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# Localization, Whitehead groups and the Atiyah conjecture

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Let  $K_1^w(\mathbb{Z}G)$  be the  $K_1$ -group of square matrices over  $\mathbb{Z}G$  which are not necessarily invertible but induce weak isomorphisms after passing to Hilbert space completions. Let  $\mathfrak{D}(G; \mathbb{Q})$  be the division closure of  $\mathbb{Q}G$  in the algebra  $\mathfrak{U}(G)$  of operators affiliated to the group von Neumann algebra. Let  $\mathcal{C}$  be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Let  $G$  be a torsionfree group which belongs to  $\mathcal{C}$ . Then we prove that  $K_1^w(\mathbb{Z}(G))$  is isomorphic to  $K_1(\mathfrak{D}(G; \mathbb{Q}))$ . Furthermore we show that  $\mathfrak{D}(G; \mathbb{Q})$  is a skew field and hence  $K_1(\mathfrak{D}(G; \mathbb{Q}))$  is the abelianization of the multiplicative group of units in  $\mathfrak{D}(G; \mathbb{Q})$ .

## 0. Introduction

In [Friedl and Lück 2017] we introduced the *universal  $L^2$ -torsion*  $\rho_u^{(2)}(X; \mathcal{N}(G))$  of an  $L^2$ -acyclic finite  $G$ -CW-complex  $X$  and discussed its applications. It takes values in a certain abelian group  $\text{Wh}^w(G)$ , which is the quotient of the  $K_1$ -group  $K_1^w(\mathbb{Z}G)$  by the subgroup given by trivial units  $\{\pm g \mid g \in G\}$ . Elements  $[A]$  of  $K_1^w(\mathbb{Z}G)$  are given by  $(n, n)$ -matrices  $A$  over  $\mathbb{Z}G$  which are not necessarily invertible but for which the operator  $r_A^{(2)}: L^2(G)^n \rightarrow L^2(G)^n$  given by right multiplication with  $A$  is a weak isomorphism, i.e., it is injective and has dense image. We require for such square matrices  $A, B$  the following relations in  $K_1^w(\mathbb{Z}G)$ :

$$[AB] = [A] \cdot [B]; \quad \left[ \begin{pmatrix} A & * \\ 0 & B \end{pmatrix} \right] = [A] \cdot [B].$$

More details about  $\text{Wh}^w(G)$  and  $K_1^w(\mathbb{Z}G)$  will be given in Section 3.

Let  $\mathfrak{D}(G; \mathbb{Q}) \subseteq \mathfrak{U}(G)$  be the smallest subring of the algebra  $\mathfrak{U}(G)$  of operators  $L^2(G) \rightarrow L^2(G)$  affiliated to the group von Neumann algebra  $\mathcal{N}(G)$  which contains  $\mathbb{Q}G$  and is division closed, i.e., any element in  $\mathfrak{D}(G; \mathbb{Q})$  which is invertible in  $\mathfrak{U}(G)$  is already invertible in  $\mathfrak{D}(G; \mathbb{Q})$ . (These notions will be explained in detail in Section 2A.)

The main result of this paper is:

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**Theorem 0.1** ( $K_1^w(G)$  and units in  $\mathfrak{D}(G; \mathbb{Q})$ ). *Let  $\mathcal{C}$  be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Let  $G$  be a torsionfree group which belongs to  $\mathcal{C}$ . Then  $\mathfrak{D}(G; \mathbb{Q})$  is a skew field and there are isomorphisms*

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathfrak{D}(G; \mathbb{Q})) \xrightarrow{\cong} \mathfrak{D}(G; \mathbb{Q})^\times / [\mathfrak{D}(G; \mathbb{Q})^\times, \mathfrak{D}(G; \mathbb{Q})^\times].$$

In the special case that  $G = \mathbb{Z}$ , the right side reduces to the multiplicative abelian group of nontrivial elements in the field  $\mathbb{Q}(z, z^{-1})$  of rational functions with rational coefficients in one variable. This reflects the fact that in the case  $G = \mathbb{Z}$  the universal  $L^2$ -torsion is closely related to Alexander polynomials.

## 1. Universal localization

**1A. Review of universal localization.** Let  $R$  be a (associative) ring (with unit) and let  $\Sigma$  be a set of homomorphisms between finitely generated projective (left)  $R$ -modules. A ring homomorphism  $f : R \rightarrow S$  is called  $\Sigma$ -*inverting* if for every element  $\alpha : M \rightarrow N$  of  $\Sigma$  the induced map  $S \otimes_R \alpha : S \otimes_R M \rightarrow S \otimes_R N$  is an isomorphism. A  $\Sigma$ -inverting ring homomorphism  $i : R \rightarrow R_\Sigma$  is called *universal  $\Sigma$ -inverting* if for any  $\Sigma$ -inverting ring homomorphism  $f : R \rightarrow S$  there is precisely one ring homomorphism  $f_\Sigma : R_\Sigma \rightarrow S$  satisfying  $f_\Sigma \circ i = f$ . If  $f : R \rightarrow R_\Sigma$  and  $f' : R \rightarrow R'_\Sigma$  are two universal  $\Sigma$ -inverting homomorphisms, then by the universal property there is precisely one isomorphism  $g : R_\Sigma \rightarrow R'_\Sigma$  with  $g \circ f = f'$ . This shows the uniqueness of the universal  $\Sigma$ -inverting homomorphism. The universal  $\Sigma$ -inverting ring homomorphism exists; see [Schofield 1985, Section 4]. If  $\Sigma$  is a set of matrices, a model for  $R_\Sigma$  is given by considering the free  $R$ -ring generated by the set of symbols  $\{\bar{a}_{i,j} \mid A = (a_{i,j}) \in \Sigma\}$  and dividing out the relations given in matrix form by  $\bar{A}A = A\bar{A} = 1$ , where  $\bar{A}$  stands for  $(\bar{a}_{i,j})$  for  $A = (a_{i,j})$ . The map  $i : R \rightarrow R_\Sigma$  does not need to be injective and the functor  $R_\Sigma \otimes_R -$  does not need to be exact in general.

A special case of a universal localization is the Ore localization  $S^{-1}R$  of a ring  $R$  for a multiplicative closed subset  $S \subseteq R$  which satisfies the Ore condition, namely take  $\Sigma$  to be the set of  $R$ -homomorphisms  $r_s : R \rightarrow R$ ,  $r \mapsto rs$ , where  $s$  runs through  $S$ . For the Ore localization the functor  $S^{-1}R \otimes_R -$  is exact and the kernel of the canonical map  $R \rightarrow S^{-1}R$  is  $\{r \in R \mid \exists s \in S \text{ with } rs = 0\}$ .

Let  $R$  be a ring and let  $\Sigma$  be a set of homomorphisms between finitely generated projective  $R$ -modules. We call  $\Sigma$  *saturated* if for any two elements  $f_0 : P_0 \rightarrow Q_0$  and  $f_1 : P_1 \rightarrow Q_1$  of  $\Sigma$  and any  $R$ -homomorphism  $g_0 : P_0 \rightarrow Q_1$  and  $g_1 : P_1 \rightarrow Q_0$  the  $R$ -homomorphisms

$$\begin{pmatrix} f_0 & 0 \\ g_0 & f_1 \end{pmatrix} : P_0 \oplus P_1 \rightarrow Q_0 \oplus Q_1 \quad \text{and} \quad \begin{pmatrix} f_0 & g_1 \\ 0 & f_1 \end{pmatrix} : P_0 \oplus P_1 \rightarrow Q_0 \oplus Q_1$$

belong to  $\Sigma$  and, for every  $R$ -homomorphism  $f_0 : P_0 \rightarrow Q_0$  which becomes invertible over  $R_\Sigma$ , there is an element  $f_1 : P_1 \rightarrow Q_1$  in  $\Sigma$ , finitely generated projective  $R$ -modules  $X$  and  $Y$ , and  $R$ -isomorphisms  $u : P_0 \oplus X \xrightarrow{\cong} P_1 \oplus Y$  and  $v : Q_0 \oplus X \xrightarrow{\cong} Q_1 \oplus Y$  satisfying  $(f_1 \oplus \text{id}_Y) \circ u = v \circ (f_0 \oplus \text{id}_X)$ . We can always find for  $\Sigma$  another set  $\Sigma'$  with  $\Sigma \subseteq \Sigma'$  such that  $\Sigma'$  is saturated and the canonical map  $R_\Sigma \rightarrow R_{\Sigma'}$  is bijective. Moreover, in nearly all cases we will consider sets  $\Sigma$  which are already saturated. Indeed if  $\Sigma'$  denotes the set of all maps between finitely generated projective (left) modules which become invertible over  $R_\Sigma$ , then  $\Sigma \subseteq \Sigma'$ ,  $\Sigma'$  is saturated, and the canonical map  $R_\Sigma \rightarrow R_{\Sigma'}$  is an isomorphism; see [Cohn 1985, Exercise 7.2.8 on page 394]. Therefore we can assume without harm in the sequel that  $\Sigma$  is saturated.

**1B.  $K_1$  of universal localizations.** Let  $R$  be a ring and let  $\Sigma$  be a (saturated) set of homomorphisms between finitely generated projective  $R$ -modules.

**Definition 1.1.** Let  $K_1(R, \Sigma)$  be the abelian group defined in terms of generators and relations as follows: Generators  $[f]$  are (conjugacy classes of)  $R$ -endomorphisms  $f : P \rightarrow P$  of finitely generated projective  $R$ -modules  $P$  such that  $\text{id}_{R_\Sigma} \otimes_R f : R_\Sigma \otimes_R P \rightarrow R_\Sigma \otimes_R P$  is an isomorphism. If  $f, g : P \rightarrow P$  are  $R$ -endomorphisms of the same finitely generated projective  $R$ -module  $P$  such that  $\text{id}_{R_\Sigma} \otimes_R f$  and  $\text{id}_{R_\Sigma} \otimes_R g$  are bijective, then we require the relation

$$[g \circ f] = [g] + [f].$$

If we have a commutative diagram of finitely generated projective  $R$ -modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0 & \xrightarrow{i} & P_1 & \xrightarrow{p} & P_2 & \longrightarrow & 0 \\ & & f_0 \downarrow & & f_1 \downarrow & & f_2 \downarrow & & \\ 0 & \longrightarrow & P_0 & \xrightarrow{i} & P_1 & \xrightarrow{p} & P_2 & \longrightarrow & 0 \end{array}$$

such that  $\text{id}_{R_\Sigma} \otimes_R f_0$ ,  $\text{id}_{R_\Sigma} \otimes_R f_2$  (and hence  $\text{id}_{R_\Sigma} \otimes_R f_1$ ) are bijective, then we require the relation

$$[f_1] = [f_0] + [f_2].$$

If the set  $\Sigma$  consists of all isomorphisms  $R^n \xrightarrow{\cong} R^n$  for all  $n \geq 0$ , then for an  $R$ -endomorphism  $f : P \rightarrow P$  of a finitely generated projective  $R$ -module  $P$ , the induced map  $\text{id}_{R_\Sigma} \otimes f$  is bijective if and only if  $f$  itself is already bijective and hence  $K_1(R, \Sigma)$  is just the classical first  $K$ -group  $K_1(R)$ .

The main result of this section is:

**Theorem 1.2** ( $K_1(R, \Sigma)$  and  $K_1(R_\Sigma)$ ). *Suppose that every element in  $\Sigma$  is given by an endomorphism of a finitely generated projective  $R$ -module and that the*

canonical map  $i : R \rightarrow R_\Sigma$  is injective. Then the homomorphism

$$\alpha : K_1(R, \Sigma) \xrightarrow{\cong} K_1(R_\Sigma), \quad [f : P \rightarrow P] \mapsto [\text{id}_{R_\Sigma} \otimes_R f : R_\Sigma \otimes_R P \rightarrow R_\Sigma \otimes_R P],$$

is bijective.

*Proof.* We construct an inverse

$$\beta : K_1(R_\Sigma) \rightarrow K_1(R, \Sigma) \tag{1.3}$$

as follows: Consider an element  $x$  in  $K_1(R_\Sigma)$ . Then we can choose a finitely generated projective  $R$ -module  $Q$  (actually, we could choose it to be finitely generated free) and an  $R_\Sigma$ -automorphism

$$a : R_\Sigma \otimes_R Q \xrightarrow{\cong} R_\Sigma \otimes_R Q$$

such that  $x = [a]$ . Now the key ingredient is Cramer's rule; see [Schofield 1985, Theorem 4.3 on page 53]. It implies the existence of a finitely generated projective  $R$ -module  $P$ , two  $R$ -homomorphisms  $b, b' : P \oplus Q \rightarrow P \oplus Q$  and an  $R_\Sigma$ -homomorphism  $a' : R_\Sigma \otimes_R Q \rightarrow R_\Sigma \otimes_R P$  such that  $\text{id}_{R_\Sigma} \otimes_R b$  is bijective, and for the  $R_\Sigma$ -homomorphism

$$A = \begin{pmatrix} \text{id}_{R_\Sigma \otimes_R P} & a' \\ 0 & a \end{pmatrix} : R_\Sigma \otimes_R P \oplus R_\Sigma \otimes_R Q \rightarrow R_\Sigma \otimes_R P \oplus R_\Sigma \otimes_R Q$$

the composite

$$\begin{aligned} R_\Sigma \oplus (P \oplus Q) &\xrightarrow{i} R_\Sigma \otimes_R P \oplus R_\Sigma \otimes_R Q \xrightarrow{A} R_\Sigma \otimes_R P \oplus R_\Sigma \otimes_R Q \\ &\xrightarrow{i^{-1}} R_\Sigma \oplus (P \oplus Q) \xrightarrow{\text{id}_{R_\Sigma} \otimes_R b} R_\Sigma \oplus (P \oplus Q) \end{aligned}$$

agrees with  $\text{id}_{R_\Sigma} \otimes_R b'$ , where  $i$  is the canonical  $R_\Sigma$ -isomorphism. Then also  $\text{id}_{R_\Sigma} \otimes_R b$  is bijective. We want to define

$$\beta(x) := [b'] - [b]. \tag{1.4}$$

The main problem is to show that this is independent of the various choices. Given a finitely generated projective  $R$ -module  $P$  and an  $R_\Sigma$ -automorphism

$$a : R_\Sigma \otimes_R Q \xrightarrow{\cong} R_\Sigma \otimes_R Q$$

and two such choices  $(P, b, b', a')$  and  $(\bar{P}, \bar{b}, \bar{b}', \bar{a}')$ , we next show

$$[b] - [b'] := [\bar{b}] - [\bar{b}']. \tag{1.5}$$

We can write

$$\begin{aligned} b &= \begin{pmatrix} b_{P,P} & b_{Q,P} \\ b_{P,Q} & b_{Q,Q} \end{pmatrix}, & b' &= \begin{pmatrix} b'_{P,P} & b'_{Q,P} \\ b'_{P,Q} & b'_{Q,Q} \end{pmatrix}, \\ \bar{b} &= \begin{pmatrix} \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} \\ \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} \end{pmatrix}, & \bar{b}' &= \begin{pmatrix} \bar{b}'_{\bar{P},\bar{P}} & \bar{b}'_{Q,\bar{P}} \\ \bar{b}'_{\bar{P},Q} & \bar{b}'_{Q,Q} \end{pmatrix}, \end{aligned}$$

for  $R$ -homomorphisms  $b_{P,P} : P \rightarrow P$ ,  $b_{P,Q} : P \rightarrow Q$ ,  $b_{Q,P} : Q \rightarrow P$  and  $b_{Q,Q} : Q \rightarrow Q$ , and analogously for  $b'$ ,  $\bar{b}$  and  $\bar{b}'$ . Then the relation between  $b$  and  $b'$  and  $\bar{b}$  and  $\bar{b}'$  becomes

$$\begin{pmatrix} \text{id}_{R_\Sigma} \otimes_R b_{P,P} & \text{id}_{R_\Sigma} \otimes_R b_{Q,P} \\ \text{id}_{R_\Sigma} \otimes_R b_{P,Q} & \text{id}_{R_\Sigma} \otimes_R b_{Q,Q} \end{pmatrix} \circ \begin{pmatrix} \text{id}_{R_\Sigma \otimes_R P} & a' \\ 0 & a \end{pmatrix} = \begin{pmatrix} \text{id}_{R_\Sigma} \otimes_R b'_{P,P} & \text{id}_{R_\Sigma} \otimes_R b'_{Q,P} \\ \text{id}_{R_\Sigma} \otimes_R b'_{P,Q} & \text{id}_{R_\Sigma} \otimes_R b'_{Q,Q} \end{pmatrix},$$

and analogously for  $\bar{b}$  and  $\bar{b}'$ . This implies  $\text{id}_{R_\Sigma} \otimes_R b_{P,P} = \text{id}_{R_\Sigma} \otimes_R b'_{P,P}$  and hence  $b_{P,P} = b'_{P,P}$  because of the injectivity of  $i : R \rightarrow R_\Sigma$ . Analogously we get  $b_{P,Q} = b'_{P,Q}$ ,  $\bar{b}_{\bar{P},\bar{P}} = \bar{b}'_{\bar{P},\bar{P}}$  and  $\bar{b}_{\bar{P},Q} = \bar{b}'_{\bar{P},Q}$ .

The argument in [Schofield 1985, page 64–65] based on Malcolmson's criterion [ibid., Theorem 4.2 on page 53] implies that there exist finitely generated projective  $R$ -modules  $X_0$  and  $X_1$ , and  $R$ -homomorphisms

$$\begin{aligned} d_1 : X_1 &\rightarrow X_1, & d_2 : X_2 &\rightarrow X_2, \\ e_1 : X_1 &\rightarrow Q, & e_2 : X_2 &\rightarrow P, \\ \mu : P \oplus Q \oplus \bar{P} \oplus Q \oplus X_1 \oplus X_2 \oplus Q &\rightarrow P \oplus Q \oplus \bar{P} \oplus Q \oplus X_1 \oplus X_2 \oplus Q, \\ \nu : P \oplus Q \oplus \bar{P} \oplus Q \oplus X_1 \oplus X_2 &\rightarrow P \oplus Q \oplus \bar{P} \oplus Q \oplus X_1 \oplus X_2, \\ \tau : P \oplus Q \oplus \bar{P} \oplus Q \oplus X_1 \oplus X_2 &\rightarrow Q, \end{aligned}$$

such that  $\text{id}_{R_\Sigma} \otimes_R d_1$ ,  $\text{id}_{R_\Sigma} \otimes_R d_2$ ,  $\text{id}_{R_\Sigma} \otimes_R \mu$  and  $\text{id}_{R_\Sigma} \otimes_R \nu$  are  $R_\Sigma$ -isomorphisms and, for the four  $R$ -homomorphisms

$$P \oplus Q \oplus \bar{P} \oplus Q \oplus X_1 \oplus X_2 \oplus Q \rightarrow P \oplus Q \oplus \bar{P} \oplus Q \oplus X_1 \oplus X_2 \oplus Q$$

given by

$$\alpha = \begin{pmatrix} b_{P,P} & b_{Q,P} & 0 & 0 & 0 & 0 & 0 \\ b_{P,Q} & b_{Q,Q} & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{b}_{Q,\bar{P}} & \bar{b}'_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & 0 & 0 & \bar{b}'_{Q,\bar{P}} \\ 0 & \bar{b}_{Q,Q} & \bar{b}'_{\bar{P},Q} & \bar{b}_{Q,Q} & 0 & 0 & \bar{b}'_{Q,Q} \\ 0 & 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & \text{id}_Q & e_1 & 0 & 0 \end{pmatrix}, \quad \alpha' = \begin{pmatrix} b'_{P,P} & b_{Q,P} & 0 & 0 & 0 & 0 & -b'_{Q,P} \\ b'_{P,Q} & b_{Q,Q} & 0 & 0 & 0 & 0 & -b'_{Q,Q} \\ 0 & \bar{b}_{Q,\bar{P}} & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & 0 & 0 & 0 \\ 0 & \bar{b}_{Q,Q} & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_2 & -e_2 \\ 0 & 0 & 0 & \text{id}_Q & e_1 & 0 & 0 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} \nu & 0 \\ 0 & \text{id}_Q \end{pmatrix} \quad \text{and} \quad \gamma' = \begin{pmatrix} \nu & 0 \\ \tau & \text{id}_Q \end{pmatrix}$$

we get equations of maps of  $R$ -modules

$$\mu \circ \gamma = \alpha, \quad \mu \circ \gamma' = \alpha'.$$

Since  $\text{id}_{R_\Sigma} \otimes_R \mu$ ,  $\text{id}_{R_\Sigma} \otimes_R \gamma$  and  $\text{id}_{R_\Sigma} \otimes_R \gamma'$  are isomorphisms, also  $\text{id}_{R_\Sigma} \otimes_R \alpha$  and  $\text{id}_{R_\Sigma} \otimes_R \alpha'$  are isomorphisms. Hence we get well-defined elements  $[\mu]$ ,  $[\nu]$ ,  $[\nu']$ ,

$[\alpha]$  and  $[\alpha']$  in  $K_1(R, \Sigma)$  satisfying

$$[\mu] = [\gamma] + [\alpha], \quad [\mu] = [\gamma'] + [\alpha'], \quad [\gamma] = [\gamma'].$$

This implies

$$[\alpha] = [\alpha']. \quad (1.6)$$

If we interchange in the matrix defining  $\alpha$  the fourth and the last column, we get a matrix in a suitable block form, which allows us to deduce

$$\begin{aligned} [\alpha] &= - \left[ \begin{pmatrix} b_{P,P} & b_{Q,P} & 0 & 0 & 0 & 0 & 0 \\ b_{P,Q} & b_{Q,Q} & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{b}_{Q,\bar{P}} & \bar{b}'_{\bar{P},\bar{P}} & \bar{b}'_{Q,\bar{P}} & 0 & 0 & \bar{b}_{Q,\bar{P}} \\ 0 & \bar{b}_{Q,Q} & \bar{b}'_{\bar{P},Q} & \bar{b}'_{Q,Q} & 0 & 0 & \bar{b}_{Q,Q} \\ 0 & 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 0 & e_1 & 0 & \text{id}_Q \end{pmatrix} \right] \\ &= - \left[ \begin{pmatrix} b_{P,P} & b_{Q,P} & 0 & 0 \\ b_{P,Q} & b_{Q,Q} & 0 & 0 \\ 0 & \bar{b}_{Q,\bar{P}} & \bar{b}'_{\bar{P},\bar{P}} & \bar{b}'_{Q,\bar{P}} \\ 0 & \bar{b}_{Q,Q} & \bar{b}'_{\bar{P},Q} & \bar{b}'_{Q,Q} \end{pmatrix} \right] - \left[ \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ e_1 & 0 & \text{id}_Q \end{pmatrix} \right] \\ &= - \left[ \begin{pmatrix} b_{P,P} & b_{Q,P} \\ b_{P,Q} & b_{Q,Q} \end{pmatrix} \right] - \left[ \begin{pmatrix} \bar{b}'_{\bar{P},\bar{P}} & \bar{b}'_{Q,\bar{P}} \\ \bar{b}'_{\bar{P},Q} & \bar{b}'_{Q,Q} \end{pmatrix} \right] - [d_1] - [d_2] - [\text{id}_Q] \\ &= -[b] - [\bar{b}'] - [d_1] - [d_2]. \end{aligned} \quad (1.7)$$

Similarly we get from the matrix describing  $\alpha'$  after interchanging the second and the last column, multiplying the second column with  $-1$ , interchanging the fourth and the last column and finally subtracting appropriate multiples of the last row from the third row to ensure that in the last column all entries except the one in the right lower corner is a trivial matrix in a suitable block form, which allows us to deduce

$$[\alpha'] = \left[ \begin{pmatrix} b'_{P,P} & b'_{Q,P} & 0 & 0 & 0 & 0 & b_{Q,P} \\ b'_{P,Q} & b'_{Q,Q} & 0 & 0 & 0 & 0 & b_{Q,Q} \\ 0 & 0 & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & 0 & 0 & \bar{b}_{Q,\bar{P}} \\ 0 & 0 & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} & 0 & 0 & \bar{b}_{Q,Q} \\ 0 & 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & e_2 & 0 & 0 & 0 & d_2 & -e_2 \\ 0 & 0 & 0 & \text{id}_Q & e_1 & 0 & 0 \end{pmatrix} \right]$$



$$\begin{aligned}
&= \left[ \begin{pmatrix} b'_{P,P} & b'_{Q,P} & 0 & b_{Q,P} & 0 & 0 & 0 \\ b'_{P,Q} & b'_{Q,Q} & 0 & b_{Q,Q} & 0 & 0 & 0 \\ 0 & 0 & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & 0 & 0 & \bar{b}_{Q,\bar{P}} \\ 0 & & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} & 0 & 0 & \bar{b}_{Q,Q} \\ 0 & 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & e_2 & 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 0 & e_1 & 0 & \text{id}_Q \end{pmatrix} \right] \\
&= - \left[ \begin{pmatrix} b'_{P,P} & b'_{Q,P} & 0 & b_{Q,P} & 0 & 0 & 0 \\ b'_{P,Q} & b'_{Q,Q} & 0 & b_{Q,Q} & 0 & 0 & 0 \\ 0 & 0 & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & -\bar{b}_{Q,\bar{P}} \circ e_1 & 0 & 0 \\ 0 & & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} & -\bar{b}_{Q,Q} \circ e_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1 & 0 & 0 \\ 0 & e_2 & 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 0 & e_1 & 0 & \text{id}_Q \end{pmatrix} \right] \\
&= - \left[ \begin{pmatrix} b'_{P,P} & b'_{Q,P} & 0 & b_{Q,P} & 0 \\ b'_{P,Q} & b'_{Q,Q} & 0 & b_{Q,Q} & 0 \\ 0 & 0 & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & -\bar{b}_{Q,\bar{P}} \circ e_1 \\ 0 & & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} & -\bar{b}_{Q,Q} \circ e_1 \\ 0 & 0 & 0 & 0 & d_1 \end{pmatrix} \right] - \left[ \begin{pmatrix} d_2 & -e_2 \\ 0 & \text{id}_Q \end{pmatrix} \right] \\
&= - \left[ \begin{pmatrix} b'_{P,P} & b'_{Q,P} & 0 & b_{Q,P} \\ b'_{P,Q} & b'_{Q,Q} & 0 & b_{Q,Q} \\ 0 & 0 & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} \\ 0 & & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} \end{pmatrix} \right] - [d_1] - [d_2] - [\text{id}_Q] \\
&= - \left[ \begin{pmatrix} b'_{P,P} & b'_{Q,P} \\ b'_{P,Q} & b'_{Q,Q} \end{pmatrix} \right] - \left[ \begin{pmatrix} \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} \\ \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} \end{pmatrix} \right] - [d_1] - [d_2] \\
&= -[b'] - [\bar{b}] - [d_1] - [d_2]. \tag{1.8}
\end{aligned}$$

Now (1.5) follows from equations (1.6), (1.7), and (1.8).

We conclude from (1.8) that we can assign to a finitely generated projective  $R$ -module  $P$  and an  $R_\Sigma$ -automorphism  $a : R_\Sigma \otimes_R Q \xrightarrow{\cong} R_\Sigma \otimes_R Q$  a well-defined element

$$[a] \in K_1(R, \Sigma). \tag{1.9}$$

If we have an isomorphism  $u : Q \xrightarrow{\cong} Q'$  of finitely generated projective  $R$ -modules, then one easily checks

$$[(\text{id}_{R_\Sigma} \otimes Ru) \circ a \circ (\text{id}_{R_\Sigma} \otimes Ru)^{-1}] = [a]. \tag{1.10}$$

Given two finitely generated projective  $R$ -modules  $Q$  and  $\bar{Q}$  and  $R_\Sigma$ -automorphisms  $a : R_\Sigma \otimes_R Q \xrightarrow{\cong} R_\Sigma \otimes_R Q$  and  $\bar{a} : R_\Sigma \otimes_R \bar{Q} \xrightarrow{\cong} R_\Sigma \otimes_R \bar{Q}$ , one easily checks

$$[a \oplus \bar{a}] = [a] + [\bar{a}]. \quad (1.11)$$

Obviously we get, for any finitely generated projective  $R$ -module  $Q$ ,

$$[(\text{id}_{R_\Sigma} \otimes_R \text{id}_Q)] = 0. \quad (1.12)$$

Consider a finitely generated projective  $R$ -module  $Q$  and two  $R_\Sigma$ -isomorphisms  $a, \bar{a} : R_\Sigma \otimes_R Q \xrightarrow{\cong} R_\Sigma \otimes_R Q$ . Next we want to show

$$[\bar{a} \circ a] = [\bar{a}] + [a]. \quad (1.13)$$

Make the choices  $(P, b, b', a')$  and  $(\bar{P}, \bar{b}, \bar{b}', \bar{a}')$  for  $a$  and  $\bar{a}$  as we did above in the definition of  $[a]$  and  $[\bar{a}]$ . Consider the  $R_\Sigma$ -automorphism

$$A = \begin{pmatrix} \text{id}_{R_\Sigma \otimes_R P} & 0 & 0 & a' \\ 0 & \text{id}_{R_\Sigma \otimes_R Q} & 0 & a \\ 0 & 0 & \text{id}_{R_\Sigma \otimes_R \bar{P}} & \bar{a}' a \\ 0 & 0 & 0 & \bar{a} a \end{pmatrix}$$

of  $(R_\Sigma \otimes_R P) \oplus (R_\Sigma \otimes_R Q) \oplus (R_\Sigma \otimes_R \bar{P}) \oplus (R_\Sigma \otimes_R Q)$ , and the  $R$ -endomorphisms of  $P \oplus Q \oplus \bar{P} \oplus Q$

$$B = \begin{pmatrix} b_{P,P} & b_{Q,P} & 0 & 0 \\ b_{P,Q} & b_{Q,Q} & 0 & 0 \\ 0 & -\bar{b}'_{Q,\bar{P}} & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} \\ 0 & -\bar{b}'_{Q,Q} & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} b'_{P,P} & b_{Q,P} & 0 & b'_{Q,P} \\ b'_{P,Q} & b_{Q,Q} & 0 & b'_{Q,Q} \\ 0 & -\bar{b}'_{Q,P} & \bar{b}_{P,P} & 0 \\ 0 & -\bar{b}'_{Q,Q} & \bar{b}_{P,Q} & 0 \end{pmatrix}.$$

From the block structure of  $B$  one concludes that  $(\text{id}_{R_\Sigma} \otimes B)$  is an isomorphism and we get, in  $K_1(R, \Sigma)$ ,

$$[B] = \left[ \begin{pmatrix} b_{P,P} & b_{Q,P} \\ b_{P,Q} & b_{Q,Q} \end{pmatrix} \right] + \left[ \begin{pmatrix} \bar{b}_{P,P} & \bar{b}_{Q,P} \\ \bar{b}_{P,Q} & \bar{b}_{Q,Q} \end{pmatrix} \right] = [b] + [\bar{b}]. \quad (1.14)$$

If we interchange in  $B'$  the second and last column and multiply the last column with  $-1$ , we conclude from the block structure of the resulting matrix that  $(\text{id}_{R_\Sigma} \otimes B')$  is an isomorphism and we get, in  $K_1(R, \Sigma)$ ,

$$\begin{aligned} [B'] &= \left[ \begin{pmatrix} b'_{P,P} & b'_{Q,P} & 0 & b_{Q,P} \\ b'_{P,Q} & b'_{Q,Q} & 0 & b_{Q,Q} \\ 0 & 0 & \bar{b}_{\bar{P},\bar{P}} & \bar{b}'_{Q,\bar{P}} \\ 0 & 0 & \bar{b}_{\bar{P},Q} & \bar{b}'_{Q,Q} \end{pmatrix} \right] \\ &= \left[ \begin{pmatrix} b'_{P,P} & b'_{Q,P} \\ b'_{P,Q} & b'_{Q,Q} \end{pmatrix} \right] + \left[ \begin{pmatrix} \bar{b}_{P,P} & \bar{b}'_{Q,P} \\ \bar{b}_{P,Q} & \bar{b}'_{Q,Q} \end{pmatrix} \right] = [b'] + [\bar{b}']. \end{aligned} \quad (1.15)$$

Since  $(\text{id}_{R_\Sigma} \otimes B)$  and  $(\text{id}_{R_\Sigma} \otimes B')$  are isomorphisms and  $(\text{id}_{R_\Sigma} \otimes B) \circ A = (\text{id}_{R_\Sigma} \otimes B')$ , we get, directly from the definitions,

$$[\bar{a}a] = [B'] - [B]. \quad (1.16)$$

Now (1.13) follows from equations (1.14), (1.15) and (1.16). Now one easily checks that equations (1.10), (1.11), (1.12) and (1.13) imply that the homomorphism  $\beta$  announced in (1.3) is well-defined. One easily checks that  $\beta$  is an inverse to the homomorphism  $\alpha$  appearing in the statement of Theorem 1.2. This finishes the proof of Theorem 1.2.  $\square$

**1C. Schofield's localization sequence.** The proofs of this paper are motivated by Schofield's construction of a localization sequence

$$K_1(R) \rightarrow K_1(R_\Sigma) \rightarrow K_1(\mathcal{T}) \rightarrow K_0(R) \rightarrow K_0(R_\Sigma),$$

where  $\mathcal{T}$  is the full subcategory of the category of the finitely presented  $R$ -modules whose objects are cokernels of elements in  $\Sigma$ ; see [Schofield 1985, Theorem 5.12 on page 60]. Under certain conditions this sequence has been extended to the left in [Neeman 2007; Neeman and Ranicki 2004]. Notice that in connection with potential proofs of the Atiyah conjecture it is important to figure out under which condition  $K_0(FG) \rightarrow K_0(\mathcal{D}(G; F))$  is surjective for a torsionfree group  $G$  and a subfield  $F \subseteq \mathbb{C}$ ; see [Lück 2002, Theorem 10.38 on page 387]. In this connection the question becomes interesting whether  $G$  has property (UL) — see Section 2C — and how to continue the sequence above to the right.

## 2. Groups with property (ULA)

Throughout this section let  $F$  be a field with  $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ .

**2A. Review of division and rational closure.** Let  $R$  be a subring of the ring  $S$ . The *division closure*  $\mathcal{D}(R \subseteq S) \subseteq S$  is the smallest subring of  $S$  which contains  $R$  and is division closed, i.e., any element  $x \in \mathcal{D}(R \subseteq S)$  which is invertible in  $S$  is already invertible in  $\mathcal{D}(R \subseteq S)$ . The *rational closure*  $\mathcal{R}(R \subseteq S) \subseteq S$  is the smallest subring of  $S$  which contains  $R$  and is rationally closed, i.e., for every natural number  $n$  and matrix  $A \in M_{n,n}(\mathcal{D}(R \subseteq S))$  which is invertible in  $S$ , the matrix  $A$  is already invertible over  $\mathcal{R}(R \subseteq S)$ . The division closure and the rational closure always exist. Obviously  $R \subseteq \mathcal{D}(R \subseteq S) \subseteq \mathcal{R}(R \subseteq S) \subseteq S$ .

Consider an inclusion of rings  $R \subseteq S$ . Let  $\Sigma(R \subseteq S)$  the set of all square matrices over  $R$  which become invertible over  $S$ . Then there is a canonical epimorphism of rings from the universal localization of  $R$  with respect to  $\Sigma(R \subseteq S)$  to the rational closure of  $R$  in  $S$  — see [Reich 2006, Proposition 4.10(iii)] —

$$\lambda : R_{\Sigma(R \subseteq S)} \rightarrow \mathcal{R}(R \subseteq S). \quad (2.1)$$

Recall that we have inclusions  $R \subseteq \mathfrak{D}(R \subseteq S) \rightarrow \mathfrak{R}(R \subseteq S) \subseteq S$ .

Consider a group  $G$ . Let  $\mathcal{N}(G)$  be the group von Neumann algebra, which can be identified with the algebra  $\mathfrak{B}(L^2(G), L^2(G))^G$  of bounded  $G$ -equivariant operators  $L^2(G) \rightarrow L^2(G)$ . Denote by  $\mathfrak{U}(G)$  the algebra of operators which are affiliated to the group von Neumann algebra. This is the same as the Ore localization of  $\mathcal{N}(G)$  with respect to the multiplicatively closed subset of nonzero divisors in  $\mathcal{N}(G)$ ; see [Lück 2002, Chapter 8]. By the right regular representation we can embed  $\mathbb{C}G$  and hence also  $FG$  as a subring in  $\mathcal{N}(G)$ . We will denote by  $\mathfrak{R}(G; F)$  and  $\mathfrak{D}(G; F)$  the division and the rational closure of  $FG$  in  $\mathfrak{U}(G)$ . So we get a commutative diagram of inclusions of rings

$$\begin{array}{ccc} FG & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathfrak{D}(G; F) & & \mathfrak{U}(G) \\ \downarrow & & \downarrow \\ \mathfrak{R}(G; F) & \longrightarrow & \mathfrak{U}(G) \end{array}$$

**2B. Review of the Atiyah conjecture for torsionfree groups.** Recall that there is a dimension function  $\dim_{\mathcal{N}(G)}$  defined for all (algebraic)  $\mathcal{N}(G)$ -modules; see [Lück 2002, Section 6.1].

**Definition 2.2** (Atiyah conjecture with coefficients in  $F$ ). We say that a torsionfree group  $G$  satisfies the *Atiyah conjecture with coefficients in  $F$*  if for any matrix  $A \in M_{m,n}(FG)$  the von Neumann dimension  $\dim_{\mathcal{N}(G)}(\ker(r_A))$  of the kernel of the  $\mathcal{N}(G)$ -homomorphism  $r_A : \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n$  given by right multiplication with  $A$  is an integer.

**Theorem 2.3** (status of the Atiyah conjecture). (1) *If the torsionfree group  $G$  satisfies the Atiyah conjecture with coefficients in  $F$ , then also each of its subgroups satisfy the Atiyah conjecture with coefficients in  $F$ .*

(2) *If the torsionfree group  $G$  satisfies the Atiyah conjecture with coefficients in  $\mathbb{C}$ , then  $G$  satisfies the Atiyah conjecture with coefficients in  $F$ .*

(3) *The torsionfree group  $G$  satisfies the Atiyah conjecture with coefficients in  $F$  if and only if  $\mathfrak{D}(G; F)$  is a skew field.*

*If the torsionfree group  $G$  satisfies the Atiyah conjecture with coefficients in  $F$ , then the rational closure  $\mathfrak{R}(G; F)$  agrees with the division closure  $\mathfrak{D}(G; F)$ .*

(4) *Let  $\mathcal{C}$  be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Suppose that  $G$  is a torsionfree group which belongs to  $\mathcal{C}$ .*

*Then  $G$  satisfies the Atiyah conjecture with coefficients in  $\mathbb{C}$ .*

(5) *Let  $G$  be an infinite group which is the fundamental group of a compact connected orientable irreducible 3-manifold  $M$  with empty or toroidal boundary. Suppose that one of the following conditions is satisfied:*

- *$M$  is not a closed graph manifold.*
- *$M$  is a closed graph manifold which admits a Riemannian metric of non-positive sectional curvature.*

*Then  $G$  is torsionfree and belongs to  $\mathcal{C}$ . In particular  $G$  satisfies the Atiyah conjecture with coefficients in  $\mathbb{C}$ .*

(6) *Let  $\mathcal{D}$  be the smallest class of groups such that:*

- *The trivial group belongs to  $\mathcal{D}$ .*
- *If  $p : G \rightarrow A$  is an epimorphism of a torsionfree group  $G$  onto an elementary amenable group  $A$  and if  $p^{-1}(B) \in \mathcal{D}$  for every finite group  $B \subset A$ , then  $G \in \mathcal{D}$ .*
- *$\mathcal{D}$  is closed under taking subgroups.*
- *$\mathcal{D}$  is closed under colimits and inverse limits over directed systems.*

*If the group  $G$  belongs to  $\mathcal{D}$ , then  $G$  is torsionfree and the Atiyah conjecture with coefficients in  $\overline{\mathbb{Q}}$  holds for  $G$ .*

*The class  $\mathcal{D}$  is closed under direct sums, direct products and free products. Every residually torsionfree elementary amenable group belongs to  $\mathcal{D}$ .*

*Proof.* (1) This follows from [Lück 2002, Theorem 6.29(2) on page 253].

(2) This is obvious.

(3) This is proved in the case  $F = \mathbb{C}$  in [Lück 2002, Lemma 10.39 on page 388]. The proof goes through for an arbitrary field  $F$  with  $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$  without modifications.

(4) This is due to Linnell; see for instance [Linnell 1993] or [Lück 2002, Theorem 10.19 on page 378].

(5) It suffices to show that  $G = \pi_1(M)$  belongs to the class  $\mathcal{C}$  appearing in assertion (4). As explained in [Dubois et al. 2016, Section 10], we conclude from combining [Agol 2008; 2013; Liu 2013; Przytycki and Wise 2012; 2014; Wise 2012a; 2012b] that there exists a finite normal covering  $p : \overline{M} \rightarrow M$  and a fiber bundle  $S \rightarrow \overline{M} \rightarrow S^1$  for some compact connected orientable surface  $S$ . Hence it suffices to show that  $\pi_1(S)$  belongs to  $\mathcal{C}$ . If  $S$  has nonempty boundary, this follows from the fact that  $\pi_1(S)$  is free. If  $S$  is closed, the commutator subgroup of  $\pi_1(S)$  is free and hence  $\pi_1(S)$  belongs to  $\mathcal{C}$ . Now assertion (5) follows from assertion (4).

(6) This result is due to Schick for  $\mathbb{Q}$  (see for instance [Schick 2001] or [Lück 2002, Theorem 10.22 on page 379]) and for  $\overline{\mathbb{Q}}$  due to Dodziuk, Linnell, Mathai, Schick and Yates [Dodziuk et al. 2003, Theorem 1.4] □

For more information and further explanations about the Atiyah conjecture we refer for instance to [Lück 2002, Chapter 10].

### 2C. *The property (UL).*

**Definition 2.4** (property (UL)). We say that a group  $G$  has the property (UL) with respect to  $F$  if the canonical epimorphism

$$\lambda : FG_{\Sigma(FG \subseteq \mathcal{U}(G, F))} \rightarrow \mathcal{R}(G; F)$$

defined in (2.1) is bijective.

Next we investigate which groups  $G$  are known to have property (UL).

Let  $\mathcal{A}$  denote the class of groups consisting of the finitely generated free groups and the amenable groups. If  $\mathcal{Y}$  and  $\mathcal{X}$  are classes of groups, define

$$\begin{aligned} L(\mathcal{Y}) &= \{G \mid \text{every finite subset of } G \text{ is contained in a } \mathcal{Y}\text{-group}\}, \\ \mathcal{Y}\mathcal{X} &= \{G \mid \text{there exists } H \triangleleft G \text{ such that } H \in \mathcal{Y} \text{ and } G/H \in \mathcal{X}\}. \end{aligned}$$

Now define  $\mathcal{X}$  to be the smallest class of groups which contains  $\mathcal{A}$  and is closed under directed unions and group extension. Next, for each ordinal  $a$ , define a class of groups  $\mathcal{X}_a$  as follows:

- $\mathcal{X}_0 = \{1\}$ .
- $\mathcal{X}_a = L(\mathcal{X}_{a-1}\mathcal{A})$  if  $a$  is a successor ordinal.
- $\mathcal{X}_a = \bigcup_{b < a} \mathcal{X}_b$  if  $a$  is a limit ordinal.

**Lemma 2.5.** (1) *Each  $\mathcal{X}_a$  is subgroup closed.*

$$(2) \mathcal{X} = \bigcup_{a \geq 0} \mathcal{X}_a.$$

(3)  *$\mathcal{X}$  is subgroup closed.*

*Proof.* (1) This is easily proved by induction on  $a$ .

(2) Set  $\mathcal{Y} = \bigcup_{a \geq 0} \mathcal{X}_a$ . Obviously  $\mathcal{X} \supseteq \mathcal{Y}$ . We prove the reverse inclusion by showing that  $\mathcal{Y}$  is closed under directed unions and group extension. The former is obvious, because if the group  $G$  is the directed union of subgroups  $G_i$  and  $a_i$  is the least ordinal such that  $G_i \in \mathcal{X}_{a_i}$ , we set  $a = \sup_i a_i$  and then  $G \in \mathcal{X}_{a+1}$ . For the latter, we show that  $\mathcal{X}_a\mathcal{X}_b \subseteq \mathcal{X}_{a+b}$  by induction on  $b$ , the case  $b = 0$  being obvious. If  $b$  is a successor ordinal, write  $b = c + 1$ . Then

$$\begin{aligned} \mathcal{X}_a\mathcal{X}_b &= \mathcal{X}_a(L(\mathcal{X}_c)\mathcal{A}) \subseteq L(\mathcal{X}_a\mathcal{X}_c)\mathcal{A} \\ &\subseteq L(\mathcal{X}_{a+c})\mathcal{A} && \text{by induction} \\ &\subseteq \mathcal{X}_{a+c+1} = \mathcal{X}_{a+b}. \end{aligned}$$

On the other hand, if  $b$  is a limit ordinal, then

$$\begin{aligned} \mathcal{X}_a \mathcal{X}_b &= \mathcal{X}_a \left( \bigcup_{c < b} \mathcal{X}_c \right) = \bigcup_{c < b} \mathcal{X}_a \mathcal{X}_c \\ &\subseteq \bigcup_{c < b} \mathcal{X}_{a+c} && \text{by induction} \\ &\subseteq \mathcal{X}_{a+b}, \end{aligned}$$

as required.

(3) This follows from assertions (1) and (2).  $\square$

**Lemma 2.6.** *Let  $G = \bigcup_{i \in I} G_i$  be groups such that, given  $i, j \in I$ , there exists  $l \in I$  such that  $G_i, G_j \subseteq G_l$ . Write  $\Sigma = \Sigma(FG \subseteq \mathcal{U}(G))$  and  $\Sigma_i = \Sigma(FG_i \subseteq \mathcal{U}(G_i))$  for  $i \in I$ . Suppose the identity map on  $FG_i$  extends to an isomorphism  $\lambda_i : (FG_i)_{\Sigma_i} \rightarrow \mathcal{R}(G_i; F)$  for all  $i \in I$ .*

*Then the identity map on  $FG$  extends to an isomorphism  $\lambda : FG_{\Sigma} \rightarrow \mathcal{R}(G; F)$ .*

*Proof.* By definition, the identity map on  $FG$  extends to an epimorphism  $\lambda : FG_{\Sigma} \rightarrow \mathcal{R}(G; F)$ . We need to show that  $\lambda$  is injective, and here we follow the proof of [Linnell 1998, Lemma 13.5]. Clearly  $\Sigma_i \subseteq \Sigma$  for all  $i \in I$  and thus the inclusion map  $FG_i \hookrightarrow FG$  extends to a map  $\mu_i : (FG_i)_{\Sigma_i} \rightarrow FG_{\Sigma}$  for all  $i \in I$ . Since  $\lambda_i$  is an isomorphism, we may define  $\nu_i = \mu_i \circ \lambda_i^{-1} : \mathcal{R}(G_i; F) \rightarrow FG_{\Sigma}$  for all  $i \in I$ . If  $G_i \subseteq G_j$ , then  $\mathcal{R}(G_i; F) \subseteq \mathcal{R}(G_j; F)$  and we let  $\psi_{ij} : \mathcal{R}(G_i; F) \rightarrow \mathcal{R}(G_j; F)$  denote the natural inclusion. Observe that  $\mu_i(x) = \mu_j \lambda_j^{-1} \psi_{ij} \lambda_i(x)$  for all  $x$  in the image of  $FG_i$  in  $(FG_i)_{\Sigma_i}$  and therefore, by the universal property,  $\mu_i = \mu_j \lambda_j^{-1} \psi_{ij} \lambda_i$  and hence  $\mu_i \lambda_i^{-1} = \mu_j \lambda_j^{-1} \psi_{ij}$ . Thus  $\nu_i = \nu_j \psi_{ij}$  and the  $\nu_i$  fit together to give a map  $\nu : \bigcup_{i \in I} \mathcal{R}(G_i; F) \rightarrow FG_{\Sigma}$ . It is easily checked that  $\nu \circ \lambda : FG_{\Sigma} \rightarrow FG_{\Sigma}$  is a map which is the identity on the image of  $FG$  in  $FG_{\Sigma}$  and hence by the universal property of localization,  $\nu \circ \lambda$  is the identity. This proves that  $\lambda$  is injective, as required.  $\square$

If  $G$  is a group and  $\alpha$  is an automorphism of  $G$ , then  $\alpha$  extends to an automorphism of  $\mathcal{U}(G)$ , which we shall also denote by  $\alpha$ . This is not only an algebraic automorphism, but is also a homeomorphism with respect to the various topologies on  $\mathcal{U}(G)$ .

**Lemma 2.7.** *If  $\alpha$  is an automorphism of  $G$ , then  $\alpha(\mathcal{D}(G; F)) = \mathcal{D}(G; F)$ .*

*Proof.* This is clear, because  $\alpha(FG) = FG$ .  $\square$

**Lemma 2.8.** *Let  $H \triangleleft G$  be groups and let  $\mathcal{D}(H; F)G$  denote the subring of  $\mathcal{D}(G; F)$  generated by  $\mathcal{D}(H; F)$  and  $G$ .*

*Then, for a suitable crossed product,  $\mathcal{D}(H; F)G \cong \mathcal{D}(H; F) * G/H$  by a map which extends the identity on  $\mathcal{D}(H; F)$  and, for  $g \in G$ , sends  $\mathcal{D}(H; F) \cdot g$  to  $\mathcal{D}(H; F) * Hg$ .*

*Proof.* Let  $T$  be a transversal for  $H$  in  $G$ . Since  $h \mapsto tht^{-1}$  is an automorphism of  $H$ , we see that  $t \cdot \mathfrak{D}(H; F) \cdot t^{-1} = \mathfrak{D}(H; F)$  for all  $t \in T$  by Lemma 2.7 and so  $\mathfrak{D}(H; F)G = \sum_{t \in T} \mathfrak{D}(H; F)G \cdot t$ . This sum is direct because the sum  $\sum_{t \in T} \mathfrak{U}(H) \cdot t$  is direct, and the result is established.  $\square$

In the sequel recall that  $\mathfrak{R}(G; F) = \mathfrak{D}(G; F)$  holds if  $\mathfrak{D}(G; F)$  is a skew field.

**Lemma 2.9.** *Let  $H \triangleleft G$  be groups such that  $G/H$  is finite and  $H$  is torsion-free. Assume that  $\mathfrak{D}(H; F)$  is a skew field. Set  $\Sigma = \Sigma(FG \subseteq \mathfrak{U}(G))$  and  $\Phi = \Sigma(FH \subseteq \mathfrak{U}(H))$ , and let  $\mu : FH_{\Phi} \rightarrow \mathfrak{D}(H; F)$  and  $\lambda : FG_{\Sigma} \rightarrow \mathfrak{D}(G; F)$  denote the corresponding localization maps.*

*Then  $\mathfrak{D}(G; F)$  is a semisimple artinian ring and agrees with  $\mathfrak{R}(G; F)$ . Furthermore, if  $\mu$  is an isomorphism, then so is  $\lambda$ .*

*Proof.* Let  $\mathfrak{D}(H; F)G$  denote the subring of  $\mathfrak{D}(G; F)$  generated by  $\mathfrak{D}(H; F)$  and  $G$ . Then Lemma 2.8 shows, that for a suitable crossed product, there is an isomorphism  $\theta : \mathfrak{D}(H; F) * G/H \rightarrow \mathfrak{D}(H; F)G$  which extends the identity map on  $\mathfrak{D}(H; F)$ . This ring has dimension  $|G/H|$  over the skew field  $\mathfrak{D}(H; F)$  and is therefore artinian. Since every matrix over an artinian ring is either a zero-divisor or invertible (in particular every element is either a zero-divisor or invertible), we see that  $\mathfrak{R}(G; F) = \mathfrak{D}(G; F) = \mathfrak{D}(H; F)G$ . Furthermore, by Maschke's theorem,  $\mathfrak{D}(H; F)G$  semisimple artinian. Now assume that  $\mu$  is an isomorphism. We may identify  $FG$  with the subring  $FH * G/H$  and then by [Linnell 1993, Lemma 4.5] there is an isomorphism  $\psi : \mathfrak{D}(H; F) * G/H \rightarrow FG_{\Phi}$  which extends the identity map on  $FG$ . Also  $\Phi \subseteq \Sigma$ , so the identity map on  $FG$  extends to a map  $\rho : FG_{\Phi} \rightarrow FG_{\Sigma}$ . Then  $\rho \circ \psi \circ \theta^{-1} \circ \lambda : FG_{\Sigma} \rightarrow FG_{\Sigma}$  is a map extending the identity on  $FG$ , hence is the identity, and the result follows.  $\square$

Recall that the group  $G$  is locally indicable if for every nontrivial finitely generated subgroup  $H$  there exists  $N \triangleleft H$  such that  $N/H$  is infinite cyclic. Also if  $R$  is a subring of the skew field  $D$  such that  $\mathfrak{D}(R \subseteq D) = D$ , then we say that  $D$  is a field of fractions for  $R$  ( $D$  will be noncommutative, i.e., a skew field in general).

**Definition 2.10.** Let  $K$  be a skew field, let  $G$  be a locally indicable group, let  $K * G$  be a crossed product, and let  $D$  be a field of fractions for  $K * G$ . Then we say that  $D$  is a *Hughes-free* [Hughes 1970, §2; Lewin 1974, pp. 340, 342; Lück 2002, Lemma 10.81; [Dicks et al. 2004, p. 1128]] field of fractions for  $K * G$  if whenever  $N \triangleleft H \leq G$ ,  $H/N$  is infinite cyclic and  $t \in H$  such that  $\langle Nt \rangle = H/N$  (i.e.,  $t$  generates  $H \bmod N$ ), then  $\{t^i \mid i \in \mathbb{Z}\}$  is linearly independent over  $\mathfrak{D}(K * N \subseteq D)$ .

A key result here is that of Ian Hughes [1970, Theorem, page 182] (see also [Dicks et al. 2004, Theorem 7.1]), which states:

**Theorem 2.11** (Hughes's theorem). *Let  $K$  be a skew field, let  $G$  be a locally indicable group, let  $K * G$  be a crossed product, and let  $D_1$  and  $D_2$  be Hughes-free*



field of fractions for  $K * G$ . Then there is an isomorphism  $D_1 \rightarrow D_2$  which is the identity on  $K * G$ .

Recall that a ring  $R$  is called a *fir* (free ideal ring [Cohn 1995, §1.6]) if every left ideal is a free left  $R$ -module of unique rank, and every right ideal is a free right  $R$ -module of unique rank. Also,  $R$  is called a *semifir* if the above condition is only satisfied for all finitely generated left and right ideals. It is easy to see that if  $K$  is a skew field,  $G$  is the infinite cyclic group and  $K * G$  is a crossed product, then every nonzero left or right ideal is free of rank one and hence  $K * G$  is a fir. We can now apply [Cohn 1995, Theorem 5.3.9] (a result essentially due to Bergman [1974]) to deduce that if  $G$  is a free group and  $K * G$  is a crossed product, then  $K * G$  is a fir.

We also need the concept of a *universal* field of fractions; this is described in [Cohn 1985, §7.2; 1995, §4.5]. It is proven in [Cohn 1985, Corollary 7.5.11; 1995, Corollary 4.5.9] that if  $R$  is a semifir, then it has a universal field of fractions  $D$ . Furthermore the inclusion  $R \subseteq D$  is an honest map [Cohn 1985, p. 250; 1995, p. 177], fully inverting [Cohn 1985, p. 415; 1995, p. 177], and the localization map  $R_{\mathcal{D}(R \subseteq D)} \rightarrow D$  is an isomorphism. We can now state a crucial result of Jacques Lewin [1974, Proposition 6].

**Theorem 2.12** (Lewin's theorem). *Let  $K$  be a skew field, let  $G$  be a free group, let  $K * G$  be a crossed product, and let  $D$  be the universal field of fractions for  $K * G$ . Then  $D$  is Hughes-free.*

Actually Lewin only proves the result for  $K$  a field and  $K * G$  the group algebra  $KG$  over  $K$ . However, with the remarks above, in particular that  $K * G$  is a fir, we can follow Lemmas 1–6 and Theorem 1 of [Lewin 1974] verbatim to deduce Theorem 2.12.

**Lemma 2.13.** *Let  $H \triangleleft G$  be groups and let  $G/H \in \mathcal{A}$ . Assume that  $\mathcal{D}(G; F)$  is a skew field. Write  $\Sigma = \Sigma(FG \subseteq \mathcal{U}(G))$  and  $\Phi = \Sigma(FH \subseteq \mathcal{U}(H))$ . Let  $\mu : FH_\Phi \rightarrow \mathcal{R}(H; F)$  and  $\lambda : FG_\Sigma \rightarrow \mathcal{R}(G; F)$  be the localization maps which extend the identity on  $FH$  and  $FG$ , respectively. Suppose that  $\mu$  is an isomorphism.*

*Then  $\mathcal{D}(G; F) = \mathcal{R}(G; F)$ , and  $\lambda$  is an isomorphism.*

*Proof.* We already know that  $\mathcal{D}(G; F) = \mathcal{R}(G; F)$  because we are assuming that  $\mathcal{D}(G; F)$  is a skew field, and clearly  $\lambda$  is an epimorphism. We need to show that  $\lambda$  is injective. Lemma 2.8 shows that  $\mathcal{D}(H; F)G \cong \mathcal{D}(H; F) * G/H$  and we will use the corresponding isomorphism to identify these two rings without further comment. Since we are assuming that  $\mathcal{D}(G; F)$  is a skew field,  $\mathcal{D}(H; F) * G/H$  is a domain. Furthermore,  $FG_\Phi \cong (FH * G/H)_\Phi \cong FH_\Phi * G/H$  by Lemma 2.7 and [Linnell 1993, Lemma 4.5], and we deduce that the localization map  $FG_\Phi \rightarrow \mathcal{D}(H; F) * G/H$  is an isomorphism, because we are assuming that  $\mu$  is an isomorphism. Let  $\Psi =$

$\Sigma(\mathcal{D}(H; F)G \subseteq \mathcal{D}(G; F))$ . The proof of [Schofield 1985, Theorem 4.6] shows that  $(FG_\Phi)_\Psi \cong FG_{\Sigma'}$  for a suitable set of matrices  $\Sigma'$  over  $FG$  (where we have identified  $FG_\Phi$  with  $\mathcal{D}(H; F)G$  by the above isomorphisms). All the matrices in  $\Sigma'$  become invertible over  $\mathcal{R}(G; F)$ , so by [Cohn 1985, Exercise 7.2.8] we may replace  $\Sigma'$  by its saturation. It remains to prove that the localization map  $\mathcal{D}(H; F)G_\Psi \rightarrow \mathcal{R}(G; F)$  is injective.

We have two cases to consider, namely  $G/H$  amenable and  $G/H$  finitely generated free. For the former we apply [Dodziuk et al. 2003, Theorem 6.3] (essentially a result of Tamari [1957]). We deduce that  $\mathcal{D}(H; F) * G/H$  satisfies the Ore condition for the multiplicatively closed subset of nonzero elements of  $\mathcal{D}(H; F) * G/H$  and it follows that the localization map  $\mathcal{D}(H; F)G_\Psi \rightarrow \mathcal{R}(G; F)$  is an isomorphism.

For the latter case, let  $L \triangleleft M$  be subgroups of  $G$  containing  $H$  such that  $M/L$  is infinite cyclic and let  $t \in M$  be a generator for  $M \bmod L$ . Since the sum  $\sum_{i \in \mathbb{Z}} \mathcal{U}(L)t^i$  is direct, we see that the sum  $\sum_{i \in \mathbb{Z}} \mathcal{D}(L; F)t^i$  is also direct and we deduce that  $\mathcal{D}(G; F)$  is a Hughes-free field of fractions for  $\mathcal{D}(H; F) * G/H$ . It now follows from Theorems 2.11 and 2.12 that  $\mathcal{D}(G; F)$  is a universal field of fractions for  $\mathcal{D}(H; L)G$  and, in particular, the localization map  $\mathcal{D}(H; F)G_\Psi \rightarrow \mathcal{R}(G; F)$  is injective. This finishes the proof.  $\square$

**Theorem 2.14.** *Let  $H \triangleleft G$  be groups with  $H \in \mathcal{X}$ ,  $H$  torsionfree and  $G/H$  finite. Let  $\Sigma = \Sigma(FG \subseteq \mathcal{U}(G))$ . Assume that  $\mathcal{D}(H; F)$  is a skew field.*

*Then  $\mathcal{D}(G; F) = \mathcal{R}(G; F)$ , and  $H$  has the property (UL) with respect to  $F$ , i.e., the localization map  $FG_\Sigma \rightarrow \mathcal{R}(G; F)$  is an isomorphism.*

*Proof.* We first consider the special case  $G = H$  (so  $G$  is torsionfree). We use the description of the class of groups  $\mathcal{X}$  given in Lemma 2.5(2) and prove the result by transfinite induction. The result is obvious if  $G \in \mathcal{X}_0$ , because then  $G = 1$ . The induction step is done as follows. Consider an ordinal  $b$  with  $b \neq 0$  and a group  $G \in \mathcal{X}_b$  such that the claim is already known for all groups  $H \in \mathcal{X}_a$  for all ordinals  $a < b$ . We have to show the claim for  $G$ . If  $b$  is a limit ordinal, this is obvious since  $G$  belongs to  $\mathcal{X}_a$  for every ordinal  $a < b$ . It remains to treat the case where  $b$  is not a limit ordinal. Then  $G \in L(\mathcal{X}_a \mathcal{A})$  for some ordinal  $a < b$ . By Lemma 2.6, it is sufficient to consider the case  $G \in \mathcal{X}_a \mathcal{A}$ . Now apply Lemma 2.13.

The general case, when  $G$  is not necessarily equal to  $H$ , now follows from Lemma 2.9.  $\square$

There are many groups for which Theorem 2.14 can be applied, some of which we now describe. Let  $N$  be either an Artin pure braid group, or a RAAG, or a subgroup of finite index in a right-angled Coxeter group. Let  $\overline{\mathbb{Q}}$  denote the field of all algebraic numbers. We can now state:

**Theorem 2.15.** *Let  $G$  be a group which contains  $N$  as a normal subgroup such that  $G/N$  is elementary amenable, and let  $\Sigma = \Sigma(FG \subseteq \mathcal{U}(G))$ . Assume that  $G$  contains a torsionfree subgroup of finite index and that  $F$  is a subfield of  $\overline{\mathbb{Q}}$ . Then the localization map  $FG_{\Sigma} \rightarrow \mathcal{R}(G; F)$  is an isomorphism, i.e.,  $G$  has property (UL) with respect to  $F$ .*

*Proof.* First we recall some group-theoretic results. An Artin pure braid group is polyfree (see [Rolfsen 2010, §2.4], for example) and RAAGs are polyfree by [Hermiller and Šunić 2007, Theorem A]. Finally right-angled Coxeter groups have a characteristic subgroup of index a power of 2 which is isomorphic to a subgroup of a right-angled Artin group [Linnell et al. 2012, Proposition 5(2)] and therefore this subgroup is polyfree. This shows that in all cases  $G \in \mathcal{X}$  and hence any subgroup of  $G$  is in  $\mathcal{X}$ , because  $\mathcal{X}$  is subgroup closed by Lemma 2.5 (3).

Now let  $H$  be a torsionfree normal subgroup of finite index in  $G$ . We need to show that  $H$  satisfies the Atiyah conjecture with coefficients in  $F$ . We may assume that  $F = \overline{\mathbb{Q}}$ . For the case  $N$  is an Artin pure braid group, this follows from [Linnell and Schick 2007, Corollary 5.41]. For the case  $N$  is a RAAG, this follows from [Linnell et al. 2012, Theorem 2]. Finally if  $N$  is a subgroup of finite index in a right-angled Coxeter group, this follows from [Linnell et al. 2012, Theorem 2 and Proposition 5(2)] and [Schreive 2014, Theorem 1.1].  $\square$

## 2D. The property (ULA).

**Definition 2.16** (property (ULA)). We say that a torsionfree group  $G$  has the property (ULA) with respect to the subfield  $F \subseteq \mathbb{C}$  if the canonical epimorphism

$$\lambda : R_{\Sigma}(FG \subseteq \mathcal{R}(G; F)) \rightarrow \mathcal{R}(G; F)$$

is bijective and  $\mathcal{D}(G; F)$  is a skew field.

Given a torsionfree group  $G$ , recall from Theorem 2.3(3) that  $\mathcal{D}(G; F)$  is a skew field if and only if  $G$  satisfies the Atiyah conjecture with coefficients in  $F$  and that we have  $\mathcal{D}(G; F) = \mathcal{R}(G; F)$  provided that  $\mathcal{D}(G; F)$  is a skew field. So  $G$  satisfies condition (ULA) with respect to  $F$  if and only if  $G$  satisfies both condition (UL) with respect to  $F$  and the Atiyah conjecture with coefficients in  $F$ .

**Theorem 2.17** (groups in  $\mathcal{C}$  have property (ULA)). *Let  $\mathcal{C}$  be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Suppose that  $G$  is a torsionfree group which belongs to  $\mathcal{C}$ .*

*Then  $G$  has property (ULA).*

*Proof.* This follows from Theorem 2.3(3)–(4) and Theorem 2.14 since obviously  $\mathcal{C} \subseteq \mathcal{X}$ .  $\square$

### 3. Proof of the main theorem

Next we explain why we are interested in group with properties (ULA) by proving our main theorem, Theorem 0.1, which will be a direct consequence of Theorems 2.17 and 3.5.

**Definition 3.1.** Let  $G$  be a group, let  $R$  be a ring with  $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$ , and denote by  $F \subseteq \mathbb{C}$  its quotient field. Let

$$K_1^w(RG)$$

be the abelian group defined in terms of generators and relations as follows: Generators  $[f]$  are given by (conjugacy classes of)  $RG$ -endomorphisms  $f : P \rightarrow P$  of finitely generated projective  $RG$ -modules  $P$  such that  $\omega_* f : \omega_* P \rightarrow \omega_* P$  is a  $\mathcal{D}(G; F)$ -isomorphism for the inclusion  $\omega : RG \rightarrow \mathcal{D}(G; F)$ . If  $f, g : P \rightarrow P$  are  $RG$ -endomorphisms of the same finitely generated projective  $RG$ -module  $P$  such that  $\omega_* f$  and  $\omega_* g$  are bijective, then we require the relation

$$[g \circ f] = [g] + [f].$$

If we have a commutative diagram of finitely generated projective  $RG$ -modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0 & \xrightarrow{i} & P_1 & \xrightarrow{p} & P_2 & \longrightarrow & 0 \\ & & f_0 \downarrow & & f_1 \downarrow & & f_2 \downarrow & & \\ 0 & \longrightarrow & P_0 & \xrightarrow{i} & P_1 & \xrightarrow{p} & P_2 & \longrightarrow & 0 \end{array}$$

such that  $\omega_* f_0$ ,  $\omega_* f_1$  and  $\omega_* f_2$  are bijective, then we require the relation

$$[f_1] = [f_0] + [f_2].$$

Furthermore, define

$$\tilde{K}_1^w(RG) := \operatorname{coker}(\{\pm 1\} \xrightarrow{\cong} K_1(\mathbb{Z}) \rightarrow K_1(\mathbb{Z}G) \rightarrow K_1^w(RG));$$

$$\operatorname{Wh}^w(G; R) = \operatorname{coker}(\{\pm g \mid g \in G\} \rightarrow K_1(\mathbb{Z}G) \rightarrow K_1^w(RG));$$

$$\operatorname{Wh}^w(G) = \operatorname{Wh}^w(G; \mathbb{Z});$$

$$\tilde{K}_1(\mathcal{R}(G; F)) := \operatorname{coker}(\{\pm 1\} \xrightarrow{\cong} K_1(\mathbb{Z}) \rightarrow K_1(\mathbb{Z}G) \rightarrow K_1(\mathcal{R}(G; F)));$$

$$\operatorname{Wh}(\mathcal{R}(G; F)) = \operatorname{coker}(\{\pm g \mid g \in G\} \rightarrow K_1(\mathbb{Z}G) \rightarrow K_1(\mathcal{R}(G; F))).$$

**Remark 3.2.** Let  $A$  be a square matrix over  $RG$ . Then the square matrix  $\omega(A)$  over  $\mathcal{D}(G; F)$  is invertible if and only if the operator  $r_A^{(2)} : L^2(G)^n \rightarrow L^2(G)^n$  given by right multiplication with  $A$  is a weak isomorphism, i.e., it is injective and has dense image. This follows from the conclusion of [Lück 2002, Theorem 6.24 on page 249 and Theorem 8.22(5) on page 327] that  $r_A^{(2)}$  is a weak isomorphism if and only if it becomes invertible in  ${}^{\mathcal{U}}\mathcal{U}(G)$ .

There is a Dieudonné determinant for invertible matrices over a skew field  $D$  which takes values in the abelianization of the group of units  $D^\times/[D^\times, D^\times]$  and induces an isomorphism — see [Silvester 1981, Corollary 43 on page 133] —

$$\det_D : K_1(D) \xrightarrow{\cong} D^\times/[D^\times, D^\times]. \quad (3.3)$$

The inverse

$$J_D : D^\times/[D^\times, D^\times] \xrightarrow{\cong} K_1(D) \quad (3.4)$$

sends the class of a unit in  $D$  to the class of the corresponding  $(1, 1)$ -matrix.

**Theorem 3.5** ( $K_1^w(FG)$  for groups with property (ULA) with respect to  $F$ ). *Let  $R$  be a ring with  $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$ . Denote by  $F \subseteq \mathbb{C}$  the quotient field of  $R$ . Let  $G$  be a torsionfree group with the property (ULA) with respect to  $F$ .*

*Then the canonical maps sending  $[f]$  to  $[\omega_* f]$ ,*

$$\begin{aligned} \omega_* : K_1^w(RG) &\xrightarrow{\cong} K_1(\mathfrak{D}(G; F)), \\ \omega_* : \tilde{K}_1^w(RG) &\xrightarrow{\cong} \tilde{K}_1(\mathfrak{D}(G; F)), \\ \omega_* : \text{Wh}^w(G; R) &\xrightarrow{\cong} \text{Wh}(\mathfrak{D}(G; F)), \end{aligned}$$

*are bijective. Moreover,  $\mathfrak{D}(G; F)$  is a skew field and the Dieudonné determinant induces an isomorphism*

$$\det_D : K_1(\mathfrak{D}(G; F)) \xrightarrow{\cong} \mathfrak{D}(G; F)^\times/[\mathfrak{D}(G; F)^\times, \mathfrak{D}(G; F)^\times].$$

*Proof.* This follows directly from Theorem 1.2. □

Finally we can give the proof of Theorem 0.1.

*Proof of Theorem 0.1.* Because of Theorem 2.17 the group  $G$  has property (ULA) and we can apply Theorem 1.2. It remains to explain why in the special case  $R = \mathbb{Z}$  the group  $K_1^w(\mathbb{Z}G)$  appearing in Theorem 1.2, namely as introduced in Definition 3.1, agrees with the group  $K_1^w(\mathbb{Z}G)$  appearing in the introduction. This boils down to explaining why, for a  $(n, n)$ -matrix  $A$  over  $\mathbb{Z}G$ , the operator  $r_A^{(2)} : L^2(G)^n \rightarrow L^2(G)^n$  is a weak isomorphism if and only if  $A$  becomes invertible in  $\mathfrak{D}(G; \mathbb{Q})$ . By definition  $A$  is invertible in  $\mathfrak{D}(G; \mathbb{Q})$  if and only if it is invertible in  $\mathfrak{U}(G)$ . Now apply [Lück 2002, Theorem 6.24 on page 249 and Theorem 8.22(5) on page 327]. □

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
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