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Algebraic K-theory and a semifinite Fuglede–Kadison determinant

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In this paper we apply algebraic K-theory techniques to construct a Fuglede–Kadison type determinant for a semifinite von Neumann algebra equipped with a fixed trace. Our construction is based on the approach to determinants for Banach algebras developed by Skandalis and de la Harpe. This approach can be extended to the semifinite case since the first topological K-group of the trace ideal in a semifinite von Neumann algebra is trivial. Along the way we also improve the methods of Skandalis and de la Harpe by considering relative K-groups with respect to an ideal instead of the usual absolute K-groups. Our construction recovers the determinant homomorphism introduced by Brown, but all the relevant algebraic properties are automatic due to the algebraic K-theory framework.

1. Introduction

One first encounters the relationship between algebraic K-theory and determinants in the isomorphism between the first algebraic K-group of the complex numbers and the complex multiplicative group. This isomorphism is implemented by the determinant of an invertible matrix. In the present paper we will expand on this relationship in the context of Banach algebras and, in particular, we will see how to recover the Fuglede–Kadison determinant for semifinite von Neumann algebras as introduced by Brown [Brown 1986; Fuglede and Kadison 1952]. Brown based his construction on ideas of Grothendieck [1956] and Fack [1982; 1983], who defined a determinant function as an analogue of the product of the eigenvalues up to a given cutoff.

The main advantage of applying an algebraic K-theory approach to determinants is that all the algebraic properties of determinants follow as a direct consequence of the definitions. Moreover, when determinants are interpreted as invariants of algebraic K-theory, they can be used to detect nontrivial elements in these generally rather complicated abelian groups. On the other hand, basing the construction of determinants purely on functional analytic methods requires a substantial amount

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of work for proving the main algebraic properties, and the more conceptual framework provided by algebraic K-theory is entirely lost.

The key property that we investigate in this text is the relationship between the operator trace, the logarithm and the determinant as expressed by the identity

$$\log(\det(g)) = \mathrm{Tr}(\log(g)).$$

In order to expand on this basic relationship in a K-theoretic context one considers a unital Banach algebra A together with the homomorphism

$$\mathrm{GL}(A) \to \mathrm{GL}^{\mathrm{top}}(A),$$

where $\mathrm{GL}(A)$ denotes the general linear group (over A) equipped with the discrete topology, and $\mathrm{GL}^{\mathrm{top}}(A)$ is the same algebraic group but with the topology coming from the unital Banach algebra A. Passing to classifying spaces and applying Quillen's plus construction [1973], one obtains a continuous map

$$\mathrm{BGL}(A)^+ \to \mathrm{BGL}^{\mathrm{top}}(A)$$

(which is unique up to homotopy). By taking homotopy fibres and homotopy groups this gives rise to a long exact sequence of abelian groups,

$$K_{*+1}^{\mathrm{top}}(A) \xrightarrow{\partial} K_*^{\mathrm{rel}}(A) \xrightarrow{\partial} K_*^{\mathrm{alg}}(A) \longrightarrow K_*^{\mathrm{top}}(A),$$

which is related to the SBI-sequence in continuous cyclic homology by means of Chern characters, resulting in the commutative diagram

$$\begin{tikzcd}
K_{*+1}^{\mathrm{top}}(A) \arrow[r, "\partial"] \arrow[d, "\mathrm{ch}^{\mathrm{top}}"'] & K_*^{\mathrm{rel}}(A) \arrow[r] \arrow[d, "\mathrm{ch}^{\mathrm{rel}}"'] & K_*^{\mathrm{alg}}(A) \arrow[r] \arrow[d, "\mathrm{ch}^{\mathrm{alg}}"'] & K_*^{\mathrm{top}}(A) \arrow[d, "\mathrm{ch}^{\mathrm{top}}"'] \\
\mathrm{HP}_{*+1}(A) \arrow[r, "S"] & \mathrm{HC}_{*-1}(A) \arrow[r, "B"] & \mathrm{HN}_*(A) \arrow[r, "I"] & \mathrm{HP}_*(A)
\end{tikzcd} \tag{1.1}$$

of abelian groups; see [Karoubi 1987; Connes and Karoubi 1988].

In this paper we focus on the low degree (and more explicit) version of this commutative diagram. More precisely, supposing that the unital Banach algebra A comes equipped with a tracial functional $\tau : A \to \mathbb{C}$, one obtains an invariant of the continuous cyclic homology group $\mathrm{HC}_0(A)$, and hence by precomposition with the relative Chern character we obtain a homomorphism

$$\tau \circ \mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(A) \to \mathbb{C}.$$

Supposing furthermore that $K_1^{\mathrm{top}}(A) = \{0\}$, it follows from the commutative diagram in (1.1) combined with Bott-periodicity in topological K-theory that the character $\tau \circ \mathrm{ch}^{\mathrm{rel}}$ induces a homomorphism

$$\det_\tau : K_1^{\mathrm{alg}}(A) \to \mathbb{C}/(2\pi i \cdot \mathrm{Im}(\underline{\tau})),$$

where $\underline{\tau} : K_0^{\mathrm{top}}(A) \to \mathbb{C}$ is the character on even topological K-theory induced by our tracial functional. In this way we recover the determinant defined by Skandalis and de la Harpe [de la Harpe and Skandalis 1984; de la Harpe 2013].

We extend this framework for defining determinants by incorporating that the tracial functional τ might only be defined on an ideal J sitting inside the unital Banach algebra A (where J is not required to be closed in the norm-topology of A). In this context, we assume that $\tau : J \to \mathbb{C}$ is a hypertrace in the sense that $\tau(ja) = \tau(aj)$ for all $a \in A$, $j \in J$. The correct K-groups to consider are then relative versions of relative K-theory and algebraic K-theory, and similarly one considers relative versions of the cyclic homology groups appearing in the SBI-sequence (we do not use relative topological K-theory because of excision). The idea of applying relative K-groups in relation to determinant-type invariants of algebraic K-theory was (among other things) developed in the Ph.D. thesis of the second author [Kaad 2009].

In the setting of a semifinite von Neumann algebra N equipped with a fixed normal, faithful and semifinite trace $\tau : N_+ \to [0, \infty]$, it is relevant to look at the trace ideal

$$\mathscr{L}^1_\tau(N) := \{x \in N : \tau(|x|) < \infty\}$$

sitting inside the von Neumann algebra N. Using the facts that $K_1^{\mathrm{top}}(\mathscr{L}^1_\tau(N)) = \{0\}$ and $\mathrm{Im}(\underline{\tau} : K_0^{\mathrm{top}}(\mathscr{L}^1_\tau(N)) \to \mathbb{C}) \subseteq \mathbb{R}$, we obtain an algebraic K-theory invariant[1]

$$\det_\tau : K_1^{\mathrm{alg}}(\mathscr{L}^1_\tau(N), N) \to \mathbb{C}/i\mathbb{R},$$

which recovers the Fuglede–Kadison determinant in the context of semifinite von Neumann algebras; see [Brown 1986; Fuglede and Kadison 1952]. We emphasize one more time that all the relevant algebraic properties of this determinant follow immediately from its construction. Moreover, we show that $\det_\tau$ is given by the explicit formula

$$\det_\tau(g) = \tau(\log(|g|)) + i\mathbb{R} \qquad (g \in \mathrm{GL}_n(N),\ g - \mathbb{1}_n \in M_n(\mathscr{L}^1_\tau(N)). \tag{1.2}$$

Here, τ is extended to $M_n(N)$ in the obvious way by taking the sum over the diagonal.

Recently, the Fuglede–Kadison determinant was generalized in another direction by Dykema, Sukochev and Zanin to operator bimodules over II_1-factors [Dykema et al. 2017]. They define this determinant using functional analytic methods via an

[1] In the main text, we denote this map by $\widetilde{\det}_\tau$, and use the notation $\det_\tau$ for the composition with the isomorphism $\mathbb{C}/i\mathbb{R} \cong (0, \infty)$ given by $z + i\mathbb{R} \mapsto e^{(z+\bar{z})/2}$.

expression analogous to (1.2). It then requires an elaborate argument to prove that this determinant is multiplicative [Dykema et al. 2017, Theorem 1.3].

The present paper is organized as follows. In Section 2 we introduce the relevant K-groups and in Section 3 we develop the low degree version of the long exact sequence which compares relative algebraic K-theory to topological K-theory. In Section 4 we introduce the low degree version of the relative Chern character in the presence of an ideal $J \subseteq A$. In Section 5 we present our relative approach to the construction of Skandalis–de la Harpe determinants. In Section 6 we show that the first topological K-group of the trace ideal in a semifinite von Neumann algebra is trivial, and in Section 7 we apply this fact to construct the semifinite Fuglede–Kadison determinant.

2. *K*-theory for relative pairs of Banach algebras

2.1. Definition. Let $(A, \|\cdot_A\|)$ be a unital Banach algebra and $J \subset A$ be a (not necessarily closed) ideal. We call (J, A) a *relative pair of Banach algebras* when the following hold:

(1) J is a Banach algebra in its own right. Thus, J is endowed with a norm $\|\cdot\|_J : J \to [0, \infty)$ such that $(J, \|\cdot\|_J)$ is a Banach algebra.

(2) For all $a, b \in A$ and $j \in J$ we have

$$\|ajb\|_J \leq \|a\|_A \, \|j\|_J \, \|b\|_A \quad \text{and} \quad \|j\|_A \leq \|j\|_J \,.$$

2.2. For a relative pair of Banach algebras (J, A) we obtain for all $n \in \mathbb{N}$ a relative pair of Banach algebras $(M_n(J), M_n(A))$, where the $n \times n$ matrices in $M_n(J)$ are equipped with the norm $\|j\|_{M_n(J)} := \sum_{k,l=1}^{n} \|j_{kl}\|_J$, and similarly for $M_n(A)$.

2.3. The rest of this section is a reminder on various K-groups for relative pairs of Banach algebras. A standard reference for topological K-theory is [Blackadar 1998]. Very good treatments of algebraic K-theory can be found in [Rosenberg 1994; Weibel 2013]. The probably less common relative K-theory of Banach algebras has been introduced in [Karoubi 1987; Connes and Karoubi 1988].

2.4. Definition. Let A be a Banach algebra. If A has a unit, we denote the group of invertible elements in $M_n(A)$ by $\mathrm{GL}_n(A)$. If A has no unit, we define for all $n \in \mathbb{N}$ the group

$$\mathrm{GL}_n(A) := \{g \in \mathrm{GL}_n(A^+) : g - \mathbb{1}_n \in M_n(A)\} \subset \mathrm{GL}_n(A^+),$$

where A^+ is the unitization of A and $\mathbb{1}_n$ the unit of $\mathrm{GL}_n(A^+)$. The group $\mathrm{GL}_n(A)$ becomes a topological group when equipped with the topology coming from the metric $d(g, h) := \|g - h\|_{M_n(A)}$.

2.5. Definition. The *topological K-groups* of the pair (J, A) can be defined to be the usual topological K-groups of J, i.e.,

$$K_i^{\mathrm{top}}(J, A) := K_i^{\mathrm{top}}(J) \qquad (i = 0, 1).$$

This is due to the fact that topological K-theory satisfies excision [Blackadar 1998, Theorem 5.4.2]. For our purposes, it will be useful to know another realization of $K_0^{\mathrm{top}}(J)$, namely $K_2^{\mathrm{top}}(J)$, which may be defined by

$$K_2^{\mathrm{top}}(J) = \lim_{n\to\infty} \pi_1(\mathrm{GL}_n(J), \mathbb{1}_n) = \lim_{n\to\infty} \{[\gamma] \in C^\infty(S^1, \mathrm{GL}_n(J))/\sim\, : \gamma(1) = \mathbb{1}_n\},$$

where the equivalence relation $\sim$ is given by smooth basepoint preserving homotopies and the group operation is given by the pointwise product of invertible matrices; see [Blackadar 1998, Section 9.1].

The fact that $K_0^{\mathrm{top}}(J)$ and $K_2^{\mathrm{top}}(J)$ are isomorphic is known as Bott periodicity, [Blackadar 1998, Theorem 9.2.1]. An explicit isomorphism is given by

$$\beta_J : K_0^{\mathrm{top}}(J) \to K_2^{\mathrm{top}}(J), \qquad [e] - [f] \mapsto [\gamma_e \gamma_f^{-1}],$$

where $e, f \in M_n(J^+)$ are idempotents with $e - f \in M_n(J)$. The so-called idempotent loops γ_e are defined by $\gamma_e(z) := ze + \mathbb{1}_n - e$ for $z \in S^1$.

2.6. Definition. The *first algebraic K-theory* of the pair (J, A) is defined by

$$K_1^{\mathrm{alg}}(J, A) := \lim_{n\to\infty} (\mathrm{GL}_n(J)/[\mathrm{GL}_n(J), \mathrm{GL}_n(A)]),$$

where

$$[\mathrm{GL}_n(J), \mathrm{GL}_n(A)] := \langle ghg^{-1}h^{-1} : g \in \mathrm{GL}_n(J),\ h \in \mathrm{GL}_n(A)\rangle$$

is a normal subgroup of $\mathrm{GL}_n(J)$.

2.7. Definition. Let A be a Banach algebra. For all $n \in \mathbb{N}$, we let $R_n(A)$ denote the group of smooth paths $\sigma : [0, 1] \to \mathrm{GL}_n(A)$ such that $\sigma(0) = \mathbb{1}_n$. The group operation is given by pointwise multiplication.

Now, let (J, A) be a relative pair of Banach algebras. From the compatibility of the norms on J and A (see Definition 2.1) it follows that

$$\sigma\tau\sigma^{-1}\tau^{-1} \in R_n(J) \qquad (\sigma \in R_n(J),\ \tau \in R_n(A)).$$

We thus have the normal subgroup

$$[R_n(J), R_n(A)] := \langle \sigma\tau\sigma^{-1}\tau^{-1} \mid \sigma \in R_n(J),\ \tau \in R_n(A)\rangle$$

of $R_n(J)$. On $R_n(J)$ we may consider the equivalence relation $\sim$ of being homotopic with fixed endpoints through a smooth homotopy. Denote the quotient map by $q : R_n(J) \to R_n(J)/\sim$. We define

$$K_1^{\mathrm{rel}}(J, A) := \lim_{n\to\infty} \big((R_n(J)/\sim)/q\big([R_n(J), R_n(A)]\big)\big).$$

3. The comparison sequence

3.1. Definition. We define the following group homomorphisms:

$$\begin{aligned}
\partial &: K_2^{\mathrm{top}}(J) \to K_1^{\mathrm{rel}}(J, A), && [\gamma] \mapsto [t \mapsto \gamma(e^{2\pi i t})],\\
\theta &: K_1^{\mathrm{rel}}(J, A) \to K_1^{\mathrm{alg}}(J, A), && [\sigma] \mapsto [\sigma(1)^{-1}],\\
p &: K_1^{\mathrm{alg}}(J, A) \to K_1^{\mathrm{top}}(J), && [g] \mapsto [g].
\end{aligned}$$

3.2. Lemma. *The sequence*

$$K_2^{\mathrm{top}}(J) \xrightarrow{\partial} K_1^{\mathrm{rel}}(J, A) \xrightarrow{\theta} K_1^{\mathrm{alg}}(J, A) \xrightarrow{p} K_1^{\mathrm{top}}(J) \longrightarrow 0$$

is exact.

Proof. The only nontrivial thing to check is exactness at $K_1^{\mathrm{rel}}(J, A)$. It is clear that $\theta \circ \partial = 0$. On the other hand, let $\sigma \in R_n(J)$ and suppose that $[\sigma(1)^{-1}]$ is trivial in $K_1^{\mathrm{alg}}(J, A)$. Then there are $g_i \in \mathrm{GL}_m(J)$ and $h_i \in \mathrm{GL}_m(A)$ such that

$$\sigma(1)^{-1} = \prod_{i=1}^{n} [g_i, h_i].$$

By Whitehead's lemma [Rosenberg 1994, Theorem 2.5.3], we may assume that g_i and h_i lie in the connected component of the identity. Thus, there are smooth paths $\alpha_i \in R_m(J)$ connecting $\mathbb{1}_m$ and g_i, and $\beta_i \in R_m(A)$ connecting $\mathbb{1}_m$ and h_i. Then

$$\tau := \prod_{i=1}^{n} [\alpha_i, \beta_i] \in [R_m(J), R_m(A)]$$

is a path from $\mathbb{1}_m$ to $\sigma(1)^{-1}$. Hence $\gamma := \sigma \cdot \tau^{-1}$ is a smooth loop at $\mathbb{1}_m$ and $\partial([\gamma]) = [\sigma]$ since $[\tau^{-1}]$ is trivial in $K_1^{\mathrm{rel}}(J, A)$. □

4. The relative Chern character

4.1. Let (J, A) be a relative pair of Banach algebras. By $J \otimes_\pi A$ we denote the projective tensor product of J and A. The compatibility of the norms on J and A ensures that the multiplication operator

$$m : J \otimes_\pi A \to J, \qquad j \otimes a \mapsto ja$$

is bounded.

4.2. Definition. We define the Hochschild boundary map

$$b : J \otimes_\pi A \to J, \qquad j \otimes a \mapsto ja - aj$$

and the zeroth relative continuous cyclic homology of the pair (J, A) by

$$\mathrm{HC}_0(J, A) := J/\operatorname{Im}(b).$$

Since $\operatorname{Im}(b) \subset J$ might not be closed we regard $\mathrm{HC}_0(J, A)$ simply as a vector space without further topological structure.

4.3. Definition. Recall from 2.2 that $(M_n(J), M_n(A))$ is a relative pair of Banach algebras for all $n \in \mathbb{N}$. We thus have for each $n \in \mathbb{N}$ the relative continuous cyclic homology groups $\mathrm{HC}_0(M_n(J), M_n(A))$, and we may consider the direct limit of vector spaces

$$\lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A)).$$

This direct limit is linked to $\mathrm{HC}_0(J, A)$ via the linear map

$$\mathrm{TR} : \lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A)) \to \mathrm{HC}_0(J, A),$$

which is induced by the "trace" $\mathrm{TR} : M_n(J) \to J$ mapping a matrix to the sum of its diagonal entries. To verify that TR is indeed well-defined at the level of relative continuous cyclic homology, one may translate the proof of [Loday 1998, Corollary 1.2.3] to our current setting.

4.4. Our next task is to construct the relative Chern character. This will be a group homomorphism

$$\mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(J, A) \to \mathrm{HC}_0(J, A)$$

induced by

$$R_n(J) \ni \sigma \mapsto \mathrm{TR}\biggl(\int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\,dt\biggr) \in J.$$

We shall express $\mathrm{ch}^{\mathrm{rel}}$ as the composition of two homomorphisms: a generalized logarithm

$$\log : K_1^{\mathrm{rel}}(J, A) \to \lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A))$$

and the generalized trace as defined in Definition 4.3. We now introduce the generalized logarithm:

4.5. Proposition. *There is a well-defined homomorphism*

$$\log : K_1^{\mathrm{rel}}(J, A) \to \lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A)), \qquad [\sigma] \mapsto \biggl[\int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\,dt\biggr].$$

Proof. Suppose first that $\sigma_0, \sigma_1 \in R_n(J)$ are homotopic through a smooth homotopy $H : [0, 1] \times [0, 1] \to \mathrm{GL}_n(J)$ with fixed endpoints. So, $H(t, j) = \sigma_j(t)$ for $j = 0, 1$.

We will show that

$$\int_0^1 \frac{d\sigma_1}{dt}\sigma_1^{-1}\,dt - \int_0^1 \frac{d\sigma_0}{dt}\sigma_0^{-1}\,dt \in \operatorname{Im}(b),$$

where $b : M_n(J) \otimes_\pi M_n(A) \to M_n(J)$ is the Hochschild boundary map associated to the relative pair $(M_n(J), M_n(A))$.

Define

$$L(H) := -\int_0^1 \int_0^1 \frac{\partial H}{\partial t} H^{-1} \otimes \frac{\partial H}{\partial s} H^{-1}\, dt\, ds.$$

We consider $L(H)$ as an element of $M_n(J) \otimes_\pi M_n(A)$ (in fact we even end up in $M_n(J) \otimes_\pi M_n(J)$, which we may then map to $M_n(J) \otimes_\pi M_n(A)$ via the inclusion $M_n(J) \to M_n(A)$). Applying the Hochschild boundary b, we see that

$$b(L(H)) = -\int_0^1\int_0^1 \Big[\frac{\partial H}{\partial t} H^{-1}, \frac{\partial H}{\partial s} H^{-1}\Big]\, dt\, ds.$$

An easy calculation shows that

$$\begin{aligned}\Big[\frac{\partial H}{\partial t} H^{-1}, \frac{\partial H}{\partial s} H^{-1}\Big] &= -\frac{\partial H}{\partial t}\frac{\partial H^{-1}}{\partial s} + \frac{\partial H}{\partial s}\frac{\partial H^{-1}}{\partial t} \\ &= \frac{\partial}{\partial t}\Big(\frac{\partial H}{\partial s} H^{-1}\Big) - \frac{\partial}{\partial s}\Big(\frac{\partial H}{\partial t} H^{-1}\Big).\end{aligned}$$

By the fundamental theorem of calculus, we conclude

$$\begin{aligned} b(L(H)) &= \int_0^1\int_0^1 \frac{\partial}{\partial s}\Big(\frac{\partial H}{\partial t} H^{-1}\Big)\, ds\, dt - \int_0^1\int_0^1 \frac{\partial}{\partial t}\Big(\frac{\partial H}{\partial s} H^{-1}\Big)\, dt\, ds \\ &= \int_0^1 \Big(\frac{\partial H}{\partial t}(t,1) H(t,1)^{-1} - \frac{\partial H}{\partial s}(t,0) H(t,0)^{-1}\Big)\, dt \\ &\qquad - \int_0^1 \Big(\frac{\partial H}{\partial s}(1,s) H(1,s)^{-1} - \frac{\partial H}{\partial s}(0,s) H(0,s)^{-1}\Big)\, ds \\ &= \int_0^1 \frac{d\sigma_1}{dt}\sigma_1^{-1}\, dt - \int_0^1 \frac{d\sigma_0}{dt}\sigma_0^{-1}\, dt. \end{aligned}$$

The second term in the next to last line of our computation vanishes, since our homotopy has fixed endpoints.

We have thus proved that the assignment

$$\log : R_n(J) \to M_n(J), \qquad \sigma \mapsto \int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\, dt$$

descends to a well-defined map $\log : (R_n(J)/\sim) \to \mathrm{HC}_0(M_n(J), M_n(A))$. Furthermore, since log is compatible with direct limits, we obtain a well-defined map

$$\log : \lim_{n\to\infty} (R_n(J)/\sim) \to \lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A)).$$

We now show that $\log([\sigma_0 \cdot \sigma_1]) = \log([\sigma_0]) + \log([\sigma_1])$ for all $\sigma_0, \sigma_1 \in R_n(J)$. Choose a smooth function $\phi : \mathbb{R} \to [0,1]$ such that

$$\phi((-\infty, 0]) = \{0\} \quad \text{and} \quad \phi\big(\big[\tfrac{1}{2}, \infty\big)\big) = \{1\}.$$

Define the smooth function $\psi : \mathbb{R} \to [0, 1]$ by $\psi(t) := \phi\big(t - \frac{1}{2}\big)$. We then have that

$$\sigma_0\sigma_1 \sim (\sigma_0 \circ \psi) \cdot (\sigma_1 \circ \phi),$$

and it thus suffices to verify that $\log((\sigma_0 \circ \psi) \cdot (\sigma_1 \circ \phi)) = \log(\sigma_0) + \log(\sigma_1)$. But this identity follows by a change of variables:

$$\begin{aligned}\log\big((\sigma_0 \circ \psi) \cdot (\sigma_1 \circ \phi)\big)& \\ &= \int_0^{1/2} \frac{d(\sigma_1 \circ \phi)}{dt}(\sigma_1 \circ \phi)^{-1}\, dt + \int_{1/2}^{1} \frac{d(\sigma_0 \circ \psi)}{dt}(\sigma_0 \circ \psi)^{-1}\, dt \\ &= \log(\sigma_0) + \log(\sigma_1).\end{aligned}$$

To finish the proof of the proposition we only need to show that $\log([\sigma\tau\sigma^{-1}]) = \log([\tau])$ whenever $\sigma \in R_n(A)$ and $\tau \in R_n(J)$. To this end, we consider the smooth homotopy with fixed endpoints

$$H(s, t) := \sigma(f(s, t))\tau(t)\sigma(f(s, t))^{-1}, \qquad f(s, t) := ts + 1 - s = s(t - 1) + 1$$

between $\sigma\tau\sigma^{-1}$ and $\sigma(1)\tau\sigma(1)^{-1}$. This proves that

$$\log([\sigma\tau\sigma^{-1}]) = \log([\sigma(1)\tau\sigma(1)^{-1}]) = \log([\tau]),$$

where we have used the fact that $\sigma(1)x\sigma(1)^{-1}$ and x determine the same element in $\mathrm{HC}_0(M_n(J), M_n(A))$ for all $x \in M_n(J)$. $\square$

4.6. Definition. By the *relative Chern character* $\mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(J, A) \to \mathrm{HC}_0(J, A)$, we understand the homomorphism obtained as the composition

$$\mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(J, A) \xrightarrow{\log} \lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A)) \xrightarrow{\mathrm{TR}} \mathrm{HC}_0(J, A)$$

of the generalized logarithm and the generalized trace.

5. The relative Skandalis–de la Harpe determinant

5.1. Analogous to the determinant of Skandalis and de la Harpe, we are now in a position to define such a determinant purely by means of K-theory for relative pairs of Banach algebras. In particular, we are able to deal with the presence of a not necessarily closed ideal J inside a unital Banach algebra A.

5.2. Definition. Let (J, A) be a relative pair of Banach algebras. In this section we assume $\tau : J \to \mathbb{C}$ to be a continuous linear functional which additionally satisfies

$$\tau(ja) = \tau(aj) \qquad (a \in A,\ j \in J).$$

The latter property means that τ is a *hypertrace*. For such a trace there is a well-defined map (also denoted by τ):

$$\tau : \mathrm{HC}_0(J, A) \to \mathbb{C}, \qquad j + \mathrm{Im}(b) \mapsto \tau(j).$$

Furthermore, we let

$$\tilde{\tau} := -\tau \circ \mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(J, A) \to \mathbb{C},$$

with $\mathrm{ch}^{\mathrm{rel}}$ as in Definition 4.6. Note that $\tilde{\tau}$ is a homomorphism into the additive group $\mathbb{C}$.

5.3. Recall (Lemma 3.2) that there is an exact sequence in relative K-theory:

$$K_2^{\mathrm{top}}(J) \xrightarrow{\partial} K_1^{\mathrm{rel}}(J, A) \xrightarrow{\theta} K_1^{\mathrm{alg}}(J, A) \xrightarrow{p} K_1^{\mathrm{top}}(J) \longrightarrow 0.$$

This allows us to define the *relative Skandalis–de la Harpe determinant*

$$\widetilde{\det}_\tau : \mathrm{Ker}(p) \to \mathbb{C}/\,\mathrm{Im}(\tilde{\tau} \circ \partial)$$

by

$$\widetilde{\det}_\tau([g]) := \tilde{\tau}([\sigma]) + \mathrm{Im}(\tilde{\tau} \circ \partial),$$

where $[\sigma] \in K_1^{\mathrm{rel}}(J, A)$ satisfies $\theta([\sigma]) = [\sigma(1)^{-1}] = [g] \ni K_1^{\mathrm{alg}}(J, A)$. Such a lift always exists since $\mathrm{Ker}(p) = \mathrm{Im}(\theta)$. Furthermore, this assignment is well-defined since if $[\sigma_0]$ and $[\sigma_1]$ are lifts of the same element $[g]$ then

$$[\sigma_0][\sigma_1]^{-1} \in \mathrm{Ker}(\theta) = \mathrm{Im}(\partial).$$

It follows that $\tilde{\tau}([\sigma_0]) = \tilde{\tau}([\sigma_1])$ modulo $\mathrm{Im}(\tilde{\tau} \circ \partial)$.

Compare this with the definition on page 245 of [de la Harpe and Skandalis 1984], where absolute K-theory is used rather than relative K-theory.

5.4. Lemma. *We have the following equality of subgroups of* $(\mathbb{C}, +)$:

$$2\pi i \cdot \mathrm{Im}(\underline{\tau} : K_0^{\mathrm{top}}(J) \to \mathbb{C}) = \mathrm{Im}(\tilde{\tau} \circ \partial : K_2^{\mathrm{top}}(J) \to \mathbb{C}).$$

By $\underline{\tau} : K_0^{\mathrm{top}}(J) \to \mathbb{C}$ *we understand the map induced by* τ.

Proof. The claim follows from commutativity of the following diagram:

$$\begin{tikzcd} K_0^{\mathrm{top}}(J) \arrow[r, "\beta_J", "\cong"'] \arrow[dr, "-2\pi i \cdot \underline{\tau}"'] & K_2^{\mathrm{top}}(J) \arrow[d, "\tilde{\tau} \circ \partial"] \\ & \mathbb{C} \end{tikzcd} \tag{5.5}$$

By β_J we mean the Bott isomorphism map, as in Definition 2.5.

To prove commutativity of (5.5), we note that for an idempotent $f \in M_n(J^+)$,

$$\mathrm{ch}^{\mathrm{rel}}(\partial([\gamma_f])) = \mathrm{TR}\biggl(2\pi i \int_0^1 e^{2\pi i t} f (e^{-2\pi i t} f + \mathbb{1}_n - f)\, dt\biggr) = 2\pi i\, \mathrm{TR}(f).$$

If now $e, f \in M_n(J^+)$ are idempotents satisfying $e - f \in M_n(J)$, then

$$\tilde{\tau}\bigl(\partial(\beta_J([e] - [f]))\bigr) = \tilde{\tau}(\partial([\gamma_e \gamma_f^{-1}])) = -2\pi i \cdot \tau(\mathrm{TR}(e - f)).$$

So (5.5) indeed commutes. $\square$

5.6. Together with Lemma 5.4 we see that the following diagram commutes:

$$\begin{tikzcd}
K_2^{\mathrm{top}}(J) \arrow[r, "\partial"] \arrow[d, "\tilde{\tau}\circ\partial"] & K_1^{\mathrm{rel}}(J, A) \arrow[r, two heads, "\theta"] \arrow[d, "\tilde{\tau}"] & \operatorname{Ker} p \arrow[r, "p"] \arrow[d, "\widetilde{\det}_\tau"] & 0 \\
2\pi i \cdot \operatorname{Im}(\underline{\tau}) \arrow[r, tail] & \mathbb{C} \arrow[r, two heads] & \mathbb{C}/(2\pi i \cdot \operatorname{Im}(\underline{\tau})) &
\end{tikzcd}$$

In the next section this will be applied to the case that the kernel of p is all of $K_1^{\mathrm{alg}}(J, A)$. In that case we get a determinant

$$\widetilde{\det}_\tau : K_1^{\mathrm{alg}}(J, A) \to \mathbb{C}/(2\pi i \cdot \operatorname{Im}(\underline{\tau})).$$

6. Topological K-theory of trace ideals

6.1. In the following, $N \subset \mathscr{L}(H)$ always denotes a semifinite von Neumann algebra equipped with a fixed normal, faithful and semifinite trace $\tau : N_+ \to [0, \infty]$. A good reference for traces on von Neumann algebras is [Dixmier 1981, I.6.1, I.6.10].

6.2. We let $\|\cdot\| : N \to [0, \infty)$ denote the operator norm on N and we let

$$\mathscr{L}^1_\tau(N) := \{x \in N : \tau(|x|) < \infty\} \subset N$$

denote the trace ideal in N. We recall that $\mathscr{L}^1_\tau(N) \subset N$ is indeed a $*$-ideal and that $\mathscr{L}^1_\tau(N)$ becomes a Banach $*$-algebra in its own right when equipped with the norm

$$\|x\|_{1,\infty} := \|x\| + \tau(|x|) \qquad (x \in \mathscr{L}^1_\tau(N)).$$

Moreover, it holds that $(\mathscr{L}^1_\tau(N), N)$ is a relative pair of Banach algebras in the sense of Definition 2.1.

6.3. For each $n \in \mathbb{N}$ we have that $M_n(N) \subset \mathscr{L}(H^{\oplus n})$ is a semifinite von Neumann algebra. Indeed, we may define the normal, faithful and semifinite trace $\tau_n : M_n(N)_+ \to [0, \infty]$ by $\tau_n(x) := \sum_{i=1}^n \tau(x_{ii})$. The inclusion $M_n(\mathscr{L}^1_\tau(N)) \to M_n(N)$ then induces an isomorphism

$$M_n(\mathscr{L}^1_\tau(N)) \cong \mathscr{L}^1_{\tau_n}(M_n(N))$$

of Banach $*$-algebras. This isomorphism is, however, not an isometry since (by convention) $M_n(\mathscr{L}^1_\tau(N))$ is equipped with the norm defined as in 2.2.

6.4. Lemma. *The group* $\mathrm{GL}_n(\mathscr{L}^1_\tau(N))$ *is path connected for all* $n \in \mathbb{N}$. *In particular, it holds that*

$$K_1^{\mathrm{top}}(\mathscr{L}^1_\tau(N)) = \{0\}.$$

Proof. Since $M_n(\mathscr{L}^1_\tau(N)) \cong \mathscr{L}^1_{\tau_n}(M_n(N))$ by 6.3, it suffices to verify the lemma for $n = 1$. Thus, let $g \in \mathrm{GL}_1(\mathscr{L}^1_\tau(N))$. Using polar decomposition we may suppose that $g^*g = \mathbb{1} = gg^*$ or that $g = |g|$. In the first case we may find an $x \in \mathscr{L}^1_\tau(N)$

with $x = -x^*$ such that $g = e^x$. In the second case we may find an $x \in \mathscr{L}^1_\tau(N)$ with $x = x^*$ such that $g = e^x$. In both cases we obtain the smooth path $t \mapsto e^{tx}$ connecting $\mathbb{1}$ and g. $\square$

7. The semifinite Fuglede–Kadison determinant

7.1. We are now going to use K-theory for relative pairs of Banach algebras to define our determinant. From 6.2 we know that $(\mathscr{L}^1_\tau(N), N)$ is a relative pair of Banach algebras and that $\tau : \mathscr{L}^1_\tau(N) \to \mathbb{C}$ is continuous with respect to $\|\cdot\|_{1,\infty}$. Since $\tau : \mathscr{L}^1_\tau(N) \to \mathbb{C}$ is moreover a hypertrace, we get (as defined in 5.6) a determinant

$$\widetilde{\det}_\tau : K_1^{\mathrm{alg}}(\mathscr{L}^1_\tau(N), N) \to \mathbb{C}/(2\pi i \cdot \mathrm{Im}(\underline{\tau})).$$

Note that our determinant is defined on all of $K_1^{\mathrm{alg}}(\mathscr{L}^1_\tau(N), N)$ by Lemma 6.4.

7.2. Lemma. *We have*

$$\mathrm{Im}\big(\underline{\tau} : K_0^{\mathrm{top}}(\mathscr{L}^1_\tau(N)) \to \mathbb{C}\big) \subset \mathbb{R}.$$

Proof. Since $M_n(\mathscr{L}^1_\tau(N)^+) \subset M_n(N)$ is closed under holomorphic functional calculus for all $n \in \mathbb{N}$, every idempotent in $M_n(\mathscr{L}^1_\tau(N)^+)$ is similar to a projection in $M_n(\mathscr{L}^1_\tau(N)^+)$; see [Blackadar 1998, Proposition 4.6.2]. And for projections $p, q \in M_n(\mathscr{L}^1_\tau(N)^+)$ with $p - q \in M_n(\mathscr{L}^1_\tau(N))$ we see that

$$\underline{\tau}([p] - [q]) = \tau(\mathrm{TR}(p - q)) \in \mathbb{R},$$

where we have used that all the diagonal entries $(p - q)_{jj}$ are self-adjoint. $\square$

7.3. We thus have a well-defined homomorphism

$$\widetilde{\det}_\tau : K_1^{\mathrm{alg}}(\mathscr{L}^1_\tau(N), N) \to \mathbb{C}/i\mathbb{R}, \qquad \widetilde{\det}_\tau : [g] \mapsto \tilde{\tau}([\sigma]) + i\mathbb{R},$$

where $[\sigma] \in K_1^{\mathrm{rel}}(\mathscr{L}^1_\tau(N), N)$ is any lift of $[g]$, by which we mean that $\theta([\sigma]) = [\sigma(1)^{-1}] = [g]$.

Note that there is an isomorphism of abelian groups

$$\mathbb{C}/i\mathbb{R} \to (0, \infty), \qquad z + i\mathbb{R} \mapsto e^{\Re(z)},$$

where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$. This gives rise to the following definition:

7.4. Definition. We define the *semifinite Fuglede–Kadison determinant*

$$\det_\tau : K_1^{\mathrm{alg}}(\mathscr{L}^1_\tau(N), N) \to (0, \infty)$$

by

$$\det_\tau([g]) := e^{\Re(\widetilde{\det}_\tau([g]))}.$$

More explicitly, we have

$$\det_\tau([g]) = \exp((\Re\circ\tilde{\tau})[\sigma]) = \exp\biggl(-(\Re\circ\tau\circ\mathrm{TR})\biggl(\int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\,dt\biggr)\biggr),$$

where $[\sigma]\in K_1^{\mathrm{rel}}(\mathscr{L}_\tau^1(N),N)$ is any lift of $[g]\in K_1^{\mathrm{alg}}(\mathscr{L}_\tau^1(N),N)$, meaning that $[\sigma(1)^{-1}]=[g]$.

7.5. Proposition. *The semifinite Fuglede–Kadison determinant* $\det_\tau$ *has the following properties*:

(1) $\det_\tau([gh]) = \det_\tau([g])\det_\tau([h])$ *for all* $g,h\in \mathrm{GL}_n(\mathscr{L}_\tau^1(N))$.

(2) $\det_\tau([hgh^{-1}]) = \det_\tau([g])$ *for all* $g\in\mathrm{GL}_n(\mathscr{L}_\tau^1(N))$ *and* $h\in\mathrm{GL}_n(N)$.

(3) $\det_\tau([e^x]) = (\exp\circ\Re\circ\tau\circ\mathrm{TR})(x)$ *for* $x\in M_n(\mathscr{L}_\tau^1(N))$.

These properties follow directly from the definition of the determinant and the fact that $\widetilde{\det}_\tau$ is a homomorphism.

In [Brown 1986, Section 1], the equality in the following proposition is the definition of the determinant.

7.6. Proposition. *The following explicit formula holds*:

$$\det_\tau([g]) = e^{\tau(\log|g|)}, \qquad (g\in\mathrm{GL}_1(\mathscr{L}_\tau^1(N))).$$

Proof. Let $g\in\mathrm{GL}_1(\mathscr{L}_\tau^1(N))$. Using the polar decomposition $g=u|g|$, we may compute

$$\det_\tau([g]) = \det_\tau([u])\det_\tau([|g|]).$$

Since $u\in\mathrm{GL}_1(\mathscr{L}_\tau^1(N))$ is unitary in the ambient von Neumann algebra, we may write $u=e^{ix}$ for some $x\in\mathscr{L}_\tau^1(N)$ with $x=x^*$. Moreover, we have $\log|g|\in\mathscr{L}_\tau^1(N)$. By Proposition 7.5(3) we thus have that

$$\det_\tau([g]) = e^{\Re(\tau(ix))}\cdot e^{\tau(\log|g|)} = e^{\tau(\log|g|)}. \qquad \square$$

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