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Suslin's moving lemma with modulus

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The moving lemma of Suslin (also known as the generic equidimensionality theorem) states that a cycle on $X \times \mathbb{A}^n$ meeting all faces properly can be moved so that it becomes equidimensional over $\mathbb{A}^n$. This leads to an isomorphism of motivic Borel–Moore homology and higher Chow groups.

In this short paper we formulate and prove a variant of this. It leads to a modulus version of the isomorphism, in an appropriate pro setting.

1. Introduction

Suslin [2000] proved that, roughly speaking, a cycle on $X \times \mathbb{A}^n$ meeting all faces properly can be moved so that it becomes equidimensional over $\mathbb{A}^n$. Here X is an affine variety over a base field k. As a consequence he shows that the inclusion

$$z_r^{equi}(X, \bullet) \hookrightarrow z_r(X, \bullet) \tag{1.1}$$

of the cycle complex of equidimensional cycles into Bloch's cycle complex is a quasi-isomorphism for $r \geq 0$. This result is significant in incorporating Bloch's higher Chow groups into the Voevodsky–Suslin–Friedlander theory of mixed motives. Namely, for smooth schemes X over a field, we have an inclusion of complexes

$$C_*\big(z_{equi}(\mathbb{A}^i, 0)(X)\big) \hookrightarrow z^i(X, \bullet).$$

The left side is a sheaf of complexes defining the Voevodsky–Suslin–Friedlander motivic cohomology (at least over perfect fields, see [Mazza et al. 2006, Theorem 16.7]). The inclusion is a quasi-isomorphism by *Suslin's moving lemma* when X is the spectrum of a field. *Voevodsky's injectivity theorem* [Mazza et al. 2006, Corollary 11.2] for homotopy invariant sheaves with transfers then implies that the inclusion is a quasi-isomorphism locally on an arbitrary smooth X.

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Recently the context has been extended to *cycles with modulus*. Binda and Saito [2014] introduced the cycle complex with modulus $z_r(\bar{X}|Y, \bullet)$ for $r \geq 0$ and a pair $(\bar{X}, Y)$ of a finite-type k-scheme $\bar{X}$ and an effective Cartier divisor Y on it. We usually write $X := \bar{X} \setminus Y$. This generalizes Bloch's cycle complex in the sense that $z_r(\bar{X}|\varnothing, \bullet) = z_r(\bar{X}, \bullet)$. The homology group $\mathrm{CH}_r(\bar{X}|Y, n) := \mathrm{H}_n(z_r(\bar{X}|Y, \bullet))$ is called the higher Chow group with modulus. Moreover, we can construct a generalization of the inclusion (1.1):

$$z_r^{equi}(\bar{X}|Y, \bullet) \hookrightarrow z_r(\bar{X}|Y, \bullet).$$

The reader will find all the definitions of these objects in Section 2.

Our future aim is to extend the comparison between the higher Chow group and motivic cohomology group to the modulus setting. For this, we need to generalize (i) Suslin's moving lemma, and (ii) Voevodsky's injectivity theorem. The generalization of (ii) is expected to be done by the developing theory of *motives with modulus*, which was introduced by Kahn, Saito and Yamazaki [2015] as a generalization of Voevodsky's theory of motives.

In this paper, we generalize (i). In other words, we prove a variant of Suslin's moving lemma which takes the modulus condition into account (Theorem 3.11 below). Suslin's moving method does not preserve the so-called modulus condition on cycles, but instead we can show that the moved cycle satisfies the modulus condition to a lesser extent, and we have explicit control of the loss. It leads to the following:

Theorem 1.2 (Theorem 4.1). *Suppose $\bar{X}$ is affine and X is an open set of $\bar{X}$ such that $\bar{X} \setminus X$ is the support of an effective Cartier divisor Y. Let $r \geq 0$. Then the inclusions for $m \geq 0$,*

$$z_r^{equi}(\bar{X}|mY, \bullet) \hookrightarrow z_r(\bar{X}|mY, \bullet),$$

induce an isomorphism of inverse limits of their homology groups:

$$\varprojlim_m \mathrm{H}_n(z_r^{equi}(\bar{X}|mY, \bullet)) \cong \varprojlim_m \mathrm{CH}_r(\bar{X}|mY, n).$$

Note that it is quite natural and might be even necessary that inverse limits appear in the isomorphism. Indeed, we have several comparison isomorphisms in the theory of modulus which hold after taking inverse limits. A typical example is [Kerz and Saito 2016, Theorem III] which describes $\pi_1^{\mathrm{ab}}(X)^\circ$ as the inverse limit $\varprojlim_Y \mathrm{CH}_0(\bar{X}|Y)^\circ$, where $\bar{X}$ is a proper normal compactification of a smooth variety X over a finite field and the limit runs over effective Cartier divisors Y such that $\bar{X} \setminus Y = X$, and the superscript $(-)^\circ$ means the degree zero part. This is a higher dimensional analogue of the class field theory. Another example is [Rülling and Saito 2016, Theorem 2], a comparison isomorphism between the inverse limits of the Chow group with modulus and the relative motivic cohomology group of certain degree. This would be the first part of an isomorphism we aim to prove in the future. Moreover, Krishna and Park [2015, Theorem 1.0.7] prove a description of the crystalline cohomology group in terms of additive higher Chow groups, hypercohomology and inverse limits. Here, the additive higher Chow group is a special case of the higher Chow group with modulus, which can be obtained by taking a special pair of the form $(X \times \mathbb{A}^1, m(X \times \{0\}))$, $m \geq 1$ in our setting. Also, see Morrow's article [2016, §4] — the relative cohomology groups we consider in Section 4C echo his proposal for the definition of compact support K-groups.

We remark that the isomorphism in Theorem 1.2 actually comes from an isomorphism of pro-abelian groups. We can also give an explicit "pro bound" to annihilate the levelwise kernel and cokernel of the map (see Remark 4.2 (1)).

2. Definitions

We set $\square^n := (\mathbb{P}^1 \setminus \{\infty\})^n = \mathrm{Spec}(k[y_1, \ldots, y_n])$ in this paper, unlike some authors who prefer 1 as the point at infinity. With this convention our computations look simpler. We set a divisor on $(\mathbb{P}^1)^n$:

$$F_n = \sum_{i=1}^{n} (\mathbb{P}^1)^{i-1} \times \{\infty\} \times (\mathbb{P}^1)^{n-i}.$$

The *faces* of $\square^n$ are $\{y_i = 0\}$, $\{y_i = 1\}$ and their intersections.

Definition 2.1 [Binda and Saito 2014; Kahn et al. 2015]. (1) Let $\underline{z}_r(\bar{X}|Y, n)$ be the group of $(r+n)$-dimensional cycles on $X \times \square^n$ whose components V meet all faces of $\square^n$ properly, and *have modulus* Y, i.e.:

> Let $\overline{V}^N$ be the normalization of $\overline{V} \subset \bar{X} \times (\mathbb{P}^1)^n$, the closure of V. Let $\varphi_V : \overline{V}^N \to \bar{X} \times (\mathbb{P}^1)^n$ be the natural map. Then the inequality of Cartier divisors
>
> $$\varphi_V^{-1}(Y \times (\mathbb{P}^1)^n) \leq \varphi_V^{-1}(\bar{X} \times F_n)$$
>
> holds. (When $n = 0$ the condition reads: the closure $\overline{V} \subset \bar{X}$ of V is contained in X i.e., $V = \overline{V}$.)

Let $\partial_{i,\epsilon} : \square^{n-1} \hookrightarrow \square^n$, where $i \in \{1, \ldots, n\}$ and $\epsilon \in \{0, 1\}$, be the embedding of the face $\{y_i = \epsilon\}$:

$$\partial_{i,\epsilon} : (y_1, \ldots, y_{n-1}) \mapsto (y_1, \ldots, \overset{i}{\check{\epsilon}}, y_i, \ldots, y_{n-1}).$$

The groups $\underline{z}_r(\bar{X}|Y, n)$ form a complex with the differentials

$$\sum_{i=1}^{n} (-1)^i (\partial_{i,1}^* - \partial_{i,0}^*) : \underline{z}_r(\bar{X}|Y, n) \to \underline{z}_r(\bar{X}|Y, n-1).$$

(2) Let $\underline{z}_r^{equi}(\overline{X}|Y, n)$ be the subgroup of $\underline{z}_r(\overline{X}|Y, n)$ consisting of cycles that are equidimensional over $\square^n$, necessarily of relative dimension r. They define a subcomplex $\underline{z}_r^{equi}(\overline{X}|Y, \bullet)$ of $\underline{z}_r(\overline{X}|Y, \bullet)$.

Remark 2.2. The condition that V has modulus Y makes sense for any closed subset V of $X \times \square^n$. In that setting, normalization of a closed subset means the disjoint union of the normalizations of its reduced irreducible components.

Definition 2.3. We define the *degenerate part* $\underline{z}_r(\overline{X}|Y, n)_{\mathrm{degn}} \subset \underline{z}_r(\overline{X}|Y, n)$ as the subgroup generated by the cycles of the form

$$(\mathrm{id}_{\overline{X}} \times \mathrm{pr}_i)^*(V), \quad \text{where } V \in \underline{z}_r(\overline{X}|Y, n-1)$$

and

$$\mathrm{pr}_i : \square^n \to \square^{n-1}, \quad (y_1, \dots, y_n) \mapsto (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$$

for some $i = 1, \dots, n$. We also define the *degenerate part* $\underline{z}_r^{equi}(\overline{X}|Y, n)_{\mathrm{degn}} \subset \underline{z}_r^{equi}(\overline{X}|Y, n)$ in a similar way. We set

$$z_r(\overline{X}|Y, n) := \underline{z}_r(\overline{X}|Y, n)/\underline{z}_r(\overline{X}|Y, n)_{\mathrm{degn}},$$
$$z_r^{equi}(\overline{X}|Y, n) := \underline{z}_r^{equi}(\overline{X}|Y, n)/\underline{z}_r^{equi}(\overline{X}|Y, n)_{\mathrm{degn}}.$$

Noting that the differentials $\partial_{i,\epsilon}$ preserve degenerate parts, we can see that $z_r(\overline{X}|Y, n)$ and $z_r^{equi}(\overline{X}|Y, n)$ also form complexes. We define the *higher Chow group with modulus* by

$$\mathrm{CH}_r(\overline{X}|Y, n) := \mathrm{H}_n(z_r(\overline{X}|Y, \bullet)).$$

We also consider the homology groups of the latter:

$$\mathrm{H}_n(z_r^{equi}(\overline{X}|Y, \bullet)).$$

Voevodsky–Suslin–Friedlander give no particular name to its counterpart without modulus. In this paper, we would like to call it the *Suslin homology group with compact support with modulus*. The term "with compact support" reflects the fact that we are using z^{equi} instead of c^{equi}, where the latter is used to define the usual Suslin homology.

Remark 2.4. In this remark, we explain that we can use another complex to define the higher Chow group with modulus. This is a general fact on cubical objects (see, for example, [Levine 2009, §1.2]). The subgroups

$$\underline{z}_r(\overline{X}|Y, n)_0 := \bigcap_{i=1}^{n} \ker(\partial_{i,0}^*) \subset \underline{z}_r(\overline{X}|Y, n)$$

form a subcomplex. One checks that the composite

$$\underline{z}_r(\overline{X}|Y, \bullet)_0 \to \underline{z}_r(\overline{X}|Y, \bullet) \to z_r(\overline{X}|Y, \bullet)$$

is an isomorphism, where the first map is the natural inclusion and the latter is the quotient map. This implies that we have a direct sum decomposition

$$\underline{z}_r(\bar{X}|Y,\bullet) = z_r(\bar{X}|Y,\bullet) \oplus \underline{z}_r(\bar{X}|Y,\bullet)_{\mathrm{degn}}$$

of a complex, and that $\mathrm{CH}_r(\bar{X}|Y,n) \cong \mathrm{H}_n(\underline{z}_r(\bar{X}|Y,\bullet)_0)$. We have a similar decomposition of $\underline{z}_r^{equi}(\bar{X}|Y,\bullet)$, and the inclusion $\underline{z}_r^{equi}(\bar{X}|Y,\bullet) \hookrightarrow \underline{z}_r(\bar{X}|Y,\bullet)$ is compatible with the decompositions.

3. Equidimensionality theorem

Let k be an infinite base field. We will formulate and prove a variant of Suslin's equidimensionality Theorem 3.11 for modulus pairs $(\bar{X}, Y)$, i.e., a k-scheme $\bar{X}$ of finite type equipped with an effective Cartier divisor Y, for which $\bar{X}$ is affine.

Recall a *face* of $\square^n = \mathrm{Spec}(k[y_1,\dots,y_n])$ is a closed subscheme of the form $\{y_i = 0\}$, $\{y_i = 1\}$ or an intersection of them. Define a Cartier divisor $\partial\square^n = \sum \partial_{i,\epsilon}(\square^{n-1})$, where the sum is over all $1 \le i \le n$ and $\epsilon = 0, 1$. Recall the map $\partial_{i,\epsilon} : \square^{n-1} \hookrightarrow \square^n$ denotes the embedding corresponding to the equation $y_i = \epsilon$ for each i, ϵ. The divisor $\partial\square^n$ is defined by the equation

$$h(\underline{y}) = y_1(1-y_1)\cdots y_n(1-y_n). \tag{3.1}$$

We need the following version of Suslin's moving lemma where we control the degrees of the map Φ^n.

Theorem 3.2. *Let $\bar{X} = \mathrm{Spec}(R)$ be an affine k-scheme of finite type and $V \subset \bar{X} \times \square^n$ be a closed subset of dimension $n+t$ for some $t \ge 0$. Suppose an $\bar{X}$-morphism*

$$\Phi' : \bar{X} \times \partial\square^n \to \bar{X} \times \square^n$$

is given and there is an integer $d \ge 2$ such that for any codimension 1 face

$$\partial_{l,\epsilon} : \square^{n-1} \hookrightarrow \square^n,$$

the composite $\Phi' \circ (\mathrm{id}_{\bar{X}} \times \partial_{l,\epsilon})$ is defined by polynomials $\Phi'_{i,l,\epsilon} \in R[y_1,\dots,y_{n-1}]$ $(1 \le i \le n)$ whose degrees with respect to y_j are at most d for each j.

Then we can find an $\bar{X}$-map

$$\Phi^n : \bar{X} \times \square^n \to \bar{X} \times \square^n$$

extending Φ' such that $(\Phi^n)^{-1}(V) \subset \bar{X} \times \square^n$ has fibers of dimension $\le t$ over $\square^n \setminus \partial\square^n$, and moreover, the functions $\Phi^n_i \in R[y_1,\dots,y_n]$ defining Φ^n $(1 \le i \le n)$ have degrees $\le d$ with respect to each y_j.

Proof. The map Φ' is determined by R-coefficient polynomials $f_i(y_1,\dots,y_n)$ mod $h(\underline{y})$ $(1 \le i \le n)$. If we substitute $y_j = 0$ or $y_j = 1$ to f_i we get a polynomial which has degree $\le d$ with respect to each y_k by the hypothesis.

Lemma 3.3. *Let $d \geq 1$ be an integer. Suppose given a polynomial $f(y_1, \dots, y_n) \in R[y_1, \dots, y_n]$ such that for each j, if we substitute any of $y_j = 0$ or $y_j = 1$, the resulting polynomial has degree $\leq d$ with respect to each y_k. Then $f \mod h(\underline{y})$ has a (unique) representative which has degree $\leq d$ with respect to each y_j (where we keep the notation $h(\underline{y}) = y_1(1 - y_1) \cdots y_n(1 - y_n)$ introduced in (3.1)).*

Proof. For each i denote by $y_i(-|_{y_i=1})$ the operator which sends a polynomial f to $y_i \cdot (f|_{y_i=1})$ and define $(1 - y_i)(-|_{y_i=0})$ similarly. Note that for different i and j the operators $y_i(-|_{y_i=1})$ and $y_j(-|_{y_j=1})$ commute (and similarly for other pairs). Put $\alpha_i := 1 - y_i(-|_{y_i=1}) - (1 - y_i)(-|_{y_i=0})$. Then one can see the polynomial

$$f - (\alpha_1 \cdots \alpha_n f)$$

is the desired representative. □

By the previous lemma, we take representatives $f_i(\underline{y})$ having degrees $\leq d$ with respect to each y_j.

Take a finite set $\{x_1, \dots, x_m\}$ of generators of the k-algebra R. We are going to define the asserted map Φ^n by setting its components ($1 \leq i \leq n$) to be

$$\Phi_i^n(\underline{y}) := f_i(\underline{y}) + h(\underline{y}) F_i(\underline{x}),$$

where $F_i(t_1, \dots, t_m) \in k[t_1, \dots, t_m]$ are homogeneous polynomials in variables $t_1, \dots, t_m$ of some uniform degree N. From this form, the functions Φ_i^n have degrees $\leq d$ with respect to each y_j.

Now, in his proof of the generic equidimensionality theorem, Suslin [2000, Theorem 1.1] actually introduces the following specific statement in the first two paragraphs and proves it in [loc. cit., §§(1.2)–(1.8)].

Specific statement 3.4 [Suslin 2000, proof of Theorem 1.1]. *Let R be a k-algebra of finite type and let $x_1, \dots, x_m \in R$ be a finite set of generators over k. Let $H(\underline{y}) \in k[y_1, \dots, y_n]$ and $f_i(\underline{y}) \in R[y_1, \dots, y_n]$, $1 \leq i \leq n$, be polynomials in variables $y_1, \dots, y_n$. Let V be a closed subset in $\mathbb{A}_R^n = \operatorname{Spec}(R[y_1, \dots, y_n])$ of dimension $\leq n + t$ for some nonnegative integer t.*

Consider R-morphisms $\Phi : \mathbb{A}_R^n \to \mathbb{A}_R^n$ defined by polynomials of the form

$$\Phi_i(\underline{y}) = f_i(\underline{y}) + H(\underline{y}) F_i(\underline{x}), \quad 1 \leq i \leq n,$$

where $F_i(\underline{t}) \in k[t_1, \dots, t_m]$ are homogeneous polynomials in variables $t_1, \dots, t_m$ of some uniform degree N.

Then if N is large enough, for almost all tuples $(F_i)_{i=1}^n$, the fibers of the projection $\Phi^{-1}(V) \subset \mathbb{A}_R^n \to \mathbb{A}_k^n$ have dimensions $\leq t$ over $\mathbb{A}_k^n \setminus \{H(\underline{y}) = 0\}$.

(For a fixed N, the tuples of polynomials $(F_i)_i$ are parametrized by the rational points of an affine space of dimension $\binom{N+m-1}{m-1} n$. The statement means that the set of tuples $(F_i)_i$ where the stated condition fails is contained in a proper closed subset of the affine space.)

Thus if N is large enough, a general choice of $(F_i)_{i=1}^n$ makes our assertion on fiber dimension true. This completes the proof of Theorem 3.2. $\square$

Now, to understand the Suslin moving lemma in the context of modulus, first recall the following:

Lemma 3.5 (containment lemma [Krishna and Park 2012, Proposition 2.4]). *Let $V \subset \bar{X} \times \square^n$ be a closed subset which has modulus Y and $V' \subset V$ be a smaller closed subset. Then V' also has modulus Y.*

Proposition 3.6. *Let $(\bar{X}, Y)$ be a modulus pair with $\bar{X} = \mathrm{Spec}(R)$ affine. Let d be a positive integer and $V \subset \bar{X} \times \square^n$ be a closed subset having modulus $nd \cdot Y$. Suppose*

$$\Phi : \bar{X} \times \square^{n'} \to \bar{X} \times \square^n$$

is an $\bar{X}$-morphism defined by polynomials $\Phi_j \in R[y_1, \dots, y_{n'}]$ $(1 \le j \le n)$ having degrees $\le d$ with respect to each y_i. Then the closed subset $\Phi^{-1}(V)$ of $\bar{X} \times \square^{n'}$ has modulus Y.

Proof. Since the assertion is local on $\bar{X}$, we may assume Y is principal and defined by $u \in R$. Let V' denote any one of the irreducible components of $\Phi^{-1}(V)$ and let $\overline{V'}^N$ be the normalization of its closure $\overline{V'}$ in $\bar{X} \times (\mathbb{P}^1)^{n'}$;

$$\begin{tikzcd}
\overline{V'}^N \arrow[d] & & & \\
\overline{V'} & V' \arrow[l, hook'] \arrow[r, hook] & \Phi^{-1}(V) \arrow[r, hook] \arrow[d] & \bar{X} \times \square^{n'} \arrow[d, "\Phi"] \\
 & & V \arrow[r, hook] & \bar{X} \times \square^n
\end{tikzcd}$$

Thanks to the containment lemma (Lemma 3.5), the closure of $\Phi(V')$ in V has modulus ndY. By replacing V by the closure of $\Phi(V')$ in V, we may assume the map $V' \to V$ is dominant.

Claim 3.7. Let $\overline{V'}^{N\circ}$ be the domain of definition of the rational map

$$\overline{V'}^N \to \bar{X} \times (\mathbb{P}^1)^{n'} \overset{\Phi}{\dashrightarrow} \bar{X} \times (\mathbb{P}^1)^n.$$

Then the complement of $\overline{V'}^{N\circ}$ in $\overline{V'}^N$ has codimension ≥ 2.

Proof. Let v be a point of $\overline{V'}^N$ of codimension 1. Since the generic point η of $\overline{V'}^N$ lands in $\bar{X} \times \square^{n'}$ we have a commutative diagram

$$\begin{tikzcd} \eta \arrow[r] \arrow[d, hook] & \bar{X} \times \square^{n'} \arrow[r, "\Phi"] & \bar{X} \times (\mathbb{P}^1)^n \arrow[d] \\ \mathrm{Spec}\mathcal{O}_v \arrow[rr] & & \bar{X} \end{tikzcd}$$

The assertion follows from the valuative criterion of properness applied to the projective morphism $\bar{X} \times (\mathbb{P}^1)^n \to \bar{X}$. $\square$

By Claim 3.7, we find that a Cartier divisor on $\overline{V'}^N$ is effective if and only if its restriction to $\overline{V'}^{N\circ}$ is effective, since $\overline{V'}^N$ is normal.

Write $\mathrm{pr}_j : \bar{X} \times (\mathbb{P}^1)^n \to \mathbb{P}^1$ for the projection to the j-th $\mathbb{P}^1$ and Φ_j for the composite rational map

$$\bar{X} \times (\mathbb{P}^1)^{n'} \overset{\Phi}{\dashrightarrow} \bar{X} \times (\mathbb{P}^1)^n \xrightarrow{\mathrm{pr}_j} \mathbb{P}^1,$$

also seen as a rational function on $\bar{X} \times (\mathbb{P}^1)^{n'}$. We will denote the pull-backs of Φ and Φ_j to $\overline{V'}^{N\circ}$ by Φ^V and Φ^V_j. By definition of $\overline{V'}^{N\circ}$ they are well-defined morphisms from $\overline{V'}^{N\circ}$ to $\bar{X} \times (\mathbb{P}^1)^n$ and to $\mathbb{P}^1$ respectively. There is a uniquely induced morphism $\overline{V'}^{N\circ} \to \overline{V}^N$ because now we are assuming $V' \to V$ is dominant.

For any given point of $\overline{V'}^{N\circ}$, we can find an affine open set $\mathrm{Spec}(A) \subset \overline{V}^N$ and an affine neighborhood $\mathrm{Spec}(B) \subset \overline{V'}^{N\circ}$ of the point over which Φ^V restricts to a morphism $\Phi^V : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r, hook] & \overline{V'}^{N\circ} \arrow[r] \arrow[d, "\Phi^V"] & \bar{X} \times (\mathbb{P}^1)^{n'} \arrow[d, dashed, "\Phi"] \\ \mathrm{Spec}(A) \arrow[r, hook] & \overline{V}^N \arrow[r] & \bar{X} \times (\mathbb{P}^1)^n \end{tikzcd}$$

By shrinking $\mathrm{Spec}(A)$ if necessary, we may assume y_j or $1/y_j$ is regular on $\mathrm{Spec}(A)$ for each j. Denote by $J \subset \{1, \dots, n\}$ the set of j's for which $1/y_j$ is regular. The divisor F_n is defined by the equation $1/\prod_{j \in J} y_j = 0$ on $\mathrm{Spec}(A)$. Since V has modulus ndY, the rational function $\left(1/\prod_{j\in J} y_j\right)/u^{nd}$ on $\mathrm{Spec}(A)$ is regular. Pulling it back by Φ^V, we find that the rational function

$$\frac{1}{\prod_{j\in J} \Phi^V_j} \Big/ u^{nd} \tag{3.8}$$

on $\mathrm{Spec}(B)$ is regular.

Shrinking $\mathrm{Spec}(B)$ if necessary, we may assume y_i or $1/y_i$ is regular on $\mathrm{Spec}(B)$ for each i. Let $I \subset \{1, \dots, n'\}$ be the set of i's for which $1/y_i$ is regular on $\mathrm{Spec}(B)$; the divisor $F_{n'}$ is defined by $1/\prod_{i\in I} y_i = 0$ on $\mathrm{Spec}(B)$.

Claim 3.9. The rational function $\Phi_j^V/\prod_{i\in I} y_i^d$ on $\mathrm{Spec}(B)$ is regular for each $j \in \{1, \dots, n\}$, i.e., it is a morphism from $\mathrm{Spec}(B)$ into $\mathbb{A}^1 \subset \mathbb{P}^1$.

Proof. The function is the restriction of the meromorphic function $\Phi_j/\prod_{i\in I} y_i^d$ on $\bar{X} \times (\mathbb{P}^1)^{n'}$. It is written as an R-coefficient polynomial in the variables $1/y_i$ ($i \in I$) and y_i ($i \in I^c$) by the assumption on Φ. So it is regular around the (image of the) considered point on $\bar{X} \times (\mathbb{P}^1)^{n'}$. $\square$

By the regularity of the function (3.8) and Claim 3.9, the function

$$\left(\frac{1}{\prod_{j\in J}\Phi_j^V}\Big/u^{nd}\right)\cdot\prod_{j\in J}\frac{\Phi_j^V}{\prod_{i\in I} y_i^d} = \frac{1}{\prod_{i\in I} y_i^{d\cdot\#J}}\Big/u^{nd}$$

is regular on $\mathrm{Spec}(B)$. This shows a relation of Cartier divisors on $\mathrm{Spec}(B)$:

$$nd\left(\prod_{i\in I}\frac{1}{y_i}\right) - nd(u) \geq 0,$$

which implies the relation

$$(\text{pullback of } F_{n'}) - (\text{pullback of } Y) \geq 0$$

on $\mathrm{Spec}(B)$, hence on $\overline{V'}^{N\circ}$, which is valid on $\overline{V'}^{N}$ as well by Claim 3.7. This completes the proof of Proposition 3.6. $\square$

Remark 3.10. Under the hypotheses of Proposition 3.6, we can prove that the morphism Φ is admissible [Kahn et al. 2015, Definition 1.1] for the pair

$$((\mathbb{P}^1_R)^{n'}, ndF_{n'}), \quad ((\mathbb{P}^1_R)^n, F_n).$$

Here, for pairs (X, D), (Y, E) of schemes and effective Cartier divisors, a morphism $f : X \setminus D \to Y \setminus E$ is said to be *admissible* if the following holds: Let $\overline{\Gamma}_f$ be the closure of the graph of f in $X \times Y$ and $\overline{\Gamma}_f^N$ be its normalization. Let $\varphi : \overline{\Gamma}_f^N \to X \times Y$ be the natural map. Then the inequality of Cartier divisors $\varphi^{-1}(D \times Y) \geq \varphi^{-1}(X \times E)$ on $\overline{\Gamma}_f^N$ holds.

It gives an alternative proof of Proposition 3.6 thanks to [Krishna and Park 2012, Lemma 2.2]. Here we sketch the proof of the admissibility. We use the fact that admissibility can be checked after replacing the source by an open cover (for a trivial reason), and after blowing up $(\mathbb{P}^1)^{n'}$ by a closed subset outside $\square^{n'}$ (by [Krishna and Park 2012, Lemma 2.2] again). Set $\eta_i = 1/y_i$. The scheme $(\mathbb{P}^1)^{n'}$ is covered by open subsets $U_I = \mathrm{Spec}(R[\eta_i, y_{i'}\ {}_{i\in I, i'\notin I}])$, where I runs though the subsets of $\{1, \dots, n'\}$. On the region U_I, the rational function $\Phi_j^{(I)}$ defined by the next equation is regular, by the assumption on Φ_j:

$$\Phi_j = \frac{\Phi_j^{(I)}(\eta_i, y_{i'})}{\prod_{i\in I}\eta_i^d}.$$

We blow up U_I by the ideal $\left(\Phi_j^{(I)}, \prod_{i\in I}\eta_i^d\right)$. We perform this blow up for all $j \in \{1,\dots,n\}$. The resulting scheme is covered by the 2^n open subsets

$$U_{IJ} = \operatorname{Spec}\left(R\left[\eta_i, y_{i'}\ {}_{i\in I, i'\notin I}, \frac{\prod_{i\in I}\eta_i^d}{\Phi_j^{(I)}(\eta_i, y_{i'})}, \frac{\Phi_{j'}^{(I)}(\eta_i, y_{i'})}{\prod_{i\in I}\eta_i^d}\ {}_{j\in J, j'\notin J}\right]\right),$$

where J runs through the subsets of $\{1,\dots,n\}$. The morphism Φ naturally extends to a morphism $\Phi : U_{IJ} \to U_J \subset (\mathbb{P}^1)^n$.

On U_{IJ}, the pull-back of F_n by Φ is represented by the function

$$\prod_{j\in J}\frac{\prod_{i\in I}\eta_i^d}{\Phi_j^{(I)}(\eta_i, y_{i'})}.$$

The divisor $ndF_{n'}$ is represented by $\prod_{i\in I}\eta_i^{nd}$. Hence the difference $ndF_{n'|U_{IJ}} - \Phi^*_{|U_{IJ}}F_n$ is defined by the function

$$\prod_i \eta_i^{(n-\#J)d}\cdot\prod_{j\in J}\Phi_j^{(I)},$$

which is a regular function on U_{IJ}. This proves the admissibility.

From Theorem 3.2 and Proposition 3.6, we get:

Theorem 3.11. *Let $(\bar{X}, Y)$ be a modulus pair with $\bar{X}$ affine, and $V \subset \bar{X}\times\square^n$ be a purely $(n+t)$-dimensional closed subset for some $t \geq 0$. Suppose V has modulus $2n\cdot Y$. Then there is a series of maps*

$$\Phi^\bullet : \bar{X}\times\square^\bullet \to \bar{X}\times\square^\bullet$$

compatible with face maps, i.e., for any codimension 1 face $\partial_{i,\epsilon} : \square^m \hookrightarrow \square^{m+1}$, the following diagram commutes:

$$\begin{tikzcd} \bar{X}\times\square^m \arrow[r, "\Phi^m"] \arrow[d, hook, "\partial_{i,\epsilon}"] & \bar{X}\times\square^m \arrow[d, hook, "\partial_{i,\epsilon}"] \\ \bar{X}\times\square^{m+1} \arrow[r, "\Phi^{m+1}"] & \bar{X}\times\square^{m+1} \end{tikzcd}$$

such that the closed subset

$$(\Phi^n)^{-1}(V) \subset \bar{X}\times\square^n$$

is equidimensional over $\square^n$ of relative dimension t, and has modulus Y. Moreover, the defining polynomials Φ_i^m can be taken to have degree ≤ 2 for each variable y_j.

It is proved by induction on m, starting with $\Phi^0 = \mathrm{id}$ which has degree 0 and with V replaced by its restrictions to faces. Note that given a series of maps with the indicated compatibility and a cycle α on $\bar{X} \times \square^m$, the following equality of cycles on $\bar{X} \times \square^{m-1}$ holds whenever the relevant cycles are well-defined:

$$d((\Phi^m)^*\alpha) = (\Phi^{m-1})^*(d\alpha). \tag{3.12}$$

4. Suslin homology with compact support with modulus and higher Chow groups with modulus

In this section, let $\bar{X}$ be an *affine* finite-type scheme over an arbitrary field k and X be an open subset such that $\bar{X} \setminus X$ is the support of an effective Cartier divisor. The letter Y will denote effective Cartier divisors with support $\bar{X} \setminus X$. The aim of this section is to prove the following theorem.

Theorem 4.1. *Let $r \geq 0$. The inclusions*

$$\underline{z}_r^{equi}(\bar{X}|Y, \bullet) \subset \underline{z}_r(\bar{X}|Y, \bullet)$$

induce isomorphisms on the homology pro-groups for each n:

$$\left\{\mathrm{H}_n(\underline{z}_r^{equi}(\bar{X}|Y, \bullet))\right\}_Y \xrightarrow{\cong} \left\{\mathrm{H}_n(\underline{z}_r(\bar{X}|Y, \bullet))\right\}_Y,$$

where Y runs through effective Cartier divisors with support $\bar{X} \setminus X$.

Remark 4.2. (1) An explicit pro bound to annihilate the levelwise kernel and cokernel of the map will be indicated in Lemma 4.5. Theorem 4.1 implies Theorem 1.2 in the introduction, in light of Remark 2.4.

(2) In the terminology of [Fausk and Isaksen 2007, §6], the above theorem can be expressed as: the map $\{\underline{z}_r^{equi}(\bar{X}|Y, \bullet)\}_Y \to \{\underline{z}_r(\bar{X}|Y, \bullet)\}_Y$ is a weak equivalence in the $\mathcal{H}_*$-model category of pro-complexes of abelian groups.

4A. ***Construction of weak homotopy.*** Temporarily assume k is an infinite field, so that we can use the results in Section 3.

Fix an effective Cartier divisor Y with support $\bar{X} \setminus X$. Suppose we are given a cycle $V \in \underline{z}_r(\bar{X}|2nY, n)$. Apply Theorem 3.11 to $|V|$ and get a series of $\bar{X}$-maps $\Phi^\bullet : \bar{X} \times \square^\bullet \to \bar{X} \times \square^\bullet$.

Repeated application of Theorem 3.2 gives another series of $\bar{X}$-maps

$$\widetilde{\Phi}^\bullet : \bar{X} \times \square^\bullet \times \mathbb{A}^1 \to \bar{X} \times \square^\bullet \times \mathbb{A}^1$$

satisfying:

(1) The following diagrams commute:

$$\begin{tikzcd}
\bar{X} \times \square^n \arrow[r, "\mathrm{id}"] \arrow[d, "i_0"'] & \bar{X} \times \square^n \arrow[d, "i_0"] \\
\bar{X} \times \square^n \times \mathbb{A}^1 \arrow[r, "\widetilde{\Phi}^n"] & \bar{X} \times \square^n \times \mathbb{A}^1
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{X} \times \square^n \arrow[r, "\Phi^n"] \arrow[d, "i_1"'] & \bar{X} \times \square^n \arrow[d, "i_1"] \\
\bar{X} \times \square^n \times \mathbb{A}^1 \arrow[r, "\widetilde{\Phi}^n"] & \bar{X} \times \square^n \times \mathbb{A}^1
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{X} \times \square^{n-1} \times \mathbb{A}^1 \arrow[r, "\widetilde{\Phi}^{n-1}"] \arrow[d, "1_{\bar{X}} \times \partial_{j,\epsilon} \times 1_{\mathbb{A}^1}"'] & \bar{X} \times \square^{n-1} \times \mathbb{A}^1 \arrow[d, "1_{\bar{X}} \times \partial_{j,\epsilon} \times 1_{\mathbb{A}^1}"] \\
\bar{X} \times \square^n \times \mathbb{A}^1 \arrow[r, "\widetilde{\Phi}^n"] & \bar{X} \times \square^n \times \mathbb{A}^1
\end{tikzcd}$$

(2) The dimensions of the fibers of the map

$$(\widetilde{\Phi}^n)^{-1}(|V| \times \mathbb{A}^1) \hookrightarrow \bar{X} \times \square^n \times \mathbb{A}^1 \to \square^n \times \mathbb{A}^1$$

are $\leq r$ over $\square^n \times (\mathbb{A}^1 \setminus \{0\})$. (Consequently if V happens to be in $\underline{z}_r^{equi}$, then $(\widetilde{\Phi}^n)^{-1}(|V| \times \mathbb{A}^1)$ is equidimensional over $\square^n \times \mathbb{A}^1$.)

(3) The map $\widetilde{\Phi}^n$ is defined by $n+1$ polynomials belonging to $\mathcal{O}(\bar{X})[y_1, \ldots, y_n, t]$ having degrees ≤ 2 in each variable, where t is the coordinate of $\mathbb{A}^1$.

We explain a little more about the construction of $\widetilde{\Phi}^n$. It is done by induction on n. Suppose we have constructed $\widetilde{\Phi}^{n-1}$, with $|V|$ in condition (2) replaced by the union of its restrictions to the faces.

Set a Cartier divisor $Z := (\square^n \times 0) + (\square^n \times 1) + (\partial\square^n \times \mathbb{A}^1)$ on $\square^n \times \mathbb{A}^1$. Via the isomorphism $\square^n \times \mathbb{A}^1 \cong \square^{n+1}$, we have $Z \cong \partial\square^{n+1}$. Condition (1) for $\widetilde{\Phi}^{n-1}$ implies that there exists a unique $\bar{X}$-map

$$\bar{X} \times Z \to \bar{X} \times \square^n \times \mathbb{A}^1$$

whose restrictions to the faces isomorphic to $\bar{X} \times \square^n$ are the maps already defined: either id, Φ^n or $\widetilde{\Phi}^{n-1}$. This existence follows from the next elementary fact proved by induction and the snake lemma: Let R be a commutative ring with unit and let $x_1, \ldots, x_n$ be elements of R which form a regular sequence, no matter how they are ordered. Then the set of elements $x_1, \ldots, x_{n-2}, x_{n-1}x_n$ has the same property, and we have an isomorphism

$$R/x_1\cdots x_nR \xrightarrow{\sim} \varprojlim\Big[\prod_i R/x_iR \rightrightarrows \prod_{i<j} R/(x_i,x_j)R\Big].$$

By the induction hypothesis and the choice of $\Phi^\bullet$, the maps id, Φ^n, $\widetilde{\Phi}^{n-1}$ are defined by polynomials whose degrees are ≤ 2 in each variable. Then by Theorem 3.2, we obtain $\widetilde{\Phi}^n$ having degrees ≤ 2 and satisfying (1)–(2).

We note a compatibility property satisfied by the pull-back operation $(\widetilde{\Phi}^n)^*$. Suppose we are given a cycle α on $\bar{X}\times\square^m$. We can consider its differential $d(\alpha)$ on $\bar{X}\times\square^{m-1}$ if it is well-defined. On the other hand, suppose we are given a cycle β on $\bar{X}\times\square^m\times\mathbb{A}^1$. Via the isomorphism $\bar{X}\times\square^m\times\mathbb{A}^1\cong\bar{X}\times\square^{m+1}$ we view it as a cycle on the latter, and consider its differential which is a cycle on $\bar{X}\times\square^m$. We denote it by $\tilde{d}\beta$.

Thanks to condition (1) on $\widetilde{\Phi}^\bullet$, the following equality of cycles on $\bar{X}\times\square^m$ holds whenever the relevant cycles are all well-defined:

$$\tilde{d}((\widetilde{\Phi}^m)^*(\alpha\times\mathbb{A}^1)) = (\widetilde{\Phi}^{m-1})^*(d(\alpha)\times\mathbb{A}^1)+(-1)^{m+1}((\Phi^m)^*\alpha-\alpha). \qquad (4.3)$$

This applies in particular to $\alpha=V$: all terms are indeed well-defined, for example, by the choice of $\Phi^\bullet$ and $\widetilde{\Phi}^\bullet$, the irreducible components of $(\widetilde{\Phi}^n)^{-1}(|V|\times\mathbb{A}^1)$ have dimensions at most $r+n+1$, which is the lowest possible due to the fact that $\widetilde{\Phi}^n$ is an $\bar{X}$-endomorphism of a smooth $\bar{X}$-scheme. So the term $(\widetilde{\Phi}^n)^*(V\times\mathbb{A}^1)$ is well-defined. Similarly for other terms.

4B. ***Proof of the comparison theorem.*** Finally we can prove Theorem 4.1. Let $f^Y:\underline{z}_r^{equi}(\bar{X}|Y,\bullet)\to\underline{z}_r(\bar{X}|Y,\bullet)$ denote the natural inclusion. It suffices to prove that

$$\big\{\mathrm{H}_n\underline{z}_r^{equi}(\bar{X}|Y,\bullet)\big\}_Y \xrightarrow{\{\mathrm{H}_nf^Y\}_Y} \big\{\mathrm{H}_n\underline{z}_r(\bar{X}|Y,\bullet)\big\}_Y$$

is an isomorphism in the category of pro-abelian groups pro-**Ab**. Its kernel and cokernel are $\{\mathrm{Ker}(\mathrm{H}_nf^Y)\}_Y$ and $\{\mathrm{Coker}(\mathrm{H}_nf^Y)\}_Y$ [Artin and Mazur 1969, Appendix, Proposition 4.1]. We prove that they are zero objects in pro-**Ab**. Now we recall the following elementary lemma:

Lemma 4.4. *An object $A=\{A^\gamma\}_{\gamma\in\Gamma}\in$ pro-**Ab** is the zero object if and only if for any $\gamma\in\Gamma$ there exists $\gamma'>\gamma$ such that the projection map $p_\gamma^{\gamma'}:A^{\gamma'}\to A^\gamma$ is the zero map.*

Therefore, the problem is reduced to showing the following:

Lemma 4.5. *For any effective Cartier divisor Y and $n\geq 0$, the projections*

$$\mathrm{Ker}(\mathrm{H}_nf^{2(n+1)Y})\to\mathrm{Ker}(\mathrm{H}_nf^Y) \quad \textit{and} \quad \mathrm{Coker}(\mathrm{H}_nf^{2nY})\to\mathrm{Coker}(\mathrm{H}_nf^Y)$$

are the zero maps.

Proof. Assume first k is infinite. We first prove that $\mathrm{Coker}(\mathrm{H}_n f^{2nY}) \to \mathrm{Coker}(\mathrm{H}_n f^Y)$ is the zero map for any $n \geq 0$. Take any element $W \in \mathrm{H}_n(\underline{z}_r(\bar{X}|2nY, \bullet))$. Apply the construction in Section 4A to W and get a cycle $(\widetilde{\Phi}^n)^* W \in \underline{z}_r^{equi}(\bar{X}|Y, n)$. Thanks to Equation (3.12), it is annihilated by the differential. Equation (4.3) now reads

$$\tilde{d}((\widetilde{\Phi}^n)^*(W \times \mathbb{A}^1)) = (-1)^{n+1}((\Phi^n)^* W - W)$$

in $\underline{z}_r(\bar{X}|Y, n)$, hence we have $W = (\Phi^n)^* W$ in $\mathrm{H}_n(\underline{z}_r(\bar{X}|Y, \bullet))$. This proves the assertion for the cokernel.

Next we prove that $\mathrm{Ker}(\mathrm{H}_n f^{(2n+2)Y}) \to \mathrm{Ker}(\mathrm{H}_n f^Y)$ is the zero map. Take any cycle V representing an element in $\mathrm{Ker}(\mathrm{H}_n f^{(2n+2)Y})$. Then, there exists $W \in \underline{z}_r(\bar{X}|(2n+2)Y, n+1)$ such that $V = dW$ as cycles.

Apply the construction in Section 4A to W (n replaced with $n+1$) and get a cycle $(\Phi^{n+1})^* W \in \underline{z}_r^{equi}(\bar{X}|Y, n+1)$ and $(\tilde{\Phi}^{n+1})^*(W \times \mathbb{A}^1) \in \underline{z}_r(\bar{X}|Y, n+2)$ whose modulus condition follows from Proposition 3.6. Equation (4.3) for $\alpha = W$ reads

$$\tilde{d}((\widetilde{\Phi}^{n+1})^*(W \times \mathbb{A}^1)) = (\tilde{\Phi}^n)^*(V \times \mathbb{A}^1) + (-1)^n((\Phi^{n+1})^* W - W).$$

Differentiate it to get $0 = d(\widetilde{\Phi}^n)^*(V \times \mathbb{A}^1) + (-1)^n(d(\Phi^{n+1})^* W - V)$. Hence

$$V = d(\Phi^{n+1})^* W + (-1)^n d(\widetilde{\Phi}^n)^*(V \times \mathbb{A}^1).$$

Thanks to the choice of $\widetilde{\Phi}^\bullet$ and the fact that V is equidimensional, both $(\Phi^{n+1})^* W$ and $(\widetilde{\Phi}^n)^*(V \times \mathbb{A}^1)$ are equidimensional cycles. So V is zero in $\mathrm{H}_n(\underline{z}_r^{equi}(\bar{X}|Y, \bullet))$. This proves the assertion for the kernel, hence completes the proof for infinite fields.

Finally, suppose that k is finite. This case is settled by a trace (norm) argument. Let $l \in \{2, 3\}$ and k_l be an infinite pro-l extension of k. Given any $V \in \mathrm{Coker}(\mathrm{H}_n f^{2nY})$, its image in $\mathrm{Coker}(\mathrm{H}_n f^Y)_{k_l}$ is zero by the infinite field case (the subscript means the group is computed after the scalar extension k_l/k). Since the latter group is the direct limit of $\mathrm{Coker}(\mathrm{H}_n f^Y)_{k'_l}$, where k'_l runs through the finite subextensions of k_l/k, the element V vanishes in some $\mathrm{Coker}(\mathrm{H}_n f^Y)_{k'_l}$. The finite push-forward map $\mathrm{Coker}(\mathrm{H}_n f^Y)_{k'_l} \to \mathrm{Coker}(\mathrm{H}_n f^Y)$ has the property that its composite with the scalar extension map

$$\mathrm{Coker}(\mathrm{H}_n f^Y) \to \mathrm{Coker}(\mathrm{H}_n f^Y)_{k'_l} \to \mathrm{Coker}(\mathrm{H}_n f^Y)$$

is the multiplication by $[k'_l : k]$. Therefore the image of V in $\mathrm{Coker}(\mathrm{H}_n f^Y)$ is annihilated by $[k'_l : k]$, which is a power of l. Since $[k'_2 : k]$ and $[k'_3 : k]$ are relatively prime, the image of V itself is zero. The proof for $\{\mathrm{Ker}(\mathrm{H}_n f^Y)\}_Y$ is the same. $\square$

4C. ***A consequence on the relative motivic cohomologies.*** In this final subsection $\bar{X}$ can be any algebraic scheme. Let X be an open set of $\bar{X}$ such that the complement $\bar{X} \setminus X$ is the support of an effective Cartier divisor Y.

Consider the presheaf of complexes on the small Zariski site $\bar{X}_{\mathrm{Zar}}$,

$$z_r(\bar{X}|Y, \bullet)_{\mathrm{Zar}} : (\bar{U} \subset \bar{X}) \mapsto z_r(\bar{U}|Y \cap \bar{U}, \bullet),$$

which turns out to be a sheaf, as well as $z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}}$ similarly defined. We have a natural inclusion of sheaves $z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}} \subset z_r(\bar{X}|Y, \bullet)_{\mathrm{Zar}}$. The induced maps on homology sheaves

$$\left\{\mathrm{H}_n(z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y \xrightarrow{\{f_n^Y\}_Y} \left\{\mathrm{CH}_r(\bar{X}|Y, n)_{\mathrm{Zar}}\right\}_Y \tag{4.6}$$

are pro-isomorphisms of Zariski sheaves for all n. Indeed, by Lemma 4.5, the maps of sheaves

$$\mathrm{Coker}(f_n^{2nY}) \to \mathrm{Coker}(f_n^Y), \quad \mathrm{Ker}(f_n^{(2n+2)Y}) \to \mathrm{Ker}(f_n^Y)$$

are zero.

As a general fact on pro-categories, the functors $\mathrm{H}^n_{\mathrm{Zar}}(\bar{X}, -)$ extend to functors

$$\text{pro-sheaves} \to \text{pro-abelian groups}, \quad \{F_i\}_i \mapsto \{\mathrm{H}^n_{\mathrm{Zar}}(\bar{X}, F_i)\}_i. \tag{4.7}$$

We have hypercohomology spectral sequences in the abelian category of pro-abelian groups:

$$E_2^{pq} = \left\{\mathrm{H}^p_{\mathrm{Zar}}(\bar{X}, \mathrm{H}_{-q}(z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}}))\right\}_Y \Rightarrow \left\{\mathbf{H}^{p+q}_{\mathrm{Zar}}(\bar{X}, z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y$$
$$'E_2^{pq} = \left\{\mathrm{H}^p_{\mathrm{Zar}}(\bar{X}, \mathrm{CH}_r(\bar{X}|Y, -q)_{\mathrm{Zar}}))\right\}_Y \Rightarrow \left\{\mathbf{H}^{p+q}_{\mathrm{Zar}}(\bar{X}, z_r(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y$$

which are bounded to the range $0 \le p \le \dim \bar{X}$ and $q \le 0$. Since the natural map $E \to {}'E$ of spectral sequences induces isomorphisms on E_2-terms by equations (4.6) and (4.7), we get isomorphisms

$$\left\{\mathbf{H}^n_{\mathrm{Zar}}(\bar{X}, z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y \to \left\{\mathbf{H}^n_{\mathrm{Zar}}(\bar{X}, z_r(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y.$$

So we have proved:

Theorem 4.8. *Let $r \ge 0$ and $n \in \mathbb{Z}$. For any algebraic scheme $\bar{X}$ and an effective Cartier divisor Y_0 on $\bar{X}$, the natural map of pro-abelian groups*

$$\left\{\mathbf{H}^n_{\mathrm{Zar}}(\bar{X}, z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y \to \left\{\mathbf{H}^n_{\mathrm{Zar}}(\bar{X}, z_r(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y$$

are isomorphisms, where Y runs through effective Cartier divisors with support $|Y_0|$.

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