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Equivariant noncommutative motives

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Given a finite group G, we develop a theory of G-equivariant noncommutative motives. This theory provides a well-adapted framework for the study of G-schemes, Picard groups of schemes, G-algebras, 2-cocycles, G-equivariant algebraic K-theory, etc. Among other results, we relate our theory with its commutative counterpart as well as with Panin's theory. As a first application, we extend Panin's computations, concerning twisted projective homogeneous varieties, to a large class of invariants. As a second application, we prove that whenever the category of perfect complexes of a G-scheme X admits a full exceptional collection of G-invariant ($\neq$ G-equivariant) objects, the G-equivariant Chow motive of X is of Lefschetz type. Finally, we construct a G-equivariant motivic measure with values in the Grothendieck ring of G-equivariant noncommutative Chow motives.

1. Introduction

A *differential graded* (dg) *category* $\mathcal{A}$, over a base field k, is a category enriched over dg k-vector spaces; see Section 2A. Every (dg) k-algebra A naturally gives rise to a dg category with a single object. Schemes provide another source of examples, since the category of perfect complexes $\mathrm{perf}(X)$ of every quasicompact quasiseparated k-scheme X admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$; see Section 2B.

Let G be a finite group. A dg category $\mathcal{A}$ equipped with a G-action is denoted by $\mathrm{G} \circlearrowright \mathcal{A}$ and called a G-*dg category*. For example, every G-scheme X, subgroup $\mathrm{G} \subseteq \mathrm{Pic}(X)$ of the Picard group of a scheme X, G-algebra A, or cohomology class $[\alpha] \in H^2(\mathrm{G}, k^\times)$, naturally gives rise to a G-dg category. The associated dg categories of G-equivariant objects $\mathcal{A}^{\mathrm{G}}$ are given, respectively, by equivariant perfect complexes $\mathrm{perf}^{\mathrm{G}}_{\mathrm{dg}}(X)$, perfect complexes $\mathrm{perf}_{\mathrm{dg}}(Y)$ on a $|\mathrm{G}|$-fold cover over X, semidirect product algebras $A \rtimes \mathrm{G}$, and twisted group algebras $k_\alpha[\mathrm{G}]$.

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By precomposition with the functor $G \circlearrowright \mathcal{A} \mapsto \mathcal{A}^{G}$, all invariants of dg categories E can be promoted to invariants of G-dg categories E^{G}. For example, algebraic K-theory leads to equivariant algebraic K-theory in the sense of Thomason [1987]; see Section 4A. In order to study all these invariants simultaneously, we develop in Section 3 a theory of G-equivariant noncommutative motives. Among other results, we construct a symmetric monoidal functor $U^{G} : \text{G-dgcat}_{\text{sp}}(k) \to \text{NChow}^{G}(k)$, from smooth proper G-dg categories to G-equivariant noncommutative Chow motives, which is "initial" among all such invariants E^{G}. The morphisms of $\text{NChow}^{G}(k)$ are given in terms of the G-equivariant Grothendieck group of certain triangulated categories of bimodules. In particular, the ring of endomorphisms of the $\otimes$-unit $U^{G}(G \circlearrowright_1 0k)$ identifies with the representation ring $R(G)$ of the group G.

I. Panin [1994] constructed a certain motivic category $\mathcal{C}^{G}(k)$, which mixes smooth projective G-schemes with (noncommutative) separable algebras, and performed therein several computations concerning twisted projective homogeneous varieties. In Theorem 5.3 we construct a fully faithful symmetric monoidal functor from $\mathcal{C}^{G}(k)$ to $\text{NChow}^{G}(k)$. As a byproduct, we extend Panin's computations to all the aforementioned invariants E^{G}; see Theorem 5.10.

Making use of results of [Edidin and Graham 1998] on equivariant intersection theory, [Laterveer 1998; Iyer and Müller-Stach 2009] extended the theory of Chow motives to the G-equivariant setting. In Theorem 6.4, we relate this latter theory with that of G-equivariant noncommutative motives. Concretely, we construct a $\mathbb{Q}$-linear, fully faithful, symmetric monoidal Φ making the diagram

$$\begin{tikzcd} \text{SmProj}^{G}(k)^{\text{op}} \arrow[r, "X \mapsto G \circlearrowright \text{perf}_{\text{dg}}(X)"] \arrow[d, "\mathfrak{h}^{G}(-)_{\mathbb{Q}}"'] & \text{G-dgcat}_{\text{sp}}(k) \arrow[d, "U^{G}(-)_{\mathbb{Q}}"] \\ \text{Chow}^{G}(k)_{\mathbb{Q}} \arrow[d, "\pi"'] & \text{NChow}^{G}(k)_{\mathbb{Q}} \arrow[d, "(-)_{I_{\mathbb{Q}}}"] \\ \text{Chow}^{G}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)} \arrow[r, "\Phi"'] & \text{NChow}^{G}(k)_{\mathbb{Q}, I_{\mathbb{Q}}} \end{tikzcd} \tag{1.1}$$

commute, where $\text{Chow}^{G}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$ is the orbit category (see Section 6B) and $(-)_{I_{\mathbb{Q}}}$ the localization functor associated to the augmentation ideal $I \subset R(G) \overset{\text{rank}}{\twoheadrightarrow} \mathbb{Z}$. Intuitively speaking, the commutative diagram (1.1) shows that after "$\otimes$-trivializing" the G-equivariant Tate motive $\mathbb{Q}(1)$ and localizing at the augmentation ideal $I_{\mathbb{Q}}$, the commutative world embeds fully faithfully into the noncommutative world.

The Grothendieck ring of varieties admits a G-equivariant analogue $K_0 \,\text{Var}^{G}(k)$. Although very important, the structure of this latter ring is quite mysterious. In order to capture some of its flavor, several G-equivariant motivic measures have been built. In Theorem 8.2, we prove that the assignment $X \mapsto U^{G}(G \circlearrowright \text{perf}_{\text{dg}}(X))$, with X a smooth projective G-variety, gives rise to a G-equivariant motivic measure $\mu_{\mathrm{nc}}^{\mathrm{G}} : K_0 \operatorname{Var}^{\mathrm{G}}(k) \to K_0(\mathrm{NChow}^{\mathrm{G}}(k))$ with values in the Grothendieck ring of the category of G-equivariant noncommutative Chow motives. It turns out that $\mu_{\mathrm{nc}}^{\mathrm{G}}$ contains a lot of interesting information. For example, when $k \subseteq \mathbb{C}$, the G-equivariant motivic measure $K_0 \operatorname{Var}^{\mathrm{G}}(k) \to R_{\mathbb{C}}(\mathrm{G})$, $X \mapsto \sum_i (-1)^i H_c^i(X^{\mathrm{an}}, \mathbb{C})$, factors through $\mu_{\mathrm{nc}}^{\mathrm{G}}$; see Proposition 8.3.

Applications. Let X be a smooth projective G-scheme. In order to study it, we can proceed in two distinct directions. In one direction, we can associate to X its G-equivariant Chow motive $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$. In another direction, we can associate to X its G-category of perfect complexes $\mathrm{G} \circlearrowright \operatorname{perf}(X)$. Making use of the bridge (1.1), we establish the following relation[1] between these two distinct mathematical objects.

Theorem 1.2. *If* $\operatorname{perf}(X)$ *admits a full exceptional collection* $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ *of length* n *of* G-invariant objects, *i.e.*, $\sigma^*(\mathcal{E}_i) \simeq \mathcal{E}_i$ *for every* $\sigma \in \mathrm{G}$, *then there exists a choice of integers* $r_1, \dots, r_n \in \{0, \dots, \dim(X)\}$ *such that*

$$\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}} \simeq \mathbb{L}^{\otimes r_1} \oplus \cdots \oplus \mathbb{L}^{\otimes r_n}, \tag{1.3}$$

where $\mathbb{L} \in \mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ *stands for the* G*-equivariant Lefschetz motive.*

Remark 1.4. A G-equivariant object is G-invariant, but the converse does not hold!

Theorem 1.2 can be applied to any G-action on projective spaces, quadrics, Grassmannians, etc; see Section 7B. Among other ingredients, its proof makes use of the language of G-dg categories and of the theory of G-equivariant noncommutative Chow motives. Intuitively speaking, Theorem 1.2 shows that the existence of a full exceptional collection of G-invariant objects “quasidetermines” the G-equivariant Chow motive $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$. The unique indeterminacy is the sequence $r_1, \dots, r_n$ of length n. Note that this indeterminacy cannot be refined. For example, the categories $\operatorname{perf}(\operatorname{Spec}(k) \amalg \operatorname{Spec}(k))$ and $\operatorname{perf}(\mathbb{P}^1)$ (equipped with the trivial G-action) admit full exceptional collections of length 2 but the corresponding G-equivariant Chow motives are distinct:

$$\mathfrak{h}^{\mathrm{G}}(\operatorname{Spec}(k) \amalg \operatorname{Spec}(k))_{\mathbb{Q}} \simeq \mathfrak{h}^{\mathrm{G}}(\operatorname{Spec}(k))_{\mathbb{Q}}^{\oplus 2} \not\simeq \mathfrak{h}^{\mathrm{G}}(\operatorname{Spec}(k))_{\mathbb{Q}} \oplus \mathbb{L} \simeq \mathfrak{h}^{\mathrm{G}}(\mathbb{P}^1)_{\mathbb{Q}}.$$

Corollary 1.5. *For every good* G*-cohomology theory* H_{G}^* (in the sense of Laterveer [1998, Definition 1.10]), *we have* $H_{\mathrm{G}}^i(X) = 0$ *if* i *is odd and* $\sum_i \dim H_{\mathrm{G}}^i(X) = n$.

Proof. It is proved in [Laterveer 1998, Proposition 1.12] that H_{G}^* factors through $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$. Using Theorem 1.2, we conclude $H_{\mathrm{G}}^*(X) \simeq H_{\mathrm{G}}^*(\mathbb{L})^{\otimes r_1} \oplus \cdots \oplus H_{\mathrm{G}}^*(\mathbb{L})^{\otimes r_n}$. The proof now follows from the facts that $\dim H_{\mathrm{G}}^2(\mathbb{L}) = 1$ and that $H_{\mathrm{G}}^i(\mathbb{L}) \simeq 0$ for $i \neq 2$. □

[1] In the particular case where G is the trivial group, Theorem 1.2 was proved in [Marcolli and Tabuada 2015, Theorem 1.1].

Remark 1.6. Corollary 1.5 implies that the length of a hypothetical full exceptional collection of G-invariant objects is equal to $\sum_i \dim H^i_{\mathrm{G}}(X)$. Moreover, if $H^i_{\mathrm{G}}(X) \not\simeq 0$ for some odd integer i, then such a full exceptional collection cannot exist.

Theorem 1.2 also shows that the G-equivariant Chow motive $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$ loses all the information concerning the G-action on X. In contrast, the G-equivariant noncommutative Chow motive $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ keeps track of some of the G-action! Concretely, as proved in Proposition 7.8, there exist (nontrivial) cohomology classes $[\alpha_1], \dots, [\alpha_n] \in H^2(\mathrm{G}, k^\times)$ such that

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_1} k) \oplus \cdots \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_n} k). \tag{1.7}$$

This implies, in particular, that all the invariants $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ can be computed in terms of twisted group algebras $\bigoplus_{i=1}^n E(k_{\alpha_i}[\mathrm{G}])$. Taking into account the decompositions (1.3) and (1.7), the G-equivariant Chow motive $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$ and the G-equivariant noncommutative Chow motive $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ should be considered as complementary. While the former keeps track of the Tate twists but not of the G-action, the latter keeps track of the G-action but not of the Tate twists.

Remark 1.8. In Section 7C we also discuss the case of full exceptional collections where the objects are not G-invariant but rather permuted by the G-action.

Notation. Throughout the article, k will denote a base field and G a finite group. We will write $1 \in \mathrm{G}$ for the unit element and $|\mathrm{G}|$ for the order of G. Except in Section 2, we will always assume that $\mathrm{char}(k) \nmid |\mathrm{G}|$. All schemes will be defined over $\mathrm{Spec}(k)$, and all adjunctions will be displayed vertically with the left adjoint on the left side, and the right adjoint on the right.

2. Preliminaries

In this section we recall the main notions concerning dg categories, (twisted) equivariant perfect complexes, and group actions on dg categories. This gives us the opportunity to fix some notation which will be used throughout the article.

2A. ***Dg categories.*** Let $(\mathcal{C}(k), \otimes, k)$ be the symmetric monoidal category of dg k-vector spaces; we use cohomological notation. A *dg category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$, and a *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller's ICM survey [2006]. Let $\mathrm{dgcat}(k)$ be the category of small dg categories.

Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects and $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. The categories $\mathrm{Z}^0(\mathcal{A})$ and $\mathrm{H}^0(\mathcal{A})$ have the same objects, and $\mathrm{Z}^0(\mathcal{A})(x, y) := Z^0(\mathcal{A}(x, y))$ and $\mathrm{H}^0(\mathcal{A})(x, y) := H^0(\mathcal{A}(x, y))$, where $Z^0(-)$ denotes the 0th-cycles functor and $H^0(-)$ the 0th-cohomology functor.

Recall from [Keller 2006, §2.3] the definition of the dg category of dg functors $\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$. Given dg functors $F, G : \mathcal{A} \to \mathcal{B}$, a natural transformation of dg functors $\epsilon : F \Rightarrow G$ corresponds to an element of $\mathrm{Z}^0(\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B}))(F, G)$. When ϵ is invertible, we call it a *natural isomorphism of dg functors*. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *dg equivalence* if there exists a dg functor $G : \mathcal{B} \to \mathcal{A}$ and natural isomorphisms of dg functors $F \circ G \Rightarrow \mathrm{id}$ and $\mathrm{id} \Rightarrow G \circ F$.

For a dg category $\mathcal{A}$, a (right) *dg $\mathcal{A}$-module* is a dg functor $M : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category of dg k-vector spaces. Let us write $\mathcal{C}(\mathcal{A})$ for the category of dg $\mathcal{A}$-modules and $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ for the dg category $\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}}, \mathcal{C}_{\mathrm{dg}}(k))$. By construction, we have $\mathrm{Z}^0(\mathcal{C}_{\mathrm{dg}}(\mathcal{A})) \simeq \mathcal{C}(\mathcal{A})$. The dg category $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ comes equipped with the Yoneda dg functor $\mathcal{A} \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$, $x \mapsto \mathcal{A}(-, x)$. Following [Keller 2006, §3.2], the *derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$* is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the (objectwise) quasi-isomorphisms. This category is triangulated and admits arbitrary direct sums. Let us write $\mathcal{D}_c(\mathcal{A})$ for the full subcategory of compact objects. In the same vein, let $\mathcal{C}_{c,\mathrm{dg}}(\mathcal{A})$ be the full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ consisting of those dg $\mathcal{A}$-modules which belong to $\mathcal{D}_c(\mathcal{A})$. By construction, we have $\mathrm{H}^0(\mathcal{C}_{c,\mathrm{dg}}(\mathcal{A})) \simeq \mathcal{D}_c(\mathcal{A})$.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if the restriction functor $\mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ is an equivalence of (triangulated) categories. An example is given by the Yoneda dg functor $\mathcal{A} \to \mathcal{C}_{c,\mathrm{dg}}(\mathcal{A})$. As proved in [Tabuada 2005, Théorème 5.3], the category $\mathrm{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let $\mathrm{Hmo}(k)$ be the associated homotopy category.

Given dg categories $\mathcal{A}$ and $\mathcal{B}$, let us write $\mathcal{A} \times \mathcal{B}$, $\mathcal{A} \amalg \mathcal{B}$, and $\mathcal{A} \otimes \mathcal{B}$ for their product, coproduct, and tensor product, respectively.

A *dg $\mathcal{A}$-$\mathcal{B}$-bimodule* is a dg functor $\mathrm{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$, or equivalently, a dg $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$-module. An example is the dg $\mathcal{A}$-$\mathcal{B}$-bimodule

$$ {}_F\mathrm{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k), \quad (x, z) \mapsto \mathcal{B}(z, F(x)) \tag{2.1} $$

associated to a dg functor $F : \mathcal{A} \to \mathcal{B}$. Let us write $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ for the full triangulated subcategory $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of those dg $\mathcal{A}$-$\mathcal{B}$-bimodules B such that for every $x \in \mathcal{A}$ the dg $\mathcal{B}$-module $\mathrm{B}(x, -)$ belongs to $\mathcal{D}_c(\mathcal{B})$. In the same vein, let $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ be the full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of those dg $\mathcal{A}$-$\mathcal{B}$-bimodules which belong to $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. By construction, $\mathrm{H}^0(\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})) \simeq \mathrm{rep}(\mathcal{A}, \mathcal{B})$.

Following [Kontsevich 1998; 2005; 2009; 2010], a dg category $\mathcal{A}$ is called *smooth* if the dg $\mathcal{A}$-$\mathcal{A}$-bimodule ${}_{\mathrm{id}}\mathrm{B}$ belongs to the triangulated category $\mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})$ and *proper* if $\sum_i \dim H^i \mathcal{A}(x, y) < \infty$ for any ordered pair of objects (x, y). Examples include the finite dimensional k-algebras of finite global dimension (when k is perfect) as well as the dg categories $\mathrm{perf}_{\mathrm{dg}}(X)$ associated to smooth proper schemes X. Given smooth proper dg categories $\mathcal{A}$ and $\mathcal{B}$, the associated dg categories $\mathcal{A} \times \mathcal{B}$, $\mathcal{A} \amalg \mathcal{B}$, and $\mathcal{A} \otimes \mathcal{B}$ are also smooth proper. Finally, let us write $\mathrm{dgcat}_{\mathrm{sp}}(k)$ for the full subcategory of $\mathrm{dgcat}(k)$ consisting of the smooth proper dg categories.

2B. ***(Twisted) equivariant perfect complexes.*** Let $\mathcal{E}$ be an abelian (or exact) category. Following [Keller 2006, §4.4], the *derived dg category* $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$ *of* $\mathcal{E}$ is defined as the dg quotient $\mathcal{C}_{\mathrm{dg}}(\mathcal{E})/\mathcal{A}c_{\mathrm{dg}}(\mathcal{E})$ of the dg category of complexes over $\mathcal{E}$ by its full dg subcategory of acyclic complexes. Given a quasicompact quasiseparated scheme X, we write $\mathrm{Mod}(X)$ for the Grothendieck category of $\mathcal{O}_X$-modules, $\mathcal{D}(X)$ for the derived category $\mathcal{D}(\mathrm{Mod}(X))$, and $\mathcal{D}_{\mathrm{dg}}(X)$ for the dg category $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$ with $\mathcal{E} := \mathrm{Mod}(X)$. In the same vein, we write $\mathrm{perf}(X)$ for the full triangulated subcategory, and $\mathrm{perf}_{\mathrm{dg}}(X)$ for the full dg subcategory, of perfect complexes.

Given a quasicompact quasiseparated G-scheme X, we write $\mathrm{Mod}^{\mathrm{G}}(X)$ for the Grothendieck category of G-equivariant $\mathcal{O}_X$-modules, $\mathcal{D}^{\mathrm{G}}(X)$ for the derived category $\mathcal{D}(\mathrm{Mod}^{\mathrm{G}}(X))$, and $\mathcal{D}^{\mathrm{G}}_{\mathrm{dg}}(X)$ for the dg category $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$ with $\mathcal{E} := \mathrm{Mod}^{\mathrm{G}}(X)$. In the same vein, we write $\mathrm{perf}^{\mathrm{G}}(X)$ for the full triangulated subcategory, and $\mathrm{perf}^{\mathrm{G}}_{\mathrm{dg}}(X)$ for the full dg subcategory, of G-equivariant perfect complexes.

Definition 2.2. A map $\alpha : \mathrm{G} \times \mathrm{G} \to k^{\times}$ is called a 2-*cocycle* if $\alpha(1, \sigma) = \alpha(\sigma, 1) = 1$ and $\alpha(\rho, \alpha)\alpha(\tau, \rho\sigma) = \alpha(\tau, \rho)\alpha(\tau\rho, \sigma)$ for every $\sigma, \rho, \tau \in \mathrm{G}$.

Given a quasicompact quasiseparated G-scheme X and a 2-*cocycle* α, we write $\mathrm{Mod}^{\mathrm{G},\alpha}(X)$ for the Grothendieck category of α-twisted G-equivariant $\mathcal{O}_X$-modules, $\mathcal{D}^{\mathrm{G},\alpha}(X)$ for the derived category $\mathcal{D}(\mathrm{Mod}^{\mathrm{G},\alpha}(X))$, and $\mathcal{D}^{\mathrm{G},\alpha}_{\mathrm{dg}}(X)$ for the dg category $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$ with $\mathcal{E} := \mathrm{Mod}^{\mathrm{G},\alpha}(X)$. In the same vein, we write $\mathrm{perf}^{\mathrm{G},\alpha}(X)$ for the full triangulated subcategory, and $\mathrm{perf}^{\mathrm{G},\alpha}_{\mathrm{dg}}(X)$ for the full dg subcategory, of G-equivariant perfect complexes.

2C. ***Group actions on dg categories.*** Following [Deligne 1997; Elagin 2014], we introduce the following notion:

Definition 2.3. A (left) G-*action on a dg category* $\mathcal{A}$ consists of the data

(i) a family of dg equivalences $\phi_\sigma : \mathcal{A} \to \mathcal{A}$ for $\sigma \in \mathrm{G}$, with $\phi_1 = \mathrm{id}$;

(ii) a family of natural isomorphisms of dg functors $\epsilon_{\rho,\sigma} : \phi_\rho \circ \phi_\sigma \Rightarrow \phi_{\rho\sigma}$ for $\sigma, \rho \in \mathrm{G}$, with $\epsilon_{1,\sigma} = \epsilon_{\sigma,1} = \mathrm{id}$, such that the equality $\epsilon_{\tau\rho,\sigma} \circ (\epsilon_{\tau,\rho} \circ \phi_\sigma) = \epsilon_{\tau,\rho\sigma} \circ (\phi_\tau \circ \epsilon_{\rho,\sigma})$ holds for every $\sigma, \rho, \tau \in \mathrm{G}$.

Throughout the article, a dg category $\mathcal{A}$ equipped with a G-action will be denoted by $\mathrm{G} \circlearrowright \mathcal{A}$ and will be called a G-*dg category*.

Example 2.4 (G-schemes). Given a quasicompact quasiseparated G-scheme X, the dg category $\mathrm{perf}_{\mathrm{dg}}(X)$ inherits a G-action induced by the pull-back dg equivalences $\phi_\sigma := \sigma^*$; consult [Elagin 2014; Sosna 2012] for details.

Example 2.5 (line bundles). Let X be a quasicompact quasiseparated scheme. In the case where G can be realized as a subgroup of the Picard group $\mathrm{Pic}(X)$, the dg category $\mathrm{perf}_{\mathrm{dg}}(X)$ inherits a G-action induced by the dg equivalences $\phi_\sigma := - \otimes_{\mathcal{O}_X} \mathcal{L}_\sigma$, where $\mathcal{L}_\sigma$ stands for the invertible line bundle associated to $\sigma \in \mathrm{G}$; consult [Elagin 2014; Sosna 2012] for details.

Example 2.6 (G-algebras). Given a G-action on a (dg) algebra A, the associated dg category with a single object inherits a G-action with $\epsilon_{\rho,\sigma} := \mathrm{id}$.

Example 2.7 (2-cocycles). Given a 2-cocycle $\alpha : \mathrm{G} \times \mathrm{G} \to k^\times$, the dg category k inherits a G-action given by $\phi_\sigma := \mathrm{id}$ and $\epsilon_{\rho,\sigma} := \alpha(\rho, \sigma)$. We will denote this G-dg category by $\mathrm{G} \circlearrowright_\alpha k$. Note that these are all the possible G-actions.

Remark 2.8. Given a G-dg category $\mathrm{G} \circlearrowright \mathcal{A}$, $\mathcal{A}^{\mathrm{op}}$ inherits a G-action given by the dg equivalences ϕ_σ and by the natural isomorphisms of dg functors $\epsilon_{\rho,\sigma}^{-1}$.

Given G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$, the product $\mathcal{A} \times \mathcal{B}$ inherits a G-action given by the dg equivalences $\phi_\sigma \times \phi_\sigma$ and by the natural isomorphisms of dg equivalences $\epsilon_{\rho,\sigma} \times \epsilon_{\rho,\sigma}$, and likewise the tensor product $\mathcal{A} \otimes \mathcal{B}$ inherits a G-action by dg equivalences $\phi_\sigma \otimes \phi_\sigma$ and natural isomorphisms of dg equivalences $\epsilon_{\rho,\sigma} \otimes \epsilon_{\rho,\sigma}$. In the same vein, the dg category of dg functors $\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ inherits a G-action given by the dg equivalences $F \mapsto \phi_\sigma \circ F \circ \phi_{\sigma^{-1}}$ and by the natural isomorphisms of dg functors induced from $\epsilon_{\sigma^{-1},\rho^{-1}}$ and $\epsilon_{\rho,\sigma}$.

Let $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$ be two G-dg categories, and $\mathcal{C}_{\mathrm{dg}}(k)$ the dg category of dg k-vector spaces equipped with the trivial G-action. Thanks to the above considerations, $\mathcal{C}_{\mathrm{dg}}(\mathcal{A}) := \mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}}, \mathcal{C}_{\mathrm{dg}}(k))$ inherits a G-action, which restricts to $\mathcal{C}_{c,\mathrm{dg}}(\mathcal{A})$. Similarly, the dg category of dg $\mathcal{A}$-$\mathcal{B}$-bimodules $\mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}) := \mathrm{Fun}_{\mathrm{dg}}(\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}, \mathcal{C}_{\mathrm{dg}}(k))$ inherits a G-action, which restricts to $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$.

Definition 2.9. A G-*equivariant dg functor* $\mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$ consists of the data

(i) a dg functor $F : \mathcal{A} \to \mathcal{B}$;

(ii) a family of natural isomorphisms of dg functors $\eta_\sigma : F \circ \phi_\sigma \Rightarrow \phi_\sigma \circ F$, for $\sigma \in \mathrm{G}$, such that $\eta_{\rho\sigma} \circ (F \circ \epsilon_{\rho,\sigma}) = (\epsilon_{\rho,\sigma} \circ F) \circ (\phi_\rho \circ \eta_\sigma) \circ (\eta_\rho \circ \phi_\sigma)$ for every $\sigma, \rho \in \mathrm{G}$.

A G-equivariant dg functor with a Morita equivalence F is called a G-*equivariant Morita equivalence*. For example, given a small G-dg category $\mathrm{G} \circlearrowright \mathcal{A}$, the Yoneda dg functor $\mathcal{A} \to \mathcal{C}_{c,\mathrm{dg}}(\mathcal{A})$, $x \mapsto \mathcal{A}(-, x)$, is a G-equivariant Morita equivalence.

Let us denote by G-dgcat(k) the category whose objects are the small G-dg categories and whose morphisms are the G-equivariant dg functors. Given G-equivariant dg functors $F : \mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$ and $G : \mathrm{G} \circlearrowright \mathcal{B} \to \mathrm{G} \circlearrowright \mathcal{C}$, their composition is defined as $(G \circ F, (\eta_\sigma \circ F) \circ (G \circ \eta_\sigma))$. The category G-dgcat$(k)$ carries a symmetric monoidal structure given by $(\mathrm{G} \circlearrowright \mathcal{A}) \otimes (\mathrm{G} \circlearrowright \mathcal{B}) := \mathrm{G} \circlearrowright (\mathcal{A} \otimes \mathcal{B})$. This monoidal structure is closed, with internal-Homs given by $\mathrm{G} \circlearrowright \mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$.

Equivariant objects. Let $\mathrm{G} \circlearrowright \mathcal{A}$ be a G-dg category. A G-*equivariant object in* $\mathrm{G} \circlearrowright \mathcal{A}$ consists of an object $x \in \mathcal{A}$ and a family of closed degree zero isomorphisms $\theta_\sigma : x \to \phi_\sigma(x)$ for $\sigma \in \mathrm{G}$, with $\theta_1 = \mathrm{id}$, such that the compositions

$$x \xrightarrow{\theta_\rho} \phi_\rho(x) \xrightarrow{\phi_\rho(\theta_\sigma)} \phi_\rho(\phi_\sigma(x)) \xrightarrow{\epsilon_{\rho,\sigma}(x)} \phi_{\rho\sigma}(x)$$

are equal to $\theta_{\rho\sigma} : x \to \phi_{\rho\sigma}(x)$ for every $\sigma, \rho \in \mathrm{G}$. A *morphism of* G-*equivariant objects* $(x, \theta_\sigma) \to (y, \theta_\sigma)$ is an element f of the dg k-vector space $\mathcal{A}(x, y)$ such that $\theta_\sigma \circ f = \phi_\sigma(f) \circ \theta_\sigma$ for every $\sigma \in \mathrm{G}$. Let us write $\mathcal{A}^{\mathrm{G}}$ for the dg category of G-equivariant objects in $\mathrm{G} \circlearrowright \mathcal{A}$. From a topological viewpoint, the dg category $\mathcal{A}^{\mathrm{G}}$ may be understood as the "homotopy fixed points" of the G-action on $\mathcal{A}$.

Example 2.10 (equivariant perfect complexes). Let $\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)$ be equipped with the G-action of Example 2.4. When $\mathrm{char}(k) \nmid |G|$, Elagin [2011, Theorem 9.6; 2014, Theorem 1.1] proved that $\mathrm{perf}_{\mathrm{dg}}(X)^{\mathrm{G}}$ is Morita equivalent to the dg category $\mathrm{perf}^{\mathrm{G}}_{\mathrm{dg}}(X)$; see Section 2B.

Example 2.11 (covering spaces). Let $\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)$ be as in Example 2.5. Consider the relative spectrum $Y := \mathrm{Spec}_X\big(\bigoplus_{\sigma \in \mathrm{G}} \mathcal{L}_\sigma^{-1}\big)$, which is a nonramified $|G|$-fold cover of X. When $\mathrm{char}(k) \nmid |G|$, Elagin [2014, Theorem 1.2] proved that $\mathrm{perf}_{\mathrm{dg}}(X)^{\mathrm{G}}$ is Morita equivalent to $\mathrm{perf}_{\mathrm{dg}}(Y)$.

Example 2.12 (semidirect product algebras). Let $\mathrm{G} \circlearrowright A$ be as in Example 2.6. As mentioned in Remark 2.8, the dg category $\mathcal{C}_{c,\mathrm{dg}}(A)$ inherits a G-action. When $\mathrm{char}(k) \nmid |G|$, the dg category $\mathcal{C}_{c,\mathrm{dg}}(A)^{\mathrm{G}}$ is Morita equivalent to the semidirect product (dg) algebra $A \rtimes \mathrm{G}$.

Example 2.13 (twisted group algebras). Let $\mathrm{G} \circlearrowright_\alpha k$ be as in Example 2.7. Similarly to Example 2.12, when $\mathrm{char}(k) \nmid |G|$, the dg category $\mathcal{C}_{c,\mathrm{dg}}(k)^{\mathrm{G}}$ is Morita equivalent to the twisted group algebra $k_\alpha[\mathrm{G}]$.

Remark 2.14 (G-equivariant dg functors). Let $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$ be two dg categories. The assignment $(F, \eta_\sigma) \mapsto (F, (\eta_\sigma \circ \phi_{\sigma^{-1}}) \circ (F \circ \epsilon^{-1}_{\sigma,\sigma^{-1}}))$ establishes a bijection between the set of G-equivariant dg functors $\mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$ and the set of G-equivariant objects in $\mathrm{G} \circlearrowright \mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$; see Remark 2.8. Its inverse is given by the assignment $(F, \theta_\sigma) \mapsto (F, (\phi_\sigma \circ F \circ \epsilon_{\sigma^{-1},\sigma}) \circ (\theta_\sigma \circ \phi_\sigma))$.

Given a G-equivariant dg functor $F : \mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$, the assignment $(x, \theta_\sigma) \mapsto (F(x), \eta_\sigma \circ F(\theta_\sigma))$ yields a dg functor $F^{\mathrm{G}} : \mathcal{A}^{\mathrm{G}} \to \mathcal{B}^{\mathrm{G}}$. We hence obtain a functor

$$\mathrm{G}\text{-dgcat}(k) \to \mathrm{dgcat}(k), \quad \mathrm{G} \circlearrowright \mathcal{A} \mapsto \mathcal{A}^{\mathrm{G}}. \tag{2.15}$$

Twisted equivariant objects. Let $\alpha : \mathrm{G} \times \mathrm{G} \to k^\times$ be a 2-cocycle and $\mathrm{G} \circlearrowright \mathcal{A}$ a G-dg category. An α-*twisted* G-*equivariant object in* $\mathrm{G} \circlearrowright \mathcal{A}$ consists of an object $x \in \mathcal{A}$ and a family of closed degree zero isomorphisms $\theta_\sigma : x \to \phi_\sigma(x)$ for $\sigma \in \mathrm{G}$, with $\theta_1 = \mathrm{id}$, such that the compositions

$$x \xrightarrow{\theta_\rho} \phi_\rho(x) \xrightarrow{\phi_\rho(\theta_\sigma)} \phi_\rho(\phi_\sigma(x)) \xrightarrow{\epsilon_{\rho,\sigma}(x)} \phi_{\rho\sigma}(x)$$

are equal to $\alpha(\rho,\sigma)\theta_{\rho\sigma} : x \to \phi_{\rho\sigma}(x)$ for every $\sigma, \rho \in \mathrm{G}$. A *morphism of α-twisted* G*-equivariant objects* $(x,\theta_\sigma) \to (y,\theta_\sigma)$ is an element f of the dg k-vector space $\mathcal{A}(x,y)$ such that $\theta_\sigma \circ f = \phi_\sigma(f) \circ \theta_\sigma$ for every $\sigma \in \mathrm{G}$. Let us write $\mathcal{A}^{\mathrm{G},\alpha}$ for the dg category of α-twisted G-equivariant objects in $\mathrm{G} \circlearrowright \mathcal{A}$. Note that $\mathcal{A}^{\mathrm{G},\alpha}$ identifies with the dg category of G-equivariant objects in $(\mathrm{G} \circlearrowright \mathcal{A}) \otimes (\mathrm{G} \circlearrowright_{\alpha^{-1}} k)$.

Example 2.16 (twisted equivariant perfect complexes). Let $\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)$ be as in Example 2.4. Similarly to Example 2.10, $\mathrm{perf}_{\mathrm{dg}}(X)^{\mathrm{G},\alpha}$ is Morita equivalent to the dg category of α-twisted G-equivariant perfect complexes $\mathrm{perf}_{\mathrm{dg}}^{\mathrm{G},\alpha}(X)$.

3. Equivariant noncommutative motives

In this section we introduce the category of equivariant noncommutative Chow motives. We start by recalling its nonequivariant predecessor.

3A. ***Noncommutative Chow motives.*** Recall from Section 2A that $\mathrm{Hmo}(k)$ is the localization of $\mathrm{dgcat}(k)$ at the class of Morita equivalences. As proved in [Tabuada 2005, Corollaire 5.10], there is a canonical bijection between $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A},\mathcal{B})$ and the set of isomorphism classes of the triangulated category $\mathrm{rep}(\mathcal{A},\mathcal{B})$. Under this bijection, the composition law of $\mathrm{Hmo}(k)$ is induced by the triangulated bifunctors

$$\mathrm{rep}(\mathcal{A},\mathcal{B}) \times \mathrm{rep}(\mathcal{B},\mathcal{C}) \to \mathrm{rep}(\mathcal{A},\mathcal{C}), \quad (\mathrm{B},\mathrm{B}') \mapsto \mathrm{B} \otimes_{\mathcal{B}} \mathrm{B}' \tag{3.1}$$

and the localization functor from $\mathrm{dgcat}(k)$ to $\mathrm{Hmo}(k)$ is given by

$$\mathrm{dgcat}(k) \to \mathrm{Hmo}(k), \quad \mathcal{A} \mapsto \mathcal{A}, \quad (\mathcal{A} \xrightarrow{F} \mathcal{B}) \mapsto {}_F\mathrm{B}. \tag{3.2}$$

The *additivization* of $\mathrm{Hmo}(k)$ is the additive category $\mathrm{Hmo}_0(k)$ with the same objects and with morphisms $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A},\mathcal{B})$ given by $K_0\,\mathrm{rep}(\mathcal{A},\mathcal{B})$. The composition law is induced by the triangulated bifunctors (3.1). By construction, $\mathrm{Hmo}_0(k)$ comes equipped with the functor

$$\mathrm{Hmo}(k) \to \mathrm{Hmo}_0(k), \quad \mathcal{A} \mapsto \mathcal{A}, \quad \mathrm{B} \mapsto [\mathrm{B}]. \tag{3.3}$$

Let us denote by $U : \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ the composition (3.3)$\circ$(3.2). As proved in [Tabuada 2005, Lemme 6.1], the category $\mathrm{Hmo}_0(k)$ carries a symmetric monoidal structure induced by the tensor product of dg categories and by the triangulated bifunctors

$$\mathrm{rep}(\mathcal{A},\mathcal{B}) \times \mathrm{rep}(\mathcal{C},\mathcal{D}) \to \mathrm{rep}(\mathcal{A}\otimes\mathcal{C},\mathcal{B}\otimes\mathcal{D}), \quad (\mathrm{B},\mathrm{B}') \mapsto \mathrm{B}\otimes\mathrm{B}'.$$

By construction, the functor U is symmetric monoidal.

The category $\mathrm{NChow}(k)$ of *noncommutative Chow motives*[2] is defined as the idempotent completion of the full subcategory of $\mathrm{Hmo}_0(k)$ consisting of the objects $U(\mathcal{A})$ with $\mathcal{A}$ a smooth proper dg category. The category $\mathrm{NChow}(k)$ is additive, idempotent complete, and rigid symmetric monoidal.

3B. ***Equivariant noncommutative Chow motives.*** Let $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$ be two small G-dg categories. As mentioned in Remark 2.8, the dg category $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ inherits a G-action. As a consequence, we obtain an induced G-action on the triangulated category

$$\mathrm{H}^0(\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})) \simeq \mathrm{rep}(\mathcal{A}, \mathcal{B}).$$

Due to [Elagin 2014, Theorem 8.7], the category of G-equivariant objects $\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}}$ is also triangulated.

Given small G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$, $\mathrm{G} \circlearrowright \mathcal{B}$, and $\mathrm{G} \circlearrowright \mathcal{C}$, consider the G-equivariant dg functor

$$\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B}) \times \mathrm{rep}_{\mathrm{dg}}(\mathcal{B}, \mathcal{C}) \to \mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{C}), \quad (\mathrm{B}, \mathrm{B}') \mapsto \mathrm{B} \otimes_{\mathcal{B}} \mathrm{B}'$$

(G acts diagonally on the left-hand side). By first applying $\mathrm{H}^0(-)$ and then $(-)^{\mathrm{G}}$, we obtain an induced triangulated bifunctor

$$\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \times \mathrm{rep}(\mathcal{B}, \mathcal{C})^{\mathrm{G}} \to \mathrm{rep}(\mathcal{A}, \mathcal{C})^{\mathrm{G}}. \tag{3.4}$$

Let $\mathrm{Hmo}^{\mathrm{G}}(k)$ be the category with the same objects as $\mathrm{G}\text{-dgcat}(k)$ and with morphisms $\mathrm{Hom}_{\mathrm{Hmo}^{\mathrm{G}}(k)}(\mathrm{G} \circlearrowright \mathcal{A}, \mathrm{G} \circlearrowright \mathcal{B})$ given by the set of isomorphism classes of the category $\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}}$. The composition law is induced by the triangulated bifunctors (3.4). Thanks to Remark 2.14, we have the functor

$$\mathrm{G}\text{-dgcat}(k) \to \mathrm{Hmo}^{\mathrm{G}}(k), \quad \mathrm{G} \circlearrowright \mathcal{A} \mapsto \mathrm{G} \circlearrowright \mathcal{A}, \quad (\mathrm{G} \circlearrowright \mathcal{A} \xrightarrow{F} \mathrm{G} \circlearrowright \mathcal{B}) \mapsto {}_F\mathrm{B}. \tag{3.5}$$

Lemma 3.6. *The functor* (3.5) *inverts* G*-equivariant Morita equivalences.*

Proof. Let $\mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$ be a G-equivariant Morita equivalence. Thanks to the Yoneda lemma, it suffices to show that for every object $\mathrm{G} \circlearrowright \mathcal{C}$ the homomorphism

$$\mathrm{Hom}_{\mathrm{Hmo}^{\mathrm{G}}(k)}(\mathrm{G} \circlearrowright \mathcal{C}, \mathrm{G} \circlearrowright \mathcal{A}) \to \mathrm{Hom}_{\mathrm{Hmo}^{\mathrm{G}}(k)}(\mathrm{G} \circlearrowright \mathcal{C}, \mathrm{G} \circlearrowright \mathcal{B}) \tag{3.7}$$

is invertible. Since $\mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$ is a G-equivariant Morita equivalence, we have an induced G-equivariant equivalence of categories $\mathrm{rep}(\mathcal{C}, \mathcal{A}) \to \mathrm{rep}(\mathcal{C}, \mathcal{B})$, and consequently an equivalence of categories $\mathrm{rep}(\mathcal{C}, \mathcal{A})^{\mathrm{G}} \to \mathrm{rep}(\mathcal{C}, \mathcal{B})^{\mathrm{G}}$. □

The *additivization* of $\mathrm{Hmo}^{\mathrm{G}}(k)$ is the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ with the same objects and with abelian groups of morphisms $\mathrm{Hom}_{\mathrm{Hmo}_0^{\mathrm{G}}(k)}(\mathrm{G} \circlearrowright \mathcal{A}, \mathrm{G} \circlearrowright \mathcal{B})$ given by $K_0\,\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}}$, where $K_0\,\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}}$ stands for the Grothendieck group of the tri-

[2] For further information concerning noncommutative (Chow) motives, consult [Tabuada 2015].

angulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}}$. The composition law is induced by the triangulated bifunctors (3.4). By construction, $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ comes equipped with the functor

$$\mathrm{Hmo}^{\mathrm{G}}(k) \to \mathrm{Hmo}_0^{\mathrm{G}}(k), \quad \mathrm{G} \circlearrowright \mathcal{A} \mapsto \mathrm{G} \circlearrowright \mathcal{A}, \quad \mathrm{B} \mapsto [\mathrm{B}]. \tag{3.8}$$

Let us denote by $U^{\mathrm{G}} : \mathrm{G}\text{-dgcat}(k) \to \mathrm{Hmo}_0^{\mathrm{G}}(k)$ the composition (3.8) $\circ$ (3.5).

Given small G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$, $\mathrm{G} \circlearrowright \mathcal{B}$, $\mathrm{G} \circlearrowright \mathcal{C}$, and $\mathrm{G} \circlearrowright \mathcal{D}$, consider the G-equivariant dg functor

$$\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B}) \times \mathrm{rep}_{\mathrm{dg}}(\mathcal{C}, \mathcal{D}) \to \mathrm{rep}_{\mathrm{dg}}(\mathcal{A} \otimes \mathcal{C}, \mathcal{B} \otimes \mathcal{D}), \quad (\mathrm{B}, \mathrm{B}') \mapsto \mathrm{B} \otimes \mathrm{B}'$$

(G acts diagonally on the left-hand side). By first applying $\mathrm{H}^0(-)$ and then $(-)^{\mathrm{G}}$, we obtain an induced triangulated bifunctor

$$\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \times \mathrm{rep}(\mathcal{C}, \mathcal{D})^{\mathrm{G}} \to \mathrm{rep}(\mathcal{A} \otimes \mathcal{C}, \mathcal{B} \otimes \mathcal{D})^{\mathrm{G}}. \tag{3.9}$$

The assignment $(\mathrm{G} \circlearrowright \mathcal{A}, \mathrm{G} \circlearrowright \mathcal{B}) \mapsto \mathrm{G} \circlearrowright (\mathcal{A} \otimes \mathcal{B})$, combined with the triangulated bifunctors (3.9), gives rise to a symmetric monoidal structure on $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ with $\otimes$-unit $U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)$. By construction, the functor U^{G} is symmetric monoidal.

Proposition 3.10. *The category* $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ *is additive. Moreover, we have*

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{B}) \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A} \times \mathcal{B})) \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A} \amalg \mathcal{B})) \tag{3.11}$$

for any two small G-*dg categories* $\mathrm{G} \circlearrowright \mathcal{A}$ *and* $\mathrm{G} \circlearrowright \mathcal{B}$.

Proof. By construction, the morphism sets of $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ are abelian groups and the composition law is bilinear. Hence, it suffices to show the isomorphisms (3.11), which imply in particular that the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ admits direct sums. Given a small G-dg category $\mathrm{G} \circlearrowright \mathcal{C}$, we have equivalences of categories

$$\mathrm{rep}(\mathcal{C}, \mathcal{A} \times \mathcal{B})^{\mathrm{G}} \simeq \mathrm{rep}(\mathcal{C}, \mathcal{A})^{\mathrm{G}} \times \mathrm{rep}(\mathcal{C}, \mathcal{B})^{\mathrm{G}},$$
$$\mathrm{rep}(\mathcal{A} \amalg \mathcal{B}, \mathcal{C})^{\mathrm{G}} \simeq \mathrm{rep}(\mathcal{A}, \mathcal{C})^{\mathrm{G}} \times \mathrm{rep}(\mathcal{B}, \mathcal{C})^{\mathrm{G}}.$$

By passing to the Grothendieck group K_0, we conclude that $U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A} \times \mathcal{B}))$ is the product, and $U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A} \amalg \mathcal{B}))$ the coproduct, in $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ of $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})$ with $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{B})$. Using the fact that the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ is $\mathbb{Z}$-linear, we obtain finally the isomorphisms (3.11). □

Definition 3.12. The category $\mathrm{NChow}^{\mathrm{G}}(k)$ of G-*equivariant noncommutative Chow motives* is the idempotent completion of the full subcategory of $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ consisting of the objects $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})$ with $\mathcal{A}$ a smooth proper dg category.

Since the smooth proper dg categories are stable under (co)products, it follows from the isomorphisms (3.11) that the category $\mathrm{NChow}^{\mathrm{G}}(k)$ is also additive.

Proposition 3.13. *The symmetric monoidal category* $\mathrm{NChow}^{\mathrm{G}}(k)$ *is rigid.*

Proof. By construction of $\mathrm{NChow}^{\mathrm{G}}(k)$, it suffices to show that $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})$, with $\mathcal{A}$ a smooth proper dg category $\mathcal{A}$, is strongly dualizable. Take for the dual of $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})$ the object $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}^{\mathrm{op}})$ (see Remark 2.8). The dg $\mathcal{A}$-$\mathcal{A}$-bimodule

$$_{\mathrm{id}}\mathrm{B} : \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k), \quad (x, y) \mapsto \mathcal{A}(y, x) \tag{3.14}$$

associated to the identity dg functor $\mathrm{id} : \mathcal{A} \to \mathcal{A}$ is canonically a G-equivariant object. Moreover, since $\mathcal{A}$ is smooth proper, the dg $\mathcal{A}$-$\mathcal{A}$-bimodule (3.14) belongs to the triangulated categories $\mathrm{rep}(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}, k)^{\mathrm{G}}$ and $\mathrm{rep}(k, \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})^{\mathrm{G}}$. Let us then take for the evaluation morphism the Grothendieck class of (3.14) in

$$\mathrm{Hom}_{\mathrm{NChow}^{\mathrm{G}}(k)}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})), U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)) \simeq K_0 \, \mathrm{rep}(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}, k)^{\mathrm{G}},$$

and for the coevaluation morphism the Grothendieck class of (3.14) in

$$\mathrm{Hom}_{\mathrm{NChow}^{\mathrm{G}}(k)}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k), U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}))) \simeq K_0 \, \mathrm{rep}(k, \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})^{\mathrm{G}}.$$

This data satisfies the axioms of a strongly dualizable object. □

Proposition 3.15. *For every cohomology class* $[\alpha] \in H^2(\mathrm{G}, k^\times)$, *the ring of endomorphisms*

$$\mathrm{End}_{\mathrm{NChow}^{\mathrm{G}}(k)}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)) \tag{3.16}$$

(*where multiplication is given by composition*) *is isomorphic to the representation ring*[3] $R(\mathrm{G})$ *of the group* G.

Proof. By construction of $\mathrm{NChow}^{\mathrm{G}}(k)$, we have canonical ring identifications

$$\mathrm{End}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)) = K_0(\mathrm{rep}(k, k)^{\mathrm{G}, \alpha\alpha^{-1}}) \simeq K_0 \, \mathrm{rep}(k, k)^{\mathrm{G}} = \mathrm{End}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)).$$

Hence, it suffices to prove the particular case $\alpha = 1$. As mentioned in Example 2.10, the category $\mathrm{rep}(k, k)^{\mathrm{G}} \simeq \mathcal{D}_c(k)^{\mathrm{G}} \simeq \mathrm{perf}(\mathrm{Spec}(k))^{\mathrm{G}}$ is equivalent to $\mathrm{perf}^{\mathrm{G}}(\mathrm{Spec}(k))$. This implies that the abelian group (3.16), with $\alpha = 1$, is isomorphic to the G-equivariant Grothendieck group $K_0(\mathrm{perf}^{\mathrm{G}}(\mathrm{Spec}(k)))$ of $\mathrm{Spec}(k)$. In what concerns the ring structure, the Eckmann–Hilton argument, combined with the fact that $U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)$ is the $\otimes$-unit of $\mathrm{NChow}^{\mathrm{G}}(k)$, implies that the multiplication on (3.16) given by composition agrees with the multiplication on (3.16) induced by the symmetric monoidal structure on $\mathrm{perf}^{\mathrm{G}}(\mathrm{Spec}(k))$. The proof follows now from the definition of $R(\mathrm{G})$ as the G-equivariant Grothendieck ring of $\mathrm{Spec}(k)$. □

Proposition 3.15 gives rise automatically to the following enhancement:

Corollary 3.17. *The category* $\mathrm{NChow}^{\mathrm{G}}(k)$ (*and* $\mathrm{Hmo}_0^{\mathrm{G}}(k)$) *is* $R(\mathrm{G})$-*linear.*

[3] Consult Serre's book [1977] for a detailed study of the representation ring.

3C. *Coefficients.* Given a commutative ring R, let $\mathrm{Hmo}_0^{\mathrm{G}}(k)_R$ be the R-linear additive category obtained from $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ by tensoring each abelian group of morphisms with R. By construction, $\mathrm{Hmo}_0^{\mathrm{G}}(k)_R$ inherits a symmetric monoidal structure making the functor $(-)_R : \mathrm{Hmo}_0^{\mathrm{G}}(k) \to \mathrm{Hmo}_0^{\mathrm{G}}(k)_R$ symmetric monoidal. The category $\mathrm{NChow}^{\mathrm{G}}(k)_R$ of G-*equivariant noncommutative Chow motives with R-coefficients* is the idempotent completion of the subcategory of $\mathrm{Hmo}_0^{\mathrm{G}}(k)_R$ consisting of the objects $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})_R$ with $\mathcal{A}$ a smooth proper dg category.

4. Equivariant and enhanced additive invariants

Given a small dg category $\mathcal{A}$, let $T(\mathcal{A})$ be the dg category of pairs (i, x), where $i \in \{1, 2\}$ and $x \in \mathcal{A}$. The dg k-vector spaces $T(\mathcal{A})((i, x), (j, y))$ are given by $\mathcal{A}(x, y)$ if $j \geq i$ and are zero otherwise. Note that we have two inclusion dg functors $\iota_1, \iota_2 : \mathcal{A} \to T(\mathcal{A})$. A functor $E : \mathrm{dgcat}(k) \to \mathrm{D}$, with values in an additive category, is called an *additive invariant* if it satisfies the following two conditions:

(i) it sends Morita equivalences to isomorphisms;

(ii) given a small dg category $\mathcal{A}$, the dg functors ι_1, ι_2 induce an isomorphism[4]

$$[E(\iota_1)\ E(\iota_2)] : E(\mathcal{A}) \oplus E(\mathcal{A}) \to E(T(\mathcal{A})).$$

Examples of additive invariants include algebraic K-theory, Hochschild homology HH, cyclic homology HC, periodic cyclic homology HP, negative cyclic homology HN, etc.; consult [Tabuada 2015, §2.2] for details. As proved in [Tabuada 2005, Théorèmes 5.3 et 6.3], the functor $U : \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ is the *universal additive invariant*, i.e., given any additive category D we have an induced equivalence of categories

$$U^* : \mathrm{Fun}_{\mathrm{additive}}(\mathrm{Hmo}_0(k), \mathrm{D}) \to \mathrm{Fun}_{\mathrm{add}}(\mathrm{dgcat}(k), \mathrm{D}), \tag{4.1}$$

where the left-hand side denotes the category of additive functors and the right-hand side the category of additive invariants.

Remark 4.2 (additive invariants of twisted group algebras). Let $\alpha : \mathrm{G} \times \mathrm{G} \to k^\times$ be a 2-cocycle and $k_\alpha[\mathrm{G}]$ the associated twisted group algebra. Recall that a conjugacy class $\langle\sigma\rangle$ of G is called *α-regular* if $\alpha(\sigma, \rho) = \alpha(\rho, \sigma)$ for every element ρ of the centralizer $C_{\mathrm{G}}(\sigma)$. Thanks to the (generalized) Maschke theorem, the algebra $k_\alpha[\mathrm{G}]$ is semisimple. Moreover, the number of simple $k_\alpha[\mathrm{G}]$-modules is equal to the number $|\langle\mathrm{G}\rangle_\alpha|$ of α-regular conjugacy classes of G. Let $E : \mathrm{dgcat}(k) \to \mathrm{D}$ be an additive invariant. Making use of [Tabuada and Van den Bergh 2015b, Corollary 3.20 and Remark 3.21], we obtain the following computations:

[4]Condition (ii) can be equivalently formulated in terms of semiorthogonal decompositions in the sense of Bondal and Orlov [1995]; consult [Tabuada 2005, Théorème 6.3(4)] for details.

(i) We have $E(k_\alpha[\mathrm{G}]) \simeq \bigoplus_{i=1}^{|\langle \mathrm{G}\rangle_\alpha|} E(D_i)$, where $D_i := \mathrm{End}_{k_\alpha[\mathrm{G}]}(S_i)$ is the division algebra associated to the simple (right) $k_\alpha[\mathrm{G}]$-module S_i.

(ii) When D is $\mathbb{Q}$-linear, we have $E(k_\alpha[\mathrm{G}]) \simeq \bigoplus_{i=1}^{|\langle \mathrm{G}\rangle_\alpha|} E(l_i)$ where l_i (a finite field extension of k) is the center of D_i.

(iii) When k is algebraically closed, we have $E(k_\alpha[\mathrm{G}]) \simeq E(k)^{\oplus|\langle \mathrm{G}\rangle_\alpha|}$.

4A. ***Equivariant additive invariants.*** Given an additive invariant E, the associated G-*equivariant additive invariant* is defined as the composition

$$E^{\mathrm{G}} : \mathrm{G}\text{-dgcat}(k) \xrightarrow{(2.15)} \mathrm{dgcat}(k) \xrightarrow{E} \mathrm{D}. \tag{4.3}$$

From a topological viewpoint, $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})$ may be understood as the value of E at the "homotopy fixed points" of the G-action on $\mathcal{A}$. Here are some examples:

Example 4.4. (i) Let $\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)$ be as in Example 2.4. Due to Example 2.10, we have an identification between $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ and $E(\mathrm{perf}^{\mathrm{G}}_{\mathrm{dg}}(X))$.

(ii) Let $\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)$ be as in Example 2.5. Due to Example 2.11, we have an identification between $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ and $E(\mathrm{perf}_{\mathrm{dg}}(Y))$.

(iii) Let $\mathrm{G} \circlearrowright A$ be as in Example 2.6. Due to Example 2.12, we have an identification between $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{C}_{c,\mathrm{dg}}(A))$ and $E(A \rtimes \mathrm{G})$.

(iv) Let $\mathrm{G} \circlearrowright_\alpha k$ be as in Example 2.7. Due to Example 2.13, we have an identification between $E^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha \mathcal{C}_{c,\mathrm{dg}}(k))$ and $E(k_\alpha[\mathrm{G}])$.

Example 4.5 (equivariant algebraic K-theory). The composed functor (4.3) with $E := K$ is called G-*equivariant algebraic K-theory*. Recall that a quasicompact quasiseparated G-scheme X has the *resolution property* if every G-equivariant coherent $\mathcal{O}_X$-module is a quotient of a G-bundle. For example, the existence of an ample family of line G-bundles implies the resolution property. As explained in [Krishna and Ravi 2015, Corollary 3.9], whenever X has the resolution property, $K^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \simeq K(\mathrm{perf}^{\mathrm{G}}_{\mathrm{dg}}(X))$ agrees with the G-equivariant algebraic K-theory $K^{\mathrm{G}}(X)$ of X in the sense of Thomason [1987, §1.4].

Example 4.6 (equivariant Hochschild, cyclic, periodic, and negative homology). The composed functors (4.3) with $E := HH$, HC, HP, and HN, are called G-*equivariant Hochschild, cyclic, periodic, and negative homology*, respectively. Consult [Feĭgin and Tsygan 1987a, §A.6; 1987b, §4] for the computations of these G-equivariant additive invariants at the small G-dg categories $\mathrm{G} \circlearrowright \mathcal{C}_{c,\mathrm{dg}}(A)$; see Example 4.4(iii).

Proposition 4.7. *Given a* G*-equivariant additive invariant* E^{G}, *there exists an additive functor* $\overline{E^{\mathrm{G}}} : \mathrm{Hmo}^{\mathrm{G}}_0(k) \to \mathrm{D}$ *such that* $\overline{E^{\mathrm{G}}} \circ U^{\mathrm{G}} \simeq E^{\mathrm{G}}$.

Proof. Let us denote by $\overline{E} : \mathrm{Hmo}_0(k) \to \mathrm{D}$ the additive functor corresponding to E under the equivalence of categories (4.1). By precomposing it with the functor (4.9) of Lemma 4.8 below, we obtain the desired additive functor $\overline{E^{\mathrm{G}}}$. □

Lemma 4.8. *The functor* (2.15) *gives rise to an additive functor*

$$\mathrm{Hmo}_0^{\mathrm{G}}(k) \to \mathrm{Hmo}_0(k), \quad U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \mapsto U(\mathcal{A}^{\mathrm{G}}) \tag{4.9}$$

such that (4.9) $\circ\, U^{\mathrm{G}} \simeq U \circ$ (2.15).

Proof. Given two small G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$, consider the dg functor $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to \mathrm{rep}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{G}}, \mathcal{B}^{\mathrm{G}})$ that sends $(\mathrm{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k), \theta_\sigma)$ to

$$\mathcal{A}^{\mathrm{G}} \otimes (\mathcal{B}^{\mathrm{G}})^{\mathrm{op}} = \mathcal{A}^{\mathrm{G}} \otimes (\mathcal{B}^{\mathrm{op}})^{\mathrm{G}} \xrightarrow{(a)} (\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}})^{\mathrm{G}} \xrightarrow{\mathrm{B}^{\mathrm{G}}} \mathcal{C}_{\mathrm{dg}}(k)^{\mathrm{G}} \xrightarrow{(b)} \mathcal{C}_{\mathrm{dg}}(k),$$

where (a) stands for the canonical dg functor and (b) for the dg functor which sends a G-representation (M, θ_σ) to the dg k-vector space of G-invariants M^{G}; since $\mathrm{char}(k) \nmid |\mathrm{G}|$ the latter dg functor is well-defined. By first taking the left dg Kan extension (see [Kelly 1982, §4]) of $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to \mathrm{rep}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{G}}, \mathcal{B}^{\mathrm{G}})$ along the Yoneda dg functor $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to \mathcal{C}_{c,\mathrm{dg}}(\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})^{\mathrm{G}})$ and then the functor $\mathrm{H}^0(-)$, we obtain an induced triangulated functor $\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to \mathrm{rep}(\mathcal{A}^{\mathrm{G}}, \mathcal{B}^{\mathrm{G}})$; see [Elagin 2014, Theorem 8.7]. Consequently, by passing K_0, we obtain an induced homomorphism

$$K_0\, \mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to K_0\, \mathrm{rep}(\mathcal{A}^{\mathrm{G}}, \mathcal{B}^{\mathrm{G}}). \tag{4.10}$$

The additive functor (4.9) is given by the assignments $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \mapsto U(\mathcal{A}^{\mathrm{G}})$ and (4.10). By construction, we have (4.9) $\circ\, U^{\mathrm{G}} \simeq U \circ$ (2.15). □

4B. ***Enhanced additive invariants.*** Given an additive invariant E, the associated G-*enhanced additive invariant* is defined by

$$E^{\circlearrowright} : \text{G-dgcat}(k) \to \mathrm{D}^{\mathrm{G}}, \quad \mathrm{G} \circlearrowright \mathcal{A} \mapsto (E(\mathcal{A}), E(\phi_\sigma)),$$

where D^{G} stands for the category of G-equivariant objects in D (with respect to the trivial G-action); since E sends Morita equivalences to isomorphisms, $E^{\circlearrowright}$ is well-defined. When E is symmetric monoidal, $E^{\circlearrowright}$ is also symmetric monoidal.

Proposition 4.11. *Given a* G*-enhanced additive invariant* $E^{\circlearrowright}$, *there exists an additive functor* $\overline{E^{\circlearrowright}} : \mathrm{Hmo}_0^{\mathrm{G}}(k) \to \mathrm{D}^{\mathrm{G}}$ *such that* $\overline{E^{\circlearrowright}} \circ U^{\mathrm{G}} \simeq E^{\circlearrowright}$.

Proof. Given small G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$, the composition

$$K_0\, \mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to K_0\, \mathrm{rep}(\mathcal{A}, \mathcal{B}) \to \mathrm{Hom}_{\mathrm{D}}(E(\mathcal{A}), E(\mathcal{B})), \tag{4.12}$$

where the first homomorphism is induced by the restriction functor and the second homomorphism by the additive functor $\overline{E}$, takes values in the abelian subgroup $\mathrm{Hom}_{\mathrm{D}^{\mathrm{G}}}\big((E(\mathcal{A}), E(\phi_\sigma)), (E(\mathcal{B}), E(\phi_\sigma))\big)$. Therefore, $\overline{E^{\circlearrowright}}$ is defined by the assignments $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \mapsto (E(\mathcal{A}), E(\phi_\sigma))$ and (4.12). □

5. Relation with Panin's motivic category

Let H be an algebraic group scheme over k. Recall from [Panin 1994, §6], and from [Merkurjev 2005, §2.3], the construction of the motivic category[5] $\mathcal{C}^{\mathrm{H}}(k)$. The objects are the pairs (X, A), where X is a smooth projective H-scheme and A is a separable algebra, and the morphisms are given by the Grothendieck groups

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{H}}(k)}((X, A), (Y, B)) := K_0 \,\mathrm{Vect}^{\mathrm{H}}(X \times Y, A^{\mathrm{op}} \otimes B),$$

where $\mathrm{Vect}^{\mathrm{H}}(X \times Y, A^{\mathrm{op}} \otimes B)$ stands for the exact category of those H-equivariant right $(\mathcal{O}_{X\times Y} \otimes (A^{\mathrm{op}} \otimes B))$-modules which are locally free and of finite rank as $\mathcal{O}_{X\times Y}$-modules. Given

$$[\mathcal{F}] \in K_0 \,\mathrm{Vect}^{\mathrm{H}}(X \times Y, A^{\mathrm{op}} \otimes B) \quad \text{and} \quad [\mathcal{G}] \in K_0 \,\mathrm{Vect}^{\mathrm{H}}(Y \times Z, B^{\mathrm{op}} \otimes C),$$

their composition is defined by the formula

$$(\pi_{XZ})_*(\pi_{XY}^*([\mathcal{F}]) \otimes_B \pi_{YZ}^*([\mathcal{G}])) \in K_0 \,\mathrm{Vect}^{\mathrm{H}}(X \times Z, A^{\mathrm{op}} \otimes C),$$

where π_{ST} stands for the projection of $X \times Y \times Z$ into $S \times T$. The category $\mathcal{C}^{\mathrm{H}}(k)$ carries a symmetric monoidal structure induced by $(X, A) \otimes (Y, B) := (X \times Y, A \otimes B)$. Moreover, it comes equipped with two symmetric monoidal functors

$$\mathrm{SmProj}^{\mathrm{H}}(k)^{\mathrm{op}} \to \mathcal{C}^{\mathrm{H}}(k), \quad X \mapsto (X, k), \tag{5.1}$$

$$\mathrm{Sep}(k) \to \mathcal{C}^{\mathrm{H}}(k), \quad A \mapsto (\mathrm{Spec}(k), A), \tag{5.2}$$

defined on the category of smooth projective H-schemes and separable algebras, respectively. Let us denote by $\mathrm{G}\text{-dgcat}_{\mathrm{sp}}(k) \subset \mathrm{G}\text{-dgcat}(k)$ the full subcategory of those small G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$ with $\mathcal{A}$ smooth proper.

Theorem 5.3. *When* $\mathrm{H} = \mathrm{G}$ *is a (constant) finite algebraic group scheme, there exists an additive, fully faithful, symmetric monoidal functor* $\Psi : \mathcal{C}^{\mathrm{G}}(k) \to \mathrm{NChow}^{\mathrm{G}}(k)$ *making the following diagrams commute*:

$$\begin{tikzcd}
\mathrm{SmProj}^{\mathrm{G}}(k)^{\mathrm{op}} \arrow[r, "X \mapsto \mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)"] \arrow[d, "(5.1)"'] & \mathrm{G}\text{-dgcat}_{\mathrm{sp}}(k) \arrow[d, "U^{\mathrm{G}}"] \\
\mathcal{C}^{\mathrm{G}}(k) \arrow[r, "\Psi"'] & \mathrm{NChow}^{\mathrm{G}}(k)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Sep}(k) \arrow[r, "A \mapsto \mathrm{G} \circlearrowright_1 A"] \arrow[d, "(5.2)"'] & \mathrm{G}\text{-dgcat}_{\mathrm{sp}}(k) \arrow[d, "U^{\mathrm{G}}"] \\
\mathcal{C}^{\mathrm{G}}(k) \arrow[r, "\Psi"'] & \mathrm{NChow}^{\mathrm{G}}(k)
\end{tikzcd}$$

Proof. Given a smooth projective G-scheme X and a separable algebra A, let us write $\mathrm{Mod}(X, A)$ for the Grothendieck category of right $(\mathcal{O}_X \otimes A)$-modules, $\mathcal{D}(X, A)$ for the derived category $\mathcal{D}(\mathrm{Mod}(X, A))$, and $\mathcal{D}_{\mathrm{dg}}(X, A)$ for the dg category $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$ with $\mathcal{E} := \mathrm{Mod}(X, A)$. In the same vein, let us write $\mathrm{perf}(X, A)$ for

[5] Panin and Merkurjev denoted this motivic category by $\mathbb{A}^{\mathrm{H}}$ and $\mathcal{C}(\mathrm{H})$, respectively.

the full triangulated subcategory, and $\mathrm{perf}_{\mathrm{dg}}(X, A)$ for the full dg subcategory, of those complexes of right $(\mathcal{O}_X \otimes A)$-modules which are perfect as complexes of $\mathcal{O}_X$-modules. As proved in [Tabuada 2014, Lemma 6.4], the dg category $\mathrm{perf}_{\mathrm{dg}}(X, A)$ is smooth proper.

Let X and Y be smooth projective G-schemes and A and B separable algebras. Consider the inclusion functor

$$\mathrm{Vect}(X \times Y, A^{\mathrm{op}} \otimes B) \to \mathrm{perf}(X \times Y, A^{\mathrm{op}} \otimes B) \tag{5.4}$$

as well as the functor

$$\mathrm{perf}(X \times Y, A^{\mathrm{op}} \otimes B) \to \mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}(X, A), \mathrm{perf}_{\mathrm{dg}}(Y, B)), \quad \mathcal{F} \mapsto {}_{\Phi_{\mathcal{F}}}\mathrm{B}, \tag{5.5}$$

where $\Phi_{\mathcal{F}}$ stands for the Fourier–Mukai dg functor

$$\mathrm{perf}_{\mathrm{dg}}(X, A) \to \mathrm{perf}_{\mathrm{dg}}(Y, B), \quad \mathcal{G} \mapsto (\pi_Y)_*(\pi_X^*(\mathcal{G}) \otimes_A \mathcal{F}).$$

Both functors (5.4)–(5.5) are G-equivariant. Consequently, making use of the identification $\mathrm{perf}^{\mathrm{G}}(X \times Y, A^{\mathrm{op}} \otimes B) \simeq \mathrm{perf}(X \times Y, A^{\mathrm{op}} \otimes B)^{\mathrm{G}}$ (see Example 2.10), we obtain induced group homomorphisms

$$K_0 \,\mathrm{Vect}^{\mathrm{G}}(X \times Y, A^{\mathrm{op}} \otimes B) \to K_0 \,\mathrm{perf}^{\mathrm{G}}(X \times Y, A^{\mathrm{op}} \otimes B), \tag{5.6}$$

$$K_0 \,\mathrm{perf}^{\mathrm{G}}(X \times Y, A^{\mathrm{op}} \otimes B) \to K_0 \,\mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}(X, A), \mathrm{perf}_{\mathrm{dg}}(Y, B))^{\mathrm{G}}. \tag{5.7}$$

The assignments $(X, A) \mapsto U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X, A))$, combined with the group homomorphisms (5.7) $\circ$ (5.6), gives rise to an additive symmetric monoidal functor $\Psi : \mathcal{C}^{\mathrm{G}}(k) \to \mathrm{NChow}^{\mathrm{G}}(k)$, similarly to [Tabuada 2014, Theorem 6.10]. As explained on page 30 of that article, the functor (5.5) is an equivalence. This implies that (5.7) is invertible. Since $X \times Y$ admits an ample family of line G-bundles, the homomorphism (5.6) is also invertible. We hence conclude that the functor Ψ is, moreover, fully faithful. Finally, the commutativity of the two diagrams follows from the identifications $\mathrm{perf}_{\mathrm{dg}}(X, k) = \mathrm{perf}_{\mathrm{dg}}(X)$ and $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k), A) = \mathcal{C}_{c,\mathrm{dg}}(A)$ and from the fact that the Yoneda dg functor $A \to \mathcal{D}_{c,\mathrm{dg}}(A)$ is a G-equivariant Morita equivalence. □

Corollary 5.8. *Given $X, Y \in \mathrm{SmProj}^{\mathrm{G}}(k)$, we have a group isomorphism*

$$\mathrm{Hom}_{\mathrm{NChow}^{\mathrm{G}}(k)}\big(U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)), U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Y))\big) \simeq K_0^{\mathrm{G}}(X \times Y).$$

Proof. Combine Thomason's definition $K_0^{\mathrm{G}}(X \times Y) := K_0 \,\mathrm{Vect}^{\mathrm{G}}(X \times Y)$ of the G-equivariant Grothendieck group of $X \times Y$ with Theorem 5.3. □

5A. ***Twisted projective homogeneous varieties.*** Let H be a split semisimple algebraic group scheme over k, $P \subset \mathrm{H}$ a parabolic subgroup, and $\gamma : \mathrm{Gal}(k_{\mathrm{sep}}/k) \to \mathrm{H}(k_{\mathrm{sep}})$

a 1-cocycle. Out of this data, we can construct the projective homogeneous H-variety H/P as well as its twisted form ${}_{\gamma}\mathrm{H}/P$. Let $\widetilde{\mathrm{H}}$ and $\widetilde{P}$ be the universal covers of H and P, $R(\widetilde{\mathrm{H}})$ and $R(\widetilde{P})$ the associated representation rings, n the index $[W(\widetilde{\mathrm{H}}) : W(\widetilde{P})]$ of the Weyl groups, $\widetilde{Z}$ the center of $\widetilde{\mathrm{H}}$, and $\mathrm{Ch} := \mathrm{Hom}(\widetilde{Z}, \mathbb{G}_m)$ the character group. Under this notation, Panin [1994, Theorem 4.2] proved that every Ch-homogeneous basis $\rho_1, \dots, \rho_n$ of $R(\widetilde{P})$ over $R(\widetilde{\mathrm{H}})$ gives rise to an isomorphism

$$({}_{\gamma}\mathrm{H}/P, k) \simeq \bigoplus_{i=1}^{n} (\mathrm{Spec}(k), A_i) \tag{5.9}$$

in $\mathcal{C}^{\mathrm{H}}(k)$, where A_i stands for the Tits' central simple algebra associated to ρ_i.

Theorem 5.10. *Let* H, P, γ *be as above*, *and* G_k *the* (*constant*) *algebraic group scheme associated to the finite group* G. *For every homomorphism* $\mathrm{G}_k \to \mathrm{H}$ *and* G-*equivariant additive invariant* E^{G}, *we have an induced isomorphism*

$$E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}({}_{\gamma}\mathrm{H}/P)) \simeq \bigoplus_{i=1}^{n} E(A_i[\mathrm{G}]), \tag{5.11}$$

where ${}_{\gamma}\mathrm{H}/P$ *is considered as a* G-*scheme.*

Proof. Via $\mathrm{G}_k \to \mathrm{H}$, Panin's computation (5.9) holds also in the motivic category $\mathcal{C}^{\mathrm{G}}(k)$. Making use of Theorem 5.3 and Lemma 3.6, we conclude that

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}({}_{\gamma}\mathrm{H}/P)) \simeq \bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 A_i) \simeq \bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 \mathcal{C}_{c,\mathrm{dg}}(A_i)).$$

The proof then follows from Proposition 4.7 and Example 4.4(iii). $\square$

Remark 5.12 (G-equivariant Hochschild homology). When E^{G} is G-equivariant Hochschild homology HH^{G}, the right-hand side of (5.11) reduces to

$$\bigoplus_{i=1}^{n} HH(A_i[\mathrm{G}]) \overset{(a)}{\simeq} \bigoplus_{i=1}^{n} HH(k[\mathrm{G}]) \otimes HH_0(A_i) \overset{(b)}{\simeq} \bigoplus_{i=1}^{n} HH(k[\mathrm{G}]), \tag{5.13}$$

where (a) follows from [Loday 1998, Corollary 1.2.14] and (b) from the fact that $HH_0(A) \simeq k$ for every central simple k-algebra A. In the particular case where k is algebraically closed, (5.13) reduces moreover to $\bigoplus_{i=1}^{n} HH(k)^{\oplus |\langle \mathrm{G} \rangle|}$; see Remark 4.2(iii).

5B. ***Quasisplit case.*** When the algebraic group scheme H is a *quasi*split, Panin [1994, Theorem 12.4] proved that a computation similar to (5.9) also holds. In this generality, the algebras A_i are no longer central simple but only separable. The analogue of Theorem 5.10 (with the same proof) holds similarly. Moreover, when $E^{\mathrm{G}} := HH^{\mathrm{G}}$, the right-hand side of (5.11) reduces to

$$\bigoplus_{i=1}^{n} HH(k[\mathrm{G}]) \otimes A_i/[A_i, A_i].$$

6. Relation with equivariant motives

6A. ***Equivariant motives.*** Given a smooth projective G-scheme X and an integer $i \in \mathbb{Z}$, let us write $\mathrm{CH}^i_{\mathrm{G}}(X)_{\mathbb{Q}}$ for the i-codimensional G-equivariant Chow group of X in the sense of Edidin and Graham [1998]. Since the group G is finite, we have $\mathrm{CH}^i_{\mathrm{G}}(X)_{\mathbb{Q}} = 0$ whenever $i \notin \{0, \ldots, \dim(X)\}$; see [Edidin and Graham 2000, Proposition 5.2].

Let X and Y be smooth projective G-schemes, $X = \coprod_j X_j$ the decomposition of X into its connected components, and r an integer. The $\mathbb{Q}$-vector space $\mathrm{Corr}^r_{\mathrm{G}}(X, Y) := \bigoplus_j \mathrm{CH}^{\dim(X_j)+r}_{\mathrm{G}}(X_j \times Y)_{\mathbb{Q}}$ is called the space of G-*equivariant correspondences of degree r from X to Y*. Given G-equivariant correspondences $f \in \mathrm{Corr}^r_{\mathrm{G}}(X, Y)$ and $g \in \mathrm{Corr}^s_{\mathrm{G}}(Y, Z)$, their composition is defined by the formula

$$(\pi_{XZ})_*(\pi^*_{XY}(f) \cdot \pi^*_{YZ}(g)) \in \mathrm{Corr}^{r+s}_{\mathrm{G}}(X, Z). \tag{6.1}$$

Recall from [Laterveer 1998], and from [Iyer and Müller-Stach 2009], the construction of the category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ of G-*equivariant Chow motives with* $\mathbb{Q}$-*coefficients*. The objects are the triples (X, p, m), where X is a smooth projective G-scheme, $p^2 = p \in \mathrm{Corr}^0_{\mathrm{G}}(X, X)$ is an idempotent endomorphism, and m is an integer. The $\mathbb{Q}$-vector spaces of morphisms are given by

$$\mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}((X, p, m), (Y, q, n)) := q \circ \mathrm{Corr}^{n-m}_{\mathrm{G}}(X, Y) \circ p,$$

and the composition law is induced by the composition (6.1) of correspondences. By construction, the category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ is $\mathbb{Q}$-linear, additive, and idempotent complete. Moreover, it carries a symmetric monoidal structure induced by the formula $(X, p, m) \otimes (Y, q, n) := (X \times Y, p \otimes q, m + n)$. The G-equivariant Lefschetz motive $(\mathrm{Spec}(k), \mathrm{id}, -1)$ will be denoted by $\mathbb{L}$ and the G-equivariant Tate motive $(\mathrm{Spec}(k), \mathrm{id}, 1)$ by $\mathbb{Q}(1)$; in both cases G acts trivially. Finally, the category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ comes equipped with the symmetric monoidal functor

$$\mathfrak{h}^{\mathrm{G}}(-)_{\mathbb{Q}} : \mathrm{SmProj}^{\mathrm{G}}(k)^{\mathrm{op}} \to \mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}, \quad X \mapsto (X, \mathrm{id}, 0).$$

The category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ is additive and rigid symmetric monoidal.

6B. ***Orbit categories.*** Let $\mathcal{C}$ be an additive symmetric monoidal category and $\mathcal{O} \in \mathcal{C}$ a $\otimes$-invertible object. The *orbit category* $\mathcal{C}/_{-\otimes\mathcal{O}}$ has the same objects as $\mathcal{C}$ and abelian groups of morphisms $\mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(a, b) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(a, b \otimes \mathcal{O}^{\otimes i})$. Given objects a, b, and c, and morphisms

$$\mathrm{f} = \{f_i\}_{i \in \mathbb{Z}} \in \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(a, b \otimes \mathcal{O}^{\otimes i}), \quad \mathrm{g} = \{g_i\}_{i \in \mathbb{Z}} \in \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(b, c \otimes \mathcal{O}^{\otimes i}),$$

the i'-th component of $\mathrm{g} \circ \mathrm{f}$ is defined as $\sum_i (g_{i'-i} \otimes \mathcal{O}^{\otimes i}) \circ f_i$. The functor

$$\pi : \mathcal{C} \to \mathcal{C}/_{-\otimes\mathcal{O}}, \quad a \mapsto a, \quad f \mapsto \mathrm{f} = \{f_i\}_{i \in \mathbb{Z}},$$

where $f_0 = f$ and $f_i = 0$ if $i \neq 0$, is endowed with a natural isomorphism of functors $\pi \circ (- \otimes \mathcal{O}) \Rightarrow \pi$ and is 2-universal among all such functors; see [Tabuada 2013, §7]. The category $\mathcal{C}/_{-\otimes\mathcal{O}}$ is additive and, as proved in [Tabuada 2013, Lemma 7.3], inherits from $\mathcal{C}$ a symmetric monoidal structure making π symmetric monoidal.

6C. *Localization at the augmentation ideal.* Let I be the kernel of the rank homomorphism $R(\mathrm{G}) \twoheadrightarrow \mathbb{Z}$ and $R(\mathrm{G})_I$ the localization of $R(\mathrm{G})$ at the ideal I. Recall from Corollary 3.17 that the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ is $R(\mathrm{G})$-linear. Let us denote by $\mathrm{Hmo}_0^{\mathrm{G}}(k)_I$ the $R(\mathrm{G})_I$-linear additive category obtained from $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ by applying the functor $(-)_I := - \otimes_{R(\mathrm{G})} R(\mathrm{G})_I$ to each $R(\mathrm{G})$-module of morphisms. By construction, $\mathrm{Hmo}_0^{\mathrm{G}}(k)_I$ inherits from $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ a symmetric monoidal structure making the functor $(-)_I : \mathrm{Hmo}_0^{\mathrm{G}}(k) \to \mathrm{Hmo}_0^{\mathrm{G}}(k)_I$ symmetric monoidal. The category $\mathrm{NChow}^{\mathrm{G}}(k)_I$ of *I-localized* G*-equivariant noncommutative Chow motives* is defined as the idempotent completion of the subcategory of $\mathrm{Hmo}_0^{\mathrm{G}}(k)_I$ consisting of the objects $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})_I$ with $\mathcal{A}$ a smooth proper dg category.

Proposition 6.2. *Given any two cohomology classes $[\alpha], [\beta] \in H^2(\mathrm{G}, k^\times)$, we have an isomorphism $U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)_I \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\beta k)_I$ in* $\mathrm{NChow}^{\mathrm{G}}(k)_I$.

Proof. By construction of $\mathrm{NChow}^{\mathrm{G}}(k)$, we have group isomorphisms

$$\mathrm{Hom}_{\mathrm{NChow}^{\mathrm{G}}(k)}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k), U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\beta k)) \simeq K_0(\mathcal{D}_c(k)^{\mathrm{G},\alpha\beta^{-1}}),$$
$$\mathrm{Hom}_{\mathrm{NChow}^{\mathrm{G}}(k)}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\beta k), U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)) \simeq K_0(\mathcal{D}_c(k)^{\mathrm{G},\beta\alpha^{-1}}).$$

Consider the $\alpha\beta^{-1}$-twisted G-equivariant object $k_{\alpha\beta^{-1}}\mathrm{G} \in \mathcal{D}_c(k)^{\mathrm{G},\alpha\beta^{-1}}$ defined as $\big(\bigoplus_{\rho\in\mathrm{G}} \phi_\rho(k), \theta_\sigma\big)$, where $\phi_\rho(k) = k$ and θ_σ is given by the collection of units $(\alpha^{-1}\beta)(\sigma,\rho) \in k^\times$. Similarly, consider the $\beta\alpha^{-1}$-twisted G-equivariant object $k_{\beta\alpha^{-1}}\mathrm{G} \in \mathcal{D}_c(k)^{\mathrm{G},\beta\alpha^{-1}}$ defined as $\big(\bigoplus_{\rho\in\mathrm{G}} \phi_\rho(k), \theta_\sigma\big)$, where θ_σ is given by the units $(\beta^{-1}\alpha)(\sigma,\rho)$. The associated Grothendieck classes then correspond to morphisms

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k) \xrightarrow{f} U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\beta k) \quad \text{and} \quad U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\beta k) \xrightarrow{g} U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)$$

in the category $\mathrm{NChow}^{\mathrm{G}}(k)$. Since the rank of the elements $g \circ f, f \circ g \in R(\mathrm{G})$ is nonzero (see Proposition 3.15), we conclude from the definition of $\mathrm{NChow}^{\mathrm{G}}(k)_I$ that the morphisms f_I and g_I are invertible. This completes the proof. □

Remark 6.3 (groups of central type). Note that the group algebra $k[\mathrm{G}]$ is not simple: it contains the nontrivial augmentation ideal. In the case where G is of central type, there exist cohomology classes $[\alpha] \in H^2(\mathrm{G}, k^\times)$ for which the twisted group algebra $k_\alpha[\mathrm{G}]$ is simple! For example, the group $\mathrm{G} := \mathrm{H} \times \widehat{\mathrm{H}}$ (with H abelian) is of central type and the twisted group algebra $k_\alpha[\mathrm{G}]$ associated to the 2-cocycle $\alpha((\sigma,\chi),(\rho,\psi)) := \chi(\rho)$ is simple. Combining Remark 4.2 with Example 4.4(iv) and Proposition 4.7, we conclude that $U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k) \not\simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)$ in $\mathrm{NChow}^{\mathrm{G}}(k)$. This shows that Proposition 6.2 is false before I-localization.

6D. *Bridges.* The next result relates the category of G-equivariant noncommutative motives with the category of G-equivariant motives.

Theorem 6.4. *There exists a $\mathbb{Q}$-linear, fully faithful, symmetric monoidal functor Φ making the following diagram commute*:

$$\begin{tikzcd}
\mathrm{SmProj}^{\mathrm{G}}(k)^{\mathrm{op}} \arrow[rr, "X\mapsto \mathrm{G}\circlearrowright\mathrm{perf}_{\mathrm{dg}}(X)"] \arrow[d, "\mathfrak{h}^{\mathrm{G}}(-)_{\mathbb{Q}}"'] & & \mathrm{G}\text{-}\mathrm{dgcat}_{\mathrm{sp}}(k) \arrow[d, "U^{\mathrm{G}}(-)_{\mathbb{Q}}"] \\
\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}} \arrow[d, "\pi"'] & & \mathrm{NChow}^{\mathrm{G}}(k)_{\mathbb{Q}} \arrow[d, "(-)_{I_{\mathbb{Q}}}"] \\
\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)} \arrow[rr, "\Phi"'] & & \mathrm{NChow}^{\mathrm{G}}(k)_{\mathbb{Q},I_{\mathbb{Q}}}
\end{tikzcd} \tag{6.5}$$

Proof. Let us denote by $\mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q}}$ the idempotent completion of the full subcategory of $\mathcal{C}^{\mathrm{G}}(k)_{\mathbb{Q}}$ (see Section 5) consisting of the objects $(X,k)_{\mathbb{Q}}$. Given smooth projective G-schemes X and Y, we have isomorphisms

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q}}}(\mathfrak{h}^{\mathrm{G}}_0(X)_{\mathbb{Q}}, \mathfrak{h}^{\mathrm{G}}_0(Y)_{\mathbb{Q}}) = K_0\,\mathrm{Vect}^{\mathrm{G}}(X\times Y)_{\mathbb{Q}} \simeq K^{\mathrm{G}}_0(X\times Y)_{\mathbb{Q}}.$$

Moreover, given $[\mathcal{F}]_{\mathbb{Q}} \in K^{\mathrm{G}}_0(X\times Y)_{\mathbb{Q}}$ and $[\mathcal{G}]_{\mathbb{Q}} \in K^{\mathrm{G}}_0(Y\times Z)_{\mathbb{Q}}$, their composition is defined by the formula $(\pi_{XZ})_*(\pi^*_{XY}([\mathcal{F}]_{\mathbb{Q}})\otimes \pi^*_{YZ}([\mathcal{G}]_{\mathbb{Q}}))$. Furthermore, $\mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q}}$ comes equipped with the symmetric monoidal functor

$$\mathfrak{h}^{\mathrm{G}}_0(-) : \mathrm{SmProj}^{\mathrm{G}}(k)^{\mathrm{op}} \to \mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q}}, \quad X \mapsto (X,k)_{\mathbb{Q}}.$$

Similarly to Section 6C, we can also consider the $I_{\mathbb{Q}}$-localized category $\mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q},I_{\mathbb{Q}}}$.

Let us now construct a functor Φ_1 making the diagram

$$\begin{tikzcd}
\mathrm{SmProj}^{\mathrm{G}}(k)^{\mathrm{op}} \arrow[r, equal] \arrow[d, "\mathfrak{h}^{\mathrm{G}}(-)_{\mathbb{Q}}"'] & \mathrm{SmProj}(k)^{\mathrm{op}} \arrow[r, "X\mapsto \mathrm{G}\circlearrowright\mathrm{perf}_{\mathrm{dg}}(X)"] \arrow[d, "\mathfrak{h}^{\mathrm{G}}_0(-)_{\mathbb{Q}}"] & \mathrm{G}\text{-}\mathrm{dgcat}_{\mathrm{sp}}(k) \arrow[d, "U^{\mathrm{G}}(-)_{\mathbb{Q}}"] \\
\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}} \arrow[d, "\pi"'] & \mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q}} \arrow[d, "(-)_{I_{\mathbb{Q}}}"] & \mathrm{NChow}^{\mathrm{G}}(k)_{\mathbb{Q}} \arrow[d, "(-)_{I_{\mathbb{Q}}}"] \\
\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)} & \mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q},I_{\mathbb{Q}}} \arrow[l, "\Phi_1"] \arrow[r, "\Phi_2"'] & \mathrm{NChow}^{\mathrm{G}}(k)_{\mathbb{Q},I_{\mathbb{Q}}}
\end{tikzcd} \tag{6.6}$$

commute, where Φ_2 stands for the $\mathbb{Q}$-linear, fully faithful, symmetric monoidal functor naturally induced from Ψ; see Theorem 5.3. As proved in [Edidin and Graham 2000, Corollary 5.1], we have a Riemann–Roch isomorphism $\tau_X : K^{\mathrm{G}}_0(X)_{\mathbb{Q},I_{\mathbb{Q}}} \to \bigoplus_{i=0}^{\dim(X)} \mathrm{CH}^i_{\mathrm{G}}(X)_{\mathbb{Q}}$ for every smooth projective G-scheme X. This isomorphism preserves the multiplicative structures. Moreover, given any G-equivariant map $f : X \to Y$, the following squares are commutative (we assume that f is proper on the right-hand side):

$$\begin{tikzcd} K_0^{\mathrm{G}}(X)_{\mathbb{Q},I_{\mathbb{Q}}} \arrow[r, "\tau_X"] & \bigoplus_{i=0}^{\dim(X)} \mathrm{CH}^i_{\mathrm{G}}(X)_{\mathbb{Q}} \\ K_0^{\mathrm{G}}(Y)_{\mathbb{Q},I_{\mathbb{Q}}} \arrow[u, "f^*"] \arrow[r, "\tau_Y"'] & \bigoplus_{i=0}^{\dim(Y)} \mathrm{CH}^i_{\mathrm{G}}(Y)_{\mathbb{Q}} \arrow[u, "f^*"'] \end{tikzcd} \qquad \begin{tikzcd} K_0^{\mathrm{G}}(X)_{\mathbb{Q},I_{\mathbb{Q}}} \arrow[r, "\tau_X"] \arrow[d, "f_*"'] & \bigoplus_{i=0}^{\dim(X)} \mathrm{CH}^i_{\mathrm{G}}(X)_{\mathbb{Q}} \arrow[d, "f_*"] \\ K_0^{\mathrm{G}}(Y)_{\mathbb{Q},I_{\mathbb{Q}}} \arrow[r, "\tau_Y"'] & \bigoplus_{i=0}^{\dim(Y)} \mathrm{CH}^i_{\mathrm{G}}(Y)_{\mathbb{Q}} \end{tikzcd}$$

By construction of the orbit category, we have isomorphisms

$$\mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}}(\pi(\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}), \pi(\mathfrak{h}^{\mathrm{G}}(Y)_{\mathbb{Q}})) \simeq \bigoplus_{i=0}^{\dim(X\times Y)} \mathrm{CH}^i_{\mathrm{G}}(X\times Y)_{\mathbb{Q}}.$$

Therefore, we conclude from the preceding considerations that the assignments

$$\mathfrak{h}_0^{\mathrm{G}}(X)_{\mathbb{Q}} \mapsto \mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}} \quad \text{and} \quad K_0^{\mathrm{G}}(X\times Y)_{\mathbb{Q},I_{\mathbb{Q}}} \xrightarrow{\tau_{X\times Y}} \bigoplus_{i=0}^{\dim(X\times Y)} \mathrm{CH}^i_{\mathrm{G}}(X\times Y)_{\mathbb{Q}}$$

give rise to a functor $\Phi_1 : \mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q},I_{\mathbb{Q}}} \to \mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$ making the diagram (6.6) commute. The functor Φ_1 is $\mathbb{Q}$-linear, fully faithful, and symmetric monoidal. Since the objects (X, p, m) and $(X, p, 0)$ become isomorphic in the orbit category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$, the functor Φ_1 is, moreover, essentially surjective, and hence an equivalence of categories. Now, choose a (quasi-)inverse functor Φ_1^{-1} of Φ_1 and define Φ as the composition $\Phi_2 \circ \Phi_1^{-1}$. By construction, Φ is $\mathbb{Q}$-linear, fully faithful, symmetric monoidal, and makes the upper rectangle of (6.5) commute. □

7. Full exceptional collections

7A. ***Full exceptional collections.*** Let $\mathcal{T}$ be a k-linear triangulated category. Recall from [Bondal and Orlov 1995, Definition 2.4; Huybrechts 2006, §1.4] that a *semiorthogonal decomposition of length n*, denoted by $\mathcal{T} = \langle \mathcal{T}_1, \dots, \mathcal{T}_n\rangle$, consists of full triangulated subcategories $\mathcal{T}_1, \dots, \mathcal{T}_n \subset \mathcal{T}$ satisfying the following conditions: the inclusions $\mathcal{T}_i \subset \mathcal{T}$ admit left and right adjoints, the triangulated category $\mathcal{T}$ is generated by the objects of $\mathcal{T}_1, \dots, \mathcal{T}_n$, and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_j, \mathcal{T}_i) = 0$ when $i < j$. An object $\mathcal{E} \in \mathcal{T}$ is called *exceptional* if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, \mathcal{E}) = k$ and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, \mathcal{E}[m]) = 0$ when $m \neq 0$. A *full exceptional collection of length n*, denoted by $\mathcal{T} = (\mathcal{E}_1, \dots, \mathcal{E}_n)$, is a sequence of exceptional objects $\mathcal{E}_1, \dots, \mathcal{E}_n$ which generate the triangulated category $\mathcal{T}$ and for which we have $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}_j, \mathcal{E}_i[m]) = 0, m \in \mathbb{Z}$, when $i < j$. Every full exceptional collection gives rise to a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{D}_c(k), \dots, \mathcal{D}_c(k)\rangle$.

Proposition 7.1. *Let $\mathcal{A}$ be a small* G*-dg category and $\mathcal{A}_i \subseteq \mathcal{A}$, $1 \leq i \leq n$, full dg subcategories. Assume that $\sigma^*(\mathcal{A}_i) \subseteq \mathcal{A}_i$ for every $\sigma \in \mathrm{G}$, and that $\mathcal{D}_c(\mathcal{A})$ admits a semiorthogonal decomposition $\langle \mathcal{D}_c(\mathcal{A}_1), \dots, \mathcal{D}_c(\mathcal{A}_n)\rangle$. Under these assumptions, we have an isomorphism $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \simeq \bigoplus_{i=1}^n U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}_i)$ in* $\mathrm{Hmo}_0^{\mathrm{G}}(k)$.

Proof. The inclusions of dg categories $\mathcal{A}_i \subseteq \mathcal{A}$ give rise to a morphism

$$\bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}_i) \to U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \tag{7.2}$$

in the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$. In order to show that (7.2) is an isomorphism, it suffices by the Yoneda lemma to show that the induced group homomorphism

$$\mathrm{Hom}\Big(U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{B}), \bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}_i)\Big) \to \mathrm{Hom}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{B}), U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}))$$

is invertible for every small G-dg category $\mathrm{G} \circlearrowright \mathcal{B}$. By construction of the additive category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$, the preceding homomorphism identifies with

$$\bigoplus_{i=1}^{n} K_0 \operatorname{rep}(\mathcal{B}, \mathcal{A}_i)^{\mathrm{G}} \to K_0 \operatorname{rep}(\mathcal{B}, \mathcal{A})^{\mathrm{G}}. \tag{7.3}$$

Since $\mathcal{D}_c(\mathcal{A}) = \langle \mathcal{D}_c(\mathcal{A}_1), \dots, \mathcal{D}_c(\mathcal{A}_n)\rangle$, we have a semiorthogonal decomposition

$$\operatorname{rep}(\mathcal{B}, \mathcal{A}) = \langle \operatorname{rep}(\mathcal{B}, \mathcal{A}_1), \dots, \operatorname{rep}(\mathcal{B}, \mathcal{A}_n)\rangle.$$

Using first the fact that the functor $(-)^{\mathrm{G}}$ preserves semiorthogonal decompositions, and then the fact that the functor $K_0(-)$ sends semiorthogonal decompositions to direct sums, we conclude that the group homomorphism (7.3) is invertible. $\square$

7B. *Invariant objects.* Let $\mathrm{G} \circlearrowright \mathcal{A}$ be a small G-dg category. An object $M \in \mathcal{D}(\mathcal{A})$ is called G-*invariant* if $\phi_\sigma(M) \simeq M$ for every $\sigma \in \mathrm{G}$. Every G-equivariant object in $\mathrm{G} \circlearrowright \mathcal{D}(\mathcal{A})$ is G-invariant, but the converse does not hold.

Remark 7.4 (strictification). Given a G-invariant object $M \in \mathcal{D}(\mathcal{A})$, let us fix an isomorphism $\theta_\sigma : M \to \phi_\sigma(M)$ for every $\sigma \in \mathrm{G}$. If $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(M, M) \simeq k$, then $\phi_\rho(\theta_\sigma) \circ \theta_\rho$ and $\theta_{\rho\sigma}$ differ by multiplication with an invertible element $\alpha(\rho, \sigma) \in k^\times$. Moreover, these invertible elements define a 2-cocycle α whose cohomology class $[\alpha] \in H^2(\mathrm{G}, k^\times)$ is independent of the choice of the θ_σ. Consequently, $M \in \mathcal{D}(\mathcal{A})^{\mathrm{G},\alpha}$. Furthermore, $M^{\otimes n} \in \mathcal{D}(\mathcal{A})^{\mathrm{G},\alpha^n}$. Roughly speaking, every "simple" G-invariant object can be strictified into a twisted G-equivariant object.

Proposition 7.5. *Let $\mathcal{A}$ be a small* G*-dg category such that $\mathcal{D}_c(\mathcal{A})$ admits a full exceptional collection $(\mathcal{E}_1, \dots, \mathcal{E}_n)$. Suppose $\mathcal{E}_i \in \mathcal{D}_c(\mathcal{A})^{\mathrm{G},\alpha_i}$, with $[\alpha_i] \in H^2(\mathrm{G}, k^\times)$. Then we have $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \simeq \bigoplus_{i=1}^n U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_i} k)$ in $\mathrm{Hmo}_0^{\mathrm{G}}(k)$.*

Proof. By construction, the set of morphisms $\mathrm{Hom}_{\mathrm{Hmo}^{\mathrm{G}}(k)}(\mathrm{G} \circlearrowright_{\alpha_i} k, \mathrm{G} \circlearrowright \mathcal{A})$ is given by the set of isomorphism classes of the triangulated category $\operatorname{rep}(k, \mathcal{A})^{\mathrm{G},\alpha_i} \simeq \mathcal{D}_c(\mathcal{A})^{\mathrm{G},\alpha_i}$. Consequently, the object $\mathcal{E}_i \in \mathcal{D}_c(\mathcal{A})^{\mathrm{G},\alpha_i}$ corresponds to a morphism $\mathcal{E}_i : \mathrm{G} \circlearrowright_{\alpha_i} k \to \mathrm{G} \circlearrowright \mathcal{A}$ in $\mathrm{Hmo}^{\mathrm{G}}(k)$. Consider the associated morphism

$$([\mathcal{E}_1] \oplus \cdots \oplus [\mathcal{E}_i] \oplus \cdots \oplus [\mathcal{E}_n]) : \bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_i} k) \to U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \tag{7.6}$$

in the additive category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$. In order to show that (7.6) is an isomorphism, we can now follow *mutatis mutandis* the proof of Proposition 7.1. $\square$

Corollary 7.7. *Given a* G*-dg category* $\mathrm{G} \circlearrowright \mathcal{A}$ *as in Proposition 7.5, we have*

(i) $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \simeq \bigoplus_{i=1}^n E(k_{\alpha_i}[\mathrm{G}])$ *for every* G*-equivariant additive invariant*;

(ii) $E^{\circlearrowright}(\mathrm{G} \circlearrowright \mathcal{A}) \simeq \bigoplus_{i=1}^n (E(k), \mathrm{id})$ *for every* G*-enhanced additive invariant.*

Proof. Item (i) follows from the combination of Propositions 4.7 and 7.5 with Example 4.4(iv). Item (ii) follows from the combination of Propositions 4.11 and 7.5 with the fact that $E^{\circlearrowright}(\mathrm{G} \circlearrowright_\alpha k) \simeq (E(k), \mathrm{id})$ for every $[\alpha] \in H^2(\mathrm{G}, k^\times)$. $\square$

Proposition 7.8. *Let X be a quasicompact quasiseparated* G*-scheme such that* $\mathrm{perf}(X)$ *admits a full exceptional collection* $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ *of* G*-invariant objects. Let us denote by* $[\alpha_i] \in H^2(\mathrm{G}, k^\times)$ *the cohomology class of Remark 7.4 associated to the exceptional object* $\mathcal{E}_i$*. Under these assumptions and notations, we have an isomorphism* $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \simeq \bigoplus_{i=1}^n U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_i} k)$ *in* $\mathrm{Hmo}_0^{\mathrm{G}}(k)$.

Proof. Apply Proposition 7.5 to the dg category $\mathrm{perf}_{\mathrm{dg}}(X)$. $\square$

Example 7.9 (projective spaces). Let $\mathbb{P}^n$ be the n-th projective space. As proved in [Beĭlinson 1978], $\mathrm{perf}(\mathbb{P}^n)$ admits a full exceptional collection $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n))$. Moreover, the objects $\mathcal{O}(i)$ are G-invariant for any G-action on $\mathbb{P}^n$. Let us denote by $[\alpha]$ the cohomology class of Remark 7.4 associated to the exceptional object $\mathcal{O}(1)$. In this notation, Proposition 7.8 yields an isomorphism

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^n)) \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k) \oplus \cdots \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^n} k).$$

Example 7.10 (odd dimensional quadrics). Assume that $\mathrm{char}(k) \neq 2$. Let (V, q) be a nondegenerate quadratic form of odd dimension $n \geq 3$ and $Q_q \subset \mathbb{P}(V)$ the associated smooth projective quadric of dimension $d := n - 2$. As proved in [Kapranov 1988], $\mathrm{perf}(Q_q)$ admits a full exceptional collection $(\mathcal{S}, \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(d-1))$, where $\mathcal{S}$ denotes the spinor bundle. Moreover, the objects $\mathcal{O}(i)$ and $\mathcal{S}$ are G-invariant for any G-action on Q_q; see [Elagin 2012, §3.2]. Let us denote by $[\alpha]$ and $[\beta]$ the cohomology classes of Remark 7.4 associated to the exceptional object $\mathcal{O}(1)$ and $\mathcal{S}$, respectively. Under these notations, Proposition 7.8 yields an isomorphism between $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Q_q))$ and the direct sum

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\beta k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k) \oplus \cdots \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^{(d-1)}} k).$$

Example 7.11 (Grassmannians). Assume that $\mathrm{char}(k) = 0$. Let V be a k-vector space of dimension d, $n \leq d$ a positive integer, and $\mathrm{Gr} := \mathrm{Gr}(n, V)$ the Grassmannian of n-dimensional subspaces in V. As proved in [Kapranov 1988], $\mathrm{perf}(\mathrm{Gr})$ admits a full exceptional collection $(\mathcal{O}, \mathcal{U}^\vee, \dots, \Sigma^\lambda_{n(d-n)}\mathcal{U}^\vee)$, where $\mathcal{U}^\vee$ denotes the dual of the tautological vector bundle on Gr and Σ^λ_i the Schur functor associated to a Young diagram λ with $|\lambda| = i$ having at most n rows and $d - n$

columns. Moreover, the objects $\Sigma_i^\lambda \mathcal{U}^\vee$ are G-invariant for any G-action on Q_q which is induced by an homomorphism $\mathrm{G} \to \mathrm{PGL}(V)$. Let us denote by $[\alpha]$ the cohomology class of Remark 7.4 associated to the exceptional object $\mathcal{U}^\vee$. In this notation, Proposition 7.8 yields an isomorphism

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(\mathrm{Gr})) \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k) \oplus \cdots \oplus \Big(\bigoplus_\lambda U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^{n(d-n)}} k)\Big).$$

Proof of Theorem 1.2. To simplify the exposition, we write $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}(i)$ instead of $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}} \otimes \mathbb{Q}(1)^{\otimes i}$. Following Remark 7.4, let us denote by $[\alpha_i] \in H^2(\mathrm{G}, k^\times)$ the cohomology class associated to the exceptional object $\mathcal{E}_i$. By combining Propositions 6.2 and 7.8, we obtain induced isomorphisms

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q},I_{\mathbb{Q}}} \simeq \bigoplus_{i=1}^n U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_i} k)_{\mathbb{Q},I_{\mathbb{Q}}} \simeq \bigoplus_{i=1}^n U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)_{\mathbb{Q},I_{\mathbb{Q}}} \tag{7.12}$$

in the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)_{\mathbb{Q},I_{\mathbb{Q}}}$. Since $\mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}$ (with trivial G-action) is the $\otimes$-unit of $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ and $U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)_{\mathbb{Q},I_{\mathbb{Q}}}$ the $\otimes$-unit of $\mathrm{NChow}^{\mathrm{G}}(k)_{\mathbb{Q},I_{\mathbb{Q}}}$, we conclude from Theorem 6.4 that $\pi(\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}})$ is isomorphic to $\bigoplus_{j=1}^n \pi(\mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}})$ in the orbit category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$. Let us now "lift" this isomorphism to the category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$. Since the functor π is additive, there exist morphisms

$$\mathrm{f} = \{f_i\}_{i\in\mathbb{Z}} \in \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}\Big(\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}, \bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}(i)\Big),$$
$$\mathrm{g} = \{g_i\}_{i\in\mathbb{Z}} \in \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}\Big(\bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}, \mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}(i)\Big)$$

verifying the equalities $\mathrm{g} \circ \mathrm{f} = \mathrm{id} = \mathrm{f} \circ \mathrm{g}$. Moreover, as explained in Section 6, we have

$$\mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}\Big(\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}, \bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}(i)\Big) \simeq \bigoplus_{j=1}^n \mathrm{CH}_{\mathrm{G}}^{\dim(X)+i}(X)_{\mathbb{Q}},$$
$$\mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}\Big(\bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}, \mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}(i)\Big) \simeq \bigoplus_{j=1}^n \mathrm{CH}_{\mathrm{G}}^i(X)_{\mathbb{Q}}.$$

This implies that $f_i = 0$ when $i \notin \{-\dim(X), \dots, 0\}$ and that $g_i = 0$ when $i \notin \{0, \dots, \dim(X)\}$. The sets $\{f_{-r} \mid 0 \le r \le \dim(X)\}$ and $\{g_r(-r) \mid 0 \le r \le \dim(X)\}$ then give rise to morphisms in the category of G-equivariant Chow motives:

$$\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}} \to \bigoplus_{r=0}^{\dim(X)} \bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}(-r), \tag{7.13}$$
$$\bigoplus_{r=0}^{\dim(X)} \bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}(-r) \to \mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}. \tag{7.14}$$

The composition (7.14)∘(7.13) agrees with the 0th component of $\mathrm{g} \circ \mathrm{f} = \mathrm{id}$, i.e., with the identity of $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$. Thus, since $\mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}(-r) = \mathbb{L}^{\otimes r}$, the G-equivariant

Chow motive $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$ is a direct summand of $\bigoplus_{r=0}^{\dim(X)}\bigoplus_{j=1}^{n}\mathbb{L}^{\otimes r}$. By definition of the G-equivariant Lefschetz motive $\mathbb{L}$, we have $\mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}(\mathbb{L}^{\otimes p}, \mathbb{L}^{\otimes q}) = \delta_{pq}\cdot\mathbb{Q}$, where δ_{pq} stands for the Kronecker symbol. This implies that $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$ is a subsum of $\bigoplus_{r=0}^{\dim(X)}\bigoplus_{j=1}^{n}\mathbb{L}^{\otimes r}$. Using the fact that $\pi(\mathbb{L}^{\otimes r})$ is isomorphic to $\pi(\mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}})$, and $\pi(\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}})$ to $\bigoplus_{j=1}^{n}\pi(\mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}})$, we conclude finally that there exists a choice of integers $r_1,\dots,r_n\in\{0,\dots,\dim(X)\}$ such that $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}\simeq\mathbb{L}^{\otimes r_1}\oplus\cdots\oplus\mathbb{L}^{\otimes r_n}$. This concludes the proof. □

Remark 7.15. The above proof of Theorem 1.2 is divided into two steps. In the first step, we established the isomorphism (7.12). In the second step, we explained how (7.12) leads to the desired isomorphism $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}\simeq\mathbb{L}^{\otimes r_1}\oplus\cdots\oplus\mathbb{L}^{\otimes r_n}$. The proof of the second step is similar to that of [Marcolli and Tabuada 2015, Theorem 1.1].

7C. *Permutations.* Given a subgroup $\mathrm{H}\subseteq\mathrm{G}$, consider the small G-dg category $\mathrm{G}\circlearrowright\coprod_{\bar\rho\in\mathrm{G}/\mathrm{H}}k$, where G acts by permutation of the components.

Proposition 7.16. *Let* $\mathrm{G}\circlearrowright\mathcal{A}$ *be a small* G*-dg category such that* $\mathcal{D}_c(\mathcal{A})$ *admits a full exceptional collection* $(\mathcal{E}_1,\dots,\mathcal{E}_n)$. *Assume that the induced* G*-action on* $\mathcal{D}_c(\mathcal{A})$ *transitively permutes the objects* $\mathcal{E}_1,\dots,\mathcal{E}_n$ *(up to isomorphism) and that* $\mathrm{Hom}(\mathcal{E}_i,\mathcal{E}_j[m])=0$ *for every* $m\in\mathbb{Z}$ *and* $i\neq j$. *Let* $\mathrm{H}\subseteq\mathrm{G}$ *be the stabilizer of* $\mathcal{E}_1$. *If the cohomology group* $H^2(\mathrm{H},k^\times)$ *is trivial (e.g.,* $k=\mathbb{C}$ *and* H *is cyclic), then we have an isomorphism* $\mathrm{G}\circlearrowright\mathcal{A}\simeq\mathrm{G}\circlearrowright\coprod_{\bar\rho\in\mathrm{G}/\mathrm{H}}k$ *in* $\mathrm{Hmo}^{\mathrm{G}}(k)$.

Proof. We have the following equivalence of categories:

$$\Big(\prod_{\bar\rho\in\mathrm{G}/\mathrm{H}}\mathcal{D}_c(\mathcal{A})\Big)^{\mathrm{G}}\to\mathcal{D}_c(\mathcal{A})^{\mathrm{H}},\qquad(\{\mathrm{B}_{\bar\rho}\}_{\bar\rho\in\mathrm{G}/\mathrm{H}},\{\theta_\sigma\}_{\sigma\in\mathrm{G}})\mapsto(\mathrm{B}_{\bar 1},\{\theta_\sigma\}_{\sigma\in\mathrm{H}}).$$

Consequently, we obtain an induced identification

$$\mathrm{Hom}\Big(U^{\mathrm{G}}\Big(\mathrm{G}\circlearrowright\coprod_{\bar\rho\in\mathrm{G}/\mathrm{H}}k\Big),U^{\mathrm{G}}(\mathrm{G}\circlearrowright\mathcal{A})\Big)\simeq\mathrm{Hom}(U^{\mathrm{H}}(\mathrm{H}\circlearrowright_1 k),U^{\mathrm{H}}(\mathrm{H}\circlearrowright\mathcal{A})).\quad(7.17)$$

Since by assumption the cohomology group $H^2(\mathrm{H},k^\times)$ is trivial, the H-invariant object $\mathcal{E}_1$ is H-equivariant, i.e., it belongs to $\mathcal{D}_c(\mathcal{A})^{\mathrm{H}}$; see Remark 7.4. Via the identification (7.17), $\mathcal{E}_1$ corresponds then to a morphism $\mathrm{G}\circlearrowright\coprod_{\bar\rho\in\mathrm{G}/\mathrm{H}}k\to\mathrm{G}\circlearrowright\mathcal{A}$ in $\mathrm{Hmo}^{\mathrm{G}}(k)$. Using the fact that $\mathrm{Hom}_{\mathcal{D}_c(\mathcal{A})}(\mathcal{E}_i,\mathcal{E}_j[m])=0$ for every $m\in\mathbb{Z}$ and $i\neq j$, we observe that this morphism is a G-equivariant Morita equivalence. Therefore, the proof now follows automatically from Lemma 3.6. □

Proposition 7.18. *Let* X *be a quasicompact quasiseparated* G*-scheme such that* $\mathrm{perf}(X)$ *admits a full exceptional collection*

$$\big(\mathcal{E}_1^1,\dots,\mathcal{E}_1^{s_1},\dots,\mathcal{E}_i^1,\dots,\mathcal{E}_i^{s_i},\dots,\mathcal{E}_n^1,\dots,\mathcal{E}_n^{s_n}\big).\quad(7.19)$$

For every fixed $i\in\{1,\dots,n\}$, *assume that the* G*-action on* $\mathrm{perf}(X)$ *transitively permutes the objects* $\mathcal{E}_i^1,\dots,\mathcal{E}_i^{s_i}$ *(up to isomorphism) and that* $\mathrm{Hom}(\mathcal{E}_i^j,\mathcal{E}_i^l[m])=0$

for every $m \in \mathbb{Z}$ *and* $j \neq l$. *Let* $\mathrm{H}_i \subseteq \mathrm{G}$ *be the stabilizer of* $\mathcal{E}_i^1$. *If* $\mathrm{H}_i \neq \mathrm{G}$, *assume that the cohomology group* $H^2(\mathrm{H}_i, k^\times)$ *is trivial. If* $\mathrm{H}_i = \mathrm{G}$, *denote by* $[\alpha_i] \in H^2(\mathrm{G}, k^\times)$ *the cohomology class of Remark 7.4 associated to the exceptional object* $\mathcal{E}_i^1$. *Under these assumptions, we have an isomorphism*

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \simeq \bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)_i)$$

in $\mathrm{Hmo}_0^{\mathrm{G}}(k)$, *where*

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)_i) \simeq \begin{cases} U^{\mathrm{G}}\big(\mathrm{G} \circlearrowright \coprod_{\bar{\rho} \in \mathrm{G}/\mathrm{H}_i} k\big) & \text{if } \mathrm{H}_i \neq \mathrm{G}, \\ U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_i} k) & \text{if } \mathrm{H}_i = \mathrm{G}. \end{cases}$$

Remark 7.20. Note that in the case where $s_1 = \cdots = s_n = 1$, Proposition 7.18 reduces to Proposition 7.8.

Proof. Let us denote by $\mathrm{perf}(X)_i$ the smallest triangulated subcategory of $\mathrm{perf}(X)$ generated by the exceptional objects $\mathcal{E}_i^1, \ldots, \mathcal{E}_i^{s_i}$. In the same vein, let us write $\mathrm{perf}_{\mathrm{dg}}(X)_i$ for the full dg subcategory of $\mathrm{perf}_{\mathrm{dg}}(X)$ consisting of those objects belonging to $\mathrm{perf}(X)_i$. With this notation, the full exceptional collection (7.19) can be written as a semiorthogonal decomposition $\mathrm{perf}(X) = \langle \mathrm{perf}(X)_1, \ldots, \mathrm{perf}(X)_n \rangle$. Using Proposition 7.1, we hence obtain an isomorphism between $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ and $\bigoplus_{i=1}^n U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)_i)$ in $\mathrm{Hmo}_0^{\mathrm{G}}(k)$. The proof follows now from application of Propositions 7.16 and 7.8 to each one of the G-dg categories such that $\mathrm{H}_i \neq \mathrm{G}$ and $\mathrm{H}_i = \mathrm{G}$, respectively. □

Example 7.21 (even dimensional quadrics). Let Q_q be a smooth projective quadric of even dimension d; consult Example 7.10. As proved in [Kapranov 1988], $\mathrm{perf}(Q_q)$ admits a full exceptional collection $(S_-, S_+, \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(d-1))$, where $\mathcal{S}_+$ and $\mathcal{S}_-$ are the spinor bundles. Moreover, we have $\mathrm{Hom}(S_-, S_+[m]) = 0$ for every $m \in \mathbb{Z}$. Similarly to Example 7.10, the objects $\mathcal{O}(i)$ are G-invariant for any G-action on Q_q. Regarding the spinor bundles, they are G-invariant or sent to each other by the quotient $\mathrm{G}/\mathrm{H} \simeq C_2$; see [Elagin 2012, §3.2]. In the former case, we obtain a motivic decomposition similar to that of Example 7.10. In the latter case, assuming that $H^2(\mathrm{H}, k^\times)$ is trivial, Proposition 7.18 yields an isomorphism between $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Q_q))$ and the direct sum

$$U^{\mathrm{G}}\Big(\mathrm{G} \circlearrowright \coprod_{\bar{\rho} \in C_2} k\Big) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k) \oplus \cdots \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^{(d-1)}} k),$$

where $[\alpha]$ stands for the cohomology class of Remark 7.4 associated to $\mathcal{O}(1)$.

Example 7.22 (del Pezzo surfaces). Assume that $\mathrm{char}(k) = 0$. Let X be the del Pezzo surface obtained by blowing up $\mathbb{P}^2$ at two distinct points x and y. As proved in [Orlov 1992, §4], $\mathrm{perf}(X)$ admits a full exceptional collection of length five $(\mathcal{O}_{E_1}(-1), \mathcal{O}_{E_2}(-1), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$, where $E_1 := \pi^{-1}(x)$ and $E_2 := \pi^{-1}(y)$ denote the exceptional divisors of the blowup $\pi : X \to \mathbb{P}^2$. Moreover, we have

$\mathrm{Hom}(\mathcal{O}_{E_1}(-1), \mathcal{O}_{E_2}(-1)[m]) = 0$ for every $m \in \mathbb{Z}$. The objects $\mathcal{O}(i)$ are G-invariant for every G-action on X. $\mathcal{O}_{E_1}(-1)$ and $\mathcal{O}_{E_2}(-1)$ are G-invariant or sent to each other by the quotient $\mathrm{G}/\mathrm{H} \simeq C_2$; see [Elagin 2012, §3.3]. In the former case, Proposition 7.8 yields an isomorphism between $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ and

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\gamma} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\beta} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{1} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^2} k),$$

where $[\alpha]$, $[\beta]$, and $[\gamma]$, stand for the cohomology classes of Remark 7.4 associated to the exceptional objects $\mathcal{O}(1)$, $\mathcal{O}_{E_2}(-1)$, and $\mathcal{O}_{E_1}(-1)$, respectively. In the latter case, assuming that the cohomology group $H^2(\mathrm{H}, k^\times)$ is trivial, Proposition 7.18 yields an isomorphism between $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ and the direct sum

$$U^{\mathrm{G}}\Big(\mathrm{G} \circlearrowright \coprod_{\bar{\rho} \in C_2} k\Big) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{1} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^2} k).$$

Remark 7.23 (direct summands). Let X be a smooth projective G-scheme as in Proposition 7.18. A proof similar to Theorem 1.2 shows that $\mathfrak{h}(X)_{\mathbb{Q}}$ is a direct summand of the G-equivariant Chow motive

$$\bigoplus_{r=0}^{\dim(X)} \bigoplus_{i=0}^{n} \mathfrak{h}^{\mathrm{G}}\Big(\coprod_{\bar{\rho} \in \mathrm{G}/\mathrm{H}_i} \mathrm{Spec}(k)\Big)_{\mathbb{Q}}(-r),$$

where G acts by permutation of the components.

8. Equivariant motivic measures

In this section, by a *variety* we mean a reduced separated k-scheme of finite type. Let us write $\mathrm{Var}^{\mathrm{G}}(k)$ for the category of G-*varieties*, i.e., varieties which are equipped with a G-action such that every orbit is contained in an affine open set; this condition is automatically satisfied whenever X is quasiprojective. The *Grothendieck ring of* G-*varieties* $K_0 \mathrm{Var}^{\mathrm{G}}(k)$ is defined to be the quotient of the free abelian group on the set of isomorphism classes of G-varieties $[X]$ by the relations $[X] = [Y] + [X \setminus Y]$, where Y is a closed G-subvariety of X. The multiplication is induced by the product of G-varieties (with diagonal G-action). A G-*equivariant motivic measure* is a ring homomorphism $\mu^{\mathrm{G}} : K_0 \mathrm{Var}^{\mathrm{G}}(k) \to R$.

Example 8.1. (i) When $k \subseteq \mathbb{C}$, the topological Euler characteristic χ (with compact support) gives rise to a G-equivariant motivic measure

$$\mu^{\mathrm{G}}_{\chi} : K_0 \mathrm{Var}^{\mathrm{G}}(k) \to R_{\mathbb{Q}}(\mathrm{G}), \quad [X] \mapsto \sum_i (-1)^i H^i_c(X^{\mathrm{an}}, \mathbb{Q}),$$

where $H^i_c(X^{\mathrm{an}}, \mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-linear G-representation.

(ii) When $\mathrm{char}(k)=0$, the characteristic polynomial $P_X(t) := \sum_i H^i_{dR}(X)t^i$, with X a smooth projective G-variety, gives rise to a G-equivariant motivic measure $\mu_P^{\mathrm{G}} : K_0\,\mathrm{Var}^{\mathrm{G}}(k) \to R(\mathrm{G})[t]$, where $H^i_{dR}(X)$ is considered as a finite dimensional k-linear G-representation.

Let us denote by $K_0(\mathrm{NChow}^{\mathrm{G}}(k))$ the Grothendieck ring of the additive symmetric monoidal category of G-equivariant noncommutative Chow motives.

Theorem 8.2. *When* $\mathrm{char}(k)=0$, *the assignment* $X \mapsto [U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))]$, *with X a smooth projective* G-*variety, gives rise to a* G-*equivariant motivic measure*

$$\mu_{\mathrm{nc}}^{\mathrm{G}} : K_0\,\mathrm{Var}^{\mathrm{G}}(k) \to K_0(\mathrm{NChow}^{\mathrm{G}}(k)).$$

Proof. Thanks to Bittner's presentation [2004, Lemma 7.1] of the ring $K_0\,\mathrm{Var}^{\mathrm{G}}(k)$, it suffices to verify the following two conditions:

(i) Given smooth projective G-schemes X and Y, we have

$$[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X \times Y))] = [U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \otimes U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Y))].$$

(ii) Let X be a smooth projective G-variety, Y a closed smooth G-subvariety of codimension c, $\mathrm{Bl}_Y(X)$ the blowup of X along Y, and E the exceptional divisor of this blowup. With this notation, the difference

$$\big[U^{\mathrm{G}}\big(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(\mathrm{Bl}_Y(X))\big)\big] - \big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(E))\big]$$

is equal to the difference

$$\big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))\big] - \big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Y))\big].$$

As proved in [Tabuada and Van den Bergh 2015a, Lemma 4.26], we have the G-equivariant Morita equivalence

$$\mathrm{perf}_{\mathrm{dg}}(X) \otimes \mathrm{perf}_{\mathrm{dg}}(Y) \to \mathrm{perf}_{\mathrm{dg}}(X \times Y), \quad (\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \boxtimes \mathcal{G}.$$

Thus, (i) follows from Lemma 3.6 and the fact that the functor U^{G} is symmetric monoidal. For (ii), recall from [Orlov 1992, Theorem 4.3] that $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Bl}_Y(X))$ contains full G-dg subcategories $\mathrm{perf}_{\mathrm{dg}}(X), \mathrm{perf}_{\mathrm{dg}}(Y)_0, \ldots, \mathrm{perf}_{\mathrm{dg}}(Y)_{c-2}$ inducing a semiorthogonal decomposition $\mathrm{perf}(\mathrm{Bl}_Y(X)) = \langle \mathrm{perf}(X), \mathrm{perf}(Y)_0, \ldots, \mathrm{perf}(Y)_{c-2}\rangle$. Moreover, we have an isomorphism $\mathrm{perf}_{\mathrm{dg}}(Y)_i \simeq \mathrm{perf}_{\mathrm{dg}}(Y)$ in $\mathrm{Hmo}^{\mathrm{G}}(k)$ for every i. Making use of Proposition 7.1, we obtain the equality

$$\big[U^{\mathrm{G}}\big(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(\mathrm{Bl}_Y(X))\big)\big] = \big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))\big] + (c-1)\big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Y))\big].$$

Similarly, recall from [Orlov 1992, Theorem 2.6] that $\mathrm{perf}_{\mathrm{dg}}(E)$ contains full G-dg subcategories $\mathrm{perf}_{\mathrm{dg}}(Y)_0, \ldots, \mathrm{perf}_{\mathrm{dg}}(Y)_{c-1}$ inducing a semiorthogonal decomposition $\mathrm{perf}(E) = \langle \mathrm{perf}(Y)_0, \ldots, \mathrm{perf}(Y)_{c-1}\rangle$. Moreover, $\mathrm{perf}_{\mathrm{dg}}(Y)_i \simeq \mathrm{perf}_{\mathrm{dg}}(Y)$ in

$\mathrm{Hmo}^{\mathrm{G}}(k)$ for every i. Making use of Proposition 7.1, we conclude that

$$\big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(E))\big] = c\big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Y))\big].$$

Condition (ii) now follows automatically from the preceding two equalities. $\square$

Proposition 8.3. *The motivic measure $\mu_{\chi}^{\mathrm{G}} \otimes_{\mathbb{Q}} \mathbb{C}$ factors through $\mu_{\mathrm{nc}}^{\mathrm{G}}$.*

Proof. Hochschild homology $HH : \mathrm{dgcat}(k) \to \mathcal{D}(k)$ is an example of a symmetric monoidal additive invariant. Thanks to Proposition 4.11, it then gives rise to an additive symmetric monoidal functor $\overline{HH^{\circlearrowright}} : \mathrm{Hmo}_0^{\mathrm{G}}(k) \to \mathcal{D}(k)^{\mathrm{G}}$ such that $\overline{HH^{\circlearrowright}} \circ U^{\mathrm{G}} \simeq HH^{\circlearrowright}$. Consider the composition

$$\mathrm{Hmo}_0^{\mathrm{G}}(k) \xrightarrow{\overline{HH^{\circlearrowright}}} \mathcal{D}(k)^{\mathrm{G}} \xrightarrow{-\otimes_k \mathbb{C}} \mathcal{D}(\mathbb{C})^{\mathrm{G}}. \tag{8.4}$$

It is well-known that an object of $\mathcal{D}(k)$ is strongly dualizable if and only if it is compact. Since the category of G-equivariant noncommutative Chow motives is rigid (see Proposition 3.13), the composition (8.4) yields a ring homomorphism

$$K_0(\mathrm{NChow}^{\mathrm{G}}(k)) \to K_0(\mathcal{D}_c(\mathbb{C})^{\mathrm{G}}) \simeq R_{\mathbb{C}}(\mathrm{G}). \tag{8.5}$$

We claim that $\mu_{\chi}^{\mathrm{G}} \otimes_{\mathbb{Q}} \mathbb{C}$ agrees with the composition of $\mu_{\mathrm{nc}}^{\mathrm{G}}$ with (8.5). Let X be a smooth projective G-variety. Thanks to Bittner's presentation of $K_0 \,\mathrm{Var}^{\mathrm{G}}(k)$, it suffices to verify that the class of $HH^{\circlearrowright}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \otimes_k \mathbb{C}$ in the representation ring $R_{\mathbb{C}}(\mathrm{G})$ agrees with $\sum_i (-1)^i H_c^i(X^{\mathrm{an}}, \mathbb{C})$. This follows from the identifications

$$\begin{aligned}
[HH^{\circlearrowright}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \otimes_k \mathbb{C}] &= \sum_i (-1)^i HH_i(\mathrm{perf}_{\mathrm{dg}}(X)) \otimes_k \mathbb{C} \\
&= \sum_i (-1)^i \bigoplus_{p-q=i} H^q(X, \Omega_X^p) \otimes_k \mathbb{C} \qquad (8.6)\\
&= \sum_{p,q} (-1)^{p-q} H^q(X, \Omega_X^p) \otimes_k \mathbb{C} \\
&= \sum_{p,q} (-1)^{p+q} H^q(X, \Omega_X^p) \otimes_k \mathbb{C} \\
&= \sum_i (-1)^i H_c^i(X^{\mathrm{an}}, \mathbb{C}),
\end{aligned}$$

where (8.6) is a consequence of the (functorial) Hochschild–Kostant–Rosenberg isomorphism $HH_i(\mathrm{perf}_{\mathrm{dg}}(X)) \simeq \bigoplus_{p-q=i} H^q(X, \Omega_X^p)$. $\square$

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