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A fixed point theorem on noncompact manifolds

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We generalise the Atiyah–Segal–Singer fixed point theorem to noncompact manifolds. Using KK-theory, we extend the equivariant index to the noncompact setting, and obtain a fixed point formula for it. The fixed point formula is the explicit cohomological expression from Atiyah–Segal–Singer's result. In the noncompact case, however, we show in examples that this expression yields characters of infinite-dimensional representations. In one example, we realise characters of discrete series representations on the regular elements of a maximal torus, in terms of the index we define. Further results are a fixed point formula for the index pairing between equivariant K-theory and K-homology, and a nonlocalised expression for the index we use, in terms of deformations of principal symbols. The latter result is one of several links we find to indices of deformed symbols and operators studied by various authors.

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1. Introduction

In the second part of the *Index of elliptic operators* series, Atiyah and Segal [1968] proved a fixed point formula for compact groups and manifolds. This allows one to compute the equivariant index of an elliptic operator (or an elliptic complex) in terms of data on the fixed point sets of the group elements. In [Atiyah and Singer 1968b], a cohomological version of this formula was obtained, which we call the Atiyah–Segal–Singer fixed point theorem. It has applications, for example, in representation theory. Indeed, Atiyah and Bott [1968] used a fixed point formula

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(which equals the Atiyah–Segal–Singer theorem in the case considered) to prove the Weyl character formula.

Our goals in this paper are to generalise the Atiyah–Segal–Singer theorem to noncompact manifolds, and to apply this generalisation in relevant situations.

The main result and some applications. We define an index on possibly noncompact manifolds, which generalises the equivariant index for compact groups and manifolds (see Definition 2.7). Assuming the fixed point set of a group element g is compact, we show that this index is given by exactly the same cohomological expression as in the Atiyah–Segal–Singer theorem. This is our main result, Theorem 2.16. We also obtain a fixed point formula for the index pairing between equivariant K-theory and K-homology in Theorem 2.18. In the nonequivariant setting, very general expressions for this pairing were given in [Carey et al. 2014]; Theorem 2.18 is an equivariant version of these results for the operators considered here.

While the cohomological expression for the index is the same as in the compact case, in noncompact examples we see that it gives rise to characters of *infinite-dimensional* representations. These can never occur as indices of elliptic operators on compact manifolds, so that the theory really gives us something new. For example, we use the fixed point theorem in Section 6E to express the character of a representation in the discrete series of a semisimple Lie group in terms of our index, on the regular elements of a maximal torus. Other examples and applications are:

- a holomorphic linearisation theorem, related to [Guillemin et al. 2002, Chapter 4] and [Braverman 2002, Theorem 7.2];
- explicit computations for actions by the circle on the plane and the two-sphere;
- a relation with kernels of Fredholm operators, in particular Callias-type Dirac operators [Anghel 1993; Braverman and Shi 2016; Bunke 1995; Callias 1978; Kucerovsky 2001];
- a relation with Braverman's index of deformed Dirac operators [Braverman 2002];
- a relative index theorem, in the spirit of [Gromov and Lawson 1983, Theorem 4.18];
- some geometric consequences in the cases of the Hodge–Dirac and Spin-Dirac operators.

In all cases we consider, we find that the index can be expressed explicitly in terms of the kernel of a deformation of the operator in question. (In the discrete series example, the operator does not even have to be deformed.) On noncompact manifolds, one can often obtain a well-defined index of a Dirac operator by applying a deformation, with suitable growth behaviour. See, for example, [Anghel

1993; Braverman 2002; 2015; Bunke 1995; Callias 1978; Hochs and Mathai 2015; Hochs and Song 2017a; Kucerovsky 2001; Ma and Zhang 2014; Paradan 2011]. This index then depends on the deformation used. While we do not use a deformation to define our index, we see in examples that it equals an index defined via a deformation. One could speculate that this means that the index we use implicitly includes a canonical choice of such a deformation. For the Callias-type operators studied in [Anghel 1993; Braverman and Shi 2016; Bunke 1995; Callias 1978; Kucerovsky 2001], their equivariant indices can be expressed as the index we define, plus a term representing the dependence on the deformation used, in terms of its behaviour "at infinity". (Previously, Callias-type operators were not studied in combination with group actions, so only nonequivariant indices were computed.)

The relation to index theory of deformed Dirac operators is strengthened in the last section of this paper, which is independent of the fixed point formula. There we give an expression for the index of elliptic operators involving deformations of their principal symbols.

Other generalised fixed point theorems include [Berline and Vergne 1996a, Main Theorem 1; 1996b, Theorem 20] (for transversally elliptic operators), [Braverman 2002, Theorem 7.5] (for deformed Dirac operators on noncompact manifolds), the results in [Dell'Ambrogio et al. 2014] (for *correspondences*, generalising self-maps on manifolds), [Emerson 2011, Theorem 2.7] (for groupoids) and [Wang and Wang 2016, Theorem 6.1] (for orbifolds).

Idea of the proof. Let us sketch some technical steps involved in defining the index and proving the fixed point formula. We consider a Riemannian manifold M, and an elliptic operator D on a vector bundle $E \to M$. Let G be a compact Lie group acting on E, preserving D. Under assumptions about grading and self-adjointness, we have a class $[D]$ in the equivariant K-homology group $K_0^G(M)$ of M. Let $g \in G$. Then we may replace G by the compact Abelian group generated by g, and still retain all information about the action by the element g. A localisation theorem in K-homology allows us to construct the *g-index* map

$$\mathrm{index}_g : K_0^G(M) \to \mathbb{C}.$$

This localisation theorem is closely related to a localisation theorem in KK-theory: Theorem 3.2 in [Rosenberg 1999]. The g-index of the operator D is defined as the g-index of its class $[D]$ in $K_0^G(M)$. If M is compact, this is the usual equivariant index of D, evaluated at g.

If M is compact, the principal symbol σ_D of D defines a class in the equivariant topological K-theory group $K_G^0(TM)$. In our setting, M may be noncompact. Then we have a class

$$[\sigma_D] \in KK_G(M, TM)$$

in the equivariant KK-theory of the pair $(C_0(M), C_0(TM))$. The Dolbeault–Dirac operator on TM defines a class

$$[D_{TM}] \in KK_G(TM, \mathrm{pt}).$$

An index theorem by Kasparov implies that, with respect to the Kasparov product $\otimes_{TM}$ over $C_0(TM)$, we have

$$[D] = [\sigma_D] \otimes_{TM} [D_{TM}] \in KK_G(M, \mathrm{pt}) = K_0^G(M).$$

This generalises the Atiyah–Singer index theorem.

The proof of the fixed point formula for the g-index of D is a KK-theoretic generalisation of the proof by Atiyah and Segal [1968] for the compact case. This generalisation involves Kasparov's index theorem, localisation theorems in KK-theory, and KK-theoretic versions of the Gysin wrong-way maps in K-theory. Another ingredient is a class

$$\sigma_g^D \in K_G^0(TM)_g \tag{1.1}$$

associated to σ_D, in the equivariant topological K-theory of TM, localised (in the algebraic sense) at g. Using these techniques, and keeping track of what happens in both entries in KK-theory, allows us to obtain an expression for the g-index of D in terms of data on the fixed point set of g. While all constructions in the proof are KK-theoretic in nature, the end result is a purely cohomological expression. An explicit description of the class (1.1) in terms of a deformation of the symbol σ_D allows us to prove a nonlocalised expression for the g-index, independent of the fixed point formula.

Outline. The g-index is introduced in Section 2. It allows us to state the fixed point formula in Theorem 2.16.

In Section 3, we prove the localisation results, which imply that the g-index is well-defined. In Section 4, we review an index theorem by Kasparov. This result, and related techniques, are used in the proof of the fixed point theorem in Section 5.

The applications and examples mentioned above are discussed in Section 6. In Section 7, we obtain a nonlocalised expression for the g-index of an elliptic operator.

Notation. If A is a subset of a set B, then we denote the inclusion map $A \hookrightarrow B$ by j_A^B. We denote the one-point set by pt. For any set A, we write p^A for the map from A to pt.

If U is an open subset of a locally compact Hausdorff space X, then we denote by k_U^X the inclusion map $C_0(U) \hookrightarrow C_0(X)$ defined by extending functions by zero outside U. If Y is another locally compact Hausdorff space, we write

$$KK(X, Y) := KK(C_0(X), C_0(Y)),$$

and similarly for equivariant KK-theory. The Kasparov product $\otimes_{C_0(X)}$ over $C_0(X)$ will also be denoted by $\otimes_X$. If X has a Borel measure, and $E \to X$ is a Hermitian vector bundle, then the $*$-homomorphism $\pi_X : C_0(X) \to \mathcal{B}(L^2(E))$ is given by the pointwise multiplication on L^2-sections of E. If H is a locally compact group acting on X, and $H' < H$ is a subgroup, we write $H \times_{H'} X$ for the quotient of $H \times X$ by the action by H' given by

$$h' \cdot (h, x) = (hh'^{-1}, h'x),$$

for $h' \in H'$, $h \in H$ and $x \in X$.

Throughout this paper, G will be a compact Abelian group containing an element g whose powers are dense in G. The only exception is Section 6E, where G denotes a semisimple Lie group. There, a compact Cartan subgroup $T < G$ will play the role that G plays in the rest of this paper.

If M is a manifold, its tangent bundle projection $TM \to M$ is denoted by τ_M. If a Riemannian metric is given, we will often tacitly use it to identify the tangent bundle of M with the cotangent bundle. The complexification of a vector space or vector bundle is denoted by a subscript $\mathbb{C}$.

2. The fixed point formula

Our goal in this paper is to generalise the Atiyah–Segal–Singer fixed point theorem ([Atiyah and Singer 1968b, Theorem 3.9], based on [Atiyah and Segal 1968, Theorem 2.12]) to noncompact manifolds, and to find interesting applications of this generalisation. This leads us to define the g-index in Section 2B. The key to defining the g-index is a localisation theorem, which is stated in Section 2A. The main result of this paper is the fixed point formula in Theorem 2.16, stated in Section 2D. This formula is entirely cohomological, and does not involve KK-theory. Some properties of the g-index are given in Section 2C.

Throughout this paper, M will be a Riemannian manifold. We consider an isometric diffeomorphism g from M to itself. Suppose the closure of the powers of g in the isometry group (with respect to the compact-open topology) is a compact group G. Equivalently, suppose g is an element of a compact group H acting isometrically on M, and let $G < H$ be the closed subgroup generated by g. In any case, G is Abelian. Let M^g be the fixed point set of g.

Let $E = E^+ \oplus E^- \to M$ be a $\mathbb{Z}_2$-graded, Hermitian vector bundle. Let D be an odd, essentially self-adjoint, elliptic differential operator, with principal symbol σ_D. (For example, D can be a Dirac-type operator on a complete manifold.) We will also write D for the self-adjoint closure of D. Then we have the element

$$[D] := \left[L^2(E), \frac{D}{\sqrt{D^2+1}}, \pi_M\right] \tag{2.1}$$

of the equivariant *K-homology* group $KK_G(M, \mathrm{pt}) := KK_G(C_0(M), \mathbb{C})$. Here $\pi_M : C_0(M) \to \mathcal{B}(L^2(E))$ is given by pointwise multiplication. For background material on KK-theory, see [Blackadar 1998, Chapter VIII].

2A. *Localisation.* Let $R(G)_g := R(G)_{I_g}$ be the localisation of the representation ring $R(G)$ at the prime ideal

$$I_g := \{\chi \in R(G) : \chi(g) = 0\}.$$

For any module $\mathcal{M}$ over $R(G)$, we write $\mathcal{M}_g := \mathcal{M}_{I_g}$ for the corresponding localised module over $R(G)_g$. Similarly, if $m \in \mathcal{M}$, and $\varphi : \mathcal{M} \to \mathcal{M}'$ is a module homomorphism to another such module, we write $m_g \in \mathcal{M}_g$ and

$$\varphi_g : \mathcal{M}_g \to \mathcal{M}'_g$$

for the corresponding localised versions.

For any two G-C^*-algebras A and B, the group $KK_G(A, B)$ is a module over the ring $R(G) = KK_G(\mathbb{C}, \mathbb{C})$, via the exterior Kasparov product. Fix a G-C^*-algebra A. The inclusion map

$$j^M_{M^g} : M^g \hookrightarrow M$$

induces

$$(j^M_{M^g})^*_g : KK_G(A, C_0(M))_g \to KK_G(A, C_0(M^g))_g.$$

Theorem 2.2. *If A is separable, the map $(j^M_{M^g})^*_g$ is an isomorphism of Abelian groups. This is still true if $M \setminus M^g$ is a manifold, rather than all of M.*

Remark 2.3. If $A = \mathbb{C}$, then this reduces to [Atiyah and Segal 1968, Theorem 1.1]. We need this more general statement, because in the noncompact case, principal symbols define classes in $KK_G(C_0(M), C_0(TM))$ as in (4.4), rather than in $KK_G(\mathbb{C}, C_0(TM))$ when M is compact.

We will also use an analogue of Theorem 2.2 for the first entry in KK-theory. Its formulation is slightly more subtle.

Theorem 2.4. *Suppose that M^g is compact and that A is σ-unital. Let $U, V \subset M$ be two G-invariant, relatively compact open neighbourhoods of M^g, such that $\overline{U} \subset V$. Then the map*

$$((j^V_{\overline{U}})_*)_g : KK_G(C_0(\overline{U}), A)_g \to KK_G(C_0(V), A)_g$$

is an isomorphism of Abelian groups. This is still true if M is only a locally compact Hausdorff topological space rather than a manifold, as long as the open subset $M \setminus M^g$ is a manifold.

Theorems 2.2 and 2.4 will be proved in Section 3 for graded KK-theory, i.e., the combination of even and odd KK-theory. We will only apply the even versions,

however. The cases where only $M \setminus M^g$ is a manifold were included because we will also apply Theorem 2.4 to one-point compactifications of manifolds. Theorems 2.2 and 2.4 are similar to Theorem 3.2 in [Rosenberg 1999].

2B. *The g-index.* Suppose the fixed point set M^g is compact. Let U, V be as in Theorem 2.4. Consider the proper map $p^{\overline{U}} : \overline{U} \to \mathrm{pt}$, and the inclusion map $k_V^M : C_0(V) \to C_0(M)$ given by extending function by zero outside V. Let A be a σ-unital G-C^*-algebra. By Theorem 2.4, we have the maps

$$\begin{aligned} KK_G(C_0(M), A)_g &\xrightarrow{(k_V^M)_g^*} KK_G(C_0(V), A)_g \\ &\xrightarrow{((j_{\overline{U}}^V)_*)_g^{-1}} KK_G(C(\overline{U}), A)_g \xrightarrow{(p_*^{\overline{U}})_g} KK_G(\mathbb{C}, A)_g. \end{aligned} \tag{2.5}$$

Lemma 2.6. *The composition* (2.5) *is independent of the sets U and V.*

Proof. To prove independence of U, let U' be a G-invariant, relatively compact neighbourhood of M^g such that $\overline{U'} \subset U$. Then we have the commutative diagram

$$\begin{tikzcd} & KK_G(C(\overline{U'}), A) \arrow[dl, "(j_{\overline{U'}}^V)_*"'] \arrow[d, "(j_{\overline{U'}}^{\overline{U}})_*"] \arrow[dr, "p_*^{\overline{U'}}"] & \\ KK_G(C_0(V), A) & KK_G(C(\overline{U}), A) \arrow[l, "(j_{\overline{U}}^V)_*"] \arrow[r, "p_*^{\overline{U}}"'] & KK_G(\mathbb{C}, A) \end{tikzcd}$$

Commutativity of this diagram implies that

$$(p_*^{\overline{U'}})_g \circ ((j_{\overline{U'}}^V)_*)_g^{-1} = (p_*^{\overline{U}})_g \circ ((j_{\overline{U}}^V)_*)_g^{-1}.$$

So (2.5) is indeed independent of U.

To prove independence of V, let V' be a G-invariant, relatively compact open subset of M containing V. Then the following diagram commutes:

$$\begin{tikzcd} & KK_G(C_0(V'), A) \arrow[d, "(k_V^{V'})^*"] & \\ KK_G(C_0(M), A) \arrow[ur, "(k_{V'}^M)^*"] \arrow[r, "(k_V^M)^*"'] & KK_G(C_0(V), A) & KK_G(C(\overline{U}), A) \arrow[ul, "(j_{\overline{U}}^{V'})_*"'] \arrow[l, "(j_{\overline{U}}^V)_*"] \end{tikzcd}$$

Therefore, we have

$$((j_{\overline{U}}^{V'})_*)_g^{-1} \circ (k_{V'}^M)_g^* = ((j_{\overline{U}}^V)_*)_g^{-1} \circ (k_V^M)_g^*,$$

so that (2.5) is independent of V. □

To define the g-index, we only need the case of Lemma 2.6 where $A = \mathbb{C}$. Later we will also use the general case, however.

Let

$$\mathrm{ev}_g : R(G) \to \mathbb{C}$$

be defined by evaluating characters at g, i.e., $\mathrm{ev}_g(\chi) := \chi(g)$, for $\chi \in R(G)$. In view of Lemma 2.6, we obtain a well-defined index as follows.

Definition 2.7. The *g-index* is the map

$$\mathrm{index}_g : KK_G(M, \mathrm{pt}) \to \mathbb{C}$$

defined as the composition

$$KK_G(M, \mathrm{pt}) \hookrightarrow KK_G(M, \mathrm{pt})_g \xrightarrow{(p_*^{\overline{U}})_g \circ ((j_{\overline{U}}^V)_*)_g^{-1} \circ (k_V^M)_g^*} KK_G(\mathrm{pt}, \mathrm{pt})_g \cong R(G)_g \xrightarrow{(\mathrm{ev}_g)_g} \mathbb{C}. \quad (2.8)$$

We will write

$$\mathrm{index}_g(D) := \mathrm{index}_g[D],$$

where $[D] \in KK_G(M, \mathrm{pt})$ is the class (2.1).

Note that $(k_V^M)_g^*[D]_g$ is simply the localisation at g of the K-homology class of the restriction of D to V.

Remark 2.9. The g-index of D could also have been called the D-Lefschetz number of g.

2C. *Properties of the g-index.* If M is *compact*, then we may take $U = V = M$ in Definition 2.7. Furthermore, the map $p^M : M \to \mathrm{pt}$ is proper. In that case, the composition (2.5) simply equals the map

$$(p_*^M)_g : KK_G(C_0(M), A)_g \to KK_G(\mathbb{C}, A)_g.$$

If $A = \mathbb{C}$, then it follows that for compact M, the g-index of D equals

$$\mathrm{index}_g(D) = \mathrm{index}_G(D)(g), \quad (2.10)$$

the usual equivariant index of D evaluated at g. Note that on the right-hand side of (2.10), G can be any compact Lie group acting isometrically on M if the action lifts to E, commutes with D, and contains g.

In general, however, the g-indices on noncompact manifolds give us something more general than the equivariant index in the compact case. In the examples in Section 6, we will see that the g-index can be used to describe characters of infinite-dimensional representations. These cannot be realised as indices on compact manifolds. And even on compact manifolds, an equivariant index can be decomposed into g-indices which individually correspond to infinite-dimensional representations. See Section 6D.

The g-index has an excision property.

Lemma 2.11. *Let V be a G-invariant, relatively compact, open neighbourhood of M^g. Suppose there is a G-equivariant open embedding $V \hookrightarrow \widetilde{M}$ into a G-manifold $\widetilde{M}$. Suppose the action by G on $\widetilde{M}$ has no fixed points outside V. Suppose there is a Hermitian, $\mathbb{Z}_2$-graded G-vector bundle $\widetilde{E} \to \widetilde{M}$ and an odd, self-adjoint, elliptic differential operator $\widetilde{D}$ on $\widetilde{E}$ such that $\widetilde{E}|_V = E|_V$ and $\widetilde{D}|_V = D|_V$. Then*

$$\operatorname{index}_g(D) = \operatorname{index}_g(\widetilde{D}).$$

Proof. By Proposition 10.8.8 in [Higson and Roe 2000], we have

$$(k_V^M)^*[D] = (k_V^{\widetilde{M}})^*[\widetilde{D}] \in KK_G(V, \mathrm{pt}).$$

This implies the claim. □

Example 2.12. Suppose M has a G-equivariant Spin-structure, and let D be the Spin-Dirac operator. Let $M \hookrightarrow \widetilde{M}$ be a G-equivariant open embedding into a compact G-manifold $\widetilde{M}$ with a G-equivariant Spin-structure. If G is connected and $\operatorname{index}_g(D) \neq 0$, then g must have a fixed point in $\widetilde{M} \setminus M$. Indeed, Atiyah and Hirzebruch [1970] showed that the g-index of the Spin-Dirac operator on $\widetilde{M}$ is zero in this case. So the claim follows by Lemma 2.11.

Another property of the g-index is multiplicativity. Let D_1 and D_2 be operators like D on manifolds M_1 and M_2, respectively, and consider the product operator

$$D_1 \times D_2 := D_1 \otimes 1 + 1 \otimes D_2$$

on $M_1 \times M_2$ (where graded tensor products are used). Then functoriality of the Kasparov product implies that

$$\operatorname{index}_g(D_1 \times D_2) = \operatorname{index}_g(D_1) \operatorname{index}_g(D_2).$$

In the index theory of deformed Dirac operators developed in [Braverman 2002], the deformation used means that an analogous multiplicativity property is highly nontrivial; see [Hochs and Song 2017b; Ma and Zhang 2014; Paradan 2011].

2D. *Fixed points.* Having defined the g-index, we can state the main result of this paper. We will use the fact that the connected components of the fixed point set M^g are smooth submanifolds of M, possibly of different dimensions.

Since M^g is compact, the restriction to TM^g of the principal symbol σ_D of D defines a class

$$[\sigma_D|_{TM^g}] \in KK_G(\mathrm{pt}, TM^g). \tag{2.13}$$

Let $N \to M^g$ be the union of the normal bundles to each of the components of M^g. Consider the topological K-theory class

$$\left[\textstyle\bigwedge N_{\mathbb{C}}\right] := \left[\bigoplus_j \bigwedge^{2j} N \otimes \mathbb{C}\right] - \left[\bigoplus_j \bigwedge^{2j+1} N \otimes \mathbb{C}\right] \in KK_G(\mathrm{pt}, M^g). \tag{2.14}$$

For any trivial G-space X, we have

$$KK_G(\mathrm{pt}, X) \cong KK(\mathrm{pt}, X) \otimes R(G).$$

We can evaluate the factor in $R(G)$ of any class $a \in KK_G(\mathrm{pt}, X)$ at g, to obtain $a(g) \in KK(\mathrm{pt}, X) \otimes \mathbb{C}$. In this way, evaluating the classes (2.13) and (2.14) at g gives the classes

$$[\sigma_D|_{TM^g}](g) \in KK(\mathrm{pt}, TM^g) \otimes \mathbb{C}$$

and

$$\left[\bigwedge N_{\mathbb{C}}\right](g) \in KK(\mathrm{pt}, M^g) \otimes \mathbb{C}, \tag{2.15}$$

respectively.

Consider the Chern characters

$$\mathrm{ch} : KK(\mathrm{pt}, TM^g) \to H^*(TM^g),$$
$$\mathrm{ch} : KK(\mathrm{pt}, M^g) \to H^*(M^g),$$

defined on each smooth component of M^g separately. By [Atiyah and Segal 1968, Lemma 2.7], the K-theory class (2.15) is invertible; hence so is its Chern character. An explicit expression for the inverse

$$\frac{1}{\mathrm{ch}\left(\left[\bigwedge N_{\mathbb{C}}\right](g)\right)} \in H^*(M^g) \otimes \mathbb{C}$$

of this element is given in [Atiyah and Singer 1968b, (3.8)]. The cohomology group $H^*(M^g)$ acts on $H^*(TM^g)$ via the pullback along the tangent bundle projection τ_{M^g}. Let $\mathrm{Todd}(TM^g \otimes \mathbb{C})$ be the cohomology class on M^g obtained by putting together the Todd-classes of the complexified tangent bundles to all components of M^g.

Theorem 2.16 (fixed point formula). *The g-index of D equals*

$$\mathrm{index}_g(D) = \int_{TM^g} \frac{\mathrm{ch}([\sigma_D|_{TM^g}](g))\,\mathrm{Todd}(TM^g \otimes \mathbb{C})}{\mathrm{ch}\left(\left[\bigwedge N_{\mathbb{C}}\right](g)\right)}. \tag{2.17}$$

The integral in this expression is the sum of the integrals over all connected components of TM^g of the integrand corresponding to each component.

If M is compact, then (2.10) implies that Theorem 2.16 reduces to the Atiyah–Segal–Singer fixed point formula [Atiyah and Singer 1968b, Theorem 3.9].

2E. ***The index pairing.*** In the course of the proof of Theorem 2.16, we will also find a fixed point formula for the index pairing (i.e., the Kasparov product)

$$KK_G(\mathrm{pt}, M) \times KK_G(M, \mathrm{pt}) \to KK_G(\mathrm{pt}, \mathrm{pt}).$$

Note that any element of the equivariant topological K-theory group $KK_G(\mathrm{pt}, M)$ can be represented by a formal difference $[F_0] - [F_1]$, for two G-equivariant vector bundles $F_0, F_1 \to M$ that are equal outside a compact set. We will write

$F := F_0 \oplus F_1$, with the $\mathbb{Z}_2$-grading where F_0 is the even part and F_1 the odd part, and $[F] := [F_0] - [F_1] \in KK_G(\mathrm{pt}, M)$.

Theorem 2.18 (fixed point formula for the index pairing). *We have*

$$([F] \otimes_M [D])(g) = \int_{TM^g} \frac{\mathrm{ch}([F|_{M^g}](g))\,\mathrm{ch}([\sigma_D|_{TM^g}](g))\,\mathrm{Todd}(TM^g \otimes \mathbb{C})}{\mathrm{ch}\big(\big[\bigwedge N_{\mathbb{C}}\big](g)\big)}.$$

Recall that M^g was assumed to be compact, and that we use the action by the cohomology of M^g on the cohomology of TM^g via the pullback along τ_{M^g}.

Theorem 3.33 in [Carey et al. 2014] is a nonequivariant index formula for the index pairing in a more general context. Theorem 2.18 is an equivariant version of this result, for operators like D.

The proof of Theorem 2.18 is simpler than that of Theorem 2.16, because it does not involve localisation in the first entry of KK-theory. Theorem 2.16 is needed for the examples and applications in Section 6, such as the relation with characters of discrete series representations. The reason for this is that Theorem 2.16 provides an expression for an index of the operator D itself, without the need to twist it by a K-theory class.

3. Localisation

We now turn to a proof of Theorems 2.2 and 2.4. This involves certain module structures discussed in Section 3A, which are used to prove vanishing results in Section 3B. In this section, we consider *graded* KK-theory, i.e., the direct sum of even and odd KK-theory.

3A. *Module structures.* Let H be a locally compact group, and let $H' < H$ be a compact subgroup.

Proposition 3.1. *Let Y be a locally compact, Hausdorff, proper H-space for which there is an equivariant, continuous map $Y \to H/H'$. Then for any H-C^*-algebra A, the groups*

$$KK_H(A, C_0(Y)) \quad \textit{and} \quad KK_H(C_0(Y), A)$$

have structures of unital $R(H')$-modules.

Proposition 3.1 follows from the fact that vector bundles, even on noncompact spaces, define classes in KK-theory in the following way. This is probably well-known, but we include a proof for completeness' sake.

Let X be a locally compact Hausdorff space on which H acts properly. Let $E \to X$ be a Hermitian H-vector bundle. The space $\Gamma_0(E)$ of continuous sections of E vanishing at infinity is a right Hilbert $C_0(X)$-module by pointwise

multiplication and inner products. Let $\pi_X : C_0(X) \to \mathcal{B}(\Gamma_0(E))$ be given by pointwise multiplication.

Lemma 3.2. *The triple*

$$(\Gamma_0(E), 0, \pi_X) \tag{3.3}$$

is an H-equivariant Kasparov $(C_0(X), C_0(X))$-cycle.

For compact spaces X, this fact is noted for example in (3.1) in [Rosenberg 1999]. In general, we will denote the class in $KK_H(X, X)$ defined by (3.3) by $[E]$.

Proof. We show that for all $f \in C_0(X)$, the operator $\pi_X(f)$ on $\Gamma_0(E)$ is compact. This implies the claim.

Let $U \subset X$ be a relatively compact open subset admitting an orthonormal frame $\{e_1, \dots, e_r\}$ of $E|_U$. Let $s \in \Gamma_0(E)$. Then

$$s|_U = \sum_{j=1}^{r} (e_j, s)_E \, e_j.$$

Here $(-\,,-)_E$ is the metric on E. So if $f \in C_0(X)$ is supported inside U, then

$$\pi_X(f)s = \sum_{j=1}^{r} (e_j, fs)e_j = \sum_{j=1}^{r} (\bar{f}e_j, s)e_j.$$

By extending the sections e_j outside U to elements of $\Gamma_0(E)$, we find that $\pi_X(f)$ is a finite-rank operator.

For a general $f \in C_c(X)$, there is a finite open cover $\{U_j\}_{j=1}^n$ of $\operatorname{supp}(f)$ such that every set U_j admits a local orthonormal frame for E. Let $\{\varphi_j\}_{j=1}^n$ be functions such that $\operatorname{supp}(\varphi_j) \subset U_j$, and $\sum_{j=1}^n \varphi_j$ equals one on $\operatorname{supp}(f)$. Then, by the preceding argument,

$$\pi_X(f) = \sum_{j=1}^{n} \pi_X(\varphi_j f)$$

is a finite-rank operator. Hence, for all $f \in C_0(X)$, the operator $\pi_X(f)$ on $\Gamma_0(E)$ is indeed compact. □

Now consider the situation of Proposition 3.1. Let $p : Y \to H/H'$ be an equivariant, continuous map. Let V be a finite-dimensional representation space of H'. We have the H-vector bundles

$$H \times_{H'} V \to H/H'$$

and

$$E_V := p^*(H \times_{H'} V) \to Y.$$

By Lemma 3.2, this vector bundle defines a class

$$[E_V] \in KK_H(Y, Y).$$

Lemma 3.4. *The map from $R(H')$ to $KK_H(Y, Y)$ given by*

$$[V] \mapsto [E_V],$$

with $[E_V]$ defined as above, is a ring homomorphism.

Proof. This follows from the fact that in the setting of Lemma 3.2, for any two Hermitian H-vector bundles $E, E' \to X$, one has

$$[E] \otimes_X [E'] = [E \otimes E']. \qquad \square$$

The ring homomorphism of Lemma 3.4 defines the module structures sought in Proposition 3.1, which has therefore been proved. If $A = \mathbb{C}$ and Y is compact, the $R(H')$-module structure on $KK_H(\mathbb{C}, C_0(Y))$ defined in this way is the one used in [Atiyah and Segal 1968].

3B. *Vanishing results.* We will prove Theorems 2.2 and 2.4 by generalising Atiyah and Segal's proof of [Atiyah and Segal 1968, Theorem 1.1]. An important step is the following generalisation of [Atiyah and Segal 1968, Corollary 1.4].

Proposition 3.5. *Let $H < G$ be a closed subgroup such that $g \notin H$. Let Y be a compact G-space for which there is an equivariant map $Y \to G/H$, and A a G-C^*-algebra. Then*

$$KK_G(A, C_0(Y))_g = KK_G(C_0(Y), A)_g = 0.$$

Proof. By [Atiyah and Segal 1968, Corollary 1.3], we have $R(H)_g = 0$. As Atiyah and Segal argued below that corollary, it is therefore enough to show that $KK_G(A, C_0(Y))$ and $KK_G(C_0(Y), A)$ are unital $R(H)$-modules. Hence, the claim follows from Proposition 3.1. $\square$

We will deduce Theorems 2.2 and 2.4 from the following special cases.

Proposition 3.6. *In the setting of Theorem 2.2, suppose g has no fixed points in M. Then, if A is separable, we have*

$$KK_G(A, C_0(M))_g = 0. \tag{3.7}$$

If A is σ-unital, then for all G-invariant, relatively compact open subsets $U \subset M$,

$$KK_G(C_0(U), A)_g = 0. \tag{3.8}$$

If $A = \mathbb{C}$, then (3.7) is precisely [Atiyah and Segal 1968, Proposition 1.5]. By a generalisation of the arguments in [Atiyah and Segal 1968, Section 1], we will deduce Proposition 3.6 from Proposition 3.5.

By Palais' slice theorem [1961, Proposition 2.2.2], there is an open cover $\{U_j\}_{j=1}^{\infty}$ of M by G-invariant open sets such that for all j,

$$\overline{U_j} \cong G \times_{H_j} \overline{S_j}$$

(via the action map), for the stabiliser $H_j < G$ of a point in U_j, and an H_j-invariant submanifold $S_j \subset M$. Suppose that g has no fixed points. Then it does not lie in any of the stabilisers H_j. Therefore, Proposition 3.5 implies that

$$KK_G(A, C_0(\overline{U_j}))_g = KK_G(C_0(\overline{U_j}), A)_g = 0.$$

Let $X \subset M$ be any G-invariant, compact subset. The proof of Proposition 3.6 is based on the following fact.

Lemma 3.9. *If A is separable, then*

$$KK_G(A, C_0(X))_g = 0. \tag{3.10}$$

If A is σ-unital, then

$$KK_G(C_0(X), A)_g = 0. \tag{3.11}$$

Proof. We use an induction argument based on exact sequences in KK-theory. We work out the details for (3.10). Then (3.11) can be proved in the same way, with exact sequences in the second entry in KK-theory replaced by the corresponding exact sequences in the first entry. The conditions that A is separable or σ-unital imply that these exact sequences exist.

For $j, n \in \mathbb{N}$, write $X_j := \overline{U_j} \cap X$, and $Y_n := X_1 \cup \cdots \cup X_n$. Fix $n \in \mathbb{N}$, and consider the exact sequence of C^*-algebras

$$0 \to C_0(X_{n+1} \setminus Y_n) \to C_0(X_{n+1}) \to C_0(X_{n+1} \cap Y_n) \to 0.$$

It induces the exact triangle

$$\begin{tikzcd} KK_G(A, C_0(X_{n+1})) \arrow[r] & KK_G(A, C_0(X_{n+1} \cap Y_n)) \arrow[d, "\partial"] \\ & KK_G(A, C_0(X_{n+1} \setminus Y_n)) \arrow[ul] \end{tikzcd}$$

(See, e.g., [Blackadar 1998, Theorem 19.5.7].) By Proposition 3.5, we have

$$KK_G(A, C_0(X_{n+1}))_g = KK_G(A, C_0(X_{n+1} \cap Y_n))_g = 0.$$

Since localisation at g preserves exactness, we find that

$$KK_G(A, C_0(X_{n+1} \setminus Y_n))_g = 0. \tag{3.12}$$

Using the exact sequence

$$0 \to C_0(Y_{n+1} \setminus Y_n) \to C_0(Y_{n+1}) \to C_0(Y_n) \to 0$$

in a similar way, we obtain the exact triangle

$$\begin{tikzcd}
KK_G(A, C_0(Y_{n+1}))_g \arrow[r] & KK_G(A, C_0(Y_n))_g \arrow[d, "\partial"] \\
& KK_G(A, C_0(Y_{n+1} \setminus Y_n))_g \arrow[ul]
\end{tikzcd}$$

Since $Y_{n+1} \setminus Y_n = X_{n+1} \setminus Y_n$, the vanishing of (3.12) implies that

$$KK_G(A, C_0(Y_{n+1}))_g = KK_G(A, C_0(Y_n))_g.$$

Because $Y_1 = X_1$, Proposition 3.5 implies that

$$KK_G(A, C_0(Y_1))_g = 0.$$

Since X is compact, it can be covered by finitely many of the sets X_j. Hence, the claim follows by induction on n. □

Proof of Proposition 3.6. Let $U \subset M$ be a G-invariant, relatively compact open subset. Consider the exact sequence

$$0 \to C_0(U) \to C_0(\overline{U}) \to C_0(\partial U) \to 0.$$

If A is σ-unital, this induces the localised exact triangle

$$\begin{tikzcd}
KK_G(C_0(\overline{U}), A)_g \arrow[dr] & KK_G(C_0(\partial U), A)_g \arrow[l] \\
& KK_G(C_0(U), A)_g \arrow[u, "\partial"']
\end{tikzcd}$$

Lemma 3.9 implies that

$$KK_G(C_0(\overline{U}), A)_g = KK_G(C_0(\partial U), A)_g = 0.$$

So we find that $KK_G(C_0(U), A)_g = 0$.

Similarly, if A is separable, we have the exact triangle

$$\begin{tikzcd}
KK_G(A, C_0(\overline{U}))_g \arrow[r] & KK_G(A, C_0(\partial U))_g \arrow[d, "\partial"] \\
& KK_G(A, C_0(U))_g \arrow[ul]
\end{tikzcd}$$

Applying Lemma 3.9 in the same way, we find that $KK_G(A, C_0(U))_g = 0$. The equality (3.7) follows, because M is the direct limit of sets U as above, and because KK-theory commutes with direct limits in the second entry. □

Remark 3.13. The reason why (3.8) does not hold if U is replaced by M, and hence why Theorem 2.4 has to be stated more subtly than Theorem 2.2, is that KK-theory does not commute with direct limits in the first entry. For example, the

domain of the analytic assembly map in the Baum–Connes conjecture [Baum et al. 1994] is the *representable K-homology* group

$$RK_*^H(X) := \varinjlim_{Y \subset X;\ Y/H \text{ cpt}} KK_H(C_0(Y), \mathbb{C}),$$

for a locally compact Hausdorff space X on which a locally compact group H acts properly. This does not equal the usual K-homology group $KK_H(C_0(X), \mathbb{C})$ in general.

3C. *Proofs of localisation results.*

Proof of Theorem 2.2. Consider the exact sequence

$$0 \to C_0(M \setminus M^g) \to C_0(M) \xrightarrow{(j_{M^g}^M)^*} C_0(M^g) \to 0.$$

It induces the exact triangle

$$\begin{tikzcd} KK_G(A, C_0(M)) \arrow[r, "(j_{M^g}^M)^*"] & KK_G(A, C_0(M^g)) \arrow[d, "\partial"] \\ & KK_G(A, C_0(M \setminus M^g)) \arrow[ul] \end{tikzcd}$$

After localisation at g, the first part of Proposition 3.6 yields the exact triangle

$$\begin{tikzcd} KK_G(A, C_0(M))_g \arrow[r, "(j_{M^g}^M)_g^*"] & KK_G(A, C_0(M^g))_g \arrow[d, "\partial"] \\ & 0 \arrow[ul] \end{tikzcd}$$

□

Proof of Theorem 2.4. Let U and V be as in Theorem 2.4. Similarly to the proof of Theorem 2.2, we have an exact triangle

$$\begin{tikzcd} KK_G(C_0(V), A)_g \arrow[dr] & KK_G(C_0(\overline{U}), A)_g \arrow[l, "((j_{\overline{U}}^V)_*)_g"'] \\ & KK_G(C_0(V \setminus \overline{U}), A)_g \arrow[u, "\partial"'] \end{tikzcd}$$

Because $V \setminus \overline{U}$ is a relatively compact subset of $M \setminus M^g$, the second part of Proposition 3.6 implies that the bottom localised KK-group in this triangle equals zero. □

4. Kasparov's index theorem

In the proof of the Atiyah–Segal–Singer fixed point theorem, the Atiyah–Singer index theorem is used to relate topological and analytical indices to each other. In the noncompact case discussed in this paper, a roughly similar role is played by an index theorem of Kasparov. We state Kasparov's index theorem in Section 4A. In Section 4B, we discuss the fibrewise Bott element for the normal bundle of a submanifold in KK-theory, which is dual to the class of the Dolbeault–Dirac operator in a sense. This Bott element will play an important role in the proof of Theorem 2.16. In Section 4D, we show how the Bott element can be used to deduce the Atiyah–Singer index theorem from Kasparov's index theorem in the compact case. (The main step in the argument used there will be used in the proof of Theorem 2.16.)

Most of the material in this section is based on [Atiyah and Singer 1968b; Kasparov 2016] and explanations to the authors by Kasparov. Although the results here are not ours, we found it worthwhile to include the details, because they have not appeared in print yet.

4A. ***The index theorem.*** To state the theorem, we recall the definition of the Dolbeault operator class

$$[D_{TM}] \in KK_G(TM, \mathrm{pt}) \tag{4.1}$$

in [Kasparov 2016, Definition 2.8]. The tangent bundle $T(TM)$ of TM has a natural almost complex structure J. For $m \in M$ and $v \in T_m M$, we have

$$T_v(TM) = T_m M \oplus T_v(T_m M) = T_m M \oplus T_m M.$$

With respect to this decomposition, the almost complex structure J is given by the matrix $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Let $\bar\partial + \bar\partial^*$ be the Dolbeault–Dirac operator on smooth sections of the vector bundle $\bigwedge^{0,*} T^*(TM) \to TM$, for this almost complex structure. We will identify this vector bundle with $\tau_M^* \bigwedge TM_{\mathbb{C}} \to TM$. The class (4.1) is the class of this operator, as in (2.1). In our arguments however, it will be more convenient to use the Spin^c-Dirac operator D_{TM}, on the same vector bundle. This defines the same K-homology class as $\bar\partial + \bar\partial^*$.

Definition 4.2. The *topological index* is the map

$$\mathrm{index}_t : KK_G(M, TM) \to KK_G(M, \mathrm{pt})$$

given by the Kasparov product with $[D_{TM}]$.

Consider the principal symbol $\tilde\sigma_D := \sigma_D/\sqrt{\sigma_D^2 + 1}$ of the operator $D/\sqrt{D^2+1}$. For $f \in C_0(M)$, we have for all $m \in M$ and $v \in T_m M$,

$$f(m)(1 - \tilde\sigma_D(v)^2) = f(m)(\sigma_D(v)^2 + 1)^{-1}.$$

Since the operator D is elliptic and of positive order, this expression tends to zero as m or v tends to infinity. It therefore defines a compact operator on the Hilbert $C_0(TM)$-module $\Gamma_0(\tau_M^* E)$, analogously to the proof of Lemma 3.2. Therefore, the triple

$$(\Gamma_0(\tau_M^* E), \tilde{\sigma}_D, \pi_{TM} \circ \tau_M^*) \tag{4.3}$$

is a G-equivariant Kasparov $(C_0(M), C_0(TM))$-cycle. Here, $\pi_{TM} : C_b(TM) \to \mathcal{B}(\Gamma_0(\tau_M^* E))$ is given by pointwise multiplication. Denote by

$$[\sigma_D] \in KK_G(M, TM) \tag{4.4}$$

the class of (4.3). In view of the following lemma, this symbol class is a natural generalisation of the K-theory symbol class defined in [Atiyah and Singer 1968a] when M is compact.

Lemma 4.5. *If M is compact, consider the map p^M from M to a point. The image*

$$p_*^M[\sigma_D] \in K_G^*(TM)$$

is the usual symbol class.

Proof. Since $\pi_{TM} \circ \tau_M^* \circ (p^M)^*$ is the representation of $\mathbb{C}$ in $\Gamma_0(\tau_M^* E)$ by scalar multiplication, we have

$$p_*^M[\sigma_D] = [\Gamma_0(\tau_M^* E), \tilde{\sigma}_D] \in KK_G(\mathrm{pt}, TM).$$

This corresponds to the class

$$[\sigma_{D^+} : \tau_M^* E^+ \to \tau_M^* E^-] \in K_G^0(TM)$$

in the sense of [Lawson and Michelsohn 1989, Chapter III, (1.7)], where TM is identified with the open unit ball bundle BM over M. (Restricting σ_{D^+} to BM and then identifying $BM \cong TM$ has the same effect as normalising σ_{D^+}.) The lemma is then proved. □

We conclude this subsection by stating Kasparov's index theorem, which will be used to obtain a cohomological formula for the g-index.

Theorem 4.6 (Kasparov's index theorem [2016, Theorem 4.2]). *The K-homology class of the elliptic operator in* (2.1) *is equal to the topological index of its symbol class* (4.4), *i.e.*,

$$[D] = \mathrm{index}_t[\sigma_D] \in KK_G(M, \mathbb{C}). \tag{4.7}$$

Remark 4.8. In [Kasparov 2016, Theorem 4.2], the operator in question is assumed to be properly supported, which is not true for the operator $D/\sqrt{1+D^2}$ in general. However, let $\{\chi_j\}_{j=1}^\infty$ be a sequence of G-invariant, compactly supported functions, such that $\{\chi_j^2\}_{j=1}^\infty$ is a partition of unity. (This exists since G is compact.)

Then the operator

$$\sum_{j=1}^{\infty} \chi_j \frac{D}{\sqrt{1+D^2}} \chi_j$$

is properly supported, and also satisfies the other assumptions of [Kasparov 2016, Theorem 4.2]. Since this operator defines the same K-homology class as $D/\sqrt{1+D^2}$, we can apply [Kasparov 2016, Theorem 4.2] to the class of the latter operator in this way.

4B. ***The Bott element.*** If S is a closed (as a topological subspace, i.e., not necessarily compact), G-invariant submanifold of M, then the Dolbeault operator classes on TS and on a tubular neighbourhood of TS in TM are related by a *(fibrewise) Bott element*. This is a technical tool that will be used several times in the paper. The material here is analogous to Definition 2.6 and Theorem 2.7 in [Kasparov 2016].

Consider the tangent bundle projections

$$\tau_S : TS \to S,$$
$$\tau_N : TN \to N.$$

Denote by $\pi : N \to S$ the normal bundle of S in M. Let $T\pi : TN \to TS$ be the tangent map of π. It again defines a vector bundle. The following diagram commutes:

$$\begin{array}{ccc} TN & \xrightarrow{\tau_N} & N \\ \downarrow{\scriptstyle T\pi} & & \downarrow{\scriptstyle \pi} \\ TS & \xrightarrow{\tau_S} & S \end{array} \tag{4.9}$$

This defines a vector bundle $TN \to S$. Consider the vector bundle

$$\bigwedge \widetilde{N}_{\mathbb{C}} := T\pi^*\big(\tau_S^* \bigwedge N \otimes \mathbb{C}\big) \to TN.$$

Let $s \in S$. Then

$$(TN)_s := T\pi^{-1}(\tau_S^{-1}(s)) = \tau_N^{-1}(N_s) = T_s S \times N_s \times N_s.$$

Let $w \in (TN)_s$, and let $(\eta, \zeta) \in N_s \times N_s$ be the projection of w according to this decomposition. Note that

$$\big(\bigwedge \widetilde{N}_{\mathbb{C}}\big)_w = \bigwedge N_s \otimes \mathbb{C}.$$

We define the vector bundle endomorphism B of $\bigwedge \widetilde{N}_{\mathbb{C}}$ by

$$B_w = \operatorname{ext}\big(\zeta + \sqrt{-1}\,\eta\big) - \operatorname{int}\big(\zeta + \sqrt{-1}\,\eta\big),$$

for all s, w, η and ζ as above. Here ext denotes the wedge product, and int denotes contraction. With respect to the grading of $\bigwedge \widetilde{N}_{\mathbb{C}}$ by even and odd exterior powers, the operator B is odd.

As B is fibrewise selfadjoint, we have the bounded operator $B(1+B^2)^{-1/2}$ on $\Gamma_0\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big)$. The space $\Gamma_0\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big)$ is a right Hilbert $C_0(TN)$-module in the usual way, with respect to pointwise multiplication by functions and the pointwise inner product. Consider the representation

$$\tilde{\pi}_{TS} := \pi_{TN} \circ T\pi^* : C_0(TS) \to \mathcal{B}\big(\Gamma_0\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big)\big),$$

where π_{TN} is given by pointwise multiplication by functions in $C_b(TN)$.

Lemma 4.10. *The triple*

$$\big(\Gamma_0\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big), B(1+B^2)^{-1/2}, \tilde{\pi}_{TS}\big) \tag{4.11}$$

is a G-equivariant Kasparov $(C_0(TS), C_0(TN))$-cycle.

Proof. Let $f \in C_0(TS)$. Since $B(1+B^2)^{-1/2}$ is a vector bundle endomorphism, it commutes with $\tilde{\pi}_{TS}(f)$. Moreover, we have for all $w \in (TN)_s$ as above,

$$\big(\tilde{\pi}_{TS}(f)\big(1-[B(1+B^2)^{-1/2}]^2\big)\big)_w = \frac{f(v)}{1+\|\eta\|^2+\|\zeta\|^2},$$

with $v := T\pi(w) \in T_sS$. This defines a function in $C_0(TN)$, and hence acts on the Hilbert $C_0(TN)$-module $\Gamma_0\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big)$ as a compact operator. As G preserves the metric on TN, the operator $B(1+B^2)^{-1/2}$ is G-equivariant. □

Definition 4.12. The *(fibrewise) Bott element* of the normal bundle $N \to S$ is the class

$$\beta_N \in KK_G(TS, TN)$$

of the cycle (4.11).

4C. *The Bott element and Dolbeault classes.* The Bott element is useful to us because of the following property. This was used in [Kasparov 2016, second paragraph on p. 1326]; we work out some details of the proof in this subsection.

Proposition 4.13. *Under the Kasparov product*

$$KK_G(TS, TN) \times KK_G(TN, \mathrm{pt}) \to KK_G(TS, \mathrm{pt}),$$

one has

$$\beta_N \otimes_{TN} [D_{TN}] = [D_{TS}].$$

To prove this proposition, one can use the part D_1 of the Spin^c-Dirac operator D_{TN} acting in the fibre directions of $TN \to TS$. For $s \in S$ and $v \in T_sS$, we have $T\pi^{-1}(v) = N_s \oplus T_vN$. Let a be the rank of N, and let $\{f_1, \dots, f_a\}$ be a local orthonormal frame of $N \to S$. This defines coordinate functions κ_j and λ_j on the parts N_s and T_vN of the fibres $T\pi^{-1}(v)$ of $TN \to TS$, respectively. For $j = 1, \dots, a$, consider the vector bundle endomorphisms

$$e_j := \mathrm{ext}(f_j) - \mathrm{int}(f_j) \quad \text{and} \quad \epsilon_j := \mathrm{ext}(f_j) + \mathrm{int}(f_j)$$

of $\bigwedge N \otimes \mathbb{C} \to S$, pulled back along (4.9) to endomorphisms of $\bigwedge \widetilde{N}_{\mathbb{C}} \to TN$. Then D_1 is the operator

$$D_1 := \sum_{j=1}^{a} e_j \frac{\partial}{\partial \kappa_j} + \sqrt{-1}\epsilon_j \frac{\partial}{\partial \lambda_j}$$

on $\Gamma^\infty\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big)$. This can be viewed as a family of operators on the fibres of $TN \to TS$.

It defines a class in KK-theory as follows. Let $\Gamma_c\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big)$ be the space of continuous compactly supported sections of $\bigwedge \widetilde{N}_{\mathbb{C}}$. Let $\mathcal{E}_0$ be the completion of this space into a Hilbert $C_0(TS)$-module with respect to the $C_0(TS)$-valued inner product

$$\langle f, h\rangle(v) := \int_{T\pi^{-1}(v)} \overline{f(t)} h(t)\, dt \tag{4.14}$$

for $f, h \in \Gamma_c\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big)$ and $v \in TS$. The operator D_1 gives rise to the class

$$[D_1] := \big[\mathcal{E}_0, D_1(1 + D_1^2)^{-1/2}, \pi_{TN}\big] \in KK_G(TN, TS). \tag{4.15}$$

Lemma 4.16. *We have*

$$[D_1] \otimes_{TS} [D_{TS}] = [D_{TN}] \in KK_G(TN, \mathrm{pt}).$$

Proof. Regarding N as an open subset of M, we identify their tangent bundles when restricted to S, i.e., $TN|_S = TM|_S$. Therefore, as vector bundles over TN, we have

$$\begin{aligned}\bigwedge \widetilde{N}_{\mathbb{C}} \otimes T\pi^*\tau_S^* \bigwedge TS_{\mathbb{C}} &= T\pi^*\tau_S^* \bigwedge N_{\mathbb{C}} \otimes T\pi^*\tau_S^* \bigwedge TS_{\mathbb{C}} \\ &= T\pi^*\tau_S^* \bigwedge (N \oplus TS)_{\mathbb{C}} = T\pi^*\tau_S^* \bigwedge (TM|_S)_{\mathbb{C}} \\ &= T\pi^*\tau_S^* \bigwedge (TN|_S)_{\mathbb{C}} = \tau_N^* \bigwedge TN_{\mathbb{C}}.\end{aligned}$$

The last equality follows from commutativity of (4.9). Thus, as Hilbert spaces with representations of $C_0(TN)$,

$$\mathcal{E}_0 \otimes_{C_0(TS)} L^2\big(TS, \tau_S^* \bigwedge TS_{\mathbb{C}}\big) \cong L^2\big(TN, \tau_N^* \bigwedge TN_{\mathbb{C}}\big). \tag{4.17}$$

Under this identification, we have

$$D_1 \otimes 1 + 1 \otimes D_{TS} = D_{TN}.$$

(Here we use graded tensor products.) Consider the bounded operator

$$F := \frac{D_1 \otimes 1 + 1 \otimes D_{TS}}{\sqrt{1 + D_1^2 \otimes 1 + 1 \otimes D_{TS}^2}} \tag{4.18}$$

on $\mathcal{E}_0 \otimes_{C_0(TS)} L^2\big(TS, \tau_S^* \bigwedge TS_{\mathbb{C}}\big)$. Write

$$F_1 := \frac{D_1}{\sqrt{1+D_1^2}}, \qquad F_{TS} := \frac{D_{TS}}{\sqrt{1+D_{TS}^2}}.$$

We can verify that F is an $1 \otimes F_{TS}$-connection, and the graded commutator $[F_1 \otimes 1, F]$ is positive modulo compact operators. Thus, by [Blackadar 1998, Definition 18.4.1], the Kasparov product $[D_1] \otimes_{TS} [D_{TS}]$ is represented by the operator F on the space $L^2\big(TN, \tau_N^* \bigwedge TN_{\mathbb{C}}\big)$. The lemma is then proved. □

Lemma 4.19. *The product*

$$\beta_N \otimes_{TN} [D_1]$$

is the identity element of the ring $KK_G(TS, TS)$.

Proof. The idea is that in this product, we pair fibrewise Bott classes and Dolbeault classes, and thus obtain the trivial line bundle over TS. To see this, observe first the isomorphism

$$\Gamma_c\big(TN, \textstyle\bigwedge \widetilde{N}_{\mathbb{C}}\big) \otimes_{C_c(TN)} \Gamma_c\big(TN, \textstyle\bigwedge \widetilde{N}_{\mathbb{C}}\big) \cong \Gamma_c\big(TN, \textstyle\bigwedge \widetilde{N}_{\mathbb{C}} \otimes \bigwedge \widetilde{N}_{\mathbb{C}}\big) \tag{4.20}$$

as $C_c(TS)$-modules. Denote by $\mathcal{E}'$ the completion of the right-hand side under the $C_0(TS)$-valued inner product defined in a similar way as (4.14). It can be checked that

$$F_0 := \frac{B \otimes 1 + 1 \otimes D_1}{\sqrt{1 + B^2 \otimes 1 + 1 \otimes D_1^2}} \tag{4.21}$$

is a $1 \otimes D_1/\sqrt{1+D_1^2}$-connection, and that for all a in $C_0(TS)$, the operator

$$\tilde{\pi}_{TS}(a)[B(1+B^2)^{-1/2} \otimes 1, F_0]\tilde{\pi}_{TS}(a)^*$$

is positive modulo compact operators on $\mathcal{E}'$. Hence, the Kasparov product of β_N, given by (4.11), and the class $[D_1]$, given by (4.15), is equal to

$$[\mathcal{E}', F_0, \tilde{\pi}_{TS}] \in KK_G(TS, TS). \tag{4.22}$$

As in the proof of [Kasparov 2016, Theorem 2.7(2)], we apply the rotation homotopy

$$F_t := \frac{(B + \sin(t) D_1) \otimes 1 + 1 \otimes \cos(t) D_1}{\sqrt{1 + (B^2 + \sin(t)^2 D_1^2) \otimes 1 + 1 \otimes \cos(t)^2 D_1^2}},$$

for $t \in \big[0, \frac{\pi}{2}\big]$. Then the operator F_0 in the cycle (4.22) is transformed into $F_{\pi/2} = F' \otimes 1$, where

$$F' := (B + D_1)(1 + B^2 + D_1^2)^{-1/2}.$$

In summary, β_N and $[D_1]$ are families of operators indexed by TS whose Kasparov product is represented by F'. At every $v \in T_s S$, the square of $B + D_1$ is the harmonic oscillator operator

$$\sum_{j=1}^{a}\Big(\kappa_j^2+\lambda_j^2-\frac{\partial}{\partial\kappa_j^2}-\frac{\partial^2}{\partial\lambda_j^2}\Big)+2\deg-a$$

on $T\pi^{-1}(v)\cong N_s\times N_s$. (Here deg is the degree in $\bigwedge N$.) It has a one-dimensional kernel, concentrated in degree zero, generated by

$$(\eta,\zeta)\mapsto e^{-(\|\eta\|^2+\|\zeta\|^2)/2}\in C_0(N_s\times N_s). \tag{4.23}$$

Thus, over each fibre, F' is a Fredholm operator with index 1, and $\beta_N\otimes_{TN}[D_1]$ is equal to the exterior product of this Fredholm operator in $KK_G(\mathbb{C},\mathbb{C})$ and the class $[C_0(TS),0,\pi_{TS}]\in KK_G(TS,TS)$, both representing the respective ring identities. Hence the claim follows. □

Proof of Proposition 4.13. Using Lemmas 4.16 and 4.19, and associativity of the Kasparov product, we find that

$$\beta_N\otimes_{TN}[D_{TN}]=(\beta_N\otimes_{TN}[D_1])\otimes_{TS}[D_{TS}]=[D_{TS}].$$

This finishes the proof. □

We will later need the restriction of the Bott element to TS. Consider the class

$$\big[\tau_S^*\textstyle\bigwedge N_{\mathbb{C}}\big]:=\big[\bigoplus_j\bigwedge^{2j}\tau_S^*N\otimes\mathbb{C}\big]-\big[\bigoplus_j\bigwedge^{2j+1}\tau_S^*N\otimes\mathbb{C}\big]\in KK_G(TS,TS),$$

defined as in Lemma 3.2.

Lemma 4.24. *We have*

$$(j_{TS}^{TN})^*\beta_N=[\tau_S^*\textstyle\bigwedge N_{\mathbb{C}}]\in KK_G(TS,TS).$$

Proof. The Hilbert $C_0(TS)$-module in $(j_{TS}^{TN})^*\beta_N$ is $\Gamma_0\big(TS,\tau_S^*\bigwedge N_{\mathbb{C}}\big)$. Because $B|_{TS}$ is the zero operator, the claim follows. □

4D. *The Atiyah–Singer index theorem.* Suppose for now that M is compact and G is trivial. Then Kasparov's index theorem reduces to the Atiyah–Singer index theorem; see [Kasparov 2016, Remark 4.5]. We provide the details of this implication here, because these will be used in the proof of Theorem 2.16.

Consider the Atiyah–Singer topological index map

$$\mathrm{index}_t^{\mathrm{AS}}:KK(\mathrm{pt},TM)\to\mathbb{Z},$$

which maps a class $\sigma\in KK(\mathrm{pt},TM)$ to

$$\int_{TM}\mathrm{ch}(\sigma)\,\mathrm{Todd}(TM\otimes\mathbb{C}). \tag{4.25}$$

Note that we do not have the factor $(-1)^{\dim M}$ in (4.25) as in [Atiyah and Singer 1968b, Theorem 2.12], because we use a different almost complex structure on TM than in [Atiyah and Singer 1968b, p. 554], giving the opposite orientation.

Lemma 4.26. *As a map $KK(\mathrm{pt}, TM) \to KK(\mathrm{pt}, \mathrm{pt})$, right multiplication by $[D_{TM}]$ is the Atiyah–Singer topological index.*

Because of Lemma 4.26, Theorem 4.6 implies the Atiyah–Singer index theorem. Indeed, since M is compact, the map $p^M : M \to \mathrm{pt}$ is proper. By functoriality of the Kasparov product, Lemma 4.26 implies that the following diagram commutes:

$$\begin{array}{ccc} KK(M, TM) & \xrightarrow{\mathrm{index}_t} & KK(M, \mathrm{pt}) \\ \Big\downarrow p_*^M & & \Big\downarrow p_*^M \\ KK(\mathrm{pt}, TM) & \xrightarrow{\mathrm{index}_t^{\mathrm{AS}}} & \mathbb{Z} = KK(\mathrm{pt}, \mathrm{pt}) \end{array}$$

By Lemma 4.5, applying the map p_*^M to both sides of (4.7), and using commutativity of the above diagram, one obtains the Atiyah–Singer index theorem.

Proof of Lemma 4.26. The proof is a reformulation of the arguments in [Atiyah and Singer 1968b], using KK-theory. There are embeddings $M \hookrightarrow \mathbb{R}^n$ with normal bundle N of rank a, and $TM \hookrightarrow T\mathbb{R}^n = \mathbb{C}^n$ with normal bundle TN. As N is homeomorphic to a tubular neighbourhood of M in $\mathbb{R}^n$, we can identify TN with an open neighbourhood of TM in $\mathbb{C}^n$. (Note that here, the submanifold S of M in Section 4B is replaced by the submanifold M of $\mathbb{R}^n$.)

Denote by

$$\beta_N \in KK(TM, TN)$$

the fibrewise Bott element over TM in TN, in the sense of Definition 4.12. Then by Proposition 4.13,

$$[D_{TM}] = \beta_N \otimes [D_{TN}]. \tag{4.27}$$

The Chern character is compatible with the pairing of K-theory and K-homology. The Chern character of the Bott generator β of $K^0(\mathbb{R}^2)$ is a generator of $H^2(\mathbb{R}^2)$. As the Dolbeault class $[D_{\mathbb{R}^2}]$ of $\mathbb{R}^2$ is dual to β, its Chern character is the Poincaré dual of $\mathrm{ch}(\beta)$. So $\mathrm{ch}[D_{\mathbb{R}^2}]$ is the fundamental class $[\mathbb{R}^2]$ of $\mathbb{R}^2$. Similarly, working with the exterior Kasparov product of n copies of β, we conclude that $\mathrm{ch}[D_{\mathbb{R}^{2n}}] = [\mathbb{R}^{2n}]$. Noting that $T\mathbb{R}^n = \mathbb{R}^{2n}$, by functoriality of the Chern character we have

$$\begin{aligned} \mathrm{ch}[D_{TN}] &= \mathrm{ch}\big(\big(k_{TN}^{T\mathbb{R}^n}\big)_*[D_{T\mathbb{R}^n}]\big) = \big(k_{TN}^{T\mathbb{R}^n}\big)_* \,\mathrm{ch}[D_{T\mathbb{R}^n}] \\ &= \big(k_{TN}^{T\mathbb{R}^n}\big)_*[T\mathbb{R}^n] = [TN]. \end{aligned} \tag{4.28}$$

Thus, the Chern character of $[D_{TN}]$ is the fundamental class $[TN] \in H_{2n}(TN)$. Let $\sigma \in KK(\mathrm{pt}, TM)$ be given. Then (4.27) and (4.28) imply that

$$\sigma \otimes_{TM} [D_{TM}] = \int_{TN} \mathrm{ch}(\sigma) \wedge \mathrm{ch}(\beta_N). \tag{4.29}$$

The Thom isomorphism $\psi_{TN} : H^*(TM) \to H^*(TN)$ (mapping between compactly supported cohomologies) is an isomorphism of $H^*(TM)$-modules. So we can rewrite the integral (4.29) as

$$\begin{aligned}\int_{TN} \mathrm{ch}(\sigma) \wedge \mathrm{ch}(\beta_N) &= \int_{TM} \psi_{TN}^{-1}(\mathrm{ch}(\sigma) \wedge \mathrm{ch}(\beta_N)) \\ &= \int_{TM} \mathrm{ch}(\sigma) \wedge \psi_{TN}^{-1}(\mathrm{ch}(\beta_N)). \end{aligned} \tag{4.30}$$

To calculate $u := \psi_{TN}^{-1}(\mathrm{ch}(\beta_{TN}))$, we make use of the following diagram:

$$\begin{tikzcd} K^*(TM) \arrow[r, "\psi_{TN}"] \arrow[d, "\mathrm{ch}"'] & K^*(TN) \arrow[r, "(j_{TM}^{TN})^*"] \arrow[d, "\mathrm{ch}"'] & K^*(TM) \arrow[d, "\mathrm{ch}"'] \\ H^*(TM) \arrow[r, "\psi_{TN}"] & H^*(TN) \arrow[r, "(j_{TM}^{TN})^*"] & H^*(TM) \end{tikzcd}$$

Note that in the second line, the composition is equal to the exterior product by the Euler class $e(TN)$. In the above diagram, we have

$$\begin{tikzcd} & \beta_N \arrow[r, mapsto, "(j_{TM}^{TN})^*"] \arrow[d, mapsto, "\mathrm{ch}"'] & \sum_j (-1)^j \bigwedge^j TN \arrow[d, mapsto, "\mathrm{ch}"'] \\ u \arrow[r, mapsto, "\psi_{TN}"] & \mathrm{ch}(\beta_N) \arrow[r, mapsto, "(j_{TM}^{TN})^*"] & u \cdot e(TN) \end{tikzcd}$$

by Lemma 4.24. As the above square commutes by functoriality of the Chern character, and since $TN = \tau_M^* N_{\mathbb{C}}$ and $N_{\mathbb{C}} \oplus (TM \otimes \mathbb{C}) = M \times \mathbb{C}^n$, we obtain

$$u = \frac{\mathrm{ch}\big(\sum_j (-1)^j \bigwedge^j TN\big)}{e(TN)} = \tau_M^* \left(\frac{e(TM)}{\mathrm{ch}\big(\sum_j (-1)^j \bigwedge^j TM)\big)} \right) = \tau_M^*(\mathrm{Todd}(TM \otimes \mathbb{C})).$$

Therefore, together with (4.29) and (4.30) one has

$$\sigma_{TM} \otimes [D_{TM}] = \int_{TM} \mathrm{ch}(\sigma)\, \mathrm{Todd}(TM \otimes \mathbb{C}),$$

and the lemma is proved. □

5. Proof of the fixed point formula

After proving Theorems 2.2 and 2.4 and discussing Kasparov's index theorem, we are ready to prove Theorem 2.16.

We start in Section 5A, by generalising Gysin maps, or wrong-way functoriality maps in K-theory, that play a key role in [Atiyah and Singer 1968a]. We use these generalised Gysin maps in Section 5B to set up the commutative diagrams we need.

We discuss a map defined by evaluating characters at g in Section 5C. Then we introduce a class in the topological K-theory of TM, localised at g, defined by the principal symbol of D. The properties of that class allow us to finish the proof of Theorem 2.16.

5A. ***Gysin maps.*** Let $S \subset M$ be a G-invariant submanifold, with inclusion map $j_S^M : S \hookrightarrow M$. (In the applications of what follows, S will be a connected component of the fixed point set M^g.) Let $N \to S$ be the normal bundle of S in M. The inclusion map $j_{TS}^{TN} : TS \hookrightarrow TN$ induces a map

$$(j_{TS}^{TN})^* : C_0(TN) \to C_0(TS)$$

by restriction. We identify TN with an open neighbourhood of TS in TM, via a G-equivariant embedding $TN \hookrightarrow TM$. In this way, we have the injective map

$$k_{TN}^{TM} : C_0(TN) \hookrightarrow C_0(TM),$$

defined by extending functions by zero.

Definition 5.1. Let A be any G-C^*-algebra. The map

$$(j_{TS}^{TM})_! : KK_G(A, C_0(TS)) \to KK_G(A, C_0(TM))$$

is the composition

$$KK_G(A, C_0(TS)) \xrightarrow{-\otimes_{C_0(TS)}\beta_N} KK_G(A, C_0(TN)) \xrightarrow{(k_{TN}^{TM})_*} KK_G(A, C_0(TM)).$$

Here $\beta_N \in KK_G(TS, TN)$ is the Bott element, as in Definition 4.12.

We also have the usual map

$$(j_{TS}^{TM})^* : KK_G(A, C_0(TM)) \to KK_G(A, C_0(TS)).$$

Lemma 5.2. *The map*

$$(j_{TS}^{TM})^* \circ (j_{TS}^{TM})_! : KK_G(A, C_0(TS)) \to KK_G(A, C_0(TS))$$

is given by the Kasparov product from the right with

$$(j_{TS}^{TN})^* \beta_N \in KK_G(C_0(TS), C_0(TS)).$$

Proof. For all $x \in KK_G(A, C_0(TS))$, functoriality of the Kasparov product implies that

$$\begin{aligned}(j_{TS}^{TM})^* \circ (j_{TS}^{TM})_!(x) &= (j_{TS}^{TM})^* \circ (k_{TN}^{TM})_*(x \otimes_{C_0(TS)} \beta_N) \\ &= x \otimes_{C_0(TS)} ((j_{TS}^{TM})^* \circ (k_{TN}^{TM})_* \beta_N).\end{aligned}$$

Since $(j_{TS}^{TM})^* \circ (k_{TN}^{TM})_* = (j_{TS}^{TN})^*$, the claim follows. □

Lemma 5.3. *For any G-invariant closed subset $X \subset M$, and any G-invariant neighbourhood V of X, the following diagram commutes*:

$$\begin{tikzcd}
KK_G(X, TS) \arrow[r, "-\otimes_{TS}[D_{TS}]"] \arrow[d, "(j_X^V)_*"] & KK_G(X, \mathrm{pt}) \arrow[dd, "(j_X^V)_*"] \\
KK_G(V, TS) \arrow[d, "(j_{TS}^{TM})_!"] & \\
KK_G(V, TM) \arrow[r, "-\otimes_{TM}[D_{TM}]"] & KK_G(V, \mathrm{pt})
\end{tikzcd}$$

Proof. For all $a \in KK_G(X, TS)$, functoriality and associativity of the Kasparov product imply that

$$\begin{aligned}((j_{TS}^{TM})_! \circ (j_X^V)_*(a)) \otimes_{TM} [D_{TM}] &= (k_{TN}^{TM})_*((j_X^V)_*(a) \otimes_{TS} \beta_N) \otimes_{TM} [D_{TM}] \\ &= (j_X^V)_*(a) \otimes_{TS} ((k_{TN}^{TM})_*(\beta_N) \otimes_{TM} [D_{TM}]).\end{aligned}$$

Now $(k_{TN}^{TM})^*[D_{TM}] = [D_{TN}]$, so

$$(k_{TN}^{TM})_*(\beta_N) \otimes_{TM} [D_{TM}] = \beta_N \otimes_{TN} (k_{TN}^{TM})^*[D_{TM}] = \beta_N \otimes_{TN} [D_{TN}] = [D_{TS}],$$

where the last equality was proved in Proposition 4.13. □

5B. *Localisation and Gysin maps.* Let U and V be as in Theorem 2.4. Consider the following diagram:

$$\begin{tikzcd}
KK_G(M, TM) \arrow[r, "\mathrm{index}_t"] \arrow[d, "(k_V^M)^*"'] & KK_G(M, \mathrm{pt}) \arrow[d, "(k_V^M)^*"] \\
KK_G(V, TM) \arrow[r, "-\otimes_{TM}[D_{TM}]"] \arrow[d, "(j_{TM^g}^{TM})^*"'] & KK_G(V, \mathrm{pt}) \\
KK_G(V, TM^g) & \\
KK_G(V, TM^g) \arrow[u, "-\otimes_{TM^g}(j_{TM^g}^{TN})^*\beta_N"] \arrow[uu, bend right, "(j_{TM^g}^{TM})_!"'] & \\
KK_G(\overline{U}, TM^g) \arrow[u, "(j_{\overline{U}}^V)_*"] \arrow[r, "-\otimes_{TM^g}[D_{TM^g}]"] & KK_G(\overline{U}, \mathrm{pt}) \arrow[uuu, "(j_{\overline{U}}^V)_*"']
\end{tikzcd} \tag{5.4}$$

The top part of this diagram commutes because of functoriality of the Kasparov product. The part with the product with $(j_{TM^g}^{TM})^*\beta_N$ in it commutes by Lemma 5.2, applied with $A = C_0(V)$, and S running over the connected components of M^g.

The remaining part of the diagram commutes by Lemma 5.3, applied in a similar way with S a connected component of M^g, and $X = \overline{U}$.

Diagram (5.4) can be extended as follows:

$$\begin{tikzcd}
 & & KK_G(M, TM) \arrow[r, "\mathrm{index}_t"] \arrow[d, "(k_V^M)^*"'] & KK_G(M, \mathrm{pt}) \arrow[d, "(k_V^M)^*"'] \\
KK_G(\mathrm{pt}, TM) \arrow[d, "(j_{TM^g}^{TM})^*"'] & KK_G(\overline{U}, TM) \arrow[l, "p_*^{\overline{U}}"'] \arrow[r, "(j_{\overline{U}}^V)_*"] \arrow[d, "(j_{TM^g}^{TM})^*"'] & KK_G(V, TM) \arrow[r, "-\otimes_{TM}[D_{TM}]"] \arrow[d, "(j_{TM^g}^{TM})^*"'] & KK_G(V, \mathrm{pt}) \\
KK_G(\mathrm{pt}, TM^g) \arrow[d, "-\otimes_{TM^g}[D_{TM^g}]"'] & KK_G(\overline{U}, TM^g) \arrow[l, "p_*^{\overline{U}}"'] \arrow[r, "(j_{\overline{U}}^V)_*"] \arrow[d, "-\otimes_{TM^g}[D_{TM^g}]"'] & KK_G(V, TM^g) \arrow[r, bend left, "(j_{TM^g}^{TM})_!"] & \\
KK_G(\mathrm{pt}, TM^g) \arrow[u, "-\otimes_{TM^g}(j_{TM^g}^{TN})^*\beta_N"] & KK_G(\overline{U}, TM^g) \arrow[l, "p_*^{\overline{U}}"'] \arrow[r, "(j_{\overline{U}}^V)_*"] \arrow[u, "-\otimes_{TM^g}(j_{TM^g}^{TN})^*\beta_N"] & KK_G(V, TM^g) \arrow[u, "-\otimes_{TM^g}(j_{TM^g}^{TN})^*\beta_N"] & \\
KK_G(\mathrm{pt}, \mathrm{pt}) & KK_G(\overline{U}, \mathrm{pt}) \arrow[l, "p_*^{\overline{U}}"] \arrow[uurr, bend right, "(j_{\overline{U}}^V)_*"'] & &
\end{tikzcd} \tag{5.5}$$

The right-hand part of this diagram is diagram (5.4), and hence commutes. The other parts commute by functoriality of KK-theory and the Kasparov product.

Theorem 2.4 implies that the maps $(j_{\overline{U}}^V)_*$ become invertible after localisation at g. We will also use inverses of the localised classes

$$((j_{TM^g}^{TN})^*\beta_N)_g \in KK_G(TM^g, TM^g)_g. \tag{5.6}$$

Lemma 5.7. *The element* (5.6) *is invertible.*

Proof. By Lemma 4.24, we have

$$(j_{TM^g}^{TN})^*\beta_N = \left[\tau_{M^g}^* \bigwedge N_{\mathbb{C}}\right].$$

Atiyah and Segal [1968, Lemma 2.7] showed that $\left[\bigwedge N_{\mathbb{C}}\right]$ is invertible in $K_G^0(M^g)_g$. The map

$$\tau_{M^g}^* : K_G^0(M^g) \to KK_G(TM^g, TM^g)$$

sending a class $[E] \in K_G^0(M^g)$ to $[\tau_{M^g}^* E]$ is a unital ring homomorphism. Hence, so is its localisation at g. Therefore, the class

$$\left[\tau_{M^g}^* \bigwedge N_{\mathbb{C}}\right]_g = (\tau_{M^g}^*)_g \left[\bigwedge N_{\mathbb{C}}\right]_g \in KK_G(TM^g, TM^g)_g$$

is invertible. $\square$

5C. *Evaluation.* Let X and Y be locally compact Hausdorff spaces with *trivial* actions by a compact group G. Then the exterior Kasparov product

$$KK(X, Y) \times KK_G(\mathrm{pt}, \mathrm{pt}) \to KK_G(X, Y)$$

defines an isomorphism

$$KK(X, Y) \otimes R(G) \cong KK_G(X, Y). \tag{5.8}$$

If X is a point, this is a classical fact. We will also apply this isomorphism to the class $[D_{TM^g}] \in KK_G(TM^g, \mathrm{pt})$. There it is trivial, since G acts trivially on the Hilbert space in question. In the only other case where we will use the isomorphism (5.8), we will have $X = Y$, and this space has finitely many connected components. (To be precise, we will have $X = Y = TM^g$.) Let us work out the isomorphism explicitly in that case, for the cycles we will apply it to. These are G-equivariant Kasparov $(C_0(X), C_0(X))$-cycles of the form $(\Gamma_0(E), F, \pi)$, where $E \to X$ is a vector bundle (of finite rank). Let $a \in KK_G(X, X)$ be the class of a cycle of this form, and let $b \in KK(X, X)$ be the class defined by the same cycle, where the group action is ignored. As G acts trivially on X, each fibre of E is a representation space of G. Suppose for simplicity that X is connected; the general case follows by applying the arguments to its connected components. (This works since there are finitely many of them.) Since X is connected, the representations by G on all fibres of E are equivalent. Let V be any one of these fibres, viewed as a representation space of G. Denote by 1_G the ring identity of $R(G)$, i.e., the trivial representation of G. Let $E_0 := X \times V \to X$ be the trivial bundle with fibre V. Consider the representations

$$\pi_X^X : C_0(X) \to \mathcal{B}(C_0(X)),$$
$$\pi_X^{E_0} : C_0(X) \to \mathcal{B}(\Gamma_0(E_0)),$$

defined by pointwise multiplication. Then

$$\begin{aligned}\big(\big[C_0(X), 0, \pi_X^X\big] \otimes [V]\big) + (b \otimes 1_G)& \\ = \big(\big[\Gamma_0(E_0), 0, \pi_X^{E_0}\big] \otimes 1_G\big) + a \in KK_G(X, X).&\end{aligned} \tag{5.9}$$

In fact, both sides of (5.9) are represented by the cycle

$$\big(\Gamma_0(E_0 \oplus E), 0 \oplus F, \pi_X^{E_0} \oplus \pi\big), \tag{5.10}$$

but, initially, with *different* G-actions. Namely, for the left-hand side of (5.9), G acts on the first summand E_0 in (5.10), while for the right-hand side of (5.9), G acts on the second summand E in (5.10). As G acts trivially on X, representations of G commute with those of $C_0(X)$. Since, in addition, F is G-invariant, these two actions by G can be connected by a rotation homotopy, so (5.9) follows. In that equality, a is represented as an element of $KK(X, Y) \otimes R(G)$.

In general, using (5.8), one can apply the evaluation $\mathrm{ev}_g = 1 \otimes \mathrm{ev}_g$ as a map

$$\mathrm{ev}_g : KK_G(X, Y) \to KK(X, Y) \otimes \mathbb{C}. \tag{5.11}$$

This map is compatible with localisation at g, in the sense that the following diagram commutes:

$$\begin{tikzcd} KK_G(X, Y) \arrow[r, "\mathrm{ev}_g"] \arrow[d] & KK(X, Y) \otimes \mathbb{C} \\ KK_G(X, Y)_g \arrow[ur, "(\mathrm{ev}_g)_g"'] & \end{tikzcd}$$

If $a \in KK_G(X, Y)$, we will also write

$$a(g) := \mathrm{ev}_g(a) \in KK(X, Y) \otimes \mathbb{C}.$$

The evaluation map (5.11) is compatible with Kasparov products. This follows from the facts that the isomorphism (5.8) is compatible with the product, that Kasparov products in $R(G)$ coincide with tensor products of representations, and that the character of the tensor product of two finite-dimensional representations is the product of the characters of the individual representations.

Hence, we can attach the following commutative diagram to the lower left-hand side of (5.5):

$$\begin{tikzcd} KK(\mathrm{pt}, TM^g) \otimes \mathbb{C} & KK_G(\mathrm{pt}, TM^g) \arrow[l, "\mathrm{ev}_g"'] \\ KK(\mathrm{pt}, TM^g) \otimes \mathbb{C} \arrow[u, "(-\otimes_{TM^g}(j^{TM}_{TM^g})^*\beta_N(g))\otimes 1"] \arrow[d, "(-\otimes_{TM^g}[D_{TM^g}])\otimes 1"'] & KK_G(\mathrm{pt}, TM^g) \arrow[l, "\mathrm{ev}_g"'] \arrow[u, "-\otimes_{TM^g}(j^{TN}_{TM^g})^*\beta_N"'] \arrow[d, "-\otimes_{TM^g}[D_{TM^g}]"'] \\ \mathbb{C} & KK_G(\mathrm{pt}, \mathrm{pt}) \arrow[l, "\mathrm{ev}_g"] \end{tikzcd} \tag{5.12}$$

Here, $[D_{TM^g}] \in KK(TM^g, \mathrm{pt})$ is identified with $[D_{TM^g}] \otimes 1 \in KK(TM^g, \mathrm{pt}) \otimes R(G)$, so that $\mathrm{ev}_g([D_{TM^g}]) = [D_{TM^g}] \otimes 1$. In particular, when $M^g = \mathrm{pt}$, the vertical map on the lower left corner is the identity.

By Lemma 4.26 and compactness of M^g, the map

$$- \otimes_{TM^g} [D_{TM^g}] : KK(\mathrm{pt}, TM^g) \to KK(\mathrm{pt}, \mathrm{pt})$$

is the Atiyah–Singer topological index map $\mathrm{index}_t^{\mathrm{AS}}$. We will use the same notation for its extension to a map $KK(\mathrm{pt}, TM^g) \otimes \mathbb{C} \to \mathbb{C}$.

Using commutativity of (5.5) and (5.12), and invertibility of the localised maps $((j^V_{\overline{U}})_*)_g$ and classes (5.6), we obtain the commutative diagram

$$\begin{tikzcd}
 & KK_G(M, TM)_g \arrow[r, "(\mathrm{index}_t)_g"] \arrow[d, "(p_*^{\overline{U}})_g \circ ((j_{\overline{U}}^V)_*)_g^{-1} \circ (k_V^M)_g^*"] & KK_G(M, \mathrm{pt})_g \arrow[dd, "((j_{\overline{U}}^V)_*)_g^{-1} \circ (k_V^M)_g^*"] \\
 & KK_G(\mathrm{pt}, TM)_g \arrow[d, "(j_{TM^g}^{TM})^*"] & \\
KK(\mathrm{pt}, TM^g) \otimes \mathbb{C} \arrow[d, "\mathrm{index}_t^{\mathrm{AS}}(-\otimes_{TM^g}((j_{TM^g}^{TN})^*\beta_N)^{-1}(g))"] & KK_G(\mathrm{pt}, TM^g)_g \arrow[l, "(\mathrm{ev}_g)_g"'] \arrow[d, "-\otimes_{TM^g}((j_{TM^g}^{TN})^*\beta_N)_g^{-1} \otimes_{TM^g}[D_{TM^g}]_g"] & \\
\mathbb{C} & KK_G(\mathrm{pt}, \mathrm{pt})_g \arrow[l, "(\mathrm{ev}_g)_g"] & KK_G(\overline{U}, \mathrm{pt})_g \arrow[l, "(p_*^{\overline{U}})_g"]
\end{tikzcd} \tag{5.13}$$

5D. *The g-symbol class.* Recall that in (4.4) we defined the class

$$[\sigma_D] \in KK_G(M, TM).$$

The last ingredient of the proof of Theorem 2.16 is a class defined by σ_D in the topological K-theory of TM, localised at g. In Section 7, we will describe this class more explicitly, and use it to obtain another expression for the g-index.

Definition 5.14. The *g-symbol class* of D is the class σ_g^D in the localised topological K-theory of TM defined by

$$\sigma_g^D := (p_*^{\overline{U}})_g \circ ((j_{\overline{U}}^V)_*)_g^{-1} \circ (k_V^M)_g^*[\sigma_D]_g \in KK_G(\mathrm{pt}, TM)_g. \tag{5.15}$$

The g-symbol class generalises the usual symbol class in the compact case.

Lemma 5.16. *If M is compact, then σ_g^D is the localisation at g of the usual class of σ_D in $KK_G(\mathrm{pt}, TM)$.*

Proof. If M is compact, then we can choose $U = V = M$. Then, since the map $p^M : M \to \mathrm{pt}$ is proper, we have

$$\sigma_g^D = (p_*^M[\sigma_D])_g,$$

which is the usual symbol class by Lemma 4.5. □

We now prove some properties of the g-symbol class that will be used in the proof of Theorem 2.16. As before, we write $\tilde{\sigma}_D := \sigma_D/\sqrt{\sigma_D^2 + 1}$.

Lemma 5.17. *The class*

$$(k_V^M)_g^*[\sigma_D]_g \in KK_G(V, TM)_g$$

is the localisation at g of the class

$$[\sigma_D|_V]_{TM} := [\Gamma_0(\tau_V^*(E|_V)), \tilde{\sigma}_D|_{TV}, \pi_V] \in KK_G(V, TM).$$

Here, the $C_0(TM)$-valued inner product on $\Gamma_0(E|_V)$ is defined by the natural $C_0(TV)$-valued inner product, composed with the inclusion k_{TV}^{TM}.

Proof. The class

$$(k_V^M)^*[\sigma_D] \in KK_G(V, TM)$$

is represented by the Kasparov cycle

$$(\Gamma_0(\tau_M^* E), \tilde{\sigma}_D, (k_V^M)^*\pi_M) = (\Gamma_0(\tau_V^*(E|_V)), \tilde{\sigma}_D|_{TV}, \pi_V) \oplus (\Gamma_0(\tau_{M\setminus V}^*(E|_{M\setminus V})), \tilde{\sigma}_D|_{TM\setminus TV}, 0).$$

The second summand on the right-hand side is a degenerate cycle, so the claim follows. □

Consider the class

$${}_{\overline{U}}[\sigma_D|_{TM^g}] := \left[\Gamma_0(\tau_{M^g}^*(E|_{M^g})), \tilde{\sigma}_D|_{TM^g}, \left(j_{M^g}^{\overline{U}}\right)_* \pi_{M_g}\right] \in KK_G(\overline{U}, TM^g).$$

Lemma 5.18. *We have*

$$\left(j_{\overline{U}}^V\right)_* \left({}_{\overline{U}}[\sigma_D|_{TM^g}]\right) = (j_{TM^g}^{TM})^*[\sigma_D|_V]_{TM} \in KK_G(V, TM^g).$$

Proof. By definition,

$$(j_{TM^g}^{TM})^*[\sigma_D|_V]_{TM} = \left[\Gamma_0(\tau_V^*(E|_V)) \otimes_{j_{TM^g}^{TM}} C_0(TM^g), \tilde{\sigma}_D|_V \otimes 1, \pi_V \otimes 1\right].$$

The map

$$\Gamma_0(\tau_V^*(E|_V)) \otimes_{j_{TM^g}^{TM}} C_0(TM^g) \to \Gamma_0(\tau_{M^g}^*(E|_{M^g}))$$

that maps $s \otimes \varphi$ to $\varphi s|_{TM^g}$, for $s \in \Gamma_0(\tau_V^*(E|_V))$ and $\varphi \in C_0(TM^g)$, is an isomorphism of Hilbert $C_0(TM^g)$-modules. It intertwines the operators $\tilde{\sigma}_D|_V \otimes 1$ and $\tilde{\sigma}_D|_{TM^g}$, and the representations $\pi_V \otimes 1$ and

$$(j_{M^g}^V)_* \pi_{M^g} = \left(j_{\overline{U}}^V\right)_* \left(j_{M^g}^{\overline{U}}\right)_* \pi_{M^g}.$$

The lemma is then proved. □

Proposition 5.19. *The class*

$$(j_{TM^g}^{TM})_g^* \sigma_g^D \in KK_G(\mathrm{pt}, TM^g)_g$$

is the localisation at g of the usual class $[\sigma_D|_{TM^g}]$ in the equivariant topological K-theory of TM^g.

Proof. By commutativity of (the top left part of) diagram (5.5), we have

$$(j_{TM^g}^{TM})_g^* \sigma_g^D = \left(p_*^{\overline{U}}\right)_g \circ \left(\left(j_{\overline{U}}^V\right)_*\right)_g^{-1} \circ (j_{TM^g}^{TM})_g^* \circ (k_V^M)_g^* [\sigma_D]_g.$$

By Lemma 5.17, we have

$$(k_V^M)_g^* [\sigma_D]_g = ([\sigma_D|_V]_{TM})_g.$$

By Lemma 5.18 we have

$$\left(\left(j_{\overline{U}}^{V}\right)_*\right)_g^{-1} \circ (j_{TM^g}^{TM})_g^*([\sigma_D|_V]_{TM})_g =_{\overline{U}} [\sigma_D|_{TM^g}]_g.$$

By Lemma 4.5, we have

$$p_*^{\overline{U}}\left({}_{\overline{U}}[\sigma_D|_{TM^g}]\right) = [\sigma_D|_{TM^g}] \in KK_G(\mathrm{pt}, TM^g).$$

So the claim follows. □

We have now finished all preparation needed to prove Theorem 2.16.

Proof of Theorem 2.16. Using Kasparov's index theorem, Theorem 4.6, and commutativity of (5.13), we find that

$$\begin{aligned}\mathrm{index}_g(D) &= (\mathrm{ev}_g)_g \circ \left(p_*^{\overline{U}}\right)_g \circ \left(\left(j_{\overline{U}}^{V}\right)_*\right)_g^{-1} \circ (k_V^M)_g^*[D] \\ &= (\mathrm{ev}_g)_g \circ \left(p_*^{\overline{U}}\right)_g \circ \left(\left(j_{\overline{U}}^{V}\right)_*\right)_g^{-1} \circ (k_V^M)_g^* \circ (\mathrm{index}_t)_g[\sigma_D]_g \\ &= \mathrm{index}_t^{\mathrm{AS}}\left(((j_{TM^g}^{TM})^*\sigma_g^D)(g) \otimes_{TM^g} ((j_{TM^g}^{TN})^*\beta_N)^{-1}(g)\right).\end{aligned}$$

By Lemma 4.24 and Proposition 5.19, the latter expression equals

$$\mathrm{index}_t^{\mathrm{AS}}\left([\sigma_D|_{TM^g}](g) \otimes_{TM^g} \left[\bigwedge N_{\mathbb{C}}\right]^{-1}(g)\right).$$

Furthermore,

$$[\sigma_D|_{TM^g}](g) \otimes_{TM^g} \left[\tau_{M^g}^* \bigwedge N_{\mathbb{C}}\right]^{-1}(g) = [\sigma_D|_{TM^g}](g) \cdot \left[\bigwedge N_{\mathbb{C}}\right]^{-1}(g),$$

where the dot means the right $K_G^0(M^g)$-module structure of $K_G^0(TM^g)$. We conclude that

$$\mathrm{index}_g(D) = \mathrm{index}_t^{\mathrm{AS}}\left([\sigma_D|_{TM^g}](g) \cdot \left[\bigwedge N_{\mathbb{C}}\right]^{-1}(g)\right).$$

Theorem 2.16 now follows from the definition of the topological index map (4.25), and multiplicativity of the Chern character. □

5E. *The index pairing.* The arguments used to prove Theorem 2.16 also imply Theorem 2.18 about the index pairing. In fact, the parts of the proof of Theorem 2.16 about localisation in the first entry of KK-theory are not needed in the proof of Theorem 2.18.

The key step is a localisation property of the K-homology class of D, localised at g.

Proposition 5.20. *We have*

$$[D]_g = (j_{TM^g}^{TM})_g^*[\sigma_D]_g \otimes_{TM^g} \left[\tau_{M^g}^* \bigwedge N_{\mathbb{C}}\right]_g^{-1} \otimes_{TM^g} [D_{TM^g}]_g \in KK_G(M, \mathrm{pt})_g.$$

Proof. Lemmas 5.2 and 5.3 imply that the following diagram commutes:

$$\begin{tikzcd}
KK_G(M, TM) \arrow[r, "-\otimes_{TM}[D_{TM}]"] \arrow[d, "(j^{TM}_{TM^g})^*"'] & KK_G(M, \mathrm{pt}) \\
KK_G(M, TM^g) & KK_G(M, TM^g) \arrow[l, "-\otimes_{TM^g}(j^{TN}_{TM^g})^*\beta_N"] \arrow[ul, "(j^{TM}_{TM^g})_!"'] \arrow[u, "-\otimes_{TM^g}[D_{TM^g}]"']
\end{tikzcd}$$

Therefore, the claim follows from Lemmas 4.24 and 5.7, and Theorem 4.6. □

Proof of Theorem 2.18. Let $[F] \in KK_G(\mathrm{pt}, M)$ be as in Section 2E. By compatibility of the Kasparov product with localisation and evaluation, Proposition 5.20 implies that

$$\begin{aligned}
&([F] \otimes_M [D])(g) \\
&\qquad = ([F]_g \otimes_M [D]_g)(g) \\
&\qquad = ([F]_g \otimes_M (j^{TM}_{TM^g})^*_g[\sigma_D]_g)(g) \otimes_{TM^g} \left[\tau^*_{M^g} \textstyle\bigwedge N_{\mathbb{C}}\right](g)^{-1} \otimes_{TM^g} [D_{TM^g}](g).
\end{aligned}$$

Now

$$([F]_g \otimes_M (j^{TM}_{TM^g})^*_g[\sigma_D]_g)(g) = [\tau^*_{M^g}(F|_{M^g})](g) \otimes [\sigma_D|_{TM^g}](g) \in KK(\mathrm{pt}, TM^g) \otimes \mathbb{C},$$

where on the right-hand side, the tensor product denotes the ring structure on the topological K-theory of TM^g. Therefore, and because $[D_{TM^g}](g) = [D_{TM^g}] \otimes 1$ is in $KK(TM^g, \mathrm{pt}) \otimes \mathbb{C}$, the claim follows from Lemma 4.26. □

6. Examples and applications

The g-index was defined in terms of KK-theory, but Theorem 2.16 allows us to express it entirely in cohomological terms. Using this theorem, we can compute the g-index explicitly in examples, and show how it is related to other indices.

For finite fixed point sets, Theorem 2.16 has a simpler form, as discussed in Section 6A. In Section 6B, we give a linearisation theorem for the g-index of a twisted Dolbeault–Dirac operator on a complex manifold, in the case of a finite fixed point set. We then work out the example of the Dolbeault–Dirac operator on the complex plane, acted on by the circle, in Section 6C. An illustration of the linearisation theorem is given in Section 6D, where we apply it to the two-sphere, to decompose the usual equivariant index. In Section 6E, we realise characters of discrete series representations of semisimple Lie groups on regular points of a maximal torus, in terms of the g-index. For Fredholm operators, and in particular Callias-type deformations of Dirac operators, we describe the relation between the g-index and the character of the action by g on the kernel of such an operator, in Section 6F. We then give a relation with an index studied by Braverman in Section 6G, and a relative index theorem along the lines of work by Gromov and

Lawson in Section 6H. In Section 6I, we mention some geometric consequences of the vanishing or nonvanishing of the g-index of a Hodge–Dirac or Spin-Dirac operator.

6A. ***Finite fixed point sets.*** If the fixed point set M^g is zero-dimensional, then $TM^g = M^g$, τ_{M^g} is the identity map, $\mathrm{Todd}(TM^g \otimes \mathbb{C})$ is trivial and

$$\mathrm{ch}([\sigma_D|_{TM^g}](g)) = \mathrm{Tr}(g|_{E^+}) - \mathrm{Tr}(g|_{E^-}).$$

Furthermore, since M^g only consists of isolated points, we have

$$K^0(M^g) = \bigoplus_{m \in M^g} \mathbb{Z} = H^*(M^g),$$

and the Chern character is the identity map. So we now have, at a fixed point $m \in M^g$,

$$\mathrm{ch}\big(\big[\textstyle\bigwedge N_{\mathbb{C}}\big](g)\big)_m = \mathrm{ch}\big(\big[\textstyle\bigwedge TM_{\mathbb{C}}|_{M^g}\big](g)\big)_m = \det_{\mathbb{R}}(1 - g|_{T_m M}).$$

The last equality is obtained by evaluating the virtual character of $\bigwedge T_m M_{\mathbb{C}}$ at g, so one obtains

$$\mathrm{Tr}_{\mathbb{C}}(g|_{\bigwedge^{\mathrm{even}} T_m M_{\mathbb{C}}}) - \mathrm{Tr}_{\mathbb{C}}(g|_{\bigwedge^{\mathrm{odd}} T_m M_{\mathbb{C}}}).$$

Therefore, Theorem 2.16 implies the following generalisation of Atiyah and Bott's fixed point theorem [1968, Theorem A] to noncompact manifolds, but for compact G.

Corollary 6.1. *When M^g is a finite set of points,*

$$\mathrm{index}_g(D) = \sum_{m \in M^g} \frac{\mathrm{Tr}(g|_{E_m^+}) - \mathrm{Tr}(g|_{E_m^-})}{\det_{\mathbb{R}}(1 - g^{-1}|_{T_m M})}. \tag{6.2}$$

Remark 6.3. In the statement of the Atiyah–Bott fixed point theorem, the denominator is $|\det_{\mathbb{R}}(1 - g|_{T_m M})|$. In our case, g is contained in a compact group G, so the real eigenvalues of g are 1 or -1. Thus $\det_{\mathbb{R}}(1 - g^{-1}|_{T_m M})$ is always positive. See also page 186 in [Berline et al. 1992]. Also, the fact that g acts orthogonally on $T_m M$ implies that $\det_{\mathbb{R}}(1 - g^{-1}|_{T_m M}) = \det_{\mathbb{R}}(1 - g|_{T_m M})$.

Now suppose that M is a complex manifold, and g is holomorphic. Let $F \to M$ be a holomorphic vector bundle, and consider the Dolbeault–Dirac operator $\bar\partial_F + \bar\partial_F^*$ on M, coupled to F.

Corollary 6.4. *If M^g is a finite set of points, then*

$$\mathrm{index}_g(\bar\partial_F + \bar\partial_F^*) = \sum_{m \in M^g} \frac{\mathrm{Tr}_{\mathbb{C}}(g|_{F_m})}{\det_{\mathbb{C}}(1 - g^{-1}|_{T_m M})}. \tag{6.5}$$

For equivalent expressions, note that

$$\det_{\mathbb{C}}(1-g^{-1}|_{T_m^{1,0}M}) = \det_{\mathbb{C}}(1-g^{-1}|_{T_mM}) = \det_{\mathbb{C}}(1-g|_{T_m^{0,1}M})$$

in (6.5).

Proof. In Theorem 4.12 of [Atiyah and Bott 1968], it is shown that in this situation, the right-hand side of (6.2) equals the right-hand side of (6.5). The key observation is that the supertrace of $g|_{\bigwedge^*(T^{0,1}M)}$ is cancelled by the second factor in

$$\det_{\mathbb{R}}(1-g^{-1}|_{T_mM}) = \det_{\mathbb{C}}(1-g^{-1}|_{T_m^{1,0}M})\det_{\mathbb{C}}(1-g^{-1}|_{T_m^{0,1}M}).$$

(See also [Berline et al. 1992, Corollary 6.8].) □

6B. *A holomorphic linearisation theorem.* A tool used in some index problems is a *linearisation theorem*, relating an index to indices on vector spaces. See for example Chapter 4 of [Guillemin et al. 2002] and Theorem 7.2 in [Braverman 2002]. A version for Callias-type operators can be deduced from Theorem 2.16 in [Braverman and Shi 2016]. In those references, cobordism arguments are used to prove linearisation theorems. We will use the excision property of the g-index to obtain an analogous result. (So we do not use Theorem 2.16 here.) We will state and prove this result in the setting of Corollary 6.4, where M is a complex manifold, D is the Dolbeault–Dirac operator coupled to a holomorphic vector bundle $F \to M$, and M^g is finite. A more general statement, where M^g is not finite or D is not a Dolbeault–Dirac operator, is possible, but would be less explicit.

Under these assumptions, for any $m \in M^g$, let $\bar\partial^{T_mM}$ be the Dolbeault operator on the complex vector space T_mM.

Corollary 6.6 (holomorphic linearisation theorem). *We have*

$$\operatorname{index}_g(\bar\partial_F + \bar\partial_F^*) = \sum_{m\in M^g} \operatorname{Tr}_{\mathbb{C}}(g|_{F_m}) \operatorname{index}_g(\bar\partial^{T_mM} + (\bar\partial^{T_mM})^*).$$

Proof. By Lemma 2.11, the g-index of $\bar\partial_F + \bar\partial_F^*$ equals the g-index of the Dolbeault–Dirac operator on the union over $m \in M^g$ of the tangent spaces T_mM, coupled to the vector bundle which on every space T_mM is trivial with fibre F_m. It follows directly from the definition that the g-index is additive with respect to disjoint unions. Hence

$$\begin{aligned}\operatorname{index}_g(\bar\partial_F + \bar\partial_F^*) &= \sum_{m\in M^g} \operatorname{index}_g(\bar\partial^{T_mM} \otimes 1_{F_m} + (\bar\partial^{T_mM})^* \otimes 1_{F_m}) \\ &= \sum_{m\in M^g} \operatorname{Tr}_{\mathbb{C}}(g|_{F_m}) \operatorname{index}_g(\bar\partial^{T_mM} + (\bar\partial^{T_mM})^*).\end{aligned}$$ □

An example on computing and explicitly realising an index of the form

$$\operatorname{index}_g(\bar\partial^{T_mM} + (\bar\partial^{T_mM})^*),$$

as in Corollary 6.6, is given in the next subsection. An example showing that the linearisation theorem gives a natural result if M is compact is given in Section 6D.

6C. ***The circle acting on the plane.*** Consider the usual action by the circle $\mathbb{T}^1 = \mathrm{U}(1)$ on the complex plane $\mathbb{C}$, and the (untwisted) Dolbeault–Dirac operator $\bar{\partial} + \bar{\partial}^*$ on $\mathbb{C}$. We will compute the distribution Θ on $\mathbb{T}^1$ given by the function

$$g \mapsto \mathrm{index}_g(\bar{\partial} + \bar{\partial}^*). \tag{6.7}$$

This function is defined on the set of elements $g \in \mathbb{T}^1$ with dense powers, i.e., the elements of the form $g = e^{\sqrt{-1}\alpha}$, where $\alpha \in \mathbb{R} \setminus 2\pi\mathbb{Q}$. So the function is defined almost everywhere.

By Corollary 6.4, we have for such g

$$\mathrm{index}_g(\bar{\partial} + \bar{\partial}^*) = \frac{1}{1-g^{-1}}.$$

So the function (6.7) is given by $g \mapsto 1/(1-g^{-1})$ almost everywhere. One can deduce that the sum of functions

$$\sum_{k=0}^{\infty} (g \mapsto g^{-k}) \tag{6.8}$$

converges as a distribution on $\mathbb{T}^1$ to Θ.

This allows us to describe the g-index of $\bar{\partial}+\bar{\partial}^*$ in terms of its kernel. Indeed, consider the Euclidean density $dz = dx\,dy$ on $\mathbb{C}$, and the corresponding space $L^2(\mathbb{C})$. Let $\mathcal{O}(\mathbb{C})$ be the space of holomorphic functions on $\mathbb{C}$. Let $\psi \in C^\infty(\mathbb{C})$ be a positive, $\mathbb{T}^1$-invariant function. Let $L^2(\mathbb{C}, \psi)$ be the completion of $C_c^\infty(\mathbb{C})$ to a Hilbert space with respect to the inner product

$$(f_1, f_2)_\psi := (\psi f_1, \psi f_2)_{L^2(\mathbb{C})}. \tag{6.9}$$

Let π be the representation of $\mathbb{T}^1$ in $L^2(\mathbb{C}, \psi)$ given by

$$(\pi(g)f)(z) = f(g^{-1}z),$$

for all $g \in \mathbb{T}^1$, $f \in L^2(\mathbb{C}, \psi)$ and $z \in \mathbb{C}$.

Set

$$\mathcal{O}_{L^2}(\mathbb{C}, \psi) := \mathcal{O}(\mathbb{C}) \cap L^2(\mathbb{C}, \psi).$$

For $k \in \mathbb{Z}_{\geq 0}$, let $e^k \in \mathcal{O}(\mathbb{C})$ be the function $z \mapsto z^k$. Then for all $k \in \mathbb{Z}_{\geq 0}$ and $z \in \mathbb{C}$,

$$\pi(g)e^k = g^{-k}e^k. \tag{6.10}$$

Suppose ψ was chosen so that $e^k \in L^2(\mathbb{C}, \psi)$ for all k. For example, one can take $\psi(z) = e^{-|z|^2/2}$.

Let $\Omega^{0,*}_{L^2}(\mathbb{C})$ be the Hilbert space of square-integrable forms of type $(0,*)$. Let $\Omega^{0,*}_{L^2}(\mathbb{C},\psi)$ be the analogous Hilbert space with the inner product weighted by ψ as in (6.9). Set

$$\ker_{L^2,\psi}(\bar\partial+\bar\partial^*)^\pm := \ker(\bar\partial+\bar\partial^*)^\pm \cap \Omega^{0,*}_{L^2}(\mathbb{C},\psi).$$

We can realise the distribution Θ given by the g-indices of $\bar\partial+\bar\partial^*$ in terms of the representation of $\mathbb{T}^1$ in this space.

Proposition 6.11. *The restriction of the representation π of $\mathbb{T}^1$ to $\ker_{L^2,\psi}(\bar\partial+\bar\partial^*)^\pm$ has a distributional character $\chi^\pm$, and we have*

$$\Theta = \chi^+ - \chi^- \in \mathcal{D}'(\mathbb{T}^1).$$

Proof. First note that

$$\ker(\bar\partial+\bar\partial^*)^+ = \mathcal{O}(\mathbb{C}),$$
$$\ker(\bar\partial+\bar\partial^*)^- = 0.$$

So we only need to consider the even part of $\ker_{L^2,\psi}(\bar\partial+\bar\partial^*)$, which equals

$$\ker_{L^2,\psi}(\bar\partial+\bar\partial^*)^+ = \mathcal{O}_{L^2}(\mathbb{C},\psi). \tag{6.12}$$

The functions $\{e^k\}_{k\geq 0}$ form an orthogonal basis of $\mathcal{O}_{L^2}(\mathbb{C},\psi)$. By (6.10), the character of the representation π on the space (6.12) equals the series (6.8), which converges to Θ. □

Remark 6.13. The $L^2(\mathbb{C},\psi)$-kernel of $\bar\partial+\bar\partial^*$ can be identified as the L^2-kernel of a deformed operator. For example, let $\psi(z)=e^{-|z|^2/2}$. Recall that $\bar\partial+\bar\partial^*$ is an operator on $\Omega^{0,*}(\mathbb{C})$, given by

$$\bar\partial+\bar\partial^* = c(dz)\frac{\partial}{\partial z} + c(d\bar z)\frac{\partial}{\partial \bar z},$$

where now $c(d\bar z)=\frac{1}{\sqrt2}\operatorname{ext}(d\bar z)$ and $c(dz)=-\frac{1}{\sqrt2}\operatorname{int}(dz)$. (See [Berline et al. 1992, Section 3.6].) Set

$$b := \frac{1}{2} z c(d\bar z).$$

Then $b^* = -\frac{1}{2}\bar z c(dz)$. We have the deformed operator

$$\bar\partial + b = c(d\bar z)\Big(\frac{\partial}{\partial \bar z}+\frac{z}{2}\Big) : \Omega^{0,0}(\mathbb{C}) \to \Omega^{0,1}(\mathbb{C}),$$
$$(\bar\partial+b)^* = c(dz)\Big(\frac{\partial}{\partial z}-\frac{\bar z}{2}\Big) : \Omega^{0,1}(\mathbb{C}) \to \Omega^{0,0}(\mathbb{C}).$$

The operator $U:\Omega^{0,*}(\mathbb{C})\to\Omega^{0,*}(\mathbb{C},\psi)$ given by $U(\alpha)=\psi^{-1}\alpha$ is a unitary isomorphism. We have

$$\bar\partial U(f) = \bar\partial(\psi^{-1}f) = \psi^{-1}\Big(\bar\partial+\frac{z}{2}\Big)f = U\Big(\Big(\bar\partial+\frac{z}{2}\Big)f\Big).$$

Similarly, U intertwines $\bar{\partial}^*$ and $(\bar{\partial}+b)^*$. It then follows that

$$\ker_{L^2}(\bar{\partial}+b) \cong \ker_{L^2,\psi}(\bar{\partial}),$$
$$\ker_{L^2}(\bar{\partial}+b)^* \cong \ker_{L^2,\psi}(\bar{\partial}^*) = 0.$$

6D. ***The circle acting on the two-sphere.*** As in Section 6C, we consider the circle group $\mathbb{T}^1$, this time acting by rotations on the two-sphere S^2. In this compact setting, the usual index theory, and the Atiyah–Segal–Singer theorem apply. But we can use the g-index to decompose indices in this case.

We embed $\mathbb{T}^1 \cong \mathrm{SO}(2)$ into $\mathrm{SO}(3)$ in the top-left corner. Then $S^2 = \mathrm{SO}(3)/\mathbb{T}^1$. Identifying this space with $\mathbb{P}^1(\mathbb{C})$, we obtain a complex structure on it. Fix $n \in \mathbb{Z}_{\geq 0}$. Let $\mathbb{C}_n$ be the space of complex numbers, on which $\mathbb{T}^1$ acts by

$$g \cdot z = g^n z,$$

for $g \in \mathbb{T}^1$ and $z \in \mathbb{C}_n$. We have the line bundle

$$L_n := \mathrm{SO}(3) \times_{\mathbb{T}^1} \mathbb{C}_n \to S^2.$$

Let $\bar{\partial}_n + \bar{\partial}_n^*$ be the Dolbeault–Dirac operator on S^2, coupled to L_n. Since S^2 is compact, we have the equivariant index

$$\mathrm{index}_{\mathrm{SO}(3)}(\bar{\partial}_n + \bar{\partial}_n^*) \in R(\mathrm{SO}(3)).$$

By the Borel–Weil–Bott theorem, this index is the irreducible representation V_n of $\mathrm{SO}(3)$ with highest weight n (with respect to the positive root corresponding to the identification of S^2 with $\mathbb{P}^1(\mathbb{C})$).

Fix an element $g \in \mathbb{T}^1$ with dense powers. By the Atiyah–Segal–Singer theorem, or Corollary 6.4, the character of V_n evaluated at g equals

$$\mathrm{index}_{\mathbb{T}^1}(\bar{\partial}_n + \bar{\partial}_n^*)(g) = \frac{g^n}{1-g^{-1}} + \frac{g^{-n}}{1-g}. \tag{6.14}$$

The two terms on the right-hand side correspond to the two fixed points of the action by $\mathbb{T}^1$. This expression can be rewritten as the finite sum

$$\sum_{j=0}^{2n} g^{j-n}.$$

This is the usual decomposition of $V_n|_{\mathbb{T}^1}$ into irreducible representations of $\mathbb{T}^1$.

So far, we have done nothing new in this example. But let $\bar{\partial}^{\mathbb{C}} + (\bar{\partial}^{\mathbb{C}})^*$ be the Dolbeault–Dirac operator on $\mathbb{C}$. Then the linearisation theorem, Corollary 6.6, implies that

$$\mathrm{index}_{\mathbb{T}^1}(\bar{\partial}_n + \bar{\partial}_n^*)(g) = \mathrm{index}_g(\bar{\partial}^{\mathbb{C}} + (\bar{\partial}^{\mathbb{C}})^*)g^n + \mathrm{index}_{g^{-1}}(\bar{\partial}^{\mathbb{C}} + (\bar{\partial}^{\mathbb{C}})^*)g^{-n}.$$

As we saw in Section 6C, Corollary 6.4 implies that

$$\operatorname{index}_g(\bar{\partial}^{\mathbb{C}} + (\bar{\partial}^{\mathbb{C}})^*) = \frac{1}{1 - g^{-1}},$$

and likewise with g replaced by g^{-1}. This agrees with (6.14). Using Proposition 6.11, we can realise the latter index as the character of the representation of $\mathbb{T}^1$ in

$$\ker_{L^2,\psi}(\bar{\partial} + \bar{\partial}^*)^+,$$

with ψ as in Section 6C.

6E. ***Discrete series characters.*** In this subsection only, we use the letter G to denote a connected, semisimple Lie group. Let $T < G$ be a maximal torus, and suppose it is a Cartan subgroup of G, i.e., G has discrete series representations. (The torus T plays the role that the group G plays in the rest of this paper; we have changed notation because this is standard in the current setting.) Let $K < G$ be a maximal compact subgroup containing T. We denote the normalisers of T in G and K by $N_G(T)$ and $N_K(T)$, respectively.

Lemma 6.15. *The fixed point set of the action by T on G/T is $N_K(T)/T$, the Weyl group W_c of $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.*

Proof. Since

$$(G/T)^T = N_G(T)/T,$$

it is enough to show that

$$N_G(T) = N_K(T).$$

To prove this, let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition of $\mathfrak{g}$. Suppose $X \in \mathfrak{p}$, such that $\exp(tX) \in N_G(T)$ for all $t \in \mathbb{R}$. Then for all $H \in \mathfrak{t}$,

$$\exp(tX)\exp(H)\exp(-tX) = \exp(\operatorname{Ad}(\exp(tX))H) \in T.$$

So $[X, H] \in \mathfrak{t}$. Because $X \in \mathfrak{p}$ and $H \in \mathfrak{t} \subset \mathfrak{k}$, we have $[X, H] \in \mathfrak{p}$. Hence $[X, H] = 0$. Since $\mathfrak{t}$ is maximal commutative, we find that $X \in \mathfrak{t}$, so that $X = 0$. Therefore, an element $Y \in \mathfrak{g}$ such that $\exp(tY) \in N_G(T)$ for all $t \in \mathbb{R}$ must lie in $\mathfrak{k}$. Since G is connected, the claim follows. □

Example 6.16. If $G = SL(2, \mathbb{R})$, then a strongly elliptic coadjoint orbit of G is equivariantly diffeomorphic to G/T. This is now a hyperbolic plane, on which T acts by rotations. This action has one fixed point, corresponding to the trivial Weyl group of $K = T$.

Let $\lambda \in i\mathfrak{t}^*$ be regular (in the sense that $(\alpha, \lambda) \neq 0$ for all roots α, for a Weyl group invariant inner product). Fix a set R^+ of positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ by defining a root α to be positive if $(\alpha, \lambda) > 0$. Let ρ be half the sum of the positive roots.

The choice of positive roots determines a G-invariant complex structure on the manifold G/T, defined by

$$T_{eT}^{0,1}(G/T) = (\mathfrak{g}/\mathfrak{t})^{0,1} := \bigoplus_{\alpha \in R^+} (\mathfrak{g}_{\mathbb{C}})_{-\alpha}. \tag{6.17}$$

Suppose $\lambda + \rho$ is an integral weight. Then $\lambda - \rho$ is integral as well, and we have the holomorphic line bundle

$$L_{\lambda-\rho} := G \times_T \mathbb{C}_{\lambda-\rho} \to G/T,$$

where T acts on $\mathbb{C}_{\lambda-\rho} := \mathbb{C}$ via the weight $e^{\lambda-\rho}$. Let

$$\bar{\partial}_{L_{\lambda-\rho}} + \bar{\partial}^*_{L_{\lambda-\rho}}$$

be the Dolbeault–Dirac operator on G/T, coupled to $L_{\lambda-\rho}$.

Let Θ_λ be the distributional character of the discrete series representation of G with infinitesimal character λ.

Proposition 6.18. *Let $g \in T$ be such that the powers of g are dense in T.* (*Then in particular*, *g is a regular element.*) *One has*

$$\operatorname{index}_g(\bar{\partial}_{L_{\lambda-\rho}} + \bar{\partial}^*_{L_{\lambda-\rho}}) = (-1)^{\dim(G/K)/2}\Theta_\lambda(g).$$

Proof. The proof is analogous to Atiyah and Bott's derivation of the Weyl character formula from their fixed point theorem in [Atiyah and Bott 1968, Section 5]. By Corollary 6.4 and Lemma 6.15, we have

$$\operatorname{index}_g(\bar{\partial}_{L_{\lambda-\rho}} + \bar{\partial}^*_{L_{\lambda-\rho}}) = \sum_{aT \in N_K(T)/T} \frac{e^{\lambda-\rho}(a^{-1}ga)}{\det(1 - \operatorname{Ad}^{0,1}_{\mathfrak{g}/\mathfrak{t}}(a^{-1}ga))}. \tag{6.19}$$

Here $\operatorname{Ad}^{0,1}_{\mathfrak{g}/\mathfrak{t}} : T \to \operatorname{GL}((\mathfrak{g}/\mathfrak{t})^{0,1})$ is induced by the adjoint representation. Because of (6.17), we have

$$\det(1 - \operatorname{Ad}^{0,1}_{\mathfrak{g}/\mathfrak{t}}(a^{-1}ga)) = \prod_{\alpha \in R^+} (1 - e^{-\alpha}(a^{-1}ga)).$$

Since in the identification $N_K(T)/T = W_c$, the normaliser $N_K(T)$ acts on $i\mathfrak{t}^*$ via the coadjoint action, we find that (6.19) equals

$$\sum_{w \in W_c} \frac{e^{w\cdot(\lambda-\rho)}}{\prod_{\alpha \in R^+}(1 - e^{-w\cdot\alpha})}(g). \tag{6.20}$$

Consider the Weyl denominator

$$\Delta := e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha}).$$

One has, for all $w \in W_c$,

$$w \cdot \Delta := e^{w\cdot\rho} \prod_{\alpha\in R^+} (1 - e^{-w\cdot\alpha}) = \varepsilon(w)\Delta,$$

where $\varepsilon(w) = \det w$ is the sign of w. Hence we find that (6.20) equals

$$\frac{\sum_{w\in W_c} \varepsilon(w) e^{w\cdot\lambda}}{\Delta}(g).$$

(This expression still makes sense if ρ is not an integral weight.) By Harish-Chandra's character formula for the discrete series (see [Harish-Chandra 1966, Theorem 16] or [Knapp 2001, Theorem 12.7]), this is $(-1)^{\dim(G/K)/2}\Theta_\lambda(g)$. □

Note that Proposition 6.18 only relates the value of the character Θ_λ at g to the g-index of $\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}}$ if g is a regular element of some maximal torus. Such elements form an open subset of G, and characters are not determined by their restrictions to this set. However, we can still use Proposition 6.18 to give a description of the g-index in terms of the kernel of $\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}}$.

Proposition 6.21. *Suppose G is a linear group. Then the representation of G in the L^2-kernel of $(\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}})^\pm$ has a distributional character $\Theta^\pm$ that can be evaluated at g, and one has*

$$\operatorname{index}_g(\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}}) = \Theta^+(g) - \Theta^-(g).$$

Proof. This follows from Proposition 6.18 and Schmid's realisation of the discrete series in the L^2-Dolbeault cohomology of G/T with values in $L_{\lambda-\rho}$, in [Schmid 1976, Theorem 1.5]. Schmid's result implies that the space

$$\ker_{L^2}(\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}})^\pm$$

equals zero if $\pm = -(-1)^{\dim(G/K)/2}$, and the representation of G in this space is the discrete series representation with infinitesimal character λ if $\pm = (-1)^{\dim(G/K)/2}$. (The integer k in Schmid's result now equals $\dim(G/K)/2$, and his λ is our $\lambda - \rho$.) Hence,

$$\Theta^+ - \Theta^- = (-1)^{\dim(G/K)/2}\Theta_\lambda.$$

So the claim follows from Proposition 6.18. □

Paradan [2003] gave a realisation of restrictions of discrete series representations to maximal compact subgroups, as an equivariant index of a deformation of the operator $\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}}$. That realisation allowed him to apply the *quantisation commutes with reduction* principle to find a geometric formula for multiplicities of K-types.

In the paper [Hochs and Wang 2017], we further explore the relation between index theory and characters of the discrete series.

6F. ***Fredholm operators.*** We return to the notation used in the rest of this paper except for Section 6E, where G is a compact Abelian group generated topologically by an element $g \in G$.

For Fredholm operators, it is a natural question how the g-index of such an operator is related to the traces of g acting on even and odd parts of its kernel. This depends on the behaviour of the operator "towards infinity". To make this more explicit, let M^+ be the one-point compactification of M. The point at infinity is fixed by g. Let $U, V \subset M$ be as in Section 2B. Let $U', V' \subset M^+$ be g-invariant neighbourhoods of the point at infinity, such that $\overline{U'} \subset V'$, and $V \cap V' = \varnothing$. Then $U \sqcup U'$ and $V \sqcup V'$ are neighbourhoods of $(M^+)^g$ as in (2.5). Lemma 2.6 therefore implies that for any σ-unital G-C^* algebra A, the following diagram commutes:

$$\begin{tikzcd}
KK_G(C(M^+), A)_g \arrow[r, "(k_V^{M^+})_g^* \oplus (k_{V'}^{M^+})_g^*"'] \arrow[dd, "(p_*^{M^+})_g"'] & KK_G(C_0(V), A)_g \oplus KK_G(C_0(V'), A)_g \arrow[d, "((j_{\overline{U}}^V)_*)_g^{-1} \oplus ((j_{\overline{U'}}^{V'})_*)_g^{-1}"'] \\
 & KK_G(C(\overline{U}), A)_g \oplus KK_G(C(\overline{U'}), A)_g \arrow[d, "(p_*^{\overline{U}})_g \oplus (p_*^{\overline{U'}})_g"'] \\
KK_G(\mathbb{C}, A)_g & KK_G(\mathbb{C}, A)_g \oplus KK_G(\mathbb{C}, A)_g \arrow[l, "+"']
\end{tikzcd} \tag{6.22}$$

Indeed, since M^+ is compact, one can apply Lemma 2.6 to the pairs of neighbourhoods $U \sqcup U' \subset V \sqcup V'$ and $M^+ \subset M^+$ of $(M^+)^g$.

Now suppose that $(D^2+1)^{-1}$ is a compact operator. Then $F := D/\sqrt{D^2+1}$ is Fredholm, so $\ker_{L^2}(D)$ is finite-dimensional. Let the representation

$$\pi_{M^+} : C(M^+) \to \mathcal{B}(L^2(E))$$

be defined for $f \in C_0(M)$ and $z \in \mathbb{C}$ by

$$\pi_{M^+}(f+z) = \pi_M(f) + z. \tag{6.23}$$

Then the triple $(L^2(E), F, \pi_{M^+})$ is a Kasparov $(C(M^+), \mathbb{C})$-module. Let

$${}_{M^+}[D] \in KK_G(M^+, \mathrm{pt}) \tag{6.24}$$

be its class. In this case, we will write $\mathrm{index}_g^\infty(D)$ for a version of the g-index of D that captures the behaviour of D at infinity:

$$\mathrm{index}_g^\infty(D) := (\mathrm{ev}_g) \circ \left(p_*^{\overline{U'}}\right)_g \circ \left(\left(j_{\overline{U'}}^{V'}\right)_*\right)_g^{-1} \circ (k_{V'}^{M^+})_g^*({}_{M^+}[D]_g). \tag{6.25}$$

Proposition 6.26. *If $(D^2+1)^{-1}$ is compact, then*

$$\mathrm{Tr}(g \text{ on } \ker_{L^2}(D^+)) - \mathrm{Tr}(g \text{ on } \ker_{L^2}(D^-)) = \mathrm{index}_g(D) + \mathrm{index}_g^\infty(D). \tag{6.27}$$

Proof. By commutativity of (6.22), with $A = \mathbb{C}$, we have

$$(\mathrm{ev}_g)_g \circ (p_*^{M^+})_g[D] = \mathrm{index}_g(D) + \mathrm{index}_g^\infty(D). \tag{6.28}$$

Now

$$p_*^{M^+}[D] = [L^2(E), F] = [\ker F, 0] \in KK_G(\mathrm{pt}, \mathrm{pt}),$$

so the left-hand sides of (6.27) and (6.28) are equal. □

In concrete situations, knowledge of $\mathrm{index}_g^\infty(D)$ then allows one to use the fixed point formula in Theorem 2.16 to compute the left-hand side of (6.27).

This can be made more explicit in a situation relevant to the treatment of Callias-type deformations of Dirac operators in the context of KK-theory in [Bunke 1995; Kucerovsky 2001]. Suppose that $\Phi \in \mathrm{End}(E)^G$ is an odd, self-adjoint vector bundle endomorphism. Suppose that $\Phi^2 - 1_E$ tends to zero at infinity, so that it is a compact operator on $\Gamma_0(E)$. Then $(\Gamma_0(E), \Phi, \pi_{M^+})$ is an equivariant Kasparov $(C(M^+), C_0(M))$-cycle. Let $[\Phi] \in KK_G(M^+, M)$ be its class. Now we do not assume that $(D^2+1)^{-1}$ itself is compact, but that

$$[D_\Phi] := [\Phi] \otimes_M [D] \in KK_G(M^+, \mathrm{pt})$$

is the class of an elliptic operator D_Φ as in (6.24). Then $(D_\Phi^2+1)^{-1}$ is compact. (The idea is that $D_\Phi = D + \Phi$ if $D\Phi + \Phi D$ is sufficiently small; compare this with [Bunke 1995, Proposition 2.18].) By functoriality of the Kasparov product, we have for $U', V' \subset M$ as above,

$$\mathrm{index}_g^\infty(D_\Phi) = (\mathrm{ev}_g)_g\Big(\Big(\big(p_*^{\overline{U'}}\big)_g \circ \big(\big(j_{\overline{U'}}^{V'}\big)_*\big)_g^{-1} \circ (k_{V'}^{M^+})_g^*[\Phi]_g\Big) \otimes_M [D]_g\Big). \tag{6.29}$$

This expression has the advantage that Φ is a vector bundle endomorphism, which makes (6.29) easier to evaluate than (6.25). In particular, if $\Phi^2 = 1_E$ on $V' \cap M$, then $(k_{V'}^{M^+})^*[\Phi] = 0$. In that case, Theorem 2.16 and Proposition 6.26 imply that

$$\begin{aligned}\mathrm{Tr}(g \text{ on } \ker_{L^2}(D_\Phi^+)) - \mathrm{Tr}(g \text{ on } \ker_{L^2}(D_\Phi^-)) \\ = \int_{TM^g} \frac{\mathrm{ch}([\sigma_{D_\Phi}|_{TM^g}](g))\,\mathrm{Todd}(TM^g \otimes \mathbb{C})}{\mathrm{ch}\big(\big[\bigwedge N_{\mathbb{C}}\big](g)\big)}.\end{aligned} \tag{6.30}$$

Example 6.31. Let $M = \mathbb{C}^n$, and let g be the diagonal action by n nontrivial elements of $\mathrm{U}(1)$. Then $M^g = \{0\}$, and $N = \mathbb{C}^n$. Let $\beta_{\mathbb{C}^n} \in KK_G(\mathrm{pt}, \mathbb{C}^{2n})$ be the Bott element as in Definition 4.12. Now the class $[D_1] \in KK_G(\mathbb{C}^{2n}, \mathrm{pt})$ as in (4.15) is the Dolbeault class of $\mathbb{C}^{2n}$. The Kasparov product

$$\beta_{\mathbb{C}^n} \otimes_{\mathbb{C}^{2n}} [D_1] \in KK_G(\mathrm{pt}, \mathrm{pt})$$

is represented by the elliptic operator $D_B := B \otimes 1 + 1 \otimes D_1$ as in (4.21). Hence $(D_B^2+1)^{-1}$ is a compact operator. In the proof of Lemma 4.19, we saw that the

L^2-kernel of D_B is spanned by the g-invariant function (4.23). So

$$\mathrm{Tr}(g \text{ on } \ker_{L^2}(D_B^+)) - \mathrm{Tr}(g \text{ on } \ker_{L^2}(D_B^-)) = 1. \tag{6.32}$$

On the other hand, let $b \in C^\infty(\mathbb{R})$ be an odd function, with values in $[-1, 1]$, such that $b(x) = 1$ for all $x \geq 1$. If we replace $B(1+B^2)^{-1/2}$ by $b(B)$ in (4.11), then the resulting class in $KK_G(\mathrm{pt}, \mathbb{C}^{2n})$ is the same class $\beta_{\mathbb{C}^n}$. But with this normalisation function, we have $b(B)^2 = 1$ outside the unit ball in $\mathbb{C}^n$. So

$$\mathrm{index}_g^\infty(D_B) = 0. \tag{6.33}$$

Finally, by Corollary 6.4, with $F = \bigwedge \widetilde{N}_{\mathbb{C}} = \bigwedge \mathbb{C}^{2n}$, we have

$$\mathrm{index}_g(D_B) = 1. \tag{6.34}$$

The equalities (6.32), (6.33) and (6.34) illustrate Proposition 6.26 in this case.

Example 6.35. In the setting of Theorem 2.18, the index pairing $[F] \otimes_M [D]$ in $KK_G(\mathrm{pt}, \mathrm{pt})$ is represented by a Fredholm operator D_F. Analogously to (6.29), we have $\mathrm{index}_g^\infty(D_F) = 0$, so that Proposition 6.26 and Theorem 2.16 yield an expression for $\mathrm{Tr}(g \text{ on } \ker_{L^2}(D_F^+)) - \mathrm{Tr}(g \text{ on } \ker_{L^2}(D_F^-))$. But in this setting, the same expression follows directly from Theorem 2.18.

See Remark 7.10 for the construction of a Fredholm operator D_{fv} as a deformation of *any* elliptic operator D, with $\mathrm{index}_g^\infty(D_{fv}) = 0$.

6G. *Braverman's index.* Suppose $X \in \mathfrak{g}$ such that $g = \exp X$. Let X^M be the vector field on M defined by X. Suppose D is a Dirac-type operator, and consider the deformed operator

$$D_X^f := D + \sqrt{-1}\, f c(X^M).$$

Here $f \in C^\infty(M)^G$, and $c : TM \to \mathrm{End}(E)$ is a given Clifford action, used to define the Dirac operator D. Braverman [2002, Theorem 7.5] obtained a fixed point theorem for such operators. This implies that the g-index equals Braverman's index in this case.

Corollary 6.36. *If f is admissible* [Braverman 2002, Definition 2.6], *then the representation of G in* $\ker_{L^2}(D_X^f)^\pm$ *has a character $\chi^\pm$ that can be evaluated at g, and one has*

$$\mathrm{index}_g(D) = \chi^+(g) - \chi^-(g).$$

Proof. The fixed point formula for $\mathrm{index}_g(D)$ in Theorem 2.16 is precisely the evaluation at g of the right-hand side of the formula in [Braverman 2002, Theorem 7.5]. (This equality also shows that $\ker_{L^2}(D_X^f)$ has a character that can be evaluated at g.) □

Remark 6.37. In the above construction, the element $X \in \mathfrak{g}$, which represents the *taming map* used in [Braverman 2002], depends on the group element g. So the g-index of D is not the character of the Braverman index of D deformed by a single taming map, but the taming map depends on g.

6H. ***A relative index theorem.*** Gromov and Lawson [1983, Theorem 4.18] obtain a relative index formula for pairs of elliptic operators that coincide outside compact sets. (See Theorem 2.18 in [Braverman and Shi 2016] for a version for Callias-type operators.) There is an analogue of this result for the g-index.

For $j = 0, 1$, let M_j be a manifold with the same structure and properties as M. Let $E_j \to M_j$ be a vector bundle like $E \to M$, and let D_j be an operator on E_j like D on E. Suppose there are relatively compact neighbourhoods U_j of M_j^g outside of which M_j, E_j and D_j can be identified. As on page 38 of [Gromov and Lawson 1983], we compactify M_j to a manifold $\widetilde{M}_j$, by taking a neighbourhood V_j of U_j, and attaching a compact manifold X to it. Since $M_0 \setminus V_0 = M_1 \setminus V_1$, the same manifold X can be used for $j = 0, 1$. Extend the vector bundles E_j and the operators D_j to vector bundles $\widetilde{E}_j \to \widetilde{M}_j$ and elliptic operators $\widetilde{D}_j$ on $\widetilde{E}_j$. Suppose the map g extends to $\widetilde{M}_j$ and $\widetilde{E}_j$, commuting with $\widetilde{D}_j$; this extends to continuous actions by G on $\widetilde{M}_j$ and $\widetilde{E}_j$ preserving $\widetilde{D}_j$.

Proposition 6.38 (relative index theorem). *We have*

$$\operatorname{index}_g(D_1) - \operatorname{index}_g(D_0) = \operatorname{index}_G(\widetilde{D}_1)(g) - \operatorname{index}_G(\widetilde{D}_0)(g).$$

Since the manifolds $\widetilde{M}_j$ are compact, the indices on the right-hand side of this equality are given by the usual equivariant index.

Proof. By the Atiyah–Segal–Singer fixed point theorem (or Theorem 2.16), we have, for $j = 0, 1$,

$$\operatorname{index}_G(\widetilde{D}_j)(g) = \int_{TM_j^g} \frac{\operatorname{ch}\big([\sigma_{\widetilde{D}_j}|_{TM_j^g}](g)\big)\operatorname{Todd}(TM_j^g \otimes \mathbb{C})}{\operatorname{ch}\big(\big[\bigwedge(N_j)_{\mathbb{C}}\big](g)\big)} + \int_{TX^g} \frac{\operatorname{ch}\big([\sigma_{\widetilde{D}_j}|_{TX^g}](g)\big)\operatorname{Todd}(TX^g \otimes \mathbb{C})}{\operatorname{ch}\big(\big[\bigwedge(N_X)_{\mathbb{C}}\big](g)\big)}.$$

Here $N_j \to M_j^g$ and $N_X \to X^g$ are normal bundles to fixed point sets. Since $\sigma_{\widetilde{D}_1}|_{TX^g} = \sigma_{\widetilde{D}_0}|_{TX^g}$, Theorem 2.16 implies the claim. □

6I. ***Some geometric consequences.*** The g-index of a G-equivariant elliptic operator is a topological invariant of the group action that can be used to detect geometric properties of the action. We illustrate this in the cases of the Hodge–de Rham and Spin-Dirac operators.

Let $D = d + d^* : \Omega_{\mathbb{C}}^{\mathrm{even}}(M) \to \Omega_{\mathbb{C}}^{\mathrm{odd}}(M)$ be the Hodge–de Rham operator on M, acting on complex differential forms. The symbol class of this operator is $\left[\tau_M^* \bigwedge TM_{\mathbb{C}}\right]$, whose restriction to TM^g equals

$$[\sigma_D|_{TM^g}] = \left[\tau_{M^g}^* \bigwedge N_{\mathbb{C}}\right] \otimes \left[\tau_{M^g}^* \bigwedge TM_{\mathbb{C}}^g\right]. \tag{6.39}$$

Let D_{M^g} be the componentwise defined Hodge–de Rham operator on M^g. Then Theorem 2.16 and (6.39) imply that

$$\mathrm{index}_g(d+d^*) = \int_{TM^g} \mathrm{ch}(\sigma_{D_{M^g}})\,\mathrm{Todd}(TM^g \otimes \mathbb{C}) = \mathrm{index}(D_{M^g}) = \chi(M^g), \tag{6.40}$$

the Euler characteristic of M^g. (See also [Lawson and Michelsohn 1989, p. 262].)

Corollary 6.41. *If* $\mathrm{index}_g(d+d^*) \neq 0$, *then every g-invariant vector field on M has a zero on M^g.*

Proof. A g-invariant vector field on M restricts to a vector field on M^g. If it does not vanish there, then $\chi(M^g) = 0$. So the claim follows from (6.40). □

Next, suppose that M is a Spin manifold, and that g lifts to the spinor bundle. Let D be the Spin-Dirac operator.

Corollary 6.42. *If G is connected, M is noncompact, and* $\mathrm{index}_g(D) = 0$, *then the one-point compactification M^+ of M is not a G-equivariant* Spin *manifold.*

Proof. If M^+ is a G-equivariant Spin manifold with Dirac operator D_{M^+}, then the vanishing result of Atiyah and Hirzebruch [1970] and Theorem 2.16 imply that

$$0 = \mathrm{index}_g(D_{M^+}) = \mathrm{index}_g(D) + a_\infty.$$

Here a_∞ is the contribution to the right-hand side of (2.17) of the fixed point at infinity, which is nonzero by [Atiyah and Bott 1968, Theorem 8.35]. □

7. A nonlocalised index formula

In the compact case, the Kirillov formula is a nonlocalised expression for the equivariant index of an elliptic operator; see [Berline et al. 1992, Theorem 8.2]. This can be deduced from the fixed point formula in the compact case. In the case of noncompact manifolds, there is also a nonlocalised expression for the g-index, Proposition 7.8 below. This follows from Kasparov's index theorem and the properties of the g-symbol class introduced in Section 5D, rather than from Theorem 2.16.

A potentially interesting feature of this nonlocalised formula is that it involves the same kind of deformed symbols as the ones used for Dirac operators on symplectic manifolds in [Paradan 2011]. Those deformed symbols are *transversally elliptic* rather than elliptic. Berline and Vergne obtained a generalisation of the

Atiyah–Segal–Singer fixed point formula to transversally elliptic operators or symbols; see [Berline and Vergne 1996a, Main Theorem 1; 1996b, Theorem 20]. This formula involves a distribution on the group. It was pointed out to the authors by Vergne that this formula implies that for the deformed symbols we will consider, at points g where this distribution is given by a function, it is given by the g-index.

The index of such a deformed symbol was shown to equal the index of a deformed Dirac operator in Theorem 5.5 in [Braverman 2002]. In Theorem 1.5 in [Ma and Zhang 2014], this index is proved to be equal to another index of deformed Dirac operators, defined using the Atiyah–Patodi–Singer index on manifolds with boundary. In contrast to [Braverman 2002; Ma and Zhang 2014; Paradan 2011], the expression for the g-index in terms of deformed symbols is independent of the choices made in this deformation. Furthermore, it applies to more general elliptic operators than Dirac operators.

We shall describe the g-symbol class σ_g^D of Definition 5.14 more explicitly, using a deformed symbol. Let v be a G-invariant vector field on M that does not vanish outside V.

Example 7.1. If $X \in \mathfrak{g}$ such that $g = \exp(X)$, one can take the vector field v induced by X. This vector field obviously depends on g.

Example 7.2. If H is a compact Lie group acting on M, containing G, then it can be possible to choose a single vector field v that works for all elements of H. Indeed, suppose there is an H-equivariant map $\psi : M \to \mathfrak{h}$, and consider the Kirwan vector field v, defined by

$$v_m := \frac{d}{dt}\Big|_{t=0} \exp(t\psi(m)) \cdot m,$$

for $m \in M$. Suppose this vector field is nonzero outside a compact set. Then ψ is a taming map as in Definition 2.4 in [Braverman 2002]. In this case, the vector field v can be used for any element of H.

Let $f : V \to \mathbb{R}_{\geq 0}$ be a G-invariant continuous function, such that $f(m) = 0$ when $m \in U$ and $\lim_{m \to m'} f(m) = \infty$ if $m' \in \partial V$. Consider the deformed symbol $\sigma_{D,fv} \in \mathrm{End}(\tau_V^*(E|_V))$, given by

$$\sigma_{D,fv}(u) := \sigma_D(u + f(m)v_m) \tag{7.3}$$

for $m \in V$ and $u \in T_m M$. Set

$$\tilde{\sigma}_{D,fv} := \frac{\sigma_{D,fv}}{\sqrt{\sigma_{D,fv}^2 + 1}}.$$

This defines an odd, self-adjoint, bounded operator on the Hilbert $C_0(TV)$-module $\Gamma_0(\tau_V^*(E|_V))$. Furthermore, we have for every vector field u on M, and every

$m' \in \partial V$,

$$\lim_{m \to m'} \tilde{\sigma}_{D,fv}(u_m) = \operatorname{sgn}(\sigma_D(v_{m'})).$$

We extend $\tilde{\sigma}_{D,fv}$ to a continuous vector bundle endomorphism of $\tau_M^* E$ by setting

$$\tilde{\sigma}_{D,fv}(u) := \operatorname{sgn}(\sigma_D(v_m))$$

for all $u \in T_m M$, where $m \in M \setminus V$. (Since $v_m \neq 0$ if $m \in M \setminus V$, this operator is invertible outside V.)

Note that

$$\tilde{\sigma}_{D,fv}(u)^2 - 1 \to 0 \tag{7.4}$$

as $u \to \infty$ in TM. Indeed, let $m \in M$ and $u \in T_m M$ be given. If $m \notin V$, then $v_m \neq 0$ and $\tilde{\sigma}_{D,fv}(u)^2 = 1$. And if $m \in V$, then

$$\tilde{\sigma}_{D,fv}(u)^2 - 1 = (\sigma_D(u + f(m)v_m)^2 + 1)^{-1}.$$

Since D is elliptic and has first order, this goes to zero as $u \to \infty$ in TV. We therefore find that $(\Gamma_0(\tau_M^* E), \tilde{\sigma}_{D,fv})$ is a Kasparov $(\mathbb{C}, C_0(TM))$-cycle. Let

$${}_{\mathrm{pt}}[\sigma_{D,fv}] \in KK_G(\mathrm{pt}, TM)$$

be its class, which will be called the deformed symbol class.

Lemma 7.5. *The localisation of the deformed symbol class at g is σ_g^D, i.e.,*

$${}_{\mathrm{pt}}[\sigma_{D,fv}]_g = \sigma_g^D \in KK_G(\mathrm{pt}, TM)_g.$$

Proof. As in Section 6F, let M^+ be the one-point compactification of M. Let $U, V, U', V' \subset M^+$ be as in that subsection. Consider the class

$${}_{M^+}[\sigma_{D,fv}] := [\Gamma_0(\tau_M^* E), \tilde{\sigma}_{D,fv}, \pi_{M^+}] \in KK_G(M^+, TM),$$

where π_{M^+} is as in (6.23). Then by commutativity of (6.22), for $A = C_0(TM)$, we have

$$\begin{aligned} {}_{\mathrm{pt}}[\sigma_{D,fv}]_g &= \left(p_*^{M^+}\right)_g ({}_{M^+}[\sigma_{D,fv}]) \\ &= \left(p_*^{\overline{U}}\right)_g \circ \left(\left(j_{\overline{U}}^{V}\right)_*\right)_g^{-1} \circ \left(k_V^{M^+}\right)_g^* ({}_{M^+}[\sigma_{D,fv}]_g) \\ &\quad + (p_*^{\overline{U'}})_g \circ \left(\left(j_{\overline{U'}}^{V'}\right)_*\right)_g^{-1} \circ \left(k_{V'}^{M^+}\right)_g^* ({}_{M^+}[\sigma_{D,fv}]_g). \end{aligned} \tag{7.6}$$

Now since $f = 0$ on V, we have

$$\left(k_V^{M^+}\right)^* ({}_{M^+}[\sigma_{D,fv}]) = \left(k_V^M\right)^* [\sigma_D].$$

So the first term in (7.6) equals σ_g^D. Furthermore,

$$\left(k_{V'}^{M^+}\right)^* ({}_{M^+}[\sigma_{D,fv}]) = [\Gamma^\infty(E|_{V'}), \operatorname{sgn}(\sigma_D(v)), \pi_{V'}] = 0,$$

since this class is represented by a degenerate cycle. □

Remark 7.7. Instead of (7.3), we could have used a more general deformed symbol of the form

$$\sigma_{D,f\Phi}(u) := \sigma_D(u) + f(m)\Phi_m,$$

for $m \in M$, $u \in T_m M$ and a G-equivariant, fibrewise self-adjoint, odd vector bundle endomorphism Φ of E, which is invertible outside V. We have used the natural choice $\Phi = \sigma_D(v)$.

The realisation of the g-symbol class in Lemma 7.5 leads to the following nonlocal formula for the g-index.

Proposition 7.8 (nonlocalised formula for the g-index). *The g-index of D is calculated by*

$$\mathrm{index}_g(D) = ({}_{\mathrm{pt}}[\sigma_{D,fv}] \otimes_{TM} [D_{TM}])(g). \tag{7.9}$$

Proof. It follows from Definitions (2.8) and 5.14, and Theorem 4.6, that

$$\mathrm{index}_g(D) = (\sigma_g^D \otimes_{TM} [D_{TM}]_g)(g).$$

The claim therefore follows from Lemma 7.5. □

Remark 7.10. Recall that when M is noncompact, $\mathrm{index}_g(D)$ is defined using KK-functorial maps. In Proposition 7.8, the class

$${}_{\mathrm{pt}}[\sigma_{D,fv}] \otimes_{TM} [D_{TM}] \in KK_G(\mathrm{pt}, \mathrm{pt})$$

is represented by a Fredholm operator D_{fv}, defined in terms of the deformed symbol $\sigma_{D,fv}$ and the Dolbeault–Dirac operator D_{TM}. Proposition 7.8 states that

$$\mathrm{index}_g(D) = \mathrm{Tr}(g \text{ on } \ker_{L^2}(D_{fv}^+)) - \mathrm{Tr}(g \text{ on } \ker_{L^2}(D_{fv}^-)). \tag{7.11}$$

Then Theorem 2.16 yields a cohomological expression for the right-hand side of (7.11). (Note the analogy with (6.30); we now have $\mathrm{index}_g^\infty(D_{fv}) = 0$.)

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