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Hochschild homology, lax codescent, and duplicial structure

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We study the duplicial objects of Dwyer and Kan, which generalize the cyclic objects of Connes. We describe duplicial objects in terms of the decalage comonads, and we give a conceptual account of the construction of duplicial objects due to Böhm and Ştefan. This is done in terms of a 2-categorical generalization of Hochschild homology. We also study duplicial structure on nerves of categories, bicategories, and monoidal categories.

1. Introduction

The cyclic category Λ was introduced by Connes [1983] as part of his program to study noncommutative geometry. Cyclic objects, given by functors with domain Λ, have been studied by too many authors to list here, but many of these can be found in the reference list of the classic book [Loday 1992].

Various generalizations of cyclic structure have been considered; in particular the notion of *duplicial object* was studied in [Dwyer and Kan 1985]. These are given by functors with domain $\boldsymbol{K}^{\mathrm{op}}$, for a certain category $\boldsymbol{K}$ of which Λ is a quotient. Like cyclic objects, duplicial objects are simplicial objects equipped with extra structure. In both cases, the extra structure involves an endomorphism $t_n : X_n \to X_n$ of the object of n-simplices, for each n, subject to various conditions relating it to the simplicial structure. The difference between the two notions is that in the case of cyclic structure, the map t_n is an automorphism of order $n+1$, so that $t_n^{n+1} = 1$.

There is also an intermediate notion, in which the t_n are required to be invertible but the condition that $t_n^{n+1} = 1$ is dropped. This was called *paracyclic structure* in [Getzler and Jones 1993], and also studied in [Elmendorf 1993], where the indexing category was called the "linear category". Somewhat confusingly, the name paracyclic has also been used by some authors to refer to what is called duplicial by Dwyer and Kan.

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In this paper we provide a new perspective on duplicial structure, and analyze ways in which it arises. As explained, for example, in [Mac Lane 1971], a comonad on a category gives rise to simplicial structure on each object of that category, and this is the starting point for many homology theories. Just as simplicial structure can be used to define homology, cyclic (or duplicial or paracyclic) structure can be used to define cyclic homology. In a series of papers, Böhm and Ştefan [2008; 2009; 2012] looked at what further structure than a comonad is needed to equip the induced simplicial object with duplicial structure; the main extra ingredient turned out to be a second comonad with a distributive law [Beck 1969] between the two. They also showed that their machinery could be used to construct the cyclic homology of bialgebroids. This was further studied in the papers [Krähmer and Slevin 2016; Kowalzig et al. 2015] by the third of us, along with various coauthors.

In the case of comonads and simplicial structure, there is a universal nature to the construction, once again explained in [Mac Lane 1971], and also in Section 2 below. There is no analogue given in the analysis of Böhm–Ştefan, and our first goal is to provide one.

As well as the construction of simplicial structure from comonads, we also consider a second way that simplicial structure arises, namely as nerves of categories or other (possibly higher) categorical structures. Our second main goal is to analyze when the simplicial sets arising as nerves can be given duplicial structure.

The third main achievement of the paper actually arose as a byproduct of our investigations towards the first goal. It is a connection between duplicial structure, especially as arising via the Böhm–Ştefan construction, and Hochschild homology and cohomology. We present this first. We consider some very simple aspects of Hochschild homology and cohomology, only involving the zeroth homology and cohomology, and we generalize it to a 2-categorical context in a "lax" way. The resulting theory allows us to recapture the Böhm–Ştefan construction as a sort of cap product in a very special case.

We end this introduction by remarking briefly on the two roles of 2-categories in this paper. On the one hand, 2-categories appear at a fairly accessible point in the ever-expanding zoo of higher categorical structures: in what is now becoming common terminology they are the "(2,2)-categories", where an (m, n)-category has no nontrivial morphisms above dimension m, and no noninvertible morphisms above dimension n. This is relevant to the lax version of Hochschild theory we begin to develop here. On the other hand, 2-categories have a key organizational role. Collections of categories naturally form themselves into 2-categories, and higher dimensional categories can also often usefully be formed into 2-categories, as seen for example in Joyal's approach to quasicategory theory. It is this organizational role which is most important in the current paper, and lies behind our analysis of comonads, distributive laws, duplicial structure, and so on.

2. Simplicial structure, comonads, and decalage

In this section we recall some ideas related to simplicial structure, most of which are well-known, although the notation used varies. The one new result is Proposition 2.4, which reformulates the notion of duplicial structure in terms of decalage comonads.

2A. ***Simplicial structure arising from comonads.*** Let $\mathbb{M}$ be the strict monoidal category of finite ordinals and order-preserving maps, with tensor product given by ordinal sum and the empty ordinal serving as the unit. This is sometimes known as the "algebraists' Δ", and is denoted by Δ in [Mac Lane 1971] and Δ_+ in many other sources, such as [Verity 2008].

The full subcategory of $\mathbb{M}$ consisting of the nonempty finite ordinals is isomorphic to the usual Δ (the "topologists' Δ"). A contravariant functor defined on Δ is a simplicial object, while a contravariant functor defined on (the underlying category of) $\mathbb{M}$ is an augmented simplicial object.

$\mathbb{M}$ is the "universal monoidal category containing a monoid", in the sense that for any strict monoidal category $\mathcal{C}$, there is a bijection between monoids in $\mathcal{C}$ and strict monoidal functors from $\mathbb{M}$ to $\mathcal{C}$. (Similarly, if $\mathcal{C}$ is a general monoidal category then to give a monoid in $\mathcal{C}$ is equivalent, in a suitable sense, to giving a strong monoidal functor from $\mathbb{M}$ to $\mathcal{C}$.)

Dually, there is a bijection between comonoids in $\mathcal{C}$ and strict monoidal functors from $\mathbb{M}^{\mathrm{op}}$ to $\mathcal{C}$, and so any comonoid in $\mathcal{C}$ determines an augmented simplicial object in $\mathcal{C}$. In particular, we could take $\mathcal{C}$ to be the strict monoidal category $[X, X]$ of endofunctors of a category X, so that a comonoid in $\mathcal{C}$ is just a comonad on X. Then any comonad g on X determines a unique strict monoidal functor $\mathbb{M}^{\mathrm{op}} \to [X, X]$. We may now transpose this so as to obtain a functor $X \to [\mathbb{M}^{\mathrm{op}}, X]$ sending each object of X to an augmented simplicial object in X called its *bar resolution* with respect to g.

When, in the introduction, we referred to the "universal nature" of the construction of simplicial objects from comonads, it was precisely this analysis, using the universal property of $\mathbb{M}$, which we had in mind, and which we shall extend so as to explain the Böhm–Ştefan construction.

Remark 2.1. There is an automorphism of $\mathbb{M}$ which arises from the fact that the opposite of the ordinal

$$n = \{0 < \cdots < n-1\}$$

is isomorphic to n itself. The automorphism fixes the objects, and sends an order-preserving map $f : m \to n$ to f^{rev}, where $f^{\mathrm{rev}}(i) = m - 1 - f(n-1-i)$. This automorphism reverses the monoidal structure, in the sense that $n + n' = n' + n$ on objects, while for morphisms $f : m \to n$ and $f' : m' \to n'$ we have

$$(f + f')^{\mathrm{rev}} = (f')^{\mathrm{rev}} + f^{\mathrm{rev}}.$$

2B. ***The decalage comonads.*** The monoidal structure on $\mathbb{M}$ extends, via Day convolution [Day 1970], to a monoidal structure on the category $[\mathbb{M}^{\mathrm{op}}, \mathbf{Set}]$ of augmented simplicial sets. The resulting structure is nonsymmetric, but closed on both sides, so that there is both a left and a right internal hom.

Since the ordinal 1 is a monoid in $\mathbb{M}$, the representable $\mathbb{M}(-,1)$ is a monoid in $[\mathbb{M}^{\mathrm{op}}, \mathbf{Set}]$, and so the internal hom out of $\mathbb{M}(-,1)$ becomes a comonad; or rather, there are two such comonads depending on whether one uses the left or right internal hom. These are called the decalage comonads, and they both restrict to give comonads, also called decalage, on the category $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ of simplicial sets.

As well as this abstract description, there is also a straightforward explicit description, which we now give for the case of augmented simplicial sets.

Given an augmented simplicial set X as in the diagram

$$\cdots\; X_2 \;\begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xrightarrow{d_2} \end{smallmatrix}\; X_1 \;\begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \end{smallmatrix}\; X_0 \xrightarrow{d_0} X_{-1}$$

the right decalage $\mathrm{Dec}_{\mathrm{r}}(X)$ of X is the augmented simplicial set

$$\cdots\; X_3 \;\begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xrightarrow{d_2} \end{smallmatrix}\; X_2 \;\begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \end{smallmatrix}\; X_1 \xrightarrow{d_0} X_0$$

obtained by discarding X_{-1} and the last face and degeneracy map in each degree. There is a canonical map $\varepsilon : \mathrm{Dec}_{\mathrm{r}}(X) \to X$ defined using the discarded face maps, so that $\varepsilon_n : \mathrm{Dec}_{\mathrm{r}}(X)_n \to X_n$ is d_{n+1}; and a canonical map $\delta : \mathrm{Dec}_{\mathrm{r}}(X) \to \mathrm{Dec}_{\mathrm{r}}(\mathrm{Dec}_{\mathrm{r}}(X))$ defined via the discarded degeneracy maps, so that $\delta_n : \mathrm{Dec}_{\mathrm{r}}(X)_n \to \mathrm{Dec}_{\mathrm{r}}(\mathrm{Dec}_{\mathrm{r}}(X))_n$ is s_{n+1}. These maps δ and ε define the comultiplication and counit of the comonad.

Similarly, the left decalage $\mathrm{Dec}_{\mathrm{l}}(X)$ of X is the augmented simplicial set

$$\cdots\; X_3 \;\begin{smallmatrix} \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xrightarrow{d_2} \\ \xleftarrow{s_2} \\ \xrightarrow{d_3} \end{smallmatrix}\; X_2 \;\begin{smallmatrix} \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xrightarrow{d_2} \end{smallmatrix}\; X_1 \xrightarrow{d_1} X_0$$

obtained by discarding X_{-1} and the first face and degeneracy map in each degree.

We have described the decalage comonads for simplicial and augmented simplicial sets, but in much the same way, there are decalage comonads $\mathrm{Dec}_{\mathrm{r}}$ and $\mathrm{Dec}_{\mathrm{l}}$ on the categories $[\Delta^{\mathrm{op}}, P]$ and $[\mathbb{M}^{\mathrm{op}}, P]$ of simplicial and augmented simplicial objects in P for any category P, although in general there will no longer be a monoidal structure with respect to which decalage is given by an internal hom.

2C. ***Duplicial structure.*** Here we recall the definition of duplicial structure, and give a reformulation using the decalage comonads. As stated already in the introduction, a duplicial object in a category is a simplicial object X, equipped with a map $t_n : X_n \to X_n$ for each $n \geqslant 0$, subject to various conditions which we now state explicitly:

$$d_i t_{n+1} = \begin{cases} t_n d_{i-1} & \text{if } 1 \le i \le n+1, \\ d_{n+1} & \text{if } i = 0; \end{cases} \tag{2.2}$$

$$s_i t_n = \begin{cases} t_{n+1} s_{i-1} & \text{if } 1 \le i \le n, \\ t_{n+1}^2 s_n & \text{if } i = 0. \end{cases} \tag{2.3}$$

There is also a formulation of this structure which uses an "extra degeneracy map" $s_{-1} : X_n \to X_{n+1}$ in each degree instead of the t_n; this s_{-1} may be constructed as the composite $t_{n+1}s_n$. As in the introduction, X is called *paracyclic* if each t_n is invertible, and *cyclic* if additionally $t_n^{n+1} = 1$.

The indexing category for cyclic structure is Connes' cyclic category Λ, which is a sort of wreath product of Δ and the cyclic groups. This is explained for example in [Loday 1992, Chapter 6], where the more general notion of crossed simplicial group can also be found. This involves replacing the cyclic groups by some other family of groups indexed by the natural numbers, and equipped with suitable actions of Δ which allow the formation of the wreath product. The indexing category for paracyclic structure can be obtained in this way on taking all the groups to be $\mathbb{Z}$ [Loday 1992, Proposition 6.3.4(c)]. Using the presentation for duplicial structure given above, it is straightforward to modify this argument to see that the indexing category $\boldsymbol{K}$ for duplicial structure is once again a wreath product, but this time by a "crossed simplicial monoid", involving the monoid $\mathbb{N}$ in each degree.

Proposition 2.4. *Giving duplicial structure to a simplicial object X is equivalent to giving a simplicial map $t : \mathrm{Dec_r}\, X \to \mathrm{Dec_l}\, X$ making the following diagrams commute*:

$$\begin{tikzcd} \mathrm{Dec_r}\, X \arrow[r, "t"] \arrow[dr, "\varepsilon"'] & \mathrm{Dec_l}\, X \arrow[d, "\varepsilon"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Dec_r}\, X \arrow[r, "t"] \arrow[d, "\delta"'] & \mathrm{Dec_l}\, X \arrow[r, "\delta"] & \mathrm{Dec_l}^2\, X \\ \mathrm{Dec_r}^2\, X \arrow[r, "\mathrm{Dec_r}\, t"'] & \mathrm{Dec_r}\,\mathrm{Dec_l}\, X \arrow[r, equal] & \mathrm{Dec_l}\,\mathrm{Dec_r}\, X \arrow[u, "\mathrm{Dec_l}\, t"'] \end{tikzcd}$$

Proof. The data of a simplicial map $t : \mathrm{Dec_r}\, X \to \mathrm{Dec_l}\, X$ comprises a sequence of maps $t_n : X_n \to X_n$ for each $n > 0$ satisfying certain conditions. Compatibility of t with face maps gives the cases where $i > 0$ of (2.2), while those where $i = 0$ are the compatibility condition with ε. Likewise, compatibility of t with degeneracy maps yields the cases $i, n > 0$ of (2.3), while the cases where $n > 0$ but $i = 0$ are the compatibility condition with δ.

The one thing which remains is to see that a map $t_0 : X_0 \to X_0$ satisfying (2.3) for $n = 0$ can be uniquely recovered from the remaining data and axioms. In order to have $s_0 t_0 = t_1^2 s_0$, we must have $t_0 = d_0 s_0 t_0 = d_0 t_1^2 s_0 = d_1 t_1 s_0$. So we just need to check that, defining t_0 in this way, it satisfies the required relations; but this is indeed the case as the following calculations show:

$$(d_1t_1s_0)d_0 = d_1t_1d_0s_1 = d_1d_1t_2s_1 = d_1d_2t_2s_1 = d_1t_1d_1s_1 = d_1t_1 \quad \text{and}$$
$$s_0(d_1t_1s_0) = d_2s_0t_1s_0 = d_2t_2^2s_1s_0 = t_1d_1t_2s_1s_0 = t_1^2d_0s_1s_0 = t_1^2d_0s_0s_0 = t_1^2s_0. \qquad \square$$

2D. ***The Böhm–Ştefan construction.*** We now describe the construction in [Böhm and Ştefan 2008; 2009]. The original formulation involves monads and coduplicial structure, but we work dually with comonads so as to obtain duplicial structure. Let A and P be categories, and suppose that we have a comonad (g, δ, ε) on A and a functor $f : A \to P$. As explained in Section 2A, we obtain from g a functor $A \to [\mathbb{M}^{\mathrm{op}}, A]$ sending each object to its bar resolution with respect to g, and postcomposing with f yields a functor $f^g : A \to [\mathbb{M}^{\mathrm{op}}, P]$. Explicitly, f^g takes x in A to the augmented simplicial object $f^g(x)$ with $f^g(x)_n = fg^{n+1}x$ and with face and degeneracy maps:

$$d_i = fg^i\varepsilon g^{n-i}x : f^g(x)_n \to f^g(x)_{n-1} \quad \text{and}$$
$$s_j = fg^j\delta g^{n-j}x : f^g(x)_n \to f^g(x)_{n+1}.$$

The basic construction of [Böhm and Ştefan 2008] uses additional data to equip objects of the form $f^g(x)$ with duplicial structure. We suppose given another comonad h on A, and a *distributive law* [Beck 1969] $\lambda : gh \to hg$ — a natural transformation satisfying four axioms relating it to the comonad structures. We suppose moreover that the functor $f : A \to P$ is equipped with a natural transformation $\varphi : fh \to fg$ rendering commutative the diagrams

$$\begin{tikzcd} fh \arrow[rrr, "\varphi"] \arrow[d, "f\delta"'] & & & fg \arrow[d, "f\delta"] \\ fh^2 \arrow[r, "\varphi h"'] & fgh \arrow[r, "f\lambda"'] & fhg \arrow[r, "\varphi g"'] & fg^2 \end{tikzcd} \qquad \begin{tikzcd} fh \arrow[rr, "\varphi"] \arrow[dr, "f\varepsilon"'] & & fg. \arrow[dl, "f\varepsilon"] \\ & f & \end{tikzcd} \tag{2.5}$$

This was called *left* λ*-coalgebra* structure on f in [Kowalzig et al. 2015], and the totality $(A, P, g, h, f, \lambda, \varphi)$ of the structure considered so far was called an *admissible septuple* in [Böhm and Ştefan 2008]. Finally, we assume given an object $x \in A$ equipped with a map $\xi : gx \to hx$ rendering commutative the diagrams

$$\begin{tikzcd} gx \arrow[rrr, "\xi"] \arrow[d, "\delta x"'] & & & hx \arrow[d, "\delta x"] \\ g^2x \arrow[r, "g\xi"'] & ghx \arrow[r, "\lambda x"'] & hgx \arrow[r, "h\xi"'] & h^2x \end{tikzcd} \qquad \begin{tikzcd} gx \arrow[rr, "\xi"] \arrow[dr, "\varepsilon x"'] & & hx. \arrow[dl, "\varepsilon x"] \\ & x & \end{tikzcd} \tag{2.6}$$

This was called *right* λ*-coalgebra* structure in [Krähmer and Slevin 2016], and a "transposition map" in [Böhm and Ştefan 2008], though the notion itself goes back to [Burroni 1973]. Under these assumptions, it was shown in [Böhm and Ştefan

2008] that the simplicial object $f^g(x)$ admits a duplicial structure. The duplicial operator $t_n : f^g(x)_n \to f^g(x)_n$ is given by the composite

$$fg^{n+1}x \xrightarrow{fg^n\xi x} fg^nhx \xrightarrow{f\lambda^n x} fhg^nx \xrightarrow{\varphi g^n x} fg^{n+1}x,$$

where the natural transformation $\lambda^n : g^nh \to hg^n$ denotes the composite

$$g^nh \xrightarrow{g^{n-1}\lambda} g^{n-1}hg \xrightarrow{g^{n-2}\lambda g} g^{n-2}hg^2 \longrightarrow \cdots \longrightarrow ghg^{n-1} \xrightarrow{\lambda g^{n-1}} hg^n.$$

In [Böhm and Ştefan 2008], this construction was used to obtain, among other things, the cyclic cohomology and homology of bialgebroids.

There is an automorphism $\Phi : [\mathbb{M}^{\mathrm{op}}, P] \to [\mathbb{M}^{\mathrm{op}}, P]$, induced by the automorphism in Remark 2.1, that maps a simplicial object X to the simplicial object associated to X, obtained by reversing the order of all face and degeneracy maps. In [Kowalzig et al. 2015] it is explained that $\Phi f^h(x)$ is duplicial, and that there are two duplicial maps

$$f^g(x) \xrightarrow{R} \Phi f^h(x), \qquad \Phi f^h(x) \xrightarrow{L} f^g(x),$$

defined by iteration of φ and ξ, respectively, which are mutual inverses if and only if both objects are cyclic.

2E. ***Zeroth Hochschild homology and cohomology.*** Let A be a ring, and X a bimodule over A. There is an induced simplicial abelian group, part of which looks like

$$\cdots\ A\otimes A\otimes X \underset{\xrightarrow{d_2}}{\overset{\xrightarrow{d_0}}{\xrightarrow{d_1}}} A\otimes X \underset{\xrightarrow{d_1}}{\overset{\xrightarrow{d_0}}{\xleftarrow{s_0}}} X$$

with the maps given as follows:

$$\begin{aligned} d_0(a\otimes x) &= xa, & d_0(a\otimes b\otimes x) &= b\otimes xa,\\ d_1(a\otimes x) &= ax, & d_1(a\otimes b\otimes x) &= ab\otimes x,\\ s_0(x) &= 1\otimes x, & d_2(a\otimes b\otimes x) &= a\otimes bx, \end{aligned}$$

and which is defined analogously in higher degrees. We call this simplicial object the *Hochschild complex* of X, although often that name refers to the corresponding (normalized or otherwise) chain complex.

The *zeroth homology of A with coefficients in X* is the colimit $H_0(A, X)$ of this diagram, which can more simply be computed as the coequalizer of the two maps $A\otimes X \rightrightarrows X$; more explicitly still, this is the quotient of X by the subgroup generated by all elements of the form $ax - xa$.

Dually there is a cosimplicial object, part of which looks like

$$X \underset{\xrightarrow{\delta_1}}{\overset{\xrightarrow{\delta_0}}{\xleftarrow{\sigma_0}}} [A, X] \underset{\xrightarrow{\delta_2}}{\overset{\xrightarrow{\delta_0}}{\xrightarrow{\delta_1}}} [A\otimes A, X]\ \cdots$$

with the maps given as follows:

$$\begin{aligned} \delta_0(x)(a) &= xa, & \delta_0(f)(a\otimes b) &= f(a)b, \\ \delta_1(x)(a) &= ax, & \delta_1(f)(a\otimes b) &= f(ab), \\ \sigma_0(f) &= f(1), & \delta_2(f)(a\otimes b) &= af(b), \end{aligned}$$

and now the *zeroth Hochschild cohomology of A with coefficients in X* is the limit $H^0(A, X)$ (really an equalizer) of this diagram, given explicitly by the subgroup of X consisting of those x for which $xa = ax$ for all $a \in A$.

2F. ***Universality of zeroth Hochschild homology and cohomology.*** There are universal characterizations for both $H^0(A, X)$ and $H_0(A, X)$. For any A-bimodule X and any abelian group P, there is an induced bimodule structure on $[X, P]$ given by $(af)(x) = f(xa)$ and $(fa)(x) = f(ax)$, and this construction gives a functor $[X, -] : \mathbf{Ab} \to A\text{-}\mathbf{Mod}\text{-}A$. In particular, we may take $X = A$ with its regular left and right actions.

Proposition 2.7. *The functor $[A, -] : \mathbf{Ab} \to A\text{-}\mathbf{Mod}\text{-}A$ has a left adjoint sending an A-bimodule X to $H_0(A, X)$.*

Similarly, there is for any A-bimodule X and abelian group P an induced bimodule structure on $X \otimes P$ given by $a(x \otimes p) = ax \otimes p$ and $(x \otimes p)a = xa \otimes p$, and this gives a functor $X \otimes (-) : \mathbf{Ab} \to A\text{-}\mathbf{Mod}\text{-}A$. Considering again the case $X = A$, we have:

Proposition 2.8. *The functor $A \otimes (-) : \mathbf{Ab} \to A\text{-}\mathbf{Mod}\text{-}A$ has a right adjoint sending an A-bimodule X to $H^0(A, X)$.*

3. Bimodules

We described above the Hochschild complex of a ring A with coefficients in an A-bimodule. A ring is the same thing as a monoid in the monoidal category **Ab** of abelian groups, and more generally the Hochschild complex and the zeroth homology and cohomology can be constructed if A is a monoid in a suitable symmetric monoidal closed category $\mathcal{V}$. In particular, we could do this for the cartesian closed category **Cat**. But **Cat** is in fact a 2-category, which opens the way to consider *lax* variants of the theory, and it is such a variant that we now present. While it would be possible to develop this theory in the context of a general symmetric monoidal closed bicategory $\mathcal{V}$, it is only the case $\mathcal{V} = \mathbf{Cat}$ which we need, and so we restrict ourselves to that.

The first step, carried out in this section, is to describe in detail the notion of bimodule that will play the role of coefficient object for our lax homology and cohomology. We describe a certain 2-category A-**Mod**-A of bimodules, which involves a combination of strict and lax notions. The precise choice of what should be strict and what should be lax might at first seem arbitrary; we have made these choices so that our cohomology $H^0(A,-)$ and homology $H_0(A,-)$ can be defined via universal properties.

3A. *Monoids.* A monoid in **Cat** is precisely a strict monoidal category. It is not particularly difficult to adapt the theory that follows to deal with nonstrict monoidal categories, but we do not need this extra generality, and feel that the complications that it causes might distract from the story we wish to tell. It is probably also possible to extend the theory to deal with skew monoidal categories [Szlachányi 2012; Lack and Street 2012], although we have not checked this in detail.

We shall therefore consider a strict monoidal category (A, m, i). We shall write $a \otimes b$ or sometimes just ab for the image under the tensor functor $m : A \times A \to A$ of a pair (a, b).

3B. *Modules.* Next we need a notion of module over our monoid (strict monoidal category) A. There is a well-developed (pseudo) notion of an action of a monoidal category on a category, sometimes called an actegory. Here, however, we deal only with the strict case, which does not use the 2-category structure of **Cat**; once again it would not be difficult to extend our theory to deal with pseudo (or possibly skew) actions, but this is not needed for our applications so we have not done so. Giving a strict left action of A on a category X is equivalent to giving a strict monoidal functor from A to the strict monoidal category $\mathrm{End}(X)$ of endofunctors of X. The image under the corresponding functor $\alpha : A \times X \to X$ of an object (a, x) is written ax. Similarly there are (strict) right actions involving functors $\beta : X \times A \to X : (x, a) \mapsto xa$ satisfying strict associativity and unit conditions.

In fact, we also make use of a slightly more general notion. It is possible to consider actions of monoids not just on sets, but also on objects of other categories; in the same way, it is possible to consider actions of monoidal categories on objects of other 2-categories. If X is an object of a 2-category $\mathcal{K}$, then an action of A on X is a strict monoidal functor from A to the strict monoidal category $\mathcal{K}(X, X)$ of endomorphisms of X.

If the 2-category $\mathcal{K}$ admits copowers, then there is an equivalent formulation as follows. Recall that the copower of an object X by a category P is an object $P \cdot X$ equipped with isomorphisms of categories

$$\mathcal{K}(P \cdot X, Y) \cong \mathbf{Cat}(P, \mathcal{K}(X, Y))$$

2-natural in the variable $Y \in \mathcal{K}$. If $\mathcal{K}$ has all copowers, then there are 2-natural isomorphisms $(P \times Q) \cdot X \cong P \cdot (Q \cdot X)$ and $1 \cdot X \cong X$. In this case, a strict (left) action of A on X is equivalently a morphism $\alpha : A \cdot X \to X$ in $\mathcal{K}$ for which the diagrams

$$\begin{tikzcd}
(A \times A) \cdot X \arrow[rr, "m \cdot 1"] \arrow[d] & & A \cdot X \arrow[d, "\alpha"] \\
A \cdot (A \cdot X) \arrow[r, "1 \cdot \alpha"'] & A \cdot X \arrow[r, "\alpha"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \cdot X \arrow[r, "i \cdot 1"] \arrow[dr] & A \cdot X \arrow[d, "\alpha"] \\
& X
\end{tikzcd}$$

commute, where the unnamed maps are the isomorphisms just described. (There are also still more general notions of action of A; see [Kelly and Lack 1997, Section 2].)

Note that the 2-category **Cat** admits copowers, with $A \cdot X$ given by the cartesian product $A \times X$, so that in this case our more general notion of action of A on $X \in \mathbf{Cat}$ reduces to the initial one.

Example 3.1. Our running example throughout this section and the next takes A to be the strict monoidal category $\mathbb{M}^{\mathrm{op}}$; it is this example which will be used to explain the Böhm–Ştefan construction. Since a strict monoidal functor $\mathbb{M}^{\mathrm{op}} \to \mathcal{K}(X, X)$ is precisely a comonoid in $\mathcal{K}(X, X)$, a left $\mathbb{M}^{\mathrm{op}}$-module is a comonad in the 2-category $\mathcal{K}$, in the sense of [Street 1972]. On the other hand, a right $\mathbb{M}^{\mathrm{op}}$-module is also just a comonad in $\mathcal{K}$, as follows from Remark 2.1.

In the case $\mathcal{K} = \mathbf{Cat}$, a comonad in **Cat** is a category X equipped with a comonad g. For an object n of $\mathbb{M}^{\mathrm{op}}$ and an object $x \in X$, the value nx of the corresponding left $\mathbb{M}^{\mathrm{op}}$-action is given by $g^n x$.

3C. ***Morphisms of modules.*** When it comes to morphisms of modules, once again there is a question of how lax they should be, and this time we deviate from the completely strict situation. If X and Y are (strict, as ever) left A-modules in **Cat**, we define a *lax A-morphism* to be a functor $p : X \to Y$, equipped with a natural transformation

$$\begin{tikzcd}
A \times X \arrow[r, "1 \times p"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Downarrow \varrho"] & A \times Y \arrow[d, "\alpha"] \\
X \arrow[r, "p"'] & Y
\end{tikzcd}$$

whose components have the form

$$a.p(x) \xrightarrow{\varrho_{a,x}} p(ax)$$

for $a \in A$ and $x \in X$, and which satisfy two coherence conditions. The first asks that $\varrho_{i,x} : p(x) = i.p(x) \to p(ix) = p(x)$ is the identity. The second asks that the composite

$$ab.p(x) \xrightarrow{a\varrho_{b,x}} a.p(bx) \xrightarrow{\varrho_{a,bx}} p(abx)$$

be equal to $\varrho_{ab,x}$. Often we omit the subscripts and simply write ϱ for $\varrho_{a,x}$. When ϱ is an identity, we say that the A-morphism is *strict*.

For actions on objects of a general 2-category $\mathcal{K}$ given by strict monoidal functors $A \to \mathcal{K}(X, X)$ and $A \to \mathcal{K}(Y, Y)$, a lax A-morphism is a morphism $p : X \to Y$ in $\mathcal{K}$ together with a natural transformation

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow"] & \mathcal{K}(X,X) \arrow[d, "\mathcal{K}(X,p)"] \\ \mathcal{K}(Y,Y) \arrow[r, "\mathcal{K}(p,Y)"'] & \mathcal{K}(X,Y) \end{tikzcd}$$

satisfying an associativity and a unit axiom generalizing those above. If $\mathcal{K}$ admits copowers, then the natural transformation displayed above determines and is determined by a 2-cell

$$\begin{tikzcd} A \cdot X \arrow[r, "1 \cdot p"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Downarrow \varrho"] & A \cdot Y \arrow[d, "\alpha"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$

in $\mathcal{K}$, satisfying associativity and unit conditions.

If (p, ϱ) and (p', ϱ') are lax A-morphisms from X to Y, an A-transformation from (p, ϱ) to (p', ϱ') is a 2-cell $\tau : p \to p'$ satisfying the evident compatibility condition; in the case $\mathcal{K} = \mathbf{Cat}$, this says that the diagram

$$\begin{tikzcd} a.px \arrow[r, "1.\tau"] \arrow[d, "\varrho_{a,x}"'] & a.p'x \arrow[d, "\varrho'_{a,x}"] \\ p(ax) \arrow[r, "\tau"'] & p'(ax) \end{tikzcd}$$

commutes for all objects $a \in A$ and $x \in X$.

There is a 2-category A-**Mod** whose objects are the A-modules (in **Cat**), whose morphisms are the lax A-morphisms, and whose 2-cells are the A-transformations. This 2-category admits copowers, with $B \cdot X$ given by the category $X \times B$ equipped with the action $\alpha \times 1 : A \times X \times B \to X \times B$, where $\alpha : A \times X \to X$ is the action on X.

Example 3.2. In the case $A = \mathbb{M}^{\mathrm{op}}$, we saw that an A-module was precisely a category X equipped with a comonad g. A lax A-morphism is what was called a *comonad opfunctor* in [Street 1972], and indeed $\mathbb{M}^{\mathrm{op}}$-**Mod** is the 2-category called $\mathrm{Mnd}^*_*(\mathbf{Cat}^*_*)$ in that paper.

3D. ***Bimodules.*** As usual, a bimodule is an object which is both a left and right module with suitable compatibility between the two actions. Although our notion of action is strict, the compatibility between the actions is not. There is clearly a notion of (A, B)-bimodule for different A and B, but we only need the case where $A = B$. A succinct definition of A-bimodule is an object of A-**Mod** equipped with a right A-module structure, but we can also spell out what this means.

First of all, there is a category X with a strict left action $\alpha : A \times X \to X$. The right action involves a functor $\beta : X \times A \to X$ defining a strict right action, but this should be not just a functor, but a lax A-module morphism $A \cdot X \to X$. This lax A-morphism structure consists of maps

$$a(xb) \xrightarrow{\lambda_{a,x,b}} (ax)b$$

natural in the variables $a \in A$, $x \in X$, $b \in A$, and making each diagram

$$\begin{tikzcd} aa'(xb) \arrow[r, "1\lambda_{a',x,b}"] \arrow[dr, "\lambda_{aa',x,b}"'] & a((a'x)b) \arrow[d, "\lambda_{a,a'x,b}"] \\ & ((aa')x)b \end{tikzcd} \qquad \begin{tikzcd} i(xb) \arrow[r, equal] \arrow[dr, "\lambda_{i,x,b}"'] & xb \arrow[d, equal] \\ & (ix)b \end{tikzcd}$$

commute. Finally, the associative and unit laws required for the right action defined by $\beta : X \times A \to X$ should hold not just as equations between functors, but as equations between lax A-morphisms. Explicitly, this means that each diagram

$$\begin{tikzcd} a(xbb') \arrow[r, "\lambda_{a,xb,b'}"] \arrow[dr, "\lambda_{a,x,bb'}"'] & (a(xb))b' \arrow[d, "\lambda_{a,x,b}1"] \\ & (ax)(bb') \end{tikzcd} \qquad \begin{tikzcd} a(xi) \arrow[r, equal] \arrow[dr, "\lambda_{a,x,i}"'] & ax \arrow[d, equal] \\ & (ax)i \end{tikzcd}$$

should commute.

Example 3.3. Returning to our running example $A = \mathbb{M}^{\mathrm{op}}$, we have already seen that the 2-category A-**Mod** is just Street's 2-category $\mathrm{Mnd}^*_*(\mathbf{Cat}^*_*)$ of comonads and comonad opfunctors, and that a right $\mathbb{M}^{\mathrm{op}}$-action in a 2-category is a comonad in that 2-category. So an A-bimodule will be a comonad in $\mathrm{Mnd}^*_*(\mathbf{Cat}^*_*)$, which as explained in [Street 1972] amounts to a category X equipped with comonads g and h and a distributive law $\lambda : gh \to hg$ between them.

3E. *Morphisms of bimodules.* While our morphisms of left modules are lax, we shall consider only strict morphisms of right modules, but these should again be defined relative to the 2-category A-**Mod**. The reason for these choices will become clear in Theorem 4.5 below. This means that a morphism $(X, \alpha, \beta) \to (Y, \alpha, \beta)$ of bimodules will be a lax A-morphism $(p, \varrho) : (X, \alpha) \to (Y, \alpha)$ of the underlying left modules, for which the diagram

$$\begin{tikzcd} X \times A \arrow[r, "p\times 1"] \arrow[d, "\beta"'] & Y \times A \arrow[d, "\beta"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$

of categories and functors commutes, and for which moreover the diagram

$$\begin{tikzcd} & a.(px.b) \arrow[r, "\lambda_{a,px,b}"] & (a.px).b \arrow[dr, "\varrho_{a,x}.1"] & \\ a.p(xb) \arrow[ur, equal] \arrow[dr, "\varrho_{a,xb}"'] & & & p(ax).b \\ & p(a(xb)) \arrow[r, "p(\lambda_{a,x,b})"'] & p((ax)b) \arrow[ur, equal] & \end{tikzcd} \tag{3.4}$$

commutes for all $a, b \in A$ and $x \in X$.

The bimodules and their morphisms constitute the objects and morphisms of a 2-category A-**Mod**-A; a 2-cell $(p, \varrho) \to (p', \varrho')$ is a natural transformation $\tau : p \to p'$ which is a 2-cell relative to both the left and right actions.

Example 3.5. For an A-bimodule X and an arbitrary category P, the functor category $[X, P]$ has left and right actions of A, given by $(af)(x) = f(xa)$ and $(fa)(x) = f(ax)$, and these define a bimodule structure on $[X, P]$. This forms the object part of a 2-functor $[X, -] : \mathbf{Cat} \to A$-**Mod**-$A$. We shall be particularly interested in the case where X is A with its standard bimodule structure; in this case, since the left and right actions on A are strictly compatible, so too are those on $[A, P]$.

Example 3.6. Dually, for an A-bimodule X and an arbitrary category P, the product category $P \times X$ has left and right actions inherited from X, and this forms the object part of a 2-functor $(-) \times X : \mathbf{Cat} \to A$-**Mod**-$A$.

4. Lax cohomology and homology

4A. ***The Hochschild complex.*** Let A be a strict monoidal category and X a bimodule over A, in the sense of the previous section. Then we can define maps

$$\cdots\ A \times A \times X \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{array} A \times X \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \end{array} X \tag{4.1}$$

exactly as in Section 2E, except that, because of the lax compatibility between the actions, the simplicial identity $d_1 d_0 = d_0 d_2$ no longer holds; instead, there is a natural transformation $\lambda : d_1 d_0 \to d_0 d_2$ whose component at an object (b, a, x) in

$A \times A \times X$ is the map $\lambda_{a,x,b} : a(xb) \to (ax)b$. Similarly, each simplicial identity involving a first face map and a last face map is replaced by a natural transformation. The various coherence conditions on λ appearing in the definition of A-bimodule imply various coherence conditions on these natural transformations; the entire structure determines a **Cat**-valued presheaf on a 2-category which is obtained by a "blowing up" of the category Δ, similar in nature to that in [Lack 2000].

Similarly, there are maps

$$X \overset{\xrightarrow{\delta_0}}{\underset{\xrightarrow{\delta_1}}{\xleftarrow{\sigma_0}}} [A, X] \overset{\xrightarrow{\delta_0}}{\underset{\xrightarrow{\delta_2}}{\xrightarrow{\delta_1}}} [A \times A, X] \cdots \tag{4.2}$$

defined as in Section 2E once again; this time the cosimplicial identity $\delta_2\delta_0 = \delta_1\delta_0$ becomes a natural transformation $\delta_2\delta_0 \to \delta_1\delta_0$, whose components are once again induced by the lax compatibilities $\lambda_{a,x,b}$.

4B. *Cohomology.* In Section 2E, we defined the zeroth Hochschild cohomology group $H^0(A, X)$ of a bimodule over a ring as the equalizer of the pair of maps $\delta_0, \delta_1 : X \rightrightarrows [A, X]$. In the case of the lax cohomology of a bimodule over a strict monoidal category A, we define the zeroth Hochschild cohomology $H^0(A, X)$ by taking a "lax version" of an equalizer, involving all of the data displayed in (4.2), called a *lax descent object*; this is a mild variant from [Lack 2002] of a notion introduced in [Street 1987]. Interpreting this for (4.2) yields that $H^0(A, X)$ is the universal category Y equipped with a functor $y : Y \to X$ and a natural transformation $\xi : \delta_1 y \to \delta_0 y$ such that $\sigma_0\xi : x = \sigma_0\delta_1 y \to \sigma_0\delta_0 y = y$ is the identity and the diagram

$$\begin{tikzcd}
 & \delta_2\delta_0 y \arrow[r, "\lambda y"] & \delta_0\delta_1 y \arrow[dr, "\delta_0\xi"] & \\
\delta_2\delta_1 y \arrow[ur, "\delta_2\xi"] \arrow[dr, equal] & & & \delta_0\delta_0 y \\
 & \delta_1\delta_1 y \arrow[r, "\delta_1\xi"'] & \delta_1\delta_0 y \arrow[ur, equal] &
\end{tikzcd}$$

commutes. Explicitly, an object of $H^0(A, X)$ is an object $x \in X$ equipped with maps $\xi_a : ax \to xa$ natural in $a \in A$, and satisfying $\xi_i = 1$ as well as the cocycle condition asserting that the diagram

$$\begin{tikzcd}
 & a(xb) \arrow[r, "\lambda x"] & (ax)b \arrow[dr, "\xi b"] & \\
a(bx) \arrow[ur, "a\xi"] \arrow[dr] & & & (xa)b \\
 & (ab)x \arrow[r, "\xi"'] & x(ab) \arrow[ur] &
\end{tikzcd}$$

commutes for all $a, b \in A$.

Example 4.3. In the case of classical Hochschild cohomology, for a ring A the zeroth cohomology group $H^0(A, A)$ is the centre of the ring; similarly, for a strict monoidal category A, the lax cohomology $H^0(A, A)$ is the *lax centre* of A in the sense of [Day et al. 2007], originally introduced in [Schauenburg 2000] with the name *weak centre*.

Example 4.4. Consider our running example of $A = \mathbb{M}^{\mathrm{op}}$, so that an A-bimodule X is a category equipped with comonads g and h and a distributive law $\lambda : gh \to hg$. Explicit calculation shows that an object of $H^0(A, X)$ is an object $x \in X$ equipped with a map $\xi : gx \to hx$ making the diagrams (2.6) commute, so we recover the notion of right λ-coalgebra of Section 2D.

The next result justifies the definition of the lax cohomology $H^0(A, X)$ analogously to Proposition 2.8 for the usual Hochschild cohomology.

Theorem 4.5. *The 2-functor $(-) \times A : \mathbf{Cat} \to A\text{-}\mathbf{Mod}\text{-}A$ has a right adjoint sending an A-bimodule X to $H^0(A, X)$.*

Proof. Let X be an A-bimodule and P a category. Giving a (strict) right A-module morphism $p : P \times A \to X$ is equivalent to giving a functor $f : P \to X$; here $f(y) = p(y, 1)$ and $p(y, a) = f(y)a$. (It is here that the strictness of the right action is necessary.) To enrich such a morphism of modules into a morphism (p, ϱ) of bimodules, we should give suitably natural and coherent maps

$$\varrho_{a,y,b} : a.p(y, b) \to p(y, ab)$$

for all $a \in A$ and $(y, b) \in P \times A$. By the compatibility condition (3.4), the map $\varrho_{a,y,b}$ can be constructed as

$$ap(y, b) = a(p(y, 1)b) \xrightarrow{\lambda_{a,p(y,1),b}} (ap(y, 1))b \xrightarrow{\varrho_{a,y,1}1} p(y, a)b = p(y, ab)$$

and so the general ϱ is determined by those of the form $\varrho_{a,y,1}$, and these have the form $\xi_{a,y} : af(y) \to f(y)a$. The unit condition asserting that each $\varrho_{1,y,b}$ is the identity says that $\xi_{1,y}$ is the identity. The cocycle condition on the ϱ is equivalent to the cocycle condition asserting that $\xi_{a,y}$ makes each $f(y)$ into an object of $H^0(A, X)$. Naturality of $\xi_{a,y}$ in y implies that for each morphism $\psi : y \to y'$ in P, the map $f(\psi)$ defines a morphism $(f(y), \xi_{a,y}) \to (f(y'), \xi_{a,y'})$ in $H^0(A, X)$.

This gives the desired bijection between bimodule morphisms $P \times A \to X$ and functors $P \to H^0(A, X)$; it is straightforward to check that this carries over to 2-cells, and so defines an isomorphism of categories

$$A\text{-}\mathbf{Mod}\text{-}A(P \times A, X) \cong \mathbf{Cat}(P, H^0(A, X))$$

exhibiting $H^0(A, X)$ as the value at X of a right adjoint to $(-) \times A$. □

4C. *Homology.* In Section 2E, the zeroth Hochschild homology group was defined as the coequalizer of the maps $d_0, d_1 : A \otimes X \rightrightarrows X$. For lax homology, we define $H_0(A, X)$ of an A-bimodule X to be the *lax codescent object* of the data displayed in (4.1). Lax codescent objects are the colimit notion corresponding to the lax descent objects used to define lax cohomology.

Spelling this out, $H_0(A, X)$ is the universal category Y equipped with a functor $f : X \to Y$ and a natural transformation $\varphi : fd_0 \to fd_1$ satisfying the normalization condition $\varphi s_0 = 1$ and the cocycle condition

$$\begin{tikzcd}
& fd_1d_0 \arrow[r, "f\lambda"] & fd_0d_2 \arrow[dr, "\varphi d_2"] & \\
fd_0d_0 \arrow[ur, "\varphi d_0"] \arrow[dr, equal] & & & fd_1d_2. \\
& fd_0d_1 \arrow[r, "\varphi d_1"'] & fd_1d_1 \arrow[ur, equal] &
\end{tikzcd}$$

Explicitly, $H_0(A, X)$ is obtained from X by adjoining morphisms $xa \to ax$ satisfying naturality conditions in both variables, with $xi \to ix$ required to be the identity, and obeying the cocycle condition which requires the diagram

$$\begin{tikzcd}
& b(xa) \arrow[r, "\lambda"] & (bx)a \arrow[dr, "\varphi_{bx,a}"] & \\
(xa)b \arrow[ur, "\varphi_{xa,b}"] \arrow[dr] & & & a(bx) \\
& x(ab) \arrow[r, "\varphi_{x,ab}"'] & (ab)x \arrow[ur] &
\end{tikzcd}$$

to commute.

Example 4.6. Let $A = \mathbb{M}^{\mathrm{op}}$, and let X have A-bimodule structure corresponding to comonads g and h and a distributive law $\lambda : gh \to hg$. By the defining universal property of the category $H_0(A, X)$, giving a functor $H_0(A, X) \to P$ is the same as giving a functor $f : A \to P$ and natural transformation $\varphi : fh \to fg$ making the diagrams (2.5) commute, so we recover the notion of left λ-coalgebra from Section 2D.

Example 4.7. Again with $A = \mathbb{M}^{\mathrm{op}}$, the "regular" A-bimodule structure on A corresponds to the two decalage comonads equipped with the identity distributive law between them. The full subcategory of $\mathbb{M}^{\mathrm{op}}$ given by the nonempty finite ordinals is a sub-bimodule; since it is also isomorphic to Δ^{op}, there is an induced bimodule structure on Δ^{op}. By the preceding example and the description of duplicial structure given in Proposition 2.4, a functor $H_0(\mathbb{M}^{\mathrm{op}}, \Delta^{\mathrm{op}}) \to P$ is precisely a duplicial object in P, so that $H_0(\mathbb{M}^{\mathrm{op}}, \Delta^{\mathrm{op}})$ itself is the category $\boldsymbol{K}^{\mathrm{op}}$ indexing duplicial structure. Similarly, a functor $H_0(\mathbb{M}^{\mathrm{op}}, \mathbb{M}^{\mathrm{op}}) \to P$ is an augmented duplicial object in P, and $H_0(\mathbb{M}^{\mathrm{op}}, \mathbb{M}^{\mathrm{op}})$ is the category indexing augmented duplicial structure.

Just as before, the lax zeroth Hochschild homology has a universal characterization paralleling Proposition 2.7.

Theorem 4.8. *The* 2*-functor* $[A, -] : \mathbf{Cat} \to A\text{-}\mathbf{Mod}\text{-}A$ *has a left adjoint sending an* A*-bimodule* X *to* $H_0(A, X)$.

Proof. Let X be an A-bimodule and P a category. Just as in the classical case, giving a (strict) morphism of right A-modules $p : X \to [A, P]$ is equivalent to giving a morphism $f : X \to P$ with $f(x) = p(x)(1)$ and $p(x)(a) = f(xa)$. In order to enrich such a p into a morphism $(p, \varrho) : X \to [A, P]$ of bimodules, we should give a suitably coherent map $\varrho_{a,x} : a.p(x) \to p(ax)$ in $[A, P]$ for all $a \in A$ and $x \in X$. Thus for $b \in A$ we should give

$$f(x(ba)) = p(x)(ba) = (a.p(x))b \xrightarrow{\varrho_{a,x}(b)} p(ax)(b) = f((ax)b).$$

Commutativity of (3.4) means that the general $\varrho_{a,x}(b)$ is equal to the composite

$$a.p(xb) \xrightarrow{\varrho_{a,xb}(1)} p(a(xb)) \xrightarrow{p\lambda_{a,x,b}} p((ax)b) = p(ax)b.$$

Thus ϱ is determined by the maps $\varrho_{a,x}(1) : f(xa) \to f(ax)$, which we can regard as defining a natural transformation $\varphi : fd_0 \to fd_1$. The normalization condition asserting that $\varrho_{1,x}$ is an identity now says that

$$f = fd_0 s_0 \xrightarrow{\varphi s_0} fd_1 s_0 = f$$

is an identity. The cocycle condition on ϱ is equivalent to the cocycle condition on φ, and so we have a bijection between bimodule morphisms $X \to [A, P]$ and functors $H_0(A, X) \to P$. It is straightforward to extend this to 2-cells, and so to obtain an isomorphism of categories

$$A\text{-}\mathbf{Mod}\text{-}A(X, [A, P]) \cong \mathbf{Cat}(H_0(A, X), P)$$

exhibiting $H_0(A, X)$ as the value at X of a left adjoint to $[A, -]$. □

4D. ***The universal coefficient theorem and the cap product.*** In this section we develop a few very simple ingredients of classical Hochschild theory in our lax context. The first of these is the universal coefficient theorem. In its more general forms this involves short exact sequences connecting homology and cohomology, but in degree zero it is particularly simple.

Proposition 4.9 (universal coefficient theorem). *For any bimodule* X *and category* P *there is an isomorphism of categories*

$$\mathbf{Cat}(H_0(A, X), P) \cong H^0(A, [X, P])$$

natural in X *and* P.

Proof. By the universal property of $H_0(A, X)$ as a lax codescent object, an object of the left-hand side amounts to a functor $f : X \to P$ equipped with a natural transformation $\varphi : fd_0 \to fd_1$ satisfying the normalization and cocycle conditions. But the functor f can be seen as an object of $[X, P]$, while $\delta_0(f) : A \to [X, P]$ and $\delta_1(f)$ correspond under the adjunction $- \times A \dashv \mathbf{Cat}(A, -)$ to $fd_0 : A \times X \to P$ and fd_1, so that giving $\varphi : fd_0 \to fd_1$ is equivalent to giving $\xi : \delta_0(f) \to \delta_1(f)$. A straightforward calculation shows that the normalization and cocycle conditions for φ to make f into a functor $H_0(A, X) \to P$ are equivalent to the normalization and cocycle conditions for ξ to make f into an object of $H^0(A, [X, P])$.

This proves that we have a bijection on objects; the case of morphisms is similar but easier, and is left to the reader. □

Construction 4.10 (cap product). Given any bimodule X, the unit of the adjunction $H_0(A, -) \dashv [A, -]$ of Theorem 4.8 has the form $\chi : X \to [A, H_0(A, X)]$. Applying the cohomology 2-functor $H^0(A, -)$, we obtain a functor

$$H^0(A, X) \xrightarrow{H^0(A,\chi)} H^0(A, [A, H_0(A, X)]),$$

and composing with the "universal coefficient" isomorphism $H^0(A, [A, P]) \cong \mathbf{Cat}(H_0(A, A), P)$ of Proposition 4.9, we obtain a functor

$$H^0(A, X) \longrightarrow \mathbf{Cat}(H_0(A, A), H_0(A, X))$$

whose adjoint transpose

$$H^0(A, X) \times H_0(A, A) \longrightarrow H_0(A, X)$$

can be seen as a special case of the cap product for our lax homology and cohomology. But we choose instead to transpose again to obtain a functor

$$H_0(A, A) \xrightarrow{\mathrm{BS}} \mathbf{Cat}(H^0(A, X), H_0(A, X)),$$

which we call the *Böhm–Ştefan* map.

Example 4.11. We now analyze this Böhm–Ştefan map in the case of our running example. Suppose then that $A = \mathbb{M}^{\mathrm{op}}$, and X is an A-bimodule, with the bimodule structure corresponding to comonads g and h and a distributive law $\lambda : gh \to hg$. Let $p : H_0(A, X) \to P$ be an arbitrary functor, and let $y \in H^0(A, X)$. As in Example 4.6, giving p is equivalent to giving a functor $f : X \to P$ equipped with left λ-coalgebra structure $\varphi : fh \to fg$, while as in Example 4.4, giving y is equivalent to giving an object $x \in X$ equipped with right λ-coalgebra structure $\xi : gx \to hx$. There is now an induced functor

$$H_0(A, A) \xrightarrow{\mathrm{BS}} \mathbf{Cat}(H^0(A, X), H_0(A, X)) \xrightarrow{\mathrm{ev}_y} H_0(A, X) \xrightarrow{p} P$$

which by Example 4.7 picks out an augmented duplicial object in P. This object is precisely the one constructed in [Böhm and Ştefan 2008] as recalled in Section 2D above. This construction was generalized slightly in [Böhm and Ştefan 2012] to include right λ-coalgebra structures on arbitrary functors $Y \to X$, rather than just objects of X; in this case y becomes a functor $Y \to H^0(A, X)$ and the composite

$$H_0(A, A) \xrightarrow{\text{BS}} \mathbf{Cat}(H^0(A, X), H_0(A, X)) \xrightarrow{\mathbf{Cat}(y,p)} \mathbf{Cat}(Y, P)$$

defines an augmented duplicial object in $\mathbf{Cat}(Y, P)$.

5. Duplicial structure on nerves

In this section we turn to our second main goal, which is to analyze duplicial structure on nerves of various sorts of categorical structures; specifically, on categories, on monoidal categories, and on bicategories.

A monoidal category can of course be seen as a one-object bicategory, and a category can be seen as a bicategory with no nonidentity 2-cells, so in principle we could pass straight to the case of bicategories, and then merely read off the results for the other two cases, but instead we have chosen to do the case of categories first, as a sort of warm-up.

5A. ***Duplicial structure on categories.*** The nerve functor from $\mathbf{Cat}$ to $[\Delta^{\text{op}}, \mathbf{Set}]$ is of course fully faithful, so that we may identify (small) categories with certain simplicial sets. It therefore makes sense to speak of duplicial structure borne by a category. The decalage comonads on $[\Delta^{\text{op}}, \mathbf{Set}]$ restrict to $\mathbf{Cat}$, and so we may analyze duplicial structure on categories using Proposition 2.4.

The right decalage comonad sends a category C to the coproduct $\sum_x C/x$ over all objects $x \in C$ of the corresponding slice categories. The counit is the functor induced by the domain functors $C/x \to C$, while the comultiplication $\sum_x C/x \to \sum_{f:w\to x} C/w$ sends the x-component to the 1_x-component via the identity functor $C/x \to C/x$. Dually, the left decalage comonad sends a category C to the coproduct $\sum_x x/C$, with similar descriptions available for the counit and comultiplication.

Since both C/x and x/C are connected categories, a functor $\sum_x C/x \to \sum_x x/C$ is necessarily given by an assignment $c \mapsto tc$ on objects together with a functor $t : C/x \to tx/C$ for each x. Compatibility with the counit (on objects) means that the image under t of an object $f : a \to x$ of C/x should have the form $tf : tx \to a$. Functoriality, together with counit compatibility on morphisms means that if $fg = h$ then $g.th = tf$. Compatibility with the comultiplication requires a slightly more complicated calculation.

An object of the right decalage $\text{Dec}_\text{r}(C)$ has the form $f : a \to x$, and the comultiplication $\text{Dec}_\text{r}(C) \to \text{Dec}_\text{r}(\text{Dec}_\text{r}(C))$ sends it to the composable pair $(1_x, f)$.

Now $\mathrm{Dec_r}(t) : \mathrm{Dec_r}(\mathrm{Dec_r}(C)) \to \mathrm{Dec_r}(\mathrm{Dec_l}(C))$ sends this to the composable pair (f, tf), which, as we have seen, must have composite $t1_x$. This composable pair can equally be seen as lying in $\mathrm{Dec_l}(\mathrm{Dec_r}(C))$, and finally applying $\mathrm{Dec_l}(t)$ gives the composable pair $(tf, t^2 1_x)$. Compatibility with comultiplication says that this should be equal to the composable pair $(tf, 1_{tx})$, and this clearly says that $t^2(1_x) = 1_{tx}$ for all objects x. We have only checked compatibility with the comultiplication on objects, but in fact no further condition is needed for compatibility on morphisms. We summarize this calculation as follows.

Proposition 5.1. *Giving duplicial structure to a small category C is equivalent to giving*

- *for each object x an object tx,*
- *for each morphism $f : a \to x$ a morphism $tf : tx \to a$,*

subject to the conditions that

- *$t^2(1_x) = 1_{tx}$ for all objects x,*
- *$f.t(gf) = tg$ for any composable pair (g, f),*

which we call the identity and functoriality conditions, respectively.

The next result gives a cleaner reformulation of these conditions. In its statement, recall that the inclusion 2-functor **Gpd** $\hookrightarrow$ **Cat** has a left 2-adjoint Π_1, whose counit at a small category C is the functor $p : C \to \Pi_1(C)$ which freely adjoins an inverse for every arrow of C. The 2-dimensional aspect of the universal property means that, for any category D, the functor $[\Pi_1(C), D) \to [C, D]$ given by composition with p is fully faithful.

Theorem 5.2. *Giving duplicial structure to a small category C is equivalent to giving a left adjoint in* **Cat** *for the functor $p : C \to \Pi_1(C)$.*

Proof. First suppose that p has a left adjoint $i : \Pi_1(C) \to C$ with counit $\varepsilon : ip \to 1$ and unit $\eta : 1 \to pi$; since $\Pi_1(C)$ is a groupoid, η is invertible, and therefore i is fully faithful. For each object $y \in C$, define ty to be ipy, and for each morphism $f : x \to y$, define $tf : ipy \to x$ to be the composite

$$ipy \xrightarrow{i(pf)^{-1}} ipx \xrightarrow{\varepsilon_x} x.$$

Then $t(1_x) = \varepsilon_x$ and so using the triangle identities twice yields

$$t^2(1_x) = t(\varepsilon_x) = \varepsilon_{tx}.i(p\varepsilon_x)^{-1} = \varepsilon_{ipx}.i\eta_{px} = 1_{ipx},$$

while for a composable pair (g, f) we have

$$f.t(gf) = f.\varepsilon_x.i(p(gf)^{-1}) = \varepsilon_y.ip(f).i(pf)^{-1}.i(pg)^{-1} = \varepsilon_y.i(pg)^{-1} = t(g);$$

so this defines duplicial structure on C.

Conversely, if C is equipped with duplicial structure there is an induced functor $G : C \to C$ sending an object x to tx and a morphism $f : x \to y$ to $t^2 f : tx \to ty$. This preserves identity morphisms because $t^2(1_x) = 1_{tx}$ by assumption, and preserves composition by three applications of the fact that if $h = gf$ then $f.th = tg$. (The functor G can be seen as a simplicial endomorphism of the nerve of C; as such it is the "curious natural transformation" of [Dwyer and Kan 1985].) For each $x \in C$, write ε_x for the morphism $t(1_x) : tx \to x$. Now $f.tf = t(1_y)$ by the functoriality condition, since $1_y f = f$; and replacing f by tf we also have $tf.t^2 f = t(1_x)$. Combining these, $\varepsilon_y.Gf = t1_y.t^2 f = f.tf.t^2 f = f.t1_x = f.\varepsilon_x$ and so the ε_x are indeed natural. Furthermore, $G\varepsilon_x = t^2(\varepsilon_x) = t^3(1_x) = t(1_{tx}) = \varepsilon_{Gx}$ and so (G, ε) is a well-copointed endofunctor in the sense of [Kelly 1980].

Next we show that for any $f : x \to y$, the morphism $Gf := t^2 f$ is invertible, with inverse $t(f.\varepsilon_x)$. First observe that $\varepsilon_x.t(f.\varepsilon_x) = tf$ by the functoriality condition once again. Consequently, we have

$$t(f.\varepsilon_x).t^2(f) = t(f.\varepsilon_x).t(\varepsilon_x.t(f.\varepsilon_x)) = t(\varepsilon_x) = t^2(1_x) = 1_{tx},$$

using the functoriality condition again at the second step; this gives one of the inverse laws. By naturality of ε and the functoriality condition yet again, we have

$$t^2 f.t(f.\varepsilon_x) = t^2 f.t(\varepsilon_y.t^2(f)) = t(\varepsilon_y) = t^2(1_y) = 1_{ty},$$

giving the other. Thus each Gf is invertible. By the universal property of $\Pi_1(C)$, therefore, there is a unique functor $i : \Pi_1(C) \to C$ with $ip = G$. By the 2-dimensional aspect of the universal property of $\Pi_1(C)$, there is a unique natural transformation $\eta : 1 \to pi$ with $\eta p : p \to pip$ equal to $(p\varepsilon)^{-1}$, and so satisfying the triangle equation $p\varepsilon.\eta p = 1$. By the 2-dimensional aspect of the universal property once again, the other triangle equation $\varepsilon i.i\eta = 1$ holds if and only if $\varepsilon ip.i\eta p = 1$ does, but by the calculation

$$\varepsilon ip.i\eta p = \varepsilon ip.(ip\varepsilon)^{-1} = ip\varepsilon.(ip\varepsilon)^{-1} = 1$$

this is indeed the case, and so p does have a left adjoint.

It remains to show that these two processes are mutually inverse. First suppose that C has duplicial structure t, and then construct a left adjoint $i \dashv p$ as above. The duplicial structure that this induces sends an object x to $ipx = ix = tx$, and a morphism $f : x \to y$ to $\varepsilon_x.i(pf)^{-1}$, where $i(pf)^{-1} = t(f.\varepsilon_x)$. But now $\varepsilon_x.i(pf)^{-1} = \varepsilon_x.t(f.\varepsilon_x) = tf$ by the functoriality condition, and so we have recovered the original duplicial structure.

For the other direction, suppose first that p has a left adjoint i with counit ε. Construct the induced duplicial structure t, and the left adjoint i' and counit ε' induced by that. By the universal property of $\Pi_1(C)$ once again it suffices to show that $ip = i'p$ and $\varepsilon = \varepsilon'$. For an object x, we have $\varepsilon'_x = t(1_x) = \varepsilon_x.i(p1_x)^{-1} = \varepsilon_x$, and so $\varepsilon = \varepsilon'$; this includes the fact that ip and $i'p$ agree on objects, and so it remains only to show that they agree on morphisms. To see this, let $f : x \to y$ be a morphism, so that $i'pf : i'px \to i'py$ is given by $t^2(f) : tx \to ty$. Now $tf = \varepsilon_x.i(pf)^{-1}$, so

$$ip(tf)^{-1} = ipipf.i(p\varepsilon_x)^{-1} = ipipf.i\eta px = i\eta py.ipf,$$

and so finally $i'pf = t^2 f = \varepsilon_{ipy}.i\eta py.ipf = ipf$. □

Example 5.3. If C is a groupoid, then $p : C \to \Pi_1(C)$ is invertible, and so has a canonical left adjoint $p^{-1} : \Pi_1(C) \to C$. So every groupoid has a canonical duplicial structure.

Example 5.4. Suppose that there is a groupoid G and a functor $i : G \to C$ with a right adjoint $r : C \to G$. By the universal property of $\Pi_1(C)$, there is a unique induced functor $q : \Pi_1(C) \to G$ with $qp = r$. By [Gabriel and Zisman 1967, Proposition 1.3], this q is an equivalence. Thus p also has a left adjoint, and so C has a duplicial structure.

Remark 5.5. We have seen that a category C has duplicial structure just when $p : C \to \Pi_1(C)$ has a left adjoint. This is paracyclic just when each t_n is invertible, or equivalently just when each t_n^{n+1} is invertible. Now the t_n^{n+1} define the functor $ip : C \to C$; since p is bijective on objects and i is fully faithful, the composite ip is invertible if and only if i and p are both invertible, and this can happen only if C is a groupoid.

For a groupoid, giving duplicial structure is equivalent to giving a left adjoint to the invertible $p : C \to \Pi_1(C)$; of course such a left adjoint is necessarily isomorphic to p^{-1} and so in particular an equivalence. The duplicial structure is paracyclic just when this left adjoint is in fact an invertible functor, and cyclic just when it is p^{-1} as above. Thus, for a category C, the existence of paracyclic structure implies the existence of cyclic structure, but this does not mean that paracyclic structure on a category is necessarily cyclic. Furthermore, a groupoid can admit multiple cyclic structures, since there can be multiple choices of unit and counit for an adjunction $p^{-1} \dashv p$; in fact such choices correspond to choices of a natural isomorphism $1_G \cong 1_G$.

5B. ***Duplicial structure on bicategories.*** We next consider what it means to give duplicial structure on the nerve of a bicategory B [Street 1996]. Recall that this nerve is the simplicial set NB, in which

- the 0-simplices are the objects of B;
- the 1-simplices are the arrows $f : x \to y$ of B;

- the 2-simplices are the 2-cells in B of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z; \end{tikzcd}$$

- the 3-simplices are the commuting diagrams of 2-cells of the form

$$\begin{tikzcd} (hg)f \arrow[r, "\cong"] \arrow[d, "\alpha f"'] & h(gf) \arrow[r, "h\beta"] & hk \arrow[d, "\gamma"] \\ \ell f \arrow[rr, "\delta"'] & & m \end{tikzcd}$$

  in which the unnamed isomorphism is the relevant associativity constraint of B.

The face and degeneracy maps are as expected, and the higher simplices are determined by 3-coskeletality. The assignment $B \mapsto NB$ is the object part of a fully faithful functor $N : \mathbf{NLax} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$, where **NLax** is the category of bicategories and *normal lax functors* between them — ones preserving identities on the nose, but binary composition only up to noninvertible 2-cells $Fg.Ff \Rightarrow F(gf)$. The first appearance in print we could find of the fact that this nerve functor is fully faithful was in [Bullejos et al. 2005].

Once again, the decalage comonads on $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ restrict to the full subcategory **NLax**, and so it makes sense to speak of duplicial structure *on a bicategory*. Indeed the description of these restricted comonads is similar to the case of **Cat**, except that rather than slice categories now we use "lax slices". For an object x of a bicategory B, we write B/x for the bicategory whose objects are morphisms $f : a \to x$ with codomain x, whose morphisms from $f : a \to x$ to $g : b \to x$ have the form

$$\begin{tikzcd} a \arrow[rr, "s"] \arrow[dr, "f"'] & \arrow[d, phantom, "\overset{\Leftarrow}{\sigma}"] & b \arrow[dl, "g"] \\ & x & \end{tikzcd}$$

and whose 2-cells are defined in the evident way. Similarly the "lax coslice" x/B has objects of the form $f : x \to a$, and morphisms from $f : x \to a$ to $g : x \to b$ of the form

$$\begin{tikzcd} & x \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\overset{\Rightarrow}{\sigma}"] & \\ a \arrow[rr, "s"'] & {} & b. \end{tikzcd}$$

We now define $\mathrm{Dec}_{\mathrm{r}}(B) = \sum_x B/x$ and $\mathrm{Dec}_{\mathrm{l}}(B) = \sum_x x/B$, with the actions on normal lax functors, and the counits and comultiplications given by a straightforward generalization of the corresponding definitions for **Cat**.

Before giving our characterization result, let us recall that a 2-cell in a bicategory as on the left in

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] & \\ b \arrow[rr, "h"'] & & c \end{tikzcd} \qquad \begin{tikzcd} & a \arrow[dl, "k"'] \arrow[dr, "g"] \arrow[d, phantom, "\overset{\beta}{\Longrightarrow}"] & \\ b \arrow[rr, "h"'] & & c \end{tikzcd}$$

is said to *exhibit f as a right lifting of g through h* [Street and Walters 1978] if every 2-cell as on the right above factors as $\alpha.h\bar{\beta}$ for a unique 2-cell $\bar{\beta} : k \Rightarrow f$.

Theorem 5.6. *Equipping a bicategory B with duplicial structure is equivalent to giving*

(a) *for each object $x \in B$ an object $tx \in B$ and a morphism $\varepsilon_x : tx \to x$;*

(b) *for each morphism $f : a \to x$ in B a morphism $tf : tx \to a$ and a 2-cell*

$$\begin{tikzcd} & tx \arrow[dl, "tf"'] \arrow[dr, "\varepsilon_x"] \arrow[d, phantom, "\overset{\varepsilon_f}{\Longrightarrow}"] & \\ a \arrow[rr, "f"'] & & x \end{tikzcd}$$

exhibiting tf as a right lifting of ε_x through f;

all subject to the conditions that

(c) $t1_x = \varepsilon_x$;

(d) $t^2 1_x = 1_{tx}$;

(e) ε_{1_x} *is the left identity isomorphism* $1_x.t1_x \to t1_x$;

(f) ε_{t1_x} *is the right identity isomorphism* $t1_x.1_{tx} \to t1_x$.

In the case where B is a category, (a) and (c) correspond to giving tx and $t1_x : tx \to x$ for each x, while (b) says that for each $f : a \to x$ there is a unique map tf with $f.tf = t1_x$; condition (d) now follows from the uniqueness, and conditions (e) and (f) are automatic. It is now not hard to see that this is equivalent to the conditions in Proposition 5.1.

Proof. By redefining the composition with identity 1-cells, any bicategory may be made isomorphic in **NLax** to one in which identities are strict. Thus without loss of generality we may suppose that B has strict identities; then the conditions in (e) and (f) become $\varepsilon_{1_x} = 1_{\varepsilon_x}$ and $\varepsilon_{t1_x} = 1_{t(1_x)}$.

Duplicial structure consists of a normal lax functor $t : \mathrm{Dec}_\mathrm{r}(B) \to \mathrm{Dec}_\mathrm{l}(B)$ which is compatible with the counit and comultiplication maps. As in the case of **Cat**, since each B/x and x/B is connected, t must be given by an assignment $x \mapsto tx$ on objects and normal lax functors $B/x \to tx/B$.

To give t on objects compatibly with the counits is to give, for each $f : a \to x$, a morphism $tf : tx \to a$. To give t on morphisms compatibly with the counits is to give, for each triangle as on the left below, a triangle as on the right:

$$\begin{tikzcd} a \arrow[rr, "s"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Leftarrow\sigma"] & b \arrow[dl, "g"] \\ & x & \end{tikzcd} \qquad \begin{tikzcd} & tx \arrow[dl, "tf"'] \arrow[dr, "tg"] \arrow[d, phantom, "t_s(\sigma)\Rightarrow"] & \\ a \arrow[rr, "s"'] & & b \end{tikzcd}$$

The action of t on 2-cells is unique if it exists, given the counit condition; it exists just when, for all $\sigma : gs \to f$ and $\tau : s' \to s$, the diagram on the left commutes, where σ' is defined as in the diagram on the right:

$$\begin{tikzcd} s'.tf \arrow[r, "\tau.tf"] \arrow[dr, "t_{s'}(\sigma')"'] & s.tf \arrow[d, "t_s(\sigma)"] \\ & tg \end{tikzcd} \qquad \begin{tikzcd} g.s' \arrow[r, "g.\tau"] \arrow[dr, "\sigma'"'] & g.s \arrow[d, "\sigma"] \\ & f \end{tikzcd}$$

or, more compactly:

$$t_{s'}(\sigma \circ (g.\tau)) = t_s(\sigma) \circ (\tau.tf). \tag{5.7}$$

Since the components $\mathrm{Dec}_\mathrm{r}(B) \to B$ and $\mathrm{Dec}_\mathrm{l}(B) \to B$ of the counit are strict morphisms of bicategories, it follows that $t : \mathrm{Dec}_\mathrm{r}(B) \to \mathrm{Dec}_\mathrm{l}(B)$ is also strict, which amounts to the requirements

$$t_{1_a}(1_f) = 1_{tf} \quad \text{and} \quad t_{s'}(\sigma') \circ (s'.t_s(\sigma)) = t_{s's}(\sigma \circ \sigma' s) \tag{5.8}$$

for all $\sigma : gs \to f$ and $\sigma' : hs' \to g$.

It remains to see what the comultiplication axiom imposes. As in the case for **Cat**, the only new condition appears at the level of objects of $\mathrm{Dec}_\mathrm{r}(B)$, where it says that for any $f : a \to x$, we have

$$t^2 1_x = 1_{tx} \quad \text{and} \quad (t_{tf}(t_f(1_f)) : tf.tt1_x \to tf) = 1_{tf}. \tag{5.9}$$

So duplicial structure on a bicategory B amounts to the assignments $x \mapsto tx$, $(f : a \to x) \mapsto (tf : tx \to a)$, and $(s, \sigma : gs \to f) \mapsto (t_s\sigma : s.tf \to tg)$, subject to the conditions expressed in (5.7), (5.8), and (5.9). We now relate this to the structure in the statement of the theorem.

For any $x \in B$, we define $\varepsilon_x = t(1_x) : tx \to x$, and for any $f : a \to x$ in B, we define $\varepsilon_f = t_f(1_f) : f.tf \to t1_x = \varepsilon_x$. Now in the conditions appearing in the theorem, (c) holds by construction, (d) holds by the first half of (5.9), while (e) holds by taking $f = 1_x$ in the first half of (5.8). For (f), take $f = 1_x$ in the definition of ε_f, the second half of (5.9), and the first half of (5.8), to deduce that $\varepsilon_{t1_x} = t_{t1_x}(1_{t1_x}) = t_{t1_x}(t_{1_x}(1_{1_x})) = 1_{t1_x}$.

Thus, in order to show that a duplicial bicategory has all of the structure in the theorem, it remains only to show that $t_f(1_f)$ exhibits tf as a right lifting of $t1_x$ through f; in other words, that for any $g : tx \to a$ and any $\varphi : fg \to t1_x$, there is a unique $\psi : g \to tf$ which gives φ when pasted with ε_f. But we may consider the pair (g, φ) as a morphism in B/x from $t1_x$ to f, and so obtain $t_g(\varphi) : g.t^2 1_x \to tf$, and since $t^2 1_x = 1_{tx}$, this gives our $\psi : g \to tf$. Pasting it with ε_f gives

$$\begin{aligned}
\varepsilon_f \circ f\psi &= t_f(1_f) \circ (f.t_g(\varphi)) \\
&= t_{fg}(\varphi) && \text{(by (5.8))} \\
&= t_{fg}(1_{t1_x} \circ (1_x.\varphi)) \\
&= t_{t1_x}(1_{t1_x}) \circ (\varphi.t^2 1_x) && \text{(by (5.7))} \\
&= t_{t1_x}(1_{t1_x}) \circ \varphi && \text{(by (5.9))} \\
&= \varphi, && \text{(by (f))}
\end{aligned}$$

which proves the existence of ψ. As for uniqueness, suppose that $\psi : g \to tf$ satisfies $\varepsilon_f \circ f\psi = \varphi$; that is, $t_f(1_f) \circ (f.\psi) = \varphi$. Then

$$\begin{aligned}
t_g(\varphi) &= t_g(t_f(1_f) \circ (f.\psi)) \\
&= t_{tf}(t_f(1_f)) \circ (\psi.t^2 1_x) && \text{(by (5.7))} \\
&= \psi.t^2 1_x && \text{(by (5.9))} \\
&= \psi, && \text{(by (5.9))}
\end{aligned}$$

giving uniqueness as required.

Thus, a duplicial bicategory satisfies the conditions in the theorem. For the converse, suppose that B is equipped with structure as in the theorem; then we are given the assignments $x \mapsto tx$ and $(f : a \to x) \mapsto (tf : tx \to a)$, as well as the 2-cells $t_f(1_f) : f.tf \to \varepsilon_x$ satisfying the universal property of (b) and the conditions (c), (d), (e), and (f). Given $\sigma : gs \to f$, if we are to have (5.8) and then (5.7), then

$$\varepsilon_g \circ (g.t_s(\sigma)) = t_g(1_g) \circ (g.t_s(\sigma)) = t_{gs}(\sigma) = \varepsilon_f \circ (\sigma.tf),$$

and so $t_s(\sigma)$ is uniquely determined using the universal property of the right lifting 2-cell ε_g. It remains to check that if we define $t_s(\sigma)$ in this way, then (5.7), (5.8), and (5.9) do indeed hold.

Since $\varepsilon_g \circ (g.t_s(\sigma)) \circ (g.\tau.tf) = \varepsilon_f \circ (\sigma.tf) \circ (g.\tau.tf)$, the composite $t_s(\sigma) \circ (\tau.tf)$ satisfies the defining property of $t_{s'}(\sigma \circ (g.\tau))$, and so (5.7) holds. Similarly,

$$\begin{aligned}\varepsilon_h \circ (h.t_{s'}(\sigma')) \circ (h.s'.t_s(\sigma)) &= \varepsilon_g \circ (\sigma'.tg) \circ (h.s'.t_s(\sigma)) \\ &= \varepsilon_g \circ (g.t_s(\sigma)) \circ (\sigma'.s.tf) \\ &= \varepsilon_f \circ (\sigma.tf) \circ (\sigma'.s.tf),\end{aligned}$$

and so $t_{s'}(\sigma') \circ (s'.t_s(\sigma))$ satisfies the defining property of $t_{s's}(\sigma \circ \sigma' s)$, while 1_{tf} clearly satisfies the defining property of $t_{1_a}(1_f)$. Thus (5.8) holds.

The first half of (5.9) is just (d); as for the second half, it says that $t_{tf}(\varepsilon_f) = 1_{tf}$, and the defining property of $t_{tf}(\varepsilon_f)$ is that $\varepsilon_f \circ (f.t_{tf}(\varepsilon_f)) = \varepsilon_{t1_x} \circ (\varepsilon_f.t^2 1_x)$; but $t^2 1_x = 1_{tx}$ by (d), and $\varepsilon_{t1_x} = 1_{t1_x}$ by (e). Thus the right-hand side becomes ε_f, and clearly $\varepsilon_f \circ 1_{tf} = \varepsilon_f$, whence the result. □

5C. ***Duplicial structure on monoidal categories.*** A monoidal category can be thought of as a one-object bicategory, and as such it has a nerve: there is a unique 0-simplex, the 1-simplices are the objects of the monoidal category, the 2-simplices consist of three objects X, Y, Z and a morphism $f : X \otimes Y \to Z$, and so on. Thus the monoidal categories determine a full subcategory of $[\Delta^{\mathrm{op}}, \mathbf{Set}]$, with the morphisms being the (lax) monoidal functors which are strict with respect to the unit. It is not the case that the decalage comonads restrict to this full subcategory: the decalage of a one-object bicategory will generally have many objects, indeed an object of the decalage will be a morphism of the monoidal category. Nonetheless, we can ask what it is to have duplicial structure on a monoidal category, thought of as a one-object bicategory.

Reading off directly from Theorem 5.6, we see that, for a monoidal category C with tensor product $\otimes$ and unit i, duplicial structure on C consists of the following:

(a) an object d (corresponding to ε_x for the unique object x of the bicategory);

(b) for each object x, a right internal hom $[x, d]$, by which we mean an object equipped with a morphism $\varepsilon_x : x \otimes [x, d] \to d$ inducing a bijection

$$C(x \otimes -, d) \cong C(-, [x, d])$$

subject to conditions which we now enumerate. First of all, we require that the internal hom $[i, d]$ be d itself. This is not a restriction in practice, since in any monoidal category and any object x the internal hom $[i, x]$ exists and may be taken to be x. The more serious requirement is that the (chosen) hom $[d, d]$ is i, with counit $d \otimes i \to d$ given by the unit isomorphism of the monoidal category. In fact the real condition here is that the map $i \to [d, d]$ induced by the unit isomorphism $d \otimes i \to d$ is invertible; when this is the case we may always redefine $[d, d]$ as required.

One formulation of the notion of (not necessarily symmetric) $*$-autonomous category [Barr 1995, Definition 2.3] is a monoidal category C equipped with an equivalence $(-)^* : C \to C^{\mathrm{op}}$ and natural isomorphism $C(x, y^*) \cong C(i, (x \otimes y)^*)$, with i the unit. Using the natural isomorphism, we may construct further isomorphisms $C(x, y^*) \cong C(i, (x \otimes y)^*) \cong C(i, (x \otimes y \otimes i)^*) \cong C(x \otimes y, i^*)$, and so y^* must in fact be given by $[y, i^*]$. Conversely, if C is a monoidal category with all (right) internal homs $[x, d]$ for a given object d, then there is a functor $(-)^* : C \to C^{\mathrm{op}}$ sending x to $[x, d]$, and a natural isomorphism $C(x, y^*) \cong C(i, (x \otimes y)^*)$; thus C will be $*$-autonomous when this functor $(-)^*$ is an equivalence.

A *compact closed category* is a symmetric monoidal category C in which every object has a monoidal dual. In this case, the functor $C \to C^{\mathrm{op}}$ sending each object to its monoidal dual is an equivalence. Thus, every compact closed category is $*$-autonomous; the dualizing object d is the unit object i in this case. In a general $*$-autonomous category, x^* need not be the monoidal dual of x.

Both duplicial structure and $*$-autonomous structure on a monoidal category C involve an object d for which the right internal homs $[x, d]$ exist. The difference is that $*$-autonomous categories require the functor $[-, d]$ to be an equivalence, while duplicial monoidal categories require the canonical map $i \to [d, d]$ to be invertible. But in fact, for a $*$-autonomous category the canonical map $i \to [d, d]$ is always invertible [Barr 1995, Section 6] and so any $*$-autonomous category has duplicial structure.

Theorem 5.10. *Any monoidal category with paracyclic structure possesses a $*$-autonomous structure. Conversely, any monoidal category with $*$-autonomous structure is monoidally equivalent to one with paracyclic structure.*

Proof. If C is a monoidal category with paracyclic structure, then there is an object d for which the right internal homs $[-, d]$ exist, and the resulting functor $C \to C^{\mathrm{op}}$ is not just an equivalence but an isomorphism. This gives C a $*$-autonomous structure.

For the converse, let C be a $*$-autonomous monoidal category with dualizing object d. We shall construct another $*$-autonomous monoidal category $\widetilde{C}$ which is monoidally equivalent to C, for which the induced duality functor $\widetilde{C} \to \widetilde{C}^{\mathrm{op}}$ can be chosen to be an isomorphism.

An object x of $\widetilde{C}$ is a $\mathbb{Z}$-indexed family $(x_n)_{n \in \mathbb{Z}}$ of objects of C, together with an isomorphism $\theta_n : x_n \cong x^*_{n+1}$ for each n. A morphism $x \to y$ is just a morphism $f : x_0 \to y_0$ in C. There is an evident equivalence of categories $\widetilde{C} \to C$ sending x to x_0.

We may transport the monoidal structure across this equivalence to obtain a monoidal structure on $\widetilde{C}$. The resulting $\widetilde{C}$ is clearly still $*$-autonomous, but now we may define the functor $\widetilde{C} \to \widetilde{C}^{\mathrm{op}}$ in such a way that it is an isomorphism of categories, by setting $(x^*)_n = x_{n-1}$. In order to make this functorial, observe that for any morphism $f : x_0 \to y_0$, we may use the θ_n to define morphisms $f_{2n} : x_{2n} \to y_{2n}$ and $f_{2n+1} : y_{2n+1} \to x_{2n+1}$ which are compatible in the evident sense. □

It turns out that if C is $*$-autonomous, then the pseudoinverse $C^{\mathrm{op}} \to C$ to $(-)^*$ gives rise to a *left* internal hom $d^{(-)}$, characterized by a natural isomorphism $C(a, d^b) \cong C(b \otimes a, d)$. If the monoidal category C actually has cyclic structure, then applying $[-, d]$ twice gives the identity, and so in particular the left and right homs d^b and $[b, d]$ are isomorphic; in other words, $[-, d]$ is also a left internal hom. In this case, d is said to be a *cyclic dualizing object*.

Conversely, if C is $*$-autonomous with cyclic dualizing object d, then applying $[-, d]$ twice is isomorphic to the identity. Once again, though, for a cyclic structure we need it to be equal to the identity.

Theorem 5.11. *A monoidal category with cyclic structure has a $*$-autonomous structure with cyclic dualizing object. Conversely, any $*$-autonomous monoidal category with cyclic dualizing object is monoidally equivalent to one with cyclic structure.*

Proof. The first half follows from the discussion before the theorem. For the second, let C be a $*$-autonomous monoidal category with cyclic dualizing object d. As in the previous proposition, we construct another $*$-autonomous monoidal category $\overline{C}$ which is monoidally equivalent to C. An object x of $\overline{C}$ consists of a pair (x_+, x_-) of objects of C equipped with an isomorphism $\theta : x_+ \cong x_-^*$. A morphism $f : x \to y$ consists of a morphism $f_+ : x_+ \to y_+$; once again, there is an associated $f_- : y_- \to x_-$ suitably compatible with the θ. There is again an evident equivalence $\overline{C} \to C$ sending x to x_+, and we may transport the monoidal structure across this equivalence.

Since d is a cyclic dualizing object, any isomorphism $\theta : x_+ \cong x_-^*$ has a corresponding $\theta' : x_- \cong {}^*x_+ \cong x_+^*$. Thus we may define $\overline{C} \to \overline{C}^{\mathrm{op}}$ to send (x_+, x_-, θ) to (x_-, x_+, θ'), and applying this twice clearly gives the identity. □

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References

[Barr 1995] M. Barr, "Nonsymmetric *-autonomous categories", *Theoret. Comput. Sci.* **139**:1–2 (1995), 115–130. MR Zbl

[Beck 1969] J. Beck, "Distributive laws", pp. 119–140 in *Seminar on triples and categorical homology theory* (Zürich, 1966/67), edited by B. Eckmann, Lecture Notes in Math. **80**, Springer, 1969. MR Zbl

[Böhm and Ştefan 2008] G. Böhm and D. Ştefan, "(Co)cyclic (co)homology of bialgebroids: an approach via (co)monads", *Comm. Math. Phys.* **282**:1 (2008), 239–286. MR Zbl

[Böhm and Ştefan 2009] G. Böhm and D. Ştefan, "Examples of para-cocyclic objects induced by BD-laws", *Algebr. Represent. Theory* **12**:2–5 (2009), 153–180. MR Zbl

[Böhm and Ştefan 2012] G. Böhm and D. Ştefan, "A categorical approach to cyclic duality", *J. Noncommut. Geom.* **6**:3 (2012), 481–538. MR Zbl

[Bullejos et al. 2005] M. Bullejos, E. Faro, and V. Blanco, "A full and faithful nerve for 2-categories", *Appl. Categ. Structures* **13**:3 (2005), 223–233. MR Zbl

[Burroni 1973] E. Burroni, "Algèbres non déterministiques et D-catégories", *Cahiers Topologie Géom. Différentielle* **14**:4 (1973), 417–475. MR Zbl

[Connes 1983] A. Connes, "Cohomologie cyclique et foncteurs Ext^n", *C. R. Acad. Sci. Paris Sér. I Math.* **296**:23 (1983), 953–958. MR Zbl

[Day 1970] B. Day, "On closed categories of functors", pp. 1–38 in *Reports of the Midwest Category Seminar, IV*, edited by S. Mac Lane, Lecture Notes in Math. **137**, Springer, 1970. MR Zbl

[Day et al. 2007] B. Day, E. Panchadcharam, and R. Street, "Lax braidings and the lax centre", pp. 1–17 in *Hopf algebras and generalizations*, edited by L. H. Kauffman et al., Contemp. Math. **441**, American Mathematical Society, Providence, RI, 2007. MR Zbl

[Dwyer and Kan 1985] W. G. Dwyer and D. M. Kan, "Normalizing the cyclic modules of Connes", *Comment. Math. Helv.* **60**:4 (1985), 582–600. MR Zbl

[Elmendorf 1993] A. D. Elmendorf, "A simple formula for cyclic duality", *Proc. Amer. Math. Soc.* **118**:3 (1993), 709–711. MR Zbl

[Gabriel and Zisman 1967] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik **35**, Springer, 1967. MR Zbl

[Getzler and Jones 1993] E. Getzler and J. D. S. Jones, "The cyclic homology of crossed product algebras", *J. Reine Angew. Math.* **445** (1993), 161–174. MR Zbl

[Kelly 1980] G. M. Kelly, "A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on", *Bull. Austral. Math. Soc.* **22**:1 (1980), 1–83. MR Zbl

[Kelly and Lack 1997] G. M. Kelly and S. Lack, "On property-like structures", *Theory Appl. Categ.* **3** (1997), No. 9, 213–250. MR Zbl

[Kowalzig et al. 2015] N. Kowalzig, U. Krähmer, and P. Slevin, "Cyclic homology arising from adjunctions", *Theory Appl. Categ.* **30** (2015), 1067–1095. MR Zbl

[Krähmer and Slevin 2016] U. Krähmer and P. Slevin, "Factorisations of distributive laws", *J. Pure Appl. Algebra* **220**:4 (2016), 1403–1418. MR Zbl

[Lack 2000] S. Lack, "A coherent approach to pseudomonads", *Adv. Math.* **152**:2 (2000), 179–202. MR Zbl

[Lack 2002] S. Lack, "Codescent objects and coherence", *J. Pure Appl. Algebra* **175**:1-3 (2002), 223–241. MR Zbl

[Lack and Street 2012] S. Lack and R. Street, "Skew monoidales, skew warpings and quantum categories", *Theory Appl. Categ.* **26** (2012), No. 15, 385–402. MR Zbl

[Loday 1992] J.-L. Loday, *Cyclic homology*, Grundlehren der Math. Wissenschaften **301**, Springer, 1992. MR Zbl

[Mac Lane 1971] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Math. **5**, Springer, 1971. MR Zbl

[Schauenburg 2000] P. Schauenburg, "Duals and doubles of quantum groupoids ($\times_R$-Hopf algebras)", pp. 273–299 in *New trends in Hopf algebra theory* (La Falda, 1999), edited by N. Andruskiewitsch et al., Contemp. Math. **267**, American Mathematical Society, Providence, RI, 2000. MR Zbl

[Street 1972] R. Street, "The formal theory of monads", *J. Pure Appl. Algebra* **2**:2 (1972), 149–168. MR Zbl

[Street 1987] R. Street, "Correction to "Fibrations in bicategories"", *Cahiers Topologie Géom. Différentielle Catég.* **28**:1 (1987), 53–56. MR Zbl

[Street 1996] R. Street, "Categorical structures", pp. 529–577 in *Handbook of algebra*, vol. 1, edited by M. Hazewinkel, Elsevier, Amsterdam, 1996. MR Zbl

[Street and Walters 1978] R. Street and R. Walters, "Yoneda structures on 2-categories", *J. Algebra* **50**:2 (1978), 350–379. MR Zbl

[Szlachányi 2012] K. Szlachányi, "Skew-monoidal categories and bialgebroids", *Adv. Math.* **231**:3-4 (2012), 1694–1730. MR Zbl

[Verity 2008] D. Verity, *Complicial sets characterising the simplicial nerves of strict ω-categories*, Mem. Amer. Math. Soc. **905**, 2008. MR Zbl

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