b}_{2j+1}^{\vee} \to \bigoplus_{m \in \mathbb{Z}} \mathbb{Q}\beta_l^{-m} \star \beta_r^m$$

given by $(T_{\mathbb{Q}}^{\vee})^m(\beta^n \bar{b}_1^{\vee}) \mapsto \beta_l^{-m} \star \beta_r^m$ (the notation is consistent with the one used in the proof of Theorem 10.2). The limit $\varinjlim_n \bigoplus_{j \geq 0} \mathbb{Z}\beta^{n-j}\bar{b}_{2j+1}^{\vee}$ is the union of the images for the canonical morphisms

$$\phi_n : \bigoplus_{j \geq 0} \mathbb{Z}\beta^{n-j}\bar{b}_{2j+1}^{\vee} \to \bigoplus_{m \in \mathbb{Z}} \mathbb{Q}\beta_l^{-m} \star \beta_r^m.$$

We claim that these morphisms are given by

$$\phi_n(\beta^{n-j}\bar{b}_{2j+1}^{\vee}) = \frac{(\beta_l^{-m} \star \beta_r^m) \prod_{i=0}^{j-1} (\beta_l^{-1} \star \beta_r - a_{2i+1})}{\prod_{i=1}^{j} c'_{2i+1}},$$

where the multiplication on the right-hand side is componentwise, i.e.,

$$(\beta_l^{-n} \star \beta_r^n)(\beta_l^{-m} \star \beta_r^m) = \beta_l^{-n-m} \star \beta_r^{n+m}.$$

Indeed, for $j = 0$ we have $\phi_n(\beta^n \bar{b}_1^{\vee}) = \beta_l^n \star \beta_r^{-n}$. The general case follows from the equalities

$$\begin{aligned} &\phi_{n+1}(a_{2j-1}\beta^{n+1-j}\bar{b}_{2j-1}^{\vee} + c'_{2j+1}\beta^{n-j}\bar{b}_{2j+1}^{\vee}) \\ &\qquad = \phi_{n+1}(S_{\mathbb{Z}}^{\vee}(\beta^{n-j+1}\bar{b}_{2j-1}^{\vee})) = \phi_n(\beta^{n-j+1}\bar{b}_{2j-1}^{\vee}). \end{aligned}$$

The claim of the theorem follows. □

Remark 10.5. It follows from the above theorem applied to $k = \mathbb{R}$ (or any other field satisfying $\mathrm{W}(k) = \mathbb{Z}$) that M is an algebra for the usual multiplication of polynomials, i.e., that products $f_{j_1,n_1} f_{j_2,n_2}$ can be expressed as linear combinations of the $f_{j,n}$. For example we have

$$f_{1,0}^2 = 9f_{1,-1} + 198f_{2,-1} + 720f_{3,-1}.$$

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ALEXEY ANANYEVSKIY: alseang@gmail.com
Institute for Advanced Study, Princeton, NJ, United States

and

Chebyshev Laboratory, St. Petersburg State University, St. Petersburg, Russia

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