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We find conditions such that cup products induce isomorphisms in low degrees for extensions between stable polynomial representations of the general linear group. We apply this result to prove generalizations and variants of the Steinberg tensor product theorem. Our connectedness bounds for cup product maps depend on numerical invariants which seem also relevant to other problems, such as the cohomological behavior of the Schur functor.

1. Introduction

Let $\Bbbk$ be a field of positive characteristic p, and let G be an algebraic group over $\Bbbk$. The category of rational representations of G (as in [Jantzen 2003]) is equipped with a tensor product, which induces a cup product on extension groups:

$$\operatorname{Ext}_G^*(M, N) \otimes \operatorname{Ext}_G^*(P, Q) \xrightarrow{\cup} \operatorname{Ext}_G^*(M \otimes P, N \otimes Q).$$

Of course the cup product is injective (but usually not surjective) in cohomological degree zero, and in general it is neither injective nor surjective in higher degrees. If $G = GL_n(\Bbbk)$, it was observed in [Touzé 2010] that the cup product is injective in all degrees when M, N, P, Q are *stable polynomial representations*, i.e., when M, N, P, Q are polynomial representations in the usual sense [Green 2007; Martin 1993] and furthermore when n is big enough with respect to their degrees. This surprising fact is easily proved by using the description of stable polynomial representations in terms of the strict polynomial functors of Friedlander and Suslin [1997].

The first main result of this article is Theorem 3.6, which establishes conditions under which cup products are not only injective, but also surjective in low degrees. Theorem 3.6 actually applies to the case where N and Q are representations twisted r times by the Frobenius morphism, i.e., for cup products of the form

$$\operatorname{Ext}_G^*(M, N) \otimes \operatorname{Ext}_G^*(P^{(r)}, Q^{(r)}) \xrightarrow{\cup} \operatorname{Ext}_G^*(M \otimes P^{(r)}, N \otimes Q^{(r)}).$$

As for injectivity in [Touzé 2010], the natural home for stating and proving this connectedness property of cup products is the category of strict polynomial functors. We note that already in degrees 0 and 1, our theorem looks much stronger than what was previously known for the behavior of cup products; see Remark 4.7.

We then give concrete applications of Theorem 3.6 to the representation theory of $GL_n(\Bbbk)$. Namely, we prove the following two new generalizations of Steinberg's tensor product theorem.

- We call *tensor products of Steinberg type* the stable polynomial representations of the form $M \otimes N^{(r)}$, where all the composition factors of M have p^r-restricted highest weights. Representations of this form appear naturally, e.g., in the theory of good p-filtrations [Andersen 2001].

  In Theorem 5.8, we describe the structure of the abelian subcategory generated by these tensor products of Steinberg type (with r and deg M fixed). In particular, we prove that the $GL_n(\Bbbk)$-module $M \otimes N^{(r)}$ has the same structure as the $GL_n(\Bbbk) \times GL_n(\Bbbk)$-module $M \otimes N$. This is interesting because the latter is much easier to study. (The classical Steinberg tensor product theorem corresponds to the very special case where M and N are simple. Indeed, in that case $M \otimes N$ is simple as a $GL_n(\Bbbk) \times GL_n(\Bbbk)$-module, and thus by our theorem the $GL_n(\Bbbk)$-module $M \otimes N^{(r)}$ is simple too).

- As made explicit in [Krause 2013], stable polynomial representations are equipped with an internal tensor product (Day convolution product), dual to the internal Hom used in Ext-computations, e.g., in [Touzé 2013b; 2014]. In Theorem 6.2 we explain how to reduce the computation of internal tensor products of simples to the case of simples with p-restricted highest weights. Thus, Theorem 6.2 plays the same role for understanding internal tensor products of simples as the classical Steinberg tensor product theorem does for understanding ordinary tensor products of simples.

In Appendix B we show that Theorem 3.6 can also be used to derive new proofs of two well-known fundamental theorems for simple representations of $GL_n(\Bbbk)$: Steinberg's tensor product theorem and Clausen and James' theorem. We note that another functor technology proof of Steinberg's tensor product theorem is given in [Kuhn 2002]. The proof given here seems quite different; see Remark B.11.

The bounds for connectedness given in Theorem 3.6 depend on some cohomological constants $p(M, r)$ and $i(N, r)$. To be more specific, a projective stable polynomial module is p^r-bounded if its socle is a direct sum of simples with p^r-restricted highest weights; see Corollary 4.2. The integer $p(M, r)$ is characterized by:

> $p(M, r) \geq k$ if and only if there exists a resolution P of M in which the first k terms $P_0, \ldots, P_{k-1}$ are p^r-bounded projectives.

The integer $i(N, r)$ is defined dually; see Definition 3.4. Although we use this definition for stable polynomial representations, it makes sense for unstable polynomial representations as well. We are not aware of previous occurrences of these constants in the literature. We study their basic properties and give characterizations of these constants, as well as elementary computation rules and examples. In most examples, we limit ourselves to giving estimates for these constants rather than exact values, and leave the following challenging problem open.

Problem. Compute (or obtain a reasonable understanding of) the exact value of $p(M, r)$ and $i(M, r)$ for the most important $\mathrm{GL}_n(\Bbbk)$-modules (simple modules, standard or costandard modules).

One further motivation for this problem is that the constants $p(M, r)$ and $i(M, r)$ seem to be related to other problems of interest. Let us give two examples.

- In Theorem 8.2, we prove that the constants $p(M, 1)$ and $i(M, 1)$ govern the connectedness of the Schur functor on the level of extensions. The cohomological behavior of the Schur functor was already studied in a series of papers [Doty et al. 2004; Kleshchev and Nakano 2001; Kleshchev and Sheth 1999]. Our Theorem 8.2 gives a simpler and effective approach to this problem. For example, with our first computations of $i(F, 1)$ and $p(F, 1)$, we recover and generalize many results from [Kleshchev and Nakano 2001].

- It seems that the values of $p(L, r)$ capture some interesting concrete properties of simple functors L. Clausen and James' theorem [Clausen 1980; James 1980] says that $p(L, 1) > 0$ if and only if the highest weight of L is p-restricted. Reischuk [2016] has proved that $p(L, 1) > 1$ if and only if $Q^d \underline{\otimes} L$ is simple, where Q^d is the simple head of S^d (see Section 6 and in particular Theorem 6.2 and Corollaries 6.6 and 6.9 to understand why this particular internal tensor product is interesting). It would be interesting to understand if higher inequalities $p(L, 1) > k$ (of cohomological nature) are directly connected to some nonhomological representation-theoretic properties of L.

We finish by explaining a wider perspective behind the work presented here. Functor category techniques have proved useful for studying representations and cohomology of many variants of classical matrix groups. See, e.g., [Touzé 2010] for symplectic and orthogonal group schemes, [Axtell 2013; Drupieski 2016] for super Schur algebras, [Hong and Yacobi 2017] for quantum GL_n, [Franjou et al. 1999; Djament and Vespa 2010] for finite classical groups or more generally [Djament 2012] for discrete unitary groups. In these examples, the functor categories in play share many properties with the category of strict polynomial functors used here. So we expect that the techniques and results developed in this article can be adapted to these cases. For example, we prove in [Touzé 2017a] an analogue of Theorem 3.6 for polynomial representations of orthogonal and symplectic group schemes.

This article has been written in such a way that the main thread of ideas and proofs is self-contained. In particular, only very basic facts of the representation theory of general linear groups are used (the highest weight category structure is used only for the results of Section 7B) and no combinatorics of the symmetric group is used (except a result of Bessenrodt and Kleshchev [2000] in Corollary 6.6). These basic facts are recalled in Section 2. In the same spirit, we have also added an appendix on representations of tensor products of finite dimensional algebras, whose results are used in Section 5.

2. Background

2A. *Strict polynomial functors and Schur algebras.* In this article $\Bbbk$ is a field of positive characteristic p, and $\mathcal{P}_{d,\Bbbk}$ denotes the category of homogeneous strict polynomial functors of degree d over $\Bbbk$ (with possibly infinite dimensional values). We refer, e.g., to [Friedlander 2003], [Friedlander and Suslin 1997] or [Krause 2013] for an introduction to strict polynomial functors. If one considers an infinite ground field $\Bbbk$, strict polynomial functors have a nice description like the one in [Macdonald 1995] (where they are simply called "polynomial functors"). Namely, strict polynomial functors are functors from finite dimensional $\Bbbk$-vector spaces to $\Bbbk$-vector spaces, such that for all pairs of finite dimensional vector spaces (V, W), the map

$$\mathrm{Hom}_{\Bbbk}(V, W) \to \mathrm{Hom}_{\Bbbk}(F(V), F(W)),$$

$$f \mapsto F(f)$$

is given by a homogeneous polynomial of degree d (that is, given by an element of $S^d(\mathrm{Hom}_{\Bbbk}(V, W)^*) \otimes \mathrm{Hom}_{\Bbbk}(F(V), F(W)))$.

For example, the category $\mathcal{P}_{0,\Bbbk}$ is equivalent to the category of constant functors, which is equivalent to the category of $\Bbbk$-vector spaces. Typical examples of homogeneous functors of higher degree d are the tensor product functors $\otimes^d : V \mapsto V^{\otimes d}$, the symmetric power functors $S^d : V \mapsto (V^{\otimes d})_{\mathfrak{S}_d}$ and the divided power functors $\Gamma^d : V \mapsto (V^{\otimes d})^{\mathfrak{S}_d}$. (Here the symmetric group $\mathfrak{S}_d$ acts on $V^{\otimes d}$ by permuting the factors of the tensor product). Note that $S^0 = \otimes^0 = \Gamma^0 = \Bbbk$ and $S^1 = \otimes^1 = \Gamma^1$, but for $d \geq p$ the functor S^d is not isomorphic to Γ^d.

We denote by $\mathcal{P}_{\Bbbk}$ the category of strict polynomial functors (of bounded degree), that is, $\mathcal{P}_{\Bbbk} = \bigoplus_{d \geq 0} \mathcal{P}_{d,\Bbbk}$. Thus, objects of $\mathcal{P}_{\Bbbk}$ decompose as *finite* direct sums of homogeneous functors, and there are no nonzero morphisms between homogeneous functors of different degrees. All functors of vector spaces considered in this article will actually be strict polynomial functors of bounded degree, and hence we will always omit the terms "of bounded degree", and we will often omit the terms "strict polynomial".

By evaluating a strict polynomial functor F on $\Bbbk^n$, one obtains a polynomial $\mathrm{GL}_n(\Bbbk)$-module $F(\Bbbk^n)$. Restricting to homogeneous strict polynomial functors of degree d, one obtains a functor

$$\mathcal{P}_{d,\Bbbk} \xrightarrow{\mathrm{ev}_{\Bbbk^n}} \mathrm{Pol}_{d,\mathrm{GL}_n(\Bbbk)} \simeq S(n,d)\text{-Mod},$$

where $\mathrm{Pol}_{d,\mathrm{GL}_n(\Bbbk)}$ denotes the category of homogeneous polynomial representations of $\mathrm{GL}_n(\Bbbk)$ of degree d, and $S(n,d)$-Mod the equivalent category of modules over the Schur algebra $S(n,d)$ (which is finite dimensional). It is an equivalence of categories, provided $n \geq d$. (Friedlander and Sulin [1997] proved it for functors with finite dimensional values, but their proof actually works in the general case; see also [Krause 2013].) In particular, $\mathcal{P}_{\Bbbk}$ has nice properties similar to those of modules over a finite dimensional algebra. We shall use the following ones in the sequel.

(1) Simple functors are homogeneous functors, and their values are finite dimensional vector spaces. A functor has a finite composition series if and only if it has finite dimensional values; such functors are called *finite*. Finally, every functor is the union of its finite subfunctors.

(2) Arbitrary direct sums of injective functors are injective, and every functor can be embedded into a direct sum of finite injectives.

(3) Any nonzero strict polynomial functor has a nonzero socle, a nonzero head and a finite Loewy length.

2B. *Frobenius twists.* let $\Bbbk$ be a field of positive characteristic p. For all $r \geq 0$, we denote by $I^{(r)}$ the r-th Frobenius twist functor. The functor $I^{(0)} = I = S^1 = \Gamma^1 = \Lambda^1$ is the identity functor. More generally, for all $r \geq 0$ the functor $I^{(r)}$ is the unique simple additive functor of degree p^r (up to isomorphism).

Notation 2.1. We use the traditional notation $F^{(r)} = F \circ I^{(r)}$. We also denote by $F \otimes G^{(r)}$ the tensor product of F and $G^{(r)}$, i.e., Frobenius twists have a priority higher than tensor products in our notations.

The effect on Ext^* of precomposition by Frobenius twist is now well understood in all degrees [Touzé 2013a; Chałupnik 2015]. In particular, in degrees $i = 0$ or $i = 1$, the $\Bbbk$-linear morphism

$$\mathrm{Ext}^i_{\mathcal{P}_{\Bbbk}}(F,G) \to \mathrm{Ext}^i_{\mathcal{P}_{\Bbbk}}(F^{(r)}, G^{(r)})$$

induced by precomposition by $I^{(r)}$ is an isomorphism. This description of the effect of precomposition by Frobenius twists in degrees 0 and 1 can be proved by very elementary means; see, e.g., [Breen et al. 2016, Appendix A]. We will not need to know about higher degrees, except in the proof of Proposition 7.3.

2C. *Elementary facts regarding simple functors.* Traditionally, simple polynomial $GL_n(\Bbbk)$-modules are classified by examining the action of a maximal torus on $GL_n(\Bbbk)$-modules, that is, using the concept of highest weights; see, e.g., [Martin 1993, Chapter 1]. In the sequel of the article, we shall use the following consequences of this classification.

(1) Isomorphism classes of simple functors are in bijective correspondence with partitions. For each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ we fix a simple functor L_λ in the corresponding isomorphism class. Then L_λ is homogeneous of degree $\sum \lambda_i$. We call λ the *highest weight* of L_λ. Indeed, by evaluating on $\Bbbk^n$, we obtain a simple polynomial module $L_\lambda(\Bbbk^n)$ with highest weight λ. For example, the only simple functor of degree 0 is $L_{(0)} = \Bbbk$.

(2) Simple functors are self-dual. To be more specific, each simple functor L is isomorphic to its dual $L^\sharp$, defined by $L^\sharp(V) := L(V^*)^*$.

(3) Simple functors have endomorphism rings of dimension 1.

(4) For all partitions λ and μ and all $r \geq 0$, $L_{\lambda + p^r \mu}$ is a composition factor of $L_\lambda \otimes L_\mu^{(r)}$.

Remark 2.2. Actually, one needs the fact that $\Bbbk$ is algebraically closed to obtain easily (by Schur's lemma) that the endomorphism ring of a simple functor has dimension one. When $\Bbbk$ is not algebraically closed, this can be proved using the fact that Schur algebras are quasihereditary; see, e.g., [Martin 1993, Chapter 3].

2D. *Bifunctors and sum-diagonal adjunction.* We will need strict polynomial functors with several variables for intermediate computations, as well as in the study of tensor products of Steinberg type in Section 5. Definitions and basic properties of strict polynomial functors extend without problem to the case of functors with several variables, and we refer to [Suslin et al. 1997, Section 2], [Touzé 2010, Section 2] or [Touzé 2017b, Section 3] for details. We recall here the main features of the theory in the context of bifunctors, and leave to the reader the obvious formulas with three variables or more.

Given two nonnegative integers d_1 and d_2, we denote by $\mathcal{P}_{d_1, d_2, \Bbbk}$ the category of homogeneous strict polynomial bifunctors of bidegree (d_1, d_2) (with possibly infinite dimensional values). Typical examples of objects of this category are the *bifunctors of separable type*, which are the bifunctors of the form

$$F \boxtimes G : (V, W) \mapsto F(V) \otimes G(W),$$

where F and G are homogeneous strict polynomial functors of degree d_1 and d_2, respectively. Just as in the one variable case, evaluating bifunctors on a pair of vector spaces $(\Bbbk^n, \Bbbk^m)$ yields a functor

$$\mathcal{P}_{d_1, d_2, \Bbbk} \to S(n, d_1) \otimes S(m, d_2)\text{-Mod},$$

where $S(m, d_1)$ and $S(m, d_2)$ are Schur algebras (which are finite dimensional). Moreover, this functor is an equivalence of categories if $n \geq d_1$ and $m \geq d_2$. In particular $\mathcal{P}_{d_1, d_2, \Bbbk}$ satisfies the three properties mentioned at the end of Section 2A. We have a Künneth morphism

$$\text{Ext}^*_{\mathcal{P}_{d_1, \Bbbk}}(F_1, G_1) \otimes \text{Ext}^*_{\mathcal{P}_{d_2, \Bbbk}}(F_2, G_2) \xrightarrow{\kappa} \text{Ext}^*_{\mathcal{P}_{d_1, d_2, \Bbbk}}(F_1 \boxtimes F_2, G_1 \boxtimes G_2),$$

which is an isomorphism if the quadruple (F_1, G_1, F_2, G_2) satisfies the following condition.

Condition 2.3 (Künneth condition). In the quadruple (F_1, G_1, F_2, G_2), F_1 and F_2 are both finite functors, or F_1 and G_1 are both finite functors.

We also denote by $\mathcal{P}_{d, \Bbbk}(2)$ the category of homogeneous strict polynomial bifunctors of total degree d, and by $\mathcal{P}_{\Bbbk}(2)$ the category of strict polynomial functors of bounded degree, with possibly infinite dimensional values. We have decompositions

$$\mathcal{P}_{\Bbbk}(2) = \bigoplus_{d \geq 0} \mathcal{P}_{d, \Bbbk}(2), \qquad \mathcal{P}_{d, \Bbbk}(2) = \bigoplus_{d_1 + d_2 = d} \mathcal{P}_{d_1, d_2, \Bbbk}.$$

In particular, each bifunctor B decomposes uniquely as a direct sum $B = \bigoplus B^{(d_1, d_2)}$, where $B^{(d_1, d_2)}$ is a homogeneous strict polynomial bifunctor of bidegree (d_1, d_2). We shall refer to $B^{(d_1, d_2)}$ as the *homogeneous component of bidegree (d_1, d_2) of B*. A typical example of (degree d homogeneous) bifunctor is the bifunctor

$$F_{\boxplus} : (V, W) \mapsto F(V \oplus W),$$

where F is a (degree d homogeneous) strict polynomial functor of degree d. Conversely, from a (degree d homogeneous) bifunctor B of total degree d one can construct a (degree d homogeneous) strict polynomial functor with one variable by *diagonal evaluation*:

$$B_\Delta : V \mapsto B(V, V).$$

These two constructions are exact and adjoint to each other on both sides. Hence we have graded isomorphisms

$$\text{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(B, F_{\boxplus}) \simeq \text{Ext}^*_{\mathcal{P}_{\Bbbk}}(B_\Delta, F),$$
$$\text{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F_{\boxplus}, B) \simeq \text{Ext}^*_{\mathcal{P}_{\Bbbk}}(F, B_\Delta).$$

These two isomorphisms were first used in the context of strict polynomial functors in [Franjou et al. 1999]. In this article, they will be the key tool for Theorem 3.6. As in [Franjou et al. 1999], we will often use them when B is of separable type $B = G \boxtimes H$, and hence when $B_\Delta = G \otimes H$.

2E. *The internal tensor product.* The category $\mathcal{P}_{d,\Bbbk}$ is endowed with a closed symmetric monoidal structure. We denote this internal tensor product by $\underline{\otimes}$, and by $\underline{\mathrm{Hom}}$ the associated internal hom. We refer the reader to [Krause 2013] for a presentation of this internal tensor product. We study the internal tensor product of simple functors in Section 6. For this purpose, we will use the following facts.

(1) If F is a functor, we denote by F^V the parametrized functor

$$W \mapsto F(\mathrm{Hom}_{\Bbbk}(V, W)).$$

Then the internal Hom is the functor given by

$$\underline{\mathrm{Hom}}(F, G)(V) = \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F^V, G).$$

(2) The study of internal tensor products can be reduced to the study of internal Hom by using the isomorphism natural with respect to F, G:

$$(F \underline{\otimes} G)^{\sharp} \simeq \underline{\mathrm{Hom}}(F, G^{\sharp}).$$

Here $^{\sharp}$ is the duality defined by $F^{\sharp}(V) = F(V^*)^*$, where $*$ is the $\Bbbk$-linear duality of vector spaces.

Remark 2.4. Schur algebras do not have a Hopf algebra structure in general. (Indeed, Schur algebras have finite global dimension, and a Hopf algebra structure would make them self-injective in addition, and hence semisimple.) Thus the internal tensor product on $\mathcal{P}_{d,\Bbbk}$ is an example of a monoidal product which does not come from a Hopf algebra structure.

2F. *Connection with representations of symmetric groups.* The Schur functors relate strict polynomial functors to representations of the symmetric groups $\mathfrak{S}_d$. We will use these Schur functors in Sections 6 and 8. Let d be a positive integer. Consider the right action of the symmetric group $\mathfrak{S}_d$ on $\otimes^d$ given by permuting the factors of the tensor product. The Schur functor is the functor

$$f_d := \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(\otimes^d, -) : \mathcal{P}_{d,\Bbbk} \to \Bbbk\mathfrak{S}_d\text{-Mod}.$$

Since $\otimes^d$ is projective, the Schur functor f_d is exact. It has adjoints on both sides. To be more specific, the left adjoint ℓ_d is defined by $\ell_d(M) = (\otimes^d) \otimes_{\mathfrak{S}_d} M$, while the right adjoint r_d is defined by $r_d(M) = ((\otimes^d) \otimes M)^{\mathfrak{S}_d}$. The unit and counit of adjunction induce natural isomorphisms

$$M \xrightarrow{\sim} f_d(\ell_d(M)), \qquad f_d(r_d(M)) \xrightarrow{\sim} M.$$

In particular, the Schur functor f_d is a quotient functor.

3. Exts in low degrees between tensor products

3A. *Definition of $i(F, r)$ and $p(F, r)$.* For all tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$ of nonnegative integers, we let

$$\Gamma^\lambda := \Gamma^{\lambda_1} \otimes \cdots \otimes \Gamma^{\lambda_n} \quad \text{and} \quad S^\lambda := S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_n}.$$

Let $\mathcal{T}$ denote the set of all tuples of nonnegative integers. Then the family $(\Gamma^\lambda)_{\lambda \in \mathcal{T}}$ forms a projective generator of $\mathcal{P}_\Bbbk$, while the family $(S^\lambda)_{\lambda \in \mathcal{T}}$ forms an injective cogenerator of $\mathcal{P}_\Bbbk$.

Definition 3.1. Let r be a nonnegative integer. A tuple of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_n)$ is p^r*-bounded* if $\lambda_k < p^r$ for all k. A *basic p^r-bounded projective* (resp. *injective*) is a functor of the form Γ^λ (resp. S^λ), where λ is p^r-bounded. A strict polynomial functor F is *left p^r-bounded* if it is a quotient of a direct sum of basic p^r-bounded projectives. Similarly, F is *right p^r-bounded* if it embeds in a product of basic p^r-bounded injectives.

Remark 3.2. If $r = 0$, the tuple $(0, \ldots, 0)$ is the only p^r-bounded tuple. Since $\Gamma^0 = S^0 = \Bbbk$, a functor is p^0-bounded if and only if it is constant.

The following lemma collects elementary facts on p^r-bounded functors.

Lemma 3.3. (1) *The following statements are equivalent*:
  (i) *F is right p^r-bounded*,
  (ii) *$\mathrm{Soc}(F)$ is right p^r-bounded*,
  (iii) *F embeds into a direct sum of basic p^r-bounded injectives.*

 (2) *The following statements are equivalent*:
  (i′) *F is left p^r-bounded*,
  (ii′) *$\mathrm{Head}(F)$ is left p^r-bounded*,
  (iii′) *F is the union of finite left p^r-bounded subfunctors.*

Proof. (1) It is clear that (iii)$\Rightarrow$(i)$\Rightarrow$(ii). If (ii) holds, then each simple summand of $\mathrm{Soc}(F)$ embeds into a basic p^r-bounded injective. Thus $\mathrm{Soc}(F)$ embeds into a direct sum of basic p^r-bounded injectives J. Since J is injective, the monomorphism $\mathrm{Soc}(F) \hookrightarrow J$ extends to a map $\phi : F \to J$. But $\mathrm{Soc}(\ker \phi) \subset \ker \phi \cap \mathrm{Soc}(F) = 0$ so ϕ is injective. This proves (iii).

(2) It is clear that (i′)$\Rightarrow$(ii′). The proof of (ii′)$\Rightarrow$(i′) is dual to the one of (ii)$\Rightarrow$(iii). Let us prove (i′)$\Leftrightarrow$(iii′). If F is left p^r-bounded, there is a map $\pi : \bigoplus_{\lambda \in A} \Gamma^\lambda \twoheadrightarrow F$. Thus F is the union of the images of the $\pi\left(\bigoplus_{\lambda \in B} \Gamma^\lambda\right)$, where B is a finite subset of A. Conversely, if F is the union of a family of finite left p^r-bounded functors F_α, then F is a quotient of $\bigoplus F_\alpha$. Hence F is left p^r-bounded. $\square$

Definition 3.4. Let r be a nonnegative integer, and let F be a strict polynomial functor.

(1) We define $p(F, r) \in [0, +\infty]$ as the supremum of all the integers $n \geq 0$ such that F admits a projective resolution P in which the first n objects $P_0, \ldots, P_{n-1}$ are left p^r-bounded.

(2) We define $i(F, r) \in [0, +\infty]$ as the supremum of all the integers $n \geq 0$ such that F admits an injective resolution J in which the first n objects $J^0, \ldots, J^{n-1}$ are right p^r-bounded.

Remark 3.5. (i) By definition $p(F, r) > 0$ if and only if F is left p^r-bounded, and $i(F, r) > 0$ if and only if F is right p^r-bounded.

(ii) If $p^r > \deg F$, then all projectives or injectives appearing in any resolution of F are p^r-bounded, so $p(F, r) = i(F, r) = +\infty$. In particular, if F is constant, it is homogeneous of degree 0 and $i(F, r) = p(F, r) = +\infty$ for all $r \geq 0$.

(iii) In the definition, $p(F, r)$ and $i(F, r)$ belong to $[0, +\infty]$. However, the category $\mathcal{P}_{d, \Bbbk}$ has finite global dimension $\mathrm{gldim}(d, \Bbbk)$, which is explicitly computed in [Totaro 1997]. If F is homogeneous of degree d, then $p(F, r)$ and $i(F, r)$ actually belong to $[0, \ldots, \mathrm{gldim}(d, \Bbbk)] \cup \{+\infty\}$.

3B. *Application to the connectedness of cup products.* The tensor product on $\mathcal{P}_{\Bbbk}$, $\otimes : \mathcal{P}_{\Bbbk} \times \mathcal{P}_{\Bbbk} \to \mathcal{P}_{\Bbbk}$, induces a cup product on extension groups in the usual way; see, e.g., [Benson 1998, Section 3.2]. The purpose of this section is to prove the following result.

Theorem 3.6. *Let (F, G, X, Y) be a quadruple of homogeneous strict polynomial functors satisfying the Künneth condition (Condition 2.3), and let $r \geq 0$. The cup product induces a graded injective map*

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F, G) \otimes \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(X^{(r)}, Y^{(r)}) \hookrightarrow \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F \otimes X^{(r)}, G \otimes Y^{(r)}).$$

Moreover, this graded injective map is an isomorphism in degree k in the following situations:

(1) *when $\deg F < \deg G$, and $k < i(G, r)$;*

(2) *when $\deg F > \deg G$, and $k < p(F, r)$;*

(3) *when $\deg F = \deg G$, and $k < p(F, r) + i(G, r)$.*

Remark 3.7. If $\deg F \neq \deg G$ then the domain of the cup product is zero, as there is no nonzero Ext between homogeneous functors of different degrees. Thus, in cases (1) and (2), Theorem 3.6 merely says that the codomain of the cup product is zero in low degrees.

The remainder of Section 3 is devoted to the proof of Theorem 3.6. Observe that we have a factorization of cup products

$$\text{Ext}^*_{\mathcal{P}_\Bbbk}(F, G) \otimes \text{Ext}^*_{\mathcal{P}_\Bbbk}(X^{(r)}, Y^{(r)}) \xrightarrow{\ \cup\ } \text{Ext}^*_{\mathcal{P}_\Bbbk}(F \otimes X^{(r)}, G \otimes Y^{(r)})$$

$$\downarrow{\scriptstyle \kappa} \qquad\qquad {\scriptstyle -_\Delta}$$

$$\text{Ext}^*_{\mathcal{P}_\Bbbk(2)}(F \boxtimes X^{(r)}, G \boxtimes Y^{(r)})$$

In particular, Theorem 3.6 is a consequence of the following slightly more general statement, in which the Künneth condition is removed.

Theorem 3.8. *Let F, G, X, Y be homogeneous functors, and let $r \geq 0$. Diagonal evaluation induces a graded injective map*

$$\text{Ext}^*_{\mathcal{P}_\Bbbk(2)}(F \boxtimes X^{(r)}, G \boxtimes Y^{(r)}) \hookrightarrow \text{Ext}^*_{\mathcal{P}_\Bbbk}(F \otimes X^{(r)}, G \otimes Y^{(r)}).$$

Moreover, this graded injective map is an isomorphism in degree k in the situations listed in Theorem 3.6.

The proof of Theorem 3.8 relies on a series of lemmas. The proofs of these lemmas are all based upon the sum-diagonal adjunction technique recalled in Section 2D.

Lemma 3.9. *Let F, G, F', G' be homogeneous functors satisfying $\deg F = \deg G$ and $\deg F' = \deg G'$. Diagonal evaluation yields an injective map*

$$\text{Ext}^*_{\mathcal{P}_\Bbbk(2)}(F \boxtimes F', G \boxtimes G') \hookrightarrow \text{Ext}^*_{\mathcal{P}_\Bbbk}(F \otimes F', G \otimes G')$$

whose cokernel is isomorphic to the following direct sum, indexed by the tuples of nonnegative integers (d_1, d_2, e_1, e_2) such that $d_2 > 0$ and $e_1 > 0$:

$$\bigoplus_{\substack{0 < d_2, e_1 \\ 0 \leq d_1, e_2}} \text{Ext}^*_{\mathcal{P}_\Bbbk(2)}\big(F \boxtimes F', (G_\boxplus)^{(d_1, d_2)} \otimes (G'_\boxplus)^{(e_1, e_2)}\big).$$

This cokernel is also isomorphic to the direct sum

$$\bigoplus_{\substack{0 < d_2, e_1 \\ 0 \leq d_1, e_2}} \text{Ext}^*_{\mathcal{P}_\Bbbk(2)}\big((F_\boxplus)^{(d_1, d_2)} \otimes (F'_\boxplus)^{(e_1, e_2)}, G \boxtimes G'\big).$$

Proof. We recall the proof of injectivity from [Touzé 2010] and prove the first description of the cokernel. The proof of the second description is similar. The map given by diagonal evaluation is equal to the composite of the map

$$\eta_* : \text{Ext}^*_{\mathcal{P}_\Bbbk(2)}(F \boxtimes F', G \boxtimes G') \to \text{Ext}^*_{\mathcal{P}_\Bbbk(2)}(F \boxtimes F', (G \otimes G')_\boxplus)$$

induced by the canonical map $\eta : G \boxtimes G' \to (G \otimes G')_\boxplus$, together with the adjunction isomorphism

$$\text{Ext}^*_{\mathcal{P}_\Bbbk(2)}(F \boxtimes F', (G \otimes G')_\boxplus) \simeq \text{Ext}^*_{\mathcal{P}_\Bbbk}(F \otimes F', G \otimes G').$$

Thus, to prove Lemma 3.9, it suffices to prove that η_* is injective and to identify its cokernel. But

$$(G \otimes G')_{\boxplus} = G_{\boxplus} \otimes G'_{\boxplus},$$

and there is a decomposition

$$(G \otimes G')_{\boxplus} = G \boxtimes G' \oplus \bigoplus_{d_2 > 0 \text{ or } e_1 > 0} (G_{\boxplus})^{(d_1, d_2)} \otimes (G'_{\boxplus})^{(e_1, e_2)}.$$

The map η identifies with the inclusion of $G \boxtimes G'$ into the right-hand side, and since the decomposition is a direct sum, it follows that η_* admits a section, and the cokernel of η_* is isomorphic to

$$\bigoplus_{d_2 > 0 \text{ or } e_1 > 0} \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F \boxtimes F', (G_{\boxplus})^{(d_1, d_2)} \otimes (G'_{\boxplus})^{(e_1, e_2)}).$$

This is almost the description of the cokernel given in Lemma 3.9, but the summation index is different. Since there are no nonzero extensions between homogeneous bifunctors of different degrees, all the terms in the direct sum are zero, except the ones satisfying $d_1 + e_1 = \deg F$ and $d_2 + e_2 = \deg F'$. Since $d_1 + d_2 = \deg G = \deg F$, the nonzero terms in the direct sum satisfy $e_1 = d_2$. Thus we can replace the summation index "$d_2 > 0$ or $e_1 > 0$" by "$e_1 > 0$ and $d_2 > 0$" and we are done. $\qquad\square$

The proof of the next lemma is omitted since it is very similar to the proof of Lemma 3.9.

Lemma 3.10. *Let F, F', G, G' be homogeneous functors. If $\deg F > \deg G$, then $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F \otimes F', G \otimes G')$ is isomorphic to the following direct sum, indexed by the tuples of nonnegative integers (d_1, d_2, e_1, e_2) such that $e_1 > 0$:*

$$\bigoplus_{\substack{0 < e_1 \\ 0 \le d_1, d_2, e_2}} \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F \boxtimes F', (G_{\boxplus})^{(d_1, d_2)} \otimes (G'_{\boxplus})^{(e_1, e_2)}).$$

If $\deg F < \deg G$, then it is isomorphic to

$$\bigoplus_{\substack{0 < e_1 \\ 0 \le d_1, d_2, e_2}} \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}((F_{\boxplus})^{(d_1, d_2)} \otimes (F'_{\boxplus})^{(e_1, e_2)}, G \boxtimes G').$$

The next two vanishing lemmas are analogues of the key vanishing result (i.e., Pirashvili's vanishing) of [Friedlander and Suslin 1997, Theorem 2.13].

Lemma 3.11. *Let F and G be homogeneous functors with $\deg G > 0$ and let λ be a p^r-bounded tuple. Then*

$$\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F \otimes G^{(r)}, S^{\lambda}) = 0 = \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(\Gamma^{\lambda}, F \otimes G^{(r)}).$$

Proof. We prove the first equality. The proof of the second one is similar. We will use the fact that for all homogeneous G of positive degree and for all p^r-bounded tuples ν,

$$\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(G^{(r)}, S^{\nu}) = 0. \tag{$*$}$$

This is proved when G has finite dimensional values in [Touzé 2012, Lemma 2.3], and it holds for an arbitrary G because any functor is the colimit of its finite subfunctors. (Alternatively, one could also prove this vanishing by sum-diagonal adjunction.) To reduce the equality of Lemma 3.11 to formula $(*)$, we proceed as follows. First, sum-diagonal adjunction yields an isomorphism:

$$\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F \otimes G^{(r)}, S^{\lambda}) \simeq \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F \boxtimes G^{(r)}, (S^{\lambda})_{\boxplus}).$$

We observe that $(S^{\lambda})_{\boxplus}$ decomposes as a direct sum of tensor products of the form $S^{\mu} \boxtimes S^{\nu}$ such that μ and ν are p^r-bounded. Thus Lemma 3.11 will be proved if we can prove that $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F \boxtimes G^{(r)}, S^{\mu} \boxtimes S^{\nu})$ is zero when μ and ν are p^r-bounded. So let $\phi : F \boxtimes G^{(r)} \to S^{\mu} \boxtimes S^{\nu}$ be a morphism. By freezing the first variable of the bifunctors, we obtain for all V a morphism of functors

$$\phi_V : F(V) \otimes G^{(r)}(-) \to S^{\mu}(V) \otimes S^{\nu}(-).$$

By formula $(*)$, ϕ_V must be zero for all V. In particular, ϕ must be zero. $\qquad\square$

Lemma 3.12. *Let r be a positive integer, let J be a be a right p^r-bounded injective functor, let P be a left p^r-bounded projective functor, let Z be a homogeneous functor and let B and C be two homogeneous bifunctors. If $\deg C = (e_1, e_2)$ with $e_1 > 0$, and $C^{(r)}$ denotes the bifunctor $(V, W) \mapsto C(V^{(r)}, W^{(r)})$, then*

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(B \otimes C^{(r)}, J \boxtimes Z) = 0 = \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(P \boxtimes Z, B \otimes C^{(r)}).$$

Proof. We prove the first equality. The proof of the second one is similar. If J_Z is an injective resolution of the functor Z, then $J \boxtimes J_Z$ is an injective resolution of the bifunctor $J \boxtimes Z$. Thus, it is sufficient to do the proof in degree zero (i.e., for Hom) and when Z is injective, the general case will follow by taking resolutions. So let us take a morphism of bifunctors $\phi : B \otimes C^{(r)} \to J \boxtimes Z$. Then by freezing the first variable of the bifunctors, we obtain for all V a morphism of functors:

$$\phi_V : B(V, -) \otimes C^{(r)}(V, -) \to J(V) \otimes Z(-).$$

But by Lemma 3.11, ϕ_V is zero for all V. In particular, ϕ must be zero. $\qquad\square$

Proof of Theorem 3.8. By Lemma 3.9, diagonal evaluation yields an injective morphism on the Ext-level (if $\deg F \neq \deg G$ or $\deg X \neq \deg Y$, the source of the cup product morphism is zero for degree reasons, so that injectivity is trivial). Hence, it remains to prove the cancellation in low degrees of the cokernel, described in Lemmas 3.9 and 3.10.

Assume that $\deg F \geq \deg G$. Take a finite resolution of F of the form:

$$0 \to \bar{F} \to F_{p(F,r)-1} \to \cdots \to F_0 \to F \to 0$$

where the functors F_k with $k < p(F,r)$ are left p^r-bounded projective functors. Take B and C as in Lemma 3.12. By using long exact sequences, we obtain that for all $k \in \mathbb{Z}$ (with the convention that Ext are zero in nonpositive degrees):

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F \boxtimes X^{(r)}, B \otimes C^{(r)}) \simeq \mathrm{Ext}^{*-p(F,r)}_{\mathcal{P}_{\Bbbk}(2)}(\bar{F} \boxtimes X^{(r)}, B \otimes C^{(r)}). \qquad (*)$$

In particular the Ext on the left-hand side is $(p(F,r)-1)$-connected, i.e., zero in degrees $* < p(F,r)$. By Lemmas 3.9 and 3.10, the case where $B = (F_{\boxplus})^{(d_1,d_2)}$ and $C^{(r)} = (Y^{(r)}_{\boxplus})^{(e_1,e_2)}$ with $e_1 > 0$ implies that the cup product induces an isomorphism in degrees less than $p(F,r)$. A similar argument shows that the cup product is an isomorphism in degrees less than $i(G,r)$ if $\deg F \leq \deg G$. Assume now that $\deg F = \deg G$. By Lemma 3.9 and isomorphism $(*)$, the statement of Theorem 3.8 is equivalent to showing that

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}\big(\bar{F} \boxtimes X^{(r)}, (G_{\boxplus})^{(d_1,d_2)} \otimes (Y^{(r)}_{\boxplus})^{(e_1,e_2)}\big)$$

is $(i(G,r)-1)$-connected for $d_2 > 0$ and $e_1 > 0$. By Lemma 3.9 again, this is equivalent to showing that the cup product

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(\bar{F}, G) \otimes \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(X^{(r)}, Y^{(r)}) \to \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(\bar{F} \otimes X^{(r)}, G \otimes Y^{(r)})$$

is an isomorphism in degrees less than $i(G,r)$. But we have already proved that the latter holds, since $\deg \bar{F} \leq \deg G$. $\qquad \square$

4. An equivalent definition of $p(F,r)$ and $i(F,r)$

The next proposition gives an equivalent definition of $p(F,r)$ and $i(F,r)$. While the proof of Theorem 3.6 really relies on Definition 3.1, this new definition is useful for applying Theorem 3.6 in concrete situations. In particular, the translation of Theorem 3.6 in low degrees given in Corollary 4.4 will be used in Sections 5 and 6.

Recall that a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is p^r-restricted (for some nonnegative integer r) if $\lambda_n < p^r$ and for $i < n$, $\lambda_i - \lambda_{i+1} < p^r$. By convention, the partition (0) is p^r-restricted for all $r \geq 0$. Using euclidean division, one sees that any partition λ can be written in a unique way as a sum $\lambda = \lambda^0 + p^r \lambda^1$, where λ^0 is p^r-restricted. A simple indexed by a p^r-restricted partition will be loosely called a p^r-restricted simple. The proof of Proposition 4.1 relies on two classical fundamental results on simple polynomial representations in positive characteristic. We state them here with references to the literature, but we prove in Appendix B that both of them can actually be derived from Theorem 3.6.

(1) Steinberg's tensor product theorem [Jantzen 2003, II.3.17]. *If λ is p^r-restricted and μ is an arbitrary partition, then $L_\lambda \otimes L_\mu^{(r)}$ is isomorphic to $L_{\lambda+p^r\mu}$.*

(2) Clausen and James' theorem [Clausen 1980; James 1980]. *A partition λ of d is p-restricted if and only if $\mathrm{Hom}_{\mathcal{P}}(L_\lambda, \otimes^d) = \mathrm{Hom}_{\mathcal{P}}(\otimes^d, L_\lambda)$ is nonzero.*

Proposition 4.1. *Let r be a nonnegative integer, and let F be a functor.*

(1) *The integer $p(F, r)$ is the supremum of all $n \geq 0$ such that F admits a projective resolution P, in which the first n objects $P_0, \ldots, P_{n-1}$ are direct sums of projective covers of p^r-restricted simples.*

(2) *The integer $i(F, r)$ is the supremum of all $n \geq 0$ such that F admits an injective resolution J, in which the first n objects $J^0, \ldots, J^{n-1}$ are direct sums of injective envelopes of p^r-restricted simples.*

Proof. We restrict ourselves to proving the second statement; the proof of the first one is similar. Let us denote by J_μ the injective envelope of L_μ. We have to prove that

(i) for all p^r-restricted partitions μ, there is a p^r-bounded tuple λ such that J_μ is a direct summand of S^λ, and

(ii) for all p^r-bounded tuples λ, the indecomposable direct summands of S^λ are all isomorphic to J_μ with μ p^r-restricted.

Write $\mu = \sum_{i=0}^n p^i \mu^i$ for p-restricted partitions μ^i. By Steinberg's tensor product theorem, L_μ is isomorphic to

$$L_{\mu^0}^{(0)} \otimes \cdots \otimes L_{\mu^n}^{(n)}.$$

By Clausen and James' theorem, L_μ is a subfunctor of $(I^{(0)})^{\otimes|\mu^0|} \otimes \cdots \otimes (I^{(n)})^{\otimes|\mu^n|}$. Since $I^{(i)} \subset S^{p^i}$, we obtain that L_μ is a subfunctor of $\bigotimes_{0 \leq i \leq n} (S^{p^i})^{\otimes|\mu^i|}$. If μ is p^r-restricted then $n < r$, so L_μ (hence also J_μ) is a subfunctor of S^λ with λ p^r-bounded. This proves (i). Let λ be a p^r-bounded tuple, and let μ be a partition such that μ is not p^r-restricted. By Steinberg's tensor product theorem, L_μ is isomorphic to $L_{\mu'} \otimes L_{\mu''}^{(r)}$ for a p^r-restricted partition μ' and a nonzero partition μ''. So by Lemma 3.11, $\mathrm{Hom}_{\mathcal{P}}(L_\mu, S^\lambda)$ is zero, and hence J_μ is not a direct summand of S^λ. This proves (ii). $\square$

Corollary 4.2. *For all F, $i(F, r) > 0$ if and only if $\mathrm{Soc}(F)$ is a direct sum of p^r-restricted simples. Likewise, $p(F, r) > 0$ if and only if $\mathrm{Head}(F)$ is a direct sum of p^r-restricted simples.*

If $(d_0, \ldots, d_k)$ is a tuple of nonnegative integers, we let

$$T^{(d_0, \ldots, d_k)} = (\otimes^{d_0}) \otimes (\otimes^{d_1})^{(1)} \otimes \cdots \otimes (\otimes^{d_k})^{(k)}.$$

Corollary 4.3. *Let L be a simple functor. The following are equivalent*:

(i) $p(L, r) > 0$,

(ii) $i(L, r) > 0$,

(iii) L is p^r-restricted,

(iv) there exists a tuple $(d_0, \ldots, d_{r-1})$ such that L is a quotient of $T^{(d_0, \ldots, d_{r-1})}$, and

(v) there exists a tuple $(d_0, \ldots, d_{r-1})$ such that L embeds into $T^{(d_0, \ldots, d_{r-1})}$.

Proof. We have (i)⇔ (ii) ⇔ (iii) by Corollary 4.2, (iv) ⇔ (v) by Kuhn duality (both L and $T^{(d_0, \ldots, d_{r-1})}$ are self-dual) and (iii) ⇒ (iv) by Steinberg's tensor product theorem and Clausen and James' theorem (as used in the proof of Proposition 4.1). Finally, the functor $T^{(d_0, \ldots, d_{r-1})}$ is a quotient of $(\Gamma^1)^{\otimes d_0} \otimes \cdots \otimes (\Gamma^{p^{r-1}})^{\otimes d_r}$, so that $p(T^{(d_0, \ldots, d_{r-1})}, r) > 0$. Hence (iv)⇒(iii) by Corollary 4.2. □

The next corollary is a translation of Theorem 3.6 in low degrees in terms of p^r-restricted weights. It will be used in Sections 5 and 6.

Corollary 4.4. *Let (F, G, X, Y) be a quadruple of homogeneous strict polynomial functors satisfying the Künneth condition (Condition 2.3), and let $r \geq 0$. Precomposing by $I^{(r)}$ and taking cup products induces injective morphisms*

$$\operatorname{Hom}_{\mathcal{P}_{\Bbbk}}(F, G) \otimes \operatorname{Hom}_{\mathcal{P}_{\Bbbk}}(X, Y) \hookrightarrow \operatorname{Hom}_{\mathcal{P}_{\Bbbk}}(F \otimes X^{(r)}, G \otimes Y^{(r)}), \quad (4.5)$$

$$\begin{aligned}\operatorname{Hom}_{\mathcal{P}_{\Bbbk}}(F, G) \otimes \operatorname{Ext}^1_{\mathcal{P}_{\Bbbk}}(X, Y) \\ \oplus \operatorname{Ext}^1_{\mathcal{P}_{\Bbbk}}(F, G) \otimes \operatorname{Hom}_{\mathcal{P}_{\Bbbk}}(X, Y)\end{aligned} \hookrightarrow \operatorname{Ext}^1_{\mathcal{P}_{\Bbbk}}(F \otimes X^{(r)}, G \otimes Y^{(r)}). \quad (4.6)$$

If one of the conditions (C1) *or* (C2) *below holds, morphism* (4.5) *is an isomorphism. If both* (C1) *and* (C2) *hold, then morphism* (4.6) *is also an isomorphism.*

(C1) $\deg F \leq \deg G$ *and* $\operatorname{Head}(G)$ *is a direct sum of p^r-restricted simples.*

(C2) $\deg F \geq \deg G$ *and* $\operatorname{Soc}(F)$ *is a direct sum of p^r-restricted simples.*

Proof. Recall from Section 2B that precomposing by $I^{(r)}$ yields a $\Bbbk$-linear isomorphism on the level of Hom and Ext^1. Thus the statement of Corollary 4.4 is equivalent to the statement where X and Y are replaced by $X^{(r)}$ and $Y^{(r)}$ at the source of the cup product maps. By Corollary 4.2, (C1) means that $i(G, r) > 0$, and (C2) that $i(F, r) > 0$. Thus Corollary 4.2 follows directly from Theorem 3.6. □

Remark 4.7. In Sections 5 and 6 we will use Corollary 4.4 for quite general functors F and G. However, this result is already interesting in the very special case where F and G are p^r-restricted simples. Indeed the isomorphism given by Corollary 4.4 is then a stronger form, albeit valid only for stable polynomial representations of GL_n, of formulas of Donkin [1982] and Andersen [1984]; see also [Jantzen 2003, II.10.16, II.10.17]. For example, Corollary 4.4 implies that if $\lambda \neq \lambda'$ are partitions of d and $G = \operatorname{GL}_n$ with n big enough, then the number of L_μ

in the socle of the tensor product $\operatorname{Ext}^1_{G_r}(L_\lambda, L_{\lambda'})^{(-r)} \otimes L'_\mu$ is zero, unless $\mu = \mu'$, in which case it equals the dimension of $\operatorname{Ext}^1_G(L_\lambda, L_{\lambda'})$.

5. Tensor products of Steinberg type

Recall that a simple functor L is a composition factor of an arbitrary functor F if L is a subquotient of F. Equivalently, if $0 = F^{-1} \subset F^0 \subset \cdots \subset F$ is an exhaustive filtration of F whose successive quotients are semisimple (e.g., the Loewy filtration of F), then L appears as a direct summand in one of these successive quotients.

Definition 5.1. A tensor product of Steinberg type is a strict polynomial functor isomorphic to a tensor product $F \otimes G^{(r)}$, where r is a nonnegative integer and F is a functor whose composition factors are all p^r-restricted.

The purpose of the present section is to explore the structure of these tensor products of Steinberg type. Note that by Steinberg's tensor product theorem (applied to the tensor product of the socle filtration of F by the socle filtration of $G^{(r)}$), all composition factors of $F \otimes G^{(r)}$ are of the form $L_\lambda \otimes L_\mu^{(r)}$, with L_λ a composition factor of F and L_μ a composition factor of G. This observation motivates the following definition.

Definition 5.2. Let e, d, r be nonnegative integers. We let $\mathcal{St}(d, e, r)$ be the full subcategory of $\mathcal{P}_{d+ep^r, \Bbbk}$ supported by the strict polynomial functors whose composition factors are all of the form $L_\lambda \otimes L_\mu^{(r)}$ for p^r-restricted partition λ of d and partitions μ of e.

Lemma 5.3. *The category $\mathcal{St}(d, e, r)$ contains all the tensor products of Steinberg type $F \otimes G^{(r)}$, where F is homogeneous of degree d and G is homogeneous of degree e. Moreover, it is localizing and colocalizing, i.e., it is closed under sums, products, subobjects, quotients and extensions.*

Proof. Everything is straightforward from the definition of $\mathcal{St}(d, e, r)$ except maybe that $\mathcal{St}(d, e, r)$ is closed under arbitrary products. Let L be a composition factor of a product $\prod X_\alpha$. Then there is a nonzero map $P_L \to \prod X_\alpha$, where P_L denotes the projective cover of L. Thus there is an α such that $\operatorname{Hom}_{\mathcal{P}_\Bbbk}(P_L, X_\alpha) \neq 0$, so that $L = L_\lambda \otimes L_\mu^{(r)}$ with λ p^r-restricted. $\qquad\square$

The next lemma makes critical use of Corollary 4.4.

Lemma 5.4. *In the category $\mathcal{St}(d, e, r)$, any object X has a presentation $P_1 \to P_0 \to X \to 0$ in which the P_i are direct sums of tensor products of Steinberg type $F \otimes G^{(r)}$ with F and G finite. Similarly, X has a copresentation $0 \to X \to Q^0 \to Q^1$ in which the Q^i are products of such tensor products.*

Proof. It suffices to prove that all the objects of $\mathcal{St}(d, e, r)$ are quotients of direct sums of tensor products of Steinberg type with values in finite dimensional spaces

(then using the duality $\sharp$, they will also embed into products of such functors). Let X be an object of $\mathcal{S}t(d, e, r)$, and let X^i denote the i-th term of the socle filtration of X. Assume that X^{i-1} is a quotient of P^{i-1}, where P^{i-1} has the required form. Then X^i/X^{i-1} is a direct sum of $L_\lambda \otimes L_\mu^{(r)}$, and each of these functors is a homomorphic quotient of $P_\lambda \otimes P_\mu^{(r)}$, where P_μ and P_λ are projective functors, and P_λ is left p^r-bounded. Using Corollary 4.4 and the projectivity of P_μ and P_λ, we obtain $\mathrm{Ext}^1_{\mathcal{P}_{\mathbb{k}}}(P_\lambda \otimes P_\mu^{(r)}, X^{i-1}) = 0$. Hence the map $P_\lambda \otimes P_\mu^{(r)} \to X^i/X^{i-1}$ lifts to $f : P_\lambda \otimes P_\mu^{(r)} \to X^i$. The functor P_λ has a unique largest quotient P_λ' whose composition factors are p^r-restricted. Let K_λ be the kernel of the quotient map $P_\lambda \twoheadrightarrow P_\lambda'$. By Corollary 4.4, $\mathrm{Hom}_{\mathcal{P}_{\mathbb{k}}}(K_\lambda \otimes P_\mu^{(r)}, X^i) = 0$. Therefore, f induces a map $P_\lambda' \otimes P_\mu^{(r)} \to X^i$. In particular, if we define $P^i := P^{i-1} \oplus \bigoplus P_\lambda' \otimes P_\mu^{(r)}$, then P^i is a direct sum of tensor products of Steinberg type with values in finite dimensional vector spaces, and X^i is a quotient of P^i. Since homogeneous strict polynomial functors have finite socle filtrations, this proves the lemma. $\square$

We will prove that the categories $\mathcal{S}t(d, e, r)$ have an alternative description in terms of bifunctors. To be more specific, we denote by

$$\Phi : \mathcal{P}_{d,e,\mathbb{k}}(2) \to \mathcal{P}_{d+p^r e,\mathbb{k}}$$

the functor such that $\Phi(B)(V) = B(V, V^{(r)})$. We observe that Φ is exact, but it is not an equivalence of categories. For example, if $d = p^r$ and $e = 1$, the bifunctor $I^{(r)} \boxtimes I$ is simple, while its image $\otimes^{2(r)}$ is not ($\Lambda^{2(r)}$ is a proper subfunctor). However, Φ behaves better if we suitably restrict its source and target categories.

Definition 5.5. Let e, d, r be nonnegative integers. We denote by $\mathcal{S}t'(d, e, r)$ the full subcategory of $\mathcal{P}_{d,e,\mathbb{k}}(2)$ supported by the strict polynomial bifunctors whose composition factors are all of the form $L_\lambda \boxtimes L_\mu$, where λ is a p^r-restricted partition of weight d and μ is a partition of weight e.

We have the following analogues of Lemmas 5.3 and 5.4.

Lemma 5.6. *The subcategory $\mathcal{S}t'(d, e, r)$ contains all the separable functors $F \boxtimes G$, where F is homogeneous of degree d with p^r-restricted composition factors and G is homogeneous of degree e. Moreover, $\mathcal{S}t'(d, e, r)$ is closed under sums, products, subobjects, quotients and extensions.*

Lemma 5.7. *In the category $\mathcal{S}t'(d, e, r)$, any object B has a presentation $P_1 \to P_0 \to X \to 0$ in which the P_i are direct sums of tensor products of separable type $F \boxtimes G$, where F and G are finite and the composition factors of F are p^r-restricted. Similarly, B has a copresentation $0 \to X \to Q^0 \to Q^1$ in which the Q^i are products of such tensor products.*

We can now state the central theorem of this section.

Theorem 5.8. *The functor* Φ *restricts to an equivalence of categories*:

$$\Phi : \mathcal{S}t'(d, e, r) \xrightarrow{\simeq} \mathcal{S}t(d, e, r).$$

Proof. We first prove that Φ is fully faithful. Let $\mathcal{T}$ be the full subcategory of $\mathcal{S}t'(d, e, r)$ supported by the bifunctors of separable type $F \boxtimes G$ with F and G finite. By Lemma 5.7 and exactness of Φ, it suffices to prove that the restriction of Φ to $\mathcal{T}$ is fully faithful. This follows from Corollary 4.4. To prove that Φ is essentially surjective, we consider the functor $\Psi : \mathcal{P}_{d+ep^r, \Bbbk} \to \mathcal{P}_{d,e,\Bbbk}(2)$ which sends a functor F to the bifunctor

$$(\Psi F)(V, W) = \mathrm{Hom}_{\mathcal{P}_{d+p^r e, \Bbbk}}(\Gamma^{d, V} \otimes (\Gamma^{e, W})^{(r)}, F).$$

If $F \otimes G^{(r)}$ is a tensor product of Steinberg type, by Corollary 4.2 F is right p^r-bounded, so that Corollary 4.4 and [Friedlander and Suslin 1997, Theorem 2.10] yield isomorphisms of strict polynomial functors:

$$(\Phi\Psi(F \otimes G^{(r)}))(V) \simeq \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(\Gamma^{d,V}, F) \otimes \mathrm{Hom}_{\mathcal{P}_{e,\Bbbk}}(\Gamma^{e, V^{(r)}}, G)$$
$$\simeq F(V) \otimes G(V^{(r)}).$$

Thus $\Phi \circ \Psi$ is the identity functor on the tensor products of Steinberg type. By Lemma 5.4, all the functors of $\mathcal{S}t(d, e, r)$ are kernels of products of tensor products of Steinberg type. Thus by left exactness of $\Phi \circ \Psi$, the restriction of $\Phi \circ \Psi$ to the whole category $\mathcal{S}t(d, e, r)$ is isomorphic to the identity functor. Hence Φ is essentially surjective (and Ψ is the inverse of Φ). $\square$

Theorem 5.8 generalizes the Steinberg tensor product theorem. Indeed, external tensor products $L_\lambda \boxtimes L_\mu$ of simple functors are simple bifunctors, so that Theorem 5.8 and the stability of $\mathcal{S}t(d, e, r)$ by subobjects imply that the functors $\Phi(L_\lambda \boxtimes L_\mu) = L_\lambda \otimes L_\mu^{(r)}$ are simple. More generally, Theorem 5.8 can be used to convert any question about the structure of the tensor products of Steinberg type (socle length, submodule lattices, or even Ext^1 issues) into similar questions about the structure of bifunctors of separable type which are much easier to study. To illustrate this, we give new properties of tensor products of Steinberg type, obtained by translating some general properties of representations of tensor products of finite dimensional algebras given in Appendix A (recall that the category $\mathcal{P}_{d,e,\Bbbk}(2)$ is equivalent to the category of $S(d, d) \otimes S(e, e)$-modules).

Remark 5.9. In the following corollaries, we do not assume that F and G are homogeneous. In each case, the proof reduces easily to the homogeneous case by additivity of tensor products. We also observe that each of these corollaries is a stronger statement than the classical Steinberg tensor product theorem.

Corollary 5.10 (socle series). *If the composition factors of F are p^r-restricted, then for all G, the socle filtration of $F \otimes G^{(r)}$ is the tensor product of the socle filtration of F by the socle filtration of G, precomposed by $I^{(r)}$.*

Corollary 5.11 (subfunctors). *Assume that the composition factors of F are p^r-restricted. Let G be any functor. Assume that F or G is multiplicity free. Then the subfunctors $S \subset F \otimes G^{(r)}$ are of the form*

$$S = \sum_\alpha F_\alpha \otimes G_\alpha^{(r)}$$

for some families of subfunctors $F_\alpha \subset F$ and $G_\alpha \subset G$.

Corollary 5.12 (diagrams). *Assume that F and G are multiplicity-free and the composition factors of F are p^r-restricted. Then the diagram associated to $F \otimes G^{(r)}$ as defined in [Alperin 1980] has the functors $L_\lambda \otimes L_\mu^{(r)}$ as vertices, where L_λ is a composition factor of F and L_μ is a composition factor of G, and there is an edge $L_\lambda \otimes L_\mu^{(r)} \to L_\lambda' \otimes L_\mu'^{(r)}$ if and only if one of the following two conditions holds:*

(i) *$L_\lambda = L_\lambda'$ and there is an edge $L_\mu \to L_\mu'$ in the diagram of G,*

(ii) *$L_\mu = L_\mu'$ and there is an edge $L_\lambda \to L_\lambda'$ in the diagram of F.*

The next statement follows from Proposition A.8. It uses the fact that all simple strict polynomial functors satisfy $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(L, L) = 0$, which follows from the fact that the Schur algebras are quasihereditary.

Corollary 5.13 (tensor products on the left). *Let λ be a p^r-restricted partition. Let $L_\lambda \otimes \mathcal{P}_\Bbbk^{(r)}$ denote the full subcategory of $\mathcal{P}_\Bbbk$ whose objects are the functors isomorphic to tensor products of the form $L_\lambda \otimes F^{(r)}$. Then*

(1) *the subcategory $L_\lambda \otimes \mathcal{P}_\Bbbk^{(r)}$ is localizing and colocalizing,*

(2) *precomposing by $I^{(r)}$ and tensoring by L_λ yields an equivalence of categories $\mathcal{P}_\Bbbk \simeq L_\lambda \otimes \mathcal{P}_\Bbbk^{(r)}$.*

6. Application to internal tensor products

The purpose of this section is to study the internal tensor product of simple functors. In particular, Theorem 6.2 plays a role for internal tensor products similar to the role of the Steinberg theorem for ordinary tensor products.

6A. *Internal tensor products of simple functors.* Let F_1 and G_1 be two homogeneous functors of degree d, and F_2 and G_2 homogeneous functors of degree e. The internal tensor product is equipped with a coproduct

$$(F_1 \otimes F_2) \underline{\otimes} (G_1 \otimes G_2) \to (F_1 \underline{\otimes} G_1) \otimes (F_2 \underline{\otimes} G_2).$$

To be more specific, this coproduct coincides on the standard projectives with the following composite (where the first map is the canonical inclusion and the second one is the canonical projection):

$$(\Gamma^{d,T} \otimes \Gamma^{e,U}) \underline{\otimes} (\Gamma^{d,V} \otimes \Gamma^{e,W}) \hookrightarrow (\Gamma^{d+e,T\oplus U} \underline{\otimes} \Gamma^{d+e,V\oplus W})$$

$$= \Gamma^{d+e,(T\oplus U)\otimes(V\oplus W)}$$

$$\twoheadrightarrow \Gamma^{d,T\otimes V} \otimes \Gamma^{e,U\otimes W}$$

$$= (\Gamma^{d,T} \underline{\otimes} \Gamma^{d,V}) \otimes (\Gamma^{e,U} \underline{\otimes} \Gamma^{e,W}).$$

The following proposition is a consequence of Corollary 4.4.

Proposition 6.1. *Let F, G, X, Y be homogeneous strict polynomial functors, and let $r \geq 0$. If $\deg F < \deg G$ and G is left p^r-bounded, or if $\deg F > \deg G$ and F is left p^r-bounded, then*

$$(F \otimes X^{(r)}) \underline{\otimes} (G \otimes Y^{(r)}) = 0.$$

If $\deg F = \deg G$ and F or G is left p^r-bounded, then the coproduct induces an isomorphism

$$(F \otimes X^{(r)}) \underline{\otimes} (G \otimes Y^{(r)}) \simeq (F \underline{\otimes} G) \otimes (X \underline{\otimes} Y)^{(r)}.$$

Proof. In this proof, we assume that $\deg F \geq \deg G$ and F is p^r-bounded (the proof is similar if $\deg F \leq \deg G$ and G is p^r-bounded). Since the internal tensor product is right exact and commutes with arbitrary direct sums, it suffices to prove Proposition 6.1 when G and Y are finite.

Since F is left p^r-bounded, the parametrized functor F^V also is. Hence, if $\deg F > \deg G$, Corollary 4.4 implies that

$$\underline{\mathrm{Hom}}(F \otimes X^{(r)}, G^\sharp \otimes (Y^\sharp)^{(r)}) = 0. \tag{$*$}$$

Since G and Y are finite, $G^\sharp \otimes (Y^\sharp)^{(r)}$ is isomorphic to $(G \otimes Y^{(r)})^\sharp$. Hence the equality $(*)$ can be reinterpreted as

$$((F \otimes X^{(r)}) \underline{\otimes} (G \otimes Y^{(r)}))^\sharp = 0.$$

This proves the asserted cancellation. Assume now that $\deg F = \deg G$. Then by Corollary 4.4 the cup product induces an isomorphism

$$\underline{\mathrm{Hom}}(F, G) \otimes \underline{\mathrm{Hom}}(X, Y)^{(r)} \simeq \underline{\mathrm{Hom}}(F \otimes X^{(r)}, Y \otimes Y^{(r)}). \tag{$**$}$$

But the coproduct is dual to the cup product; that is, for all functors F, G, H and K there is a commutative diagram in which the horizontal isomorphisms are the canonical isomorphisms recalled in Section 2E:

$$(F \underline{\otimes} H)^{\sharp} \otimes (G \underline{\otimes} K)^{\sharp} \xrightarrow{\;\simeq\;} \underline{\mathrm{Hom}}(F, H^{\sharp}) \otimes \underline{\mathrm{Hom}}(G, K^{\sharp})$$

$$\Big\downarrow \text{can} \qquad\qquad\qquad\qquad \Big\downarrow \cup$$

$$((F \underline{\otimes} H) \otimes (G \underline{\otimes} K))^{\sharp} \qquad\qquad \underline{\mathrm{Hom}}(F \otimes H, G^{\sharp} \otimes K^{\sharp})$$

$$\Big\downarrow \text{coproduct}^{\sharp} \qquad\qquad\qquad \Big\downarrow \text{can}$$

$$((F \otimes H) \underline{\otimes} (G \otimes K))^{\sharp} \xrightarrow{\;\simeq\;} \underline{\mathrm{Hom}}(F \otimes H, (G \otimes K)^{\sharp})$$

If the functors G and K are finite, so is $G \underline{\otimes} K$ and the canonical maps denoted "can" in the diagram above are isomorphisms. Thus, the isomorphism of Proposition 6.1 can be deduced from the diagram above with $H = X^{(r)}$ and $K = Y^{(r)}$, and from the isomorphism $(\ast\ast)$. $\square$

The following theorem reduces the study of internal tensor products of simple functors to the case of p-restricted simple functors. In other terms, it plays the same role for internal tensor products as the classical Steinberg tensor product theorem does for ordinary tensor products.

Theorem 6.2. *Let $\lambda^0, \ldots, \lambda^r$ and $\mu^0, \ldots, \mu^s$ be p-restricted partitions, and let $\lambda = \sum p^i \lambda^i$ and $\mu = \sum p^i \mu^i$.*

(1) *If $r = s$ and μ^i and λ^i have the same weight for all i, then $L_\lambda \underline{\otimes} L_\mu$ is nonzero and there is an isomorphism*

$$L_\lambda \underline{\otimes} L_\mu \simeq (L_{\lambda^0} \underline{\otimes} L_{\mu^0}) \otimes (L_{\lambda^1} \underline{\otimes} L_{\mu^1})^{(1)} \otimes \cdots \otimes (L_{\lambda^r} \underline{\otimes} L_{\mu^r})^{(r)}.$$

(2) *Otherwise, $L_\lambda \underline{\otimes} L_\mu$ is zero.*

Proof. The classical Steinberg tensor product theorem shows that

$$L_\lambda = L_{\lambda^0} \otimes \cdots \otimes L_{\lambda^r}^{(r)} \quad \text{and} \quad L_\mu = L_{\mu^0} \otimes \cdots \otimes L_{\mu^s}^{(s)},$$

where the L_{λ^i} and the L_{μ^j} are p-restricted, hence right p-bounded by Corollary 4.2. Thus the result follows by applying Proposition 6.1. $\square$

6B. *The case of p-restricted simple functors.* To investigate internal tensor products of p-restricted simple functors, we rely on the Schur functor.

Lemma 6.3. *For all strict polynomial functors F, there are isomorphisms of functors, natural with respect to F:*

$$F \underline{\otimes} \otimes^d \simeq \underline{\mathrm{Hom}}(\otimes^d, F) \simeq \otimes^d \otimes f_d(F).$$

Moreover, if we consider the action of $\mathfrak{S}_d$ on the left-hand side induced by the left action of $\mathfrak{S}_d$ on $\otimes^d$, the action on the middle term induced by the right action of $\mathfrak{S}_d$ on $\otimes^d$, and the diagonal action of $\mathfrak{S}_d$ on the right-hand side, then these isomorphisms are $\mathfrak{S}_d$-equivariant.

Proof. We have isomorphisms of strict polynomial functors, natural with respect to V and F:

$$\underline{\mathrm{Hom}}(\Gamma^{d,V}, F) \simeq F_V \simeq F \otimes \Gamma_V^d. \tag{$*$}$$

Take $V = \Bbbk^d$ and let the torus $(\mathbb{G}_m)^{\times d}$ act on $\Bbbk^d$ by $(\lambda_1, \ldots, \lambda_d) \cdot (x_1, \ldots, x_d) = (\lambda_1 x_1, \ldots, \lambda_d x_d)$. Then the summand of weight $(1, \ldots, 1)$ of the right-hand side of isomorphism $(*)$ is $F \otimes \otimes^d$, and it is isomorphic to the summand of weight $(1, \ldots, 1)$ of the left-hand side, which is $\underline{\mathrm{Hom}}(\otimes^d, F)$. Moreover, $\underline{\mathrm{Hom}}(\otimes^d, F)$ is isomorphic to the functor $U \mapsto \mathrm{Hom}_{\mathcal{P}_\Bbbk}((\otimes^d)^U, F)$. Since $(\otimes^d)^U$ is isomorphic to $(U^*)^{\otimes d} \otimes \otimes^d$, we get an isomorphism of strict polynomial functors with variable U:

$$\underline{\mathrm{Hom}}(\otimes^d, F) \simeq \mathrm{Hom}_{\mathcal{P}_\Bbbk}((U^*)^{\otimes d} \otimes \otimes^d, F) \simeq U^{\otimes d} \otimes f_d(F).$$

Finally, one easily checks that these explicit constructions of the isomorphisms of Lemma 6.3 yield $\mathfrak{S}_d$-equivariant isomorphisms. □

Proposition 6.4. *For all functors F, G, there is an isomorphism of $\Bbbk\mathfrak{S}_d$-modules*

$$f_d(F \otimes G) \simeq f_d(F) \otimes f_d(G),$$

where the tensor product on the right is the Kronecker product of $f_d(F)$ and $f_d(G)$ (i.e., $\mathfrak{S}_d$ acts diagonally).

Proof. Lemma 6.3 yields a chain of isomorphisms:

$$F \otimes (G \otimes \otimes^d) \simeq F \otimes (\otimes^d \otimes f_d(G)) \simeq (F \otimes \otimes^d) \otimes f_d(G) \simeq \otimes^d \otimes f_d(F) \otimes f_d(G).$$

Thus the evaluation of $F \otimes (G \otimes \otimes^d)$ at $\Bbbk$ is isomorphic to $f_d(F) \otimes f_d(G)$. On the other hand, $F \otimes (G \otimes \otimes^d)$ is isomorphic to $(F \otimes G) \otimes \otimes^d$ and Lemma 6.3 shows that the evaluation of the latter at $\Bbbk$ is isomorphic to $f_d(F \otimes G)$. □

The following corollary shows that in the first case of Theorem 6.2, the internal tensor product is always nonzero.

Corollary 6.5. *Let L and L' be two p-restricted simples. Then $L \otimes L'$ is nonzero.*

Proof. By Clausen and James' theorem, $f_d(L)$ and $f_d(L')$ are nonzero. Hence, by Proposition 6.4, $f_d(L \otimes L')$ is nonzero. Thus $L \otimes L'$ is nonzero. □

Given two p-restricted simples L and L', a natural question is to determine if the analogue of Theorem B.12 holds, i.e., if the nonzero functor $L \otimes L'$ is simple. In fact, Bessenrodt and Kleshchev [2000] have proved that the Kronecker product of two simple representations of symmetric groups is almost never simple. In particular, Proposition 6.4 has the following consequence in odd characteristic.

Corollary 6.6. *Assume that p is odd. Let L and L' be two p-restricted simples such that $f_d(L)$ and $f_d(L')$ both have dimension at least two. Then $L \otimes L'$ is not simple.*

Proof. Since the right adjoint of f_d satisfies $f_d \circ r_d = \mathrm{Id}$, $f_d(L)$ sends simple functors either to simple $\Bbbk\mathfrak{S}_d$-modules or to zero. But $f_d(L \underline{\otimes} L') \simeq f_d(L) \otimes f_d(L')$ is a Kronecker product of two simple $\Bbbk\mathfrak{S}_d$-modules, so is not simple by [Bessenrodt and Kleshchev 2000, Theorem 2]. Thus $L \underline{\otimes} L'$ cannot be simple. $\qquad\square$

Remark 6.7. Corollary 6.6 uses [Bessenrodt and Kleshchev 2000, Theorem 2], which is a nontrivial result on symmetric groups. It would be interesting to find a more elementary proof of Corollary 6.6, in the spirit of the proof of Theorem B.12.

To solve completely (in odd characteristic) the problem of knowing if an internal tensor product $L \underline{\otimes} L'$ can be simple, it remains to study the case where $f_d(L')$ has dimension 1. The remainder of the section is devoted to this study. In our discussion below, we show in Corollary 6.10 that when $f_d(L')$ has dimension one, $L \underline{\otimes} L'$ may sometimes be simple and sometimes not, and in Corollary 6.9 we show that it suffices to study the case $L' = Q^d$. The latter case is studied in [Reischuk 2016], where the simplicity of $L \underline{\otimes} Q^d$ is shown to be equivalent to $p(L, 1) > 1$.

There are two $\Bbbk\mathfrak{S}_d$-modules of dimension 1, namely the signature module $\Bbbk^{\mathrm{alt}}$ and the trivial module $\Bbbk$. The signature module is the image by the Schur functor of $\Lambda^d = L_{(1,\dots,1)}$. Since $\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(\otimes^d, S^d)$ has dimension 1 and since S^d is a quotient of $\otimes^d$, the head of S^d is a p-restricted simple functor. This functor is known under the name of truncated symmetric powers, and we denote it by Q^d as in [Breen et al. 2016]. Then $f_d(Q^d)$ is the trivial $\Bbbk\mathfrak{S}_d$-module. Thus, to solve completely (in odd characteristic) the problem of knowing if an internal tensor product $L \underline{\otimes} L'$ can be simple, it remains to investigate the internal tensor products $L \underline{\otimes} Q^d$ and $L \underline{\otimes} \Lambda^d$ for p-restricted simples L.

Proposition 6.8. *Let F be a homogeneous functor of degree d. Consider the right action of $\mathfrak{S}_d$ on $\otimes^d$ given by permuting the factors of the tensor product. If $p \neq 2$ then*

$$F \underline{\otimes} \Lambda^d \simeq (\otimes^d) \otimes_{\mathfrak{S}_d} (\Bbbk^{\mathrm{alt}} \otimes f_d(F)).$$

If $\mathrm{Head}(F)$ *is a direct sum of p-restricted simples (and p arbitrary), then*

$$F \underline{\otimes} Q^d \simeq (\otimes^d) \otimes_{\mathfrak{S}_d} f_d(F).$$

Proof. Lemma 6.3 yields an $\mathfrak{S}_d$-equivariant isomorphism $F \underline{\otimes} \otimes^d \simeq \otimes^d \otimes f_d(F)$. Taking the coinvariants under the signed action of $\mathfrak{S}_d$ and using right exactness of internal tensor products, we obtain the first isomorphism. For the second, let R^d be the radical of S^d. Since $f_d(S^d) = f_d(Q^d)$ and the Schur functor is exact, we have $f_d(R^d) = 0$. Hence, by Lemma 6.3, $R^d \underline{\otimes} \otimes^d$ is zero. But if P is left p-bounded projective, it is a direct summand in a direct sum of copies of $\otimes^d$, and hence $R^d \underline{\otimes} P$ is zero. Now F is left p-bounded by Corollary 4.2, so $R^d \underline{\otimes} F = F \underline{\otimes} R^d = 0$. By right exactness of tensor products we thus obtain an isomorphism $F \underline{\otimes} S^d \simeq F \underline{\otimes} Q^d$. Then the computation of $F \underline{\otimes} S^d$ is done in the same fashion as that of $F \underline{\otimes} \Lambda^d$. $\qquad\square$

If M is a simple $\mathfrak{S}_d$-module, then $M \otimes \Bbbk^{\text{alt}}$ is also simple. Let L_μ be the simple p-restricted functor such that $f_d(L_\mu) = M$. We denote by $m(\mu)$ the p-restricted partition such that $f_d(L_{m(\mu)}) = M \otimes \Bbbk^{\text{alt}}$. The involution $\mu \mapsto m(\mu)$ (or rather $\mu' \mapsto m(\mu')$ where μ' stands for the conjugate partition of μ) is known as the Mullineux correspondence [Martin 1993, Section 4.2], and its combinatorial description has been proved by Ford and Kleshchev [1997]; see also the work of Brundan and Kujawa [2003] for a more recent and different proof. Proposition 6.8 has the following consequence.

Corollary 6.9. *Let μ be a p-restricted partition. Then*

$$L_\mu \underline{\otimes} \Lambda^d \simeq L_{m(\mu)} \underline{\otimes} Q^d.$$

As another consequence of Proposition 6.8, we obtain that the internal tensor product of two simple functors may sometimes be simple and sometimes not. The problem of knowing exactly for which p-restricted partitions μ the functor $L_\mu \underline{\otimes} \Lambda^d$ is simple is studied in [Reischuk 2016].

Corollary 6.10. *Assume that p is odd. Then $Q^d \underline{\otimes} \Lambda^d$ is isomorphic to Λ^d, and $\Lambda^d \underline{\otimes} \Lambda^d$ is isomorphic to S^d.*

7. Estimates for $p(F, r)$ and $i(F, r)$

7A. *Basic properties of $p(F, r)$ and $i(F, r)$.* Let r be a positive integer. We introduce the following two homogeneous functors of degree d, where $T^{(d_0,...,d_k)} = (\otimes^{d_0}) \otimes (\otimes^{d_1})^{(1)} \otimes \cdots \otimes (\otimes^{d_k})^{(k)}$ as in Corollary 4.3:

$$L(d, r) = \bigoplus_{\substack{\lambda \text{ not } p^r\text{-restricted} \\ \text{and } |\lambda| = d}} L_\lambda, \qquad T(d, r) = \bigoplus_{\substack{\sum_{0 \le i < r} p^i d_i < d \\ \sum_{0 \le i} p^i d_i = d}} T^{(d_0,...,d_k)}.$$

These functors are defined so that they contain all the simples of degree d, or all the twisted tensor powers of degree d, which have at least one factor precomposed by $I^{(s)}$ with $s \ge r$. Hence they are nonzero if and only if $d \ge p^r$. By Corollary 4.3, $L(d, r)$ is a quotient of $T(d, r)$. Since these two functors are self-dual, it follows that $L(d, r)$ is also a subfunctor of $T(d, r)$.

Proposition 7.1. *Let F be a homogeneous functor of degree d, and let $G(d, r)$ be equal to either $L(d, r)$ or $T(d, r)$. Then $p(F, r)$ is the lowest (possibly $+\infty$) degree k such that the vector space $\text{Ext}^k_{\mathcal{P}_\Bbbk}(F, G(d, r))$ is nonzero, and $i(F, r)$ is the lowest k such that $\text{Ext}^k_{\mathcal{P}_\Bbbk}(G(d, r), F)$ is nonzero.*

Proof. Let P be a degree d homogeneous p^r-bounded projective. Then Theorem 3.6 implies that $\text{Ext}^*_{\mathcal{P}_\Bbbk}(P, G(d, r))$ is zero. Take a projective resolution Q of F whose first $p(F, r)$-terms (i.e., up to index $p(F, r) - 1$) are left p^r-bounded projectives,

and let K be the kernel of $Q_{p(F,r)-1} \to Q_{p(F,r)-2}$. By definition of $p(F, r)$, K is not p^r-bounded. By Corollary 4.2, this means that there exists a nonzero map $K \to L(d, r)$, and hence also a nonzero map $K \to T(d, r)$. By dimension shifting,

$$\mathrm{Ext}^i_{\mathcal{P}_{\Bbbk}}(F, G(d, r)) \simeq \begin{cases} 0 & \text{if } i < p(F, r), \\ \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(K, G(d, r)) \neq 0 & \text{if } i = p(F, r). \end{cases}$$

The proof for $i(F, r)$ is similar. $\qquad\square$

Since $T(d, r)$ is a self-dual functor, $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(T(d, r), F^{\sharp})$ is always isomorphic to $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F, T(d, r))$. Thus we obtain the following corollary.

Corollary 7.2. *For all functors F, we have $i(F^{\sharp}, r) = p(F, r)$.*

We now indicate how $i(F, r)$ behaves with respect to some usual operations on strict polynomial functors. There are similar statements for $p(F, r)$ which can be deduced from the formula $p(F, r) = i(F^{\sharp}, r)$ or by repeating the proofs with projective resolutions. We leave this to the reader.

Proposition 7.3. *Let F and G be two functors. The following hold*:

(a) $i(F_V, r) = i(F, r)$.

(b) $i(F, r) = i(F^{(s)}, r + s)$.

(c) $i(F \otimes G, r) = \min\{i(F, r), i(G, r)\}$.

(d) $i(F \oplus G, r) = \min\{i(F, r), i(G, r)\}$.

(e) $i(F, r) \geq \min\{i(S, r) : S \text{ is finite and } S \subset F\}$.

Proof. Statement (d) is straightforward from the characterization of $i(F, r)$ in terms of Ext^* provided by Proposition 7.1, and implies that for the remaining statements, we can assume that F and G are homogeneous. We let $d := \deg F$ and $g := \deg G$. Statement (e) follows from the interpretation of $i(F, r)$ given in Proposition 7.1 and the fact that $\mathrm{Ext}^*(T(d, r), -)$ commutes with directed colimits. To prove (a), observe that F is a direct summand in F^V so that $i(F, r) \geq i(F_V, r)$. Moreover, if J is a standard p^r-bounded injective then J_V is a direct sum of standard p^r-bounded injectives. Hence if Q is an injective resolution of F whose first $i(F, r)$ terms are left p^r-bounded injectives, then Q_V is an injective resolution of F_V whose first $i(F, r)$ terms are left p^r-bounded injectives, so that $i(F_V, r) \geq i(F, r)$.

To prove (b), we use the isomorphisms

$$\mathrm{Ext}^*_{\mathcal{P},\Bbbk}(T(d, r + s), F^{(s)}) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(T(d, r)^{(s)}, F^{(s)}) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(T(d, r), F_{E_s}).$$

The first isomorphism is induced by the inclusion $T(d, r)^{(s)} \subset T(d, r + s)$; the cokernel of this split inclusion is easily seen to be zero by using the sum diagonal adjunction. The second isomorphism is proved in [Touzé 2013a; Chałupnik 2015].

In this formula F_{E_s} is a nonnegatively graded functor, and the degree on the right-hand side is the total degree. The graded functor F_{E_s} equals $F_{\Bbb{k}^{p^s}}$ in an ungraded way, so that the lowest nonzero degree k on the right-hand side of the isomorphism is greater or equal to $i(F_{\Bbb{k}^{p^s}}, r) = i(F, r)$. Hence $i(F^{(s)}, r + s) \geq i(F, r)$. Conversely, the degree zero component of F_{E_s} is isomorphic to F, so that the lowest nonzero degree k on the right-hand side of the isomorphism is lower or equal to $i(F, r)$, and hence $i(F^{(s)}, r + s) \leq i(F, r)$.

It remains to prove (c). Assume for example that $i(F, r) \leq i(G, r)$. If Q and Q' are injective resolutions of F and G, respectively, whose first $i(F, r)$ terms are p^r-bounded, then $Q \otimes Q'$ is an injective resolution of $F \otimes G$ whose first $i(F, r)$ terms are p^r-bounded, and hence $i(F \otimes G, r) \geq i(F, r)$. Conversely, let x be a nonzero extension in $\mathrm{Ext}^{i(F,r)}(T(d, r), F)$. Let L be a simple subfunctor of G. As L is a quotient of a functor $T^{(d_0, \ldots, d_\ell)}$ for some tuple $(d_0, \ldots, d_\ell)$ by Corollary 4.3, there is a nonzero map $f : T^{(d_0, \ldots, d_\ell)} \to G$. Since cup products are injective (by Theorem 3.6 with $r = 0$ or by Lemma 3.9), $x \cup f$ is a nonzero element of $\mathrm{Ext}^{i(F,r)}(T(d, r) \otimes T^{(d_0, \ldots, d_\ell)}, F \otimes G)$. But $T(d, r) \otimes T^{(d_0, \ldots, d_\ell)}$ is a direct summand of $T(d + g, r)$, so that $i(F \otimes G, r) \leq i(F, r)$. □

7B. *A few examples.*

Proposition 7.4. *Let r be a nonnegative integer. The following hold:*

(1) *If $\deg F < p^r$, then $i(F, r) = +\infty$.*

(2) *If $d \geq p^r$, then $i(S^d, r) = 0$.*

(3) *If $d \geq p^r$, then $i(\Lambda^d, r) = p^r - 1$.*

(4) *If $d \geq p^r$, then $i(\Gamma^d, r) = 2(p^r - 1)$.*

Proof. The first statement follows from the fact that when $d < p^r$, all basic injectives of degree d are p^r-bounded. If $d \geq p^r$, the multiplication of the symmetric algebra and the natural inclusion $I^{(r)} \hookrightarrow S^{p^r}$ induce a nonzero map $\otimes^{d-p^r} \otimes I^{(r)} \to S^d$. Hence, by Proposition 7.1, $i(S^d, r) = 0$. Let us prove that $i(\Lambda^d, p^r) = p^r - 1$. The homogeneous part of degree d of the reduced bar construction of the symmetric algebra S provides an injective resolution of Λ^d whose first $p^r - 1$ terms are basic p^r-bounded injectives; see, e.g., [Totaro 1997]. Thus $i(\Lambda^d, r) \geq p^r - 1$. Conversely, using sum-diagonal adjunction one obtains that $\mathrm{Ext}^*_{\mathcal{P}_{\Bbb{k}}}(\otimes^{d-p^r} \otimes I^{(r)}, \Lambda^d)$ is isomorphic to the tensor product

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbb{k}}}(\otimes^{d-p^r}, \Lambda^{d-p^r}) \otimes \mathrm{Ext}^*_{\mathcal{P}_{\Bbb{k}}}(I^{(r)}, \Lambda^{p^r}).$$

The factor on the left-hand side of the tensor product is concentrated in degree zero (as $\otimes^{d-p^r}$ is projective) and one-dimensional by [Friedlander and Suslin 1997, Corollary 2.12], and by [Friedlander and Suslin 1997, (4.5.1), p. 251], the factor on the right-hand side of the tensor product is one-dimensional and concentrated in

degree $p^r - 1$. Now $\otimes^{d-p^r} \otimes I^{(r)}$ is a direct summand in $T(d, r)$, so that $i(\Lambda^d, r) \leq p^r - 1$ by Proposition 7.1. A similar argument applies to (4): the homogeneous part of degree d of the twofold reduced bar construction of the symmetric algebra yields an injective resolution whose first $2(p^r - 1)$ terms are basic p^r-bounded injectives, and on the other hand, one can compute that $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(\otimes^{d-p^r} \otimes I^{(r)}, \Gamma^d)$ is one-dimensional and concentrated in degree $2(p^r - 1)$. $\square$

Let us denote by S_λ the Schur functor associated to a partition λ and by W_λ the Weyl functor associated to λ. These are finite homogeneous strict polynomial functors, whose degree is the weight of the partition λ, and we have $W_\lambda = S_\lambda^\sharp$. They generalize the functors S^d, Λ^d and Γ^d. Indeed,

$$W_{(d,0,0,\dots)} = \Gamma^d, \qquad S_{(d,0,0,\dots)} = S^d, \qquad S_{(1,\dots,1)} = W_{(1,\dots,1)} = \Lambda^d.$$

The S_λ are the costandard, and the W_λ the standard, objects of the highest weight category structure of $\mathcal{P}_\Bbbk$. In particular $\mathrm{Soc}(S_\lambda) = L_\lambda = \mathrm{Head}(W_\lambda)$. We refer the reader to [Touzé 2013b, Section 6.1.1] or [Krause 2017] for more details and references on these functors. The following lemma may be useful for computations.

Lemma 7.5. *Let λ be a partition and let λ' be the dual partition. For all tuples $(d_0, \dots, d_k)$ there is a graded isomorphism (where* Ext *is understood as zero in negative degrees):*

$$\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(T^{(d_0,\dots,d_k)}, S_\lambda) \simeq \mathrm{Ext}^{*+s}_{\mathcal{P}_\Bbbk}(T^{(d_0,\dots,d_k)}, W_{\lambda'}),$$

where $s = d_1(p-1) + d_2(p^2 - 1) + \cdots + d_k(p^k - 1)$.

Proof. We use Ringel duality Θ, which is an autoequivalence of $\mathbf{D}(\mathcal{P}_{d,\Bbbk})$. See, e.g., [Touzé 2013b, Section 3; Chałupnik 2008, Section 2]. We have $\Theta S_\lambda = W_{\lambda'}$ and $\Theta T^{(d_0,\dots,d_k)} = T^{(d_0,\dots,d_k)}[-s]$, so that the lemma follows from interpreting morphisms of degree s in the derived category as extensions of degree s. $\square$

Proposition 7.6. *Let λ be a partition and λ' the dual partition. Then we have*

$$i(S_\lambda, r) + p^r - 1 \leq i(W_{\lambda'}, r).$$

Assume, moreover, that $\lambda = \lambda^0 + p\lambda^1 + \cdots + p^k\lambda^k$, where each λ^k is a p-restricted partition of d_k, and $k \geq r$. Then

$$i(W_{\lambda'}, r) \leq \sum_{i=1}^{k} d_i(p^i - 1).$$

Proof. We use the isomorphism of Lemma 7.5. If $T^{(d_0,\dots,d_r)}$ is a direct summand of $T(d, r)$, then the associated shift s is always greater than or equal to $p^r - 1$. This proves the first inequality. As regards the second inequality, we have $L_\lambda = L_{\lambda^0} \otimes \cdots \otimes L_{\lambda^k}^{(k)}$ by the Steinberg tensor product theorem. By Clausen and James' theorem, L_λ is then a quotient of $T^{(d_0,\dots,d_k)}$. Thus we get a nonzero element

in $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(T^{(d_0,\ldots,d_k)}, S_\lambda)$ by composing the quotient map $T^{(d_0,\ldots,d_k)} \to L_\lambda$ with the inclusion $L_\lambda \subset S_\lambda$. Therefore, by Lemma 7.5, there is a nonzero extension of degree $\sum_{i=1}^k d_i(p^i - 1)$ between $T^{(d_0,\ldots,d_k)}$ (hence $T(d, r)$) and $W_{\lambda'}$. □

We finish this section by computing the integers $i(F, r)$ when F is any Schur or Weyl functor of degree 4 in characteristic $p = 2$. The result is already known for S^4, Λ^4 and Γ^4 by Proposition 7.4. For the three remaining partitions, the computation relies on the following short exact sequences.

Lemma 7.7. *Let $\Bbbk$ be a field of characteristic $p = 2$. There are short exact sequences*

(1) $0 \to \Lambda^4 \to \Lambda^3 \otimes \Lambda^1 \to S_{(2,1,1)} \to 0,$

(2) $0 \to S_{(3,1)} \to S^3 \otimes S^1 \to S^4 \to 0,$

(3) $0 \to S_{(2,2)} \to S^2 \otimes S^2 \to S_{(3,1)} \oplus S^4 \to 0.$

Proof. The first two sequences are the standard presentation and copresentation of Schur functors and are valid over an arbitrary ring [Akin et al. 1982]. Only the last one is specific to the characteristic 2 case and needs to be proved. As proved in [Akin et al. 1982], the Schur functor $S_{(2,2)}$ has copresentation given by

$$0 \to S_{(2,2)} \to S^2 \otimes S^2 \xrightarrow{(\phi,\mathrm{mult})} S^3 \otimes S^1 \oplus S^4, \qquad (*)$$

where mult denotes the map induced by the multiplication for the symmetric algebra and ϕ is defined as the composite map

$$S^2 \otimes S^2 \xrightarrow{S^2 \otimes \Delta} S^2 \otimes \otimes^2 \xrightarrow{\mathrm{mult} \otimes S^1} S^3 \otimes S^1,$$

for Δ induced by the comultiplication of the symmetric algebra. Since the field has characteristic 2, there is a surjective map $\pi : S^2 \to \Lambda^2$, and ϕ factors in a unique way as $\phi = \psi \circ (S^2 \otimes \pi)$. Now the composite $\mathrm{mult} \circ \psi : S^2 \otimes \Lambda^2 \to S^4$ is zero, so that the image of ψ is contained in $S_{(3,1)}$. Thus the copresentation $(*)$ induces a copresentation

$$0 \to S_{(2,2)} \to S^2 \otimes S^2 \to S_{(3,1)} \oplus S^4.$$

The last map on the right is surjective for Euler characteristic reasons (the dimensions being independent of the characteristic, one can do the computation in characteristic zero, where $S^2 \otimes S^2$ is isomorphic to $S_{(2,2)} \oplus S_{(3,1)} \oplus S^4$ by the Pieri rule). □

The extension groups between $\otimes^2 \otimes I^{(1)}$, $I^{(1)} \otimes I^{(1)}$ or $I^{(2)}$ on the one hand, and tensor products of symmetric or exterior powers on the other, are easy to compute. Now one can completely compute the extension groups between $T(4, r)$ and the Schur functors simply by inspecting the (not very) long exact $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(T(d, r), -)$-sequences associated to the short exact sequences of Lemma 7.7. One can then obtain the corresponding computations with Weyl functors by Lemma 7.5. We

record the resulting computations of $i(F, r)$ in the following proposition. Since $p^3 = 2^3 > 4 = d$, only the cases $r = 1$ and $r = 2$ are interesting.

Proposition 7.8. *Let $\Bbbk$ be a field of characteristic* 2. *The following computations hold.*

F	Γ^4	$W_{(3,1)}$	$W_{(2,2)}$	$W_{(2,1,1)}$	Λ^4	$S_{(2,1,1)}$	$S_{(2,2)}$	$S_{(3,1)}$	S^4
$i(F,1)$	2	2	2	1	1	1	0	0	0
$i(F,2)$	6	5	4	4	2	2	1	1	0

Remark 7.9. One sees in this example that the integers $i(F, r)$ are increasing with respect to the dominance order for Weyl functors, and decreasing with respect to the dominance order for Schur functors. It would be quite interesting to know if this is the shadow of some general phenomenon.

8. Application to symmetric groups

Lemma 8.1. *The Schur functor sends p-bounded projectives and injectives to projective and injective $\Bbbk\mathfrak{S}_d$-modules, respectively. Moreover, if F is a p-bounded projective or if G is a p-bounded injective, then the Schur functor induces an isomorphism*
$$\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F, G) \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{S}_d}(f_d(F), f_d(G)).$$

Proof. The left adjoint of f_d sends $\Bbbk\mathfrak{S}_d$ to $\otimes^d$. Thus $f_d(\otimes^d) = f_d(\ell_d(\Bbbk\mathfrak{S}_d)) \simeq \Bbbk\mathfrak{S}_d$ is projective. Moreover, the map induced by f_d,
$$\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(\otimes^d, G) \to \mathrm{Hom}_{\mathfrak{S}_d}(\Bbbk\mathfrak{S}_d, f_d(G)),$$
is an isomorphism because it identifies with the adjunction isomorphism for (ℓ_d, f_d). This proves Lemma 8.1 for the p-bounded projective $\otimes^d$. If $i < p$ then Γ^i is a direct summand of $\otimes^i$ (the retract of the canonical inclusion $\Gamma^i \hookrightarrow \otimes^i$ is the natural transformation which sends $v_1 \otimes \cdots \otimes v_i$ to $\sum_{\sigma \in \mathfrak{S}_i} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}$.). Thus p-bounded projectives are direct summands of direct sums of copies of $\otimes^d$. As f_d commutes with arbitrary direct sums, this implies that Lemma 8.1 holds for all p-bounded projectives. The proof for p-bounded injectives is similar, using the right adjoint r_d. $\qquad\square$

The next theorem generalizes many theorems in [Kleshchev and Nakano 2001]. In particular, Theorem 8.2 does not require any restriction on the characteristic, and works for all F and all G. As regards concrete computations, the explicit bounds for $i(G, 1)$ for Weyl functors G given in Section 7B yield connectedness bounds which are at least as good as those given in [Kleshchev and Nakano 2001]. However, we have not investigated estimates for $i(G, 1)$ when G is simple. Hence, unlike

[Kleshchev and Nakano 2001], we don't have concrete connectedness estimates for simple functors.

Theorem 8.2. *Let F and G be homogeneous strict polynomial functors of degree d. The map induced by the Schur functor*

$$\text{Ext}^k_{\mathcal{P}_{d,\Bbbk}}(F, G) \to \text{Ext}^k_{\Bbbk\mathfrak{S}_d}(f_d(F), f_d(G))$$

is an isomorphism in degrees $k < p(F, 1) + i(G, 1) - 1$, and it is injective in degree $k = p(F, 1) + i(G, 1) - 1$.

Proof. Assume that there is a short exact sequence $0 \to H \to J \to H' \to 0$, where J is a p-bounded injective. The Schur functor induces a morphism from the induced $\text{Ext}^*_{\mathcal{P}_{d,\Bbbk}}(F, -)$-long exact sequence to the induced $\text{Ext}^*_{\mathfrak{S}_d}(f_d(F), f_d(-))$-long exact sequence. Using Lemma 8.1 together with the five lemma, we see that the Schur functor is k-connected for the pair (F, H) (i.e., an isomorphism in Ext-degree $< k$ and injective in Ext-degree k) if and only if it is $(k - 1)$-connected for the pair (F, H'). Using this argument, we reduce the proof of Theorem 8.2 to the case where $i(G, 1) = 0$. By a similar argument applied to the contravariant variable of Ext, we reduce the proof further to the case where $i(F, 1) = 1$. In the latter case, F is a quotient of a p-bounded projective P and we have a commutative diagram

$$
\begin{array}{ccc}
\text{Hom}_{\mathcal{P}_{d,\Bbbk}}(P, G) & \xrightarrow[\simeq]{f_d} & \text{Hom}_{\mathfrak{S}_d}(f_d(P), f_d(G)) \\
\downarrow & & \uparrow \\
\text{Hom}_{\mathcal{P}_{d,\Bbbk}}(F, G) & \xrightarrow{f_d} & \text{Hom}_{\mathfrak{S}_d}(f_d(F), f_d(G))
\end{array}
$$

which proves that for the pair (F, G) the Schur functor is indeed $p(F, 1) - 1$ connected (i.e., injective in degree zero). $\square$

The following examples show that the bounds in Theorem 8.2 are optimal.

Example 8.3. Let Q^p be the socle of Γ^p. Then Q^p is the simple functor with highest weight $(p - 1, 1)$. In particular, $i(Q^p, 1) \geq 1$ by Corollary 4.2. Since Γ^p is the middle term of a nonsplit extension

$$0 \to Q^p \to \Gamma^p \to I^{(1)} \to 0,$$

we have $\text{Ext}^1_{\mathcal{P}_\Bbbk}(I^{(1)}, Q^p) \neq 0$, which proves that $i(Q^p, 1) \leq 1$ by Proposition 7.1. Thus $i(Q^p, 1) = 1$. We claim that the following map is not an isomorphism:

$$\text{Ext}^{i(Q^p,1)-1}_{\mathcal{P}_{p,\Bbbk}}(\Gamma^p, Q^p) \to \text{Ext}^{i(Q^p,1)-1}_{\Bbbk\mathfrak{S}_p}(f_p(Q^p), f_p(\Gamma^p)).$$

Indeed, the domain $\text{Hom}_{\mathcal{P}_\Bbbk}(\Gamma^p, Q^p)$ is zero as $\text{Head}(\Gamma^p) = I^{(1)} \neq Q^p$. But $f_p(\Gamma^p) = f_p(Q^p) = \Bbbk$. Thus the codomain $\text{Hom}_{\Bbbk\mathfrak{S}_p}(f_p(\Gamma^p), f_p(Q^p))$ has dimension one.

Example 8.4. Let F be a homogeneous functor of degree d. By Proposition 7.1, $\operatorname{Ext}^{i(F,1)}_{\mathcal{P}_{\Bbbk,d}}(T(d,1),F)$ is nonzero. On the other hand $f_d(T(d,1))=0$ by Corollary 4.4, so that the following map is not injective:

$$\operatorname{Ext}^{i(F,1)}_{\mathcal{P}_{d,\Bbbk}}(T(d,1),F) \to \operatorname{Ext}^{i(F,1)}_{\Bbbk\mathfrak{S}_d}(f_d(T(d,1)),f_d(F)).$$

Appendix A. Representations of tensor product algebras

This appendix collects some results about representations of tensor product algebras. All these results are standard (except maybe Proposition A.6), but they are scattered in the literature and not always stated under the form that we want to use.

In the remainder of the section, we fix two finite dimensional algebras A and B over a ground field $\Bbbk$. We assume furthermore that $\Bbbk$ is a splitting field for these two algebras; that is, the endomorphism rings of simple modules have dimension one over $\Bbbk$. (This hypothesis is satisfied for quasihereditary algebras, and of course for all algebras if $\Bbbk$ is algebraically closed).

If M is an A-module and N a B-module, we denote by $M \boxtimes N$ their tensor product, viewed as an $A \otimes B$-module. The tensor product yields a Künneth morphism

$$\operatorname{Ext}^*_A(M,M') \otimes \operatorname{Ext}^*_B(N,N') \xrightarrow{\kappa} \operatorname{Ext}^*_{A\otimes B}(M \boxtimes N, M' \boxtimes N').$$

Proposition A.1. *The Künneth morphism κ is an isomorphism if M and M' have finite dimension or if M and N have finite dimension.*

Proof. If M has finite dimension, then it has a projective resolution by finite dimensional projective A-modules. Thus, it suffices to prove the result in degree zero (i.e., for Hom), the general result follows formally by taking resolutions. Using semi-exactness and additivity of Hom and $\boxtimes$ with respect to their first variable, one reduces furthermore to the case $M = A$. If M' has finite dimension, the Künneth morphism in degree zero identifies with the map

$$M' \otimes \operatorname{Hom}_B(N,N') \to \operatorname{Hom}_B(N,M' \otimes N'),$$

which is an isomorphism since M' has finite dimension. If N has finite dimension, one can also reduce to the case $N = B$, and in the latter case it is clear that κ is an isomorphism. $\qquad\square$

Proposition A.2. *Up to isomorphism, the simple $A \otimes B$-modules are the tensor products $L_1 \boxtimes L_2$ where L_1 is a simple module over A and L_2 a simple module over B. Moreover, two such simple modules $L_1 \boxtimes L_2$ and $L'_1 \boxtimes L'_2$ are isomorphic if and only if $L_1 \simeq L'_1$ and $L_2 \simeq L'_2$.*

Proof. The fact that $L_1 \boxtimes L_2$ is simple if L_1 and L_2 are simple follows from the density theorem [Curtis and Reiner 1981, (3.27)]. If two such tensor products $L_1 \boxtimes L_2$ and $L'_1 \boxtimes L'_2$ are isomorphic, then $L_1 \simeq L'_1$ and $L_2 \simeq L'_2$ because

$\mathrm{Hom}_{A\otimes B}(L_1 \boxtimes L_2, L'_1 \boxtimes L'_2)$ is isomorphic to $\mathrm{Hom}_A(L_1, L'_1) \otimes \mathrm{Hom}_B(L_2, L'_2)$. It remains to prove that any simple $A \otimes B$-module is of the form $L_1 \boxtimes L_2$. The Jacobson radical $J(A \otimes B)$ of $A \otimes B$ contains $J(A) \otimes B + A \otimes J(B)$, since the latter is a nilpotent ideal [Curtis and Reiner 1981, (5.15)]. Thus we have a surjective morphism

$$\pi : A/J(A) \otimes B/J(B) \twoheadrightarrow A \otimes B/J(A \otimes B).$$

Since the quotient $C/J(C)$ of a $\Bbbk$-algebra C is a semisimple ring [Curtis and Reiner 1981, (5.19)] with the same simple modules as C, it follows form the Wedderburn theorem and dimension counting that π is an isomorphism and that all simple $A \otimes B$-modules have the form $L_1 \boxtimes L_2$. □

Lemma A.3. *For all modules M and N, $\mathrm{Soc}(M) \boxtimes \mathrm{Soc}(N) = \mathrm{Soc}(M \boxtimes N)$.*

Proof. By Proposition A.2, $\mathrm{Soc}(M) \boxtimes \mathrm{Soc}(N)$ is a semisimple submodule of $M \boxtimes N$. Moreover, for all simple modules $L_1 \boxtimes L_2$, we have

$$\begin{aligned}
\mathrm{Hom}_{A\otimes B}(L_1 \boxtimes L_2, M \boxtimes N) &= \mathrm{Hom}_A(L_1, M) \otimes \mathrm{Hom}_B(L_2, N) \\
&= \mathrm{Hom}_A(L_1, \mathrm{Soc}(M)) \otimes \mathrm{Hom}_B(L_2, \mathrm{Soc}(N)) \\
&= \mathrm{Hom}_{A\otimes B}(L_1 \boxtimes L_2, \mathrm{Soc}(M) \boxtimes \mathrm{Soc}(N)).
\end{aligned}$$

Consequently, all simple submodules of $M \boxtimes N$ are submodules of $\mathrm{Soc}(M) \boxtimes \mathrm{Soc}(N)$. This proves the lemma. □

Lemma A.4. *For all modules M and N, $\mathrm{Head}(M \boxtimes N) = \mathrm{Head}(M) \boxtimes \mathrm{Head}(N)$.*

Proof. If M and N have finite dimension, the proof is dual to the proof of Lemma A.3. By additivity of $\boxtimes$ with respect to both variables, the result is then true when M and N are arbitrary projectives. In general, let P and Q be projective covers of $\mathrm{Head}(M)$ and $\mathrm{Head}(N)$, respectively. One has quotient maps

$$P \boxtimes Q \twoheadrightarrow M \boxtimes N \twoheadrightarrow \mathrm{Head}(M) \boxtimes \mathrm{Head}(N),$$

and the result follows by taking heads of these modules. □

Lemma A.3 can be applied iteratively to identify the socle filtration of $M \boxtimes N$. We index socle filtrations of modules so that the (-1)-th term is zero and the zeroth term is the socle of the modules.

Proposition A.5. *For all modules M and N, the socle filtration of $M \boxtimes N$ is the tensor product of the socle filtration of M with the socle filtration of N.*

Proof. Let M^i, N^i and $(M \boxtimes N)^i$ be the terms of the socle filtrations of M, N and $M \boxtimes N$, and let $F^n := \sum_{i+j=n} M^i \boxtimes N^j$. We prove $F^n = (M \boxtimes N)^n$ by induction on n. We have $F^0 = M^0 \boxtimes N^0 = (M \boxtimes N)^0$ by Lemma A.3. Assume

that $F^n = (M \boxtimes N)^n$. Let ι be the canonical inclusion

$$\bigoplus_{i+j=n+1} (M^i/M^{i-1}) \boxtimes (N^j/N^{j-1}) = F^{n+1}/F^n \hookrightarrow (M \boxtimes N)/F^n.$$

Let ϕ denote the canonical inclusion

$$(M \boxtimes N)/F^n \hookrightarrow \bigoplus_{i+j=n+1} (M/M^{j-1}) \boxtimes (N/N^{j-1}).$$

The composite $\phi \circ \iota$ is the direct sum of the canonical inclusions

$$(M^{i+1}/M^i) \boxtimes (N^{j+1}/N^j) \hookrightarrow (M/M^i) \boxtimes (N/N^j).$$

Thus, it follows from Lemma A.3 that $\phi \circ \iota$ maps the semisimple module F^{n+1}/F^n isomorphically onto the socle of the target of ϕ. In particular, the inclusion ι is in fact an isomorphism. $\square$

Recall that a finite module is multiplicity free if it has a composition series whose composition factors are pairwise nonisomorphic.

Proposition A.6. *Assume that one of the modules M or N is multiplicity free. Then for all submodules $S \subset M \boxtimes N$, there are submodules U_α of M and submodules V_α of N such that $S = \sum U_\alpha \boxtimes V_\alpha$.*

Proof. Since any module over a finite dimensional algebra is the sum of its finite submodules, it suffices to prove Proposition A.6 when all modules have finite dimension. Assume for example that M is multiplicity free, and fix a submodule $S \subset M \boxtimes N$.

Let T be a submodule of S such that $T/\mathrm{Rad}(T) \simeq L_1 \boxtimes L_2$ is simple. There is a submodule $U \subset M$ such that $\mathrm{Head}(U) \simeq L_1$. We claim that $T \subset U \boxtimes N$. Indeed, since M is multiplicity free, $L_1 \boxtimes L_2$ is not a composition factor of $(M/U) \boxtimes N$. Since $\mathrm{Head}(T) = L_1 \boxtimes L_2$, no nontrivial homomorphic image of T can be contained in $(M/U) \boxtimes N$. Thus $T \subset U \boxtimes N$.

We now construct a strictly decreasing sequence of modules $V_0 = N \supset V_1 \supset \cdots \supset V_n$ such that $U \boxtimes V_n = T$. Assume that V_i is constructed such that $T \subset U \boxtimes V_i$. If the inclusion is an equality then the construction is finished. Otherwise, the canonical map $\phi : \mathrm{Head}(T) \to \mathrm{Head}(U \boxtimes V_i)$ is not surjective. By Lemma A.4, $\mathrm{Head}(U \boxtimes V_i) = \mathrm{Head}(U) \boxtimes \mathrm{Head}(V_i)$ and by using the Künneth formula, we see that all submodules of $\mathrm{Head}(U \boxtimes V_i)$ are of the form $\mathrm{Head}(U) \boxtimes W$, where W is a submodule of $\mathrm{Head}(V_i)$. In particular, $\mathrm{Im}\,\phi$ is of the form $\mathrm{Head}(U) \boxtimes W_\phi$. The inverse image of $\mathrm{Im}\,\phi$ by the quotient map

$$\pi \boxtimes \pi_i : U \boxtimes V_i \twoheadrightarrow \mathrm{Head}(U) \boxtimes \mathrm{Head}(V_i)$$

is $\mathrm{Rad}(U) \boxtimes V_i + U \boxtimes \pi_i^{-1}(W_\phi)$. This is a submodule of $U \boxtimes V_i$ which contains T.

But $\mathrm{Head}(T) \simeq L_1 \boxtimes L_2$ is not a composition factor of

$$\mathrm{Rad}(U) \boxtimes (V_i/\pi_i^{-1}(W_\phi)) \simeq \frac{\mathrm{Rad}(U) \boxtimes V_i + U \boxtimes \pi_i^{-1}(W_\phi)}{U \boxtimes \pi_i^{-1}(W_\phi)}.$$

Thus T is actually a submodule of $U \boxtimes \pi_i^{-1}(W_\phi)$. We define $V_{i+1} := \pi_i^{-1}(W_\phi)$. Since ϕ is not surjective, V_{i+1} is a strict submodule of V_i and $T \subset U \boxtimes V_{i+1}$. Since $V_0 = N$ has finite dimension, we cannot indefinitely repeat this construction and decrease the dimension of the submodules V_i. Hence there must be an integer n such that $U \boxtimes V_n = T$.

We have proved so far that all submodules of $T \subset S$ with simple head are of the form $U \boxtimes V$ for some submodules $U \subset M$ and $V \subset N$. But for each composition factor L_α of S we can find a T_α with $T_\alpha/\mathrm{Rad}(T_\alpha) = L_\alpha$. Then $S = \sum T_\alpha = \sum U_\alpha \boxtimes V_\alpha$ and we are done. $\square$

The submodule lattice of multiplicity free modules can be described in terms of certain oriented diagrams [Alperin 1980]. To be more specific, the diagram $D(M)$ associated to a module M has the composition factors of M as vertices, and there is an edge $L \to L'$ if and only if there is a submodule $U \subset M$ such that $\mathrm{Head}(U) \simeq L$ and L' is a homomorphic image of $\mathrm{Rad}(U)$ (such a module U is unique [Alperin 1980, Lemma 4]). The following proposition describes the diagrams of tensor products $M \boxtimes N$.

Proposition A.7. *The tensor product $M \boxtimes N$ is multiplicity free if and only if both M and N are multiplicity free. If this happens, then the vertices of $D(M \boxtimes N)$ are the tensor products $L_1 \boxtimes L_2$, where L_1 is a composition factor of M and L_2 a composition factor of N. Moreover, there is an edge $L_1 \boxtimes L_2 \to L_1' \boxtimes L_2'$ if and only if either $L_1 = L_1'$ and there is an edge $L_2 \to L_2'$ in $D(N)$ or if $L_2 = L_2'$ and there is an edge $L_1 \to L_1'$ in $D(M)$.*

Proof. We only prove the statement about the edges of $D(M \boxtimes N)$. Let $L_1 \boxtimes L_2$ be a composition factor of $M \boxtimes N$ and let $U \subset M$ such that $\mathrm{Head}(U) = L_1$ and $V \subset N$ such that $\mathrm{Head}(V) = L_2$. Then by Lemma A.4, $\mathrm{Head}(U \boxtimes V) = L_1 \boxtimes L_2$. Thus there is an edge $L_1 \boxtimes L_2 \to L_1' \boxtimes L_2'$ if and only if $L_1' \boxtimes L_2'$ is a homomorphic image of $\mathrm{Rad}(U) \boxtimes V + U \boxtimes \mathrm{Rad}(V)$, that is, if and only if $\mathrm{Hom}_{A \boxtimes B}(\mathrm{Rad}(U) \boxtimes V, L_1' \boxtimes L_2') \neq 0$ or $\mathrm{Hom}_{A \boxtimes B}(U \boxtimes \mathrm{Rad}(V), L_1' \boxtimes L_2') \neq 0$. By the Künneth formula, the first condition is equivalent to the fact that L_1' is a homomorphic image of $\mathrm{Rad}(U)$ and that $L_2' = L_2$ while the second one is equivalent to the fact that L_2' is a homomorphic image of $\mathrm{Rad}(V)$ and that $L_1' = L_1$. $\square$

Proposition A.8. *Let L be a simple A-module satisfying $\mathrm{Ext}_A^1(L, L) = 0$, let $\mathcal{C}$ be a localizing and colocalizing subcategory of B-Mod, and let $L \boxtimes \mathcal{C}$ denote the full subcategory of $A \otimes B$-Mod whose objects are isomorphic to tensor products of the form $L \boxtimes M$, where M is an object of $\mathcal{C}$. Then*

(i) $L \boxtimes \mathcal{C}$ is a localizing and colocalizing subcategory of $A \otimes B\text{-}Mod$,

(ii) tensor product by L induces an equivalence of categories $\mathcal{C} \simeq L \boxtimes \mathcal{C}$.

Proof. The second statement follows from the Künneth formula and the fact that $\operatorname{End}_A(L) = \Bbbk$. Let us prove (i). The stability of $L \boxtimes \mathcal{C}$ by arbitrary direct sums is obvious, and since L is finite dimensional the canonical morphism $L \boxtimes \prod M_i \to \prod L \boxtimes M_i$ is an isomorphism, which proves the stability by direct products. If $S \subset L \boxtimes N$ then $S = \sum U_\alpha \boxtimes V_\alpha$ by Proposition A.6. But the only nonzero submodule of L is L itself, so that $S = \sum L \boxtimes V_\alpha \simeq L \boxtimes \left(\sum V_\alpha \right)$ is an object of $L \boxtimes \mathcal{C}$. The stability by quotients follows from the stability by subobjects. Finally, since $\operatorname{Ext}^1_A(L, L) = 0$ and $\operatorname{End}_A(L) = \Bbbk$, the Künneth formula shows that $\operatorname{Ext}^1_{A \otimes B}(L \boxtimes N, L \boxtimes N')$ is isomorphic to $\operatorname{Ext}^1_B(N, N')$. Thus, all extensions of $L \boxtimes N$ by $L \boxtimes N'$ are of the form $L \boxtimes E$, where E is an extension of N by N'. Hence $L \boxtimes \mathcal{C}$ is stable by extensions. $\qquad\square$

Appendix B. On theorems of Steinberg and Clausen–James

In this appendix, we give new proofs of Steinberg's tensor product theorem for GL_n and Clausen and James' theorem, based on Theorem 3.6.

Lemma B.1. *A strict polynomial functor is simple if and only if it is self-dual and its endomorphism ring has dimension one.*

Proof. The condition is necessary by facts (2) and (3) from Section 2C. We prove it is sufficient. Let L be a simple subfunctor of F. The composite

$$F \simeq F^\sharp \twoheadrightarrow L^\sharp \simeq L \hookrightarrow F$$

is a nonzero endomorphism of F. Since the endomorphism ring of F has dimension one, this morphism must be a nonzero multiple of the identity, and hence an isomorphism. Thus one must have $L = F$. $\qquad\square$

Proposition B.2 (weak Steinberg theorem). *Let $r \geq 0$, let L_1 be a left p^r-bounded simple functor, and let L_2 be any simple functor. Then $L_1 \otimes L_2^{(r)}$ is simple.*

Proof. Self-duality of L_1, L_2 and $I^{(r)}$ and general properties of duality imply that $L_1 \otimes L_2^{(r)}$ is self-dual. Moreover, since L_1 is left p^r-bounded, Theorem 3.6 yields an isomorphism:

$$\operatorname{End}_{\mathcal{P}_\Bbbk}(L_1 \otimes L_2^{(r)}) \simeq \operatorname{End}_{\mathcal{P}_\Bbbk}(L_1) \otimes \operatorname{End}_{\mathcal{P}_\Bbbk}(L_2) \simeq \Bbbk \otimes \Bbbk = \Bbbk.$$

Hence, $L_1 \otimes L_2^{(r)}$ is simple by Lemma B.1. $\qquad\square$

Our next task is to prove that the p-restricted simple functors are left p-bounded. Our proof will use the following proposition, which extends the classification of additive strict polynomial functors proved in [Touzé 2017b].

Proposition B.3. *Let $F \in \mathcal{P}_{d_0,d_1,\ldots,d_n,\Bbbk}$ be a strict polynomial functor with $1 + n$ variables, such that F is nonzero and additive with respect to each of the last n variables. Let G be the strict polynomial functor defined by $G(V) = F(V, \Bbbk, \ldots, \Bbbk)$. Then the d_is, $1 \leq i \leq n$, are powers of p, i.e., $d_i = p^{r_i}$ and there is an isomorphism*

$$F \simeq G \boxtimes I^{(r_1)} \boxtimes \cdots \boxtimes I^{(r_n)}.$$

Proof. By induction, we can reduce ourselves to proving that $d_n = p^{r_n}$ and that F is isomorphic to $\overline{F} \boxtimes I^{(r_n)}$, where $\overline{F}(V_0, \ldots, V_{n-1}) := F(V_0, \ldots, V_{n-1}, \Bbbk)$. The functors with $n + 1$ variables of the form $P \boxtimes \Gamma^\mu$, where P is a projective functor with n variables, homogeneous of multidegree $(d_0, \ldots, d_{n-1})$, and $\mu = (\mu_1, \ldots, \mu_k)$ is a tuple with $\sum \mu_i = d_n$, form a projective generator of $\mathcal{P}_{d_0,d_1,\ldots,d_n,\Bbbk}$. Thus F is a quotient of a direct sum $\bigoplus P_i \boxtimes \Gamma^{\mu^i}$.

Observe that if μ has more than one nonzero coefficient, then there are no nonzero morphisms from a functor of the form $P \boxtimes \Gamma^\mu$ to F. Indeed, for some n-tuple $\overline{V} = (V_0, \ldots, V_{n-1})$, such a nonzero morphism would induce a nonzero morphism of strict polynomial functors from $P(\overline{V}) \otimes \Gamma^\mu(-)$ to the additive functor $F(\overline{V}, -)$. This would contradict [Friedlander and Suslin 1997, Theorem 2.13].

In particular, F is in fact a quotient of $\bigoplus P_i \boxtimes \Gamma^{d_n} = P \boxtimes \Gamma^{d_n}$ with $P = \bigoplus P_i$. Moreover, the following composite is zero, where $\phi = P \boxtimes \mathrm{mult}$, with "mult" referring to the multiplication of the divided power algebra:

$$\bigoplus_{k=1}^{d_n} P \boxtimes (\Gamma^k \otimes \Gamma^{d_n-k}) \xrightarrow{\phi} P \boxtimes \Gamma^{d_n} \to F.$$

Hence F is a quotient of $P \boxtimes (\mathrm{Coker}\, \phi)$. But $\mathrm{Coker}\, \phi$ is nonzero if and only if $d_n = p^{r_n}$ for some r_n, and in this case it is equal to $I^{(r_n)}$. Thus $d_n = p^{r_n}$, and we have a surjective map $\psi : P \boxtimes I^{(r_n)} \twoheadrightarrow F$. By replacing the last variable by $\Bbbk$, we obtain a surjective map $\overline{\psi} : P \twoheadrightarrow \overline{F}$. We then take a projective functor with n variables Q and a map $\chi : Q \to P$ whose image is $\mathrm{Ker}\, \overline{\psi}$. Then using additivity with respect with the last variable, one sees that we have a right exact sequence:

$$Q \boxtimes I^{(r_n)} \xrightarrow{\chi \boxtimes I^{(r)}} P \boxtimes I^{(r_n)} \xrightarrow{\psi} F \to 0.$$

This implies that F is isomorphic to $\overline{F} \boxtimes I^{(r_n)}$. $\square$

Corollary B.4. *If L is a simple functor, there exists nonnegative integers $d_0, \ldots, d_r$ such that L is a quotient of the functor $T^{(d_0,\ldots,d_r)} = \bigotimes_{0 \leq i \leq r} (\otimes^{d_i})^{(i)}$.*

Proof. If L has degree zero, then L is the constant functor $\Bbbk$. Hence it is a quotient of $T^{(0)} = \otimes^0 = \Bbbk$. Assume L is not constant. Then there exists a positive integer n, the Eilenberg–Mac Lane degree of L, such that the functor with n variables

$$L_{\boxplus_n} : (V_1, \ldots, V_n) \mapsto L(V_1 \oplus \cdots \oplus V_n)$$

contains a nonzero homogeneous direct summand F which is additive with respect to each of its variables (see, e.g., [Touzé 2017b, Section 2] for more details on Eilenberg–Mac Lane degrees for strict polynomial functors). By Proposition B.3, F is of the form

$$F = G \boxtimes I^{(r_1)} \boxtimes \cdots \boxtimes I^{(r_n)}$$

where G is a homogeneous functor of degree zero, i.e., a constant functor. In particular, F (hence also $L_{\boxplus_n}$) contains $I^{(r_1)} \boxtimes \cdots \boxtimes I^{(r_n)}$ as a direct summand. Thus we have

$$0 \neq \mathrm{Hom}_{\mathcal{P}_{\Bbbk}(n)}(I^{(r_1)} \boxtimes \cdots \boxtimes I^{(r_n)}, L_{\boxplus_n}) \simeq \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(I^{(r_1)} \otimes \cdots \otimes I^{(r_n)}, L).$$

Since L is simple, any nonzero morphism with target L is surjective. Thus the inequality above proves that L is a quotient of $I^{(r_1)} \otimes \cdots \otimes I^{(r_n)}$. By reordering the factors of this tensor product (and using that $(I^{(k)})^{\otimes d_k} = (\otimes^{d_k})^{(k)}$), we obtain the result. $\square$

We now consider two assertions, indexed by a nonnegative integer k.

- $A(k)$ If L is a p-restricted functor of degree d with $d \leq k$, then L is a quotient of $\otimes^d$.

- $B(k)$ Let d be a nonnegative integer and let T be a homogeneous functor of positive degree e. If $d + pe \leq k + 1$, then no p-restricted simple functor occurs as a composition factor of the tensor product $\otimes^d \otimes T^{(1)}$.

Lemma B.5. *Assertion $A(0)$ is true.*

Proof. If L is a simple functor of degree 0, then L is the constant functor $\Bbbk$. Hence it is a quotient of $\otimes^0 = \Bbbk$. $\square$

Lemma B.6. *If $A(k)$ is true, then $B(k)$ is true.*

Proof. The functor $\otimes^d \otimes T^{(1)}$ admits a filtration whose successive quotients are direct sums of functors of the form $L_\lambda \otimes T^{(1)}$, where L_λ is a simple functor of degree d. Thus, it suffices to prove that these tensor products $L_\lambda \otimes T^{(1)}$ have no p-restricted composition factors. Let us write $\lambda = \alpha + p\beta$, where α is a p-restricted partition and β is a partition. Since $|\alpha| \leq d \leq k$, assertion $A(k)$ implies that the simple functor L_α is left p-bounded. By the weak Steinberg theorem of Proposition B.2, $L_\lambda \simeq L_\alpha \otimes L_\beta^{(1)}$ is simple, and it is isomorphic to L_λ by elementary highest weight theory (see item (4) in Section 2C). Hence,

$$L_\lambda \otimes T^{(1)} \simeq L_\alpha \otimes (L_\beta \otimes T)^{(1)}.$$

The functor $(L_\beta \otimes T)^{(1)}$ has composition factors of the form $(L_\gamma)^{(1)}$ with $\gamma \neq (0)$. Since L_α is left p-bounded, Proposition B.2 implies that the composition factors of $L_\lambda \otimes T^{(1)}$ have the form $L_\alpha \otimes L_\gamma^{(1)} = L_{\alpha+p\gamma}$, and hence are not p-restricted. $\square$

Lemma B.7. *If $A(k)$ and $B(k)$ are true, then $A(k+1)$ is true.*

Proof. By elementary highest weight theory (see (4) in Section 2C), L_λ is a composition factor of

$$L_{\lambda^0} \otimes L_{\lambda^1}^{(1)} \otimes \cdots \otimes L_{\lambda^r}^{(r)}.$$

Thus it suffices to prove that the latter is a simple functor. Since $A(k)$ is true, it remains to prove that a p-restricted functor L of degree $k+1$ is necessarily a quotient of $\otimes^{k+1}$. By Corollary B.4, there exists a tuple of nonnegative integers $(d_0, \ldots, d_r)$ such that L is a quotient of a tensor product of the form $T^{(d_0,\ldots,d_r)}$. But assertion $B(k)$ says that such tensor products have no p-restricted composition factor except maybe if $(d_0, \ldots, d_r) = (k+1, 0, \ldots, 0)$. $\square$

Lemmas B.5, B.6 and B.7 imply that $A(k)$ is true for all $k \geq 0$. We are now ready to prove:

Theorem B.8 (Steinberg's tensor product theorem). *Let $\lambda^0, \ldots, \lambda^r$ be p-restricted partitions, and let $\lambda = \sum_{i=0}^r p^i \lambda^i$. There is an isomorphism*

$$L_\lambda \simeq L_{\lambda^0} \otimes L_{\lambda^1}^{(1)} \otimes \cdots \otimes L_{\lambda^r}^{(r)}.$$

Proof. Since $A(k)$ is true for all $k \geq 0$, p-restricted simples are quotients of tensor powers $\otimes^d$. Moreover $(\otimes^d)^{(i)} = (I^{(i)})^{\otimes d}$ is a quotient of $(\Gamma^{p^i})^{\otimes d}$. Thus, for all $k \leq r$, $\bigotimes_{i<k} L_{\lambda^i}^{(i)}$ is left p^k-bounded. An induction on k using Proposition B.2 shows that each tensor product $\bigotimes_{i \leq k} L_{\lambda^i}^{(i)}$ is simple. $\square$

Remark B.9. If $\lambda = (\lambda_1, \ldots, \lambda_k)$, then $L_\lambda(\Bbbk^n)$ is a simple polynomial $\mathrm{GL}_n(\Bbbk)$-module if $n \geq k$ and is zero if $n < k$ (This follows from the properties of the deflating Schur functor $d_{N,n}$ given in [Martin 1993, pp. 109–110], and the fact that $\mathrm{ev}_{\Bbbk^n} = d_{N,n} \circ \mathrm{ev}_{\Bbbk^N}$.) Thus, Theorem B.8 actually implies the Steinberg tensor product theorem for polynomial representations of $\mathrm{GL}_n(\Bbbk)$, for all values of n (and in particular without requiring that the representations are stable). Finally, all simple rational representations of $\mathrm{GL}_n(\Bbbk)$ can be obtained by tensoring simple polynomial representations of $\mathrm{GL}_n(\Bbbk)$ by a power of the determinant representation. Thus, Theorem B.8 implies the classical Steinberg tensor product theorem as in [Jantzen 2003, II.3.17].

Theorem B.10 (Clausen and James' theorem). *A simple functor L is p-restricted if and only if $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(L, \otimes^d) = \mathrm{Hom}_{\mathcal{P}_\Bbbk}(\otimes^d, L)$ is nonzero.*

Proof. Property $A(k)$ gives the "only if" part. Conversely, assume that the highest weight λ of L is not p-restricted. Using euclidean division, we write $\lambda = \lambda^0 + p\lambda^1$ with λ^0 p-restricted and λ^1 nonzero. Thus $L \simeq L_{\lambda^0} \otimes L_{\lambda^1}^{(1)}$ by Steinberg's tensor product theorem. By property $A(k)$, L_{λ^0} is left and right p-bounded, so that by Theorem 3.6, $\mathrm{Hom}_{\mathcal{P}}(L, \otimes^d) = \mathrm{Hom}_{\mathcal{P}}(\otimes^d, L) = 0$. $\square$

Remark B.11. There already exists a functorial proof of Steinberg's tensor product theorem in the literature [Kuhn 2002, Theorem 7.11]. However, the proof given in this appendix is quite different from that in [Kuhn 2002]. Let us stress two differences. First, the proof in [Kuhn 2002] uses finite fields, while the size of the ground field plays no role in our proof. Second, to obtain a concrete form of [Kuhn 2002, Theorem 7.11], one needs to know the classification of simple representations of symmetric groups. On the contrary, our proof does not use any knowledge of representations of symmetric groups. Better still, our reasoning also proves Clausen and James' theorem, so we can actually use our approach to derive the classification of simple representations of symmetric groups from the classification of simple representations of GL_n.

Steinberg's tensor product theorem tells us that if L_λ is simple and p-restricted and L_μ is simple, then $L_\lambda \otimes L_\mu^{(1)}$ is simple. The following statement completes the picture regarding tensor products of simple objects.

Theorem B.12. *Let L and L' be both simple and p-restricted. Then $L \otimes L'$ is not simple, unless one of the two is the constant functor $\Bbbk$.*

The remainder of the section is devoted to the proof of Theorem B.12.

Lemma B.13. *Let d be a positive integer, and let L be a simple quotient of $\otimes^d$. The following injection induced by the tensor product is not surjective*:

$$\mathrm{Hom}_{\mathcal{P}_\Bbbk}(\otimes^d, L) \otimes \mathrm{Hom}_{\mathcal{P}_\Bbbk}(I, I) \hookrightarrow \mathrm{Hom}_{\mathcal{P}_\Bbbk}(\otimes^{d+1}, L \otimes I).$$

Proof. Fix a vector space V equipped with an isomorphism $\Bbbk^d \oplus \Bbbk \simeq V$. Let $\iota_1 : \Bbbk \hookrightarrow V$, $\iota_2 : \Bbbk^d \hookrightarrow V$, $\pi_1 : V \to \Bbbk$ and $\pi_2 : V \to \Bbbk^d$ be the associated canonical maps. Since $\mathrm{End}_{\mathcal{P}_\Bbbk}(I) \simeq \Bbbk$, any nonzero map ϕ in the image of the injection of Lemma B.13 is of the form $\phi = f \otimes \mathrm{Id}$ for a nonzero f. Thus the following composite is nonzero (it equals the map induced by f):

$$(\Bbbk^d)^{\otimes d} \otimes \Bbbk \xrightarrow{(\iota_2)^{\otimes d} \otimes \iota_1} V^{\otimes d+1} \xrightarrow{\phi} L(V) \otimes V \xrightarrow{L(\pi_2) \otimes \pi_1} L(\Bbbk^d) \otimes \Bbbk. \qquad (*)$$

For all morphisms $f : \otimes^d \to L$, we define a morphism $\psi_f : \otimes^{d+1} \to L \otimes I$ by $\psi_f(x_1 \otimes \cdots \otimes x_{d+1}) = f(x_2 \otimes \cdots \otimes x_{d+1}) \otimes x_1$. If f is nonzero, then ψ_f is nonzero, while for $\phi = \psi_f$ the composite $(*)$ is zero. In particular ψ_f is not in the image of the inclusion. $\qquad \square$

Lemma B.14. *Let d be a positive integer, and let L be a simple quotient of $\otimes^d$. Let $L^{(d-1,1)}$ be the homogeneous summand of bidegree $(d-1,1)$ of the bifunctor $(V, W) \mapsto L(V \oplus W)$. There is an isomorphism $L^{(d-1,1)} \simeq F_L \boxtimes I$, where F_L is a nonzero homogeneous functor of degree $d-1$.*

Proof. Proposition B.3 provides an isomorphism $L^{(d-1,1)} \simeq F_L \boxtimes I$. We have to prove that F_L is nonzero. By using the sum-diagonal adjunction and the Künneth formula, we obtain that $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(\otimes^{d+1}, L \otimes I)$ is isomorphic to

$$\mathrm{Hom}_{\mathcal{P}_\Bbbk}(\otimes^d, L) \otimes \mathrm{End}_{\mathcal{P}_\Bbbk}(I) \oplus \mathrm{Hom}_{\mathcal{P}_\Bbbk}(\otimes^d, F_L \otimes I) \otimes \mathrm{End}_{\mathcal{P}_\Bbbk}(I).$$

For dimension reasons, Lemma B.13 implies that $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(\otimes^d, F_L \otimes I)$ is nonzero. Hence F_L is nonzero. $\square$

Proof of Theorem B.12. We will show that the dimension of $\mathrm{End}_{\mathcal{P}_\Bbbk}(L \otimes L')$ is not one. To this purpose, we use the sum-diagonal adjunction and the Künneth formula. We obtain that the vector space $\mathrm{End}_{\mathcal{P}_\Bbbk}(L \otimes L')$ contains

$$\mathrm{End}_{\mathcal{P}_\Bbbk}(L) \otimes \mathrm{End}_{\mathcal{P}_\Bbbk}(L') \oplus \mathrm{Hom}_{\mathcal{P}_\Bbbk}(L, F_L \otimes I) \otimes \mathrm{Hom}_{\mathcal{P}_\Bbbk}(L', F_{L'} \otimes I)$$

as a direct summand, with F_L and $F_{L'}$ defined as in Lemma B.14. Again using the sum-diagonal adjunction and the Künneth formula, we get that $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(L, F_L \otimes I)$ contains $\mathrm{End}_{\mathcal{P}_\Bbbk}(F_L) \otimes \mathrm{End}_{\mathcal{P}_\Bbbk}(I)$ as a direct summand (and similarly for L'). But Lemma B.14 asserts that F_L and $F_{L'}$ are nonzero, so that the dimension of the corresponding endomorphism spaces is at least one. So, the dimension of $\mathrm{End}_{\mathcal{P}_\Bbbk}(L \otimes L')$ is at least two. $\square$

Acknowledgments

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