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Hochschild homology, lax codescent, and duplicial structure

Richard Garner, Stephen Lack and Paul Slevin

We study the duplicial objects of Dwyer and Kan, which generalize the cyclic objects of Connes. We describe duplicial objects in terms of the decalage comonads, and we give a conceptual account of the construction of duplicial objects due to Böhm and Ştefan. This is done in terms of a 2-categorical generalization of Hochschild homology. We also study duplicial structure on nerves of categories, bicategories, and monoidal categories.

1. Introduction

The cyclic category Λ was introduced by Connes [1983] as part of his program to study noncommutative geometry. Cyclic objects, given by functors with domain Λ, have been studied by too many authors to list here, but many of these can be found in the reference list of the classic book [Loday 1992].

Various generalizations of cyclic structure have been considered; in particular the notion of *duplicial object* was studied in [Dwyer and Kan 1985]. These are given by functors with domain $\boldsymbol{K}^{\mathrm{op}}$, for a certain category $\boldsymbol{K}$ of which Λ is a quotient. Like cyclic objects, duplicial objects are simplicial objects equipped with extra structure. In both cases, the extra structure involves an endomorphism $t_n : X_n \to X_n$ of the object of n-simplices, for each n, subject to various conditions relating it to the simplicial structure. The difference between the two notions is that in the case of cyclic structure, the map t_n is an automorphism of order $n+1$, so that $t_n^{n+1} = 1$.

There is also an intermediate notion, in which the t_n are required to be invertible but the condition that $t_n^{n+1} = 1$ is dropped. This was called *paracyclic structure* in [Getzler and Jones 1993], and also studied in [Elmendorf 1993], where the indexing category was called the "linear category". Somewhat confusingly, the name paracyclic has also been used by some authors to refer to what is called duplicial by Dwyer and Kan.

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In this paper we provide a new perspective on duplicial structure, and analyze ways in which it arises. As explained, for example, in [Mac Lane 1971], a comonad on a category gives rise to simplicial structure on each object of that category, and this is the starting point for many homology theories. Just as simplicial structure can be used to define homology, cyclic (or duplicial or paracyclic) structure can be used to define cyclic homology. In a series of papers, Böhm and Ştefan [2008; 2009; 2012] looked at what further structure than a comonad is needed to equip the induced simplicial object with duplicial structure; the main extra ingredient turned out to be a second comonad with a distributive law [Beck 1969] between the two. They also showed that their machinery could be used to construct the cyclic homology of bialgebroids. This was further studied in the papers [Krähmer and Slevin 2016; Kowalzig et al. 2015] by the third of us, along with various coauthors.

In the case of comonads and simplicial structure, there is a universal nature to the construction, once again explained in [Mac Lane 1971], and also in Section 2 below. There is no analogue given in the analysis of Böhm–Ştefan, and our first goal is to provide one.

As well as the construction of simplicial structure from comonads, we also consider a second way that simplicial structure arises, namely as nerves of categories or other (possibly higher) categorical structures. Our second main goal is to analyze when the simplicial sets arising as nerves can be given duplicial structure.

The third main achievement of the paper actually arose as a byproduct of our investigations towards the first goal. It is a connection between duplicial structure, especially as arising via the Böhm–Ştefan construction, and Hochschild homology and cohomology. We present this first. We consider some very simple aspects of Hochschild homology and cohomology, only involving the zeroth homology and cohomology, and we generalize it to a 2-categorical context in a "lax" way. The resulting theory allows us to recapture the Böhm–Ştefan construction as a sort of cap product in a very special case.

We end this introduction by remarking briefly on the two roles of 2-categories in this paper. On the one hand, 2-categories appear at a fairly accessible point in the ever-expanding zoo of higher categorical structures: in what is now becoming common terminology they are the "(2,2)-categories", where an (m, n)-category has no nontrivial morphisms above dimension m, and no noninvertible morphisms above dimension n. This is relevant to the lax version of Hochschild theory we begin to develop here. On the other hand, 2-categories have a key organizational role. Collections of categories naturally form themselves into 2-categories, and higher dimensional categories can also often usefully be formed into 2-categories, as seen for example in Joyal's approach to quasicategory theory. It is this organizational role which is most important in the current paper, and lies behind our analysis of comonads, distributive laws, duplicial structure, and so on.

2. Simplicial structure, comonads, and decalage

In this section we recall some ideas related to simplicial structure, most of which are well-known, although the notation used varies. The one new result is Proposition 2.4, which reformulates the notion of duplicial structure in terms of decalage comonads.

2A. ***Simplicial structure arising from comonads.*** Let $\mathbb{M}$ be the strict monoidal category of finite ordinals and order-preserving maps, with tensor product given by ordinal sum and the empty ordinal serving as the unit. This is sometimes known as the "algebraists' Δ", and is denoted by Δ in [Mac Lane 1971] and Δ_+ in many other sources, such as [Verity 2008].

The full subcategory of $\mathbb{M}$ consisting of the nonempty finite ordinals is isomorphic to the usual Δ (the "topologists' Δ"). A contravariant functor defined on Δ is a simplicial object, while a contravariant functor defined on (the underlying category of) $\mathbb{M}$ is an augmented simplicial object.

$\mathbb{M}$ is the "universal monoidal category containing a monoid", in the sense that for any strict monoidal category $\mathcal{C}$, there is a bijection between monoids in $\mathcal{C}$ and strict monoidal functors from $\mathbb{M}$ to $\mathcal{C}$. (Similarly, if $\mathcal{C}$ is a general monoidal category then to give a monoid in $\mathcal{C}$ is equivalent, in a suitable sense, to giving a strong monoidal functor from $\mathbb{M}$ to $\mathcal{C}$.)

Dually, there is a bijection between comonoids in $\mathcal{C}$ and strict monoidal functors from $\mathbb{M}^{\mathrm{op}}$ to $\mathcal{C}$, and so any comonoid in $\mathcal{C}$ determines an augmented simplicial object in $\mathcal{C}$. In particular, we could take $\mathcal{C}$ to be the strict monoidal category $[X, X]$ of endofunctors of a category X, so that a comonoid in $\mathcal{C}$ is just a comonad on X. Then any comonad g on X determines a unique strict monoidal functor $\mathbb{M}^{\mathrm{op}} \to [X, X]$. We may now transpose this so as to obtain a functor $X \to [\mathbb{M}^{\mathrm{op}}, X]$ sending each object of X to an augmented simplicial object in X called its *bar resolution* with respect to g.

When, in the introduction, we referred to the "universal nature" of the construction of simplicial objects from comonads, it was precisely this analysis, using the universal property of $\mathbb{M}$, which we had in mind, and which we shall extend so as to explain the Böhm–Ştefan construction.

Remark 2.1. There is an automorphism of $\mathbb{M}$ which arises from the fact that the opposite of the ordinal

$$n = \{0 < \cdots < n-1\}$$

is isomorphic to n itself. The automorphism fixes the objects, and sends an order-preserving map $f : m \to n$ to f^{rev}, where $f^{\mathrm{rev}}(i) = m - 1 - f(n-1-i)$. This automorphism reverses the monoidal structure, in the sense that $n + n' = n' + n$ on objects, while for morphisms $f : m \to n$ and $f' : m' \to n'$ we have

$$(f + f')^{\mathrm{rev}} = (f')^{\mathrm{rev}} + f^{\mathrm{rev}}.$$

2B. ***The decalage comonads.*** The monoidal structure on $\mathbb{M}$ extends, via Day convolution [Day 1970], to a monoidal structure on the category $[\mathbb{M}^{\mathrm{op}}, \mathbf{Set}]$ of augmented simplicial sets. The resulting structure is nonsymmetric, but closed on both sides, so that there is both a left and a right internal hom.

Since the ordinal 1 is a monoid in $\mathbb{M}$, the representable $\mathbb{M}(-,1)$ is a monoid in $[\mathbb{M}^{\mathrm{op}}, \mathbf{Set}]$, and so the internal hom out of $\mathbb{M}(-,1)$ becomes a comonad; or rather, there are two such comonads depending on whether one uses the left or right internal hom. These are called the decalage comonads, and they both restrict to give comonads, also called decalage, on the category $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ of simplicial sets.

As well as this abstract description, there is also a straightforward explicit description, which we now give for the case of augmented simplicial sets.

Given an augmented simplicial set X as in the diagram

$$\cdots\ X_2 \;\begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xrightarrow{d_2} \end{smallmatrix}\; X_1 \;\begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \end{smallmatrix}\; X_0 \xrightarrow{d_0} X_{-1}$$

the right decalage $\mathrm{Dec}_{\mathrm{r}}(X)$ of X is the augmented simplicial set

$$\cdots\ X_3 \;\begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xrightarrow{d_2} \end{smallmatrix}\; X_2 \;\begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \end{smallmatrix}\; X_1 \xrightarrow{d_0} X_0$$

obtained by discarding X_{-1} and the last face and degeneracy map in each degree. There is a canonical map $\varepsilon : \mathrm{Dec}_{\mathrm{r}}(X) \to X$ defined using the discarded face maps, so that $\varepsilon_n : \mathrm{Dec}_{\mathrm{r}}(X)_n \to X_n$ is d_{n+1}; and a canonical map $\delta : \mathrm{Dec}_{\mathrm{r}}(X) \to \mathrm{Dec}_{\mathrm{r}}(\mathrm{Dec}_{\mathrm{r}}(X))$ defined via the discarded degeneracy maps, so that $\delta_n : \mathrm{Dec}_{\mathrm{r}}(X)_n \to \mathrm{Dec}_{\mathrm{r}}(\mathrm{Dec}_{\mathrm{r}}(X))_n$ is s_{n+1}. These maps δ and ε define the comultiplication and counit of the comonad.

Similarly, the left decalage $\mathrm{Dec}_{\mathrm{l}}(X)$ of X is the augmented simplicial set

$$\cdots\ X_3 \;\begin{smallmatrix} \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xrightarrow{d_2} \\ \xleftarrow{s_2} \\ \xrightarrow{d_3} \end{smallmatrix}\; X_2 \;\begin{smallmatrix} \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xrightarrow{d_2} \end{smallmatrix}\; X_1 \xrightarrow{d_1} X_0$$

obtained by discarding X_{-1} and the first face and degeneracy map in each degree.

We have described the decalage comonads for simplicial and augmented simplicial sets, but in much the same way, there are decalage comonads $\mathrm{Dec}_{\mathrm{r}}$ and $\mathrm{Dec}_{\mathrm{l}}$ on the categories $[\Delta^{\mathrm{op}}, P]$ and $[\mathbb{M}^{\mathrm{op}}, P]$ of simplicial and augmented simplicial objects in P for any category P, although in general there will no longer be a monoidal structure with respect to which decalage is given by an internal hom.

2C. ***Duplicial structure.*** Here we recall the definition of duplicial structure, and give a reformulation using the decalage comonads. As stated already in the introduction, a duplicial object in a category is a simplicial object X, equipped with a map $t_n : X_n \to X_n$ for each $n \geqslant 0$, subject to various conditions which we now state explicitly:

$$d_i t_{n+1} = \begin{cases} t_n d_{i-1} & \text{if } 1 \leq i \leq n+1, \\ d_{n+1} & \text{if } i = 0; \end{cases} \tag{2.2}$$

$$s_i t_n = \begin{cases} t_{n+1} s_{i-1} & \text{if } 1 \leq i \leq n, \\ t_{n+1}^2 s_n & \text{if } i = 0. \end{cases} \tag{2.3}$$

There is also a formulation of this structure which uses an "extra degeneracy map" $s_{-1}: X_n \to X_{n+1}$ in each degree instead of the t_n; this s_{-1} may be constructed as the composite $t_{n+1} s_n$. As in the introduction, X is called *paracyclic* if each t_n is invertible, and *cyclic* if additionally $t_n^{n+1} = 1$.

The indexing category for cyclic structure is Connes' cyclic category Λ, which is a sort of wreath product of Δ and the cyclic groups. This is explained for example in [Loday 1992, Chapter 6], where the more general notion of crossed simplicial group can also be found. This involves replacing the cyclic groups by some other family of groups indexed by the natural numbers, and equipped with suitable actions of Δ which allow the formation of the wreath product. The indexing category for paracyclic structure can be obtained in this way on taking all the groups to be $\mathbb{Z}$ [Loday 1992, Proposition 6.3.4(c)]. Using the presentation for duplicial structure given above, it is straightforward to modify this argument to see that the indexing category $\boldsymbol{K}$ for duplicial structure is once again a wreath product, but this time by a "crossed simplicial monoid", involving the monoid $\mathbb{N}$ in each degree.

Proposition 2.4. *Giving duplicial structure to a simplicial object X is equivalent to giving a simplicial map $t: \mathrm{Dec_r}\, X \to \mathrm{Dec_l}\, X$ making the following diagrams commute*:

$$\begin{tikzcd} \mathrm{Dec_r}\, X \arrow[r, "t"] \arrow[dr, "\varepsilon"'] & \mathrm{Dec_l}\, X \arrow[d, "\varepsilon"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Dec_r}\, X \arrow[r, "t"] \arrow[d, "\delta"'] & \mathrm{Dec_l}\, X \arrow[r, "\delta"] & \mathrm{Dec_l}^2\, X \\ \mathrm{Dec_r}^2\, X \arrow[r, "\mathrm{Dec_r}\, t"'] & \mathrm{Dec_r}\,\mathrm{Dec_l}\, X \arrow[r, equal] & \mathrm{Dec_l}\,\mathrm{Dec_r}\, X \arrow[u, "\mathrm{Dec_l}\, t"'] \end{tikzcd}$$

Proof. The data of a simplicial map $t: \mathrm{Dec_r}\, X \to \mathrm{Dec_l}\, X$ comprises a sequence of maps $t_n: X_n \to X_n$ for each $n > 0$ satisfying certain conditions. Compatibility of t with face maps gives the cases where $i > 0$ of (2.2), while those where $i = 0$ are the compatibility condition with ε. Likewise, compatibility of t with degeneracy maps yields the cases $i, n > 0$ of (2.3), while the cases where $n > 0$ but $i = 0$ are the compatibility condition with δ.

The one thing which remains is to see that a map $t_0: X_0 \to X_0$ satisfying (2.3) for $n = 0$ can be uniquely recovered from the remaining data and axioms. In order to have $s_0 t_0 = t_1^2 s_0$, we must have $t_0 = d_0 s_0 t_0 = d_0 t_1^2 s_0 = d_1 t_1 s_0$. So we just need to check that, defining t_0 in this way, it satisfies the required relations; but this is indeed the case as the following calculations show:

$$(d_1t_1s_0)d_0 = d_1t_1d_0s_1 = d_1d_1t_2s_1 = d_1d_2t_2s_1 = d_1t_1d_1s_1 = d_1t_1 \quad \text{and}$$
$$s_0(d_1t_1s_0) = d_2s_0t_1s_0 = d_2t_2^2s_1s_0 = t_1d_1t_2s_1s_0 = t_1^2d_0s_1s_0 = t_1^2d_0s_0s_0 = t_1^2s_0. \quad \square$$

2D. ***The Böhm–Ştefan construction.*** We now describe the construction in [Böhm and Ştefan 2008; 2009]. The original formulation involves monads and coduplicial structure, but we work dually with comonads so as to obtain duplicial structure. Let A and P be categories, and suppose that we have a comonad (g, δ, ε) on A and a functor $f : A \to P$. As explained in Section 2A, we obtain from g a functor $A \to [\mathbb{M}^{\mathrm{op}}, A]$ sending each object to its bar resolution with respect to g, and postcomposing with f yields a functor $f^g : A \to [\mathbb{M}^{\mathrm{op}}, P]$. Explicitly, f^g takes x in A to the augmented simplicial object $f^g(x)$ with $f^g(x)_n = fg^{n+1}x$ and with face and degeneracy maps:

$$d_i = fg^i\varepsilon g^{n-i}x : f^g(x)_n \to f^g(x)_{n-1} \quad \text{and}$$
$$s_j = fg^j\delta g^{n-j}x : f^g(x)_n \to f^g(x)_{n+1}.$$

The basic construction of [Böhm and Ştefan 2008] uses additional data to equip objects of the form $f^g(x)$ with duplicial structure. We suppose given another comonad h on A, and a *distributive law* [Beck 1969] $\lambda : gh \to hg$ — a natural transformation satisfying four axioms relating it to the comonad structures. We suppose moreover that the functor $f : A \to P$ is equipped with a natural transformation $\varphi : fh \to fg$ rendering commutative the diagrams

$$\begin{tikzcd} fh \arrow[rrr, "\varphi"] \arrow[d, "f\delta"'] & & & fg \arrow[d, "f\delta"] \\ fh^2 \arrow[r, "\varphi h"'] & fgh \arrow[r, "f\lambda"'] & fhg \arrow[r, "\varphi g"'] & fg^2 \end{tikzcd} \qquad \begin{tikzcd} fh \arrow[rr, "\varphi"] \arrow[dr, "f\varepsilon"'] & & fg. \arrow[dl, "f\varepsilon"] \\ & f & \end{tikzcd} \tag{2.5}$$

This was called *left* λ*-coalgebra* structure on f in [Kowalzig et al. 2015], and the totality $(A, P, g, h, f, \lambda, \varphi)$ of the structure considered so far was called an *admissible septuple* in [Böhm and Ştefan 2008]. Finally, we assume given an object $x \in A$ equipped with a map $\xi : gx \to hx$ rendering commutative the diagrams

$$\begin{tikzcd} gx \arrow[rrr, "\xi"] \arrow[d, "\delta x"'] & & & hx \arrow[d, "\delta x"] \\ g^2x \arrow[r, "g\xi"'] & ghx \arrow[r, "\lambda x"'] & hgx \arrow[r, "h\xi"'] & h^2x \end{tikzcd} \qquad \begin{tikzcd} gx \arrow[rr, "\xi"] \arrow[dr, "\varepsilon x"'] & & hx. \arrow[dl, "\varepsilon x"] \\ & x & \end{tikzcd} \tag{2.6}$$

This was called *right* λ*-coalgebra* structure in [Krähmer and Slevin 2016], and a "transposition map" in [Böhm and Ştefan 2008], though the notion itself goes back to [Burroni 1973]. Under these assumptions, it was shown in [Böhm and Ştefan

2008] that the simplicial object $f^g(x)$ admits a duplicial structure. The duplicial operator $t_n : f^g(x)_n \to f^g(x)_n$ is given by the composite

$$fg^{n+1}x \xrightarrow{fg^n\xi x} fg^n hx \xrightarrow{f\lambda^n x} fhg^n x \xrightarrow{\varphi g^n x} fg^{n+1}x,$$

where the natural transformation $\lambda^n : g^n h \to hg^n$ denotes the composite

$$g^n h \xrightarrow{g^{n-1}\lambda} g^{n-1}hg \xrightarrow{g^{n-2}\lambda g} g^{n-2}hg^2 \longrightarrow \cdots \longrightarrow ghg^{n-1} \xrightarrow{\lambda g^{n-1}} hg^n.$$

In [Böhm and Ştefan 2008], this construction was used to obtain, among other things, the cyclic cohomology and homology of bialgebroids.

There is an automorphism $\Phi : [\mathbb{M}^{\mathrm{op}}, P] \to [\mathbb{M}^{\mathrm{op}}, P]$, induced by the automorphism in Remark 2.1, that maps a simplicial object X to the simplicial object associated to X, obtained by reversing the order of all face and degeneracy maps. In [Kowalzig et al. 2015] it is explained that $\Phi f^h(x)$ is duplicial, and that there are two duplicial maps

$$f^g(x) \xrightarrow{R} \Phi f^h(x), \qquad \Phi f^h(x) \xrightarrow{L} f^g(x),$$

defined by iteration of φ and ξ, respectively, which are mutual inverses if and only if both objects are cyclic.

2E. *Zeroth Hochschild homology and cohomology.* Let A be a ring, and X a bimodule over A. There is an induced simplicial abelian group, part of which looks like

$$\cdots\ A\otimes A\otimes X \;\substack{\xrightarrow{d_0}\\ \xrightarrow{d_1}\\ \xrightarrow{d_2}}\; A\otimes X \;\substack{\xrightarrow{d_0}\\ \xleftarrow{s_0}\\ \xrightarrow{d_1}}\; X$$

with the maps given as follows:

$$\begin{aligned} d_0(a\otimes x) &= xa, & d_0(a\otimes b\otimes x) &= b\otimes xa,\\ d_1(a\otimes x) &= ax, & d_1(a\otimes b\otimes x) &= ab\otimes x,\\ s_0(x) &= 1\otimes x, & d_2(a\otimes b\otimes x) &= a\otimes bx, \end{aligned}$$

and which is defined analogously in higher degrees. We call this simplicial object the *Hochschild complex* of X, although often that name refers to the corresponding (normalized or otherwise) chain complex.

The *zeroth homology of A with coefficients in X* is the colimit $H_0(A, X)$ of this diagram, which can more simply be computed as the coequalizer of the two maps $A\otimes X \rightrightarrows X$; more explicitly still, this is the quotient of X by the subgroup generated by all elements of the form $ax - xa$.

Dually there is a cosimplicial object, part of which looks like

$$X \;\substack{\xrightarrow{\delta_0}\\ \xleftarrow{\sigma_0}\\ \xrightarrow{\delta_1}}\; [A, X] \;\substack{\xrightarrow{\delta_0}\\ \xrightarrow{\delta_1}\\ \xrightarrow{\delta_2}}\; [A\otimes A, X]\ \cdots$$

with the maps given as follows:

$$\begin{aligned} \delta_0(x)(a) &= xa, & \delta_0(f)(a\otimes b) &= f(a)b, \\ \delta_1(x)(a) &= ax, & \delta_1(f)(a\otimes b) &= f(ab), \\ \sigma_0(f) &= f(1), & \delta_2(f)(a\otimes b) &= af(b), \end{aligned}$$

and now the *zeroth Hochschild cohomology of A with coefficients in X* is the limit $H^0(A, X)$ (really an equalizer) of this diagram, given explicitly by the subgroup of X consisting of those x for which $xa = ax$ for all $a \in A$.

2F. ***Universality of zeroth Hochschild homology and cohomology.*** There are universal characterizations for both $H^0(A, X)$ and $H_0(A, X)$. For any A-bimodule X and any abelian group P, there is an induced bimodule structure on $[X, P]$ given by $(af)(x) = f(xa)$ and $(fa)(x) = f(ax)$, and this construction gives a functor $[X, -] : \mathbf{Ab} \to A\text{-}\mathbf{Mod}\text{-}A$. In particular, we may take $X = A$ with its regular left and right actions.

Proposition 2.7. *The functor $[A, -] : \mathbf{Ab} \to A\text{-}\mathbf{Mod}\text{-}A$ has a left adjoint sending an A-bimodule X to $H_0(A, X)$.*

Similarly, there is for any A-bimodule X and abelian group P an induced bimodule structure on $X \otimes P$ given by $a(x \otimes p) = ax \otimes p$ and $(x \otimes p)a = xa \otimes p$, and this gives a functor $X \otimes (-) : \mathbf{Ab} \to A\text{-}\mathbf{Mod}\text{-}A$. Considering again the case $X = A$, we have:

Proposition 2.8. *The functor $A \otimes (-) : \mathbf{Ab} \to A\text{-}\mathbf{Mod}\text{-}A$ has a right adjoint sending an A-bimodule X to $H^0(A, X)$.*

3. Bimodules

We described above the Hochschild complex of a ring A with coefficients in an A-bimodule. A ring is the same thing as a monoid in the monoidal category $\mathbf{Ab}$ of abelian groups, and more generally the Hochschild complex and the zeroth homology and cohomology can be constructed if A is a monoid in a suitable symmetric monoidal closed category $\mathcal{V}$. In particular, we could do this for the cartesian closed category $\mathbf{Cat}$. But $\mathbf{Cat}$ is in fact a 2-category, which opens the way to consider *lax* variants of the theory, and it is such a variant that we now present. While it would be possible to develop this theory in the context of a general symmetric monoidal closed bicategory $\mathcal{V}$, it is only the case $\mathcal{V} = \mathbf{Cat}$ which we need, and so we restrict ourselves to that.

The first step, carried out in this section, is to describe in detail the notion of bimodule that will play the role of coefficient object for our lax homology and cohomology. We describe a certain 2-category $A\text{-}\mathbf{Mod}\text{-}A$ of bimodules, which involves a combination of strict and lax notions. The precise choice of what should be strict and what should be lax might at first seem arbitrary; we have made these choices so that our cohomology $H^0(A,-)$ and homology $H_0(A,-)$ can be defined via universal properties.

3A. ***Monoids.*** A monoid in **Cat** is precisely a strict monoidal category. It is not particularly difficult to adapt the theory that follows to deal with nonstrict monoidal categories, but we do not need this extra generality, and feel that the complications that it causes might distract from the story we wish to tell. It is probably also possible to extend the theory to deal with skew monoidal categories [Szlachányi 2012; Lack and Street 2012], although we have not checked this in detail.

We shall therefore consider a strict monoidal category (A, m, i). We shall write $a \otimes b$ or sometimes just ab for the image under the tensor functor $m : A \times A \to A$ of a pair (a, b).

3B. ***Modules.*** Next we need a notion of module over our monoid (strict monoidal category) A. There is a well-developed (pseudo) notion of an action of a monoidal category on a category, sometimes called an actegory. Here, however, we deal only with the strict case, which does not use the 2-category structure of **Cat**; once again it would not be difficult to extend our theory to deal with pseudo (or possibly skew) actions, but this is not needed for our applications so we have not done so. Giving a strict left action of A on a category X is equivalent to giving a strict monoidal functor from A to the strict monoidal category $\mathrm{End}(X)$ of endofunctors of X. The image under the corresponding functor $\alpha : A \times X \to X$ of an object (a, x) is written ax. Similarly there are (strict) right actions involving functors $\beta : X \times A \to X : (x, a) \mapsto xa$ satisfying strict associativity and unit conditions.

In fact, we also make use of a slightly more general notion. It is possible to consider actions of monoids not just on sets, but also on objects of other categories; in the same way, it is possible to consider actions of monoidal categories on objects of other 2-categories. If X is an object of a 2-category $\mathcal{K}$, then an action of A on X is a strict monoidal functor from A to the strict monoidal category $\mathcal{K}(X, X)$ of endomorphisms of X.

If the 2-category $\mathcal{K}$ admits copowers, then there is an equivalent formulation as follows. Recall that the copower of an object X by a category P is an object $P \cdot X$ equipped with isomorphisms of categories

$$\mathcal{K}(P \cdot X, Y) \cong \mathbf{Cat}(P, \mathcal{K}(X, Y))$$

2-natural in the variable $Y \in \mathcal{K}$. If $\mathcal{K}$ has all copowers, then there are 2-natural isomorphisms $(P \times Q) \cdot X \cong P \cdot (Q \cdot X)$ and $1 \cdot X \cong X$. In this case, a strict (left) action of A on X is equivalently a morphism $\alpha : A \cdot X \to X$ in $\mathcal{K}$ for which the diagrams

$$
\begin{tikzcd}
(A \times A) \cdot X \arrow[rr, "m \cdot 1"] \arrow[d] & & A \cdot X \arrow[d, "\alpha"] \\
A \cdot (A \cdot X) \arrow[r, "1 \cdot \alpha"'] & A \cdot X \arrow[r, "\alpha"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \cdot X \arrow[r, "i \cdot 1"] \arrow[dr] & A \cdot X \arrow[d, "\alpha"] \\
& X
\end{tikzcd}
$$

commute, where the unnamed maps are the isomorphisms just described. (There are also still more general notions of action of A; see [Kelly and Lack 1997, Section 2].)

Note that the 2-category **Cat** admits copowers, with $A \cdot X$ given by the cartesian product $A \times X$, so that in this case our more general notion of action of A on $X \in \mathbf{Cat}$ reduces to the initial one.

Example 3.1. Our running example throughout this section and the next takes A to be the strict monoidal category $\mathbb{M}^{\mathrm{op}}$; it is this example which will be used to explain the Böhm–Ştefan construction. Since a strict monoidal functor $\mathbb{M}^{\mathrm{op}} \to \mathcal{K}(X, X)$ is precisely a comonoid in $\mathcal{K}(X, X)$, a left $\mathbb{M}^{\mathrm{op}}$-module is a comonad in the 2-category $\mathcal{K}$, in the sense of [Street 1972]. On the other hand, a right $\mathbb{M}^{\mathrm{op}}$-module is also just a comonad in $\mathcal{K}$, as follows from Remark 2.1.

In the case $\mathcal{K} = \mathbf{Cat}$, a comonad in **Cat** is a category X equipped with a comonad g. For an object n of $\mathbb{M}^{\mathrm{op}}$ and an object $x \in X$, the value nx of the corresponding left $\mathbb{M}^{\mathrm{op}}$-action is given by $g^n x$.

3C. ***Morphisms of modules.*** When it comes to morphisms of modules, once again there is a question of how lax they should be, and this time we deviate from the completely strict situation. If X and Y are (strict, as ever) left A-modules in **Cat**, we define a *lax A-morphism* to be a functor $p : X \to Y$, equipped with a natural transformation

$$
\begin{tikzcd}
A \times X \arrow[r, "1 \times p"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Downarrow \varrho"] & A \times Y \arrow[d, "\alpha"] \\
X \arrow[r, "p"'] & Y
\end{tikzcd}
$$

whose components have the form

$$a.p(x) \xrightarrow{\varrho_{a,x}} p(ax)$$

for $a \in A$ and $x \in X$, and which satisfy two coherence conditions. The first asks that $\varrho_{i,x} : p(x) = i.p(x) \to p(ix) = p(x)$ is the identity. The second asks that the composite

$$ab.p(x) \xrightarrow{a\varrho_{b,x}} a.p(bx) \xrightarrow{\varrho_{a,bx}} p(abx)$$

be equal to $\varrho_{ab,x}$. Often we omit the subscripts and simply write ϱ for $\varrho_{a,x}$. When ϱ is an identity, we say that the A-morphism is *strict*.

For actions on objects of a general 2-category $\mathcal{K}$ given by strict monoidal functors $A \to \mathcal{K}(X, X)$ and $A \to \mathcal{K}(Y, Y)$, a lax A-morphism is a morphism $p : X \to Y$ in $\mathcal{K}$ together with a natural transformation

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow"] & \mathcal{K}(X, X) \arrow[d, "\mathcal{K}(X,p)"] \\ \mathcal{K}(Y, Y) \arrow[r, "\mathcal{K}(p,Y)"'] & \mathcal{K}(X, Y) \end{tikzcd}$$

satisfying an associativity and a unit axiom generalizing those above. If $\mathcal{K}$ admits copowers, then the natural transformation displayed above determines and is determined by a 2-cell

$$\begin{tikzcd} A \cdot X \arrow[r, "1 \cdot p"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Downarrow \varrho"] & A \cdot Y \arrow[d, "\alpha"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$

in $\mathcal{K}$, satisfying associativity and unit conditions.

If (p, ϱ) and (p', ϱ') are lax A-morphisms from X to Y, an A-transformation from (p, ϱ) to (p', ϱ') is a 2-cell $\tau : p \to p'$ satisfying the evident compatibility condition; in the case $\mathcal{K} = \mathbf{Cat}$, this says that the diagram

$$\begin{tikzcd} a.px \arrow[r, "1.\tau"] \arrow[d, "\varrho_{a,x}"'] & a.p'x \arrow[d, "\varrho'_{a,x}"] \\ p(ax) \arrow[r, "\tau"'] & p'(ax) \end{tikzcd}$$

commutes for all objects $a \in A$ and $x \in X$.

There is a 2-category A-**Mod** whose objects are the A-modules (in **Cat**), whose morphisms are the lax A-morphisms, and whose 2-cells are the A-transformations. This 2-category admits copowers, with $B \cdot X$ given by the category $X \times B$ equipped with the action $\alpha \times 1 : A \times X \times B \to X \times B$, where $\alpha : A \times X \to X$ is the action on X.

Example 3.2. In the case $A = \mathbb{M}^{\mathrm{op}}$, we saw that an A-module was precisely a category X equipped with a comonad g. A lax A-morphism is what was called a *comonad opfunctor* in [Street 1972], and indeed $\mathbb{M}^{\mathrm{op}}$-**Mod** is the 2-category called $\mathrm{Mnd}^*_*(\mathbf{Cat}^*_*)$ in that paper.

3D. *Bimodules.* As usual, a bimodule is an object which is both a left and right module with suitable compatibility between the two actions. Although our notion of action is strict, the compatibility between the actions is not. There is clearly a notion of (A, B)-bimodule for different A and B, but we only need the case where $A = B$. A succinct definition of A-bimodule is an object of A-**Mod** equipped with a right A-module structure, but we can also spell out what this means.

First of all, there is a category X with a strict left action $\alpha : A \times X \to X$. The right action involves a functor $\beta : X \times A \to X$ defining a strict right action, but this should be not just a functor, but a lax A-module morphism $A \cdot X \to X$. This lax A-morphism structure consists of maps

$$a(xb) \xrightarrow{\lambda_{a,x,b}} (ax)b$$

natural in the variables $a \in A$, $x \in X$, $b \in A$, and making each diagram

$$\begin{tikzcd} aa'(xb) \arrow[r, "1\lambda_{a',x,b}"] \arrow[dr, "\lambda_{aa',x,b}"'] & a((a'x)b) \arrow[d, "\lambda_{a,a'x,b}"] \\ & ((aa')x)b \end{tikzcd} \qquad \begin{tikzcd} i(xb) \arrow[r, equal] \arrow[dr, "\lambda_{i,x,b}"'] & xb \arrow[d, equal] \\ & (ix)b \end{tikzcd}$$

commute. Finally, the associative and unit laws required for the right action defined by $\beta : X \times A \to X$ should hold not just as equations between functors, but as equations between lax A-morphisms. Explicitly, this means that each diagram

$$\begin{tikzcd} a(xbb') \arrow[r, "\lambda_{a,xb,b'}"] \arrow[dr, "\lambda_{a,x,bb'}"'] & (a(xb))b' \arrow[d, "\lambda_{a,x,b}1"] \\ & (ax)(bb') \end{tikzcd} \qquad \begin{tikzcd} a(xi) \arrow[r, equal] \arrow[dr, "\lambda_{a,x,i}"'] & ax \arrow[d, equal] \\ & (ax)i \end{tikzcd}$$

should commute.

Example 3.3. Returning to our running example $A = \mathbb{M}^{\mathrm{op}}$, we have already seen that the 2-category A-**Mod** is just Street's 2-category $\mathrm{Mnd}^*_*(\mathbf{Cat}^*_*)$ of comonads and comonad opfunctors, and that a right $\mathbb{M}^{\mathrm{op}}$-action in a 2-category is a comonad in that 2-category. So an A-bimodule will be a comonad in $\mathrm{Mnd}^*_*(\mathbf{Cat}^*_*)$, which as explained in [Street 1972] amounts to a category X equipped with comonads g and h and a distributive law $\lambda : gh \to hg$ between them.

3E. ***Morphisms of bimodules.*** While our morphisms of left modules are lax, we shall consider only strict morphisms of right modules, but these should again be defined relative to the 2-category A-**Mod**. The reason for these choices will become clear in Theorem 4.5 below. This means that a morphism $(X, \alpha, \beta) \to (Y, \alpha, \beta)$ of bimodules will be a lax A-morphism $(p, \varrho) : (X, \alpha) \to (Y, \alpha)$ of the underlying left modules, for which the diagram

$$\begin{tikzcd} X \times A \arrow[r, "p\times 1"] \arrow[d, "\beta"'] & Y \times A \arrow[d, "\beta"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$

of categories and functors commutes, and for which moreover the diagram

$$\begin{tikzcd} & a.(px.b) \arrow[r, "\lambda_{a,px,b}"] & (a.px).b \arrow[dr, "\varrho_{a,x}.1"] & \\ a.p(xb) \arrow[ur, equal] \arrow[dr, "\varrho_{a,xb}"'] & & & p(ax).b \\ & p(a(xb)) \arrow[r, "p(\lambda_{a,x,b})"'] & p((ax)b) \arrow[ur, equal] & \end{tikzcd} \tag{3.4}$$

commutes for all $a, b \in A$ and $x \in X$.

The bimodules and their morphisms constitute the objects and morphisms of a 2-category A-**Mod**-A; a 2-cell $(p, \varrho) \to (p', \varrho')$ is a natural transformation $\tau : p \to p'$ which is a 2-cell relative to both the left and right actions.

Example 3.5. For an A-bimodule X and an arbitrary category P, the functor category $[X, P]$ has left and right actions of A, given by $(af)(x) = f(xa)$ and $(fa)(x) = f(ax)$, and these define a bimodule structure on $[X, P]$. This forms the object part of a 2-functor $[X, -] : \mathbf{Cat} \to A$-**Mod**-$A$. We shall be particularly interested in the case where X is A with its standard bimodule structure; in this case, since the left and right actions on A are strictly compatible, so too are those on $[A, P]$.

Example 3.6. Dually, for an A-bimodule X and an arbitrary category P, the product category $P \times X$ has left and right actions inherited from X, and this forms the object part of a 2-functor $(-) \times X : \mathbf{Cat} \to A$-**Mod**-$A$.

4. Lax cohomology and homology

4A. ***The Hochschild complex.*** Let A be a strict monoidal category and X a bimodule over A, in the sense of the previous section. Then we can define maps

$$\cdots\ A \times A \times X \ \begin{smallmatrix} \xrightarrow{d_0} \\ \xrightarrow{d_1} \\ \xrightarrow{d_2} \end{smallmatrix}\ A \times X \ \begin{smallmatrix} \xrightarrow{d_0} \\ \xleftarrow{s_0} \\ \xrightarrow{d_1} \end{smallmatrix}\ X \tag{4.1}$$

exactly as in Section 2E, except that, because of the lax compatibility between the actions, the simplicial identity $d_1 d_0 = d_0 d_2$ no longer holds; instead, there is a natural transformation $\lambda : d_1 d_0 \to d_0 d_2$ whose component at an object (b, a, x) in

$A \times A \times X$ is the map $\lambda_{a,x,b} : a(xb) \to (ax)b$. Similarly, each simplicial identity involving a first face map and a last face map is replaced by a natural transformation. The various coherence conditions on λ appearing in the definition of A-bimodule imply various coherence conditions on these natural transformations; the entire structure determines a **Cat**-valued presheaf on a 2-category which is obtained by a "blowing up" of the category Δ, similar in nature to that in [Lack 2000].

Similarly, there are maps

$$X \underset{\xrightarrow{\delta_1}}{\overset{\xrightarrow{\delta_0}}{\xleftarrow{\sigma_0}}} [A, X] \underset{\xrightarrow{\delta_2}}{\overset{\xrightarrow{\delta_0}}{\xrightarrow{\delta_1}}} [A \times A, X] \cdots \tag{4.2}$$

defined as in Section 2E once again; this time the cosimplicial identity $\delta_2\delta_0 = \delta_1\delta_0$ becomes a natural transformation $\delta_2\delta_0 \to \delta_1\delta_0$, whose components are once again induced by the lax compatibilities $\lambda_{a,x,b}$.

4B. ***Cohomology.*** In Section 2E, we defined the zeroth Hochschild cohomology group $H^0(A, X)$ of a bimodule over a ring as the equalizer of the pair of maps $\delta_0, \delta_1 : X \rightrightarrows [A, X]$. In the case of the lax cohomology of a bimodule over a strict monoidal category A, we define the zeroth Hochschild cohomology $H^0(A, X)$ by taking a "lax version" of an equalizer, involving all of the data displayed in (4.2), called a *lax descent object*; this is a mild variant from [Lack 2002] of a notion introduced in [Street 1987]. Interpreting this for (4.2) yields that $H^0(A, X)$ is the universal category Y equipped with a functor $y : Y \to X$ and a natural transformation $\xi : \delta_1 y \to \delta_0 y$ such that $\sigma_0\xi : x = \sigma_0\delta_1 y \to \sigma_0\delta_0 y = y$ is the identity and the diagram

$$\begin{tikzcd}
 & \delta_2\delta_0 y \arrow[r, "\lambda y"] & \delta_0\delta_1 y \arrow[dr, "\delta_0\xi"] & \\
\delta_2\delta_1 y \arrow[ur, "\delta_2\xi"] \arrow[dr, equal] & & & \delta_0\delta_0 y \\
 & \delta_1\delta_1 y \arrow[r, "\delta_1\xi"'] & \delta_1\delta_0 y \arrow[ur, equal] &
\end{tikzcd}$$

commutes. Explicitly, an object of $H^0(A, X)$ is an object $x \in X$ equipped with maps $\xi_a : ax \to xa$ natural in $a \in A$, and satisfying $\xi_i = 1$ as well as the cocycle condition asserting that the diagram

$$\begin{tikzcd}
 & a(xb) \arrow[r, "\lambda x"] & (ax)b \arrow[dr, "\xi b"] & \\
a(bx) \arrow[ur, "a\xi"] \arrow[dr] & & & (xa)b \\
 & (ab)x \arrow[r, "\xi"'] & x(ab) \arrow[ur] &
\end{tikzcd}$$

commutes for all $a, b \in A$.

Example 4.3. In the case of classical Hochschild cohomology, for a ring A the zeroth cohomology group $H^0(A, A)$ is the centre of the ring; similarly, for a strict monoidal category A, the lax cohomology $H^0(A, A)$ is the *lax centre* of A in the sense of [Day et al. 2007], originally introduced in [Schauenburg 2000] with the name *weak centre*.

Example 4.4. Consider our running example of $A = \mathbb{M}^{\mathrm{op}}$, so that an A-bimodule X is a category equipped with comonads g and h and a distributive law $\lambda : gh \to hg$. Explicit calculation shows that an object of $H^0(A, X)$ is an object $x \in X$ equipped with a map $\xi : gx \to hx$ making the diagrams (2.6) commute, so we recover the notion of right λ-coalgebra of Section 2D.

The next result justifies the definition of the lax cohomology $H^0(A, X)$ analogously to Proposition 2.8 for the usual Hochschild cohomology.

Theorem 4.5. *The* 2*-functor* $(-) \times A : \mathbf{Cat} \to A\text{-}\mathbf{Mod}\text{-}A$ *has a right adjoint sending an* A*-bimodule* X *to* $H^0(A, X)$.

Proof. Let X be an A-bimodule and P a category. Giving a (strict) right A-module morphism $p : P \times A \to X$ is equivalent to giving a functor $f : P \to X$; here $f(y) = p(y, 1)$ and $p(y, a) = f(y)a$. (It is here that the strictness of the right action is necessary.) To enrich such a morphism of modules into a morphism (p, ϱ) of bimodules, we should give suitably natural and coherent maps

$$\varrho_{a,y,b} : a.p(y, b) \to p(y, ab)$$

for all $a \in A$ and $(y, b) \in P \times A$. By the compatibility condition (3.4), the map $\varrho_{a,y,b}$ can be constructed as

$$ap(y, b) = a(p(y, 1)b) \xrightarrow{\lambda_{a,p(y,1),b}} (ap(y, 1))b \xrightarrow{\varrho_{a,y,1}1} p(y, a)b = p(y, ab)$$

and so the general ϱ is determined by those of the form $\varrho_{a,y,1}$, and these have the form $\xi_{a,y} : af(y) \to f(y)a$. The unit condition asserting that each $\varrho_{1,y,b}$ is the identity says that $\xi_{1,y}$ is the identity. The cocycle condition on the ϱ is equivalent to the cocycle condition asserting that $\xi_{a,y}$ makes each $f(y)$ into an object of $H^0(A, X)$. Naturality of $\xi_{a,y}$ in y implies that for each morphism $\psi : y \to y'$ in P, the map $f(\psi)$ defines a morphism $(f(y), \xi_{a,y}) \to (f(y'), \xi_{a,y'})$ in $H^0(A, X)$.

This gives the desired bijection between bimodule morphisms $P \times A \to X$ and functors $P \to H^0(A, X)$; it is straightforward to check that this carries over to 2-cells, and so defines an isomorphism of categories

$$A\text{-}\mathbf{Mod}\text{-}A(P \times A, X) \cong \mathbf{Cat}(P, H^0(A, X))$$

exhibiting $H^0(A, X)$ as the value at X of a right adjoint to $(-) \times A$. □

4C. *Homology.* In Section 2E, the zeroth Hochschild homology group was defined as the coequalizer of the maps $d_0, d_1 : A \otimes X \rightrightarrows X$. For lax homology, we define $H_0(A, X)$ of an A-bimodule X to be the *lax codescent object* of the data displayed in (4.1). Lax codescent objects are the colimit notion corresponding to the lax descent objects used to define lax cohomology.

Spelling this out, $H_0(A, X)$ is the universal category Y equipped with a functor $f : X \to Y$ and a natural transformation $\varphi : fd_0 \to fd_1$ satisfying the normalization condition $\varphi s_0 = 1$ and the cocycle condition

$$\begin{tikzcd}[column sep=small]
 & fd_1d_0 \arrow[r, "f\lambda"] & fd_0d_2 \arrow[dr, "\varphi d_2"] & \\
fd_0d_0 \arrow[ur, "\varphi d_0"] \arrow[dr, equal] & & & fd_1d_2. \\
 & fd_0d_1 \arrow[r, "\varphi d_1"'] & fd_1d_1 \arrow[ur, equal] &
\end{tikzcd}$$

Explicitly, $H_0(A, X)$ is obtained from X by adjoining morphisms $xa \to ax$ satisfying naturality conditions in both variables, with $xi \to ix$ required to be the identity, and obeying the cocycle condition which requires the diagram

$$\begin{tikzcd}[column sep=small]
 & b(xa) \arrow[r, "\lambda"] & (bx)a \arrow[dr, "\varphi_{bx,a}"] & \\
(xa)b \arrow[ur, "\varphi_{xa,b}"] \arrow[dr] & & & a(bx) \\
 & x(ab) \arrow[r, "\varphi_{x,ab}"'] & (ab)x \arrow[ur] &
\end{tikzcd}$$

to commute.

Example 4.6. Let $A = \mathbb{M}^{\mathrm{op}}$, and let X have A-bimodule structure corresponding to comonads g and h and a distributive law $\lambda : gh \to hg$. By the defining universal property of the category $H_0(A, X)$, giving a functor $H_0(A, X) \to P$ is the same as giving a functor $f : A \to P$ and natural transformation $\varphi : fh \to fg$ making the diagrams (2.5) commute, so we recover the notion of left λ-coalgebra from Section 2D.

Example 4.7. Again with $A = \mathbb{M}^{\mathrm{op}}$, the "regular" A-bimodule structure on A corresponds to the two decalage comonads equipped with the identity distributive law between them. The full subcategory of $\mathbb{M}^{\mathrm{op}}$ given by the nonempty finite ordinals is a sub-bimodule; since it is also isomorphic to Δ^{op}, there is an induced bimodule structure on Δ^{op}. By the preceding example and the description of duplicial structure given in Proposition 2.4, a functor $H_0(\mathbb{M}^{\mathrm{op}}, \Delta^{\mathrm{op}}) \to P$ is precisely a duplicial object in P, so that $H_0(\mathbb{M}^{\mathrm{op}}, \Delta^{\mathrm{op}})$ itself is the category $\boldsymbol{K}^{\mathrm{op}}$ indexing duplicial structure. Similarly, a functor $H_0(\mathbb{M}^{\mathrm{op}}, \mathbb{M}^{\mathrm{op}}) \to P$ is an augmented duplicial object in P, and $H_0(\mathbb{M}^{\mathrm{op}}, \mathbb{M}^{\mathrm{op}})$ is the category indexing augmented duplicial structure.

Just as before, the lax zeroth Hochschild homology has a universal characterization paralleling Proposition 2.7.

Theorem 4.8. *The* 2*-functor* $[A, -] : \mathbf{Cat} \to A\text{-}\mathbf{Mod}\text{-}A$ *has a left adjoint sending an* A*-bimodule* X *to* $H_0(A, X)$.

Proof. Let X be an A-bimodule and P a category. Just as in the classical case, giving a (strict) morphism of right A-modules $p : X \to [A, P]$ is equivalent to giving a morphism $f : X \to P$ with $f(x) = p(x)(1)$ and $p(x)(a) = f(xa)$. In order to enrich such a p into a morphism $(p, \varrho) : X \to [A, P]$ of bimodules, we should give a suitably coherent map $\varrho_{a,x} : a.p(x) \to p(ax)$ in $[A, P]$ for all $a \in A$ and $x \in X$. Thus for $b \in A$ we should give

$$f(x(ba)) = p(x)(ba) = (a.p(x))b \xrightarrow{\varrho_{a,x}(b)} p(ax)(b) = f((ax)b).$$

Commutativity of (3.4) means that the general $\varrho_{a,x}(b)$ is equal to the composite

$$a.p(xb) \xrightarrow{\varrho_{a,xb}(1)} p(a(xb)) \xrightarrow{p\lambda_{a,x,b}} p((ax)b) = p(ax)b.$$

Thus ϱ is determined by the maps $\varrho_{a,x}(1) : f(xa) \to f(ax)$, which we can regard as defining a natural transformation $\varphi : fd_0 \to fd_1$. The normalization condition asserting that $\varrho_{1,x}$ is an identity now says that

$$f = fd_0s_0 \xrightarrow{\varphi s_0} fd_1s_0 = f$$

is an identity. The cocycle condition on ϱ is equivalent to the cocycle condition on φ, and so we have a bijection between bimodule morphisms $X \to [A, P]$ and functors $H_0(A, X) \to P$. It is straightforward to extend this to 2-cells, and so to obtain an isomorphism of categories

$$A\text{-}\mathbf{Mod}\text{-}A(X, [A, P]) \cong \mathbf{Cat}(H_0(A, X), P)$$

exhibiting $H_0(A, X)$ as the value at X of a left adjoint to $[A, -]$. □

4D. ***The universal coefficient theorem and the cap product.*** In this section we develop a few very simple ingredients of classical Hochschild theory in our lax context. The first of these is the universal coefficient theorem. In its more general forms this involves short exact sequences connecting homology and cohomology, but in degree zero it is particularly simple.

Proposition 4.9 (universal coefficient theorem). *For any bimodule* X *and category* P *there is an isomorphism of categories*

$$\mathbf{Cat}(H_0(A, X), P) \cong H^0(A, [X, P])$$

natural in X *and* P.

Proof. By the universal property of $H_0(A, X)$ as a lax codescent object, an object of the left-hand side amounts to a functor $f : X \to P$ equipped with a natural transformation $\varphi : fd_0 \to fd_1$ satisfying the normalization and cocycle conditions. But the functor f can be seen as an object of $[X, P]$, while $\delta_0(f) : A \to [X, P]$ and $\delta_1(f)$ correspond under the adjunction $- \times A \dashv \mathbf{Cat}(A, -)$ to $fd_0 : A \times X \to P$ and fd_1, so that giving $\varphi : fd_0 \to fd_1$ is equivalent to giving $\xi : \delta_0(f) \to \delta_1(f)$. A straightforward calculation shows that the normalization and cocycle conditions for φ to make f into a functor $H_0(A, X) \to P$ are equivalent to the normalization and cocycle conditions for ξ to make f into an object of $H^0(A, [X, P])$.

This proves that we have a bijection on objects; the case of morphisms is similar but easier, and is left to the reader. □

Construction 4.10 (cap product). Given any bimodule X, the unit of the adjunction $H_0(A, -) \dashv [A, -]$ of Theorem 4.8 has the form $\chi : X \to [A, H_0(A, X)]$. Applying the cohomology 2-functor $H^0(A, -)$, we obtain a functor

$$H^0(A, X) \xrightarrow{H^0(A,\chi)} H^0(A, [A, H_0(A, X)]),$$

and composing with the "universal coefficient" isomorphism $H^0(A, [A, P]) \cong \mathbf{Cat}(H_0(A, A), P)$ of Proposition 4.9, we obtain a functor

$$H^0(A, X) \longrightarrow \mathbf{Cat}(H_0(A, A), H_0(A, X))$$

whose adjoint transpose

$$H^0(A, X) \times H_0(A, A) \longrightarrow H_0(A, X)$$

can be seen as a special case of the cap product for our lax homology and cohomology. But we choose instead to transpose again to obtain a functor

$$H_0(A, A) \xrightarrow{\mathrm{BS}} \mathbf{Cat}(H^0(A, X), H_0(A, X)),$$

which we call the *Böhm–Ştefan* map.

Example 4.11. We now analyze this Böhm–Ştefan map in the case of our running example. Suppose then that $A = \mathbb{M}^{\mathrm{op}}$, and X is an A-bimodule, with the bimodule structure corresponding to comonads g and h and a distributive law $\lambda : gh \to hg$. Let $p : H_0(A, X) \to P$ be an arbitrary functor, and let $y \in H^0(A, X)$. As in Example 4.6, giving p is equivalent to giving a functor $f : X \to P$ equipped with left λ-coalgebra structure $\varphi : fh \to fg$, while as in Example 4.4, giving y is equivalent to giving an object $x \in X$ equipped with right λ-coalgebra structure $\xi : gx \to hx$. There is now an induced functor

$$H_0(A, A) \xrightarrow{\mathrm{BS}} \mathbf{Cat}(H^0(A, X), H_0(A, X)) \xrightarrow{\mathrm{ev}_y} H_0(A, X) \xrightarrow{p} P$$

which by Example 4.7 picks out an augmented duplicial object in P. This object is precisely the one constructed in [Böhm and Ştefan 2008] as recalled in Section 2D above. This construction was generalized slightly in [Böhm and Ştefan 2012] to include right λ-coalgebra structures on arbitrary functors $Y \to X$, rather than just objects of X; in this case y becomes a functor $Y \to H^0(A, X)$ and the composite

$$H_0(A, A) \xrightarrow{\mathrm{BS}} \mathbf{Cat}(H^0(A, X), H_0(A, X)) \xrightarrow{\mathbf{Cat}(y,p)} \mathbf{Cat}(Y, P)$$

defines an augmented duplicial object in $\mathbf{Cat}(Y, P)$.

5. Duplicial structure on nerves

In this section we turn to our second main goal, which is to analyze duplicial structure on nerves of various sorts of categorical structures; specifically, on categories, on monoidal categories, and on bicategories.

A monoidal category can of course be seen as a one-object bicategory, and a category can be seen as a bicategory with no nonidentity 2-cells, so in principle we could pass straight to the case of bicategories, and then merely read off the results for the other two cases, but instead we have chosen to do the case of categories first, as a sort of warm-up.

5A. ***Duplicial structure on categories.*** The nerve functor from $\mathbf{Cat}$ to $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ is of course fully faithful, so that we may identify (small) categories with certain simplicial sets. It therefore makes sense to speak of duplicial structure borne by a category. The decalage comonads on $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ restrict to $\mathbf{Cat}$, and so we may analyze duplicial structure on categories using Proposition 2.4.

The right decalage comonad sends a category C to the coproduct $\sum_x C/x$ over all objects $x \in C$ of the corresponding slice categories. The counit is the functor induced by the domain functors $C/x \to C$, while the comultiplication $\sum_x C/x \to \sum_{f:w\to x} C/w$ sends the x-component to the 1_x-component via the identity functor $C/x \to C/x$. Dually, the left decalage comonad sends a category C to the coproduct $\sum_x x/C$, with similar descriptions available for the counit and comultiplication.

Since both C/x and x/C are connected categories, a functor $\sum_x C/x \to \sum_x x/C$ is necessarily given by an assignment $c \mapsto tc$ on objects together with a functor $t : C/x \to tx/C$ for each x. Compatibility with the counit (on objects) means that the image under t of an object $f : a \to x$ of C/x should have the form $tf : tx \to a$. Functoriality, together with counit compatibility on morphisms means that if $fg = h$ then $g.th = tf$. Compatibility with the comultiplication requires a slightly more complicated calculation.

An object of the right decalage $\mathrm{Dec_r}(C)$ has the form $f : a \to x$, and the comultiplication $\mathrm{Dec_r}(C) \to \mathrm{Dec_r}(\mathrm{Dec_r}(C))$ sends it to the composable pair $(1_x, f)$.

Now $\mathrm{Dec_r}(t) : \mathrm{Dec_r}(\mathrm{Dec_r}(C)) \to \mathrm{Dec_r}(\mathrm{Dec_l}(C))$ sends this to the composable pair (f, tf), which, as we have seen, must have composite $t1_x$. This composable pair can equally be seen as lying in $\mathrm{Dec_l}(\mathrm{Dec_r}(C))$, and finally applying $\mathrm{Dec_l}(t)$ gives the composable pair $(tf, t^2 1_x)$. Compatibility with comultiplication says that this should be equal to the composable pair $(tf, 1_{tx})$, and this clearly says that $t^2(1_x) = 1_{tx}$ for all objects x. We have only checked compatibility with the comultiplication on objects, but in fact no further condition is needed for compatibility on morphisms. We summarize this calculation as follows.

Proposition 5.1. *Giving duplicial structure to a small category C is equivalent to giving*

- *for each object x an object tx,*
- *for each morphism $f : a \to x$ a morphism $tf : tx \to a$,*

subject to the conditions that

- *$t^2(1_x) = 1_{tx}$ for all objects x,*
- *$f.t(gf) = tg$ for any composable pair (g, f),*

which we call the identity and functoriality conditions, respectively.

The next result gives a cleaner reformulation of these conditions. In its statement, recall that the inclusion 2-functor **Gpd** $\hookrightarrow$ **Cat** has a left 2-adjoint Π_1, whose counit at a small category C is the functor $p : C \to \Pi_1(C)$ which freely adjoins an inverse for every arrow of C. The 2-dimensional aspect of the universal property means that, for any category D, the functor $[\Pi_1(C), D) \to [C, D]$ given by composition with p is fully faithful.

Theorem 5.2. *Giving duplicial structure to a small category C is equivalent to giving a left adjoint in* **Cat** *for the functor $p : C \to \Pi_1(C)$.*

Proof. First suppose that p has a left adjoint $i : \Pi_1(C) \to C$ with counit $\varepsilon : ip \to 1$ and unit $\eta : 1 \to pi$; since $\Pi_1(C)$ is a groupoid, η is invertible, and therefore i is fully faithful. For each object $y \in C$, define ty to be ipy, and for each morphism $f : x \to y$, define $tf : ipy \to x$ to be the composite

$$ipy \xrightarrow{i(pf)^{-1}} ipx \xrightarrow{\varepsilon_x} x.$$

Then $t(1_x) = \varepsilon_x$ and so using the triangle identities twice yields

$$t^2(1_x) = t(\varepsilon_x) = \varepsilon_{tx}.i(p\varepsilon_x)^{-1} = \varepsilon_{ipx}.i\eta_{px} = 1_{ipx},$$

while for a composable pair (g, f) we have

$$f.t(gf) = f.\varepsilon_x.i(p(gf)^{-1}) = \varepsilon_y.ip(f).i(pf)^{-1}.i(pg)^{-1} = \varepsilon_y.i(pg)^{-1} = t(g);$$

so this defines duplicial structure on C.

Conversely, if C is equipped with duplicial structure there is an induced functor $G : C \to C$ sending an object x to tx and a morphism $f : x \to y$ to $t^2 f : tx \to ty$. This preserves identity morphisms because $t^2(1_x) = 1_{tx}$ by assumption, and preserves composition by three applications of the fact that if $h = gf$ then $f.th = tg$. (The functor G can be seen as a simplicial endomorphism of the nerve of C; as such it is the "curious natural transformation" of [Dwyer and Kan 1985].) For each $x \in C$, write ε_x for the morphism $t(1_x) : tx \to x$. Now $f.tf = t(1_y)$ by the functoriality condition, since $1_y f = f$; and replacing f by tf we also have $tf.t^2 f = t(1_x)$. Combining these, $\varepsilon_y.Gf = t1_y.t^2 f = f.tf.t^2 f = f.t1_x = f.\varepsilon_x$ and so the ε_x are indeed natural. Furthermore, $G\varepsilon_x = t^2(\varepsilon_x) = t^3(1_x) = t(1_{tx}) = \varepsilon_{Gx}$ and so (G, ε) is a well-copointed endofunctor in the sense of [Kelly 1980].

Next we show that for any $f : x \to y$, the morphism $Gf := t^2 f$ is invertible, with inverse $t(f.\varepsilon_x)$. First observe that $\varepsilon_x.t(f.\varepsilon_x) = tf$ by the functoriality condition once again. Consequently, we have

$$t(f.\varepsilon_x).t^2(f) = t(f.\varepsilon_x).t(\varepsilon_x.t(f.\varepsilon_x)) = t(\varepsilon_x) = t^2(1_x) = 1_{tx},$$

using the functoriality condition again at the second step; this gives one of the inverse laws. By naturality of ε and the functoriality condition yet again, we have

$$t^2 f.t(f.\varepsilon_x) = t^2 f.t(\varepsilon_y.t^2(f)) = t(\varepsilon_y) = t^2(1_y) = 1_{ty},$$

giving the other. Thus each Gf is invertible. By the universal property of $\Pi_1(C)$, therefore, there is a unique functor $i : \Pi_1(C) \to C$ with $ip = G$. By the 2-dimensional aspect of the universal property of $\Pi_1(C)$, there is a unique natural transformation $\eta : 1 \to pi$ with $\eta p : p \to pip$ equal to $(p\varepsilon)^{-1}$, and so satisfying the triangle equation $p\varepsilon.\eta p = 1$. By the 2-dimensional aspect of the universal property once again, the other triangle equation $\varepsilon i.i\eta = 1$ holds if and only if $\varepsilon ip.i\eta p = 1$ does, but by the calculation

$$\varepsilon ip.i\eta p = \varepsilon ip.(ip\varepsilon)^{-1} = ip\varepsilon.(ip\varepsilon)^{-1} = 1$$

this is indeed the case, and so p does have a left adjoint.

It remains to show that these two processes are mutually inverse. First suppose that C has duplicial structure t, and then construct a left adjoint $i \dashv p$ as above. The duplicial structure that this induces sends an object x to $ipx = ix = tx$, and a morphism $f : x \to y$ to $\varepsilon_x.i(pf)^{-1}$, where $i(pf)^{-1} = t(f.\varepsilon_x)$. But now $\varepsilon_x.i(pf)^{-1} = \varepsilon_x.t(f.\varepsilon_x) = tf$ by the functoriality condition, and so we have recovered the original duplicial structure.

For the other direction, suppose first that p has a left adjoint i with counit ε. Construct the induced duplicial structure t, and the left adjoint i' and counit ε' induced by that. By the universal property of $\Pi_1(C)$ once again it suffices to show that $ip = i'p$ and $\varepsilon = \varepsilon'$. For an object x, we have $\varepsilon'_x = t(1_x) = \varepsilon_x.i(p1_x)^{-1} = \varepsilon_x$, and so $\varepsilon = \varepsilon'$; this includes the fact that ip and $i'p$ agree on objects, and so it remains only to show that they agree on morphisms. To see this, let $f : x \to y$ be a morphism, so that $i'pf : i'px \to i'py$ is given by $t^2(f) : tx \to ty$. Now $tf = \varepsilon_x.i(pf)^{-1}$, so

$$ip(tf)^{-1} = ipipf.i(p\varepsilon_x)^{-1} = ipipf.i\eta px = i\eta py.ipf,$$

and so finally $i'pf = t^2 f = \varepsilon_{ipy}.i\eta py.ipf = ipf$. □

Example 5.3. If C is a groupoid, then $p : C \to \Pi_1(C)$ is invertible, and so has a canonical left adjoint $p^{-1} : \Pi_1(C) \to C$. So every groupoid has a canonical duplicial structure.

Example 5.4. Suppose that there is a groupoid G and a functor $i : G \to C$ with a right adjoint $r : C \to G$. By the universal property of $\Pi_1(C)$, there is a unique induced functor $q : \Pi_1(C) \to G$ with $qp = r$. By [Gabriel and Zisman 1967, Proposition 1.3], this q is an equivalence. Thus p also has a left adjoint, and so C has a duplicial structure.

Remark 5.5. We have seen that a category C has duplicial structure just when $p : C \to \Pi_1(C)$ has a left adjoint. This is paracyclic just when each t_n is invertible, or equivalently just when each t_n^{n+1} is invertible. Now the t_n^{n+1} define the functor $ip : C \to C$; since p is bijective on objects and i is fully faithful, the composite ip is invertible if and only if i and p are both invertible, and this can happen only if C is a groupoid.

For a groupoid, giving duplicial structure is equivalent to giving a left adjoint to the invertible $p : C \to \Pi_1(C)$; of course such a left adjoint is necessarily isomorphic to p^{-1} and so in particular an equivalence. The duplicial structure is paracyclic just when this left adjoint is in fact an invertible functor, and cyclic just when it is p^{-1} as above. Thus, for a category C, the existence of paracyclic structure implies the existence of cyclic structure, but this does not mean that paracyclic structure on a category is necessarily cyclic. Furthermore, a groupoid can admit multiple cyclic structures, since there can be multiple choices of unit and counit for an adjunction $p^{-1} \dashv p$; in fact such choices correspond to choices of a natural isomorphism $1_G \cong 1_G$.

5B. ***Duplicial structure on bicategories.*** We next consider what it means to give duplicial structure on the nerve of a bicategory B [Street 1996]. Recall that this nerve is the simplicial set NB, in which

- the 0-simplices are the objects of B;
- the 1-simplices are the arrows $f : x \to y$ of B;

- the 2-simplices are the 2-cells in B of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z; \end{tikzcd}$$

- the 3-simplices are the commuting diagrams of 2-cells of the form

$$\begin{tikzcd} (hg)f \arrow[r, "\cong"] \arrow[d, "\alpha f"'] & h(gf) \arrow[r, "h\beta"] & hk \arrow[d, "\gamma"] \\ \ell f \arrow[rr, "\delta"'] & & m \end{tikzcd}$$

in which the unnamed isomorphism is the relevant associativity constraint of B.

The face and degeneracy maps are as expected, and the higher simplices are determined by 3-coskeletality. The assignment $B \mapsto NB$ is the object part of a fully faithful functor $N : \mathbf{NLax} \to [\Delta^{\mathrm{op}}, \mathbf{Set}]$, where **NLax** is the category of bicategories and *normal lax functors* between them — ones preserving identities on the nose, but binary composition only up to noninvertible 2-cells $Fg.Ff \Rightarrow F(gf)$. The first appearance in print we could find of the fact that this nerve functor is fully faithful was in [Bullejos et al. 2005].

Once again, the decalage comonads on $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ restrict to the full subcategory **NLax**, and so it makes sense to speak of duplicial structure *on a bicategory*. Indeed the description of these restricted comonads is similar to the case of **Cat**, except that rather than slice categories now we use "lax slices". For an object x of a bicategory B, we write B/x for the bicategory whose objects are morphisms $f : a \to x$ with codomain x, whose morphisms from $f : a \to x$ to $g : b \to x$ have the form

$$\begin{tikzcd} a \arrow[rr, "s"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\overset{\Leftarrow}{\sigma}"] & b \arrow[dl, "g"] \\ & x & \end{tikzcd}$$

and whose 2-cells are defined in the evident way. Similarly the "lax coslice" x/B has objects of the form $f : x \to a$, and morphisms from $f : x \to a$ to $g : x \to b$ of the form

$$\begin{tikzcd} & x \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\overset{\Rightarrow}{\sigma}"] & \\ a \arrow[rr, "s"'] & {} & b. \end{tikzcd}$$

We now define $\mathrm{Dec_r}(B) = \sum_x B/x$ and $\mathrm{Dec_l}(B) = \sum_x x/B$, with the actions on normal lax functors, and the counits and comultiplications given by a straightforward generalization of the corresponding definitions for **Cat**.

Before giving our characterization result, let us recall that a 2-cell in a bicategory as on the left in

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\overset{\alpha}{\Longrightarrow}"] & \\ b \arrow[rr, "h"'] & & c \end{tikzcd} \qquad \begin{tikzcd} & a \arrow[dl, "k"'] \arrow[dr, "g"] \arrow[d, phantom, "\overset{\beta}{\Longrightarrow}"] & \\ b \arrow[rr, "h"'] & & c \end{tikzcd}$$

is said to *exhibit f as a right lifting of g through h* [Street and Walters 1978] if every 2-cell as on the right above factors as $\alpha.h\bar{\beta}$ for a unique 2-cell $\bar{\beta} : k \Rightarrow f$.

Theorem 5.6. *Equipping a bicategory B with duplicial structure is equivalent to giving*

(a) *for each object $x \in B$ an object $tx \in B$ and a morphism $\varepsilon_x : tx \to x$;*

(b) *for each morphism $f : a \to x$ in B a morphism $tf : tx \to a$ and a 2-cell*

$$\begin{tikzcd} & tx \arrow[dl, "tf"'] \arrow[dr, "\varepsilon_x"] \arrow[d, phantom, "\overset{\varepsilon_f}{\Longrightarrow}"] & \\ a \arrow[rr, "f"'] & & x \end{tikzcd}$$

exhibiting tf as a right lifting of ε_x through f;

all subject to the conditions that

(c) $t1_x = \varepsilon_x$;

(d) $t^2 1_x = 1_{tx}$;

(e) ε_{1_x} *is the left identity isomorphism* $1_x.t1_x \to t1_x$;

(f) ε_{t1_x} *is the right identity isomorphism* $t1_x.1_{tx} \to t1_x$.

In the case where B is a category, (a) and (c) correspond to giving tx and $t1_x : tx \to x$ for each x, while (b) says that for each $f : a \to x$ there is a unique map tf with $f.tf = t1_x$; condition (d) now follows from the uniqueness, and conditions (e) and (f) are automatic. It is now not hard to see that this is equivalent to the conditions in Proposition 5.1.

Proof. By redefining the composition with identity 1-cells, any bicategory may be made isomorphic in **NLax** to one in which identities are strict. Thus without loss of generality we may suppose that B has strict identities; then the conditions in (e) and (f) become $\varepsilon_{1_x} = 1_{\varepsilon_x}$ and $\varepsilon_{t1_x} = 1_{t(1_x)}$.

Duplicial structure consists of a normal lax functor $t : \mathrm{Dec}_{\mathrm{r}}(B) \to \mathrm{Dec}_{\mathrm{l}}(B)$ which is compatible with the counit and comultiplication maps. As in the case of **Cat**, since each B/x and x/B is connected, t must be given by an assignment $x \mapsto tx$ on objects and normal lax functors $B/x \to tx/B$.

To give t on objects compatibly with the counits is to give, for each $f : a \to x$, a morphism $tf : tx \to a$. To give t on morphisms compatibly with the counits is to give, for each triangle as on the left below, a triangle as on the right:

$$\begin{tikzcd} a \arrow[rr, "s"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Leftarrow \sigma"] & b \arrow[dl, "g"] \\ & x & \end{tikzcd} \qquad \begin{tikzcd} & tx \arrow[dl, "tf"'] \arrow[dr, "tg"] \arrow[d, phantom, "\Rightarrow t_s(\sigma)"] & \\ a \arrow[rr, "s"'] & {} & b \end{tikzcd}$$

The action of t on 2-cells is unique if it exists, given the counit condition; it exists just when, for all $\sigma : gs \to f$ and $\tau : s' \to s$, the diagram on the left commutes, where σ' is defined as in the diagram on the right:

$$\begin{tikzcd} s'.tf \arrow[r, "\tau.tf"] \arrow[dr, "t_{s'}(\sigma')"'] & s.tf \arrow[d, "t_s(\sigma)"] \\ & tg \end{tikzcd} \qquad \begin{tikzcd} g.s' \arrow[r, "g.\tau"] \arrow[dr, "\sigma'"'] & g.s \arrow[d, "\sigma"] \\ & f \end{tikzcd}$$

or, more compactly:

$$t_{s'}(\sigma \circ (g.\tau)) = t_s(\sigma) \circ (\tau.tf). \tag{5.7}$$

Since the components $\mathrm{Dec}_{\mathrm{r}}(B) \to B$ and $\mathrm{Dec}_{\mathrm{l}}(B) \to B$ of the counit are strict morphisms of bicategories, it follows that $t : \mathrm{Dec}_{\mathrm{r}}(B) \to \mathrm{Dec}_{\mathrm{l}}(B)$ is also strict, which amounts to the requirements

$$t_{1_a}(1_f) = 1_{tf} \quad \text{and} \quad t_{s'}(\sigma') \circ (s'.t_s(\sigma)) = t_{s's}(\sigma \circ \sigma' s) \tag{5.8}$$

for all $\sigma : gs \to f$ and $\sigma' : hs' \to g$.

It remains to see what the comultiplication axiom imposes. As in the case for **Cat**, the only new condition appears at the level of objects of $\mathrm{Dec}_{\mathrm{r}}(B)$, where it says that for any $f : a \to x$, we have

$$t^2 1_x = 1_{tx} \quad \text{and} \quad (t_{tf}(t_f(1_f)) : tf.tt1_x \to tf) = 1_{tf}. \tag{5.9}$$

So duplicial structure on a bicategory B amounts to the assignments $x \mapsto tx$, $(f : a \to x) \mapsto (tf : tx \to a)$, and $(s, \sigma : gs \to f) \mapsto (t_s\sigma : s.tf \to tg)$, subject to the conditions expressed in (5.7), (5.8), and (5.9). We now relate this to the structure in the statement of the theorem.

For any $x \in B$, we define $\varepsilon_x = t(1_x) : tx \to x$, and for any $f : a \to x$ in B, we define $\varepsilon_f = t_f(1_f) : f.tf \to t1_x = \varepsilon_x$. Now in the conditions appearing in the theorem, (c) holds by construction, (d) holds by the first half of (5.9), while (e) holds by taking $f = 1_x$ in the first half of (5.8). For (f), take $f = 1_x$ in the definition of ε_f, the second half of (5.9), and the first half of (5.8), to deduce that $\varepsilon_{t1_x} = t_{t1_x}(1_{t1_x}) = t_{t1_x}(t_{1_x}(1_{1_x})) = 1_{t1_x}$.

Thus, in order to show that a duplicial bicategory has all of the structure in the theorem, it remains only to show that $t_f(1_f)$ exhibits tf as a right lifting of $t1_x$ through f; in other words, that for any $g : tx \to a$ and any $\varphi : fg \to t1_x$, there is a unique $\psi : g \to tf$ which gives φ when pasted with ε_f. But we may consider the pair (g, φ) as a morphism in B/x from $t1_x$ to f, and so obtain $t_g(\varphi) : g.t^2 1_x \to tf$, and since $t^2 1_x = 1_{tx}$, this gives our $\psi : g \to tf$. Pasting it with ε_f gives

$$\begin{aligned}
\varepsilon_f \circ f\psi &= t_f(1_f) \circ (f.t_g(\varphi)) \\
&= t_{fg}(\varphi) && \text{(by (5.8))} \\
&= t_{fg}(1_{t1_x} \circ (1_x.\varphi)) \\
&= t_{t1_x}(1_{t1_x}) \circ (\varphi.t^2 1_x) && \text{(by (5.7))} \\
&= t_{t1_x}(1_{t1_x}) \circ \varphi && \text{(by (5.9))} \\
&= \varphi, && \text{(by (f))}
\end{aligned}$$

which proves the existence of ψ. As for uniqueness, suppose that $\psi : g \to tf$ satisfies $\varepsilon_f \circ f\psi = \varphi$; that is, $t_f(1_f) \circ (f.\psi) = \varphi$. Then

$$\begin{aligned}
t_g(\varphi) &= t_g(t_f(1_f) \circ (f.\psi)) \\
&= t_{tf}(t_f(1_f)) \circ (\psi.t^2 1_x) && \text{(by (5.7))} \\
&= \psi.t^2 1_x && \text{(by (5.9))} \\
&= \psi, && \text{(by (5.9))}
\end{aligned}$$

giving uniqueness as required.

Thus, a duplicial bicategory satisfies the conditions in the theorem. For the converse, suppose that B is equipped with structure as in the theorem; then we are given the assignments $x \mapsto tx$ and $(f : a \to x) \mapsto (tf : tx \to a)$, as well as the 2-cells $t_f(1_f) : f.tf \to \varepsilon_x$ satisfying the universal property of (b) and the conditions (c), (d), (e), and (f). Given $\sigma : gs \to f$, if we are to have (5.8) and then (5.7), then

$$\varepsilon_g \circ (g.t_s(\sigma)) = t_g(1_g) \circ (g.t_s(\sigma)) = t_{gs}(\sigma) = \varepsilon_f \circ (\sigma.tf),$$

and so $t_s(\sigma)$ is uniquely determined using the universal property of the right lifting 2-cell ε_g. It remains to check that if we define $t_s(\sigma)$ in this way, then (5.7), (5.8), and (5.9) do indeed hold.

Since $\varepsilon_g \circ (g.t_s(\sigma)) \circ (g.\tau.tf) = \varepsilon_f \circ (\sigma.tf) \circ (g.\tau.tf)$, the composite $t_s(\sigma) \circ (\tau.tf)$ satisfies the defining property of $t_{s'}(\sigma \circ (g.\tau))$, and so (5.7) holds. Similarly,

$$\begin{aligned}\varepsilon_h \circ (h.t_{s'}(\sigma')) \circ (h.s'.t_s(\sigma)) &= \varepsilon_g \circ (\sigma'.tg) \circ (h.s'.t_s(\sigma)) \\ &= \varepsilon_g \circ (g.t_s(\sigma)) \circ (\sigma'.s.tf) \\ &= \varepsilon_f \circ (\sigma.tf) \circ (\sigma'.s.tf),\end{aligned}$$

and so $t_{s'}(\sigma') \circ (s'.t_s(\sigma))$ satisfies the defining property of $t_{s's}(\sigma \circ \sigma's)$, while 1_{tf} clearly satisfies the defining property of $t_{1_a}(1_f)$. Thus (5.8) holds.

The first half of (5.9) is just (d); as for the second half, it says that $t_{tf}(\varepsilon_f) = 1_{tf}$, and the defining property of $t_{tf}(\varepsilon_f)$ is that $\varepsilon_f \circ (f.t_{tf}(\varepsilon_f)) = \varepsilon_{t1_x} \circ (\varepsilon_f.t^2 1_x)$; but $t^2 1_x = 1_{tx}$ by (d), and $\varepsilon_{t1_x} = 1_{t1_x}$ by (e). Thus the right-hand side becomes ε_f, and clearly $\varepsilon_f \circ 1_{tf} = \varepsilon_f$, whence the result. □

5C. ***Duplicial structure on monoidal categories.*** A monoidal category can be thought of as a one-object bicategory, and as such it has a nerve: there is a unique 0-simplex, the 1-simplices are the objects of the monoidal category, the 2-simplices consist of three objects X, Y, Z and a morphism $f : X \otimes Y \to Z$, and so on. Thus the monoidal categories determine a full subcategory of $[\Delta^{\mathrm{op}}, \mathbf{Set}]$, with the morphisms being the (lax) monoidal functors which are strict with respect to the unit. It is not the case that the decalage comonads restrict to this full subcategory: the decalage of a one-object bicategory will generally have many objects, indeed an object of the decalage will be a morphism of the monoidal category. Nonetheless, we can ask what it is to have duplicial structure on a monoidal category, thought of as a one-object bicategory.

Reading off directly from Theorem 5.6, we see that, for a monoidal category C with tensor product $\otimes$ and unit i, duplicial structure on C consists of the following:

(a) an object d (corresponding to ε_x for the unique object x of the bicategory);

(b) for each object x, a right internal hom $[x, d]$, by which we mean an object equipped with a morphism $\varepsilon_x : x \otimes [x, d] \to d$ inducing a bijection

$$C(x \otimes -, d) \cong C(-, [x, d])$$

subject to conditions which we now enumerate. First of all, we require that the internal hom $[i, d]$ be d itself. This is not a restriction in practice, since in any monoidal category and any object x the internal hom $[i, x]$ exists and may be taken to be x. The more serious requirement is that the (chosen) hom $[d, d]$ is i, with counit $d \otimes i \to d$ given by the unit isomorphism of the monoidal category. In fact the real condition here is that the map $i \to [d, d]$ induced by the unit isomorphism $d \otimes i \to d$ is invertible; when this is the case we may always redefine $[d, d]$ as required.

One formulation of the notion of (not necessarily symmetric) $*$-autonomous category [Barr 1995, Definition 2.3] is a monoidal category C equipped with an equivalence $(-)^* : C \to C^{\mathrm{op}}$ and natural isomorphism $C(x, y^*) \cong C(i, (x \otimes y)^*)$, with i the unit. Using the natural isomorphism, we may construct further isomorphisms $C(x, y^*) \cong C(i, (x \otimes y)^*) \cong C(i, (x \otimes y \otimes i)^*) \cong C(x \otimes y, i^*)$, and so y^* must in fact be given by $[y, i^*]$. Conversely, if C is a monoidal category with all (right) internal homs $[x, d]$ for a given object d, then there is a functor $(-)^* : C \to C^{\mathrm{op}}$ sending x to $[x, d]$, and a natural isomorphism $C(x, y^*) \cong C(i, (x \otimes y)^*)$; thus C will be $*$-autonomous when this functor $(-)^*$ is an equivalence.

A *compact closed category* is a symmetric monoidal category C in which every object has a monoidal dual. In this case, the functor $C \to C^{\mathrm{op}}$ sending each object to its monoidal dual is an equivalence. Thus, every compact closed category is $*$-autonomous; the dualizing object d is the unit object i in this case. In a general $*$-autonomous category, x^* need not be the monoidal dual of x.

Both duplicial structure and $*$-autonomous structure on a monoidal category C involve an object d for which the right internal homs $[x, d]$ exist. The difference is that $*$-autonomous categories require the functor $[-, d]$ to be an equivalence, while duplicial monoidal categories require the canonical map $i \to [d, d]$ to be invertible. But in fact, for a $*$-autonomous category the canonical map $i \to [d, d]$ is always invertible [Barr 1995, Section 6] and so any $*$-autonomous category has duplicial structure.

Theorem 5.10. *Any monoidal category with paracyclic structure possesses a $*$-autonomous structure. Conversely, any monoidal category with $*$-autonomous structure is monoidally equivalent to one with paracyclic structure.*

Proof. If C is a monoidal category with paracyclic structure, then there is an object d for which the right internal homs $[-, d]$ exist, and the resulting functor $C \to C^{\mathrm{op}}$ is not just an equivalence but an isomorphism. This gives C a $*$-autonomous structure.

For the converse, let C be a $*$-autonomous monoidal category with dualizing object d. We shall construct another $*$-autonomous monoidal category $\widetilde{C}$ which is monoidally equivalent to C, for which the induced duality functor $\widetilde{C} \to \widetilde{C}^{\mathrm{op}}$ can be chosen to be an isomorphism.

An object x of $\widetilde{C}$ is a $\mathbb{Z}$-indexed family $(x_n)_{n \in \mathbb{Z}}$ of objects of C, together with an isomorphism $\theta_n : x_n \cong x^*_{n+1}$ for each n. A morphism $x \to y$ is just a morphism $f : x_0 \to y_0$ in C. There is an evident equivalence of categories $\widetilde{C} \to C$ sending x to x_0.

We may transport the monoidal structure across this equivalence to obtain a monoidal structure on $\widetilde{C}$. The resulting $\widetilde{C}$ is clearly still $*$-autonomous, but now we may define the functor $\widetilde{C} \to \widetilde{C}^{\mathrm{op}}$ in such a way that it is an isomorphism of categories, by setting $(x^*)_n = x_{n-1}$. In order to make this functorial, observe that for any morphism $f : x_0 \to y_0$, we may use the θ_n to define morphisms $f_{2n} : x_{2n} \to y_{2n}$ and $f_{2n+1} : y_{2n+1} \to x_{2n+1}$ which are compatible in the evident sense. $\square$

It turns out that if C is $*$-autonomous, then the pseudoinverse $C^{\mathrm{op}} \to C$ to $(-)^*$ gives rise to a *left* internal hom $d^{(-)}$, characterized by a natural isomorphism $C(a, d^b) \cong C(b \otimes a, d)$. If the monoidal category C actually has cyclic structure, then applying $[-, d]$ twice gives the identity, and so in particular the left and right homs d^b and $[b, d]$ are isomorphic; in other words, $[-, d]$ is also a left internal hom. In this case, d is said to be a *cyclic dualizing object*.

Conversely, if C is $*$-autonomous with cyclic dualizing object d, then applying $[-, d]$ twice is isomorphic to the identity. Once again, though, for a cyclic structure we need it to be equal to the identity.

Theorem 5.11. *A monoidal category with cyclic structure has a $*$-autonomous structure with cyclic dualizing object. Conversely, any $*$-autonomous monoidal category with cyclic dualizing object is monoidally equivalent to one with cyclic structure.*

Proof. The first half follows from the discussion before the theorem. For the second, let C be a $*$-autonomous monoidal category with cyclic dualizing object d. As in the previous proposition, we construct another $*$-autonomous monoidal category $\overline{C}$ which is monoidally equivalent to C. An object x of $\overline{C}$ consists of a pair (x_+, x_-) of objects of C equipped with an isomorphism $\theta : x_+ \cong x_-^*$. A morphism $f : x \to y$ consists of a morphism $f_+ : x_+ \to y_+$; once again, there is an associated $f_- : y_- \to x_-$ suitably compatible with the θ. There is again an evident equivalence $\overline{C} \to C$ sending x to x_+, and we may transport the monoidal structure across this equivalence.

Since d is a cyclic dualizing object, any isomorphism $\theta : x_+ \cong x_-^*$ has a corresponding $\theta' : x_- \cong {}^*x_+ \cong x_+^*$. Thus we may define $\overline{C} \to \overline{C}^{\mathrm{op}}$ to send (x_+, x_-, θ) to (x_-, x_+, θ'), and applying this twice clearly gives the identity. □

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RICHARD GARNER: richard.garner@mq.edu.au
Department of Mathematics, Macquarie University, North Ryde, NSW, Australia

STEPHEN LACK: steve.lack@mq.edu.au
Department of Mathematics, Macquarie University, North Ryde, NSW, Australia

PAUL SLEVIN: p.slevin.1@research.gla.ac.uk
School of Mathematics and Statistics, University of Glasgow, Glasgow, United Kingdom

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Localization, Whitehead groups and the Atiyah conjecture

Wolfgang Lück and Peter Linnell

Let $K_1^w(\mathbb{Z}G)$ be the K_1-group of square matrices over $\mathbb{Z}G$ which are not necessarily invertible but induce weak isomorphisms after passing to Hilbert space completions. Let $\mathcal{D}(G;\mathbb{Q})$ be the division closure of $\mathbb{Q}G$ in the algebra $\mathcal{U}(G)$ of operators affiliated to the group von Neumann algebra. Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Let G be a torsionfree group which belongs to $\mathcal{C}$. Then we prove that $K_1^w(\mathbb{Z}(G))$ is isomorphic to $K_1(\mathcal{D}(G;\mathbb{Q}))$. Furthermore we show that $\mathcal{D}(G;\mathbb{Q})$ is a skew field and hence $K_1(\mathcal{D}(G;\mathbb{Q}))$ is the abelianization of the multiplicative group of units in $\mathcal{D}(G;\mathbb{Q})$.

0. Introduction

In [Friedl and Lück 2017] we introduced the *universal L^2-torsion* $\rho_u^{(2)}(X;\mathcal{N}(G))$ of an L^2-acyclic finite G-CW-complex X and discussed its applications. It takes values in a certain abelian group $\mathrm{Wh}^w(G)$, which is the quotient of the K_1-group $K_1^w(\mathbb{Z}G)$ by the subgroup given by trivial units $\{\pm g \mid g \in G\}$. Elements $[A]$ of $K_1^w(\mathbb{Z}G)$ are given by (n,n)-matrices A over $\mathbb{Z}G$ which are not necessarily invertible but for which the operator $r_A^{(2)} : L^2(G)^n \to L^2(G)^n$ given by right multiplication with A is a weak isomorphism, i.e., it is injective and has dense image. We require for such square matrices A, B the following relations in $K_1^w(\mathbb{Z}G)$:

$$[AB] = [A]\cdot[B]; \qquad \left[\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}\right] = [A]\cdot[B].$$

More details about $\mathrm{Wh}^w(G)$ and $K_1^w(\mathbb{Z}G)$ will be given in Section 3.

Let $\mathcal{D}(G;\mathbb{Q}) \subseteq \mathcal{U}(G)$ be the smallest subring of the algebra $\mathcal{U}(G)$ of operators $L^2(G) \to L^2(G)$ affiliated to the group von Neumann algebra $\mathcal{N}(G)$ which contains $\mathbb{Q}G$ and is division closed, i.e., any element in $\mathcal{D}(G;\mathbb{Q})$ which is invertible in $\mathcal{U}(G)$ is already invertible in $\mathcal{D}(G;\mathbb{Q})$. (These notions will be explained in detail in Section 2A.)

The main result of this paper is:

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Keywords: localization, algebraic K-theory, Atiyah conjecture.

Theorem 0.1 ($K_1^w(G)$ and units in $\mathcal{D}(G;\mathbb{Q})$). *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Let G be a torsionfree group which belongs to $\mathcal{C}$. Then $\mathcal{D}(G;\mathbb{Q})$ is a skew field and there are isomorphisms*

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G;\mathbb{Q})) \xrightarrow{\cong} \mathcal{D}(G;\mathbb{Q})^\times/[\mathcal{D}(G;\mathbb{Q})^\times, \mathcal{D}(G;\mathbb{Q})^\times].$$

In the special case that $G = \mathbb{Z}$, the right side reduces to the multiplicative abelian group of nontrivial elements in the field $\mathbb{Q}(z, z^{-1})$ of rational functions with rational coefficients in one variable. This reflects the fact that in the case $G = \mathbb{Z}$ the universal L^2-torsion is closely related to Alexander polynomials.

1. Universal localization

1A. *Review of universal localization.* Let R be a (associative) ring (with unit) and let Σ be a set of homomorphisms between finitely generated projective (left) R-modules. A ring homomorphism $f : R \to S$ is called *Σ-inverting* if for every element $\alpha : M \to N$ of Σ the induced map $S \otimes_R \alpha : S \otimes_R M \to S \otimes_R N$ is an isomorphism. A Σ-inverting ring homomorphism $i : R \to R_\Sigma$ is called *universal Σ-inverting* if for any Σ-inverting ring homomorphism $f : R \to S$ there is precisely one ring homomorphism $f_\Sigma : R_\Sigma \to S$ satisfying $f_\Sigma \circ i = f$. If $f : R \to R_\Sigma$ and $f' : R \to R'_\Sigma$ are two universal Σ-inverting homomorphisms, then by the universal property there is precisely one isomorphism $g : R_\Sigma \to R'_\Sigma$ with $g \circ f = f'$. This shows the uniqueness of the universal Σ-inverting homomorphism. The universal Σ-inverting ring homomorphism exists; see [Schofield 1985, Section 4]. If Σ is a set of matrices, a model for R_Σ is given by considering the free R-ring generated by the set of symbols $\{\bar{a}_{i,j} \mid A = (a_{i,j}) \in \Sigma\}$ and dividing out the relations given in matrix form by $\bar{A}A = A\bar{A} = 1$, where $\bar{A}$ stands for $(\bar{a}_{i,j})$ for $A = (a_{i,j})$. The map $i : R \to R_\Sigma$ does not need to be injective and the functor $R_\Sigma \otimes_R -$ does not need to be exact in general.

A special case of a universal localization is the Ore localization $S^{-1}R$ of a ring R for a multiplicative closed subset $S \subseteq R$ which satisfies the Ore condition, namely take Σ to be the set of R-homomorphisms $r_s : R \to R,\ r \mapsto rs$, where s runs through S. For the Ore localization the functor $S^{-1}R \otimes_R -$ is exact and the kernel of the canonical map $R \to S^{-1}R$ is $\{r \in R \mid \exists s \in S \text{ with } rs = 0\}$.

Let R be a ring and let Σ be a set of homomorphisms between finitely generated projective R-modules. We call Σ *saturated* if for any two elements $f_0 : P_0 \to Q_0$ and $f_1 : P_1 \to Q_1$ of Σ and any R-homomorphism $g_0 : P_0 \to Q_1$ and $g_1 : P_1 \to Q_0$ the R-homomorphisms

$$\begin{pmatrix} f_0 & 0 \\ g_0 & f_1 \end{pmatrix} : P_0 \oplus P_1 \to Q_0 \oplus Q_1 \quad \text{and} \quad \begin{pmatrix} f_0 & g_1 \\ 0 & f_1 \end{pmatrix} : P_0 \oplus P_1 \to Q_0 \oplus Q_1$$

belong to Σ and, for every R-homomorphism $f_0 : P_0 \to Q_0$ which becomes invertible over R_Σ, there is an element $f_1 : P_1 \to Q_1$ in Σ, finitely generated projective R-modules X and Y, and R-isomorphisms $u : P_0 \oplus X \xrightarrow{\cong} P_1 \oplus Y$ and $v : Q_0 \oplus X \xrightarrow{\cong} Q_1 \oplus Y$ satisfying $(f_1 \oplus \mathrm{id}_Y) \circ u = v \circ (f_0 \oplus \mathrm{id}_X)$. We can always find for Σ another set Σ' with $\Sigma \subseteq \Sigma'$ such that Σ' is saturated and the canonical map $R_\Sigma \to R_{\Sigma'}$ is bijective. Moreover, in nearly all cases we will consider sets Σ which are already saturated. Indeed if Σ' denotes the set of all maps between finitely generated projective (left) modules which become invertible over R_Σ, then $\Sigma \subseteq \Sigma'$, Σ' is saturated, and the canonical map $R_\Sigma \to R_{\Sigma'}$ is an isomorphism; see [Cohn 1985, Exercise 7.2.8 on page 394]. Therefore we can assume without harm in the sequel that Σ is saturated.

1B. *K_1 of universal localizations.* Let R be a ring and let Σ be a (saturated) set of homomorphisms between finitely generated projective R-modules.

Definition 1.1. Let $K_1(R, \Sigma)$ be the abelian group defined in terms of generators and relations as follows: Generators $[f]$ are (conjugacy classes of) R-endomorphisms $f : P \to P$ of finitely generated projective R-modules P such that $\mathrm{id}_{R_\Sigma} \otimes_R f : R_\Sigma \otimes_R P \to R_\Sigma \otimes_R P$ is an isomorphism. If $f, g : P \to P$ are R-endomorphisms of the same finitely generated projective R-module P such that $\mathrm{id}_{R_\Sigma} \otimes_R f$ and $\mathrm{id}_{R_\Sigma} \otimes_R g$ are bijective, then we require the relation

$$[g \circ f] = [g] + [f].$$

If we have a commutative diagram of finitely generated projective R-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P_0 \arrow[r, "i"] \arrow[d, "f_0"'] & P_1 \arrow[r, "p"] \arrow[d, "f_1"'] & P_2 \arrow[r] \arrow[d, "f_2"'] & 0 \\
0 \arrow[r] & P_0 \arrow[r, "i"] & P_1 \arrow[r, "p"] & P_2 \arrow[r] & 0
\end{tikzcd}$$

such that $\mathrm{id}_{R_\Sigma} \otimes_R f_0$, $\mathrm{id}_{R_\Sigma} \otimes_R f_2$ (and hence $\mathrm{id}_{R_\Sigma} \otimes_R f_1$) are bijective, then we require the relation

$$[f_1] = [f_0] + [f_2].$$

If the set Σ consists of all isomorphisms $R^n \xrightarrow{\cong} R^n$ for all $n \geq 0$, then for an R-endomorphism $f : P \to P$ of a finitely generated projective R-module P, the induced map $\mathrm{id}_{R_\Sigma} \otimes f$ is bijective if and only if f itself is already bijective and hence $K_1(R, \Sigma)$ is just the classical first K-group $K_1(R)$.

The main result of this section is:

Theorem 1.2 ($K_1(R, \Sigma)$ and $K_1(R_\Sigma)$). *Suppose that every element in Σ is given by an endomorphism of a finitely generated projective R-module and that the* *canonical map $i : R \to R_\Sigma$ is injective. Then the homomorphism*

$$\alpha : K_1(R, \Sigma) \xrightarrow{\cong} K_1(R_\Sigma), \quad [f : P \to P] \mapsto [\mathrm{id}_{R_\Sigma} \otimes_R f : R_\Sigma \otimes_R P \to R_\Sigma \otimes_R P],$$

is bijective.

Proof. We construct an inverse

$$\beta : K_1(R_\Sigma) \to K_1(R, \Sigma) \tag{1.3}$$

as follows: Consider an element x in $K_1(R_\Sigma)$. Then we can choose a finitely generated projective R-module Q (actually, we could choose it to be finitely generated free) and an R_Σ-automorphism

$$a : R_\Sigma \otimes_R Q \xrightarrow{\cong} R_\Sigma \otimes_R Q$$

such that $x = [a]$. Now the key ingredient is Cramer's rule; see [Schofield 1985, Theorem 4.3 on page 53]. It implies the existence of a finitely generated projective R-module P, two R-homomorphisms $b, b' : P \oplus Q \to P \oplus Q$ and an R_Σ-homomorphism $a' : R_\Sigma \otimes_R Q \to R_\Sigma \otimes_R P$ such that $\mathrm{id}_{R_\Sigma} \otimes_R b$ is bijective, and for the R_Σ-homomorphism

$$A = \begin{pmatrix} \mathrm{id}_{R_\Sigma \otimes_R P} & a' \\ 0 & a \end{pmatrix} : R_\Sigma \otimes_R P \oplus R_\Sigma \otimes_R Q \to R_\Sigma \otimes_R P \oplus R_\Sigma \otimes_R Q$$

the composite

$$R_\Sigma \oplus (P \oplus Q) \xrightarrow{i} R_\Sigma \otimes_R P \oplus R_\Sigma \otimes_R Q \xrightarrow{A} R_\Sigma \otimes_R P \oplus R_\Sigma \otimes_R Q$$
$$\xrightarrow{i^{-1}} R_\Sigma \oplus (P \oplus Q) \xrightarrow{\mathrm{id}_{R_\Sigma} \otimes_R b} R_\Sigma \oplus (P \oplus Q)$$

agrees with $\mathrm{id}_{R_\Sigma} \otimes_R b'$, where i is the canonical R_Σ-isomorphism. Then also $\mathrm{id}_{R_\Sigma} \otimes_R b$ is bijective. We want to define

$$\beta(x) := [b'] - [b]. \tag{1.4}$$

The main problem is to show that this is independent of the various choices. Given a finitely generated projective R-module P and an R_Σ-automorphism

$$a : R_\Sigma \otimes_R Q \xrightarrow{\cong} R_\Sigma \otimes_R Q$$

and two such choices (P, b, b', a') and $(\bar{P}, \bar{b}, \bar{b}', \bar{a}')$, we next show

$$[b] - [b] := [\bar{b}] - [\bar{b}']. \tag{1.5}$$

We can write

$$b = \begin{pmatrix} b_{P,P} & b_{Q,P} \\ b_{P,Q} & b_{Q,Q} \end{pmatrix}, \quad b' = \begin{pmatrix} b'_{P,P} & b_{Q,P} \\ b'_{P,Q} & b'_{Q,Q} \end{pmatrix},$$
$$\bar{b} = \begin{pmatrix} \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} \\ \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} \end{pmatrix}, \quad \bar{b}' = \begin{pmatrix} \bar{b}'_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} \\ \bar{b}'_{\bar{P},Q} & \bar{b}'_{Q,Q} \end{pmatrix},$$

for R-homomorphisms $b_{P,P}\colon P\to P$, $b_{P,Q}\colon P\to Q$, $b_{Q,P}\colon Q\to P$ and $b_{Q,Q}\colon Q\to Q$, and analogously for b', $\bar{b}$ and $\bar{b}'$. Then the relation between b and b' and $\bar{b}$ and $\bar{b}'$ becomes

$$\begin{pmatrix}\mathrm{id}_{R_\Sigma}\otimes_R b_{P,P} & \mathrm{id}_{R_\Sigma}\otimes_R b_{Q,P}\\ \mathrm{id}_{R_\Sigma}\otimes_R b_{P,Q} & \mathrm{id}_{R_\Sigma}\otimes_R b_{Q,Q}\end{pmatrix}\circ\begin{pmatrix}\mathrm{id}_{R_\Sigma\otimes_R P} & a'\\ 0 & a\end{pmatrix}=\begin{pmatrix}\mathrm{id}_{R_\Sigma}\otimes_R b'_{P,P} & \mathrm{id}_{R_\Sigma}\otimes_R b'_{Q,P}\\ \mathrm{id}_{R_\Sigma}\otimes_R b'_{P,Q} & \mathrm{id}_{R_\Sigma}\otimes_R b'_{Q,Q}\end{pmatrix},$$

and analogously for $\bar{b}$ and $\bar{b}'$. This implies $\mathrm{id}_{R_\Sigma}\otimes_R b_{P,P}=\mathrm{id}_{R_\Sigma}\otimes_R b'_{P,P}$ and hence $b_{P,P}=b'_{P,P}$ because of the injectivity of $i\colon R\to R_\Sigma$. Analogously we get $b_{P,Q}=b'_{P,Q}$, $\bar{b}_{\bar{P},\bar{P}}=\bar{b}'_{\bar{P},\bar{P}}$ and $\bar{b}_{\bar{P},Q}=\bar{b}'_{\bar{P},Q}$.

The argument in [Schofield 1985, page 64–65] based on Malcolmson's criterion [ibid., Theorem 4.2 on page 53] implies that there exist finitely generated projective R-modules X_0 and X_1, and R-homomorphisms

$$\begin{aligned}
d_1\colon X_1\to X_1,&\quad d_2\colon X_2\to X_2,\\
e_1\colon X_1\to Q,&\quad e_2\colon X_2\to P,\\
\mu\colon P\oplus Q\oplus\bar{P}\oplus Q\oplus X_1\oplus X_2\oplus Q&\to P\oplus Q\oplus\bar{P}\oplus Q\oplus X_1\oplus X_2\oplus Q,\\
\nu\colon P\oplus Q\oplus\bar{P}\oplus Q\oplus X_1\oplus X_2&\to P\oplus Q\oplus\bar{P}\oplus Q\oplus X_1\oplus X_2,\\
\tau\colon P\oplus Q\oplus\bar{P}\oplus Q\oplus X_1\oplus X_2&\to Q,
\end{aligned}$$

such that $\mathrm{id}_{R_\Sigma}\otimes_R d_1$, $\mathrm{id}_{R_\Sigma}\otimes_R d_2$, $\mathrm{id}_{R_\Sigma}\otimes_R \mu$ and $\mathrm{id}_{R_\Sigma}\otimes_R \nu$ are R_Σ-isomorphisms and, for the four R-homomorphisms

$$P\oplus Q\oplus\bar{P}\oplus Q\oplus X_1\oplus X_2\oplus Q\to P\oplus Q\oplus\bar{P}\oplus Q\oplus X_1\oplus X_2\oplus Q$$

given by

$$\alpha=\begin{pmatrix}
b_{P,P} & b_{Q,P} & 0 & 0 & 0 & 0 & 0\\
b_{P,Q} & b_{Q,Q} & 0 & 0 & 0 & 0 & 0\\
0 & \bar{b}_{Q,\bar{P}} & \bar{b}'_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & 0 & 0 & \bar{b}'_{Q,\bar{P}}\\
0 & \bar{b}_{Q,Q} & \bar{b}'_{\bar{P},Q} & \bar{b}_{Q,Q} & 0 & 0 & \bar{b}'_{Q,Q}\\
0 & 0 & 0 & 0 & d_1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & d_2 & 0\\
0 & 0 & 0 & \mathrm{id}_Q & e_1 & 0 & 0
\end{pmatrix},\quad
\alpha'=\begin{pmatrix}
b'_{P,P} & b_{Q,P} & 0 & 0 & 0 & 0 & -b'_{Q,P}\\
b'_{P,Q} & b_{Q,Q} & 0 & 0 & 0 & 0 & -b'_{Q,Q}\\
0 & \bar{b}_{Q,\bar{P}} & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & 0 & 0 & 0\\
0 & \bar{b}_{Q,Q} & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & d_1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & d_2 & -e_2\\
0 & 0 & 0 & \mathrm{id}_Q & e_1 & 0 & 0
\end{pmatrix},$$

$$\gamma=\begin{pmatrix}\nu & 0\\ 0 & \mathrm{id}_Q\end{pmatrix}\quad\text{and}\quad\gamma'=\begin{pmatrix}\nu & 0\\ \tau & \mathrm{id}_Q\end{pmatrix}$$

we get equations of maps of R-modules

$$\mu\circ\gamma=\alpha,\quad \mu\circ\gamma'=\alpha'.$$

Since $\mathrm{id}_{R_\Sigma}\otimes_R\mu$, $\mathrm{id}_{R_\Sigma}\otimes_R\gamma$ and $\mathrm{id}_{R_\Sigma}\otimes_R\gamma'$ are isomorphisms, also $\mathrm{id}_{R_\Sigma}\otimes_R\alpha$ and $\mathrm{id}_{R_\Sigma}\otimes_R\alpha'$ are isomorphisms. Hence we get well-defined elements $[\mu]$, $[\nu]$, $[\nu']$,

$[\alpha]$ and $[\alpha']$ in $K_1(R,\Sigma)$ satisfying

$$[\mu] = [\gamma] + [\alpha], \quad [\mu] = [\gamma'] + [\alpha'], \quad [\gamma] = [\gamma'].$$

This implies

$$[\alpha] = [\alpha']. \tag{1.6}$$

If we interchange in the matrix defining α the fourth and the last column, we get a matrix in a suitable block form, which allows us to deduce

$$\begin{aligned}
[\alpha] &= -\left[\begin{pmatrix}
b_{P,P} & b_{Q,P} & 0 & 0 & 0 & 0 & 0\\
b_{P,Q} & b_{Q,Q} & 0 & 0 & 0 & 0 & 0\\
0 & \bar{b}_{Q,\bar{P}} & \bar{b}'_{\bar{P},\bar{P}} & \bar{b}'_{Q,\bar{P}} & 0 & 0 & \bar{b}_{Q,\bar{P}}\\
0 & \bar{b}_{Q,Q} & \bar{b}'_{\bar{P},Q} & \bar{b}'_{Q,Q} & 0 & 0 & \bar{b}_{Q,Q}\\
0 & 0 & 0 & 0 & d_1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & d_2 & 0\\
0 & 0 & 0 & 0 & e_1 & 0 & \mathrm{id}_Q
\end{pmatrix}\right]\\
&= -\left[\begin{pmatrix}
b_{P,P} & b_{Q,P} & 0 & 0\\
b_{P,Q} & b_{Q,Q} & 0 & 0\\
0 & \bar{b}_{Q,\bar{P}} & \bar{b}'_{\bar{P},\bar{P}} & \bar{b}'_{Q,\bar{P}}\\
0 & \bar{b}_{Q,Q} & \bar{b}'_{\bar{P},Q} & \bar{b}'_{Q,Q}
\end{pmatrix}\right] - \left[\begin{pmatrix}
d_1 & 0 & 0\\
0 & d_2 & 0\\
e_1 & 0 & \mathrm{id}_Q
\end{pmatrix}\right]\\
&= -\left[\begin{pmatrix}
b_{P,P} & b_{Q,P}\\
b_{P,Q} & b_{Q,Q}
\end{pmatrix}\right] - \left[\begin{pmatrix}
\bar{b}'_{\bar{P},\bar{P}} & \bar{b}'_{Q,\bar{P}}\\
\bar{b}'_{\bar{P},Q} & \bar{b}'_{Q,Q}
\end{pmatrix}\right] - [d_1] - [d_2] - [\mathrm{id}_Q]\\
&= -[b] - [\bar{b}'] - [d_1] - [d_2].
\end{aligned} \tag{1.7}$$

Similarly we get from the matrix describing α' after interchanging the second and the last column, multiplying the second column with -1, interchanging the fourth and the last column and finally subtracting appropriate multiples of the last row from the third row to ensure that in the last column all entries except the one in the right lower corner is a trivial matrix in a suitable block form, which allows us to deduce

$$[\alpha'] = \left[\begin{pmatrix}
b'_{P,P} & b'_{Q,P} & 0 & 0 & 0 & 0 & b_{Q,P}\\
b'_{P,Q} & b'_{Q,Q} & 0 & 0 & 0 & 0 & b_{Q,Q}\\
0 & 0 & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & 0 & 0 & \bar{b}_{Q,\bar{P}}\\
0 & & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} & 0 & 0 & \bar{b}_{Q,Q}\\
0 & 0 & 0 & 0 & d_1 & 0 & 0\\
0 & e_2 & 0 & 0 & 0 & d_2 & -e_2\\
0 & 0 & 0 & \mathrm{id}_Q & e_1 & 0 & 0
\end{pmatrix}\right]$$

$$
\begin{aligned}
&= \left[\begin{pmatrix}
b'_{P,P} & b'_{Q,P} & 0 & b_{Q,P} & 0 & 0 & 0 \\
b'_{P,Q} & b'_{Q,Q} & 0 & b_{Q,Q} & 0 & 0 & 0 \\
0 & 0 & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & 0 & 0 & \bar{b}_{Q,\bar{P}} \\
0 & & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} & 0 & 0 & \bar{b}_{Q,Q} \\
0 & 0 & 0 & 0 & d_1 & 0 & 0 \\
0 & e_2 & 0 & 0 & 0 & d_2 & 0 \\
0 & 0 & 0 & 0 & e_1 & 0 & \mathrm{id}_Q
\end{pmatrix}\right] \\
&= -\left[\begin{pmatrix}
b'_{P,P} & b'_{Q,P} & 0 & b_{Q,P} & 0 & 0 & 0 \\
b'_{P,Q} & b'_{Q,Q} & 0 & b_{Q,Q} & 0 & 0 & 0 \\
0 & 0 & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & -\bar{b}_{Q,\bar{P}} \circ e_1 & 0 & 0 \\
0 & & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} & -\bar{b}_{Q,Q} \circ e_1 & 0 & 0 \\
0 & 0 & 0 & 0 & d_1 & 0 & 0 \\
0 & e_2 & 0 & 0 & 0 & d_2 & 0 \\
0 & 0 & 0 & 0 & e_1 & 0 & \mathrm{id}_Q
\end{pmatrix}\right] \\
&= -\left[\begin{pmatrix}
b'_{P,P} & b'_{Q,P} & 0 & b_{Q,P} & 0 \\
b'_{P,Q} & b'_{Q,Q} & 0 & b_{Q,Q} & 0 \\
0 & 0 & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} & -\bar{b}_{Q,\bar{P}} \circ e_1 \\
0 & & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} & -\bar{b}_{Q,Q} \circ e_1 \\
0 & 0 & 0 & 0 & d_1
\end{pmatrix}\right] - \left[\begin{pmatrix} d_2 & -e_2 \\ 0 & \mathrm{id}_Q \end{pmatrix}\right] \\
&= -\left[\begin{pmatrix}
b'_{P,P} & b'_{Q,P} & 0 & b_{Q,P} \\
b'_{P,Q} & b'_{Q,Q} & 0 & b_{Q,Q} \\
0 & 0 & \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} \\
0 & & \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q}
\end{pmatrix}\right] - [d_1] - [d_2] - [\mathrm{id}_Q] \\
&= -\left[\begin{pmatrix} b'_{P,P} & b'_{Q,P} \\ b'_{P,Q} & b'_{Q,Q} \end{pmatrix}\right] - \left[\begin{pmatrix} \bar{b}_{\bar{P},\bar{P}} & \bar{b}_{Q,\bar{P}} \\ \bar{b}_{\bar{P},Q} & \bar{b}_{Q,Q} \end{pmatrix}\right] - [d_1] - [d_2] \\
&= -[b'] - [\bar{b}] - [d_1] - [d_2]. \qquad (1.8)
\end{aligned}
$$

Now (1.5) follows from equations (1.6), (1.7), and (1.8).

We conclude from (1.8) that we can assign to a finitely generated projective R-module P and an R_Σ-automorphism $a : R_\Sigma \otimes_R Q \xrightarrow{\cong} R_\Sigma \otimes_R Q$ a well-defined element

$$[a] \in K_1(R, \Sigma). \qquad (1.9)$$

If we have an isomorphism $u : Q \xrightarrow{\cong} Q'$ of finitely generated projective R-modules, then one easily checks

$$[(\mathrm{id}_{R_\Sigma} \otimes_R u) \circ a \circ (\mathrm{id}_{R_\Sigma} \otimes_R u)^{-1}] = [a]. \qquad (1.10)$$

Given two finitely generated projective R-modules Q and $\overline{Q}$ and R_Σ-automorphisms $a : R_\Sigma \otimes_R Q \xrightarrow{\cong} R_\Sigma \otimes_R Q$ and $\bar{a} : R_\Sigma \otimes_R \overline{Q} \xrightarrow{\cong} R_\Sigma \otimes_R \overline{Q}$, one easily checks

$$[a \oplus \bar{a}] = [a] + [\bar{a}]. \tag{1.11}$$

Obviously we get, for any finitely generated projective R-module Q,

$$[(\mathrm{id}_{R_\Sigma} \otimes_R \mathrm{id}_Q)] = 0. \tag{1.12}$$

Consider a finitely generated projective R-module Q and two R_Σ-isomorphisms $a, \bar{a} : R_\Sigma \otimes_R Q \xrightarrow{\cong} R_\Sigma \otimes_R Q$. Next we want to show

$$[\bar{a} \circ a] = [\bar{a}] + [a]. \tag{1.13}$$

Make the choices (P, b, b', a') and $(\overline{P}, \bar{b}, \bar{b}', \bar{a}')$ for a and $\bar{a}$ as we did above in the definition of $[a]$ and $[\bar{a}]$. Consider the R_Σ-automorphism

$$A = \begin{pmatrix} \mathrm{id}_{R_\sigma \otimes_R P} & 0 & 0 & a' \\ 0 & \mathrm{id}_{R_\sigma \otimes_R Q} & 0 & a \\ 0 & 0 & \mathrm{id}_{R_\sigma \otimes_R \overline{P}} & \bar{a}'a \\ 0 & 0 & 0 & \bar{a}a \end{pmatrix}$$

of $(R_\Sigma \otimes_R P) \oplus (R_\Sigma \otimes_R Q) \oplus (R_\Sigma \otimes_R \overline{P}) \oplus (R_\Sigma \otimes_R Q)$, and the R-endomorphisms of $P \oplus Q \oplus \overline{P} \oplus Q$

$$B = \begin{pmatrix} b_{P,P} & b_{Q,P} & 0 & 0 \\ b_{P,Q} & b_{Q,Q} & 0 & 0 \\ 0 & -\bar{b}'_{Q,\overline{P}} & \bar{b}_{\overline{P},\overline{P}} & \bar{b}_{Q,\overline{P}} \\ 0 & -\bar{b}'_{Q,Q} & \bar{b}_{\overline{P},Q} & \bar{b}_{Q,Q} \end{pmatrix} \quad \text{and} \quad B' = \begin{pmatrix} b'_{P,P} & b_{Q,P} & 0 & b'_{Q,P} \\ b'_{P,Q} & b_{Q,Q} & 0 & b'_{Q,Q} \\ 0 & -\bar{b}'_{Q,P} & \bar{b}_{P,P} & 0 \\ 0 & -\bar{b}'_{Q,Q} & \bar{b}_{P,Q} & 0 \end{pmatrix}.$$

From the block structure of B one concludes that $(\mathrm{id}_{R_\Sigma} \otimes B)$ is an isomorphism and we get, in $K_1(R, \Sigma)$,

$$[B] = \left[\begin{pmatrix} b_{P,P} b_{Q,P} \\ b_{P,Q} b_{Q,Q} \end{pmatrix}\right] + \left[\begin{pmatrix} \bar{b}_{P,P} \bar{b}_{Q,P} \\ \bar{b}_{P,Q} \bar{b}_{Q,Q} \end{pmatrix}\right] = [b] + [\bar{b}]. \tag{1.14}$$

If we interchange in B'' the second and last column and multiply the last column with -1, we conclude from the block structure of the resulting matrix that $(\mathrm{id}_{R_\Sigma} \otimes B')$ is an isomorphism and we get, in $K_1(R, \Sigma)$,

$$\begin{aligned} [B'] &= \left[\begin{pmatrix} b'_{P,P} & b'_{Q,P} & 0 & b_{Q,P} \\ b'_{P,Q} & b'_{Q,Q} & 0 & b_{Q,Q} \\ 0 & 0 & \bar{b}_{\overline{P},\overline{P}} & \bar{b}'_{Q,\overline{P}} \\ 0 & 0 & \bar{b}_{\overline{P},Q} & \bar{b}'_{Q,Q} \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} b'_{P,P} & b'_{Q,P} \\ b'_{P,Q} & b'_{Q,Q} \end{pmatrix}\right] + \left[\begin{pmatrix} \bar{b}_{P,P} & \bar{b}'_{Q,P} \\ \bar{b}_{P,Q} & \bar{b}'_{Q,Q} \end{pmatrix}\right] = [b'] + [\bar{b}']. \end{aligned} \tag{1.15}$$

Since $(\mathrm{id}_{R_\Sigma} \otimes B)$ and $(\mathrm{id}_{R_\Sigma} \otimes B')$ are isomorphisms and $(\mathrm{id}_{R_\Sigma} \otimes B) \circ A = (\mathrm{id}_{R_\Sigma} \otimes B')$, we get, directly from the definitions,

$$[\bar{a}a] = [B'] - [B]. \tag{1.16}$$

Now (1.13) follows from equations (1.14), (1.15) and (1.16). Now one easily checks that equations (1.10), (1.11), (1.12) and (1.13) imply that the homomorphism β announced in (1.3) is well-defined. One easily checks that β is an inverse to the homomorphism α appearing in the statement of Theorem 1.2. This finishes the proof of Theorem 1.2. □

1C. *Schofield's localization sequence.* The proofs of this paper are motivated by Schofield's construction of a localization sequence

$$K_1(R) \to K_1(R_\Sigma) \to K_1(\mathcal{T}) \to K_0(R) \to K_0(R_\Sigma),$$

where $\mathcal{T}$ is the full subcategory of the category of the finitely presented R-modules whose objects are cokernels of elements in Σ; see [Schofield 1985, Theorem 5.12 on page 60]. Under certain conditions this sequence has been extended to the left in [Neeman 2007; Neeman and Ranicki 2004]. Notice that in connection with potential proofs of the Atiyah conjecture it is important to figure out under which condition $K_0(FG) \to K_0(\mathcal{D}(G; F))$ is surjective for a torsionfree group G and a subfield $F \subseteq \mathbb{C}$; see [Lück 2002, Theorem 10.38 on page 387]. In this connection the question becomes interesting whether G has property (UL) — see Section 2C — and how to continue the sequence above to the right.

2. Groups with property (ULA)

Throughout this section let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.

2A. *Review of division and rational closure.* Let R be a subring of the ring S. The *division closure* $\mathcal{D}(R \subseteq S) \subseteq S$ is the smallest subring of S which contains R and is division closed, i.e., any element $x \in \mathcal{D}(R \subseteq S)$ which is invertible in S is already invertible in $\mathcal{D}(R \subseteq S)$. The *rational closure* $\mathcal{R}(R \subseteq S) \subseteq S$ is the smallest subring of S which contains R and is rationally closed, i.e., for every natural number n and matrix $A \in M_{n,n}(\mathcal{D}(R \subseteq S))$ which is invertible in S, the matrix A is already invertible over $\mathcal{R}(R \subseteq S)$. The division closure and the rational closure always exist. Obviously $R \subseteq \mathcal{D}(R \subseteq S) \subseteq \mathcal{R}(R \subseteq S) \subseteq S$.

Consider an inclusion of rings $R \subseteq S$. Let $\Sigma(R \subseteq S)$ the set of all square matrices over R which become invertible over S. Then there is a canonical epimorphism of rings from the universal localization of R with respect to $\Sigma(R \subseteq S)$ to the rational closure of R in S — see [Reich 2006, Proposition 4.10(iii)] —

$$\lambda : R_{\Sigma(R \subseteq S)} \to \mathcal{R}(R \subseteq S). \tag{2.1}$$

Recall that we have inclusions $R \subseteq \mathcal{D}(R \subseteq S) \to \mathcal{R}(R \subseteq S) \subseteq S$.

Consider a group G. Let $\mathcal{N}(G)$ be the group von Neumann algebra, which can be identified with the algebra $\mathcal{B}(L^2(G), L^2(G))^G$ of bounded G-equivariant operators $L^2(G) \to L^2(G)$. Denote by $\mathcal{U}(G)$ the algebra of operators which are affiliated to the group von Neumann algebra. This is the same as the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of nonzero divisors in $\mathcal{N}(G)$; see [Lück 2002, Chapter 8]. By the right regular representation we can embed $\mathbb{C}G$ and hence also FG as a subring in $\mathcal{N}(G)$. We will denote by $\mathcal{R}(G; F)$ and $\mathcal{D}(G; F)$ the division and the rational closure of FG in $\mathcal{U}(G)$. So we get a commutative diagram of inclusions of rings

$$\begin{tikzcd} FG \arrow[r] \arrow[d] & \mathcal{N}(G) \arrow[dd] \\ \mathcal{D}(G;F) \arrow[d] & \\ \mathcal{R}(G;F) \arrow[r] & \mathcal{U}(G) \end{tikzcd}$$

2B. *Review of the Atiyah conjecture for torsionfree groups.* Recall that there is a dimension function $\dim_{\mathcal{N}(G)}$ defined for all (algebraic) $\mathcal{N}(G)$-modules; see [Lück 2002, Section 6.1].

Definition 2.2 (Atiyah conjecture with coefficients in F). We say that a torsionfree group G satisfies the *Atiyah conjecture with coefficients in F* if for any matrix $A \in M_{m,n}(FG)$ the von Neumann dimension $\dim_{\mathcal{N}(G)}(\ker(r_A))$ of the kernel of the $\mathcal{N}(G)$-homomorphism $r_A : \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ given by right multiplication with A is an integer.

Theorem 2.3 (status of the Atiyah conjecture). (1) *If the torsionfree group G satisfies the Atiyah conjecture with coefficients in F, then also each of its subgroups satisfy the Atiyah conjecture with coefficients in F.*

(2) *If the torsionfree group G satisfies the Atiyah conjecture with coefficients in $\mathbb{C}$, then G satisfies the Atiyah conjecture with coefficients in F.*

(3) *The torsionfree group G satisfies the Atiyah conjecture with coefficients in F if and only if $\mathcal{D}(G; F)$ is a skew field.*

If the torsionfree group G satisfies the Atiyah conjecture with coefficients in F, then the rational closure $\mathcal{R}(G; F)$ agrees with the division closure $\mathcal{D}(G; F)$.

(4) *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Suppose that G is a torsionfree group which belongs to $\mathcal{C}$.*

Then G satisfies the Atiyah conjecture with coefficients in $\mathbb{C}$.

(5) *Let G be an infinite group which is the fundamental group of a compact connected orientable irreducible 3-manifold M with empty or toroidal boundary. Suppose that one of the following conditions is satisfied*:

- *M is not a closed graph manifold.*
- *M is a closed graph manifold which admits a Riemannian metric of nonpositive sectional curvature.*

Then G is torsionfree and belongs to $\mathcal{C}$. In particular G satisfies the Atiyah conjecture with coefficients in $\mathbb{C}$.

(6) *Let $\mathcal{D}$ be the smallest class of groups such that*:

- *The trivial group belongs to $\mathcal{D}$.*
- *If $p : G \to A$ is an epimorphism of a torsionfree group G onto an elementary amenable group A and if $p^{-1}(B) \in \mathcal{D}$ for every finite group $B \subset A$, then $G \in \mathcal{D}$.*
- *$\mathcal{D}$ is closed under taking subgroups.*
- *$\mathcal{D}$ is closed under colimits and inverse limits over directed systems.*

If the group G belongs to $\mathcal{D}$, then G is torsionfree and the Atiyah conjecture with coefficients in $\overline{\mathbb{Q}}$ holds for G.

The class $\mathcal{D}$ is closed under direct sums, direct products and free products. Every residually torsionfree elementary amenable group belongs to $\mathcal{D}$.

Proof. (1) This follows from [Lück 2002, Theorem 6.29(2) on page 253].

(2) This is obvious.

(3) This is proved in the case $F = \mathbb{C}$ in [Lück 2002, Lemma 10.39 on page 388]. The proof goes through for an arbitrary field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ without modifications.

(4) This is due to Linnell; see for instance [Linnell 1993] or [Lück 2002, Theorem 10.19 on page 378].

(5) It suffices to show that $G = \pi_1(M)$ belongs to the class $\mathcal{C}$ appearing in assertion (4). As explained in [Dubois et al. 2016, Section 10], we conclude from combining [Agol 2008; 2013; Liu 2013; Przytycki and Wise 2012; 2014; Wise 2012a; 2012b] that there exists a finite normal covering $p : \overline{M} \to M$ and a fiber bundle $S \to \overline{M} \to S^1$ for some compact connected orientable surface S. Hence it suffices to show that $\pi_1(S)$ belongs to $\mathcal{C}$. If S has nonempty boundary, this follows from the fact that $\pi_1(S)$ is free. If S is closed, the commutator subgroup of $\pi_1(S)$ is free and hence $\pi_1(S)$ belongs to $\mathcal{C}$. Now assertion (5) follows from assertion (4).

(6) This result is due to Schick for $\mathbb{Q}$ (see for instance [Schick 2001] or [Lück 2002, Theorem 10.22 on page 379]) and for $\overline{\mathbb{Q}}$ due to Dodziuk, Linnell, Mathai, Schick and Yates [Dodziuk et al. 2003, Theorem 1.4] □

For more information and further explanations about the Atiyah conjecture we refer for instance to [Lück 2002, Chapter 10].

2C. *The property (UL).*

Definition 2.4 (property (UL)). We say that a group G has the property (UL) with respect to F if the canonical epimorphism

$$\lambda : FG_{\Sigma(FG\subseteq\mathcal{U}(G,F))} \to \mathcal{R}(G;F)$$

defined in (2.1) is bijective.

Next we investigate which groups G are known to have property (UL).

Let $\mathcal{A}$ denote the class of groups consisting of the finitely generated free groups and the amenable groups. If $\mathcal{Y}$ and $\mathcal{Z}$ are classes of groups, define

$$\mathrm{L}(\mathcal{Y}) = \{G \mid \text{every finite subset of } G \text{ is contained in a } \mathcal{Y}\text{-group}\},$$
$$\mathcal{Y}\mathcal{Z} = \{G \mid \text{there exists } H \lhd G \text{ such that } H \in \mathcal{Y} \text{ and } G/H \in \mathcal{Z}\}.$$

Now define $\mathcal{X}$ to be the smallest class of groups which contains $\mathcal{A}$ and is closed under directed unions and group extension. Next, for each ordinal a, define a class of groups $\mathcal{X}_a$ as follows:

- $\mathcal{X}_0 = \{1\}$.
- $\mathcal{X}_a = \mathrm{L}(\mathcal{X}_{a-1}\mathcal{A})$ if a is a successor ordinal.
- $\mathcal{X}_a = \bigcup_{b<a}\mathcal{X}_b$ if a is a limit ordinal.

Lemma 2.5. (1) *Each $\mathcal{X}_a$ is subgroup closed.*

(2) $\mathcal{X} = \bigcup_{a\geq 0}\mathcal{X}_a$.

(3) *$\mathcal{X}$ is subgroup closed.*

Proof. (1) This is easily proved by induction on a.

(2) Set $\mathcal{Y} = \bigcup_{a\geq 0}\mathcal{X}_a$. Obviously $\mathcal{X} \supseteq \mathcal{Y}$. We prove the reverse inclusion by showing that $\mathcal{Y}$ is closed under directed unions and group extension. The former is obvious, because if the group G is the directed union of subgroups G_i and a_i is the least ordinal such that $G_i \in \mathcal{X}_{a_i}$, we set $a = \sup_i a_i$ and then $G \in \mathcal{X}_{a+1}$. For the latter, we show that $\mathcal{X}_a\mathcal{X}_b \subseteq \mathcal{X}_{a+b}$ by induction on b, the case $b = 0$ being obvious. If b is a successor ordinal, write $b = c + 1$. Then

$$\begin{aligned}\mathcal{X}_a\mathcal{X}_b = \mathcal{X}_a(\mathrm{L}(X_c)\mathcal{A}) &\subseteq \mathrm{L}(\mathcal{X}_a\mathcal{X}_c)\mathcal{A} \\ &\subseteq \mathrm{L}(\mathcal{X}_{a+c})\mathcal{A} && \text{by induction} \\ &\subseteq \mathcal{X}_{a+c+1} = \mathcal{X}_{a+b}.\end{aligned}$$

On the other hand, if b is a limit ordinal, then

$$\begin{aligned}\mathcal{X}_a\mathcal{X}_b = \mathcal{X}_a\Big(\bigcup_{c<b}\mathcal{X}_c\Big) &= \bigcup_{c<b}\mathcal{X}_a\mathcal{X}_c \\ &\subseteq \bigcup_{c<b}\mathcal{X}_{a+c} && \text{by induction} \\ &\subseteq \mathcal{X}_{a+b},\end{aligned}$$

as required.

(3) This follows from assertions (1) and (2). $\square$

Lemma 2.6. *Let $G = \bigcup_{i\in I} G_i$ be groups such that, given $i, j \in I$, there exists $l \in I$ such that $G_i, G_j \subseteq G_l$. Write $\Sigma = \Sigma(FG \subseteq \mathcal{U}(G))$ and $\Sigma_i = \Sigma(FG_i \subseteq \mathcal{U}(G_i))$ for $i \in I$. Suppose the identity map on FG_i extends to an isomorphism $\lambda_i : (FG_i)_{\Sigma_i} \to \mathcal{R}(G_i; F)$ for all $i \in I$.*

Then the identity map on FG extends to an isomorphism $\lambda : FG_\Sigma \to \mathcal{R}(G; F)$.

Proof. By definition, the identity map on FG extends to an epimorphism $\lambda : FG_\Sigma \to \mathcal{R}(G; F)$. We need to show that λ is injective, and here we follow the proof of [Linnell 1998, Lemma 13.5]. Clearly $\Sigma_i \subseteq \Sigma$ for all $i \in I$ and thus the inclusion map $FG_i \hookrightarrow FG$ extends to a map $\mu_i : (FG_i)_{\Sigma_i} \to FG_\Sigma$ for all $i \in I$. Since λ_i is an isomorphism, we may define $\nu_i = \mu_i \circ \lambda_i^{-1} : \mathcal{R}(G_i; F) \to FG_\Sigma$ for all $i \in I$. If $G_i \subseteq G_j$, then $\mathcal{R}(G_i; F) \subseteq \mathcal{R}(G_j; F)$ and we let $\psi_{ij} : \mathcal{R}(G_i; F) \to \mathcal{R}(G_j; F)$ denote the natural inclusion. Observe that $\mu_i(x) = \mu_j\lambda_j^{-1}\psi_{ij}\lambda_i(x)$ for all x in the image of FG_i in $(FG_i)_{\Sigma_i}$ and therefore, by the universal property, $\mu_i = \mu_j\lambda_j^{-1}\psi_{ij}\lambda_i$ and hence $\mu_i\lambda_i^{-1} = \mu_j\lambda_j^{-1}\psi_{ij}$. Thus $\nu_i = \nu_j\psi_{ij}$ and the ν_i fit together to give a map $\nu : \bigcup_{i\in I}\mathcal{R}(G_i; F) \to FG_\Sigma$. It is easily checked that $\nu \circ \lambda : FG_\Sigma \to FG_\Sigma$ is a map which is the identity on the image of FG in FG_Σ and hence by the universal property of localization, $\nu \circ \lambda$ is the identity. This proves that λ is injective, as required. $\square$

If G is a group and α is an automorphism of G, then α extends to an automorphism of $\mathcal{U}(G)$, which we shall also denote by α. This is not only an algebraic automorphism, but is also a homeomorphism with respect to the various topologies on $\mathcal{U}(G)$.

Lemma 2.7. *If α is an automorphism of G, then $\alpha(\mathcal{D}(G; F)) = \mathcal{D}(G; F)$.*

Proof. This is clear, because $\alpha(FG) = FG$. $\square$

Lemma 2.8. *Let $H \lhd G$ be groups and let $\mathcal{D}(H; F)G$ denote the subring of $\mathcal{D}(G; F)$ generated by $\mathcal{D}(H; F)$ and G.*

Then, for a suitable crossed product, $\mathcal{D}(H; F)G \cong \mathcal{D}(H; F) * G/H$ by a map which extends the identity on $\mathcal{D}(H; F)$ and, for $g \in G$, sends $\mathcal{D}(H; F) \cdot g$ to $\mathcal{D}(H; F) * Hg$.

Proof. Let T be a transversal for H in G. Since $h \mapsto tht^{-1}$ is an automorphism of H, we see that $t \cdot \mathcal{D}(H; F) \cdot t^{-1} = \mathcal{D}(H; F)$ for all $t \in T$ by Lemma 2.7 and so $\mathcal{D}(H; F)G = \sum_{t \in T} \mathcal{D}(H; F)G \cdot t$. This sum is direct because the sum $\sum_{t \in T} \mathcal{U}(H) \cdot t$ is direct, and the result is established. □

In the sequel recall that $\mathcal{R}(G; F) = \mathcal{D}(G; F)$ holds if $\mathcal{D}(G; F)$ is a skew field.

Lemma 2.9. *Let $H \lhd G$ be groups such that G/H is finite and H is torsion-free. Assume that $\mathcal{D}(H; F)$ is a skew field. Set $\Sigma = \Sigma(FG \subseteq \mathcal{U}(G))$ and $\Phi = \Sigma(FH \subseteq \mathcal{U}(H))$, and let $\mu : FH_\Phi \to \mathcal{D}(H; F)$ and $\lambda : FG_\Sigma \to \mathcal{D}(G; F)$ denote the corresponding localization maps.*

Then $\mathcal{D}(G; F)$ is a semisimple artinian ring and agrees with $\mathcal{R}(G; F)$. Furthermore, if μ is an isomorphism, then so is λ.

Proof. Let $\mathcal{D}(H; F)G$ denote the subring of $\mathcal{D}(G; F)$ generated by $\mathcal{D}(H; F)$ and G. Then Lemma 2.8 shows, that for a suitable crossed product, there is an isomorphism $\theta : \mathcal{D}(H; F) * G/H \to \mathcal{D}(H; F)G$ which extends the identity map on $\mathcal{D}(H; F)$. This ring has dimension $|G/H|$ over the skew field $\mathcal{D}(H; F)$ and is therefore artinian. Since every matrix over an artinian ring is either a zero-divisor or invertible (in particular every element is either a zero-divisor or invertible), we see that $\mathcal{R}(G; F) = \mathcal{D}(G; F) = \mathcal{D}(H; F)G$. Furthermore, by Maschke's theorem, $\mathcal{D}(H; F)G$ semisimple artinian. Now assume that μ is an isomorphism. We may identify FG with the subring $FH * G/H$ and then by [Linnell 1993, Lemma 4.5] there is an isomorphism $\psi : \mathcal{D}(H; F) * G/H \to FG_\Phi$ which extends the identity map on FG. Also $\Phi \subseteq \Sigma$, so the identity map on FG extends to a map $\rho : FG_\Phi \to FG_\Sigma$. Then $\rho \circ \psi \circ \theta^{-1} \circ \lambda : FG_\Sigma \to FG_\Sigma$ is a map extending the identity on FG, hence is the identity, and the result follows. □

Recall that the group G is locally indicable if for every nontrivial finitely generated subgroup H there exists $N \lhd H$ such that N/H is infinite cyclic. Also if R is a subring of the skew field D such that $\mathcal{D}(R \subseteq D) = D$, then we say that D is a field of fractions for R (D will be noncommutative, i.e., a skew field in general).

Definition 2.10. Let K be a skew field, let G be a locally indicable group, let $K * G$ be a crossed product, and let D be a field of fractions for $K * G$. Then we say that D is a *Hughes-free* [Hughes 1970, §2; Lewin 1974, pp. 340, 342; Lück 2002, Lemma 10.81; [Dicks et al. 2004, p. 1128]] field of fractions for $K * G$ if whenever $N \lhd H \leq G$, H/N is infinite cyclic and $t \in H$ such that $\langle Nt \rangle = H/N$ (i.e., t generates H mod N), then $\{t^i \mid i \in \mathbb{Z}\}$ is linearly independent over $\mathcal{D}(K * N \subseteq D)$.

A key result here is that of Ian Hughes [1970, Theorem, page 182] (see also [Dicks et al. 2004, Theorem 7.1]), which states:

Theorem 2.11 (Hughes's theorem). *Let K be a skew field, let G be a locally indicable group, let $K * G$ be a crossed product, and let D_1 and D_2 be Hughes-free* *field of fractions for $K * G$. Then there is an isomorphism $D_1 \to D_2$ which is the identity on $K * G$.*

Recall that a ring R is called a *fir* (free ideal ring [Cohn 1995, §1.6]) if every left ideal is a free left R-module of unique rank, and every right ideal is a free right R-module of unique rank. Also, R is called a semifir if the above condition is only satisfied for all finitely generated left and right ideals. It is easy to see that if K is a skew field, G is the infinite cyclic group and $K * G$ is a crossed product, then every nonzero left or right ideal is free of rank one and hence $K * G$ is a fir. We can now apply [Cohn 1995, Theorem 5.3.9] (a result essentially due to Bergman [1974]) to deduce that if G is a free group and $K * G$ is a crossed product, then $K * G$ is a fir.

We also need the concept of a *universal* field of fractions; this is described in [Cohn 1985, §7.2; 1995, §4.5]. It is proven in [Cohn 1985, Corollary 7.5.11; 1995, Corollary 4.5.9] that if R is a semifir, then it has a universal field of fractions D. Furthermore the inclusion $R \subseteq D$ is an honest map [Cohn 1985, p. 250; 1995, p. 177], fully inverting [Cohn 1985, p. 415; 1995, p. 177], and the localization map $R_{\mathfrak{D}(R\subseteq D)} \to D$ is an isomorphism. We can now state a crucial result of Jacques Lewin [1974, Proposition 6].

Theorem 2.12 (Lewin's theorem). *Let K be a skew field, let G be a free group, let $K * G$ be a crossed product, and let D be the universal field of fractions for $K * G$. Then D is Hughes-free.*

Actually Lewin only proves the result for K a field and $K * G$ the group algebra KG over K. However, with the remarks above, in particular that $K * G$ is a fir, we can follow Lemmas 1–6 and Theorem 1 of [Lewin 1974] verbatim to deduce Theorem 2.12.

Lemma 2.13. *Let $H \lhd G$ be groups and let $G/H \in \mathcal{A}$. Assume that $\mathfrak{D}(G; F)$ is a skew field. Write $\Sigma = \Sigma(FG \subseteq \mathfrak{U}(G))$ and $\Phi = \Sigma(FH \subseteq \mathfrak{U}(H))$. Let $\mu : FH_\Phi \to \mathfrak{R}(H; F)$ and $\lambda : FG_\Sigma \to \mathfrak{R}(G; F)$ be the localization maps which extend the identity on FH and FG, respectively. Suppose that μ is an isomorphism.*

Then $\mathfrak{D}(G; F) = \mathfrak{R}(G; F)$, and λ is an isomorphism.

Proof. We already know that $\mathfrak{D}(G; F) = \mathfrak{R}(G; F)$ because we are assuming that $\mathfrak{D}(G; F)$ is a skew field, and clearly λ is an epimorphism. We need to show that λ is injective. Lemma 2.8 shows that $\mathfrak{D}(H; F)G \cong \mathfrak{D}(H; F) * G/H$ and we will use the corresponding isomorphism to identify these two rings without further comment. Since we are assuming that $\mathfrak{D}(G; F)$ is a skew field, $\mathfrak{D}(H; F) * G/H$ is a domain. Furthermore, $FG_\Phi \cong (FH * G/H)_\Phi \cong FH_\Phi * G/H$ by Lemma 2.7 and [Linnell 1993, Lemma 4.5], and we deduce that the localization map $FG_\Phi \to \mathfrak{D}(H; F) * G/H$ is an isomorphism, because we are assuming that μ is an isomorphism. Let $\Psi =$

$\Sigma(\mathscr{D}(H;F)G \subseteq \mathscr{D}(G;F))$. The proof of [Schofield 1985, Theorem 4.6] shows that $(FG_\Phi)_\Psi \cong FG_{\Sigma'}$ for a suitable set of matrices Σ' over FG (where we have identified FG_Φ with $\mathscr{D}(H;F)G$ by the above isomorphisms). All the matrices in Σ' become invertible over $\mathscr{R}(G;F)$, so by [Cohn 1985, Exercise 7.2.8] we may replace Σ' by its saturation. It remains to prove that the localization map $\mathscr{D}(H;F)G_\Psi \to \mathscr{R}(G;F)$ is injective.

We have two cases to consider, namely G/H amenable and G/H finitely generated free. For the former we apply [Dodziuk et al. 2003, Theorem 6.3] (essentially a result of Tamari [1957]). We deduce that $\mathscr{D}(H;F) * G/H$ satisfies the Ore condition for the multiplicatively closed subset of nonzero elements of $\mathscr{D}(H;F) * G/H$ and it follows that the localization map $\mathscr{D}(H;F)G_\Psi \to \mathscr{R}(G;F)$ is an isomorphism.

For the latter case, let $L \lhd M$ be subgroups of G containing H such that M/L is infinite cyclic and let $t \in M$ be a generator for M mod L. Since the sum $\sum_{i\in\mathbb{Z}} \mathscr{U}(L)t^i$ is direct, we see that the sum $\sum_{i\in\mathbb{Z}} \mathscr{D}(L;F)t^i$ is also direct and we deduce that $\mathscr{D}(G;F)$ is a Hughes-free field of fractions for $\mathscr{D}(H;F) * G/H$. It now follows from Theorems 2.11 and 2.12 that $\mathscr{D}(G;F)$ is a universal field of fractions for $\mathscr{D}(H;L)G$ and, in particular, the localization map $\mathscr{D}(H;F)G_\Psi \to \mathscr{R}(G;F)$ is injective. This finishes the proof. $\square$

Theorem 2.14. *Let $H \lhd G$ be groups with $H \in \mathscr{X}$, H torsionfree and G/H finite. Let $\Sigma = \Sigma(FG \subseteq \mathscr{U}(G))$. Assume that $\mathscr{D}(H;F)$ is a skew field.*

Then $\mathscr{D}(G;F) = \mathscr{R}(G;F)$, and H has the property (UL) *with respect to F, i.e., the localization map $FG_\Sigma \to \mathscr{R}(G;F)$ is an isomorphism.*

Proof. We first consider the special case $G = H$ (so G is torsionfree). We use the description of the class of groups $\mathscr{X}$ given in Lemma 2.5(2) and prove the result by transfinite induction. The result is obvious if $G \in \mathscr{X}_0$, because then $G = 1$. The induction step is done as follows. Consider an ordinal b with $b \neq 0$ and a group $G \in \mathscr{X}_b$ such that the claim is already known for all groups $H \in \mathscr{X}_a$ for all ordinals $a < b$. We have to show the claim for G. If b is a limit ordinal, this is obvious since G belongs to $\mathscr{X}_a$ for every ordinal $a < b$. It remains to treat the case where b is not a limit ordinal. Then $G \in \mathrm{L}(\mathscr{X}_a\mathscr{A})$ for some ordinal $a < b$. By Lemma 2.6, it is sufficient to consider the case $G \in \mathscr{X}_a\mathscr{A}$. Now apply Lemma 2.13.

The general case, when G is not necessarily equal to H, now follows from Lemma 2.9. $\square$

There are many groups for which Theorem 2.14 can be applied, some of which we now describe. Let N be either an Artin pure braid group, or a RAAG, or a subgroup of finite index in a right-angled Coxeter group. Let $\overline{\mathbb{Q}}$ denote the field of all algebraic numbers. We can now state:

Theorem 2.15. *Let G be a group which contains N as a normal subgroup such that G/N is elementary amenable, and let $\Sigma = \Sigma(FG \subseteq \mathcal{U}(G))$. Assume that G contains a torsionfree subgroup of finite index and that F is a subfield of $\overline{\mathbb{Q}}$. Then the localization map $FG_\Sigma \to \mathcal{R}(G; F)$ is an isomorphism, i.e., G has property* (UL) *with respect to F.*

Proof. First we recall some group-theoretic results. An Artin pure braid group is polyfree (see [Rolfsen 2010, §2.4], for example) and RAAGs are polyfree by [Hermiller and Šunić 2007, Theorem A]. Finally right-angled Coxeter groups have a characteristic subgroup of index a power of 2 which is isomorphic to a subgroup of a right-angled Artin group [Linnell et al. 2012, Proposition 5(2)] and therefore this subgroup is polyfree. This shows that in all cases $G \in \mathcal{X}$ and hence any subgroup of G is in $\mathcal{X}$, because $\mathcal{X}$ is subgroup closed by Lemma 2.5 (3).

Now let H be a torsionfree normal subgroup of finite index in G. We need to show that H satisfies the Atiyah conjecture with coefficients in F. We may assume that $F = \overline{\mathbb{Q}}$. For the case N is an Artin pure braid group, this follows from [Linnell and Schick 2007, Corollary 5.41]. For the case N is a RAAG, this follows from [Linnell et al. 2012, Theorem 2]. Finally if N is a subgroup of finite index in a right-angled Coxeter group, this follows from [Linnell et al. 2012, Theorem 2 and Proposition 5(2)] and [Schreve 2014, Theorem 1.1]. □

2D. *The property (ULA).*

Definition 2.16 (property (ULA)). We say that a torsionfree group G has the property (ULA) with respect to the subfield $F \subseteq \mathbb{C}$ if the canonical epimorphism

$$\lambda : R_{\Sigma(FG \subseteq \mathcal{R}(G;F))} \to \mathcal{R}(G; F)$$

is bijective and $\mathcal{D}(G; F)$ is a skew field.

Given a torsionfree group G, recall from Theorem 2.3(3) that $\mathcal{D}(G; F)$ is a skew field if and only G satisfies the Atiyah conjecture with coefficients in F and that we have $\mathcal{D}(G; F) = \mathcal{R}(G; F)$ provided that $\mathcal{D}(G; F)$ is a skew field. So G satisfies condition (ULA) with respect to F if and only if G satisfies both condition (UL) with respect to F and the Atiyah conjecture with coefficients in F.

Theorem 2.17 (groups in $\mathcal{C}$ have property (ULA)). *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Suppose that G is a torsionfree group which belongs to $\mathcal{C}$.*

Then G has property (ULA).

Proof. This follows from Theorem 2.3(3)–(4) and Theorem 2.14 since obviously $\mathcal{C} \subseteq \mathcal{X}$. □

3. Proof of the main theorem

Next we explain why we are interested in group with properties (ULA) by proving our main theorem, Theorem 0.1, which will be a direct consequence of Theorems 2.17 and 3.5.

Definition 3.1. Let G be a group, let R be a ring with $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$, and denote by $F \subseteq \mathbb{C}$ its quotient field. Let

$$K_1^w(RG)$$

be the abelian group defined in terms of generators and relations as follows: Generators $[f]$ are given by (conjugacy classes of) RG-endomorphisms $f : P \to P$ of finitely generated projective RG-modules P such that $\omega_* f : \omega_* P \to \omega_* P$ is a $\mathcal{D}(G; F)$-isomorphism for the inclusion $\omega : RG \to \mathcal{D}(G; F)$. If $f, g : P \to P$ are RG-endomorphisms of the same finitely generated projective RG-module P such that $\omega_* f$ and $\omega_* g$ are bijective, then we require the relation

$$[g \circ f] = [g] + [f].$$

If we have a commutative diagram of finitely generated projective RG-modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & P_0 \arrow[r, "i"] \arrow[d, "f_0"'] & P_1 \arrow[r, "p"] \arrow[d, "f_1"'] & P_2 \arrow[r] \arrow[d, "f_2"'] & 0 \\
0 \arrow[r] & P_0 \arrow[r, "i"] & P_1 \arrow[r, "p"] & P_2 \arrow[r] & 0
\end{tikzcd}$$

such that $\omega_* f_0$, $\omega_* f_1$ and $\omega_* f_2$ are bijective, then we require the relation

$$[f_1] = [f_0] + [f_2].$$

Furthermore, define

$$\begin{aligned}
\widetilde{K}_1^w(RG) &:= \operatorname{coker}\bigl(\{\pm 1\} \xrightarrow{\cong} K_1(\mathbb{Z}) \to K_1(\mathbb{Z}G) \to K_1^w(RG)\bigr);\\
\mathrm{Wh}^w(G; R) &= \operatorname{coker}\bigl(\{\pm g \mid g \in G\} \to K_1(\mathbb{Z}G) \to K_1^w(RG)\bigr);\\
\mathrm{Wh}^w(G) &= \mathrm{Wh}^w(G; \mathbb{Z});\\
\widetilde{K}_1(\mathcal{R}(G; F)) &:= \operatorname{coker}\bigl(\{\pm 1\} \xrightarrow{\cong} K_1(\mathbb{Z}) \to K_1(\mathbb{Z}G) \to K_1(\mathcal{R}(G; F))\bigr);\\
\mathrm{Wh}(\mathcal{R}(G; F)) &= \operatorname{coker}\bigl(\{\pm g \mid g \in G\} \to K_1(\mathbb{Z}G) \to K_1(\mathcal{R}(G; F))\bigr).
\end{aligned}$$

Remark 3.2. Let A be a square matrix over RG. Then the square matrix $\omega(A)$ over $\mathcal{D}(G; F)$ is invertible if and only if the operator $r_A^{(2)} : L^2(G)^n \to L^2(G)^n$ given by right multiplication with A is a weak isomorphism, i.e., it is injective and has dense image. This follows from the conclusion of [Lück 2002, Theorem 6.24 on page 249 and Theorem 8.22(5) on page 327] that $r_A^{(2)}$ is a weak isomorphisms if and only if it becomes invertible in $\mathcal{U}(G)$.

There is a Dieudonné determinant for invertible matrices over a skew field D which takes values in the abelianization of the group of units $D^\times/[D^\times, D^\times]$ and induces an isomorphism — see [Silvester 1981, Corollary 43 on page 133] —

$$\det_D : K_1(D) \xrightarrow{\cong} D^\times/[D^\times, D^\times]. \tag{3.3}$$

The inverse

$$J_D : D^\times/[D^\times, D^\times] \xrightarrow{\cong} K_1(D) \tag{3.4}$$

sends the class of a unit in D to the class of the corresponding $(1, 1)$-matrix.

Theorem 3.5 ($K_1^w(FG)$ for groups with property (ULA) with respect to F). *Let R be a ring with $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$. Denote by $F \subseteq \mathbb{C}$ the quotient field of R. Let G be a torsionfree group with the property* (ULA) *with respect to F.*

Then the canonical maps sending $[f]$ to $[\omega_* f]$,

$$\omega_* : K_1^w(RG) \xrightarrow{\cong} K_1(\mathcal{D}(G; F)),$$
$$\omega_* : \widetilde{K}_1^w(RG) \xrightarrow{\cong} \widetilde{K}_1(\mathcal{D}(G; F)),$$
$$\omega_* : \mathrm{Wh}^w(G; R) \xrightarrow{\cong} \mathrm{Wh}(\mathcal{D}(G; F)),$$

are bijective. Moreover, $\mathcal{D}(G; F)$ is a skew field and the Dieudonné determinant induces an isomorphism

$$\det_D : K_1(\mathcal{D}(G; F)) \xrightarrow{\cong} \mathcal{D}(G; F)^\times/[\mathcal{D}(G; F)^\times, \mathcal{D}(G; F)^\times].$$

Proof. This follows directly from Theorem 1.2. □

Finally we can give the proof of Theorem 0.1.

Proof of Theorem 0.1. Because of Theorem 2.17 the group G has property (ULA) and we can apply Theorem 1.2. It remains to explain why in the special case $R = \mathbb{Z}$ the group $K_1^w(\mathbb{Z}G)$ appearing in Theorem 1.2, namely as introduced in Definition 3.1, agrees with the group $K_1^w(\mathbb{Z}G)$ appearing in the introduction. This boils down to explaining why, for a (n, n)-matrix A over $\mathbb{Z}G$, the operator $r_A^{(2)} : L^2(G)^n \to L^2(G)^n$ is a weak isomorphism if and only if A becomes invertible in $\mathcal{D}(G; \mathbb{Q})$. By definition A is invertible in $\mathcal{D}(G; \mathbb{Q})$ if and only if it is invertible in $\mathcal{U}(G)$. Now apply [Lück 2002, Theorem 6.24 on page 249 and Theorem 8.22(5) on page 327]. □

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WOLFGANG LÜCK: wolfgang.lueck@him.uni-bonn.de
Mathematisches Institut, Rheinische Wilhelms-Universität Bonn, Endenicher Allee 60, D-53115 Bonn, Germany
http://www.him.uni-bonn.de/lueck

PETER LINNELL: plinnell@math.vt.edu
Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, United States
http://www.math.vt.edu/people/plinnell/

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Suslin's moving lemma with modulus

Wataru Kai and Hiroyasu Miyazaki

The moving lemma of Suslin (also known as the generic equidimensionality theorem) states that a cycle on $X \times \mathbb{A}^n$ meeting all faces properly can be moved so that it becomes equidimensional over $\mathbb{A}^n$. This leads to an isomorphism of motivic Borel–Moore homology and higher Chow groups.

In this short paper we formulate and prove a variant of this. It leads to a modulus version of the isomorphism, in an appropriate pro setting.

1. Introduction

Suslin [2000] proved that, roughly speaking, a cycle on $X \times \mathbb{A}^n$ meeting all faces properly can be moved so that it becomes equidimensional over $\mathbb{A}^n$. Here X is an affine variety over a base field k. As a consequence he shows that the inclusion

$$z_r^{equi}(X, \bullet) \hookrightarrow z_r(X, \bullet) \tag{1.1}$$

of the cycle complex of equidimensional cycles into Bloch's cycle complex is a quasi-isomorphism for $r \geq 0$. This result is significant in incorporating Bloch's higher Chow groups into the Voevodsky–Suslin–Friedlander theory of mixed motives. Namely, for smooth schemes X over a field, we have an inclusion of complexes

$$C_*\big(z_{equi}(\mathbb{A}^i, 0)(X)\big) \hookrightarrow z^i(X, \bullet).$$

The left side is a sheaf of complexes defining the Voevodsky–Suslin–Friedlander motivic cohomology (at least over perfect fields, see [Mazza et al. 2006, Theorem 16.7]). The inclusion is a quasi-isomorphism by *Suslin's moving lemma* when X is the spectrum of a field. *Voevodsky's injectivity theorem* [Mazza et al. 2006, Corollary 11.2] for homotopy invariant sheaves with transfers then implies that the inclusion is a quasi-isomorphism locally on an arbitrary smooth X.

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Recently the context has been extended to *cycles with modulus*. Binda and Saito [2014] introduced the cycle complex with modulus $z_r(\bar{X}|Y, \bullet)$ for $r \geq 0$ and a pair $(\bar{X}, Y)$ of a finite-type k-scheme $\bar{X}$ and an effective Cartier divisor Y on it. We usually write $X := \bar{X} \setminus Y$. This generalizes Bloch's cycle complex in the sense that $z_r(\bar{X}|\varnothing, \bullet) = z_r(\bar{X}, \bullet)$. The homology group $\mathrm{CH}_r(\bar{X}|Y, n) := \mathrm{H}_n(z_r(\bar{X}|Y, \bullet))$ is called the higher Chow group with modulus. Moreover, we can construct a generalization of the inclusion (1.1):

$$z_r^{equi}(\bar{X}|Y, \bullet) \hookrightarrow z_r(\bar{X}|Y, \bullet).$$

The reader will find all the definitions of these objects in Section 2.

Our future aim is to extend the comparison between the higher Chow group and motivic cohomology group to the modulus setting. For this, we need to generalize (i) Suslin's moving lemma, and (ii) Voevodsky's injectivity theorem. The generalization of (ii) is expected to be done by the developing theory of *motives with modulus*, which was introduced by Kahn, Saito and Yamazaki [2015] as a generalization of Voevodsky's theory of motives.

In this paper, we generalize (i). In other words, we prove a variant of Suslin's moving lemma which takes the modulus condition into account (Theorem 3.11 below). Suslin's moving method does not preserve the so-called modulus condition on cycles, but instead we can show that the moved cycle satisfies the modulus condition to a lesser extent, and we have explicit control of the loss. It leads to the following:

Theorem 1.2 (Theorem 4.1). *Suppose $\bar{X}$ is affine and X is an open set of $\bar{X}$ such that $\bar{X} \setminus X$ is the support of an effective Cartier divisor Y. Let $r \geq 0$. Then the inclusions for $m \geq 0$,*

$$z_r^{equi}(\bar{X}|mY, \bullet) \hookrightarrow z_r(\bar{X}|mY, \bullet),$$

induce an isomorphism of inverse limits of their homology groups:

$$\varprojlim_m \mathrm{H}_n(z_r^{equi}(\bar{X}|mY, \bullet)) \cong \varprojlim_m \mathrm{CH}_r(\bar{X}|mY, n).$$

Note that it is quite natural and might be even necessary that inverse limits appear in the isomorphism. Indeed, we have several comparison isomorphisms in the theory of modulus which hold after taking inverse limits. A typical example is [Kerz and Saito 2016, Theorem III] which describes $\pi_1^{\mathrm{ab}}(X)^\circ$ as the inverse limit $\varprojlim_Y \mathrm{CH}_0(\bar{X}|Y)^\circ$, where $\bar{X}$ is a proper normal compactification of a smooth variety X over a finite field and the limit runs over effective Cartier divisors Y such that $\bar{X} \setminus Y = X$, and the superscript $(-)^\circ$ means the degree zero part. This is a higher dimensional analogue of the class field theory. Another example is [Rülling and Saito 2016, Theorem 2], a comparison isomorphism between the inverse limits of the Chow group with modulus and the relative motivic cohomology group of certain degree. This would be the first part of an isomorphism we aim to prove in the future. Moreover, Krishna and Park [2015, Theorem 1.0.7] prove a description of the crystalline cohomology group in terms of additive higher Chow groups, hypercohomology and inverse limits. Here, the additive higher Chow group is a special case of the higher Chow group with modulus, which can be obtained by taking a special pair of the form $(X \times \mathbb{A}^1, m(X \times \{0\}))$, $m \geq 1$ in our setting. Also, see Morrow's article [2016, §4] — the relative cohomology groups we consider in Section 4C echo his proposal for the definition of compact support K-groups.

We remark that the isomorphism in Theorem 1.2 actually comes from an isomorphism of pro-abelian groups. We can also give an explicit "pro bound" to annihilate the levelwise kernel and cokernel of the map (see Remark 4.2 (1)).

2. Definitions

We set $\square^n := (\mathbb{P}^1 \setminus \{\infty\})^n = \operatorname{Spec}(k[y_1, \dots, y_n])$ in this paper, unlike some authors who prefer 1 as the point at infinity. With this convention our computations look simpler. We set a divisor on $(\mathbb{P}^1)^n$:

$$F_n = \sum_{i=1}^{n} (\mathbb{P}^1)^{i-1} \times \{\infty\} \times (\mathbb{P}^1)^{n-i}.$$

The *faces* of $\square^n$ are $\{y_i = 0\}$, $\{y_i = 1\}$ and their intersections.

Definition 2.1 [Binda and Saito 2014; Kahn et al. 2015]. (1) Let $\underline{z}_r(\bar{X}|Y, n)$ be the group of $(r+n)$-dimensional cycles on $X \times \square^n$ whose components V meet all faces of $\square^n$ properly, and *have modulus* Y, i.e.:

> Let $\bar{V}^N$ be the normalization of $\bar{V} \subset \bar{X} \times (\mathbb{P}^1)^n$, the closure of V. Let $\varphi_V : \bar{V}^N \to \bar{X} \times (\mathbb{P}^1)^n$ be the natural map. Then the inequality of Cartier divisors
> $$\varphi_V^{-1}(Y \times (\mathbb{P}^1)^n) \leq \varphi_V^{-1}(\bar{X} \times F_n)$$
> holds. (When $n = 0$ the condition reads: the closure $\bar{V} \subset \bar{X}$ of V is contained in X i.e., $V = \bar{V}$.)

Let $\partial_{i,\epsilon} : \square^{n-1} \hookrightarrow \square^n$, where $i \in \{1, \dots, n\}$ and $\epsilon \in \{0, 1\}$, be the embedding of the face $\{y_i = \epsilon\}$:

$$\partial_{i,\epsilon} : (y_1, \dots, y_{n-1}) \mapsto (y_1, \dots, \overset{i}{\check{\epsilon}}, y_i, \dots, y_{n-1}).$$

The groups $\underline{z}_r(\bar{X}|Y, n)$ form a complex with the differentials

$$\sum_{i=1}^{n} (-1)^i (\partial_{i,1}^* - \partial_{i,0}^*) : \underline{z}_r(\bar{X}|Y, n) \to \underline{z}_r(\bar{X}|Y, n-1).$$

(2) Let $\underline{z}_r^{equi}(\overline{X}|Y,n)$ be the subgroup of $\underline{z}_r(\overline{X}|Y,n)$ consisting of cycles that are equidimensional over $\square^n$, necessarily of relative dimension r. They define a subcomplex $\underline{z}_r^{equi}(\overline{X}|Y,\bullet)$ of $\underline{z}_r(\overline{X}|Y,\bullet)$.

Remark 2.2. The condition that V has modulus Y makes sense for any closed subset V of $X \times \square^n$. In that setting, normalization of a closed subset means the disjoint union of the normalizations of its reduced irreducible components.

Definition 2.3. We define the *degenerate part* $\underline{z}_r(\overline{X}|Y,n)_{\mathrm{degn}} \subset \underline{z}_r(\overline{X}|Y,n)$ as the subgroup generated by the cycles of the form

$$(\mathrm{id}_{\overline{X}} \times \mathrm{pr}_i)^*(V), \quad \text{where } V \in \underline{z}_r(\overline{X}|Y,n-1)$$

and

$$\mathrm{pr}_i : \square^n \to \square^{n-1}, \quad (y_1,\dots,y_n) \mapsto (y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)$$

for some $i = 1,\dots,n$. We also define the *degenerate part* $\underline{z}_r^{equi}(\overline{X}|Y,n)_{\mathrm{degn}} \subset \underline{z}_r^{equi}(\overline{X}|Y,n)$ in a similar way. We set

$$z_r(\overline{X}|Y,n) := \underline{z}_r(\overline{X}|Y,n)/\underline{z}_r(\overline{X}|Y,n)_{\mathrm{degn}},$$
$$z_r^{equi}(\overline{X}|Y,n) := \underline{z}_r^{equi}(\overline{X}|Y,n)/\underline{z}_r^{equi}(\overline{X}|Y,n)_{\mathrm{degn}}.$$

Noting that the differentials $\partial_{i,\epsilon}$ preserve degenerate parts, we can see that $z_r(\overline{X}|Y,n)$ and $z_r^{equi}(\overline{X}|Y,n)$ also form complexes. We define the *higher Chow group with modulus* by

$$\mathrm{CH}_r(\overline{X}|Y,n) := \mathrm{H}_n(z_r(\overline{X}|Y,\bullet)).$$

We also consider the homology groups of the latter:

$$\mathrm{H}_n(z_r^{equi}(\overline{X}|Y,\bullet)).$$

Voevodsky–Suslin–Friedlander give no particular name to its counterpart without modulus. In this paper, we would like to call it the *Suslin homology group with compact support with modulus*. The term “with compact support” reflects the fact that we are using z^{equi} instead of c^{equi}, where the latter is used to define the usual Suslin homology.

Remark 2.4. In this remark, we explain that we can use another complex to define the higher Chow group with modulus. This is a general fact on cubical objects (see, for example, [Levine 2009, §1.2]). The subgroups

$$\underline{z}_r(\overline{X}|Y,n)_0 := \bigcap_{i=1}^{n} \ker(\partial_{i,0}^*) \subset \underline{z}_r(\overline{X}|Y,n)$$

form a subcomplex. One checks that the composite

$$\underline{z}_r(\overline{X}|Y,\bullet)_0 \to \underline{z}_r(\overline{X}|Y,\bullet) \to z_r(\overline{X}|Y,\bullet)$$

is an isomorphism, where the first map is the natural inclusion and the latter is the quotient map. This implies that we have a direct sum decomposition

$$\underline{z}_r(\bar{X}|Y,\bullet) = z_r(\bar{X}|Y,\bullet) \oplus \underline{z}_r(\bar{X}|Y,\bullet)_{\mathrm{degn}}$$

of a complex, and that $\mathrm{CH}_r(\bar{X}|Y,n) \cong \mathrm{H}_n(\underline{z}_r(\bar{X}|Y,\bullet)_0)$. We have a similar decomposition of $\underline{z}_r^{equi}(\bar{X}|Y,\bullet)$, and the inclusion $\underline{z}_r^{equi}(\bar{X}|Y,\bullet) \hookrightarrow \underline{z}_r(\bar{X}|Y,\bullet)$ is compatible with the decompositions.

3. Equidimensionality theorem

Let k be an infinite base field. We will formulate and prove a variant of Suslin's equidimensionality Theorem 3.11 for modulus pairs $(\bar{X}, Y)$, i.e., a k-scheme $\bar{X}$ of finite type equipped with an effective Cartier divisor Y, for which $\bar{X}$ is affine.

Recall a *face* of $\square^n = \mathrm{Spec}(k[y_1, \dots, y_n])$ is a closed subscheme of the form $\{y_i = 0\}, \{y_i = 1\}$ or an intersection of them. Define a Cartier divisor $\partial\square^n = \sum \partial_{i,\epsilon}(\square^{n-1})$, where the sum is over all $1 \le i \le n$ and $\epsilon = 0, 1$. Recall the map $\partial_{i,\epsilon} : \square^{n-1} \hookrightarrow \square^n$ denotes the embedding corresponding to the equation $y_i = \epsilon$ for each i, ϵ. The divisor $\partial\square^n$ is defined by the equation

$$h(\underline{y}) = y_1(1-y_1)\cdots y_n(1-y_n). \tag{3.1}$$

We need the following version of Suslin's moving lemma where we control the degrees of the map Φ^n.

Theorem 3.2. *Let $\bar{X} = \mathrm{Spec}(R)$ be an affine k-scheme of finite type and $V \subset \bar{X} \times \square^n$ be a closed subset of dimension $n+t$ for some $t \ge 0$. Suppose an $\bar{X}$-morphism*

$$\Phi' : \bar{X} \times \partial\square^n \to \bar{X} \times \square^n$$

is given and there is an integer $d \ge 2$ such that for any codimension 1 face

$$\partial_{l,\epsilon} : \square^{n-1} \hookrightarrow \square^n,$$

the composite $\Phi' \circ (\mathrm{id}_{\bar{X}} \times \partial_{l,\epsilon})$ is defined by polynomials $\Phi'_{i,l,\epsilon} \in R[y_1, \dots, y_{n-1}]$ $(1 \le i \le n)$ whose degrees with respect to y_j are at most d for each j.

Then we can find an $\bar{X}$-map

$$\Phi^n : \bar{X} \times \square^n \to \bar{X} \times \square^n$$

extending Φ' such that $(\Phi^n)^{-1}(V) \subset \bar{X} \times \square^n$ has fibers of dimension $\le t$ over $\square^n \setminus \partial\square^n$, and moreover, the functions $\Phi^n_i \in R[y_1, \dots, y_n]$ defining Φ^n $(1 \le i \le n)$ have degrees $\le d$ with respect to each y_j.

Proof. The map Φ' is determined by R-coefficient polynomials $f_i(y_1, \dots, y_n)$ mod $h(\underline{y})$ $(1 \le i \le n)$. If we substitute $y_j = 0$ or $y_j = 1$ to f_i we get a polynomial which has degree $\le d$ with respect to each y_k by the hypothesis.

Lemma 3.3. *Let $d \geq 1$ be an integer. Suppose given a polynomial $f(y_1, \dots, y_n) \in R[y_1, \dots, y_n]$ such that for each j, if we substitute any of $y_j = 0$ or $y_j = 1$, the resulting polynomial has degree $\leq d$ with respect to each y_k. Then $f \bmod h(\underline{y})$ has a (unique) representative which has degree $\leq d$ with respect to each y_j (where we keep the notation $h(\underline{y}) = y_1(1-y_1) \cdots y_n(1-y_n)$ introduced in (3.1)).*

Proof. For each i denote by $y_i(-|_{y_i=1})$ the operator which sends a polynomial f to $y_i \cdot (f|_{y_i=1})$ and define $(1-y_i)(-|_{y_i=0})$ similarly. Note that for different i and j the operators $y_i(-|_{y_i=1})$ and $y_j(-|_{y_j=1})$ commute (and similarly for other pairs). Put $\alpha_i := 1 - y_i(-|_{y_i=1}) - (1-y_i)(-|_{y_i=0})$. Then one can see the polynomial

$$f - (\alpha_1 \cdots \alpha_n f)$$

is the desired representative. □

By the previous lemma, we take representatives $f_i(\underline{y})$ having degrees $\leq d$ with respect to each y_j.

Take a finite set $\{x_1, \dots, x_m\}$ of generators of the k-algebra R. We are going to define the asserted map Φ^n by setting its components ($1 \leq i \leq n$) to be

$$\Phi_i^n(\underline{y}) := f_i(\underline{y}) + h(\underline{y}) F_i(\underline{x}),$$

where $F_i(t_1, \dots, t_m) \in k[t_1, \dots, t_m]$ are homogeneous polynomials in variables $t_1, \dots, t_m$ of some uniform degree N. From this form, the functions Φ_i^n have degrees $\leq d$ with respect to each y_j.

Now, in his proof of the generic equidimensionality theorem, Suslin [2000, Theorem 1.1] actually introduces the following specific statement in the first two paragraphs and proves it in [loc. cit., §§(1.2)–(1.8)].

Specific statement 3.4 [Suslin 2000, proof of Theorem 1.1]. *Let R be a k-algebra of finite type and let $x_1, \dots, x_m \in R$ be a finite set of generators over k. Let $H(\underline{y}) \in k[y_1, \dots, y_n]$ and $f_i(\underline{y}) \in R[y_1, \dots, y_n]$, $1 \leq i \leq n$, be polynomials in variables $y_1, \dots, y_n$. Let V be a closed subset in $\mathbb{A}^n_R = \operatorname{Spec}(R[y_1, \dots, y_n])$ of dimension $\leq n + t$ for some nonnegative integer t.*

Consider R-morphisms $\Phi : \mathbb{A}^n_R \to \mathbb{A}^n_R$ defined by polynomials of the form

$$\Phi_i(\underline{y}) = f_i(\underline{y}) + H(\underline{y}) F_i(\underline{x}), \quad 1 \leq i \leq n,$$

where $F_i(\underline{t}) \in k[t_1, \dots, t_m]$ are homogeneous polynomials in variables $t_1, \dots, t_m$ of some uniform degree N.

Then if N is large enough, for almost all tuples $(F_i)_{i=1}^n$, the fibers of the projection $\Phi^{-1}(V) \subset \mathbb{A}^n_R \to \mathbb{A}^n_k$ have dimensions $\leq t$ over $\mathbb{A}^n_k \setminus \{H(\underline{y}) = 0\}$.

(For a fixed N, the tuples of polynomials $(F_i)_i$ are parametrized by the rational points of an affine space of dimension $\binom{N+m-1}{m-1} n$. The statement means that the set of tuples $(F_i)_i$ where the stated condition fails is contained in a proper closed subset of the affine space.)

Thus if N is large enough, a general choice of $(F_i)_{i=1}^n$ makes our assertion on fiber dimension true. This completes the proof of Theorem 3.2. $\square$

Now, to understand the Suslin moving lemma in the context of modulus, first recall the following:

Lemma 3.5 (containment lemma [Krishna and Park 2012, Proposition 2.4]). *Let $V \subset \bar{X} \times \square^n$ be a closed subset which has modulus Y and $V' \subset V$ be a smaller closed subset. Then V' also has modulus Y.*

Proposition 3.6. *Let $(\bar{X}, Y)$ be a modulus pair with $\bar{X} = \mathrm{Spec}(R)$ affine. Let d be a positive integer and $V \subset \bar{X} \times \square^n$ be a closed subset having modulus $nd \cdot Y$. Suppose*

$$\Phi : \bar{X} \times \square^{n'} \to \bar{X} \times \square^n$$

is an $\bar{X}$-morphism defined by polynomials $\Phi_j \in R[y_1, \ldots, y_{n'}]$ $(1 \le j \le n)$ having degrees $\le d$ with respect to each y_i. Then the closed subset $\Phi^{-1}(V)$ of $\bar{X} \times \square^{n'}$ has modulus Y.

Proof. Since the assertion is local on $\bar{X}$, we may assume Y is principal and defined by $u \in R$. Let V' denote any one of the irreducible components of $\Phi^{-1}(V)$ and let $\overline{V'}^N$ be the normalization of its closure $\overline{V'}$ in $\bar{X} \times (\mathbb{P}^1)^{n'}$;

$$\begin{tikzcd}
\overline{V'}^N \arrow[d] & & & \\
\overline{V'} & V' \arrow[l, hook'] \arrow[r, hook] & \Phi^{-1}(V) \arrow[r, hook] \arrow[d] & \bar{X} \times \square^{n'} \arrow[d, "\Phi"] \\
& & V \arrow[r, hook] & \bar{X} \times \square^n
\end{tikzcd}$$

Thanks to the containment lemma (Lemma 3.5), the closure of $\Phi(V')$ in V has modulus ndY. By replacing V by the closure of $\Phi(V')$ in V, we may assume the map $V' \to V$ is dominant.

Claim 3.7. Let $\overline{V'}^{N\circ}$ be the domain of definition of the rational map

$$\overline{V'}^N \to \bar{X} \times (\mathbb{P}^1)^{n'} \overset{\Phi}{\dashrightarrow} \bar{X} \times (\mathbb{P}^1)^n.$$

Then the complement of $\overline{V'}^{N\circ}$ in $\overline{V'}^N$ has codimension ≥ 2.

Proof. Let v be a point of $\overline{V'}^N$ of codimension 1. Since the generic point η of $\overline{V'}^N$ lands in $\bar{X} \times \square^{n'}$ we have a commutative diagram

$$\begin{tikzcd} \eta \arrow[r] \arrow[d, hook] & \bar{X} \times \square^{n'} \arrow[r, "\Phi"] & \bar{X} \times (\mathbb{P}^1)^n \arrow[d] \\ \mathrm{Spec}\mathcal{O}_v \arrow[rr] & & \bar{X} \end{tikzcd}$$

The assertion follows from the valuative criterion of properness applied to the projective morphism $\bar{X} \times (\mathbb{P}^1)^n \to \bar{X}$. $\square$

By Claim 3.7, we find that a Cartier divisor on $\overline{V'}^N$ is effective if and only if its restriction to $\overline{V'}^{N\circ}$ is effective, since $\overline{V'}^N$ is normal.

Write $\mathrm{pr}_j : \bar{X} \times (\mathbb{P}^1)^n \to \mathbb{P}^1$ for the projection to the j-th $\mathbb{P}^1$ and Φ_j for the composite rational map

$$\bar{X} \times (\mathbb{P}^1)^{n'} \overset{\Phi}{\dashrightarrow} \bar{X} \times (\mathbb{P}^1)^n \xrightarrow{\mathrm{pr}_j} \mathbb{P}^1,$$

also seen as a rational function on $\bar{X} \times (\mathbb{P}^1)^{n'}$. We will denote the pull-backs of Φ and Φ_j to $\overline{V'}^{N\circ}$ by Φ^V and Φ^V_j. By definition of $\overline{V'}^{N\circ}$ they are well-defined morphisms from $\overline{V'}^{N\circ}$ to $\bar{X} \times (\mathbb{P}^1)^n$ and to $\mathbb{P}^1$ respectively. There is a uniquely induced morphism $\overline{V'}^{N\circ} \to \overline{V}^N$ because now we are assuming $V' \to V$ is dominant.

For any given point of $\overline{V'}^{N\circ}$, we can find an affine open set $\mathrm{Spec}(A) \subset \overline{V}^N$ and an affine neighborhood $\mathrm{Spec}(B) \subset \overline{V'}^{N\circ}$ of the point over which Φ^V restricts to a morphism $\Phi^V : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

$$\begin{tikzcd} \mathrm{Spec}(B) \arrow[r, hook] & \overline{V'}^{N\circ} \arrow[r] \arrow[d, "\Phi^V"] & \bar{X} \times (\mathbb{P}^1)^{n'} \arrow[d, dashed, "\Phi"] \\ \mathrm{Spec}(A) \arrow[r, hook] & \overline{V}^N \arrow[r] & \bar{X} \times (\mathbb{P}^1)^n \end{tikzcd}$$

By shrinking $\mathrm{Spec}(A)$ if necessary, we may assume y_j or $1/y_j$ is regular on $\mathrm{Spec}(A)$ for each j. Denote by $J \subset \{1, \dots, n\}$ the set of j's for which $1/y_j$ is regular. The divisor F_n is defined by the equation $1/\prod_{j\in J} y_j = 0$ on $\mathrm{Spec}(A)$. Since V has modulus ndY, the rational function $\big(1/\prod_{j\in J} y_j\big)/u^{nd}$ on $\mathrm{Spec}(A)$ is regular. Pulling it back by Φ^V, we find that the rational function

$$\frac{1}{\prod_{j\in J} \Phi^V_j} \Big/ u^{nd} \tag{3.8}$$

on $\mathrm{Spec}(B)$ is regular.

Shrinking $\mathrm{Spec}(B)$ if necessary, we may assume y_i or $1/y_i$ is regular on $\mathrm{Spec}(B)$ for each i. Let $I \subset \{1, \dots, n'\}$ be the set of i's for which $1/y_i$ is regular on $\mathrm{Spec}(B)$; the divisor $F_{n'}$ is defined by $1/\prod_{i\in I} y_i = 0$ on $\mathrm{Spec}(B)$.

Claim 3.9. The rational function $\Phi_j^V/\prod_{i\in I} y_i^d$ on $\mathrm{Spec}(B)$ is regular for each $j \in \{1,\dots,n\}$, i.e., it is a morphism from $\mathrm{Spec}(B)$ into $\mathbb{A}^1 \subset \mathbb{P}^1$.

Proof. The function is the restriction of the meromorphic function $\Phi_j/\prod_{i\in I} y_i^d$ on $\bar{X}\times(\mathbb{P}^1)^{n'}$. It is written as an R-coefficient polynomial in the variables $1/y_i$ ($i \in I$) and y_i ($i \in I^c$) by the assumption on Φ. So it is regular around the (image of the) considered point on $\bar{X}\times(\mathbb{P}^1)^{n'}$. □

By the regularity of the function (3.8) and Claim 3.9, the function

$$\left(\frac{1}{\prod_{j\in J}\Phi_j^V}\Big/ u^{nd}\right)\cdot\prod_{j\in J}\frac{\Phi_j^V}{\prod_{i\in I} y_i^d} = \frac{1}{\prod_{i\in I} y_i^{d\cdot\#J}}\Big/ u^{nd}$$

is regular on $\mathrm{Spec}(B)$. This shows a relation of Cartier divisors on $\mathrm{Spec}(B)$:

$$nd\left(\prod_{i\in I}\frac{1}{y_i}\right) - nd(u) \geq 0,$$

which implies the relation

$$(\text{pullback of } F_{n'}) - (\text{pullback of } Y) \geq 0$$

on $\mathrm{Spec}(B)$, hence on $\overline{V'}^{N\circ}$, which is valid on $\overline{V'}^{N}$ as well by Claim 3.7. This completes the proof of Proposition 3.6. □

Remark 3.10. Under the hypotheses of Proposition 3.6, we can prove that the morphism Φ is admissible [Kahn et al. 2015, Definition 1.1] for the pair

$$((\mathbb{P}^1_R)^{n'}, ndF_{n'}), \quad ((\mathbb{P}^1_R)^n, F_n).$$

Here, for pairs (X, D), (Y, E) of schemes and effective Cartier divisors, a morphism $f : X \setminus D \to Y \setminus E$ is said to be *admissible* if the following holds: Let $\overline{\Gamma}_f$ be the closure of the graph of f in $X \times Y$ and $\overline{\Gamma}_f^N$ be its normalization. Let $\varphi : \overline{\Gamma}_f^N \to X \times Y$ be the natural map. Then the inequality of Cartier divisors $\varphi^{-1}(D\times Y) \geq \varphi^{-1}(X \times E)$ on $\overline{\Gamma}_f^N$ holds.

It gives an alternative proof of Proposition 3.6 thanks to [Krishna and Park 2012, Lemma 2.2]. Here we sketch the proof of the admissibility. We use the fact that admissibility can be checked after replacing the source by an open cover (for a trivial reason), and after blowing up $(\mathbb{P}^1)^{n'}$ by a closed subset outside $\square^{n'}$ (by [Krishna and Park 2012, Lemma 2.2] again). Set $\eta_i = 1/y_i$. The scheme $(\mathbb{P}^1)^{n'}$ is covered by open subsets $U_I = \mathrm{Spec}(R[\eta_i, y_{i'}\ {}_{i\in I, i'\notin I}])$, where I runs though the subsets of $\{1,\dots,n'\}$. On the region U_I, the rational function $\Phi_j^{(I)}$ defined by the next equation is regular, by the assumption on Φ_j:

$$\Phi_j = \frac{\Phi_j^{(I)}(\eta_i, y_{i'})}{\prod_{i\in I}\eta_i^d}.$$

We blow up U_I by the ideal $\big(\Phi_j^{(I)}, \prod_{i\in I}\eta_i^d\big)$. We perform this blow up for all $j\in\{1,\dots,n\}$. The resulting scheme is covered by the 2^n open subsets

$$U_{IJ} = \operatorname{Spec}\left(R\left[\eta_i, y_{i'}\ {}_{i\in I, i'\notin I}, \frac{\prod_{i\in I}\eta_i^d}{\Phi_j^{(I)}(\eta_i, y_{i'})}, \frac{\Phi_{j'}^{(I)}(\eta_i, y_{i'})}{\prod_{i\in I}\eta_i^d}\ {}_{j\in J, j'\notin J}\right]\right),$$

where J runs through the subsets of $\{1,\dots,n\}$. The morphism Φ naturally extends to a morphism $\Phi : U_{IJ}\to U_J\subset(\mathbb{P}^1)^n$.

On U_{IJ}, the pull-back of F_n by Φ is represented by the function

$$\prod_{j\in J}\frac{\prod_{i\in I}\eta_i^d}{\Phi_j^{(I)}(\eta_i, y_{i'})}.$$

The divisor $ndF_{n'}$ is represented by $\prod_{i\in I}\eta_i^{nd}$. Hence the difference $ndF_{n'|U_{IJ}} - \Phi^*_{|U_{IJ}}F_n$ is defined by the function

$$\prod_i \eta_i^{(n-\#J)d}\cdot\prod_{j\in J}\Phi_j^{(I)},$$

which is a regular function on U_{IJ}. This proves the admissibility.

From Theorem 3.2 and Proposition 3.6, we get:

Theorem 3.11. *Let $(\bar X, Y)$ be a modulus pair with $\bar X$ affine, and $V\subset\bar X\times\square^n$ be a purely $(n+t)$-dimensional closed subset for some $t\geq 0$. Suppose V has modulus $2n\cdot Y$. Then there is a series of maps*

$$\Phi^\bullet : \bar X\times\square^\bullet\to\bar X\times\square^\bullet$$

compatible with face maps, i.e., for any codimension 1 face $\partial_{i,\epsilon}:\square^m\hookrightarrow\square^{m+1}$, the following diagram commutes:

$$\begin{tikzcd}
\bar X\times\square^m \arrow[r, "\Phi^m"] \arrow[d, hook, "\partial_{i,\epsilon}"] & \bar X\times\square^m \arrow[d, hook, "\partial_{i,\epsilon}"] \\
\bar X\times\square^{m+1} \arrow[r, "\Phi^{m+1}"] & \bar X\times\square^{m+1}
\end{tikzcd}$$

such that the closed subset

$$(\Phi^n)^{-1}(V)\subset\bar X\times\square^n$$

is equidimensional over $\square^n$ of relative dimension t, and has modulus Y. Moreover, the defining polynomials Φ_i^m can be taken to have degree ≤ 2 for each variable y_j.

It is proved by induction on m, starting with $\Phi^0=\mathrm{id}$ which has degree 0 and with V replaced by its restrictions to faces. Note that given a series of maps with the indicated compatibility and a cycle α on $\overline{X} \times \square^m$, the following equality of cycles on $\overline{X} \times \square^{m-1}$ holds whenever the relevant cycles are well-defined:

$$d((\Phi^m)^*\alpha) = (\Phi^{m-1})^*(d\alpha). \tag{3.12}$$

4. Suslin homology with compact support with modulus and higher Chow groups with modulus

In this section, let $\overline{X}$ be an *affine* finite-type scheme over an arbitrary field k and X be an open subset such that $\overline{X} \setminus X$ is the support of an effective Cartier divisor. The letter Y will denote effective Cartier divisors with support $\overline{X} \setminus X$. The aim of this section is to prove the following theorem.

Theorem 4.1. *Let $r \geq 0$. The inclusions*

$$\underline{z}_r^{equi}(\overline{X}|Y, \bullet) \subset \underline{z}_r(\overline{X}|Y, \bullet)$$

induce isomorphisms on the homology pro-groups for each n:

$$\left\{\mathrm{H}_n(\underline{z}_r^{equi}(\overline{X}|Y, \bullet))\right\}_Y \xrightarrow{\cong} \left\{\mathrm{H}_n(\underline{z}_r(\overline{X}|Y, \bullet))\right\}_Y,$$

where Y runs through effective Cartier divisors with support $\overline{X} \setminus X$.

Remark 4.2. (1) An explicit pro bound to annihilate the levelwise kernel and cokernel of the map will be indicated in Lemma 4.5. Theorem 4.1 implies Theorem 1.2 in the introduction, in light of Remark 2.4.

(2) In the terminology of [Fausk and Isaksen 2007, §6], the above theorem can be expressed as: the map $\{\underline{z}_r^{equi}(\overline{X}|Y, \bullet)\}_Y \to \{\underline{z}_r(\overline{X}|Y, \bullet)\}_Y$ is a weak equivalence in the $\mathcal{H}_*$-model category of pro-complexes of abelian groups.

4A. ***Construction of weak homotopy.*** Temporarily assume k is an infinite field, so that we can use the results in Section 3.

Fix an effective Cartier divisor Y with support $\overline{X} \setminus X$. Suppose we are given a cycle $V \in \underline{z}_r(\overline{X}|2nY, n)$. Apply Theorem 3.11 to $|V|$ and get a series of $\overline{X}$-maps $\Phi^\bullet : \overline{X} \times \square^\bullet \to \overline{X} \times \square^\bullet$.

Repeated application of Theorem 3.2 gives another series of $\overline{X}$-maps

$$\widetilde{\Phi}^\bullet : \overline{X} \times \square^\bullet \times \mathbb{A}^1 \to \overline{X} \times \square^\bullet \times \mathbb{A}^1$$

satisfying:

(1) The following diagrams commute:

$$\begin{tikzcd}
\bar{X} \times \square^n \arrow[r, "\mathrm{id}"] \arrow[d, "i_0"'] & \bar{X} \times \square^n \arrow[d, "i_0"] \\
\bar{X} \times \square^n \times \mathbb{A}^1 \arrow[r, "\widetilde{\Phi}^n"] & \bar{X} \times \square^n \times \mathbb{A}^1
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{X} \times \square^n \arrow[r, "\Phi^n"] \arrow[d, "i_1"'] & \bar{X} \times \square^n \arrow[d, "i_1"] \\
\bar{X} \times \square^n \times \mathbb{A}^1 \arrow[r, "\widetilde{\Phi}^n"] & \bar{X} \times \square^n \times \mathbb{A}^1
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{X} \times \square^{n-1} \times \mathbb{A}^1 \arrow[r, "\widetilde{\Phi}^{n-1}"] \arrow[d, "1_{\bar{X}} \times \partial_{j,\epsilon} \times 1_{\mathbb{A}^1}"'] & \bar{X} \times \square^{n-1} \times \mathbb{A}^1 \arrow[d, "1_{\bar{X}} \times \partial_{j,\epsilon} \times 1_{\mathbb{A}^1}"] \\
\bar{X} \times \square^n \times \mathbb{A}^1 \arrow[r, "\widetilde{\Phi}^n"] & \bar{X} \times \square^n \times \mathbb{A}^1
\end{tikzcd}$$

(2) The dimensions of the fibers of the map

$$(\widetilde{\Phi}^n)^{-1}(|V| \times \mathbb{A}^1) \hookrightarrow \bar{X} \times \square^n \times \mathbb{A}^1 \to \square^n \times \mathbb{A}^1$$

are $\le r$ over $\square^n \times (\mathbb{A}^1 \setminus \{0\})$. (Consequently if V happens to be in $\underline{z}_r^{equi}$, then $(\widetilde{\Phi}^n)^{-1}(|V| \times \mathbb{A}^1)$ is equidimensional over $\square^n \times \mathbb{A}^1$.)

(3) The map $\widetilde{\Phi}^n$ is defined by $n+1$ polynomials belonging to $\mathcal{O}(\bar{X})[y_1, \dots, y_n, t]$ having degrees ≤ 2 in each variable, where t is the coordinate of $\mathbb{A}^1$.

We explain a little more about the construction of $\widetilde{\Phi}^n$. It is done by induction on n. Suppose we have constructed $\widetilde{\Phi}^{n-1}$, with $|V|$ in condition (2) replaced by the union of its restrictions to the faces.

Set a Cartier divisor $Z := (\square^n \times 0) + (\square^n \times 1) + (\partial\square^n \times \mathbb{A}^1)$ on $\square^n \times \mathbb{A}^1$. Via the isomorphism $\square^n \times \mathbb{A}^1 \cong \square^{n+1}$, we have $Z \cong \partial\square^{n+1}$. Condition (1) for $\widetilde{\Phi}^{n-1}$ implies that there exists a unique $\bar{X}$-map

$$\bar{X} \times Z \to \bar{X} \times \square^n \times \mathbb{A}^1$$

whose restrictions to the faces isomorphic to $\bar{X} \times \square^n$ are the maps already defined: either id, Φ^n or $\widetilde{\Phi}^{n-1}$. This existence follows from the next elementary fact proved by induction and the snake lemma: Let R be a commutative ring with unit and let $x_1, \dots, x_n$ be elements of R which form a regular sequence, no matter how they are ordered. Then the set of elements $x_1, \dots, x_{n-2}, x_{n-1}x_n$ has the same property, and we have an isomorphism

$$R/x_1 \cdots x_n R \xrightarrow{\sim} \varprojlim \Big[\prod_i R/x_i R \rightrightarrows \prod_{i<j} R/(x_i, x_j) R \Big].$$

By the induction hypothesis and the choice of $\Phi^\bullet$, the maps id, Φ^n, $\widetilde{\Phi}^{n-1}$ are defined by polynomials whose degrees are ≤ 2 in each variable. Then by Theorem 3.2, we obtain $\widetilde{\Phi}^n$ having degrees ≤ 2 and satisfying (1)–(2).

We note a compatibility property satisfied by the pull-back operation $(\widetilde{\Phi}^n)^*$. Suppose we are given a cycle α on $\bar{X} \times \square^m$. We can consider its differential $d(\alpha)$ on $\bar{X} \times \square^{m-1}$ if it is well-defined. On the other hand, suppose we are given a cycle β on $\bar{X} \times \square^m \times \mathbb{A}^1$. Via the isomorphism $\bar{X} \times \square^m \times \mathbb{A}^1 \cong \bar{X} \times \square^{m+1}$ we view it as a cycle on the latter, and consider its differential which is a cycle on $\bar{X} \times \square^m$. We denote it by $\tilde{d}\beta$.

Thanks to condition (1) on $\widetilde{\Phi}^\bullet$, the following equality of cycles on $\bar{X} \times \square^m$ holds whenever the relevant cycles are all well-defined:

$$\tilde{d}((\widetilde{\Phi}^m)^*(\alpha \times \mathbb{A}^1)) = (\tilde{\Phi}^{m-1})^*(d(\alpha) \times \mathbb{A}^1) + (-1)^{m+1}((\Phi^m)^*\alpha - \alpha). \tag{4.3}$$

This applies in particular to $\alpha = V$: all terms are indeed well-defined, for example, by the choice of $\Phi^\bullet$ and $\widetilde{\Phi}^\bullet$, the irreducible components of $(\widetilde{\Phi}^n)^{-1}(|V| \times \mathbb{A}^1)$ have dimensions at most $r + n + 1$, which is the lowest possible due to the fact that $\widetilde{\Phi}^n$ is an $\bar{X}$-endomorphism of a smooth $\bar{X}$-scheme. So the term $(\widetilde{\Phi}^n)^*(V \times \mathbb{A}^1)$ is well-defined. Similarly for other terms.

4B. *Proof of the comparison theorem.* Finally we can prove Theorem 4.1. Let $f^Y : \underline{z}_r^{equi}(\bar{X}|Y, \bullet) \to \underline{z}_r(\bar{X}|Y, \bullet)$ denote the natural inclusion. It suffices to prove that

$$\big\{\mathrm{H}_n \underline{z}_r^{equi}(\bar{X}|Y, \bullet)\big\}_Y \xrightarrow{\{\mathrm{H}_n f^Y\}_Y} \big\{\mathrm{H}_n \underline{z}_r(\bar{X}|Y, \bullet)\big\}_Y$$

is an isomorphism in the category of pro-abelian groups pro-**Ab**. Its kernel and cokernel are $\{\mathrm{Ker}(\mathrm{H}_n f^Y)\}_Y$ and $\{\mathrm{Coker}(\mathrm{H}_n f^Y)\}_Y$ [Artin and Mazur 1969, Appendix, Proposition 4.1]. We prove that they are zero objects in pro-**Ab**. Now we recall the following elementary lemma:

Lemma 4.4. *An object $A = \{A^\gamma\}_{\gamma \in \Gamma} \in$ pro-**Ab** is the zero object if and only if for any $\gamma \in \Gamma$ there exists $\gamma' > \gamma$ such that the projection map $p_\gamma^{\gamma'} : A^{\gamma'} \to A^\gamma$ is the zero map.*

Therefore, the problem is reduced to showing the following:

Lemma 4.5. *For any effective Cartier divisor Y and $n \geq 0$, the projections*

$$\mathrm{Ker}(\mathrm{H}_n f^{2(n+1)Y}) \to \mathrm{Ker}(\mathrm{H}_n f^Y) \quad \textit{and} \quad \mathrm{Coker}(\mathrm{H}_n f^{2nY}) \to \mathrm{Coker}(\mathrm{H}_n f^Y)$$

are the zero maps.

Proof. Assume first k is infinite. We first prove that $\mathrm{Coker}(\mathrm{H}_n f^{2nY}) \to \mathrm{Coker}(\mathrm{H}_n f^Y)$ is the zero map for any $n \geq 0$. Take any element $W \in \mathrm{H}_n(\underline{z}_r(\bar{X}|2nY, \bullet))$. Apply the construction in Section 4A to W and get a cycle $(\widetilde{\Phi}^n)^* W \in \underline{z}_r^{equi}(\bar{X}|Y, n)$. Thanks to Equation (3.12), it is annihilated by the differential. Equation (4.3) now reads

$$\tilde{d}((\widetilde{\Phi}^n)^*(W \times \mathbb{A}^1)) = (-1)^{n+1}((\Phi^n)^* W - W)$$

in $\underline{z}_r(\bar{X}|Y, n)$, hence we have $W = (\Phi^n)^* W$ in $\mathrm{H}_n(\underline{z}_r(\bar{X}|Y, \bullet))$. This proves the assertion for the cokernel.

Next we prove that $\mathrm{Ker}(\mathrm{H}_n f^{(2n+2)Y}) \to \mathrm{Ker}(\mathrm{H}_n f^Y)$ is the zero map. Take any cycle V representing an element in $\mathrm{Ker}(\mathrm{H}_n f^{(2n+2)Y})$. Then, there exists $W \in \underline{z}_r(\bar{X}|(2n+2)Y, n+1)$ such that $V = dW$ as cycles.

Apply the construction in Section 4A to W (n replaced with $n+1$) and get a cycle $(\Phi^{n+1})^* W \in \underline{z}_r^{equi}(\bar{X}|Y, n+1)$ and $(\tilde{\Phi}^{n+1})^*(W \times \mathbb{A}^1) \in \underline{z}_r(\bar{X}|Y, n+2)$ whose modulus condition follows from Proposition 3.6. Equation (4.3) for $\alpha = W$ reads

$$\tilde{d}((\widetilde{\Phi}^{n+1})^*(W \times \mathbb{A}^1)) = (\tilde{\Phi}^n)^*(V \times \mathbb{A}^1) + (-1)^n((\Phi^{n+1})^* W - W).$$

Differentiate it to get $0 = d(\widetilde{\Phi}^n)^*(V \times \mathbb{A}^1) + (-1)^n(d(\Phi^{n+1})^* W - V)$. Hence

$$V = d(\Phi^{n+1})^* W + (-1)^n d(\widetilde{\Phi}^n)^*(V \times \mathbb{A}^1).$$

Thanks to the choice of $\widetilde{\Phi}^\bullet$ and the fact that V is equidimensional, both $(\Phi^{n+1})^* W$ and $(\widetilde{\Phi}^n)^*(V \times \mathbb{A}^1)$ are equidimensional cycles. So V is zero in $\mathrm{H}_n(\underline{z}_r^{equi}(\bar{X}|Y, \bullet))$. This proves the assertion for the kernel, hence completes the proof for infinite fields.

Finally, suppose that k is finite. This case is settled by a trace (norm) argument. Let $l \in \{2, 3\}$ and k_l be an infinite pro-l extension of k. Given any $V \in \mathrm{Coker}(\mathrm{H}_n f^{2nY})$, its image in $\mathrm{Coker}(\mathrm{H}_n f^Y)_{k_l}$ is zero by the infinite field case (the subscript means the group is computed after the scalar extension k_l/k). Since the latter group is the direct limit of $\mathrm{Coker}(\mathrm{H}_n f^Y)_{k'_l}$, where k'_l runs through the finite subextensions of k_l/k, the element V vanishes in some $\mathrm{Coker}(\mathrm{H}_n f^Y)_{k'_l}$. The finite push-forward map $\mathrm{Coker}(\mathrm{H}_n f^Y)_{k'_l} \to \mathrm{Coker}(\mathrm{H}_n f^Y)$ has the property that its composite with the scalar extension map

$$\mathrm{Coker}(\mathrm{H}_n f^Y) \to \mathrm{Coker}(\mathrm{H}_n f^Y)_{k'_l} \to \mathrm{Coker}(\mathrm{H}_n f^Y)$$

is the multiplication by $[k'_l : k]$. Therefore the image of V in $\mathrm{Coker}(\mathrm{H}_n f^Y)$ is annihilated by $[k'_l : k]$, which is a power of l. Since $[k'_2 : k]$ and $[k'_3 : k]$ are relatively prime, the image of V itself is zero. The proof for $\{\mathrm{Ker}(\mathrm{H}_n f^Y)\}_Y$ is the same. □

4C. ***A consequence on the relative motivic cohomologies.*** In this final subsection $\bar{X}$ can be any algebraic scheme. Let X be an open set of $\bar{X}$ such that the complement $\bar{X} \setminus X$ is the support of an effective Cartier divisor Y.

Consider the presheaf of complexes on the small Zariski site $\bar{X}_{\mathrm{Zar}}$,

$$z_r(\bar{X}|Y, \bullet)_{\mathrm{Zar}} : (\bar{U} \subset \bar{X}) \mapsto z_r(\bar{U}|Y \cap \bar{U}, \bullet),$$

which turns out to be a sheaf, as well as $z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}}$ similarly defined. We have a natural inclusion of sheaves $z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}} \subset z_r(\bar{X}|Y, \bullet)_{\mathrm{Zar}}$. The induced maps on homology sheaves

$$\left\{\mathrm{H}_n(z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y \xrightarrow{\{f_n^Y\}_Y} \left\{\mathrm{CH}_r(\bar{X}|Y, n)_{\mathrm{Zar}}\right\}_Y \tag{4.6}$$

are pro-isomorphisms of Zariski sheaves for all n. Indeed, by Lemma 4.5, the maps of sheaves

$$\mathrm{Coker}(f_n^{2nY}) \to \mathrm{Coker}(f_n^Y), \quad \mathrm{Ker}(f_n^{(2n+2)Y}) \to \mathrm{Ker}(f_n^Y)$$

are zero.

As a general fact on pro-categories, the functors $\mathrm{H}^n_{\mathrm{Zar}}(\bar{X}, -)$ extend to functors

$$\text{pro-sheaves} \to \text{pro-abelian groups}, \quad \{F_i\}_i \mapsto \{\mathrm{H}^n_{\mathrm{Zar}}(\bar{X}, F_i)\}_i. \tag{4.7}$$

We have hypercohomology spectral sequences in the abelian category of pro-abelian groups:

$$E_2^{pq} = \left\{\mathrm{H}^p_{\mathrm{Zar}}(\bar{X}, \mathrm{H}_{-q}(z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}}))\right\}_Y \Rightarrow \left\{\mathbf{H}^{p+q}_{\mathrm{Zar}}(\bar{X}, z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y$$
$${}'E_2^{pq} = \left\{\mathrm{H}^p_{\mathrm{Zar}}(\bar{X}, \mathrm{CH}_r(\bar{X}|Y, -q)_{\mathrm{Zar}})\right\}_Y \Rightarrow \left\{\mathbf{H}^{p+q}_{\mathrm{Zar}}(\bar{X}, z_r(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y$$

which are bounded to the range $0 \le p \le \dim \bar{X}$ and $q \le 0$. Since the natural map $E \to {}'E$ of spectral sequences induces isomorphisms on E_2-terms by equations (4.6) and (4.7), we get isomorphisms

$$\left\{\mathbf{H}^n_{\mathrm{Zar}}(\bar{X}, z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y \to \left\{\mathbf{H}^n_{\mathrm{Zar}}(\bar{X}, z_r(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y.$$

So we have proved:

Theorem 4.8. *Let $r \ge 0$ and $n \in \mathbb{Z}$. For any algebraic scheme $\bar{X}$ and an effective Cartier divisor Y_0 on $\bar{X}$, the natural map of pro-abelian groups*

$$\left\{\mathbf{H}^n_{\mathrm{Zar}}(\bar{X}, z_r^{equi}(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y \to \left\{\mathbf{H}^n_{\mathrm{Zar}}(\bar{X}, z_r(\bar{X}|Y, \bullet)_{\mathrm{Zar}})\right\}_Y$$

are isomorphisms, where Y runs through effective Cartier divisors with support $|Y_0|$.

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WATARU KAI: kiwtraiaau@gmail.com
Fakultät für Mathematik, Universität Duisburg-Essen, D-45127 Essen, Germany

HIROYASU MIYAZAKI: miyazaki@ms.u-tokyo.ac.jp
Graduate School of Mathematical Sciences, University of Tokyo, Tokyo 153-8914, Japan

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Abstract tilting theory for quivers and related categories

Moritz Groth and Jan Šťovíček

We generalize the construction of reflection functors from classical representation theory of quivers to arbitrary small categories with freely attached sinks or sources. These reflection morphisms are shown to induce equivalences between the corresponding representation theories with values in arbitrary stable homotopy theories, including representations over fields, rings or schemes as well as differential-graded and spectral representations.

Specializing to representations over a field and to specific shapes, this recovers derived equivalences of Happel for finite, acyclic quivers. However, even over a field our main result leads to new derived equivalences, for example, for not necessarily finite or acyclic quivers.

Our results rely on a careful analysis of the compatibility of gluing constructions for small categories with homotopy Kan extensions and homotopical epimorphisms, and on a study of the combinatorics of amalgamations of categories.

1. Introduction

Happel [1987] considered derived categories of finite-dimensional algebras over fields. Interesting special cases of such algebras are path algebras of finite and

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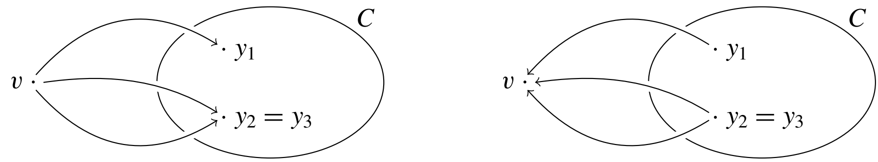
acyclic quivers. Let us recall that a quiver is simply an oriented graph and that a quiver is acyclic if it admits no nontrivial oriented cycles. Given such an acyclic quiver Q and a source $q_0 \in Q$ (no edge ends at q_0) there is the reflected quiver Q' obtained by turning the source into a sink. Bernšteĭn, Gel'fand, and Ponomarev [Bernšteĭn et al. 1973] showed that the corresponding abelian categories of representations are related by reflection functors. If one works with representations of a *finite, acyclic* quiver over a field, then Happel [1987] proved that derived reflection functors yield exact equivalences between the corresponding bounded derived categories of the path algebras.

The main goal of this paper is to generalize this result in two different directions. First, we show that one obtains similar equivalences if one drops the assumption of working over a field. More precisely, we construct such exact equivalences of derived or homotopy categories of representations over a ring, of representations in quasicoherent modules on arbitrary schemes, of differential-graded representations over differential-graded algebras, and of spectral representations. In fact, we obtain equivalences of homotopy theories of representations and we show that the existence of such equivalences is a formal consequence of *stability* only. Hence there are many additional variants for representations with values in other *stable homotopy theories* arising in algebra, geometry, and topology (for more details about what we mean by a stable homotopy theory see further below).

Second, we generalize this result, in that we obtain such equivalences for a significantly larger class of shapes. Given an *arbitrary small category* C and a finite string $y_1, y_2, \ldots, y_n$ of objects in C, then we can form new categories C^- and C^+ by freely adjoining a source or a sink to these objects in C. The string of objects may have some repetition, so that the generic picture to have in mind is as in Figure 1. In this situation we show that the categories C^- and C^+ have equivalent homotopy theories of representations with values in arbitrary stable homotopy theories, i.e., that they are *strongly stably equivalent* in a sense made precise in (1.1).

To illustrate this abstract statement let us turn to some special cases, which we explore further in [Groth and Šťovíček $\geq$ 2018]. As a first example, if we specialize to a finite, acyclic quiver and consider representations over a field, then we recover the derived equivalences of Happel [1987] (actually also a version for unbounded chain complexes). However, even for representations over a field and of quivers, the main result leads to new classes of derived equivalences.

For example, dropping the finiteness assumption, we see that reflection functors induce derived equivalences between the infinite-dimensional (possibly nonunital) algebras associated to infinite, acyclic quivers. Alternatively, we can drop the acyclicity assumption. As long as there are sources or sinks in a finite quiver, corresponding reflection functors yield derived equivalences between infinite-dimensional path algebras. Combining these two, we can also drop both the finiteness and the



The category C^{-} The category C^{+}

Figure 1. Adjoining a source and a sink to $C \in \mathcal{C}at$.

acyclicity assumption. As soon as an arbitrary quiver has sources or sinks, there are associated derived equivalences given by reflection functors.

Choosing other examples of stable homotopy theories, we see that all these equivalences also have variants if we do not work over a field but with more general abstract representations. As a further specialization we deduce that finite oriented trees can be reoriented arbitrarily without affecting the abstract representation theory, thereby reproducing the main result of [Groth and Šťovíček 2016b]. To mention an additional instance, if one considers representations of a poset in Grothendieck abelian categories, then our main result reestablishes a special case of a result of Ladkani [2007], but also extends it for example to differential-graded and spectral representations. And there are additional such statements starting with more general small categories instead.

These abstract equivalences are realized by general reflection morphisms between homotopy theories of representations. The arguments involved in their construction are rather formal as they rely only on the existence of a well-behaved calculus of restrictions and (homotopy) Kan extensions of diagrams in stable homotopy theories. Besides being fairly transparent, there are two additional advantages of this method of construction.

(i) First, this leads to equivalences of *homotopy theories* of abstract representations as opposed to mere equivalences of *homotopy categories* of representations. Since equivalences of homotopy theories *are* exact, the corresponding functors between derived categories or homotopy categories can be turned into exact equivalences with respect to classical triangulations [Groth 2013]. However, in general, the existence of exact equivalences of triangulated categories of representations does not imply that there are equivalences of homotopy theories in the background. While this is the case for representations over rings by [Dugger and Shipley 2004], as soon as one passes to differential-graded or spectral representations it is in general a stronger result to have equivalences of homotopy theories.

(ii) Second, in this way the equivalences of homotopy theories of representations with values in stable homotopy theories are seen to be compatible with exact morphisms of stable homotopy theories. In particular, these equivalences hence interact nicely with restriction and (co)induction of scalar morphisms, with localizations and colocalizations, with derived tensor and hom morphisms, and more general exact morphisms.

Let us now be more specific about what we mean by abstract (stable) homotopy theories. By now there are various ways of axiomatizing (stable) homotopy theories, including Quillen model categories [Quillen 1967; Hovey 1999], quasicategories or ∞-categories [Lurie 2009; 2016; Groth 2010], derivators [Grothendieck 1991; Heller 1988; Franke 1996], as well as the more classical triangulated categories. In this paper we use the language of derivators, which by definition can be thought of as minimal, purely categorical extensions of the more classical derived or homotopy categories to a framework with a well-behaved calculus of homotopy (co)limits and homotopy Kan extensions. In this approach to abstract homotopy theory, homotopy (co)limits and homotopy Kan extensions are defined and characterized by ordinary universal properties, thereby making their calculus accessible by elementary categorical techniques.

The basic idea about derivators is as follows. Given an abelian category $\mathcal{A}$, the derived category $D(\mathcal{A})$ is rather ill-behaved. In particular, the calculus of derived (co)limits and derived Kan extensions is not visible to $D(\mathcal{A})$ *alone*. Hence, if one agrees on the relevance of this calculus (and some evidence for this is for example provided by the observation that classical triangulations simply encode certain shadows of iterated derived cokernel constructions), why not simply encode derived categories of diagram categories $D(\mathcal{A}^B)$ for various small categories B together with restriction functors between them? Pursuing this more systematically, one is lead to consider the *derivator* of $\mathcal{A}$, a certain 2-functor

$$\mathscr{D}_{\mathcal{A}} : B \mapsto \mathscr{D}_{\mathcal{A}}(B) = D(\mathcal{A}^B),$$

and derived Kan extensions now are merely adjoints to (derived) restriction functors. The values of $\mathscr{D}_{\mathcal{A}}$ are considered as plain categories, but *exactness properties* of the derivator can be used to construct canonical triangulations and canonical higher triangulations in the sense of Maltsiniotis [2005]. In fact, this holds more generally for strong, stable derivators (see [Franke 1996; Maltsiniotis 2001; Groth 2013; 2016a]), such as homotopy derivators of stable model categories or stable ∞-categories. Let us recall that a derivator is stable if it admits a zero object and if a square is cartesian if and only if it is cocartesian (see [Groth et al. 2014b; Groth and Šťovíček 2016c] for alternative characterizations). While stability is invisible to ordinary category theory, there is a ubiquity of stable derivators arising in algebra, geometry, and topology [Groth and Šťovíček 2016c, §5].

Now, the connection to abstract representation theory or abstract tilting theory is provided by the following observation. Given a derivator $\mathscr{D}$ and a small category B, there is the derivator $\mathscr{D}^B$ of coherent diagrams of shape B in $\mathscr{D}$. This exponentiation is compatible with the formation of exponentials at the level of abelian categories, (nice) model categories, and ∞-categories. For example, given a Grothendieck abelian category $\mathcal{A}$ and a small category B there is an equivalence of derivators

$$\mathscr{D}_{\mathcal{A}}^B \simeq \mathscr{D}_{\mathcal{A}^B}.$$

Specializing further, this shows that the passage to category algebras (like path algebras, incidence algebras, and group algebras) can be modeled by this shifting operation at the level of derivators.

To state the main result of this paper more precisely, let $\mathcal{DER}_{\mathrm{St,ex}}$ be the 2-category of stable derivators, exact morphisms, and all natural transformations. For every small category B, exponentiation by B defines a 2-functor

$$(-)^B : \mathcal{DER}_{\mathrm{St,ex}} \to \mathcal{DER} : \mathscr{D} \mapsto \mathscr{D}^B,$$

where $\mathcal{DER}$ is the 2-category of derivators. Denoting again by C an arbitrary small category and by C^-, C^+ the categories obtained from C by freely attaching a source or a sink to a prescribed string of objects (see again Figure 1), we show that these two categories are *strongly stably equivalent* in the sense of [Groth and Šťovíček 2016c]. Thus, we show that there is a pseudonatural equivalence of 2-functors

$$\Phi : (-)^{C^-} \simeq (-)^{C^+} : \mathcal{DER}_{\mathrm{St,ex}} \to \mathcal{DER}, \tag{1.1}$$

and in this precise sense C^-, C^+ have equivalent abstract representation theories.

In the sequel [Groth and Šťovíček $\geq$ 2018] we study these general reflection morphisms further. We show that unrelated reflections commute, leading to abstract Coxeter morphisms for finite, acyclic quivers. Moreover, the reflections are shown to be realized by explicitly constructed invertible spectral bimodules, and this yields nontrivial elements in spectral Picard groupoids. We also obtain a spectral Serre duality result for acyclic quivers and, more generally, strongly homotopy finite categories.

While here and in the sequel we state and prove the above results using the language of derivators, it is completely formal to also deduce implications for model categories and ∞-categories of abstract representations. For concreteness, given a stable, combinatorial model category $\mathcal{M}$, the existence of the strong stable equivalence (1.1) implies by [Renaudin 2009] that the model categories $\mathcal{M}^{C^-}$ and $\mathcal{M}^{C^+}$ are connected by a zigzag of Quillen equivalences. Similarly, there is a variant for stable, presentable ∞-categories of representations.

This paper belongs to a series of papers on abstract representation theory and abstract tilting theory, and can be considered as sequel to [Groth and Šťovíček 2016a; 2016b; 2016c]. This project relies both on a basic formal understanding of stability [Groth 2013; Groth et al. 2014b] as well as on a basic understanding of the interaction of monoidality and stability [Groth et al. 2014a; Ponto and Shulman 2016]. We intend to come back to further applications to abstract representation theory elsewhere.

The content of the sections is as follows. In Sections 2 and 3 we recall some basics concerning derivators. In Section 4 we outline the strategy of the construction of the general reflection morphisms leading to the desired strong stable equivalence. In Section 5-6 we introduce free oriented gluing constructions of small categories and study their compatibility with Kan extensions and homotopical epimorphisms. This allows us in Section 7 to construct reflection equivalences in the special case of separated sources and sinks. In Section 8 we establish two simple detection criteria for homotopical epimorphisms, which we use in Section 9 to conclude the construction of reflection equivalences in the general case. In Section 10 we deduce some consequences of our abstract tilting result. Finally, in the Appendix we collect some results concerning the combinatorics of amalgamations of small categories which are useful in Section 9.

2. Review of stable derivators and strong stable equivalences

In this section we include a short review of stable derivators. For more details we refer the reader to [Groth 2013; Groth et al. 2014b]. The key idea behind a derivator is that they enhance the more classical derived categories of abelian categories and homotopy categories of model categories by also keeping track of homotopy categories of diagram categories together with the calculus of homotopy Kan extensions. Like stable model categories and stable ∞-categories, stable derivators provide an enhancement of triangulated categories.

To make this precise, let $\mathcal{C}at$ be the 2-category of small categories and $\mathcal{CAT}$ the 2-category of not necessarily small categories. We refer the reader to [Borceux 1994] for basic 2-categorical terminology.

Definition 2.1. A *prederivator* is a 2-functor $\mathscr{D} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT}$. *Morphisms* of prederivators are pseudonatural transformations, and *transformations* between these morphisms are modifications, yielding the 2-category $\mathcal{PDER}$ of prederivators.

Given a prederivator $\mathscr{D}$ we call objects in $\mathscr{D}(A)$ *coherent diagrams (of shape A)*. For every functor $u : A \to B$ there is a *restriction functor* $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$. In the special case that $A = \mathbb{1}$ is the terminal category and hence $u = b : \mathbb{1} \to B$ classifies an object $b \in B$, we refer to $b^* : \mathscr{D}(B) \to \mathscr{D}(\mathbb{1})$ as an *evaluation functor*. Evaluating a morphism $f : X \to Y$ in $\mathscr{D}(B)$ we obtain induced morphisms $f_b : X_b \to Y_b$, $b \in B$, in the *underlying category* $\mathscr{D}(\mathbb{1})$.

If a restriction functor $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ admits a left adjoint, then we refer to it as a *left Kan extension functor* and denote it by $u_! : \mathscr{D}(A) \to \mathscr{D}(B)$. In the special case that $u = \pi_A : A \to \mathbb{1}$ collapses A to a point, such a left adjoint is also denoted by $(\pi_A)_! = \mathrm{colim}_A : \mathscr{D}(A) \to \mathscr{D}(\mathbb{1})$ and referred to as a *colimit functor*. Dually, we speak of *right Kan extension functors* $u_* : \mathscr{D}(A) \to \mathscr{D}(B)$ and *limit functors* $(\pi_A)_* = \lim_A : \mathscr{D}(A) \to \mathscr{D}(\mathbb{1})$.

For *derivators* we ask for the existence of such Kan extension functors and that they can be calculated *pointwise* (see [Mac Lane 1998, X.3.1] for the classical context of ordinary categories). To express this purely 2-categorically, we consider the *slice squares*

$$\begin{tikzcd} (u/b) \arrow[r, "p"] \arrow[d, "\pi_{(u/b)}"'] \arrow[dr, phantom, "\swArrow"] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "b"'] & B \end{tikzcd} \qquad \begin{tikzcd} (b/u) \arrow[r, "q"] \arrow[d, "\pi_{(b/u)}"'] \arrow[dr, phantom, "\neArrow"] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "b"'] & B \end{tikzcd} \tag{2.2}$$

coming with transformations $u \circ p \to b \circ \pi$ and $b \circ \pi \to u \circ q$, respectively. Here, objects in the *slice category* (u/b) are pairs (a, f) consisting of an object $a \in A$ and a morphism $f : u(a) \to b$ in B. A morphism $(a, f) \to (a', f')$ is a map $a \to a'$ in A making the obvious triangles commute. The functor $p : (u/b) \to A$ is the obvious projection and the component of the transformation $u \circ p \to b \circ \pi$ at (a, f) is f. The square on the right in (2.2) is defined dually.

Definition 2.3. A prederivator $\mathscr{D} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT}$ is a *derivator*[1] if the following properties are satisfied.

(Der1) $\mathscr{D} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT}$ takes coproducts to products, i.e., the canonical map $\mathscr{D}\big(\coprod A_i\big) \to \prod \mathscr{D}(A_i)$ is an equivalence. In particular, $\mathscr{D}(\varnothing)$ is equivalent to the terminal category.

(Der2) For any $A \in \mathcal{C}at$, a morphism $f : X \to Y$ in $\mathscr{D}(A)$ is an isomorphism if and only if the morphisms $f_a : X_a \to Y_a$, $a \in A$, are isomorphisms in $\mathscr{D}(\mathbb{1})$.

(Der3) Each functor $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ has both a left adjoint $u_!$ and a right adjoint u_*.

(Der4) For any functor $u : A \to B$ and any $b \in B$ the canonical transformations

$$\pi_! p^* \xrightarrow{\eta} \pi_! p^* u^* u_! \to \pi_! \pi^* b^* u_! \xrightarrow{\epsilon} b^* u_! \quad \text{and}$$
$$b^* u_* \xrightarrow{\eta} \pi_* \pi^* b^* u_* \to \pi_* q^* u^* u_* \xrightarrow{\epsilon} \pi_* q^*$$

associated to the slice squares (2.2) are isomorphisms.

[1] We emphasize that $\mathcal{C}at^{\mathrm{op}}$ is obtained from $\mathcal{C}at$ by changing the orientation of functors but not of natural transformations. Thus, following [Heller 1988; Franke 1996], our convention for derivators is based on *diagrams*. There is an equivalent approach using *presheaves*, i.e., contravariant functors; see for example [Grothendieck 1991; Cisinski 2003].

Axiom (Der4) thus says that for $u : A \to B, b \in B$, and $X \in \mathscr{D}(A)$, certain canonical maps

$$\operatorname{colim}_{(u/b)} p^* X \to u_!(X)_b \quad \text{and} \quad u_*(X)_b \to \lim_{(b/u)} q^* X$$

are isomorphisms. We say a bit more about the formalism related to (Der4) in Section 3.

Morphisms and *transformations* of derivators are morphisms and transformations of underlying prederivators, yielding the sub-2-category $\mathcal{DER} \subseteq \mathcal{PDER}$ of derivators. Given a (pre)derivator, we often write $X \in \mathscr{D}$ if there is a small category A such that $X \in \mathscr{D}(A)$.

Examples 2.4. (i) Let $\mathcal{C}$ be an ordinary category. The 2-functor

$$y_{\mathcal{C}} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT} : A \mapsto \mathcal{C}^A$$

is a derivator if and only if $\mathcal{C}$ is complete and cocomplete. Kan extension functors in such a *represented derivator* are *ordinary Kan extensions* from classical category theory. The underlying category of $y_{\mathcal{C}}$ is isomorphic to $\mathcal{C}$.

(ii) Let $\mathcal{A}$ be a Grothendieck abelian category and let $\mathrm{Ch}(\mathcal{A})$ be the category of unbounded chain complexes in $\mathcal{A}$. For every $A \in \mathcal{C}at$ we denote by W^A the class of levelwise quasi-isomorphisms in $\mathrm{Ch}(\mathcal{A})^A$. The 2-functor

$$\mathscr{D}_{\mathcal{A}} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT} : A \mapsto \mathrm{Ch}(\mathcal{A})^A[(W^A)^{-1}]$$

is a derivator. Kan extension functors in $\mathscr{D}_{\mathcal{A}}$ are *derived Kan extensions* in the sense of homological algebra. The underlying category of $\mathscr{D}_{\mathcal{A}}$ is isomorphic to the *derived category* $D(\mathcal{A})$ of $\mathcal{A}$. As interesting examples we obtain derivators associated to fields, rings, and schemes.

(iii) Let $\mathcal{M}$ be a Quillen model category [Quillen 1967; Hovey 1999] with weak equivalences W. Denoting by W^A the levelwise weak equivalences in $\mathcal{M}^A$, there is an associated *homotopy derivator*

$$\mathscr{H}o_{\mathcal{M}} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT} : A \mapsto \mathcal{M}^A[(W^A)^{-1}];$$

see [Cisinski 2003] for the general case and [Groth 2013, Proposition 1.30] for an easy proof in the case of combinatorial model categories. Kan extension functors in $\mathscr{H}o_{\mathcal{M}}$ are *homotopy Kan extensions*. The underlying category of $\mathscr{H}o_{\mathcal{M}}$ is isomorphic to the *homotopy category* $\mathrm{Ho}(\mathcal{M})$. Similarly, there are homotopy derivators associated to complete and cocomplete ∞-categories or quasicategories [Joyal $\geq$ 2018; 2008; Lurie 2009; Groth 2010]; see [Groth et al. 2014b] for a proof sketch. These two classes give rise to a plethora of additional examples of derivators.

Thus, derivators encode key formal properties of the calculus of Kan extensions, derived Kan extensions, and homotopy Kan extensions, as it is available in typical situations arising in nature. It turns out that many constructions are combinations of such Kan extensions, including the general reflection functors we construct in this paper; see Sections 4, 7, and 9.

Let $[1]$ be the poset $(0 < 1)$ considered as a category and let $\square = [1] \times [1]$ be the commutative square. We denote by $i_\ulcorner : \ulcorner \to \square$, $i_\lrcorner : \lrcorner \to \square$ the full subcategories obtained by removing the final and initial object, respectively. A square $X \in \mathscr{D}(\square)$ is *cartesian* if it lies in the essential image of $(i_\lrcorner)_* : \mathscr{D}(\lrcorner) \to \mathscr{D}(\square)$. Dually, we define *cocartesian squares*.

Definition 2.5. A derivator is *pointed* if the underlying category has a zero object. A pointed derivator is *stable* if a square is cartesian if and only if it is cocartesian.

Examples 2.6. (i) The derivator of a Grothendieck abelian category is stable. In particular, fields, rings, and schemes have associated stable derivators.

(ii) Homotopy derivators of stable model categories and stable ∞-categories are stable.

(iii) The derivator of differential graded modules over a differential graded algebra is stable.

(iv) The derivator of module spectra over a symmetric ring spectrum is stable. In particular, the derivator of spectra itself is stable.

We refer the reader to [Groth and Šťovíček 2016c, Examples 5.4] for many additional examples of stable derivators arising in algebra, geometry, and topology. It can be shown that the values of (strong) stable derivators are canonically triangulated categories [Franke 1996; Maltsiniotis 2001; Groth 2013, Theorem 4.16 and Corollary 4.19] and even higher triangulated categories [Groth and Šťovíček 2016a, Theorem 13.6, Corollary 13.11, and Remark 13.12] in the sense of Maltsiniotis [2005].

In a pointed derivator $\mathscr{D}$ one can define *suspensions, loops, cofibers, and fibers* (see [Groth 2013, §3]), yielding adjunctions

$$(\Sigma, \Omega) : \mathscr{D}(\mathbb{1}) \rightleftarrows \mathscr{D}(\mathbb{1}) \quad \text{and} \quad (\mathsf{cof}, \mathsf{fib}) : \mathscr{D}([1]) \rightleftarrows \mathscr{D}([1]).$$

We recall from [Groth and Šťovíček 2016c, §8] some basic notation and terminology related to n-cubes $[1]^n = [1] \times \cdots \times [1]$. The poset $[1]^n$ is isomorphic to the power set of $\{1, \ldots, n\}$, and this isomorphism is used implicitly in what follows. We denote by $i_{\geq k} : [1]^n_{\geq k} \to [1]^n$, $0 \leq k \leq n$, the full subcategory spanned by all subsets of cardinality at least k. This notation has obvious variants, for example the full subcategory $i_{=n-1} : [1]^n_{=n-1} \to [1]^n$ is the discrete category $n \cdot \mathbb{1} = \mathbb{1} \sqcup \cdots \sqcup \mathbb{1}$ on n objects.

Definition 2.7. Let $\mathscr{D}$ be a derivator. An n-cube $X \in \mathscr{D}([1]^n)$ is *strongly cartesian* if it lies in the essential image of $(i_{\geq n-1})_* : \mathscr{D}([1]^n_{\geq n-1}) \to \mathscr{D}([1]^n)$. An n-cube $X \in \mathscr{D}([1]^n)$ is *cartesian* if it lies in the essential image of $(i_{\geq 1})_*$.

Dually, one defines *(strongly) cocartesian n-cubes*. Following ideas of Goodwillie [1991], one shows the following.

Theorem 2.8 [Groth and Šťovíček 2016c, Theorem 8.3, Corollary 8.8]. *An n-cube, $n \geq 2$, in a derivator is strongly cartesian if and only if all subcubes are cartesian if and only if all subsquares are cartesian.*

Stable derivators admit the following different characterizations.

Theorem 2.9 [Groth et al. 2014b, Theorem 7.1; 2016c, Corollary 8.9]. *The following are equivalent for a pointed derivator $\mathscr{D}$.*

(i) *The adjunction* $(\Sigma, \Omega) : \mathscr{D}(\mathbb{1}) \to \mathscr{D}(\mathbb{1})$ *is an equivalence.*

(ii) *The adjunction* $(\mathsf{cof}, \mathsf{fib}) : \mathscr{D}([1]) \to \mathscr{D}([1])$ *is an equivalence.*

(iii) *The derivator $\mathscr{D}$ is stable.*

(iv) *An n-cube in $\mathscr{D}$, $n \geq 2$, is strongly cartesian if and only if it is strongly cocartesian.*

An n-cube which is simultaneously strongly cartesian and strongly cocartesian is *strongly bicartesian*. In the case of $n = 2$ this reduces to the classical notion of a *bicartesian square*. Strongly bicartesian n-cubes in stable derivators satisfy the 2-out-of-3 property with respect to composition and cancellation (see [Groth and Šťovíček 2016c, §8] for the case of n-cubes).

The natural domains for Kan extensions with parameters are given by shifted derivators in the sense of the following proposition. This exponential construction is central to abstract representation theory.

Proposition 2.10 [Groth 2013, Theorem 1.25 and Proposition 4.3]. *Let $\mathscr{D}$ be a derivator and let $B \in \mathcal{C}at$. The 2-functor*

$$\mathscr{D}^B : \mathcal{C}at^{\mathrm{op}} \to \mathcal{CAT} : A \mapsto \mathscr{D}(B \times A)$$

is again a derivator, the derivator of coherent diagrams of shape B, *which is pointed or stable as soon as $\mathscr{D}$ is.*

This shifting operation also applies to morphisms and natural transformations in either variable, thereby defining a two-variable pseudofunctor

$$\mathcal{C}at^{\mathrm{op}} \times \mathcal{DER} \to \mathcal{DER} : (A, \mathscr{D}) \mapsto \mathscr{D}^A.$$

In abstract representation theory we are interested in suitable restrictions of related 2-functors. To begin with, as special cases of morphisms of derivators preserving certain (co)limits [Groth 2013, §2.2] there are the following definitions.

Definition 2.11. (i) A morphism of derivators is *right exact* if it preserves initial objects and cocartesian squares.

(ii) A morphism of derivators is *left exact* if it preserves terminal objects and cartesian squares.

(iii) A morphism of derivators is *exact* if it is right exact and left exact.

A morphism between stable derivators is right exact if and only if it is left exact if and only if it is exact. In particular, adjunctions and equivalences between stable derivators give rise to exact morphisms. (Adjunctions and equivalences of derivators are defined internally to the 2-category $\mathcal{DER}$; see [Groth 2013, §2] for details including explicit reformulations.)

Identity morphisms are exact and exact morphisms are closed under compositions, and there is thus the 2-category $\mathcal{DER}_{\mathrm{St,ex}} \subseteq \mathcal{DER}$ of stable derivators, exact morphisms, and arbitrary natural transformations. Hence, for every $A \in \mathcal{C}at$ we obtain an induced 2-functor $(-)^A : \mathcal{DER} \to \mathcal{DER}$ which can be restricted to

$$(-)^A : \mathcal{DER}_{\mathrm{St,ex}} \to \mathcal{DER}.$$

Definition 2.12 [Groth and Šťovíček 2016c, Definition 5.1]. Two small categories A and A' are *strongly stably equivalent*, in notation $A \overset{s}{\sim} A'$, if there is a pseudonatural equivalence between the 2-functors

$$\Phi : (-)^A \simeq (-)^{A'} : \mathcal{DER}_{\mathrm{St,ex}} \to \mathcal{DER}.$$

Such a pseudonatural equivalence is called a *strong stable equivalence*.

This definition makes precise the idea that the categories A and A' have the same representation theories in arbitrary stable derivators. More formally, a strong stable equivalence $\Phi : A \overset{s}{\sim} A'$ consists of

(i) an equivalence of derivators $\Phi_{\mathscr{D}} : \mathscr{D}^A \simeq \mathscr{D}^{A'}$ for *every* stable derivator $\mathscr{D}$, and

(ii) associated to every exact morphism of stable derivators $F : \mathscr{D} \to \mathscr{E}$, a natural isomorphism $\gamma_F : F \circ \Phi_{\mathscr{D}} \to \Phi_E \circ F$,

$$\begin{tikzcd}
\mathscr{D}^A \arrow[r, "\Phi_{\mathscr{D}}", "\simeq"'] \arrow[d, "F"'] \arrow[dr, phantom, "\swarrow\cong"] & \mathscr{D}^{A'} \arrow[d, "F"] \\
\mathscr{E}^A \arrow[r, "\simeq", "\Phi_{\mathscr{E}}"'] & \mathscr{E}^{A'}
\end{tikzcd}$$

satisfying the usual coherence properties of a pseudonatural transformation.

The motivation for this definition is the following example of the shifting operation; see [Groth and Šťovíček 2016c, §5].

Example 2.13. Let $\mathcal{A}$ be a Grothendieck abelian category and $B \in \mathcal{C}at$. There is an equivalence of derivators

$$\mathscr{D}_{\mathcal{A}}^{B} \simeq \mathscr{D}_{\mathcal{A}^{B}}.$$

In particular, if B, B' are strongly stably equivalent, then there is a chain of equivalences of derivators

$$\mathscr{D}_{\mathcal{A}^{B}} \simeq \mathscr{D}_{\mathcal{A}}^{B} \simeq \mathscr{D}_{\mathcal{A}}^{B'} \simeq \mathscr{D}_{\mathcal{A}^{B'}}.$$

Specializing to the Grothendieck abelian category of modules over a ring R and assuming that $B = Q$, $B' = Q'$ are quivers with finitely many vertices, we obtain equivalences

$$\mathscr{D}_{RQ} \simeq \mathscr{D}_{RQ'}$$

of the derivators of the respective path algebras. Since equivalences of derivators are exact, this yields exact equivalences of derived categories

$$D(RQ) \overset{\Delta}{\simeq} D(RQ'),$$

showing that strongly stably equivalent quivers are derived equivalent over arbitrary rings. A priori, however, it is a much stronger result if we know that two quivers are strongly stably equivalent, since this means that the quivers have the same homotopy theories of abstract representations. We expand a bit on this in Section 10.

3. Review of homotopy exact squares

In this section we review some results concerning the calculus of homotopy exact squares. This calculus is arguably the most important technical tool in the theory of derivators and it allows us to establish many useful manipulation rules for Kan extensions in derivators. For more details, see for example [Ayoub 2007; Maltsiniotis 2012; Groth 2013; Groth et al. 2014b; Groth and Šťovíček 2016c].

To begin with let us consider a natural transformation $\alpha : up \to vq$ living in a square of small categories

$$\begin{tikzcd} D \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\swarrow_{\alpha}"] & A \arrow[d, "u"] \\ B \arrow[r, "v"'] & C \end{tikzcd} \tag{3.1}$$

The square (3.1) is *homotopy exact* if one of the canonical mates

$$q_! p^* \to q_! p^* u^* u_! \xrightarrow{\alpha^*} q_! q^* v^* v_! \to v^* u_! \quad \text{and} \tag{3.2}$$

$$u^* v_* \to p_* p^* u^* v_* \xrightarrow{\alpha^*} p_* q^* v^* v_* \to p_* q^* \tag{3.3}$$

is a natural isomorphism. It turns out that (3.2) is an isomorphism if and only if (3.3) is an isomorphism.

Using this terminology, note that axiom (Der4) from Definition 2.3 precisely says that slice squares (2.2) are homotopy exact. Although it may seem from the definition that the notion of homotopy exactness depends on the theory of derivators, this is only seemingly the case. Homotopy exact squares can be characterized by means of the classical homotopy theory of (diagrams of) topological spaces. In fact, a square is homotopy exact if and only if the canonical mate is an isomorphism for the homotopy derivator of topological spaces, and this even admits a combinatorial reformulation; see [Groth et al. 2014b, §3].

For later reference, we collect a few additional examples of homotopy exact squares and make explicit what they tell us about Kan extensions.

Examples 3.4. (i) *Kan extensions along fully faithful functors are fully faithful.* If $u : A \to B$ is fully faithful, then the square

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & A \arrow[d, "u"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

is homotopy exact, which is to say that the unit $\eta : \mathrm{id} \to u^*u_!$ and the counit $\epsilon : u^*u_* \to \mathrm{id}$ are isomorphisms [Groth 2013, Proposition 1.20]. Thus,

$$u_!, u_* : \mathscr{D}(A) \to \mathscr{D}(B)$$

are fully faithful.

(ii) *Kan extensions and restrictions in unrelated variables commute.* Given functors $u : A \to B$ and $v : C \to D$, the commutative square

$$\begin{tikzcd} A \times C \arrow[r, "u\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times v"'] & B \times C \arrow[d, "\mathrm{id}\times v"] \\ A \times D \arrow[r, "u\times\mathrm{id}"'] & B \times D \end{tikzcd}$$

is homotopy exact [Groth 2013, Proposition 2.5]. Thus, the canonical mate transformation $(\mathrm{id} \times v)_!(u \times \mathrm{id})^* \to (u \times \mathrm{id})^*(1 \times v)_!$ is an isomorphism, and similarly for right Kan extensions.

(iii) *Right adjoint functors are homotopy final.* If $u : A \to B$ is a right adjoint, then the square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\swarrow_{\mathrm{id}}"] & B \arrow[d, "\pi_B"] \\ \mathbb{1} \arrow[r, "\mathrm{id}"'] & \mathbb{1} \end{tikzcd}$$

is homotopy exact, i.e., the canonical mate $\operatorname{colim}_A u^* \to \operatorname{colim}_B$ is an isomorphism [Groth 2013, Proposition 1.18]. In particular, if $b \in B$ is a terminal object, then there is a canonical isomorphism $b^* \cong \operatorname{colim}_B$.

(iv) *Homotopy exact squares are compatible with pasting.* Since the passage to the canonical mates (3.2) and (3.3) is functorial with respect to horizontal and vertical pasting, such pastings of homotopy exact squares are again homotopy exact [Groth 2013, Lemma 1.14].

It follows from Examples 3.4(ii) that there are Kan extension morphisms of derivators. In fact, given a derivator $\mathscr{D}$ and a functor $u : A \to B$, there are adjunctions of derivators given by *parametrized Kan extensions*,

$$(u_!, u^*) : \mathscr{D}^A \rightleftarrows \mathscr{D}^B \quad \text{and} \quad (u^*, u_*) : \mathscr{D}^B \rightleftarrows \mathscr{D}^A.$$

If u is fully faithful, then $u_!, u_* : \mathscr{D}^A \to \mathscr{D}^B$ are fully faithful morphisms of derivators and as such they induce equivalences onto their respective essential images. In particular, these essential images are again derivators [Groth and Šťovíček 2016c, §3].

The point of the following lemma is that to check whether an object $X \in \mathscr{D}^B$ is in the essential image of $u_!$, it suffices to test objects in $B - u(A)$ only.

Lemma 3.5 [Groth 2013, Lemma 1.21]. *Let $\mathscr{D}$ be a derivator and $u : A \to B$ a fully faithful functor between small categories. A coherent diagram $X \in \mathscr{D}^B$ lies in the essential image of $u_! : \mathscr{D}^A \to \mathscr{D}^B$ if and only if $\epsilon_b : u_! u^*(X)_b \to X_b$ is an isomorphism for all $b \in B - u(A)$.*

This lemma takes a particular simple form for certain Kan extensions in pointed derivators. Recall that a fully faithful functor $u : A \to B$ is a *sieve* if for every morphism $b \to u(a')$ in B with target in the image of u it follows that $b = u(a)$ for some $a \in A$. There is the dual notion of a *cosieve*.

Proposition 3.6 [Groth 2013, Proposition 3.6]. *Let $\mathscr{D}$ be a pointed derivator and $u : A \to B$ a sieve. The morphism $u_* : \mathscr{D}^A \to \mathscr{D}^B$ is fully faithful and $X \in \mathscr{D}^B$ lies in the essential image of u_* if and only if $u_b \cong 0$ for all $b \in B - u(A)$.*

We refer to right Kan extension morphisms along sieves as *right extensions by zero*. Dually, left Kan extensions along cosieves are *left extensions by zero*.

Remark 3.7. If $\mathscr{D}$ is not pointed, then Proposition 3.6 yields right extensions by terminal objects and left extensions by initial objects in the obvious sense [Groth 2013, Proposition 1.23].

By Examples 3.4 there is an easy criterion guaranteeing that Kan extensions are fully faithful. The case of restrictions is more subtle. Inspired by the notion of a *homological epimorphism* introduced by Geigle and Lenzing [1991, §4], there is the following definition; see [Groth and Šťovíček 2016b, §6] and, in particular, Remark 6.4 in [loc. cit.].

Definition 3.8. A functor $u : A \to B$ is a *homotopical epimorphism* if for every derivator $\mathscr{D}$ the restriction functor $u^* : \mathscr{D}(B) \to \mathscr{D}(A)$ is fully faithful.

If u is a homotopical epimorphism then $u^* : \mathscr{D}^B \to \mathscr{D}^A$ induces an equivalence onto its essential image. Basic examples and closure properties are collected in [Groth and Šťovíček 2016b, §6–7]. Here it suffices to note that $u : A \to B$ is a homotopical epimorphism if and only if the square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "u"'] & B \arrow[d, "\mathrm{id}"] \\ B \arrow[r, "\mathrm{id}"'] & B \end{tikzcd}$$

is homotopy exact. We will get back to this in Sections 6 and 8.

4. A pictorial guide to general reflection morphisms

In this section we describe the strategy behind the construction of the general reflection morphisms as carried out in Sections 7 and 9. While some main steps follow the lines of the construction in [Groth and Šťovíček 2016b, §5], they have to be adapted significantly to cover the more general class of examples we consider in this paper.

Let $C \in \mathcal{C}at$ and let C^- be the category obtained from C by freely attaching a new object v together with n morphisms from v to objects in C; see Figure 1. Performing a similar construction but this time adding morphisms pointing to v we obtain the category C^+. Thus, the categories C^-, C^+ are obtained from C by attaching a source and sink, respectively, to the same objects in C, and the picture to have in mind is as in Figure 1.

One of our main goals is to show that for *every* small category C the categories C^- and C^+ are strongly stably equivalent, i.e., that for every stable derivator $\mathscr{D}$ there is an equivalence $\mathscr{D}^{C^-} \simeq \mathscr{D}^{C^+}$ which is pseudonatural with respect to exact morphisms (Definition 2.12). Mimicking the classical construction of reflection functors [Bernšteĭn et al. 1973], we obtain reflection morphisms $s^- : \mathscr{D}^{C^-} \to \mathscr{D}^{C^+}$ and $s^+ : \mathscr{D}^{C^+} \to \mathscr{D}^{C^-}$, which we show to define such a strong stable equivalence. As a first approximation, the rough strategy behind the construction of s^- and s^+ is as follows (see Figure 2).

(i) Take a representation of C^- and separate the morphisms adjacent to the new source by inserting new morphisms, one point being that the shape D^- of this new representation contains an isomorphic copy of the source of valence n. Moreover, we know precisely which representations of D^- arise this way, namely those which populate the new morphisms by isomorphisms. If we

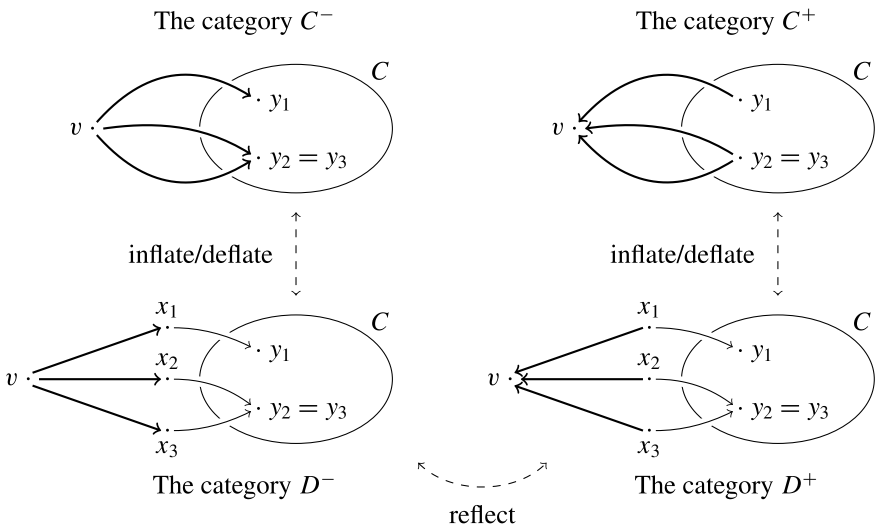


Figure 2. Rough strategy behind construction of reflection functors.

write $\mathscr{D}^{D^-,\mathrm{ex}} \subseteq \mathscr{D}^{D^-}$ for the full subprederivator spanned by such representations, then this yields an equivalence $\mathscr{D}^{C^-} \simeq \mathscr{D}^{D^-,\mathrm{ex}}$ (thereby also implying that $\mathscr{D}^{D^-,\mathrm{ex}}$ is a derivator).

(ii) Show that the reflection morphisms for sources and sinks of valence n as constructed in [Groth and Šťovíček 2016b] yield similar reflection morphisms in this more general situation. Thus, if D^+ is the category obtained from D^- by turning the source into a sink, then we construct certain morphisms of derivators $\mathscr{D}^{D^-} \to \mathscr{D}^{D^+}$, which restrict to equivalences $\mathscr{D}^{D^-,\mathrm{ex}} \to \mathscr{D}^{D^+,\mathrm{ex}}$. We expand on this step further below.

(iii) Finally, it is sufficient to show that we can restrict representations of D^+ to representations of C^+, thereby possibly identifying some of the sources of morphisms adjacent to the new sink. If we only consider representations of D^+ satisfying certain exactness properties, then this step induces an equivalence of derivators. Note that the situation in this step differs from the one in step (i) since here the arrows point in different directions. It turns out that this step is not formally dual and, instead, is more involved than the similar looking first step.

The first and third steps are taken care of in Sections 8–9, while the second step is addressed in Sections 5–7. We now expand on this second step, which performs the actual reflection and is motivated by the classical reflection functors from representation theory; see [Gabriel 1972; Bernšteĭn et al. 1973; Happel 1986] and also the discussion in [Groth and Šťovíček 2016b, §5]. Let $v \to x_i$, $i = 1, \ldots, n$, be the morphisms in D^- which are adjacent to the source v. Given an abstract representation $X \in \mathscr{D}^{D^-}$, we consider the morphism $X_v \to \bigoplus_{i=1}^n X_{x_i}$ induced by the structure maps and pass to its cofiber. However, in order to obtain a representation of the reflected category D^+, we have to take some care in setting up coherent biproduct diagrams appropriately.

To begin with, we recall from [Groth and Šťovíček 2016b, §4 and §7] that finite biproduct objects in stable derivators can be modeled by n-cubes of length two. In more detail, let us consider the diagram in $\mathcal{C}at$

$$n \cdot \mathbb{1} = [1]^n_{=n-1} \xrightarrow{i_1} [1]^n_{\geq n-1} \xrightarrow{i_2} [1]^n \xrightarrow{i_3} I \xrightarrow{i_4} [2]^n \xrightarrow{q} R^n, \tag{4.1}$$

in which we ignore the functor $q : [2]^n \to R^n$ for now. The functors i_1, i_2 are the obvious fully faithful inclusion functors, and the composition $i_4 i_3 : [1]^n \to [2]^n$ is the inclusion as the n-cube $[1, 2]^n$, i.e., the convex hull of $(1, \ldots, 1), (2, \ldots, 2) \in [2]^n$. Let $I \subseteq [2]^n$ be the full subcategory spanned by $[1, 2]^n$ and the corners

$$(0, 2, \ldots, 2), \quad (2, 0, 2, \ldots, 2), \quad \ldots, \quad (2, \ldots, 2, 0),$$

and let $i_3 : [1]^n \to I$ and $i_4 : I \to [2]^n$ be the corresponding factorization of $i_4 i_3$. The associated Kan extension morphisms

$$\mathscr{D}^{n\cdot\mathbb{1}} = \mathscr{D}^{[1]^n_{=n-1}} \xrightarrow{(i_1)_*} \mathscr{D}^{[1]^n_{\geq n-1}} \xrightarrow{(i_2)_*} \mathscr{D}^{[1]^n} \xrightarrow{(i_3)_!} \mathscr{D}^I \xrightarrow{(i_4)_*} \mathscr{D}^{[2]^n} \tag{4.2}$$

are fully faithful and the essential image is in the stable case as follows. For every stable derivator $\mathscr{D}$ we denote by $\mathscr{D}^{[2]^n,\mathrm{ex}} \subseteq \mathscr{D}^{[2]^n}$ the full subderivator spanned by the diagrams such that

(i) all subcubes are strongly bicartesian,

(ii) the values at all corners are trivial, and

(iii) the maps $(i_1, \ldots, i_{k-1}, 0, i_{k+1}, \ldots, i_n) \to (i_1, \ldots, i_{k-1}, 2, i_{k+1}, \ldots, i_n)$ are sent to isomorphisms for all $i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n$ and k.

We note that (iii) is a consequence of (i) and (ii) together with isomorphisms being stable under base change [Groth 2013, Proposition 3.12], but it is included here for emphasis. As discussed in [Groth and Šťovíček 2016b, §4] such diagrams model coherent finite biproduct diagrams together with all the inclusion and projection morphisms. The following result justifies referring to $\mathscr{D}^{[2]^n,\mathrm{ex}}$ as a derivator.

Proposition 4.3 [Groth and Šťovíček 2016b, Proposition 4.9]. *Let $\mathscr{D}$ be a stable derivator and $n \geq 2$. The morphisms (4.2) are fully faithful and induce an equivalence $\mathscr{D}^{n\cdot\mathbb{1}} \simeq \mathscr{D}^{[2]^n,\mathrm{ex}}$, which is pseudonatural with respect to exact morphisms. The derivator $\mathscr{D}^{[2]^n,\mathrm{ex}}$ is the* derivator of biproduct n-cubes.

Note that property (iii) of the characterization of biproduct n-cubes suggests that such diagrams arise via restriction from a "larger shape where the length two morphisms are invertible". This turns out to be true and will be taken care of by the remaining functor in (4.1).

In fact, let $p : [2] \to R$ be the localization functor inverting the length two morphism $0 \to 2$ in $[2]$, so that R corepresents pairs of composable morphisms such that the composition is an isomorphism; see [Groth and Šťovíček 2016b, §7] for a precise description of R. We know that p is a homotopical epimorphism [Groth and Šťovíček 2016b, Proposition 7.3], and it is completely formal to see that the same is true for the n-fold product $q : [2]^n \to R^n$.

Corollary 4.4 [Groth and Šťovíček 2016b, Corollary 7.4]. *Let $\mathscr{D}$ be a derivator and $n \geq 1$. The functor $q : [2]^n \to R^n$ is a homotopical epimorphism and $q^* : \mathscr{D}^{R^n} \to \mathscr{D}^{[2]^n}$ induces an equivalence onto the full subderivator of $\mathscr{D}^{[2]^n}$ spanned by all diagrams X such that*

$$X_{i_1,\dots,i_{k-1},0,i_{k+1},\dots,i_n} \to X_{i_1,\dots,i_{k-1},2,i_{k+1},\dots,i_n}$$

is an isomorphism for all $i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n$ and k.

Thus, in the stable case, there is the following result concerning the morphisms

$$\mathscr{D}^{n\cdot\mathbb{1}} = \mathscr{D}^{[1]^n_{=n-1}} \xrightarrow{(i_1)_*} \mathscr{D}^{[1]^n_{\geq n-1}} \xrightarrow{(i_2)_*} \mathscr{D}^{[1]^n} \xrightarrow{(i_3)_!} \mathscr{D}^{I} \xrightarrow{(i_4)_*} \mathscr{D}^{[2]^n} \xrightarrow{q_!} \mathscr{D}^{R^n}. \tag{4.5}$$

Let $\mathscr{D}^{R^n,\mathrm{ex}} \subseteq \mathscr{D}^{R^n}$ be the full subderivator spanned by all $X \in \mathscr{D}^{R^n}$ such that q^*X is a biproduct n-cube, i.e., such that $q^*X \in \mathscr{D}^{[2]^n,\mathrm{ex}}$.

Corollary 4.6 [Groth and Šťovíček 2016b, Corollary 7.5]. *Let $\mathscr{D}$ be a stable derivator and $n \geq 2$. The morphisms (4.5) are fully faithful and induce an equivalence $\mathscr{D}^{n\cdot\mathbb{1}} \simeq \mathscr{D}^{R^n,\mathrm{ex}}$, which is pseudonatural with respect to exact morphisms. The derivator $\mathscr{D}^{R^n,\mathrm{ex}}$ is the* derivator of invertible biproduct n-cubes.

With this preparation we now describe in more detail the second step in the above strategy behind the construction of general reflection morphisms (see Figure 3). The above-mentioned morphism $\mathscr{D}^{D^-} \to \mathscr{D}^{D^+}$ is roughly obtained as follows.

(i) Starting with an abstract representation $X \in \mathscr{D}^{D^-}$, we glue in a coherent biproduct n-cube centered at $\bigoplus_{i=1}^n X_{x_i}$. The corresponding morphism $\mathscr{D}^{D^-} \to \mathscr{D}^{E_1^-}$ is obtained by adapting the respective morphisms in (4.2), and this step relies on the discussion of "free oriented gluing constructions" in Section 5.

(ii) Next, using a variant of the functor $q : [2]^n \to R^n$, we invert the biproduct n-cubes, thereby constructing a restriction morphism $\mathscr{D}^{E_2^-} \to \mathscr{D}^{E_1^-}$. To understand this morphism, we study the compatibility of homotopical epimorphisms with "free oriented gluing constructions"; see Section 6.

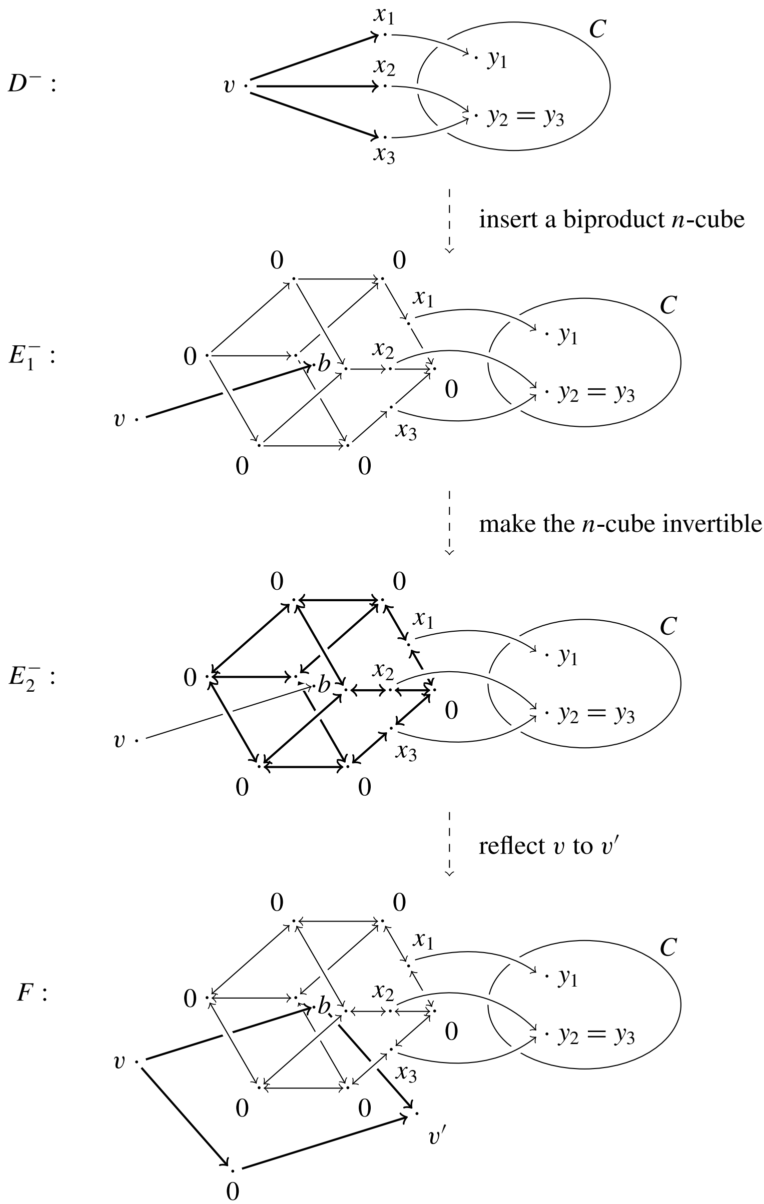


Figure 3. Intermediate steps in the construction of reflection functors. Changes from step to step are drawn in bold.

(iii) As a next step, given a representation $X \in \mathscr{D}^{E_2^-}$, we extend it by passing from $X_v \to \bigoplus_{i=1}^n X_{x_i}$ to the corresponding cofiber square. To get our hands on the resulting morphism of derivators $\mathscr{D}^{E_2^-} \to \mathscr{D}^F$ we again apply results from Section 5.

(iv) The steps so far yield a morphism of derivators $\mathscr{D}^{D^-} \to \mathscr{D}^F$. One observes that the category F also comes with a functor $D^+ \to F$. Dualizing the steps so far, we show that there is a similar morphism of derivators $\mathscr{D}^{D^+} \to \mathscr{D}^F$, and that the span $\mathscr{D}^{D^-} \to \mathscr{D}^F \leftarrow \mathscr{D}^{D^+}$ restricts to the desired equivalence.

These steps are carried out in detail in Section 7, and combined with the above inflation and deflation steps, they are shown in Section 9 to yield the intended general reflection morphisms $\mathscr{D}^{C^-} \to \mathscr{D}^{C^+}$ and $\mathscr{D}^{C^+} \to \mathscr{D}^{C^-}$, showing that the categories C^- and C^+ are strongly stably equivalent; see Theorem 9.11. In the following two sections we first develop some of the necessary techniques.

5. Free oriented gluing constructions

In this section we study in more detail the gluing construction alluded to in Section 4. In particular, we see that these gluing constructions behave well with Kan extension morphisms. The results of this section and Section 6 are central to the construction of the reflection morphisms in Section 7.

To begin, let us consider the following construction (which is a special case of pushouts of small categories; see the Appendix).

Construction 5.1. Let $A_1, A_2 \in \mathcal{C}at$ be small categories. Let $n \in \mathbb{N}$, $s_1, \ldots, s_n \in A_1$, and $t_1, \ldots, t_n \in A_2$. Moreover, let $[1]$ again be the poset $(0 < 1)$ considered as a category. The category $[1]$ comes with a functor $(0, 1) : \mathbb{1} \sqcup \mathbb{1} \to [1]$ classifying the objects 0 and 1. Using this notation, we define the category A to be the pushout

$$\begin{tikzcd} \coprod_{i=1,\ldots,n} \mathbb{1} \sqcup \mathbb{1} \arrow[r, "s \sqcup t"] \arrow[d] & A_1 \sqcup A_2 \arrow[d, "(i_1, i_2)"] \\ \coprod_{i=1,\ldots,n} [1] \arrow[r, "\beta"'] & A \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{5.2}$$

and call it the *free oriented gluing construction* associated to (A_1, A_2, s, t). Given $k \in \{1, \ldots, n\}$, we denote the image of the morphism $0 \to 1$ in the k-th copy of $[1]$ by $\beta_k : i_1(s_k) \to i_2(t_k)$.

This construction clearly enjoys the following properties.

Lemma 5.3. *In the situation of (5.2) the following properties are satisfied.*

(i) *The functors $i_1 : A_1 \to A$ and $i_2 : A_2 \to A$ are fully faithful with disjoint images.*

(ii) *Every object in A lies either in $i_1(A_1)$ or in $i_2(A_2)$.*

(iii) *There are no morphisms in A from an object in $i_2(A_2)$ to an object in $i_1(A_1)$.*

(iv) *For every morphism* $f : i_1(a_1) \to i_2(a_2)$ *there is a unique* $k \in \{1, \ldots, n\}$ *and a unique factorization of* f *as*

$$f : i_1(a_1) \xrightarrow{i_1(f')} i_1(s_k) \xrightarrow{\beta_k} i_2(t_k) \xrightarrow{i_2(f'')} i_2(a_2).$$

Proof. This is immediate from the construction of the pushout category in (5.2) (see also Lemma A.12). □

Definition 5.4. We refer to the factorizations in Lemma 5.3(iv) as *standard factorizations* and call the unique number $k \in \{1, \ldots, n\}$ the *type* of f.

Example 5.5. Let $C \in \mathcal{C}at$ and let $y_1, \ldots, y_n \in C$ be a list of objects (possibly with repetition). Let $t = y : n \cdot \mathbb{1} \to C$ be the corresponding functor. Moreover, note that $[1]^n_{\leq 1}$ is the source of valence n which comes with the functor $s : n \cdot \mathbb{1} \to [1]^n_{\leq 1}$ classifying the objects different from the source. The pushout square

$$\begin{tikzcd}
\coprod_{i=1,\ldots,n} \mathbb{1} \sqcup \mathbb{1} \arrow[r, "s \sqcup t"] \arrow[d] & {[1]^n_{\leq 1}} \sqcup C \arrow[d] \\
\coprod_{i=1,\ldots,n} [1] \arrow[r] & D^-
\end{tikzcd}$$

exhibits the category D^- showing up in the outline of the strategy of the construction of general reflection morphisms (see Figure 2) as an instance of a free oriented gluing construction. There is a similar description of the category D^+ in Figure 2.

Example 5.6. As a special case of Construction 5.1 we recover the one-point extensions of [Groth and Šťovíček 2016b, §8]. In fact, this is the case for the free oriented gluing construction associated to (A_1, A_2, s, t) in the case where $n = 1$ and A_1 or A_2 is the terminal category $\mathbb{1}$.

Construction 5.7. We now consider two free oriented gluing constructions A and A' which are associated to (A_1, A_2, s, t) and (A'_1, A'_2, s', t'), respectively. Let us assume that the second summands $A_2 = A'_2$ as well as the targets $t = t'$ agree while there is a functor $u_1 : A_1 \to A'_1$ such that $s' = u_1 \circ s$. This situation may be summarized by the following commutative diagram:

$$\begin{tikzcd}
\coprod_{i=1,\ldots,n} \mathbb{1} \sqcup \mathbb{1} \arrow[r, "s \sqcup t"] \arrow[d] \arrow[rrd, bend right, "s' \sqcup t"'] & A_1 \sqcup A_2 \arrow[rd, "u_1 \sqcup \mathrm{id}"] \arrow[dd] & \\
 & & A'_1 \sqcup A_2 \arrow[dd, "{(i'_1, i'_2)}"] \\
\coprod_{i=1,\ldots,n} [1] \arrow[r, "\beta"'] \arrow[rrd, bend right, "\beta'"'] & A \arrow[rd, "u"] & \\
 & & A'
\end{tikzcd} \tag{5.8}$$

Here, both the front and the back face are the pushout squares defining the respective gluing constructions and $u\colon A\to A'$ is induced by the universal property of the back pushout square. We refer to the situation described in (5.8) as *two compatible (free oriented) gluing constructions* (see Figure 4 for an illustration).

Combining the face on the right in (5.8) with the inclusions of the respective first summands we obtain a commutative square of small categories, which we consider in two ways as a square populated by the identity transformation:

$$
\begin{tikzcd}
A_1 \arrow[r, "i_1"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\mathrm{id}\,\Nearrow"] & A \arrow[d, "u"] \\
A_1' \arrow[r, "i_1'"'] & A'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A_1 \arrow[r, "i_1"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\Swarrow\,\mathrm{id}"] & A \arrow[d, "u"] \\
A_1' \arrow[r, "i_1'"'] & A'
\end{tikzcd}
\tag{5.9}
$$

The following proposition guarantees that Kan extensions along u and Kan extensions along u_1 interact as expected.

Proposition 5.10. *If* (5.8) *are two compatible gluing constructions, then both squares in* (5.9) *are homotopy exact, i.e., in every derivator the canonical mates*

$$(i_1')^*u_*\to(u_1)_*(i_1)^*\quad\textit{and}\quad(u_1)_!(i_1)^*\to(i_1')^*u_!$$

are isomorphisms.

Proof. We first show that the square on the left in (5.9) is homotopy exact, and show that the canonical mate $(i_1)_!u_1^*\to u^*(i_1')_!$ is an isomorphism. Since the functors $i_1\colon A_1\to A$ and $i_2\colon A_2\to A$ are jointly surjective, it suffices by (Der2) to show

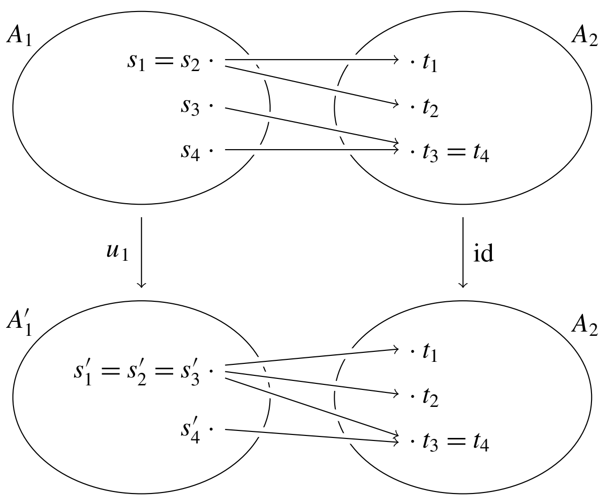


Figure 4. Two compatible (free oriented) gluing constructions.

that the restrictions of the canonical mate with i_1^*, i_2^* are isomorphisms. For the first case we consider the pastings

$$\begin{tikzcd} A_1 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swArrow_{\mathrm{id}}"] & A_1 \arrow[r, "u_1"] \arrow[d, "i_1"] \arrow[dr, phantom, "\swArrow_{\mathrm{id}}"] & A_1' \arrow[d, "i_1'"] \\ A_1 \arrow[r, "i_1"'] & A \arrow[r, "u"'] & A' \end{tikzcd} \quad = \quad \begin{tikzcd} A_1 \arrow[r, "u_1"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swArrow_{\mathrm{id}}"] & A_1' \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\swArrow_{\mathrm{id}}"] & A_1' \arrow[d, "i_1'"] \\ A_1 \arrow[r, "u_1"'] & A_1' \arrow[r, "i_1'"'] & A' \end{tikzcd}$$

Since i_1, i_1' are fully faithful, the square to the very left and the square to the very right are homotopy exact (Examples 3.4). Moreover, the second square from the right is constant and hence homotopy exact. The functoriality of mates with respect to pasting implies that the restricted canonical mate $i_1^*(i_1)_!u_1^* \to i_1^*u^*(i_1')_!$ is an isomorphism.

Now, given an object $i_2(a_2) \in A$ we consider the pasting

$$\begin{tikzcd} \coprod_k A_2(t_k, a_2) \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\swArrow_{\mathrm{id}}"] & (i_1/i_2a_2) \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\swArrow"] & A_1 \arrow[r, "u_1"] \arrow[d, "i_1"] \arrow[dr, phantom, "\swArrow_{\mathrm{id}}"] & A_1' \arrow[d, "i_1'"] \\ \mathbb{1} \arrow[r, "\mathrm{id}"'] & \mathbb{1} \arrow[r, "i_2a_2"'] & A \arrow[r, "u"'] & A' \end{tikzcd}$$

in which the square in the middle is a slice square. The functor r sends a morphism $t_k \to a_2$ to the pair $(s_k, i_1s_k \to i_2t_k \to i_2a_2) \in (i_1/i_2a_2)$. Using Lemma 5.3 the reader can easily check that this functor is a right adjoint so that the above square on the left is homotopy exact by the homotopy finality of right adjoints (Examples 3.4). Note that the above pasting agrees with the pasting

$$\begin{tikzcd} \coprod_k A_2(t_k, a_2) \arrow[r, "r'"] \arrow[d] \arrow[dr, phantom, "\swArrow_{\mathrm{id}}"] & (i_1'/i_2'a_2) \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\swArrow"] & A_1' \arrow[d, "i_1'"] \\ \mathbb{1} \arrow[r, "\mathrm{id}"'] & \mathbb{1} \arrow[r, "i_2'a_2"'] & A' \end{tikzcd}$$

given by a slice square and a similarly defined right adjoint functor r'. The functoriality of mates with pasting hence implies that $(i_1)_!u_1^* \to u^*(i_1')^*$ is an isomorphism at i_2a_2.

We now turn to the second claim and show that the canonical mate $u^*(i_1')_* \to (i_1)_*u_1^*$ is an isomorphism. Using again that i_1, i_2 are jointly surjective, it suffices to show that the corresponding restrictions of the canonical mate are invertible. Since i_1, i_1' are sieves, both right Kan extensions are right extensions by terminal objects (Remark 3.7), and the above canonical mate is hence automatically an isomorphism on objects of the form i_2a_2. It remains to show that its restriction along i_1^* is an isomorphism and for that purpose we consider the diagram

$$\begin{tikzcd} A_1 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow\mathrm{id}"] & A_1 \arrow[d, "i_1"] \\ A_1 \arrow[r, "i_1"'] \arrow[d, "u_1"'] \arrow[dr, phantom, "\swarrow\mathrm{id}"] & A \arrow[d, "u"] \\ A_1' \arrow[r, "i_1'"'] & A' \end{tikzcd} \quad = \quad \begin{tikzcd} A_1 \arrow[r, "\mathrm{id}"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\swarrow\mathrm{id}"] & A_1 \arrow[d, "u_1"] \\ A_1' \arrow[r, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow\mathrm{id}"] & A_1' \arrow[d, "i_1'"] \\ A_1' \arrow[r, "i_1'"'] & A' \end{tikzcd}$$

Using the same arguments as in the first part of the proof, we conclude that $i_1^* u^* (i_1')_* \to i_1^*(i_1)_* u_1^*$ is an isomorphism, concluding the proof. □

In the case that u_1 and, hence, u are fully faithful, there is the following convenient result.

Corollary 5.11. *Let (5.8) be two compatible gluing constructions such that u_1 and, hence, u are fully faithful, and let $\mathscr{D}$ be a derivator.*

(i) *The right Kan extension morphism $u_* : \mathscr{D}^A \to \mathscr{D}^{A'}$ is fully faithful with essential image given by those X such that $(i_1')^* X$ lies in the essential image of $(u_1)_* : \mathscr{D}^{A_1} \to \mathscr{D}^{A_1'}$.*

(ii) *The left Kan extension morphism $u_! : \mathscr{D}^A \to \mathscr{D}^{A'}$ is fully faithful with essential image given by those X such that $(i_1')^* X$ lies in the essential image of $(u_1)_! : \mathscr{D}^{A_1} \to \mathscr{D}^{A_1'}$.*

Proof. We give a proof of (i); the case of (ii) is dual. Since both u_1 and u are fully faithful, the respective right Kan extension morphisms are fully faithful (Examples 3.4). Thus, the corresponding essential images consist precisely of those diagrams on which the respective units $\eta_1 : \mathrm{id} \to (u_1)_* u_1^*$ and $\eta : \mathrm{id} \to u_* u^*$ are isomorphisms. To express this differently we consider the following pastings:

$$\begin{tikzcd} A_1 \arrow[r, "i_1"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A \arrow[r, "u"] \arrow[d, "u"] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A' \arrow[d, "="] \\ A_1' \arrow[r, "i_1'"'] & A' \arrow[r, "="'] & A' \end{tikzcd} \quad = \quad \begin{tikzcd} A_1 \arrow[r, "u_1"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A_1' \arrow[r, "i_1'"] \arrow[d, "="] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A' \arrow[d, "="] \\ A_1' \arrow[r, "="'] & A_1' \arrow[r, "i_1'"'] & A' \end{tikzcd}$$

By Lemma 3.5, $X \in \mathscr{D}^{A'}$ lies in the essential image of u_* if and only if $(i_1')^* \eta$ is an isomorphism on X. Using the compatibility of mates with pasting and the homotopy exactness of the square to the very left (Proposition 5.10), this is the case if and only if the canonical mate associated to the pasting on the left is an isomorphism on X. But since the above two pastings agree, this is the case if and only if the canonical mate of the pasting on the right is an isomorphism on X. As the square on the right is constant and hence homotopy exact, this is to say that η_1 is an isomorphism on $(i_1')^*X$, i.e., that $(i_1')^*X$ is in the essential image of $(u_1)_*$ (by an additional application of Lemma 3.5). □

As we shall see in Section 7, the results of this section allow us to add the desired biproduct n-cubes and (co)fiber squares needed for the reflection morphisms. To also be able to pass to the invertible n-cube we include the following section.

6. Gluing constructions and homotopical epimorphisms

In this section we continue the study of free oriented gluing constructions as defined in Section 5 and show that they are compatible with homotopical epimorphisms (Definition 3.8). The goal is to establish Theorem 6.5 showing that if we have a pair of compatible gluing constructions (5.8) such that u_1 is a homotopical epimorphism then so is u. Moreover, the essential images of the corresponding restriction morphisms u_1^* and u^* are related as desired.

In the situation of two compatible gluing constructions (5.8), the respective inclusions of the second summands induce the following commutative square, which we consider as being populated by the identity transformation as indicated in

$$\begin{tikzcd} A_2 \arrow[r, "i_2"] \arrow[d, "="'] & A \arrow[d, "u"] \\ A_2 \arrow[r, "i_2'"'] \arrow[ur, phantom, "\nearrow\mathrm{id}"] & A' \end{tikzcd} \tag{6.1}$$

Proposition 6.2. *Given two compatible oriented gluing constructions as in (5.8), the commutative square (6.1) is homotopy exact.*

Proof. To reformulate the claimed homotopy exactness of the square (6.1), we consider the pasting on the left in

$$\begin{tikzcd} \mathbb{1} \arrow[r, "a_2"] \arrow[d, "="'] & A_2 \arrow[r, "i_2"] \arrow[d, "="'] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "a_2"'] \arrow[ur, phantom, "\nearrow\mathrm{id}"] & A_2 \arrow[r, "i_2'"'] \arrow[ur, phantom, "\nearrow\mathrm{id}"] & A' \end{tikzcd} = \begin{tikzcd} \mathbb{1} \arrow[r, "i_2a_2"] \arrow[d, "="'] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "i_2'a_2"'] \arrow[ur, phantom, "\nearrow\mathrm{id}"] & A' \end{tikzcd} = \begin{tikzcd} \mathbb{1} \arrow[r, "{(i_2a_2,\mathrm{id})}"] \arrow[d, "="'] & (i_2'a_2/u) \arrow[r, "q"] \arrow[d, "\pi"] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "="'] \arrow[ur, phantom, "\nearrow\mathrm{id}"] & \mathbb{1} \arrow[r, "i_2'a_2"'] \arrow[ur, phantom, "\nearrow"] & A' \end{tikzcd}$$

in which the left square is constant and hence homotopy exact. Using (Der2) and the compatibility of mates with pasting we conclude that (6.1) is homotopy exact if and only if the above pasting is homotopy exact for every $a_2 \in A_2$. Note that this pasting is simply the above commutative square in the middle, which in turn can be written as the above pasting on the right. In that pasting, the square on the right is a slice square and hence homotopy exact. The square on the left is given by the functor classifying the initial object $(i_2a_2, \mathrm{id}: i_2'a_2 \to ui_2a_2)$ in the slice category $(i_2'a_2/u)$, and that square is hence homotopy exact by the homotopy initiality of left adjoint functors (Examples 3.4). The compatibility of homotopy exact squares with pasting concludes the proof. □

We again consider two compatible gluing constructions as in (5.8). In that notation, by Proposition 5.10 there is a homotopy exact square

$$\begin{tikzcd} A_1 \arrow[r, "i_1"] \arrow[d, "u_1"'] & A \arrow[d, "u"] \\ A_1' \arrow[r, "i_1'"'] & A' \end{tikzcd}$$

of small categories.

Proposition 6.3. *Given two compatible gluing constructions as in (5.8) such that $u_1 : A_1 \to A_1'$ is a homotopical epimorphism, $u : A \to A'$ is also a homotopical epimorphism.*

Proof. By assumption, $u_1 : A_1 \to A_1'$ is a homotopical epimorphism, i.e., the unit $\eta_1 : \mathrm{id} \to (u_1)_* u_1^*$ is an isomorphism. We have to show that the unit $\eta : \mathrm{id} \to u_*u^*$ is as well. Using that the inclusions $i_1' : A_1' \to A'$ and $i_2' : A_2' \to A'$ are jointly surjective, (Der2) implies that it is enough to show that $(i_1')^*\eta$ and $(i_2')^*\eta$ are isomorphisms. As for the first restriction, let us consider the pasting on the left in

$$\begin{tikzcd} A_1 \arrow[r, "i_1"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A \arrow[r, "u"] \arrow[d, "u"] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A' \arrow[d, "="] \\ A_1' \arrow[r, "i_1'"'] & A' \arrow[r, "="'] & A' \end{tikzcd} = \begin{tikzcd} A_1 \arrow[r, "u_1"] \arrow[d, "u_1"'] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A_1' \arrow[r, "i_1'"] \arrow[d, "="] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A' \arrow[d, "="] \\ A_1' \arrow[r, "="'] & A_1' \arrow[r, "i_1'"'] & A' \end{tikzcd}$$

The square to the left is homotopy exact by Proposition 5.10, and the compatibility of homotopy exact squares with pasting implies that $(i_1')^*\eta$ is an isomorphism if and only if the pasting on the left is homotopy exact. Note that this pasting agrees with the pasting on the right in which the square to the right is constant and hence homotopy exact. Moreover, the homotopy exactness of the square on the left is equivalent to u_1 being a homotopical epimorphism, showing that $(i_1')^*\eta$ is an isomorphism.

In order to show that also the restriction $(i_2')^*\eta$ is an isomorphism, let us consider the pasting on the left in

$$
\begin{tikzcd}
A_2 \arrow[r, "i_2"] \arrow[d, "="'] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A \arrow[r, "u"] \arrow[d, "u"] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A' \arrow[d, "="] \\
A'_2 \arrow[r, "i'_2"'] & A' \arrow[r, "="'] & A'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A_2 \arrow[r, "i'_2"] \arrow[d, "="'] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A' \arrow[d, "="] \\
A'_1 \arrow[r, "i'_2"'] & A'
\end{tikzcd}
$$

Using similar arguments as in the previous case together with the homotopy exactness of the square to the very left (Proposition 6.2), we deduce that $(i'_2)^*\eta$ is an isomorphism if and only if the pasting on the left is homotopy exact. Since this pasting agrees with the constant square on the very right, we conclude by the homotopy exactness of constant squares. □

In the situation of Proposition 6.3, both restriction morphisms $u^* : \mathscr{D}^{A'} \to \mathscr{D}^A$ and $u_1^* : \mathscr{D}^{A'_1} \to \mathscr{D}^{A_1}$ are fully faithful for every derivator $\mathscr{D}$. To show that the essential images are related as desired (see Theorem 6.5) we establish the following result.

Lemma 6.4. *Let (5.8) be two compatible gluing constructions such that $u_1 : A_1 \to A'_1$ is a homotopical epimorphism, and let $\mathscr{D}$ be a derivator. A diagram $X \in \mathscr{D}^A$ lies in the essential image of $u^* : \mathscr{D}^{A'} \to \mathscr{D}^A$ if and only if $i_1^*\epsilon : i_1^* u^* u_* X \to i_1^* X$ is an isomorphism.*

Proof. By Proposition 6.3 the functor $u : A \to A'$ is also a homotopical epimorphism and $u^* : \mathscr{D}^{A'} \to \mathscr{D}^A$ is hence a fully faithful morphism of derivators. A diagram $X \in \mathscr{D}^A$ lies in the essential image of u^* if and only if the counit $\epsilon : u^* u_* X \to X$ is an isomorphism. Using the joint surjectivity of $i_1 : A_1 \to A$ and $i_2 : A_2 \to A$, by (Der2) this is the case if and only if the restricted counits $i_1^*\epsilon$, $i_2^*\epsilon$ are isomorphisms on X. Hence, to conclude the proof it suffices to show that $i_2^*\epsilon$ is always an isomorphism, and to this end we consider the pasting on the left in

$$
\begin{tikzcd}
A_2 \arrow[r, "i_2"] \arrow[d, "="'] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A \arrow[r, "="] \arrow[d, "="] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A \arrow[d, "u"] \\
A_2 \arrow[r, "i_2"'] & A \arrow[r, "u"'] & A'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A_2 \arrow[r, "i_2"] \arrow[d, "="'] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A \arrow[d, "u"] \\
A_2 \arrow[r, "i'_2"'] & A'
\end{tikzcd}
$$

The homotopy exactness of constant squares and the compatibility of canonical mates with pasting implies that $i_2^*\epsilon$ is always an isomorphism if and only if the pasting on the left is homotopy exact. However, this pasting agrees with the square on the right, which is homotopy exact by Proposition 6.2. □

Theorem 6.5. *Given two compatible gluing constructions as in (5.8) such that $u_1 : A_1 \to A'_1$ is a homotopical epimorphism, $u : A \to A'$ is also a homotopical epimorphism. Moreover, $X \in \mathscr{D}^A$ lies in the essential image of $u^* : \mathscr{D}^{A'} \to \mathscr{D}^A$ if and only if $i_1^* X \in \mathscr{D}^{A_1}$ lies in the essential image of $u_1^* : \mathscr{D}^{A'_1} \to \mathscr{D}^{A_1}$.*

Proof. By Proposition 6.3, the functor $u : A \to A'$ is a homotopical epimorphism and $u^* : \mathscr{D}^{A'} \to \mathscr{D}^A$, as a fully faithful morphism of derivators, induces an equivalence onto its essential image. A coherent diagram $X \in \mathscr{D}^A$ lies by Lemma 6.4 in this essential image if and only if $i_1^*\epsilon : i_1^* u^* u_* X \to i_1^* X$ is an isomorphism. But, using the homotopy exactness of constant squares, this is the case if and only if the canonical mate associated to the pasting on the left in

$$\begin{tikzcd} A_1 \arrow[r, "i_1"] \arrow[d, "="'] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A \arrow[r, "="] \arrow[d, "="] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A \arrow[d, "u"] \\ A_1 \arrow[r, "i_1"'] & A \arrow[r, "u"'] & A' \end{tikzcd} \quad = \quad \begin{tikzcd} A_1 \arrow[r, "="] \arrow[d, "="'] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A_1 \arrow[r, "i_1"] \arrow[d, "u_1"] \arrow[dr, phantom, "\mathrm{id}\nearrow"] & A \arrow[d, "u"] \\ A_1 \arrow[r, "u_1"'] & A_1' \arrow[r, "i_1'"'] & A' \end{tikzcd}$$

is an isomorphism on X. Since the above two pastings agree, the compatibility of mates with respect to pasting together with the homotopy exactness of the square to the very right (Proposition 5.10) implies that $X \in \mathscr{D}^A$ lies in the essential image of u^* if and only if the canonical mate $\epsilon_1 i_1^* : u_1^*(u_1)_* i_1^* \to i_1^*$ is an isomorphism on X. Since $u_1^* : \mathscr{D}^{A_1'} \to \mathscr{D}^{A_1}$ is fully faithful, the counit ϵ_1 is an isomorphism on $i_1^* X$ if and only if $i_1^* X$ lies in the essential image of u_1^*. □

In the construction of reflection morphisms in Section 7 we will see that the results of this section allow us to pass from biproduct n-cubes to invertible biproduct n-cubes (compare again with the strategy outlined in Section 4).

7. Reflection morphisms: the separated case

In this section we construct the reflection morphisms in abstract stable derivators and show them to be strong stable equivalences. The strategy behind the construction is described in Section 4. Here we deal only with the part of the construction depicted in the lower half of Figure 2, which is described in more detail in Figure 3. Thus, we shall assume that the source/sink is "separated" from the category C by freely added morphisms. The inflation/deflation steps indicated by the vertical dashed arrows in Figure 2 are postponed to Section 9.

More precisely, the goal is the following. Let $C \in \mathcal{C}at$, and let $y_1, \ldots, y_n \in C$ be objects (not necessarily distinct). We can view this data as a functor $y : n \cdot \mathbb{1} \to C$. We obtain two new categories D^- and D^+ by attaching a source of valence n and a sink of valence n, respectively, to C by means of the free oriented gluing construction in the sense of Section 5 (see the first line of Figure 3). Formally, we consider the two pushout diagrams in $\mathcal{C}at$

$$\begin{tikzcd} \coprod_{i=1}^n \mathbb{1} \sqcup \mathbb{1} \arrow[r, "\mathsf{inc} \sqcup y"] \arrow[d] & ([1]^n_{=n-1})^{\lhd} \sqcup C \arrow[d] \\ \coprod_{i=1}^n [1] \arrow[r, "k"'] & \ulcorner D^- \end{tikzcd} \qquad \begin{tikzcd} \coprod_{i=1}^n \mathbb{1} \sqcup \mathbb{1} \arrow[r, "\mathsf{inc} \sqcup y"] \arrow[d] & ([1]^n_{=n-1})^{\rhd} \sqcup C \arrow[d] \\ \coprod_{i=1}^n [1] \arrow[r, "k"'] & \ulcorner D^+ \end{tikzcd} \qquad (7.1)$$

where inc stands for the obvious inclusions $n \cdot \mathbb{1} \to (n \cdot \mathbb{1})^{\lhd} = ([1]^n_{=n-1})^{\lhd}$ and $n \cdot \mathbb{1} \to (n \cdot \mathbb{1})^{\rhd} = ([1]^n_{=n-1})^{\rhd}$. (Given a small category A, we denote by $A^{\rhd}$ the cocone on A, i.e., the category obtained from A by freely adjoining a new terminal object ∞, and, dually, by $A^{\lhd}$ the cone on A.)

Here we carry out the individual steps of the construction of a strong stable equivalence of D^- and D^+; see Figure 3. Starting with a representation $X \in \mathscr{D}^{D^-}$ in a stable derivator $\mathscr{D}$, this roughly amounts to the following:

(i) Glue in a biproduct n-cube centered at $\bigoplus_{i=1}^n X_{x_i}$.

(ii) Pass to the invertible biproduct n-cube.

(iii) Add a cofiber square to the resulting morphism $X_v \to \bigoplus_{i=1}^n X_{x_i}$.

At the level of shapes this corresponds to considering the first three functors in

$$D^- \to E_1^- \to E_2^- \to F \leftarrow E_2^+ \leftarrow E_1^+ \leftarrow D^+, \tag{7.2}$$

precise definitions of which are given below.

As we discuss further below, the category F is symmetric in the following sense. If we begin with a representation $X \in \mathscr{D}^{D^+}$ and perform similar steps then we end up with a representation of the same category $F \in \mathcal{C}at$. At the level of shapes this amounts to considering the remaining three functors in (7.2).

We now turn to the first step, which essentially amounts to gluing an n-cube $[2]^n$ to D^-, yielding the functor $D^- \to E_1^-$ in (7.2); see again Figure 3. To define this functor, we consider the diagram of small categories

$$\begin{tikzcd}
{[1]^n_{=n-1}} \arrow[r] \arrow[d] & {[1]^n_{\geq n-1}} \arrow[r] \arrow[d] & {[1]^n} \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & {[2]^n} \arrow[d] \\
{([1]^n_{=n-1})^{\lhd}} \arrow[r, "i_1"'] & {([1]^n_{\geq n-1})^{\lhd}} \arrow[r, "i_2"'] & {([1]^n)^{\lhd}} \arrow[r, "i_3"'] & \ulcorner I_1 \arrow[r, "i_4"'] & \ulcorner I_2
\end{tikzcd} \tag{7.3}$$

in which the two pushout squares to the right define the categories I_1, I_2, the top row is as in (4.1), and the two squares to the left are naturality squares. The functor $D^- \to E_1^-$ is obtained by an application of the free oriented gluing construction to the bottom row in (7.3). Thus, we consider the following diagram consisting of pushout squares:

$$\begin{tikzcd}
{([1]^n_{=n-1})^{\lhd} \sqcup C} \arrow[r] \arrow[d] & {([1]^n_{\geq n-1})^{\lhd} \sqcup C} \arrow[r] \arrow[d] & {([1]^n)^{\lhd} \sqcup C} \arrow[r] \arrow[d] & I_1 \sqcup C \arrow[r] \arrow[d] & I_2 \sqcup C \arrow[d] \\
D^- \arrow[r, "j_1"'] & \ulcorner A_1 \arrow[r, "j_2"'] & \ulcorner A_2 \arrow[r, "j_3"'] & \ulcorner A_3 \arrow[r, "j_4"'] & \ulcorner E_1^-
\end{tikzcd} \tag{7.4}$$

Associated to the bottom row in this diagram there are the fully faithful Kan extension morphisms

$$\mathscr{D}^{D^-} \xrightarrow{(j_1)_*} \mathscr{D}^{A_1} \xrightarrow{(j_2)_*} \mathscr{D}^{A_2} \xrightarrow{(j_3)_!} \mathscr{D}^{A_3} \xrightarrow{(j_4)_*} \mathscr{D}^{E_1^-}. \tag{7.5}$$

We note that the category E_1^- comes by definition with a functor

$$l : [2]^n \to I_2 \to E_1^-$$

(see (7.3) and (7.4)). For every stable derivator $\mathscr{D}$ we denote by $\mathscr{D}^{E_1^-,\mathrm{ex}} \subseteq \mathscr{D}^{E_1^-}$ the full subderivator spanned by all $X \in \mathscr{D}^{E_1^-}$ for which the n-cube $l^*X \in \mathscr{D}^{[2]^n}$ is a biproduct n-cube (see Proposition 4.3). The following proposition implies that $\mathscr{D}^{E_1^-,\mathrm{ex}}$ is indeed a derivator.

Proposition 7.6. *Let $\mathscr{D}$ be a stable derivator. The morphisms in (7.5) are fully faithful and induce an equivalence $\mathscr{D}^{D^-} \simeq \mathscr{D}^{E_1^-,\mathrm{ex}}$. This equivalence is pseudonatural with respect to exact morphisms.*

Proof. The first part of this proof is very similar to the proof of Proposition 4.3; see [Groth and Šťovíček 2016b, Proposition 4.9]. We begin by considering the functors in the bottom row of (7.3). Since these functors are fully faithful, the associated Kan extension morphisms

$$\mathscr{D}^{([1]^n_{=n-1})^{\lhd}} \xrightarrow{(i_1)_*} \mathscr{D}^{([1]^n_{\geq n-1})^{\lhd}} \xrightarrow{(i_2)_*} \mathscr{D}^{([1]^n)^{\lhd}} \xrightarrow{(i_3)_!} \mathscr{D}^{I_1} \xrightarrow{(i_4)_*} \mathscr{D}^{I_2} \tag{7.7}$$

are also fully faithful. We now describe the essential images of the respective morphisms, and show that they induce the following pseudonatural equivalences:

(i) Since i_1 is a sieve, the morphism $(i_1)_*$ is right extension by zero and hence induces an equivalence onto the full subderivator of $\mathscr{D}^{([1]^n_{\geq n-1})^{\lhd}}$ defined by this vanishing condition.

(ii) One easily checks that $(i_2)_*$ precisely amounts to adding a strongly cartesian n-cube, hence induces a corresponding equivalence of derivators.

(iii) The functor i_3 is a cosieve and $(i_3)_!$ is hence left extension by zero, yielding an equivalence onto the full subderivator of $\mathscr{D}^{I_1}$ defined by this vanishing condition.

(iv) The morphism $(i_4)_*$ precisely amounts to adding strongly cartesian n-cubes. In fact, this follows as in the case of Proposition 4.3; see [Groth and Šťovíček 2016b, §4] for details.

Now, recall that the functors in the bottom row of (7.4) are obtained from the corresponding functors in the bottom row of (7.3) by the free oriented gluing construction. Hence, by Corollary 5.11 we can describe the respective essential images of the Kan extension morphisms in (7.5) in terms of the essential images of the corresponding morphisms in (7.7). The above explicit description of these latter essential images concludes the proof of the first statement. The pseudonaturality with respect to exact morphisms follows since exact morphisms preserve right and left extensions by zero as well as strongly cartesian and strongly cocartesian n-cubes. $\square$

The next step in this construction consists of inverting the biproduct n-cube $[2]^n$ in E_1^-, yielding the functor $E_1^- \to E_2^-$ in (7.2); see again Figure 3. To give a precise definition of this functor, we begin by observing that the category E_1^- is obtained from $[2]^n$ by two iterated free gluing constructions in the sense of Section 5. In fact, let $E_1 \in \mathcal{C}at$ be defined as the free oriented gluing construction on the left in

$$
\begin{tikzcd}
\coprod_{i=1,\ldots,n} \mathbb{1} \sqcup \mathbb{1} \arrow[r] \arrow[d] & {[2]^n \sqcup C} \arrow[d] \\
\coprod_{i=1,\ldots,n} [1] \arrow[r] & \ulcorner E_1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{1} \sqcup \mathbb{1} \arrow[r] \arrow[d] & \mathbb{1} \sqcup E_1 \arrow[d] \\
{[1]} \arrow[r] & \ulcorner E_1^-
\end{tikzcd}
\tag{7.8}
$$

obtained from $n \cdot \mathbb{1} \cong [1]^n_{=n-1} \to [1]^n \overset{[1,2]^n}{\to} [2]^n$ and $(y_1, \ldots, y_n) : n \cdot \mathbb{1} \to C$. (Here, $[1,2]^n : [1]^n \to [2]^n$ is the n-fold product of the functor $[1] \to [2] : i \mapsto i+1$.) Note that the category E_1^- is simply the free oriented gluing construction associated to the functors $\mathrm{id} : \mathbb{1} \to \mathbb{1}$, and $(1, \ldots, 1) : \mathbb{1} \to [2]^n \to E_1$, as depicted in the pushout square on the right in (7.8). In order to obtain the category E_2^- we now simply replace the n-cube $[2]^n$ by the invertible n-cube R^n, as defined prior to Corollary 4.4. In detail, we define E_2^- as the corresponding two-step free oriented gluing construction described via the pushout squares

$$
\begin{tikzcd}
\coprod_{i=1,\ldots,n} \mathbb{1} \sqcup \mathbb{1} \arrow[r] \arrow[d] & R^n \sqcup C \arrow[d] \\
\coprod_{i=1,\ldots,n} [1] \arrow[r] & \ulcorner E_2
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{1} \sqcup \mathbb{1} \arrow[r] \arrow[d] & \mathbb{1} \sqcup E_2 \arrow[d] \\
{[1]} \arrow[r] & \ulcorner E_2^-
\end{tikzcd}
\tag{7.9}
$$

Finally, the functor $r : E_1^- \to E_2^-$ is obtained by tracing the homotopical epimorphism $q : [2]^n \to R^n$ (Corollary 4.4) through the above constructions, thereby first obtaining a functor $E_1 \to E_2$ and then $r : E_1^- \to E_2^-$ ((7.8) and (7.9) yield two pairs of compatible oriented gluing constructions in the sense of Section 5).

To perform the next step of the construction of reflection functors we now consider the commutative square

$$
\begin{tikzcd}
{[2]^n} \arrow[r, "i"] \arrow[d, "q"'] & E_1^- \arrow[d, "r"] \\
R^n \arrow[r, "j"'] & E_2^-
\end{tikzcd}
$$

to which we apply our results from Section 6.

Proposition 7.10. *The functor $r : E_1^- \to E_2^-$ is a homotopical epimorphism. Furthermore, for every derivator $\mathscr{D}$, a diagram $X \in \mathscr{D}^{E_1^-}$ lies in the essential image of $r^* : \mathscr{D}^{E_2^-} \to \mathscr{D}^{E_1^-}$ if and only if $i^*X \in \mathscr{D}^{[2]^n}$ lies in the essential image of $q^* : \mathscr{D}^{R^n} \to \mathscr{D}^{[2]^n}$.*

Proof. The following diagram expresses that $r : E_1^- \to E_2^-$ is obtained in two steps as a free oriented gluing construction starting with $q : [2]^n \to R^n$:

$$\begin{tikzcd} {[2]^n} \arrow[r] \arrow[d, "q"'] & E_1 \arrow[r] \arrow[d] & E_1^- \arrow[d, "r"] \\ R^n \arrow[r] & E_2 \arrow[r] & E_2^- \end{tikzcd}$$

Since q is a homotopical epimorphism and we have a description of the essential image of $q^* : \mathscr{D}^{[2]^n} \to \mathscr{D}^{R^n}$ (Corollary 4.4), the result follows from two applications of Theorem 6.5. □

The morphism r^* induces an equivalence onto its essential image defined by invertibility conditions (Corollary 4.4). We are interested in the following restriction of this equivalence. Note that the category E_2^- comes by construction with a functor $j : R^n \to E_2^-$ (see (7.9)). For every stable derivator $\mathscr{D}$, we denote by $\mathscr{D}^{E_2^-,\mathrm{ex}} \subseteq \mathscr{D}^{E_2^-}$ the full subderivator spanned by all diagrams $X \in \mathscr{D}^{E_2^-}$ for which the n-cube $j^*X \in \mathscr{D}^{R^n}$ is an invertible biproduct n-cube in the sense of Corollary 4.6. Recall also the definition of the derivator $\mathscr{D}^{E_1^-,\mathrm{ex}}$ as considered in Proposition 7.6.

Corollary 7.11. *Let $\mathscr{D}$ be a stable derivator. The morphism $r^* : \mathscr{D}^{E_2^-} \to \mathscr{D}^{E_1^-}$ induces an equivalence of derivators $\mathscr{D}^{E_2^-,\mathrm{ex}} \simeq \mathscr{D}^{E_1^-,\mathrm{ex}}$ which is pseudonatural with respect to exact morphisms of derivators.*

Proof. This is immediate from Corollary 4.4 and Proposition 7.10. □

The third step in the construction of reflection morphisms amounts to extending the morphisms $X_v \to \bigoplus_{i=1}^n X_{x_i}$ in abstract representations to cofiber squares, as will be made precise by the functor $E_2^- \to F$ in (7.2); see again Figure 3. We recall that cofiber squares in pointed derivators are constructed as follows (see [Groth 2013, §3.3]). Let the functor $[1] \to \square = [1] \times [1]$ classify the top horizontal morphism $(0,0) \to (1,0)$ and let $[1] \xrightarrow{i} \ulcorner \xrightarrow{j} \square$ be the obvious factorization of it. For every pointed derivator $\mathscr{D}$ the corresponding Kan extension morphisms

$$\mathscr{D}^{[1]} \xrightarrow{i_*} \mathscr{D}^{\ulcorner} \xrightarrow{j_!} \mathscr{D}^{\square} \tag{7.12}$$

are fully faithful. Since i is a sieve, i_* is right extension by zero (Proposition 3.6). It follows that (7.12) induces an equivalence of derivators $\mathscr{D}^{[1]} \simeq \mathscr{D}^{\square,\mathrm{ex}}$, where

$\mathscr{D}^{\square,\mathrm{ex}} \subseteq \mathscr{D}^{\square}$ is the full subderivator spanned by the *cofiber squares*, i.e., those coherent squares $X \in \mathscr{D}^{\square}$ having the following properties:

(i) The square vanishes at the lower left corner, $X_{0,1} \cong 0$.

(ii) The square is cocartesian.

This construction is clearly pseudonatural with respect to right exact morphisms. Given a coherent morphism $X = (f : x \to y) \in \mathscr{D}^{[1]}$ the corresponding cofiber square looks like

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \arrow[d, "\mathsf{cof}(f)"] \\ 0 \arrow[r] & z \end{tikzcd}$$

To prepare the corresponding relative construction, we consider the diagram of small categories

$$\begin{tikzcd} \mathbb{1} \arrow[r, "1"] \arrow[d, "{(1,\dots,1)}"'] & {[1]} \arrow[r, "i"] \arrow[d] & \ulcorner \arrow[r, "j"] \arrow[d, "l_1"] & \square \arrow[d, "l_2"] \\ R^n \arrow[r] & B_1 \arrow[r, "i_1"'] & B_2 \arrow[r, "i_2"'] & B \end{tikzcd} \tag{7.13}$$

consisting of pushout squares. The square to the left exhibits B_1 as a one-point extension of R^n (Example 5.6). And the category B is obtained from the invertible n-cube R^n by attaching a new morphism with target the center $(1, \dots, 1) \in R^n$ and a square containing this morphism as top horizontal morphism. (The category F as well as $E_2^- \to F$ in (7.2) will be obtained from (7.13) by a free oriented gluing construction.) We begin by considering a pointed derivator $\mathscr{D}$ and the Kan extension morphisms

$$\mathscr{D}^{B_1} \xrightarrow{(i_1)_*} \mathscr{D}^{B_2} \xrightarrow{(i_2)_!} \mathscr{D}^{B}. \tag{7.14}$$

Let $\mathscr{D}^{B_2,\mathrm{ex}} \subseteq \mathscr{D}^{B_2}$ be the full subderivator spanned by all $X \in \mathscr{D}^{B_2}$ such that $l_1^* X$ vanishes at $(0, 1)$. Similarly, let $\mathscr{D}^{B,\mathrm{ex}} \subseteq \mathscr{D}^{B}$ be the full subderivator spanned by those diagrams $X \in \mathscr{D}^{B}$ such that $l_2^* X$ is a cofiber square.

Lemma 7.15. *Let $\mathscr{D}$ be a pointed derivator.*

(i) *The morphism $(i_1)_*$ is fully faithful and induces $\mathscr{D}^{B_1} \simeq \mathscr{D}^{B_2,\mathrm{ex}}$.*

(ii) *The morphism $(i_2)_!$ is fully faithful with essential image the full subderivator of $\mathscr{D}^{B}$ spanned by all X such that $l_2^* X$ is cocartesian.*

(iii) *The morphisms in (7.14) induce an equivalence $\mathscr{D}^{B_1} \simeq \mathscr{D}^{B,\mathrm{ex}}$.*

These equivalences are pseudonatural with respect to right exact morphisms.

Proof. We leave it to the reader to work out the necessary homotopy (co)finality arguments and apply [Groth 2013, Proposition 3.10]. $\square$

We note that the category E_2^- can be obtained as a free oriented gluing construction from B_1. In fact, associated to the functor

$$n \cdot \mathbb{1} = [1]^n_{=n-1} \longrightarrow [1]^n \xrightarrow{[1,2]^n} [2]^n \xrightarrow{q} R^n \longrightarrow B_1$$

and $y = (y_1, \ldots, y_n) : n \cdot \mathbb{1} \to C$ there is the free oriented gluing construction given by the pushout square on the left in

$$\begin{tikzcd}
\coprod_{i=1,\ldots,n} \mathbb{1} \sqcup \mathbb{1} \arrow[r] \arrow[d] & B_1 \sqcup C \arrow[r] \arrow[d] & B_2 \sqcup C \arrow[r] \arrow[d] & B \sqcup C \arrow[d] \\
\coprod_{i=1,\ldots,n} [1] \arrow[r] & E_2^- \arrow[r, "j_1"'] & F_1 \arrow[r, "j_2"'] & F
\end{tikzcd} \tag{7.16}$$

The remaining two pushout squares are induced by the bottom row in (7.13). Thus, in the terminology of Section 5 we have two pairs of compatible oriented gluing constructions. For every derivator $\mathscr{D}$ the Kan extension morphisms

$$\mathscr{D}^{E_2^-} \xrightarrow{(j_1)_*} \mathscr{D}^{F_1} \xrightarrow{(j_2)_!} \mathscr{D}^F \tag{7.17}$$

are fully faithful. Note that the category F comes with a functor $l : \square \to B \to F$; see (7.13) and (7.16).

Proposition 7.18. *Let $\mathscr{D}$ be a pointed derivator. The morphisms (7.17) are fully faithful and induce an equivalence onto the full subderivator of $\mathscr{D}^F$ spanned by all $X \in \mathscr{D}^F$ such that $l^*X \in \mathscr{D}^\square$ is a cofiber square. This equivalence is pseudonatural with respect to right exact morphisms.*

Proof. Since we are in the context of two pairs of free oriented gluing constructions, this is immediate from two applications of Corollary 5.11 to Lemma 7.15. □

We are interested in the following induced equivalence. Note that associated to the category F there are functors

$$l : \square \to F \quad \text{and} \quad m : R^n \to F;$$

see (7.13) and (7.16). Given a stable derivator $\mathscr{D}$, we denote by $\mathscr{D}^{F,\mathrm{ex}} \subseteq \mathscr{D}^F$ the full subderivator spanned by all $X \in \mathscr{D}^F$ satisfying the following properties:

(i) The square $l^*X \in \mathscr{D}^\square$ is a cofiber square.

(ii) The n-cube $m^*X \in \mathscr{D}^{R^n}$ is an invertible biproduct n-cube.

Recall also the definition of the derivator $\mathscr{D}^{E_2^-,\mathrm{ex}}$ as considered in Corollary 7.11.

Corollary 7.19. *Let $\mathscr{D}$ be a stable derivator. The morphisms (7.17) induce an equivalence of derivators $\mathscr{D}^{E_2^-,\mathrm{ex}} \simeq \mathscr{D}^{F,\mathrm{ex}}$ which is pseudonatural with respect to exact morphisms.*

Proof. This is immediate from Proposition 7.18 and the defining exactness and vanishing conditions of $\mathscr{D}^{E_2^-,\mathrm{ex}}$ and $\mathscr{D}^{F,\mathrm{ex}}$. □

It now suffices to assemble the above individual steps in order to settle the reflection morphisms in the separated case.

Theorem 7.20. *Let $C \in \mathcal{C}at$, let $y_1, \ldots, y_n \in C$ (not necessarily distinct), and let $D^-, D^+ \in \mathcal{C}at$ be as in (7.1). The categories D^- and D^+ are strongly stably equivalent.*

Proof. As discussed at the beginning of this section, the functors in (7.2) correspond to the respective steps in the construction of the strong stable equivalence. Proposition 7.6, Corollary 7.11, and Corollary 7.19 take care of the first three steps. In fact, they show that for every stable derivator $\mathscr{D}$, there are equivalences of derivators

$$\mathscr{D}^{D^-} \simeq \mathscr{D}^{E_1^-,\mathrm{ex}} \simeq \mathscr{D}^{E_2^-,\mathrm{ex}} \simeq \mathscr{D}^{F,\mathrm{ex}}$$

which are pseudonatural with respect to exact morphisms.

If we start with an abstract representation of D^+ instead, then, as indicated by the remaining three functors in (7.2), we can perform similar constructions to again obtain an abstract representation of F. We leave it to the reader to verify that in this way we in fact construct a category isomorphic to F. (The arguments for this are essentially the same as in the case of [Groth and Šťovíček 2016b, Lemma 9.15].) At the level of derivators of representations, this amounts to additional pseudonatural equivalences

$$\mathscr{D}^{D^+} \simeq \mathscr{D}^{E_1^+,\mathrm{ex}} \simeq \mathscr{D}^{E_2^+,\mathrm{ex}} \simeq \mathscr{D}^{F,\mathrm{ex}},$$

which are similar to Proposition 7.6, Corollary 7.11, and Corollary 7.19. These steps amount to gluing in a biproduct n-cube, inverting the n-cube, and adding a fiber square, respectively. Since cofiber squares and fiber squares agree in stable derivators, it follows that the essential image of these three steps is again given by the derivator $\mathscr{D}^{F,\mathrm{ex}}$ as described prior to Corollary 7.19. Putting these pseudonatural equivalences together,

$$\mathscr{D}^{D^-} \simeq \mathscr{D}^{E_1^-,\mathrm{ex}} \simeq \mathscr{D}^{E_2^-,\mathrm{ex}} \simeq \mathscr{D}^{F,\mathrm{ex}} \simeq \mathscr{D}^{E_2^+,\mathrm{ex}} \simeq \mathscr{D}^{E_1^+,\mathrm{ex}} \simeq \mathscr{D}^{D^+},$$

we obtain the desired strong stable equivalence $\mathscr{D}^{D^-} \simeq \mathscr{D}^{D^+}$. □

8. Detection criteria for homotopical epimorphisms

The aim of this section is to establish two simple detection results for homotopical epimorphisms. These will be used in Section 9 to construct reflection morphisms in the general case and thereby to complete the plan from Section 4.

The first criterion is completely straightforward; we show that (co)reflective (co)localizations are homotopical epimorphisms (compare to [Groth and Šťovíček 2016b, Proposition 6.5]).

Proposition 8.1. *Let $(l, r) : A \rightleftarrows B$ be an adjunction of small categories with unit $\eta : \mathrm{id} \to rl$ and counit $\varepsilon : lr \to \mathrm{id}$.*

(i) *For every prederivator $\mathscr{D}$ there is an adjunction*

$$(r^*, l^*, \eta^* : \mathrm{id} \to l^*r^*, \varepsilon^* : r^*l^* \to \mathrm{id}) : \mathscr{D}^A \rightleftarrows \mathscr{D}^B.$$

(ii) *If l is a reflective localization, i.e., r is fully faithful, then l is a homotopical epimorphism. Moreover, $X \in \mathscr{D}^A$ lies in the essential image of l^* if and only if $X_{\eta_a} : X_a \to X_{rla}$ is an isomorphism for all $a \in A - r(B)$.*

(iii) *If r is a coreflective colocalization, i.e., l is fully faithful, then r is a homotopical epimorphism. Moreover, $Y \in \mathscr{D}^B$ lies in the essential image of r^* if and only if $Y_{\varepsilon_b} : Y_{lrb} \to Y_b$ is an isomorphism for all $b \in B - l(A)$.*

Proof. The first statement is immediate from the fact that every prederivator $\mathscr{D}$ defines a 2-functor

$$\mathscr{D}^{(-)} : \mathcal{C}at^{\mathrm{op}} \to \mathcal{PDER} : A \mapsto \mathscr{D}^A$$

and since 2-functors preserve adjunctions. By duality it suffices to establish the second statement. Since r is fully faithful, the counit $\varepsilon : lr \to \mathrm{id}$ is an isomorphism, and hence so is the counit $\varepsilon^* : r^*l^* \to \mathrm{id}$. But this means that $l^* : \mathscr{D}^B \to \mathscr{D}^A$ is fully faithful, i.e., that $l : A \to B$ is a homotopical epimorphism. The essential image of l^* consists precisely of those $X \in \mathscr{D}^A$ such that the unit $\eta^* : X \to l^*r^*X$ is an isomorphism. By (Der2) this is the case if and only if η^*_a is an isomorphism for every $a \in A$. Now, the triangular identity

$$\mathrm{id} = \varepsilon^* r^* \circ r^* \eta^* : r^* \to r^*l^*r^* \to r^*$$

and the invertibility of ε^* implies that $r^*\eta^*$ is an isomorphism. Hence to characterize the essential image of l^* it suffices to check η^* at all objects $a \in A - r(B)$. $\square$

This first criterion is already enough for one of the inflation and deflation steps in Section 9. For the remaining one we establish the following additional criterion, which will be applied to more general localization functors. While these functors do not necessarily admit adjoints, they are still essentially surjective, thereby making the first condition in the coming proposition automatic.

Proposition 8.2. *Let $u : A \to B$ be essentially surjective, let $\mathscr{D}$ be a derivator, and let $u^* : \mathscr{D}^B \to \mathscr{D}^A$ be the restriction morphism. Let us assume further that $\mathscr{E} \subseteq \mathscr{D}^A$ is a full subprederivator such that*

(i) *the essential image $\mathrm{im}(u^*)$ lies in $\mathscr{E}$, i.e., $\mathrm{im}(u^*) \subseteq \mathscr{E} \subseteq \mathscr{D}^A$, and*

(ii) *the unit* $\eta\colon X\to u^*u_!X$ *is an isomorphism for all* $X\in\mathscr{E}$.

Then $u^*\colon\mathscr{D}^B\to\mathscr{D}^A$ *is fully faithful and* $\mathrm{im}(u^*)=\mathscr{E}$. *In particular,* $\mathscr{E}$ *is a derivator.*

Proof. To prove that u^* is fully faithful it suffices to show that $\varepsilon\colon u_!u^*\to\mathrm{id}$ is a natural isomorphism. The assumptions imply that ηu^* is a natural isomorphism. Hence, by the triangular identity

$$\mathrm{id}=u^*\varepsilon\circ\eta u^*\colon u^*\xrightarrow{\eta u^*}u^*u_!u^*\xrightarrow{u^*\varepsilon}u^*,$$

it follows that also $u^*\varepsilon$ is an isomorphism. In order to conclude that ε is a natural isomorphism, it suffices by (Der2) to show that $b^*\varepsilon$ is an isomorphism for every $b\in B$. This follows immediately from the essential surjectivity of u and the fact that $u^*\varepsilon$ is invertible.

Since u^* is fully faithful, its essential image consists precisely of those $X\in\mathscr{D}^A$ such that the unit $\eta\colon X\to u^*u_!X$ is an isomorphism. The assumptions (i) and (ii) immediately imply that this is the case if and only if $X\in\mathscr{E}$. Finally, $\mathscr{E}$ is also a derivator by the invariance of derivators under equivalences. □

Thus, once we make an educated guess of an $\mathscr{E}$ satisfying the above assumptions, we get an equivalence onto $\mathscr{E}$. The relation to homotopical epimorphisms is as follows.

Remark 8.3. In our later applications the subprederivator $\mathscr{E}\subseteq\mathscr{D}^A$ is a full subprederivator $\mathscr{D}^{A,\mathrm{ex}}$ determined by some exactness conditions. Recall from [Groth and Šťovíček 2016c, §3] that such exactness conditions are formalized by certain (co)cones in A to be populated by (co)limiting (co)cones. As a special case this includes the assumption that certain morphisms are populated by isomorphisms.

In such a situation we hence start with a full subprederivator $\mathscr{D}^{A,\mathrm{ex}}\subseteq\mathscr{D}^A$ for *every* derivator $\mathscr{D}$. If the assumptions of Proposition 8.2 are satisfied, then this implies first that $u\colon A\to B$ is a homotopical epimorphism and second that the essential image of u^* is $\mathrm{im}(u^*)=\mathscr{D}^{A,\mathrm{ex}}$.

To be able to apply Proposition 8.2 in specific situations, it is useful to have better control over the adjunction unit $\eta\colon\mathrm{id}\to u^*u_!$.

Construction 8.4. Let $\mathscr{D}$ be a derivator, $A\in\mathcal{C}at$, and let $a\in A$. Associated to the square

$$\begin{tikzcd} \mathbb{1} \arrow[r, "a"] \arrow[d] & A \arrow[d, "\pi_A"] \\ \mathbb{1} \arrow[r] & \mathbb{1} \end{tikzcd}$$

there is the canonical mate

$$a^*\to\mathrm{colim}_A\,. \tag{8.5}$$

As a special case relevant in later applications, given a functor $u\colon A \to B$ and $a \in A$ there is the functor $p\colon (u/ua) \to A$. Whiskering the mate (8.5) in the case of $(a, \mathrm{id}\colon ua \to ua) \in (u/ua)$ with p^* we obtain a canonical map

$$a^* = (a, \mathrm{id}_{ua})^* p^* \to \mathrm{colim}_{(u/ua)}\, p^*. \tag{8.6}$$

Lemma 8.7. *Let $\mathscr{D}$ be a derivator, $u\colon A \to B$, and $a \in A$. The component of the unit $a^*\eta\colon a^* \to a^*u^*u_!$ is isomorphic to $a^* \to \mathrm{colim}_{(u/ua)}\, p^*$ (8.6). In particular, η_a is an isomorphism if and only if this is the case for (8.6).*

Proof. To reformulate that the adjunction unit η_a is an isomorphism we consider the pasting on the left in

$$\begin{tikzcd} \mathbb{1} \arrow[r, "a"] \arrow[d] & A \arrow[r] \arrow[d] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "a"'] & A \arrow[r, "u"'] & B \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "(a,\mathrm{id}_{ua})"] \arrow[d] & (u/ua) \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\swArrow"] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r] & \mathbb{1} \arrow[r, "ua"'] & B \end{tikzcd}$$

in which the square to the left is constant and hence homotopy exact. Note that this pasting agrees with the pasting on the right in which the square to the right is a slice square and hence also homotopy exact. The functoriality of canonical mates with pasting concludes the proof. □

We will later apply the previous lemma in situations in which the slice category admits homotopy final functors from certain simpler shapes. For this purpose we collect the following result.

Lemma 8.8. *Let $u\colon A \to B$ be a homotopy final functor and let $a \in A$.*

(i) *The map $u(a)^* \to \mathrm{colim}_B$ (8.5) is naturally isomorphic to $a^*u^* \to \mathrm{colim}_A\, u^*$, the whiskering of an instance of (8.5) with u^*.*

(ii) *If A admits a terminal object ∞, then the map $a^* \to \mathrm{colim}_A$ (8.5) is naturally isomorphic to $a^* \to \infty^*$.*

Proof. Using the functoriality of canonical mates, for the first statement it suffices to observe that the two pastings

$$\begin{tikzcd} \mathbb{1} \arrow[r, "a"] \arrow[d] & A \arrow[r, "u"] \arrow[d] & B \arrow[d] \\ \mathbb{1} \arrow[r] & \mathbb{1} \arrow[r] & \mathbb{1} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "ua"] \arrow[d] & B \arrow[d] \\ \mathbb{1} \arrow[r] & \mathbb{1} \end{tikzcd}$$

agree and that the square in the middle is homotopy exact by assumption on u. For the second statement it suffices to unravel the definition of (8.5) using ∞^* as a model for colim_A. □

We finish the section with another lemma related to Construction 8.4 which will be useful when dealing with a more complicated instance of Proposition 8.2 in the next section.

Lemma 8.9. *Let $\mathscr{D}$ be a derivator, $u : A \to B$ be fully faithful, and $a \in A$. The map $a^* \to \mathrm{colim}_A$ (8.5) at $X \in \mathscr{D}^A$ is isomorphic to $u(a)^* \to \mathrm{colim}_B$ (8.5) at $u_!X$.*

Proof. Considering the pasting on the left in the diagram

$$\begin{tikzcd} & A \arrow[r] \arrow[d] & A \arrow[d, "u"] \\ \mathbb{1} \arrow[r, "a"] \arrow[d] & A \arrow[r, "u"] \arrow[d] & B \arrow[d] \\ \mathbb{1} \arrow[r] & \mathbb{1} \arrow[r] & \mathbb{1} \end{tikzcd} \qquad \begin{tikzcd} a^* \arrow[r, "\eta", "\cong"'] \arrow[d] & a^*u^*u_! \arrow[d] \\ \mathrm{colim}_A \arrow[r, "\cong"'] & \mathrm{colim}_B\, u_! \end{tikzcd}$$

it is immediate from the functoriality of mates with pasting that the square on the right commutes. □

9. General reflection morphisms

In this section we implement the remaining steps of the strategy outlined in Section 4, namely the inflation and deflation steps from Figure 2. This will allow us to finish the construction of a strong stable equivalence between the categories C^+ and C^- depicted in Figure 1 (see Theorem 9.11).

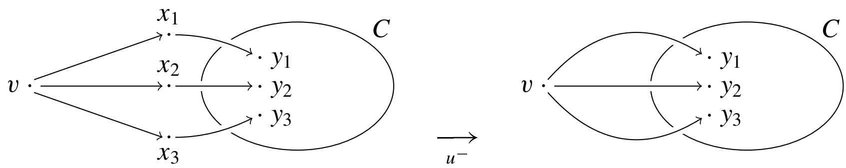
We start by formalizing the construction of the categories C^- and C^+. Let $C \in \mathcal{C}at$, and let $y_1, \ldots, y_n \in C$ be objects. We denote by $y : n \cdot \mathbb{1} \to C$ the resulting functor. For all preparatory results before Corollary 9.10, we adopt the following hypothesis which will allow us to apply results from the Appendix.

Hypothesis 9.1. The functor $y : n \cdot \mathbb{1} \to C$ is injective on objects. Equivalently, $y_1, y_2, \ldots, y_n$ are pairwise distinct objects of C.

We obtain C^- and C^+ by attaching a source of valence n and a sink of valence n to C, respectively. More precisely, the source of valence n is the cone $(n \cdot \mathbb{1})^{\triangleleft}$ obtained from $n \cdot \mathbb{1}$ by adjoining an initial object, and dually for the sink $(n \cdot \mathbb{1})^{\triangleright}$. Using the obvious inclusion functors $n \cdot \mathbb{1} \to (n \cdot \mathbb{1})^{\triangleleft}$ and $n \cdot \mathbb{1} \to (n \cdot \mathbb{1})^{\triangleright}$ we define C^- and C^+ as the respective pushouts in

$$\begin{tikzcd} n \cdot \mathbb{1} \arrow[r, "y"] \arrow[d] & C \arrow[d] \\ (n \cdot \mathbb{1})^{\triangleleft} \arrow[r] & C^- \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} n \cdot \mathbb{1} \arrow[r, "y"] \arrow[d] & C \arrow[d] \\ (n \cdot \mathbb{1})^{\triangleright} \arrow[r] & C^+ \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{9.2}$$

Assuming Hypothesis 9.1, note that $C \to C^+$ and $C \to C^-$ are fully faithful by Proposition A.11, and we view these functors as inclusions.



The category D^- The category C^-

Figure 5. The functor $u^- : D^- \to C^-$, which contracts the edges $x_i \to y_i$. It is used to separate the source of C^-.

As already mentioned in Section 4, the two inflation and deflation steps are not dual to each other. Starting with a representation of C^- we separate the morphisms adjacent to the source by adding morphisms pointing in the *same* direction, while in the other case we add morphisms pointing in the *opposite* direction.

Let us start with the easier case and consider the functor $u^- : D^- \to C^-$ as shown in Figure 5. Formally, we can construct the functor by means of the following pushout squares in $\mathcal{C}at$, where we use the inclusion of the target object $1 : \mathbb{1} \to [1]$ and the collapse functor $\pi : [1] \to \mathbb{1}$ in the upper line:

$$\begin{tikzcd}
n \cdot \mathbb{1} \arrow[r] \arrow[d, "y"'] & (n \cdot [1])^{\lhd} \arrow[r] \arrow[d] & (n \cdot \mathbb{1})^{\lhd} \arrow[d] \\
C \arrow[r, "j^-"'] & D^- \arrow[r, "u^-"'] & C^-
\end{tikzcd} \tag{9.3}$$

The functor j^- is fully faithful by Proposition A.11, and for every derivator $\mathscr{D}$, the restriction morphism $(u^-)^* : \mathscr{D}^{C^-} \to \mathscr{D}^{D^-}$ separates the objects adjacent to the source. We denote by $\mathscr{D}^{D^-,\mathrm{ex}} \subseteq \mathscr{D}^{D^-}$ the full subderivator spanned by all diagrams $X \in \mathscr{D}^{D^-}$ such that $k^*X \in \mathscr{D}^{n\cdot[1]}$ consists of isomorphisms, where

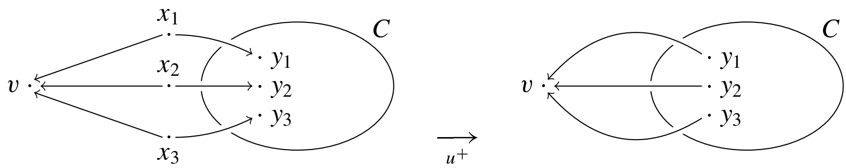
$$k : n \cdot [1] \to (n \cdot [1])^{\lhd} \to D^-$$

is the obvious functor.

Proposition 9.4. *The functor $u^- : D^- \to C^-$ is a homotopical epimorphism. Moreover, for every derivator $\mathscr{D}$ the essential image of $(u^-)^* : \mathscr{D}^{C^-} \to \mathscr{D}^{D^-}$ is $\mathscr{D}^{D^-,\mathrm{ex}}$ and the resulting equivalence $(u^-)^* : \mathscr{D}^{C^-} \simeq \mathscr{D}^{D^-,\mathrm{ex}}$ is pseudonatural with respect to arbitrary morphisms of derivators.*

Proof. This is an immediate application of Proposition 8.1. In fact, the functor $u^- : D^- \to C^-$ is a reflective localization, a fully faithful right adjoint being given by the obvious functor $r : C^- \to D^-$ which sends v to v and which is the identity on C. Let us denote the resulting adjunction by

$$(u^-, r, \eta : \mathrm{id} \to r \circ u^-, \varepsilon = \mathrm{id} : u^- \circ r \to \mathrm{id}).$$



The category D^+ The category C^+

Figure 6. The functor $u^+ : D^+ \to C^+$, which contracts the edges $x_i \to y_i$. It is used to separate the sink of C^+.

The only nonidentity components of the adjunction unit η are those at $x_i \in D^-$ for $i = 1, \ldots, n$, in which case they are given by

$$\eta_{x_i} : x_i \to y_i, \qquad i = 1, \ldots, n.$$

By Proposition 8.1 we conclude that u^- is a homotopical epimorphism and that $X \in \mathscr{D}^{D^-}$ lies in the essential image of $(u^-)^*$ if and only if $X_{x_i} \to X_{y_i}$ is an isomorphism, which is to say that $X \in \mathscr{D}^{D^-,\mathrm{ex}}$. $\square$

The other inflation and deflation step turns out to be a bit more involved, and the situation is shown in Figure 6. We again have defining pushout squares

$$\begin{tikzcd} n \cdot \mathbb{1} \arrow[r] \arrow[d, "y"'] & Z_n \arrow[r, "q"] \arrow[d, "t"] & (n \cdot \mathbb{1})^{\triangleright} \arrow[d, "y^{\triangleright}"] \\ C \arrow[r, "j^+"'] & \ulcorner D^+ \arrow[r, "u^+"'] & \ulcorner C^+ \end{tikzcd} \tag{9.5}$$

where Z_n is the free category generated by the quiver

$$Z_n : \quad \begin{tikzcd}[column sep=small] & x_1 \arrow[dl] \arrow[drr] & x_2 \arrow[dl] \arrow[dr] & \cdots & x_{n-1} \arrow[dl] \arrow[dr] & x_n \arrow[dll] \arrow[dr] & \\ y_1 & y_2 & & v & & y_{n-1} & y_n \end{tikzcd}$$

where $n \cdot \mathbb{1} \to Z_n$ classifies $y_1, \ldots, y_n$, and where $q : Z_n \to (n \cdot \mathbb{1})^{\triangleright}$ sends each x_i and y_i to the i-th copy of $\mathbb{1}$ and v to the terminal object ∞. Assuming Hypothesis 9.1, both j^+ and $u^+ j^+$ are fully faithful, and we again view $u^+ j^+$ as an inclusion. As it will be important in further computations, we spell out what morphisms in D^+ and C^+ look like.

Lemma 9.6.

(i) *Every nonidentity morphism in the category C^+ has a unique expression of one of the forms γ, ω, $\omega\gamma$, where γ stands for a nonidentity morphism of C and ω stand for a nonidentity morphism of $(n \cdot \mathbb{1})^{\triangleright}$.*

(ii) *Every nonidentity morphism in the category D^+ has a unique expression of one of the forms γ, ω, $\gamma\omega$, where γ stands for a nonidentity morphism of C and ω stand for a nonidentity morphism of Z_n.*

Proof. This is an immediate consequence of Lemma A.12. □

For every derivator $\mathscr{D}$ we denote by $\mathscr{D}^{D^+,\mathrm{ex}} \subseteq \mathscr{D}^{D^+}$ the full subderivator formed by the coherent diagrams X such that $X_{x_i} \to X_{y_i}$ is an isomorphism for every $i = 1, \ldots, n$.

Proposition 9.7. *If $y : n \cdot \mathbb{1} \to C$ is injective on objects, then $u^+ : D^+ \to C^+$ (9.5) is a homotopical epimorphism. Moreover, for $\mathscr{D} \in \mathcal{DER}$ the essential image of $(u^+)^* : \mathscr{D}^{C^+} \to \mathscr{D}^{D^+}$ is $\mathscr{D}^{D^+,\mathrm{ex}}$ and the resulting equivalence $(u^+)^* : \mathscr{D}^{C^+} \simeq \mathscr{D}^{D^+,\mathrm{ex}}$ is pseudonatural with respect to arbitrary morphisms of derivators.*

Proof. Let us fix a derivator $\mathscr{D}$ and let $\mathscr{E} = \mathscr{D}^{D^+,\mathrm{ex}}$. We show that Proposition 8.2 applies. Clearly u^+ is essentially surjective on objects and $\mathrm{im}((u^+)^*) \subseteq \mathscr{E}$. It remains to verify the assumption Proposition 8.2(ii), and by (Der2) it suffices to check the invertibility of the unit η at every $d \in D^+$. By Lemma 8.7 this is the case if and only if the instance

$$(d, \mathrm{id}_{u^+d})^* p^* \to \mathrm{colim}_{(u^+/u^+d)}\, p^* \tag{9.8}$$

of (8.6) is invertible for every $d \in D^+$ and on $\mathscr{D}^{D^+,\mathrm{ex}}$. Here, $p : (u^+/u^+d) \to D^+$ is the canonical functor, and there are the following three cases.

First, let $d = j^+(c)$, $c \in C$, so that $u^+d = c$. Since $(j^+c, \mathrm{id}_c) \in (u^+/c)$ is a terminal object, by Lemma 8.8 the corresponding morphism (9.8) is an isomorphism on $\mathscr{D}^{D^+,\mathrm{ex}}$ if and only if $(j^+c, \mathrm{id})^* p^* \to (j^+c, \mathrm{id})^* p^*$ is an isomorphism on $\mathscr{D}^{D^+,\mathrm{ex}}$, and this is even true for all $X \in \mathscr{D}^{D^+}$.

Suppose next that $d = x_i$ for some $i = 1, \ldots, n$. In this case $u^+d = y_i \in C^+$ and it is easy to see that (u^+/y_i) admits

$$(x_i, \mathrm{id}_{y_i}) \to (y_i, \mathrm{id}_{y_i})$$

as homotopy final subcategory, where the map is given by the freely attached map from Z_n. Two applications of Lemma 8.8 imply that we have to show that $x_i^* \to y_i^*$ is an isomorphism on $\mathscr{D}^{D^+,\mathrm{ex}}$, which is true by the defining exactness properties.

The remaining case is $d = v$. With the aid of Lemma 9.6, we divide the objects

$$w = (d', g : u^+(d') \to v)$$

of (u^+/v) into five disjoint classes, according to what d' is and whether the structure morphism g factors through a nonidentity morphism in C. Each object $w \in (u^+/v)$ has exactly one of the following forms (where unlabeled arrows $y_i \to v$ always stand for the maps in C^+ coming from $(n \cdot \mathbb{1})^{\triangleright}$ in (9.2)):

(i) $w = (v, \mathrm{id}_v)$,

(ii) $w = (x_i, y_i \to v)$ for some $i \in \{1, \ldots, n\}$,

(iii) $w = (y_i, y_i \to v)$ for some $i \in \{1, \ldots, n\}$,

(iv) $w = (j^+(c), c \xrightarrow{h} y_i \to v)$ for some $c \in C$ and nonidentity map h in C, or

(v) $w = (x_i, y_i \xrightarrow{h} y_j \to v)$ for $i, j \in \{1, \ldots, n\}$ and nonidentity map h in C.

Let $H \subseteq (u^+/v)$ be the full subcategory spanned by the objects of type (i)–(iii). This category is a free category generated by the following quiver, where the object from which we wish to inspect the map (9.8) is in the box (for brevity we denote the objects only by the corresponding object of D^+):

$$H: \quad \begin{tikzcd} & x_1 \arrow[dl] \arrow[drrr] & & x_2 \arrow[dl] \arrow[dr] & \cdots & x_{n-1} \arrow[dl] \arrow[dr] & & x_n \arrow[dlll] \arrow[dr] & \\ y_1 & & y_2 & & \boxed{v} & & y_{n-1} & & y_n \end{tikzcd} \tag{9.9}$$

Another short computation reveals that every object of type (v) admits a unique map in (u^+/v) to the object of type (iv) with $c = y_i$ and the same morphism h in C, and that every object of type (iv) admits a unique map in (u^+/v) to an object of type (iii) obtained by stripping off h from the structure morphism. In particular, the inclusion $H \to (u^+/v)$ is a right adjoint and hence homotopy final, so that Lemma 8.8 applies. As an upshot so far, the decoration of the objects in (9.9) defines a functor $i : H \to D^+$, and it remains to show that the map,

$$v^* i^*(X) \to \mathrm{colim}_H\, i^* X$$

which is an instance of (8.5), is an isomorphism for all $X \in \mathscr{D}^{D^+, \mathrm{ex}}$.

To this end, let $j : H' \to H$ be the full subcategory of H obtained by removing $y_i, i = 1, \ldots, n$. It is straightforward to show that $j_! : \mathscr{D}^{H'} \to \mathscr{D}^H$ is fully faithful with essential image precisely those $Y \in \mathscr{D}^H$ such that $Y_{x_i} \to Y_{y_i}$ is invertible (compare to [Groth 2013, Proposition 3.12(1)]). In particular, for $X \in \mathscr{D}^{D^+, \mathrm{ex}}$ the restriction $i^* X$ belongs to this essential image, and Lemma 8.9 thus reduces our task to show that $v^* \to \mathrm{colim}_{H'}$ (8.5) is an isomorphism on $j^* i^* X \in \mathscr{D}^{H'}$. By Lemma 8.8 this is even the case for every diagram in $\mathscr{D}^{H'}$, since $v \in H'$ is a terminal object.

To summarize, all assumptions of Proposition 8.2 are satisfied and u^+ is hence a homotopical epimorphism with essential image $\mathscr{D}^{D^+, \mathrm{ex}}$. □

Now we shall revoke Hypothesis 9.1.

Corollary 9.10. *Let $C \in \mathcal{C}at$, let $y_1, \ldots, y_n \in C$ (not necessarily distinct), and consider the functors $u^- : D^- \to C^-$ and $u^+ : D^+ \to C^+$ constructed again by the pushouts (9.3) and (9.5), respectively. Then u^- and u^+ are still homotopical* *epimorphisms and the essential images are* $\mathscr{D}^{D^-,\mathrm{ex}}$ *and* $\mathscr{D}^{D^+,\mathrm{ex}}$, *defined by the same exactness conditions as in Proposition 9.4 and Proposition 9.7, respectively.*

Proof. We discuss only u^+, the case of u^- being similar. Suppose $y\colon n\cdot\mathbb{1}\to C$ is any functor. Thanks to Lemma A.2(i) there is a factorization $y = p\tilde{y}$ such that $p\colon\widetilde{C}\to C$ is an equivalence of categories and $\tilde{y}_1,\ldots,\tilde{y}_n$ are pairwise distinct objects in $\widetilde{C}$. Replacing y by $\tilde{y}$ in (9.5), we obtain Proposition A.7 and Lemma A.2(ii), a diagram whose lower row changes only up to equivalence. □

Finally, we can establish the main result of this paper.

Theorem 9.11. *Let* $C\in\mathcal{C}at$, *let* $y_1,\ldots,y_n\in C$ (*not necessarily distinct*), *and let* $C^-, C^+\in\mathcal{C}at$ *be as in* (9.2). *The categories* C^- *and* C^+ *are strongly stably equivalent.*

Proof. In Theorem 7.20 we constructed a pseudonatural equivalence $\mathscr{D}^{D^-}\simeq\mathscr{D}^{D^+}$. It is direct from the construction of this equivalence that it restricts to a pseudonatural equivalence $\mathscr{D}^{D^-,\mathrm{ex}}\simeq\mathscr{D}^{D^+,\mathrm{ex}}$. Invoking Corollary 9.10, we obtain a chain

$$\mathscr{D}^{C^-}\simeq\mathscr{D}^{D^-,\mathrm{ex}}\simeq\mathscr{D}^{D^+,\mathrm{ex}}\simeq\mathscr{D}^{C^+}$$

of pseudonatural equivalences. Putting them together, we obtain the strong stable equivalence

$$(s^-,s^+)\colon\mathscr{D}^{C^-}\simeq\mathscr{D}^{C^+},\tag{9.12}$$

concluding the proof. □

Definition 9.13. Let $\mathscr{D}$ be a stable derivator, let $C\in\mathcal{C}at$, let $y_1,\ldots,y_n\in C$ (not necessarily distinct), and let $C^-, C^+\in\mathcal{C}at$ be as in (9.2). The components s^-, s^+ of the strong stable equivalence in (9.12), witnessing that $C^-\overset{s}{\sim}C^+$, are *(general) reflection morphisms*.

10. Applications to abstract representation theory

In this section we draw some consequences from the main theorem in this paper (Theorem 9.11). Since the categories C^+ and C^- are strongly stably equivalent, we obtain abstract tilting results for various contexts. To begin with, let us specialize to representations over a ring.

Example 10.1. Let R be a (possibly noncommutative) ring. Let $C\in\mathcal{C}at$ with $y_1,\ldots,y_n\in C$ (not necessarily distinct), and let $C^-, C^+\in\mathcal{C}at$ be as in (9.2).

(i) There is an exact equivalence of categories $\mathscr{D}_R^{C^-}(\mathbb{1})\overset{\Delta}{\simeq}\mathscr{D}_R^{C^+}(\mathbb{1})$.

(ii) If C has only finitely many objects, then the category algebras RC^- and RC^+ are derived equivalent over R:

$$D(RC^-)\overset{\Delta}{\simeq}D(RC^+).$$

In fact, the first statement is [Groth 2013, Proposition 4.18], while the second statement follows from Example 2.13. However, having a strong stable equivalence is a stronger result in the following three senses.

(i) Simply by choosing specific stable derivators, this yields exact equivalences of derived or homotopy categories of representations over rings or schemes, of differential graded representations, of spectral representations, and of other types of representations; see [Groth and Šťovíček 2016c, §5].

(ii) There are equivalences of derivators of representations, as opposed to having mere equivalences of underlying categories. For example, in the case of homotopy derivators of combinatorial, stable model categories $\mathcal{M}$ it is a formal consequence of the existence of an equivalence $\mathscr{H}o_{\mathcal{M}}^{Q} \sim \mathscr{H}o_{\mathcal{M}}^{Q'}$ and [Renaudin 2009] that the corresponding model categories of representations $\mathcal{M}^{Q}, \mathcal{M}^{Q'}$ are related by a zigzag of Quillen equivalences.

(iii) The equivalences are pseudonatural with respect to exact morphisms, and hence commute with various types of morphisms like restriction of scalars, induction and coinduction of scalars, derived tensor and hom functors, localizations and colocalizations.

With this added generality in mind, for the rest of the section we mostly focus on the shapes C^-, C^+. As a first instance, we recover the main result of [Groth and Šťovíček 2016b].

Theorem 10.2 [Groth and Šťovíček 2016b, Corollary 9.23]. *Let T be a finite oriented tree and let T' be a reorientation of T. The trees T and T' are strongly stably equivalent.*

Proof. By an inductive argument, it suffices to show that if T is as above and $t_0 \in T$ is a source, then the reflected tree $T' = \sigma_{t_0} T$ and T are strongly stably equivalent. But obviously $T = C^-$ and $T' = C^+$ for the full subcategory $C \subseteq T$ of T obtained by removing t_0. Hence Theorem 9.11 concludes the proof. □

Increasing the class of shapes, we obtain the following.

Theorem 10.3. *Let Q be a finite acyclic quiver, let $q_0 \in Q$ be a source or a sink, and let $Q' = \sigma_{q_0} Q$ be the reflected quiver. The two quivers Q and Q' are strongly stably equivalent.*

Proof. Assuming without loss of generality that q_0 is a source, we observe that $Q = C^-$ for the full subcategory $C \subseteq Q$ obtained by removing q_0. In this case one notes that $Q' = C^+$ and Theorem 9.11 applies. □

Remark 10.4. Specializing to the derivator $\mathscr{D}_k$ of a field k, Theorem 10.3 yields exact equivalences of derived categories:

$$D(kQ) \overset{\Delta}{\simeq} D(kQ').$$

The classical representation theory is more concerned with bounded derived categories of finite dimensional representations. However, as shown in [Rickard 1989, Corollary 8.3] (and its proof), any exact equivalence between the unbounded derived categories restricts to an exact equivalence of the corresponding bounded derived categories:

$$D^b(kQ) \overset{\Delta}{\simeq} D^b(kQ')$$

Hence, the reflection functors yield such an equivalence and we recover a theorem of Happel [1987, §1.7].

In contrast to the case of trees, already for acyclic quivers it is not true that such quivers can be reoriented arbitrarily without affecting the abstract representation theory. If Q, Q' are finite and without oriented cycles, then Q, Q' being strongly stably equivalent still implies that Q and Q' have the same underlying graph [Groth and Šťovíček 2016c, Proposition 5.3], but this condition is no longer sufficient. Let us consider the simplest case, where Q is an orientation of an n-cycle:

$$\begin{tikzcd}
 & & n \arrow[dll, dash] \arrow[drr, dash] & & \\
1 \arrow[r, dash] & 2 \arrow[r, dash] & \cdots \arrow[r, dash] & n-2 \arrow[r, dash] & n-1
\end{tikzcd}$$

In representation theory one says that Q is a *Euclidean* (or *extended Dynkin*) quiver of type $\widetilde{A}_{n-1}$ [Ringel 1984; Simson and Skowroński 2007]. Given such Q, put $c(Q) = \{p, q\}$, where p is the number of arrows oriented clockwise and q is the number of arrows oriented counterclockwise. Then one obtains the following.

Proposition 10.5. *Let Q, Q' be two orientations of an n-cycle, $n \geq 1$. Then $Q \overset{s}{\sim} Q'$ if and only if $c(Q) = c(Q')$.*

Proof. The sufficiency of the "clock condition" $c(Q) = c(Q')$ is easy. One quickly convinces oneself that given Q with $c(Q) = \{p, q\}$, $p \leq q$, after finitely many reflections at sinks or sources one gets a quiver isomorphic to

$$\widetilde{A}_{p,q}: \quad \begin{tikzcd}
 & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[dr] & \\
\bullet \arrow[ur] \arrow[dr] & & & & & \bullet \\
 & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[r] & \bullet \arrow[ur] &
\end{tikzcd}$$

with p arrows above and q arrows below. Hence, if $c(Q) = C(Q')$, one gets for any stable derivator $\mathscr{D}$ a strong stable equivalence $\mathscr{D}^Q \simeq \mathscr{D}^{\widetilde{A}_{p,q}} \simeq \mathscr{D}^{Q'}$ by composing finitely many general reflection morphisms (Theorem 9.11).

To prove the necessity, let k be a field, $\mathscr{D} = \mathscr{D}_k$ be the derivator of k, and suppose that $\mathscr{D}_k^Q \simeq \mathscr{D}_k^{Q'}$. We shall appeal to results from representation theory and show that then $c(Q) = c(Q')$. The equivalence of derivators gives an equivalence of the underlying categories, which in turn gives an equivalence of the subcategories of compact objects. In our case this means that the bounded derived categories of finitely generated modules of the corresponding path algebras are equivalent:

$$D^b(kQ) \simeq D^b(kQ').$$

Now kQ is a finite dimensional algebra over k if and only if not all arrows have the same orientation if and only if $c(Q) \neq \{0, n\}$ if and only if all objects of $D^b(\mathrm{mod}\, kQ)$ have finite dimensional endomorphism rings. Thus $c(Q) = \{0, n\}$ if and only if $c(Q') = \{0, n\}$.

Suppose now that $c(Q), c(Q') \neq \{0, n\}$. Then kQ is finite dimensional and we can construct a so-called Auslander–Reiten quiver of $D^b(kQ)$. This is an infinite quiver which is a useful combinatorial invariant of $D^b(kQ)$, and its general shape is described in [Happel 1987, Corollary 4.5(ii)]. A more precise description can be extracted from [Ringel 1984, Theorem 3.6.5, p. 158] or [Simson and Skowroński 2007, Proposition XII.2.8]. In particular, the numbers p, q, where $c(Q) = \{p, q\}$, can be read off the Auslander–Reiten quiver since it contains so-called tubes of ranks precisely 1, p, and q. Of course one can do the same for Q', and hence $c(Q) = c(Q')$. □

Remark 10.6. The existence of reflection equivalences in Theorem 9.11 applies to more general shapes than finite, acyclic quivers.

(i) First, neither the finiteness nor the acyclicity is needed. In fact, given an arbitrary quiver Q with a source or a sink, Theorem 9.11 yields a strong stable equivalence between Q and the reflected quiver Q'. In particular, if Q has finitely many objects only, the infinite-dimensional path algebras kQ and kQ' are derived equivalent for arbitrary fields k, and there are variants if we use rings as coefficients instead.

(ii) More generally, as noted in Example 10.1, Theorem 9.11 yields strong stable equivalences for shapes which are more general than quivers. To the best of the authors' knowledge, even in the case that $R = k$ is a field, the result that the category algebras kC^- and kC^+ are derived equivalent does not appear in the published literature.

Appendix: Amalgamation of categories

As is illustrated by the construction of abstract reflection functors, performing more complicated constructions in derivators often means that we need to "glue together" various small categories or diagram shapes. Formally, we are speaking of pushouts of categories, which is a fairly complicated construction. As we need to understand some of these pushouts rather explicitly (for example, in order to be able to compute slice categories), here we discuss some basic properties of pushouts and amalgamations of small categories. We fix the following notation for the rest of the appendix:

$$\begin{tikzcd} W \arrow[r, "f_Y"] \arrow[d, "f_X"'] & Y \arrow[d, "g_Y"] \\ X \arrow[r, "g_X"'] & Z \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{A.1}$$

Often one is only interested in categories up to equivalences, but pushouts of small categories are, in general, not well behaved with equivalences. To address this issue, we include the following lemma.

Lemma A.2. *Let $f_X : W \to X$ be a functor in Cat.*

(i) *There exists a factorization $f_X = p \circ f_{\widetilde{X}}$ such that $f_{\widetilde{X}} : W \to \widetilde{X}$ is injective on objects and $p : \widetilde{X} \to X$ is surjective on objects and an equivalence.*

(ii) *If f_X is injective on objects and $f_Y : W \to Y$ in (A.1) is an equivalence, then also $g_X : X \to Z$ is an equivalence.*

Proof. Both are easy consequences of the existence of an (in fact unique) model structure on *Cat* with weak equivalences being the equivalences. This is a special case of a more general result in [Joyal and Tierney 1991], and (i) is simply a factorization of f_X into a cofibration followed by a trivial fibration. Meanwhile, (ii) means that this model structure is left proper, which follows from the fact that every small category is cofibrant [Hirschhorn 2003, Corollary 13.1.3]. □

For the rest of the section we adopt the following assumption and convention.

Hypothesis A.3. Assume that f_X and f_Y are honest inclusions of categories, that is, injective on objects and faithful. We will view f_X and f_Y as (not necessarily full) inclusions $W \subseteq X$ and $W \subseteq Y$, respectively.

Definition A.4. The pushout (A.1) is called an *amalgamation* if also g_X and g_Y are injective on objects and faithful. In this case we also view g_X and g_Y as inclusions $X \subseteq Z$ and $Y \subseteq Z$, respectively.

Remark A.5. In the usual terminology of model theory, an amalgamation of the span

$$X \xleftarrow{f_X} W \xrightarrow{f_Y} Y$$

would in fact mean *any* commutative square like (A.1) (i.e., not necessarily a pushout) for which g_X and g_Y are inclusions. But if such a square exists, the pushout square is also an amalgamation in this sense.

As shown in [MacDonald and Scull 2009, Example 4.4], not every pushout of inclusions is an amalgamation. On the other hand, a sufficient condition for the existence of amalgamations is given in the same paper.

Definition A.6. A functor $f : W \to Y$ has the 3-*for*-2 *property* if, whenever α and β are two composable morphisms in Y and two of α, β, $\beta\alpha$ belong to the honest (not just essential) image of f, then so does the third.

Proposition A.7 [MacDonald and Scull 2009, Theorem 3.3]. *Suppose $f_X : W \to X$ and $f_Y : W \to Y$ are functors in Cat which are injective on objects, faithful, and have the* 3-*for*-2 *property. Then their pushout* (A.1) *is an amalgamation.*

Remark A.8. The result is rather subtle in that it is *not* enough to assume that only one of f_X and f_Y has the 3-for-2 property; see [MacDonald and Scull 2009, Example 4.4] again. Note that $f : W \to Y$ has the 3-for-2 property, for example, if f is fully faithful or if W is a groupoid (so in particular if W is a discrete category as in Sections 5, 6, and 9).

For practical purposes it will be convenient to know that the 3-for-2 property transfers via amalgamations, i.e., that also the functors g_X and g_Y have it. Once we know this, we can iterate the amalgamation process. Here we need to refine the argument in [MacDonald and Scull 2009].

We first recall details about the construction of a pushout in *Cat*. At the level of objects, we simply construct the pushout of sets. The morphisms in the pushout are more interesting; see [MacDonald and Scull 2009, §2] for details. To this end, we denote by $\bar{Z}$ the pushout of the sets of morphisms of X and Y over the set of morphisms of W. In particular, an element of $\bar{Z}$ which comes from both X and Y comes already from W by our standing assumption. Every morphism in Z is represented by a finite sequence

$$(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

of length $n \geq 1$ in $\bar{Z}$, subject to the condition that the codomain of α_{i+1} always agrees with the domain of α_i. The composition of morphisms is simply given by concatenation. Of course we must identify some of these sequences. To do so, we first define a partial order on the set of allowable sequences of elements of $\bar{Z}$ which is generated by the *elementary reductions*

$$(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}, \ldots, \alpha_n) > (\alpha_1, \ldots, \alpha_i\alpha_{i+1}, \ldots, \alpha_n),$$

where either both α_i and α_{i+1} are morphisms from X and the composition on the right takes place in X, or symmetrically α_i and α_{i+1} are from Y and we compose them in Y. This *reduction order* is of course a binary relation, and by taking its symmetric and transitive closure, we obtain an equivalence relation. The morphisms in Z are then precisely the equivalence classes of allowable sequences in $\bar{Z}$.

For convenience, we introduce the following notation. Given an allowable sequence $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, denote the equivalence class of γ by $[\alpha_1, \alpha_2, \ldots, \alpha_n]$, and view this equivalence class as a partially ordered set with the restriction of the reduction order above. The following is a key observation.

Lemma A.9. *Suppose that $\gamma = (\alpha_1)$ consist of a single element of $\bar{Z}$. Then γ is the unique minimal element of $[\alpha_1]$ with respect to the reduction order.*

Proof. This is exactly what the first paragraph of the proof of [MacDonald and Scull 2009, Theorem 3.3] asserts. For a very detailed proof we refer to the rest of the proof of Theorem 3.3 and to §5 in [op. cit.]. □

Now we can complement Proposition A.7 with the promised result, which will allow for iterated amalgamations.

Proposition A.10. *Suppose that $f_X : W \to X$ and $f_Y : W \to Y$ are injective on objects and faithful functors with the 3-for-2 property. Then, in their pushout amalgamation (A.1), also g_X and g_Y have the 3-for-2 property.*

Proof. By symmetry we only need to treat g_X. Suppose that α_1, β are composable morphisms in Z and that α_1 and $\alpha_1\beta$ both belong to X. We must show that β belongs there as well.

To this end, β can be represented by a suitable sequence $\gamma = (\alpha_2, \dots, \alpha_n)$ of elements of $\bar{Z}$. Then $\alpha_1\beta$ is represented by $\delta = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and, by Lemma A.9, $[\alpha_1, \alpha_2, \dots, \alpha_n]$ has the unique minimal element $(\alpha_1\beta)$ with respect to the reduction order. We shall prove by induction on n that β is in X.

Suppose first that $n = 2$. In this case $\beta = \alpha_2$ belongs either to X or Y. If β is in X, we are done. If β is in Y, we know by the above that $(\alpha_1, \beta) > (\alpha_1\beta)$ in the reduction order on $[\alpha_1\beta]$. By definition of the reduction order, the latter must be an elementary reduction, and hence all $\alpha_1, \beta, \alpha_1\beta$ belong to X or all three belong to Y. In the first case we are done and in the second case we know that $\alpha_1, \alpha_1\beta \in X \cap Y = W$. Hence $\beta \in W \subseteq X$ by the 3-for-2 property of $f_Y : W \xrightarrow{\subseteq} Y$.

If now $n > 2$, there is an elementary reduction

$$(\alpha_1, \alpha_2, \dots, \alpha_i, \alpha_{i+1}, \dots, \alpha_n) > (\alpha_1, \alpha_2, \dots, \alpha_i\alpha_{i+1}, \dots, \alpha_n).$$

Let us choose such a reduction with maximal possible i. Two situations may occur. If $i > 1$, then by the very definition of elementary reductions we have that $(\alpha_2, \dots, \alpha_n) > (\alpha_2, \dots, \alpha_i\alpha_{i+1}, \dots, \alpha_n)$ and also that β is in X by the induction hypothesis.

Suppose on the other hand that $i = 1$. We claim that in such a case α_2 is in X. To this end, assume by way of contradiction that $\alpha_2 \in Y \setminus W$. Then $\alpha_1 \in W$, since we have the reduction $(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n) > (\alpha_1\alpha_2, \alpha_3, \dots, \alpha_n)$. Consequently, $\alpha_1\alpha_2 \in Y \setminus W$, since otherwise $\alpha_1, \alpha_1\alpha_2 \in W$ would imply $\alpha_2 \in W$. Finally, since the sequence $(\alpha_1\alpha_2, \alpha_3, \dots, \alpha_n)$ must reduce further, the maximality of $i = 1$ implies

$$(\alpha_1\alpha_2, \alpha_3, \dots, \alpha_n) > (\alpha_1\alpha_2\alpha_3, \dots, \alpha_n).$$

Now $\alpha_1\alpha_2 \in Y \setminus W$, so $\alpha_3 \in Y$ in order for the reduction to be defined. However, then we also have an elementary reduction

$$(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) > (\alpha_1, \alpha_2\alpha_3, \ldots, \alpha_n),$$

contradicting the maximality of i. This proves the claim.

To summarize, we have $\alpha_1, \alpha_2 \in X$. Now let $\alpha' = \alpha_1\alpha_2 \in X$ and β' be the equivalence class $[\alpha_3, \ldots, \alpha_n]$. Then $\alpha', \alpha'\beta' \in X$ and we infer by the inductive hypothesis that $\beta' \in X$. Then clearly $\beta = \alpha_2\beta'$ is in X, which finishes the induction.

The case when α, β are composable in Z and $\beta, \alpha\beta$ are in X is similar. □

As pointed out in [MacDonald and Scull 2009], a special case when a functor has the 3-for-2 property is when it is fully faithful. Under our usual assumptions, it turns out that also full faithfulness is compatible with amalgamations. This has been observed already in [Trnková 1965], and we include a short proof for the convenience of the reader.

Proposition A.11. *Suppose that $f_X : W \to X$ and $f_Y : W \to Y$ are injective on objects. If f_X is fully faithful and f_Y is faithful and has the* 3*-for-*2 *property, then in the pushout amalgamation* (A.1), *$g_Y : Y \to Z$ is fully faithful.*

Proof. We only need to prove that g_Y is full. Suppose that we are given a morphism in Z, represented by a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\bar{Z}$ such that the domain of α_n and the codomain of α_1 belong to Y. By possibly reducing this sequence, we may assume that α_i belongs to Y for i odd and to X for i even. If i is even, the domain and the codomain of α_i must be objects in $X \cap Y = W$. Since f_X is full, α_i is a morphism in W, and hence also in Y. Thus all the α_i in fact belong to Y and so does their composition. □

Finally, we consider the case where W is a discrete category (recall Remark A.8). The main advantage is that, analogous to the situation with free products of monoids, all morphisms of a pushout amalgamation of two categories over a discrete category have unique reduced factorizations to morphisms of the original categories (see Lemma 5.3(iv) for an illustration). To state this precisely, we call an allowable sequence $(\alpha_1, \ldots, \alpha_n)$ of elements of $\bar{Z}$ *reduced* if it is minimal with respect to the reduction order. For W discrete, the following stronger version of Lemma A.9 holds.

Lemma A.12. *Suppose that W is a discrete category and that $f_X : W \to X$ and $f_Y : W \to Y$ are injective on objects. Given any morphism in Z represented by a sequence $\gamma = (\alpha_1, \ldots, \alpha_n)$ in $\bar{Z}$, the equivalence class $[\alpha_1, \ldots, \alpha_n]$ has a unique minimal element in the reduction order. In other words, each nonidentity morphism β in Z uniquely factors as $\beta = \alpha_1 \cdots \alpha_n$, where each α_i belongs to X or Y, but no composition $\alpha_i\alpha_{i+1}$ belongs to X or Y.*

Proof. Suppose that we have two elementary reductions of our sequence γ,

$$(\alpha_1, \dots, \alpha_i\alpha_{i+1}, \dots, \alpha_n) < \gamma > (\alpha_1, \dots, \alpha_j\alpha_{j+1}, \dots, \alpha_n), \tag{A.13}$$

where $i \le j$ without loss of generality. We claim that there is a common predecessor. This is clear if $i = j$ and easy if $j - i \ge 2$, as then both the reductions further reduce to $(\alpha_1, \dots, \alpha_i\alpha_{i+1}, \dots, \alpha_j\alpha_{j+1}, \dots, \alpha_n)$. If $j = i+1$, there are two cases. First, all of $\alpha_i, \alpha_{i+1}, \alpha_{i+2}$ may belong to one of X or Y. Then $(\alpha_1, \dots, \alpha_i\alpha_{i+1}\alpha_{i+2}, \dots, \alpha_n)$ is the common predecessor which we are looking for. Second, two of $\alpha_i, \alpha_{i+1}, \alpha_{i+2}$ may belong to X and one to Y, or vice versa. Then, since both the reductions from (A.13) were possible, it is easy to check that in all possible distributions of $\alpha_i, \alpha_{i+1}, \alpha_{i+2}$ among X and Y, we always get that one of $\alpha_i, \alpha_{i+1}, \alpha_{i+2}$ belongs to $W = X \cap Y$, so it is the identity morphism. If $\alpha_{i+1} = \mathrm{id}$, the original reductions are equal, and in the remaining cases $(\alpha_1, \dots, \alpha_i\alpha_{i+1}\alpha_{i+2}, \dots, \alpha_n)$ is a common predecessor of the two. This proves the claim.

An easy induction argument shows now that $([\alpha_1, \dots, \alpha_n], <)$ is a downwards directed poset. Together with the obvious fact that the reduction order satisfies the descending chain condition, it follows that $([\alpha_1, \dots, \alpha_n], <)$ has a unique minimal ($=$ reduced) element. □

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MORITZ GROTH: mgroth@math.uni-bonn.de
Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, Germany

JAN ŠŤOVÍČEK: stovicek@karlin.mff.cuni.cz
Department of Algebra, Charles University in Prague, Praha, Czech Republic

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Equivariant noncommutative motives

Gonçalo Tabuada

Given a finite group G, we develop a theory of G-equivariant noncommutative motives. This theory provides a well-adapted framework for the study of G-schemes, Picard groups of schemes, G-algebras, 2-cocycles, G-equivariant algebraic K-theory, etc. Among other results, we relate our theory with its commutative counterpart as well as with Panin's theory. As a first application, we extend Panin's computations, concerning twisted projective homogeneous varieties, to a large class of invariants. As a second application, we prove that whenever the category of perfect complexes of a G-scheme X admits a full exceptional collection of G-invariant ($\neq$ G-equivariant) objects, the G-equivariant Chow motive of X is of Lefschetz type. Finally, we construct a G-equivariant motivic measure with values in the Grothendieck ring of G-equivariant noncommutative Chow motives.

1. Introduction

A *differential graded* (dg) *category* $\mathcal{A}$, over a base field k, is a category enriched over dg k-vector spaces; see Section 2A. Every (dg) k-algebra A naturally gives rise to a dg category with a single object. Schemes provide another source of examples, since the category of perfect complexes $\mathrm{perf}(X)$ of every quasicompact quasiseparated k-scheme X admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$; see Section 2B.

Let G be a finite group. A dg category $\mathcal{A}$ equipped with a G-action is denoted by $\mathrm{G} \circlearrowright \mathcal{A}$ and called a G-*dg category*. For example, every G-scheme X, subgroup $\mathrm{G} \subseteq \mathrm{Pic}(X)$ of the Picard group of a scheme X, G-algebra A, or cohomology class $[\alpha] \in H^2(\mathrm{G}, k^\times)$, naturally gives rise to a G-dg category. The associated dg categories of G-equivariant objects $\mathcal{A}^{\mathrm{G}}$ are given, respectively, by equivariant perfect complexes $\mathrm{perf}^{\mathrm{G}}_{\mathrm{dg}}(X)$, perfect complexes $\mathrm{perf}_{\mathrm{dg}}(Y)$ on a $|\mathrm{G}|$-fold cover over X, semidirect product algebras $A \rtimes \mathrm{G}$, and twisted group algebras $k_\alpha[\mathrm{G}]$.

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By precomposition with the functor $G \circlearrowright \mathcal{A} \mapsto \mathcal{A}^G$, all invariants of dg categories E can be promoted to invariants of G-dg categories E^G. For example, algebraic K-theory leads to equivariant algebraic K-theory in the sense of Thomason [1987]; see Section 4A. In order to study all these invariants simultaneously, we develop in Section 3 a theory of G-equivariant noncommutative motives. Among other results, we construct a symmetric monoidal functor $U^G \colon \mathrm{G\text{-}dgcat}_{\mathrm{sp}}(k) \to \mathrm{NChow}^G(k)$, from smooth proper G-dg categories to G-equivariant noncommutative Chow motives, which is "initial" among all such invariants E^G. The morphisms of $\mathrm{NChow}^G(k)$ are given in terms of the G-equivariant Grothendieck group of certain triangulated categories of bimodules. In particular, the ring of endomorphisms of the $\otimes$-unit $U^G(G \circlearrowright_1 0k)$ identifies with the representation ring $R(G)$ of the group G.

I. Panin [1994] constructed a certain motivic category $\mathcal{C}^G(k)$, which mixes smooth projective G-schemes with (noncommutative) separable algebras, and performed therein several computations concerning twisted projective homogeneous varieties. In Theorem 5.3 we construct a fully faithful symmetric monoidal functor from $\mathcal{C}^G(k)$ to $\mathrm{NChow}^G(k)$. As a byproduct, we extend Panin's computations to all the aforementioned invariants E^G; see Theorem 5.10.

Making use of results of [Edidin and Graham 1998] on equivariant intersection theory, [Laterveer 1998; Iyer and Müller-Stach 2009] extended the theory of Chow motives to the G-equivariant setting. In Theorem 6.4, we relate this latter theory with that of G-equivariant noncommutative motives. Concretely, we construct a $\mathbb{Q}$-linear, fully faithful, symmetric monoidal Φ making the diagram

$$\begin{tikzcd}
\mathrm{SmProj}^G(k)^{\mathrm{op}} \arrow[rr, "X \mapsto G \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)"] \arrow[d, "\mathfrak{h}^G(-)_{\mathbb{Q}}"'] & & \mathrm{G\text{-}dgcat}_{\mathrm{sp}}(k) \arrow[d, "U^G(-)_{\mathbb{Q}}"] \\
\mathrm{Chow}^G(k)_{\mathbb{Q}} \arrow[d, "\pi"'] & & \mathrm{NChow}^G(k)_{\mathbb{Q}} \arrow[d, "(-)_{I_{\mathbb{Q}}}"] \\
\mathrm{Chow}^G(k)_{\mathbb{Q}}/_{-\otimes \mathbb{Q}(1)} \arrow[rr, "\Phi"'] & & \mathrm{NChow}^G(k)_{\mathbb{Q}, I_{\mathbb{Q}}}
\end{tikzcd} \tag{1.1}$$

commute, where $\mathrm{Chow}^G(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$ is the orbit category (see Section 6B) and $(-)_{I_{\mathbb{Q}}}$ the localization functor associated to the augmentation ideal $I \subset R(G) \overset{\mathrm{rank}}{\twoheadrightarrow} \mathbb{Z}$. Intuitively speaking, the commutative diagram (1.1) shows that after "$\otimes$-trivializing" the G-equivariant Tate motive $\mathbb{Q}(1)$ and localizing at the augmentation ideal $I_{\mathbb{Q}}$, the commutative world embeds fully faithfully into the noncommutative world.

The Grothendieck ring of varieties admits a G-equivariant analogue $K_0 \mathrm{Var}^G(k)$. Although very important, the structure of this latter ring is quite mysterious. In order to capture some of its flavor, several G-equivariant motivic measures have been built. In Theorem 8.2, we prove that the assignment $X \mapsto U^G(G \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$, with X a smooth projective G-variety, gives rise to a G-equivariant motivic measure $\mu_{\mathrm{nc}}^{\mathrm{G}} : K_0 \operatorname{Var}^{\mathrm{G}}(k) \to K_0(\mathrm{NChow}^{\mathrm{G}}(k))$ with values in the Grothendieck ring of the category of G-equivariant noncommutative Chow motives. It turns out that $\mu_{\mathrm{nc}}^{\mathrm{G}}$ contains a lot of interesting information. For example, when $k \subseteq \mathbb{C}$, the G-equivariant motivic measure $K_0 \operatorname{Var}^{\mathrm{G}}(k) \to R_{\mathbb{C}}(\mathrm{G})$, $X \mapsto \sum_i (-1)^i H_c^i(X^{\mathrm{an}}, \mathbb{C})$, factors through $\mu_{\mathrm{nc}}^{\mathrm{G}}$; see Proposition 8.3.

Applications. Let X be a smooth projective G-scheme. In order to study it, we can proceed in two distinct directions. In one direction, we can associate to X its G-equivariant Chow motive $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$. In another direction, we can associate to X its G-category of perfect complexes $\mathrm{G} \circlearrowright \mathrm{perf}(X)$. Making use of the bridge (1.1), we establish the following relation[1] between these two distinct mathematical objects.

Theorem 1.2. *If* $\mathrm{perf}(X)$ *admits a full exceptional collection* $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ *of length* n *of* G-invariant objects, *i.e.*, $\sigma^*(\mathcal{E}_i) \simeq \mathcal{E}_i$ *for every* $\sigma \in \mathrm{G}$, *then there exists a choice of integers* $r_1, \dots, r_n \in \{0, \dots, \dim(X)\}$ *such that*

$$\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}} \simeq \mathbb{L}^{\otimes r_1} \oplus \cdots \oplus \mathbb{L}^{\otimes r_n}, \tag{1.3}$$

where $\mathbb{L} \in \mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ *stands for the* G-*equivariant Lefschetz motive.*

Remark 1.4. A G-equivariant object is G-invariant, but the converse does not hold!

Theorem 1.2 can be applied to any G-action on projective spaces, quadrics, Grassmannians, etc; see Section 7B. Among other ingredients, its proof makes use of the language of G-dg categories and of the theory of G-equivariant noncommutative Chow motives. Intuitively speaking, Theorem 1.2 shows that the existence of a full exceptional collection of G-invariant objects “quasidetermines” the G-equivariant Chow motive $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$. The unique indeterminacy is the sequence $r_1, \dots, r_n$ of length n. Note that this indeterminacy cannot be refined. For example, the categories $\mathrm{perf}(\mathrm{Spec}(k) \amalg \mathrm{Spec}(k))$ and $\mathrm{perf}(\mathbb{P}^1)$ (equipped with the trivial G-action) admit full exceptional collections of length 2 but the corresponding G-equivariant Chow motives are distinct:

$$\mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k) \amalg \mathrm{Spec}(k))_{\mathbb{Q}} \simeq \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}^{\oplus 2} \not\simeq \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}} \oplus \mathbb{L} \simeq \mathfrak{h}^{\mathrm{G}}(\mathbb{P}^1)_{\mathbb{Q}}.$$

Corollary 1.5. *For every good* G-*cohomology theory* H_{G}^* (in the sense of Laterveer [1998, Definition 1.10]), *we have* $H_{\mathrm{G}}^i(X) = 0$ *if i is odd and* $\sum_i \dim H_{\mathrm{G}}^i(X) = n$.

Proof. It is proved in [Laterveer 1998, Proposition 1.12] that H_{G}^* factors through $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$. Using Theorem 1.2, we conclude $H_{\mathrm{G}}^*(X) \simeq H_{\mathrm{G}}^*(\mathbb{L})^{\otimes r_1} \oplus \cdots \oplus H_{\mathrm{G}}^*(\mathbb{L})^{\otimes r_n}$. The proof now follows from the facts that $\dim H_{\mathrm{G}}^2(\mathbb{L}) = 1$ and that $H_{\mathrm{G}}^i(\mathbb{L}) \simeq 0$ for $i \neq 2$. $\square$

[1]In the particular case where G is the trivial group, Theorem 1.2 was proved in [Marcolli and Tabuada 2015, Theorem 1.1].

Remark 1.6. Corollary 1.5 implies that the length of a hypothetical full exceptional collection of G-invariant objects is equal to $\sum_i \dim H^i_{\mathrm{G}}(X)$. Moreover, if $H^i_{\mathrm{G}}(X) \not\simeq 0$ for some odd integer i, then such a full exceptional collection cannot exist.

Theorem 1.2 also shows that the G-equivariant Chow motive $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$ loses all the information concerning the G-action on X. In contrast, the G-equivariant noncommutative Chow motive $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ keeps track of some of the G-action! Concretely, as proved in Proposition 7.8, there exist (nontrivial) cohomology classes $[\alpha_1], \ldots, [\alpha_n] \in H^2(\mathrm{G}, k^{\times})$ such that

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_1} k) \oplus \cdots \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_n} k). \tag{1.7}$$

This implies, in particular, that all the invariants $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ can be computed in terms of twisted group algebras $\bigoplus_{i=1}^n E(k_{\alpha_i}[\mathrm{G}])$. Taking into account the decompositions (1.3) and (1.7), the G-equivariant Chow motive $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$ and the G-equivariant noncommutative Chow motive $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ should be considered as complementary. While the former keeps track of the Tate twists but not of the G-action, the latter keeps track of the G-action but not of the Tate twists.

Remark 1.8. In Section 7C we also discuss the case of full exceptional collections where the objects are not G-invariant but rather permuted by the G-action.

Notation. Throughout the article, k will denote a base field and G a finite group. We will write $1 \in \mathrm{G}$ for the unit element and $|\mathrm{G}|$ for the order of G. Except in Section 2, we will always assume that $\mathrm{char}(k) \nmid |\mathrm{G}|$. All schemes will be defined over $\mathrm{Spec}(k)$, and all adjunctions will be displayed vertically with the left adjoint on the left side, and the right adjoint on the right.

2. Preliminaries

In this section we recall the main notions concerning dg categories, (twisted) equivariant perfect complexes, and group actions on dg categories. This gives us the opportunity to fix some notation which will be used throughout the article.

2A. ***Dg categories.*** Let $(\mathcal{C}(k), \otimes, k)$ be the symmetric monoidal category of dg k-vector spaces; we use cohomological notation. A *dg category* $\mathcal{A}$ is a category enriched over $\mathcal{C}(k)$, and a *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller's ICM survey [2006]. Let $\mathrm{dgcat}(k)$ be the category of small dg categories.

Let $\mathcal{A}$ be a dg category. The opposite dg category $\mathcal{A}^{\mathrm{op}}$ has the same objects and $\mathcal{A}^{\mathrm{op}}(x, y) := \mathcal{A}(y, x)$. The categories $\mathrm{Z}^0(\mathcal{A})$ and $\mathrm{H}^0(\mathcal{A})$ have the same objects, and $\mathrm{Z}^0(\mathcal{A})(x, y) := Z^0(\mathcal{A}(x, y))$ and $\mathrm{H}^0(\mathcal{A})(x, y) := H^0(\mathcal{A}(x, y))$, where $Z^0(-)$ denotes the 0th-cycles functor and $H^0(-)$ the 0th-cohomology functor.

Recall from [Keller 2006, §2.3] the definition of the dg category of dg functors $\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$. Given dg functors $F, G : \mathcal{A} \to \mathcal{B}$, a natural transformation of dg functors $\epsilon : F \Rightarrow G$ corresponds to an element of $\mathrm{Z}^0(\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B}))(F, G)$. When ϵ is invertible, we call it a *natural isomorphism of dg functors*. A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *dg equivalence* if there exists a dg functor $G : \mathcal{B} \to \mathcal{A}$ and natural isomorphisms of dg functors $F \circ G \Rightarrow \mathrm{id}$ and $\mathrm{id} \Rightarrow G \circ F$.

For a dg category $\mathcal{A}$, a (right) *dg $\mathcal{A}$-module* is a dg functor $M : \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$ with values in the dg category of dg k-vector spaces. Let us write $\mathcal{C}(\mathcal{A})$ for the category of dg $\mathcal{A}$-modules and $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ for the dg category $\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}}, \mathcal{C}_{\mathrm{dg}}(k))$. By construction, we have $\mathrm{Z}^0(\mathcal{C}_{\mathrm{dg}}(\mathcal{A})) \simeq \mathcal{C}(\mathcal{A})$. The dg category $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ comes equipped with the Yoneda dg functor $\mathcal{A} \to \mathcal{C}_{\mathrm{dg}}(\mathcal{A})$, $x \mapsto \mathcal{A}(-, x)$. Following [Keller 2006, §3.2], the *derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$* is defined as the localization of $\mathcal{C}(\mathcal{A})$ with respect to the (objectwise) quasi-isomorphisms. This category is triangulated and admits arbitrary direct sums. Let us write $\mathcal{D}_c(\mathcal{A})$ for the full subcategory of compact objects. In the same vein, let $\mathcal{C}_{c,\mathrm{dg}}(\mathcal{A})$ be the full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(\mathcal{A})$ consisting of those dg $\mathcal{A}$-modules which belong to $\mathcal{D}_c(\mathcal{A})$. By construction, we have $\mathrm{H}^0(\mathcal{C}_{c,\mathrm{dg}}(\mathcal{A})) \simeq \mathcal{D}_c(\mathcal{A})$.

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a *Morita equivalence* if the restriction functor $\mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ is an equivalence of (triangulated) categories. An example is given by the Yoneda dg functor $\mathcal{A} \to \mathcal{C}_{c,\mathrm{dg}}(\mathcal{A})$. As proved in [Tabuada 2005, Théorème 5.3], the category $\mathrm{dgcat}(k)$ admits a Quillen model structure whose weak equivalences are the Morita equivalences. Let $\mathrm{Hmo}(k)$ be the associated homotopy category.

Given dg categories $\mathcal{A}$ and $\mathcal{B}$, let us write $\mathcal{A} \times \mathcal{B}$, $\mathcal{A} \amalg \mathcal{B}$, and $\mathcal{A} \otimes \mathcal{B}$ for their product, coproduct, and tensor product, respectively.

A *dg $\mathcal{A}$-$\mathcal{B}$-bimodule* is a dg functor $\mathrm{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k)$, or equivalently, a dg $(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$-module. An example is the dg $\mathcal{A}$-$\mathcal{B}$-bimodule

$$ {}_F\mathrm{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k), \quad (x, z) \mapsto \mathcal{B}(z, F(x)) \tag{2.1} $$

associated to a dg functor $F : \mathcal{A} \to \mathcal{B}$. Let us write $\mathrm{rep}(\mathcal{A}, \mathcal{B})$ for the full triangulated subcategory $\mathcal{D}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of those dg $\mathcal{A}$-$\mathcal{B}$-bimodules B such that for every $x \in \mathcal{A}$ the dg $\mathcal{B}$-module $\mathrm{B}(x, -)$ belongs to $\mathcal{D}_c(\mathcal{B})$. In the same vein, let $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ be the full dg subcategory of $\mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ consisting of those dg $\mathcal{A}$-$\mathcal{B}$-bimodules which belong to $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. By construction, $\mathrm{H}^0(\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})) \simeq \mathrm{rep}(\mathcal{A}, \mathcal{B})$.

Following [Kontsevich 1998; 2005; 2009; 2010], a dg category $\mathcal{A}$ is called *smooth* if the dg $\mathcal{A}$-$\mathcal{A}$-bimodule ${}_{\mathrm{id}}\mathrm{B}$ belongs to the triangulated category $\mathcal{D}_c(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})$ and *proper* if $\sum_i \dim H^i \mathcal{A}(x, y) < \infty$ for any ordered pair of objects (x, y). Examples include the finite dimensional k-algebras of finite global dimension (when k is perfect) as well as the dg categories $\mathrm{perf}_{\mathrm{dg}}(X)$ associated to smooth proper schemes X. Given smooth proper dg categories $\mathcal{A}$ and $\mathcal{B}$, the associated dg categories $\mathcal{A} \times \mathcal{B}$, $\mathcal{A} \amalg \mathcal{B}$, and $\mathcal{A} \otimes \mathcal{B}$ are also smooth proper. Finally, let us write $\mathrm{dgcat}_{\mathrm{sp}}(k)$ for the full subcategory of $\mathrm{dgcat}(k)$ consisting of the smooth proper dg categories.

2B. ***(Twisted) equivariant perfect complexes.*** Let $\mathcal{E}$ be an abelian (or exact) category. Following [Keller 2006, §4.4], the *derived dg category* $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$ *of* $\mathcal{E}$ is defined as the dg quotient $\mathcal{C}_{\mathrm{dg}}(\mathcal{E})/\mathcal{A}c_{\mathrm{dg}}(\mathcal{E})$ of the dg category of complexes over $\mathcal{E}$ by its full dg subcategory of acyclic complexes. Given a quasicompact quasiseparated scheme X, we write $\mathrm{Mod}(X)$ for the Grothendieck category of $\mathcal{O}_X$-modules, $\mathcal{D}(X)$ for the derived category $\mathcal{D}(\mathrm{Mod}(X))$, and $\mathcal{D}_{\mathrm{dg}}(X)$ for the dg category $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$ with $\mathcal{E} := \mathrm{Mod}(X)$. In the same vein, we write $\mathrm{perf}(X)$ for the full triangulated subcategory, and $\mathrm{perf}_{\mathrm{dg}}(X)$ for the full dg subcategory, of perfect complexes.

Given a quasicompact quasiseparated G-scheme X, we write $\mathrm{Mod}^{\mathrm{G}}(X)$ for the Grothendieck category of G-equivariant $\mathcal{O}_X$-modules, $\mathcal{D}^{\mathrm{G}}(X)$ for the derived category $\mathcal{D}(\mathrm{Mod}^{\mathrm{G}}(X))$, and $\mathcal{D}^{\mathrm{G}}_{\mathrm{dg}}(X)$ for the dg category $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$ with $\mathcal{E} := \mathrm{Mod}^{\mathrm{G}}(X)$. In the same vein, we write $\mathrm{perf}^{\mathrm{G}}(X)$ for the full triangulated subcategory, and $\mathrm{perf}^{\mathrm{G}}_{\mathrm{dg}}(X)$ for the full dg subcategory, of G-equivariant perfect complexes.

Definition 2.2. A map $\alpha : \mathrm{G} \times \mathrm{G} \to k^{\times}$ is called a 2-*cocycle* if $\alpha(1, \sigma) = \alpha(\sigma, 1) = 1$ and $\alpha(\rho, \alpha)\alpha(\tau, \rho\sigma) = \alpha(\tau, \rho)\alpha(\tau\rho, \sigma)$ for every $\sigma, \rho, \tau \in \mathrm{G}$.

Given a quasicompact quasiseparated G-scheme X and a 2-*cocycle* α, we write $\mathrm{Mod}^{\mathrm{G},\alpha}(X)$ for the Grothendieck category of α-twisted G-equivariant $\mathcal{O}_X$-modules, $\mathcal{D}^{\mathrm{G},\alpha}(X)$ for the derived category $\mathcal{D}(\mathrm{Mod}^{\mathrm{G},\alpha}(X))$, and $\mathcal{D}^{\mathrm{G},\alpha}_{\mathrm{dg}}(X)$ for the dg category $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$ with $\mathcal{E} := \mathrm{Mod}^{\mathrm{G},\alpha}(X)$. In the same vein, we write $\mathrm{perf}^{\mathrm{G},\alpha}(X)$ for the full triangulated subcategory, and $\mathrm{perf}^{\mathrm{G},\alpha}_{\mathrm{dg}}(X)$ for the full dg subcategory, of G-equivariant perfect complexes.

2C. ***Group actions on dg categories.*** Following [Deligne 1997; Elagin 2014], we introduce the following notion:

Definition 2.3. A (left) G-*action on a dg category* $\mathcal{A}$ consists of the data

(i) a family of dg equivalences $\phi_\sigma : \mathcal{A} \to \mathcal{A}$ for $\sigma \in \mathrm{G}$, with $\phi_1 = \mathrm{id}$;

(ii) a family of natural isomorphisms of dg functors $\epsilon_{\rho,\sigma} : \phi_\rho \circ \phi_\sigma \Rightarrow \phi_{\rho\sigma}$ for $\sigma, \rho \in \mathrm{G}$, with $\epsilon_{1,\sigma} = \epsilon_{\sigma,1} = \mathrm{id}$, such that the equality $\epsilon_{\tau\rho,\sigma} \circ (\epsilon_{\tau,\rho} \circ \phi_\sigma) = \epsilon_{\tau,\rho\sigma} \circ (\phi_\tau \circ \epsilon_{\rho,\sigma})$ holds for every $\sigma, \rho, \tau \in \mathrm{G}$.

Throughout the article, a dg category $\mathcal{A}$ equipped with a G-action will be denoted by $\mathrm{G} \circlearrowright \mathcal{A}$ and will be called a G-*dg category*.

Example 2.4 (G-schemes). Given a quasicompact quasiseparated G-scheme X, the dg category $\mathrm{perf}_{\mathrm{dg}}(X)$ inherits a G-action induced by the pull-back dg equivalences $\phi_\sigma := \sigma^*$; consult [Elagin 2014; Sosna 2012] for details.

Example 2.5 (line bundles). Let X be a quasicompact quasiseparated scheme. In the case where G can be realized as a subgroup of the Picard group $\mathrm{Pic}(X)$, the dg category $\mathrm{perf}_{\mathrm{dg}}(X)$ inherits a G-action induced by the dg equivalences $\phi_\sigma := - \otimes_{\mathcal{O}_X} \mathcal{L}_\sigma$, where $\mathcal{L}_\sigma$ stands for the invertible line bundle associated to $\sigma \in \mathrm{G}$; consult [Elagin 2014; Sosna 2012] for details.

Example 2.6 (G-algebras). Given a G-action on a (dg) algebra A, the associated dg category with a single object inherits a G-action with $\epsilon_{\rho,\sigma} := \mathrm{id}$.

Example 2.7 (2-cocycles). Given a 2-cocycle $\alpha : \mathrm{G} \times \mathrm{G} \to k^\times$, the dg category k inherits a G-action given by $\phi_\sigma := \mathrm{id}$ and $\epsilon_{\rho,\sigma} := \alpha(\rho, \sigma)$. We will denote this G-dg category by $\mathrm{G} \circlearrowright_\alpha k$. Note that these are all the possible G-actions.

Remark 2.8. Given a G-dg category $\mathrm{G} \circlearrowright \mathcal{A}$, $\mathcal{A}^{\mathrm{op}}$ inherits a G-action given by the dg equivalences ϕ_σ and by the natural isomorphisms of dg functors $\epsilon_{\rho,\sigma}^{-1}$.

Given G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$, the product $\mathcal{A} \times \mathcal{B}$ inherits a G-action given by the dg equivalences $\phi_\sigma \times \phi_\sigma$ and by the natural isomorphisms of dg equivalences $\epsilon_{\rho,\sigma} \times \epsilon_{\rho,\sigma}$, and likewise the tensor product $\mathcal{A} \otimes \mathcal{B}$ inherits a G-action by dg equivalences $\phi_\sigma \otimes \phi_\sigma$ and natural isomorphisms of dg equivalences $\epsilon_{\rho,\sigma} \otimes \epsilon_{\rho,\sigma}$. In the same vein, the dg category of dg functors $\mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ inherits a G-action given by the dg equivalences $F \mapsto \phi_\sigma \circ F \circ \phi_{\sigma^{-1}}$ and by the natural isomorphisms of dg functors induced from $\epsilon_{\sigma^{-1},\rho^{-1}}$ and $\epsilon_{\rho,\sigma}$.

Let $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$ be two G-dg categories, and $\mathcal{C}_{\mathrm{dg}}(k)$ the dg category of dg k-vector spaces equipped with the trivial G-action. Thanks to the above considerations, $\mathcal{C}_{\mathrm{dg}}(\mathcal{A}) := \mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}}, \mathcal{C}_{\mathrm{dg}}(k))$ inherits a G-action, which restricts to $\mathcal{C}_{c,\mathrm{dg}}(\mathcal{A})$. Similarly, the dg category of dg $\mathcal{A}$-$\mathcal{B}$-bimodules $\mathcal{C}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}) := \mathrm{Fun}_{\mathrm{dg}}(\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}, \mathcal{C}_{\mathrm{dg}}(k))$ inherits a G-action, which restricts to $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$.

Definition 2.9. A G-*equivariant dg functor* $\mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$ consists of the data

(i) a dg functor $F : \mathcal{A} \to \mathcal{B}$;

(ii) a family of natural isomorphisms of dg functors $\eta_\sigma : F \circ \phi_\sigma \Rightarrow \phi_\sigma \circ F$, for $\sigma \in \mathrm{G}$, such that $\eta_{\rho\sigma} \circ (F \circ \epsilon_{\rho,\sigma}) = (\epsilon_{\rho,\sigma} \circ F) \circ (\phi_\rho \circ \eta_\sigma) \circ (\eta_\rho \circ \phi_\sigma)$ for every $\sigma, \rho \in \mathrm{G}$.

A G-equivariant dg functor with a Morita equivalence F is called a G-*equivariant Morita equivalence*. For example, given a small G-dg category $\mathrm{G} \circlearrowright \mathcal{A}$, the Yoneda dg functor $\mathcal{A} \to \mathcal{C}_{c,\mathrm{dg}}(\mathcal{A})$, $x \mapsto \mathcal{A}(-, x)$, is a G-equivariant Morita equivalence.

Let us denote by G-dgcat(k) the category whose objects are the small G-dg categories and whose morphisms are the G-equivariant dg functors. Given G-equivariant dg functors $F : \mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$ and $G : \mathrm{G} \circlearrowright \mathcal{B} \to \mathrm{G} \circlearrowright \mathcal{C}$, their composition is defined as $(G \circ F, (\eta_\sigma \circ F) \circ (G \circ \eta_\sigma))$. The category G-dgcat$(k)$ carries a symmetric monoidal structure given by $(\mathrm{G} \circlearrowright \mathcal{A}) \otimes (\mathrm{G} \circlearrowright \mathcal{B}) := \mathrm{G} \circlearrowright (\mathcal{A} \otimes \mathcal{B})$. This monoidal structure is closed, with internal-Homs given by $\mathrm{G} \circlearrowright \mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$.

Equivariant objects. Let $\mathrm{G} \circlearrowright \mathcal{A}$ be a G-dg category. A G-*equivariant object in* $\mathrm{G} \circlearrowright \mathcal{A}$ consists of an object $x \in \mathcal{A}$ and a family of closed degree zero isomorphisms $\theta_\sigma : x \to \phi_\sigma(x)$ for $\sigma \in \mathrm{G}$, with $\theta_1 = \mathrm{id}$, such that the compositions

$$x \xrightarrow{\theta_\rho} \phi_\rho(x) \xrightarrow{\phi_\rho(\theta_\sigma)} \phi_\rho(\phi_\sigma(x)) \xrightarrow{\epsilon_{\rho,\sigma}(x)} \phi_{\rho\sigma}(x)$$

are equal to $\theta_{\rho\sigma} : x \to \phi_{\rho\sigma}(x)$ for every $\sigma, \rho \in \mathrm{G}$. A *morphism of* G-*equivariant objects* $(x, \theta_\sigma) \to (y, \theta_\sigma)$ is an element f of the dg k-vector space $\mathcal{A}(x, y)$ such that $\theta_\sigma \circ f = \phi_\sigma(f) \circ \theta_\sigma$ for every $\sigma \in \mathrm{G}$. Let us write $\mathcal{A}^{\mathrm{G}}$ for the dg category of G-equivariant objects in $\mathrm{G} \circlearrowright \mathcal{A}$. From a topological viewpoint, the dg category $\mathcal{A}^{\mathrm{G}}$ may be understood as the "homotopy fixed points" of the G-action on $\mathcal{A}$.

Example 2.10 (equivariant perfect complexes). Let $\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)$ be equipped with the G-action of Example 2.4. When $\mathrm{char}(k) \nmid |G|$, Elagin [2011, Theorem 9.6; 2014, Theorem 1.1] proved that $\mathrm{perf}_{\mathrm{dg}}(X)^{\mathrm{G}}$ is Morita equivalent to the dg category $\mathrm{perf}^{\mathrm{G}}_{\mathrm{dg}}(X)$; see Section 2B.

Example 2.11 (covering spaces). Let $\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)$ be as in Example 2.5. Consider the relative spectrum $Y := \mathrm{Spec}_X\big(\bigoplus_{\sigma \in \mathrm{G}} \mathcal{L}_\sigma^{-1}\big)$, which is a nonramified $|G|$-fold cover of X. When $\mathrm{char}(k) \nmid |G|$, Elagin [2014, Theorem 1.2] proved that $\mathrm{perf}_{\mathrm{dg}}(X)^{\mathrm{G}}$ is Morita equivalent to $\mathrm{perf}_{\mathrm{dg}}(Y)$.

Example 2.12 (semidirect product algebras). Let $\mathrm{G} \circlearrowright A$ be as in Example 2.6. As mentioned in Remark 2.8, the dg category $\mathcal{C}_{c,\mathrm{dg}}(A)$ inherits a G-action. When $\mathrm{char}(k) \nmid |G|$, the dg category $\mathcal{C}_{c,\mathrm{dg}}(A)^{\mathrm{G}}$ is Morita equivalent to the semidirect product (dg) algebra $A \rtimes \mathrm{G}$.

Example 2.13 (twisted group algebras). Let $\mathrm{G} \circlearrowright_\alpha k$ be as in Example 2.7. Similarly to Example 2.12, when $\mathrm{char}(k) \nmid |G|$, the dg category $\mathcal{C}_{c,\mathrm{dg}}(k)^{\mathrm{G}}$ is Morita equivalent to the twisted group algebra $k_\alpha[\mathrm{G}]$.

Remark 2.14 (G-equivariant dg functors). Let $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$ be two dg categories. The assignment $(F, \eta_\sigma) \mapsto (F, (\eta_\sigma \circ \phi_{\sigma^{-1}}) \circ (F \circ \epsilon^{-1}_{\sigma,\sigma^{-1}}))$ establishes a bijection between the set of G-equivariant dg functors $\mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$ and the set of G-equivariant objects in $\mathrm{G} \circlearrowright \mathrm{Fun}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$; see Remark 2.8. Its inverse is given by the assignment $(F, \theta_\sigma) \mapsto (F, (\phi_\sigma \circ F \circ \epsilon_{\sigma^{-1},\sigma}) \circ (\theta_\sigma \circ \phi_\sigma))$.

Given a G-equivariant dg functor $F : \mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$, the assignment $(x, \theta_\sigma) \mapsto (F(x), \eta_\sigma \circ F(\theta_\sigma))$ yields a dg functor $F^{\mathrm{G}} : \mathcal{A}^{\mathrm{G}} \to \mathcal{B}^{\mathrm{G}}$. We hence obtain a functor

$$\mathrm{G}\text{-dgcat}(k) \to \mathrm{dgcat}(k), \quad \mathrm{G} \circlearrowright \mathcal{A} \mapsto \mathcal{A}^{\mathrm{G}}. \tag{2.15}$$

Twisted equivariant objects. Let $\alpha : \mathrm{G} \times \mathrm{G} \to k^\times$ be a 2-cocycle and $\mathrm{G} \circlearrowright \mathcal{A}$ a G-dg category. An α-*twisted* G-*equivariant object in* $\mathrm{G} \circlearrowright \mathcal{A}$ consists of an object $x \in \mathcal{A}$ and a family of closed degree zero isomorphisms $\theta_\sigma : x \to \phi_\sigma(x)$ for $\sigma \in \mathrm{G}$, with $\theta_1 = \mathrm{id}$, such that the compositions

$$x \xrightarrow{\theta_\rho} \phi_\rho(x) \xrightarrow{\phi_\rho(\theta_\sigma)} \phi_\rho(\phi_\sigma(x)) \xrightarrow{\epsilon_{\rho,\sigma}(x)} \phi_{\rho\sigma}(x)$$

are equal to $\alpha(\rho,\sigma)\theta_{\rho\sigma} : x \to \phi_{\rho\sigma}(x)$ for every $\sigma, \rho \in \mathrm{G}$. A *morphism of α-twisted* G*-equivariant objects* $(x, \theta_\sigma) \to (y, \theta_\sigma)$ is an element f of the dg k-vector space $\mathcal{A}(x, y)$ such that $\theta_\sigma \circ f = \phi_\sigma(f) \circ \theta_\sigma$ for every $\sigma \in \mathrm{G}$. Let us write $\mathcal{A}^{\mathrm{G},\alpha}$ for the dg category of α-twisted G-equivariant objects in $\mathrm{G} \circlearrowright \mathcal{A}$. Note that $\mathcal{A}^{\mathrm{G},\alpha}$ identifies with the dg category of G-equivariant objects in $(\mathrm{G} \circlearrowright \mathcal{A}) \otimes (\mathrm{G} \circlearrowright_{\alpha^{-1}} k)$.

Example 2.16 (twisted equivariant perfect complexes). Let $\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)$ be as in Example 2.4. Similarly to Example 2.10, $\mathrm{perf}_{\mathrm{dg}}(X)^{\mathrm{G},\alpha}$ is Morita equivalent to the dg category of α-twisted G-equivariant perfect complexes $\mathrm{perf}_{\mathrm{dg}}^{\mathrm{G},\alpha}(X)$.

3. Equivariant noncommutative motives

In this section we introduce the category of equivariant noncommutative Chow motives. We start by recalling its nonequivariant predecessor.

3A. ***Noncommutative Chow motives.*** Recall from Section 2A that $\mathrm{Hmo}(k)$ is the localization of $\mathrm{dgcat}(k)$ at the class of Morita equivalences. As proved in [Tabuada 2005, Corollaire 5.10], there is a canonical bijection between $\mathrm{Hom}_{\mathrm{Hmo}(k)}(\mathcal{A}, \mathcal{B})$ and the set of isomorphism classes of the triangulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$. Under this bijection, the composition law of $\mathrm{Hmo}(k)$ is induced by the triangulated bifunctors

$$\mathrm{rep}(\mathcal{A}, \mathcal{B}) \times \mathrm{rep}(\mathcal{B}, \mathcal{C}) \to \mathrm{rep}(\mathcal{A}, \mathcal{C}), \quad (\mathrm{B}, \mathrm{B}') \mapsto \mathrm{B} \otimes_{\mathcal{B}} \mathrm{B}' \tag{3.1}$$

and the localization functor from $\mathrm{dgcat}(k)$ to $\mathrm{Hmo}(k)$ is given by

$$\mathrm{dgcat}(k) \to \mathrm{Hmo}(k), \quad \mathcal{A} \mapsto \mathcal{A}, \quad (\mathcal{A} \xrightarrow{F} \mathcal{B}) \mapsto {}_F\mathrm{B}. \tag{3.2}$$

The *additivization* of $\mathrm{Hmo}(k)$ is the additive category $\mathrm{Hmo}_0(k)$ with the same objects and with morphisms $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(\mathcal{A}, \mathcal{B})$ given by $K_0\,\mathrm{rep}(\mathcal{A}, \mathcal{B})$. The composition law is induced by the triangulated bifunctors (3.1). By construction, $\mathrm{Hmo}_0(k)$ comes equipped with the functor

$$\mathrm{Hmo}(k) \to \mathrm{Hmo}_0(k), \quad \mathcal{A} \mapsto \mathcal{A}, \quad \mathrm{B} \mapsto [\mathrm{B}]. \tag{3.3}$$

Let us denote by $U : \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ the composition (3.3)$\circ$(3.2). As proved in [Tabuada 2005, Lemme 6.1], the category $\mathrm{Hmo}_0(k)$ carries a symmetric monoidal structure induced by the tensor product of dg categories and by the triangulated bifunctors

$$\mathrm{rep}(\mathcal{A}, \mathcal{B}) \times \mathrm{rep}(\mathcal{C}, \mathcal{D}) \to \mathrm{rep}(\mathcal{A} \otimes \mathcal{C}, \mathcal{B} \otimes \mathcal{D}), \quad (\mathrm{B}, \mathrm{B}') \mapsto \mathrm{B} \otimes \mathrm{B}'.$$

By construction, the functor U is symmetric monoidal.

The category $\mathrm{NChow}(k)$ of *noncommutative Chow motives*[2] is defined as the idempotent completion of the full subcategory of $\mathrm{Hmo}_0(k)$ consisting of the objects $U(\mathcal{A})$ with $\mathcal{A}$ a smooth proper dg category. The category $\mathrm{NChow}(k)$ is additive, idempotent complete, and rigid symmetric monoidal.

3B. ***Equivariant noncommutative Chow motives.*** Let $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$ be two small G-dg categories. As mentioned in Remark 2.8, the dg category $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ inherits a G-action. As a consequence, we obtain an induced G-action on the triangulated category

$$\mathrm{H}^0(\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})) \simeq \mathrm{rep}(\mathcal{A}, \mathcal{B}).$$

Due to [Elagin 2014, Theorem 8.7], the category of G-equivariant objects $\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}}$ is also triangulated.

Given small G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$, $\mathrm{G} \circlearrowright \mathcal{B}$, and $\mathrm{G} \circlearrowright \mathcal{C}$, consider the G-equivariant dg functor

$$\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B}) \times \mathrm{rep}_{\mathrm{dg}}(\mathcal{B}, \mathcal{C}) \to \mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{C}), \quad (\mathrm{B}, \mathrm{B}') \mapsto \mathrm{B} \otimes_{\mathcal{B}} \mathrm{B}'$$

(G acts diagonally on the left-hand side). By first applying $\mathrm{H}^0(-)$ and then $(-)^{\mathrm{G}}$, we obtain an induced triangulated bifunctor

$$\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \times \mathrm{rep}(\mathcal{B}, \mathcal{C})^{\mathrm{G}} \to \mathrm{rep}(\mathcal{A}, \mathcal{C})^{\mathrm{G}}. \tag{3.4}$$

Let $\mathrm{Hmo}^{\mathrm{G}}(k)$ be the category with the same objects as $\mathrm{G}\text{-dgcat}(k)$ and with morphisms $\mathrm{Hom}_{\mathrm{Hmo}^{\mathrm{G}}(k)}(\mathrm{G} \circlearrowright \mathcal{A}, \mathrm{G} \circlearrowright \mathcal{B})$ given by the set of isomorphism classes of the category $\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}}$. The composition law is induced by the triangulated bifunctors (3.4). Thanks to Remark 2.14, we have the functor

$$\mathrm{G}\text{-dgcat}(k) \to \mathrm{Hmo}^{\mathrm{G}}(k), \quad \mathrm{G} \circlearrowright \mathcal{A} \mapsto \mathrm{G} \circlearrowright \mathcal{A}, \quad (\mathrm{G} \circlearrowright \mathcal{A} \xrightarrow{F} \mathrm{G} \circlearrowright \mathcal{B}) \mapsto {}_F\mathrm{B}. \tag{3.5}$$

Lemma 3.6. *The functor* (3.5) *inverts* G*-equivariant Morita equivalences.*

Proof. Let $\mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$ be a G-equivariant Morita equivalence. Thanks to the Yoneda lemma, it suffices to show that for every object $\mathrm{G} \circlearrowright \mathcal{C}$ the homomorphism

$$\mathrm{Hom}_{\mathrm{Hmo}^{\mathrm{G}}(k)}(\mathrm{G} \circlearrowright \mathcal{C}, \mathrm{G} \circlearrowright \mathcal{A}) \to \mathrm{Hom}_{\mathrm{Hmo}^{\mathrm{G}}(k)}(\mathrm{G} \circlearrowright \mathcal{C}, \mathrm{G} \circlearrowright \mathcal{B}) \tag{3.7}$$

is invertible. Since $\mathrm{G} \circlearrowright \mathcal{A} \to \mathrm{G} \circlearrowright \mathcal{B}$ is a G-equivariant Morita equivalence, we have an induced G-equivariant equivalence of categories $\mathrm{rep}(\mathcal{C}, \mathcal{A}) \to \mathrm{rep}(\mathcal{C}, \mathcal{B})$, and consequently an equivalence of categories $\mathrm{rep}(\mathcal{C}, \mathcal{A})^{\mathrm{G}} \to \mathrm{rep}(\mathcal{C}, \mathcal{B})^{\mathrm{G}}$. □

The *additivization* of $\mathrm{Hmo}^{\mathrm{G}}(k)$ is the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ with the same objects and with abelian groups of morphisms $\mathrm{Hom}_{\mathrm{Hmo}_0^{\mathrm{G}}(k)}(\mathrm{G} \circlearrowright \mathcal{A}, \mathrm{G} \circlearrowright \mathcal{B})$ given by $K_0\,\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}}$, where $K_0\,\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}}$ stands for the Grothendieck group of the tri-

[2] For further information concerning noncommutative (Chow) motives, consult [Tabuada 2015].

angulated category $\mathrm{rep}(\mathcal{A},\mathcal{B})^{\mathrm{G}}$. The composition law is induced by the triangulated bifunctors (3.4). By construction, $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ comes equipped with the functor

$$\mathrm{Hmo}^{\mathrm{G}}(k) \to \mathrm{Hmo}_0^{\mathrm{G}}(k), \quad \mathrm{G} \circlearrowright \mathcal{A} \mapsto \mathrm{G} \circlearrowright \mathcal{A}, \quad \mathrm{B} \mapsto [\mathrm{B}]. \tag{3.8}$$

Let us denote by $U^{\mathrm{G}} : \mathrm{G}\text{-dgcat}(k) \to \mathrm{Hmo}_0^{\mathrm{G}}(k)$ the composition (3.8) $\circ$ (3.5).

Given small G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$, $\mathrm{G} \circlearrowright \mathcal{B}$, $\mathrm{G} \circlearrowright \mathcal{C}$, and $\mathrm{G} \circlearrowright \mathcal{D}$, consider the G-equivariant dg functor

$$\mathrm{rep}_{\mathrm{dg}}(\mathcal{A},\mathcal{B}) \times \mathrm{rep}_{\mathrm{dg}}(\mathcal{C},\mathcal{D}) \to \mathrm{rep}_{\mathrm{dg}}(\mathcal{A}\otimes\mathcal{C},\mathcal{B}\otimes\mathcal{D}), \quad (\mathrm{B},\mathrm{B}') \mapsto \mathrm{B}\otimes\mathrm{B}'$$

(G acts diagonally on the left-hand side). By first applying $\mathrm{H}^0(-)$ and then $(-)^{\mathrm{G}}$, we obtain an induced triangulated bifunctor

$$\mathrm{rep}(\mathcal{A},\mathcal{B})^{\mathrm{G}} \times \mathrm{rep}(\mathcal{C},\mathcal{D})^{\mathrm{G}} \to \mathrm{rep}(\mathcal{A}\otimes\mathcal{C},\mathcal{B}\otimes\mathcal{D})^{\mathrm{G}}. \tag{3.9}$$

The assignment $(\mathrm{G} \circlearrowright \mathcal{A}, \mathrm{G} \circlearrowright \mathcal{B}) \mapsto \mathrm{G} \circlearrowright (\mathcal{A}\otimes\mathcal{B})$, combined with the triangulated bifunctors (3.9), gives rise to a symmetric monoidal structure on $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ with $\otimes$-unit $U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)$. By construction, the functor U^{G} is symmetric monoidal.

Proposition 3.10. *The category* $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ *is additive. Moreover, we have*

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{B}) \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A}\times\mathcal{B})) \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A}\amalg\mathcal{B})) \tag{3.11}$$

for any two small G*-dg categories* $\mathrm{G} \circlearrowright \mathcal{A}$ *and* $\mathrm{G} \circlearrowright \mathcal{B}$.

Proof. By construction, the morphism sets of $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ are abelian groups and the composition law is bilinear. Hence, it suffices to show the isomorphisms (3.11), which imply in particular that the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ admits direct sums. Given a small G-dg category $\mathrm{G} \circlearrowright \mathcal{C}$, we have equivalences of categories

$$\mathrm{rep}(\mathcal{C},\mathcal{A}\times\mathcal{B})^{\mathrm{G}} \simeq \mathrm{rep}(\mathcal{C},\mathcal{A})^{\mathrm{G}} \times \mathrm{rep}(\mathcal{C},\mathcal{B})^{\mathrm{G}},$$
$$\mathrm{rep}(\mathcal{A}\amalg\mathcal{B},\mathcal{C})^{\mathrm{G}} \simeq \mathrm{rep}(\mathcal{A},\mathcal{C})^{\mathrm{G}} \times \mathrm{rep}(\mathcal{B},\mathcal{C})^{\mathrm{G}}.$$

By passing to the Grothendieck group K_0, we conclude that $U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A}\times\mathcal{B}))$ is the product, and $U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A}\amalg\mathcal{B}))$ the coproduct, in $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ of $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})$ with $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{B})$. Using the fact that the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ is $\mathbb{Z}$-linear, we obtain finally the isomorphisms (3.11). □

Definition 3.12. The category $\mathrm{NChow}^{\mathrm{G}}(k)$ of G-*equivariant noncommutative Chow motives* is the idempotent completion of the full subcategory of $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ consisting of the objects $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})$ with $\mathcal{A}$ a smooth proper dg category.

Since the smooth proper dg categories are stable under (co)products, it follows from the isomorphisms (3.11) that the category $\mathrm{NChow}^{\mathrm{G}}(k)$ is also additive.

Proposition 3.13. *The symmetric monoidal category* $\mathrm{NChow}^{\mathrm{G}}(k)$ *is rigid.*

Proof. By construction of $\mathrm{NChow}^{\mathrm{G}}(k)$, it suffices to show that $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})$, with $\mathcal{A}$ a smooth proper dg category $\mathcal{A}$, is strongly dualizable. Take for the dual of $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})$ the object $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}^{\mathrm{op}})$ (see Remark 2.8). The dg $\mathcal{A}$-$\mathcal{A}$-bimodule

$$ {}_{\mathrm{id}}\mathrm{B} : \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k), \quad (x, y) \mapsto \mathcal{A}(y, x) \tag{3.14} $$

associated to the identity dg functor $\mathrm{id} : \mathcal{A} \to \mathcal{A}$ is canonically a G-equivariant object. Moreover, since $\mathcal{A}$ is smooth proper, the dg $\mathcal{A}$-$\mathcal{A}$-bimodule (3.14) belongs to the triangulated categories $\mathrm{rep}(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}, k)^{\mathrm{G}}$ and $\mathrm{rep}(k, \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})^{\mathrm{G}}$. Let us then take for the evaluation morphism the Grothendieck class of (3.14) in

$$ \mathrm{Hom}_{\mathrm{NChow}^{\mathrm{G}}(k)}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}})), U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)) \simeq K_0 \, \mathrm{rep}(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}, k)^{\mathrm{G}}, $$

and for the coevaluation morphism the Grothendieck class of (3.14) in

$$ \mathrm{Hom}_{\mathrm{NChow}^{\mathrm{G}}(k)}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k), U^{\mathrm{G}}(\mathrm{G} \circlearrowright (\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}))) \simeq K_0 \, \mathrm{rep}(k, \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})^{\mathrm{G}}. $$

This data satisfies the axioms of a strongly dualizable object. □

Proposition 3.15. *For every cohomology class* $[\alpha] \in H^2(\mathrm{G}, k^\times)$, *the ring of endomorphisms*

$$ \mathrm{End}_{\mathrm{NChow}^{\mathrm{G}}(k)}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)) \tag{3.16} $$

(*where multiplication is given by composition*) *is isomorphic to the representation ring*[3] $R(\mathrm{G})$ *of the group* G.

Proof. By construction of $\mathrm{NChow}^{\mathrm{G}}(k)$, we have canonical ring identifications

$$ \mathrm{End}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)) = K_0(\mathrm{rep}(k, k)^{\mathrm{G}, \alpha\alpha^{-1}}) \simeq K_0 \, \mathrm{rep}(k, k)^{\mathrm{G}} = \mathrm{End}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)). $$

Hence, it suffices to prove the particular case $\alpha = 1$. As mentioned in Example 2.10, the category $\mathrm{rep}(k, k)^{\mathrm{G}} \simeq \mathcal{D}_c(k)^{\mathrm{G}} \simeq \mathrm{perf}(\mathrm{Spec}(k))^{\mathrm{G}}$ is equivalent to $\mathrm{perf}^{\mathrm{G}}(\mathrm{Spec}(k))$. This implies that the abelian group (3.16), with $\alpha = 1$, is isomorphic to the G-equivariant Grothendieck group $K_0(\mathrm{perf}^{\mathrm{G}}(\mathrm{Spec}(k)))$ of $\mathrm{Spec}(k)$. In what concerns the ring structure, the Eckmann–Hilton argument, combined with the fact that $U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)$ is the $\otimes$-unit of $\mathrm{NChow}^{\mathrm{G}}(k)$, implies that the multiplication on (3.16) given by composition agrees with the multiplication on (3.16) induced by the symmetric monoidal structure on $\mathrm{perf}^{\mathrm{G}}(\mathrm{Spec}(k))$. The proof follows now from the definition of $R(\mathrm{G})$ as the G-equivariant Grothendieck ring of $\mathrm{Spec}(k)$. □

Proposition 3.15 gives rise automatically to the following enhancement:

Corollary 3.17. *The category* $\mathrm{NChow}^{\mathrm{G}}(k)$ (*and* $\mathrm{Hmo}_0^{\mathrm{G}}(k)$) *is* $R(\mathrm{G})$*-linear.*

[3] Consult Serre's book [1977] for a detailed study of the representation ring.

3C. *Coefficients.* Given a commutative ring R, let $\mathrm{Hmo}_0^{\mathrm{G}}(k)_R$ be the R-linear additive category obtained from $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ by tensoring each abelian group of morphisms with R. By construction, $\mathrm{Hmo}_0^{\mathrm{G}}(k)_R$ inherits a symmetric monoidal structure making the functor $(-)_R : \mathrm{Hmo}_0^{\mathrm{G}}(k) \to \mathrm{Hmo}_0^{\mathrm{G}}(k)_R$ symmetric monoidal. The category $\mathrm{NChow}^{\mathrm{G}}(k)_R$ of G-*equivariant noncommutative Chow motives with R-coefficients* is the idempotent completion of the subcategory of $\mathrm{Hmo}_0^{\mathrm{G}}(k)_R$ consisting of the objects $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})_R$ with $\mathcal{A}$ a smooth proper dg category.

4. Equivariant and enhanced additive invariants

Given a small dg category $\mathcal{A}$, let $T(\mathcal{A})$ be the dg category of pairs (i, x), where $i \in \{1, 2\}$ and $x \in \mathcal{A}$. The dg k-vector spaces $T(\mathcal{A})((i, x), (j, y))$ are given by $\mathcal{A}(x, y)$ if $j \geq i$ and are zero otherwise. Note that we have two inclusion dg functors $\iota_1, \iota_2 : \mathcal{A} \to T(\mathcal{A})$. A functor $E : \mathrm{dgcat}(k) \to \mathrm{D}$, with values in an additive category, is called an *additive invariant* if it satisfies the following two conditions:

(i) it sends Morita equivalences to isomorphisms;

(ii) given a small dg category $\mathcal{A}$, the dg functors ι_1, ι_2 induce an isomorphism[4]

$$[E(\iota_1)\ E(\iota_2)] : E(\mathcal{A}) \oplus E(\mathcal{A}) \to E(T(\mathcal{A})).$$

Examples of additive invariants include algebraic K-theory, Hochschild homology HH, cyclic homology HC, periodic cyclic homology HP, negative cyclic homology HN, etc.; consult [Tabuada 2015, §2.2] for details. As proved in [Tabuada 2005, Théorèmes 5.3 et 6.3], the functor $U : \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ is the *universal additive invariant*, i.e., given any additive category D we have an induced equivalence of categories

$$U^* : \mathrm{Fun}_{\mathrm{additive}}(\mathrm{Hmo}_0(k), \mathrm{D}) \to \mathrm{Fun}_{\mathrm{add}}(\mathrm{dgcat}(k), \mathrm{D}), \tag{4.1}$$

where the left-hand side denotes the category of additive functors and the right-hand side the category of additive invariants.

Remark 4.2 (additive invariants of twisted group algebras). Let $\alpha : \mathrm{G} \times \mathrm{G} \to k^\times$ be a 2-cocycle and $k_\alpha[\mathrm{G}]$ the associated twisted group algebra. Recall that a conjugacy class $\langle \sigma \rangle$ of G is called *α-regular* if $\alpha(\sigma, \rho) = \alpha(\rho, \sigma)$ for every element ρ of the centralizer $C_{\mathrm{G}}(\sigma)$. Thanks to the (generalized) Maschke theorem, the algebra $k_\alpha[\mathrm{G}]$ is semisimple. Moreover, the number of simple $k_\alpha[\mathrm{G}]$-modules is equal to the number $|\langle \mathrm{G} \rangle_\alpha|$ of α-regular conjugacy classes of G. Let $E : \mathrm{dgcat}(k) \to \mathrm{D}$ be an additive invariant. Making use of [Tabuada and Van den Bergh 2015b, Corollary 3.20 and Remark 3.21], we obtain the following computations:

[4]Condition (ii) can be equivalently formulated in terms of semiorthogonal decompositions in the sense of Bondal and Orlov [1995]; consult [Tabuada 2005, Théorème 6.3(4)] for details.

(i) We have $E(k_\alpha[\mathrm{G}]) \simeq \bigoplus_{i=1}^{|\langle \mathrm{G}\rangle_\alpha|} E(D_i)$, where $D_i := \mathrm{End}_{k_\alpha[\mathrm{G}]}(S_i)$ is the division algebra associated to the simple (right) $k_\alpha[\mathrm{G}]$-module S_i.

(ii) When D is $\mathbb{Q}$-linear, we have $E(k_\alpha[\mathrm{G}]) \simeq \bigoplus_{i=1}^{|\langle \mathrm{G}\rangle_\alpha|} E(l_i)$ where l_i (a finite field extension of k) is the center of D_i.

(iii) When k is algebraically closed, we have $E(k_\alpha[\mathrm{G}]) \simeq E(k)^{\oplus |\langle \mathrm{G}\rangle_\alpha|}$.

4A. ***Equivariant additive invariants.*** Given an additive invariant E, the associated G-*equivariant additive invariant* is defined as the composition

$$E^{\mathrm{G}} : \mathrm{G}\text{-dgcat}(k) \xrightarrow{(2.15)} \mathrm{dgcat}(k) \xrightarrow{E} \mathrm{D}. \tag{4.3}$$

From a topological viewpoint, $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})$ may be understood as the value of E at the "homotopy fixed points" of the G-action on $\mathcal{A}$. Here are some examples:

Example 4.4. (i) Let $\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)$ be as in Example 2.4. Due to Example 2.10, we have an identification between $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ and $E(\mathrm{perf}^{\mathrm{G}}_{\mathrm{dg}}(X))$.

(ii) Let $\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)$ be as in Example 2.5. Due to Example 2.11, we have an identification between $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ and $E(\mathrm{perf}_{\mathrm{dg}}(Y))$.

(iii) Let $\mathrm{G} \circlearrowright A$ be as in Example 2.6. Due to Example 2.12, we have an identification between $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{C}_{c,\mathrm{dg}}(A))$ and $E(A \rtimes \mathrm{G})$.

(iv) Let $\mathrm{G} \circlearrowright_\alpha k$ be as in Example 2.7. Due to Example 2.13, we have an identification between $E^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha \mathcal{C}_{c,\mathrm{dg}}(k))$ and $E(k_\alpha[\mathrm{G}])$.

Example 4.5 (equivariant algebraic K-theory). The composed functor (4.3) with $E := K$ is called G-*equivariant algebraic K-theory*. Recall that a quasicompact quasiseparated G-scheme X has the *resolution property* if every G-equivariant coherent $\mathcal{O}_X$-module is a quotient of a G-bundle. For example, the existence of an ample family of line G-bundles implies the resolution property. As explained in [Krishna and Ravi 2015, Corollary 3.9], whenever X has the resolution property, $K^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \simeq K(\mathrm{perf}^{\mathrm{G}}_{\mathrm{dg}}(X))$ agrees with the G-equivariant algebraic K-theory $K^{\mathrm{G}}(X)$ of X in the sense of Thomason [1987, §1.4].

Example 4.6 (equivariant Hochschild, cyclic, periodic, and negative homology). The composed functors (4.3) with $E := HH$, HC, HP, and HN, are called G-*equivariant Hochschild, cyclic, periodic, and negative homology*, respectively. Consult [Feĭgin and Tsygan 1987a, §A.6; 1987b, §4] for the computations of these G-equivariant additive invariants at the small G-dg categories $\mathrm{G} \circlearrowright \mathcal{C}_{c,\mathrm{dg}}(A)$; see Example 4.4(iii).

Proposition 4.7. *Given a* G*-equivariant additive invariant* E^{G}, *there exists an additive functor* $\overline{E^{\mathrm{G}}} : \mathrm{Hmo}^{\mathrm{G}}_0(k) \to \mathrm{D}$ *such that* $\overline{E^{\mathrm{G}}} \circ U^{\mathrm{G}} \simeq E^{\mathrm{G}}$.

Proof. Let us denote by $\overline{E} : \mathrm{Hmo}_0(k) \to \mathrm{D}$ the additive functor corresponding to E under the equivalence of categories (4.1). By precomposing it with the functor (4.9) of Lemma 4.8 below, we obtain the desired additive functor $\overline{E^{\mathrm{G}}}$. □

Lemma 4.8. *The functor* (2.15) *gives rise to an additive functor*

$$\mathrm{Hmo}_0^{\mathrm{G}}(k) \to \mathrm{Hmo}_0(k), \quad U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \mapsto U(\mathcal{A}^{\mathrm{G}}) \tag{4.9}$$

such that (4.9) $\circ\, U^{\mathrm{G}} \simeq U \circ$ (2.15).

Proof. Given two small G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$, consider the dg functor $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to \mathrm{rep}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{G}}, \mathcal{B}^{\mathrm{G}})$ that sends $(\mathrm{B} : \mathcal{A} \otimes \mathcal{B}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{dg}}(k), \theta_\sigma)$ to

$$\mathcal{A}^{\mathrm{G}} \otimes (\mathcal{B}^{\mathrm{G}})^{\mathrm{op}} = \mathcal{A}^{\mathrm{G}} \otimes (\mathcal{B}^{\mathrm{op}})^{\mathrm{G}} \xrightarrow{(a)} (\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}})^{\mathrm{G}} \xrightarrow{\mathrm{B}^{\mathrm{G}}} \mathcal{C}_{\mathrm{dg}}(k)^{\mathrm{G}} \xrightarrow{(b)} \mathcal{C}_{\mathrm{dg}}(k),$$

where (a) stands for the canonical dg functor and (b) for the dg functor which sends a G-representation (M, θ_σ) to the dg k-vector space of G-invariants M^{G}; since $\mathrm{char}(k) \nmid |\mathrm{G}|$ the latter dg functor is well-defined. By first taking the left dg Kan extension (see [Kelly 1982, §4]) of $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to \mathrm{rep}_{\mathrm{dg}}(\mathcal{A}^{\mathrm{G}}, \mathcal{B}^{\mathrm{G}})$ along the Yoneda dg functor $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to \mathcal{C}_{c,\mathrm{dg}}(\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})^{\mathrm{G}})$ and then the functor $\mathrm{H}^0(-)$, we obtain an induced triangulated functor $\mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to \mathrm{rep}(\mathcal{A}^{\mathrm{G}}, \mathcal{B}^{\mathrm{G}})$; see [Elagin 2014, Theorem 8.7]. Consequently, by passing K_0, we obtain an induced homomorphism

$$K_0\, \mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to K_0\, \mathrm{rep}(\mathcal{A}^{\mathrm{G}}, \mathcal{B}^{\mathrm{G}}). \tag{4.10}$$

The additive functor (4.9) is given by the assignments $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \mapsto U(\mathcal{A}^{\mathrm{G}})$ and (4.10). By construction, we have (4.9) $\circ\, U^{\mathrm{G}} \simeq U \circ$ (2.15). □

4B. ***Enhanced additive invariants.*** Given an additive invariant E, the associated G-*enhanced additive invariant* is defined by

$$E^{\circlearrowright} : \mathrm{G}\text{-}\mathrm{dgcat}(k) \to \mathrm{D}^{\mathrm{G}}, \quad \mathrm{G} \circlearrowright \mathcal{A} \mapsto (E(\mathcal{A}), E(\phi_\sigma)),$$

where D^{G} stands for the category of G-equivariant objects in D (with respect to the trivial G-action); since E sends Morita equivalences to isomorphisms, $E^{\circlearrowright}$ is well-defined. When E is symmetric monoidal, $E^{\circlearrowright}$ is also symmetric monoidal.

Proposition 4.11. *Given a* G*-enhanced additive invariant* $E^{\circlearrowright}$, *there exists an additive functor* $\overline{E^{\circlearrowright}} : \mathrm{Hmo}_0^{\mathrm{G}}(k) \to \mathrm{D}^{\mathrm{G}}$ *such that* $\overline{E^{\circlearrowright}} \circ U^{\mathrm{G}} \simeq E^{\circlearrowright}$.

Proof. Given small G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$ and $\mathrm{G} \circlearrowright \mathcal{B}$, the composition

$$K_0\, \mathrm{rep}(\mathcal{A}, \mathcal{B})^{\mathrm{G}} \to K_0\, \mathrm{rep}(\mathcal{A}, \mathcal{B}) \to \mathrm{Hom}_{\mathrm{D}}(E(\mathcal{A}), E(\mathcal{B})), \tag{4.12}$$

where the first homomorphism is induced by the restriction functor and the second homomorphism by the additive functor $\overline{E}$, takes values in the abelian subgroup $\mathrm{Hom}_{\mathrm{D}^{\mathrm{G}}}\big((E(\mathcal{A}), E(\phi_\sigma)), (E(\mathcal{B}), E(\phi_\sigma))\big)$. Therefore, $\overline{E^{\circlearrowright}}$ is defined by the assignments $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \mapsto (E(\mathcal{A}), E(\phi_\sigma))$ and (4.12). □

5. Relation with Panin's motivic category

Let H be an algebraic group scheme over k. Recall from [Panin 1994, §6], and from [Merkurjev 2005, §2.3], the construction of the motivic category[5] $\mathcal{C}^{\mathrm{H}}(k)$. The objects are the pairs (X, A), where X is a smooth projective H-scheme and A is a separable algebra, and the morphisms are given by the Grothendieck groups

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{H}}(k)}((X, A), (Y, B)) := K_0 \,\mathrm{Vect}^{\mathrm{H}}(X \times Y, A^{\mathrm{op}} \otimes B),$$

where $\mathrm{Vect}^{\mathrm{H}}(X \times Y, A^{\mathrm{op}} \otimes B)$ stands for the exact category of those H-equivariant right $(\mathcal{O}_{X\times Y} \otimes (A^{\mathrm{op}} \otimes B))$-modules which are locally free and of finite rank as $\mathcal{O}_{X\times Y}$-modules. Given

$$[\mathcal{F}] \in K_0 \,\mathrm{Vect}^{\mathrm{H}}(X \times Y, A^{\mathrm{op}} \otimes B) \quad \text{and} \quad [\mathcal{G}] \in K_0 \,\mathrm{Vect}^{\mathrm{H}}(Y \times Z, B^{\mathrm{op}} \otimes C),$$

their composition is defined by the formula

$$(\pi_{XZ})_*(\pi_{XY}^*([\mathcal{F}]) \otimes_B \pi_{YZ}^*([\mathcal{G}])) \in K_0 \,\mathrm{Vect}^{\mathrm{H}}(X \times Z, A^{\mathrm{op}} \otimes C),$$

where π_{ST} stands for the projection of $X \times Y \times Z$ into $S \times T$. The category $\mathcal{C}^{\mathrm{H}}(k)$ carries a symmetric monoidal structure induced by $(X, A) \otimes (Y, B) := (X \times Y, A \otimes B)$. Moreover, it comes equipped with two symmetric monoidal functors

$$\mathrm{SmProj}^{\mathrm{H}}(k)^{\mathrm{op}} \to \mathcal{C}^{\mathrm{H}}(k), \quad X \mapsto (X, k), \tag{5.1}$$

$$\mathrm{Sep}(k) \to \mathcal{C}^{\mathrm{H}}(k), \quad A \mapsto (\mathrm{Spec}(k), A), \tag{5.2}$$

defined on the category of smooth projective H-schemes and separable algebras, respectively. Let us denote by $\mathrm{G}\text{-}\mathrm{dgcat}_{\mathrm{sp}}(k) \subset \mathrm{G}\text{-}\mathrm{dgcat}(k)$ the full subcategory of those small G-dg categories $\mathrm{G} \circlearrowright \mathcal{A}$ with $\mathcal{A}$ smooth proper.

Theorem 5.3. *When* $\mathrm{H} = \mathrm{G}$ *is a* (*constant*) *finite algebraic group scheme*, *there exists an additive*, *fully faithful*, *symmetric monoidal functor* $\Psi : \mathcal{C}^{\mathrm{G}}(k) \to \mathrm{NChow}^{\mathrm{G}}(k)$ *making the following diagrams commute*:

$$\begin{tikzcd} \mathrm{SmProj}^{\mathrm{G}}(k)^{\mathrm{op}} \arrow[r, "X \mapsto \mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)"] \arrow[d, "(5.1)"'] & \mathrm{G}\text{-}\mathrm{dgcat}_{\mathrm{sp}}(k) \arrow[d, "U^{\mathrm{G}}"] \\ \mathcal{C}^{\mathrm{G}}(k) \arrow[r, "\Psi"'] & \mathrm{NChow}^{\mathrm{G}}(k) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Sep}(k) \arrow[r, "A \mapsto \mathrm{G} \circlearrowright_1 A"] \arrow[d, "(5.2)"'] & \mathrm{G}\text{-}\mathrm{dgcat}_{\mathrm{sp}}(k) \arrow[d, "U^{\mathrm{G}}"] \\ \mathcal{C}^{\mathrm{G}}(k) \arrow[r, "\Psi"'] & \mathrm{NChow}^{\mathrm{G}}(k) \end{tikzcd}$$

Proof. Given a smooth projective G-scheme X and a separable algebra A, let us write $\mathrm{Mod}(X, A)$ for the Grothendieck category of right $(\mathcal{O}_X \otimes A)$-modules, $\mathcal{D}(X, A)$ for the derived category $\mathcal{D}(\mathrm{Mod}(X, A))$, and $\mathcal{D}_{\mathrm{dg}}(X, A)$ for the dg category $\mathcal{D}_{\mathrm{dg}}(\mathcal{E})$ with $\mathcal{E} := \mathrm{Mod}(X, A)$. In the same vein, let us write $\mathrm{perf}(X, A)$ for

[5] Panin and Merkurjev denoted this motivic category by $\mathbb{A}^{\mathrm{H}}$ and $\mathcal{C}(\mathrm{H})$, respectively.

the full triangulated subcategory, and $\mathrm{perf}_{\mathrm{dg}}(X, A)$ for the full dg subcategory, of those complexes of right $(\mathcal{O}_X \otimes A)$-modules which are perfect as complexes of $\mathcal{O}_X$-modules. As proved in [Tabuada 2014, Lemma 6.4], the dg category $\mathrm{perf}_{\mathrm{dg}}(X, A)$ is smooth proper.

Let X and Y be smooth projective G-schemes and A and B separable algebras. Consider the inclusion functor

$$\mathrm{Vect}(X \times Y, A^{\mathrm{op}} \otimes B) \to \mathrm{perf}(X \times Y, A^{\mathrm{op}} \otimes B) \tag{5.4}$$

as well as the functor

$$\mathrm{perf}(X \times Y, A^{\mathrm{op}} \otimes B) \to \mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}(X, A), \mathrm{perf}_{\mathrm{dg}}(Y, B)), \quad \mathcal{F} \mapsto {}_{\Phi_{\mathcal{F}}}\mathrm{B}, \tag{5.5}$$

where $\Phi_{\mathcal{F}}$ stands for the Fourier–Mukai dg functor

$$\mathrm{perf}_{\mathrm{dg}}(X, A) \to \mathrm{perf}_{\mathrm{dg}}(Y, B), \quad \mathcal{G} \mapsto (\pi_Y)_*(\pi_X^*(\mathcal{G}) \otimes_A \mathcal{F}).$$

Both functors (5.4)–(5.5) are G-equivariant. Consequently, making use of the identification $\mathrm{perf}^{\mathrm{G}}(X \times Y, A^{\mathrm{op}} \otimes B) \simeq \mathrm{perf}(X \times Y, A^{\mathrm{op}} \otimes B)^{\mathrm{G}}$ (see Example 2.10), we obtain induced group homomorphisms

$$K_0 \,\mathrm{Vect}^{\mathrm{G}}(X \times Y, A^{\mathrm{op}} \otimes B) \to K_0 \,\mathrm{perf}^{\mathrm{G}}(X \times Y, A^{\mathrm{op}} \otimes B), \tag{5.6}$$

$$K_0 \,\mathrm{perf}^{\mathrm{G}}(X \times Y, A^{\mathrm{op}} \otimes B) \to K_0 \,\mathrm{rep}(\mathrm{perf}_{\mathrm{dg}}(X, A), \mathrm{perf}_{\mathrm{dg}}(Y, B))^{\mathrm{G}}. \tag{5.7}$$

The assignments $(X, A) \mapsto U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X, A))$, combined with the group homomorphisms (5.7) $\circ$ (5.6), gives rise to an additive symmetric monoidal functor $\Psi : \mathcal{C}^{\mathrm{G}}(k) \to \mathrm{NChow}^{\mathrm{G}}(k)$, similarly to [Tabuada 2014, Theorem 6.10]. As explained on page 30 of that article, the functor (5.5) is an equivalence. This implies that (5.7) is invertible. Since $X \times Y$ admits an ample family of line G-bundles, the homomorphism (5.6) is also invertible. We hence conclude that the functor Ψ is, moreover, fully faithful. Finally, the commutativity of the two diagrams follows from the identifications $\mathrm{perf}_{\mathrm{dg}}(X, k) = \mathrm{perf}_{\mathrm{dg}}(X)$ and $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k), A) = \mathcal{C}_{c,\mathrm{dg}}(A)$ and from the fact that the Yoneda dg functor $A \to \mathcal{D}_{c,\mathrm{dg}}(A)$ is a G-equivariant Morita equivalence. □

Corollary 5.8. *Given $X, Y \in \mathrm{SmProj}^{\mathrm{G}}(k)$, we have a group isomorphism*

$$\mathrm{Hom}_{\mathrm{NChow}^{\mathrm{G}}(k)}\big(U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)), U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Y))\big) \simeq K_0^{\mathrm{G}}(X \times Y).$$

Proof. Combine Thomason's definition $K_0^{\mathrm{G}}(X \times Y) := K_0 \,\mathrm{Vect}^{\mathrm{G}}(X \times Y)$ of the G-equivariant Grothendieck group of $X \times Y$ with Theorem 5.3. □

5A. ***Twisted projective homogeneous varieties.*** Let H be a split semisimple algebraic group scheme over k, $P \subset \mathrm{H}$ a parabolic subgroup, and $\gamma : \mathrm{Gal}(k_{\mathrm{sep}}/k) \to \mathrm{H}(k_{\mathrm{sep}})$

a 1-cocycle. Out of this data, we can construct the projective homogeneous H-variety H/P as well as its twisted form ${}_{\gamma}\mathrm{H}/P$. Let $\widetilde{\mathrm{H}}$ and $\widetilde{P}$ be the universal covers of H and P, $R(\widetilde{\mathrm{H}})$ and $R(\widetilde{P})$ the associated representation rings, n the index $[W(\widetilde{\mathrm{H}}):W(\widetilde{P})]$ of the Weyl groups, $\widetilde{Z}$ the center of $\widetilde{\mathrm{H}}$, and $\mathrm{Ch} := \mathrm{Hom}(\widetilde{Z}, \mathbb{G}_m)$ the character group. Under this notation, Panin [1994, Theorem 4.2] proved that every Ch-homogeneous basis $\rho_1, \ldots, \rho_n$ of $R(\widetilde{P})$ over $R(\widetilde{\mathrm{H}})$ gives rise to an isomorphism

$$({}_{\gamma}\mathrm{H}/P, k) \simeq \bigoplus_{i=1}^{n} (\mathrm{Spec}(k), A_i) \tag{5.9}$$

in $\mathcal{C}^{\mathrm{H}}(k)$, where A_i stands for the Tits' central simple algebra associated to ρ_i.

Theorem 5.10. *Let* H, P, γ *be as above, and* G_k *the* (*constant*) *algebraic group scheme associated to the finite group* G. *For every homomorphism* $\mathrm{G}_k \to \mathrm{H}$ *and* G-*equivariant additive invariant* E^{G}, *we have an induced isomorphism*

$$E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}({}_{\gamma}\mathrm{H}/P)) \simeq \bigoplus_{i=1}^{n} E(A_i[\mathrm{G}]), \tag{5.11}$$

where ${}_{\gamma}\mathrm{H}/P$ *is considered as a* G-*scheme.*

Proof. Via $\mathrm{G}_k \to \mathrm{H}$, Panin's computation (5.9) holds also in the motivic category $\mathcal{C}^{\mathrm{G}}(k)$. Making use of Theorem 5.3 and Lemma 3.6, we conclude that

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}({}_{\gamma}\mathrm{H}/P)) \simeq \bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 A_i) \simeq \bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 \mathcal{C}_{c,\mathrm{dg}}(A_i)).$$

The proof then follows from Proposition 4.7 and Example 4.4(iii). $\square$

Remark 5.12 (G-equivariant Hochschild homology). When E^{G} is G-equivariant Hochschild homology HH^{G}, the right-hand side of (5.11) reduces to

$$\bigoplus_{i=1}^{n} HH(A_i[\mathrm{G}]) \overset{(a)}{\simeq} \bigoplus_{i=1}^{n} HH(k[\mathrm{G}]) \otimes HH_0(A_i) \overset{(b)}{\simeq} \bigoplus_{i=1}^{n} HH(k[\mathrm{G}]), \tag{5.13}$$

where (a) follows from [Loday 1998, Corollary 1.2.14] and (b) from the fact that $HH_0(A) \simeq k$ for every central simple k-algebra A. In the particular case where k is algebraically closed, (5.13) reduces moreover to $\bigoplus_{i=1}^{n} HH(k)^{\oplus |\langle \mathrm{G} \rangle|}$; see Remark 4.2(iii).

5B. ***Quasisplit case.*** When the algebraic group scheme H is a *quasi*split, Panin [1994, Theorem 12.4] proved that a computation similar to (5.9) also holds. In this generality, the algebras A_i are no longer central simple but only separable. The analogue of Theorem 5.10 (with the same proof) holds similarly. Moreover, when $E^{\mathrm{G}} := HH^{\mathrm{G}}$, the right-hand side of (5.11) reduces to

$$\bigoplus_{i=1}^{n} HH(k[\mathrm{G}]) \otimes A_i/[A_i, A_i].$$

6. Relation with equivariant motives

6A. ***Equivariant motives.*** Given a smooth projective G-scheme X and an integer $i \in \mathbb{Z}$, let us write $\mathrm{CH}^i_{\mathrm{G}}(X)_{\mathbb{Q}}$ for the i-codimensional G-equivariant Chow group of X in the sense of Edidin and Graham [1998]. Since the group G is finite, we have $\mathrm{CH}^i_{\mathrm{G}}(X)_{\mathbb{Q}} = 0$ whenever $i \notin \{0, \dots, \dim(X)\}$; see [Edidin and Graham 2000, Proposition 5.2].

Let X and Y be smooth projective G-schemes, $X = \coprod_j X_j$ the decomposition of X into its connected components, and r an integer. The $\mathbb{Q}$-vector space $\mathrm{Corr}^r_{\mathrm{G}}(X, Y) := \bigoplus_j \mathrm{CH}_{\mathrm{G}}^{\dim(X_j)+r}(X_j \times Y)_{\mathbb{Q}}$ is called the space of G-*equivariant correspondences of degree r from X to Y*. Given G-equivariant correspondences $f \in \mathrm{Corr}^r_{\mathrm{G}}(X, Y)$ and $g \in \mathrm{Corr}^s_{\mathrm{G}}(Y, Z)$, their composition is defined by the formula

$$(\pi_{XZ})_*(\pi^*_{XY}(f) \cdot \pi^*_{YZ}(g)) \in \mathrm{Corr}^{r+s}_{\mathrm{G}}(X, Z). \tag{6.1}$$

Recall from [Laterveer 1998], and from [Iyer and Müller-Stach 2009], the construction of the category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ of G-*equivariant Chow motives with* $\mathbb{Q}$-*coefficients*. The objects are the triples (X, p, m), where X is a smooth projective G-scheme, $p^2 = p \in \mathrm{Corr}^0_{\mathrm{G}}(X, X)$ is an idempotent endomorphism, and m is an integer. The $\mathbb{Q}$-vector spaces of morphisms are given by

$$\mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}((X, p, m), (Y, q, n)) := q \circ \mathrm{Corr}^{n-m}_{\mathrm{G}}(X, Y) \circ p,$$

and the composition law is induced by the composition (6.1) of correspondences. By construction, the category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ is $\mathbb{Q}$-linear, additive, and idempotent complete. Moreover, it carries a symmetric monoidal structure induced by the formula $(X, p, m) \otimes (Y, q, n) := (X \times Y, p \otimes q, m + n)$. The G-equivariant Lefschetz motive $(\mathrm{Spec}(k), \mathrm{id}, -1)$ will be denoted by $\mathbb{L}$ and the G-equivariant Tate motive $(\mathrm{Spec}(k), \mathrm{id}, 1)$ by $\mathbb{Q}(1)$; in both cases G acts trivially. Finally, the category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ comes equipped with the symmetric monoidal functor

$$\mathfrak{h}^{\mathrm{G}}(-)_{\mathbb{Q}} : \mathrm{SmProj}^{\mathrm{G}}(k)^{\mathrm{op}} \to \mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}, \quad X \mapsto (X, \mathrm{id}, 0).$$

The category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ is additive and rigid symmetric monoidal.

6B. ***Orbit categories.*** Let $\mathcal{C}$ be an additive symmetric monoidal category and $\mathcal{O} \in \mathcal{C}$ a $\otimes$-invertible object. The *orbit category* $\mathcal{C}/_{-\otimes\mathcal{O}}$ has the same objects as $\mathcal{C}$ and abelian groups of morphisms $\mathrm{Hom}_{\mathcal{C}/_{-\otimes\mathcal{O}}}(a, b) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(a, b \otimes \mathcal{O}^{\otimes i})$. Given objects a, b, and c, and morphisms

$$\mathrm{f} = \{f_i\}_{i\in\mathbb{Z}} \in \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(a, b \otimes \mathcal{O}^{\otimes i}), \quad \mathrm{g} = \{g_i\}_{i\in\mathbb{Z}} \in \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(b, c \otimes \mathcal{O}^{\otimes i}),$$

the i'-th component of $\mathrm{g} \circ \mathrm{f}$ is defined as $\sum_i (g_{i'-i} \otimes \mathcal{O}^{\otimes i}) \circ f_i$. The functor

$$\pi : \mathcal{C} \to \mathcal{C}/_{-\otimes\mathcal{O}}, \quad a \mapsto a, \quad f \mapsto \mathrm{f} = \{f_i\}_{i\in\mathbb{Z}},$$

where $f_0 = f$ and $f_i = 0$ if $i \neq 0$, is endowed with a natural isomorphism of functors $\pi \circ (- \otimes \mathcal{O}) \Rightarrow \pi$ and is 2-universal among all such functors; see [Tabuada 2013, §7]. The category $\mathcal{C}/_{-\otimes\mathcal{O}}$ is additive and, as proved in [Tabuada 2013, Lemma 7.3], inherits from $\mathcal{C}$ a symmetric monoidal structure making π symmetric monoidal.

6C. ***Localization at the augmentation ideal.*** Let I be the kernel of the rank homomorphism $R(\mathrm{G}) \twoheadrightarrow \mathbb{Z}$ and $R(\mathrm{G})_I$ the localization of $R(\mathrm{G})$ at the ideal I. Recall from Corollary 3.17 that the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ is $R(\mathrm{G})$-linear. Let us denote by $\mathrm{Hmo}_0^{\mathrm{G}}(k)_I$ the $R(\mathrm{G})_I$-linear additive category obtained from $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ by applying the functor $(-)_I := - \otimes_{R(\mathrm{G})} R(\mathrm{G})_I$ to each $R(\mathrm{G})$-module of morphisms. By construction, $\mathrm{Hmo}_0^{\mathrm{G}}(k)_I$ inherits from $\mathrm{Hmo}_0^{\mathrm{G}}(k)$ a symmetric monoidal structure making the functor $(-)_I : \mathrm{Hmo}_0^{\mathrm{G}}(k) \to \mathrm{Hmo}_0^{\mathrm{G}}(k)_I$ symmetric monoidal. The category $\mathrm{NChow}^{\mathrm{G}}(k)_I$ of *I-localized* G-*equivariant noncommutative Chow motives* is defined as the idempotent completion of the subcategory of $\mathrm{Hmo}_0^{\mathrm{G}}(k)_I$ consisting of the objects $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A})_I$ with $\mathcal{A}$ a smooth proper dg category.

Proposition 6.2. *Given any two cohomology classes $[\alpha], [\beta] \in H^2(\mathrm{G}, k^\times)$, we have an isomorphism $U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)_I \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\beta k)_I$ in* $\mathrm{NChow}^{\mathrm{G}}(k)_I$.

Proof. By construction of $\mathrm{NChow}^{\mathrm{G}}(k)$, we have group isomorphisms

$$\mathrm{Hom}_{\mathrm{NChow}^{\mathrm{G}}(k)}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k), U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\beta k)) \simeq K_0(\mathcal{D}_c(k)^{\mathrm{G},\alpha\beta^{-1}}),$$
$$\mathrm{Hom}_{\mathrm{NChow}^{\mathrm{G}}(k)}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\beta k), U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)) \simeq K_0(\mathcal{D}_c(k)^{\mathrm{G},\beta\alpha^{-1}}).$$

Consider the $\alpha\beta^{-1}$-twisted G-equivariant object $k_{\alpha\beta^{-1}}\mathrm{G} \in \mathcal{D}_c(k)^{\mathrm{G},\alpha\beta^{-1}}$ defined as $\big(\bigoplus_{\rho\in\mathrm{G}} \phi_\rho(k), \theta_\sigma\big)$, where $\phi_\rho(k) = k$ and θ_σ is given by the collection of units $(\alpha^{-1}\beta)(\sigma, \rho) \in k^\times$. Similarly, consider the $\beta\alpha^{-1}$-twisted G-equivariant object $k_{\beta\alpha^{-1}}\mathrm{G} \in \mathcal{D}_c(k)^{\mathrm{G},\beta\alpha^{-1}}$ defined as $\big(\bigoplus_{\rho\in\mathrm{G}} \phi_\rho(k), \theta_\sigma\big)$, where θ_σ is given by the units $(\beta^{-1}\alpha)(\sigma, \rho)$. The associated Grothendieck classes then correspond to morphisms

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k) \xrightarrow{f} U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\beta k) \quad \text{and} \quad U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\beta k) \xrightarrow{g} U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)$$

in the category $\mathrm{NChow}^{\mathrm{G}}(k)$. Since the rank of the elements $g \circ f, f \circ g \in R(\mathrm{G})$ is nonzero (see Proposition 3.15), we conclude from the definition of $\mathrm{NChow}^{\mathrm{G}}(k)_I$ that the morphisms f_I and g_I are invertible. This completes the proof. □

Remark 6.3 (groups of central type). Note that the group algebra $k[\mathrm{G}]$ is not simple: it contains the nontrivial augmentation ideal. In the case where G is of central type, there exist cohomology classes $[\alpha] \in H^2(\mathrm{G}, k^\times)$ for which the twisted group algebra $k_\alpha[\mathrm{G}]$ is simple! For example, the group $\mathrm{G} := \mathrm{H} \times \widehat{\mathrm{H}}$ (with H abelian) is of central type and the twisted group algebra $k_\alpha[\mathrm{G}]$ associated to the 2-cocycle $\alpha((\sigma, \chi), (\rho, \psi)) := \chi(\rho)$ is simple. Combining Remark 4.2 with Example 4.4(iv) and Proposition 4.7, we conclude that $U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k) \not\simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k)$ in $\mathrm{NChow}^{\mathrm{G}}(k)$. This shows that Proposition 6.2 is false before I-localization.

6D. *Bridges.* The next result relates the category of G-equivariant noncommutative motives with the category of G-equivariant motives.

Theorem 6.4. *There exists a $\mathbb{Q}$-linear, fully faithful, symmetric monoidal functor Φ making the following diagram commute*:

$$\begin{tikzcd}
\mathrm{SmProj}^{\mathrm{G}}(k)^{\mathrm{op}} \arrow[rr, "X\mapsto \mathrm{G}\circlearrowright\mathrm{perf}_{\mathrm{dg}}(X)"] \arrow[d, "\mathfrak{h}^{\mathrm{G}}(-)_{\mathbb{Q}}"'] & & \mathrm{G}\text{-}\mathrm{dgcat}_{\mathrm{sp}}(k) \arrow[d, "U^{\mathrm{G}}(-)_{\mathbb{Q}}"] \\
\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}} \arrow[d, "\pi"'] & & \mathrm{NChow}^{\mathrm{G}}(k)_{\mathbb{Q}} \arrow[d, "(-)_{I_{\mathbb{Q}}}"] \\
\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)} \arrow[rr, "\Phi"'] & & \mathrm{NChow}^{\mathrm{G}}(k)_{\mathbb{Q},I_{\mathbb{Q}}}
\end{tikzcd} \tag{6.5}$$

Proof. Let us denote by $\mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q}}$ the idempotent completion of the full subcategory of $\mathcal{C}^{\mathrm{G}}(k)_{\mathbb{Q}}$ (see Section 5) consisting of the objects $(X,k)_{\mathbb{Q}}$. Given smooth projective G-schemes X and Y, we have isomorphisms

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q}}}(\mathfrak{h}^{\mathrm{G}}_0(X)_{\mathbb{Q}}, \mathfrak{h}^{\mathrm{G}}_0(Y)_{\mathbb{Q}}) = K_0\,\mathrm{Vect}^{\mathrm{G}}(X\times Y)_{\mathbb{Q}} \simeq K^{\mathrm{G}}_0(X\times Y)_{\mathbb{Q}}.$$

Moreover, given $[\mathcal{F}]_{\mathbb{Q}}\in K^{\mathrm{G}}_0(X\times Y)_{\mathbb{Q}}$ and $[\mathcal{G}]_{\mathbb{Q}}\in K^{\mathrm{G}}_0(Y\times Z)_{\mathbb{Q}}$, their composition is defined by the formula $(\pi_{XZ})_*(\pi^*_{XY}([\mathcal{F}]_{\mathbb{Q}})\otimes\pi^*_{YZ}([\mathcal{G}]_{\mathbb{Q}}))$. Furthermore, $\mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q}}$ comes equipped with the symmetric monoidal functor

$$\mathfrak{h}^{\mathrm{G}}_0(-):\mathrm{SmProj}^{\mathrm{G}}(k)^{\mathrm{op}}\to\mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q}},\quad X\mapsto(X,k)_{\mathbb{Q}}.$$

Similarly to Section 6C, we can also consider the $I_{\mathbb{Q}}$-localized category $\mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q},I_{\mathbb{Q}}}$.

Let us now construct a functor Φ_1 making the diagram

$$\begin{tikzcd}
\mathrm{SmProj}^{\mathrm{G}}(k)^{\mathrm{op}} \arrow[r, equal] \arrow[d, "\mathfrak{h}^{\mathrm{G}}(-)_{\mathbb{Q}}"'] & \mathrm{SmProj}(k)^{\mathrm{op}} \arrow[r, "X\mapsto \mathrm{G}\circlearrowright\mathrm{perf}_{\mathrm{dg}}(X)"] \arrow[d, "\mathfrak{h}^{\mathrm{G}}_0(-)_{\mathbb{Q}}"] & \mathrm{G}\text{-}\mathrm{dgcat}_{\mathrm{sp}}(k) \arrow[d, "U^{\mathrm{G}}(-)_{\mathbb{Q}}"] \\
\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}} \arrow[d, "\pi"'] & \mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q}} \arrow[d, "(-)_{I_{\mathbb{Q}}}"] & \mathrm{NChow}^{\mathrm{G}}(k)_{\mathbb{Q}} \arrow[d, "(-)_{I_{\mathbb{Q}}}"] \\
\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)} & \mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q},I_{\mathbb{Q}}} \arrow[l, "\Phi_1"] \arrow[r, "\Phi_2"'] & \mathrm{NChow}^{\mathrm{G}}(k)_{\mathbb{Q},I_{\mathbb{Q}}}
\end{tikzcd} \tag{6.6}$$

commute, where Φ_2 stands for the $\mathbb{Q}$-linear, fully faithful, symmetric monoidal functor naturally induced from Ψ; see Theorem 5.3. As proved in [Edidin and Graham 2000, Corollary 5.1], we have a Riemann–Roch isomorphism $\tau_X:K^{\mathrm{G}}_0(X)_{\mathbb{Q},I_{\mathbb{Q}}}\to\bigoplus_{i=0}^{\dim(X)}\mathrm{CH}^i_{\mathrm{G}}(X)_{\mathbb{Q}}$ for every smooth projective G-scheme X. This isomorphism preserves the multiplicative structures. Moreover, given any G-equivariant map $f:X\to Y$, the following squares are commutative (we assume that f is proper on the right-hand side):

$$
\begin{tikzcd}
K_0^{\mathrm{G}}(X)_{\mathbb{Q},I_{\mathbb{Q}}} \arrow[r, "\tau_X"] & \bigoplus_{i=0}^{\dim(X)} \mathrm{CH}^i_{\mathrm{G}}(X)_{\mathbb{Q}} \\
K_0^{\mathrm{G}}(Y)_{\mathbb{Q},I_{\mathbb{Q}}} \arrow[r, "\tau_Y"'] \arrow[u, "f^*"] & \bigoplus_{i=0}^{\dim(Y)} \mathrm{CH}^i_{\mathrm{G}}(Y)_{\mathbb{Q}} \arrow[u, "f^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
K_0^{\mathrm{G}}(X)_{\mathbb{Q},I_{\mathbb{Q}}} \arrow[r, "\tau_X"] \arrow[d, "f_*"'] & \bigoplus_{i=0}^{\dim(X)} \mathrm{CH}^i_{\mathrm{G}}(X)_{\mathbb{Q}} \arrow[d, "f_*"] \\
K_0^{\mathrm{G}}(Y)_{\mathbb{Q},I_{\mathbb{Q}}} \arrow[r, "\tau_Y"'] & \bigoplus_{i=0}^{\dim(Y)} \mathrm{CH}^i_{\mathrm{G}}(Y)_{\mathbb{Q}}
\end{tikzcd}
$$

By construction of the orbit category, we have isomorphisms

$$\mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}}(\pi(\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}), \pi(\mathfrak{h}^{\mathrm{G}}(Y)_{\mathbb{Q}})) \simeq \bigoplus_{i=0}^{\dim(X\times Y)} \mathrm{CH}^i_{\mathrm{G}}(X\times Y)_{\mathbb{Q}}.$$

Therefore, we conclude from the preceding considerations that the assignments

$$\mathfrak{h}_0^{\mathrm{G}}(X)_{\mathbb{Q}} \mapsto \mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}} \quad \text{and} \quad K_0^{\mathrm{G}}(X\times Y)_{\mathbb{Q},I_{\mathbb{Q}}} \xrightarrow{\tau_{X\times Y}} \bigoplus_{i=0}^{\dim(X\times Y)} \mathrm{CH}^i_{\mathrm{G}}(X\times Y)_{\mathbb{Q}}$$

give rise to a functor $\Phi_1 : \mathcal{C}^{\mathrm{G}}_{\mathrm{sp}}(k)_{\mathbb{Q},I_{\mathbb{Q}}} \to \mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$ making the diagram (6.6) commute. The functor Φ_1 is $\mathbb{Q}$-linear, fully faithful, and symmetric monoidal. Since the objects (X, p, m) and $(X, p, 0)$ become isomorphic in the orbit category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$, the functor Φ_1 is, moreover, essentially surjective, and hence an equivalence of categories. Now, choose a (quasi-)inverse functor Φ_1^{-1} of Φ_1 and define Φ as the composition $\Phi_2 \circ \Phi_1^{-1}$. By construction, Φ is $\mathbb{Q}$-linear, fully faithful, symmetric monoidal, and makes the upper rectangle of (6.5) commute. □

7. Full exceptional collections

7A. ***Full exceptional collections.*** Let $\mathcal{T}$ be a k-linear triangulated category. Recall from [Bondal and Orlov 1995, Definition 2.4; Huybrechts 2006, §1.4] that a *semiorthogonal decomposition of length n*, denoted by $\mathcal{T} = \langle \mathcal{T}_1, \ldots, \mathcal{T}_n\rangle$, consists of full triangulated subcategories $\mathcal{T}_1, \ldots, \mathcal{T}_n \subset \mathcal{T}$ satisfying the following conditions: the inclusions $\mathcal{T}_i \subset \mathcal{T}$ admit left and right adjoints, the triangulated category $\mathcal{T}$ is generated by the objects of $\mathcal{T}_1, \ldots, \mathcal{T}_n$, and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_j, \mathcal{T}_i) = 0$ when $i < j$. An object $\mathcal{E} \in \mathcal{T}$ is called *exceptional* if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, \mathcal{E}) = k$ and $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}, \mathcal{E}[m]) = 0$ when $m \neq 0$. A *full exceptional collection of length n*, denoted by $\mathcal{T} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$, is a sequence of exceptional objects $\mathcal{E}_1, \ldots, \mathcal{E}_n$ which generate the triangulated category $\mathcal{T}$ and for which we have $\mathrm{Hom}_{\mathcal{T}}(\mathcal{E}_j, \mathcal{E}_i[m]) = 0, m \in \mathbb{Z}$, when $i < j$. Every full exceptional collection gives rise to a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{D}_c(k), \ldots, \mathcal{D}_c(k)\rangle$.

Proposition 7.1. *Let $\mathcal{A}$ be a small* G*-dg category and $\mathcal{A}_i \subseteq \mathcal{A}$, $1 \leq i \leq n$, full dg subcategories. Assume that $\sigma^*(\mathcal{A}_i) \subseteq \mathcal{A}_i$ for every $\sigma \in$* G*, and that $\mathcal{D}_c(\mathcal{A})$ admits a semiorthogonal decomposition $\langle \mathcal{D}_c(\mathcal{A}_1), \ldots, \mathcal{D}_c(\mathcal{A}_n)\rangle$. Under these assumptions, we have an isomorphism $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \simeq \bigoplus_{i=1}^n U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}_i)$ in* $\mathrm{Hmo}_0^{\mathrm{G}}(k)$.

Proof. The inclusions of dg categories $\mathcal{A}_i \subseteq \mathcal{A}$ give rise to a morphism

$$\bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}_i) \to U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \tag{7.2}$$

in the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$. In order to show that (7.2) is an isomorphism, it suffices by the Yoneda lemma to show that the induced group homomorphism

$$\mathrm{Hom}\Big(U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{B}), \bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}_i)\Big) \to \mathrm{Hom}(U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{B}), U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}))$$

is invertible for every small G-dg category $\mathrm{G} \circlearrowright \mathcal{B}$. By construction of the additive category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$, the preceding homomorphism identifies with

$$\bigoplus_{i=1}^{n} K_0 \,\mathrm{rep}(\mathcal{B}, \mathcal{A}_i)^{\mathrm{G}} \to K_0 \,\mathrm{rep}(\mathcal{B}, \mathcal{A})^{\mathrm{G}}. \tag{7.3}$$

Since $\mathcal{D}_c(\mathcal{A}) = \langle \mathcal{D}_c(\mathcal{A}_1), \dots, \mathcal{D}_c(\mathcal{A}_n)\rangle$, we have a semiorthogonal decomposition

$$\mathrm{rep}(\mathcal{B}, \mathcal{A}) = \langle \mathrm{rep}(\mathcal{B}, \mathcal{A}_1), \dots, \mathrm{rep}(\mathcal{B}, \mathcal{A}_n)\rangle.$$

Using first the fact that the functor $(-)^{\mathrm{G}}$ preserves semiorthogonal decompositions, and then the fact that the functor $K_0(-)$ sends semiorthogonal decompositions to direct sums, we conclude that the group homomorphism (7.3) is invertible. □

7B. *Invariant objects.* Let $\mathrm{G} \circlearrowright \mathcal{A}$ be a small G-dg category. An object $M \in \mathcal{D}(\mathcal{A})$ is called G-*invariant* if $\phi_\sigma(M) \simeq M$ for every $\sigma \in \mathrm{G}$. Every G-equivariant object in $\mathrm{G} \circlearrowright \mathcal{D}(\mathcal{A})$ is G-invariant, but the converse does not hold.

Remark 7.4 (strictification). Given a G-invariant object $M \in \mathcal{D}(\mathcal{A})$, let us fix an isomorphism $\theta_\sigma : M \to \phi_\sigma(M)$ for every $\sigma \in \mathrm{G}$. If $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(M, M) \simeq k$, then $\phi_\rho(\theta_\sigma) \circ \theta_\rho$ and $\theta_{\rho\sigma}$ differ by multiplication with an invertible element $\alpha(\rho, \sigma) \in k^\times$. Moreover, these invertible elements define a 2-cocycle α whose cohomology class $[\alpha] \in H^2(\mathrm{G}, k^\times)$ is independent of the choice of the θ_σ. Consequently, $M \in \mathcal{D}(\mathcal{A})^{\mathrm{G},\alpha}$. Furthermore, $M^{\otimes n} \in \mathcal{D}(\mathcal{A})^{\mathrm{G},\alpha^n}$. Roughly speaking, every "simple" G-invariant object can be strictified into a twisted G-equivariant object.

Proposition 7.5. *Let $\mathcal{A}$ be a small* G*-dg category such that $\mathcal{D}_c(\mathcal{A})$ admits a full exceptional collection $(\mathcal{E}_1, \dots, \mathcal{E}_n)$. Suppose $\mathcal{E}_i \in \mathcal{D}_c(\mathcal{A})^{\mathrm{G},\alpha_i}$, with $[\alpha_i] \in H^2(\mathrm{G}, k^\times)$. Then we have $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \simeq \bigoplus_{i=1}^n U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_i} k)$ in* $\mathrm{Hmo}_0^{\mathrm{G}}(k)$.

Proof. By construction, the set of morphisms $\mathrm{Hom}_{\mathrm{Hmo}^{\mathrm{G}}(k)}(\mathrm{G} \circlearrowright_{\alpha_i} k, \mathrm{G} \circlearrowright \mathcal{A})$ is given by the set of isomorphism classes of the triangulated category $\mathrm{rep}(k, \mathcal{A})^{\mathrm{G},\alpha_i} \simeq \mathcal{D}_c(\mathcal{A})^{\mathrm{G},\alpha_i}$. Consequently, the object $\mathcal{E}_i \in \mathcal{D}_c(\mathcal{A})^{\mathrm{G},\alpha_i}$ corresponds to a morphism $\mathcal{E}_i : \mathrm{G} \circlearrowright_{\alpha_i} k \to \mathrm{G} \circlearrowright \mathcal{A}$ in $\mathrm{Hmo}^{\mathrm{G}}(k)$. Consider the associated morphism

$$([\mathcal{E}_1] \oplus \cdots \oplus [\mathcal{E}_i] \oplus \cdots \oplus [\mathcal{E}_n]) : \bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_i} k) \to U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \tag{7.6}$$

in the additive category $\mathrm{Hmo}_0^{\mathrm{G}}(k)$. In order to show that (7.6) is an isomorphism, we can now follow *mutatis mutandis* the proof of Proposition 7.1. ☐

Corollary 7.7. *Given a* G*-dg category* $\mathrm{G} \circlearrowright \mathcal{A}$ *as in Proposition 7.5, we have*

(i) $E^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathcal{A}) \simeq \bigoplus_{i=1}^{n} E(k_{\alpha_i}[\mathrm{G}])$ *for every* G*-equivariant additive invariant*;

(ii) $E^{\circlearrowright}(\mathrm{G} \circlearrowright \mathcal{A}) \simeq \bigoplus_{i=1}^{n} (E(k), \mathrm{id})$ *for every* G*-enhanced additive invariant.*

Proof. Item (i) follows from the combination of Propositions 4.7 and 7.5 with Example 4.4(iv). Item (ii) follows from the combination of Propositions 4.11 and 7.5 with the fact that $E^{\circlearrowright}(\mathrm{G} \circlearrowright_{\alpha} k) \simeq (E(k), \mathrm{id})$ for every $[\alpha] \in H^2(\mathrm{G}, k^{\times})$. ☐

Proposition 7.8. *Let* X *be a quasicompact quasiseparated* G*-scheme such that* $\mathrm{perf}(X)$ *admits a full exceptional collection* $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ *of* G*-invariant objects. Let us denote by* $[\alpha_i] \in H^2(\mathrm{G}, k^{\times})$ *the cohomology class of Remark 7.4 associated to the exceptional object* $\mathcal{E}_i$*. Under these assumptions and notations, we have an isomorphism* $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \simeq \bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_i} k)$ *in* $\mathrm{Hmo}_0^{\mathrm{G}}(k)$*.*

Proof. Apply Proposition 7.5 to the dg category $\mathrm{perf}_{\mathrm{dg}}(X)$. ☐

Example 7.9 (projective spaces). Let $\mathbb{P}^n$ be the n-th projective space. As proved in [Beĭlinson 1978], $\mathrm{perf}(\mathbb{P}^n)$ admits a full exceptional collection $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n))$. Moreover, the objects $\mathcal{O}(i)$ are G-invariant for any G-action on $\mathbb{P}^n$. Let us denote by $[\alpha]$ the cohomology class of Remark 7.4 associated to the exceptional object $\mathcal{O}(1)$. In this notation, Proposition 7.8 yields an isomorphism

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(\mathbb{P}^n)) \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha} k) \oplus \cdots \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^n} k).$$

Example 7.10 (odd dimensional quadrics). Assume that $\mathrm{char}(k) \neq 2$. Let (V, q) be a nondegenerate quadratic form of odd dimension $n \geq 3$ and $Q_q \subset \mathbb{P}(V)$ the associated smooth projective quadric of dimension $d := n - 2$. As proved in [Kapranov 1988], $\mathrm{perf}(Q_q)$ admits a full exceptional collection $(\mathcal{S}, \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(d-1))$, where $\mathcal{S}$ denotes the spinor bundle. Moreover, the objects $\mathcal{O}(i)$ and $\mathcal{S}$ are G-invariant for any G-action on Q_q; see [Elagin 2012, §3.2]. Let us denote by $[\alpha]$ and $[\beta]$ the cohomology classes of Remark 7.4 associated to the exceptional object $\mathcal{O}(1)$ and $\mathcal{S}$, respectively. Under these notations, Proposition 7.8 yields an isomorphism between $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Q_q))$ and the direct sum

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\beta} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha} k) \oplus \cdots \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^{(d-1)}} k).$$

Example 7.11 (Grassmannians). Assume that $\mathrm{char}(k) = 0$. Let V be a k-vector space of dimension d, $n \leq d$ a positive integer, and $\mathrm{Gr} := \mathrm{Gr}(n, V)$ the Grassmannian of n-dimensional subspaces in V. As proved in [Kapranov 1988], $\mathrm{perf}(\mathrm{Gr})$ admits a full exceptional collection $(\mathcal{O}, \mathcal{U}^{\vee}, \dots, \Sigma^{\lambda}_{n(d-n)}\mathcal{U}^{\vee})$, where $\mathcal{U}^{\vee}$ denotes the dual of the tautological vector bundle on Gr and Σ^{λ}_i the Schur functor associated to a Young diagram λ with $|\lambda| = i$ having at most n rows and $d - n$

columns. Moreover, the objects $\Sigma_i^\lambda \mathcal{U}^\vee$ are G-invariant for any G-action on Q_q which is induced by an homomorphism $\mathrm{G} \to \mathrm{PGL}(V)$. Let us denote by $[\alpha]$ the cohomology class of Remark 7.4 associated to the exceptional object $\mathcal{U}^\vee$. In this notation, Proposition 7.8 yields an isomorphism

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(\mathrm{Gr})) \simeq U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_\alpha k) \oplus \cdots \oplus \Big(\bigoplus_\lambda U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^{n(d-n)}} k)\Big).$$

Proof of Theorem 1.2. To simplify the exposition, we write $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}(i)$ instead of $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}} \otimes \mathbb{Q}(1)^{\otimes i}$. Following Remark 7.4, let us denote by $[\alpha_i] \in H^2(\mathrm{G}, k^\times)$ the cohomology class associated to the exceptional object $\mathcal{E}_i$. By combining Propositions 6.2 and 7.8, we obtain induced isomorphisms

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))_{\mathbb{Q},I_{\mathbb{Q}}} \simeq \bigoplus_{i=1}^n U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_i} k)_{\mathbb{Q},I_{\mathbb{Q}}} \simeq \bigoplus_{i=1}^n U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)_{\mathbb{Q},I_{\mathbb{Q}}} \quad (7.12)$$

in the category $\mathrm{Hmo}_0^{\mathrm{G}}(k)_{\mathbb{Q},I_{\mathbb{Q}}}$. Since $\mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}$ (with trivial G-action) is the $\otimes$-unit of $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$ and $U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k)_{\mathbb{Q},I_{\mathbb{Q}}}$ the $\otimes$-unit of $\mathrm{NChow}^{\mathrm{G}}(k)_{\mathbb{Q},I_{\mathbb{Q}}}$, we conclude from Theorem 6.4 that $\pi(\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}})$ is isomorphic to $\bigoplus_{j=1}^n \pi(\mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}})$ in the orbit category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)}$. Let us now "lift" this isomorphism to the category $\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}$. Since the functor π is additive, there exist morphisms

$$\mathrm{f} = \{f_i\}_{i\in\mathbb{Z}} \in \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}\Big(\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}, \bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}(i)\Big),$$
$$\mathrm{g} = \{g_i\}_{i\in\mathbb{Z}} \in \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}\Big(\bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}, \mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}(i)\Big)$$

verifying the equalities $\mathrm{g} \circ \mathrm{f} = \mathrm{id} = \mathrm{f} \circ \mathrm{g}$. Moreover, as explained in Section 6, we have

$$\mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}\Big(\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}, \bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}(i)\Big) \simeq \bigoplus_{j=1}^n \mathrm{CH}_{\mathrm{G}}^{\dim(X)+i}(X)_{\mathbb{Q}},$$
$$\mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}\Big(\bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}, \mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}(i)\Big) \simeq \bigoplus_{j=1}^n \mathrm{CH}_{\mathrm{G}}^{i}(X)_{\mathbb{Q}}.$$

This implies that $f_i = 0$ when $i \notin \{-\dim(X), \ldots, 0\}$ and that $g_i = 0$ when $i \notin \{0, \ldots, \dim(X)\}$. The sets $\{f_{-r} \mid 0 \leq r \leq \dim(X)\}$ and $\{g_r(-r) \mid 0 \leq r \leq \dim(X)\}$ then give rise to morphisms in the category of G-equivariant Chow motives:

$$\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}} \to \bigoplus_{r=0}^{\dim(X)} \bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}(-r), \quad (7.13)$$
$$\bigoplus_{r=0}^{\dim(X)} \bigoplus_{j=1}^n \mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}(-r) \to \mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}. \quad (7.14)$$

The composition (7.14)$\circ$(7.13) agrees with the 0th component of $\mathrm{g}\circ\mathrm{f} = \mathrm{id}$, i.e., with the identity of $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$. Thus, since $\mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}}(-r) = \mathbb{L}^{\otimes r}$, the G-equivariant

Chow motive $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$ is a direct summand of $\bigoplus_{r=0}^{\dim(X)}\bigoplus_{j=1}^{n}\mathbb{L}^{\otimes r}$. By definition of the G-equivariant Lefschetz motive $\mathbb{L}$, we have $\mathrm{Hom}_{\mathrm{Chow}^{\mathrm{G}}(k)_{\mathbb{Q}}}(\mathbb{L}^{\otimes p},\mathbb{L}^{\otimes q})=\delta_{pq}\cdot\mathbb{Q}$, where δ_{pq} stands for the Kronecker symbol. This implies that $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}$ is a subsum of $\bigoplus_{r=0}^{\dim(X)}\bigoplus_{j=1}^{n}\mathbb{L}^{\otimes r}$. Using the fact that $\pi(\mathbb{L}^{\otimes r})$ is isomorphic to $\pi(\mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}})$, and $\pi(\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}})$ to $\bigoplus_{j=1}^{n}\pi(\mathfrak{h}^{\mathrm{G}}(\mathrm{Spec}(k))_{\mathbb{Q}})$, we conclude finally that there exists a choice of integers $r_1,\dots,r_n\in\{0,\dots,\dim(X)\}$ such that $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}\simeq\mathbb{L}^{\otimes r_1}\oplus\cdots\oplus\mathbb{L}^{\otimes r_n}$. This concludes the proof. □

Remark 7.15. The above proof of Theorem 1.2 is divided into two steps. In the first step, we established the isomorphism (7.12). In the second step, we explained how (7.12) leads to the desired isomorphism $\mathfrak{h}^{\mathrm{G}}(X)_{\mathbb{Q}}\simeq\mathbb{L}^{\otimes r_1}\oplus\cdots\oplus\mathbb{L}^{\otimes r_n}$. The proof of the second step is similar to that of [Marcolli and Tabuada 2015, Theorem 1.1].

7C. *Permutations.* Given a subgroup $\mathrm{H}\subseteq\mathrm{G}$, consider the small G-dg category $\mathrm{G}\circlearrowright\coprod_{\bar\rho\in\mathrm{G}/\mathrm{H}}k$, where G acts by permutation of the components.

Proposition 7.16. *Let* $\mathrm{G}\circlearrowright\mathcal{A}$ *be a small* G*-dg category such that* $\mathcal{D}_c(\mathcal{A})$ *admits a full exceptional collection* $(\mathcal{E}_1,\dots,\mathcal{E}_n)$. *Assume that the induced* G*-action on* $\mathcal{D}_c(\mathcal{A})$ *transitively permutes the objects* $\mathcal{E}_1,\dots,\mathcal{E}_n$ (*up to isomorphism*) *and that* $\mathrm{Hom}(\mathcal{E}_i,\mathcal{E}_j[m])=0$ *for every* $m\in\mathbb{Z}$ *and* $i\neq j$. *Let* $\mathrm{H}\subseteq\mathrm{G}$ *be the stabilizer of* $\mathcal{E}_1$. *If the cohomology group* $H^2(\mathrm{H},k^\times)$ *is trivial* (*e.g.*, $k=\mathbb{C}$ *and* H *is cyclic*), *then we have an isomorphism* $\mathrm{G}\circlearrowright\mathcal{A}\simeq\mathrm{G}\circlearrowright\coprod_{\bar\rho\in\mathrm{G}/\mathrm{H}}k$ *in* $\mathrm{Hmo}^{\mathrm{G}}(k)$.

Proof. We have the following equivalence of categories:

$$\Big(\prod_{\bar\rho\in\mathrm{G}/\mathrm{H}}\mathcal{D}_c(\mathcal{A})\Big)^{\mathrm{G}}\to\mathcal{D}_c(\mathcal{A})^{\mathrm{H}},\qquad(\{\mathrm{B}_{\bar\rho}\}_{\bar\rho\in\mathrm{G}/\mathrm{H}},\{\theta_\sigma\}_{\sigma\in\mathrm{G}})\mapsto(\mathrm{B}_{\bar 1},\{\theta_\sigma\}_{\sigma\in\mathrm{H}}).$$

Consequently, we obtain an induced identification

$$\mathrm{Hom}\Big(U^{\mathrm{G}}\Big(\mathrm{G}\circlearrowright\coprod_{\bar\rho\in\mathrm{G}/\mathrm{H}}k\Big),U^{\mathrm{G}}(\mathrm{G}\circlearrowright\mathcal{A})\Big)\simeq\mathrm{Hom}(U^{\mathrm{H}}(\mathrm{H}\circlearrowright_1 k),U^{\mathrm{H}}(\mathrm{H}\circlearrowright\mathcal{A})).\tag{7.17}$$

Since by assumption the cohomology group $H^2(\mathrm{H},k^\times)$ is trivial, the H-invariant object $\mathcal{E}_1$ is H-equivariant, i.e., it belongs to $\mathcal{D}_c(\mathcal{A})^{\mathrm{H}}$; see Remark 7.4. Via the identification (7.17), $\mathcal{E}_1$ corresponds then to a morphism $\mathrm{G}\circlearrowright\coprod_{\bar\rho\in\mathrm{G}/\mathrm{H}}k\to\mathrm{G}\circlearrowright\mathcal{A}$ in $\mathrm{Hmo}^{\mathrm{G}}(k)$. Using the fact that $\mathrm{Hom}_{\mathcal{D}_c(\mathcal{A})}(\mathcal{E}_i,\mathcal{E}_j[m])=0$ for every $m\in\mathbb{Z}$ and $i\neq j$, we observe that this morphism is a G-equivariant Morita equivalence. Therefore, the proof now follows automatically from Lemma 3.6. □

Proposition 7.18. *Let* X *be a quasicompact quasiseparated* G*-scheme such that* $\mathrm{perf}(X)$ *admits a full exceptional collection*

$$\big(\mathcal{E}_1^1,\dots,\mathcal{E}_1^{s_1},\dots,\mathcal{E}_i^1,\dots,\mathcal{E}_i^{s_i},\dots,\mathcal{E}_n^1,\dots,\mathcal{E}_n^{s_n}\big).\tag{7.19}$$

For every fixed $i\in\{1,\dots,n\}$, *assume that the* G*-action on* $\mathrm{perf}(X)$ *transitively permutes the objects* $\mathcal{E}_i^1,\dots,\mathcal{E}_i^{s_i}$ (*up to isomorphism*) *and that* $\mathrm{Hom}(\mathcal{E}_i^j,\mathcal{E}_i^l[m])=0$

for every $m \in \mathbb{Z}$ and $j \neq l$. Let $\mathrm{H}_i \subseteq \mathrm{G}$ be the stabilizer of $\mathcal{E}_i^1$. If $\mathrm{H}_i \neq \mathrm{G}$, assume that the cohomology group $H^2(\mathrm{H}_i, k^\times)$ is trivial. If $\mathrm{H}_i = \mathrm{G}$, denote by $[\alpha_i] \in H^2(\mathrm{G}, k^\times)$ the cohomology class of Remark 7.4 associated to the exceptional object $\mathcal{E}_i^1$. Under these assumptions, we have an isomorphism

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf_{dg}}(X)) \simeq \bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf_{dg}}(X)_i)$$

in $\mathrm{Hmo}_0^{\mathrm{G}}(k)$, where

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf_{dg}}(X)_i) \simeq \begin{cases} U^{\mathrm{G}}\big(\mathrm{G} \circlearrowright \coprod_{\bar{\rho} \in \mathrm{G}/\mathrm{H}_i} k\big) & \text{if } \mathrm{H}_i \neq \mathrm{G}, \\ U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha_i} k) & \text{if } \mathrm{H}_i = \mathrm{G}. \end{cases}$$

Remark 7.20. Note that in the case where $s_1 = \cdots = s_n = 1$, Proposition 7.18 reduces to Proposition 7.8.

Proof. Let us denote by $\mathrm{perf}(X)_i$ the smallest triangulated subcategory of $\mathrm{perf}(X)$ generated by the exceptional objects $\mathcal{E}_i^1, \ldots, \mathcal{E}_i^{s_i}$. In the same vein, let us write $\mathrm{perf_{dg}}(X)_i$ for the full dg subcategory of $\mathrm{perf_{dg}}(X)$ consisting of those objects belonging to $\mathrm{perf}(X)_i$. With this notation, the full exceptional collection (7.19) can be written as a semiorthogonal decomposition $\mathrm{perf}(X) = \langle \mathrm{perf}(X)_1, \ldots, \mathrm{perf}(X)_n \rangle$. Using Proposition 7.1, we hence obtain an isomorphism between $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf_{dg}}(X))$ and $\bigoplus_{i=1}^{n} U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf_{dg}}(X)_i)$ in $\mathrm{Hmo}_0^{\mathrm{G}}(k)$. The proof follows now from application of Propositions 7.16 and 7.8 to each one of the G-dg categories such that $\mathrm{H}_i \neq \mathrm{G}$ and $\mathrm{H}_i = \mathrm{G}$, respectively. □

Example 7.21 (even dimensional quadrics). Let Q_q be a smooth projective quadric of even dimension d; consult Example 7.10. As proved in [Kapranov 1988], $\mathrm{perf}(Q_q)$ admits a full exceptional collection $(S_-, S_+, \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(d-1))$, where $\mathcal{S}_+$ and $\mathcal{S}_-$ are the spinor bundles. Moreover, we have $\mathrm{Hom}(S_-, S_+[m]) = 0$ for every $m \in \mathbb{Z}$. Similarly to Example 7.10, the objects $\mathcal{O}(i)$ are G-invariant for any G-action on Q_q. Regarding the spinor bundles, they are G-invariant or sent to each other by the quotient $\mathrm{G}/\mathrm{H} \simeq C_2$; see [Elagin 2012, §3.2]. In the former case, we obtain a motivic decomposition similar to that of Example 7.10. In the latter case, assuming that $H^2(\mathrm{H}, k^\times)$ is trivial, Proposition 7.18 yields an isomorphism between $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf_{dg}}(Q_q))$ and the direct sum

$$U^{\mathrm{G}}\Big(\mathrm{G} \circlearrowright \coprod_{\bar{\rho} \in C_2} k\Big) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_1 k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha} k) \oplus \cdots \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^{(d-1)}} k),$$

where $[\alpha]$ stands for the cohomology class of Remark 7.4 associated to $\mathcal{O}(1)$.

Example 7.22 (del Pezzo surfaces). Assume that $\mathrm{char}(k) = 0$. Let X be the del Pezzo surface obtained by blowing up $\mathbb{P}^2$ at two distinct points x and y. As proved in [Orlov 1992, §4], $\mathrm{perf}(X)$ admits a full exceptional collection of length five $(\mathcal{O}_{E_1}(-1), \mathcal{O}_{E_2}(-1), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$, where $E_1 := \pi^{-1}(x)$ and $E_2 := \pi^{-1}(y)$ denote the exceptional divisors of the blowup $\pi : X \to \mathbb{P}^2$. Moreover, we have

$\mathrm{Hom}(\mathcal{O}_{E_1}(-1), \mathcal{O}_{E_2}(-1)[m]) = 0$ for every $m \in \mathbb{Z}$. The objects $\mathcal{O}(i)$ are G-invariant for every G-action on X. $\mathcal{O}_{E_1}(-1)$ and $\mathcal{O}_{E_2}(-1)$ are G-invariant or sent to each other by the quotient $\mathrm{G}/\mathrm{H} \simeq C_2$; see [Elagin 2012, §3.3]. In the former case, Proposition 7.8 yields an isomorphism between $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ and

$$U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\gamma} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\beta} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{1} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^2} k),$$

where $[\alpha]$, $[\beta]$, and $[\gamma]$, stand for the cohomology classes of Remark 7.4 associated to the exceptional objects $\mathcal{O}(1)$, $\mathcal{O}_{E_2}(-1)$, and $\mathcal{O}_{E_1}(-1)$, respectively. In the latter case, assuming that the cohomology group $H^2(\mathrm{H}, k^{\times})$ is trivial, Proposition 7.18 yields an isomorphism between $U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))$ and the direct sum

$$U^{\mathrm{G}}\Big(\mathrm{G} \circlearrowright \coprod_{\bar{\rho} \in C_2} k\Big) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{1} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha} k) \oplus U^{\mathrm{G}}(\mathrm{G} \circlearrowright_{\alpha^2} k).$$

Remark 7.23 (direct summands). Let X be a smooth projective G-scheme as in Proposition 7.18. A proof similar to Theorem 1.2 shows that $\mathfrak{h}(X)_{\mathbb{Q}}$ is a direct summand of the G-equivariant Chow motive

$$\bigoplus_{r=0}^{\dim(X)} \bigoplus_{i=0}^{n} \mathfrak{h}^{\mathrm{G}}\Big(\coprod_{\bar{\rho} \in \mathrm{G}/\mathrm{H}_i} \mathrm{Spec}(k)\Big)_{\mathbb{Q}}(-r),$$

where G acts by permutation of the components.

8. Equivariant motivic measures

In this section, by a *variety* we mean a reduced separated k-scheme of finite type. Let us write $\mathrm{Var}^{\mathrm{G}}(k)$ for the category of G-*varieties*, i.e., varieties which are equipped with a G-action such that every orbit is contained in an affine open set; this condition is automatically satisfied whenever X is quasiprojective. The *Grothendieck ring of* G-*varieties* $K_0 \mathrm{Var}^{\mathrm{G}}(k)$ is defined to be the quotient of the free abelian group on the set of isomorphism classes of G-varieties $[X]$ by the relations $[X] = [Y] + [X \setminus Y]$, where Y is a closed G-subvariety of X. The multiplication is induced by the product of G-varieties (with diagonal G-action). A G-*equivariant motivic measure* is a ring homomorphism $\mu^{\mathrm{G}} : K_0 \mathrm{Var}^{\mathrm{G}}(k) \to R$.

Example 8.1. (i) When $k \subseteq \mathbb{C}$, the topological Euler characteristic χ (with compact support) gives rise to a G-equivariant motivic measure

$$\mu_{\chi}^{\mathrm{G}} : K_0 \mathrm{Var}^{\mathrm{G}}(k) \to R_{\mathbb{Q}}(\mathrm{G}), \quad [X] \mapsto \sum_i (-1)^i H_c^i(X^{\mathrm{an}}, \mathbb{Q}),$$

where $H_c^i(X^{\mathrm{an}}, \mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-linear G-representation.

(ii) When $\mathrm{char}(k) = 0$, the characteristic polynomial $P_X(t) := \sum_i H^i_{dR}(X)t^i$, with X a smooth projective G-variety, gives rise to a G-equivariant motivic measure $\mu_P^{\mathrm{G}} : K_0 \mathrm{Var}^{\mathrm{G}}(k) \to R(\mathrm{G})[t]$, where $H^i_{dR}(X)$ is considered as a finite dimensional k-linear G-representation.

Let us denote by $K_0(\mathrm{NChow}^{\mathrm{G}}(k))$ the Grothendieck ring of the additive symmetric monoidal category of G-equivariant noncommutative Chow motives.

Theorem 8.2. *When* $\mathrm{char}(k) = 0$, *the assignment* $X \mapsto [U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))]$, *with* X *a smooth projective* G*-variety, gives rise to a* G*-equivariant motivic measure*

$$\mu_{\mathrm{nc}}^{\mathrm{G}} : K_0 \mathrm{Var}^{\mathrm{G}}(k) \to K_0(\mathrm{NChow}^{\mathrm{G}}(k)).$$

Proof. Thanks to Bittner's presentation [2004, Lemma 7.1] of the ring $K_0 \mathrm{Var}^{\mathrm{G}}(k)$, it suffices to verify the following two conditions:

(i) Given smooth projective G-schemes X and Y, we have

$$[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X \times Y))] = [U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \otimes U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Y))].$$

(ii) Let X be a smooth projective G-variety, Y a closed smooth G-subvariety of codimension c, $\mathrm{Bl}_Y(X)$ the blowup of X along Y, and E the exceptional divisor of this blowup. With this notation, the difference

$$\big[U^{\mathrm{G}}\big(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(\mathrm{Bl}_Y(X))\big)\big] - \big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(E))\big]$$

is equal to the difference

$$\big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))\big] - \big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Y))\big].$$

As proved in [Tabuada and Van den Bergh 2015a, Lemma 4.26], we have the G-equivariant Morita equivalence

$$\mathrm{perf}_{\mathrm{dg}}(X) \otimes \mathrm{perf}_{\mathrm{dg}}(Y) \to \mathrm{perf}_{\mathrm{dg}}(X \times Y), \quad (\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} \boxtimes \mathcal{G}.$$

Thus, (i) follows from Lemma 3.6 and the fact that the functor U^{G} is symmetric monoidal. For (ii), recall from [Orlov 1992, Theorem 4.3] that $\mathrm{perf}_{\mathrm{dg}}(\mathrm{Bl}_Y(X))$ contains full G-dg subcategories $\mathrm{perf}_{\mathrm{dg}}(X), \mathrm{perf}_{\mathrm{dg}}(Y)_0, \ldots, \mathrm{perf}_{\mathrm{dg}}(Y)_{c-2}$ inducing a semiorthogonal decomposition $\mathrm{perf}(\mathrm{Bl}_Y(X)) = \langle \mathrm{perf}(X), \mathrm{perf}(Y)_0, \ldots, \mathrm{perf}(Y)_{c-2} \rangle$. Moreover, we have an isomorphism $\mathrm{perf}_{\mathrm{dg}}(Y)_i \simeq \mathrm{perf}_{\mathrm{dg}}(Y)$ in $\mathrm{Hmo}^{\mathrm{G}}(k)$ for every i. Making use of Proposition 7.1, we obtain the equality

$$\big[U^{\mathrm{G}}\big(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(\mathrm{Bl}_Y(X))\big)\big] = \big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X))\big] + (c-1)\big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Y))\big].$$

Similarly, recall from [Orlov 1992, Theorem 2.6] that $\mathrm{perf}_{\mathrm{dg}}(E)$ contains full G-dg subcategories $\mathrm{perf}_{\mathrm{dg}}(Y)_0, \ldots, \mathrm{perf}_{\mathrm{dg}}(Y)_{c-1}$ inducing a semiorthogonal decomposition $\mathrm{perf}(E) = \langle \mathrm{perf}(Y)_0, \ldots, \mathrm{perf}(Y)_{c-1} \rangle$. Moreover, $\mathrm{perf}_{\mathrm{dg}}(Y)_i \simeq \mathrm{perf}_{\mathrm{dg}}(Y)$ in

$\mathrm{Hmo}^{\mathrm{G}}(k)$ for every i. Making use of Proposition 7.1, we conclude that

$$\big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(E))\big] = c\big[U^{\mathrm{G}}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(Y))\big].$$

Condition (ii) now follows automatically from the preceding two equalities. □

Proposition 8.3. *The motivic measure $\mu_\chi^{\mathrm{G}} \otimes_{\mathbb{Q}} \mathbb{C}$ factors through $\mu_{\mathrm{nc}}^{\mathrm{G}}$.*

Proof. Hochschild homology $HH : \mathrm{dgcat}(k) \to \mathcal{D}(k)$ is an example of a symmetric monoidal additive invariant. Thanks to Proposition 4.11, it then gives rise to an additive symmetric monoidal functor $\overline{HH^{\circlearrowright}} : \mathrm{Hmo}_0^{\mathrm{G}}(k) \to \mathcal{D}(k)^{\mathrm{G}}$ such that $\overline{HH^{\circlearrowright}} \circ U^{\mathrm{G}} \simeq HH^{\circlearrowright}$. Consider the composition

$$\mathrm{Hmo}_0^{\mathrm{G}}(k) \xrightarrow{\overline{HH^{\circlearrowright}}} \mathcal{D}(k)^{\mathrm{G}} \xrightarrow{-\otimes_k \mathbb{C}} \mathcal{D}(\mathbb{C})^{\mathrm{G}}. \tag{8.4}$$

It is well-known that an object of $\mathcal{D}(k)$ is strongly dualizable if and only if it is compact. Since the category of G-equivariant noncommutative Chow motives is rigid (see Proposition 3.13), the composition (8.4) yields a ring homomorphism

$$K_0(\mathrm{NChow}^{\mathrm{G}}(k)) \to K_0(\mathcal{D}_c(\mathbb{C})^{\mathrm{G}}) \simeq R_{\mathbb{C}}(\mathrm{G}). \tag{8.5}$$

We claim that $\mu_\chi^{\mathrm{G}} \otimes_{\mathbb{Q}} \mathbb{C}$ agrees with the composition of $\mu_{\mathrm{nc}}^{\mathrm{G}}$ with (8.5). Let X be a smooth projective G-variety. Thanks to Bittner's presentation of $K_0\,\mathrm{Var}^{\mathrm{G}}(k)$, it suffices to verify that the class of $HH^{\circlearrowright}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \otimes_k \mathbb{C}$ in the representation ring $R_{\mathbb{C}}(\mathrm{G})$ agrees with $\sum_i (-1)^i H_c^i(X^{\mathrm{an}}, \mathbb{C})$. This follows from the identifications

$$\begin{aligned}
[HH^{\circlearrowright}(\mathrm{G} \circlearrowright \mathrm{perf}_{\mathrm{dg}}(X)) \otimes_k \mathbb{C}] &= \sum_i (-1)^i HH_i(\mathrm{perf}_{\mathrm{dg}}(X)) \otimes_k \mathbb{C} \\
&= \sum_i (-1)^i \bigoplus_{p-q=i} H^q(X, \Omega_X^p) \otimes_k \mathbb{C} \qquad (8.6)\\
&= \sum_{p,q} (-1)^{p-q} H^q(X, \Omega_X^p) \otimes_k \mathbb{C} \\
&= \sum_{p,q} (-1)^{p+q} H^q(X, \Omega_X^p) \otimes_k \mathbb{C} \\
&= \sum_i (-1)^i H_c^i(X^{\mathrm{an}}, \mathbb{C}),
\end{aligned}$$

where (8.6) is a consequence of the (functorial) Hochschild–Kostant–Rosenberg isomorphism $HH_i(\mathrm{perf}_{\mathrm{dg}}(X)) \simeq \bigoplus_{p-q=i} H^q(X, \Omega_X^p)$. □

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GONÇALO TABUADA: tabuada@math.mit.edu
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, United States

and

Departamento de Matemática, Faculdade de Ciência e Tecnologia, Universidade Nova de Lisboa, Portugal

and

Centro de Matemática e Aplicações, Faculdade de Ciência e Tecnologia, Universidade Nova de Lisboa, Portugal

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Cohomologie non ramifiée de degré 3 : variétés cellulaires et surfaces de del Pezzo de degré au moins 5

Yang Cao

Dans cet article, où le corps de base est un corps de caractéristique zéro quelconque, pour X une variété géométriquement cellulaire, on étudie le quotient du troisième groupe de cohomologie non ramifiée $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$ par sa partie constante. Pour X une compactification lisse d'un torseur universel sur une surface géométriquement rationnelle, on montre que ce quotient est fini. Pour X une surface de del Pezzo de degré ≥ 5, on montre que ce quotient est trivial, sauf si X est une surface de del Pezzo de degré 8 d'un type particulier.

We consider geometrically cellular varieties X over an arbitrary field of characteristic zero. We study the quotient of the third unramified cohomology group $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$ by its constant part. For X a smooth compactification of a universal torsor over a geometrically rational surface, we show that this quotient is finite. For X a del Pezzo surface of degree ≥ 5, we show that this quotient is zero, unless X is a del Pezzo surface of degree 8 of a special type.

1. Introduction

Soient k un corps de caractéristique 0, $\bar{k}$ une clôture algébrique et Γ_k le groupe de Galois de $\bar{k}$ sur k. Pour une variété lisse X sur k et un faisceau étale F sur X,

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on rappelle que *la cohomologie non ramifiée* de X de degré n est le groupe

$$H^n_{\mathrm{nr}}(X, F) := H^0_{\mathrm{Zariski}}(X, \mathcal{H}^n(X, F)),$$

où $\mathcal{H}^n(X, F)$ est le faisceau Zariski associé au préfaisceau $\{U \subset X\} \mapsto H^n_{\text{ét}}(U, F)$. Soit $F = \mathbb{Q}/\mathbb{Z}(j)$ le faisceau des racines de l'unité tordu j fois. Les groupes $H^n_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(j))$ sont des invariants k-birationnels des k-variétés projectives lisses géométriquement connexes, réduits à $H^n(k, \mathbb{Q}/\mathbb{Z}(j))$ pour X k-rationnelle, c'est-à-dire k-birationnelle à un espace projectif. (cf. [Colliot-Thélène 1995, théorème 4.1.1 et proposition 4.1.4]). Le groupe $H^2_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(1))$ n'est autre que le groupe de Brauer de X, il a été fort étudié. On s'est intéressé plus récemment au groupe $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$. Le cas des coniques fut traité par Suslin. En dimension quelconque, le quotient $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))/H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ est trivial pour toute quadrique lisse qui n'est pas une quadrique d'Albert (Kahn, Rost, Sujatha, voir [Kahn 2008, théorème 10.2.4(b)]).

Notons

$$\bar{H}^n_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(j)) := \mathrm{coker}\big(H^n(k, \mathbb{Q}/\mathbb{Z}(j)) \to H^n_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(j))\big).$$

Dans cet article, nous nous intéressons aux surfaces géométriquement rationnelles les plus simples, les surfaces de del Pezzo de degré au moins 5. Rappelons que l'indice $I(X)$ d'une k-variété X est le pgcd des degrés sur k des points fermés. Si une surface de del Pezzo X de degré au moins 5 a un indice $I(X) = 1$, alors elle a un k-point et elle est k-rationnelle (cf. théorème 3.1). On a donc alors $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.

Nous nous intéressons ici au cas où $X(k)$ est éventuellement vide. Nous montrons :

Théorème 1.1 (théorème 5.2). *Soit X une k-surface de del Pezzo de degré ≥ 5. Alors $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$, sauf peut-être si $\deg(X) = 8$, $I(X) = 4$ et il existe des coniques lisses C_1, C_2 sur k telles que $X \xrightarrow{\sim} C_1 \times C_2$.*

On construit une surface de del Pezzo X de degré 8 sur le corps $k = \mathbb{C}(x, y, z)$ pour laquelle $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) \neq 0$ (exemple 5.4).

Pour les surfaces géométriquement rationnelles générales, nous montrons :

Théorème 1.2 (théorème 2.11). *Soit X une k-surface projective, lisse, géométriquement rationnelle. Soit $\mathcal{T} \to X$ un torseur universel sur X et soit $\mathcal{T}^c$ une k-compactification lisse de $\mathcal{T}$. Alors le groupe $\bar{H}^3_{\mathrm{nr}}(\mathcal{T}^c, \mathbb{Q}/\mathbb{Z}(2))$ est fini.*

Pour le faisceau $\mathbb{Z}/n(i) = \mu_n^{\otimes i}$ ou pour le complexe de faisceau $\mathbb{Z}(i)$ dont la définition est rappelée plus bas, on note $H^j(-, -)$ la cohomologie étale. Pour une courbe conique lisse C sur k, on note $[C] \in \mathrm{Br}(k)$ sa classe dans le groupe de Brauer de k.

2. Sur les variétés cellulaires et leur cohomologie non ramifiée

On rappelle la définition d'une variété cellulaire [Kahn 1999, Definition 3.2].

Définition 2.1. Un k-schéma de type fini X *a une décomposition cellulaire* (brièvement : est *cellulaire*) s'il existe un sous-ensemble fermé propre $Z \subset X$ tel que $X \setminus Z$ est isomorphe à un espace affine et Z a une décomposition cellulaire.

Un k-schéma de type fini X est dit *géométriquement cellulaire* si $X_{\bar{k}}$ a une décomposition cellulaire.

Proposition 2.1. *Soit k un corps algébriquement clos.*

(1) *Une surface projective, lisse, k-rationnelle est cellulaire.*

(2) *Une variété torique, lisse, projective sur k est cellulaire.*

(3) *Soient T un tore sur k et T^c une T-variété torique, lisse, projective. Soient X une variété cellulaire sur k et $Y \to X$ un T-torseur. Alors $Y^c := Y \times^T T^c$ est cellulaire.*

Démonstration. Par [Fulton 1993, Lemma, p. 103], on a l'énoncé (2).

Pour (3), par récurrence noethérienne, il suffit de montrer que si $X \xrightarrow{\sim} \mathbb{A}^n$ avec $n \in \mathbb{Z}_{\geq 0}$, alors Y^c est cellulaire. Dans ce cas, on sait que l'on a $H^1(\mathbb{A}^n, T) = 0$, $Y \xrightarrow{\sim} \mathbb{A}^n \times T$ et donc $Y^c \xrightarrow{\sim} \mathbb{A}^n \times T^c$. Le résultat découle de l'énoncé (2).

Pour (1), on sait (cf. [Kollár 1996, Theorem III.2.3]) que si la surface X est minimale, alors soit X est isomorphe à $\mathbb{P}^2$ soit X est fibrée en $\mathbb{P}^1$ au-dessus de $\mathbb{P}^1$. De telles surfaces sont cellulaires. Il suffit donc de montrer que si une surface lisse X est cellulaire, pour tout $x \in X(k)$, la surface éclatée $Y := \mathrm{Bl}_x X$ est cellulaire.

Supposons que $X = \mathbb{A}^2 \cup Z$ est une décomposition cellulaire de X. Si $x \in \mathbb{A}^2$, il suffit donc de montrer que $Y := \mathrm{Bl}_{(0,0)} \mathbb{A}^2$ est cellulaire. La variété $Y \subset \mathbb{A}^2 \times \mathbb{P}^1$ est définie par l'équation $xu = yv$, où $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$ et $\mathbb{P}^1 = \operatorname{Proj} k[u, v]$. Alors $Z(v = 0) \xrightarrow{\sim} \mathbb{A}^1$ et $D(v \neq 0) = \operatorname{Spec} k[x, y, u/v]/((u/v) \cdot x = y) \cong \mathbb{A}^2$.

Si $x \in Z$, il existe un ouvert $U \subset X$ et un fermé $V \subset X$ tels que U, V soient cellulaires, $U \cap V = \varnothing$, $x \notin U \cup V$ et X ait une décomposition cellulaire $X = U \cup \mathbb{A}^1 \cup V$ ou $X = U \cup \mathbb{A}^0 \cup V$. Ainsi $Y \times_X U$ et $Y \times_X V$ sont cellulaires. Dans le premier cas, $Y \times_X \mathbb{A}^1 = \mathbb{P}^1 \cup \mathbb{A}^1$ avec $\mathbb{P}^1 \cap \mathbb{A}^1 = \{x'\}$, où $\mathbb{P}^1$ est le diviseur exceptionnel. On a donc $\mathbb{P}^1 \setminus \{x'\} \cong \mathbb{A}^1$ et $(Y \times_X \mathbb{A}^1) \setminus (\mathbb{P}^1 \setminus \{x'\}) \cong \mathbb{A}^1$. Dans le deuxième cas, on a $Y \times_X \mathbb{A}^0 \cong \mathbb{P}^1 \cong \mathbb{A}^1 \cup \mathbb{A}^0$. Le résultat en découle. □

Soit de nouveau k un corps de caractéristique zéro quelconque. On utilise dans cet article le complexe motivique $\mathbb{Z}(n)$ de faisceaux sur les variétés lisses sur k (Voevodsky), sous la forme donnée par Bruno Kahn [2012, §2]). Pour toute k-variété lisse X, dans la catégorie dérivée, on a $\mathbb{Z}(n) = 0$ pour $n < 0$, $\mathbb{Z}(0) = \mathbb{Z}$, $\mathbb{Z}(1) \xrightarrow{\sim} \mathbb{G}_m[-1]$ et une suite exacte ([Kahn 2012, proposition 2.9])

$$0 \longrightarrow \mathrm{CH}^2(X) \longrightarrow H^4(X, \mathbb{Z}(2)) \longrightarrow H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow 0. \tag{2.2}$$

Théorème 2.3 [Kahn 2010, Theorem 2.5]. *Soit X une k-variété lisse, intègre, géométriquement cellulaire. Pour tout entier $n \geq 0$, on a une suite spectrale fonctorielle :*

$$E_2^{p,q}(X,n) = H^{p-q}\big(k, \mathrm{CH}^q(X_{\bar{k}}) \otimes \mathbb{Z}(n-q)\big) \Longrightarrow H^{p+q}(X, \mathbb{Z}(n)) \tag{2.4}$$

et on a un accouplement de suites spectrales :

$$E_r^{p,q}(m) \times E_r^{p',q'}(n) \to E_r^{p+p',q+q'}(m+n), \tag{2.5}$$

tel que, pour $r=2$, l'accouplement est le cup-produit.

On trouvera dans l'appendice des rappels sur l'accouplement de suites spectrales.

La différentielle $E_2^{1,1}(X,1) \to E_2^{3,0}(X,1)$ définit un homomorphisme :

$$d(1) : \mathrm{Pic}(X_{\bar{k}})^{\Gamma_k} \to \mathrm{Br}(k).$$

La différentielle $E_2^{2,2}(X,2) \to E_2^{4,1}(X,2)$ définit un homomorphisme :

$$d(2) : \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k} \to H^2(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times).$$

Lemme 2.6. *Soit X une k-variété lisse, géométriquement intègre, géométriquement cellulaire. Alors on a* $\mathrm{Im}(\mathrm{CH}^2(X) \to \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k}) \subset \mathrm{Ker}(d(2))$.

Démonstration. Puisque $\mathbb{Z}(n) = 0$ pour $n < 0$, dans la suite spectrale (2.4), on a $E_2^{p,q}(X,2) = 0$ pour $q > 2$. Donc on a un morphisme canonique : $H^4(X, \mathbb{Z}(2)) \xrightarrow{d_X} E_\infty^{2,2}(X,2)$ et une inclusion $E_\infty^{2,2}(X,2) \subset E_2^{2,2}(X,2)$. Alors on a un diagramme commutatif :

$$\begin{tikzcd}
\mathrm{CH}^2(X) \arrow[r, "i_X"] \arrow[d] & H^4(X, \mathbb{Z}(2)) \arrow[r, "d_X"] \arrow[d] & E_\infty^{2,2}(X,2) \arrow[r] \arrow[d] & E_2^{2,2}(X,2) = \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k} \arrow[d] \\
\mathrm{CH}^2(X_{\bar{k}}) \arrow[r, "i_{X_{\bar{k}}}"] & H^4(X_{\bar{k}}, \mathbb{Z}(2)) \arrow[r, "d_{X_{\bar{k}}}"] & E_\infty^{2,2}(X_{\bar{k}},2) \arrow[r] & E_2^{2,2}(X_{\bar{k}},2) = \mathrm{CH}^2(X_{\bar{k}})
\end{tikzcd}$$

où $\mathrm{CH}^2(X) \xrightarrow{i_X} H^4(X, \mathbb{Z}(2))$ désigne le morphisme dans la suite exacte (2.2). Puisque la suite spectrale (2.4) dégénère canoniquement lorsque $k = \bar{k}$, la composition dans la deuxième ligne est l'identité $\mathrm{id} : \mathrm{CH}^2(X_{\bar{k}}) \to \mathrm{CH}^2(X_{\bar{k}})$. Donc la composition dans la première ligne est le morphisme naturel $\mathrm{CH}^2(X) \to \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k}$ et on a

$$\mathrm{Im}\big(\mathrm{CH}^2(X) \to \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k}\big) \subset E_\infty^{2,2}(X,2) \subset \mathrm{Ker}(d(2)). \qquad \square$$

Notons désormais $\mathcal{M}(X)$ l'homologie du complexe

$$\mathrm{CH}^2(X) \to \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k} \xrightarrow{d(2)} H^2(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times). \tag{2.7}$$

On note

$$d'(2) : \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k} / \operatorname{Im} \mathrm{CH}^2(X) \xrightarrow{d(2)} H^2(k, \operatorname{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times)$$

l'application induite par $d(2)$. On a $\mathcal{M}(X) = \ker d'(2)$.

Le théorème suivant généralise les corollaires 7.1 et 7.2 de [Kahn 1996] :

Théorème 2.8. *Soit X une k-variété lisse, géométriquement intègre, géométriquement cellulaire. Si $H^1(k, \operatorname{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times) = 0$, alors les groupes $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$ et $\mathcal{M}(X)$ sont finis et on a une suite exacte :*

$$0 \longrightarrow \bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathcal{M}(X) \longrightarrow H^4(k, \mathbb{Q}/\mathbb{Z}(2)). \tag{2.9}$$

Démonstration. Par la suite exacte (2.2), on a une suite exacte :

$$\mathrm{CH}^2(X) \longrightarrow \frac{H^4(X, \mathbb{Z}(2))}{\operatorname{Im} H^4(k, \mathbb{Z}(2))} \longrightarrow \bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow 0.$$

Dans la suite spectrale (2.4), on a $E_2^{p,q}(X, 2) = 0$ pour $q > 2$ ou $q < 0$ et donc une suite exacte :

$$E_\infty^{3,1}(X, 2) \longrightarrow \frac{H^4(X, \mathbb{Z}(2))}{\operatorname{Im} H^4(k, \mathbb{Z}(2))} \longrightarrow \operatorname{Ker}(d(2)) \longrightarrow E_2^{5,0}(X, 2).$$

D'après le lemme 2.6, on a une suite exacte :

$$E_\infty^{3,1}(X, 2) \longrightarrow \bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathcal{M}(X) \longrightarrow H^4(k, \mathbb{Q}/\mathbb{Z}(2)).$$

Si $E_2^{3,1}(X, 2) = H^1(k, \operatorname{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times) = 0$, on a $E_\infty^{3,1}(X, 2) = 0$ et donc la suite exacte (2.9).

Par [Kahn 1999, Lemma 3.3], $\operatorname{Pic}(X_{\bar{k}})$ et $\mathrm{CH}^2(X_{\bar{k}})$ sont des $\mathbb{Z}$-modules libres de type fini. Puisque $\mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k}/\operatorname{Im} \mathrm{CH}^2(X)$ est un groupe de torsion, le groupe $\mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k}/\operatorname{Im} \mathrm{CH}^2(X)$ est fini et donc $\mathcal{M}(X)$ est fini. □

Remarque 2.10. Pour une k-variété lisse géométriquement connexe géométriquement cellulaire, le groupe $H^0(X_{\bar{k}}, \mathcal{K}_2)$ est uniquement divisible. Pour des généralisations du théorème 2.8 sous cette simple hypothèse, on consultera [Colliot-Thélène 2015, propositions 1.3 et 2.2].

Théorème 2.11. *Soit X une k-variété projective, lisse, géométriquement intègre et géométriquement cellulaire. Soit $\mathcal{T} \to X$ un torseur universel sur X et soit $\mathcal{T}^c$ une k-compactification lisse de $\mathcal{T}$. Alors le groupe $\bar{H}^3_{\mathrm{nr}}(\mathcal{T}^c, \mathbb{Q}/\mathbb{Z}(2))$ est fini.*

Démonstration. Soit S le k-tore de groupe des caractères du réseau $\operatorname{Pic}(X_{\bar{k}})$. D'après [Colliot-Thélène et al. 2005, corollaire 1], il existe une k-compactification torique lisse S^c de S. Comme le groupe $H^3_{\mathrm{nr}}(\mathcal{T}^c, \mathbb{Q}/\mathbb{Z}(2))$ est un invariant k-birationnel, il suffit d'établir le résultat pour $\mathcal{T}^c = \mathcal{T} \times^S S^c$. D'après la proposition 2.1, $\mathcal{T}^c$ est alors une variété géométriquement cellulaire. Par ailleurs, le module galoisien $\mathrm{Pic}(\mathcal{T}^c_{\bar{k}})$ est un module de permutation [Colliot-Thélène et Sansuc 1987, théorème 2.1.2]. On a donc $H^1(k, \mathrm{Pic}(\mathcal{T}^c_{\bar{k}}) \otimes \bar{k}^\times) = 0$. Une application du théorème 2.8 donne alors le résultat. $\square$

D'après la proposition 2.1, le théorème 1.2 est un cas spécial du théorème 2.11.

Pour appliquer le théorème 2.8 au calcul du groupe $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$, on a besoin de contrôler l'application $\mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k} \xrightarrow{d(2)} H^2(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times)$.

Soit X une k-variété lisse, intègre, géométriquement cellulaire. L'accouplement (2.5) pour $n = m = 1$ donne un diagramme commutatif (cf. l'appendice) :

$$\begin{tikzcd}
E_2^{1,1}(X,1) \otimes E_2^{1,1}(X,1) \arrow[r, "d_\otimes"] \arrow[d, "\cup"] & (E_2^{3,0}(X,1) \otimes E_2^{1,1}(X,1)) \oplus (E_2^{1,1}(X,1) \otimes E_2^{3,0}(X,1)) \arrow[d, "\cup+\cup"] \\
E_2^{2,2}(X,2) \arrow[r, "d(2)"] & E_2^{4,1}(X,2)
\end{tikzcd}$$

où $d_\otimes = d(1) \otimes \mathrm{id} + \mathrm{id} \otimes d(1)$. C'est-à-dire que l'on a un diagramme commutatif :

$$\begin{tikzcd}
\mathrm{Pic}(X_{\bar{k}})^{\Gamma_k} \otimes \mathrm{Pic}(X_{\bar{k}})^{\Gamma_k} \arrow[r, "d_\otimes"] \arrow[d, "\cup_1"] & (\mathrm{Br}(k) \otimes \mathrm{Pic}(X_{\bar{k}})^{\Gamma_k}) \oplus (\mathrm{Pic}(X_{\bar{k}})^{\Gamma_k} \otimes \mathrm{Br}(k)) \arrow[d, "\cup_2+\cup_2"] \\
\mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k} \arrow[r, "d(2)"] & H^2(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times),
\end{tikzcd} \tag{2.12}$$

où $\cup_1$ est l'intersection et $\cup_2$ est le cup-produit

$$H^2(k, \bar{k}^\times) \times H^0(k, \mathrm{Pic}(X_{\bar{k}})) \xrightarrow{\cup_2} H^2(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times).$$

3. Surfaces de del Pezzo de degré au moins 5

Une surface projective, lisse, géométriquement connexe X est appelée *surface de del Pezzo* si le faisceau anticanonique $-K_X$ est ample. Le degré d'une telle surface X est $\deg(X) := (K_X, K_X)$. Par [Kollár 1996, Exercise 3.9], X est alors géométriquement rationnelle, on a $1 \le \deg(X) \le 9$ et $\mathrm{Pic}(X_{\bar{k}}) \xrightarrow{\sim} \mathbb{Z}^{10-\deg(X)}$. Comme pour toute surface X projective, lisse, géométriquement rationnelle, le degré sur les zéro-cycles définit un isomorphisme $\mathrm{CH}^2(X_{\bar{k}}) \xrightarrow{\sim} \mathbb{Z}$, et on a

$$\frac{\mathrm{CH}^2(X_{\bar{k}})^\Gamma}{\mathrm{Im}\,\mathrm{CH}^2(X)} = \mathbb{Z}/I(X),$$

où $I(X)$ désigne l'indice de X.

Par les travaux de Enriques, Châtelet, Manin, Swinnerton-Dyer (voir [Colliot-Thélène 1999, paragraphe 4] ou [Várilly-Alvarado 2013, Theorem 2.1]), on a :

Théorème 3.1. *Soit X une surface de del Pezzo de degré ≥ 5.*

(1) *Si $X(k) \neq \varnothing$, alors X est k-rationnelle ;*

(2) *Si* $\deg(X) = 5$ *ou* 7, *alors $X(k) \neq \varnothing$.*

Soit X une surface de del Pezzo de degré ≥ 5. Si $X(k) \neq \varnothing$, l'énoncé (1) implique que l'on a $\bar{H}^i_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(j)) = 0$ pour tous entiers i et j. En particulier

$$\mathrm{Br}(X)/\operatorname{Im}\mathrm{Br}(k) = \bar{H}^2_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(1)) = 0$$

(voir aussi le lemme 3.2) et $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.

En fait, soit X une surface de del Pezzo de degré au moins 4, alors $I(X) = 1$ implique $X(k) \neq \varnothing$. La question analogue est ouverte pour les del Pezzo de degré 3, i.e., les surfaces cubiques. Ceci n'est pas utilisé dans le présent article.

Lemme 3.2. *Soit X une k-surface de del Pezzo de degré ≥ 5. Alors* $\mathrm{Pic}(X_{\bar{k}})$ *est stablement de permutation,* $H^1(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times) = 0$ *et* $\mathrm{Br}(X)/\operatorname{Im}\mathrm{Br}(k) = 0$.

Démonstration. Soit $\mathcal{C}$ la classe des surfaces X/K, pour K corps extension quelconque de k, de del Pezzo de degré ≥ 5. Par le théorème 3.1, si $X(K) \neq \varnothing$, alors X est K-rationnelle, et donc $\mathrm{Pic}(X_{\bar{K}})$ est stablement de permutation comme $\mathrm{Gal}(\bar{K}/K)$-module ([Colliot-Thélène et Sansuc 1987, proposition 2.A.1]). Pour chaque $X/K \in \mathcal{C}$, le $\mathrm{Gal}(\bar{K}/K)$-module $\mathrm{Pic}(X_{\bar{K}})$ est stablement de permutation, par [Colliot-Thélène et Sansuc 1987, théorème 2.B.1]. Alors

$$H^1(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times) = 0 \quad \text{et} \quad H^1(k, \mathrm{Pic}(X_{\bar{k}})) = 0.$$

Par la suite spectrale de Hochschild–Serre on obtient $\mathrm{Br}(X)/\operatorname{Im}\mathrm{Br}(k) = 0$, puisque $\mathrm{Br}(X_{\bar{k}}) = 0$. □

Proposition 3.3. *Soit X une k-surface de del Pezzo de degré ≥ 5. On a la suite exacte*

$$0 \to \mathcal{M}(X) \to \mathbb{Z}/I(X) \xrightarrow{d'(2)} H^2(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times) \tag{3.4}$$

et la suite exacte

$$0 \longrightarrow \bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \mathcal{M}(X) \longrightarrow H^4(k, \mathbb{Q}/\mathbb{Z}(2)), \tag{3.5}$$

où $\mathcal{M}(X)$ est l'homologie du complexe (2.7).

Démonstration. Ceci résulte du théorème 2.8 et du lemme 3.2. □

4. Formes tordues de $\mathbb{P}^1 \times \mathbb{P}^1$

Rappelons (voir par exemple [Auel et Bernardara 2015, Examples 3.1.3, 3.1.4]) que l'on a :

Proposition 4.1. *Soit X une surface de del Pezzo de degré* 8 *sur un corps k. Alors on a l'une des possibilités suivantes :*

(1) *X est un éclatement de $\mathbb{P}^2_k$ en un k-point, et dans ce cas, $X(k) \neq \varnothing$.*

(2) *Il existe des coniques lisses C_1, C_2 sur k telles que $X \xrightarrow{\sim} C_1 \times C_2$.*

(3) *Il existe une extension de corps K/k de degré 2 et une conique C sur K tels que $X \xrightarrow{\sim} R_{K/k}C$, où $R_{K/k}$ désigne la restriction à la Weil de K à k.*

De plus, $\mathrm{Pic}(X_{\bar{k}})$ *est un Γ_k-module de permutation.*

En fait, dans le cas où $X \subset \mathbb{P}^3_k$ est un quadrique lisse, on a l'extension discriminant K/k de degré 2 (peut-être $K = k \times k$) et, pour toute section plane lisse $C \subset X$, on a $X \simeq R_{K/k}C_K$. Ceci n'est pas utilisé dans le présent article.

Dans le cas (2), on a :

Proposition 4.2. *Soient C_1, C_2 deux coniques lisses sur k et $X \xrightarrow{\sim} C_1 \times C_2$. Supposons $X(k) = \varnothing$. L'image de $d(2)$ est $\mathbb{Z}/2$. Si $I(X) = 2$, alors $\mathcal{M}(X) = 0$ et $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$. Si $I(X) = 4$, alors $\mathcal{M}(X) = \mathbb{Z}/2$.*

Démonstration. On a $\mathrm{Pic}(C_{i,\bar{k}})^{\Gamma}_k \cong \mathrm{Pic}(C_{i,\bar{k}}) \cong \mathbb{Z}$ pour $i = 1, 2$. On note $p_i : X \to C_i$ la projection, et pour $p_i^* : \mathbb{Z} \cong \mathrm{Pic}(C_{i,\bar{k}}) \to \mathrm{Pic}(X_{\bar{k}})$, on note $e_i := p_i^*(1_{\mathbb{Z}})$. Alors $\mathrm{Pic}(X_{\bar{k}})^{\Gamma_k} \xrightarrow{\sim} \mathrm{Pic}(X_{\bar{k}}) \xrightarrow{\sim} \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ et $H^2(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times) \xrightarrow{\sim} \mathrm{Br}(k)e_1 \oplus \mathrm{Br}(k)e_2$.

Pour $i = 1, 2$, on applique [Kahn 1999, Theorem 4.4(i)] à $E_2^{1,1}(-, 1) \to E_2^{3,0}(-, 1)$. On obtient un diagramme commutatif :

$$\begin{tikzcd}
\mathrm{Pic}(C_{i,\bar{k}})^{\Gamma_k} \arrow[r, "d(C_i)"] \arrow[d, "p_i^*"] & \mathrm{Br}(k) \arrow[d, "="] \\
\mathrm{Pic}(X_{\bar{k}})^{\Gamma_k} \arrow[r, "d(1)"] & \mathrm{Br}(k)
\end{tikzcd}$$

Notons $[C_i] := d(C_i)(1_{\mathrm{Pic}(C_{i,\bar{k}})})$. En utilisant le diagramme (2.12), on obtient :

$$\begin{aligned} d(2)(e_1 \cup e_2) = (d(2) \circ \cup_1)(e_1 \otimes e_2) &= \cup_2([C_1] \otimes e_2) + \cup_2([C_2] \otimes e_1) \\ &= [C_1]e_2 + [C_2]e_1. \end{aligned} \tag{4.3}$$

On vérifie aisément

$$\mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k} \cong \mathrm{CH}^2(X_{\bar{k}}) \xrightarrow{\sim} \mathbb{Z}(e_1 \cup e_2).$$

On obtient : $\mathrm{Im}(d(2)) = 0$ si et seulement si $[C_1] = [C_2] = 0$ et sinon $\mathrm{Im}(d(2)) = \mathbb{Z}/2$. On conclut alors avec la proposition 3.3. □

Dans le cas (3), le lemme suivant est dû à Olivier Benoist :

Lemme 4.4. *Soient K/k une extension de corps de degré 2, C une conique lisse sur K et $X \xrightarrow{\sim} R_{K/k}C$ avec $X(k) = \varnothing$. Supposons que $[C] \in \mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(K))$. Alors $I(X) = 2$.*

Démonstration. Soit $\alpha \in \mathrm{Br}(k)$ un élément tel que $\alpha|_K = [C] \in \mathrm{Br}(K)$. Puisque $[C]$ est d'indice 2, l'indice de α est 2 ou 4.

Si α est d'indice 2, il existe une extension k' de degré 2 de k telle que $\alpha|_{k'} = 0 \in \mathrm{Br}(k')$.

Si α est d'indice 4, on le représente par un k-corps gauche D de degré 4. Ainsi D est déployé sur une extension L de degré 2 de K. Alors D contient une sous-algèbre commutative isomorphe à L, donc a fortiori une sous-algèbre commutative isomorphe à K. Par un théorème d'Albert [1932, Theorem 5], cf. [Jacobson 1996, Lemma 2.9.23], il existe une extension k' de degré 2 de k telle que D contient une sous-algèbre commutative isomorphe à $K' := k' \cdot K$.

Dans tout cas, il existe une extension k' de degré 2 de k telle que $[C]|_{K'} = 0 \in \mathrm{Br}(K')$, où $K' = k' \cdot K$. Donc $X(k') \neq \varnothing$ et $I(X) = 2$. □

Remarque 4.5. Soient K/k une extension de corps de degré 2, C une conique lisse sur K et $X \xrightarrow{\sim} R_{K/k}C$ avec $X(k) = \varnothing$. Si $[C] \notin \mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(K))$, alors $I(X) = 4$. Ceci sera montré dans la démonstration de la proposition 4.6.

Proposition 4.6. *Soient K/k une extension de corps de degré 2, C une conique lisse sur K et $X \xrightarrow{\sim} R_{K/k}C$ avec $X(k)=\varnothing$. Alors $\mathcal{M}(X)=0$ et $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))=0$.*

Démonstration. On a $X_K \xrightarrow{\sim} C \times_K C^\sigma$, où $\sigma \in \mathrm{Gal}(K/k)$, $\sigma \neq \mathrm{id}$ et

$$C^\sigma := (C \to \operatorname{Spec} K \xrightarrow{\sigma} \operatorname{Spec} K).$$

Donc $\mathrm{CH}^2(X_{\bar{k}}) \xrightarrow{\sim} \mathbb{Z}$ et $I(X)|4$. Puisque $d(2)(\mathrm{CH}^2(X)) = 0$, on a $\big(\#\,\mathrm{Im}(d(2))\big)|4$. L'hypothèse $X(k) = \varnothing$ équivaut à $C(K) = \varnothing$.

Par [Kahn 1999, Theorem 4.4(i),(iii)], on a un diagramme commutatif

$$\begin{tikzcd}
\mathbb{Z} \cong \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_K} \arrow[r, "\mathrm{tr}"] \arrow[d, "d(2)_K"] \arrow[dr, phantom, "(1)"] & \mathbb{Z} \cong \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k} \arrow[r, "\mathrm{Res}"] \arrow[d, "d(2)"] \arrow[dr, phantom, "(2)"] & \mathbb{Z} \cong \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_K} \arrow[d, "d(2)_K"] \\
H^2(K, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times) \arrow[r, "\mathrm{tr}"] & H^2(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times) \arrow[r, "\mathrm{Res}"] & H^2(K, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times)
\end{tikzcd}$$

où tr est le transfert et Res est la restriction. Par la proposition 4.2, l'image de $d(2)_K$ est $\mathbb{Z}/2$. Par le carré (2), l'image de $d(2)$ est $\mathbb{Z}/2$ ou $\mathbb{Z}/4$. Puisque

$$\mathrm{tr}(1_{\mathrm{CH}^2(X_{\bar{k}})}) = 2 \cdot 1_{\mathrm{CH}^2(X_{\bar{k}})},$$

par le carré (1), l'image de $d(2)$ est $\mathbb{Z}/2$ si et seulement si $\mathrm{tr}(\mathrm{Im}(d(2)_K)) = 0$.

On considère :

$$\mathrm{Br}(K) \oplus \mathrm{Br}(K) \xrightarrow{\cong} H^2(K, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times) \underset{\mathrm{Res}}{\overset{\mathrm{tr}}{\rightleftarrows}} H^2(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times) \xrightarrow{\cong} \mathrm{Br}(K).$$

Pour chaque $a, b \in \mathrm{Br}(K)$, l'action de σ sur $\mathrm{Br}(K) \oplus \mathrm{Br}(K)$ est définie par $\sigma(a, b) = (\sigma(b), \sigma(a))$ et $\mathrm{Res}(a) = (a, \sigma(a))$. Alors $\sigma(a, b) + (a, b) = \mathrm{Res}(a + \sigma(b))$ et, d'après [Mazza et al. 2006, Exercise 6.5] et [Milne 1980, V.1.12], on a $\mathrm{tr}(a,b) = a+\sigma(b)$. Par l'équation (4.3), $d(2)_K(1_{\mathrm{CH}^2(X_{\bar{k}})}) = ([C],[C])$. Donc $\mathrm{tr}(\mathrm{Im}(d(2)_K)) = 0$ si et seulement si $[C] = \sigma([C])$, i.e., $[C] \in \mathrm{Br}(K)^\sigma$. Puisque $\mathrm{Gal}(K/k) \cong \mathbb{Z}/2$, on a $H^3(\mathrm{Gal}(K/k), K^\times) \cong H^1(\mathrm{Gal}(K/k), K^\times) = 0$, et donc le morphisme $\mathrm{Br}(k) \to \mathrm{Br}(K)^\sigma$ est surjectif.

On a alors :

(1) Si $[C] \in \mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(K))$, l'image de $d(2)$ est $\mathbb{Z}/2$ et, par le lemme 4.4, on a $I(X) = 2$.

(2) Si $[C] \notin \mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(K))$, l'image de $d(2)$ est $\mathbb{Z}/4$ et, par le lemme 2.6, on a $I(X) = 4$.

On conclut alors avec la proposition 3.3. □

5. Calcul de $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2))$ pour une surface de del Pezzo X de degré ≥ 5

Rappelons un fait bien connu.

Lemme 5.1. *Soit X une k-surface de del Pezzo de degré 6. On a :*

(1) *Il existe une extension K_1/k de degré divisant 2, une K_1-forme X_1 de $\mathbb{P}^2$ sur K_1 et un morphisme $f_1 : X_{K_1} \to X_1$ birationnel.*

(2) *Il existe une extension K_2/k de degré divisant 3, une surface de del Pezzo X_2 de degré 8 sur K_2 et un morphisme $f_2 : X_{K_2} \to X_2$ tels que X_{K_2} est un éclatement de X_2 le long d'un sous-schéma réduit de dimension 0 et de degré 2. Donc l'indice $I(X_2)$ de la K_2-surface X_2 est 1 ou 2.*

Démonstration. Cela provient du fait que la configuration des 6 courbes exceptionnelles de $X_{\bar{k}}$ est celle d'un hexagone ([Colliot-Thélène 1972], ou voir [Várilly-Alvarado 2013, Section 2.4]). □

Théorème 5.2. *Soit X une k-surface de del Pezzo de degré ≥ 5. Alors $\mathcal{M}(X) = \mathbb{Z}/2$ si et seulement si $I(X) = 4$, $\deg(X) = 8$, et il existe des coniques lisses C_1, C_2 sur k telles que $X \xrightarrow{\sim} C_1 \times C_2$.*

Sinon, $\mathcal{M}(X) = 0$ et donc $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.

Démonstration. La dernière implication résulte de la proposition 3.3.

Si $X(k) \neq \varnothing$, le morphisme $\mathrm{CH}^2(X) \to \mathrm{CH}^2(X_{\bar{k}}) = \mathbb{Z}$ est surjectif. Donc $\mathcal{M}(X) = 0$. Si $\deg(X) = 5$ ou 7, par le théorème 3.1, $X(k) \neq \varnothing$ et donc alors $\mathcal{M}(X) = 0$. On suppose dorénavant $X(k) = \varnothing$.

Si $\deg(X) = 9$ avec $X(k) = \varnothing$, X est la variété de Severi–Brauer associée à une algèbre centrale simple A de degré 3 (cf. [Várilly-Alvarado 2013, Theorem 1.6]). Par un théorème de Kahn [1999, Theorem 7.1],

$$d(2)(1_{\mathrm{CH}^2(X_{\bar{k}})}) = 2[A] \in \mathrm{Br}(k) = H^2(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^*).$$

Puisque $X(k) = \varnothing$, on a $[A] \neq 0$, $3[A] = 0$ et $I(X) = 3$. Donc $d(2)(1_{\mathrm{CH}^2(X_{\bar{k}})}) \neq 0$ et $d'(2)$ est injectif. Alors $\mathcal{M}(X) = 0$.

Si $\deg(X) = 8$ avec $X(k) = \varnothing$, le résultat en degré 8 est donné par les propositions 4.1, 4.2 et 4.6.

Considérons le cas des surfaces de del Pezzo de degré 6.

S'il existe une surface de del Pezzo Y et un morphisme $f : X \to Y$ projectif, birationnel, alors f^* induit un morphisme des suites spectrales (2.4) pour Y et X. De plus, $f^* : \mathrm{CH}^2(Y_{\bar{k}})^{\Gamma_k} \xrightarrow{\sim} \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k}$ est un isomorphisme et $f^* : \mathrm{Pic}(Y_{\bar{k}}) \to \mathrm{Pic}(X_{\bar{k}})$ admet un inverse à gauche. Donc

$$\begin{aligned} f^* &: E_2^{4,1}(Y,2) \to E_2^{4,1}(X,2) && \text{est injectif,} \\ f^* &: \mathrm{Ker}(d(2)_Y) \to \mathrm{Ker}(d(2)_X) && \text{est un isomorphisme.} \end{aligned}$$

Donc $\mathcal{M}(X) \cong \mathcal{M}(Y)$.

Si $\deg(X) = 6$, avec $X(k) = \varnothing$, par le lemme 5.1(2), il existe une extension K_2/k de degré divisant 3 et une surface de del Pezzo X_2 de degré 8 sur K_2 et un K_2-morphisme $f_2 : X_{K_2} \to X_2$ projectif, birationnel, tels que $I(X_2) = 1$ ou 2. D'après ce que l'on a déjà établi pour les surfaces de del Pezzo de degré 8, on a $\mathcal{M}(X_2) = 0$ et, d'après le paragraphe ci-dessus, $\mathcal{M}(X_{K_2}) = 0$. Par [Kahn 1999, Theorem 4.4(3)], le transfert est bien défini pour la suite spectrale (2.4). Puisque le transfert est bien défini pour la suite exacte (2.2), le transfert est bien défini pour le complexe

$$\mathrm{CH}^2(X) \to \mathrm{CH}^2(X_{\bar{k}})^{\Gamma_k} \xrightarrow{d(2)} H^2(k, \mathrm{Pic}(X_{\bar{k}}) \otimes \bar{k}^\times),$$

et donc le transfert est bien défini pour $\mathcal{M}(X)$. Donc $\mathcal{M}(X)$ est annulé par 3.

Par le même argument (lemme 5.1(1)) et le résultat en degré 9, le groupe $\mathcal{M}(X)$ est annulé par 2. On a donc $\mathcal{M}(X) = 0$. □

Corollaire 5.3. *Soit X une k-surface de del Pezzo de degré* 8 *avec* $\mathcal{M}(X) \neq 0$. *Si la dimension cohomologique* $\mathrm{cd}(k)$ *de k est* ≤ 3, *alors* $\mathcal{M}(X) = \mathbb{Z}/2$, $I(X) = 4$ *et* $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = \mathbb{Z}/2$.

Démonstration. Par la proposition 3.3, on a $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = \mathcal{M}(X)$. Le résultat découle du théorème 5.2. □

Exemple 5.4. Soit $k := \mathbb{C}(t, x, y)$. Soient C_1 la conique correspondant à l'algèbre (t, x), C_2 la conique correspondant à l'algèbre $(t + 1, y)$ et $X := C_1 \times C_2$. Alors $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = \mathbb{Z}/2$.

Démonstration. Puisque la dimension cohomologique $\mathrm{cd}(k)$ de k est 3, par le théorème 5.2 et le corollaire 5.3, il suffit de montrer que $I(X) = 4$. On note $A = (t, x) \otimes (t + 1, y)$ l'algèbre de biquaternions. Par [Albert 1972, Theorem], A est un corps gauche si et seulement si, pour chaque point $x_1 \in C_1$ de degré 2 et chaque point $x_2 \in C_2$ de degré 2, on a $k(x_1) \ncong k(x_2)$. Donc $I(X) = 4$ si et seulement si A est un corps gauche. Par [Colliot-Thélène 2002, corollaire 4], A est un corps gauche si et seulement si t et $t+1$ sont indépendantes dans $\mathbb{C}(t)^\times/\mathbb{C}(t)^{\times 2}$, ce qui est satisfait. $\square$

Corollaire 5.5. *Soit X une k-surface de del Pezzo de degré ≥ 5. Supposons que toute forme quadratique en 6 variables sur k est isotrope. Alors $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.*

Démonstration. D'après le théorème 5.2, il suffit de montrer que, pour toute paire de coniques lisses C_1 et C_2 sur k, on a $I(C_1 \times C_2) \neq 4$. Soient (a, b) l'algèbre de quaternion correspondant à C_1 et (c, d) l'algèbre de quaternion correspondant à C_2. Par l'argument de la démonstration de l'exemple 5.4, $I(C_1 \times C_2) = 4$ si et seulement si $(a, b) \otimes (c, d)$ est un corps gauche. Par un théorème de Albert (cf. [Colliot-Thélène 2002, proposition 1]), ceci vaut si et seulement si la forme quadratique diagonale $\langle a, b, -ab, -c, -d, cd\rangle$ est anisotrope sur k. Ceci donne immédiatement le résultat annoncé. $\square$

Corollaire 5.6. *Soit X une k-surface de del Pezzo de degré ≥ 5. Supposons que k satisfait la propriété (C_2) (cf. [Serre 1965, §II.4.5]). Alors $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$.*

Démonstration. Par le corollaire 5.5 et la définition de la propriété (C_2), on a $\bar{H}^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$. D'après [Serre 1965, §II.4.5, théorème MS], la dimension cohomologique de k satisfait $\mathrm{cd}(k) \leq 2$. Alors,

$$H^3(k, \mathbb{Q}/\mathbb{Z}(2)) = 0 \quad \text{et donc} \quad H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0. \qquad \square$$

Le théorème 5.2 donne la conjecture de Hodge entière pour certaines variétés de dimension 4 (voir [Colliot-Thélène et Voisin 2012, §1]) :

Proposition 5.7. *Soit X une $\mathbb{C}$-variété projective et lisse de dimension 4 munie d'un morphisme dominant $X \xrightarrow{f} S$ de base une $\mathbb{C}$-surface projective lisse S et de fibre générique X_η une surface de del Pezzo de degré ≥ 5. Alors la conjecture de Hodge entière en degré 4 vaut sur X.*

Démonstration. Puisque $\mathbb{C}(S)$ satisfait la propriété (C_2) (cf. [Serre 1965, §II.4.5]), d'après le corollaire 5.6, $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$. D'après Colliot-Thélène et Voisin [2012, théorème 3.8], il suffit alors de montrer qu'il existe une variété projective lisse Y de dimension au plus 3 et un morphisme $Y \xrightarrow{f} X$ tels que l'application induite $\mathrm{CH}_0(Y) \xrightarrow{f_*} \mathrm{CH}_0(X)$ soit surjective. Comme X_η est une $\mathbb{C}(S)$-surface géométriquement rationnelle, il existe une surface T projective et lisse sur $\mathbb{C}$ et une application génériquement finie $T \to S$, telles que $X_\eta \times_{\mathbb{C}(S)} \mathbb{C}(T)$ soit rationnelle sur $\mathbb{C}(T)$. Il existe donc une application rationnelle dominante de $\mathbb{P}^2 \times T$ vers X. Il existe alors une surface projective et lisse T' birationnelle à T et un morphisme $T' \to X$ tels que l'application induite $\mathrm{CH}_0(T') \xrightarrow{f_*} \mathrm{CH}_0(X)$ soit surjective. $\square$

Appendice : Accouplements de suites spectrales

Soient X une variété lisse sur k et $\mathrm{Sh}(X)$ la catégorie des faisceaux étales sur X. On rappelle quelques définitions données dans [McCleary 2001, Section 2.3] :

Définition A.1. Un module bigradué différentiel de $\mathrm{Sh}(X)$ est une collection d'éléments $E^{p,q} \in \mathrm{Sh}(X)$ pour $p, q \in \mathbb{Z}$ et de morphismes $d : E^{*,*} \to E^{*,*}$ de bidegré $(s, 1-s)$ pour certains $s \in \mathbb{Z}$, tels que $d \circ d = 0$.

Le produit tensoriel de deux modules bigradués différentiels $(E^{*,*}(1), d(1))$, $(E^{*,*}(2), d(2))$ est un module bigradué différentiel $((E(1) \otimes E(2))^{*,*}, d_\otimes)$ avec

$$(E(1) \otimes E(2))^{p,q} = \bigoplus_{r+t=p,\ s+u=q} E^{r,s}(1) \otimes E^{t,u}(2)$$

et $d_\otimes(x \otimes y) = d(1)(x) \otimes y + (-1)^{r+s} x \otimes d(2)(y)$, où $x \in E^{r,s}(1)$, $y \in E^{t,u}(2)$.

Pour deux complexes A, B, par le théorème de Künneth, on a un morphisme canonique

$$\bigoplus_{s+r=n} H^r(A) \otimes H^s(B) \xrightarrow{p} H^n(A \times B).$$

Définition A.2. Soient $E_r^{*,*}(1), d_r(1)$, $E_r^{*,*}(2), d_r(2)$ et $E_r^{*,*}(3), d_r(3)$ trois suites spectrales dans $\mathrm{Sh}(X)$. Un accouplement

$$\psi : E_r^{*,*}(1) \times E_r^{*,*}(2) \to E_r^{*,*}(3)$$

est une collection de morphismes $\psi_r : E_r^{*,*}(1) \otimes E_r^{*,*}(2) \to E_r^{*,*}(3)$ pour chaque r, tel que ψ_{r+1} est la composition :

$$E_{r+1}^{*,*}(1) \otimes E_{r+1}^{*,*}(2) \xrightarrow{\sim} H(E_r^{*,*}(1)) \otimes H(E_r^{*,*}(2)) \xrightarrow{p} H((E_r(1) \otimes E_r(2))^{*,*})$$
$$\xrightarrow{H(\psi_r)} H(E_r^{*,*}(3)) \xrightarrow{\sim} E_{r+1}^{*,*}(3),$$

où p est le morphisme dans le théorème de Künneth.

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YANG CAO: yang.cao@math.u-psud.fr
Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Univ. Paris-Saclay, 91405 Orsay, France

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