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We generalize the construction of reflection functors from classical representation theory of quivers to arbitrary small categories with freely attached sinks or sources. These reflection morphisms are shown to induce equivalences between the corresponding representation theories with values in arbitrary stable homotopy theories, including representations over fields, rings or schemes as well as differential-graded and spectral representations.

Specializing to representations over a field and to specific shapes, this recovers derived equivalences of Happel for finite, acyclic quivers. However, even over a field our main result leads to new derived equivalences, for example, for not necessarily finite or acyclic quivers.

Our results rely on a careful analysis of the compatibility of gluing constructions for small categories with homotopy Kan extensions and homotopical epimorphisms, and on a study of the combinatorics of amalgamations of categories.

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1. Introduction

Happel [1987] considered derived categories of finite-dimensional algebras over fields. Interesting special cases of such algebras are path algebras of finite and

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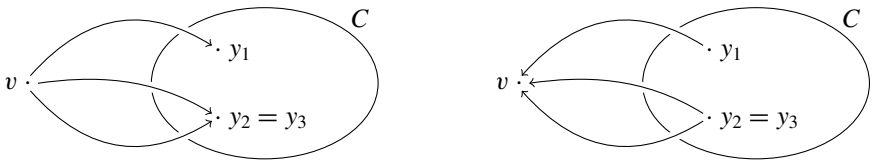
acyclic quivers. Let us recall that a quiver is simply an oriented graph and that a quiver is acyclic if it admits no nontrivial oriented cycles. Given such an acyclic quiver Q and a source $q_0 \in Q$ (no edge ends at q_0) there is the reflected quiver Q' obtained by turning the source into a sink. Bernšteĭn, Gel'fand, and Ponomarev [Bernšteĭn et al. 1973] showed that the corresponding abelian categories of representations are related by reflection functors. If one works with representations of a *finite, acyclic* quiver over a field, then Happel [1987] proved that derived reflection functors yield exact equivalences between the corresponding bounded derived categories of the path algebras.

The main goal of this paper is to generalize this result in two different directions. First, we show that one obtains similar equivalences if one drops the assumption of working over a field. More precisely, we construct such exact equivalences of derived or homotopy categories of representations over a ring, of representations in quasicoherent modules on arbitrary schemes, of differential-graded representations over differential-graded algebras, and of spectral representations. In fact, we obtain equivalences of homotopy theories of representations and we show that the existence of such equivalences is a formal consequence of *stability* only. Hence there are many additional variants for representations with values in other *stable homotopy theories* arising in algebra, geometry, and topology (for more details about what we mean by a stable homotopy theory see further below).

Second, we generalize this result, in that we obtain such equivalences for a significantly larger class of shapes. Given an *arbitrary small category* C and a finite string $y_1, y_2, \dots, y_n$ of objects in C, then we can form new categories C^- and C^+ by freely adjoining a source or a sink to these objects in C. The string of objects may have some repetition, so that the generic picture to have in mind is as in Figure 1. In this situation we show that the categories C^- and C^+ have equivalent homotopy theories of representations with values in arbitrary stable homotopy theories, i.e., that they are *strongly stably equivalent* in a sense made precise in (1.1).

To illustrate this abstract statement let us turn to some special cases, which we explore further in [Groth and Šťovíček $\geq$ 2018]. As a first example, if we specialize to a finite, acyclic quiver and consider representations over a field, then we recover the derived equivalences of Happel [1987] (actually also a version for unbounded chain complexes). However, even for representations over a field and of quivers, the main result leads to new classes of derived equivalences.

For example, dropping the finiteness assumption, we see that reflection functors induce derived equivalences between the infinite-dimensional (possibly nonunital) algebras associated to infinite, acyclic quivers. Alternatively, we can drop the acyclicity assumption. As long as there are sources or sinks in a finite quiver, corresponding reflection functors yield derived equivalences between infinite-dimensional path algebras. Combining these two, we can also drop both the finiteness and the



The category C^- The category C^+

Figure 1. Adjoining a source and a sink to $C \in \mathcal{C}at$.

acyclicity assumption. As soon as an arbitrary quiver has sources or sinks, there are associated derived equivalences given by reflection functors.

Choosing other examples of stable homotopy theories, we see that all these equivalences also have variants if we do not work over a field but with more general abstract representations. As a further specialization we deduce that finite oriented trees can be reoriented arbitrarily without affecting the abstract representation theory, thereby reproducing the main result of [Groth and Šťovíček 2016b]. To mention an additional instance, if one considers representations of a poset in Grothendieck abelian categories, then our main result reestablishes a special case of a result of Ladkani [2007], but also extends it for example to differential-graded and spectral representations. And there are additional such statements starting with more general small categories instead.

These abstract equivalences are realized by general reflection morphisms between homotopy theories of representations. The arguments involved in their construction are rather formal as they rely only on the existence of a well-behaved calculus of restrictions and (homotopy) Kan extensions of diagrams in stable homotopy theories. Besides being fairly transparent, there are two additional advantages of this method of construction.

(i) First, this leads to equivalences of *homotopy theories* of abstract representations as opposed to mere equivalences of *homotopy categories* of representations. Since equivalences of homotopy theories *are* exact, the corresponding functors between derived categories or homotopy categories can be turned into exact equivalences with respect to classical triangulations [Groth 2013]. However, in general, the existence of exact equivalences of triangulated categories of representations does not imply that there are equivalences of homotopy theories in the background. While this is the case for representations over rings by [Dugger and Shipley 2004], as soon as one passes to differential-graded or spectral representations it is in general a stronger result to have equivalences of homotopy theories.

(ii) Second, in this way the equivalences of homotopy theories of representations with values in stable homotopy theories are seen to be compatible with

exact morphisms of stable homotopy theories. In particular, these equivalences hence interact nicely with restriction and (co)induction of scalar morphisms, with localizations and colocalizations, with derived tensor and hom morphisms, and more general exact morphisms.

Let us now be more specific about what we mean by abstract (stable) homotopy theories. By now there are various ways of axiomatizing (stable) homotopy theories, including Quillen model categories [Quillen 1967; Hovey 1999], quasicategories or ∞-categories [Lurie 2009; 2016; Groth 2010], derivators [Grothendieck 1991; Heller 1988; Franke 1996], as well as the more classical triangulated categories. In this paper we use the language of derivators, which by definition can be thought of as minimal, purely categorical extensions of the more classical derived or homotopy categories to a framework with a well-behaved calculus of homotopy (co)limits and homotopy Kan extensions. In this approach to abstract homotopy theory, homotopy (co)limits and homotopy Kan extensions are defined and characterized by ordinary universal properties, thereby making their calculus accessible by elementary categorical techniques.

The basic idea about derivators is as follows. Given an abelian category $\mathcal{A}$, the derived category $D(\mathcal{A})$ is rather ill-behaved. In particular, the calculus of derived (co)limits and derived Kan extensions is not visible to $D(\mathcal{A})$ *alone*. Hence, if one agrees on the relevance of this calculus (and some evidence for this is for example provided by the observation that classical triangulations simply encode certain shadows of iterated derived cokernel constructions), why not simply encode derived categories of diagram categories $D(\mathcal{A}^B)$ for various small categories B together with restriction functors between them? Pursuing this more systematically, one is lead to consider the *derivator* of $\mathcal{A}$, a certain 2-functor

$$\mathscr{D}_{\mathcal{A}} : B \mapsto \mathscr{D}_{\mathcal{A}}(B) = D(\mathcal{A}^B),$$

and derived Kan extensions now are merely adjoints to (derived) restriction functors. The values of $\mathscr{D}_{\mathcal{A}}$ are considered as plain categories, but *exactness properties* of the derivator can be used to construct canonical triangulations and canonical higher triangulations in the sense of Maltsiniotis [2005]. In fact, this holds more generally for strong, stable derivators (see [Franke 1996; Maltsiniotis 2001; Groth 2013; 2016a]), such as homotopy derivators of stable model categories or stable ∞-categories. Let us recall that a derivator is stable if it admits a zero object and if a square is cartesian if and only if it is cocartesian (see [Groth et al. 2014b; Groth and Šťovíček 2016c] for alternative characterizations). While stability is invisible to ordinary category theory, there is a ubiquity of stable derivators arising in algebra, geometry, and topology [Groth and Šťovíček 2016c, §5].

Now, the connection to abstract representation theory or abstract tilting theory is provided by the following observation. Given a derivator $\mathscr{D}$ and a small category B,

there is the derivator $\mathscr{D}^B$ of coherent diagrams of shape B in $\mathscr{D}$. This exponentiation is compatible with the formation of exponentials at the level of abelian categories, (nice) model categories, and ∞-categories. For example, given a Grothendieck abelian category $\mathcal{A}$ and a small category B there is an equivalence of derivators

$$\mathscr{D}_{\mathcal{A}}^{B} \simeq \mathscr{D}_{\mathcal{A}^B}.$$

Specializing further, this shows that the passage to category algebras (like path algebras, incidence algebras, and group algebras) can be modeled by this shifting operation at the level of derivators.

To state the main result of this paper more precisely, let $\mathcal{D}ER_{\mathrm{St,ex}}$ be the 2-category of stable derivators, exact morphisms, and all natural transformations. For every small category B, exponentiation by B defines a 2-functor

$$(-)^B : \mathcal{D}ER_{\mathrm{St,ex}} \to \mathcal{D}ER : \mathscr{D} \mapsto \mathscr{D}^B,$$

where $\mathcal{D}ER$ is the 2-category of derivators. Denoting again by C an arbitrary small category and by C^-, C^+ the categories obtained from C by freely attaching a source or a sink to a prescribed string of objects (see again Figure 1), we show that these two categories are *strongly stably equivalent* in the sense of [Groth and Šťovíček 2016c]. Thus, we show that there is a pseudonatural equivalence of 2-functors

$$\Phi : (-)^{C^-} \simeq (-)^{C^+} : \mathcal{D}ER_{\mathrm{St,ex}} \to \mathcal{D}ER, \tag{1.1}$$

and in this precise sense C^-, C^+ have equivalent abstract representation theories.

In the sequel [Groth and Šťovíček $\geq$ 2018] we study these general reflection morphisms further. We show that unrelated reflections commute, leading to abstract Coxeter morphisms for finite, acyclic quivers. Moreover, the reflections are shown to be realized by explicitly constructed invertible spectral bimodules, and this yields nontrivial elements in spectral Picard groupoids. We also obtain a spectral Serre duality result for acyclic quivers and, more generally, strongly homotopy finite categories.

While here and in the sequel we state and prove the above results using the language of derivators, it is completely formal to also deduce implications for model categories and ∞-categories of abstract representations. For concreteness, given a stable, combinatorial model category $\mathcal{M}$, the existence of the strong stable equivalence (1.1) implies by [Renaudin 2009] that the model categories $\mathcal{M}^{C^-}$ and $\mathcal{M}^{C^+}$ are connected by a zigzag of Quillen equivalences. Similarly, there is a variant for stable, presentable ∞-categories of representations.

This paper belongs to a series of papers on abstract representation theory and abstract tilting theory, and can be considered as sequel to [Groth and Šťovíček 2016a; 2016b; 2016c]. This project relies both on a basic formal understanding of

stability [Groth 2013; Groth et al. 2014b] as well as on a basic understanding of the interaction of monoidality and stability [Groth et al. 2014a; Ponto and Shulman 2016]. We intend to come back to further applications to abstract representation theory elsewhere.

The content of the sections is as follows. In Sections 2 and 3 we recall some basics concerning derivators. In Section 4 we outline the strategy of the construction of the general reflection morphisms leading to the desired strong stable equivalence. In Section 5-6 we introduce free oriented gluing constructions of small categories and study their compatibility with Kan extensions and homotopical epimorphisms. This allows us in Section 7 to construct reflection equivalences in the special case of separated sources and sinks. In Section 8 we establish two simple detection criteria for homotopical epimorphisms, which we use in Section 9 to conclude the construction of reflection equivalences in the general case. In Section 10 we deduce some consequences of our abstract tilting result. Finally, in the Appendix we collect some results concerning the combinatorics of amalgamations of small categories which are useful in Section 9.

2. Review of stable derivators and strong stable equivalences

In this section we include a short review of stable derivators. For more details we refer the reader to [Groth 2013; Groth et al. 2014b]. The key idea behind a derivator is that they enhance the more classical derived categories of abelian categories and homotopy categories of model categories by also keeping track of homotopy categories of diagram categories together with the calculus of homotopy Kan extensions. Like stable model categories and stable ∞-categories, stable derivators provide an enhancement of triangulated categories.

To make this precise, let $\mathcal{C}at$ be the 2-category of small categories and $\mathcal{C}AT$ the 2-category of not necessarily small categories. We refer the reader to [Borceux 1994] for basic 2-categorical terminology.

Definition 2.1. A *prederivator* is a 2-functor $\mathscr{D} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$. *Morphisms* of prederivators are pseudonatural transformations, and *transformations* between these morphisms are modifications, yielding the 2-category $\mathcal{PDER}$ of prederivators.

Given a prederivator $\mathscr{D}$ we call objects in $\mathscr{D}(A)$ *coherent diagrams (of shape A)*. For every functor $u : A \to B$ there is a *restriction functor* $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$. In the special case that $A = \mathbb{1}$ is the terminal category and hence $u = b : \mathbb{1} \to B$ classifies an object $b \in B$, we refer to $b^* : \mathscr{D}(B) \to \mathscr{D}(\mathbb{1})$ as an *evaluation functor*. Evaluating a morphism $f : X \to Y$ in $\mathscr{D}(B)$ we obtain induced morphisms $f_b : X_b \to Y_b$, $b \in B$, in the *underlying category* $\mathscr{D}(\mathbb{1})$.

If a restriction functor $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ admits a left adjoint, then we refer to it as a *left Kan extension functor* and denote it by $u_! : \mathscr{D}(A) \to \mathscr{D}(B)$. In the special

case that $u = \pi_A : A \to \mathbb{1}$ collapses A to a point, such a left adjoint is also denoted by $(\pi_A)_! = \operatorname{colim}_A : \mathscr{D}(A) \to \mathscr{D}(\mathbb{1})$ and referred to as a *colimit functor*. Dually, we speak of *right Kan extension functors* $u_* : \mathscr{D}(A) \to \mathscr{D}(B)$ and *limit functors* $(\pi_A)_* = \lim_A : \mathscr{D}(A) \to \mathscr{D}(\mathbb{1})$.

For *derivators* we ask for the existence of such Kan extension functors and that they can be calculated *pointwise* (see [Mac Lane 1998, X.3.1] for the classical context of ordinary categories). To express this purely 2-categorically, we consider the *slice squares*

$$\begin{array}{ccc} (u/b) & \xrightarrow{p} & A \\ {\scriptstyle \pi_{(u/b)}}\downarrow & \swarrow & \downarrow{\scriptstyle u} \\ \mathbb{1} & \xrightarrow[b]{} & B \end{array} \qquad\qquad \begin{array}{ccc} (b/u) & \xrightarrow{q} & A \\ {\scriptstyle \pi_{(b/u)}}\downarrow & \nearrow & \downarrow{\scriptstyle u} \\ \mathbb{1} & \xrightarrow[b]{} & B \end{array} \tag{2.2}$$

coming with transformations $u \circ p \to b \circ \pi$ and $b \circ \pi \to u \circ q$, respectively. Here, objects in the *slice category* (u/b) are pairs (a, f) consisting of an object $a \in A$ and a morphism $f : u(a) \to b$ in B. A morphism $(a, f) \to (a', f')$ is a map $a \to a'$ in A making the obvious triangles commute. The functor $p : (u/b) \to A$ is the obvious projection and the component of the transformation $u \circ p \to b \circ \pi$ at (a, f) is f. The square on the right in (2.2) is defined dually.

Definition 2.3. A prederivator $\mathscr{D} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$ is a *derivator*[1] if the following properties are satisfied.

(Der1) $\mathscr{D} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT$ takes coproducts to products, i.e., the canonical map $\mathscr{D}\big(\coprod A_i\big) \to \prod \mathscr{D}(A_i)$ is an equivalence. In particular, $\mathscr{D}(\varnothing)$ is equivalent to the terminal category.

(Der2) For any $A \in \mathcal{C}at$, a morphism $f : X \to Y$ in $\mathscr{D}(A)$ is an isomorphism if and only if the morphisms $f_a : X_a \to Y_a$, $a \in A$, are isomorphisms in $\mathscr{D}(\mathbb{1})$.

(Der3) Each functor $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ has both a left adjoint $u_!$ and a right adjoint u_*.

(Der4) For any functor $u : A \to B$ and any $b \in B$ the canonical transformations

$$\pi_! p^* \xrightarrow{\eta} \pi_! p^* u^* u_! \to \pi_! \pi^* b^* u_! \xrightarrow{\epsilon} b^* u_! \quad \text{and}$$
$$b^* u_* \xrightarrow{\eta} \pi_* \pi^* b^* u_* \to \pi_* q^* u^* u_* \xrightarrow{\epsilon} \pi_* q^*$$

associated to the slice squares (2.2) are isomorphisms.

[1] We emphasize that $\mathcal{C}at^{\mathrm{op}}$ is obtained from $\mathcal{C}at$ by changing the orientation of functors but not of natural transformations. Thus, following [Heller 1988; Franke 1996], our convention for derivators is based on *diagrams*. There is an equivalent approach using *presheaves*, i.e., contravariant functors; see for example [Grothendieck 1991; Cisinski 2003].

Axiom (Der4) thus says that for $u : A \to B, b \in B$, and $X \in \mathscr{D}(A)$, certain canonical maps

$$\operatorname{colim}_{(u/b)} p^* X \to u_!(X)_b \quad \text{and} \quad u_*(X)_b \to \lim_{(b/u)} q^* X$$

are isomorphisms. We say a bit more about the formalism related to (Der4) in Section 3.

Morphisms and *transformations* of derivators are morphisms and transformations of underlying prederivators, yielding the sub-2-category $\mathcal{DER} \subseteq \mathcal{PDER}$ of derivators. Given a (pre)derivator, we often write $X \in \mathscr{D}$ if there is a small category A such that $X \in \mathscr{D}(A)$.

Examples 2.4. (i) Let $\mathcal{C}$ be an ordinary category. The 2-functor

$$y_{\mathcal{C}} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT} : A \mapsto \mathcal{C}^A$$

is a derivator if and only if $\mathcal{C}$ is complete and cocomplete. Kan extension functors in such a *represented derivator* are *ordinary Kan extensions* from classical category theory. The underlying category of $y_{\mathcal{C}}$ is isomorphic to $\mathcal{C}$.

(ii) Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathrm{Ch}(\mathcal{A})$ be the category of unbounded chain complexes in $\mathcal{A}$. For every $A \in \mathcal{C}at$ we denote by W^A the class of levelwise quasi-isomorphisms in $\mathrm{Ch}(\mathcal{A})^A$. The 2-functor

$$\mathscr{D}_{\mathcal{A}} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT} : A \mapsto \mathrm{Ch}(\mathcal{A})^A[(W^A)^{-1}]$$

is a derivator. Kan extension functors in $\mathscr{D}_{\mathcal{A}}$ are *derived Kan extensions* in the sense of homological algebra. The underlying category of $\mathscr{D}_{\mathcal{A}}$ is isomorphic to the *derived category* $D(\mathcal{A})$ of $\mathcal{A}$. As interesting examples we obtain derivators associated to fields, rings, and schemes.

(iii) Let $\mathcal{M}$ be a Quillen model category [Quillen 1967; Hovey 1999] with weak equivalences W. Denoting by W^A the levelwise weak equivalences in $\mathcal{M}^A$, there is an associated *homotopy derivator*

$$\mathscr{H}o_{\mathcal{M}} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT} : A \mapsto \mathcal{M}^A[(W^A)^{-1}];$$

see [Cisinski 2003] for the general case and [Groth 2013, Proposition 1.30] for an easy proof in the case of combinatorial model categories. Kan extension functors in $\mathscr{H}o_{\mathcal{M}}$ are *homotopy Kan extensions*. The underlying category of $\mathscr{H}o_{\mathcal{M}}$ is isomorphic to the *homotopy category* $\mathrm{Ho}(\mathcal{M})$. Similarly, there are homotopy derivators associated to complete and cocomplete ∞-categories or quasicategories [Joyal $\geq$ 2018; 2008; Lurie 2009; Groth 2010]; see [Groth et al. 2014b] for a proof sketch. These two classes give rise to a plethora of additional examples of derivators.

Thus, derivators encode key formal properties of the calculus of Kan extensions, derived Kan extensions, and homotopy Kan extensions, as it is available in typical situations arising in nature. It turns out that many constructions are combinations of such Kan extensions, including the general reflection functors we construct in this paper; see Sections 4, 7, and 9.

Let $[1]$ be the poset $(0 < 1)$ considered as a category and let $\square = [1] \times [1]$ be the commutative square. We denote by $i_{\ulcorner} : {\ulcorner} \to \square$, $i_{\lrcorner} : {\lrcorner} \to \square$ the full subcategories obtained by removing the final and initial object, respectively. A square $X \in \mathscr{D}(\square)$ is *cartesian* if it lies in the essential image of $(i_{\lrcorner})_* : \mathscr{D}({\lrcorner}) \to \mathscr{D}(\square)$. Dually, we define *cocartesian squares*.

Definition 2.5. A derivator is *pointed* if the underlying category has a zero object. A pointed derivator is *stable* if a square is cartesian if and only if it is cocartesian.

Examples 2.6. (i) The derivator of a Grothendieck abelian category is stable. In particular, fields, rings, and schemes have associated stable derivators.

(ii) Homotopy derivators of stable model categories and stable ∞-categories are stable.

(iii) The derivator of differential graded modules over a differential graded algebra is stable.

(iv) The derivator of module spectra over a symmetric ring spectrum is stable. In particular, the derivator of spectra itself is stable.

We refer the reader to [Groth and Šťovíček 2016c, Examples 5.4] for many additional examples of stable derivators arising in algebra, geometry, and topology. It can be shown that the values of (strong) stable derivators are canonically triangulated categories [Franke 1996; Maltsiniotis 2001; Groth 2013, Theorem 4.16 and Corollary 4.19] and even higher triangulated categories [Groth and Šťovíček 2016a, Theorem 13.6, Corollary 13.11, and Remark 13.12] in the sense of Maltsiniotis [2005].

In a pointed derivator $\mathscr{D}$ one can define *suspensions, loops, cofibers, and fibers* (see [Groth 2013, §3]), yielding adjunctions

$$(\Sigma, \Omega) : \mathscr{D}(\mathbb{1}) \rightleftarrows \mathscr{D}(\mathbb{1}) \quad \text{and} \quad (\mathsf{cof}, \mathsf{fib}) : \mathscr{D}([1]) \rightleftarrows \mathscr{D}([1]).$$

We recall from [Groth and Šťovíček 2016c, §8] some basic notation and terminology related to n-cubes $[1]^n = [1] \times \cdots \times [1]$. The poset $[1]^n$ is isomorphic to the power set of $\{1, \ldots, n\}$, and this isomorphism is used implicitly in what follows. We denote by $i_{\geq k} : [1]^n_{\geq k} \to [1]^n$, $0 \leq k \leq n$, the full subcategory spanned by all subsets of cardinality at least k. This notation has obvious variants, for example the full subcategory $i_{=n-1} : [1]^n_{=n-1} \to [1]^n$ is the discrete category $n \cdot \mathbb{1} = \mathbb{1} \sqcup \cdots \sqcup \mathbb{1}$ on n objects.

Definition 2.7. Let $\mathscr{D}$ be a derivator. An n-cube $X \in \mathscr{D}([1]^n)$ is *strongly cartesian* if it lies in the essential image of $(i_{\geq n-1})_* : \mathscr{D}([1]^n_{\geq n-1}) \to \mathscr{D}([1]^n)$. An n-cube $X \in \mathscr{D}([1]^n)$ is *cartesian* if it lies in the essential image of $(i_{\geq 1})_*$.

Dually, one defines *(strongly) cocartesian n-cubes*. Following ideas of Goodwillie [1991], one shows the following.

Theorem 2.8 [Groth and Šťovíček 2016c, Theorem 8.3, Corollary 8.8]. *An n-cube, $n \geq 2$, in a derivator is strongly cartesian if and only if all subcubes are cartesian if and only if all subsquares are cartesian.*

Stable derivators admit the following different characterizations.

Theorem 2.9 [Groth et al. 2014b, Theorem 7.1; 2016c, Corollary 8.9]. *The following are equivalent for a pointed derivator $\mathscr{D}$.*

(i) *The adjunction $(\Sigma, \Omega) : \mathscr{D}(\mathbb{1}) \to \mathscr{D}(\mathbb{1})$ is an equivalence.*

(ii) *The adjunction $(\mathsf{cof}, \mathsf{fib}) : \mathscr{D}([1]) \to \mathscr{D}([1])$ is an equivalence.*

(iii) *The derivator $\mathscr{D}$ is stable.*

(iv) *An n-cube in $\mathscr{D}$, $n \geq 2$, is strongly cartesian if and only if it is strongly cocartesian.*

An n-cube which is simultaneously strongly cartesian and strongly cocartesian is *strongly bicartesian*. In the case of $n = 2$ this reduces to the classical notion of a *bicartesian square*. Strongly bicartesian n-cubes in stable derivators satisfy the 2-out-of-3 property with respect to composition and cancellation (see [Groth and Šťovíček 2016c, §8] for the case of n-cubes).

The natural domains for Kan extensions with parameters are given by shifted derivators in the sense of the following proposition. This exponential construction is central to abstract representation theory.

Proposition 2.10 [Groth 2013, Theorem 1.25 and Proposition 4.3]. *Let $\mathscr{D}$ be a derivator and let $B \in \mathcal{C}at$. The* 2*-functor*

$$\mathscr{D}^B : \mathcal{C}at^{\mathrm{op}} \to \mathcal{C}AT : A \mapsto \mathscr{D}(B \times A)$$

is again a derivator, the derivator of coherent diagrams of shape B, *which is pointed or stable as soon as $\mathscr{D}$ is.*

This shifting operation also applies to morphisms and natural transformations in either variable, thereby defining a two-variable pseudofunctor

$$\mathcal{C}at^{\mathrm{op}} \times \mathcal{D}ER \to \mathcal{D}ER : (A, \mathscr{D}) \mapsto \mathscr{D}^A.$$

In abstract representation theory we are interested in suitable restrictions of related 2-functors. To begin with, as special cases of morphisms of derivators preserving certain (co)limits [Groth 2013, §2.2] there are the following definitions.

Definition 2.11. (i) A morphism of derivators is *right exact* if it preserves initial objects and cocartesian squares.

(ii) A morphism of derivators is *left exact* if it preserves terminal objects and cartesian squares.

(iii) A morphism of derivators is *exact* if it is right exact and left exact.

A morphism between stable derivators is right exact if and only if it is left exact if and only if it is exact. In particular, adjunctions and equivalences between stable derivators give rise to exact morphisms. (Adjunctions and equivalences of derivators are defined internally to the 2-category $\mathcal{DER}$; see [Groth 2013, §2] for details including explicit reformulations.)

Identity morphisms are exact and exact morphisms are closed under compositions, and there is thus the 2-category $\mathcal{DER}_{\mathrm{St,ex}} \subseteq \mathcal{DER}$ of stable derivators, exact morphisms, and arbitrary natural transformations. Hence, for every $A \in \mathcal{C}at$ we obtain an induced 2-functor $(-)^A : \mathcal{DER} \to \mathcal{DER}$ which can be restricted to

$$(-)^A : \mathcal{DER}_{\mathrm{St,ex}} \to \mathcal{DER}.$$

Definition 2.12 [Groth and Šťovíček 2016c, Definition 5.1]. Two small categories A and A' are *strongly stably equivalent*, in notation $A \overset{\mathrm{s}}{\sim} A'$, if there is a pseudonatural equivalence between the 2-functors

$$\Phi : (-)^A \simeq (-)^{A'} : \mathcal{DER}_{\mathrm{St,ex}} \to \mathcal{DER}.$$

Such a pseudonatural equivalence is called a *strong stable equivalence*.

This definition makes precise the idea that the categories A and A' have the same representation theories in arbitrary stable derivators. More formally, a strong stable equivalence $\Phi : A \overset{\mathrm{s}}{\sim} A'$ consists of

(i) an equivalence of derivators $\Phi_{\mathscr{D}} : \mathscr{D}^A \simeq \mathscr{D}^{A'}$ for *every* stable derivator $\mathscr{D}$, and

(ii) associated to every exact morphism of stable derivators $F : \mathscr{D} \to \mathscr{E}$, a natural isomorphism $\gamma_F : F \circ \Phi_{\mathscr{D}} \to \Phi_{\mathscr{E}} \circ F$,

$$\begin{array}{ccc} \mathscr{D}^A & \xrightarrow[\simeq]{\Phi_{\mathscr{D}}} & \mathscr{D}^{A'} \\ {\scriptstyle F}\downarrow & \swarrow_{\cong} & \downarrow{\scriptstyle F} \\ \mathscr{E}^A & \xrightarrow[\Phi_{\mathscr{E}}]{\simeq} & \mathscr{E}^{A'} \end{array}$$

satisfying the usual coherence properties of a pseudonatural transformation.

The motivation for this definition is the following example of the shifting operation; see [Groth and Šťovíček 2016c, §5].

Example 2.13. Let $\mathcal{A}$ be a Grothendieck abelian category and $B \in \mathcal{C}at$. There is an equivalence of derivators

$$\mathscr{D}_{\mathcal{A}}^{B} \simeq \mathscr{D}_{\mathcal{A}^B}.$$

In particular, if B, B' are strongly stably equivalent, then there is a chain of equivalences of derivators

$$\mathscr{D}_{\mathcal{A}^B} \simeq \mathscr{D}_{\mathcal{A}}^{B} \simeq \mathscr{D}_{\mathcal{A}}^{B'} \simeq \mathscr{D}_{\mathcal{A}^{B'}}.$$

Specializing to the Grothendieck abelian category of modules over a ring R and assuming that $B = Q$, $B' = Q'$ are quivers with finitely many vertices, we obtain equivalences

$$\mathscr{D}_{RQ} \simeq \mathscr{D}_{RQ'}$$

of the derivators of the respective path algebras. Since equivalences of derivators are exact, this yields exact equivalences of derived categories

$$D(RQ) \overset{\Delta}{\simeq} D(RQ'),$$

showing that strongly stably equivalent quivers are derived equivalent over arbitrary rings. A priori, however, it is a much stronger result if we know that two quivers are strongly stably equivalent, since this means that the quivers have the same homotopy theories of abstract representations. We expand a bit on this in Section 10.

3. Review of homotopy exact squares

In this section we review some results concerning the calculus of homotopy exact squares. This calculus is arguably the most important technical tool in the theory of derivators and it allows us to establish many useful manipulation rules for Kan extensions in derivators. For more details, see for example [Ayoub 2007; Maltsiniotis 2012; Groth 2013; Groth et al. 2014b; Groth and Šťovíček 2016c].

To begin with let us consider a natural transformation $\alpha : up \to vq$ living in a square of small categories

$$\begin{array}{ccc} D & \xrightarrow{p} & A \\ {\scriptstyle q}\downarrow & \swarrow_{\alpha} & \downarrow{\scriptstyle u} \\ B & \xrightarrow[v]{} & C \end{array} \tag{3.1}$$

The square (3.1) is *homotopy exact* if one of the canonical mates

$$q_! p^* \to q_! p^* u^* u_! \xrightarrow{\alpha^*} q_! q^* v^* v_! \to v^* u_! \quad \text{and} \tag{3.2}$$

$$u^* v_* \to p_* p^* u^* v_* \xrightarrow{\alpha^*} p_* q^* v^* v_* \to p_* q^* \tag{3.3}$$

is a natural isomorphism. It turns out that (3.2) is an isomorphism if and only if (3.3) is an isomorphism.

Using this terminology, note that axiom (Der4) from Definition 2.3 precisely says that slice squares (2.2) are homotopy exact. Although it may seem from the definition that the notion of homotopy exactness depends on the theory of derivators, this is only seemingly the case. Homotopy exact squares can be characterized by means of the classical homotopy theory of (diagrams of) topological spaces. In fact, a square is homotopy exact if and only if the canonical mate is an isomorphism for the homotopy derivator of topological spaces, and this even admits a combinatorial reformulation; see [Groth et al. 2014b, §3].

For later reference, we collect a few additional examples of homotopy exact squares and make explicit what they tell us about Kan extensions.

Examples 3.4. (i) *Kan extensions along fully faithful functors are fully faithful.* If $u : A \to B$ is fully faithful, then the square

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}} & A \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle u} \\ A & \xrightarrow[u]{} & B \end{array}$$

is homotopy exact, which is to say that the unit $\eta : \mathrm{id} \to u^* u_!$ and the counit $\epsilon : u^* u_* \to \mathrm{id}$ are isomorphisms [Groth 2013, Proposition 1.20]. Thus,

$$u_!, u_* : \mathscr{D}(A) \to \mathscr{D}(B)$$

are fully faithful.

(ii) *Kan extensions and restrictions in unrelated variables commute.* Given functors $u : A \to B$ and $v : C \to D$, the commutative square

$$\begin{array}{ccc} A \times C & \xrightarrow{u \times \mathrm{id}} & B \times C \\ {\scriptstyle \mathrm{id} \times v}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times v} \\ A \times D & \xrightarrow[u \times \mathrm{id}]{} & B \times D \end{array}$$

is homotopy exact [Groth 2013, Proposition 2.5]. Thus, the canonical mate transformation $(\mathrm{id} \times v)_!(u \times \mathrm{id})^* \to (u \times \mathrm{id})^*(1 \times v)_!$ is an isomorphism, and similarly for right Kan extensions.

(iii) *Right adjoint functors are homotopy final.* If $u : A \to B$ is a right adjoint, then the square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle \pi_A}\downarrow & \swarrow_{\mathrm{id}} & \downarrow{\scriptstyle \pi_B} \\ \mathbb{1} & \xrightarrow[\mathrm{id}]{} & \mathbb{1} \end{array}$$

is homotopy exact, i.e., the canonical mate $\mathrm{colim}_A u^* \to \mathrm{colim}_B$ is an isomorphism [Groth 2013, Proposition 1.18]. In particular, if $b \in B$ is a terminal object, then there is a canonical isomorphism $b^* \cong \mathrm{colim}_B$.

(iv) *Homotopy exact squares are compatible with pasting.* Since the passage to the canonical mates (3.2) and (3.3) is functorial with respect to horizontal and vertical pasting, such pastings of homotopy exact squares are again homotopy exact [Groth 2013, Lemma 1.14].

It follows from Examples 3.4(ii) that there are Kan extension morphisms of derivators. In fact, given a derivator $\mathscr{D}$ and a functor $u : A \to B$, there are adjunctions of derivators given by *parametrized Kan extensions*,

$$(u_!, u^*) : \mathscr{D}^A \rightleftarrows \mathscr{D}^B \quad \text{and} \quad (u^*, u_*) : \mathscr{D}^B \rightleftarrows \mathscr{D}^A.$$

If u is fully faithful, then $u_!, u_* \colon \mathscr{D}^A \to \mathscr{D}^B$ are fully faithful morphisms of derivators and as such they induce equivalences onto their respective essential images. In particular, these essential images are again derivators [Groth and Šťovíček 2016c, §3].

The point of the following lemma is that to check whether an object $X \in \mathscr{D}^B$ is in the essential image of $u_!$, it suffices to test objects in $B - u(A)$ only.

Lemma 3.5 [Groth 2013, Lemma 1.21]. *Let $\mathscr{D}$ be a derivator and $u : A \to B$ a fully faithful functor between small categories. A coherent diagram $X \in \mathscr{D}^B$ lies in the essential image of $u_! : \mathscr{D}^A \to \mathscr{D}^B$ if and only if $\epsilon_b : u_! u^*(X)_b \to X_b$ is an isomorphism for all $b \in B - u(A)$.*

This lemma takes a particular simple form for certain Kan extensions in pointed derivators. Recall that a fully faithful functor $u : A \to B$ is a *sieve* if for every morphism $b \to u(a')$ in B with target in the image of u it follows that $b = u(a)$ for some $a \in A$. There is the dual notion of a *cosieve*.

Proposition 3.6 [Groth 2013, Proposition 3.6]. *Let $\mathscr{D}$ be a pointed derivator and $u : A \to B$ a sieve. The morphism $u_* : \mathscr{D}^A \to \mathscr{D}^B$ is fully faithful and $X \in \mathscr{D}^B$ lies in the essential image of u_* if and only if $u_b \cong 0$ for all $b \in B - u(A)$.*

We refer to right Kan extension morphisms along sieves as *right extensions by zero*. Dually, left Kan extensions along cosieves are *left extensions by zero*.

Remark 3.7. If $\mathscr{D}$ is not pointed, then Proposition 3.6 yields right extensions by terminal objects and left extensions by initial objects in the obvious sense [Groth 2013, Proposition 1.23].

By Examples 3.4 there is an easy criterion guaranteeing that Kan extensions are fully faithful. The case of restrictions is more subtle. Inspired by the notion of a *homological epimorphism* introduced by Geigle and Lenzing [1991, §4], there is the following definition; see [Groth and Šťovíček 2016b, §6] and, in particular, Remark 6.4 in [loc. cit.].

Definition 3.8. A functor $u : A \to B$ is a *homotopical epimorphism* if for every derivator $\mathscr{D}$ the restriction functor $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ is fully faithful.

If u is a homotopical epimorphism then $u^* : \mathscr{D}^B \to \mathscr{D}^A$ induces an equivalence onto its essential image. Basic examples and closure properties are collected in [Groth and Šťovíček 2016b, §6–7]. Here it suffices to note that $u : A \to B$ is a homotopical epimorphism if and only if the square

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ B & \xrightarrow[\mathrm{id}]{} & B \end{array}$$

is homotopy exact. We will get back to this in Sections 6 and 8.

4. A pictorial guide to general reflection morphisms

In this section we describe the strategy behind the construction of the general reflection morphisms as carried out in Sections 7 and 9. While some main steps follow the lines of the construction in [Groth and Šťovíček 2016b, §5], they have to be adapted significantly to cover the more general class of examples we consider in this paper.

Let $C \in \mathcal{C}at$ and let C^- be the category obtained from C by freely attaching a new object v together with n morphisms from v to objects in C; see Figure 1. Performing a similar construction but this time adding morphisms pointing to v we obtain the category C^+. Thus, the categories C^-, C^+ are obtained from C by attaching a source and sink, respectively, to the same objects in C, and the picture to have in mind is as in Figure 1.

One of our main goals is to show that for *every* small category C the categories C^- and C^+ are strongly stably equivalent, i.e., that for every stable derivator $\mathscr{D}$ there is an equivalence $\mathscr{D}^{C^-} \simeq \mathscr{D}^{C^+}$ which is pseudonatural with respect to exact morphisms (Definition 2.12). Mimicking the classical construction of reflection functors [Bernšteĭn et al. 1973], we obtain reflection morphisms $s^- : \mathscr{D}^{C^-} \to \mathscr{D}^{C^+}$ and $s^+ : \mathscr{D}^{C^+} \to \mathscr{D}^{C^-}$, which we show to define such a strong stable equivalence. As a first approximation, the rough strategy behind the construction of s^- and s^+ is as follows (see Figure 2).

(i) Take a representation of C^- and separate the morphisms adjacent to the new source by inserting new morphisms, one point being that the shape D^- of this new representation contains an isomorphic copy of the source of valence n. Moreover, we know precisely which representations of D^- arise this way, namely those which populate the new morphisms by isomorphisms. If we

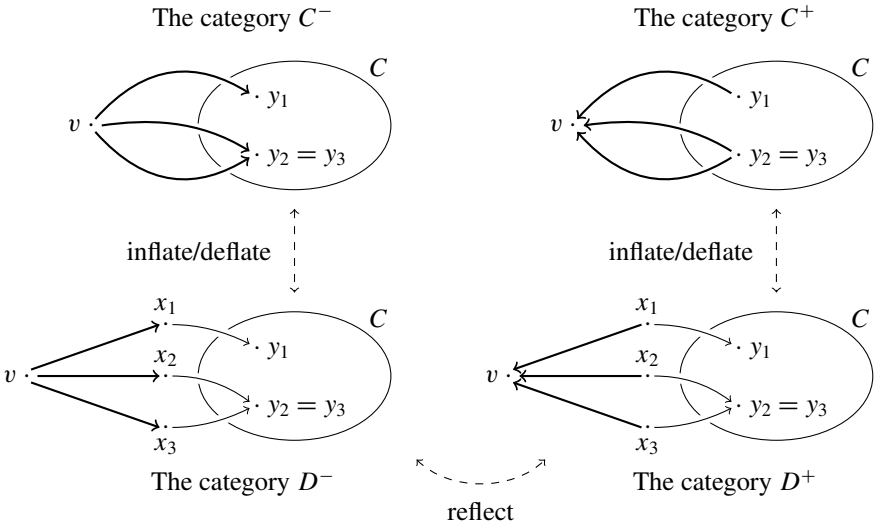


Figure 2. Rough strategy behind construction of reflection functors.

write $\mathscr{D}^{D^-,\mathrm{ex}} \subseteq \mathscr{D}^{D^-}$ for the full subprederivator spanned by such representations, then this yields an equivalence $\mathscr{D}^{C^-} \simeq \mathscr{D}^{D^-,\mathrm{ex}}$ (thereby also implying that $\mathscr{D}^{D^-,\mathrm{ex}}$ is a derivator).

(ii) Show that the reflection morphisms for sources and sinks of valence n as constructed in [Groth and Šťovíček 2016b] yield similar reflection morphisms in this more general situation. Thus, if D^+ is the category obtained from D^- by turning the source into a sink, then we construct certain morphisms of derivators $\mathscr{D}^{D^-} \to \mathscr{D}^{D^+}$, which restrict to equivalences $\mathscr{D}^{D^-,\mathrm{ex}} \to \mathscr{D}^{D^+,\mathrm{ex}}$. We expand on this step further below.

(iii) Finally, it is sufficient to show that we can restrict representations of D^+ to representations of C^+, thereby possibly identifying some of the sources of morphisms adjacent to the new sink. If we only consider representations of D^+ satisfying certain exactness properties, then this step induces an equivalence of derivators. Note that the situation in this step differs from the one in step (i) since here the arrows point in different directions. It turns out that this step is not formally dual and, instead, is more involved than the similar looking first step.

The first and third steps are taken care of in Sections 8–9, while the second step is addressed in Sections 5–7. We now expand on this second step, which performs the actual reflection and is motivated by the classical reflection functors from representation theory; see [Gabriel 1972; Bernšteĭn et al. 1973; Happel 1986] and also the discussion in [Groth and Šťovíček 2016b, §5]. Let $v \to x_i$, $i = 1, \ldots, n$,

be the morphisms in D^- which are adjacent to the source v. Given an abstract representation $X \in \mathscr{D}^{D^-}$, we consider the morphism $X_v \to \bigoplus_{i=1}^n X_{x_i}$ induced by the structure maps and pass to its cofiber. However, in order to obtain a representation of the reflected category D^+, we have to take some care in setting up coherent biproduct diagrams appropriately.

To begin with, we recall from [Groth and Šťovíček 2016b, §4 and §7] that finite biproduct objects in stable derivators can be modeled by n-cubes of length two. In more detail, let us consider the diagram in $\mathcal{C}at$

$$n \cdot \mathbb{1} = [1]^n_{=n-1} \xrightarrow{i_1} [1]^n_{\geq n-1} \xrightarrow{i_2} [1]^n \xrightarrow{i_3} I \xrightarrow{i_4} [2]^n \xrightarrow{q} R^n, \tag{4.1}$$

in which we ignore the functor $q : [2]^n \to R^n$ for now. The functors i_1, i_2 are the obvious fully faithful inclusion functors, and the composition $i_4 i_3 : [1]^n \to [2]^n$ is the inclusion as the n-cube $[1, 2]^n$, i.e., the convex hull of $(1, \dots, 1), (2, \dots, 2) \in [2]^n$. Let $I \subseteq [2]^n$ be the full subcategory spanned by $[1, 2]^n$ and the corners

$$(0, 2, \dots, 2), \quad (2, 0, 2, \dots, 2), \quad \dots, \quad (2, \dots, 2, 0),$$

and let $i_3 : [1]^n \to I$ and $i_4 : I \to [2]^n$ be the corresponding factorization of $i_4 i_3$. The associated Kan extension morphisms

$$\mathscr{D}^{n\cdot\mathbb{1}} = \mathscr{D}^{[1]^n_{=n-1}} \xrightarrow{(i_1)_*} \mathscr{D}^{[1]^n_{\geq n-1}} \xrightarrow{(i_2)_*} \mathscr{D}^{[1]^n} \xrightarrow{(i_3)_!} \mathscr{D}^I \xrightarrow{(i_4)_*} \mathscr{D}^{[2]^n} \tag{4.2}$$

are fully faithful and the essential image is in the stable case as follows. For every stable derivator $\mathscr{D}$ we denote by $\mathscr{D}^{[2]^n, \mathrm{ex}} \subseteq \mathscr{D}^{[2]^n}$ the full subderivator spanned by the diagrams such that

(i) all subcubes are strongly bicartesian,

(ii) the values at all corners are trivial, and

(iii) the maps $(i_1, \dots, i_{k-1}, 0, i_{k+1}, \dots, i_n) \to (i_1, \dots, i_{k-1}, 2, i_{k+1}, \dots, i_n)$ are sent to isomorphisms for all $i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n$ and k.

We note that (iii) is a consequence of (i) and (ii) together with isomorphisms being stable under base change [Groth 2013, Proposition 3.12], but it is included here for emphasis. As discussed in [Groth and Šťovíček 2016b, §4] such diagrams model coherent finite biproduct diagrams together with all the inclusion and projection morphisms. The following result justifies referring to $\mathscr{D}^{[2]^n, \mathrm{ex}}$ as a derivator.

Proposition 4.3 [Groth and Šťovíček 2016b, Proposition 4.9]. *Let $\mathscr{D}$ be a stable derivator and $n \geq 2$. The morphisms* (4.2) *are fully faithful and induce an equivalence $\mathscr{D}^{n\cdot\mathbb{1}} \simeq \mathscr{D}^{[2]^n, \mathrm{ex}}$, which is pseudonatural with respect to exact morphisms. The derivator $\mathscr{D}^{[2]^n, \mathrm{ex}}$ is the* derivator of biproduct n-cubes.

Note that property (iii) of the characterization of biproduct n-cubes suggests that such diagrams arise via restriction from a "larger shape where the length two morphisms are invertible". This turns out to be true and will be taken care of by the remaining functor in (4.1).

In fact, let $p : [2] \to R$ be the localization functor inverting the length two morphism $0 \to 2$ in $[2]$, so that R corepresents pairs of composable morphisms such that the composition is an isomorphism; see [Groth and Šťovíček 2016b, §7] for a precise description of R. We know that p is a homotopical epimorphism [Groth and Šťovíček 2016b, Proposition 7.3], and it is completely formal to see that the same is true for the n-fold product $q : [2]^n \to R^n$.

Corollary 4.4 [Groth and Šťovíček 2016b, Corollary 7.4]. *Let $\mathscr{D}$ be a derivator and $n \geq 1$. The functor $q : [2]^n \to R^n$ is a homotopical epimorphism and $q^* : \mathscr{D}^{R^n} \to \mathscr{D}^{[2]^n}$ induces an equivalence onto the full subderivator of $\mathscr{D}^{[2]^n}$ spanned by all diagrams X such that*

$$X_{i_1,\dots,i_{k-1},0,i_{k+1},\dots,i_n} \to X_{i_1,\dots,i_{k-1},2,i_{k+1},\dots,i_n}$$

is an isomorphism for all $i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n$ and k.

Thus, in the stable case, there is the following result concerning the morphisms

$$\mathscr{D}^{n\cdot\mathbb{1}} = \mathscr{D}^{[1]^n_{=n-1}} \xrightarrow{(i_1)_*} \mathscr{D}^{[1]^n_{\geq n-1}} \xrightarrow{(i_2)_*} \mathscr{D}^{[1]^n} \xrightarrow{(i_3)_!} \mathscr{D}^{I} \xrightarrow{(i_4)_*} \mathscr{D}^{[2]^n} \xrightarrow{q_!} \mathscr{D}^{R^n}. \tag{4.5}$$

Let $\mathscr{D}^{R^n,\mathrm{ex}} \subseteq \mathscr{D}^{R^n}$ be the full subderivator spanned by all $X \in \mathscr{D}^{R^n}$ such that q^*X is a biproduct n-cube, i.e., such that $q^*X \in \mathscr{D}^{[2]^n,\mathrm{ex}}$.

Corollary 4.6 [Groth and Šťovíček 2016b, Corollary 7.5]. *Let $\mathscr{D}$ be a stable derivator and $n \geq 2$. The morphisms* (4.5) *are fully faithful and induce an equivalence $\mathscr{D}^{n\cdot\mathbb{1}} \simeq \mathscr{D}^{R^n,\mathrm{ex}}$, which is pseudonatural with respect to exact morphisms. The derivator $\mathscr{D}^{R^n,\mathrm{ex}}$ is the* derivator of invertible biproduct n-cubes.

With this preparation we now describe in more detail the second step in the above strategy behind the construction of general reflection morphisms (see Figure 3). The above-mentioned morphism $\mathscr{D}^{D^-} \to \mathscr{D}^{D^+}$ is roughly obtained as follows.

(i) Starting with an abstract representation $X \in \mathscr{D}^{D^-}$, we glue in a coherent biproduct n-cube centered at $\bigoplus_{i=1}^n X_{x_i}$. The corresponding morphism $\mathscr{D}^{D^-} \to \mathscr{D}^{E_1^-}$ is obtained by adapting the respective morphisms in (4.2), and this step relies on the discussion of "free oriented gluing constructions" in Section 5.

(ii) Next, using a variant of the functor $q : [2]^n \to R^n$, we invert the biproduct n-cubes, thereby constructing a restriction morphism $\mathscr{D}^{E_2^-} \to \mathscr{D}^{E_1^-}$. To understand this morphism, we study the compatibility of homotopical epimorphisms with "free oriented gluing constructions"; see Section 6.

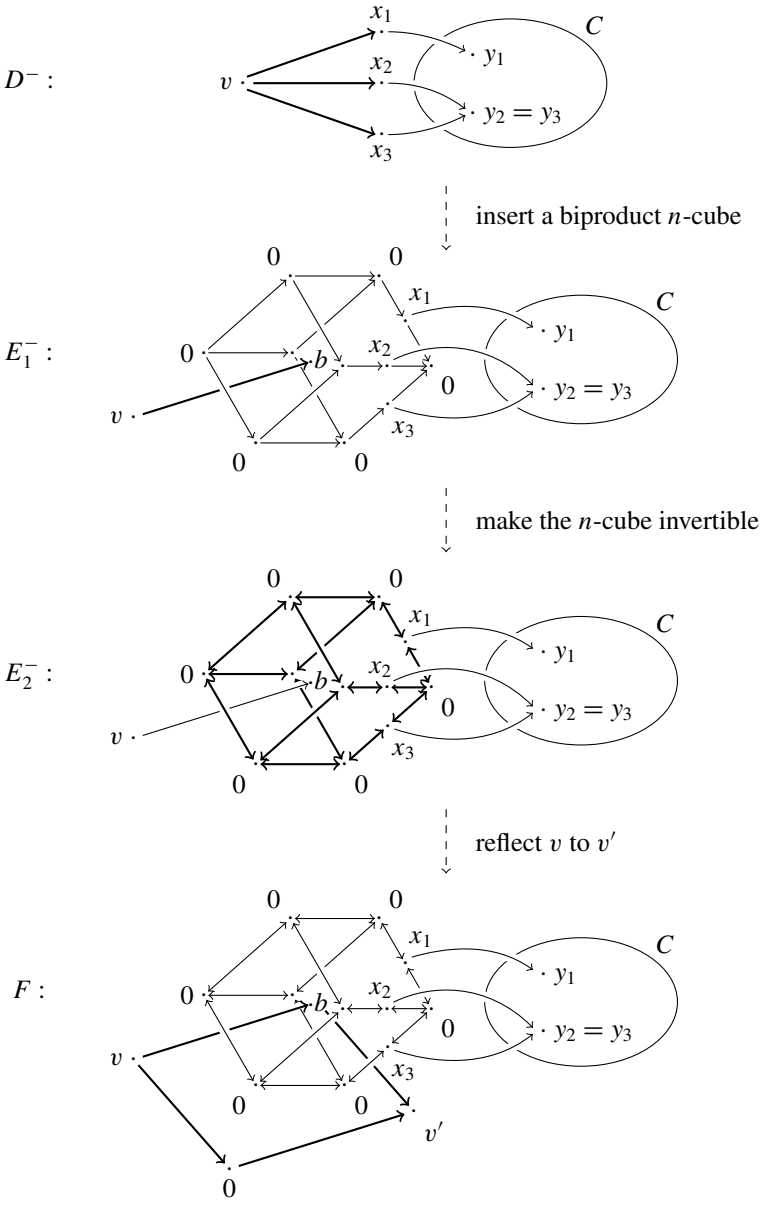


Figure 3. Intermediate steps in the construction of reflection functors. Changes from step to step are drawn in bold.

(iii) As a next step, given a representation $X \in \mathscr{D}^{E_2^-}$, we extend it by passing from $X_v \to \bigoplus_{i=1}^n X_{x_i}$ to the corresponding cofiber square. To get our hands on the resulting morphism of derivators $\mathscr{D}^{E_2^-} \to \mathscr{D}^F$ we again apply results from Section 5.

(iv) The steps so far yield a morphism of derivators $\mathscr{D}^{D^-} \to \mathscr{D}^F$. One observes that the category F also comes with a functor $D^+ \to F$. Dualizing the steps so far, we show that there is a similar morphism of derivators $\mathscr{D}^{D^+} \to \mathscr{D}^F$, and that the span $\mathscr{D}^{D^-} \to \mathscr{D}^F \leftarrow \mathscr{D}^{D^+}$ restricts to the desired equivalence.

These steps are carried out in detail in Section 7, and combined with the above inflation and deflation steps, they are shown in Section 9 to yield the intended general reflection morphisms $\mathscr{D}^{C^-} \to \mathscr{D}^{C^+}$ and $\mathscr{D}^{C^+} \to \mathscr{D}^{C^-}$, showing that the categories C^- and C^+ are strongly stably equivalent; see Theorem 9.11. In the following two sections we first develop some of the necessary techniques.

5. Free oriented gluing constructions

In this section we study in more detail the gluing construction alluded to in Section 4. In particular, we see that these gluing constructions behave well with Kan extension morphisms. The results of this section and Section 6 are central to the construction of the reflection morphisms in Section 7.

To begin, let us consider the following construction (which is a special case of pushouts of small categories; see the Appendix).

Construction 5.1. Let $A_1, A_2 \in \mathcal{C}at$ be small categories. Let $n \in \mathbb{N}$, $s_1, \dots, s_n \in A_1$, and $t_1, \dots, t_n \in A_2$. Moreover, let $[1]$ again be the poset $(0 < 1)$ considered as a category. The category $[1]$ comes with a functor $(0, 1) : \mathbb{1} \sqcup \mathbb{1} \to [1]$ classifying the objects 0 and 1. Using this notation, we define the category A to be the pushout

$$\begin{array}{ccc} \coprod_{i=1,\dots,n} \mathbb{1} \sqcup \mathbb{1} & \xrightarrow{s \sqcup t} & A_1 \sqcup A_2 \\ \downarrow & & \downarrow{\scriptstyle (i_1, i_2)} \\ \coprod_{i=1,\dots,n} [1] & \xrightarrow[\beta]{} & A \end{array} \tag{5.2}$$

and call it the *free oriented gluing construction* associated to (A_1, A_2, s, t). Given $k \in \{1, \dots, n\}$, we denote the image of the morphism $0 \to 1$ in the k-th copy of $[1]$ by $\beta_k : i_1(s_k) \to i_2(t_k)$.

This construction clearly enjoys the following properties.

Lemma 5.3. *In the situation of* (5.2) *the following properties are satisfied.*

(i) *The functors $i_1 : A_1 \to A$ and $i_2 : A_2 \to A$ are fully faithful with disjoint images.*

(ii) *Every object in A lies either in $i_1(A_1)$ or in $i_2(A_2)$.*

(iii) *There are no morphisms in A from an object in $i_2(A_2)$ to an object in $i_1(A_1)$.*

(iv) *For every morphism* $f : i_1(a_1) \to i_2(a_2)$ *there is a unique* $k \in \{1, \dots, n\}$ *and a unique factorization of* f *as*

$$f : i_1(a_1) \xrightarrow{i_1(f')} i_1(s_k) \xrightarrow{\beta_k} i_2(t_k) \xrightarrow{i_2(f'')} i_2(a_2).$$

Proof. This is immediate from the construction of the pushout category in (5.2) (see also Lemma A.12). □

Definition 5.4. We refer to the factorizations in Lemma 5.3(iv) as *standard factorizations* and call the unique number $k \in \{1, \dots, n\}$ the *type* of f.

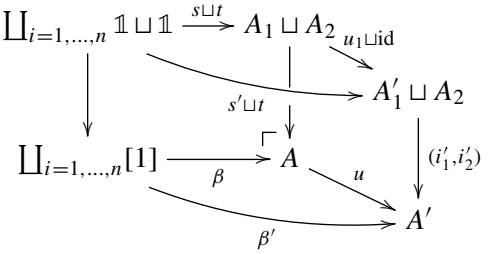
Example 5.5. Let $C \in \mathcal{C}at$ and let $y_1, \dots, y_n \in C$ be a list of objects (possibly with repetition). Let $t = y : n \cdot \mathbb{1} \to C$ be the corresponding functor. Moreover, note that $[1]^n_{\leq 1}$ is the source of valence n which comes with the functor $s : n \cdot \mathbb{1} \to [1]^n_{\leq 1}$ classifying the objects different from the source. The pushout square

$$\begin{array}{ccc} \coprod_{i=1,\dots,n} \mathbb{1} \sqcup \mathbb{1} & \xrightarrow{s \sqcup t} & [1]^n_{\leq 1} \sqcup C \\ \downarrow & & \downarrow \\ \coprod_{i=1,\dots,n} [1] & \longrightarrow & D^- \end{array}$$

exhibits the category D^- showing up in the outline of the strategy of the construction of general reflection morphisms (see Figure 2) as an instance of a free oriented gluing construction. There is a similar description of the category D^+ in Figure 2.

Example 5.6. As a special case of Construction 5.1 we recover the one-point extensions of [Groth and Šťovíček 2016b, §8]. In fact, this is the case for the free oriented gluing construction associated to (A_1, A_2, s, t) in the case where $n = 1$ and A_1 or A_2 is the terminal category $\mathbb{1}$.

Construction 5.7. We now consider two free oriented gluing constructions A and A' which are associated to (A_1, A_2, s, t) and (A'_1, A'_2, s', t'), respectively. Let us assume that the second summands $A_2 = A'_2$ as well as the targets $t = t'$ agree while there is a functor $u_1 : A_1 \to A'_1$ such that $s' = u_1 \circ s$. This situation may be summarized by the following commutative diagram:



(5.8)

Here, both the front and the back face are the pushout squares defining the respective gluing constructions and $u : A \to A'$ is induced by the universal property of the back pushout square. We refer to the situation described in (5.8) as *two compatible (free oriented) gluing constructions* (see Figure 4 for an illustration).

Combining the face on the right in (5.8) with the inclusions of the respective first summands we obtain a commutative square of small categories, which we consider in two ways as a square populated by the identity transformation:

$$\begin{array}{ccc} A_1 & \xrightarrow{i_1} & A \\ {\scriptstyle u_1}\downarrow & \swarrow\!\!\!\nearrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle u} \\ A'_1 & \xrightarrow[i'_1]{} & A' \end{array} \qquad \begin{array}{ccc} A_1 & \xrightarrow{i_1} & A \\ {\scriptstyle u_1}\downarrow & \nearrow\!\!\!\swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle u} \\ A'_1 & \xrightarrow[i'_1]{} & A' \end{array} \tag{5.9}$$

The following proposition guarantees that Kan extensions along u and Kan extensions along u_1 interact as expected.

Proposition 5.10. *If* (5.8) *are two compatible gluing constructions, then both squares in* (5.9) *are homotopy exact, i.e., in every derivator the canonical mates*

$$(i'_1)^* u_* \to (u_1)_* (i_1)^* \quad \textit{and} \quad (u_1)_! (i_1)^* \to (i'_1)^* u_!$$

are isomorphisms.

Proof. We first show that the square on the left in (5.9) is homotopy exact, and show that the canonical mate $(i_1)_! u_1^* \to u^* (i'_1)_!$ is an isomorphism. Since the functors $i_1 : A_1 \to A$ and $i_2 : A_2 \to A$ are jointly surjective, it suffices by (Der2) to show

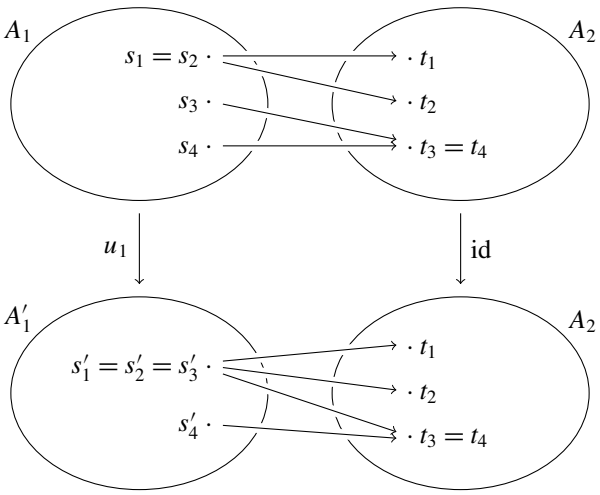
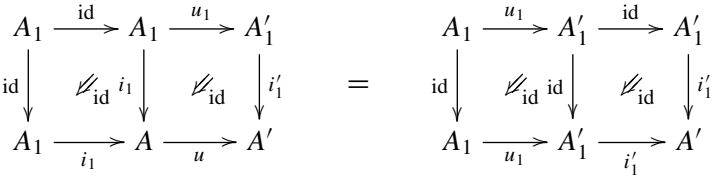


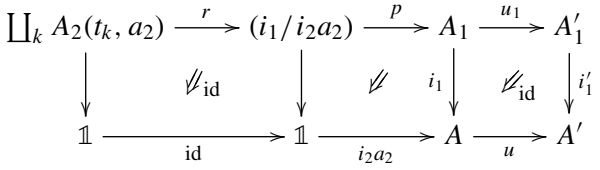
Figure 4. Two compatible (free oriented) gluing constructions.

that the restrictions of the canonical mate with i_1^*, i_2^* are isomorphisms. For the first case we consider the pastings

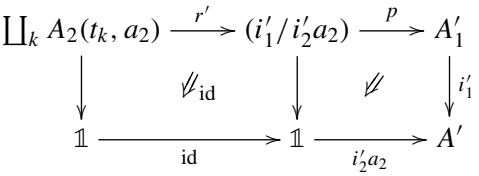


Since i_1, i_1' are fully faithful, the square to the very left and the square to the very right are homotopy exact (Examples 3.4). Moreover, the second square from the right is constant and hence homotopy exact. The functoriality of mates with respect to pasting implies that the restricted canonical mate $i_1^*(i_1)_! u_1^* \to i_1^* u^* (i_1')_!$ is an isomorphism.

Now, given an object $i_2(a_2) \in A$ we consider the pasting



in which the square in the middle is a slice square. The functor r sends a morphism $t_k \to a_2$ to the pair $(s_k, i_1 s_k \to i_2 t_k \to i_2 a_2) \in (i_1 / i_2 a_2)$. Using Lemma 5.3 the reader can easily check that this functor is a right adjoint so that the above square on the left is homotopy exact by the homotopy finality of right adjoints (Examples 3.4). Note that the above pasting agrees with the pasting



given by a slice square and a similarly defined right adjoint functor r'. The functoriality of mates with pasting hence implies that $(i_1)_! u_1^* \to u^* (i_1')^*$ is an isomorphism at $i_2 a_2$.

We now turn to the second claim and show that the canonical mate $u^*(i_1')_* \to (i_1)_* u_1^*$ is an isomorphism. Using again that i_1, i_2 are jointly surjective, it suffices to show that the corresponding restrictions of the canonical mate are invertible. Since i_1, i_1' are sieves, both right Kan extensions are right extensions by terminal objects (Remark 3.7), and the above canonical mate is hence automatically an isomorphism on objects of the form $i_2 a_2$. It remains to show that its restriction along i_1^* is an

isomorphism and for that purpose we consider the diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\mathrm{id}} & A_1 \\ {\scriptstyle \mathrm{id}}\downarrow & \swarrow_{\mathrm{id}} & \downarrow{\scriptstyle i_1} \\ A_1 & \xrightarrow[i_1]{} & A \\ {\scriptstyle u_1}\downarrow & \swarrow_{\mathrm{id}} & \downarrow{\scriptstyle u} \\ A'_1 & \xrightarrow[i'_1]{} & A' \end{array} \quad = \quad \begin{array}{ccc} A_1 & \xrightarrow{\mathrm{id}} & A_1 \\ {\scriptstyle u_1}\downarrow & \swarrow_{\mathrm{id}} & \downarrow{\scriptstyle u_1} \\ A'_1 & \xrightarrow[\mathrm{id}]{} & A'_1 \\ {\scriptstyle \mathrm{id}}\downarrow & \swarrow_{\mathrm{id}} & \downarrow{\scriptstyle i'_1} \\ A'_1 & \xrightarrow[i'_1]{} & A' \end{array}$$

Using the same arguments as in the first part of the proof, we conclude that $i_1^* u^* (i'_1)_* \to i_1^* (i_1)_* u_1^*$ is an isomorphism, concluding the proof. □

In the case that u_1 and, hence, u are fully faithful, there is the following convenient result.

Corollary 5.11. *Let* (5.8) *be two compatible gluing constructions such that u_1 and, hence, u are fully faithful, and let $\mathscr{D}$ be a derivator.*

(i) *The right Kan extension morphism $u_* : \mathscr{D}^A \to \mathscr{D}^{A'}$ is fully faithful with essential image given by those X such that $(i'_1)^* X$ lies in the essential image of $(u_1)_* : \mathscr{D}^{A_1} \to \mathscr{D}^{A'_1}$.*

(ii) *The left Kan extension morphism $u_! : \mathscr{D}^A \to \mathscr{D}^{A'}$ is fully faithful with essential image given by those X such that $(i'_1)^* X$ lies in the essential image of $(u_1)_! : \mathscr{D}^{A_1} \to \mathscr{D}^{A'_1}$.*

Proof. We give a proof of (i); the case of (ii) is dual. Since both u_1 and u are fully faithful, the respective right Kan extension morphisms are fully faithful (Examples 3.4). Thus, the corresponding essential images consist precisely of those diagrams on which the respective units $\eta_1 : \mathrm{id} \to (u_1)_* u_1^*$ and $\eta : \mathrm{id} \to u_* u^*$ are isomorphisms. To express this differently we consider the following pastings:

$$\begin{array}{ccccc} A_1 & \xrightarrow{i_1} & A & \xrightarrow{u} & A' \\ {\scriptstyle u_1}\downarrow & \swarrow_{\mathrm{id}} & \downarrow{\scriptstyle u} & \swarrow_{\mathrm{id}} & \downarrow{=} \\ A'_1 & \xrightarrow[i'_1]{} & A' & \xrightarrow[=]{} & A' \end{array} \quad = \quad \begin{array}{ccccc} A_1 & \xrightarrow{u_1} & A'_1 & \xrightarrow{i'_1} & A' \\ {\scriptstyle u_1}\downarrow & \swarrow_{\mathrm{id}} & \downarrow{=} & \swarrow_{\mathrm{id}} & \downarrow{=} \\ A'_1 & \xrightarrow[=]{} & A'_1 & \xrightarrow[i'_1]{} & A' \end{array}$$

By Lemma 3.5, $X \in \mathscr{D}^{A'}$ lies in the essential image of u_* if and only if $(i'_1)^* \eta$ is an isomorphism on X. Using the compatibility of mates with pasting and the homotopy exactness of the square to the very left (Proposition 5.10), this is the case if and only if the canonical mate associated to the pasting on the left is an isomorphism on X. But since the above two pastings agree, this is the case if and

only if the canonical mate of the pasting on the right is an isomorphism on X. As the square on the right is constant and hence homotopy exact, this is to say that η_1 is an isomorphism on $(i_1')^*X$, i.e., that $(i_1')^*X$ is in the essential image of $(u_1)_*$ (by an additional application of Lemma 3.5). □

As we shall see in Section 7, the results of this section allow us to add the desired biproduct n-cubes and (co)fiber squares needed for the reflection morphisms. To also be able to pass to the invertible n-cube we include the following section.

6. Gluing constructions and homotopical epimorphisms

In this section we continue the study of free oriented gluing constructions as defined in Section 5 and show that they are compatible with homotopical epimorphisms (Definition 3.8). The goal is to establish Theorem 6.5 showing that if we have a pair of compatible gluing constructions (5.8) such that u_1 is a homotopical epimorphism then so is u. Moreover, the essential images of the corresponding restriction morphisms u_1^* and u^* are related as desired.

In the situation of two compatible gluing constructions (5.8), the respective inclusions of the second summands induce the following commutative square, which we consider as being populated by the identity transformation as indicated in

$$\begin{array}{ccc} A_2 & \xrightarrow{i_2} & A \\ {\scriptstyle =}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle u} \\ A_2 & \xrightarrow[i_2']{} & A' \end{array} \tag{6.1}$$

Proposition 6.2. *Given two compatible oriented gluing constructions as in* (5.8)*, the commutative square* (6.1) *is homotopy exact.*

Proof. To reformulate the claimed homotopy exactness of the square (6.1), we consider the pasting on the left in

$$\begin{array}{ccccc} \mathbb{1} & \xrightarrow{a_2} & A_2 & \xrightarrow{i_2} & A \\ {\scriptstyle =}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle =} & \swarrow\!\!\!\swarrow {\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle u} \\ \mathbb{1} & \xrightarrow[a_2]{} & A_2 & \xrightarrow[i_2']{} & A' \end{array} \quad = \quad \begin{array}{ccc} \mathbb{1} & \xrightarrow{i_2a_2} & A \\ {\scriptstyle =}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle u} \\ \mathbb{1} & \xrightarrow[i_2'a_2]{} & A' \end{array} \quad = \quad \begin{array}{ccccc} \mathbb{1} & \xrightarrow{(i_2a_2,\mathrm{id})} & (i_2'a_2/u) & \xrightarrow{q} & A \\ {\scriptstyle =}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle \pi} & \swarrow & \downarrow{\scriptstyle u} \\ \mathbb{1} & \xrightarrow[=]{} & \mathbb{1} & \xrightarrow[i_2'a_2]{} & A' \end{array}$$

in which the left square is constant and hence homotopy exact. Using (Der2) and the compatibility of mates with pasting we conclude that (6.1) is homotopy exact if and only if the above pasting is homotopy exact for every $a_2 \in A_2$. Note that this pasting is simply the above commutative square in the middle, which in turn can be written as the above pasting on the right. In that pasting, the square on the right

is a slice square and hence homotopy exact. The square on the left is given by the functor classifying the initial object $(i_2a_2, \mathrm{id} : i_2'a_2 \to ui_2a_2)$ in the slice category $(i_2'a_2/u)$, and that square is hence homotopy exact by the homotopy initiality of left adjoint functors (Examples 3.4). The compatibility of homotopy exact squares with pasting concludes the proof. □

We again consider two compatible gluing constructions as in (5.8). In that notation, by Proposition 5.10 there is a homotopy exact square

$$\begin{array}{ccc} A_1 & \xrightarrow{i_1} & A \\ {\scriptstyle u_1}\downarrow & & \downarrow{\scriptstyle u} \\ A_1' & \xrightarrow[i_1']{} & A' \end{array}$$

of small categories.

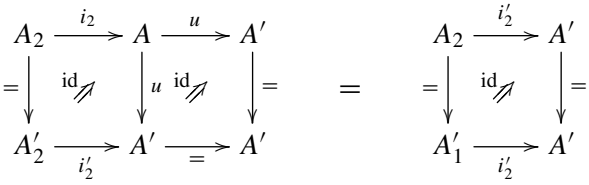
Proposition 6.3. *Given two compatible gluing constructions as in* (5.8) *such that* $u_1 : A_1 \to A_1'$ *is a homotopical epimorphism,* $u : A \to A'$ *is also a homotopical epimorphism.*

Proof. By assumption, $u_1 : A_1 \to A_1'$ is a homotopical epimorphism, i.e., the unit $\eta_1 : \mathrm{id} \to (u_1)_* u_1^*$ is an isomorphism. We have to show that the unit $\eta : \mathrm{id} \to u_* u^*$ is as well. Using that the inclusions $i_1' : A_1' \to A'$ and $i_2' : A_2' \to A'$ are jointly surjective, (Der2) implies that it is enough to show that $(i_1')^*\eta$ and $(i_2')^*\eta$ are isomorphisms. As for the first restriction, let us consider the pasting on the left in

$$\begin{array}{ccccc} A_1 & \xrightarrow{i_1} & A & \xrightarrow{u} & A' \\ {\scriptstyle u_1}\downarrow & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle u} & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle =} \\ A_1' & \xrightarrow[i_1']{} & A' & \xrightarrow[=]{} & A' \end{array} \qquad = \qquad \begin{array}{ccccc} A_1 & \xrightarrow{u_1} & A_1' & \xrightarrow{i_1'} & A' \\ {\scriptstyle u_1}\downarrow & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle =} & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle =} \\ A_1' & \xrightarrow[=]{} & A_1' & \xrightarrow[i_1']{} & A' \end{array}$$

The square to the left is homotopy exact by Proposition 5.10, and the compatibility of homotopy exact squares with pasting implies that $(i_1')^*\eta$ is an isomorphism if and only if the pasting on the left is homotopy exact. Note that this pasting agrees with the pasting on the right in which the square to the right is constant and hence homotopy exact. Moreover, the homotopy exactness of the square on the left is equivalent to u_1 being a homotopical epimorphism, showing that $(i_1')^*\eta$ is an isomorphism.

In order to show that also the restriction $(i_2')^*\eta$ is an isomorphism, let us consider the pasting on the left in

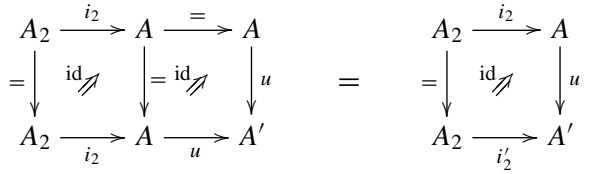


Using similar arguments as in the previous case together with the homotopy exactness of the square to the very left (Proposition 6.2), we deduce that $(i_2')^*\eta$ is an isomorphism if and only if the pasting on the left is homotopy exact. Since this pasting agrees with the constant square on the very right, we conclude by the homotopy exactness of constant squares. □

In the situation of Proposition 6.3, both restriction morphisms $u^* : \mathscr{D}^{A'} \to \mathscr{D}^{A}$ and $u_1^* : \mathscr{D}^{A_1'} \to \mathscr{D}^{A_1}$ are fully faithful for every derivator $\mathscr{D}$. To show that the essential images are related as desired (see Theorem 6.5) we establish the following result.

Lemma 6.4. *Let* (5.8) *be two compatible gluing constructions such that $u_1 : A_1 \to A_1'$ is a homotopical epimorphism, and let $\mathscr{D}$ be a derivator. A diagram $X \in \mathscr{D}^A$ lies in the essential image of $u^* : \mathscr{D}^{A'} \to \mathscr{D}^A$ if and only if $i_1^*\epsilon : i_1^*u^*u_*X \to i_1^*X$ is an isomorphism.*

Proof. By Proposition 6.3 the functor $u : A \to A'$ is also a homotopical epimorphism and $u^* : \mathscr{D}^{A'} \to \mathscr{D}^A$ is hence a fully faithful morphism of derivators. A diagram $X \in \mathscr{D}^A$ lies in the essential image of u^* if and only if the counit $\epsilon : u^*u_*X \to X$ is an isomorphism. Using the joint surjectivity of $i_1 : A_1 \to A$ and $i_2 : A_2 \to A$, by (Der2) this is the case if and only if the restricted counits $i_1^*\epsilon$, $i_2^*\epsilon$ are isomorphisms on X. Hence, to conclude the proof it suffices to show that $i_2^*\epsilon$ is always an isomorphism, and to this end we consider the pasting on the left in



The homotopy exactness of constant squares and the compatibility of canonical mates with pasting implies that $i_2^*\epsilon$ is always an isomorphism if and only if the pasting on the left is homotopy exact. However, this pasting agrees with the square on the right, which is homotopy exact by Proposition 6.2. □

Theorem 6.5. *Given two compatible gluing constructions as in* (5.8) *such that $u_1 : A_1 \to A_1'$ is a homotopical epimorphism, $u : A \to A'$ is also a homotopical epimorphism. Moreover, $X \in \mathscr{D}^A$ lies in the essential image of $u^* : \mathscr{D}^{A'} \to \mathscr{D}^A$ if and only if $i_1^*X \in \mathscr{D}^{A_1}$ lies in the essential image of $u_1^* : \mathscr{D}^{A_1'} \to \mathscr{D}^{A_1}$.*

Proof. By Proposition 6.3, the functor $u : A \to A'$ is a homotopical epimorphism and $u^* : \mathscr{D}^{A'} \to \mathscr{D}^A$, as a fully faithful morphism of derivators, induces an equivalence onto its essential image. A coherent diagram $X \in \mathscr{D}^A$ lies by Lemma 6.4 in this essential image if and only if $i_1^*\epsilon : i_1^*u^*u_*X \to i_1^*X$ is an isomorphism. But, using the homotopy exactness of constant squares, this is the case if and only if the canonical mate associated to the pasting on the left in

$$\begin{array}{ccccc} A_1 & \xrightarrow{i_1} & A & \xrightarrow{=} & A \\ {\scriptstyle =}\downarrow & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle =} & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle u} \\ A_1 & \xrightarrow[i_1]{} & A & \xrightarrow[u]{} & A' \end{array} \qquad = \qquad \begin{array}{ccccc} A_1 & \xrightarrow{=} & A_1 & \xrightarrow{i_1} & A \\ {\scriptstyle =}\downarrow & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle u_1} & \swarrow{\scriptstyle \mathrm{id}} & \downarrow{\scriptstyle u} \\ A_1 & \xrightarrow[u_1]{} & A_1' & \xrightarrow[i_1']{} & A' \end{array}$$

is an isomorphism on X. Since the above two pastings agree, the compatibility of mates with respect to pasting together with the homotopy exactness of the square to the very right (Proposition 5.10) implies that $X \in \mathscr{D}^A$ lies in the essential image of u^* if and only if the canonical mate $\epsilon_1 i_1^* : u_1^*(u_1)_* i_1^* \to i_1^*$ is an isomorphism on X. Since $u_1^* : \mathscr{D}^{A_1'} \to \mathscr{D}^{A_1}$ is fully faithful, the counit ϵ_1 is an isomorphism on i_1^*X if and only if i_1^*X lies in the essential image of u_1^*. □

In the construction of reflection morphisms in Section 7 we will see that the results of this section allow us to pass from biproduct n-cubes to invertible biproduct n-cubes (compare again with the strategy outlined in Section 4).

7. Reflection morphisms: the separated case

In this section we construct the reflection morphisms in abstract stable derivators and show them to be strong stable equivalences. The strategy behind the construction is described in Section 4. Here we deal only with the part of the construction depicted in the lower half of Figure 2, which is described in more detail in Figure 3. Thus, we shall assume that the source/sink is "separated" from the category C by freely added morphisms. The inflation/deflation steps indicated by the vertical dashed arrows in Figure 2 are postponed to Section 9.

More precisely, the goal is the following. Let $C \in \mathcal{C}at$, and let $y_1, \ldots, y_n \in C$ be objects (not necessarily distinct). We can view this data as a functor $y : n \cdot \mathbb{1} \to C$. We obtain two new categories D^- and D^+ by attaching a source of valence n and a sink of valence n, respectively, to C by means of the free oriented gluing construction in the sense of Section 5 (see the first line of Figure 3). Formally, we consider the two pushout diagrams in $\mathcal{C}at$

$$\begin{array}{ccc} \coprod_{i=1}^n \mathbb{1} \sqcup \mathbb{1} & \xrightarrow{\mathsf{inc} \sqcup y} & ([1]^n_{=n-1})^{\triangleleft} \sqcup C \\ \downarrow & & \downarrow \\ \coprod_{i=1}^n [1] & \xrightarrow[k]{} & D^- \end{array} \qquad \begin{array}{ccc} \coprod_{i=1}^n \mathbb{1} \sqcup \mathbb{1} & \xrightarrow{\mathsf{inc} \sqcup y} & ([1]^n_{=n-1})^{\triangleright} \sqcup C \\ \downarrow & & \downarrow \\ \coprod_{i=1}^n [1] & \xrightarrow[k]{} & D^+ \end{array} \tag{7.1}$$

where inc stands for the obvious inclusions $n \cdot \mathbb{1} \to (n \cdot \mathbb{1})^{\triangleleft} = ([1]^n_{=n-1})^{\triangleleft}$ and $n \cdot \mathbb{1} \to (n \cdot \mathbb{1})^{\triangleright} = ([1]^n_{=n-1})^{\triangleright}$. (Given a small category A, we denote by $A^{\triangleright}$ the cocone on A, i.e., the category obtained from A by freely adjoining a new terminal object ∞, and, dually, by $A^{\triangleleft}$ the cone on A.)

Here we carry out the individual steps of the construction of a strong stable equivalence of D^- and D^+; see Figure 3. Starting with a representation $X \in \mathscr{D}^{D^-}$ in a stable derivator $\mathscr{D}$, this roughly amounts to the following:

(i) Glue in a biproduct n-cube centered at $\bigoplus_{i=1}^n X_{x_i}$.

(ii) Pass to the invertible biproduct n-cube.

(iii) Add a cofiber square to the resulting morphism $X_v \to \bigoplus_{i=1}^n X_{x_i}$.

At the level of shapes this corresponds to considering the first three functors in

$$D^- \to E_1^- \to E_2^- \to F \leftarrow E_2^+ \leftarrow E_1^+ \leftarrow D^+, \tag{7.2}$$

precise definitions of which are given below.

As we discuss further below, the category F is symmetric in the following sense. If we begin with a representation $X \in \mathscr{D}^{D^+}$ and perform similar steps then we end up with a representation of the same category $F \in \mathcal{C}at$. At the level of shapes this amounts to considering the remaining three functors in (7.2).

We now turn to the first step, which essentially amounts to gluing an n-cube $[2]^n$ to D^-, yielding the functor $D^- \to E_1^-$ in (7.2); see again Figure 3. To define this functor, we consider the diagram of small categories

$$\begin{array}{ccccccccc}
[1]^n_{=n-1} & \longrightarrow & [1]^n_{\geq n-1} & \longrightarrow & [1]^n & \longrightarrow & I & \longrightarrow & [2]^n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
([1]^n_{=n-1})^{\triangleleft} & \xrightarrow[i_1]{} & ([1]^n_{\geq n-1})^{\triangleleft} & \xrightarrow[i_2]{} & ([1]^n)^{\triangleleft} & \xrightarrow[i_3]{} & I_1 & \xrightarrow[i_4]{} & I_2
\end{array} \tag{7.3}$$

in which the two pushout squares to the right define the categories I_1, I_2, the top row is as in (4.1), and the two squares to the left are naturality squares. The functor $D^- \to E_1^-$ is obtained by an application of the free oriented gluing construction to the bottom row in (7.3). Thus, we consider the following diagram consisting of pushout squares:

$$\begin{array}{ccccccccc}
([1]^n_{=n-1})^{\triangleleft} \sqcup C & \longrightarrow & ([1]^n_{\geq n-1})^{\triangleleft} \sqcup C & \longrightarrow & ([1]^n)^{\triangleleft} \sqcup C & \longrightarrow & I_1 \sqcup C & \longrightarrow & I_2 \sqcup C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
D^- & \xrightarrow[j_1]{} & A_1 & \xrightarrow[j_2]{} & A_2 & \xrightarrow[j_3]{} & A_3 & \xrightarrow[j_4]{} & E_1^-
\end{array} \tag{7.4}$$

Associated to the bottom row in this diagram there are the fully faithful Kan extension morphisms

$$\mathscr{D}^{D^-} \xrightarrow{(j_1)_*} \mathscr{D}^{A_1} \xrightarrow{(j_2)_*} \mathscr{D}^{A_2} \xrightarrow{(j_3)_!} \mathscr{D}^{A_3} \xrightarrow{(j_4)_*} \mathscr{D}^{E_1^-}. \tag{7.5}$$

We note that the category E_1^- comes by definition with a functor

$$l : [2]^n \to I_2 \to E_1^-$$

(see (7.3) and (7.4)). For every stable derivator $\mathscr{D}$ we denote by $\mathscr{D}^{E_1^-,\mathrm{ex}} \subseteq \mathscr{D}^{E_1^-}$ the full subderivator spanned by all $X \in \mathscr{D}^{E_1^-}$ for which the n-cube $l^*X \in \mathscr{D}^{[2]^n}$ is a biproduct n-cube (see Proposition 4.3). The following proposition implies that $\mathscr{D}^{E_1^-,\mathrm{ex}}$ is indeed a derivator.

Proposition 7.6. *Let $\mathscr{D}$ be a stable derivator. The morphisms in* (7.5) *are fully faithful and induce an equivalence $\mathscr{D}^{D^-} \simeq \mathscr{D}^{E_1^-,\mathrm{ex}}$. This equivalence is pseudonatural with respect to exact morphisms.*

Proof. The first part of this proof is very similar to the proof of Proposition 4.3; see [Groth and Šťovíček 2016b, Proposition 4.9]. We begin by considering the functors in the bottom row of (7.3). Since these functors are fully faithful, the associated Kan extension morphisms

$$\mathscr{D}^{([1]^n_{=n-1})^\triangleleft} \xrightarrow{(i_1)_*} \mathscr{D}^{([1]^n_{\geq n-1})^\triangleleft} \xrightarrow{(i_2)_*} \mathscr{D}^{([1]^n)^\triangleleft} \xrightarrow{(i_3)_!} \mathscr{D}^{I_1} \xrightarrow{(i_4)_*} \mathscr{D}^{I_2} \tag{7.7}$$

are also fully faithful. We now describe the essential images of the respective morphisms, and show that they induce the following pseudonatural equivalences:

(i) Since i_1 is a sieve, the morphism $(i_1)_*$ is right extension by zero and hence induces an equivalence onto the full subderivator of $\mathscr{D}^{([1]^n_{\geq n-1})^\triangleleft}$ defined by this vanishing condition.

(ii) One easily checks that $(i_2)_*$ precisely amounts to adding a strongly cartesian n-cube, hence induces a corresponding equivalence of derivators.

(iii) The functor i_3 is a cosieve and $(i_3)_!$ is hence left extension by zero, yielding an equivalence onto the full subderivator of $\mathscr{D}^{I_1}$ defined by this vanishing condition.

(iv) The morphism $(i_4)_*$ precisely amounts to adding strongly cartesian n-cubes. In fact, this follows as in the case of Proposition 4.3; see [Groth and Šťovíček 2016b, §4] for details.

Now, recall that the functors in the bottom row of (7.4) are obtained from the corresponding functors in the bottom row of (7.3) by the free oriented gluing construction. Hence, by Corollary 5.11 we can describe the respective essential images

of the Kan extension morphisms in (7.5) in terms of the essential images of the corresponding morphisms in (7.7). The above explicit description of these latter essential images concludes the proof of the first statement. The pseudonaturality with respect to exact morphisms follows since exact morphisms preserve right and left extensions by zero as well as strongly cartesian and strongly cocartesian n-cubes. □

The next step in this construction consists of inverting the biproduct n-cube $[2]^n$ in E_1^-, yielding the functor $E_1^- \to E_2^-$ in (7.2); see again Figure 3. To give a precise definition of this functor, we begin by observing that the category E_1^- is obtained from $[2]^n$ by two iterated free gluing constructions in the sense of Section 5. In fact, let $E_1 \in \mathcal{C}at$ be defined as the free oriented gluing construction on the left in

$$\begin{array}{ccc} \coprod_{i=1,\dots,n} \mathbb{1} \sqcup \mathbb{1} & \longrightarrow & [2]^n \sqcup C \\ \downarrow & & \downarrow \\ \coprod_{i=1,\dots,n} [1] & \longrightarrow & E_1 \end{array} \qquad \begin{array}{ccc} \mathbb{1} \sqcup \mathbb{1} & \longrightarrow & \mathbb{1} \sqcup E_1 \\ \downarrow & & \downarrow \\ [1] & \longrightarrow & E_1^- \end{array} \tag{7.8}$$

obtained from $n \cdot \mathbb{1} \cong [1]^n_{=n-1} \to [1]^n \xrightarrow{[1,2]^n} [2]^n$ and $(y_1, \dots, y_n) : n \cdot \mathbb{1} \to C$. (Here, $[1, 2]^n : [1]^n \to [2]^n$ is the n-fold product of the functor $[1] \to [2] : i \mapsto i + 1$.) Note that the category E_1^- is simply the free oriented gluing construction associated to the functors $\mathrm{id} : \mathbb{1} \to \mathbb{1}$, and $(1, \dots, 1) : \mathbb{1} \to [2]^n \to E_1$, as depicted in the pushout square on the right in (7.8). In order to obtain the category E_2^- we now simply replace the n-cube $[2]^n$ by the invertible n-cube R^n, as defined prior to Corollary 4.4. In detail, we define E_2^- as the corresponding two-step free oriented gluing construction described via the pushout squares

$$\begin{array}{ccc} \coprod_{i=1,\dots,n} \mathbb{1} \sqcup \mathbb{1} & \longrightarrow & R^n \sqcup C \\ \downarrow & & \downarrow \\ \coprod_{i=1,\dots,n} [1] & \longrightarrow & E_2 \end{array} \qquad \begin{array}{ccc} \mathbb{1} \sqcup \mathbb{1} & \longrightarrow & \mathbb{1} \sqcup E_2 \\ \downarrow & & \downarrow \\ [1] & \longrightarrow & E_2^- \end{array} \tag{7.9}$$

Finally, the functor $r : E_1^- \to E_2^-$ is obtained by tracing the homotopical epimorphism $q : [2]^n \to R^n$ (Corollary 4.4) through the above constructions, thereby first obtaining a functor $E_1 \to E_2$ and then $r : E_1^- \to E_2^-$ ((7.8) and (7.9) yield two pairs of compatible oriented gluing constructions in the sense of Section 5).

To perform the next step of the construction of reflection functors we now consider the commutative square

$$\begin{array}{ccc} [2]^n & \xrightarrow{i} & E_1^- \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ R^n & \xrightarrow[j]{} & E_2^- \end{array}$$

to which we apply our results from Section 6.

Proposition 7.10. *The functor $r : E_1^- \to E_2^-$ is a homotopical epimorphism. Furthermore, for every derivator $\mathscr{D}$, a diagram $X \in \mathscr{D}^{E_1^-}$ lies in the essential image of $r^* : \mathscr{D}^{E_2^-} \to \mathscr{D}^{E_1^-}$ if and only if $i^*X \in \mathscr{D}^{[2]^n}$ lies in the essential image of $q^* : \mathscr{D}^{R^n} \to \mathscr{D}^{[2]^n}$.*

Proof. The following diagram expresses that $r : E_1^- \to E_2^-$ is obtained in two steps as a free oriented gluing construction starting with $q : [2]^n \to R^n$:

$$\begin{array}{ccccc} [2]^n & \longrightarrow & E_1 & \longrightarrow & E_1^- \\ {\scriptstyle q}\downarrow & & \downarrow & & \downarrow{\scriptstyle r} \\ R^n & \longrightarrow & E_2 & \longrightarrow & E_2^- \end{array}$$

Since q is a homotopical epimorphism and we have a description of the essential image of $q^* : \mathscr{D}^{[2]^n} \to \mathscr{D}^{R^n}$ (Corollary 4.4), the result follows from two applications of Theorem 6.5. □

The morphism r^* induces an equivalence onto its essential image defined by invertibility conditions (Corollary 4.4). We are interested in the following restriction of this equivalence. Note that the category E_2^- comes by construction with a functor $j : R^n \to E_2^-$ (see (7.9)). For every stable derivator $\mathscr{D}$, we denote by $\mathscr{D}^{E_2^-,\mathrm{ex}} \subseteq \mathscr{D}^{E_2^-}$ the full subderivator spanned by all diagrams $X \in \mathscr{D}^{E_2^-}$ for which the n-cube $j^*X \in \mathscr{D}^{R^n}$ is an invertible biproduct n-cube in the sense of Corollary 4.6. Recall also the definition of the derivator $\mathscr{D}^{E_1^-,\mathrm{ex}}$ as considered in Proposition 7.6.

Corollary 7.11. *Let $\mathscr{D}$ be a stable derivator. The morphism $r^* : \mathscr{D}^{E_2^-} \to \mathscr{D}^{E_1^-}$ induces an equivalence of derivators $\mathscr{D}^{E_2^-,\mathrm{ex}} \simeq \mathscr{D}^{E_1^-,\mathrm{ex}}$ which is pseudonatural with respect to exact morphisms of derivators.*

Proof. This is immediate from Corollary 4.4 and Proposition 7.10. □

The third step in the construction of reflection morphisms amounts to extending the morphisms $X_v \to \bigoplus_{i=1}^n X_{x_i}$ in abstract representations to cofiber squares, as will be made precise by the functor $E_2^- \to F$ in (7.2); see again Figure 3. We recall that cofiber squares in pointed derivators are constructed as follows (see [Groth 2013, §3.3]). Let the functor $[1] \to \square = [1] \times [1]$ classify the top horizontal morphism $(0,0) \to (1,0)$ and let $[1] \xrightarrow{i} \ulcorner \xrightarrow{j} \square$ be the obvious factorization of it. For every pointed derivator $\mathscr{D}$ the corresponding Kan extension morphisms

$$\mathscr{D}^{[1]} \xrightarrow{i_*} \mathscr{D}^{\ulcorner} \xrightarrow{j_!} \mathscr{D}^{\square} \tag{7.12}$$

are fully faithful. Since i is a sieve, i_* is right extension by zero (Proposition 3.6). It follows that (7.12) induces an equivalence of derivators $\mathscr{D}^{[1]} \simeq \mathscr{D}^{\square,\mathrm{ex}}$, where

$\mathscr{D}^{\square,\mathrm{ex}} \subseteq \mathscr{D}^{\square}$ is the full subderivator spanned by the *cofiber squares*, i.e., those coherent squares $X \in \mathscr{D}^{\square}$ having the following properties:

(i) The square vanishes at the lower left corner, $X_{0,1} \cong 0$.

(ii) The square is cocartesian.

This construction is clearly pseudonatural with respect to right exact morphisms. Given a coherent morphism $X = (f : x \to y) \in \mathscr{D}^{[1]}$ the corresponding cofiber square looks like

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \downarrow{\scriptstyle \mathrm{cof}(f)} \\ 0 & \longrightarrow & z \end{array}$$

To prepare the corresponding relative construction, we consider the diagram of small categories

$$\begin{array}{ccccccc} \mathbb{1} & \xrightarrow{1} & [1] & \xrightarrow{i} & \ulcorner & \xrightarrow{j} & \square \\ {\scriptstyle (1,\dots,1)}\downarrow & & \downarrow & & \downarrow{\scriptstyle l_1} & & \downarrow{\scriptstyle l_2} \\ R^n & \longrightarrow & B_1 & \xrightarrow[i_1]{} & B_2 & \xrightarrow[i_2]{} & B \end{array} \tag{7.13}$$

consisting of pushout squares. The square to the left exhibits B_1 as a one-point extension of R^n (Example 5.6). And the category B is obtained from the invertible n-cube R^n by attaching a new morphism with target the center $(1, \dots, 1) \in R^n$ and a square containing this morphism as top horizontal morphism. (The category F as well as $E_2^- \to F$ in (7.2) will be obtained from (7.13) by a free oriented gluing construction.) We begin by considering a pointed derivator $\mathscr{D}$ and the Kan extension morphisms

$$\mathscr{D}^{B_1} \xrightarrow{(i_1)_*} \mathscr{D}^{B_2} \xrightarrow{(i_2)_!} \mathscr{D}^{B}. \tag{7.14}$$

Let $\mathscr{D}^{B_2,\mathrm{ex}} \subseteq \mathscr{D}^{B_2}$ be the full subderivator spanned by all $X \in \mathscr{D}^{B_2}$ such that $l_1^* X$ vanishes at $(0, 1)$. Similarly, let $\mathscr{D}^{B,\mathrm{ex}} \subseteq \mathscr{D}^{B}$ be the full subderivator spanned by those diagrams $X \in \mathscr{D}^{B}$ such that $l_2^* X$ is a cofiber square.

Lemma 7.15. *Let $\mathscr{D}$ be a pointed derivator.*

(i) *The morphism $(i_1)_*$ is fully faithful and induces $\mathscr{D}^{B_1} \simeq \mathscr{D}^{B_2,\mathrm{ex}}$.*

(ii) *The morphism $(i_2)_!$ is fully faithful with essential image the full subderivator of $\mathscr{D}^{B}$ spanned by all X such that $l_2^* X$ is cocartesian.*

(iii) *The morphisms in* (7.14) *induce an equivalence $\mathscr{D}^{B_1} \simeq \mathscr{D}^{B,\mathrm{ex}}$.*

These equivalences are pseudonatural with respect to right exact morphisms.

Proof. We leave it to the reader to work out the necessary homotopy (co)finality arguments and apply [Groth 2013, Proposition 3.10]. □

We note that the category E_2^- can be obtained as a free oriented gluing construction from B_1. In fact, associated to the functor

$$n \cdot \mathbb{1} = [1]^n_{=n-1} \longrightarrow [1]^n \xrightarrow{[1,2]^n} [2]^n \xrightarrow{q} R^n \longrightarrow B_1$$

and $y = (y_1, \dots, y_n) : n \cdot \mathbb{1} \to C$ there is the free oriented gluing construction given by the pushout square on the left in

$$\begin{array}{ccccccc} \coprod_{i=1,\dots,n} \mathbb{1} \sqcup \mathbb{1} & \longrightarrow & B_1 \sqcup C & \longrightarrow & B_2 \sqcup C & \longrightarrow & B \sqcup C \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \coprod_{i=1,\dots,n} [1] & \longrightarrow & E_2^- & \xrightarrow[j_1]{} & F_1 & \xrightarrow[j_2]{} & F \end{array} \tag{7.16}$$

The remaining two pushout squares are induced by the bottom row in (7.13). Thus, in the terminology of Section 5 we have two pairs of compatible oriented gluing constructions. For every derivator $\mathscr{D}$ the Kan extension morphisms

$$\mathscr{D}^{E_2^-} \xrightarrow{(j_1)_*} \mathscr{D}^{F_1} \xrightarrow{(j_2)_!} \mathscr{D}^F \tag{7.17}$$

are fully faithful. Note that the category F comes with a functor $l : \square \to B \to F$; see (7.13) and (7.16).

Proposition 7.18. *Let $\mathscr{D}$ be a pointed derivator. The morphisms* (7.17) *are fully faithful and induce an equivalence onto the full subderivator of $\mathscr{D}^F$ spanned by all $X \in \mathscr{D}^F$ such that $l^*X \in \mathscr{D}^\square$ is a cofiber square. This equivalence is pseudonatural with respect to right exact morphisms.*

Proof. Since we are in the context of two pairs of free oriented gluing constructions, this is immediate from two applications of Corollary 5.11 to Lemma 7.15. □

We are interested in the following induced equivalence. Note that associated to the category F there are functors

$$l : \square \to F \quad \text{and} \quad m : R^n \to F;$$

see (7.13) and (7.16). Given a stable derivator $\mathscr{D}$, we denote by $\mathscr{D}^{F,\mathrm{ex}} \subseteq \mathscr{D}^F$ the full subderivator spanned by all $X \in \mathscr{D}^F$ satisfying the following properties:

(i) The square $l^*X \in \mathscr{D}^\square$ is a cofiber square.

(ii) The n-cube $m^*X \in \mathscr{D}^{R^n}$ is an invertible biproduct n-cube.

Recall also the definition of the derivator $\mathscr{D}^{E_2^-,\mathrm{ex}}$ as considered in Corollary 7.11.

Corollary 7.19. *Let $\mathscr{D}$ be a stable derivator. The morphisms* (7.17) *induce an equivalence of derivators $\mathscr{D}^{E_2^-,\mathrm{ex}} \simeq \mathscr{D}^{F,\mathrm{ex}}$ which is pseudonatural with respect to exact morphisms.*

Proof. This is immediate from Proposition 7.18 and the defining exactness and vanishing conditions of $\mathscr{D}^{E_2^-,\mathrm{ex}}$ and $\mathscr{D}^{F,\mathrm{ex}}$. □

It now suffices to assemble the above individual steps in order to settle the reflection morphisms in the separated case.

Theorem 7.20. *Let $C \in \mathcal{C}at$, let $y_1, \dots, y_n \in C$ (not necessarily distinct), and let $D^-, D^+ \in \mathcal{C}at$ be as in* (7.1)*. The categories D^- and D^+ are strongly stably equivalent.*

Proof. As discussed at the beginning of this section, the functors in (7.2) correspond to the respective steps in the construction of the strong stable equivalence. Proposition 7.6, Corollary 7.11, and Corollary 7.19 take care of the first three steps. In fact, they show that for every stable derivator $\mathscr{D}$, there are equivalences of derivators

$$\mathscr{D}^{D^-} \simeq \mathscr{D}^{E_1^-,\mathrm{ex}} \simeq \mathscr{D}^{E_2^-,\mathrm{ex}} \simeq \mathscr{D}^{F,\mathrm{ex}}$$

which are pseudonatural with respect to exact morphisms.

If we start with an abstract representation of D^+ instead, then, as indicated by the remaining three functors in (7.2), we can perform similar constructions to again obtain an abstract representation of F. We leave it to the reader to verify that in this way we in fact construct a category isomorphic to F. (The arguments for this are essentially the same as in the case of [Groth and Šťovíček 2016b, Lemma 9.15].) At the level of derivators of representations, this amounts to additional pseudonatural equivalences

$$\mathscr{D}^{D^+} \simeq \mathscr{D}^{E_1^+,\mathrm{ex}} \simeq \mathscr{D}^{E_2^+,\mathrm{ex}} \simeq \mathscr{D}^{F,\mathrm{ex}},$$

which are similar to Proposition 7.6, Corollary 7.11, and Corollary 7.19. These steps amount to gluing in a biproduct n-cube, inverting the n-cube, and adding a fiber square, respectively. Since cofiber squares and fiber squares agree in stable derivators, it follows that the essential image of these three steps is again given by the derivator $\mathscr{D}^{F,\mathrm{ex}}$ as described prior to Corollary 7.19. Putting these pseudonatural equivalences together,

$$\mathscr{D}^{D^-} \simeq \mathscr{D}^{E_1^-,\mathrm{ex}} \simeq \mathscr{D}^{E_2^-,\mathrm{ex}} \simeq \mathscr{D}^{F,\mathrm{ex}} \simeq \mathscr{D}^{E_2^+,\mathrm{ex}} \simeq \mathscr{D}^{E_1^+,\mathrm{ex}} \simeq \mathscr{D}^{D^+},$$

we obtain the desired strong stable equivalence $\mathscr{D}^{D^-} \simeq \mathscr{D}^{D^+}$. □

8. Detection criteria for homotopical epimorphisms

The aim of this section is to establish two simple detection results for homotopical epimorphisms. These will be used in Section 9 to construct reflection morphisms in the general case and thereby to complete the plan from Section 4.

The first criterion is completely straightforward; we show that (co)reflective (co)localizations are homotopical epimorphisms (compare to [Groth and Šťovíček 2016b, Proposition 6.5]).

Proposition 8.1. *Let $(l, r) : A \rightleftarrows B$ be an adjunction of small categories with unit $\eta : \mathrm{id} \to rl$ and counit $\varepsilon : lr \to \mathrm{id}$.*

(i) *For every prederivator $\mathscr{D}$ there is an adjunction*

$$(r^*, l^*, \eta^* : \mathrm{id} \to l^*r^*, \varepsilon^* : r^*l^* \to \mathrm{id}) : \mathscr{D}^A \rightleftarrows \mathscr{D}^B.$$

(ii) *If l is a reflective localization, i.e., r is fully faithful, then l is a homotopical epimorphism. Moreover, $X \in \mathscr{D}^A$ lies in the essential image of l^* if and only if $X_{\eta_a} : X_a \to X_{rla}$ is an isomorphism for all $a \in A - r(B)$.*

(iii) *If r is a coreflective colocalization, i.e., l is fully faithful, then r is a homotopical epimorphism. Moreover, $Y \in \mathscr{D}^B$ lies in the essential image of r^* if and only if $Y_{\varepsilon_b} : Y_{lrb} \to Y_b$ is an isomorphism for all $b \in B - l(A)$.*

Proof. The first statement is immediate from the fact that every prederivator $\mathscr{D}$ defines a 2-functor

$$\mathscr{D}^{(-)} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{PDER} : A \mapsto \mathscr{D}^A$$

and since 2-functors preserve adjunctions. By duality it suffices to establish the second statement. Since r is fully faithful, the counit $\varepsilon : lr \to \mathrm{id}$ is an isomorphism, and hence so is the counit $\varepsilon^* : r^*l^* \to \mathrm{id}$. But this means that $l^* : \mathscr{D}^B \to \mathscr{D}^A$ is fully faithful, i.e., that $l : A \to B$ is a homotopical epimorphism. The essential image of l^* consists precisely of those $X \in \mathscr{D}^A$ such that the unit $\eta^* : X \to l^*r^*X$ is an isomorphism. By (Der2) this is the case if and only if η^*_a is an isomorphism for every $a \in A$. Now, the triangular identity

$$\mathrm{id} = \varepsilon^*r^* \circ r^*\eta^* : r^* \to r^*l^*r^* \to r^*$$

and the invertibility of ε^* implies that $r^*\eta^*$ is an isomorphism. Hence to characterize the essential image of l^* it suffices to check η^* at all objects $a \in A - r(B)$. □

This first criterion is already enough for one of the inflation and deflation steps in Section 9. For the remaining one we establish the following additional criterion, which will be applied to more general localization functors. While these functors do not necessarily admit adjoints, they are still essentially surjective, thereby making the first condition in the coming proposition automatic.

Proposition 8.2. *Let $u : A \to B$ be essentially surjective, let $\mathscr{D}$ be a derivator, and let $u^* : \mathscr{D}^B \to \mathscr{D}^A$ be the restriction morphism. Let us assume further that $\mathscr{E} \subseteq \mathscr{D}^A$ is a full subprederivator such that*

(i) *the essential image* $\mathrm{im}(u^*)$ *lies in $\mathscr{E}$, i.e.,* $\mathrm{im}(u^*) \subseteq \mathscr{E} \subseteq \mathscr{D}^A$, *and*

(ii) *the unit* $\eta : X \to u^* u_! X$ *is an isomorphism for all* $X \in \mathscr{E}$.

Then $u^* : \mathscr{D}^B \to \mathscr{D}^A$ *is fully faithful and* $\mathrm{im}(u^*) = \mathscr{E}$. *In particular,* $\mathscr{E}$ *is a derivator.*

Proof. To prove that u^* is fully faithful it suffices to show that $\varepsilon : u_! u^* \to \mathrm{id}$ is a natural isomorphism. The assumptions imply that ηu^* is a natural isomorphism. Hence, by the triangular identity

$$\mathrm{id} = u^*\varepsilon \circ \eta u^* : u^* \xrightarrow{\eta u^*} u^* u_! u^* \xrightarrow{u^*\varepsilon} u^*,$$

it follows that also $u^*\varepsilon$ is an isomorphism. In order to conclude that ε is a natural isomorphism, it suffices by (Der2) to show that $b^*\varepsilon$ is an isomorphism for every $b \in B$. This follows immediately from the essential surjectivity of u and the fact that $u^*\varepsilon$ is invertible.

Since u^* is fully faithful, its essential image consists precisely of those $X \in \mathscr{D}^A$ such that the unit $\eta : X \to u^* u_! X$ is an isomorphism. The assumptions (i) and (ii) immediately imply that this is the case if and only if $X \in \mathscr{E}$. Finally, $\mathscr{E}$ is also a derivator by the invariance of derivators under equivalences. □

Thus, once we make an educated guess of an $\mathscr{E}$ satisfying the above assumptions, we get an equivalence onto $\mathscr{E}$. The relation to homotopical epimorphisms is as follows.

Remark 8.3. In our later applications the subprederivator $\mathscr{E} \subseteq \mathscr{D}^A$ is a full subprederivator $\mathscr{D}^{A,\mathrm{ex}}$ determined by some exactness conditions. Recall from [Groth and Šťovíček 2016c, §3] that such exactness conditions are formalized by certain (co)cones in A to be populated by (co)limiting (co)cones. As a special case this includes the assumption that certain morphisms are populated by isomorphisms.

In such a situation we hence start with a full subprederivator $\mathscr{D}^{A,\mathrm{ex}} \subseteq \mathscr{D}^A$ for *every* derivator $\mathscr{D}$. If the assumptions of Proposition 8.2 are satisfied, then this implies first that $u : A \to B$ is a homotopical epimorphism and second that the essential image of u^* is $\mathrm{im}(u^*) = \mathscr{D}^{A,\mathrm{ex}}$.

To be able to apply Proposition 8.2 in specific situations, it is useful to have better control over the adjunction unit $\eta : \mathrm{id} \to u^* u_!$.

Construction 8.4. Let $\mathscr{D}$ be a derivator, $A \in \mathcal{C}at$, and let $a \in A$. Associated to the square

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{a} & A \\ \downarrow & & \downarrow{\scriptstyle \pi_A} \\ \mathbb{1} & \longrightarrow & \mathbb{1} \end{array}$$

there is the canonical mate

$$a^* \to \mathrm{colim}_A \,. \tag{8.5}$$

As a special case relevant in later applications, given a functor $u : A \to B$ and $a \in A$ there is the functor $p : (u/ua) \to A$. Whiskering the mate (8.5) in the case of $(a, \mathrm{id} : ua \to ua) \in (u/ua)$ with p^* we obtain a canonical map

$$a^* = (a, \mathrm{id}_{ua})^* p^* \to \mathrm{colim}_{(u/ua)}\, p^*. \tag{8.6}$$

Lemma 8.7. *Let $\mathscr{D}$ be a derivator, $u : A \to B$, and $a \in A$. The component of the unit $a^*\eta : a^* \to a^*u^*u_!$ is isomorphic to $a^* \to \mathrm{colim}_{(u/ua)}\, p^*$* (8.6)*. In particular, η_a is an isomorphism if and only if this is the case for* (8.6)*.*

Proof. To reformulate that the adjunction unit η_a is an isomorphism we consider the pasting on the left in

$$\begin{array}{ccccc} \mathbb{1} & \xrightarrow{a} & A & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\ \mathbb{1} & \xrightarrow[a]{} & A & \xrightarrow[u]{} & B \end{array} \qquad\qquad \begin{array}{ccccc} \mathbb{1} & \xrightarrow{(a,\mathrm{id}_{ua})} & (u/ua) & \xrightarrow{p} & A \\ \downarrow & & \downarrow & \swarrow\!\!\swarrow & \downarrow{\scriptstyle u} \\ \mathbb{1} & \longrightarrow & \mathbb{1} & \xrightarrow[ua]{} & B \end{array}$$

in which the square to the left is constant and hence homotopy exact. Note that this pasting agrees with the pasting on the right in which the square to the right is a slice square and hence also homotopy exact. The functoriality of canonical mates with pasting concludes the proof. □

We will later apply the previous lemma in situations in which the slice category admits homotopy final functors from certain simpler shapes. For this purpose we collect the following result.

Lemma 8.8. *Let $u : A \to B$ be a homotopy final functor and let $a \in A$.*

(i) *The map $u(a)^* \to \mathrm{colim}_B$* (8.5) *is naturally isomorphic to $a^*u^* \to \mathrm{colim}_A\, u^*$, the whiskering of an instance of* (8.5) *with u^*.*

(ii) *If A admits a terminal object ∞, then the map $a^* \to \mathrm{colim}_A$* (8.5) *is naturally isomorphic to $a^* \to \infty^*$.*

Proof. Using the functoriality of canonical mates, for the first statement it suffices to observe that the two pastings

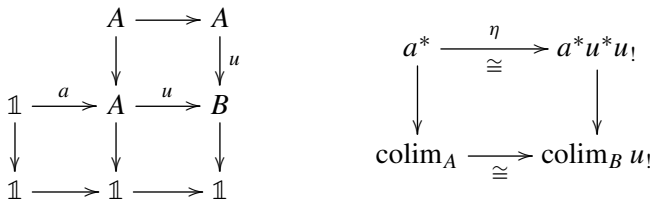
$$\begin{array}{ccccc} \mathbb{1} & \xrightarrow{a} & A & \xrightarrow{u} & B \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{1} & \longrightarrow & \mathbb{1} & \longrightarrow & \mathbb{1} \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{1} & \xrightarrow{ua} & B \\ \downarrow & & \downarrow \\ \mathbb{1} & \longrightarrow & \mathbb{1} \end{array}$$

agree and that the square in the middle is homotopy exact by assumption on u. For the second statement it suffices to unravel the definition of (8.5) using ∞^* as a model for colim_A. □

We finish the section with another lemma related to Construction 8.4 which will be useful when dealing with a more complicated instance of Proposition 8.2 in the next section.

Lemma 8.9. *Let $\mathscr{D}$ be a derivator, $u : A \to B$ be fully faithful, and $a \in A$. The map $a^* \to \operatorname{colim}_A$ (8.5) at $X \in \mathscr{D}^A$ is isomorphic to $u(a)^* \to \operatorname{colim}_B$ (8.5) at $u_! X$.*

Proof. Considering the pasting on the left in the diagram



it is immediate from the functoriality of mates with pasting that the square on the right commutes. □

9. General reflection morphisms

In this section we implement the remaining steps of the strategy outlined in Section 4, namely the inflation and deflation steps from Figure 2. This will allow us to finish the construction of a strong stable equivalence between the categories C^+ and C^- depicted in Figure 1 (see Theorem 9.11).

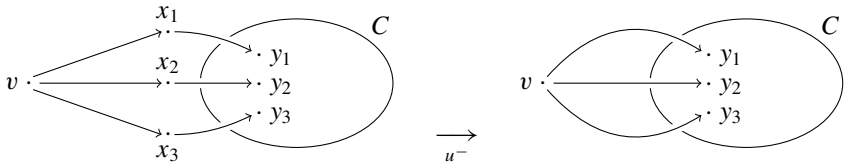
We start by formalizing the construction of the categories C^- and C^+. Let $C \in \mathcal{C}at$, and let $y_1, \ldots, y_n \in C$ be objects. We denote by $y : n \cdot \mathbb{1} \to C$ the resulting functor. For all preparatory results before Corollary 9.10, we adopt the following hypothesis which will allow us to apply results from the Appendix.

Hypothesis 9.1. The functor $y : n \cdot \mathbb{1} \to C$ is injective on objects. Equivalently, $y_1, y_2, \ldots, y_n$ are pairwise distinct objects of C.

We obtain C^- and C^+ by attaching a source of valence n and a sink of valence n to C, respectively. More precisely, the source of valence n is the cone $(n \cdot \mathbb{1})^{\triangleleft}$ obtained from $n \cdot \mathbb{1}$ by adjoining an initial object, and dually for the sink $(n \cdot \mathbb{1})^{\triangleright}$. Using the obvious inclusion functors $n \cdot \mathbb{1} \to (n \cdot \mathbb{1})^{\triangleleft}$ and $n \cdot \mathbb{1} \to (n \cdot \mathbb{1})^{\triangleright}$ we define C^- and C^+ as the respective pushouts in

(9.2)

Assuming Hypothesis 9.1, note that $C \to C^+$ and $C \to C^-$ are fully faithful by Proposition A.11, and we view these functors as inclusions.



The category D^- The category C^-

Figure 5. The functor $u^- : D^- \to C^-$, which contracts the edges $x_i \to y_i$. It is used to separate the source of C^-.

As already mentioned in Section 4, the two inflation and deflation steps are not dual to each other. Starting with a representation of C^- we separate the morphisms adjacent to the source by adding morphisms pointing in the *same* direction, while in the other case we add morphisms pointing in the *opposite* direction.

Let us start with the easier case and consider the functor $u^- : D^- \to C^-$ as shown in Figure 5. Formally, we can construct the functor by means of the following pushout squares in $\mathcal{C}at$, where we use the inclusion of the target object $1 : \mathbb{1} \to [1]$ and the collapse functor $\pi : [1] \to \mathbb{1}$ in the upper line:

$$\begin{array}{ccccc} n \cdot \mathbb{1} & \longrightarrow & (n \cdot [1])^{\triangleleft} & \longrightarrow & (n \cdot \mathbb{1})^{\triangleleft} \\ {\scriptstyle y}\downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow[j^-]{} & D^- & \xrightarrow[u^-]{} & C^- \end{array} \tag{9.3}$$

The functor j^- is fully faithful by Proposition A.11, and for every derivator $\mathscr{D}$, the restriction morphism $(u^-)^* : \mathscr{D}^{C^-} \to \mathscr{D}^{D^-}$ separates the objects adjacent to the source. We denote by $\mathscr{D}^{D^-,\mathrm{ex}} \subseteq \mathscr{D}^{D^-}$ the full subderivator spanned by all diagrams $X \in \mathscr{D}^{D^-}$ such that $k^*X \in \mathscr{D}^{n\cdot[1]}$ consists of isomorphisms, where

$$k : n \cdot [1] \to (n \cdot [1])^{\triangleleft} \to D^-$$

is the obvious functor.

Proposition 9.4. *The functor $u^- : D^- \to C^-$ is a homotopical epimorphism. Moreover, for every derivator $\mathscr{D}$ the essential image of $(u^-)^* : \mathscr{D}^{C^-} \to \mathscr{D}^{D^-}$ is $\mathscr{D}^{D^-,\mathrm{ex}}$ and the resulting equivalence $(u^-)^* : \mathscr{D}^{C^-} \simeq \mathscr{D}^{D^-,\mathrm{ex}}$ is pseudonatural with respect to arbitrary morphisms of derivators.*

Proof. This is an immediate application of Proposition 8.1. In fact, the functor $u^- : D^- \to C^-$ is a reflective localization, a fully faithful right adjoint being given by the obvious functor $r : C^- \to D^-$ which sends v to v and which is the identity on C. Let us denote the resulting adjunction by

$$(u^-, r, \eta : \mathrm{id} \to r \circ u^-, \varepsilon = \mathrm{id} : u^- \circ r \to \mathrm{id}).$$

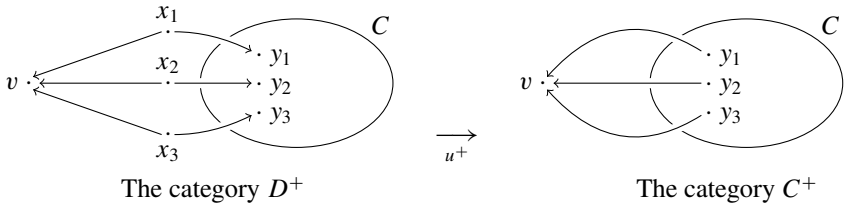


Figure 6. The functor $u^+ : D^+ \to C^+$, which contracts the edges $x_i \to y_i$. It is used to separate the sink of C^+.

The only nonidentity components of the adjunction unit η are those at $x_i \in D^-$ for $i = 1, \ldots, n$, in which case they are given by

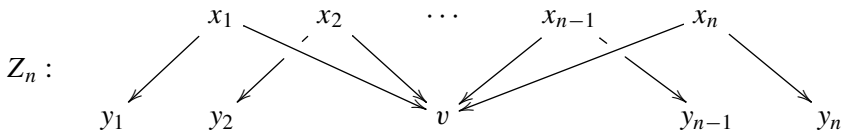
$$\eta_{x_i} : x_i \to y_i, \qquad i = 1, \ldots, n.$$

By Proposition 8.1 we conclude that u^- is a homotopical epimorphism and that $X \in \mathscr{D}^{D^-}$ lies in the essential image of $(u^-)^*$ if and only if $X_{x_i} \to X_{y_i}$ is an isomorphism, which is to say that $X \in \mathscr{D}^{D^-,\mathrm{ex}}$. □

The other inflation and deflation step turns out to be a bit more involved, and the situation is shown in Figure 6. We again have defining pushout squares

$$\begin{array}{ccccc} n \cdot \mathbb{1} & \longrightarrow & Z_n & \xrightarrow{q} & (n \cdot \mathbb{1})^{\triangleright} \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle y^{\triangleright}} \\ C & \xrightarrow[j^+]{} & D^+ & \xrightarrow[u^+]{} & C^+ \end{array} \tag{9.5}$$

where Z_n is the free category generated by the quiver



where $n \cdot \mathbb{1} \to Z_n$ classifies $y_1, \ldots, y_n$, and where $q : Z_n \to (n \cdot \mathbb{1})^{\triangleright}$ sends each x_i and y_i to the i-th copy of $\mathbb{1}$ and v to the terminal object ∞. Assuming Hypothesis 9.1, both j^+ and $u^+ j^+$ are fully faithful, and we again view $u^+ j^+$ as an inclusion. As it will be important in further computations, we spell out what morphisms in D^+ and C^+ look like.

Lemma 9.6.

(i) *Every nonidentity morphism in the category C^+ has a unique expression of one of the forms γ, ω, $\omega\gamma$, where γ stands for a nonidentity morphism of C and ω stand for a nonidentity morphism of $(n \cdot \mathbb{1})^{\triangleright}$.*

(ii) *Every nonidentity morphism in the category D^+ has a unique expression of one of the forms γ, ω, $\gamma\omega$, where γ stands for a nonidentity morphism of C and ω stand for a nonidentity morphism of Z_n.*

Proof. This is an immediate consequence of Lemma A.12. □

For every derivator $\mathscr{D}$ we denote by $\mathscr{D}^{D^+,\mathrm{ex}} \subseteq \mathscr{D}^{D^+}$ the full subderivator formed by the coherent diagrams X such that $X_{x_i} \to X_{y_i}$ is an isomorphism for every $i = 1, \dots, n$.

Proposition 9.7. *If $y : n \cdot \mathbb{1} \to C$ is injective on objects, then $u^+ : D^+ \to C^+$ (9.5) is a homotopical epimorphism. Moreover, for $\mathscr{D} \in \mathcal{DER}$ the essential image of $(u^+)^* : \mathscr{D}^{C^+} \to \mathscr{D}^{D^+}$ is $\mathscr{D}^{D^+,\mathrm{ex}}$ and the resulting equivalence $(u^+)^* : \mathscr{D}^{C^+} \simeq \mathscr{D}^{D^+,\mathrm{ex}}$ is pseudonatural with respect to arbitrary morphisms of derivators.*

Proof. Let us fix a derivator $\mathscr{D}$ and let $\mathscr{E} = \mathscr{D}^{D^+,\mathrm{ex}}$. We show that Proposition 8.2 applies. Clearly u^+ is essentially surjective on objects and $\mathrm{im}((u^+)^*) \subseteq \mathscr{E}$. It remains to verify the assumption Proposition 8.2(ii), and by (Der2) it suffices to check the invertibility of the unit η at every $d \in D^+$. By Lemma 8.7 this is the case if and only if the instance

$$(d, \mathrm{id}_{u^+d})^* p^* \to \mathrm{colim}_{(u^+/u^+d)}\, p^* \tag{9.8}$$

of (8.6) is invertible for every $d \in D^+$ and on $\mathscr{D}^{D^+,\mathrm{ex}}$. Here, $p : (u^+/u^+d) \to D^+$ is the canonical functor, and there are the following three cases.

First, let $d = j^+(c)$, $c \in C$, so that $u^+d = c$. Since $(j^+c, \mathrm{id}_c) \in (u^+/c)$ is a terminal object, by Lemma 8.8 the corresponding morphism (9.8) is an isomorphism on $\mathscr{D}^{D^+,\mathrm{ex}}$ if and only if $(j^+c, \mathrm{id})^* p^* \to (j^+c, \mathrm{id})^* p^*$ is an isomorphism on $\mathscr{D}^{D^+,\mathrm{ex}}$, and this is even true for all $X \in \mathscr{D}^{D^+}$.

Suppose next that $d = x_i$ for some $i = 1, \dots, n$. In this case $u^+d = y_i \in C^+$ and it is easy to see that (u^+/y_i) admits

$$(x_i, \mathrm{id}_{y_i}) \to (y_i, \mathrm{id}_{y_i})$$

as homotopy final subcategory, where the map is given by the freely attached map from Z_n. Two applications of Lemma 8.8 imply that we have to show that $x_i^* \to y_i^*$ is an isomorphism on $\mathscr{D}^{D^+,\mathrm{ex}}$, which is true by the defining exactness properties.

The remaining case is $d = v$. With the aid of Lemma 9.6, we divide the objects

$$w = (d', g : u^+(d') \to v)$$

of (u^+/v) into five disjoint classes, according to what d' is and whether the structure morphism g factors through a nonidentity morphism in C. Each object $w \in (u^+/v)$ has exactly one of the following forms (where unlabeled arrows $y_i \to v$ always stand for the maps in C^+ coming from $(n \cdot \mathbb{1})^\triangleright$ in (9.2)):

(i) $w = (v, \mathrm{id}_v)$,

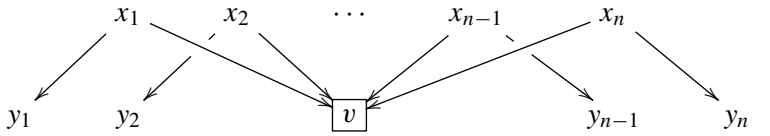
(ii) $w = (x_i, y_i \to v)$ for some $i \in \{1, \dots, n\}$,

(iii) $w = (y_i, y_i \to v)$ for some $i \in \{1, \dots, n\}$,

(iv) $w = (j^+(c), c \xrightarrow{h} y_i \to v)$ for some $c \in C$ and nonidentity map h in C, or

(v) $w = (x_i, y_i \xrightarrow{h} y_j \to v)$ for $i, j \in \{1, \dots, n\}$ and nonidentity map h in C.

Let $H \subseteq (u^+/v)$ be the full subcategory spanned by the objects of type (i)–(iii). This category is a free category generated by the following quiver, where the object from which we wish to inspect the map (9.8) is in the box (for brevity we denote the objects only by the corresponding object of D^+):

H : 

(9.9)

Another short computation reveals that every object of type (v) admits a unique map in (u^+/v) to the object of type (iv) with $c = y_i$ and the same morphism h in C, and that every object of type (iv) admits a unique map in (u^+/v) to an object of type (iii) obtained by stripping off h from the structure morphism. In particular, the inclusion $H \to (u^+/v)$ is a right adjoint and hence homotopy final, so that Lemma 8.8 applies. As an upshot so far, the decoration of the objects in (9.9) defines a functor $i : H \to D^+$, and it remains to show that the map,

$$v^* i^*(X) \to \mathrm{colim}_H\, i^* X$$

which is an instance of (8.5), is an isomorphism for all $X \in \mathscr{D}^{D^+,\mathrm{ex}}$.

To this end, let $j : H' \to H$ be the full subcategory of H obtained by removing $y_i, i = 1, \dots, n$. It is straightforward to show that $j_! : \mathscr{D}^{H'} \to \mathscr{D}^H$ is fully faithful with essential image precisely those $Y \in \mathscr{D}^H$ such that $Y_{x_i} \to Y_{y_i}$ is invertible (compare to [Groth 2013, Proposition 3.12(1)]). In particular, for $X \in \mathscr{D}^{D^+,\mathrm{ex}}$ the restriction i^*X belongs to this essential image, and Lemma 8.9 thus reduces our task to show that $v^* \to \mathrm{colim}_{H'}$ (8.5) is an isomorphism on $j^* i^* X \in \mathscr{D}^{H'}$. By Lemma 8.8 this is even the case for every diagram in $\mathscr{D}^{H'}$, since $v \in H'$ is a terminal object.

To summarize, all assumptions of Proposition 8.2 are satisfied and u^+ is hence a homotopical epimorphism with essential image $\mathscr{D}^{D^+,\mathrm{ex}}$. □

Now we shall revoke Hypothesis 9.1.

Corollary 9.10. *Let $C \in \mathcal{C}at$, let $y_1, \dots, y_n \in C$ (not necessarily distinct), and consider the functors $u^- : D^- \to C^-$ and $u^+ : D^+ \to C^+$ constructed again by the pushouts* (9.3) *and* (9.5), *respectively. Then u^- and u^+ are still homotopical*

epimorphisms and the essential images are $\mathscr{D}^{D^-,\mathrm{ex}}$ and $\mathscr{D}^{D^+,\mathrm{ex}}$, defined by the same exactness conditions as in Proposition 9.4 and Proposition 9.7, respectively.

Proof. We discuss only u^+, the case of u^- being similar. Suppose $y : n \cdot \mathbb{1} \to C$ is any functor. Thanks to Lemma A.2(i) there is a factorization $y = p\tilde{y}$ such that $p : \widetilde{C} \to C$ is an equivalence of categories and $\tilde{y}_1, \dots, \tilde{y}_n$ are pairwise distinct objects in $\widetilde{C}$. Replacing y by $\tilde{y}$ in (9.5), we obtain Proposition A.7 and Lemma A.2(ii), a diagram whose lower row changes only up to equivalence. □

Finally, we can establish the main result of this paper.

Theorem 9.11. *Let $C \in \mathcal{C}at$, let $y_1, \dots, y_n \in C$ (not necessarily distinct), and let $C^-, C^+ \in \mathcal{C}at$ be as in* (9.2)*. The categories C^- and C^+ are strongly stably equivalent.*

Proof. In Theorem 7.20 we constructed a pseudonatural equivalence $\mathscr{D}^{D^-} \simeq \mathscr{D}^{D^+}$. It is direct from the construction of this equivalence that it restricts to a pseudonatural equivalence $\mathscr{D}^{D^-,\mathrm{ex}} \simeq \mathscr{D}^{D^+,\mathrm{ex}}$. Invoking Corollary 9.10, we obtain a chain

$$\mathscr{D}^{C^-} \simeq \mathscr{D}^{D^-,\mathrm{ex}} \simeq \mathscr{D}^{D^+,\mathrm{ex}} \simeq \mathscr{D}^{C^+}$$

of pseudonatural equivalences. Putting them together, we obtain the strong stable equivalence

$$(s^-, s^+) : \mathscr{D}^{C^-} \simeq \mathscr{D}^{C^+}, \tag{9.12}$$

concluding the proof. □

Definition 9.13. Let $\mathscr{D}$ be a stable derivator, let $C \in \mathcal{C}at$, let $y_1, \dots, y_n \in C$ (not necessarily distinct), and let $C^-, C^+ \in \mathcal{C}at$ be as in (9.2). The components s^-, s^+ of the strong stable equivalence in (9.12), witnessing that $C^- \overset{\mathrm{s}}{\sim} C^+$, are *(general) reflection morphisms*.

10. Applications to abstract representation theory

In this section we draw some consequences from the main theorem in this paper (Theorem 9.11). Since the categories C^+ and C^- are strongly stably equivalent, we obtain abstract tilting results for various contexts. To begin with, let us specialize to representations over a ring.

Example 10.1. Let R be a (possibly noncommutative) ring. Let $C \in \mathcal{C}at$ with $y_1, \dots, y_n \in C$ (not necessarily distinct), and let $C^-, C^+ \in \mathcal{C}at$ be as in (9.2).

(i) There is an exact equivalence of categories $\mathscr{D}_R^{C^-}(\mathbb{1}) \overset{\Delta}{\simeq} \mathscr{D}_R^{C^+}(\mathbb{1})$.

(ii) If C has only finitely many objects, then the category algebras RC^- and RC^+ are derived equivalent over R:

$$D(RC^-) \overset{\Delta}{\simeq} D(RC^+).$$

In fact, the first statement is [Groth 2013, Proposition 4.18], while the second statement follows from Example 2.13. However, having a strong stable equivalence is a stronger result in the following three senses.

(i) Simply by choosing specific stable derivators, this yields exact equivalences of derived or homotopy categories of representations over rings or schemes, of differential graded representations, of spectral representations, and of other types of representations; see [Groth and Šťovíček 2016c, §5].

(ii) There are equivalences of derivators of representations, as opposed to having mere equivalences of underlying categories. For example, in the case of homotopy derivators of combinatorial, stable model categories $\mathcal{M}$ it is a formal consequence of the existence of an equivalence $\mathscr{H}o_{\mathcal{M}}^{Q} \sim \mathscr{H}o_{\mathcal{M}}^{Q'}$ and [Renaudin 2009] that the corresponding model categories of representations $\mathcal{M}^{Q}$, $\mathcal{M}^{Q'}$ are related by a zigzag of Quillen equivalences.

(iii) The equivalences are pseudonatural with respect to exact morphisms, and hence commute with various types of morphisms like restriction of scalars, induction and coinduction of scalars, derived tensor and hom functors, localizations and colocalizations.

With this added generality in mind, for the rest of the section we mostly focus on the shapes C^-, C^+. As a first instance, we recover the main result of [Groth and Šťovíček 2016b].

Theorem 10.2 [Groth and Šťovíček 2016b, Corollary 9.23]. *Let T be a finite oriented tree and let T' be a reorientation of T. The trees T and T' are strongly stably equivalent.*

Proof. By an inductive argument, it suffices to show that if T is as above and $t_0 \in T$ is a source, then the reflected tree $T' = \sigma_{t_0} T$ and T are strongly stably equivalent. But obviously $T = C^-$ and $T' = C^+$ for the full subcategory $C \subseteq T$ of T obtained by removing t_0. Hence Theorem 9.11 concludes the proof. □

Increasing the class of shapes, we obtain the following.

Theorem 10.3. *Let Q be a finite acyclic quiver, let $q_0 \in Q$ be a source or a sink, and let $Q' = \sigma_{q_0} Q$ be the reflected quiver. The two quivers Q and Q' are strongly stably equivalent.*

Proof. Assuming without loss of generality that q_0 is a source, we observe that $Q = C^-$ for the full subcategory $C \subseteq Q$ obtained by removing q_0. In this case one notes that $Q' = C^+$ and Theorem 9.11 applies. □

Remark 10.4. Specializing to the derivator $\mathscr{D}_k$ of a field k, Theorem 10.3 yields exact equivalences of derived categories:

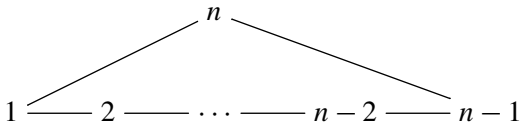
$$D(kQ) \overset{\Delta}{\simeq} D(kQ').$$

The classical representation theory is more concerned with bounded derived categories of finite dimensional representations. However, as shown in [Rickard 1989, Corollary 8.3] (and its proof), any exact equivalence between the unbounded derived categories restricts to an exact equivalence of the corresponding bounded derived categories:

$$D^b(kQ) \overset{\Delta}{\simeq} D^b(kQ')$$

Hence, the reflection functors yield such an equivalence and we recover a theorem of Happel [1987, §1.7].

In contrast to the case of trees, already for acyclic quivers it is not true that such quivers can be reoriented arbitrarily without affecting the abstract representation theory. If Q, Q' are finite and without oriented cycles, then Q, Q' being strongly stably equivalent still implies that Q and Q' have the same underlying graph [Groth and Šťovíček 2016c, Proposition 5.3], but this condition is no longer sufficient. Let us consider the simplest case, where Q is an orientation of an n-cycle:

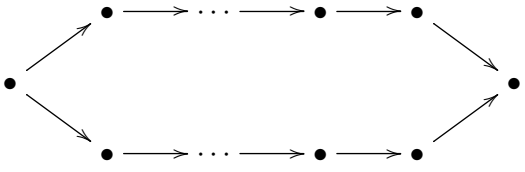


In representation theory one says that Q is a *Euclidean* (or *extended Dynkin*) quiver of type $\widetilde{A}_{n-1}$ [Ringel 1984; Simson and Skowroński 2007]. Given such Q, put $c(Q) = \{p, q\}$, where p is the number of arrows oriented clockwise and q is the number of arrows oriented counterclockwise. Then one obtains the following.

Proposition 10.5. *Let Q, Q' be two orientations of an n-cycle, $n \geq 1$. Then $Q \overset{\mathrm{s}}{\sim} Q'$ if and only if $c(Q) = c(Q')$.*

Proof. The sufficiency of the "clock condition" $c(Q) = c(Q')$ is easy. One quickly convinces oneself that given Q with $c(Q) = \{p, q\}$, $p \leq q$, after finitely many reflections at sinks or sources one gets a quiver isomorphic to

$\widetilde{A}_{p,q}$:



with p arrows above and q arrows below. Hence, if $c(Q) = C(Q')$, one gets for any stable derivator $\mathscr{D}$ a strong stable equivalence $\mathscr{D}^Q \simeq \mathscr{D}^{\widetilde{A}_{p,q}} \simeq \mathscr{D}^{Q'}$ by composing finitely many general reflection morphisms (Theorem 9.11).

To prove the necessity, let k be a field, $\mathscr{D} = \mathscr{D}_k$ be the derivator of k, and suppose that $\mathscr{D}_k^Q \simeq \mathscr{D}_k^{Q'}$. We shall appeal to results from representation theory and show

that then $c(Q) = c(Q')$. The equivalence of derivators gives an equivalence of the underlying categories, which in turn gives an equivalence of the subcategories of compact objects. In our case this means that the bounded derived categories of finitely generated modules of the corresponding path algebras are equivalent:

$$D^b(kQ) \simeq D^b(kQ').$$

Now kQ is a finite dimensional algebra over k if and only if not all arrows have the same orientation if and only if $c(Q) \neq \{0, n\}$ if and only if all objects of $D^b(\operatorname{mod} kQ)$ have finite dimensional endomorphism rings. Thus $c(Q) = \{0, n\}$ if and only if $c(Q') = \{0, n\}$.

Suppose now that $c(Q), c(Q') \neq \{0, n\}$. Then kQ is finite dimensional and we can construct a so-called Auslander–Reiten quiver of $D^b(kQ)$. This is an infinite quiver which is a useful combinatorial invariant of $D^b(kQ)$, and its general shape is described in [Happel 1987, Corollary 4.5(ii)]. A more precise description can be extracted from [Ringel 1984, Theorem 3.6.5, p. 158] or [Simson and Skowroński 2007, Proposition XII.2.8]. In particular, the numbers p, q, where $c(Q) = \{p, q\}$, can be read off the Auslander–Reiten quiver since it contains so-called tubes of ranks precisely 1, p, and q. Of course one can do the same for Q', and hence $c(Q) = c(Q')$. □

Remark 10.6. The existence of reflection equivalences in Theorem 9.11 applies to more general shapes than finite, acyclic quivers.

(i) First, neither the finiteness nor the acyclicity is needed. In fact, given an arbitrary quiver Q with a source or a sink, Theorem 9.11 yields a strong stable equivalence between Q and the reflected quiver Q'. In particular, if Q has finitely many objects only, the infinite-dimensional path algebras kQ and kQ' are derived equivalent for arbitrary fields k, and there are variants if we use rings as coefficients instead.

(ii) More generally, as noted in Example 10.1, Theorem 9.11 yields strong stable equivalences for shapes which are more general than quivers. To the best of the authors' knowledge, even in the case that $R = k$ is a field, the result that the category algebras kC^- and kC^+ are derived equivalent does not appear in the published literature.

Appendix: Amalgamation of categories

As is illustrated by the construction of abstract reflection functors, performing more complicated constructions in derivators often means that we need to "glue together" various small categories or diagram shapes. Formally, we are speaking of pushouts of categories, which is a fairly complicated construction. As we need to understand some of these pushouts rather explicitly (for example, in order to be able to

compute slice categories), here we discuss some basic properties of pushouts and amalgamations of small categories. We fix the following notation for the rest of the appendix:

$$\begin{array}{ccc} W & \xrightarrow{f_Y} & Y \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle g_Y} \\ X & \xrightarrow[g_X]{} & Z \end{array} \tag{A.1}$$

Often one is only interested in categories up to equivalences, but pushouts of small categories are, in general, not well behaved with equivalences. To address this issue, we include the following lemma.

Lemma A.2. *Let $f_X : W \to X$ be a functor in $\mathcal{C}at$.*

(i) *There exists a factorization $f_X = p \circ f_{\widetilde{X}}$ such that $f_{\widetilde{X}} : W \to \widetilde{X}$ is injective on objects and $p : \widetilde{X} \to X$ is surjective on objects and an equivalence.*

(ii) *If f_X is injective on objects and $f_Y : W \to Y$ in* (A.1) *is an equivalence, then also $g_X : X \to Z$ is an equivalence.*

Proof. Both are easy consequences of the existence of an (in fact unique) model structure on $\mathcal{C}at$ with weak equivalences being the equivalences. This is a special case of a more general result in [Joyal and Tierney 1991], and (i) is simply a factorization of f_X into a cofibration followed by a trivial fibration. Meanwhile, (ii) means that this model structure is left proper, which follows from the fact that every small category is cofibrant [Hirschhorn 2003, Corollary 13.1.3]. □

For the rest of the section we adopt the following assumption and convention.

Hypothesis A.3. Assume that f_X and f_Y are honest inclusions of categories, that is, injective on objects and faithful. We will view f_X and f_Y as (not necessarily full) inclusions $W \subseteq X$ and $W \subseteq Y$, respectively.

Definition A.4. The pushout (A.1) is called an *amalgamation* if also g_X and g_Y are injective on objects and faithful. In this case we also view g_X and g_Y as inclusions $X \subseteq Z$ and $Y \subseteq Z$, respectively.

Remark A.5. In the usual terminology of model theory, an amalgamation of the span

$$X \xleftarrow{f_X} W \xrightarrow{f_Y} Y$$

would in fact mean *any* commutative square like (A.1) (i.e., not necessarily a pushout) for which g_X and g_Y are inclusions. But if such a square exists, the pushout square is also an amalgamation in this sense.

As shown in [MacDonald and Scull 2009, Example 4.4], not every pushout of inclusions is an amalgamation. On the other hand, a sufficient condition for the existence of amalgamations is given in the same paper.

Definition A.6. A functor $f : W \to Y$ has the 3-*for*-2 *property* if, whenever α and β are two composable morphisms in Y and two of α, β, $\beta\alpha$ belong to the honest (not just essential) image of f, then so does the third.

Proposition A.7 [MacDonald and Scull 2009, Theorem 3.3]. *Suppose $f_X : W \to X$ and $f_Y : W \to Y$ are functors in $\mathcal{C}at$ which are injective on objects, faithful, and have the* 3-*for*-2 *property. Then their pushout* (A.1) *is an amalgamation.*

Remark A.8. The result is rather subtle in that it is *not* enough to assume that only one of f_X and f_Y has the 3-for-2 property; see [MacDonald and Scull 2009, Example 4.4] again. Note that $f : W \to Y$ has the 3-for-2 property, for example, if f is fully faithful or if W is a groupoid (so in particular if W is a discrete category as in Sections 5, 6, and 9).

For practical purposes it will be convenient to know that the 3-for-2 property transfers via amalgamations, i.e., that also the functors g_X and g_Y have it. Once we know this, we can iterate the amalgamation process. Here we need to refine the argument in [MacDonald and Scull 2009].

We first recall details about the construction of a pushout in $\mathcal{C}at$. At the level of objects, we simply construct the pushout of sets. The morphisms in the pushout are more interesting; see [MacDonald and Scull 2009, §2] for details. To this end, we denote by $\overline{Z}$ the pushout of the sets of morphisms of X and Y over the set of morphisms of W. In particular, an element of $\overline{Z}$ which comes from both X and Y comes already from W by our standing assumption. Every morphism in Z is represented by a finite sequence

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

of length $n \geq 1$ in $\overline{Z}$, subject to the condition that the codomain of α_{i+1} always agrees with the domain of α_i. The composition of morphisms is simply given by concatenation. Of course we must identify some of these sequences. To do so, we first define a partial order on the set of allowable sequences of elements of $\overline{Z}$ which is generated by the *elementary reductions*

$$(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n) > (\alpha_1, \ldots, \alpha_i\alpha_{i+1}, \ldots, \alpha_n),$$

where either both α_i and α_{i+1} are morphisms from X and the composition on the right takes place in X, or symmetrically α_i and α_{i+1} are from Y and we compose them in Y. This *reduction order* is of course a binary relation, and by taking its symmetric and transitive closure, we obtain an equivalence relation. The morphisms in Z are then precisely the equivalence classes of allowable sequences in $\overline{Z}$.

For convenience, we introduce the following notation. Given an allowable sequence $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, denote the equivalence class of γ by $[\alpha_1, \alpha_2, \ldots, \alpha_n]$,

and view this equivalence class as a partially ordered set with the restriction of the reduction order above. The following is a key observation.

Lemma A.9. *Suppose that $\gamma = (\alpha_1)$ consist of a single element of $\overline{Z}$. Then γ is the unique minimal element of $[\alpha_1]$ with respect to the reduction order.*

Proof. This is exactly what the first paragraph of the proof of [MacDonald and Scull 2009, Theorem 3.3] asserts. For a very detailed proof we refer to the rest of the proof of Theorem 3.3 and to §5 in [op. cit.]. □

Now we can complement Proposition A.7 with the promised result, which will allow for iterated amalgamations.

Proposition A.10. *Suppose that $f_X : W \to X$ and $f_Y : W \to Y$ are injective on objects and faithful functors with the* 3*-for-*2 *property. Then, in their pushout amalgamation* (A.1)*, also g_X and g_Y have the* 3*-for-*2 *property.*

Proof. By symmetry we only need to treat g_X. Suppose that α_1, β are composable morphisms in Z and that α_1 and $\alpha_1\beta$ both belong to X. We must show that β belongs there as well.

To this end, β can be represented by a suitable sequence $\gamma = (\alpha_2, \ldots, \alpha_n)$ of elements of $\overline{Z}$. Then $\alpha_1\beta$ is represented by $\delta = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and, by Lemma A.9, $[\alpha_1, \alpha_2, \ldots, \alpha_n]$ has the unique minimal element $(\alpha_1\beta)$ with respect to the reduction order. We shall prove by induction on n that β is in X.

Suppose first that $n = 2$. In this case $\beta = \alpha_2$ belongs either to X or Y. If β is in X, we are done. If β is in Y, we know by the above that $(\alpha_1, \beta) > (\alpha_1\beta)$ in the reduction order on $[\alpha_1\beta]$. By definition of the reduction order, the latter must be an elementary reduction, and hence all α_1, β, $\alpha_1\beta$ belong to X or all three belong to Y. In the first case we are done and in the second case we know that $\alpha_1, \alpha_1\beta \in X \cap Y = W$. Hence $\beta \in W \subseteq X$ by the 3-for-2 property of $f_Y : W \overset{\subseteq}{\to} Y$.

If now $n > 2$, there is an elementary reduction

$$(\alpha_1, \alpha_2, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n) > (\alpha_1, \alpha_2, \ldots, \alpha_i\alpha_{i+1}, \ldots, \alpha_n).$$

Let us choose such a reduction with maximal possible i. Two situations may occur. If $i > 1$, then by the very definition of elementary reductions we have that $(\alpha_2, \ldots, \alpha_n) > (\alpha_2, \ldots, \alpha_i\alpha_{i+1}, \ldots, \alpha_n)$ and also that β is in X by the induction hypothesis.

Suppose on the other hand that $i = 1$. We claim that in such a case α_2 is in X. To this end, assume by way of contradiction that $\alpha_2 \in Y \setminus W$. Then $\alpha_1 \in W$, since we have the reduction $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) > (\alpha_1\alpha_2, \alpha_3, \ldots, \alpha_n)$. Consequently, $\alpha_1\alpha_2 \in Y \setminus W$, since otherwise $\alpha_1, \alpha_1\alpha_2 \in W$ would imply $\alpha_2 \in W$. Finally, since the sequence $(\alpha_1\alpha_2, \alpha_3, \ldots, \alpha_n)$ must reduce further, the maximality of $i = 1$ implies

$$(\alpha_1\alpha_2, \alpha_3, \ldots, \alpha_n) > (\alpha_1\alpha_2\alpha_3, \ldots, \alpha_n).$$

Now $\alpha_1\alpha_2 \in Y \setminus W$, so $\alpha_3 \in Y$ in order for the reduction to be defined. However, then we also have an elementary reduction

$$(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) > (\alpha_1, \alpha_2\alpha_3, \ldots, \alpha_n),$$

contradicting the maximality of i. This proves the claim.

To summarize, we have $\alpha_1, \alpha_2 \in X$. Now let $\alpha' = \alpha_1\alpha_2 \in X$ and β' be the equivalence class $[\alpha_3, \ldots, \alpha_n]$. Then $\alpha', \alpha'\beta' \in X$ and we infer by the inductive hypothesis that $\beta' \in X$. Then clearly $\beta = \alpha_2\beta'$ is in X, which finishes the induction.

The case when α, β are composable in Z and $\beta, \alpha\beta$ are in X is similar. □

As pointed out in [MacDonald and Scull 2009], a special case when a functor has the 3-for-2 property is when it is fully faithful. Under our usual assumptions, it turns out that also full faithfulness is compatible with amalgamations. This has been observed already in [Trnková 1965], and we include a short proof for the convenience of the reader.

Proposition A.11. *Suppose that $f_X : W \to X$ and $f_Y : W \to Y$ are injective on objects. If f_X is fully faithful and f_Y is faithful and has the* 3*-for-*2 *property, then in the pushout amalgamation* (A.1), *$g_Y : Y \to Z$ is fully faithful.*

Proof. We only need to prove that g_Y is full. Suppose that we are given a morphism in Z, represented by a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\overline{Z}$ such that the domain of α_n and the codomain of α_1 belong to Y. By possibly reducing this sequence, we may assume that α_i belongs to Y for i odd and to X for i even. If i is even, the domain and the codomain of α_i must be objects in $X \cap Y = W$. Since f_X is full, α_i is a morphism in W, and hence also in Y. Thus all the α_i in fact belong to Y and so does their composition. □

Finally, we consider the case where W is a discrete category (recall Remark A.8). The main advantage is that, analogous to the situation with free products of monoids, all morphisms of a pushout amalgamation of two categories over a discrete category have unique reduced factorizations to morphisms of the original categories (see Lemma 5.3(iv) for an illustration). To state this precisely, we call an allowable sequence $(\alpha_1, \ldots, \alpha_n)$ of elements of $\overline{Z}$ *reduced* if it is minimal with respect to the reduction order. For W discrete, the following stronger version of Lemma A.9 holds.

Lemma A.12. *Suppose that W is a discrete category and that $f_X : W \to X$ and $f_Y : W \to Y$ are injective on objects. Given any morphism in Z represented by a sequence $\gamma = (\alpha_1, \ldots, \alpha_n)$ in $\overline{Z}$, the equivalence class $[\alpha_1, \ldots, \alpha_n]$ has a unique minimal element in the reduction order. In other words, each nonidentity morphism β in Z uniquely factors as $\beta = \alpha_1 \cdots \alpha_n$, where each α_i belongs to X or Y, but no composition $\alpha_i\alpha_{i+1}$ belongs to X or Y.*

Proof. Suppose that we have two elementary reductions of our sequence γ,

$$(\alpha_1, \dots, \alpha_i\alpha_{i+1}, \dots, \alpha_n) < \gamma > (\alpha_1, \dots, \alpha_j\alpha_{j+1}, \dots, \alpha_n), \tag{A.13}$$

where $i \leq j$ without loss of generality. We claim that there is a common predecessor. This is clear if $i = j$ and easy if $j - i \geq 2$, as then both the reductions further reduce to $(\alpha_1, \dots, \alpha_i\alpha_{i+1}, \dots, \alpha_j\alpha_{j+1}, \dots, \alpha_n)$. If $j = i+1$, there are two cases. First, all of $\alpha_i, \alpha_{i+1}, \alpha_{i+2}$ may belong to one of X or Y. Then $(\alpha_1, \dots, \alpha_i\alpha_{i+1}\alpha_{i+2}, \dots, \alpha_n)$ is the common predecessor which we are looking for. Second, two of $\alpha_i, \alpha_{i+1}, \alpha_{i+2}$ may belong to X and one to Y, or vice versa. Then, since both the reductions from (A.13) were possible, it is easy to check that in all possible distributions of $\alpha_i, \alpha_{i+1}, \alpha_{i+2}$ among X and Y, we always get that one of $\alpha_i, \alpha_{i+1}, \alpha_{i+2}$ belongs to $W = X \cap Y$, so it is the identity morphism. If $\alpha_{i+1} = \mathrm{id}$, the original reductions are equal, and in the remaining cases $(\alpha_1, \dots, \alpha_i\alpha_{i+1}\alpha_{i+2}, \dots, \alpha_n)$ is a common predecessor of the two. This proves the claim.

An easy induction argument shows now that $([\alpha_1, \dots, \alpha_n], <)$ is a downwards directed poset. Together with the obvious fact that the reduction order satisfies the descending chain condition, it follows that $([\alpha_1, \dots, \alpha_n], <)$ has a unique minimal (= reduced) element. □

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