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An explicit basis for the rational higher Chow groups of abelian number fields

Matt Kerr and Yu Yang

We review and simplify A. Beĭlinson's construction of a basis for the motivic cohomology of a point over a cyclotomic field, then promote the basis elements to higher Chow cycles and evaluate the KLM regulator map on them.

1. Introduction

Let $\zeta_N \in \mathbb{C}^*$ be a primitive N-th root of 1 ($N \geq 2$). The seminal article [Beĭlinson 1984] concludes with a construction of elements Ξ_b (for $b \in (\mathbb{Z}/N\mathbb{Z})^*$) in motivic cohomology

$$H^1_{\mathcal{M}}(\mathrm{Spec}(\mathbb{Q}(\zeta_N)), \mathbb{Q}(n)) \cong K^{(n)}_{2n-1}(\mathbb{Q}(\zeta_N)) \otimes \mathbb{Q}$$

mapping to $\mathrm{Li}_n(\zeta_N^b) = \sum_{k\geq 1} \zeta_N^{kb}/k^n \in \mathbb{C}/(2\pi \boldsymbol{i})^n\mathbb{R}$ under his regulator. Since by Borel's theorem [1974], we have $\mathrm{rk}\, K^{(n)}_{2n-1}(\mathbb{Q}(\zeta_N))_{\mathbb{Q}} = \frac{1}{2}\phi(N)$ (for $N \geq 3$), an immediate consequence is that the $\{\Xi_b\}$ span $K^{(n)}_{2n-1}(\mathbb{Q}(\zeta_N))_{\mathbb{Q}}$; indeed, Beĭlinson's results anticipated the eventual proofs [Rapoport 1988; Burgos Gil 2002] of the equality (for number fields) of his regulator with that of Borel [1977]. An expanded account of his construction was written up by Neukirch (with Rapoport and Schneider) in [Neukirch 1988], up to a "crucial lemma" [op. cit., Part II, Lemma 2.4] required for the regulator computation, which was subsequently proved by Esnault [1989].

The intervening years have seen some improvements in technology, especially Bloch's introduction of higher Chow groups [Bloch 1986], which yield an *integral* definition of motivic cohomology for smooth schemes X. In particular, we have[1]

$$\begin{aligned} H^1_{\mathcal{M}}(\mathrm{Spec}(\mathbb{Q}(\zeta_N)), \mathbb{Z}(n)) &\cong CH^n(\mathbb{Q}(\zeta_N), 2n-1) \\ &:= H_{2n-1}\{Z^n(\mathbb{Q}(\zeta_N), \bullet), \partial\}, \end{aligned}$$

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[1] We use the shorthand $CH^*(F, *)$ ($Z^*(F, *)$, etc.) for $CH^*(\mathrm{Spec}(F), *)$ (F a field).

and can ask for explicit cycles in $\ker(\partial) \subset Z^n(\mathbb{Q}(\zeta_N), 2n-1)$ representing (multiples of) Beĭlinson's elements Ξ_b. Another relevant development was the explicit realization of Beĭlinson's regulator in [Kerr et al. 2006; Kerr and Lewis 2007] as a morphism $\widetilde{\mathrm{AJ}}$ of complexes, from a *rationally* quasi-isomorphic subcomplex $Z^n_{\mathbb{R}}(X, \bullet)$ of $Z^n(X, \bullet)$ to a complex computing the absolute Hodge cohomology of X. Here this "KLM morphism" yields an Abel–Jacobi mapping

$$\mathrm{AJ} : CH^n(\mathbb{Q}(\zeta_N), 2n-1) \otimes \mathbb{Q} \to \mathbb{C}/(2\pi \boldsymbol{i})^n \mathbb{Q}, \tag{1.1}$$

and in the present note we shall construct (for all n) higher Chow cycles

$$\hat{\mathscr{Z}}_b \in \ker(\partial) \subset Z^n_{\mathbb{R}}(\mathbb{Q}(\zeta_N), 2n-1) \otimes \mathbb{Q}$$

satisfying

$$(n-3)N^{n-1}\hat{\mathscr{Z}}_b \in Z^n_{\mathbb{R}}(\mathbb{Q}(\zeta_N), 2n-1)) \quad \text{and} \quad \mathrm{AJ}(\hat{\mathscr{Z}}_b) = \mathrm{Li}_n(\zeta_N^b).$$

(See Theorems 3.3, 3.8, and 4.2, with $\hat{\mathscr{Z}} = (-1)^n \tilde{\mathscr{Z}}/N^{n-1}$.) This is entirely more explicit than the constructions in [Beĭlinson 1984; Neukirch 1988], and yields a brief and transparent evaluation of the regulator, which moreover allows us to dispense with some of the hypotheses of [Neukirch 1988, Part II, Lemma 2.4] or [Esnault 1989, Theorem 3.9] and thus avoid the more complicated construction of [Neukirch 1988, Part II, Lemma 3.1]. Furthermore, in concert with the anticipated extension of $\widetilde{\mathrm{AJ}}$ to the entire complex $Z^n(X, \bullet)$ (making (1.1) integral), we expect that our cycles will be useful for studying the torsion in $CH^n(\mathbb{Q}(\zeta_N), 2n-1)$, as begun in [Petras 2008; 2009]; see Remark 4.1 and Section 4E.

2. Beĭlinson's construction

In this section we show that (the graph of) the n-tuple of functions

$$\left\{1 - \zeta_N z_1 \cdots z_{n-1}, \left(\frac{z_1}{z_1 - 1}\right)^N, \ldots, \left(\frac{z_{n-1}}{z_{n-1} - 1}\right)^N\right\}$$

completes to a relative motivic cohomology class on $(\square^{n-1}, \partial\square^{n-1})$. Most of the work that follows is to show that its image under a residue map vanishes; see (2.12). It also serves to establish notation for Section 3, where we recast this class as a higher Chow cycle and compute its regulator.

2A. *Notation.* Let $N \geq 2$, and $\zeta \in \mathbb{C}$ be a primitive N-th root of unity; i.e., $\zeta = e^{2\pi \boldsymbol{i} a/N}$, where a is coprime to N. Denoting by $\Phi_N(x)$ the N-th cyclotomic polynomial, each such a yields an embedding σ of $\mathbb{F} := \mathbb{Q}[\omega]/(\Phi_N(\omega))$ into $\mathbb{C}$ (by sending $\omega \mapsto \zeta$). (If $N = 2$, then $\mathbb{F} = \mathbb{Q}$ and $\omega = \zeta = -1$.)

Working over any subfield of $\mathbb{C}$ containing ζ, write

$$\square^n := (\mathbb{P}^1 \backslash \{1\})^n \supset (\mathbb{P}^1 \backslash \{0, 1\})^n =: \mathbb{T}^n,$$

with coordinates $(z_1, \dots, z_n)$. We have isomorphisms from $\mathbb{T}^n$ to $\mathbb{G}_m^n$ (with coordinates $(t_1, \dots, t_n)$), given by $t_i := z_i/(z_i - 1)$. Define a function $f_n(\underline{z}) := 1 - \zeta^b t_1 \cdots t_n$ on $\mathbb{T}^n$ (with b coprime to N), and normal crossing subschemes

$$S^n := \{\underline{z} \in \mathbb{T}^n \mid \text{some } z_i = \infty\} \subset S^n \cup |(f_n)_0| =: \widetilde{S}^n \subset \mathbb{T}^n.$$

(Alternatively, we may view these schemes as defined over $\mathbb{F}$ by replacing ζ^b with ω^b.)

Now consider the morphism

$$\imath_n : \mathbb{T}^{n-1} \to \mathbb{T}^n, \qquad (t_1, \dots, t_{n-1}) \mapsto (t_1, \dots, t_{n-1}, (\zeta^b t_1 \cdots t_{n-1})^{-1}).$$

Lemma 2.1. *The morphism $\imath_n$ sends $\mathbb{T}^{n-1}$ isomorphically onto $|(f_n)_0|$, with*

$$\imath_n(\widetilde{S}^{n-1}) = |(f_n)_0| \cap S^n.$$

We also remark that the Zariski closure of $\imath_n(\mathbb{T}^{n-1})$ in $\square^n$ is just $\imath_n(\mathbb{T}^{n-1})$.

2B. ***Results for Betti cohomology.*** The construction just described has quite pleasant cohomological properties, as we shall now see.

Lemma 2.2. *As a $\mathbb{Q}$-MHS,*

$$H^q(\mathbb{T}^n, S^n) \cong \begin{cases} \mathbb{Q}(-n), & q = n, \\ 0, & q \neq n. \end{cases}$$

Proof. Apply the Künneth formula to $(\mathbb{T}^n, S^n) \cong (\mathbb{G}_m, \{1\})^n$. $\square$

Lemma 2.3. *As a $\mathbb{Q}$-MHS,*

$$H^q(\mathbb{T}^n, \widetilde{S}^n) \cong \begin{cases} \mathbb{Q}(0) \oplus \mathbb{Q}(-1) \oplus \cdots \oplus \mathbb{Q}(-n), & q = n, \\ 0, & q \neq n. \end{cases}$$

Proof. This is clear for $(\mathbb{T}^1, \widetilde{S}^1) \cong (\mathbb{G}_m, \{1, \bar{\zeta}\})$. Now consider the exact sequence

$$H^{*-1}(\mathbb{T}^n, S^n) \xrightarrow{\imath_n^*} \underline{H^{*-1}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1})} \xrightarrow{\delta} H^*(\mathbb{T}^n, \widetilde{S}^n) \to \underline{H^*(\mathbb{T}^n, S^n)} \xrightarrow{\imath_n^*} H^*(\mathbb{T}^{n-1}, \widetilde{S}^{n-1})$$

of $\mathbb{Q}$-MHS, associated to the inclusion $(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}) \overset{\imath_n}{\hookrightarrow} (\mathbb{T}^n, S^n)$. (This is just the relative cohomology sequence, once one notes that $((\mathbb{T}^n, S^n), \imath_n(\mathbb{T}^{n-1}, \widetilde{S}^{n-1})) = (\mathbb{T}^n, S^n \cup \imath_n(\mathbb{T}^{n-1})) = (\mathbb{T}^n, \widetilde{S}^n)$ by Lemma 2.1.) If $* \neq n$, then the underlined terms are 0 via Lemma 2.2 and induction. If $* = n$, then the *end* terms are 0 via Lemma 2.2 and induction, and

$$0 \to H^{n-1}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}) \xrightarrow{\delta} H^n(\mathbb{T}^n, \widetilde{S}^n) \to H^n(\mathbb{T}^n, S^n) \to 0 \tag{2.4}$$

is a short-exact sequence.

Now observe that:

- $H^n(\mathbb{T}^n, S^n; \mathbb{C}) = F^n H^n(\mathbb{T}^n, S^n; \mathbb{C})$ is generated by the holomorphic form
$$\eta := \frac{1}{(2\pi \boldsymbol{i})^n} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n};$$
- $H_{n-1}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{Q})$ is generated by images $\underline{e}(U_i)$ of the cells
$$\bigcup_{i=0}^{n} U_i = [0,1]^n \setminus \bigcup_{\ell=1}^{n} \Bigl\{ \sum x_i = \ell - \frac{a}{N} \Bigr\},$$
where $\underline{e} : [0,1]^n \to \mathbb{T}^n$ is defined by $(x_1, \ldots, x_n) \mapsto (e^{2\pi \boldsymbol{i} x_1}, \ldots, e^{2\pi \boldsymbol{i} x_n}) = (t_1, \ldots, t_n)$;
- $\displaystyle\int_{\underline{e}(U_i)} \eta = \int_{U_i} dx_1 \wedge \cdots \wedge dx_n \in \mathbb{Q}.$

(Writing $\mathscr{S}^1$ for the unit circle, $((\mathscr{S}^1)^n, (\mathscr{S}^1)^n \cap \widetilde{S}^n)$ is a deformation retract of $(\mathbb{T}^n, \widetilde{S}^n)$. The $\underline{e}(U_i)$ visibly yield all the relative cycles in the former, justifying the second observation.) Together these immediately imply that (2.4) is split, completing the proof. □

2C. ***Results for Deligne cohomology.*** Recall that Beĭlinson's absolute Hodge cohomology [1986] of an analytic scheme Y over $\mathbb{C}$ sits in an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{r-1}(\mathrm{Y}, \mathbb{A}(p))) \to H^r_{\mathcal{D}}(\mathrm{Y}, \mathbb{A}(p)) \\ \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^r(\mathrm{Y}, \mathbb{A}(p))) \to 0.$$

(Here we use a subscript "$\mathcal{D}$" since the construction after all is a "weight-corrected" version of Deligne cohomology; the subscript "MHS" of course means "$\mathbb{A}$-MHS".) We shall not have any use for details of its construction here, and refer the reader to [Kerr and Lewis 2007, §2].

Lemma 2.5. *The map* $\iota_n^* : H^n_{\mathcal{D}}(\mathbb{T}^n, S^n; \mathbb{A}(n)) \to H^n_{\mathcal{D}}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{A}(n))$ *is zero* ($\mathbb{A} = \mathbb{Q}$ *or* $\mathbb{R}$).

Proof. Consider the exact sequence

$$\cdots \to H^n_{\mathcal{D}}(\mathbb{T}^n, S^n; \mathbb{Q}(n)) \xrightarrow{\iota_n^*} H^n_{\mathcal{D}}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{Q}(n)) \xrightarrow{\delta_{\mathcal{D}}} H^{n+1}_{\mathcal{D}}(\mathbb{T}^n, \widetilde{S}^n; \mathbb{Q}(n)) \to \cdots.$$

It suffices to show that $\delta_{\mathcal{D}}$ is injective. Now

$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^n(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{Q}(n))) = \{0\},$$
$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^{n+1}(\mathbb{T}^n, \widetilde{S}^n; \mathbb{Q}(n))) = \{0\},$$

by Lemma 2.3, and so $\delta_{\mathcal{D}}$ is given by

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{n-1}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{Q}(n))) \xrightarrow{\delta_{\mathcal{D}}} \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^n(\mathbb{T}^n, \widetilde{S}^n; \mathbb{Q}(n))).$$

Since (2.4) is split, the corresponding sequence of Ext^1-groups is exact, and $\delta_{\mathcal{D}}$ is injective. $\square$

2D. ***Results for motivic cohomology.*** Let X be any smooth simplicial scheme (of finite type), defined over a subfield of $\mathbb{C}$. We have Deligne class maps

$$c_{\mathcal{D},\mathbb{A}} : H^r_{\mathcal{M}}(X, \mathbb{Q}(p)) \to H^r_{\mathcal{D}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{A}(p))$$

(for $\mathbb{A} = \mathbb{Q}$ or $\mathbb{R}$). The case of particular interest here is when $r = 1$, X is a point, and

$$c_{\mathcal{D},\mathbb{A}}(\mathrm{Z}) = \frac{1}{(2\pi \boldsymbol{i})^{p-1}} \int_{\mathrm{Z}^{\mathrm{an}}_{\mathbb{C}}} R_{2p-1} \in \mathbb{C}/\mathbb{A}(p), \tag{2.6}$$

where, interpreting $\log(z)$ as the 0-current with branch cut along $T_z := z^{-1}(\mathbb{R}_-)$,

$$\begin{aligned} R_{2p-1} &:= \sum_{k=1}^{2p-1} (2\pi \boldsymbol{i})^{k-1} R^{(k)}_{2p-1} \\ &:= \sum_{k=1}^{2p-1} (2\pi \boldsymbol{i})^{k-1} \log(z_k) \frac{dz_{k+1}}{z_{k+1}} \wedge \cdots \wedge \frac{dz_{2p-1}}{z_{2p-1}} \cdot \delta_{T_{z_1} \cap \cdots \cap T_{z_{k-1}}} \end{aligned} \tag{2.7}$$

is the regulator current of [Kerr et al. 2006; Kerr and Lewis 2007] belonging to $D^{2p-2}((\mathbb{P}^1)^{\times(2p-1)})$. Here it is essential that the representative higher Chow cycle Z belong to the quasi-isomorphic subcomplex $Z^p_{\mathbb{R}}(\mathrm{pt.}, \bullet)_{\mathbb{Q}} \subset Z^p(\mathrm{pt.}, \bullet)_{\mathbb{Q}}$ comprising cycles in good position with respect to certain real analytic chains; see [Kerr and Lewis 2007, §8] or Remark 3.4 below.

Now take a number field K, $[K : \mathbb{Q}] = d = r_1 + 2r_2$, and set

$$d_m = d_m(K) := \begin{cases} r_1 + r_2 - 1, & m = 1, \\ r_1 + r_2, & m > 1 \text{ odd}, \\ r_2, & m > 0 \text{ even}. \end{cases}$$

For X defined over K, write $\widetilde{X^{\mathrm{an}}_{\mathbb{C}}} := \coprod_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} ({}^{\sigma}X)^{\mathrm{an}}_{\mathbb{C}}$ and

$$\begin{tikzcd} H^r_{\mathcal{M}}(X, \mathbb{Q}(p)) \arrow[rr, "\tilde{c}_{\mathcal{D},\mathbb{R}}"] \arrow[dr, "\tilde{c}^+_{\mathcal{D},\mathbb{R}}"'] & & H^r(\widetilde{X^{\mathrm{an}}_{\mathbb{C}}}, \mathbb{R}(p)) \\ & H^r_{\mathcal{D}}(\widetilde{X^{\mathrm{an}}_{\mathbb{C}}}, \mathbb{R}(p))^+ \arrow[ur, hook] & \end{tikzcd}$$

for the map $\mathrm{Z} \mapsto (c_{\mathcal{D},\mathbb{R}}({}^{\sigma}\mathrm{Z}))_{\sigma}$, which factors through the invariants under de Rham conjugation. If $X = \mathrm{Spec}(K)$, then we have $H^1_{\mathcal{D}}(\widetilde{X^{\mathrm{an}}_{\mathbb{C}}}, \mathbb{R}(p)) \cong \mathbb{R}(p-1)^{\oplus d}$ and $H^1_{\mathcal{D}}(\widetilde{X^{\mathrm{an}}_{\mathbb{C}}}, \mathbb{R}(p))^+ \cong \mathbb{R}(p-1)^{\oplus d_p}$. Write $H^r_{\mathcal{M}}(X, \mathbb{R}(p)) = H^r_{\mathcal{M}}(X, \mathbb{Q}(p)) \otimes_{\mathbb{Q}} \mathbb{R}$.

Lemma 2.8. *For* $X = \mathrm{Spec}(K)$, $\mathbb{G}_{m,K}^{\times n}$, $(\mathbb{T}_K^n, S_K^n)$, *or* $(\mathbb{T}_K^n, \widetilde{S}_K^n)$,

$$\tilde{c}_{\mathcal{D},\mathbb{R}}^{+} \otimes \mathbb{R} : H_{\mathcal{M}}^r(X, \mathbb{R}(p)) \to H_{\mathcal{D}}^r(\widetilde{X_{\mathbb{C}}^{\mathrm{an}}}, \mathbb{R}(p))^{+}$$

is an isomorphism ($\forall r, p$).

Proof. By [Burgos Gil 2002], the composition

$$K_{2p-1}(\mathcal{O}_K) \otimes \mathbb{Q} \xrightarrow{\cong} H_{\mathcal{M}}^1(\mathrm{Spec}(K), \mathbb{Q}(p)) \xrightarrow{\tilde{c}_{\mathcal{D},\mathbb{R}}^{+}} \mathbb{R}(p-1)^{\oplus d_p} \xrightarrow{\cdot 2/(2\pi \boldsymbol{i})^{p-1}} \mathbb{R}^{d_p}$$

is exactly the Borel regulator (and the groups are zero for $r \neq 1$). The lemma follows for $X = \mathrm{Spec}(K)$.

Let Y be a smooth quasiprojective variety, defined over K, and pick $p \in \mathbb{G}_m(K)$. Write $Y \overset{\imath}{\hookrightarrow} \mathbb{G}_{m,Y} \overset{\jmath}{\hookrightarrow} \mathbb{A}_Y^1 \overset{\kappa}{\hookleftarrow} Y$ for the Cartesian products with Y of the morphisms

$$\mathrm{Spec}(K) \overset{\imath_p}{\hookrightarrow} \mathbb{G}_{m,K} \overset{\jmath}{\hookrightarrow} \mathbb{A}_K^1 \overset{\imath_0}{\hookleftarrow} \mathrm{Spec}(K).$$

Then by the homotopy property,

$$\imath^* : H_{\mathcal{K}}^r(\mathbb{G}_{m,Y}, \mathbb{R}(p)) \to H_{\mathcal{K}}^r(Y, \mathbb{R}(p)) \cong H_{\mathcal{K}}^r(\mathbb{A}_Y^1, \mathbb{R}(p))$$

splits the localization sequence

$$\cdots \xrightarrow{\kappa_*} H_{\mathcal{K}}^r(\mathbb{A}_Y^1, \mathbb{R}(p)) \xrightarrow{\jmath^*} H_{\mathcal{K}}^r(\mathbb{G}_{m,Y}, \mathbb{R}(p)) \xrightarrow{\mathrm{Res}} H_{\mathcal{K}}^{r-1}(Y, \mathbb{R}(p-1)) \xrightarrow{\kappa_*} \cdots$$

for $\mathcal{K} = \mathcal{M}, \mathcal{D}$ (in particular, $\kappa_* = 0$). It follows that

$$H_{\mathcal{K}}^r(\mathbb{G}_{m,Y}, \mathbb{R}(p)) \cong H_{\mathcal{K}}^r(Y, \mathbb{R}(p)) \oplus H_{\mathcal{K}}^{r-1}(Y, \mathbb{R}(p-1)),$$

compatibly with $c_{\mathcal{D},\mathbb{R}}$; applying this iteratively gives the lemma for $\mathbb{G}_{m,K}^{\times n}$.

Finally, both $(\mathbb{T}^n, S_K^n)$ and $(\mathbb{T}_K^n, \widetilde{S}_K^n)$ may be regarded as (co)simplicial normal crossing schemes $X^\bullet$. (That is, writing $\widetilde{S}_K^n = \bigcup Y_i$, we take $X^0 = \mathbb{T}_K^n$, $X^1 = \coprod_i Y_i$, $X^2 = \coprod_{i<j} Y_i \cap Y_j$, etc.) We have spectral sequences

$$E_1^{i,j} = H_{\mathcal{K}}^{2p+j}(X^i, \mathbb{R}(p)) \Longrightarrow H_{\mathcal{K}}^{2p+i+j}(X^\bullet, \mathbb{R}(p)),$$

compatible with $c_{\mathcal{D},\mathbb{R}}$, and all X^i are disjoint unions of powers of $\mathbb{G}_{m,K}$. The lemma is proved. □

Lemma 2.9. *The map* $\imath_n^* : H_{\mathcal{M}}^n(\mathbb{T}^n, S^n; \mathbb{A}(n)) \to H_{\mathcal{M}}^n(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{A}(n))$ *is zero* (*for* $\mathbb{A} = \mathbb{Q}$ *or* $\mathbb{R}$).

Proof. Form the obvious commutative square and use the results of Lemmas 2.5 and 2.8. □

2E. ***The Beĭlinson elements.*** To each $I \subset \{1, \dots, n\}$ and $\epsilon : I \to \{0, \infty\}$ we associate a face map $\rho_I^\epsilon : \square^{n-|I|} \hookrightarrow \square^n$, with $z_i = \epsilon(i)$ (for all $i \in I$) on the image, and degeneracy maps $\delta_i : \square^n \twoheadrightarrow \square^{n-1}$ killing the i-th coordinate. For any smooth quasiprojective variety X (say, over a field $K \supseteq \mathbb{Q}$), let $c^p(X, n)$ denote the free abelian group on subvarieties (of codimension p) of $X \times \square^n$ meeting all faces $X \times \rho_I^\epsilon(\square^{n-|I|})$ properly, and $d^p(X, n) = \sum \operatorname{im}(\mathrm{id}_X \times \delta_i^*) \subset c^p(X, n)$. Then $Z^p(X, \bullet) := c^p(X, \bullet)/d^p(X, \bullet)$ defines a complex with differential

$$\partial = \sum_{i=1}^{n} (-1)^{i-1} \big((\mathrm{id}_X \times \rho_i^0)^* - (\mathrm{id}_X \times \rho_i^\infty)^* \big),$$

whose r-th homology defines Bloch's higher Chow group

$$CH^p(X, r) \cong H_{\mathcal{M}}^{2p-r}(X, \mathbb{Z}(p)). \tag{2.10}$$

This isomorphism does not apply for singular varieties (e.g., our simplicial schemes above), and for our purposes in this paper it is the right-hand side of (2.10) that provides the correct generalization. In particular, we have

$$H_{\mathcal{M}}^r(X \times (\square^a, \partial\square^a), \mathbb{Q}(p)) \cong H_{\mathcal{M}}^{r-a}(X, \mathbb{Q}(p)),$$

where $\partial\square^a := \bigcup_{i \in \{1,\dots,a\},\, \varepsilon \in \{0,\infty\}} \rho_i^\varepsilon(\square^{a-1})$. We note here that the (rational) motivic cohomology of a cosimplicial normal-crossing scheme $X^\bullet$ can be computed via (the simple complex associated to) a double complex:

$$E_0^{a,b} := Z^p(X^a, -b)_{\mathbb{Q}}^{\#} \Longrightarrow H_{\mathcal{M}}^{2p+a+b}(X^\bullet, \mathbb{Q}(p)), \tag{2.11}$$

where # denotes cycles meeting all components of all $X^{q>a} \times \partial_I^\epsilon \square^{-b}$ properly.[2]

Continuing to write t_i for $z_i/(z_i - 1)$, we now consider

$$f(\underline{z}) = f_{n-1}(z_1, \dots, z_{n-1}) := 1 - \omega^b t_1 \cdots t_{n-1}$$

as a regular function on $\square_{\mathbb{F}}^{n-1}$, and

$$\mathcal{Z} := \{ (\underline{z};\, f(\underline{z}), t_1^N, \dots, t_{n-1}^N) \mid \underline{z} \in \square^{n-1} \backslash |(f)_0| \}$$

as an element of

$$\ker\big\{ Z^n(\square^{n-1} \backslash |(f)_0|, n)_{\mathbb{Q}}^{\#} \\ \xrightarrow{\partial \oplus \sum (\rho_i^\varepsilon)^*} Z^n(\square^{n-1} \backslash |(f)_0|, n-1) \oplus \textstyle\bigoplus_{i,\varepsilon} Z^n \big(\square^{n-2} \backslash \big| (f|_{z_i=\varepsilon})_0 \big|, n \big)_{\mathbb{Q}} \big\},$$

and hence of

$$H_{\mathcal{M}}^n(\square_{\mathbb{F}}^{n-1} \backslash |(f)_0|, \partial\square^{n-1} \backslash \partial|(f)_0|; \mathbb{Q}(n))$$

[2] See [Levine 1994, §3] and [Kerr and Lewis 2007, §8.2] for the relevant moving lemmas (and for detailed discussion of differentials, etc.).

(where $\partial|(f)_0| := \partial\square^{n-1}\cap|(f)_0| = \bigcup_{i,\varepsilon}|(f|_{z_i=\varepsilon})_0|$, and # indicates cycles meeting faces of $\partial\square^{n-1}\backslash\partial|(f)_0|$ properly). The powers t_i^N are unnecessary at this stage but will be crucial later. For simplicity, we write the class of Z in this group as a symbol $\{f_{n-1}, t_1^N, \dots, t_{n-1}^N\}$.

Using Lemma 2.1, we have a (vertical) localization exact sequence

$$\begin{tikzcd}
{} \arrow[d] & \\
H^n_{\mathcal{M}}(\square^{n-1}, \partial\square^{n-1}; \mathbb{Q}(n)) \arrow[d] \arrow[r, leftrightarrow, "\cong"] & CH^n(\mathbb{F}, 2n-1)_{\mathbb{Q}} \\
H^n_{\mathcal{M}}(\square^{n-1}\backslash|(f)_0|, \partial\square^{n-1}\backslash|(f)_0|; \mathbb{Q}(n)) \arrow[d, "\mathrm{Res}_{|(f)_0|}"] & \\
H^{n-1}_{\mathcal{M}}(\mathbb{T}^{n-2}, \widetilde{S}^{n-2}; \mathbb{Q}(n-1)) \arrow[d] & H^{n-1}_{\mathcal{M}}(\mathbb{T}^{n-1}, S^{n-1}; \mathbb{Q}(n-1)) \arrow[l, "\iota^*_{n-1}"'] \\
{} &
\end{tikzcd} \tag{2.12}$$

in which evidently

$$\mathrm{Res}_{|(f)_0|}\{f_{n-1}, t_1^N, \dots, t_{n-1}^N\} = \iota^*_{n-1}\{t_1^N, \dots, t_{n-1}^N\}.$$

Proposition 2.13. *$\mathcal{Z}$ lifts to a class $\widetilde{\Xi} \in CH^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$.*

Proof. Apply (2.12) and Lemma 2.9. □

This is essentially Beĭlinson's construction; we normalize the class by

$$\Xi := \frac{(-1)^n}{N^{n-1}}\widetilde{\Xi}.$$

3. The higher Chow cycles

3A. ***Representing Beĭlinson's elements.*** We first describe (2.11) more explicitly in the relevant cases. As above, write $\partial : Z^n(\square^r, s)^{\#}_{\mathbb{Q}} \to Z^n(\square^r, s-1)^{\#}_{\mathbb{Q}}$ for the higher Chow differential, and

$$\delta : Z^n(\square^r, s)^{\#}_{\mathbb{Q}} \to \bigoplus_{i,\varepsilon} Z^n(\square^{r-1}, s)^{\#}_{\mathbb{Q}}$$

for the cosimplicial differential $\sum_{i=1}^r (-1)^{i-1}((\rho_i^0 \times \mathrm{id}_{\square^s})^* - (\rho_i^\infty \times \mathrm{id}_{\square^s})^*)$. A complex of cocycles for the top motivic cohomology group in (2.12) is given by

$$\mathfrak{Z}^n_{\square}(k) := Z^n_{\mathcal{M}}((\square^{n-1}_{\mathbb{F}}, \partial\square^{n-1}_{\mathbb{F}}), k)_{\mathbb{Q}} := \bigoplus_{a=0}^{n-1} \bigoplus_{(I,\epsilon),\,|I|=a} Z^n(\square_{\mathbb{F}}^{n-a-1}, a+k)^{\#}_{\mathbb{Q}} \tag{3.1}$$

with differential $\mathbb{D} := \partial + (-1)^{n-a-1}\delta$. These are, of course, the simple complex and total differential associated to the natural double complex

$$E_0^{a,b} = \bigoplus_{(I,\epsilon),\, |I|=a} Z^n(\square_{\mathbb{F}}^{n-a-1}, -b)^{\#}_{\mathbb{Q}}.$$

Analogously, one defines

$$\begin{aligned}\mathfrak{Z}^n_{\square\backslash f}(k) &:= Z^n_{\mathcal{M}}\big((\square_{\mathbb{F}}^{n-1}\backslash|(f)_0|, \partial\square_{\mathbb{F}}^{n-1}\backslash\partial|(f)_0|), k\big)_{\mathbb{Q}},\\ \mathfrak{Z}^{n-1}_f(k) &:= Z^{n-1}_{\mathcal{M}}((\mathbb{T}^{n-2}, \widetilde{\mathcal{S}}^{n-2}), k)_{\mathbb{Q}},\end{aligned}$$

so that $\mathfrak{Z}^{n-1}_f(\bullet) \xrightarrow{\imath_*} \mathfrak{Z}^n_{\square}(\bullet) \to \mathfrak{Z}^n_{\square\backslash f}(\bullet)$ are morphisms of (homological) complexes.

Now define

$$\theta : \mathfrak{Z}^n_{\square}(k) \to Z^n(\mathbb{F}, n+k-1)_{\mathbb{Q}}$$

by simply adding up the cycles (with no signs) on the right-hand side of (3.1). (Use the natural maps $\square^{n-a-1} \times \square^{a+k} \to \square^{n+k-1}$ obtained by concatenating coordinates.) Then we have:

Lemma 3.2. *The map θ is a quasi-isomorphism of complexes.*

Proof. Checking that θ is a morphism of complexes is easy and left to the reader. The $a = n-1$, $(I, \epsilon) = (\{1, \dots, n-1\}, \underline{0})$ term of (3.1) is a copy of $Z^n(\mathbb{F}, n+k-1)$ in $\mathfrak{Z}^n_{\square}(k)$, which leads to a morphism $\psi : Z^n(\mathbb{F}, n+\bullet-1) \to \mathfrak{Z}^n_{\square}(\bullet)$ with $\theta \circ \psi = \mathrm{id}$. Moreover, it is elementary that ψ is a quasi-isomorphism: taking $d_0 = \partial$ gives

$$E_1^{a,b} = \bigoplus_{(I,\epsilon),\, |I|=a} CH^n(\square_{\mathbb{F}}^{n-a-1}, -b)_{\mathbb{Q}} \cong CH^n(\mathbb{F}, -b)^{\oplus 2^a\binom{n-1}{a}},$$

so $E_2^{a,b} = 0$ except for $E_2^{n-1,b} \cong CH^n(\mathbb{F}, -b)$, which is exactly the image of $\psi(\ker\partial)$.[3] □

In particular, we may view θ as yielding the isomorphism in the top row of (2.10).

By the moving lemmas of Bloch [1994] and Levine [1994], we have another quasi-isomorphism

$$\frac{\mathfrak{Z}^n_{\square}(\bullet)}{\imath_*\mathfrak{Z}^{n-1}_f(\bullet)} \xrightarrow{\simeq} \mathfrak{Z}^n_{\square\backslash f}(\bullet),$$

which enables us to replace any $\mathcal{Y}_{\square\backslash f} \in \ker(\mathbb{D}) \subset \mathfrak{Z}^n_{\square\backslash f}(n)$ by a homologous $\mathcal{Y}'_{\square\backslash f}$ arising as the restriction of some $\mathcal{Y}'_{\square} \in \mathfrak{Z}^n_{\square}(n)$ with $\mathbb{D}\mathcal{Y}'_{\square} = \imath_*(\mathcal{Y}''_f)$ and $\mathcal{Y}''_f \in \ker(\mathbb{D}) \in \mathfrak{Z}^{n-1}_f(n-1)$. This gives an "explicit" prescription for computing $\mathrm{Res}_{|(f)_0|}$ in (2.10).

Now we come to our central point: the cycle $\mathcal{Z}=\{f_{n-1}, t_1^N, \dots, t_{n-1}^N\}$ of Section 2E already belongs to $(Z^n(\square_{\mathbb{F}}^{n-1}, n)^{\#}_{\mathbb{Q}} \subseteq)\mathfrak{Z}^n_{\square}(n)$, without "moving" it by a boundary. Its restriction to $\mathfrak{Z}^n_{\square\backslash f}(n)$ is clearly $\mathbb{D}$-closed, and $\mathbb{D}\mathcal{Z} = \imath_*\{t_1^N, \dots, t_{n-1}^N\} =: \imath_*\mathcal{T}$.

[3] This is true for any field, but specifically for our $\mathbb{F} = \mathbb{Q}(\omega)$, the only nonzero term is $E_2^{n-1,n}$.

By Proposition 2.13, the class of $\mathcal{T}$ in homology of $\mathfrak{Z}_f^{n-1}(\bullet)$ is trivial, and so there exists $\mathcal{T}' \in \mathfrak{Z}_f^{n-1}(n)$ with $\mathbb{D}\mathcal{T}' = -\mathcal{T}$. Defining

$$\mathcal{W} := \imath_* \mathcal{T}', \qquad \widetilde{\mathcal{Z}} := \mathcal{Z} + \mathcal{W},$$

we now have $\mathbb{D}\widetilde{\mathcal{Z}} = 0$. This allows us to make a rather precise statement about the lift in Proposition 2.13. Denote the projection $(z_1, \dots, z_{2n-1}) \mapsto (z_1, \dots, z_{n-i})$ by $\mathsf{p}_i : \square^{2n-1} \twoheadrightarrow \square^{n-i}$.

Theorem 3.3. $\widetilde{\Xi}$ *has a representative in* $Z^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$ *of the form*

$$\widetilde{\mathscr{Z}} = \mathscr{Z} + \mathscr{W} = \mathscr{Z} + \mathscr{W}_1 + \mathscr{W}_2 + \cdots + \mathscr{W}_{n-1},$$

where $\mathscr{Z} = \theta(\mathcal{Z})$ *(i.e.,* $\mathcal{Z}$ *interpreted as an element of* $Z^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$*) and* $\mathscr{W}_i$ *is supported on* $\mathsf{p}_i^{-1}|(f_{n-i})_0|$.

Proof. Viewing $(|(f_{n-1})_0|, \partial|(f_{n-1})_0|) \cong (\mathbb{T}^{n-2}, \widetilde{S}^{n-2})$ as a simplicial subscheme $\mathfrak{X}^\bullet$ of $(\square^{n-1}, \partial\square^{n-1}) =: X^\bullet$, the subscheme $\mathfrak{X}^{i-1} \subset X^{i-1}$ comprises $2^{i-1}\binom{n-1}{i-1}$ copies of $|(f_{n-i})_0| \subset \square^{n-1}$. We may decompose

$$\mathcal{W} \in \bigoplus_{i=1}^{n} \bigoplus_{(I,\epsilon),\,|I|=i-1} \imath_* Z^{n-1}(|(f_{n-i})_0|, n+i-1)^{\#}_{\mathbb{Q}} \subset \bigoplus_{i=1}^{n-1} E_0^{i-1,-n-i+1}$$

into its constituent pieces $\mathcal{W}_i \in E_0^{i-1,-n-i+1}$, and define $\mathscr{W}_i := \theta(\mathcal{W}_i)$ and $\mathscr{W} := \theta(\mathcal{W})$. Clearly $\mathrm{supp}(\mathscr{W}_i) \subset \mathsf{p}_i^{-1}|(f_{n-i})_0|$, and $\widetilde{\mathscr{Z}} := \theta(\widetilde{\mathcal{Z}})$ is ∂-closed, giving the desired representation. $\square$

Remark 3.4. In fact, $\sigma(\mathscr{Z}) \in Z^n_{\mathbb{R}}(\mathrm{Spec}(\mathbb{C}), 2n-1)_{\mathbb{Q}}$ for any $\sigma \in \mathrm{Hom}(\mathbb{F}, \mathbb{C})$: the intersections $T_{z_1} \cap \cdots \cap T_{z_k} \cap (\rho_I^\epsilon)^* \sigma(\mathscr{Z})$ are empty excepting $T_{z_1} \cap \cdots \cap T_{z_k} \cap \sigma(\mathscr{Z})$ for $k \le n-1$ and $T_{z_1} \cap \cdots \cap T_{z_k} \cap (\rho_n^0)^* \sigma(\mathscr{Z})$ for $k \le n-2$, which are both of the expected real codimension. A trivial modification of the above argument then shows that the $\mathscr{W}_i$ may be chosen so that the $\sigma(\mathscr{W}_i)$ (and hence $\sigma(\widetilde{\mathscr{Z}})$) are in $Z^n_{\mathbb{R}}(\mathrm{Spec}(\mathbb{C}), 2n-1)_{\mathbb{Q}}$ as well. We shall henceforth assume that this has been done.

3B. ***Computing the KLM map.*** We begin by simplifying the formula (2.6) for the regulator map.

Lemma 3.5. *Let* $K \subset \mathbb{C}$ *and suppose* $Z \in \ker(\partial) \subset Z^n_{\mathbb{R}}(\mathrm{Spec}(K), 2n-1)_{\mathbb{Q}}$ *satisfies*

$$T_{z_1} \cap \cdots \cap T_{z_n} \cap Z^{\mathrm{an}}_{\mathbb{C}} = \varnothing. \tag{3.6}$$

Then

$$c_{\mathcal{D},\mathbb{Q}}(Z) = \int_{Z^{\mathrm{an}}_{\mathbb{C}} \cap T_{z_1} \cap \cdots \cap T_{z_{n-1}}} \log(z_n) \frac{dz_{n+1}}{z_{n+1}} \wedge \cdots \wedge \frac{dz_{2n-1}}{z_{2n-1}}$$

in $\mathbb{C}/\mathbb{Q}(n)$.

Proof. We have

$$c_{\mathcal{D},\mathbb{Q}}(Z) = \sum_{k=1}^{n-1} (2\pi \boldsymbol{i})^{k-n} \int_{Z_{\mathbb{C}}^{\mathrm{an}}} R_{2n-1}^{(k)} + \int_{Z_{\mathbb{C}}^{\mathrm{an}}} R_{2n-1}^{(n)} + \sum_{k=1}^{n-1} (2\pi \boldsymbol{i})^{k} \int_{Z_{\mathbb{C}}^{\mathrm{an}}} R_{2n-1}^{(n+k)}.$$

The terms $\int_{Z_{\mathbb{C}}^{\mathrm{an}}} R_{2n-1}^{(k)}$ are zero by type, since $\dim_{\mathbb{C}} Z_{\mathbb{C}} = n-1$, and the $\int_{Z_{\mathbb{C}}^{\mathrm{an}}} R_{2n-1}^{(n+k)}$ are integrals over $Z_{\mathbb{C}}^{\mathrm{an}} \cap T_{z_1} \cap \cdots \cap T_{z_{n+k-1}} = \varnothing$. So only the middle term remains. □

Lemma 3.7. *For any* $\sigma \in \mathrm{Hom}(\mathbb{F}, \mathbb{C})$, $T_{z_1} \cap \cdots \cap T_{z_n} \cap \sigma(\widetilde{\mathscr{Z}}) = \varnothing$.

Proof. From Theorem 3.3, $\sigma(\mathscr{W}_i)$ is supported over $\mathsf{p}_i^{-1}(|(f_{n-i})_0|)$; that is, on $\sigma(\mathscr{W}_i)$ we have $z_1 \cdots z_{n-i} = \bar{\zeta}^b$, and so $T_{z_1} \cap \cdots \cap T_{z_{n-i}} \cap \sigma(\mathscr{W}_i) = \varnothing$, since $\bar{\zeta}^b \notin (-1)^{n-i}\mathbb{R}_+$. On $\sigma(\mathscr{Z})$, $z_n = f_{n-1}(z_1, \ldots, z_{n-1}) = 1 - \zeta^b t_1 \cdots t_{n-1}$ (where $t_i = z_i/(z_i - 1)$), and on T_{z_i}, $t_i \in [0, 1]$. It follows that on $T_{z_1} \cap \cdots \cap T_{z_n} \cap \sigma(\mathscr{Z})$, z_n belongs to $\mathbb{R}_- \cap (1 - \zeta^b[0, 1])$, which is empty. □

We may now compute the regulator on the cycle of Theorem 3.3, independently of the choice of the $\mathscr{W}_i$.

Theorem 3.8. $$c_{\mathcal{D},\mathbb{Q}}(\sigma(\Xi)) = \mathrm{Li}_n(\zeta^b) \in \mathbb{C}/\mathbb{Q}(n).$$

Proof. By Lemmas 3.5 and 3.7, we obtain

$$c_{\mathcal{D},\mathbb{Q}}(\sigma(\widetilde{\mathscr{Z}})) = \int_{\sigma(\mathscr{Z})_{\mathbb{C}}^{\mathrm{an}} \cap T_{z_1} \cap \cdots \cap T_{z_{n-1}}} \log(z_n) \frac{dz_{n+1}}{z_{n+1}} \wedge \cdots \wedge \frac{dz_{2n-1}}{z_{2n-1}} + \sum_{i=1}^{n-1} \int_{\sigma(\mathscr{W}_i)_{\mathbb{C}}^{\mathrm{an}} \cap T_{z_1} \cap \cdots \cap T_{z_{n-1}}} \log(z_n) \frac{dz_{n+1}}{z_{n+1}} \wedge \cdots \wedge \frac{dz_{2n-1}}{z_{2n-1}},$$

in which (by the proof of Lemma 3.7) $\sigma(\mathscr{W}_i)_{\mathbb{C}}^{\mathrm{an}} \cap T_{z_1} \cap \cdots \cap T_{z_{n-1}} = \varnothing$ for all i. The remaining (first) term becomes

$$\begin{aligned}
\int_{\underline{z} \in \mathbb{R}_-^{\times(n-1)}} & \log(f_{n-1}(\underline{z})) \frac{dt_1^N}{t_1^N} \wedge \cdots \wedge \frac{dt_{n-1}^N}{t_{n-1}^N} \\
&= (-N)^{n-1} \int_{\underline{t} \in [0,1]^{\times(n-1)}} \log(1 - \zeta^b t_1 \cdots t_{n-1}) \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_{n-1}}{t_{n-1}} \\
&= (-N)^{n-1} \int_0^{\zeta^b} \int_0^{u_{n-1}} \cdots \int_0^{u_2} \log(1 - u_1) \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{n-1}}{u_{n-1}} \\
&= (-1)^n N^{n-1} \mathrm{Li}_n(\zeta^b),
\end{aligned}$$

where $u_{n-1} = \zeta^b t_{n-1},\ u_{n-2} = \zeta^b t_{n-2} t_{n-1},\ \ldots,\ u_1 = \zeta^b t_1 \cdots t_{n-1}$. □

To write the image of our cycles under the Borel regulator, we refine notation by writing σ_a (for $\sigma : \omega \mapsto e^{2\pi \boldsymbol{i} a/N}$), $f_{n-1,b} = 1 - \omega^b t_1 \cdots t_{n-1}$, Ξ_b, $\widetilde{\mathscr{Z}}_b$, $\mathscr{Z}_b$, etc. So Theorem 3.8 reads $c_{\mathcal{D},\mathbb{Q}}(\sigma_a(\Xi_b)) = \mathrm{Li}_n(e^{2\pi \boldsymbol{i} ab/N})$, and one has the following corollary.

Corollary 3.9. *Let $N \geq 3$ and set*

$$A := \left\{ a \in \mathbb{N} \mid (a, N) = 1 \text{ and } 1 \leq a \leq \left\lfloor \tfrac{N}{2} \right\rfloor \right\};$$

then for any $b \in A$,

$$\tilde{c}^{+}_{\mathcal{D},\mathbb{R}}(\Xi_b) = (\pi_n(\mathrm{Li}_n(e^{2\pi \boldsymbol{i} ab/N}))_{a\in A} \in \mathbb{R}(n-1)^{\oplus \frac{1}{2}\phi(N)},$$

where $\pi_n : \mathbb{C} \to \mathbb{R}(n-1)$ is $\boldsymbol{i}\mathrm{Im}$ for n even, and Re for n odd. If $N = 2$, then $\tilde{c}^{+}_{\mathcal{D},\mathbb{R}} = 0$ for n even and $\tilde{c}^{+}_{\mathcal{D},\mathbb{R}}(\Xi_1) = \zeta(n) \in \mathbb{R}(n-1)$ for n odd.

As an immediate consequence, we get a (rational) basis for the higher Chow cycles on a point over any abelian extension of $\mathbb{Q}$.

Corollary 3.10. *The $\{\Xi_b\}_{b\in A}$ span $CH^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$. Moreover, for any subfield $\mathbb{E} \subset \mathbb{F}$, with $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{E})$, there exists a subset $B \subset A$ (with $|B| = d_n(\mathbb{E})$) such that the $\left\{\sum_{\gamma\in\Gamma} {}^{\gamma}\,\Xi_b\right\}_{b\in B}$ span $CH^n(\mathbb{E}, 2n-1)_{\mathbb{Q}}$.*

Proof. In view of Lemma 2.8, for the first statement we need only check the linear independence of the vectors $\underline{v}^{(b)}$ in Corollary 3.9. Let χ be one of the $\frac{1}{2}\phi(N)$ Dirichlet characters modulo N with $\chi(-1) = (-1)^{n-1}$; and let $\rho_\alpha : \mathbb{C}^{|A|} \to \mathbb{C}^{|A|}$ be the permutation operator defined by $\mu(\underline{v})_j = v_{\alpha\cdot j}$, where $\alpha \in (\mathbb{Z}/N\mathbb{Z})^*$ is a generator. Then the linear combinations

$$\underline{v}^{\chi} := \sum_{b\in A} \chi(b)\underline{v}^{(b)} = \left(\frac{1}{2} \sum_{b=1}^{N} \chi(b)\pi_n(\mathrm{Li}_n(e^{2\pi \boldsymbol{i} ab/N})) \right)_{a\in A}$$

are independent (over $\mathbb{C}$) provided they are nonzero, since their eigenvalues $\overline{\chi(\alpha)}$ under ρ_α are distinct. By the computation in [Zagier 1991, pp. 420–422], if χ is induced from a primitive character χ_0 modulo $N_0 = N/M$, then (with μ being the Möbius function and $\tau(\,\cdot\,)$ the Gauss sum)

$$v_1^{\chi} = \frac{1}{2M^{n-1}} \left\{ \sum_{d|M} \mu(d)\chi_0(d)d^{n-1} \right\} \tau(\chi_0) L(\overline{\chi_0}, n),$$

the last two factors of which are nonzero by primitivity of χ_0; the bracketed term is $\prod_{p>1 \text{ prime},\, p|M} (1 - \chi_0(p)p^{n-1})$, hence also nonzero.

The second statement follows at once, since the composition of $\sum_{\gamma\in\Gamma}$ with $CH^n(\mathbb{E}, 2n-1)_{\mathbb{Q}} \hookrightarrow CH^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$ is a multiple of the identity. $\square$

4. Explicit representatives

We finally turn to the construction of the cycles described by Theorem 3.3. Here the benefit of using t_i^N (at least, if one is happy to work rationally) comes to the fore: it allows us to obtain uniform formulas for all N, and to use as few terms as possible. In fact, it turns out that *for all n* it is possible to take $\mathscr{W}_3 = \cdots =$

$\mathscr{W}_{n-1} = 0$. (While it is easy to argue abstractly that $\mathscr{W}_{n-1}$ can always be taken to be zero, this stronger statement surprised us.) For brevity, we shall use the notation $(f_1(\underline{t}, u, v), \dots, f_m(\underline{t}, u, v))$ for

$$\{(f_1(\underline{t}, u, v), \dots, f_m(\underline{t}, u, v)) \mid t_i, u, v \in \mathbb{P}^1\} \cap \square^m;$$

all precycles are defined over $\mathbb{F} = \mathbb{Q}(\omega)$, and we write $\xi := \omega^b$.

4A. K_3 *case* ($n = 2$). Let $\mathscr{Z} = (t/(t-1), 1 - \xi t, t^N)$, as dictated by Theorem 3.3; then all $\partial_i^\varepsilon \mathscr{Z} = 0$. In particular,

$$\partial_1^0 \mathscr{Z} = (1 - \xi t, t^N)|_{t/(t-1)=0} = (1, 0) = 0$$

and

$$\partial_2^0 \mathscr{Z} = \left(\frac{\xi^{-1}}{\xi^{-1} - 1}, \xi^{-N} \right) = \left(\frac{1}{1 - \xi}, 1 \right) = 0.$$

So we may take $\mathscr{W} = 0$ and $\widetilde{\mathscr{Z}} = \mathscr{Z}$.

In contrast, if we took $\mathscr{Z} = (t/(t-1), 1 - \xi t, t)$, then $\partial_2^0 \mathscr{Z} = (1/(1-\xi), \xi^{-1})$ and a nonzero $\mathscr{W}$-term is required.

4B. K_5 *case* ($n = 3$). Of course $\mathscr{Z} = (t_1/(t_1-1), t_2/(t_2-1), 1 - \xi t_1 t_2, t_1^N, t_2^N)$. Taking

$$\mathscr{W}_1 = \frac{1}{2} \left(\frac{t_1}{t_1 - 1}, \frac{1}{1 - \xi t_1}, \frac{(u - t_1^N)(u - t_1^{-N})}{(u-1)^2}, t_1^N u, \frac{u}{t_1^N} \right),$$

we note that $z_2 = 1/(1 - \xi t_1)$ implies $t_2 = (1 - \xi t_1)^{-1}/((1 - \xi t_1)^{-1} - 1) = 1/\xi t_1$, which in turn implies $f_2(t_1, t_2) = 0$. Now we have

$$\partial \mathscr{Z} = \partial_3^0 \mathscr{Z} = \left(\frac{t_1}{t_1 - 1}, \frac{t_2}{t_2 - 1}, t_1^N, t_2^N \right) \Bigg|_{1 - \xi t_1 t_2 = 0} = \left(\frac{t_1}{t_1 - 1}, \frac{1}{1 - \xi t_1}, t_1^N, \frac{1}{t_1^N} \right)$$

and

$$\partial \mathscr{W}_1 = -\partial_3^\infty \mathscr{W}_1 = -2 \cdot \frac{1}{2} \left(\frac{t_1}{t_1 - 1}, \frac{1}{1 - \xi t_1}, t_1^N, \frac{1}{t_1^N} \right) = -\partial \mathscr{Z}.$$

Therefore $\widetilde{\mathscr{Z}} = \mathscr{Z} + \mathscr{W}_1$ is closed.

Remark 4.1. See [Petras 2008, §3.1] for a detailed discussion of the properties of these cycles, especially the (integral!) distribution relations of [loc. cit., Proposition 3.1.26].

In particular, we can specialize to $N = 2$ to obtain

$$2\widetilde{\mathscr{Z}} = 2 \left(\frac{t_1}{t_1 - 1}, \frac{t_2}{t_2 - 1}, 1 + t_1 t_2, t_1^2, t_2^2 \right) + \left(\frac{t_1}{t_1 - 1}, \frac{1}{1 + t_1}, \frac{(u - t_1^2)(u - t_1^{-2})}{(u-1)^2}, t_1^2 u, \frac{u}{t_1^2} \right)$$

in $Z^3_{\mathbb{R}}(\mathbb{Q}, 5)$, spanning $CH^3(\mathbb{Q}, 5)_{\mathbb{Q}} \cong K_5(\mathbb{Q})_{\mathbb{Q}}$, with

$$c_{\mathcal{D},\mathbb{Q}}(2\widetilde{\mathscr{Z}}) = -8 \operatorname{Li}_3(-1) = 6\zeta(3) \in \mathbb{C}/\mathbb{Q}(3).$$

4C. K_7 case ($n=4$). Set

$$\mathscr{Z}=\left(\frac{t_1}{t_1-1},\frac{t_2}{t_2-1},\frac{t_3}{t_3-1},1-\xi t_1t_2t_3,t_1^N,t_2^N,t_3^N\right),$$

$$\mathscr{W}_1=\frac{1}{2}(\mathscr{W}_1^{(1)}+\mathscr{W}_1^{(2)}),$$

$$\mathscr{W}_1^{(1)}=\left(\frac{t_1}{t_1-1},\frac{t_2}{t_2-1},\frac{1}{1-\xi t_1t_2},\frac{(u-t_1^N)(u-t_2^N)}{(u-1)(u-t_1^Nt_2^N)},\frac{u}{t_1^N},\frac{u}{t_2^N},\frac{1}{u}\right),$$

$$\mathscr{W}_1^{(2)}=\left(\frac{t_1}{t_1-1},\frac{t_2}{t_2-1},\frac{1}{1-\xi t_1t_2},\frac{(u-t_1^N)(u-t_2^N)}{(u-1)(u-t_1^Nt_2^N)},\frac{t_1^N}{u},\frac{t_2^N}{u},\frac{u}{t_1^Nt_2^N}\right),$$

$$\mathscr{W}_2=-\frac{1}{2}\left(\frac{t_1}{t_1-1},\frac{1}{1-\xi t_1},\frac{(v-t_1^Nu)(v-ut_1^{-N})}{(v-u^2)(v-1)},\right.$$
$$\left.\frac{(u-t_1^N)(u-vt_1^{-N})}{(u-v)^2},\frac{vt_1^N}{u},\frac{v}{t_1^Nu},\frac{u}{v}\right).$$

Direct computation shows

$$\partial\mathscr{Z}=-\partial_4^0\mathscr{Z}=-\partial_4^\infty\mathscr{W}_1^{(1)}=-\partial_4^\infty\mathscr{W}_1^{(2)},$$
$$\partial\mathscr{W}_1=-\tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(1)}+\tfrac{1}{2}\partial_4^\infty\mathscr{W}_1^{(1)}-\tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(2)}+\tfrac{1}{2}\partial_4^\infty\mathscr{W}_1^{(2)},$$
$$\partial\mathscr{W}_2=-\partial_3^\infty\mathscr{W}_2=\tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(1)}+\tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(2)},$$

which sum to zero.

Alternately, we can take

$$\mathscr{W}_1=\left(\frac{t_1}{t_1-1},\frac{t_2}{t_2-1},\frac{1}{1-\xi t_1t_2},\frac{(u-t_1^N)(u-t_2^N)}{(u-1)(u-t_1^Nt_2^N)},\frac{t_1^N}{u},\frac{t_2^N}{u},\frac{u}{t_1^Nt_2^N}\right),$$

$$\mathscr{W}_2=\left(\frac{t_1}{t_1-1},\frac{1}{1-\xi t_1},\frac{(u-vt_1^N)(u-vt_1^{-N})}{(u-v)^2},\frac{vt_1^N}{u},\frac{v}{t_1^Nu},\frac{u}{v},v-1\right).$$

Writing

$$\mathscr{V}_1=\left(\frac{t_1}{t_1-1},\frac{t_2}{t_2-1},\frac{1}{1-\xi t_1t_2},t_1^N,t_2^N,\frac{1}{t_1^Nt_2^N}\right),$$

$$\mathscr{V}_2=\left(\frac{t_1}{t_1-1},\frac{1}{1-\xi t_1},\frac{(u-t_1^N)(u-t_1^{-N})}{(u-1)^2},\frac{t_1^N}{u},\frac{1}{t_1^Nu},u\right),$$

one has $\partial\mathscr{Z}=-\mathscr{V}_1$, $\partial\mathscr{W}_1=-\mathscr{V}_2+\mathscr{V}_1$, $\partial\mathscr{W}_2=\mathscr{V}_2$; so again $\widetilde{\mathscr{Z}}$ is a closed cycle.

We present the general n construction next, but include the $n=5$ case as an appendix (as the authors only saw the pattern after working out this case).

4D. ***General n construction* ($n \geq 4$).** To state the final result, we define

$$\mathscr{Z} := \left(\frac{t_1}{t_1-1}, \dots, \frac{t_{n-1}}{t_{n-1}-1}, 1-\xi t_1 \cdots t_{n-1}, t_1^N, \dots, t_{n-1}^N\right),$$

$$\mathscr{W}_1 := \frac{1}{n-3}\widetilde{\mathscr{W}_1}$$

$$:= \frac{(-1)^{n-1}}{n-3}\Biggl(\frac{t_1}{t_1-1}, \dots, \frac{t_{n-2}}{t_{n-2}-1}, \frac{1}{1-\xi t_1 \cdots t_{n-2}},$$

$$\frac{(u-t_1^N)\cdots(u-t_{n-2}^N)}{(u-t_1^N\cdots t_{n-2}^N)(u-1)^{n-3}}, \frac{t_1^N}{u}, \dots, \frac{t_{n-2}^N}{u}, \frac{u}{t_1^N \cdots t_{n-2}^N}\Biggr),$$

$$\mathscr{W}_2 := \frac{1}{n-3}\sum_{i=1}^{n-1}(-1)^{i-1}\mathscr{W}_2^{(i)},$$

where for $1 \leq i \leq n-2$,

$$\mathscr{W}_2^{(i)} := \Biggl(\frac{t_1}{t_1-1}, \dots, \frac{t_{n-3}}{t_{n-3}-1}, \frac{1}{1-\xi t_1 \cdots t_{n-3}}, \frac{(u-t_1^N v)\cdots(u-t_{n-3}^N v)}{(u-t_1^N\cdots t_{n-3}^N v)(u-v)^{n-4}},$$

$$\frac{vt_1^N}{u}, \dots, \frac{v}{u}, \dots, \frac{vt_{n-3}^N}{u}, \frac{u}{vt_1^N \cdots t_{n-3}^N}, v-1\Biggr),$$

(with v/u occurring in the $(n+i-1)$-st entry[4]) and

$$\mathscr{W}_2^{(n-1)} :=$$

$$\Biggl(\frac{t_1}{t_1-1}, \dots, \frac{t_{n-3}}{t_{n-3}-1}, \frac{1}{1-\xi t_1 \cdots t_{n-3}}, \frac{(u-t_1^N v)\cdots(u-t_{n-3}^N v)}{(u-t_1^{-N}\cdots t_{n-3}^{-N} v)^{-1}(u-v)^{n-2}},$$

$$\frac{vt_1^N}{u}, \dots, \frac{vt_{n-3}^N}{u}, \frac{v}{ut_1^N \cdots t_{n-3}^N}, \frac{u}{v}, v-1\Biggr).$$

Theorem 4.2. *$\widetilde{\mathscr{Z}} = \mathscr{Z} + \mathscr{W}_1 + \mathscr{W}_2$ yields a closed cycle, with the properties described in Theorem 3.3.* (*In particular, this recovers the second K_7 construction and the K_9 construction above, for $n = 4$ and* 5.)

Proof. Writing

$$\mathscr{Y}_0 := \partial_n^0 \mathscr{Z} = \left(\frac{t_1}{t_1-1}, \dots, \frac{t_{n-2}}{t_{n-2}-1}, \frac{1}{1-\xi t_1 \cdots t_{n-2}}, t_1^N, \dots, t_{n-2}^N, \frac{1}{t_1^N \cdots t_{n-2}^N}\right),$$

$\mathscr{Y}_i := \partial_{2n-1}^0 \mathscr{W}_2^{(i)}$ ($i = 1, \dots, n-1$), and $\mathscr{X}_{i,j} := \partial_j^\infty \mathscr{W}_2^{(i)}$ ($j = 1, \dots, n-2$), one computes that $\partial\mathscr{Z} = (-1)^{n-1}\mathscr{Y}_0$,

[4]That is, either before ($i = 1$), after ($i = n-2$), or in the middle of the sequence $vt_1^N/u, vt_2^N/u, \dots, vt_{n-3}^N/u$.

$$\partial\widetilde{\mathscr{W}}_1=(-1)^n\partial_n^{\infty}\widetilde{\mathscr{W}}_1+\sum_{i=1}^{n-1}(-1)^i\partial_i^{\infty}\widetilde{\mathscr{W}}_1=(-1)^n(n-3)\mathscr{Y}_0+\sum_{i=1}^{n-1}(-1)^i\mathscr{Y}_i,$$

and $\partial\mathscr{W}_2^{(i)}=\mathscr{Y}_i+\sum_{j=1}^{n-2}(-1)^j\mathscr{X}_{i,j}$. We have, therefore,

$$\partial\widetilde{\mathscr{Z}}=\frac{1}{n-3}\sum_{i=1}^{n-1}\sum_{j=1}^{n-2}(-1)^{i+j-1}\mathscr{X}_{i,j},\tag{4.3}$$

and for each $i>j$, the reader may verify that $\mathscr{X}_{i,j}=\mathscr{X}_{j,i-1}$, so that the terms on the right-hand side of (4.3) cancel in pairs. □

4E. ***Expected implications for torsion.*** One of the anticipated applications of the explicit AJ maps of [Kerr et al. 2006; Kerr and Lewis 2007] has been the detection of torsion in higher Chow groups. While they provide an explicit map of complexes from $Z^p_{\mathbb{R}}(X,\bullet)$ to the *integral* Deligne cohomology complex, the fact that $Z^p_{\mathbb{R}}(X,\bullet)\subset Z^p(X,\bullet)$ is only a *rational* quasi-isomorphism leaves open the possibility that a given cycle with (nontrivial) torsion KLM-image is bounded by a precycle in the larger complex. So far, therefore, any conclusions we can try to draw about torsion are speculative, as they depend on the (so far) conjectural extension of the KLM map to an *integrally* quasi-isomorphic subcomplex.

Let us describe what the existence of such an extension, together with the cycles just constructed, would yield. Let $f:\mathbb{Z}/N\mathbb{Z}\to\mathbb{Z}$ be a function which is zero off $(\mathbb{Z}/N\mathbb{Z})^*$, with $f(-b)=(-1)^nf(b)$, and write

$$\varepsilon_n:=\begin{cases}1, & n=2,\\ 2, & n=3,\\ n-3, & n\geq 4.\end{cases}$$

Then (fixing $\sigma(\omega)=\zeta_N=e^{2\pi\boldsymbol{i}/N}$) the cycle

$$Z_f^n(N):=\varepsilon_n\sum_{b=0}^{N-1}f(b)\sigma(\widetilde{\mathscr{Z}}_b)\in Z^n_{\mathbb{R}}(\mathbb{Q}(\zeta_N),2n-1)$$

is integral. Working up to sign, we compute (in $\mathbb{C}/\mathbb{Z}$) by Theorem 3.8

$$\begin{aligned}\tau_f^n(N)&:=\frac{\pm1}{(2\pi\boldsymbol{i})^n}c_{\mathcal{D}}(Z_f^n(N))=\frac{\pm\varepsilon_nN^{n-1}}{(2\pi\boldsymbol{i})^n}\sum_{b=0}^{N-1}f(b)\sum_{k\geq1}\frac{\zeta_N^{kb}}{k^n}\\&=\frac{\pm\varepsilon_nN^{n-1}}{2(2\pi\boldsymbol{i})^n}\sum_{b=0}^{N-1}f(b)\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{\zeta_N^{kb}}{k^n}=\frac{\pm\varepsilon_nN^{n-1}}{2\cdot n!}\sum_{b=0}^{N-1}f(b)B_n(\frac{b}{N}),\end{aligned}$$

which is evidently a rational number.[5] This (nonconjecturally) establishes that $Z_f^n(N)$ is torsion. Under our working (conjectural!) hypothesis, if $\tau_f^n(N) = \pm A_f^n(N)/C_f^n(N)$ in lowest form, we may additionally conclude that the order of $Z_f^n(N)$ is a multiple of $C_f^n(N)$.

For example, taking $N = 5$, $n = 2$, and $f(1) = f(4) = 1$, $f(2) = f(3) = 0$, we obtain $Z_f^2(5) \in Z_{\mathbb{R}}^2(\mathbb{Q}(\sqrt{5}), 3)$ with $\tau_f^2(5) = \frac{\pm 1}{120}$. This checks out with what is known (cf. Proposition 6.9 and Remark 6.10 of [Petras 2009]), and would make $Z_f^2(5)$ a generator of $CH^2(\mathbb{Q}(\sqrt{5}), 3)$.

For $N = 2$, $f(1) = 1$, and $n = 2m$ (i.e., $CH^{2m}(\mathbb{Q}, 4m-1)$), the above computation simplifies to

$$\begin{aligned} |\tau_f^{2m}(2)| &= \frac{\pm \varepsilon_{2m} 2^{2m-2}}{(2m)!} B_{2m}\left(\frac{1}{2}\right) \\ &= \frac{\pm(2m-3)(2^{2m-1}-1)}{2(2m)!} B_{2m}, \end{aligned}$$

which yields $\frac{1}{24}, \frac{7}{1440}, \frac{31}{20160}, \frac{635}{483840}$ for $m = 1, 2, 3, 4$, respectively. It is known that $CH^2(\mathbb{Q}, 3) \cong \mathbb{Z}/24\mathbb{Z}$ [Petras 2009], but the other orders seem unexpectedly large and should warrant further investigation.

Appendix: K_9 case ($n = 5$)

Begin by writing

$$\mathscr{Z} = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{t_3}{t_3-1}, \frac{t_4}{t_4-1}, 1-\xi t_1 t_2 t_3 t_4, t_1^N, t_2^N, t_3^N, t_4^N\right),$$

$$\begin{aligned} \mathscr{W}_1 = \frac{1}{2}\Bigg(&\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{t_3}{t_3-1}, \frac{1}{1-\xi t_1 t_2 t_3}, \\ &\qquad \frac{(u-t_1^l)(u-t_2^l)(u-t_3^l)}{(u-1)^2(u-t_1^l t_2^l t_3^l)}, \frac{t_1^N}{u}, \frac{t_2^N}{u}, \frac{t_3^N}{u}, \frac{u}{t_1^N t_2^N t_3^N}\Bigg), \end{aligned}$$

$$\mathscr{W}_2^{(1)} = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N v)(u-t_2^N v)}{(u-t_1^N t_2^N v)(u-v)}, \frac{v}{u}, \frac{t_1^N v}{u}, \frac{t_2^N v}{u}, \frac{u}{v t_1^N t_2^N}, v-1\right),$$

$$\mathscr{W}_2^{(2)} = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N v)(u-t_2^N v)}{(u-t_1^N t_2^N v)(u-v)}, \frac{v t_1^N}{u}, \frac{v}{u}, \frac{t_2^N v}{u}, \frac{u}{v t_1^N t_2^N}, v-1\right),$$

$$\mathscr{W}_2^{(3)} = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N v)(u-t_2^N v)}{(u-t_1^N t_2^N v)(u-v)}, \frac{v t_1^N}{u}, \frac{v t_2^N}{u}, \frac{v}{u}, \frac{u}{v t_1^N t_2^N}, v-1\right),$$

[5] $B_n(x) = \sum_{j=0}^n \binom{n}{j} B_j x^{n-j}$ is the n-th Bernoulli polynomial (and $\{B_j\}$ the Bernoulli numbers).

$$\mathscr{W}_2^{(4)} = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \right.$$
$$\left. \frac{(u-t_1^N v)(u-t_2^N v)}{(u-vt_1^{-N}t_2^{-N})^{-1}(u-v)^3}, \frac{vt_1^N}{u}, \frac{vt_2^N}{u}, \frac{v}{ut_1^N t_2^N}, \frac{u}{v}, v-1 \right),$$

$$\mathscr{W}_2 = \frac{1}{2}(\mathscr{W}_2^{(1)} - \mathscr{W}_2^{(2)} + \mathscr{W}_2^{(3)} - \mathscr{W}_2^{(4)}).$$

To compute the boundaries, introduce

$$\mathscr{U}_1 = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{t_3}{t_3-1}, \frac{1}{1-\xi t_1 t_2 t_3}, t_1^N, t_2^N, t_3^N, \frac{1}{t_1^N t_2^N t_3^N} \right),$$

$$\mathscr{U}_2 = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N)(u-t_2^N)}{(u-t_1^N t_2^N)(u-1)}, \frac{1}{u}, \frac{t_1^N}{u}, \frac{t_2^N}{u}, \frac{u}{t_1^N t_2^N} \right),$$

$$\mathscr{U}_3 = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N)(u-t_2^N)}{(u-t_1^N t_2^N)(u-1)}, \frac{t_1^N}{u}, \frac{1}{u}, \frac{t_3^N}{u}, \frac{u}{t_1^N t_2^N} \right),$$

$$\mathscr{U}_4 = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N)(u-t_2^N)}{(u-t_1^N t_2^N)(u-1)}, \frac{t_1^N}{u}, \frac{t_2^N}{u}, \frac{1}{u}, \frac{u}{t_1^N t_2^N} \right),$$

$$\mathscr{U}_5 = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N)(u-t_2^N)(u-t_1^{-N}t_2^{-N})}{(u-1)^3}, \frac{t_1^N}{u}, \frac{t_2^N}{u}, \frac{1}{ut_1^N t_2^N}, u \right),$$

and

$$\mathscr{V}_1 = \left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, \frac{(u-t_1^N v)(u-t_1^{-N} v)}{(u-v)^2}, \frac{v}{u}, \frac{t_1^N v}{u}, \frac{v}{ut_1^N}, \frac{u}{v}, v-1 \right),$$

$$\mathscr{V}_2 = \left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, \frac{(u-t_1^N v)(u-t_1^{-N} v)}{(u-v)^2}, \frac{vt_1^N}{u}, \frac{v}{u}, \frac{v}{t_1^N u}, \frac{u}{v}, v-1 \right),$$

$$\mathscr{V}_3 = \left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, \frac{(u-t_1^N v)(u-t_1^{-N} v)}{(u-v)^2}, \frac{vt_1^N}{u}, \frac{v}{ut_1^N}, \frac{v}{u}, \frac{u}{v}, v-1 \right).$$

Then $\partial \mathscr{Z} = \mathscr{U}_1$, $\partial \mathscr{W}_1 = -\mathscr{U}_1 + \frac{1}{2}(-\mathscr{U}_2 + \mathscr{U}_3 - \mathscr{U}_4 + \mathscr{U}_5)$, $\partial \mathscr{W}_2^{(1)} = -\mathscr{V}_1 + \mathscr{U}_2$, $\partial \mathscr{W}_2^{(2)} = -\mathscr{V}_2 + \mathscr{U}_3$, $\partial \mathscr{W}_2^{(3)} = -\mathscr{V}_3 + \mathscr{U}_4$, and $\partial \mathscr{W}_2^{(4)} = \mathscr{U}_5 - \mathscr{V}_1 + \mathscr{V}_2 - \mathscr{V}_3$; and so $\widetilde{\mathscr{Z}}$ is closed.

As for $n = 3$, we obtain a generator for $CH^5(\mathbb{Q}, 9)_{\mathbb{Q}} \cong K_9(\mathbb{Q})_{\mathbb{Q}}$ by setting $N = 2$ and $\xi = -1$; the integral cycle $2\widetilde{\mathscr{Z}}$ has $c_{\mathcal{D},\mathbb{Q}}(2\widetilde{\mathscr{Z}}) = 15\zeta(5)$.

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Algebraic K-theory and a semifinite Fuglede–Kadison determinant

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In this paper we apply algebraic K-theory techniques to construct a Fuglede–Kadison type determinant for a semifinite von Neumann algebra equipped with a fixed trace. Our construction is based on the approach to determinants for Banach algebras developed by Skandalis and de la Harpe. This approach can be extended to the semifinite case since the first topological K-group of the trace ideal in a semifinite von Neumann algebra is trivial. Along the way we also improve the methods of Skandalis and de la Harpe by considering relative K-groups with respect to an ideal instead of the usual absolute K-groups. Our construction recovers the determinant homomorphism introduced by Brown, but all the relevant algebraic properties are automatic due to the algebraic K-theory framework.

1. Introduction

One first encounters the relationship between algebraic K-theory and determinants in the isomorphism between the first algebraic K-group of the complex numbers and the complex multiplicative group. This isomorphism is implemented by the determinant of an invertible matrix. In the present paper we will expand on this relationship in the context of Banach algebras and, in particular, we will see how to recover the Fuglede–Kadison determinant for semifinite von Neumann algebras as introduced by Brown [Brown 1986; Fuglede and Kadison 1952]. Brown based his construction on ideas of Grothendieck [1956] and Fack [1982; 1983], who defined a determinant function as an analogue of the product of the eigenvalues up to a given cutoff.

The main advantage of applying an algebraic K-theory approach to determinants is that all the algebraic properties of determinants follow as a direct consequence of the definitions. Moreover, when determinants are interpreted as invariants of algebraic K-theory, they can be used to detect nontrivial elements in these generally rather complicated abelian groups. On the other hand, basing the construction of determinants purely on functional analytic methods requires a substantial amount

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of work for proving the main algebraic properties, and the more conceptual framework provided by algebraic K-theory is entirely lost.

The key property that we investigate in this text is the relationship between the operator trace, the logarithm and the determinant as expressed by the identity

$$\log(\det(g)) = \mathrm{Tr}(\log(g)).$$

In order to expand on this basic relationship in a K-theoretic context one considers a unital Banach algebra A together with the homomorphism

$$\mathrm{GL}(A) \to \mathrm{GL}^{\mathrm{top}}(A),$$

where $\mathrm{GL}(A)$ denotes the general linear group (over A) equipped with the discrete topology, and $\mathrm{GL}^{\mathrm{top}}(A)$ is the same algebraic group but with the topology coming from the unital Banach algebra A. Passing to classifying spaces and applying Quillen's plus construction [1973], one obtains a continuous map

$$\mathrm{BGL}(A)^+ \to \mathrm{BGL}^{\mathrm{top}}(A)$$

(which is unique up to homotopy). By taking homotopy fibres and homotopy groups this gives rise to a long exact sequence of abelian groups,

$$K_{*+1}^{\mathrm{top}}(A) \xrightarrow{\partial} K_*^{\mathrm{rel}}(A) \xrightarrow{\partial} K_*^{\mathrm{alg}}(A) \longrightarrow K_*^{\mathrm{top}}(A),$$

which is related to the SBI-sequence in continuous cyclic homology by means of Chern characters, resulting in the commutative diagram

$$\begin{tikzcd}
K_{*+1}^{\mathrm{top}}(A) \arrow[r, "\partial"] \arrow[d, "\mathrm{ch}^{\mathrm{top}}"'] & K_*^{\mathrm{rel}}(A) \arrow[r] \arrow[d, "\mathrm{ch}^{\mathrm{rel}}"'] & K_*^{\mathrm{alg}}(A) \arrow[r] \arrow[d, "\mathrm{ch}^{\mathrm{alg}}"'] & K_*^{\mathrm{top}}(A) \arrow[d, "\mathrm{ch}^{\mathrm{top}}"'] \\
\mathrm{HP}_{*+1}(A) \arrow[r, "S"] & \mathrm{HC}_{*-1}(A) \arrow[r, "B"] & \mathrm{HN}_*(A) \arrow[r, "I"] & \mathrm{HP}_*(A)
\end{tikzcd} \tag{1.1}$$

of abelian groups; see [Karoubi 1987; Connes and Karoubi 1988].

In this paper we focus on the low degree (and more explicit) version of this commutative diagram. More precisely, supposing that the unital Banach algebra A comes equipped with a tracial functional $\tau : A \to \mathbb{C}$, one obtains an invariant of the continuous cyclic homology group $\mathrm{HC}_0(A)$, and hence by precomposition with the relative Chern character we obtain a homomorphism

$$\tau \circ \mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(A) \to \mathbb{C}.$$

Supposing furthermore that $K_1^{\mathrm{top}}(A) = \{0\}$, it follows from the commutative diagram in (1.1) combined with Bott-periodicity in topological K-theory that the character $\tau \circ \mathrm{ch}^{\mathrm{rel}}$ induces a homomorphism

$$\det_\tau : K_1^{\mathrm{alg}}(A) \to \mathbb{C}/(2\pi i \cdot \mathrm{Im}(\underline{\tau})),$$

where $\underline{\tau} : K_0^{\mathrm{top}}(A) \to \mathbb{C}$ is the character on even topological K-theory induced by our tracial functional. In this way we recover the determinant defined by Skandalis and de la Harpe [de la Harpe and Skandalis 1984; de la Harpe 2013].

We extend this framework for defining determinants by incorporating that the tracial functional τ might only be defined on an ideal J sitting inside the unital Banach algebra A (where J is not required to be closed in the norm-topology of A). In this context, we assume that $\tau : J \to \mathbb{C}$ is a hypertrace in the sense that $\tau(ja) = \tau(aj)$ for all $a \in A$, $j \in J$. The correct K-groups to consider are then relative versions of relative K-theory and algebraic K-theory, and similarly one considers relative versions of the cyclic homology groups appearing in the SBI-sequence (we do not use relative topological K-theory because of excision). The idea of applying relative K-groups in relation to determinant-type invariants of algebraic K-theory was (among other things) developed in the Ph.D. thesis of the second author [Kaad 2009].

In the setting of a semifinite von Neumann algebra N equipped with a fixed normal, faithful and semifinite trace $\tau : N_+ \to [0, \infty]$, it is relevant to look at the trace ideal

$$\mathscr{L}_\tau^1(N) := \{x \in N : \tau(|x|) < \infty\}$$

sitting inside the von Neumann algebra N. Using the facts that $K_1^{\mathrm{top}}(\mathscr{L}_\tau^1(N)) = \{0\}$ and $\mathrm{Im}(\underline{\tau} : K_0^{\mathrm{top}}(\mathscr{L}_\tau^1(N)) \to \mathbb{C}) \subseteq \mathbb{R}$, we obtain an algebraic K-theory invariant[1]

$$\det_\tau : K_1^{\mathrm{alg}}(\mathscr{L}_\tau^1(N), N) \to \mathbb{C}/i\mathbb{R},$$

which recovers the Fuglede–Kadison determinant in the context of semifinite von Neumann algebras; see [Brown 1986; Fuglede and Kadison 1952]. We emphasize one more time that all the relevant algebraic properties of this determinant follow immediately from its construction. Moreover, we show that $\det_\tau$ is given by the explicit formula

$$\det_\tau(g) = \tau(\log(|g|)) + i\mathbb{R} \qquad (g \in \mathrm{GL}_n(N),\ g - \mathbb{1}_n \in M_n(\mathscr{L}_\tau^1(N)). \tag{1.2}$$

Here, τ is extended to $M_n(N)$ in the obvious way by taking the sum over the diagonal.

Recently, the Fuglede–Kadison determinant was generalized in another direction by Dykema, Sukochev and Zanin to operator bimodules over II_1-factors [Dykema et al. 2017]. They define this determinant using functional analytic methods via an

[1]In the main text, we denote this map by $\widetilde{\det}_\tau$, and use the notation $\det_\tau$ for the composition with the isomorphism $\mathbb{C}/i\mathbb{R} \cong (0, \infty)$ given by $z + i\mathbb{R} \mapsto e^{(z+\bar{z})/2}$.

expression analogous to (1.2). It then requires an elaborate argument to prove that this determinant is multiplicative [Dykema et al. 2017, Theorem 1.3].

The present paper is organized as follows. In Section 2 we introduce the relevant K-groups and in Section 3 we develop the low degree version of the long exact sequence which compares relative algebraic K-theory to topological K-theory. In Section 4 we introduce the low degree version of the relative Chern character in the presence of an ideal $J \subseteq A$. In Section 5 we present our relative approach to the construction of Skandalis–de la Harpe determinants. In Section 6 we show that the first topological K-group of the trace ideal in a semifinite von Neumann algebra is trivial, and in Section 7 we apply this fact to construct the semifinite Fuglede–Kadison determinant.

2. *K*-theory for relative pairs of Banach algebras

2.1. Definition. Let $(A, \|\cdot_A\|)$ be a unital Banach algebra and $J \subset A$ be a (not necessarily closed) ideal. We call (J, A) a *relative pair of Banach algebras* when the following hold:

(1) J is a Banach algebra in its own right. Thus, J is endowed with a norm $\|\cdot\|_J : J \to [0, \infty)$ such that $(J, \|\cdot\|_J)$ is a Banach algebra.

(2) For all $a, b \in A$ and $j \in J$ we have

$$\|ajb\|_J \leq \|a\|_A \, \|j\|_J \, \|b\|_A \quad \text{and} \quad \|j\|_A \leq \|j\|_J \, .$$

2.2. For a relative pair of Banach algebras (J, A) we obtain for all $n \in \mathbb{N}$ a relative pair of Banach algebras $(M_n(J), M_n(A))$, where the $n \times n$ matrices in $M_n(J)$ are equipped with the norm $\|j\|_{M_n(J)} := \sum_{k,l=1}^n \|j_{kl}\|_J$, and similarly for $M_n(A)$.

2.3. The rest of this section is a reminder on various K-groups for relative pairs of Banach algebras. A standard reference for topological K-theory is [Blackadar 1998]. Very good treatments of algebraic K-theory can be found in [Rosenberg 1994; Weibel 2013]. The probably less common relative K-theory of Banach algebras has been introduced in [Karoubi 1987; Connes and Karoubi 1988].

2.4. Definition. Let A be a Banach algebra. If A has a unit, we denote the group of invertible elements in $M_n(A)$ by $\mathrm{GL}_n(A)$. If A has no unit, we define for all $n \in \mathbb{N}$ the group

$$\mathrm{GL}_n(A) := \{g \in \mathrm{GL}_n(A^+) : g - \mathbb{1}_n \in M_n(A)\} \subset \mathrm{GL}_n(A^+),$$

where A^+ is the unitization of A and $\mathbb{1}_n$ the unit of $\mathrm{GL}_n(A^+)$. The group $\mathrm{GL}_n(A)$ becomes a topological group when equipped with the topology coming from the metric $d(g, h) := \|g - h\|_{M_n(A)}$.

2.5. Definition. The *topological K-groups* of the pair (J, A) can be defined to be the usual topological K-groups of J, i.e.,

$$K_i^{\text{top}}(J, A) := K_i^{\text{top}}(J) \qquad (i = 0, 1).$$

This is due to the fact that topological K-theory satisfies excision [Blackadar 1998, Theorem 5.4.2]. For our purposes, it will be useful to know another realization of $K_0^{\text{top}}(J)$, namely $K_2^{\text{top}}(J)$, which may be defined by

$$K_2^{\text{top}}(J) = \lim_{n\to\infty} \pi_1(\mathrm{GL}_n(J), \mathbb{1}_n) = \lim_{n\to\infty} \{[\gamma] \in C^\infty(S^1, \mathrm{GL}_n(J))/\sim \,:\, \gamma(1) = \mathbb{1}_n\},$$

where the equivalence relation $\sim$ is given by smooth basepoint preserving homotopies and the group operation is given by the pointwise product of invertible matrices; see [Blackadar 1998, Section 9.1].

The fact that $K_0^{\text{top}}(J)$ and $K_2^{\text{top}}(J)$ are isomorphic is known as Bott periodicity, [Blackadar 1998, Theorem 9.2.1]. An explicit isomorphism is given by

$$\beta_J : K_0^{\text{top}}(J) \to K_2^{\text{top}}(J), \qquad [e] - [f] \mapsto [\gamma_e \gamma_f^{-1}],$$

where $e, f \in M_n(J^+)$ are idempotents with $e - f \in M_n(J)$. The so-called idempotent loops γ_e are defined by $\gamma_e(z) := ze + \mathbb{1}_n - e$ for $z \in S^1$.

2.6. Definition. The *first algebraic K-theory* of the pair (J, A) is defined by

$$K_1^{\text{alg}}(J, A) := \lim_{n\to\infty} (\mathrm{GL}_n(J)/[\mathrm{GL}_n(J), \mathrm{GL}_n(A)]),$$

where

$$[\mathrm{GL}_n(J), \mathrm{GL}_n(A)] := \langle ghg^{-1}h^{-1} : g \in \mathrm{GL}_n(J),\ h \in \mathrm{GL}_n(A)\rangle$$

is a normal subgroup of $\mathrm{GL}_n(J)$.

2.7. Definition. Let A be a Banach algebra. For all $n \in \mathbb{N}$, we let $R_n(A)$ denote the group of smooth paths $\sigma : [0, 1] \to \mathrm{GL}_n(A)$ such that $\sigma(0) = \mathbb{1}_n$. The group operation is given by pointwise multiplication.

Now, let (J, A) be a relative pair of Banach algebras. From the compatibility of the norms on J and A (see Definition 2.1) it follows that

$$\sigma\tau\sigma^{-1}\tau^{-1} \in R_n(J) \qquad (\sigma \in R_n(J),\ \tau \in R_n(A)).$$

We thus have the normal subgroup

$$[R_n(J), R_n(A)] := \langle \sigma\tau\sigma^{-1}\tau^{-1} \mid \sigma \in R_n(J),\ \tau \in R_n(A)\rangle$$

of $R_n(J)$. On $R_n(J)$ we may consider the equivalence relation $\sim$ of being homotopic with fixed endpoints through a smooth homotopy. Denote the quotient map by $q : R_n(J) \to R_n(J)/\sim$. We define

$$K_1^{\text{rel}}(J, A) := \lim_{n\to\infty} \big((R_n(J)/\sim)/q\big([R_n(J), R_n(A)]\big)\big).$$

3. The comparison sequence

3.1. Definition. We define the following group homomorphisms:

$$\begin{aligned} \partial &: K_2^{\mathrm{top}}(J) \to K_1^{\mathrm{rel}}(J, A), && [\gamma] \mapsto [t \mapsto \gamma(e^{2\pi i t})], \\ \theta &: K_1^{\mathrm{rel}}(J, A) \to K_1^{\mathrm{alg}}(J, A), && [\sigma] \mapsto [\sigma(1)^{-1}], \\ p &: K_1^{\mathrm{alg}}(J, A) \to K_1^{\mathrm{top}}(J), && [g] \mapsto [g]. \end{aligned}$$

3.2. Lemma. *The sequence*

$$K_2^{\mathrm{top}}(J) \xrightarrow{\partial} K_1^{\mathrm{rel}}(J, A) \xrightarrow{\theta} K_1^{\mathrm{alg}}(J, A) \xrightarrow{p} K_1^{\mathrm{top}}(J) \longrightarrow 0$$

is exact.

Proof. The only nontrivial thing to check is exactness at $K_1^{\mathrm{rel}}(J, A)$. It is clear that $\theta \circ \partial = 0$. On the other hand, let $\sigma \in R_n(J)$ and suppose that $[\sigma(1)^{-1}]$ is trivial in $K_1^{\mathrm{alg}}(J, A)$. Then there are $g_i \in \mathrm{GL}_m(J)$ and $h_i \in \mathrm{GL}_m(A)$ such that

$$\sigma(1)^{-1} = \prod_{i=1}^{n} [g_i, h_i].$$

By Whitehead's lemma [Rosenberg 1994, Theorem 2.5.3], we may assume that g_i and h_i lie in the connected component of the identity. Thus, there are smooth paths $\alpha_i \in R_m(J)$ connecting $\mathbb{1}_m$ and g_i, and $\beta_i \in R_m(A)$ connecting $\mathbb{1}_m$ and h_i. Then

$$\tau := \prod_{i=1}^{n} [\alpha_i, \beta_i] \in [R_m(J), R_m(A)]$$

is a path from $\mathbb{1}_m$ to $\sigma(1)^{-1}$. Hence $\gamma := \sigma \cdot \tau^{-1}$ is a smooth loop at $\mathbb{1}_m$ and $\partial([\gamma]) = [\sigma]$ since $[\tau^{-1}]$ is trivial in $K_1^{\mathrm{rel}}(J, A)$. □

4. The relative Chern character

4.1. Let (J, A) be a relative pair of Banach algebras. By $J \otimes_\pi A$ we denote the projective tensor product of J and A. The compatibility of the norms on J and A ensures that the multiplication operator

$$m : J \otimes_\pi A \to J, \qquad j \otimes a \mapsto ja$$

is bounded.

4.2. Definition. We define the Hochschild boundary map

$$b : J \otimes_\pi A \to J, \qquad j \otimes a \mapsto ja - aj$$

and the zeroth relative continuous cyclic homology of the pair (J, A) by

$$\mathrm{HC}_0(J, A) := J/\operatorname{Im}(b).$$

Since $\operatorname{Im}(b) \subset J$ might not be closed we regard $\mathrm{HC}_0(J, A)$ simply as a vector space without further topological structure.

4.3. Definition. Recall from 2.2 that $(M_n(J), M_n(A))$ is a relative pair of Banach algebras for all $n \in \mathbb{N}$. We thus have for each $n \in \mathbb{N}$ the relative continuous cyclic homology groups $\mathrm{HC}_0(M_n(J), M_n(A))$, and we may consider the direct limit of vector spaces

$$\lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A)).$$

This direct limit is linked to $\mathrm{HC}_0(J, A)$ via the linear map

$$\mathrm{TR} : \lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A)) \to \mathrm{HC}_0(J, A),$$

which is induced by the "trace" $\mathrm{TR} : M_n(J) \to J$ mapping a matrix to the sum of its diagonal entries. To verify that TR is indeed well-defined at the level of relative continuous cyclic homology, one may translate the proof of [Loday 1998, Corollary 1.2.3] to our current setting.

4.4. Our next task is to construct the relative Chern character. This will be a group homomorphism

$$\mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(J, A) \to \mathrm{HC}_0(J, A)$$

induced by

$$R_n(J) \ni \sigma \mapsto \mathrm{TR}\left(\int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\, dt\right) \in J.$$

We shall express $\mathrm{ch}^{\mathrm{rel}}$ as the composition of two homomorphisms: a generalized logarithm

$$\log : K_1^{\mathrm{rel}}(J, A) \to \lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A))$$

and the generalized trace as defined in Definition 4.3. We now introduce the generalized logarithm:

4.5. Proposition. *There is a well-defined homomorphism*

$$\log : K_1^{\mathrm{rel}}(J, A) \to \lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A)), \qquad [\sigma] \mapsto \left[\int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\, dt\right].$$

Proof. Suppose first that $\sigma_0, \sigma_1 \in R_n(J)$ are homotopic through a smooth homotopy $H : [0, 1] \times [0, 1] \to \mathrm{GL}_n(J)$ with fixed endpoints. So, $H(t, j) = \sigma_j(t)$ for $j = 0, 1$.

We will show that

$$\int_0^1 \frac{d\sigma_1}{dt}\sigma_1^{-1}\, dt - \int_0^1 \frac{d\sigma_0}{dt}\sigma_0^{-1}\, dt \in \operatorname{Im}(b),$$

where $b : M_n(J) \otimes_\pi M_n(A) \to M_n(J)$ is the Hochschild boundary map associated to the relative pair $(M_n(J), M_n(A))$.

Define

$$L(H) := -\int_0^1 \int_0^1 \frac{\partial H}{\partial t} H^{-1} \otimes \frac{\partial H}{\partial s} H^{-1}\, dt\, ds.$$

We consider $L(H)$ as an element of $M_n(J) \otimes_\pi M_n(A)$ (in fact we even end up in $M_n(J) \otimes_\pi M_n(J)$, which we may then map to $M_n(J) \otimes_\pi M_n(A)$ via the inclusion $M_n(J) \to M_n(A)$). Applying the Hochschild boundary b, we see that

$$b(L(H)) = -\int_0^1 \int_0^1 \Big[\frac{\partial H}{\partial t} H^{-1}, \frac{\partial H}{\partial s} H^{-1}\Big]\, dt\, ds.$$

An easy calculation shows that

$$\begin{aligned}\Big[\frac{\partial H}{\partial t} H^{-1}, \frac{\partial H}{\partial s} H^{-1}\Big] &= -\frac{\partial H}{\partial t}\frac{\partial H^{-1}}{\partial s} + \frac{\partial H}{\partial s}\frac{\partial H^{-1}}{\partial t} \\ &= \frac{\partial}{\partial t}\Big(\frac{\partial H}{\partial s} H^{-1}\Big) - \frac{\partial}{\partial s}\Big(\frac{\partial H}{\partial t} H^{-1}\Big).\end{aligned}$$

By the fundamental theorem of calculus, we conclude

$$\begin{aligned} b(L(H)) &= \int_0^1 \int_0^1 \frac{\partial}{\partial s}\Big(\frac{\partial H}{\partial t} H^{-1}\Big)\, ds\, dt - \int_0^1 \int_0^1 \frac{\partial}{\partial t}\Big(\frac{\partial H}{\partial s} H^{-1}\Big)\, dt\, ds \\ &= \int_0^1 \Big(\frac{\partial H}{\partial t}(t,1) H(t,1)^{-1} - \frac{\partial H}{\partial s}(t,0) H(t,0)^{-1}\Big)\, dt \\ &\qquad - \int_0^1 \Big(\frac{\partial H}{\partial s}(1,s) H(1,s)^{-1} - \frac{\partial H}{\partial s}(0,s) H(0,s)^{-1}\Big)\, ds \\ &= \int_0^1 \frac{d\sigma_1}{dt} \sigma_1^{-1}\, dt - \int_0^1 \frac{d\sigma_0}{dt} \sigma_0^{-1}\, dt. \end{aligned}$$

The second term in the next to last line of our computation vanishes, since our homotopy has fixed endpoints.

We have thus proved that the assignment

$$\log : R_n(J) \to M_n(J), \qquad \sigma \mapsto \int_0^1 \frac{d\sigma}{dt} \sigma^{-1}\, dt$$

descends to a well-defined map $\log : (R_n(J)/\sim) \to \mathrm{HC}_0(M_n(J), M_n(A))$. Furthermore, since log is compatible with direct limits, we obtain a well-defined map

$$\log : \lim_{n\to\infty} (R_n(J)/\sim) \to \lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A)).$$

We now show that $\log([\sigma_0 \cdot \sigma_1]) = \log([\sigma_0]) + \log([\sigma_1])$ for all $\sigma_0, \sigma_1 \in R_n(J)$. Choose a smooth function $\phi : \mathbb{R} \to [0,1]$ such that

$$\phi((-\infty, 0]) = \{0\} \quad \text{and} \quad \phi\big(\big[\tfrac{1}{2}, \infty\big)\big) = \{1\}.$$

Define the smooth function $\psi : \mathbb{R} \to [0, 1]$ by $\psi(t) := \phi\big(t - \frac{1}{2}\big)$. We then have that

$$\sigma_0\sigma_1 \sim (\sigma_0 \circ \psi) \cdot (\sigma_1 \circ \phi),$$

and it thus suffices to verify that $\log((\sigma_0 \circ \psi) \cdot (\sigma_1 \circ \phi)) = \log(\sigma_0) + \log(\sigma_1)$. But this identity follows by a change of variables:

$$\begin{aligned}\log\big((\sigma_0 \circ \psi) \cdot (\sigma_1 \circ \phi)\big)& \\ &= \int_0^{1/2} \frac{d(\sigma_1 \circ \phi)}{dt}(\sigma_1 \circ \phi)^{-1}\,dt + \int_{1/2}^{1} \frac{d(\sigma_0 \circ \psi)}{dt}(\sigma_0 \circ \psi)^{-1}\,dt \\ &= \log(\sigma_0) + \log(\sigma_1).\end{aligned}$$

To finish the proof of the proposition we only need to show that $\log([\sigma\tau\sigma^{-1}]) = \log([\tau])$ whenever $\sigma \in R_n(A)$ and $\tau \in R_n(J)$. To this end, we consider the smooth homotopy with fixed endpoints

$$H(s, t) := \sigma(f(s, t))\tau(t)\sigma(f(s, t))^{-1}, \qquad f(s, t) := ts + 1 - s = s(t - 1) + 1$$

between $\sigma\tau\sigma^{-1}$ and $\sigma(1)\tau\sigma(1)^{-1}$. This proves that

$$\log([\sigma\tau\sigma^{-1}]) = \log([\sigma(1)\tau\sigma(1)^{-1}]) = \log([\tau]),$$

where we have used the fact that $\sigma(1)x\sigma(1)^{-1}$ and x determine the same element in $\mathrm{HC}_0(M_n(J), M_n(A))$ for all $x \in M_n(J)$. $\square$

4.6. Definition. By the *relative Chern character* $\mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(J, A) \to \mathrm{HC}_0(J, A)$, we understand the homomorphism obtained as the composition

$$\mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(J, A) \xrightarrow{\log} \lim_{n\to\infty} \mathrm{HC}_0(M_n(J), M_n(A)) \xrightarrow{\mathrm{TR}} \mathrm{HC}_0(J, A)$$

of the generalized logarithm and the generalized trace.

5. The relative Skandalis–de la Harpe determinant

5.1. Analogous to the determinant of Skandalis and de la Harpe, we are now in a position to define such a determinant purely by means of K-theory for relative pairs of Banach algebras. In particular, we are able to deal with the presence of a not necessarily closed ideal J inside a unital Banach algebra A.

5.2. Definition. Let (J, A) be a relative pair of Banach algebras. In this section we assume $\tau : J \to \mathbb{C}$ to be a continuous linear functional which additionally satisfies

$$\tau(ja) = \tau(aj) \qquad (a \in A,\ j \in J).$$

The latter property means that τ is a *hypertrace*. For such a trace there is a well-defined map (also denoted by τ):

$$\tau : \mathrm{HC}_0(J, A) \to \mathbb{C}, \qquad j + \mathrm{Im}(b) \mapsto \tau(j).$$

Furthermore, we let

$$\tilde{\tau} := -\tau \circ \mathrm{ch}^{\mathrm{rel}} : K_1^{\mathrm{rel}}(J, A) \to \mathbb{C},$$

with $\mathrm{ch}^{\mathrm{rel}}$ as in Definition 4.6. Note that $\tilde{\tau}$ is a homomorphism into the additive group $\mathbb{C}$.

5.3. Recall (Lemma 3.2) that there is an exact sequence in relative K-theory:

$$K_2^{\mathrm{top}}(J) \xrightarrow{\ \partial\ } K_1^{\mathrm{rel}}(J, A) \xrightarrow{\ \theta\ } K_1^{\mathrm{alg}}(J, A) \xrightarrow{\ p\ } K_1^{\mathrm{top}}(J) \longrightarrow 0.$$

This allows us to define the *relative Skandalis–de la Harpe determinant*

$$\widetilde{\det}_\tau : \mathrm{Ker}(p) \to \mathbb{C}/\,\mathrm{Im}(\tilde{\tau} \circ \partial)$$

by

$$\widetilde{\det}_\tau([g]) := \tilde{\tau}([\sigma]) + \mathrm{Im}(\tilde{\tau} \circ \partial),$$

where $[\sigma] \in K_1^{\mathrm{rel}}(J, A)$ satisfies $\theta([\sigma]) = [\sigma(1)^{-1}] = [g] \ni K_1^{\mathrm{alg}}(J, A)$. Such a lift always exists since $\mathrm{Ker}(p) = \mathrm{Im}(\theta)$. Furthermore, this assignment is well-defined since if $[\sigma_0]$ and $[\sigma_1]$ are lifts of the same element $[g]$ then

$$[\sigma_0][\sigma_1]^{-1} \in \mathrm{Ker}(\theta) = \mathrm{Im}(\partial).$$

It follows that $\tilde{\tau}([\sigma_0]) = \tilde{\tau}([\sigma_1])$ modulo $\mathrm{Im}(\tilde{\tau} \circ \partial)$.

Compare this with the definition on page 245 of [de la Harpe and Skandalis 1984], where absolute K-theory is used rather than relative K-theory.

5.4. Lemma. *We have the following equality of subgroups of* $(\mathbb{C}, +)$:

$$2\pi i \cdot \mathrm{Im}(\underline{\tau} : K_0^{\mathrm{top}}(J) \to \mathbb{C}) = \mathrm{Im}(\tilde{\tau} \circ \partial : K_2^{\mathrm{top}}(J) \to \mathbb{C}).$$

By $\underline{\tau} : K_0^{\mathrm{top}}(J) \to \mathbb{C}$ *we understand the map induced by* τ.

Proof. The claim follows from commutativity of the following diagram:

$$\begin{tikzcd} K_0^{\mathrm{top}}(J) \arrow[r, "\beta_J", "\cong"'] \arrow[dr, "-2\pi i \cdot \underline{\tau}"'] & K_2^{\mathrm{top}}(J) \arrow[d, "\tilde{\tau} \circ \partial"] \\ & \mathbb{C} \end{tikzcd} \tag{5.5}$$

By β_J we mean the Bott isomorphism map, as in Definition 2.5.

To prove commutativity of (5.5), we note that for an idempotent $f \in M_n(J^+)$,

$$\mathrm{ch}^{\mathrm{rel}}(\partial([\gamma_f])) = \mathrm{TR}\left(2\pi i \int_0^1 e^{2\pi i t} f(e^{-2\pi i t} f + \mathbb{1}_n - f)\, dt\right) = 2\pi i\, \mathrm{TR}(f).$$

If now $e, f \in M_n(J^+)$ are idempotents satisfying $e - f \in M_n(J)$, then

$$\tilde{\tau}\big(\partial(\beta_J([e] - [f]))\big) = \tilde{\tau}(\partial([\gamma_e \gamma_f^{-1}])) = -2\pi i \cdot \tau(\mathrm{TR}(e - f)).$$

So (5.5) indeed commutes. ∎

5.6. Together with Lemma 5.4 we see that the following diagram commutes:

$$\begin{tikzcd} K_2^{\mathrm{top}}(J) \arrow[r, "\partial"] \arrow[d, "\tilde{\tau}\circ\partial"] & K_1^{\mathrm{rel}}(J, A) \arrow[r, two heads, "\theta"] \arrow[d, "\tilde{\tau}"] & \operatorname{Ker} p \arrow[r, "p"] \arrow[d, "\widetilde{\det}_\tau"] & 0 \\ 2\pi i \cdot \mathrm{Im}(\underline{\tau}) \arrow[r, tail] & \mathbb{C} \arrow[r, two heads] & \mathbb{C}/(2\pi i \cdot \mathrm{Im}(\underline{\tau})) & \end{tikzcd}$$

In the next section this will be applied to the case that the kernel of p is all of $K_1^{\mathrm{alg}}(J, A)$. In that case we get a determinant

$$\widetilde{\det}_\tau : K_1^{\mathrm{alg}}(J, A) \to \mathbb{C}/(2\pi i \cdot \mathrm{Im}\,(\underline{\tau})).$$

6. Topological K-theory of trace ideals

6.1. In the following, $N \subset \mathscr{L}(H)$ always denotes a semifinite von Neumann algebra equipped with a fixed normal, faithful and semifinite trace $\tau : N_+ \to [0, \infty]$. A good reference for traces on von Neumann algebras is [Dixmier 1981, I.6.1, I.6.10].

6.2. We let $\|\cdot\| : N \to [0, \infty)$ denote the operator norm on N and we let

$$\mathscr{L}_\tau^1(N) := \{x \in N : \tau(|x|) < \infty\} \subset N$$

denote the trace ideal in N. We recall that $\mathscr{L}_\tau^1(N) \subset N$ is indeed a $*$-ideal and that $\mathscr{L}_\tau^1(N)$ becomes a Banach $*$-algebra in its own right when equipped with the norm

$$\|x\|_{1,\infty} := \|x\| + \tau(|x|) \qquad (x \in \mathscr{L}_\tau^1(N)).$$

Moreover, it holds that $(\mathscr{L}_\tau^1(N), N)$ is a relative pair of Banach algebras in the sense of Definition 2.1.

6.3. For each $n \in \mathbb{N}$ we have that $M_n(N) \subset \mathscr{L}(H^{\oplus n})$ is a semifinite von Neumann algebra. Indeed, we may define the normal, faithful and semifinite trace $\tau_n : M_n(N)_+ \to [0, \infty]$ by $\tau_n(x) := \sum_{i=1}^n \tau(x_{ii})$. The inclusion $M_n(\mathscr{L}_\tau^1(N)) \to M_n(N)$ then induces an isomorphism

$$M_n(\mathscr{L}_\tau^1(N)) \cong \mathscr{L}_{\tau_n}^1(M_n(N))$$

of Banach $*$-algebras. This isomorphism is, however, not an isometry since (by convention) $M_n(\mathscr{L}_\tau^1(N))$ is equipped with the norm defined as in 2.2.

6.4. Lemma. *The group* $\mathrm{GL}_n(\mathscr{L}_\tau^1(N))$ *is path connected for all* $n \in \mathbb{N}$. *In particular, it holds that*

$$K_1^{\mathrm{top}}(\mathscr{L}_\tau^1(N)) = \{0\}.$$

Proof. Since $M_n(\mathscr{L}_\tau^1(N)) \cong \mathscr{L}_{\tau_n}^1(M_n(N))$ by 6.3, it suffices to verify the lemma for $n = 1$. Thus, let $g \in \mathrm{GL}_1(\mathscr{L}_\tau^1(N))$. Using polar decomposition we may suppose that $g^*g = \mathbb{1} = gg^*$ or that $g = |g|$. In the first case we may find an $x \in \mathscr{L}_\tau^1(N)$

with $x = -x^*$ such that $g = e^x$. In the second case we may find an $x \in \mathscr{L}^1_\tau(N)$ with $x = x^*$ such that $g = e^x$. In both cases we obtain the smooth path $t \mapsto e^{tx}$ connecting $\mathbb{1}$ and g. □

7. The semifinite Fuglede–Kadison determinant

7.1. We are now going to use K-theory for relative pairs of Banach algebras to define our determinant. From 6.2 we know that $(\mathscr{L}^1_\tau(N), N)$ is a relative pair of Banach algebras and that $\tau : \mathscr{L}^1_\tau(N) \to \mathbb{C}$ is continuous with respect to $\|\cdot\|_{1,\infty}$. Since $\tau : \mathscr{L}^1_\tau(N) \to \mathbb{C}$ is moreover a hypertrace, we get (as defined in 5.6) a determinant

$$\widetilde{\det}_\tau : K_1^{\mathrm{alg}}(\mathscr{L}^1_\tau(N), N) \to \mathbb{C}/(2\pi i \cdot \mathrm{Im}(\underline{\tau})).$$

Note that our determinant is defined on all of $K_1^{\mathrm{alg}}(\mathscr{L}^1_\tau(N), N)$ by Lemma 6.4.

7.2. Lemma. *We have*

$$\mathrm{Im}\big(\underline{\tau} : K_0^{\mathrm{top}}(\mathscr{L}^1_\tau(N)) \to \mathbb{C}\big) \subset \mathbb{R}.$$

Proof. Since $M_n(\mathscr{L}^1_\tau(N)^+) \subset M_n(N)$ is closed under holomorphic functional calculus for all $n \in \mathbb{N}$, every idempotent in $M_n(\mathscr{L}^1_\tau(N)^+)$ is similar to a projection in $M_n(\mathscr{L}^1_\tau(N)^+)$; see [Blackadar 1998, Proposition 4.6.2]. And for projections $p, q \in M_n(\mathscr{L}^1_\tau(N)^+)$ with $p - q \in M_n(\mathscr{L}^1_\tau(N))$ we see that

$$\underline{\tau}([p] - [q]) = \tau(\mathrm{TR}(p - q)) \in \mathbb{R},$$

where we have used that all the diagonal entries $(p - q)_{jj}$ are self-adjoint. □

7.3. We thus have a well-defined homomorphism

$$\widetilde{\det}_\tau : K_1^{\mathrm{alg}}(\mathscr{L}^1_\tau(N), N) \to \mathbb{C}/i\mathbb{R}, \qquad \widetilde{\det}_\tau : [g] \mapsto \tilde{\tau}([\sigma]) + i\mathbb{R},$$

where $[\sigma] \in K_1^{\mathrm{rel}}(\mathscr{L}^1_\tau(N), N)$ is any lift of $[g]$, by which we mean that $\theta([\sigma]) = [\sigma(1)^{-1}] = [g]$.

Note that there is an isomorphism of abelian groups

$$\mathbb{C}/i\mathbb{R} \to (0, \infty), \qquad z + i\mathbb{R} \mapsto e^{\Re(z)},$$

where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$. This gives rise to the following definition:

7.4. Definition. We define the *semifinite Fuglede–Kadison determinant*

$$\det_\tau : K_1^{\mathrm{alg}}(\mathscr{L}^1_\tau(N), N) \to (0, \infty)$$

by

$$\det_\tau([g]) := e^{\Re(\widetilde{\det}_\tau([g]))}.$$

More explicitly, we have

$$\det_\tau([g]) = \exp((\Re \circ \tilde{\tau})[\sigma]) = \exp\left(-(\Re \circ \tau \circ \mathrm{TR})\left(\int_0^1 \frac{d\sigma}{dt}\sigma^{-1}\,dt\right)\right),$$

where $[\sigma] \in K_1^{\mathrm{rel}}(\mathscr{L}_\tau^1(N), N)$ is any lift of $[g] \in K_1^{\mathrm{alg}}(\mathscr{L}_\tau^1(N), N)$, meaning that $[\sigma(1)^{-1}] = [g]$.

7.5. Proposition. *The semifinite Fuglede–Kadison determinant* $\det_\tau$ *has the following properties*:

(1) $\det_\tau([gh]) = \det_\tau([g])\det_\tau([h])$ *for all* $g, h \in \mathrm{GL}_n(\mathscr{L}_\tau^1(N))$.

(2) $\det_\tau([hgh^{-1}]) = \det_\tau([g])$ *for all* $g \in \mathrm{GL}_n(\mathscr{L}_\tau^1(N))$ *and* $h \in \mathrm{GL}_n(N)$.

(3) $\det_\tau([e^x]) = (\exp \circ \Re \circ \tau \circ \mathrm{TR})(x)$ *for* $x \in M_n(\mathscr{L}_\tau^1(N))$.

These properties follow directly from the definition of the determinant and the fact that $\widetilde{\det}_\tau$ is a homomorphism.

In [Brown 1986, Section 1], the equality in the following proposition is the definition of the determinant.

7.6. Proposition. *The following explicit formula holds*:

$$\det_\tau([g]) = e^{\tau(\log|g|)}, \qquad (g \in \mathrm{GL}_1(\mathscr{L}_\tau^1(N))).$$

Proof. Let $g \in \mathrm{GL}_1(\mathscr{L}_\tau^1(N))$. Using the polar decomposition $g = u|g|$, we may compute

$$\det_\tau([g]) = \det_\tau([u])\det_\tau([|g|]).$$

Since $u \in \mathrm{GL}_1(\mathscr{L}_\tau^1(N))$ is unitary in the ambient von Neumann algebra, we may write $u = e^{ix}$ for some $x \in \mathscr{L}_\tau^1(N)$ with $x = x^*$. Moreover, we have $\log|g| \in \mathscr{L}_\tau^1(N)$. By Proposition 7.5(3) we thus have that

$$\det_\tau([g]) = e^{\Re(\tau(ix))} \cdot e^{\tau(\log|g|)} = e^{\tau(\log|g|)}. \qquad \square$$

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Algebraic K-theory of quotient stacks

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We prove some fundamental results like localization, excision, Nisnevich descent, and the regular blow-up formula for the algebraic K-theory of certain stack quotients of schemes with affine group scheme actions. We show that the homotopy K-theory of such stacks is homotopy invariant. This implies a similar homotopy invariance property of the algebraic K-theory with coefficients.

1. Introduction

The higher algebraic K-theories of Quillen and Thomason–Trobaugh are among the most celebrated discoveries in mathematics. Fundamental results like localization, excision, Nisnevich descent, and the blow-up formula have played pivotal roles in almost every recent breakthrough in the K-theory of schemes; see, e.g., [Cortiñas 2006; Cortiñas et al. 2008; Schlichting 2010]. The generalization of these results to equivariant K-theory is the theme of this paper.

The significance of equivariant K-theory [Thomason 1987a] in the study of the ordinary (nonequivariant) K-theory is essentially based on two principles. First, it often turns out that the presence of a group action allows one to exploit representation-theoretic tools to study equivariant K-theory. Second, there are results (see, for instance, [Merkurjev 2005, Theorem 32]) which directly connect equivariant algebraic K-theory with the ordinary K-theory of schemes with group action. These principles have been effectively used in the past to study both equivariant and ordinary algebraic K-theory; see, for instance, [Joshua and Krishna 2015; Vezzosi and Vistoli 2003]. In addition, equivariant K-theory often allows one to understand various other cohomology theories of moduli stacks and moduli spaces from the K-theoretic point of view.

However, any serious progress towards the applicability of equivariant K-theory (of vector bundles) requires analogues for quotient stacks of the fundamental results of Thomason–Trobaugh. The goal of this paper is to establish these results, so that a very crucial gap in the study of the K-theory of quotient stacks can be filled. Special cases of these results were earlier proven in [Krishna 2009; Krishna and

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Østvær 2012; Heller et al. 2015]. Here is a summary of our main results. The precise statements and the underlying notation can be found in the body of the text. We fix a field k.

Theorem 1.1. *Let $\mathcal{X}$ be a nice quotient stack over k with the resolution property. Let $\mathbb{K}$ denote the (nonconnective) K-theory presheaf on the 2-category of nice quotient stacks. Let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack with open complement $\mathcal{U}$. Then the following hold.*

(1) *There is a homotopy fibration sequence of S^1-spectra*

$$\mathbb{K}(\mathcal{X} \text{ on } \mathcal{Z}) \to \mathbb{K}(\mathcal{X}) \to \mathbb{K}(\mathcal{U}).$$

(2) *The presheaf $\mathcal{X} \mapsto \mathbb{K}(\mathcal{X})$ satisfies excision.*

(3) *The presheaf $\mathcal{X} \mapsto \mathbb{K}(\mathcal{X})$ satisfies Nisnevich descent.*

(4) *The presheaf $\mathcal{X} \mapsto \mathbb{K}(\mathcal{X})$ satisfies descent for regular blow-ups.*

Theorem 1.2. *The nonconnective homotopy K-theory presheaf KH on the 2-category of nice quotient stacks with resolution property satisfies the following.*

(1) *It is invariant under every vector bundle morphism (Thom isomorphism for stacks).*

(2) *It satisfies localization, excision, Nisnevich descent, and descent for regular blow-ups.*

(3) *If $\mathcal{X}$ is the stack quotient of a scheme by a finite nice group, then $KH(\mathcal{X})$ is invariant under infinitesimal extensions.*

The following result shows that K-theory with coefficients for quotient stacks is homotopy invariant, i.e., it satisfies the Thom isomorphism. No case of this result was yet known for stacks which are not schemes.

Theorem 1.3. *Let $\mathcal{X}$ be a nice quotient stack over k with the resolution property and let $f : \mathcal{E} \to \mathcal{X}$ be a vector bundle. Then the following hold.*

(1) *For any integer n invertible in k, the map $f^* : \mathbb{K}(\mathcal{X}; \mathbb{Z}/n) \to \mathbb{K}(\mathcal{E}; \mathbb{Z}/n)$ is a homotopy equivalence.*

(2) *For any integer n nilpotent in k, the map $f^* : \mathbb{K}(\mathcal{X}; \mathbb{Z}[1/n]) \to \mathbb{K}(\mathcal{E}; \mathbb{Z}[1/n])$ is a homotopy equivalence.*

In the above results, a nice quotient stack means a stack of the form $[X/G]$, where G is an affine group scheme over k acting on a k-scheme X such that G is nice, i.e., it is either linearly reductive over k or $\mathrm{char}(k) = 0$. Group schemes of multiplicative type (e.g., diagonalizable group schemes) are notable examples of this in positive characteristic. We refer to Section 2B for more details.

Applications. Similar to the case of schemes, one expects the above results to be of central importance in the study of the K-theory of quotient stacks. Already by now, there have been two immediate major applications: (1) the cdh-descent and, (2) Weibel's conjecture for negative KH-theory of stacks. In a sense, these applications motivated the results of this paper.

Hoyois [2017] has constructed a variant of KH-theory for nice quotient stacks and has used the main results of this paper to prove the cdh-descent for this variant. The results of this paper (and their generalizations) have also been used recently by Hoyois and the first author [Hoyois and Krishna 2017] to prove cdh-descent for the KH-theory (as defined in Section 5) of nice stacks, and to prove Weibel's conjecture for the vanishing of negative KH-theory of such stacks.

Another application of the above results is related to a rigidity type theorem for the K-theory of semilocal rings. Let A be a normal semilocal ring with isolated singularity with an action of a finite group G, and let $\widehat{A}$ denote its completion along the Jacobson radical. The rigidity question asks if the map $K'_*(G, A) \to K'_*(G, \widehat{A})$ is injective. If G is trivial, this was proven for $K'_0(G, A)$ by Kamoi and Kurano [2002] for certain type of isolated singularities. They apply this result to characterize certain semilocal rings. The main tool of [Kamoi and Kurano 2002] is Theorem 1.1 for the ordinary K-theory of singular rings. We hope that the localization theorem for the K-theory of quotient stacks can now be used to prove the equivariant version of this rigidity theorem.

2. Perfect complexes on quotient stacks

Throughout this text, we work over a fixed base field k of arbitrary characteristic. In this section, we fix notations, recall basic definitions and prove some preliminary results. We conclude the section with the proof of an excision property for the derived category of perfect complexes on stacks.

2A. ***Notations and definitions.*** Let $\mathbf{Sch}_k$ denote the category of separated schemes of finite type over k. A *scheme* in this paper will mean an object of $\mathbf{Sch}_k$. A group scheme G will mean an affine group scheme over k. Recall that a stack $\mathcal{X}$ (of finite type) over the big fppf site of k is said to be an algebraic stack over k if the diagonal of $\mathcal{X}$ is representable by algebraic spaces and $\mathcal{X}$ admits a smooth, representable and surjective morphism $U \to \mathcal{X}$ from a scheme U. Throughout this text a "stack" will always refer to an algebraic stack. We shall say that $\mathcal{X}$ is a *quotient stack* if it is a stack of the form $[X/G]$ (see, for instance, [Laumon and Moret-Bailly 2000, §2.4.2]), where G is an affine group scheme acting on a scheme X.

2B. ***Nice stacks.*** Given a group scheme G, let $\mathbf{Mod}^G(k)$ denote the category of k-modules with G-action. Recall that G is said to be linearly reductive if the "functor of G-invariants" $(-)^G : \mathbf{Mod}^G(k) \to \mathbf{Mod}(k)$, given by the submodule of G-invariant elements, is exact. If $\mathrm{char}(k) = 0$, it is well known that G is linearly reductive if and only if it is reductive. In general, it follows from [Abramovich et al. 2008, Propositions 2.5, 2.7, Theorem 2.16] that G is linearly reductive if there is an extension

$$1 \to G_1 \to G \to G_2 \to 1, \tag{2.1}$$

where each of G_1 and G_2 is either finite over k of degree prime to the exponential characteristic of k, or is of multiplicative type (étale locally diagonalizable) over k. One knows that linearly reductive group schemes in positive characteristic are closed under the operations of taking closed subgroups and base change.

Definition 2.2. We shall say that a group scheme G is *nice* if either it is linearly reductive or $\mathrm{char}(k) = 0$. If G is nice and it acts on a scheme X, we shall say that the resulting quotient stack $[X/G]$ is nice.

2C. *Perfect complexes on stacks.* Given a stack $\mathcal{X}$, let $\mathrm{Sh}(\mathcal{X})$ denote the abelian category of sheaves of abelian groups, $\mathrm{Mod}(\mathcal{X})$ the abelian category of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules, and $\mathrm{QC}(\mathcal{X})$ the abelian category of quasicoherent sheaves, each on the smooth-étale site $\text{Lis-Et}(\mathcal{X})$ of $\mathcal{X}$. Let $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ denote the category of all (possibly unbounded) chain complexes over $\mathrm{Mod}(\mathcal{X})$ whose cohomology lie in $\mathrm{QC}(\mathcal{X})$, and $\mathrm{Ch}(\mathrm{QC}(\mathcal{X}))$ the category of all chain complexes over $\mathrm{QC}(\mathcal{X})$. Let $D_{\mathrm{qc}}(\mathcal{X})$ and $D(\mathrm{QC}(\mathcal{X}))$ denote the corresponding derived categories. Let $D(\mathcal{X})$ denote the unbounded derived category of $\mathrm{Mod}(\mathcal{X})$. If $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed substack with open complement $j : \mathcal{U} \hookrightarrow \mathcal{X}$, we let

$$\mathrm{Ch}_{\mathrm{qc},\mathcal{Z}}(\mathcal{X}) = \big\{\mathcal{F} \in \mathrm{Ch}_{\mathrm{qc}}(\mathcal{X}) \mid j^*(\mathcal{F}) \xrightarrow{\text{q. iso.}} 0\big\}.$$

The derived category of $\mathrm{Ch}_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$ will be denoted by $D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$. Recall that a stack $\mathcal{X}$ is said to have the resolution property if every coherent sheaf on $\mathcal{X}$ is a quotient of a vector bundle.

Lemma 2.3. *Let $\mathcal{X}$ be the stack quotient of a scheme X with an action of a group scheme G. Then the following hold.*

(1) *Every quasicoherent sheaf on $\mathcal{X}$ is the direct limit of its coherent subsheaves.*

(2) *$\mathcal{X}$ has the resolution property if X has an ample family of G-equivariant line bundles. In particular, $\mathcal{X}$ has the resolution property if X is normal with an ample family of (nonequivariant) line bundles.*

(3) *$\mathcal{X}$ has the resolution property if X is quasi-affine.*

Proof. Part (1) is [Thomason 1987b, Lemma 1.4]. For (2), note that $[\mathrm{Spec}(k)/G]$ has the resolution property [Thomason 1987b, Lemma 2.4]. Therefore, if X has an ample family of G-equivariant line bundles, it follows from [Thomason 1987b, Lemma 2.6] that $\mathcal{X}$ has the resolution property. If X is normal with an ample family of (nonequivariant) line bundles, it follows from [Thomason 1987b, Lemmas 2.10, 2.14] that $\mathcal{X}$ has the resolution property. Part (3) is well known and follows, for example, from [Hall and Rydh 2017, Lemma 7.1]. □

Recall from [SGA 6 1971, Definition I.4.2] that a complex of $\mathcal{O}_X$-modules on a Noetherian scheme is perfect if it is Zariski locally quasi-isomorphic to a bounded complex of locally free sheaves.

Definition 2.4. Let $\mathcal{X}$ be a stack over k. A chain complex $P \in \mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ is called *perfect* if for any affine scheme $U = \mathrm{Spec}(A)$ with a smooth morphism $s : U \to \mathcal{X}$, the complex of A-modules $s^*(P) \in \mathrm{Ch}(\mathrm{Mod}(A))$ is quasi-isomorphic to a bounded complex of finitely generated projective A-modules.

We shall denote the category of perfect complexes on $\mathcal{X}$ by $\mathrm{Perf}(\mathcal{X})$ and its derived category by $D^{\mathrm{perf}}(\mathcal{X})$. For a quotient stack with the resolution property, we can characterize perfect complexes in terms of their pull-backs to the total space of the quotient map.

Lemma 2.5. *Let $f : X' \to X$ be a faithfully flat map of Noetherian schemes. Let P be a chain complex of quasicoherent sheaves on X such that $f^*(P)$ is perfect on X'. Then P is a perfect complex on X.*

Proof. By [Thomason and Trobaugh 1990, Proposition 2.2.12], a complex of quasicoherent sheaves is perfect if and only if it is cohomologically bounded above, its cohomology sheaves are coherent, and it has locally finite Tor-amplitude. But all these properties are known to descend from a faithfully flat cover. □

Proposition 2.6. *Let $\mathcal{X}$ be the stack quotient of a scheme X with an action of a group scheme G and let $u : X \to \mathcal{X}$ be the quotient map. Assume that $\mathcal{X}$ has the resolution property. Let P be a chain complex of quasicoherent $\mathcal{O}_{\mathcal{X}}$-modules. Then the following are equivalent.*

(1) *P is perfect.*

(2) *$u^*(P)$ is perfect.*

(3) *$u^*(P)$ is quasi-isomorphic to a bounded complex of G-equivariant vector bundles in* $\mathrm{Ch}(\mathrm{QC}^G(X))$, *where* $\mathrm{QC}^G(X)$ *denotes the category of G-equivariant quasicoherent sheaves on X.*

Proof. (1) ⇒ (2). We let $Q = u^*(P)$. Consider an open cover of X by affine open subsets $\{\mathrm{Spec}(A_i)\}$. Let $s : U \to [X/G]$ be an atlas and $s_i : U_i \to \mathrm{Spec}(A_i)$ its base change to $\mathrm{Spec}(A_i)$, where U_i are algebraic spaces. Take étale covers $t_i : V_i \to U_i$ of U_i, where the V_i are schemes. Let $f_i : V_i \to U$ and $g_i : V_i \to \mathrm{Spec}(A_i)$ denote the obvious composite maps. It follows from (1) that $\boldsymbol{L} g_i^*(Q|_{\mathrm{Spec}(A_i)}) \simeq \boldsymbol{L} f_i^*(s^*(P))$ is a perfect complex on V_i. Therefore, by Lemma 2.5, $Q|_{\mathrm{Spec}(A_i)}$ is a perfect complex in $\mathrm{Ch}(\mathrm{Mod}(A_i))$. Equivalently, Q is perfect.

(2) $\Rightarrow$ (3). We want to apply the inductive construction lemma [Thomason and Trobaugh 1990, Lemma 1.9.5] with $\mathcal{A}$ being $\mathrm{QC}^G(X)$, $\mathcal{D}$ the category of G-equivariant vector bundles on X and $\mathcal{C}$ the category of complexes in $\mathrm{Ch}(\mathrm{QC}^G(X))$ satisfying (2). It is enough to verify that the hypothesis [loc. cit., 1.9.5.1] holds.

Suppose $C \in \mathcal{C}$ such that $H^i(C) = 0$ for $i \geq n$, and $q : \mathcal{F} \twoheadrightarrow H^{n-1}(C)$ in $\mathrm{QC}^G(X)$. By [Thomason and Trobaugh 1990, Proposition 2.2.3], $\mathcal{G} = H^{n-1}(C)$ is a coherent $\mathcal{O}_X$-module. Therefore, $\mathcal{G}$ is a coherent G-module. By Lemma 2.3(1), we can write $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$, where $\mathcal{F}_\alpha$ are coherent G-submodules of $\mathcal{F}$. Under the forgetful functor, this gives an epimorphism $q : \varinjlim \mathcal{F}_\alpha \twoheadrightarrow \mathcal{G}$ in $\mathrm{QC}(X)$, where $\mathcal{F}_\alpha$, $\mathcal{G}$ are coherent modules.

Now, as $\mathcal{G}$ is coherent and X is Noetherian, we can find an α such that the composite map $\mathcal{F}_\alpha \hookrightarrow \mathcal{F} \xrightarrow{q} \mathcal{G}$ is surjective. By the resolution property, there exists $\mathcal{E} \in \mathcal{D}$ such that $\mathcal{E} \twoheadrightarrow \mathcal{F}_\alpha$. Hence the composite $\mathcal{E} \to \mathcal{F}_\alpha \hookrightarrow \mathcal{F} \xrightarrow{q} \mathcal{G}$ is also surjective. Applying the conclusion of [Thomason and Trobaugh 1990, Lemma 1.9.5] to $C^\bullet = P$ and $D^\bullet = 0$, we get a bounded above complex E of G-vector bundles and a quasi-isomorphism $\phi : E \xrightarrow{\sim} P$ in $\mathrm{Ch}(\mathrm{QC}^G(X))$. Therefore, $E \in \mathcal{C}$.

Since X is Noetherian, E has globally finite Tor-amplitude. To show that Q is quasi-isomorphic to a bounded complex over $\mathcal{D}$, it suffices to prove that the good truncation $\tau^{\geq a}(E)$ is a bounded complex of G-equivariant vector bundles and the map $E \to \tau^{\geq a}(E)$ is a quasi-isomorphism. It is enough to prove this claim by forgetting the G-action. But this follows exactly along the lines of the proof of [Thomason and Trobaugh 1990, Proposition 2.2.12].

(3) $\Longrightarrow$ (1) is clear. ☐

2D. ***Perfect complexes and compact objects of $D_{\mathrm{qc}}(\mathcal{X})$.*** Recall that if $\mathcal{T}$ is a triangulated category which is closed under small coproducts, then an object $E \in \mathrm{Obj}(\mathcal{T})$ is called compact if the functor $\mathrm{Hom}_{\mathcal{T}}(E, -)$ on $\mathcal{T}$ commutes with small coproducts. The full triangulated subcategory of compact objects in $\mathcal{T}$ is denoted by $\mathcal{T}^c$. If X is a scheme, one of the main results of [Thomason and Trobaugh 1990] is that a chain complex $P \in \mathrm{Ch}_{\mathrm{qc}}(X)$ is perfect if and only if it is a compact object of $D_{\mathrm{qc}}(X)$. For quotient stacks, this is a consequence of the results of [Neeman 1996; Hall and Rydh 2015]:

Proposition 2.7. *Let $\mathcal{X}$ be a nice quotient stack. Then a chain complex $P \in \mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ is perfect if and only if it is a compact object of $D_{\mathrm{qc}}(\mathcal{X})$.*

Proof. Suppose P is compact. We need to show that $s^*(P)$ is perfect on $U = \mathrm{Spec}(A)$ for every smooth map $s : U \to \mathcal{X}$. Since the compact objects of $D_{\mathrm{qc}}(U)$ are perfect, it suffices to show that $s^*(P)$ is compact. We deduce this using [Neeman 1996, Theorem 5.1].

The push-forward functor $\boldsymbol{R}s_* : D_{\mathrm{qc}}(U) \to D_{\mathrm{qc}}(\mathcal{X})$ is a right adjoint to the pull-back $\boldsymbol{L}s^* : D_{\mathrm{qc}}(\mathcal{X}) \to D_{\mathrm{qc}}(U)$. As $\boldsymbol{R}s_*$ and $\boldsymbol{L}s^*$ both preserve small coproducts (see the proof of Lemma 2.8 below), it follows from [Neeman 1996, Theorem 5.1] that $s^*(P)$ is compact.

If P is perfect, then it is a compact object of $D_{\mathrm{qc}}(\mathcal{X})$ by our assumption on $\mathcal{X}$ and [Hall and Rydh 2015, Theorem C]. $\square$

Lemma 2.8. *Let $\mathcal{X}$ be a nice quotient stack and let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack. Then the compact objects of $D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$ are exactly those which are perfect in* $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$.

Proof. It follows from Proposition 2.7 that $D^{\mathrm{perf}}_{\mathcal{Z}}(\mathcal{X}) \subseteq D^c_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$. To prove the other inclusion, let $K \in D^c_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$. We need to show that K is a perfect complex in $D_{\mathrm{qc}}(\mathcal{X})$. Let $s : V = \mathrm{Spec}(A) \to \mathcal{X}$ be any smooth morphism and set $T = s^{-1}(\mathcal{Z})$. Consider a set of objects $\{F_\alpha\}$ in $D_{\mathrm{qc},T}(V)$. Since $\mathcal{X}$ is a quotient stack, there exists a smooth atlas $u : X \to \mathcal{X}$, where $X \in \mathbf{Sch}_k$. This gives a 2-Cartesian square of stacks

$$
\begin{tikzcd}
W \arrow[r, "s'"] \arrow[d, "u'"'] & X \arrow[d, "u"] \\
V \arrow[r, "s"'] & \mathcal{X}
\end{tikzcd}
\tag{2.9}
$$

The maps u and s are Tor-independent because they are smooth. Since $\Delta_{\mathcal{X}}$ is representable and V is affine, it follows that s is representable. We conclude from [Hall and Rydh 2017, Lemma 2.5(3), Corollary 4.13] that $u^*\boldsymbol{R}s_*(F_\alpha) \xrightarrow{\simeq} \boldsymbol{R}s'_*u'^*(F_\alpha)$. It follows that $\boldsymbol{R}s_*(F_\alpha) \in D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$. Using adjointness [Krishna 2009, Lemma 3.3], we get

$$
\begin{aligned}
\mathrm{Hom}_{D_{\mathrm{qc},T}(V)}\big(s^*(K), \textstyle\bigoplus_\alpha F_\alpha\big) &\simeq \mathrm{Hom}_{D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})}\big(K, \boldsymbol{R}s_*\big(\textstyle\bigoplus_\alpha F_\alpha\big)\big) \\
&\simeq^1 \mathrm{Hom}_{D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})}\big(K, \textstyle\bigoplus_\alpha \boldsymbol{R}s_*(F_\alpha)\big) \\
&\simeq^2 \textstyle\bigoplus_\alpha \mathrm{Hom}_{D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})}(K, \boldsymbol{R}s_*(F_\alpha)) \\
&\simeq \textstyle\bigoplus_\alpha \mathrm{Hom}_{D_{\mathrm{qc},T}(V)}(s^*(K), F_\alpha),
\end{aligned}
$$

where $\simeq^1$ follows from the fact that $\boldsymbol{R}s_*$ preserves small coproducts [Hall and Rydh 2017, Lemmas 2.5(3), 2.6(3)], and $\simeq^2$ follows since $K \in D^c_{\mathrm{qc},Z}(\mathcal{X})$. This shows that $s^*(K) \in D^c_{\mathrm{qc},T}(V)$. Since V is affine, this implies that $s^*(K)$ is perfect. $\square$

2E. ***Excision for derived category.*** We now prove an excision property for the derived category of perfect complexes on stacks using the technique of Cartan–Eilenberg resolutions.

Let $\mathcal{A}$ be a Grothendieck category and let $D(\mathcal{A})$ denote the unbounded derived category of $\mathcal{A}$. Let $\mathrm{Ch}(\mathcal{A})$ denote the category of all chain complexes over $\mathcal{A}$. An object $A \in \mathrm{Ch}(\mathcal{A})$ is said to be *K-injective* if for every acyclic complex $J \in \mathrm{Ch}(\mathcal{A})$, the complex $\mathrm{Hom}^\bullet(J, A)$ is acyclic. Since $\mathcal{A}$ has enough injectives, a complex over $\mathcal{A}$ has a Cartan–Eilenberg resolution; see [EGA III$_1$ 1961, Chapitre 0, (11.4.2)].

It is known that a Cartan–Eilenberg resolution of an unbounded complex over $\mathcal{A}$ need not, in general, be a K-injective resolution. However, when $\mathcal{X}$ is a scheme or a Noetherian and separated Deligne–Mumford stack over a fixed Noetherian base scheme, it has been shown that for a complex J of $\mathcal{O}_{\mathcal{X}}$-modules with quasicoherent cohomology, the total complex of a Cartan–Eilenberg resolution does give a K-injective resolution of J; see [Keller 1998; Krishna 2009, Proposition 2.2]. Our first objective is to extend these results to all algebraic stacks. We follow the techniques of [Krishna 2009] closely. Given a double complex $J^{\bullet,\bullet}$, let $\mathrm{Tot}^{\wedge}(J)$ denote the (product) total complex.

Proposition 2.10. *Let $\mathcal{X}$ be a stack and let $K \in \mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$. Let $E \xrightarrow{\epsilon} I^{\bullet,\bullet}$ be a Cartan–Eilenberg resolution of E in $\mathrm{Ch}(\mathcal{X})$. Then $E \xrightarrow{\epsilon} \mathrm{Tot}^{\wedge}(I)$ is a K-injective resolution of E.*

Proof. Since $\mathrm{Mod}(\mathcal{X})$ is a Grothendieck category and $I^{\bullet,\bullet}$ is a Cartan–Eilenberg resolution, $\mathrm{Tot}^{\wedge}(I)$ is a K-injective complex by [Weibel 1996, A.3]. We only need to show that $E \xrightarrow{\epsilon} \mathrm{Tot}^{\wedge}(I)$ is a quasi-isomorphism. Let

$$\tau^{\geq p}(E) := 0 \to E^p/B^pE \to E^{p+1} \to \cdots$$

denote the good truncation of E. Then $\{\tau^{\geq p}(E)\}_{p\in\mathbb{Z}}$ gives an inverse system of bounded below complexes with surjective maps such that $E \xrightarrow{\simeq} \varprojlim_p \tau^{\geq p}(E)$. Let $\tau^{\geq p}(I)$ denote the double complex whose i-th row is the good truncation of the i-th row of $I^{\bullet,\bullet}$ as above.

Let $L_p^{\bullet,\bullet} = \mathrm{Ker}(\tau^{\geq p}(I) \twoheadrightarrow \tau^{\geq p+1}(I))$. Then $I^{\bullet,\bullet} \twoheadrightarrow \tau^{\geq p}(I) \twoheadrightarrow \tau^{\geq p+1}(I)$ and $I^{\bullet,\bullet} \xrightarrow{\simeq} \varprojlim_p \tau^{\geq p}(I)$. Therefore, $\mathrm{Tot}^{\wedge}(I) \xrightarrow{\simeq} \varprojlim_p \mathrm{Tot}^{\wedge}(\tau^{\geq p}(I))$. Moreover, since $\tau^{\geq p}(I)$ is a Cartan–Eilenberg resolution of the bounded below complex $\tau^{\geq p}(E)$, it is known that for each $p \in \mathbb{Z}$, $\tau^{\geq p}(E) \xrightarrow{\epsilon_p} \mathrm{Tot}^{\wedge}(\tau^{\geq p}(I))$ is a quasi-isomorphism.

Furthermore, the standard properties of Cartan–Eilenberg resolutions imply that $B^pE \to B^pI^{\bullet,\bullet}$ is an injective resolution and hence, the inclusions $B^pI^{\bullet,i} \hookrightarrow I^{\bullet,i}$ are all split. In particular, the maps $\tau^{\geq p}(I) \twoheadrightarrow \tau^{\geq p+1}(I)$ are termwise split surjective. Since $\tau^{\geq p}(I)$ are upper half plane complexes with bounded below rows, we conclude that the sequences

$$0 \to \mathrm{Tot}^{\wedge}(L_p) \to \mathrm{Tot}^{\wedge}(\tau^{\geq p}(I)) \to \mathrm{Tot}^{\wedge}(\tau^{\geq p+1}(I)) \to 0 \tag{2.11}$$

are exact and are split in each degree.

Hence, we see that $\mathrm{Tot}^{\wedge}(I) \xrightarrow{\simeq} \varprojlim_p \mathrm{Tot}^{\wedge}(\tau^{\geq p}(I))$, where each $\mathrm{Tot}^{\wedge}(\tau^{\geq p}(I))$ is a bounded below complex of injective $\mathcal{O}_{\mathcal{X}}$-modules, and ϵ is induced by a compatible system of quasi-isomorphisms ϵ_p. Furthermore, $\mathrm{Tot}^{\wedge}(\tau^{\geq p}(I)) \to \mathrm{Tot}^{\wedge}(\tau^{\geq p+1}(I))$ is degreewise split surjective with kernel $\mathrm{Tot}^{\wedge}(L_p)$, which is a bounded below complex of injective $\mathcal{O}_{\mathcal{X}}$-modules. Since $\mathcal{H}_i(E) \in \mathrm{QC}(\mathcal{X})$ and $\mathrm{QC}(\mathcal{X}) \subseteq \mathrm{Mod}(\mathcal{X})$

satisfies [Laszlo and Olsson 2008, Assumption 2.1.2], the proposition now follows from [Laszlo and Olsson 2008, Proposition 2.1.4]. □

Corollary 2.12. *Let $f:\mathcal{Y}\to\mathcal{X}$ be a morphism of stacks and let $E\in D_{\mathrm{qc}}(\mathcal{Y})$. Then the natural map $\boldsymbol{R}f_*(E)\to \varprojlim_n \boldsymbol{R}f_*(\tau^{\geq n}(E))$ is an isomorphism in $D_{\mathrm{qc}}(\mathcal{X})$.*

Proof. This is easily checked by replacing E by a Cartan–Eilenberg resolution and using properties of Cartan–Eilenberg resolutions and good truncation. □

Recall that a morphism $f:\mathcal{Y}\to\mathcal{X}$ of stacks is *representable* if for every algebraic space T and a morphism $T\to\mathcal{X}$, the fiber product $T\times_{\mathcal{X}}\mathcal{Y}$ is represented by an algebraic space. If $T\times_{\mathcal{X}}\mathcal{Y}$ is represented by a scheme whenever T is a scheme, we say that $f:\mathcal{Y}\to\mathcal{X}$ is *strongly representable*.

Proposition 2.13. *Let $f:\mathcal{Y}\to\mathcal{X}$ be a strongly representable étale morphism of stacks. Let $\mathcal{Z}\xhookrightarrow{i}\mathcal{X}$ be a closed substack such that $f:\mathcal{Z}\times_{\mathcal{X}}\mathcal{Y}\to\mathcal{Z}$ induces an isomorphism of the associated reduced stacks. Then $f^*:D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})\to D_{\mathrm{qc},\mathcal{Z}\times_{\mathcal{X}}\mathcal{Y}}(\mathcal{Y})$ is an equivalence.*

Proof. We set $\mathcal{W}=\mathcal{Z}\times_{\mathcal{X}}\mathcal{Y}$. Let us first assume that $E\in D^+_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$. We claim that the adjunction map $E\to\boldsymbol{R}f_*\circ f^*(E)$ is an isomorphism. The proof of this claim is identical to that of [Krishna and Østvær 2012, Proposition 3.4] which considers the case of schemes and Deligne–Mumford stacks. We take a smooth atlas $s:U\to\mathcal{X}$ with $U\in\mathbf{Sch}_k$ and note that $U\times_{\mathcal{X}}\mathcal{Y}\to U$ is an étale morphism in $\mathbf{Sch}_k$ because f is strongly representable. As in the proof of [Krishna and Østvær 2012, Proposition 3.4], an application of [Hall and Rydh 2017, Corollary 4.13] now reduces the problem to the case of schemes. By similar arguments, if $F\in D^-_{\mathrm{qc},\mathcal{W}}(\mathcal{Y})$, the co-adjunction map $f^*\circ\boldsymbol{R}f_*(F)\to F$ is an isomorphism (see the proof of [Krishna and Østvær 2012, Theorem 3.5] for details).

To prove the proposition, we need to show that f^* is fully faithful and essentially surjective on objects. To prove the first assertion, let $E\in D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$. Since f^* is exact, it commutes with good truncation. Applying this to the isomorphism $E\xrightarrow{\simeq}\varprojlim_n\tau^{\geq n}(E)$, we conclude from Corollary 2.12 and what we showed above for the bounded below complexes that the adjunction map $E\to\boldsymbol{R}f_*\circ f^*(E)$ is an isomorphism. If $E'\in D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$ is now another object, then

$$\begin{aligned}\mathrm{Hom}_{D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})}(E,E') &\simeq \mathrm{Hom}_{D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})}(E,\boldsymbol{R}f_*\circ f^*(E'))\\ &\simeq \mathrm{Hom}_{D_{\mathrm{qc}}(\mathcal{X})}(E,\boldsymbol{R}f_*\circ f^*(E'))\\ &\simeq^1 \mathrm{Hom}_{D_{\mathrm{qc}}(\mathcal{Y})}(f^*(E),f^*(E'))\\ &\simeq \mathrm{Hom}_{D_{\mathrm{qc},\mathcal{W}}(\mathcal{Y})}(f^*(E),f^*(E')),\end{aligned}$$

where $\simeq^1$ follows from the adjointness of $(f^*,\boldsymbol{R}f_*)$ [Krishna 2009, Lemma 3.3].

To prove the essential surjectivity of f^*, let $F \in D_{\mathrm{qc},\mathcal{W}}(\mathcal{Y})$. If $F \in D^-_{\mathrm{qc},\mathcal{W}}(\mathcal{Y})$, then we have shown above that the map $f^* \circ \boldsymbol{R} f_*(F) \to F$ is an isomorphism. The general case follows from the bounded above case using the isomorphism $\varinjlim_n \tau^{\le n}(F) \xrightarrow{\simeq} F$. □

3. Algebraic *K*-theory of nice quotient stacks

In this section, we prove Theorem 1.1. Let $\mathcal{X}$ be a stack. We begin with the definition and some preliminary results on the K-theory spectrum for stacks.

3A. ***K-theory spectrum.*** The algebraic K-theory spectrum $K(\mathcal{X})$ of $\mathcal{X}$ is defined to be the K-theory spectrum of the complicial bi-Waldhausen category of perfect complexes in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ in the sense of [Thomason and Trobaugh 1990, §1.5.2]. Here, the complicial bi-Waldhausen category structure is given with respect to the degreewise split monomorphisms as cofibrations and quasi-isomorphisms as weak equivalences. The homotopy groups of the spectrum $K(\mathcal{X})$ are defined to be the K-groups of the stack $\mathcal{X}$ and are denoted by $K_n(\mathcal{X})$. Note that these groups are 0 if $n < 0$; see [Thomason and Trobaugh 1990, §1.5.3]. We shall extend this definition to negative integers later in this section. For a closed substack $\mathcal{Z}$ of $\mathcal{X}$, $K(\mathcal{X}$ on $\mathcal{Z})$ is the K-theory spectrum of the complicial bi-Waldhausen category of those perfect complexes on $\mathcal{X}$ which are acyclic on $\mathcal{X} \setminus \mathcal{Z}$.

Lemma 3.1. *For a stack $\mathcal{X}$ with affine diagonal, the inclusion of the complicial bi-Waldhausen category of perfect complexes of quasicoherent $\mathcal{O}_{\mathcal{X}}$-modules into the category of perfect complexes in* $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ *induces a homotopy equivalence of their K-theory spectra.*

Similarly, for a closed substack $\mathcal{Z} \hookrightarrow \mathcal{X}$, $K(\mathcal{X}$ on $\mathcal{Z})$ is homotopy equivalent to the K-theory spectra of the complicial bi-Waldhausen category of perfect complexes of quasicoherent $\mathcal{O}_{\mathcal{X}}$-modules which are acyclic on $\mathcal{X} \setminus \mathcal{Z}$.

Proof. For a stack $\mathcal{X}$ with affine diagonal, by [Lurie 2005, Theorem 3.8] the inclusion functors $\Phi : \mathrm{Ch}(\mathrm{QC}(\mathcal{X})) \to \mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ and $\Phi_{\mathcal{Z}} : \mathrm{Ch}_{\mathcal{Z}}(\mathrm{QC}(\mathcal{X})) \to \mathrm{Ch}_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$ induce equivalences of their left bounded derived categories. Therefore, they restrict to the equivalences of the derived homotopy categories of the bi-Waldhausen categories of perfect complexes of quasicoherent $\mathcal{O}_{\mathcal{X}}$-modules and that of perfect complexes in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$, both with support in $\mathcal{Z}$ in the case of $\Phi_{\mathcal{Z}}$. By [Thomason and Trobaugh 1990, Theorem 1.9.8], these inclusions therefore induce homotopy equivalence of their K-theory spectra. □

Lemma 3.2. *Let $\mathcal{X}$ be a quotient stack with the resolution property. Consider the following list of complicial bi-Waldhausen categories*:

(1) *bounded complexes of vector bundles on $\mathcal{X}$*,

(2) *perfect complexes in* $\mathrm{Ch}(\mathrm{QC}(\mathcal{X}))$,

(3) *perfect complexes in* $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$.

Then the obvious inclusion functors induce homotopy equivalences of all their K-theory spectra. Furthermore, $K(\mathcal{X})$ *is homotopy equivalent to the algebraic K-theory spectrum of the exact category of vector bundles on* $\mathcal{X}$.

Proof. The inclusion of (1) in (2) induces a homotopy equivalence of K-theory spectra by Proposition 2.6 and [Thomason and Trobaugh 1990, Theorem 1.9.8]. The inclusion of (2) in (3) induces homotopy equivalence of K-theory spectra by Lemma 3.1. The last assertion follows from [Thomason and Trobaugh 1990, Theorem 1.11.7]. □

3B. *The localization and excision for K-theory.* We now establish the localization sequence and excision for the K-theory of nice quotient stacks. We begin with the following localization at the level of $D_{\mathrm{qc}}(\mathcal{X})$.

Proposition 3.3. *Let* $\mathcal{X}$ *be a nice quotient stack and let* $\mathcal{Z} \hookrightarrow X$ *be a closed substack with open complement* $j : \mathcal{U} \hookrightarrow \mathcal{X}$. *Assume that* $\mathcal{X}$ *has the resolution property. Then the following hold.*

(1) $D_{\mathrm{qc}}(\mathcal{X})$, $D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$ *and* $D_{\mathrm{qc}}(\mathcal{U})$ *are compactly generated.*

(2) *The functor*
$$j^* : \frac{D_{\mathrm{qc}}(\mathcal{X})}{D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})} \to D_{\mathrm{qc}}(\mathcal{U})$$
is an equivalence of triangulated categories.

Proof. The stack $\mathcal{U}$ has the resolution property by our assumption and [Gross 2017, Theorem A]. It follows from Proposition 2.7 that every perfect complex in $D_{\mathrm{qc}}(\mathcal{X})$ is compact, i.e., $\mathcal{X}$ is *concentrated*. Since $\mathcal{X}$ and $\mathcal{U}$ have affine diagonal with resolution property, it follows from [Hall and Rydh 2017, Proposition 8.4] that $D_{\mathrm{qc}}(\mathcal{X})$, $D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$ and $D_{\mathrm{qc}}(\mathcal{U})$ are compactly generated.

The second statement is an easy consequence of adjointness of the functors $(j^*, \boldsymbol{R}j_*)$ and works exactly like the case of schemes. One checks easily that j^* is fully faithful and $j^* \circ \boldsymbol{R}j_*$ is the identity on $D_{\mathrm{qc}}(\mathcal{U})$. □

Theorem 3.4 (localization sequence). *Let* $\mathcal{X}$ *be a nice quotient stack and let* $\mathcal{Z} \hookrightarrow X$ *be a closed substack with open complement* $j : \mathcal{U} \hookrightarrow \mathcal{X}$. *Assume that* $\mathcal{X}$ *has the resolution property. Then the morphism of spectra* $K(\mathcal{X} \text{ on } \mathcal{Z}) \to K(\mathcal{X}) \to K(\mathcal{U})$ *induce a long exact sequence*

$$\cdots \to K_i(\mathcal{X} \text{ on } \mathcal{Z}) \to K_i(\mathcal{X}) \to K_i(\mathcal{U}) \to K_{i-1}(\mathcal{X} \text{ on } \mathcal{Z})$$
$$\to \cdots \to K_0(\mathcal{X} \text{ on } \mathcal{Z}) \to K_0(\mathcal{X}) \to K_0(\mathcal{U}).$$

Proof. It follows from Proposition 2.7, Lemma 2.8 and Proposition 3.3 that there is a commutative diagram of triangulated categories

$$\begin{tikzcd} D^{\mathrm{perf}}_{\mathcal{Z}}(\mathcal{X}) \arrow[r] \arrow[d, hook] & D^{\mathrm{perf}}(\mathcal{X}) \arrow[r] \arrow[d, hook] & D^{\mathrm{perf}}(\mathcal{U}) \arrow[d, hook] \\ D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X}) \arrow[r] & D_{\mathrm{qc}}(\mathcal{X}) \arrow[r] & D_{\mathrm{qc}}(\mathcal{U}) \end{tikzcd} \tag{3.5}$$

where the bottom row is a localization sequence of triangulated categories and the top row is the sequence of full subcategories of compact objects of the corresponding categories in the bottom row. Moreover, each triangulated category in the bottom row is generated by its compact objects in the top row. We can thus apply [Neeman 1992, Theorem 2.1] to conclude that the functor

$$\frac{D^{\mathrm{perf}}(\mathcal{X})}{D^{\mathrm{perf}}_{\mathcal{Z}}(\mathcal{X})} \to D^{\mathrm{perf}}(\mathcal{U}) \tag{3.6}$$

is fully faithful, and an equivalence up to direct factors.

Let Σ be the category whose objects are perfect complexes in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$, and where a map $x \to y$ is a weak equivalence if the restriction $x|_{\mathcal{U}} \to y|_{\mathcal{U}}$ is a quasi-isomorphism in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{U})$. The cofibrations in Σ are degreewise split monomorphisms. Then it is easy to see that Σ is a complicial bi-Waldhausen model for the quotient category $D^{\mathrm{perf}}(\mathcal{X})/D^{\mathrm{perf}}_{\mathcal{Z}}(\mathcal{X})$. Thus, by the Waldhausen localization theorem [Thomason and Trobaugh 1990, Theorems 1.8.2, 1.9.8], there is a homotopy fibration of spectra $K(\mathcal{X} \text{ on } \mathcal{Z}) \to K(\mathcal{X}) \to K(\Sigma)$. It follows from (3.6) and [Neeman 1992, Lemma 0.6] that $K(\Sigma) \to K(\mathcal{U})$ is a covering map of spectra. In particular, $K_i(\Sigma) \xrightarrow{\simeq} K_i(\mathcal{U})$ for $i \geq 1$ and $K_0(\Sigma) \hookrightarrow K_0(\mathcal{U})$. □

Theorem 3.7 (excision). *Let $\mathcal{X}$ be a nice quotient stack and let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack. Let $f : \mathcal{Y} \to \mathcal{X}$ be a strongly representable étale morphism of stacks such that $f : \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Z}$ induces an isomorphism of the associated reduced stacks. Assume that $\mathcal{X}$, $\mathcal{Y}$ have the resolution property. Then f^* induces a homotopy equivalence*

$$f^* : K(\mathcal{X} \text{ on } \mathcal{Z}) \xrightarrow{\simeq} K(\mathcal{Y} \text{ on } \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y}).$$

Proof. We observe that since f is strongly representable, $\mathcal{Y}$ is also a nice quotient stack. The theorem now follows directly from Lemma 2.8 and Proposition 2.13. □

3C. *Projective bundle formula.* In order to define the nonconnective K-theory of stacks, we need the projective bundle formula for their K-theory. This formula for the equivariant K-theory was proven in [Thomason 1993a, Theorem 3.1]. We adapt the argument of Thomason to extend it to the K-theory of all stacks. Though this formula is used in this text only for quotient stacks, its most general form plays a crucial role in [Hoyois and Krishna 2017]. For details on the projective bundles over algebraic stacks, see [Laumon and Moret-Bailly 2000, Chapter 14].

Theorem 3.8. *Let $\mathcal{X}$ be a stack, $\mathcal{E}$ a vector bundle of rank d and $p : \mathbb{P}\mathcal{E} \to \mathcal{X}$ the projective bundle associated to it. Let $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ be the fundamental invertible sheaf on $\mathbb{P}\mathcal{E}$ and $\mathcal{O}_{\mathbb{P}\mathcal{E}}(i)$ its i-th power in the group of invertible sheaves over $\mathcal{X}$.*

Then the morphism of K-theory spectra induced by the exact functor that sends a sequence of d perfect complexes in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$, $(E_0, \dots, E_{d-1})$ to the perfect complex

$$p^* E_0 \oplus \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \otimes p^* E_1 \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}\mathcal{E}}(1-d) \otimes p^* E_{d-1}$$

induces a homotopy equivalence

$$\Phi : \prod_d K(\mathcal{X}) \xrightarrow{\sim} K(\mathbb{P}\mathcal{E}).$$

Similarly, for each closed substack $\mathcal{Z}$, the exact functor restricts to the subcategory of complexes acyclic on $\mathcal{X} \setminus \mathcal{Z}$ to give a homotopy equivalence

$$\Phi : \prod_d K(\mathcal{X} \text{ on } \mathcal{Z}) \xrightarrow{\sim} K(\mathbb{P}\mathcal{E} \text{ on } \mathbb{P}(\mathcal{E}|\mathcal{Z})).$$

We need the following steps to prove this theorem.

Lemma 3.9. *Under the hypothesis of Theorem 3.8, let F be a perfect complex in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ or in general a complex with quasicoherent and bounded cohomology. Then the canonical adjunction morphism (3.10) is a quasi-isomorphism*:

$$\eta : F \xrightarrow{\sim} \boldsymbol{R}p_* p^* F = \boldsymbol{R}p_*(\mathcal{O}_{\mathbb{P}\mathcal{E}} \otimes p^* F). \tag{3.10}$$

In addition, for $j = 1, 2, \dots, d-1$, we have as a result of cancellation

$$\boldsymbol{R}p_*(\mathcal{O}_{\mathbb{P}\mathcal{E}}(-j) \otimes p^* F) \simeq 0. \tag{3.11}$$

Proof. The assertion of the lemma is fppf local on $\mathcal{X}$. Let $u : U \to \mathcal{X}$ be a smooth atlas for $\mathcal{X}$, where U is a scheme. Since $p : \mathbb{P}\mathcal{E} \to \mathcal{X}$ is strongly representable, we can apply [Hall and Rydh 2017, Lemma 2.5(3), Corollary 4.13] to reduce to the case when $\mathcal{X} \in \mathbf{Sch}_k$. In this latter case, the lemma is proven in [Thomason 1993a, Lemma 3]. □

Lemma 3.12. *Under the hypothesis of Theorem 3.8, if E is a perfect complex in $\mathrm{Ch}_{\mathrm{qc}}(\mathbb{P}\mathcal{E})$, then the following hold.*

(1) *$\boldsymbol{R}p_*(E)$ is a perfect complex in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$.*

(2) *If $\boldsymbol{R}p_*(E \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}}(i))$ is acyclic on $\mathcal{X}$ for $i = 0, 1, \dots, d-1$, then E is acyclic on $\mathbb{P}\mathcal{E}$.*

Proof. Since the assertion is fppf local on $\mathcal{X}$ and the perfectness is checked by base change of $\mathcal{X}$ by smooth morphisms from affine schemes, we can use [Hall and Rydh 2017, Lemma 2.5(3), Corollary 4.13] again to replace $\mathcal{X}$ by a scheme. Part (1) then follows from [Thomason 1993a, Lemma 4] and (2) follows from [Thomason 1993a, Lemma 5]. $\square$

Proof of Theorem 3.8. The proof follows exactly along the lines of the proof of [Thomason 1993a, Theorem 1], using Lemmas 3.9 and 3.12, which generalize [Thomason 1993a, Lemmas 3, 4, 5] to stacks. $\square$

3D. *K-theory of regular blow-ups of stacks.* A closed immersion $\mathcal{Y} \to \mathcal{X}$ of stacks over k is defined to be a *regular immersion* of codimension d if there exists a smooth atlas $U \to \mathcal{X}$ of $\mathcal{X}$ such that $\mathcal{Y} \times_{\mathcal{X}} U \to U$ is a regular immersion of schemes of codimension d. This is well defined as U is Noetherian and regular immersions behave well under flat base change and satisfy fpqc descent. For a closed immersion $i : \mathcal{Y} \to \mathcal{X}$, the blow-up of $\mathcal{X}$ along $\mathcal{Y}$ is defined to be $p : \widetilde{\mathcal{X}} = \operatorname{Proj}\big(\bigoplus_{n\geq 0} \mathcal{I}^n_{\mathcal{Y}}\big) \to \mathcal{X}$. See [Laumon and Moret-Bailly 2000, Chapter 14] for relative proj construction on stacks. Note that in the case of a regular immersion, $\widetilde{\mathcal{X}} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ is a projective bundle over $\mathcal{Y}$, similar to schemes.

Theorem 3.13. *Let $i : \mathcal{Y} \to \mathcal{X}$ be a regular immersion of codimension d of stacks. Let $p : \mathcal{X}' \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ along $\mathcal{Y}$ and $j : \mathcal{Y}' = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}' \to \mathcal{X}'$, $q : \mathcal{Y}' \to \mathcal{Y}$ be the maps obtained by base change. Then the square*

$$\begin{tikzcd} K(\mathcal{X}) \arrow[r, "i^*"] \arrow[d, "p^*"'] & K(\mathcal{Y}) \arrow[d, "q^*"] \\ K(\mathcal{X}') \arrow[r, "j^*"'] & K(\mathcal{Y}') \end{tikzcd} \tag{3.14}$$

is homotopy Cartesian.

Proof. This is proved in [Cortiñas et al. 2008, Proposition 1.5] in the case of schemes and an identical proof works for the case of stacks, in the presence of the results of Section 3C and Lemma 3.16. We give some details on the strategy of the proof. For $r = 0, \ldots, d-1$, let $D^{\mathrm{perf}}_r(\mathcal{X}') \subset D^{\mathrm{perf}}(\mathcal{X}')$ be the full triangulated subcategory generated by Lp^*F and $Rj_*Lq^*G \otimes \mathcal{O}_{\mathcal{X}'}(-l)$ for $F \in D^{\mathrm{perf}}(\mathcal{X})$, $G \in D^{\mathrm{perf}}(\mathcal{Y})$ and $l = 1, \ldots, r$. Let $D^{\mathrm{perf}}_r(\mathcal{Y}') \subset D^{\mathrm{perf}}(\mathcal{Y}')$ be the full triangulated subcategory generated by $Lq^*G \otimes \mathcal{O}_{\mathcal{Y}'}(-l)$ for $G \in D^{\mathrm{perf}}(\mathcal{Y})$ and $l = 0, \ldots, r$. By Lemmas 3.9 and 3.16(1), $Lp^* : D^{\mathrm{perf}}(\mathcal{X}) \to D^{\mathrm{perf}}_0(\mathcal{X}')$ and $Lq^* : D^{\mathrm{perf}}(\mathcal{Y}) \to D^{\mathrm{perf}}_0(\mathcal{Y}')$ are equivalences. Exactly as in [Cortiñas et al. 2008, Lemma 1.2], one shows that $D^{\mathrm{perf}}_{d-1}(\mathcal{X}') = D^{\mathrm{perf}}(\mathcal{X}')$ and $D^{\mathrm{perf}}_{d-1}(\mathcal{Y}') = D^{\mathrm{perf}}(\mathcal{Y}')$ using Lemmas 3.12 and 3.16.

To prove the theorem, it is sufficient to show that Lj^* is compatible with the filtrations on $D^{\mathrm{perf}}(\mathcal{X}')$ and $D^{\mathrm{perf}}(\mathcal{Y}')$:

$$\begin{array}{ccccccccc} D^{\mathrm{perf}}(\mathcal{X}) & \xrightarrow[\sim]{Lp^*} & D_0^{\mathrm{perf}}(\mathcal{X}') & \hookrightarrow & D_1^{\mathrm{perf}}(\mathcal{X}') & \hookrightarrow \cdots \hookrightarrow & D_{d-1}^{\mathrm{perf}}(\mathcal{X}') & = & D^{\mathrm{perf}}(\mathcal{X}') \\ \downarrow {\scriptstyle Li^*} & & \downarrow {\scriptstyle Lj^*} & & \downarrow {\scriptstyle Lj^*} & & & & \downarrow {\scriptstyle Lj^*} \\ D^{\mathrm{perf}}(\mathcal{Y}) & \xrightarrow[\sim]{Lq^*} & D_0^{\mathrm{perf}}(\mathcal{Y}') & \hookrightarrow & D_1^{\mathrm{perf}}(\mathcal{Y}') & \hookrightarrow \cdots \hookrightarrow & D_{d-1}^{\mathrm{perf}}(\mathcal{Y}') & = & D^{\mathrm{perf}}(\mathcal{Y}') \end{array} \tag{3.15}$$

and that for $r = 0, \ldots, d-2$, Lj^* induces equivalences on quotient triangulated categories:

$$Lj^* : D_{r+1}^{\mathrm{perf}}(\mathcal{X}')/D_r^{\mathrm{perf}}(\mathcal{X}') \xrightarrow{\sim} D_{r+1}^{\mathrm{perf}}(\mathcal{Y}')/D_r^{\mathrm{perf}}(\mathcal{Y}').$$

Given this, it follows from [Thomason and Trobaugh 1990, Theorems 1.8.2, 1.9.8] that every square in (3.15) induces a homotopy Cartesian square of K-theory spectra.

To prove the compatibility of Lj^*, it is enough to check on generators and in this case, it can be reduced to the case of schemes using [Hall and Rydh 2017, Corollary 4.13]. To prove that Lj^* induces equivalence on quotients, we first note that the composition

$$\begin{aligned} Lj^* \circ [\mathcal{O}_{\mathcal{X}'}(-r-1) \otimes Rj_* Lq^*] : D^{\mathrm{perf}}(\mathcal{Y}) \to D_{r+1}^{\mathrm{perf}}(\mathcal{X}')/D_r^{\mathrm{perf}}(\mathcal{X}') \\ \to D_{r+1}^{\mathrm{perf}}(\mathcal{Y}')/D_r^{\mathrm{perf}}(\mathcal{Y}') \end{aligned}$$

agrees with $\mathcal{O}_{\mathcal{Y}'}(-r-1) \otimes Lq^* : D^{\mathrm{perf}}(\mathcal{Y}) \to D_{r+1}^{\mathrm{perf}}(\mathcal{Y}')/D_r^{\mathrm{perf}}(\mathcal{Y}')$, up to a natural equivalence. This follows as in the proof of [Cortiñas et al. 2008, Lemma 1.4] using [Hall and Rydh 2017, Corollary 4.13]. Therefore, it is enough to show that the functors $\mathcal{O}_{\mathcal{X}'}(-r-1) \otimes Rj_* Lq^*$ and $\mathcal{O}_{\mathcal{Y}'}(-r-1) \otimes Lq^*$ are equivalences. But the proof of this is exactly the same as the one in [Cortiñas et al. 2008, Proposition 1.5] for schemes. □

Lemma 3.16. *Under the hypotheses of Theorem 3.13, the following hold.*

(1) *Let F be a perfect complex on $\mathcal{X}$. Then the canonical adjunction morphism (3.17) is a quasi-isomorphism*:

$$\eta : F \xrightarrow{\sim} Rp_* Lp^* F = Rp_*(\mathcal{O}_{\mathcal{X}'} \otimes Lp^* F). \tag{3.17}$$

(2) *Let r be an integer such that $1 \le r \le d-1$. Let $\mathcal{A}'_r$ denote the full triangulated subcategory of $D^{\mathrm{perf}}(\mathcal{X}')$ of those complexes E for which $Rp_*(E \otimes \mathcal{O}_{\mathcal{X}'}(i)) \simeq 0$ for $0 \le i < r$. Then there exists a natural transformation ∂ of functors from $\mathcal{A}'_r$ to $D^{\mathrm{perf}}(\mathcal{X}')$*:

$$\partial : (Rj_* Lq^* Rq_*(E \otimes_{\mathcal{O}_{\mathcal{X}'}} \mathcal{O}_{\mathcal{Y}'}(r-1)) \otimes \mathcal{O}_{\mathcal{X}'}(-r))[-1] \to E. \tag{3.18}$$

Moreover, $Rp_*(\partial \otimes \mathcal{O}_{\mathcal{X}'}(i))$ is a quasi-isomorphism for $0 \le i < r+1$.

(3) *Suppose $E \in D^{\mathrm{perf}}(\mathcal{X}')$ is such that $Rp_*(E \otimes \mathcal{O}_{\mathcal{X}'}(i))$ is acyclic on $\mathcal{X}$ for $i = 0, \ldots, d-1$. Then E is acyclic on $\mathcal{X}'$.*

Proof. Statements (1) and (3) are proved in [Thomason 1993b] for schemes. The general case can be deduced from this exactly as in Lemmas 3.9 and 3.12. For (2), the existence of ∂ follows from [Thomason 1993b, Lemma 2.4(a)] as the construction of ∂ given there is natural in $\mathcal{X}$ for schemes. To check that $Rp_*(\partial \otimes \mathcal{O}_{\mathcal{X}'}(i))$ is a quasi-isomorphism for $0 \le i < r+1$, we may again assume that $\mathcal{X}$ is a scheme, and this case follows from [loc. cit., Lemma 2.4(a)]. $\square$

3E. ***Negative K-theory of stacks.*** Let $\mathcal{U} \hookrightarrow \mathcal{X}$ be an open immersion of stacks over k. As $K_0(\mathcal{X}) \to K_0(\mathcal{U})$ is not always surjective in the localization theorem, we want to introduce a nonconnective spectrum $\mathbb{K}(-)$ with $K(-)$ as its (-1)-connective cover, so that $\mathbb{K}(\mathcal{X} \text{ on } \mathcal{Z}) \to \mathbb{K}(\mathcal{X}) \to \mathbb{K}(\mathcal{U})$ is a homotopy fiber sequence for any closed substack $\mathcal{Z}$ of $\mathcal{X}$ with complement $\mathcal{U}$. We define $\mathbb{K}$ only in the absolute case below. The construction of $\mathbb{K}(\mathcal{X} \text{ on } \mathcal{Z})$ follows similarly, as shown in [Thomason and Trobaugh 1990]. We shall use the following version of the Bass fundamental theorem for stacks to define $\mathbb{K}(\mathcal{X})$. The homotopy groups of $\mathbb{K}(\mathcal{X})$ will be denoted by $\mathbb{K}_i(\mathcal{X})$.

Theorem 3.19 (Bass fundamental theorem). *Let $\mathcal{X}$ be a nice quotient stack with the resolution property and let $\mathcal{X}[T] = \mathcal{X} \times \operatorname{Spec}(k[T])$. Then the following hold.*

(1) *For $n \ge 1$, there is an exact sequence*

$$0 \to K_n(\mathcal{X}) \xrightarrow{(p_1^*, -p_2^*)} K_n(\mathcal{X}[T]) \oplus K_n(\mathcal{X}[T^{-1}]) \xrightarrow{(j_1^*, j_2^*)} K_n(\mathcal{X}[T, T^{-1}]) \xrightarrow{\partial_T} K_{n-1}(\mathcal{X}) \to 0.$$

Here p_1^*, p_2^* are induced by the projections $\mathcal{X}[T] \to \mathcal{X}$, etc., and j_1^*, j_2^* are induced by the open immersions $\mathcal{X}[T^{\pm 1}] = \mathcal{X}[T, T^{-1}] \to \mathcal{X}[T]$, etc. The sum of these exact sequences for $n = 1, 2, \dots$ is an exact sequence of graded $K_*(\mathcal{X})$-modules.

(2) *For $n \ge 0$, $\partial_T : K_{n+1}(\mathcal{X}[T^{\pm 1}]) \to K_n(\mathcal{X})$ is naturally split by a map h_T of $K_*(\mathcal{X})$-modules. Indeed, the cup product with $T \in K_1(k[T^{\pm 1}])$ splits ∂_T up to a natural automorphism of $K_n(\mathcal{X})$.*

(3) *There is an exact sequence for $n = 0$:*

$$0 \to K_n(\mathcal{X}) \xrightarrow{(p_1^*, -p_2^*)} K_n(\mathcal{X}[T]) \oplus K_n(\mathcal{X}[T^{-1}]) \xrightarrow{(j_1^*, j_2^*)} K_n(\mathcal{X}[T^{\pm 1}]).$$

Proof. It follows from [Thomason 1987b, Lemma 2.6] that $\mathbb{P}^1_{\mathcal{X}}$ and $\mathcal{X}[T]$ are nice quotient stacks with the resolution property. It follows from Theorem 3.8 that there is an isomorphism $K_*(\mathbb{P}^1_{\mathcal{X}}) \simeq K_*(\mathcal{X}) \oplus K_*(\mathcal{X})$, where the two summands are $K_*(\mathcal{X})[\mathcal{O}]$ and $K_*(\mathcal{X})[\mathcal{O}(-1)]$ with respect to the external product $K(\mathcal{X}) \wedge K(\mathbb{P}^1_k) \to K(\mathbb{P}^1_{\mathcal{X}})$ and with $[\mathcal{O}], [\mathcal{O}(-1)] \in K_0(\mathbb{P}^1_k)$. As for schemes, (1) now follows directly from Theorems 3.4 and 3.7; see also [Thomason and Trobaugh 1990, Theorem 6.1].

For (2), it suffices to show that the composite map

$$\partial_T(T \cup p^*(-)) : K_n(\mathcal{X}) \to K_{n+1}(\mathcal{X}[T^{\pm 1}]) \to K_n(\mathcal{X})$$

is an automorphism of $K_n(\mathcal{X})$ for $n \geq 0$. By naturality and the fact that ∂_T is a map of $K_*(\mathcal{X})$-modules, this reduces to showing that $\partial_T : K_1(k[T^{\pm 1}]) \to K_0(k)$ sends T to ± 1. But this is well known and (3) follows from (2) using the analogue of [Thomason and Trobaugh 1990, (6.1.5)] for stacks. □

Theorem 3.20. *For a nice quotient stack $\mathcal{X}$ with the resolution property, there is a spectrum $\mathbb{K}(\mathcal{X})$ together with a natural map of spectra $K(\mathcal{X}) \to \mathbb{K}(\mathcal{X})$ which induces isomorphism $K_i(\mathcal{X}) \xrightarrow{\simeq} \mathbb{K}_i(\mathcal{X})$ for $i \geq 0$.*

Let $\mathcal{Y}$ be a nice quotient stack with the resolution property and let $f : \mathcal{Y} \to \mathcal{X}$ be a strongly representable étale map. Let $\mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed substack such that $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Z}$ induces an isomorphism of the associated reduced stacks. Let $\pi : \mathbb{P}(\mathcal{E}) \to \mathcal{X}$ be the projective bundle associated to a vector bundle $\mathcal{E}$ on $\mathcal{X}$ of rank r. Then the following hold.

(1) *There is a homotopy fiber sequence of spectra*

$$\mathbb{K}(\mathcal{X} \text{ on } \mathcal{Z}) \to \mathbb{K}(\mathcal{X}) \to \mathbb{K}(\mathcal{X} \setminus \mathcal{Z}).$$

(2) *The map $f^* : \mathbb{K}(\mathcal{X} \text{ on } \mathcal{Z}) \to \mathbb{K}(\mathcal{Y} \text{ on } \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y})$ is a homotopy equivalence.*

(3) *The map $\prod_0^{r-1} \mathbb{K}(\mathcal{X}) \to \mathbb{K}(\mathbb{P}(\mathcal{E}))$, $(a_0, \dots, a_{r-1}) \mapsto \sum_i \mathcal{O}[-i] \otimes \pi^*(a_i)$, is a homotopy equivalence.*

Proof. The construction of the spectrum $\mathbb{K}(\mathcal{X})$ follows directly from Theorem 3.19 by the formalism given in (6.2)–(6.4) of [Thomason and Trobaugh 1990]. Like for schemes, the proof of (1), (2) and (3) is a standard deduction from Theorems 3.4, 3.7 and 3.8, using the inductive construction of $\mathbb{K}(\mathcal{X})$. □

3F. *Schlichting's negative K-theory.* Schlichting [2006] defined negative K-theory of complicial bi-Waldhausen categories. Let $\mathcal{X}$ be a nice quotient stack. Schlichting's negative K-theory spectrum $K^{\mathrm{Scl}}(\mathcal{X})$ is the K-theory spectrum of the Frobenius pair associated to the category $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$. It follows from [Schlichting 2006, Theorem 8] that $K_i^{\mathrm{Scl}}(\mathcal{X}) = K_i(\mathcal{X})$ for $i \geq 0$. The following result shows that $K_i^{\mathrm{Scl}}(\mathcal{X})$ agrees with $\mathbb{K}_i(\mathcal{X})$ for $i < 0$.

Theorem 3.21. *Let $\mathcal{X}$ be a nice quotient stack with the resolution property. Then there are natural isomorphisms between $K_i^{\mathrm{Scl}}(\mathcal{X})$ and $\mathbb{K}_i(\mathcal{X})$ for $i \leq 0$.*

Proof. Let $p : \mathbb{P}^1_{\mathcal{X}} \to \mathcal{X}$ be the projection map. Then we can prove as in Theorem 3.8 that the functors $p^* : D^{\mathrm{perf}}(\mathcal{X}) \to D^{\mathrm{perf}}(\mathbb{P}^1_{\mathcal{X}})$ and $\mathcal{O}(-1) \otimes p^* : D^{\mathrm{perf}}(\mathcal{X}) \to D^{\mathrm{perf}}(\mathbb{P}^1_{\mathcal{X}})$, which are induced by maps of their Frobenius models, induce isomorphisms

$$(p^*, \mathcal{O}(-1) \otimes p^*) : K_i^{\mathrm{Scl}}(\mathcal{X}) \oplus K_i^{\mathrm{Scl}}(\mathcal{X}) \xrightarrow{\simeq} K_i^{\mathrm{Scl}}(\mathbb{P}^1_{\mathcal{X}})$$

for $i \le 0$. It follows from the proof of Bass' fundamental theorem in [Thomason and Trobaugh 1990, Theorem 6.6(b)] that there is an exact sequence of abelian groups

$$0 \to K_i^{\mathrm{Scl}}(\mathcal{X}) \to K_i^{\mathrm{Scl}}(\mathcal{X}[T]) \oplus K_i^{\mathrm{Scl}}(\mathcal{X}[T^{-1}]) \to K_i^{\mathrm{Scl}}(\mathcal{X}[T, T^{-1}]) \to K_{i-1}^{\mathrm{Scl}}(\mathcal{X}) \to 0$$

for $i \le 0$. As $K_0^{\mathrm{Scl}}(\mathcal{Y}) = \mathbb{K}_0(\mathcal{Y})$ for any stack $\mathcal{Y}$, the negative K-groups coincide. $\square$

4. Nisnevich descent for *K*-theory of quotient stacks

In this section, we prove Nisnevich descent in a 2-category of stacks whose objects are all quotients of schemes by action of a fixed group scheme. So let G be a group scheme over k. Let $\mathbf{Sch}_k^G$ denote the category of separated schemes of finite type over k with G-action. The equivariant Nisnevich topology on $\mathbf{Sch}_k^G$ and the homotopy theory of simplicial sheaves in this topology was defined and studied in detail in [Heller et al. 2015]. As an application of Theorem 3.20, we shall show in this section that the K-theory of quotient stacks for G-actions satisfies descent in the equivariant Nisnevich topology on $\mathbf{Sch}_k^G$.

Definition 4.1 [Heller et al. 2015, Definition 2.1]. A *distinguished equivariant Nisnevich square* is a Cartesian square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ A \arrow[r, hook, "j"] & X \end{tikzcd} \tag{4.2}$$

in $\mathbf{Sch}_k^G$ such that

(1) j is an open immersion,

(2) p is étale, and

(3) the induced map $(Y \setminus B)_{\mathrm{red}} \to (X \setminus A)_{\mathrm{red}}$ of schemes (without reference to the G-action) is an isomorphism.

Remark 4.3. We remark here that given a Cartesian square of the type (4.2) in $\mathbf{Sch}_k^G$, the closed subscheme $(X \setminus A)_{\mathrm{red}}$ (or $(Y \setminus B)_{\mathrm{red}}$) may not in general be G-invariant, unless G is smooth. However, it follows from [Thomason 1987a, Lemma 2.5] that we can always find a G-invariant closed subscheme $Z \subset X$ such that $Z_{\mathrm{red}} = X \setminus A$. This closed subscheme can be assumed to be reduced if G is smooth. Using the elementary fact that a morphism of schemes is étale if and only if the induced map of the associated reduced schemes is étale, it follows immediately that the condition (3) in Definition 4.1 is equivalent to

(3′) there is a G-invariant closed subscheme $Z \subset X$ with support $X \setminus A$ such that the map $Z \times_X Y \to Z$ in $\mathbf{Sch}^G_k$ is an isomorphism.

The collection of distinguished equivariant Nisnevich squares forms a cd-structure in the sense of [Voevodsky 2010]. The associated Grothendieck topology is called the equivariant *Nisnevich* topology. It is also called the *eN*-topology. It follows from [Heller et al. 2015, Theorem 2.3] that the equivariant Nisnevich cd-structure on $\mathbf{Sch}^G_k$ is complete, regular, and bounded. We refer to [Voevodsky 2010, §2] for the definition of a complete, regular, and bounded cd-structure.

Let $\mathbf{Sch}^G_{k/\mathrm{Nis}}$ denote the category of G-schemes X, such that X admits a family of G-equivariant ample line bundles, equipped with the equivariant Nisnevich topology. Note that all objects of $\mathbf{Sch}^G_{k/\mathrm{Nis}}$ have the resolution property by Lemma 2.3. It follows from [Heller et al. 2015, Corollary 2.11] that for a sheaf $\mathcal{F}$ of abelian groups on $\mathbf{Sch}^G_{k/\mathrm{Nis}}$, we have $H^i_{G/\mathrm{Nis}}(X, \mathcal{F}) = 0$ for $i > \dim(X)$.

Definition 4.4. An equivariant morphism $Y \to X$ in $\mathbf{Sch}^G_k$ *splits* if there is a filtration of X by G-invariant closed subschemes

$$\varnothing = X_{n+1} \subsetneq X_n \subsetneq \cdots \subsetneq X_0 = X, \tag{4.5}$$

such that for each j, the map $(X_j \setminus X_{j+1}) \times_X Y \to X_j \setminus X_{j+1}$ has a G-equivariant section. If f is étale and surjective, the morphism is called an equivariant *split étale cover of* X.

4A. ***Equivariant Nisnevich covers.*** In [Heller et al. 2015, Proposition 2.15], it is shown that an equivariant étale morphism $Y \to X$ in $\mathbf{Sch}^G_k$ is an equivariant Nisnevich cover if and only if it splits. Further, when G is a finite constant group scheme, it is shown that an equivariant étale map $f : Y \to X$ in $\mathbf{Sch}^G_k$ is an equivariant Nisnevich cover if and only if for any point $x \in X$, there is a point $y \in Y$ such that $f(y) = x$ and f induces isomorphisms $k(x) \simeq k(y)$ and $S_y \simeq S_x$. Here, for a point $x \in X$, the set-theoretic stabilizer $S_x \subseteq G$ is defined by $S_x = \{g \in G \mid g.x = x\}$ [Heller et al. 2015, Proposition 2.17].

Let G^0 denote the connected component of the identity element in G. Suppose that G is of the form $G = \coprod_{i=0}^r g_i G^0$, where $\{e = g_0, g_1, \ldots, g_r\}$ are points in $G(k)$ which represent the left cosets of G^0. In the next proposition, we give an explicit description of the equivariant Nisnevich covers of reduced schemes $X \in \mathbf{Sch}^G_k$. For $x \in X$, let $\widetilde{S}_x := \{g_i \mid 0 \le i \le r,\ g_i.x = x\}$.

Proposition 4.6. *Let G be a smooth affine group scheme over k as above. A morphism $f : Y \to X$ in $\mathbf{Sch}^G_k$ is an equivariant split étale cover of a reduced scheme X if and only if for any point $x \in X$, there is a point $y \in Y$ such that $f(y) = x$ and f induces isomorphisms $k(x) \simeq k(y)$ and $\widetilde{S}_y \simeq \widetilde{S}_x$.*

Proof. It is clear that a split étale G-equivariant family of morphisms satisfies the given conditions. The heart of the proof is to show the converse.

Suppose $Y \xrightarrow{f} X$ is such that for any point $x \in X$, there is a point $y \in Y$ such that $f(y) = x$ and f induces isomorphisms $k(x) \simeq k(y)$ and $\widetilde{S}_y \simeq \widetilde{S}_x$. Let W be the regular locus of X. Then W is a G-invariant dense open subscheme of X. Set $U = Y \times_X W$. Notice that W is a disjoint union of its irreducible components, and each f_U being étale, it follows that U is regular and hence a disjoint union of its irreducible components.

Let $x \in W$ be a generic point of W. Then the closure $W_x = \overline{\{x\}}$ in W is an irreducible component of W. By our assumption, there is a point $y \in U$ such that

$$f(y) = x, \qquad k_x \xrightarrow{\simeq} k_y, \quad \text{and} \quad \widetilde{S}_y \xrightarrow{\simeq} \widetilde{S}_x. \tag{4.7}$$

Then the closure $U_y = \overline{\{y\}}$ in U is an irreducible component of U. Since $U_y \to W_x$ is étale and generically an isomorphism, it must be an open immersion. Thus f maps U_y isomorphically onto an open subset of W_x. We replace W_x by this open subset $f(U_y)$ and call it our new W_x.

Let GU_y be the image of the action morphism $\mu : G \times U_y \to U$. Notice that μ is a smooth map and hence open. This in particular implies that GU_y is a G-invariant open subscheme of U as U_y is one of the disjoint irreducible components of U and hence open. By the same reason, GW_x is a G-invariant open subscheme of W.

Since the identity component G^0 is connected, it keeps U_y invariant. Therefore, $y \in U$ is fixed by G^0 and hence G acts on this point via its quotient $\overline{G} = G/G^0$. Since each $g_j G^0$ takes U_y onto an irreducible component of U and since U has only finitely many irreducible components which are all disjoint, we see that $GU_y = U_0 \amalg U_1 \amalg \cdots \amalg U_n$ is a disjoint union of some irreducible components of U with $U_0 = U_y$. In particular, for each U_j, we have $U_j = g_{j_i} G^0 U_y = g_{j_i} U_y$ for some j_i.

Since f maps U_y isomorphically onto W_x, we conclude from the above that f maps each U_j isomorphically onto one and only one W_j such that $GW_x = f(GU_y) = W_0 \amalg W_1 \amalg \cdots \amalg W_m$ (with $m \le n$) is a disjoint union of open subsets of some irreducible components of W with $W_0 = W_x$. The morphism f will map the open subscheme GU_y isomorphically onto the open subscheme GW_x if and only if no two components of GU_y are mapped onto one component of GW_x. This is ensured by using the condition (4.7).

If two distinct components of GU_y are mapped onto one component of GW_x, we can (using the equivariance of f) apply automorphisms by the g_{j_i} and assume that one of these components is U_y. In particular, we can find $j \ge 1$ such that

$$W_x = f(U_y) = f(U_j) = f(g_{j_i} U_y) = g_{j_i} f(U_y) = g_{j_i} W_x. \tag{4.8}$$

But this implies that $g_{j_i} \in \widetilde{S}_x$ and $g_{j_i} \notin \widetilde{S}_y$. This violates the condition in (4.7) that $\widetilde{S}_y$ and $\widetilde{S}_x$ are isomorphic. We have thus shown that the morphism f has a

G-equivariant splitting over a nonempty G-invariant open subset GW_x. Letting X_1 be the complement of this open subset in X with reduced scheme structure, we see that X_1 is a proper G-invariant closed subscheme of X, and by restricting our cover to X_1, we get a cover for X_1 satisfying the given conditions. The proof of the proposition is now completed by the Noetherian induction. □

4B. *Equivariant Nisnevich descent.* It is shown in [Heller et al. 2015, §3] that the category of presheaves of S^1-spectra on $\mathbf{Sch}^G_{k/\mathrm{Nis}}$ (denoted by $\mathrm{Pres}(\mathbf{Sch}^G_{k/\mathrm{Nis}})$) is equipped with the global and local injective model structures. A morphism $f : \mathcal{E} \to \mathcal{E}'$ of presheaves of spectra is called a *global weak equivalence* if the map $\mathcal{E}(X) \to \mathcal{E}'(X)$ is a weak equivalence of S^1-spectra for every object $X \in \mathbf{Sch}^G_{k/\mathrm{Nis}}$. It is a global injective cofibration if $\mathcal{E}(X) \to \mathcal{E}'(X)$ is a cofibration of S^1-spectra for every object $X \in \mathbf{Sch}^G_{k/\mathrm{Nis}}$. The map f is called a *local weak equivalence* if it induces an isomorphism on the sheaves of stable homotopy groups of the presheaves of spectra in the *eN*-topology. A local (injective) cofibration is the same as a global injective cofibration.

A presheaf of spectra $\mathcal{E}$ on $\mathbf{Sch}^G_{k/\mathrm{Nis}}$ is said to satisfy the *equivariant Nisnevich descent* (*eN*-descent) if the fibrant replacement map $\mathcal{E} \to \mathcal{E}'$ in the local injective model structure of $\mathrm{Pres}(\mathbf{Sch}^G_{k/\mathrm{Nis}})$ is a global weak equivalence. Let $\mathcal{K}^G$ denote the presheaf of spectra on $\mathbf{Sch}^G_k$ which associates the spectrum $\mathbb{K}([X/G])$ to any $X \in \mathbf{Sch}^G_k$. As a consequence of Theorem 3.20, we obtain the following.

Theorem 4.9. *Let G be a nice group scheme over k. Then the presheaf of spectra $\mathcal{K}^G$ on $\mathbf{Sch}^G_{k/\mathrm{Nis}}$ satisfies the equivariant Nisnevich descent.*

Proof. Since the *eN*-topology is regular, complete and bounded by [Heller et al. 2015, Theorem 2.3], it suffices to show using [Voevodsky 2010, Proposition 3.8] that $\mathcal{K}^G$ takes a square of the type (4.2) to a homotopy Cartesian square of spectra. In other words, we need to show that the square

$$\begin{tikzcd} \mathbb{K}([X/G]) \arrow[r, "j^*"] \arrow[d, "p^*"'] & \mathbb{K}([A/G]) \arrow[d, "p'^*"] \\ \mathbb{K}([Y/G]) \arrow[r, "j'^*"'] & \mathbb{K}([B/G]) \end{tikzcd} \tag{4.10}$$

is homotopy Cartesian. But this is an immediate consequence of Theorem 3.20. □

Corollary 4.11. *Let G be a nice group scheme over k and let $X \in \mathbf{Sch}^G_{k/\mathrm{Nis}}$. Then there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H^p_{eN}(X, \mathcal{K}^G_q) \Rightarrow \mathbb{K}_{q-p}([X/G]).$$

Proof. This is immediate from Theorem 4.9 and [Heller et al. 2015, Theorem 2.3, Corollary 2.11]. □

5. Homotopy invariance of *K*-theory with coefficients for quotient stacks

It is known that with finite coefficients, the ordinary algebraic K-theory of schemes satisfies the homotopy invariance property (see [Weibel 1989, Theorem 1.2, Proposition 1.6] for affine schemes and [Thomason and Trobaugh 1990, Theorem 9.5] for the general case). This is a hard result which was achieved by first defining a homotopy invariant version of algebraic K-theory [Weibel 1989] and then showing that with finite coefficients, this homotopy (invariant) K-theory coincides with the algebraic K-theory.

However, the proof of the agreement between algebraic K-theory and homotopy K-theory with finite coefficients requires the knowledge of a spectral sequence relating NK-theory and homotopy K-theory; see [Weibel 1989, Remark 1.3.1]. Recall here that $NK(X)$ denotes the homotopy fiber of the pull-back map ι^*, where $\iota : X \hookrightarrow \mathbb{A}^1_k \times X$ denotes the 0-section embedding into the trivial line bundle over a scheme X. The existence of homotopy K-theory for quotient stacks is not yet known and one does not know if the above spectral sequence would exist for the homotopy K-theory of quotient stacks. In this section, we adopt a different strategy to extend the results of Weibel and Thomason–Trobaugh to the K-theory of nice quotient stacks (see Theorem 5.5).

5A. ***Homotopy K-theory of stacks.*** For $n \in \mathbb{N}$, let

$$\Delta_n = \operatorname{Spec}\bigl(k[t_0, \dots, t_n]/\bigl(\sum_i t_i - 1\bigr)\bigr).$$

Recall that $\Delta_\bullet = \{\Delta_n\}_{n\geq 0}$ forms a simplicial scheme whose face and degeneracy maps are given by the formulas

$$\partial_r(t_j) = \begin{cases} t_j & \text{if } j < r, \\ 0 & \text{if } j = r, \\ t_{j-1} & \text{if } j > r, \end{cases} \qquad \delta_r(t_j) = \begin{cases} t_j & \text{if } j < r, \\ t_j + t_{j+1} & \text{if } j = r, \\ t_{j+1} & \text{if } j > r. \end{cases}$$

Definition 5.1. For a nice quotient stack $\mathcal{X}$ with the resolution property, the *homotopy K-theory* is defined to be the spectrum

$$KH(\mathcal{X}) = \operatorname{hocolim}_n \mathbb{K}(\mathcal{X} \times \Delta_n).$$

It is clear from the definition that $KH(\mathcal{X})$ is contravariant with respect to morphisms of stacks. Furthermore, there is a natural map of spectra $\mathbb{K}(\mathcal{X}) \to KH(\mathcal{X})$. It is well known that $\mathbb{K}(\mathcal{X})$ is not a homotopy invariant functor. Our first result on $KH(\mathcal{X})$ is the following.

Theorem 5.2. *Let $\mathcal{X}$ be a nice quotient stack with the resolution property, and let $f : \mathcal{E} \to \mathcal{X}$ be a vector bundle morphism. Then the associated pull-back map $f^* : KH(\mathcal{X}) \to KH(\mathcal{E})$ is a homotopy equivalence.*

Proof. We first show that the map $KH(\mathcal{X}) \to KH(\mathcal{X} \times \Delta_n)$ is a homotopy equivalence for every $n \geq 0$. But this is essentially a direct consequence of the definition of KH-theory. By identifying Δ_n with $\mathbb{A}^n_k$ and using induction, one needs to show that the map $KH(\mathcal{X}) \to KH(\mathcal{X}[T])$ is a homotopy equivalence. Proof of this is identical to the case of the KH-theory of schemes [Weibel 1989, Theorem 1.2].

To prove the general case, we write $\mathcal{X} = [X/G]$, where G is a group scheme over k acting on a k-scheme X. We let $E = u^*(\mathcal{E})$, where $u : X \to \mathcal{X}$ is the quotient map. Then E is a G-equivariant vector bundle on X such that $[E/G] \simeq \mathcal{E}$.

We consider the standard fiberwise contraction map $H : \mathbb{A}^1_k \times E \to E$. Explicitly, for an open affine $U = \operatorname{Spec}(A) \subseteq X$ over which f is trivial (without G-action), $H|_U$ is induced by the k-algebra homomorphism $A[X_1, \ldots, X_n] \to A[X_1, \ldots, X_n, T]$ given by $X_j \mapsto TX_j$. It is then clear that this defines a unique map H as above which is G-equivariant for the trivial G-action on $\mathbb{A}^1_k$. We have the commutative diagram

$$\begin{tikzcd}
 & \{1\} \times E \arrow[dl, "\mathrm{id}"'] \arrow[d, "i_1"] \arrow[dr, "h_1"] & \\
E & \mathbb{A}^1_k \times E \arrow[l, "p"'] \arrow[r, "H"] & E \\
 & \{0\} \times E \arrow[ul, "\mathrm{id}"] \arrow[u, "i_0"'] \arrow[ur, "h_0"'] &
\end{tikzcd} \tag{5.3}$$

where $h_j = H \circ i_j$ for $j = 0, 1$ and p is the projection map.

Let $\iota : X \hookrightarrow E$ denote the 0-section embedding, so that $f \circ \iota = \mathrm{id}_X$. So we only need to show that $f^* \circ \iota^*$ is the identity on $KH([E/G])$. Since $h_0 = \iota \circ f$, it suffices to show that h_0^* is the identity.

It follows from the weaker version of homotopy invariance shown above (applied to E) that p^* is an isomorphism on the KH-theory of the stack quotients. In particular, $i_0^* = (p^*)^{-1} = i_1^*$. Since $h_1 = \mathrm{id}_E$, we get $i_1^* \circ H^* = \mathrm{id}$, which in turn yields $H^* = (i_1^*)^{-1} = p^*$ and hence $h_0^* = i_0^* \circ H^* = i_0^* \circ p^* = \mathrm{id}$. This finishes the proof. □

5B. ***Proof of Theorem 1.2.*** The proof of Theorem 1.2 is a direct consequence of the definition of $KH(\mathcal{X})$ and similar results for the $\mathbb{K}$-theory. Part (1) of the theorem is Theorem 5.2. Part (2) follows directly from Theorems 3.20 and 3.13 because the homotopy colimit preserves homotopy fiber sequences.

We now prove (3). Let G be a finite group acting on a scheme X such that X admits an ample family of line bundles. Then X is covered by G-invariant affine open subschemes. By Theorem 4.9, it suffices to prove the theorem when

$X = \mathrm{Spec}(A)$ is affine. In this case, $\mathbb{K}([X/G])$ is homotopy equivalent to the K-theory of the exact category $\mathcal{P}^G(A)$ of finitely generated G-equivariant projective A-modules (see Lemma 3.2).

Since G is also assumed to be nice, it follows from [Levine and Serpé 2008, Lemma 1.3] that $\mathcal{P}^G(A)$ is equivalent to the exact category $\mathcal{P}(A^{\mathrm{tw}}[G])$ of finitely generated projective $A^{\mathrm{tw}}[G]$-modules. Recall here that $A^{\mathrm{tw}}[G] = \bigoplus_{g \in G} Ae_g$ and the product is defined by $(r_g \cdot e_g)(r_h \cdot e_h) = r_g \cdot (r_h \star g^{-1})e_{gh}$, where $\star$ indicates the G-action on A.

If I is a nilpotent ideal of A with quotient $B = A/I$, it follows from Lemma 5.4 that the map $A^{\mathrm{tw}}[G] \to B^{\mathrm{tw}}[G]$ is surjective and its kernel is a nilpotent ideal of $A^{\mathrm{tw}}[G]$. We now apply [Weibel 1989, Theorem 2.3] to conclude that the map $KH(A^{\mathrm{tw}}[G]) \to KH((A/I)^{\mathrm{tw}}[G])$ is a homotopy equivalence. Since G acts trivially on $\Delta_\bullet$, there is a canonical isomorphism $(A[\Delta_\bullet])^{\mathrm{tw}}[G] \simeq (A^{\mathrm{tw}}[G])[\Delta_\bullet]$. We conclude that the map $KH([\mathrm{Spec}(A)/G]) \to KH([\mathrm{Spec}(B)/G])$ is a homotopy equivalence. This finishes the proof. $\square$

Lemma 5.4. *Let G be a finite group acting on commutative unital rings A and B. Let $A \twoheadrightarrow B$ be a G-equivariant surjective ring homomorphism whose kernel is nilpotent. Then the induced map $A^{\mathrm{tw}}[G] \to B^{\mathrm{tw}}[G]$ is surjective and its kernel is nilpotent.*

Proof. Let I denote the kernel of $f : A \twoheadrightarrow B$. By hypothesis, there exists an integer n such that $I^n = 0$. Since the induced map $A^{\mathrm{tw}}[G] \to B^{\mathrm{tw}}[G]$ is a G-graded homomorphism induced by f on each graded piece, it is a surjection and its kernel is given by $I^{\mathrm{tw}}[G] = \bigoplus_{g \in G} Ie_g$. Since I is a G-invariant ideal of A, each element of $(I^{\mathrm{tw}}[G])^n$ is of the form $(a_1.e_{g_1} + \cdots + a_m.e_{g_m})$, where $g_i \in G$ and $a_i \in I^n$. Therefore, $(I^{\mathrm{tw}}[G])^n = 0$. $\square$

5C. ***$\mathbb{K}$-theory of stacks with coefficients.*** For an integer $n \in \mathbb{N}$, let

$$\mathbb{K}(\mathcal{X}; \mathbb{Z}[1/n]) := \mathrm{hocolim}(\mathbb{K}(\mathcal{X}) \xrightarrow{\cdot n} \mathbb{K}(\mathcal{X}) \xrightarrow{\cdot n} \cdots),$$
$$\mathbb{K}(\mathcal{X}; \mathbb{Z}/n) := \mathbb{K}(\mathcal{X}) \wedge \mathbb{S}/n,$$

where $\mathbb{S}/n$ is the mod-n Moore spectrum. Our final result is the homotopy invariance property of $\mathbb{K}$-theory with coefficients.

The proof of Theorem 5.5 uses the notion of K-theory of dg-categories. We briefly recall its definition and refer to [Keller 2006, §5.2] for further details. Let $\mathcal{A}$ be a small dg-category. The category $D(\mathcal{A})$ is the localization of the category of dg $\mathcal{A}$-modules with respect to quasi-isomorphisms. The category of perfect objects $\mathrm{Per}(\mathcal{A})$ is the smallest triangulated subcategory of $D(\mathcal{A})$ containing the representable objects and closed under shifts, extensions and direct factors. The algebraic K-theory of $\mathcal{A}$ is defined to be the K-theory spectrum of the Waldhausen category $\mathrm{Per}(\mathcal{A})$, where the cofibrations are the degreewise split monomorphisms and the weak equivalences are the quasi-isomorphisms.

Theorem 5.5. *Let $\mathcal{X}$ be a nice quotient stack over k with the resolution property and let $f : \mathcal{E} \to \mathcal{X}$ be a vector bundle. Then the following hold.*

(1) *For any integer n invertible in k, the map $f^* : \mathbb{K}(\mathcal{X}; \mathbb{Z}/n) \to \mathbb{K}(\mathcal{E}; \mathbb{Z}/n)$ is a homotopy equivalence.*

(2) *For any integer n nilpotent in k, the map $f^* : \mathbb{K}(\mathcal{X}; \mathbb{Z}[1/n]) \to \mathbb{K}(\mathcal{E}; \mathbb{Z}[1/n])$ is a homotopy equivalence.*

Proof. The category $\mathrm{Perf}(\mathcal{X})$ has a natural dg enhancement [Cisinski and Tabuada 2012, Example 5.5] whose algebraic K-theory (in the sense of K-theory of dg-categories) coincides with $\mathbb{K}(\mathcal{X})$ by [Keller 2006, Theorem 5.1]. It follows from Proposition 2.7 and [Hall and Rydh 2017, Proposition 8.4] that $D_{\mathrm{qc}}(\mathcal{X})$ is compactly generated and every perfect complex on $\mathcal{X}$ is compact. We conclude from [Tabuada 2017, Theorem 1.2] that the theorem holds when f is the projection map $\mathcal{X}[T] \to \mathcal{X}$. To prove the general case, we use (5.3) and repeat the argument of Theorem 5.2 verbatim. □

Corollary 5.6. *Let $\mathcal{X}$ be as in Theorem 5.5. Then the following hold.*

(1) *For any integer n invertible in k, the natural map $\mathbb{K}(\mathcal{X}; \mathbb{Z}/n) \to KH(\mathcal{X}; \mathbb{Z}/n)$ is a homotopy equivalence.*

(2) *For any integer n nilpotent in k, the natural map $\mathbb{K}(\mathcal{X};\mathbb{Z}[1/n]) \to KH(\mathcal{X};\mathbb{Z}[1/n])$ is a homotopy equivalence.*

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A fixed point theorem on noncompact manifolds

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We generalise the Atiyah–Segal–Singer fixed point theorem to noncompact manifolds. Using *KK*-theory, we extend the equivariant index to the noncompact setting, and obtain a fixed point formula for it. The fixed point formula is the explicit cohomological expression from Atiyah–Segal–Singer's result. In the noncompact case, however, we show in examples that this expression yields characters of infinite-dimensional representations. In one example, we realise characters of discrete series representations on the regular elements of a maximal torus, in terms of the index we define. Further results are a fixed point formula for the index pairing between equivariant *K*-theory and *K*-homology, and a nonlocalised expression for the index we use, in terms of deformations of principal symbols. The latter result is one of several links we find to indices of deformed symbols and operators studied by various authors.

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1. Introduction

In the second part of the *Index of elliptic operators* series, Atiyah and Segal [1968] proved a fixed point formula for compact groups and manifolds. This allows one to compute the equivariant index of an elliptic operator (or an elliptic complex) in terms of data on the fixed point sets of the group elements. In [Atiyah and Singer 1968b], a cohomological version of this formula was obtained, which we call the Atiyah–Segal–Singer fixed point theorem. It has applications, for example, in representation theory. Indeed, Atiyah and Bott [1968] used a fixed point formula

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(which equals the Atiyah–Segal–Singer theorem in the case considered) to prove the Weyl character formula.

Our goals in this paper are to generalise the Atiyah–Segal–Singer theorem to noncompact manifolds, and to apply this generalisation in relevant situations.

The main result and some applications. We define an index on possibly noncompact manifolds, which generalises the equivariant index for compact groups and manifolds (see Definition 2.7). Assuming the fixed point set of a group element g is compact, we show that this index is given by exactly the same cohomological expression as in the Atiyah–Segal–Singer theorem. This is our main result, Theorem 2.16. We also obtain a fixed point formula for the index pairing between equivariant K-theory and K-homology in Theorem 2.18. In the nonequivariant setting, very general expressions for this pairing were given in [Carey et al. 2014]; Theorem 2.18 is an equivariant version of these results for the operators considered here.

While the cohomological expression for the index is the same as in the compact case, in noncompact examples we see that it gives rise to characters of *infinite-dimensional* representations. These can never occur as indices of elliptic operators on compact manifolds, so that the theory really gives us something new. For example, we use the fixed point theorem in Section 6E to express the character of a representation in the discrete series of a semisimple Lie group in terms of our index, on the regular elements of a maximal torus. Other examples and applications are:

- a holomorphic linearisation theorem, related to [Guillemin et al. 2002, Chapter 4] and [Braverman 2002, Theorem 7.2];
- explicit computations for actions by the circle on the plane and the two-sphere;
- a relation with kernels of Fredholm operators, in particular Callias-type Dirac operators [Anghel 1993; Braverman and Shi 2016; Bunke 1995; Callias 1978; Kucerovsky 2001];
- a relation with Braverman's index of deformed Dirac operators [Braverman 2002];
- a relative index theorem, in the spirit of [Gromov and Lawson 1983, Theorem 4.18];
- some geometric consequences in the cases of the Hodge–Dirac and Spin-Dirac operators.

In all cases we consider, we find that the index can be expressed explicitly in terms of the kernel of a deformation of the operator in question. (In the discrete series example, the operator does not even have to be deformed.) On noncompact manifolds, one can often obtain a well-defined index of a Dirac operator by applying a deformation, with suitable growth behaviour. See, for example, [Anghel

1993; Braverman 2002; 2015; Bunke 1995; Callias 1978; Hochs and Mathai 2015; Hochs and Song 2017a; Kucerovsky 2001; Ma and Zhang 2014; Paradan 2011]. This index then depends on the deformation used. While we do not use a deformation to define our index, we see in examples that it equals an index defined via a deformation. One could speculate that this means that the index we use implicitly includes a canonical choice of such a deformation. For the Callias-type operators studied in [Anghel 1993; Braverman and Shi 2016; Bunke 1995; Callias 1978; Kucerovsky 2001], their equivariant indices can be expressed as the index we define, plus a term representing the dependence on the deformation used, in terms of its behaviour "at infinity". (Previously, Callias-type operators were not studied in combination with group actions, so only nonequivariant indices were computed.)

The relation to index theory of deformed Dirac operators is strengthened in the last section of this paper, which is independent of the fixed point formula. There we give an expression for the index of elliptic operators involving deformations of their principal symbols.

Other generalised fixed point theorems include [Berline and Vergne 1996a, Main Theorem 1; 1996b, Theorem 20] (for transversally elliptic operators), [Braverman 2002, Theorem 7.5] (for deformed Dirac operators on noncompact manifolds), the results in [Dell'Ambrogio et al. 2014] (for *correspondences*, generalising self-maps on manifolds), [Emerson 2011, Theorem 2.7] (for groupoids) and [Wang and Wang 2016, Theorem 6.1] (for orbifolds).

Idea of the proof. Let us sketch some technical steps involved in defining the index and proving the fixed point formula. We consider a Riemannian manifold M, and an elliptic operator D on a vector bundle $E \to M$. Let G be a compact Lie group acting on E, preserving D. Under assumptions about grading and self-adjointness, we have a class $[D]$ in the equivariant K-homology group $K_0^G(M)$ of M. Let $g \in G$. Then we may replace G by the compact Abelian group generated by g, and still retain all information about the action by the element g. A localisation theorem in K-homology allows us to construct the *g-index* map

$$\mathrm{index}_g : K_0^G(M) \to \mathbb{C}.$$

This localisation theorem is closely related to a localisation theorem in KK-theory: Theorem 3.2 in [Rosenberg 1999]. The g-index of the operator D is defined as the g-index of its class $[D]$ in $K_0^G(M)$. If M is compact, this is the usual equivariant index of D, evaluated at g.

If M is compact, the principal symbol σ_D of D defines a class in the equivariant topological K-theory group $K_G^0(TM)$. In our setting, M may be noncompact. Then we have a class

$$[\sigma_D] \in KK_G(M, TM)$$

in the equivariant KK-theory of the pair $(C_0(M), C_0(TM))$. The Dolbeault–Dirac operator on TM defines a class

$$[D_{TM}] \in KK_G(TM, \mathrm{pt}).$$

An index theorem by Kasparov implies that, with respect to the Kasparov product $\otimes_{TM}$ over $C_0(TM)$, we have

$$[D] = [\sigma_D] \otimes_{TM} [D_{TM}] \in KK_G(M, \mathrm{pt}) = K_0^G(M).$$

This generalises the Atiyah–Singer index theorem.

The proof of the fixed point formula for the g-index of D is a KK-theoretic generalisation of the proof by Atiyah and Segal [1968] for the compact case. This generalisation involves Kasparov's index theorem, localisation theorems in KK-theory, and KK-theoretic versions of the Gysin wrong-way maps in K-theory. Another ingredient is a class

$$\sigma_g^D \in K_G^0(TM)_g \tag{1.1}$$

associated to σ_D, in the equivariant topological K-theory of TM, localised (in the algebraic sense) at g. Using these techniques, and keeping track of what happens in both entries in KK-theory, allows us to obtain an expression for the g-index of D in terms of data on the fixed point set of g. While all constructions in the proof are KK-theoretic in nature, the end result is a purely cohomological expression. An explicit description of the class (1.1) in terms of a deformation of the symbol σ_D allows us to prove a nonlocalised expression for the g-index, independent of the fixed point formula.

Outline. The g-index is introduced in Section 2. It allows us to state the fixed point formula in Theorem 2.16.

In Section 3, we prove the localisation results, which imply that the g-index is well-defined. In Section 4, we review an index theorem by Kasparov. This result, and related techniques, are used in the proof of the fixed point theorem in Section 5.

The applications and examples mentioned above are discussed in Section 6. In Section 7, we obtain a nonlocalised expression for the g-index of an elliptic operator.

Notation. If A is a subset of a set B, then we denote the inclusion map $A \hookrightarrow B$ by j_A^B. We denote the one-point set by pt. For any set A, we write p^A for the map from A to pt.

If U is an open subset of a locally compact Hausdorff space X, then we denote by k_U^X the inclusion map $C_0(U) \hookrightarrow C_0(X)$ defined by extending functions by zero outside U. If Y is another locally compact Hausdorff space, we write

$$KK(X, Y) := KK(C_0(X), C_0(Y)),$$

and similarly for equivariant KK-theory. The Kasparov product $\otimes_{C_0(X)}$ over $C_0(X)$ will also be denoted by $\otimes_X$. If X has a Borel measure, and $E \to X$ is a Hermitian vector bundle, then the $*$-homomorphism $\pi_X : C_0(X) \to \mathcal{B}(L^2(E))$ is given by the pointwise multiplication on L^2-sections of E. If H is a locally compact group acting on X, and $H' < H$ is a subgroup, we write $H \times_{H'} X$ for the quotient of $H \times X$ by the action by H' given by

$$h' \cdot (h, x) = (hh'^{-1}, h'x),$$

for $h' \in H'$, $h \in H$ and $x \in X$.

Throughout this paper, G will be a compact Abelian group containing an element g whose powers are dense in G. The only exception is Section 6E, where G denotes a semisimple Lie group. There, a compact Cartan subgroup $T < G$ will play the role that G plays in the rest of this paper.

If M is a manifold, its tangent bundle projection $TM \to M$ is denoted by τ_M. If a Riemannian metric is given, we will often tacitly use it to identify the tangent bundle of M with the cotangent bundle. The complexification of a vector space or vector bundle is denoted by a subscript $\mathbb{C}$.

2. The fixed point formula

Our goal in this paper is to generalise the Atiyah–Segal–Singer fixed point theorem ([Atiyah and Singer 1968b, Theorem 3.9], based on [Atiyah and Segal 1968, Theorem 2.12]) to noncompact manifolds, and to find interesting applications of this generalisation. This leads us to define the g-index in Section 2B. The key to defining the g-index is a localisation theorem, which is stated in Section 2A. The main result of this paper is the fixed point formula in Theorem 2.16, stated in Section 2D. This formula is entirely cohomological, and does not involve KK-theory. Some properties of the g-index are given in Section 2C.

Throughout this paper, M will be a Riemannian manifold. We consider an isometric diffeomorphism g from M to itself. Suppose the closure of the powers of g in the isometry group (with respect to the compact-open topology) is a compact group G. Equivalently, suppose g is an element of a compact group H acting isometrically on M, and let $G < H$ be the closed subgroup generated by g. In any case, G is Abelian. Let M^g be the fixed point set of g.

Let $E = E^+ \oplus E^- \to M$ be a $\mathbb{Z}_2$-graded, Hermitian vector bundle. Let D be an odd, essentially self-adjoint, elliptic differential operator, with principal symbol σ_D. (For example, D can be a Dirac-type operator on a complete manifold.) We will also write D for the self-adjoint closure of D. Then we have the element

$$[D] := \Big[L^2(E), \frac{D}{\sqrt{D^2+1}}, \pi_M \Big] \tag{2.1}$$

of the equivariant *K-homology* group $KK_G(M, \mathrm{pt}) := KK_G(C_0(M), \mathbb{C})$. Here $\pi_M : C_0(M) \to \mathcal{B}(L^2(E))$ is given by pointwise multiplication. For background material on KK-theory, see [Blackadar 1998, Chapter VIII].

2A. ***Localisation.*** Let $R(G)_g := R(G)_{I_g}$ be the localisation of the representation ring $R(G)$ at the prime ideal

$$I_g := \{\chi \in R(G) : \chi(g) = 0\}.$$

For any module $\mathcal{M}$ over $R(G)$, we write $\mathcal{M}_g := \mathcal{M}_{I_g}$ for the corresponding localised module over $R(G)_g$. Similarly, if $m \in \mathcal{M}$, and $\varphi : \mathcal{M} \to \mathcal{M}'$ is a module homomorphism to another such module, we write $m_g \in \mathcal{M}_g$ and

$$\varphi_g : \mathcal{M}_g \to \mathcal{M}'_g$$

for the corresponding localised versions.

For any two G-C^*-algebras A and B, the group $KK_G(A, B)$ is a module over the ring $R(G) = KK_G(\mathbb{C}, \mathbb{C})$, via the exterior Kasparov product. Fix a G-C^*-algebra A. The inclusion map

$$j^M_{M^g} : M^g \hookrightarrow M$$

induces

$$(j^M_{M^g})^*_g : KK_G(A, C_0(M))_g \to KK_G(A, C_0(M^g))_g.$$

Theorem 2.2. *If A is separable, the map $(j^M_{M^g})^*_g$ is an isomorphism of Abelian groups. This is still true if $M \setminus M^g$ is a manifold, rather than all of M.*

Remark 2.3. If $A = \mathbb{C}$, then this reduces to [Atiyah and Segal 1968, Theorem 1.1]. We need this more general statement, because in the noncompact case, principal symbols define classes in $KK_G(C_0(M), C_0(TM))$ as in (4.4), rather than in $KK_G(\mathbb{C}, C_0(TM))$ when M is compact.

We will also use an analogue of Theorem 2.2 for the first entry in KK-theory. Its formulation is slightly more subtle.

Theorem 2.4. *Suppose that M^g is compact and that A is σ-unital. Let $U, V \subset M$ be two G-invariant, relatively compact open neighbourhoods of M^g, such that $\overline{U} \subset V$. Then the map*

$$((j^V_{\overline{U}})_*)_g : KK_G(C_0(\overline{U}), A)_g \to KK_G(C_0(V), A)_g$$

is an isomorphism of Abelian groups. This is still true if M is only a locally compact Hausdorff topological space rather than a manifold, as long as the open subset $M \setminus M^g$ is a manifold.

Theorems 2.2 and 2.4 will be proved in Section 3 for graded KK-theory, i.e., the combination of even and odd KK-theory. We will only apply the even versions, however. The cases where only $M \setminus M^g$ is a manifold were included because we will also apply Theorem 2.4 to one-point compactifications of manifolds. Theorems 2.2 and 2.4 are similar to Theorem 3.2 in [Rosenberg 1999].

2B. ***The g-index.*** Suppose the fixed point set M^g is compact. Let U, V be as in Theorem 2.4. Consider the proper map $p^{\overline{U}} : \overline{U} \to \mathrm{pt}$, and the inclusion map $k_V^M : C_0(V) \to C_0(M)$ given by extending function by zero outside V. Let A be a σ-unital G-C^*-algebra. By Theorem 2.4, we have the maps

$$KK_G(C_0(M), A)_g \xrightarrow{(k_V^M)_g^*} KK_G(C_0(V), A)_g \xrightarrow{((j_{\overline{U}}^V)_*)_g^{-1}} KK_G(C(\overline{U}), A)_g \xrightarrow{(p_*^{\overline{U}})_g} KK_G(\mathbb{C}, A)_g. \quad (2.5)$$

Lemma 2.6. *The composition (2.5) is independent of the sets U and V.*

Proof. To prove independence of U, let U' be a G-invariant, relatively compact neighbourhood of M^g such that $\overline{U'} \subset U$. Then we have the commutative diagram

$$\begin{tikzcd}
 & KK_G(C(\overline{U'}), A) \arrow[dl, "(j_{\overline{U'}}^V)_*"'] \arrow[d, "(j_{\overline{U'}}^{\overline{U}})_*"] \arrow[dr, "p_*^{\overline{U'}}"] & \\
KK_G(C_0(V), A) & KK_G(C(\overline{U}), A) \arrow[l, "(j_{\overline{U}}^V)_*"] \arrow[r, "p_*^{\overline{U}}"'] & KK_G(\mathbb{C}, A)
\end{tikzcd}$$

Commutativity of this diagram implies that

$$(p_*^{\overline{U'}})_g \circ ((j_{\overline{U'}}^V)_*)_g^{-1} = (p_*^{\overline{U}})_g \circ ((j_{\overline{U}}^V)_*)_g^{-1}.$$

So (2.5) is indeed independent of U.

To prove independence of V, let V' be a G-invariant, relatively compact open subset of M containing V. Then the following diagram commutes:

$$\begin{tikzcd}
 & KK_G(C_0(V'), A) \arrow[d, "(k_V^{V'})^*"] & \\
KK_G(C_0(M), A) \arrow[ur, "(k_{V'}^M)^*"] \arrow[r, "(k_V^M)^*"'] & KK_G(C_0(V), A) & KK_G(C(\overline{U}), A) \arrow[l, "(j_{\overline{U}}^V)_*"] \arrow[ul, "(j_{\overline{U}}^{V'})_*"']
\end{tikzcd}$$

Therefore, we have

$$((j_{\overline{U}}^{V'})_*)_g^{-1} \circ (k_{V'}^M)_g^* = ((j_{\overline{U}}^V)_*)_g^{-1} \circ (k_V^M)_g^*,$$

so that (2.5) is independent of V. □

To define the g-index, we only need the case of Lemma 2.6 where $A = \mathbb{C}$. Later we will also use the general case, however.

Let

$$\mathrm{ev}_g : R(G) \to \mathbb{C}$$

be defined by evaluating characters at g, i.e., $\mathrm{ev}_g(\chi) := \chi(g)$, for $\chi \in R(G)$. In view of Lemma 2.6, we obtain a well-defined index as follows.

Definition 2.7. The g-*index* is the map

$$\mathrm{index}_g : KK_G(M, \mathrm{pt}) \to \mathbb{C}$$

defined as the composition

$$KK_G(M, \mathrm{pt}) \hookrightarrow KK_G(M, \mathrm{pt})_g \xrightarrow{(p_*^{\bar{U}})_g \circ ((j_{\bar{U}}^V)_*)_g^{-1} \circ (k_V^M)_g^*} KK_G(\mathrm{pt}, \mathrm{pt})_g \cong R(G)_g \xrightarrow{(\mathrm{ev}_g)_g} \mathbb{C}. \quad (2.8)$$

We will write

$$\mathrm{index}_g(D) := \mathrm{index}_g[D],$$

where $[D] \in KK_G(M, \mathrm{pt})$ is the class (2.1).

Note that $(k_V^M)_g^*[D]_g$ is simply the localisation at g of the K-homology class of the restriction of D to V.

Remark 2.9. The g-index of D could also have been called the D-Lefschetz number of g.

2C. *Properties of the g-index.* If M is *compact*, then we may take $U = V = M$ in Definition 2.7. Furthermore, the map $p^M : M \to \mathrm{pt}$ is proper. In that case, the composition (2.5) simply equals the map

$$(p_*^M)_g : KK_G(C_0(M), A)_g \to KK_G(\mathbb{C}, A)_g.$$

If $A = \mathbb{C}$, then it follows that for compact M, the g-index of D equals

$$\mathrm{index}_g(D) = \mathrm{index}_G(D)(g), \quad (2.10)$$

the usual equivariant index of D evaluated at g. Note that on the right-hand side of (2.10), G can be any compact Lie group acting isometrically on M if the action lifts to E, commutes with D, and contains g.

In general, however, the g-indices on noncompact manifolds give us something more general than the equivariant index in the compact case. In the examples in Section 6, we will see that the g-index can be used to describe characters of infinite-dimensional representations. These cannot be realised as indices on compact manifolds. And even on compact manifolds, an equivariant index can be decomposed into g-indices which individually correspond to infinite-dimensional representations. See Section 6D.

The g-index has an excision property.

Lemma 2.11. *Let V be a G-invariant, relatively compact, open neighbourhood of M^g. Suppose there is a G-equivariant open embedding $V \hookrightarrow \widetilde{M}$ into a G-manifold $\widetilde{M}$. Suppose the action by G on $\widetilde{M}$ has no fixed points outside V. Suppose there is a Hermitian, $\mathbb{Z}_2$-graded G-vector bundle $\widetilde{E} \to \widetilde{M}$ and an odd, self-adjoint, elliptic differential operator $\widetilde{D}$ on $\widetilde{E}$ such that $\widetilde{E}|_V = E|_V$ and $\widetilde{D}|_V = D|_V$. Then*

$$\operatorname{index}_g(D) = \operatorname{index}_g(\widetilde{D}).$$

Proof. By Proposition 10.8.8 in [Higson and Roe 2000], we have

$$(k_V^M)^*[D] = (k_V^{\widetilde{M}})^*[\widetilde{D}] \in KK_G(V, \mathrm{pt}).$$

This implies the claim. □

Example 2.12. Suppose M has a G-equivariant Spin-structure, and let D be the Spin-Dirac operator. Let $M \hookrightarrow \widetilde{M}$ be a G-equivariant open embedding into a compact G-manifold $\widetilde{M}$ with a G-equivariant Spin-structure. If G is connected and $\operatorname{index}_g(D) \neq 0$, then g must have a fixed point in $\widetilde{M} \setminus M$. Indeed, Atiyah and Hirzebruch [1970] showed that the g-index of the Spin-Dirac operator on $\widetilde{M}$ is zero in this case. So the claim follows by Lemma 2.11.

Another property of the g-index is multiplicativity. Let D_1 and D_2 be operators like D on manifolds M_1 and M_2, respectively, and consider the product operator

$$D_1 \times D_2 := D_1 \otimes 1 + 1 \otimes D_2$$

on $M_1 \times M_2$ (where graded tensor products are used). Then functoriality of the Kasparov product implies that

$$\operatorname{index}_g(D_1 \times D_2) = \operatorname{index}_g(D_1) \operatorname{index}_g(D_2).$$

In the index theory of deformed Dirac operators developed in [Braverman 2002], the deformation used means that an analogous multiplicativity property is highly nontrivial; see [Hochs and Song 2017b; Ma and Zhang 2014; Paradan 2011].

2D. *Fixed points.* Having defined the g-index, we can state the main result of this paper. We will use the fact that the connected components of the fixed point set M^g are smooth submanifolds of M, possibly of different dimensions.

Since M^g is compact, the restriction to TM^g of the principal symbol σ_D of D defines a class

$$[\sigma_D|_{TM^g}] \in KK_G(\mathrm{pt}, TM^g). \tag{2.13}$$

Let $N \to M^g$ be the union of the normal bundles to each of the components of M^g. Consider the topological K-theory class

$$\left[\bigwedge N_{\mathbb{C}}\right] := \left[\bigoplus_j \bigwedge^{2j} N \otimes \mathbb{C}\right] - \left[\bigoplus_j \bigwedge^{2j+1} N \otimes \mathbb{C}\right] \in KK_G(\mathrm{pt}, M^g). \tag{2.14}$$

For any trivial G-space X, we have

$$KK_G(\mathrm{pt}, X) \cong KK(\mathrm{pt}, X) \otimes R(G).$$

We can evaluate the factor in $R(G)$ of any class $a \in KK_G(\mathrm{pt}, X)$ at g, to obtain $a(g) \in KK(\mathrm{pt}, X) \otimes \mathbb{C}$. In this way, evaluating the classes (2.13) and (2.14) at g gives the classes

$$[\sigma_D|_{TM^g}](g) \in KK(\mathrm{pt}, TM^g) \otimes \mathbb{C}$$

and

$$\big[\textstyle\bigwedge N_{\mathbb{C}}\big](g) \in KK(\mathrm{pt}, M^g) \otimes \mathbb{C}, \tag{2.15}$$

respectively.

Consider the Chern characters

$$\begin{aligned} \mathrm{ch} : KK(\mathrm{pt}, TM^g) &\to H^*(TM^g), \\ \mathrm{ch} : KK(\mathrm{pt}, M^g) &\to H^*(M^g), \end{aligned}$$

defined on each smooth component of M^g separately. By [Atiyah and Segal 1968, Lemma 2.7], the K-theory class (2.15) is invertible; hence so is its Chern character. An explicit expression for the inverse

$$\frac{1}{\mathrm{ch}\big(\big[\bigwedge N_{\mathbb{C}}\big](g)\big)} \in H^*(M^g) \otimes \mathbb{C}$$

of this element is given in [Atiyah and Singer 1968b, (3.8)]. The cohomology group $H^*(M^g)$ acts on $H^*(TM^g)$ via the pullback along the tangent bundle projection τ_{M^g}. Let $\mathrm{Todd}(TM^g \otimes \mathbb{C})$ be the cohomology class on M^g obtained by putting together the Todd-classes of the complexified tangent bundles to all components of M^g.

Theorem 2.16 (fixed point formula). *The g-index of D equals*

$$\mathrm{index}_g(D) = \int_{TM^g} \frac{\mathrm{ch}([\sigma_D|_{TM^g}](g))\,\mathrm{Todd}(TM^g \otimes \mathbb{C})}{\mathrm{ch}\big(\big[\bigwedge N_{\mathbb{C}}\big](g)\big)}. \tag{2.17}$$

The integral in this expression is the sum of the integrals over all connected components of TM^g of the integrand corresponding to each component.

If M is compact, then (2.10) implies that Theorem 2.16 reduces to the Atiyah–Segal–Singer fixed point formula [Atiyah and Singer 1968b, Theorem 3.9].

2E. *The index pairing.* In the course of the proof of Theorem 2.16, we will also find a fixed point formula for the index pairing (i.e., the Kasparov product)

$$KK_G(\mathrm{pt}, M) \times KK_G(M, \mathrm{pt}) \to KK_G(\mathrm{pt}, \mathrm{pt}).$$

Note that any element of the equivariant topological K-theory group $KK_G(\mathrm{pt}, M)$ can be represented by a formal difference $[F_0] - [F_1]$, for two G-equivariant vector bundles $F_0, F_1 \to M$ that are equal outside a compact set. We will write

$F := F_0 \oplus F_1$, with the $\mathbb{Z}_2$-grading where F_0 is the even part and F_1 the odd part, and $[F] := [F_0] - [F_1] \in KK_G(\mathrm{pt}, M)$.

Theorem 2.18 (fixed point formula for the index pairing). *We have*

$$([F] \otimes_M [D])(g) = \int_{TM^g} \frac{\mathrm{ch}([F|_{M^g}](g))\,\mathrm{ch}([\sigma_D|_{TM^g}](g))\,\mathrm{Todd}(TM^g \otimes \mathbb{C})}{\mathrm{ch}\big(\big[\bigwedge N_{\mathbb{C}}\big](g)\big)}.$$

Recall that M^g was assumed to be compact, and that we use the action by the cohomology of M^g on the cohomology of TM^g via the pullback along τ_{M^g}.

Theorem 3.33 in [Carey et al. 2014] is a nonequivariant index formula for the index pairing in a more general context. Theorem 2.18 is an equivariant version of this result, for operators like D.

The proof of Theorem 2.18 is simpler than that of Theorem 2.16, because it does not involve localisation in the first entry of KK-theory. Theorem 2.16 is needed for the examples and applications in Section 6, such as the relation with characters of discrete series representations. The reason for this is that Theorem 2.16 provides an expression for an index of the operator D itself, without the need to twist it by a K-theory class.

3. Localisation

We now turn to a proof of Theorems 2.2 and 2.4. This involves certain module structures discussed in Section 3A, which are used to prove vanishing results in Section 3B. In this section, we consider *graded* KK-theory, i.e., the direct sum of even and odd KK-theory.

3A. ***Module structures.*** Let H be a locally compact group, and let $H' < H$ be a compact subgroup.

Proposition 3.1. *Let Y be a locally compact, Hausdorff, proper H-space for which there is an equivariant, continuous map $Y \to H/H'$. Then for any H-C^*-algebra A, the groups*

$$KK_H(A, C_0(Y)) \quad \textit{and} \quad KK_H(C_0(Y), A)$$

have structures of unital $R(H')$-modules.

Proposition 3.1 follows from the fact that vector bundles, even on noncompact spaces, define classes in KK-theory in the following way. This is probably well-known, but we include a proof for completeness' sake.

Let X be a locally compact Hausdorff space on which H acts properly. Let $E \to X$ be a Hermitian H-vector bundle. The space $\Gamma_0(E)$ of continuous sections of E vanishing at infinity is a right Hilbert $C_0(X)$-module by pointwise multiplication and inner products. Let $\pi_X : C_0(X) \to \mathcal{B}(\Gamma_0(E))$ be given by pointwise multiplication.

Lemma 3.2. *The triple*

$$(\Gamma_0(E), 0, \pi_X) \tag{3.3}$$

is an H-equivariant Kasparov $(C_0(X), C_0(X))$-cycle.

For compact spaces X, this fact is noted for example in (3.1) in [Rosenberg 1999]. In general, we will denote the class in $KK_H(X, X)$ defined by (3.3) by $[E]$.

Proof. We show that for all $f \in C_0(X)$, the operator $\pi_X(f)$ on $\Gamma_0(E)$ is compact. This implies the claim.

Let $U \subset X$ be a relatively compact open subset admitting an orthonormal frame $\{e_1, \ldots, e_r\}$ of $E|_U$. Let $s \in \Gamma_0(E)$. Then

$$s|_U = \sum_{j=1}^{r} (e_j, s)_E \, e_j.$$

Here $(-\,,-)_E$ is the metric on E. So if $f \in C_0(X)$ is supported inside U, then

$$\pi_X(f)s = \sum_{j=1}^{r} (e_j, fs)e_j = \sum_{j=1}^{r} (\bar{f}e_j, s)e_j.$$

By extending the sections e_j outside U to elements of $\Gamma_0(E)$, we find that $\pi_X(f)$ is a finite-rank operator.

For a general $f \in C_c(X)$, there is a finite open cover $\{U_j\}_{j=1}^n$ of $\operatorname{supp}(f)$ such that every set U_j admits a local orthonormal frame for E. Let $\{\varphi_j\}_{j=1}^n$ be functions such that $\operatorname{supp}(\varphi_j) \subset U_j$, and $\sum_{j=1}^n \varphi_j$ equals one on $\operatorname{supp}(f)$. Then, by the preceding argument,

$$\pi_X(f) = \sum_{j=1}^{n} \pi_X(\varphi_j f)$$

is a finite-rank operator. Hence, for all $f \in C_0(X)$, the operator $\pi_X(f)$ on $\Gamma_0(E)$ is indeed compact. $\square$

Now consider the situation of Proposition 3.1. Let $p : Y \to H/H'$ be an equivariant, continuous map. Let V be a finite-dimensional representation space of H'. We have the H-vector bundles

$$H \times_{H'} V \to H/H'$$

and

$$E_V := p^*(H \times_{H'} V) \to Y.$$

By Lemma 3.2, this vector bundle defines a class

$$[E_V] \in KK_H(Y, Y).$$

Lemma 3.4. *The map from $R(H')$ to $KK_H(Y, Y)$ given by*

$$[V] \mapsto [E_V],$$

with $[E_V]$ defined as above, is a ring homomorphism.

Proof. This follows from the fact that in the setting of Lemma 3.2, for any two Hermitian H-vector bundles $E, E' \to X$, one has

$$[E] \otimes_X [E'] = [E \otimes E']. \qquad \square$$

The ring homomorphism of Lemma 3.4 defines the module structures sought in Proposition 3.1, which has therefore been proved. If $A = \mathbb{C}$ and Y is compact, the $R(H')$-module structure on $KK_H(\mathbb{C}, C_0(Y))$ defined in this way is the one used in [Atiyah and Segal 1968].

3B. *Vanishing results.* We will prove Theorems 2.2 and 2.4 by generalising Atiyah and Segal's proof of [Atiyah and Segal 1968, Theorem 1.1]. An important step is the following generalisation of [Atiyah and Segal 1968, Corollary 1.4].

Proposition 3.5. *Let $H < G$ be a closed subgroup such that $g \notin H$. Let Y be a compact G-space for which there is an equivariant map $Y \to G/H$, and A a G-C^*-algebra. Then*

$$KK_G(A, C_0(Y))_g = KK_G(C_0(Y), A)_g = 0.$$

Proof. By [Atiyah and Segal 1968, Corollary 1.3], we have $R(H)_g = 0$. As Atiyah and Segal argued below that corollary, it is therefore enough to show that $KK_G(A, C_0(Y))$ and $KK_G(C_0(Y), A)$ are unital $R(H)$-modules. Hence, the claim follows from Proposition 3.1. $\square$

We will deduce Theorems 2.2 and 2.4 from the following special cases.

Proposition 3.6. *In the setting of Theorem 2.2, suppose g has no fixed points in M. Then, if A is separable, we have*

$$KK_G(A, C_0(M))_g = 0. \tag{3.7}$$

If A is σ-unital, then for all G-invariant, relatively compact open subsets $U \subset M$,

$$KK_G(C_0(U), A)_g = 0. \tag{3.8}$$

If $A = \mathbb{C}$, then (3.7) is precisely [Atiyah and Segal 1968, Proposition 1.5]. By a generalisation of the arguments in [Atiyah and Segal 1968, Section 1], we will deduce Proposition 3.6 from Proposition 3.5.

By Palais' slice theorem [1961, Proposition 2.2.2], there is an open cover $\{U_j\}_{j=1}^{\infty}$ of M by G-invariant open sets such that for all j,

$$\overline{U_j} \cong G \times_{H_j} \overline{S_j}$$

(via the action map), for the stabiliser $H_j < G$ of a point in U_j, and an H_j-invariant submanifold $S_j \subset M$. Suppose that g has no fixed points. Then it does not lie in any of the stabilisers H_j. Therefore, Proposition 3.5 implies that

$$KK_G(A, C_0(\overline{U_j}))_g = KK_G(C_0(\overline{U_j}), A)_g = 0.$$

Let $X \subset M$ be any G-invariant, compact subset. The proof of Proposition 3.6 is based on the following fact.

Lemma 3.9. *If A is separable, then*

$$KK_G(A, C_0(X))_g = 0. \tag{3.10}$$

If A is σ-unital, then

$$KK_G(C_0(X), A)_g = 0. \tag{3.11}$$

Proof. We use an induction argument based on exact sequences in KK-theory. We work out the details for (3.10). Then (3.11) can be proved in the same way, with exact sequences in the second entry in KK-theory replaced by the corresponding exact sequences in the first entry. The conditions that A is separable or σ-unital imply that these exact sequences exist.

For $j, n \in \mathbb{N}$, write $X_j := \overline{U_j} \cap X$, and $Y_n := X_1 \cup \cdots \cup X_n$. Fix $n \in \mathbb{N}$, and consider the exact sequence of C^*-algebras

$$0 \to C_0(X_{n+1} \setminus Y_n) \to C_0(X_{n+1}) \to C_0(X_{n+1} \cap Y_n) \to 0.$$

It induces the exact triangle

$$\begin{tikzcd}
KK_G(A, C_0(X_{n+1})) \arrow[r] & KK_G(A, C_0(X_{n+1} \cap Y_n)) \arrow[d, "\partial"] \\
 & KK_G(A, C_0(X_{n+1} \setminus Y_n)) \arrow[ul]
\end{tikzcd}$$

(See, e.g., [Blackadar 1998, Theorem 19.5.7].) By Proposition 3.5, we have

$$KK_G(A, C_0(X_{n+1}))_g = KK_G(A, C_0(X_{n+1} \cap Y_n))_g = 0.$$

Since localisation at g preserves exactness, we find that

$$KK_G(A, C_0(X_{n+1} \setminus Y_n))_g = 0. \tag{3.12}$$

Using the exact sequence

$$0 \to C_0(Y_{n+1} \setminus Y_n) \to C_0(Y_{n+1}) \to C_0(Y_n) \to 0$$

in a similar way, we obtain the exact triangle

$$
\begin{tikzcd}
KK_G(A, C_0(Y_{n+1}))_g \arrow[rr] & & KK_G(A, C_0(Y_n))_g \arrow[d, "\partial"] \\
& & KK_G(A, C_0(Y_{n+1} \setminus Y_n))_g \arrow[llu]
\end{tikzcd}
$$

Since $Y_{n+1} \setminus Y_n = X_{n+1} \setminus Y_n$, the vanishing of (3.12) implies that

$$KK_G(A, C_0(Y_{n+1}))_g = KK_G(A, C_0(Y_n))_g.$$

Because $Y_1 = X_1$, Proposition 3.5 implies that

$$KK_G(A, C_0(Y_1))_g = 0.$$

Since X is compact, it can be covered by finitely many of the sets X_j. Hence, the claim follows by induction on n. □

Proof of Proposition 3.6. Let $U \subset M$ be a G-invariant, relatively compact open subset. Consider the exact sequence

$$0 \to C_0(U) \to C_0(\overline{U}) \to C_0(\partial U) \to 0.$$

If A is σ-unital, this induces the localised exact triangle

$$
\begin{tikzcd}
KK_G(C_0(\overline{U}), A)_g \arrow[rrd] & & KK_G(C_0(\partial U), A)_g \arrow[ll] \\
& & KK_G(C_0(U), A)_g \arrow[u, "\partial"]
\end{tikzcd}
$$

Lemma 3.9 implies that

$$KK_G(C_0(\overline{U}), A)_g = KK_G(C_0(\partial U), A)_g = 0.$$

So we find that $KK_G(C_0(U), A)_g = 0$.

Similarly, if A is separable, we have the exact triangle

$$
\begin{tikzcd}
KK_G(A, C_0(\overline{U}))_g \arrow[rr] & & KK_G(A, C_0(\partial U))_g \arrow[d, "\partial"] \\
& & KK_G(A, C_0(U))_g \arrow[llu]
\end{tikzcd}
$$

Applying Lemma 3.9 in the same way, we find that $KK_G(A, C_0(U))_g = 0$. The equality (3.7) follows, because M is the direct limit of sets U as above, and because KK-theory commutes with direct limits in the second entry. □

Remark 3.13. The reason why (3.8) does not hold if U is replaced by M, and hence why Theorem 2.4 has to be stated more subtly than Theorem 2.2, is that KK-theory does not commute with direct limits in the first entry. For example, the domain of the analytic assembly map in the Baum–Connes conjecture [Baum et al. 1994] is the *representable K-homology* group

$$RK_*^H(X) := \varinjlim_{Y \subset X;\ Y/H \text{ cpt}} KK_H(C_0(Y), \mathbb{C}),$$

for a locally compact Hausdorff space X on which a locally compact group H acts properly. This does not equal the usual K-homology group $KK_H(C_0(X), \mathbb{C})$ in general.

3C. *Proofs of localisation results.*

Proof of Theorem 2.2. Consider the exact sequence

$$0 \to C_0(M \setminus M^g) \to C_0(M) \xrightarrow{(j^M_{M^g})^*} C_0(M^g) \to 0.$$

It induces the exact triangle

$$\begin{tikzcd} KK_G(A, C_0(M)) \arrow[r, "(j^M_{M^g})^*"] & KK_G(A, C_0(M^g)) \arrow[d, "\partial"] \\ & KK_G(A, C_0(M \setminus M^g)) \arrow[ul] \end{tikzcd}$$

After localisation at g, the first part of Proposition 3.6 yields the exact triangle

$$\begin{tikzcd} KK_G(A, C_0(M))_g \arrow[r, "(j^M_{M^g})^*_g"] & KK_G(A, C_0(M^g))_g \arrow[d, "\partial"] \\ & 0 \arrow[ul] \end{tikzcd}$$

□

Proof of Theorem 2.4. Let U and V be as in Theorem 2.4. Similarly to the proof of Theorem 2.2, we have an exact triangle

$$\begin{tikzcd} KK_G(C_0(V), A)_g \arrow[dr] & KK_G(C_0(\overline{U}), A)_g \arrow[l, "((j^V_{\overline{U}})_*)_g"'] \\ & KK_G(C_0(V \setminus \overline{U}), A)_g \arrow[u, "\partial"] \end{tikzcd}$$

Because $V \setminus \overline{U}$ is a relatively compact subset of $M \setminus M^g$, the second part of Proposition 3.6 implies that the bottom localised KK-group in this triangle equals zero. □

4. Kasparov's index theorem

In the proof of the Atiyah–Segal–Singer fixed point theorem, the Atiyah–Singer index theorem is used to relate topological and analytical indices to each other. In the noncompact case discussed in this paper, a roughly similar role is played by an index theorem of Kasparov. We state Kasparov's index theorem in Section 4A. In Section 4B, we discuss the fibrewise Bott element for the normal bundle of a submanifold in KK-theory, which is dual to the class of the Dolbeault–Dirac operator in a sense. This Bott element will play an important role in the proof of Theorem 2.16. In Section 4D, we show how the Bott element can be used to deduce the Atiyah–Singer index theorem from Kasparov's index theorem in the compact case. (The main step in the argument used there will be used in the proof of Theorem 2.16.)

Most of the material in this section is based on [Atiyah and Singer 1968b; Kasparov 2016] and explanations to the authors by Kasparov. Although the results here are not ours, we found it worthwhile to include the details, because they have not appeared in print yet.

4A. ***The index theorem.*** To state the theorem, we recall the definition of the Dolbeault operator class

$$[D_{TM}] \in KK_G(TM, \mathrm{pt}) \tag{4.1}$$

in [Kasparov 2016, Definition 2.8]. The tangent bundle $T(TM)$ of TM has a natural almost complex structure J. For $m \in M$ and $v \in T_mM$, we have

$$T_v(TM) = T_mM \oplus T_v(T_mM) = T_mM \oplus T_mM.$$

With respect to this decomposition, the almost complex structure J is given by the matrix $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Let $\bar\partial + \bar\partial^*$ be the Dolbeault–Dirac operator on smooth sections of the vector bundle $\bigwedge^{0,*} T^*(TM) \to TM$, for this almost complex structure. We will identify this vector bundle with $\tau_M^* \bigwedge TM_{\mathbb{C}} \to TM$. The class (4.1) is the class of this operator, as in (2.1). In our arguments however, it will be more convenient to use the Spin^c-Dirac operator D_{TM}, on the same vector bundle. This defines the same K-homology class as $\bar\partial + \bar\partial^*$.

Definition 4.2. The *topological index* is the map

$$\mathrm{index}_t : KK_G(M, TM) \to KK_G(M, \mathrm{pt})$$

given by the Kasparov product with $[D_{TM}]$.

Consider the principal symbol $\tilde\sigma_D := \sigma_D/\sqrt{\sigma_D^2 + 1}$ of the operator $D/\sqrt{D^2+1}$. For $f \in C_0(M)$, we have for all $m \in M$ and $v \in T_mM$,

$$f(m)(1 - \tilde\sigma_D(v)^2) = f(m)(\sigma_D(v)^2 + 1)^{-1}.$$

Since the operator D is elliptic and of positive order, this expression tends to zero as m or v tends to infinity. It therefore defines a compact operator on the Hilbert $C_0(TM)$-module $\Gamma_0(\tau_M^* E)$, analogously to the proof of Lemma 3.2. Therefore, the triple

$$(\Gamma_0(\tau_M^* E), \tilde{\sigma}_D, \pi_{TM} \circ \tau_M^*) \tag{4.3}$$

is a G-equivariant Kasparov $(C_0(M), C_0(TM))$-cycle. Here, $\pi_{TM} : C_b(TM) \to \mathcal{B}(\Gamma_0(\tau_M^* E))$ is given by pointwise multiplication. Denote by

$$[\sigma_D] \in KK_G(M, TM) \tag{4.4}$$

the class of (4.3). In view of the following lemma, this symbol class is a natural generalisation of the K-theory symbol class defined in [Atiyah and Singer 1968a] when M is compact.

Lemma 4.5. *If M is compact, consider the map p^M from M to a point. The image*

$$p_*^M[\sigma_D] \in K_G^*(TM)$$

is the usual symbol class.

Proof. Since $\pi_{TM} \circ \tau_M^* \circ (p^M)^*$ is the representation of $\mathbb{C}$ in $\Gamma_0(\tau_M^* E)$ by scalar multiplication, we have

$$p_*^M[\sigma_D] = [\Gamma_0(\tau_M^* E), \tilde{\sigma}_D] \in KK_G(\mathrm{pt}, TM).$$

This corresponds to the class

$$[\sigma_{D^+} : \tau_M^* E^+ \to \tau_M^* E^-] \in K_G^0(TM)$$

in the sense of [Lawson and Michelsohn 1989, Chapter III, (1.7)], where TM is identified with the open unit ball bundle BM over M. (Restricting σ_{D^+} to BM and then identifying $BM \cong TM$ has the same effect as normalising σ_{D^+}.) The lemma is then proved. □

We conclude this subsection by stating Kasparov's index theorem, which will be used to obtain a cohomological formula for the g-index.

Theorem 4.6 (Kasparov's index theorem [2016, Theorem 4.2]). *The K-homology class of the elliptic operator in (2.1) is equal to the topological index of its symbol class (4.4), i.e.,*

$$[D] = \mathrm{index}_t[\sigma_D] \in KK_G(M, \mathbb{C}). \tag{4.7}$$

Remark 4.8. In [Kasparov 2016, Theorem 4.2], the operator in question is assumed to be properly supported, which is not true for the operator $D/\sqrt{1+D^2}$ in general. However, let $\{\chi_j\}_{j=1}^{\infty}$ be a sequence of G-invariant, compactly supported functions, such that $\{\chi_j^2\}_{j=1}^{\infty}$ is a partition of unity. (This exists since G is compact.)

Then the operator

$$\sum_{j=1}^{\infty} \chi_j \frac{D}{\sqrt{1+D^2}} \chi_j$$

is properly supported, and also satisfies the other assumptions of [Kasparov 2016, Theorem 4.2]. Since this operator defines the same K-homology class as $D/\sqrt{1+D^2}$, we can apply [Kasparov 2016, Theorem 4.2] to the class of the latter operator in this way.

4B. ***The Bott element.*** If S is a closed (as a topological subspace, i.e., not necessarily compact), G-invariant submanifold of M, then the Dolbeault operator classes on TS and on a tubular neighbourhood of TS in TM are related by a *(fibrewise) Bott element*. This is a technical tool that will be used several times in the paper. The material here is analogous to Definition 2.6 and Theorem 2.7 in [Kasparov 2016].

Consider the tangent bundle projections

$$\tau_S : TS \to S,$$
$$\tau_N : TN \to N.$$

Denote by $\pi : N \to S$ the normal bundle of S in M. Let $T\pi : TN \to TS$ be the tangent map of π. It again defines a vector bundle. The following diagram commutes:

$$\begin{tikzcd} TN \arrow[r, "\tau_N"] \arrow[d, "T\pi"] & N \arrow[d, "\pi"] \\ TS \arrow[r, "\tau_S"] & S \end{tikzcd} \tag{4.9}$$

This defines a vector bundle $TN \to S$. Consider the vector bundle

$$\textstyle\bigwedge \widetilde{N}_{\mathbb{C}} := T\pi^*\big(\tau_S^* \bigwedge N \otimes \mathbb{C}\big) \to TN.$$

Let $s \in S$. Then

$$(TN)_s := T\pi^{-1}(\tau_S^{-1}(s)) = \tau_N^{-1}(N_s) = T_s S \times N_s \times N_s.$$

Let $w \in (TN)_s$, and let $(\eta, \zeta) \in N_s \times N_s$ be the projection of w according to this decomposition. Note that

$$\big(\textstyle\bigwedge \widetilde{N}_{\mathbb{C}}\big)_w = \bigwedge N_s \otimes \mathbb{C}.$$

We define the vector bundle endomorphism B of $\bigwedge \widetilde{N}_{\mathbb{C}}$ by

$$B_w = \mathrm{ext}\big(\zeta + \sqrt{-1}\,\eta\big) - \mathrm{int}\big(\zeta + \sqrt{-1}\,\eta\big),$$

for all s, w, η and ζ as above. Here ext denotes the wedge product, and int denotes contraction. With respect to the grading of $\bigwedge \widetilde{N}_{\mathbb{C}}$ by even and odd exterior powers, the operator B is odd.

As B is fibrewise selfadjoint, we have the bounded operator $B(1+B^2)^{-1/2}$ on $\Gamma_0\big(TN, \bigwedge\widetilde{N}_{\mathbb{C}}\big)$. The space $\Gamma_0\big(TN, \bigwedge\widetilde{N}_{\mathbb{C}}\big)$ is a right Hilbert $C_0(TN)$-module in the usual way, with respect to pointwise multiplication by functions and the pointwise inner product. Consider the representation

$$\tilde{\pi}_{TS} := \pi_{TN} \circ T\pi^* : C_0(TS) \to \mathcal{B}\big(\Gamma_0\big(TN, \textstyle\bigwedge\widetilde{N}_{\mathbb{C}}\big)\big),$$

where π_{TN} is given by pointwise multiplication by functions in $C_b(TN)$.

Lemma 4.10. *The triple*

$$\big(\Gamma_0\big(TN, \textstyle\bigwedge\widetilde{N}_{\mathbb{C}}\big), B(1+B^2)^{-1/2}, \tilde{\pi}_{TS}\big) \tag{4.11}$$

is a G-equivariant Kasparov $(C_0(TS), C_0(TN))$-cycle.

Proof. Let $f \in C_0(TS)$. Since $B(1+B^2)^{-1/2}$ is a vector bundle endomorphism, it commutes with $\tilde{\pi}_{TS}(f)$. Moreover, we have for all $w \in (TN)_s$ as above,

$$\big(\tilde{\pi}_{TS}(f)\big(1-[B(1+B^2)^{-1/2}]^2\big)\big)_w = \frac{f(v)}{1+\|\eta\|^2+\|\zeta\|^2},$$

with $v := T\pi(w) \in T_sS$. This defines a function in $C_0(TN)$, and hence acts on the Hilbert $C_0(TN)$-module $\Gamma_0\big(TN, \bigwedge\widetilde{N}_{\mathbb{C}}\big)$ as a compact operator. As G preserves the metric on TN, the operator $B(1+B^2)^{-1/2}$ is G-equivariant. $\square$

Definition 4.12. The *(fibrewise) Bott element* of the normal bundle $N \to S$ is the class

$$\beta_N \in KK_G(TS, TN)$$

of the cycle (4.11).

4C. *The Bott element and Dolbeault classes.* The Bott element is useful to us because of the following property. This was used in [Kasparov 2016, second paragraph on p. 1326]; we work out some details of the proof in this subsection.

Proposition 4.13. *Under the Kasparov product*

$$KK_G(TS, TN) \times KK_G(TN, \mathrm{pt}) \to KK_G(TS, \mathrm{pt}),$$

one has

$$\beta_N \otimes_{TN} [D_{TN}] = [D_{TS}].$$

To prove this proposition, one can use the part D_1 of the Spin^c-Dirac operator D_{TN} acting in the fibre directions of $TN \to TS$. For $s \in S$ and $v \in T_sS$, we have $T\pi^{-1}(v) = N_s \oplus T_vN$. Let a be the rank of N, and let $\{f_1, \ldots, f_a\}$ be a local orthonormal frame of $N \to S$. This defines coordinate functions κ_j and λ_j on the parts N_s and T_vN of the fibres $T\pi^{-1}(v)$ of $TN \to TS$, respectively. For $j = 1, \ldots, a$, consider the vector bundle endomorphisms

$$e_j := \mathrm{ext}(f_j) - \mathrm{int}(f_j) \quad \text{and} \quad \epsilon_j := \mathrm{ext}(f_j) + \mathrm{int}(f_j)$$

of $\bigwedge N \otimes \mathbb{C} \to S$, pulled back along (4.9) to endomorphisms of $\bigwedge \widetilde{N}_{\mathbb{C}} \to TN$. Then D_1 is the operator

$$D_1 := \sum_{j=1}^{a} e_j \frac{\partial}{\partial \kappa_j} + \sqrt{-1}\epsilon_j \frac{\partial}{\partial \lambda_j}$$

on $\Gamma^\infty\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big)$. This can be viewed as a family of operators on the fibres of $TN \to TS$.

It defines a class in KK-theory as follows. Let $\Gamma_c\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big)$ be the space of continuous compactly supported sections of $\bigwedge \widetilde{N}_{\mathbb{C}}$. Let $\mathcal{E}_0$ be the completion of this space into a Hilbert $C_0(TS)$-module with respect to the $C_0(TS)$-valued inner product

$$\langle f, h\rangle(v) := \int_{T\pi^{-1}(v)} \overline{f(t)} h(t)\, dt \tag{4.14}$$

for $f, h \in \Gamma_c\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big)$ and $v \in TS$. The operator D_1 gives rise to the class

$$[D_1] := \big[\mathcal{E}_0, D_1(1 + D_1^2)^{-1/2}, \pi_{TN}\big] \in KK_G(TN, TS). \tag{4.15}$$

Lemma 4.16. *We have*

$$[D_1] \otimes_{TS} [D_{TS}] = [D_{TN}] \in KK_G(TN, \mathrm{pt}).$$

Proof. Regarding N as an open subset of M, we identify their tangent bundles when restricted to S, i.e., $TN|_S = TM|_S$. Therefore, as vector bundles over TN, we have

$$\begin{aligned}\textstyle\bigwedge \widetilde{N}_{\mathbb{C}} \otimes T\pi^* \tau_S^* \bigwedge TS_{\mathbb{C}} &= T\pi^* \tau_S^* \textstyle\bigwedge N_{\mathbb{C}} \otimes T\pi^* \tau_S^* \bigwedge TS_{\mathbb{C}} \\ &= T\pi^* \tau_S^* \textstyle\bigwedge (N \oplus TS)_{\mathbb{C}} = T\pi^* \tau_S^* \bigwedge (TM|_S)_{\mathbb{C}} \\ &= T\pi^* \tau_S^* \textstyle\bigwedge (TN|_S)_{\mathbb{C}} = \tau_N^* \bigwedge TN_{\mathbb{C}}.\end{aligned}$$

The last equality follows from commutativity of (4.9). Thus, as Hilbert spaces with representations of $C_0(TN)$,

$$\mathcal{E}_0 \otimes_{C_0(TS)} L^2\big(TS, \tau_S^* \textstyle\bigwedge TS_{\mathbb{C}}\big) \cong L^2\big(TN, \tau_N^* \bigwedge TN_{\mathbb{C}}\big). \tag{4.17}$$

Under this identification, we have

$$D_1 \otimes 1 + 1 \otimes D_{TS} = D_{TN}.$$

(Here we use graded tensor products.) Consider the bounded operator

$$F := \frac{D_1 \otimes 1 + 1 \otimes D_{TS}}{\sqrt{1 + D_1^2 \otimes 1 + 1 \otimes D_{TS}^2}} \tag{4.18}$$

on $\mathcal{E}_0 \otimes_{C_0(TS)} L^2\big(TS, \tau_S^* \bigwedge TS_{\mathbb{C}}\big)$. Write

$$F_1 := \frac{D_1}{\sqrt{1+D_1^2}}, \qquad F_{TS} := \frac{D_{TS}}{\sqrt{1+D_{TS}^2}}.$$

We can verify that F is an $1 \otimes F_{TS}$-connection, and the graded commutator $[F_1 \otimes 1, F]$ is positive modulo compact operators. Thus, by [Blackadar 1998, Definition 18.4.1], the Kasparov product $[D_1] \otimes_{TS} [D_{TS}]$ is represented by the operator F on the space $L^2\big(TN, \tau_N^* \bigwedge TN_{\mathbb{C}}\big)$. The lemma is then proved. $\square$

Lemma 4.19. *The product*

$$\beta_N \otimes_{TN} [D_1]$$

is the identity element of the ring $KK_G(TS, TS)$.

Proof. The idea is that in this product, we pair fibrewise Bott classes and Dolbeault classes, and thus obtain the trivial line bundle over TS. To see this, observe first the isomorphism

$$\Gamma_c\big(TN, \textstyle\bigwedge \widetilde{N}_{\mathbb{C}}\big) \otimes_{C_c(TN)} \Gamma_c\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}}\big) \cong \Gamma_c\big(TN, \bigwedge \widetilde{N}_{\mathbb{C}} \otimes \bigwedge \widetilde{N}_{\mathbb{C}}\big) \tag{4.20}$$

as $C_c(TS)$-modules. Denote by $\mathcal{E}'$ the completion of the right-hand side under the $C_0(TS)$-valued inner product defined in a similar way as (4.14). It can be checked that

$$F_0 := \frac{B \otimes 1 + 1 \otimes D_1}{\sqrt{1 + B^2 \otimes 1 + 1 \otimes D_1^2}} \tag{4.21}$$

is a $1 \otimes D_1/\sqrt{1+D_1^2}$-connection, and that for all a in $C_0(TS)$, the operator

$$\tilde{\pi}_{TS}(a)[B(1+B^2)^{-1/2} \otimes 1, F_0]\tilde{\pi}_{TS}(a)^*$$

is positive modulo compact operators on $\mathcal{E}'$. Hence, the Kasparov product of β_N, given by (4.11), and the class $[D_1]$, given by (4.15), is equal to

$$[\mathcal{E}', F_0, \tilde{\pi}_{TS}] \in KK_G(TS, TS). \tag{4.22}$$

As in the proof of [Kasparov 2016, Theorem 2.7(2)], we apply the rotation homotopy

$$F_t := \frac{(B + \sin(t) D_1) \otimes 1 + 1 \otimes \cos(t) D_1}{\sqrt{1 + (B^2 + \sin(t)^2 D_1^2) \otimes 1 + 1 \otimes \cos(t)^2 D_1^2}},$$

for $t \in \big[0, \frac{\pi}{2}\big]$. Then the operator F_0 in the cycle (4.22) is transformed into $F_{\pi/2} = F' \otimes 1$, where

$$F' := (B + D_1)(1 + B^2 + D_1^2)^{-1/2}.$$

In summary, β_N and $[D_1]$ are families of operators indexed by TS whose Kasparov product is represented by F'. At every $v \in T_s S$, the square of $B + D_1$ is the harmonic oscillator operator

$$\sum_{j=1}^{a}\Bigl(\kappa_j^2+\lambda_j^2-\frac{\partial}{\partial\kappa_j^2}-\frac{\partial^2}{\partial\lambda_j^2}\Bigr)+2\deg-a$$

on $T\pi^{-1}(v)\cong N_s\times N_s$. (Here deg is the degree in $\bigwedge N$.) It has a one-dimensional kernel, concentrated in degree zero, generated by

$$(\eta,\zeta)\mapsto e^{-(\|\eta\|^2+\|\zeta\|^2)/2}\in C_0(N_s\times N_s). \tag{4.23}$$

Thus, over each fibre, F' is a Fredholm operator with index 1, and $\beta_N\otimes_{TN}[D_1]$ is equal to the exterior product of this Fredholm operator in $KK_G(\mathbb{C},\mathbb{C})$ and the class $[C_0(TS),0,\pi_{TS}]\in KK_G(TS,TS)$, both representing the respective ring identities. Hence the claim follows. □

Proof of Proposition 4.13. Using Lemmas 4.16 and 4.19, and associativity of the Kasparov product, we find that

$$\beta_N\otimes_{TN}[D_{TN}]=(\beta_N\otimes_{TN}[D_1])\otimes_{TS}[D_{TS}]=[D_{TS}].$$

This finishes the proof. □

We will later need the restriction of the Bott element to TS. Consider the class

$$\bigl[\tau_S^*\textstyle\bigwedge N_{\mathbb{C}}\bigr]:=\bigl[\bigoplus_j\bigwedge^{2j}\tau_S^*N\otimes\mathbb{C}\bigr]-\bigl[\bigoplus_j\bigwedge^{2j+1}\tau_S^*N\otimes\mathbb{C}\bigr]\in KK_G(TS,TS),$$

defined as in Lemma 3.2.

Lemma 4.24. *We have*

$$(j_{TS}^{TN})^*\beta_N=[\tau_S^*\textstyle\bigwedge N_{\mathbb{C}}]\in KK_G(TS,TS).$$

Proof. The Hilbert $C_0(TS)$-module in $(j_{TS}^{TN})^*\beta_N$ is $\Gamma_0\bigl(TS,\tau_S^*\bigwedge N_{\mathbb{C}}\bigr)$. Because $B|_{TS}$ is the zero operator, the claim follows. □

4D. *The Atiyah–Singer index theorem.* Suppose for now that M is compact and G is trivial. Then Kasparov's index theorem reduces to the Atiyah–Singer index theorem; see [Kasparov 2016, Remark 4.5]. We provide the details of this implication here, because these will be used in the proof of Theorem 2.16.

Consider the Atiyah–Singer topological index map

$$\mathrm{index}_t^{\mathrm{AS}}:KK(\mathrm{pt},TM)\to\mathbb{Z},$$

which maps a class $\sigma\in KK(\mathrm{pt},TM)$ to

$$\int_{TM}\mathrm{ch}(\sigma)\,\mathrm{Todd}(TM\otimes\mathbb{C}). \tag{4.25}$$

Note that we do not have the factor $(-1)^{\dim M}$ in (4.25) as in [Atiyah and Singer 1968b, Theorem 2.12], because we use a different almost complex structure on TM than in [Atiyah and Singer 1968b, p. 554], giving the opposite orientation.

Lemma 4.26. *As a map* $KK(\mathrm{pt}, TM) \to KK(\mathrm{pt}, \mathrm{pt})$, *right multiplication by* $[D_{TM}]$ *is the Atiyah–Singer topological index.*

Because of Lemma 4.26, Theorem 4.6 implies the Atiyah–Singer index theorem. Indeed, since M is compact, the map $p^M : M \to \mathrm{pt}$ is proper. By functoriality of the Kasparov product, Lemma 4.26 implies that the following diagram commutes:

$$\begin{tikzcd}
KK(M, TM) \arrow[r, "\mathrm{index}_t"] \arrow[d, "p^M_*"] & KK(M, \mathrm{pt}) \arrow[d, "p^M_*"] \\
KK(\mathrm{pt}, TM) \arrow[r, "\mathrm{index}_t^{\mathrm{AS}}"] & \mathbb{Z} = KK(\mathrm{pt}, \mathrm{pt})
\end{tikzcd}$$

By Lemma 4.5, applying the map p^M_* to both sides of (4.7), and using commutativity of the above diagram, one obtains the Atiyah–Singer index theorem.

Proof of Lemma 4.26. The proof is a reformulation of the arguments in [Atiyah and Singer 1968b], using KK-theory. There are embeddings $M \hookrightarrow \mathbb{R}^n$ with normal bundle N of rank a, and $TM \hookrightarrow T\mathbb{R}^n = \mathbb{C}^n$ with normal bundle TN. As N is homeomorphic to a tubular neighbourhood of M in $\mathbb{R}^n$, we can identify TN with an open neighbourhood of TM in $\mathbb{C}^n$. (Note that here, the submanifold S of M in Section 4B is replaced by the submanifold M of $\mathbb{R}^n$.)

Denote by

$$\beta_N \in KK(TM, TN)$$

the fibrewise Bott element over TM in TN, in the sense of Definition 4.12. Then by Proposition 4.13,

$$[D_{TM}] = \beta_N \otimes [D_{TN}]. \tag{4.27}$$

The Chern character is compatible with the pairing of K-theory and K-homology. The Chern character of the Bott generator β of $K^0(\mathbb{R}^2)$ is a generator of $H^2(\mathbb{R}^2)$. As the Dolbeault class $[D_{\mathbb{R}^2}]$ of $\mathbb{R}^2$ is dual to β, its Chern character is the Poincaré dual of $\mathrm{ch}(\beta)$. So $\mathrm{ch}[D_{\mathbb{R}^2}]$ is the fundamental class $[\mathbb{R}^2]$ of $\mathbb{R}^2$. Similarly, working with the exterior Kasparov product of n copies of β, we conclude that $\mathrm{ch}[D_{\mathbb{R}^{2n}}] = [\mathbb{R}^{2n}]$. Noting that $T\mathbb{R}^n = \mathbb{R}^{2n}$, by functoriality of the Chern character we have

$$\begin{aligned}
\mathrm{ch}[D_{TN}] &= \mathrm{ch}\big(\big(k_{TN}^{T\mathbb{R}^n}\big)_*[D_{T\mathbb{R}^n}]\big) = \big(k_{TN}^{T\mathbb{R}^n}\big)_* \mathrm{ch}[D_{T\mathbb{R}^n}] \\
&= \big(k_{TN}^{T\mathbb{R}^n}\big)_*[T\mathbb{R}^n] = [TN].
\end{aligned} \tag{4.28}$$

Thus, the Chern character of $[D_{TN}]$ is the fundamental class $[TN] \in H_{2n}(TN)$. Let $\sigma \in KK(\mathrm{pt}, TM)$ be given. Then (4.27) and (4.28) imply that

$$\sigma \otimes_{TM} [D_{TM}] = \int_{TN} \mathrm{ch}(\sigma) \wedge \mathrm{ch}(\beta_N). \tag{4.29}$$

The Thom isomorphism $\psi_{TN} : H^*(TM) \to H^*(TN)$ (mapping between compactly supported cohomologies) is an isomorphism of $H^*(TM)$-modules. So we can rewrite the integral (4.29) as

$$\begin{aligned}\int_{TN} \mathrm{ch}(\sigma) \wedge \mathrm{ch}(\beta_N) &= \int_{TM} \psi_{TN}^{-1}(\mathrm{ch}(\sigma) \wedge \mathrm{ch}(\beta_N)) \\ &= \int_{TM} \mathrm{ch}(\sigma) \wedge \psi_{TN}^{-1}(\mathrm{ch}(\beta_N)).\end{aligned} \tag{4.30}$$

To calculate $u := \psi_{TN}^{-1}(\mathrm{ch}(\beta_{TN}))$, we make use of the following diagram:

$$\begin{tikzcd} K^*(TM) \arrow[r, "\psi_{TN}"] \arrow[d, "\mathrm{ch}"'] & K^*(TN) \arrow[r, "(j_{TM}^{TN})^*"] \arrow[d, "\mathrm{ch}"'] & K^*(TM) \arrow[d, "\mathrm{ch}"'] \\ H^*(TM) \arrow[r, "\psi_{TN}"] & H^*(TN) \arrow[r, "(j_{TM}^{TN})^*"] & H^*(TM) \end{tikzcd}$$

Note that in the second line, the composition is equal to the exterior product by the Euler class $e(TN)$. In the above diagram, we have

$$\begin{tikzcd} & \beta_N \arrow[r, mapsto, "(j_{TM}^{TN})^*"] \arrow[d, mapsto, "\mathrm{ch}"'] & \sum_j (-1)^j \bigwedge^j TN \arrow[d, mapsto, "\mathrm{ch}"'] \\ u \arrow[r, mapsto, "\psi_{TN}"] & \mathrm{ch}(\beta_N) \arrow[r, mapsto, "(j_{TM}^{TN})^*"] & u \cdot e(TN) \end{tikzcd}$$

by Lemma 4.24. As the above square commutes by functoriality of the Chern character, and since $TN = \tau_M^* N_{\mathbb{C}}$ and $N_{\mathbb{C}} \oplus (TM \otimes \mathbb{C}) = M \times \mathbb{C}^n$, we obtain

$$u = \frac{\mathrm{ch}\big(\sum_j (-1)^j \bigwedge^j TN\big)}{e(TN)} = \tau_M^* \left(\frac{e(TM)}{\mathrm{ch}\big(\sum_j (-1)^j \bigwedge^j TM\big)} \right) = \tau_M^*(\mathrm{Todd}(TM \otimes \mathbb{C})).$$

Therefore, together with (4.29) and (4.30) one has

$$\sigma_{TM} \otimes [D_{TM}] = \int_{TM} \mathrm{ch}(\sigma)\, \mathrm{Todd}(TM \otimes \mathbb{C}),$$

and the lemma is proved. □

5. Proof of the fixed point formula

After proving Theorems 2.2 and 2.4 and discussing Kasparov's index theorem, we are ready to prove Theorem 2.16.

We start in Section 5A, by generalising Gysin maps, or wrong-way functoriality maps in K-theory, that play a key role in [Atiyah and Singer 1968a]. We use these generalised Gysin maps in Section 5B to set up the commutative diagrams we need.

We discuss a map defined by evaluating characters at g in Section 5C. Then we introduce a class in the topological K-theory of TM, localised at g, defined by the principal symbol of D. The properties of that class allow us to finish the proof of Theorem 2.16.

5A. *Gysin maps.* Let $S \subset M$ be a G-invariant submanifold, with inclusion map $j_S^M : S \hookrightarrow M$. (In the applications of what follows, S will be a connected component of the fixed point set M^g.) Let $N \to S$ be the normal bundle of S in M. The inclusion map $j_{TS}^{TN} : TS \hookrightarrow TN$ induces a map

$$(j_{TS}^{TN})^* : C_0(TN) \to C_0(TS)$$

by restriction. We identify TN with an open neighbourhood of TS in TM, via a G-equivariant embedding $TN \hookrightarrow TM$. In this way, we have the injective map

$$k_{TN}^{TM} : C_0(TN) \hookrightarrow C_0(TM),$$

defined by extending functions by zero.

Definition 5.1. Let A be any G-C^*-algebra. The map

$$(j_{TS}^{TM})_! : KK_G(A, C_0(TS)) \to KK_G(A, C_0(TM))$$

is the composition

$$KK_G(A, C_0(TS)) \xrightarrow{-\otimes_{C_0(TS)}\beta_N} KK_G(A, C_0(TN)) \xrightarrow{(k_{TN}^{TM})_*} KK_G(A, C_0(TM)).$$

Here $\beta_N \in KK_G(TS, TN)$ is the Bott element, as in Definition 4.12.

We also have the usual map

$$(j_{TS}^{TM})^* : KK_G(A, C_0(TM)) \to KK_G(A, C_0(TS)).$$

Lemma 5.2. *The map*

$$(j_{TS}^{TM})^* \circ (j_{TS}^{TM})_! : KK_G(A, C_0(TS)) \to KK_G(A, C_0(TS))$$

is given by the Kasparov product from the right with

$$(j_{TS}^{TN})^* \beta_N \in KK_G(C_0(TS), C_0(TS)).$$

Proof. For all $x \in KK_G(A, C_0(TS))$, functoriality of the Kasparov product implies that

$$\begin{aligned}(j_{TS}^{TM})^* \circ (j_{TS}^{TM})_!(x) &= (j_{TS}^{TM})^* \circ (k_{TN}^{TM})_*(x \otimes_{C_0(TS)} \beta_N) \\ &= x \otimes_{C_0(TS)} ((j_{TS}^{TM})^* \circ (k_{TN}^{TM})_* \beta_N).\end{aligned}$$

Since $(j_{TS}^{TM})^* \circ (k_{TN}^{TM})_* = (j_{TS}^{TN})^*$, the claim follows. □

Lemma 5.3. *For any G-invariant closed subset $X \subset M$, and any G-invariant neighbourhood V of X, the following diagram commutes*:

$$\begin{tikzcd}
KK_G(X, TS) \arrow[r, "-\otimes_{TS}[D_{TS}]"] \arrow[d, "(j_X^V)_*"] & KK_G(X, \mathrm{pt}) \arrow[dd, "(j_X^V)_*"] \\
KK_G(V, TS) \arrow[d, "(j_{TS}^{TM})_!"] & \\
KK_G(V, TM) \arrow[r, "-\otimes_{TM}[D_{TM}]"] & KK_G(V, \mathrm{pt})
\end{tikzcd}$$

Proof. For all $a \in KK_G(X, TS)$, functoriality and associativity of the Kasparov product imply that

$$\begin{aligned}
((j_{TS}^{TM})_! \circ (j_X^V)_*(a)) \otimes_{TM} [D_{TM}] &= (k_{TN}^{TM})_*((j_X^V)_*(a) \otimes_{TS} \beta_N) \otimes_{TM} [D_{TM}] \\
&= (j_X^V)_*(a) \otimes_{TS} ((k_{TN}^{TM})_*(\beta_N) \otimes_{TM} [D_{TM}]).
\end{aligned}$$

Now $(k_{TN}^{TM})^*[D_{TM}] = [D_{TN}]$, so

$$(k_{TN}^{TM})_*(\beta_N) \otimes_{TM} [D_{TM}] = \beta_N \otimes_{TN} (k_{TN}^{TM})^*[D_{TM}] = \beta_N \otimes_{TN} [D_{TN}] = [D_{TS}],$$

where the last equality was proved in Proposition 4.13. □

5B. ***Localisation and Gysin maps.*** Let U and V be as in Theorem 2.4. Consider the following diagram:

$$\begin{tikzcd}
KK_G(M, TM) \arrow[r, "\mathrm{index}_t"] \arrow[d, "(k_V^M)^*"'] & KK_G(M, \mathrm{pt}) \arrow[d, "(k_V^M)^*"] \\
KK_G(V, TM) \arrow[r, "-\otimes_{TM}[D_{TM}]"] \arrow[d, "(j_{TM^g}^{TM})^*"'] & KK_G(V, \mathrm{pt}) \\
KK_G(V, TM^g) & \\
KK_G(V, TM^g) \arrow[u, "-\otimes_{TM^g}(j_{TM^g}^{TN})^*\beta_N"] \arrow[uu, bend right=60, "(j_{TM^g}^{TM})_!"'] & \\
KK_G(\overline{U}, TM^g) \arrow[u, "(j_{\overline{U}}^V)_*"] \arrow[r, "-\otimes_{TM^g}[D_{TM^g}]"] & KK_G(\overline{U}, \mathrm{pt}) \arrow[uuu, "(j_{\overline{U}}^V)_*"']
\end{tikzcd} \tag{5.4}$$

The top part of this diagram commutes because of functoriality of the Kasparov product. The part with the product with $(j_{TM^g}^{TM})^*\beta_N$ in it commutes by Lemma 5.2, applied with $A = C_0(V)$, and S running over the connected components of M^g.

The remaining part of the diagram commutes by Lemma 5.3, applied in a similar way with S a connected component of M^g, and $X = \overline{U}$.

Diagram (5.4) can be extended as follows:

$$
\begin{tikzcd}
 & & KK_G(M, TM) \arrow[r, "\mathrm{index}_t"] \arrow[d, "(k^M_V)^*"'] & KK_G(M, \mathrm{pt}) \arrow[d, "(k^M_V)^*"] \\
KK_G(\mathrm{pt}, TM) \arrow[d, "(j^{TM}_{TM^g})^*"'] & KK_G(\overline{U}, TM) \arrow[l, "p^{\overline{U}}_*"'] \arrow[r, "(j^V_{\overline{U}})_*"] \arrow[d, "(j^{TM}_{TM^g})^*"'] & KK_G(V, TM) \arrow[r, "-\otimes_{TM}[D_{TM}]"] \arrow[d, "(j^{TM}_{TM^g})^*"'] & KK_G(V, \mathrm{pt}) \\
KK_G(\mathrm{pt}, TM^g) & KK_G(\overline{U}, TM^g) \arrow[l, "p^{\overline{U}}_*"'] \arrow[r, "(j^V_{\overline{U}})_*"] & KK_G(V, TM^g) & \\
KK_G(\mathrm{pt}, TM^g) \arrow[u, "-\otimes_{TM^g}(j^{TN}_{TM^g})^*\beta_N"] \arrow[d, "-\otimes_{TM^g}[D_{TM^g}]"'] & KK_G(\overline{U}, TM^g) \arrow[l, "p^{\overline{U}}_*"'] \arrow[r, "(j^V_{\overline{U}})_*"] \arrow[u, "-\otimes_{TM^g}(j^{TN}_{TM^g})^*\beta_N"] \arrow[d, "-\otimes_{TM^g}[D_{TM^g}]"'] & KK_G(V, TM^g) \arrow[u, "-\otimes_{TM^g}(j^{TN}_{TM^g})^*\beta_N"] \arrow[uu, bend right, "(j^{TM}_{TM^g})_!"'] & \\
KK_G(\mathrm{pt}, \mathrm{pt}) & KK_G(\overline{U}, \mathrm{pt}) \arrow[l, "p^{\overline{U}}_*"] \arrow[uuurr, bend right, "(j^V_{\overline{U}})_*"'] & &
\end{tikzcd}
\tag{5.5}
$$

The right-hand part of this diagram is diagram (5.4), and hence commutes. The other parts commute by functoriality of KK-theory and the Kasparov product.

Theorem 2.4 implies that the maps $(j^V_{\overline{U}})_*$ become invertible after localisation at g. We will also use inverses of the localised classes

$$((j^{TN}_{TM^g})^*\beta_N)_g \in KK_G(TM^g, TM^g)_g. \tag{5.6}$$

Lemma 5.7. *The element (5.6) is invertible.*

Proof. By Lemma 4.24, we have

$$(j^{TN}_{TM^g})^*\beta_N = \left[\tau^*_{M^g} \textstyle\bigwedge N_{\mathbb{C}}\right].$$

Atiyah and Segal [1968, Lemma 2.7] showed that $\left[\bigwedge N_{\mathbb{C}}\right]$ is invertible in $K^0_G(M^g)_g$. The map

$$\tau^*_{M^g} : K^0_G(M^g) \to KK_G(TM^g, TM^g)$$

sending a class $[E] \in K^0_G(M^g)$ to $[\tau^*_{M^g}E]$ is a unital ring homomorphism. Hence, so is its localisation at g. Therefore, the class

$$\left[\tau^*_{M^g} \textstyle\bigwedge N_{\mathbb{C}}\right]_g = (\tau^*_{M^g})_g \left[\textstyle\bigwedge N_{\mathbb{C}}\right]_g \in KK_G(TM^g, TM^g)_g$$

is invertible. $\square$

5C. *Evaluation.* Let X and Y be locally compact Hausdorff spaces with *trivial* actions by a compact group G. Then the exterior Kasparov product

$$KK(X, Y) \times KK_G(\mathrm{pt}, \mathrm{pt}) \to KK_G(X, Y)$$

defines an isomorphism

$$KK(X, Y) \otimes R(G) \cong KK_G(X, Y). \tag{5.8}$$

If X is a point, this is a classical fact. We will also apply this isomorphism to the class $[D_{TM^g}] \in KK_G(TM^g, \mathrm{pt})$. There it is trivial, since G acts trivially on the Hilbert space in question. In the only other case where we will use the isomorphism (5.8), we will have $X = Y$, and this space has finitely many connected components. (To be precise, we will have $X = Y = TM^g$.) Let us work out the isomorphism explicitly in that case, for the cycles we will apply it to. These are G-equivariant Kasparov $(C_0(X), C_0(X))$-cycles of the form $(\Gamma_0(E), F, \pi)$, where $E \to X$ is a vector bundle (of finite rank). Let $a \in KK_G(X, X)$ be the class of a cycle of this form, and let $b \in KK(X, X)$ be the class defined by the same cycle, where the group action is ignored. As G acts trivially on X, each fibre of E is a representation space of G. Suppose for simplicity that X is connected; the general case follows by applying the arguments to its connected components. (This works since there are finitely many of them.) Since X is connected, the representations by G on all fibres of E are equivalent. Let V be any one of these fibres, viewed as a representation space of G. Denote by 1_G the ring identity of $R(G)$, i.e., the trivial representation of G. Let $E_0 := X \times V \to X$ be the trivial bundle with fibre V. Consider the representations

$$\pi_X^X : C_0(X) \to \mathcal{B}(C_0(X)),$$
$$\pi_X^{E_0} : C_0(X) \to \mathcal{B}(\Gamma_0(E_0)),$$

defined by pointwise multiplication. Then

$$\begin{aligned}\big(\big[C_0(X), 0, \pi_X^X\big] \otimes [V]\big) + (b \otimes 1_G)& \\ = \big(\big[\Gamma_0(E_0), 0, \pi_X^{E_0}\big] \otimes 1_G\big) + a \in KK_G(X, X).&\end{aligned} \tag{5.9}$$

In fact, both sides of (5.9) are represented by the cycle

$$\big(\Gamma_0(E_0 \oplus E), 0 \oplus F, \pi_X^{E_0} \oplus \pi\big), \tag{5.10}$$

but, initially, with *different* G-actions. Namely, for the left-hand side of (5.9), G acts on the first summand E_0 in (5.10), while for the right-hand side of (5.9), G acts on the second summand E in (5.10). As G acts trivially on X, representations of G commute with those of $C_0(X)$. Since, in addition, F is G-invariant, these two actions by G can be connected by a rotation homotopy, so (5.9) follows. In that equality, a is represented as an element of $KK(X, Y) \otimes R(G)$.

In general, using (5.8), one can apply the evaluation $\mathrm{ev}_g = 1 \otimes \mathrm{ev}_g$ as a map

$$\mathrm{ev}_g : KK_G(X, Y) \to KK(X, Y) \otimes \mathbb{C}. \tag{5.11}$$

This map is compatible with localisation at g, in the sense that the following diagram commutes:

$$\begin{tikzcd} KK_G(X,Y) \arrow[r, "\mathrm{ev}_g"] \arrow[d] & KK(X,Y)\otimes\mathbb{C} \\ KK_G(X,Y)_g \arrow[ur, "(\mathrm{ev}_g)_g"'] & \end{tikzcd}$$

If $a \in KK_G(X, Y)$, we will also write

$$a(g) := \mathrm{ev}_g(a) \in KK(X, Y) \otimes \mathbb{C}.$$

The evaluation map (5.11) is compatible with Kasparov products. This follows from the facts that the isomorphism (5.8) is compatible with the product, that Kasparov products in $R(G)$ coincide with tensor products of representations, and that the character of the tensor product of two finite-dimensional representations is the product of the characters of the individual representations.

Hence, we can attach the following commutative diagram to the lower left-hand side of (5.5):

$$\begin{tikzcd} KK(\mathrm{pt}, TM^g)\otimes\mathbb{C} & KK_G(\mathrm{pt}, TM^g) \arrow[l, "\mathrm{ev}_g"'] \\ KK(\mathrm{pt}, TM^g)\otimes\mathbb{C} \arrow[u, "(-\otimes_{TM^g}(j^{TM}_{TM^g})^*\beta_N(g))\otimes 1"] \arrow[d, "(-\otimes_{TM^g}[D_{TM^g}])\otimes 1"'] & KK_G(\mathrm{pt}, TM^g) \arrow[l, "\mathrm{ev}_g"'] \arrow[u, "-\otimes_{TM^g}(j^{TN}_{TM^g})^*\beta_N"'] \arrow[d, "-\otimes_{TM^g}[D_{TM^g}]"] \\ \mathbb{C} & KK_G(\mathrm{pt}, \mathrm{pt}) \arrow[l, "\mathrm{ev}_g"] \end{tikzcd} \tag{5.12}$$

Here, $[D_{TM^g}] \in KK(TM^g, \mathrm{pt})$ is identified with $[D_{TM^g}]\otimes 1 \in KK(TM^g, \mathrm{pt})\otimes R(G)$, so that $\mathrm{ev}_g([D_{TM^g}]) = [D_{TM^g}] \otimes 1$. In particular, when $M^g = \mathrm{pt}$, the vertical map on the lower left corner is the identity.

By Lemma 4.26 and compactness of M^g, the map

$$-\otimes_{TM^g}[D_{TM^g}] : KK(\mathrm{pt}, TM^g) \to KK(\mathrm{pt}, \mathrm{pt})$$

is the Atiyah–Singer topological index map $\mathrm{index}_t^{\mathrm{AS}}$. We will use the same notation for its extension to a map $KK(\mathrm{pt}, TM^g) \otimes \mathbb{C} \to \mathbb{C}$.

Using commutativity of (5.5) and (5.12), and invertibility of the localised maps $((j^V_U)_*)_g$ and classes (5.6), we obtain the commutative diagram

$$
\begin{tikzcd}
 & KK_G(M, TM)_g \arrow[r, "(\mathrm{index}_t)_g"] \arrow[d, "(p_*^{\overline{U}})_g \circ ((j^V_{\overline{U}})_*)_g^{-1} \circ (k^M_V)^*_g"] & KK_G(M, \mathrm{pt})_g \arrow[ddd, "((j^V_{\overline{U}})_*)_g^{-1} \circ (k^M_V)^*_g"'] \\
 & KK_G(\mathrm{pt}, TM)_g \arrow[d, "(j^{TM}_{TM^g})^*"] & \\
KK(\mathrm{pt}, TM^g) \otimes \mathbb{C} \arrow[d, "\mathrm{index}_t^{\mathrm{AS}}(- \otimes_{TM^g} ((j^{TN}_{TM^g})^* \beta_N)^{-1}(g))"] & KK_G(\mathrm{pt}, TM^g)_g \arrow[l, "(\mathrm{ev}_g)_g"'] \arrow[d, "- \otimes_{TM^g} ((j^{TN}_{TM^g})^* \beta_N)_g^{-1} \otimes_{TM^g} [D_{TM^g}]_g"] & \\
\mathbb{C} & KK_G(\mathrm{pt}, \mathrm{pt})_g \arrow[l, "(\mathrm{ev}_g)_g"] & KK_G(\overline{U}, \mathrm{pt})_g \arrow[l, "(p_*^{\overline{U}})_g"]
\end{tikzcd}
\tag{5.13}
$$

5D. ***The g-symbol class.*** Recall that in (4.4) we defined the class

$$[\sigma_D] \in KK_G(M, TM).$$

The last ingredient of the proof of Theorem 2.16 is a class defined by σ_D in the topological K-theory of TM, localised at g. In Section 7, we will describe this class more explicitly, and use it to obtain another expression for the g-index.

Definition 5.14. The *g-symbol class* of D is the class σ_g^D in the localised topological K-theory of TM defined by

$$\sigma_g^D := (p_*^{\overline{U}})_g \circ ((j^V_{\overline{U}})_*)_g^{-1} \circ (k^M_V)^*_g[\sigma_D]_g \in KK_G(\mathrm{pt}, TM)_g. \tag{5.15}$$

The g-symbol class generalises the usual symbol class in the compact case.

Lemma 5.16. *If M is compact, then σ_g^D is the localisation at g of the usual class of σ_D in $KK_G(\mathrm{pt}, TM)$.*

Proof. If M is compact, then we can choose $U = V = M$. Then, since the map $p^M : M \to \mathrm{pt}$ is proper, we have

$$\sigma_g^D = (p_*^M[\sigma_D])_g,$$

which is the usual symbol class by Lemma 4.5. □

We now prove some properties of the g-symbol class that will be used in the proof of Theorem 2.16. As before, we write $\tilde{\sigma}_D := \sigma_D / \sqrt{\sigma_D^2 + 1}$.

Lemma 5.17. *The class*

$$(k^M_V)^*_g[\sigma_D]_g \in KK_G(V, TM)_g$$

is the localisation at g of the class

$$[\sigma_D|_V]_{TM} := [\Gamma_0(\tau_V^*(E|_V)), \tilde{\sigma}_D|_{TV}, \pi_V] \in KK_G(V, TM).$$

Here, the $C_0(TM)$-valued inner product on $\Gamma_0(E|_V)$ is defined by the natural $C_0(TV)$-valued inner product, composed with the inclusion k_{TV}^{TM}.

Proof. The class
$$(k_V^M)^*[\sigma_D] \in KK_G(V, TM)$$
is represented by the Kasparov cycle
$$(\Gamma_0(\tau_M^* E), \tilde{\sigma}_D, (k_V^M)^*\pi_M) = (\Gamma_0(\tau_V^*(E|_V)), \tilde{\sigma}_D|_{TV}, \pi_V) \\ \oplus (\Gamma_0(\tau_{M\setminus V}^*(E|_{M\setminus V})), \tilde{\sigma}_D|_{TM\setminus TV}, 0).$$
The second summand on the right-hand side is a degenerate cycle, so the claim follows. □

Consider the class
$$_{\overline{U}}[\sigma_D|_{TM^g}] := \big[\Gamma_0(\tau_{M^g}^*(E|_{M^g})), \tilde{\sigma}_D|_{TM^g}, \big(j_{M^g}^{\overline{U}}\big)_*\pi_{M_g}\big] \in KK_G(\overline{U}, TM^g).$$

Lemma 5.18. *We have*
$$\big(j_{\overline{U}}^V\big)_*\big(_{\overline{U}}[\sigma_D|_{TM^g}]\big) = (j_{TM^g}^{TM})^*[\sigma_D|_V]_{TM} \in KK_G(V, TM^g).$$

Proof. By definition,
$$(j_{TM^g}^{TM})^*[\sigma_D|_V]_{TM} = \big[\Gamma_0(\tau_V^*(E|_V)) \otimes_{j_{TM^g}^{TM}} C_0(TM^g), \tilde{\sigma}_D|_V \otimes 1, \pi_V \otimes 1\big].$$
The map
$$\Gamma_0(\tau_V^*(E|_V)) \otimes_{j_{TM^g}^{TM}} C_0(TM^g) \to \Gamma_0(\tau_{M^g}^*(E|_{M^g}))$$
that maps $s \otimes \varphi$ to $\varphi s|_{TM^g}$, for $s \in \Gamma_0(\tau_V^*(E|_V))$ and $\varphi \in C_0(TM^g)$, is an isomorphism of Hilbert $C_0(TM^g)$-modules. It intertwines the operators $\tilde{\sigma}_D|_V \otimes 1$ and $\tilde{\sigma}_D|_{TM^g}$, and the representations $\pi_V \otimes 1$ and
$$(j_{M^g}^V)_*\pi_{M^g} = \big(j_{\overline{U}}^V\big)_*\big(j_{M^g}^{\overline{U}}\big)_*\pi_{M^g}.$$
The lemma is then proved. □

Proposition 5.19. *The class*
$$(j_{TM^g}^{TM})_g^*\sigma_g^D \in KK_G(\mathrm{pt}, TM^g)_g$$
is the localisation at g of the usual class $[\sigma_D|_{TM^g}]$ in the equivariant topological K-theory of TM^g.

Proof. By commutativity of (the top left part of) diagram (5.5), we have
$$(j_{TM^g}^{TM})_g^*\sigma_g^D = \big(p_*^{\overline{U}}\big)_g \circ \big(\big(j_{\overline{U}}^V\big)_*\big)_g^{-1} \circ (j_{TM^g}^{TM})_g^* \circ (k_V^M)_g^*[\sigma_D]_g.$$
By Lemma 5.17, we have
$$(k_V^M)_g^*[\sigma_D]_g = ([\sigma_D|_V]_{TM})_g.$$

By Lemma 5.18 we have

$$\left((j_{\overline{U}}^V)_*\right)_g^{-1} \circ (j_{TM^g}^{TM})_g^*([\sigma_D|_V]_{TM})_g =_{\overline{U}} [\sigma_D|_{TM^g}]_g.$$

By Lemma 4.5, we have

$$p_*^{\overline{U}}\left(_{\overline{U}}[\sigma_D|_{TM^g}]\right) = [\sigma_D|_{TM^g}] \in KK_G(\mathrm{pt}, TM^g).$$

So the claim follows. □

We have now finished all preparation needed to prove Theorem 2.16.

Proof of Theorem 2.16. Using Kasparov's index theorem, Theorem 4.6, and commutativity of (5.13), we find that

$$\begin{aligned}
\mathrm{index}_g(D) &= (\mathrm{ev}_g)_g \circ \left(p_*^{\overline{U}}\right)_g \circ \left((j_{\overline{U}}^V)_*\right)_g^{-1} \circ (k_V^M)_g^*[D] \\
&= (\mathrm{ev}_g)_g \circ \left(p_*^{\overline{U}}\right)_g \circ \left((j_{\overline{U}}^V)_*\right)_g^{-1} \circ (k_V^M)_g^* \circ (\mathrm{index}_t)_g[\sigma_D]_g \\
&= \mathrm{index}_t^{\mathrm{AS}}\left(((j_{TM^g}^{TM})^*\sigma_g^D)(g) \otimes_{TM^g} ((j_{TM^g}^{TN})^*\beta_N)^{-1}(g)\right).
\end{aligned}$$

By Lemma 4.24 and Proposition 5.19, the latter expression equals

$$\mathrm{index}_t^{\mathrm{AS}}\left([\sigma_D|_{TM^g}](g) \otimes_{TM^g} \left[\textstyle\bigwedge N_{\mathbb{C}}\right]^{-1}(g)\right).$$

Furthermore,

$$[\sigma_D|_{TM^g}](g) \otimes_{TM^g} \left[\tau_{M^g}^*\textstyle\bigwedge N_{\mathbb{C}}\right]^{-1}(g) = [\sigma_D|_{TM^g}](g) \cdot \left[\textstyle\bigwedge N_{\mathbb{C}}\right]^{-1}(g),$$

where the dot means the right $K_G^0(M^g)$-module structure of $K_G^0(TM^g)$. We conclude that

$$\mathrm{index}_g(D) = \mathrm{index}_t^{\mathrm{AS}}\left([\sigma_D|_{TM^g}](g) \cdot \left[\textstyle\bigwedge N_{\mathbb{C}}\right]^{-1}(g)\right).$$

Theorem 2.16 now follows from the definition of the topological index map (4.25), and multiplicativity of the Chern character. □

5E. ***The index pairing.*** The arguments used to prove Theorem 2.16 also imply Theorem 2.18 about the index pairing. In fact, the parts of the proof of Theorem 2.16 about localisation in the first entry of KK-theory are not needed in the proof of Theorem 2.18.

The key step is a localisation property of the K-homology class of D, localised at g.

Proposition 5.20. *We have*

$$[D]_g = (j_{TM^g}^{TM})_g^*[\sigma_D]_g \otimes_{TM^g} \left[\tau_{M^g}^*\textstyle\bigwedge N_{\mathbb{C}}\right]_g^{-1} \otimes_{TM^g} [D_{TM^g}]_g \in KK_G(M, \mathrm{pt})_g.$$

Proof. Lemmas 5.2 and 5.3 imply that the following diagram commutes:

$$\begin{tikzcd} KK_G(M, TM) \arrow[r, "-\otimes_{TM}[D_{TM}]"] \arrow[d, "(j^{TM}_{TM^g})^*"'] & KK_G(M, \mathrm{pt}) \\ KK_G(M, TM^g) & KK_G(M, TM^g) \arrow[l, "-\otimes_{TM^g}(j^{TN}_{TM^g})^*\beta_N"] \arrow[ul, "(j^{TM}_{TM^g})_!"'] \arrow[u, "-\otimes_{TM^g}[D_{TM^g}]"'] \end{tikzcd}$$

Therefore, the claim follows from Lemmas 4.24 and 5.7, and Theorem 4.6. □

Proof of Theorem 2.18. Let $[F] \in KK_G(\mathrm{pt}, M)$ be as in Section 2E. By compatibility of the Kasparov product with localisation and evaluation, Proposition 5.20 implies that

$$\begin{aligned} ([F] \otimes_M [D])(g) &= ([F]_g \otimes_M [D]_g)(g) \\ &= ([F]_g \otimes_M (j^{TM}_{TM^g})^*_g[\sigma_D]_g)(g) \otimes_{TM^g} \big[\tau^*_{M^g} {\textstyle\bigwedge} N_{\mathbb{C}}\big](g)^{-1} \otimes_{TM^g} [D_{TM^g}](g). \end{aligned}$$

Now

$$([F]_g \otimes_M (j^{TM}_{TM^g})^*_g[\sigma_D]_g)(g) = [\tau^*_{M^g}(F|_{M^g})](g) \otimes [\sigma_D|_{TM^g}](g) \in KK(\mathrm{pt}, TM^g) \otimes \mathbb{C},$$

where on the right-hand side, the tensor product denotes the ring structure on the topological K-theory of TM^g. Therefore, and because $[D_{TM^g}](g) = [D_{TM^g}] \otimes 1$ is in $KK(TM^g, \mathrm{pt}) \otimes \mathbb{C}$, the claim follows from Lemma 4.26. □

6. Examples and applications

The g-index was defined in terms of KK-theory, but Theorem 2.16 allows us to express it entirely in cohomological terms. Using this theorem, we can compute the g-index explicitly in examples, and show how it is related to other indices.

For finite fixed point sets, Theorem 2.16 has a simpler form, as discussed in Section 6A. In Section 6B, we give a linearisation theorem for the g-index of a twisted Dolbeault–Dirac operator on a complex manifold, in the case of a finite fixed point set. We then work out the example of the Dolbeault–Dirac operator on the complex plane, acted on by the circle, in Section 6C. An illustration of the linearisation theorem is given in Section 6D, where we apply it to the two-sphere, to decompose the usual equivariant index. In Section 6E, we realise characters of discrete series representations of semisimple Lie groups on regular points of a maximal torus, in terms of the g-index. For Fredholm operators, and in particular Callias-type deformations of Dirac operators, we describe the relation between the g-index and the character of the action by g on the kernel of such an operator, in Section 6F. We then give a relation with an index studied by Braverman in Section 6G, and a relative index theorem along the lines of work by Gromov and

Lawson in Section 6H. In Section 6I, we mention some geometric consequences of the vanishing or nonvanishing of the g-index of a Hodge–Dirac or Spin-Dirac operator.

6A. ***Finite fixed point sets.*** If the fixed point set M^g is zero-dimensional, then $TM^g = M^g$, τ_{M^g} is the identity map, $\mathrm{Todd}(TM^g \otimes \mathbb{C})$ is trivial and

$$\mathrm{ch}([\sigma_D|_{TM^g}](g)) = \mathrm{Tr}(g|_{E^+}) - \mathrm{Tr}(g|_{E^-}).$$

Furthermore, since M^g only consists of isolated points, we have

$$K^0(M^g) = \bigoplus_{m \in M^g} \mathbb{Z} = H^*(M^g),$$

and the Chern character is the identity map. So we now have, at a fixed point $m \in M^g$,

$$\mathrm{ch}\bigl([\textstyle\bigwedge N_{\mathbb{C}}](g)\bigr)_m = \mathrm{ch}\bigl([\textstyle\bigwedge TM_{\mathbb{C}}|_{M^g}](g)\bigr)_m = \det_{\mathbb{R}}(1 - g|_{T_m M}).$$

The last equality is obtained by evaluating the virtual character of $\bigwedge T_m M_{\mathbb{C}}$ at g, so one obtains

$$\mathrm{Tr}_{\mathbb{C}}(g|_{\bigwedge^{\mathrm{even}} T_m M_{\mathbb{C}}}) - \mathrm{Tr}_{\mathbb{C}}(g|_{\bigwedge^{\mathrm{odd}} T_m M_{\mathbb{C}}}).$$

Therefore, Theorem 2.16 implies the following generalisation of Atiyah and Bott's fixed point theorem [1968, Theorem A] to noncompact manifolds, but for compact G.

Corollary 6.1. *When M^g is a finite set of points,*

$$\mathrm{index}_g(D) = \sum_{m \in M^g} \frac{\mathrm{Tr}(g|_{E_m^+}) - \mathrm{Tr}(g|_{E_m^-})}{\det_{\mathbb{R}}(1 - g^{-1}|_{T_m M})}. \tag{6.2}$$

Remark 6.3. In the statement of the Atiyah–Bott fixed point theorem, the denominator is $|\det_{\mathbb{R}}(1 - g|_{T_m M})|$. In our case, g is contained in a compact group G, so the real eigenvalues of g are 1 or -1. Thus $\det_{\mathbb{R}}(1 - g^{-1}|_{T_m M})$ is always positive. See also page 186 in [Berline et al. 1992]. Also, the fact that g acts orthogonally on $T_m M$ implies that $\det_{\mathbb{R}}(1 - g^{-1}|_{T_m M}) = \det_{\mathbb{R}}(1 - g|_{T_m M})$.

Now suppose that M is a complex manifold, and g is holomorphic. Let $F \to M$ be a holomorphic vector bundle, and consider the Dolbeault–Dirac operator $\bar\partial_F + \bar\partial_F^*$ on M, coupled to F.

Corollary 6.4. *If M^g is a finite set of points, then*

$$\mathrm{index}_g(\bar\partial_F + \bar\partial_F^*) = \sum_{m \in M^g} \frac{\mathrm{Tr}_{\mathbb{C}}(g|_{F_m})}{\det_{\mathbb{C}}(1 - g^{-1}|_{T_m M})}. \tag{6.5}$$

For equivalent expressions, note that

$$\det_{\mathbb{C}}(1-g^{-1}|_{T_m^{1,0}M}) = \det_{\mathbb{C}}(1-g^{-1}|_{T_mM}) = \det_{\mathbb{C}}(1-g|_{T_m^{0,1}M})$$

in (6.5).

Proof. In Theorem 4.12 of [Atiyah and Bott 1968], it is shown that in this situation, the right-hand side of (6.2) equals the right-hand side of (6.5). The key observation is that the supertrace of $g|_{\bigwedge^*(T^{0,1}M)}$ is cancelled by the second factor in

$$\det_{\mathbb{R}}(1-g^{-1}|_{T_mM}) = \det_{\mathbb{C}}(1-g^{-1}|_{T_m^{1,0}M})\det_{\mathbb{C}}(1-g^{-1}|_{T_m^{0,1}M}).$$

(See also [Berline et al. 1992, Corollary 6.8].) □

6B. ***A holomorphic linearisation theorem.*** A tool used in some index problems is a *linearisation theorem*, relating an index to indices on vector spaces. See for example Chapter 4 of [Guillemin et al. 2002] and Theorem 7.2 in [Braverman 2002]. A version for Callias-type operators can be deduced from Theorem 2.16 in [Braverman and Shi 2016]. In those references, cobordism arguments are used to prove linearisation theorems. We will use the excision property of the g-index to obtain an analogous result. (So we do not use Theorem 2.16 here.) We will state and prove this result in the setting of Corollary 6.4, where M is a complex manifold, D is the Dolbeault–Dirac operator coupled to a holomorphic vector bundle $F \to M$, and M^g is finite. A more general statement, where M^g is not finite or D is not a Dolbeault–Dirac operator, is possible, but would be less explicit.

Under these assumptions, for any $m \in M^g$, let $\bar{\partial}^{T_mM}$ be the Dolbeault operator on the complex vector space T_mM.

Corollary 6.6 (holomorphic linearisation theorem). *We have*

$$\mathrm{index}_g(\bar{\partial}_F + \bar{\partial}_F^*) = \sum_{m\in M^g} \mathrm{Tr}_{\mathbb{C}}(g|_{F_m})\,\mathrm{index}_g(\bar{\partial}^{T_mM} + (\bar{\partial}^{T_mM})^*).$$

Proof. By Lemma 2.11, the g-index of $\bar{\partial}_F + \bar{\partial}_F^*$ equals the g-index of the Dolbeault–Dirac operator on the union over $m \in M^g$ of the tangent spaces T_mM, coupled to the vector bundle which on every space T_mM is trivial with fibre F_m. It follows directly from the definition that the g-index is additive with respect to disjoint unions. Hence

$$\begin{aligned}\mathrm{index}_g(\bar{\partial}_F + \bar{\partial}_F^*) &= \sum_{m\in M^g} \mathrm{index}_g(\bar{\partial}^{T_mM}\otimes 1_{F_m} + (\bar{\partial}^{T_mM})^*\otimes 1_{F_m})\\ &= \sum_{m\in M^g} \mathrm{Tr}_{\mathbb{C}}(g|_{F_m})\,\mathrm{index}_g(\bar{\partial}^{T_mM} + (\bar{\partial}^{T_mM})^*). \qquad \square\end{aligned}$$

An example on computing and explicitly realising an index of the form

$$\mathrm{index}_g(\bar{\partial}^{T_mM} + (\bar{\partial}^{T_mM})^*),$$

as in Corollary 6.6, is given in the next subsection. An example showing that the linearisation theorem gives a natural result if M is compact is given in Section 6D.

6C. *The circle acting on the plane.* Consider the usual action by the circle $\mathbb{T}^1 = \mathrm{U}(1)$ on the complex plane $\mathbb{C}$, and the (untwisted) Dolbeault–Dirac operator $\bar\partial + \bar\partial^*$ on $\mathbb{C}$. We will compute the distribution Θ on $\mathbb{T}^1$ given by the function

$$g \mapsto \mathrm{index}_g(\bar\partial + \bar\partial^*). \tag{6.7}$$

This function is defined on the set of elements $g \in \mathbb{T}^1$ with dense powers, i.e., the elements of the form $g = e^{\sqrt{-1}\alpha}$, where $\alpha \in \mathbb{R} \setminus 2\pi\mathbb{Q}$. So the function is defined almost everywhere.

By Corollary 6.4, we have for such g

$$\mathrm{index}_g(\bar\partial + \bar\partial^*) = \frac{1}{1-g^{-1}}.$$

So the function (6.7) is given by $g \mapsto 1/(1-g^{-1})$ almost everywhere. One can deduce that the sum of functions

$$\sum_{k=0}^{\infty}(g \mapsto g^{-k}) \tag{6.8}$$

converges as a distribution on $\mathbb{T}^1$ to Θ.

This allows us to describe the g-index of $\bar\partial + \bar\partial^*$ in terms of its kernel. Indeed, consider the Euclidean density $dz = dx\,dy$ on $\mathbb{C}$, and the corresponding space $L^2(\mathbb{C})$. Let $\mathcal{O}(\mathbb{C})$ be the space of holomorphic functions on $\mathbb{C}$. Let $\psi \in C^\infty(\mathbb{C})$ be a positive, $\mathbb{T}^1$-invariant function. Let $L^2(\mathbb{C}, \psi)$ be the completion of $C_c^\infty(\mathbb{C})$ to a Hilbert space with respect to the inner product

$$(f_1, f_2)_\psi := (\psi f_1, \psi f_2)_{L^2(\mathbb{C})}. \tag{6.9}$$

Let π be the representation of $\mathbb{T}^1$ in $L^2(\mathbb{C}, \psi)$ given by

$$(\pi(g) f)(z) = f(g^{-1}z),$$

for all $g \in \mathbb{T}^1$, $f \in L^2(\mathbb{C}, \psi)$ and $z \in \mathbb{C}$.

Set

$$\mathcal{O}_{L^2}(\mathbb{C}, \psi) := \mathcal{O}(\mathbb{C}) \cap L^2(\mathbb{C}, \psi).$$

For $k \in \mathbb{Z}_{\geq 0}$, let $e^k \in \mathcal{O}(\mathbb{C})$ be the function $z \mapsto z^k$. Then for all $k \in \mathbb{Z}_{\geq 0}$ and $z \in \mathbb{C}$,

$$\pi(g)e^k = g^{-k}e^k. \tag{6.10}$$

Suppose ψ was chosen so that $e^k \in L^2(\mathbb{C}, \psi)$ for all k. For example, one can take $\psi(z) = e^{-|z|^2/2}$.

Let $\Omega^{0,*}_{L^2}(\mathbb{C})$ be the Hilbert space of square-integrable forms of type $(0, *)$. Let $\Omega^{0,*}_{L^2}(\mathbb{C}, \psi)$ be the analogous Hilbert space with the inner product weighted by ψ as in (6.9). Set

$$\ker_{L^2,\psi}(\bar\partial + \bar\partial^*)^{\pm} := \ker(\bar\partial + \bar\partial^*)^{\pm} \cap \Omega^{0,*}_{L^2}(\mathbb{C}, \psi).$$

We can realise the distribution Θ given by the g-indices of $\bar\partial + \bar\partial^*$ in terms of the representation of $\mathbb{T}^1$ in this space.

Proposition 6.11. *The restriction of the representation π of $\mathbb{T}^1$ to $\ker_{L^2,\psi}(\bar\partial+\bar\partial^*)^{\pm}$ has a distributional character $\chi^{\pm}$, and we have*

$$\Theta = \chi^+ - \chi^- \in \mathcal{D}'(\mathbb{T}^1).$$

Proof. First note that

$$\ker(\bar\partial + \bar\partial^*)^+ = \mathcal{O}(\mathbb{C}),$$
$$\ker(\bar\partial + \bar\partial^*)^- = 0.$$

So we only need to consider the even part of $\ker_{L^2,\psi}(\bar\partial + \bar\partial^*)$, which equals

$$\ker_{L^2,\psi}(\bar\partial + \bar\partial^*)^+ = \mathcal{O}_{L^2}(\mathbb{C}, \psi). \tag{6.12}$$

The functions $\{e^k\}_{k\geq 0}$ form an orthogonal basis of $\mathcal{O}_{L^2}(\mathbb{C}, \psi)$. By (6.10), the character of the representation π on the space (6.12) equals the series (6.8), which converges to Θ. □

Remark 6.13. The $L^2(\mathbb{C}, \psi)$-kernel of $\bar\partial + \bar\partial^*$ can be identified as the L^2-kernel of a deformed operator. For example, let $\psi(z) = e^{-|z|^2/2}$. Recall that $\bar\partial + \bar\partial^*$ is an operator on $\Omega^{0,*}(\mathbb{C})$, given by

$$\bar\partial + \bar\partial^* = c(dz)\frac{\partial}{\partial z} + c(d\bar z)\frac{\partial}{\partial \bar z},$$

where now $c(d\bar z) = \frac{1}{\sqrt{2}}\operatorname{ext}(d\bar z)$ and $c(dz) = -\frac{1}{\sqrt{2}}\operatorname{int}(dz)$. (See [Berline et al. 1992, Section 3.6].) Set

$$b := \frac{1}{2} z c(d\bar z).$$

Then $b^* = -\frac{1}{2}\bar z c(dz)$. We have the deformed operator

$$\bar\partial + b = c(d\bar z)\Big(\frac{\partial}{\partial \bar z} + \frac{z}{2}\Big) : \Omega^{0,0}(\mathbb{C}) \to \Omega^{0,1}(\mathbb{C}),$$
$$(\bar\partial + b)^* = c(dz)\Big(\frac{\partial}{\partial z} - \frac{\bar z}{2}\Big) : \Omega^{0,1}(\mathbb{C}) \to \Omega^{0,0}(\mathbb{C}).$$

The operator $U : \Omega^{0,*}(\mathbb{C}) \to \Omega^{0,*}(\mathbb{C}, \psi)$ given by $U(\alpha) = \psi^{-1}\alpha$ is a unitary isomorphism. We have

$$\bar\partial U(f) = \bar\partial(\psi^{-1} f) = \psi^{-1}\Big(\bar\partial + \frac{z}{2}\Big) f = U\Big(\Big(\bar\partial + \frac{z}{2}\Big) f\Big).$$

Similarly, U intertwines $\bar\partial^*$ and $(\bar\partial+b)^*$. It then follows that

$$\ker_{L^2}(\bar\partial+b)\cong\ker_{L^2,\psi}(\bar\partial),$$
$$\ker_{L^2}(\bar\partial+b)^*\cong\ker_{L^2,\psi}(\bar\partial^*)=0.$$

6D. ***The circle acting on the two-sphere.*** As in Section 6C, we consider the circle group $\mathbb{T}^1$, this time acting by rotations on the two-sphere S^2. In this compact setting, the usual index theory, and the Atiyah–Segal–Singer theorem apply. But we can use the g-index to decompose indices in this case.

We embed $\mathbb{T}^1\cong\mathrm{SO}(2)$ into $\mathrm{SO}(3)$ in the top-left corner. Then $S^2=\mathrm{SO}(3)/\mathbb{T}^1$. Identifying this space with $\mathbb{P}^1(\mathbb{C})$, we obtain a complex structure on it. Fix $n\in\mathbb{Z}_{\geq 0}$. Let $\mathbb{C}_n$ be the space of complex numbers, on which $\mathbb{T}^1$ acts by

$$g\cdot z=g^n z,$$

for $g\in\mathbb{T}^1$ and $z\in\mathbb{C}_n$. We have the line bundle

$$L_n:=\mathrm{SO}(3)\times_{\mathbb{T}^1}\mathbb{C}_n\to S^2.$$

Let $\bar\partial_n+\bar\partial_n^*$ be the Dolbeault–Dirac operator on S^2, coupled to L_n. Since S^2 is compact, we have the equivariant index

$$\mathrm{index}_{\mathrm{SO}(3)}(\bar\partial_n+\bar\partial_n^*)\in R(\mathrm{SO}(3)).$$

By the Borel–Weil–Bott theorem, this index is the irreducible representation V_n of $\mathrm{SO}(3)$ with highest weight n (with respect to the positive root corresponding to the identification of S^2 with $\mathbb{P}^1(\mathbb{C})$).

Fix an element $g\in\mathbb{T}^1$ with dense powers. By the Atiyah–Segal–Singer theorem, or Corollary 6.4, the character of V_n evaluated at g equals

$$\mathrm{index}_{\mathbb{T}^1}(\bar\partial_n+\bar\partial_n^*)(g)=\frac{g^n}{1-g^{-1}}+\frac{g^{-n}}{1-g}. \tag{6.14}$$

The two terms on the right-hand side correspond to the two fixed points of the action by $\mathbb{T}^1$. This expression can be rewritten as the finite sum

$$\sum_{j=0}^{2n}g^{j-n}.$$

This is the usual decomposition of $V_n|_{\mathbb{T}^1}$ into irreducible representations of $\mathbb{T}^1$.

So far, we have done nothing new in this example. But let $\bar\partial^{\mathbb{C}}+(\bar\partial^{\mathbb{C}})^*$ be the Dolbeault–Dirac operator on $\mathbb{C}$. Then the linearisation theorem, Corollary 6.6, implies that

$$\mathrm{index}_{\mathbb{T}^1}(\bar\partial_n+\bar\partial_n^*)(g)=\mathrm{index}_g(\bar\partial^{\mathbb{C}}+(\bar\partial^{\mathbb{C}})^*)g^n+\mathrm{index}_{g^{-1}}(\bar\partial^{\mathbb{C}}+(\bar\partial^{\mathbb{C}})^*)g^{-n}.$$

As we saw in Section 6C, Corollary 6.4 implies that

$$\mathrm{index}_g(\bar{\partial}^{\mathbb{C}} + (\bar{\partial}^{\mathbb{C}})^*) = \frac{1}{1-g^{-1}},$$

and likewise with g replaced by g^{-1}. This agrees with (6.14). Using Proposition 6.11, we can realise the latter index as the character of the representation of $\mathbb{T}^1$ in

$$\ker_{L^2,\psi}(\bar{\partial} + \bar{\partial}^*)^+,$$

with ψ as in Section 6C.

6E. *Discrete series characters.* In this subsection only, we use the letter G to denote a connected, semisimple Lie group. Let $T < G$ be a maximal torus, and suppose it is a Cartan subgroup of G, i.e., G has discrete series representations. (The torus T plays the role that the group G plays in the rest of this paper; we have changed notation because this is standard in the current setting.) Let $K < G$ be a maximal compact subgroup containing T. We denote the normalisers of T in G and K by $N_G(T)$ and $N_K(T)$, respectively.

Lemma 6.15. *The fixed point set of the action by T on G/T is $N_K(T)/T$, the Weyl group W_c of* $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

Proof. Since

$$(G/T)^T = N_G(T)/T,$$

it is enough to show that

$$N_G(T) = N_K(T).$$

To prove this, let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the Cartan decomposition of $\mathfrak{g}$. Suppose $X \in \mathfrak{p}$, such that $\exp(tX) \in N_G(T)$ for all $t \in \mathbb{R}$. Then for all $H \in \mathfrak{t}$,

$$\exp(tX)\exp(H)\exp(-tX) = \exp(\mathrm{Ad}(\exp(tX))H) \in T.$$

So $[X, H] \in \mathfrak{t}$. Because $X \in \mathfrak{p}$ and $H \in \mathfrak{t} \subset \mathfrak{k}$, we have $[X, H] \in \mathfrak{p}$. Hence $[X, H] = 0$. Since $\mathfrak{t}$ is maximal commutative, we find that $X \in \mathfrak{t}$, so that $X = 0$. Therefore, an element $Y \in \mathfrak{g}$ such that $\exp(tY) \in N_G(T)$ for all $t \in \mathbb{R}$ must lie in $\mathfrak{k}$. Since G is connected, the claim follows. $\square$

Example 6.16. If $G = SL(2, \mathbb{R})$, then a strongly elliptic coadjoint orbit of G is equivariantly diffeomorphic to G/T. This is now a hyperbolic plane, on which T acts by rotations. This action has one fixed point, corresponding to the trivial Weyl group of $K = T$.

Let $\lambda \in i\mathfrak{t}^*$ be regular (in the sense that $(\alpha, \lambda) \neq 0$ for all roots α, for a Weyl group invariant inner product). Fix a set R^+ of positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ by defining a root α to be positive if $(\alpha, \lambda) > 0$. Let ρ be half the sum of the positive roots.

The choice of positive roots determines a G-invariant complex structure on the manifold G/T, defined by

$$T^{0,1}_{eT}(G/T) = (\mathfrak{g}/\mathfrak{t})^{0,1} := \bigoplus_{\alpha\in R^+} (\mathfrak{g}_{\mathbb{C}})_{-\alpha}. \tag{6.17}$$

Suppose $\lambda+\rho$ is an integral weight. Then $\lambda-\rho$ is integral as well, and we have the holomorphic line bundle

$$L_{\lambda-\rho} := G \times_T \mathbb{C}_{\lambda-\rho} \to G/T,$$

where T acts on $\mathbb{C}_{\lambda-\rho} := \mathbb{C}$ via the weight $e^{\lambda-\rho}$. Let

$$\bar{\partial}_{L_{\lambda-\rho}} + \bar{\partial}^*_{L_{\lambda-\rho}}$$

be the Dolbeault–Dirac operator on G/T, coupled to $L_{\lambda-\rho}$.

Let Θ_λ be the distributional character of the discrete series representation of G with infinitesimal character λ.

Proposition 6.18. *Let $g \in T$ be such that the powers of g are dense in T. (Then in particular, g is a regular element.) One has*

$$\operatorname{index}_g(\bar{\partial}_{L_{\lambda-\rho}} + \bar{\partial}^*_{L_{\lambda-\rho}}) = (-1)^{\dim(G/K)/2}\Theta_\lambda(g).$$

Proof. The proof is analogous to Atiyah and Bott's derivation of the Weyl character formula from their fixed point theorem in [Atiyah and Bott 1968, Section 5]. By Corollary 6.4 and Lemma 6.15, we have

$$\operatorname{index}_g(\bar{\partial}_{L_{\lambda-\rho}} + \bar{\partial}^*_{L_{\lambda-\rho}}) = \sum_{aT\in N_K(T)/T} \frac{e^{\lambda-\rho}(a^{-1}ga)}{\det(1-\operatorname{Ad}^{0,1}_{\mathfrak{g}/\mathfrak{t}}(a^{-1}ga))}. \tag{6.19}$$

Here $\operatorname{Ad}^{0,1}_{\mathfrak{g}/\mathfrak{t}} : T \to \mathrm{GL}((\mathfrak{g}/\mathfrak{t})^{0,1})$ is induced by the adjoint representation. Because of (6.17), we have

$$\det(1-\operatorname{Ad}^{0,1}_{\mathfrak{g}/\mathfrak{t}}(a^{-1}ga)) = \prod_{\alpha\in R^+} (1-e^{-\alpha}(a^{-1}ga)).$$

Since in the identification $N_K(T)/T = W_c$, the normaliser $N_K(T)$ acts on $i\mathfrak{t}^*$ via the coadjoint action, we find that (6.19) equals

$$\sum_{w\in W_c} \frac{e^{w\cdot(\lambda-\rho)}}{\prod_{\alpha\in R^+}(1-e^{-w\cdot\alpha})}(g). \tag{6.20}$$

Consider the Weyl denominator

$$\Delta := e^\rho \prod_{\alpha\in R^+} (1-e^{-\alpha}).$$

One has, for all $w \in W_c$,

$$w \cdot \Delta := e^{w\cdot\rho} \prod_{\alpha\in R^+} (1 - e^{-w\cdot\alpha}) = \varepsilon(w)\Delta,$$

where $\varepsilon(w) = \det w$ is the sign of w. Hence we find that (6.20) equals

$$\frac{\sum_{w\in W_c} \varepsilon(w) e^{w\cdot\lambda}}{\Delta}(g).$$

(This expression still makes sense if ρ is not an integral weight.) By Harish-Chandra's character formula for the discrete series (see [Harish-Chandra 1966, Theorem 16] or [Knapp 2001, Theorem 12.7]), this is $(-1)^{\dim(G/K)/2}\Theta_\lambda(g)$. □

Note that Proposition 6.18 only relates the value of the character Θ_λ at g to the g-index of $\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}}$ if g is a regular element of some maximal torus. Such elements form an open subset of G, and characters are not determined by their restrictions to this set. However, we can still use Proposition 6.18 to give a description of the g-index in terms of the kernel of $\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}}$.

Proposition 6.21. *Suppose G is a linear group. Then the representation of G in the L^2-kernel of $(\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}})^\pm$ has a distributional character $\Theta^\pm$ that can be evaluated at g, and one has*

$$\mathrm{index}_g(\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}}) = \Theta^+(g) - \Theta^-(g).$$

Proof. This follows from Proposition 6.18 and Schmid's realisation of the discrete series in the L^2-Dolbeault cohomology of G/T with values in $L_{\lambda-\rho}$, in [Schmid 1976, Theorem 1.5]. Schmid's result implies that the space

$$\ker_{L^2}(\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}})^\pm$$

equals zero if $\pm = -(-1)^{\dim(G/K)/2}$, and the representation of G in this space is the discrete series representation with infinitesimal character λ if $\pm = (-1)^{\dim(G/K)/2}$. (The integer k in Schmid's result now equals $\dim(G/K)/2$, and his λ is our $\lambda - \rho$.) Hence,

$$\Theta^+ - \Theta^- = (-1)^{\dim(G/K)/2}\Theta_\lambda.$$

So the claim follows from Proposition 6.18. □

Paradan [2003] gave a realisation of restrictions of discrete series representations to maximal compact subgroups, as an equivariant index of a deformation of the operator $\bar\partial_{L_{\lambda-\rho}} + \bar\partial^*_{L_{\lambda-\rho}}$. That realisation allowed him to apply the *quantisation commutes with reduction* principle to find a geometric formula for multiplicities of K-types.

In the paper [Hochs and Wang 2017], we further explore the relation between index theory and characters of the discrete series.

6F. ***Fredholm operators.*** We return to the notation used in the rest of this paper except for Section 6E, where G is a compact Abelian group generated topologically by an element $g \in G$.

For Fredholm operators, it is a natural question how the g-index of such an operator is related to the traces of g acting on even and odd parts of its kernel. This depends on the behaviour of the operator "towards infinity". To make this more explicit, let M^+ be the one-point compactification of M. The point at infinity is fixed by g. Let $U, V \subset M$ be as in Section 2B. Let $U', V' \subset M^+$ be g-invariant neighbourhoods of the point at infinity, such that $\overline{U'} \subset V'$, and $V \cap V' = \varnothing$. Then $U \sqcup U'$ and $V \sqcup V'$ are neighbourhoods of $(M^+)^g$ as in (2.5). Lemma 2.6 therefore implies that for any σ-unital G-C^* algebra A, the following diagram commutes:

$$\begin{tikzcd}
KK_G(C(M^+), A)_g \arrow[r, "(k_V^{M^+})_g^* \oplus (k_{V'}^{M^+})_g^*"'] \arrow[dd, "(p_*^{M^+})_g"'] & KK_G(C_0(V), A)_g \oplus KK_G(C_0(V'), A)_g \arrow[d, "((j_U^V)_*)_g^{-1} \oplus ((j_{U'}^{V'})_*)_g^{-1}"'] \\
 & KK_G(C(\overline{U}), A)_g \oplus KK_G(C(\overline{U'}), A)_g \arrow[d, "(p_*^{\overline{U}})_g \oplus (p_*^{\overline{U'}})_g"'] \\
KK_G(\mathbb{C}, A)_g & KK_G(\mathbb{C}, A)_g \oplus KK_G(\mathbb{C}, A)_g \arrow[l, "+"']
\end{tikzcd} \tag{6.22}$$

Indeed, since M^+ is compact, one can apply Lemma 2.6 to the pairs of neighbourhoods $U \sqcup U' \subset V \sqcup V'$ and $M^+ \subset M^+$ of $(M^+)^g$.

Now suppose that $(D^2+1)^{-1}$ is a compact operator. Then $F := D/\sqrt{D^2+1}$ is Fredholm, so $\ker_{L^2}(D)$ is finite-dimensional. Let the representation

$$\pi_{M^+} : C(M^+) \to \mathcal{B}(L^2(E))$$

be defined for $f \in C_0(M)$ and $z \in \mathbb{C}$ by

$$\pi_{M^+}(f+z) = \pi_M(f) + z. \tag{6.23}$$

Then the triple $(L^2(E), F, \pi_{M^+})$ is a Kasparov $(C(M^+), \mathbb{C})$-module. Let

$$_{M^+}[D] \in KK_G(M^+, \mathrm{pt}) \tag{6.24}$$

be its class. In this case, we will write $\mathrm{index}_g^\infty(D)$ for a version of the g-index of D that captures the behaviour of D at infinity:

$$\mathrm{index}_g^\infty(D) := (\mathrm{ev}_g) \circ \left(p_*^{\overline{U'}}\right)_g \circ \left(\left(j_{\overline{U'}}^{V'}\right)_*\right)_g^{-1} \circ (k_{V'}^{M^+})_g^*({}_{M^+}[D]_g). \tag{6.25}$$

Proposition 6.26. *If* $(D^2+1)^{-1}$ *is compact, then*

$$\mathrm{Tr}(g \text{ on } \ker_{L^2}(D^+)) - \mathrm{Tr}(g \text{ on } \ker_{L^2}(D^-)) = \mathrm{index}_g(D) + \mathrm{index}_g^\infty(D). \tag{6.27}$$

Proof. By commutativity of (6.22), with $A = \mathbb{C}$, we have

$$(\mathrm{ev}_g)_g \circ (p_*^{M^+})_g[D] = \mathrm{index}_g(D) + \mathrm{index}_g^\infty(D). \tag{6.28}$$

Now

$$p_*^{M^+}[D] = [L^2(E), F] = [\ker F, 0] \in KK_G(\mathrm{pt}, \mathrm{pt}),$$

so the left-hand sides of (6.27) and (6.28) are equal. □

In concrete situations, knowledge of $\mathrm{index}_g^\infty(D)$ then allows one to use the fixed point formula in Theorem 2.16 to compute the left-hand side of (6.27).

This can be made more explicit in a situation relevant to the treatment of Callias-type deformations of Dirac operators in the context of KK-theory in [Bunke 1995; Kucerovsky 2001]. Suppose that $\Phi \in \mathrm{End}(E)^G$ is an odd, self-adjoint vector bundle endomorphism. Suppose that $\Phi^2 - 1_E$ tends to zero at infinity, so that it is a compact operator on $\Gamma_0(E)$. Then $(\Gamma_0(E), \Phi, \pi_{M^+})$ is an equivariant Kasparov $(C(M^+), C_0(M))$-cycle. Let $[\Phi] \in KK_G(M^+, M)$ be its class. Now we do not assume that $(D^2 + 1)^{-1}$ itself is compact, but that

$$[D_\Phi] := [\Phi] \otimes_M [D] \in KK_G(M^+, \mathrm{pt})$$

is the class of an elliptic operator D_Φ as in (6.24). Then $(D_\Phi^2 + 1)^{-1}$ is compact. (The idea is that $D_\Phi = D + \Phi$ if $D\Phi + \Phi D$ is sufficiently small; compare this with [Bunke 1995, Proposition 2.18].) By functoriality of the Kasparov product, we have for $U', V' \subset M$ as above,

$$\mathrm{index}_g^\infty(D_\Phi) = (\mathrm{ev}_g)_g\Big(\Big(\big(p_*^{\overline{U'}}\big)_g \circ \big((j_{\overline{U'}}^{V'})_*\big)_g^{-1} \circ (k_{V'}^{M^+})_g^*[\Phi]_g\Big) \otimes_M [D]_g\Big). \tag{6.29}$$

This expression has the advantage that Φ is a vector bundle endomorphism, which makes (6.29) easier to evaluate than (6.25). In particular, if $\Phi^2 = 1_E$ on $V' \cap M$, then $(k_{V'}^{M^+})^*[\Phi] = 0$. In that case, Theorem 2.16 and Proposition 6.26 imply that

$$\mathrm{Tr}(g \text{ on } \ker_{L^2}(D_\Phi^+)) - \mathrm{Tr}(g \text{ on } \ker_{L^2}(D_\Phi^-)) = \int_{TM^g} \frac{\mathrm{ch}([\sigma_{D_\Phi}|_{TM^g}](g))\,\mathrm{Todd}(TM^g \otimes \mathbb{C})}{\mathrm{ch}\big(\big[\bigwedge N_{\mathbb{C}}\big](g)\big)}. \tag{6.30}$$

Example 6.31. Let $M = \mathbb{C}^n$, and let g be the diagonal action by n nontrivial elements of U(1). Then $M^g = \{0\}$, and $N = \mathbb{C}^n$. Let $\beta_{\mathbb{C}^n} \in KK_G(\mathrm{pt}, \mathbb{C}^{2n})$ be the Bott element as in Definition 4.12. Now the class $[D_1] \in KK_G(\mathbb{C}^{2n}, \mathrm{pt})$ as in (4.15) is the Dolbeault class of $\mathbb{C}^{2n}$. The Kasparov product

$$\beta_{\mathbb{C}^n} \otimes_{\mathbb{C}^{2n}} [D_1] \in KK_G(\mathrm{pt}, \mathrm{pt})$$

is represented by the elliptic operator $D_B := B \otimes 1 + 1 \otimes D_1$ as in (4.21). Hence $(D_B^2 + 1)^{-1}$ is a compact operator. In the proof of Lemma 4.19, we saw that the

L^2-kernel of D_B is spanned by the g-invariant function (4.23). So

$$\operatorname{Tr}(g \text{ on } \ker_{L^2}(D_B^+)) - \operatorname{Tr}(g \text{ on } \ker_{L^2}(D_B^-)) = 1. \tag{6.32}$$

On the other hand, let $b \in C^\infty(\mathbb{R})$ be an odd function, with values in $[-1, 1]$, such that $b(x) = 1$ for all $x \geq 1$. If we replace $B(1+B^2)^{-1/2}$ by $b(B)$ in (4.11), then the resulting class in $KK_G(\text{pt}, \mathbb{C}^{2n})$ is the same class $\beta_{\mathbb{C}^n}$. But with this normalisation function, we have $b(B)^2 = 1$ outside the unit ball in $\mathbb{C}^n$. So

$$\operatorname{index}_g^\infty(D_B) = 0. \tag{6.33}$$

Finally, by Corollary 6.4, with $F = \bigwedge \widetilde{N}_{\mathbb{C}} = \bigwedge \mathbb{C}^{2n}$, we have

$$\operatorname{index}_g(D_B) = 1. \tag{6.34}$$

The equalities (6.32), (6.33) and (6.34) illustrate Proposition 6.26 in this case.

Example 6.35. In the setting of Theorem 2.18, the index pairing $[F] \otimes_M [D]$ in $KK_G(\text{pt}, \text{pt})$ is represented by a Fredholm operator D_F. Analogously to (6.29), we have $\operatorname{index}_g^\infty(D_F) = 0$, so that Proposition 6.26 and Theorem 2.16 yield an expression for $\operatorname{Tr}(g \text{ on } \ker_{L^2}(D_F^+)) - \operatorname{Tr}(g \text{ on } \ker_{L^2}(D_F^-))$. But in this setting, the same expression follows directly from Theorem 2.18.

See Remark 7.10 for the construction of a Fredholm operator D_{fv} as a deformation of *any* elliptic operator D, with $\operatorname{index}_g^\infty(D_{fv}) = 0$.

6G. *Braverman's index.* Suppose $X \in \mathfrak{g}$ such that $g = \exp X$. Let X^M be the vector field on M defined by X. Suppose D is a Dirac-type operator, and consider the deformed operator

$$D_X^f := D + \sqrt{-1} f c(X^M).$$

Here $f \in C^\infty(M)^G$, and $c : TM \to \operatorname{End}(E)$ is a given Clifford action, used to define the Dirac operator D. Braverman [2002, Theorem 7.5] obtained a fixed point theorem for such operators. This implies that the g-index equals Braverman's index in this case.

Corollary 6.36. *If f is admissible* [Braverman 2002, Definition 2.6], *then the representation of G in* $\ker_{L^2}(D_X^f)^\pm$ *has a character $\chi^\pm$ that can be evaluated at g, and one has*

$$\operatorname{index}_g(D) = \chi^+(g) - \chi^-(g).$$

Proof. The fixed point formula for $\operatorname{index}_g(D)$ in Theorem 2.16 is precisely the evaluation at g of the right-hand side of the formula in [Braverman 2002, Theorem 7.5]. (This equality also shows that $\ker_{L^2}(D_X^f)$ has a character that can be evaluated at g.) □

Remark 6.37. In the above construction, the element $X \in \mathfrak{g}$, which represents the *taming map* used in [Braverman 2002], depends on the group element g. So the g-index of D is not the character of the Braverman index of D deformed by a single taming map, but the taming map depends on g.

6H. ***A relative index theorem.*** Gromov and Lawson [1983, Theorem 4.18] obtain a relative index formula for pairs of elliptic operators that coincide outside compact sets. (See Theorem 2.18 in [Braverman and Shi 2016] for a version for Callias-type operators.) There is an analogue of this result for the g-index.

For $j = 0, 1$, let M_j be a manifold with the same structure and properties as M. Let $E_j \to M_j$ be a vector bundle like $E \to M$, and let D_j be an operator on E_j like D on E. Suppose there are relatively compact neighbourhoods U_j of M_j^g outside of which M_j, E_j and D_j can be identified. As on page 38 of [Gromov and Lawson 1983], we compactify M_j to a manifold $\widetilde{M}_j$, by taking a neighbourhood V_j of U_j, and attaching a compact manifold X to it. Since $M_0 \setminus V_0 = M_1 \setminus V_1$, the same manifold X can be used for $j = 0, 1$. Extend the vector bundles E_j and the operators D_j to vector bundles $\widetilde{E}_j \to \widetilde{M}_j$ and elliptic operators $\widetilde{D}_j$ on $\widetilde{E}_j$. Suppose the map g extends to $\widetilde{M}_j$ and $\widetilde{E}_j$, commuting with $\widetilde{D}_j$; this extends to continuous actions by G on $\widetilde{M}_j$ and $\widetilde{E}_j$ preserving $\widetilde{D}_j$.

Proposition 6.38 (relative index theorem). *We have*

$$\mathrm{index}_g(D_1) - \mathrm{index}_g(D_0) = \mathrm{index}_G(\widetilde{D}_1)(g) - \mathrm{index}_G(\widetilde{D}_0)(g).$$

Since the manifolds $\widetilde{M}_j$ are compact, the indices on the right-hand side of this equality are given by the usual equivariant index.

Proof. By the Atiyah–Segal–Singer fixed point theorem (or Theorem 2.16), we have, for $j = 0, 1$,

$$\mathrm{index}_G(\widetilde{D}_j)(g) = \int_{TM_j^g} \frac{\mathrm{ch}\big([\sigma_{\widetilde{D}_j}|_{TM_j^g}](g)\big)\,\mathrm{Todd}(TM_j^g \otimes \mathbb{C})}{\mathrm{ch}\big(\big[\bigwedge(N_j)_{\mathbb{C}}\big](g)\big)} + \int_{TX^g} \frac{\mathrm{ch}\big([\sigma_{\widetilde{D}_j}|_{TX^g}](g)\big)\,\mathrm{Todd}(TX^g \otimes \mathbb{C})}{\mathrm{ch}\big(\big[\bigwedge(N_X)_{\mathbb{C}}\big](g)\big)}.$$

Here $N_j \to M_j^g$ and $N_X \to X^g$ are normal bundles to fixed point sets. Since $\sigma_{\widetilde{D}_1}|_{TX^g} = \sigma_{\widetilde{D}_0}|_{TX^g}$, Theorem 2.16 implies the claim. □

6I. ***Some geometric consequences.*** The g-index of a G-equivariant elliptic operator is a topological invariant of the group action that can be used to detect geometric properties of the action. We illustrate this in the cases of the Hodge–de Rham and Spin-Dirac operators.

Let $D = d + d^* : \Omega^{\text{even}}_{\mathbb{C}}(M) \to \Omega^{\text{odd}}_{\mathbb{C}}(M)$ be the Hodge–de Rham operator on M, acting on complex differential forms. The symbol class of this operator is $\left[\tau_M^* \bigwedge TM_{\mathbb{C}}\right]$, whose restriction to TM^g equals

$$[\sigma_D|_{TM^g}] = \left[\tau_{M^g}^* \bigwedge N_{\mathbb{C}}\right] \otimes \left[\tau_{M^g}^* \bigwedge TM^g_{\mathbb{C}}\right]. \tag{6.39}$$

Let D_{M^g} be the componentwise defined Hodge–de Rham operator on M^g. Then Theorem 2.16 and (6.39) imply that

$$\operatorname{index}_g(d+d^*) = \int_{TM^g} \operatorname{ch}(\sigma_{D_{M^g}}) \operatorname{Todd}(TM^g \otimes \mathbb{C}) = \operatorname{index}(D_{M^g}) = \chi(M^g), \tag{6.40}$$

the Euler characteristic of M^g. (See also [Lawson and Michelsohn 1989, p. 262].)

Corollary 6.41. *If* $\operatorname{index}_g(d+d^*) \neq 0$, *then every g-invariant vector field on M has a zero on M^g.*

Proof. A g-invariant vector field on M restricts to a vector field on M^g. If it does not vanish there, then $\chi(M^g) = 0$. So the claim follows from (6.40). □

Next, suppose that M is a Spin manifold, and that g lifts to the spinor bundle. Let D be the Spin-Dirac operator.

Corollary 6.42. *If G is connected*, *M is noncompact*, *and* $\operatorname{index}_g(D) = 0$, *then the one-point compactification M^+ of M is not a G-equivariant* Spin *manifold.*

Proof. If M^+ is a G-equivariant Spin manifold with Dirac operator D_{M^+}, then the vanishing result of Atiyah and Hirzebruch [1970] and Theorem 2.16 imply that

$$0 = \operatorname{index}_g(D_{M^+}) = \operatorname{index}_g(D) + a_\infty.$$

Here a_∞ is the contribution to the right-hand side of (2.17) of the fixed point at infinity, which is nonzero by [Atiyah and Bott 1968, Theorem 8.35]. □

7. A nonlocalised index formula

In the compact case, the Kirillov formula is a nonlocalised expression for the equivariant index of an elliptic operator; see [Berline et al. 1992, Theorem 8.2]. This can be deduced from the fixed point formula in the compact case. In the case of noncompact manifolds, there is also a nonlocalised expression for the g-index, Proposition 7.8 below. This follows from Kasparov's index theorem and the properties of the g-symbol class introduced in Section 5D, rather than from Theorem 2.16.

A potentially interesting feature of this nonlocalised formula is that it involves the same kind of deformed symbols as the ones used for Dirac operators on symplectic manifolds in [Paradan 2011]. Those deformed symbols are *transversally elliptic* rather than elliptic. Berline and Vergne obtained a generalisation of the

Atiyah–Segal–Singer fixed point formula to transversally elliptic operators or symbols; see [Berline and Vergne 1996a, Main Theorem 1; 1996b, Theorem 20]. This formula involves a distribution on the group. It was pointed out to the authors by Vergne that this formula implies that for the deformed symbols we will consider, at points g where this distribution is given by a function, it is given by the g-index.

The index of such a deformed symbol was shown to equal the index of a deformed Dirac operator in Theorem 5.5 in [Braverman 2002]. In Theorem 1.5 in [Ma and Zhang 2014], this index is proved to be equal to another index of deformed Dirac operators, defined using the Atiyah–Patodi–Singer index on manifolds with boundary. In contrast to [Braverman 2002; Ma and Zhang 2014; Paradan 2011], the expression for the g-index in terms of deformed symbols is independent of the choices made in this deformation. Furthermore, it applies to more general elliptic operators than Dirac operators.

We shall describe the g-symbol class σ_g^D of Definition 5.14 more explicitly, using a deformed symbol. Let v be a G-invariant vector field on M that does not vanish outside V.

Example 7.1. If $X \in \mathfrak{g}$ such that $g = \exp(X)$, one can take the vector field v induced by X. This vector field obviously depends on g.

Example 7.2. If H is a compact Lie group acting on M, containing G, then it can be possible to choose a single vector field v that works for all elements of H. Indeed, suppose there is an H-equivariant map $\psi : M \to \mathfrak{h}$, and consider the Kirwan vector field v, defined by

$$v_m := \frac{d}{dt}\Big|_{t=0} \exp(t\psi(m)) \cdot m,$$

for $m \in M$. Suppose this vector field is nonzero outside a compact set. Then ψ is a taming map as in Definition 2.4 in [Braverman 2002]. In this case, the vector field v can be used for any element of H.

Let $f : V \to \mathbb{R}_{\geq 0}$ be a G-invariant continuous function, such that $f(m) = 0$ when $m \in U$ and $\lim_{m \to m'} f(m) = \infty$ if $m' \in \partial V$. Consider the deformed symbol $\sigma_{D,fv} \in \mathrm{End}(\tau_V^*(E|_V))$, given by

$$\sigma_{D,fv}(u) := \sigma_D(u + f(m)v_m) \tag{7.3}$$

for $m \in V$ and $u \in T_m M$. Set

$$\tilde{\sigma}_{D,fv} := \frac{\sigma_{D,fv}}{\sqrt{\sigma_{D,fv}^2 + 1}}.$$

This defines an odd, self-adjoint, bounded operator on the Hilbert $C_0(TV)$-module $\Gamma_0(\tau_V^*(E|_V))$. Furthermore, we have for every vector field u on M, and every

$m' \in \partial V$,

$$\lim_{m \to m'} \tilde{\sigma}_{D,fv}(u_m) = \operatorname{sgn}(\sigma_D(v_{m'})).$$

We extend $\tilde{\sigma}_{D,fv}$ to a continuous vector bundle endomorphism of $\tau_M^* E$ by setting

$$\tilde{\sigma}_{D,fv}(u) := \operatorname{sgn}(\sigma_D(v_m))$$

for all $u \in T_m M$, where $m \in M \setminus V$. (Since $v_m \neq 0$ if $m \in M \setminus V$, this operator is invertible outside V.)

Note that

$$\tilde{\sigma}_{D,fv}(u)^2 - 1 \to 0 \tag{7.4}$$

as $u \to \infty$ in TM. Indeed, let $m \in M$ and $u \in T_m M$ be given. If $m \notin V$, then $v_m \neq 0$ and $\tilde{\sigma}_{D,fv}(u)^2 = 1$. And if $m \in V$, then

$$\tilde{\sigma}_{D,fv}(u)^2 - 1 = (\sigma_D(u + f(m)v_m)^2 + 1)^{-1}.$$

Since D is elliptic and has first order, this goes to zero as $u \to \infty$ in TV. We therefore find that $(\Gamma_0(\tau_M^* E), \tilde{\sigma}_{D,fv})$ is a Kasparov $(\mathbb{C}, C_0(TM))$-cycle. Let

$${}_{\mathrm{pt}}[\sigma_{D,fv}] \in KK_G(\mathrm{pt}, TM)$$

be its class, which will be called the deformed symbol class.

Lemma 7.5. *The localisation of the deformed symbol class at g is σ_g^D, i.e.,*

$${}_{\mathrm{pt}}[\sigma_{D,fv}]_g = \sigma_g^D \in KK_G(\mathrm{pt}, TM)_g.$$

Proof. As in Section 6F, let M^+ be the one-point compactification of M. Let $U, V, U', V' \subset M^+$ be as in that subsection. Consider the class

$${}_{M^+}[\sigma_{D,fv}] := [\Gamma_0(\tau_M^* E), \tilde{\sigma}_{D,fv}, \pi_{M^+}] \in KK_G(M^+, TM),$$

where π_{M^+} is as in (6.23). Then by commutativity of (6.22), for $A = C_0(TM)$, we have

$$\begin{aligned} {}_{\mathrm{pt}}[\sigma_{D,fv}]_g &= \big(p_*^{M^+}\big)_g ({}_{M^+}[\sigma_{D,fv}]) \\ &= \big(p_*^{\overline{U}}\big)_g \circ \big(\big(j_{\overline{U}}^{V}\big)_*\big)_g^{-1} \circ \big(k_V^{M^+}\big)_g^* ({}_{M^+}[\sigma_{D,fv}]_g) \\ &\quad + (p_*^{\overline{U'}})_g \circ \big(\big(j_{\overline{U'}}^{V'}\big)_*\big)_g^{-1} \circ \big(k_{V'}^{M^+}\big)_g^* ({}_{M^+}[\sigma_{D,fv}]_g). \end{aligned} \tag{7.6}$$

Now since $f = 0$ on V, we have

$$\big(k_V^{M^+}\big)^* ({}_{M^+}[\sigma_{D,fv}]) = \big(k_V^M\big)^* [\sigma_D].$$

So the first term in (7.6) equals σ_g^D. Furthermore,

$$\big(k_{V'}^{M^+}\big)^* ({}_{M^+}[\sigma_{D,fv}]) = [\Gamma^\infty(E|_{V'}), \operatorname{sgn}(\sigma_D(v)), \pi_{V'}] = 0,$$

since this class is represented by a degenerate cycle. $\square$

Remark 7.7. Instead of (7.3), we could have used a more general deformed symbol of the form

$$\sigma_{D,f\Phi}(u) := \sigma_D(u) + f(m)\Phi_m,$$

for $m \in M$, $u \in T_m M$ and a G-equivariant, fibrewise self-adjoint, odd vector bundle endomorphism Φ of E, which is invertible outside V. We have used the natural choice $\Phi = \sigma_D(v)$.

The realisation of the g-symbol class in Lemma 7.5 leads to the following nonlocal formula for the g-index.

Proposition 7.8 (nonlocalised formula for the g-index). *The g-index of D is calculated by*

$$\mathrm{index}_g(D) = ({}_{\mathrm{pt}}[\sigma_{D,fv}] \otimes_{TM} [D_{TM}])(g). \tag{7.9}$$

Proof. It follows from Definitions (2.8) and 5.14, and Theorem 4.6, that

$$\mathrm{index}_g(D) = (\sigma_g^D \otimes_{TM} [D_{TM}]_g)(g).$$

The claim therefore follows from Lemma 7.5. □

Remark 7.10. Recall that when M is noncompact, $\mathrm{index}_g(D)$ is defined using KK-functorial maps. In Proposition 7.8, the class

$${}_{\mathrm{pt}}[\sigma_{D,fv}] \otimes_{TM} [D_{TM}] \in KK_G(\mathrm{pt}, \mathrm{pt})$$

is represented by a Fredholm operator D_{fv}, defined in terms of the deformed symbol $\sigma_{D,fv}$ and the Dolbeault–Dirac operator D_{TM}. Proposition 7.8 states that

$$\mathrm{index}_g(D) = \mathrm{Tr}(g \text{ on } \ker_{L^2}(D_{fv}^+)) - \mathrm{Tr}(g \text{ on } \ker_{L^2}(D_{fv}^-)). \tag{7.11}$$

Then Theorem 2.16 yields a cohomological expression for the right-hand side of (7.11). (Note the analogy with (6.30); we now have $\mathrm{index}_g^\infty(D_{fv}) = 0$.)

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Connectedness of cup products for polynomial representations of GL_n and applications

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We find conditions such that cup products induce isomorphisms in low degrees for extensions between stable polynomial representations of the general linear group. We apply this result to prove generalizations and variants of the Steinberg tensor product theorem. Our connectedness bounds for cup product maps depend on numerical invariants which seem also relevant to other problems, such as the cohomological behavior of the Schur functor.

1. Introduction

Let $\Bbbk$ be a field of positive characteristic p, and let G be an algebraic group over $\Bbbk$. The category of rational representations of G (as in [Jantzen 2003]) is equipped with a tensor product, which induces a cup product on extension groups:

$$\mathrm{Ext}^*_G(M,N)\otimes\mathrm{Ext}^*_G(P,Q)\xrightarrow{\cup}\mathrm{Ext}^*_G(M\otimes P,N\otimes Q).$$

Of course the cup product is injective (but usually not surjective) in cohomological degree zero, and in general it is neither injective nor surjective in higher degrees. If $G=\mathrm{GL}_n(\Bbbk)$, it was observed in [Touzé 2010] that the cup product is injective in all degrees when M, N, P, Q are *stable polynomial representations*, i.e., when M, N, P, Q are polynomial representations in the usual sense [Green 2007; Martin 1993] and furthermore when n is big enough with respect to their degrees. This surprising fact is easily proved by using the description of stable polynomial representations in terms of the strict polynomial functors of Friedlander and Suslin [1997].

The first main result of this article is Theorem 3.6, which establishes conditions under which cup products are not only injective, but also surjective in low degrees. Theorem 3.6 actually applies to the case where N and Q are representations twisted r times by the Frobenius morphism, i.e., for cup products of the form

$$\mathrm{Ext}^*_G(M,N)\otimes\mathrm{Ext}^*_G(P^{(r)},Q^{(r)})\xrightarrow{\cup}\mathrm{Ext}^*_G(M\otimes P^{(r)},N\otimes Q^{(r)}).$$

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As for injectivity in [Touzé 2010], the natural home for stating and proving this connectedness property of cup products is the category of strict polynomial functors. We note that already in degrees 0 and 1, our theorem looks much stronger than what was previously known for the behavior of cup products; see Remark 4.7.

We then give concrete applications of Theorem 3.6 to the representation theory of $\mathrm{GL}_n(\Bbbk)$. Namely, we prove the following two new generalizations of Steinberg's tensor product theorem.

- We call *tensor products of Steinberg type* the stable polynomial representations of the form $M \otimes N^{(r)}$, where all the composition factors of M have p^r-restricted highest weights. Representations of this form appear naturally, e.g., in the theory of good p-filtrations [Andersen 2001].

  In Theorem 5.8, we describe the structure of the abelian subcategory generated by these tensor products of Steinberg type (with r and $\deg M$ fixed). In particular, we prove that the $\mathrm{GL}_n(\Bbbk)$-module $M \otimes N^{(r)}$ has the same structure as the $\mathrm{GL}_n(\Bbbk) \times \mathrm{GL}_n(\Bbbk)$-module $M \otimes N$. This is interesting because the latter is much easier to study. (The classical Steinberg tensor product theorem corresponds to the very special case where M and N are simple. Indeed, in that case $M \otimes N$ is simple as a $\mathrm{GL}_n(\Bbbk) \times \mathrm{GL}_n(\Bbbk)$-module, and thus by our theorem the $\mathrm{GL}_n(\Bbbk)$-module $M \otimes N^{(r)}$ is simple too).
- As made explicit in [Krause 2013], stable polynomial representations are equipped with an internal tensor product (Day convolution product), dual to the internal Hom used in Ext-computations, e.g., in [Touzé 2013b; 2014]. In Theorem 6.2 we explain how to reduce the computation of internal tensor products of simples to the case of simples with p-restricted highest weights. Thus, Theorem 6.2 plays the same role for understanding internal tensor products of simples as the classical Steinberg tensor product theorem does for understanding ordinary tensor products of simples.

In Appendix B we show that Theorem 3.6 can also be used to derive new proofs of two well-known fundamental theorems for simple representations of $\mathrm{GL}_n(\Bbbk)$: Steinberg's tensor product theorem and Clausen and James' theorem. We note that another functor technology proof of Steinberg's tensor product theorem is given in [Kuhn 2002]. The proof given here seems quite different; see Remark B.11.

The bounds for connectedness given in Theorem 3.6 depend on some cohomological constants $p(M, r)$ and $i(N, r)$. To be more specific, a projective stable polynomial module is p^r-bounded if its socle is a direct sum of simples with p^r-restricted highest weights; see Corollary 4.2. The integer $p(M, r)$ is characterized by:

> $p(M, r) \geq k$ if and only if there exists a resolution P of M in which the first k terms $P_0, \dots, P_{k-1}$ are p^r-bounded projectives.

The integer $i(N, r)$ is defined dually; see Definition 3.4. Although we use this definition for stable polynomial representations, it makes sense for unstable polynomial representations as well. We are not aware of previous occurrences of these constants in the literature. We study their basic properties and give characterizations of these constants, as well as elementary computation rules and examples. In most examples, we limit ourselves to giving estimates for these constants rather than exact values, and leave the following challenging problem open.

Problem. Compute (or obtain a reasonable understanding of) the exact value of $p(M, r)$ and $i(M, r)$ for the most important $\mathrm{GL}_n(\Bbbk)$-modules (simple modules, standard or costandard modules).

One further motivation for this problem is that the constants $p(M, r)$ and $i(M, r)$ seem to be related to other problems of interest. Let us give two examples.

- In Theorem 8.2, we prove that the constants $p(M, 1)$ and $i(M, 1)$ govern the connectedness of the Schur functor on the level of extensions. The cohomological behavior of the Schur functor was already studied in a series of papers [Doty et al. 2004; Kleshchev and Nakano 2001; Kleshchev and Sheth 1999]. Our Theorem 8.2 gives a simpler and effective approach to this problem. For example, with our first computations of $i(F, 1)$ and $p(F, 1)$, we recover and generalize many results from [Kleshchev and Nakano 2001].
- It seems that the values of $p(L, r)$ capture some interesting concrete properties of simple functors L. Clausen and James' theorem [Clausen 1980; James 1980] says that $p(L, 1) > 0$ if and only if the highest weight of L is p-restricted. Reischuk [2016] has proved that $p(L, 1) > 1$ if and only if $Q^d \underline{\otimes} L$ is simple, where Q^d is the simple head of S^d (see Section 6 and in particular Theorem 6.2 and Corollaries 6.6 and 6.9 to understand why this particular internal tensor product is interesting). It would be interesting to understand if higher inequalities $p(L, 1) > k$ (of cohomological nature) are directly connected to some nonhomological representation-theoretic properties of L.

We finish by explaining a wider perspective behind the work presented here. Functor category techniques have proved useful for studying representations and cohomology of many variants of classical matrix groups. See, e.g., [Touzé 2010] for symplectic and orthogonal group schemes, [Axtell 2013; Drupieski 2016] for super Schur algebras, [Hong and Yacobi 2017] for quantum GL_n, [Franjou et al. 1999; Djament and Vespa 2010] for finite classical groups or more generally [Djament 2012] for discrete unitary groups. In these examples, the functor categories in play share many properties with the category of strict polynomial functors used here. So we expect that the techniques and results developed in this article can be adapted to these cases. For example, we prove in [Touzé 2017a] an analogue of Theorem 3.6 for polynomial representations of orthogonal and symplectic group schemes.

This article has been written in such a way that the main thread of ideas and proofs is self-contained. In particular, only very basic facts of the representation theory of general linear groups are used (the highest weight category structure is used only for the results of Section 7B) and no combinatorics of the symmetric group is used (except a result of Bessenrodt and Kleshchev [2000] in Corollary 6.6). These basic facts are recalled in Section 2. In the same spirit, we have also added an appendix on representations of tensor products of finite dimensional algebras, whose results are used in Section 5.

2. Background

2A. ***Strict polynomial functors and Schur algebras.*** In this article $\Bbbk$ is a field of positive characteristic p, and $\mathcal{P}_{d,\Bbbk}$ denotes the category of homogeneous strict polynomial functors of degree d over $\Bbbk$ (with possibly infinite dimensional values). We refer, e.g., to [Friedlander 2003], [Friedlander and Suslin 1997] or [Krause 2013] for an introduction to strict polynomial functors. If one considers an infinite ground field $\Bbbk$, strict polynomial functors have a nice description like the one in [Macdonald 1995] (where they are simply called "polynomial functors"). Namely, strict polynomial functors are functors from finite dimensional $\Bbbk$-vector spaces to $\Bbbk$-vector spaces, such that for all pairs of finite dimensional vector spaces (V, W), the map

$$\begin{aligned} \mathrm{Hom}_{\Bbbk}(V, W) &\to \mathrm{Hom}_{\Bbbk}(F(V), F(W)),\\ f &\mapsto F(f) \end{aligned}$$

is given by a homogeneous polynomial of degree d (that is, given by an element of $S^d(\mathrm{Hom}_{\Bbbk}(V, W)^*) \otimes \mathrm{Hom}_{\Bbbk}(F(V), F(W))$).

For example, the category $\mathcal{P}_{0,\Bbbk}$ is equivalent to the category of constant functors, which is equivalent to the category of $\Bbbk$-vector spaces. Typical examples of homogeneous functors of higher degree d are the tensor product functors $\otimes^d : V \mapsto V^{\otimes d}$, the symmetric power functors $S^d : V \mapsto (V^{\otimes d})_{\mathfrak{S}_d}$ and the divided power functors $\Gamma^d : V \mapsto (V^{\otimes d})^{\mathfrak{S}_d}$. (Here the symmetric group $\mathfrak{S}_d$ acts on $V^{\otimes d}$ by permuting the factors of the tensor product). Note that $S^0 = \otimes^0 = \Gamma^0 = \Bbbk$ and $S^1 = \otimes^1 = \Gamma^1$, but for $d \geq p$ the functor S^d is not isomorphic to Γ^d.

We denote by $\mathcal{P}_{\Bbbk}$ the category of strict polynomial functors (of bounded degree), that is, $\mathcal{P}_{\Bbbk} = \bigoplus_{d \geq 0} \mathcal{P}_{d,\Bbbk}$. Thus, objects of $\mathcal{P}_{\Bbbk}$ decompose as *finite* direct sums of homogeneous functors, and there are no nonzero morphisms between homogeneous functors of different degrees. All functors of vector spaces considered in this article will actually be strict polynomial functors of bounded degree, and hence we will always omit the terms "of bounded degree", and we will often omit the terms "strict polynomial".

By evaluating a strict polynomial functor F on $\Bbbk^n$, one obtains a polynomial $\mathrm{GL}_n(\Bbbk)$-module $F(\Bbbk^n)$. Restricting to homogeneous strict polynomial functors of degree d, one obtains a functor

$$\mathcal{P}_{d,\Bbbk} \xrightarrow{\mathrm{ev}_{\Bbbk^n}} \mathrm{Pol}_{d,\mathrm{GL}_n(\Bbbk)} \simeq S(n,d)\text{-Mod},$$

where $\mathrm{Pol}_{d,\mathrm{GL}_n(\Bbbk)}$ denotes the category of homogeneous polynomial representations of $\mathrm{GL}_n(\Bbbk)$ of degree d, and $S(n,d)$-Mod the equivalent category of modules over the Schur algebra $S(n,d)$ (which is finite dimensional). It is an equivalence of categories, provided $n \geq d$. (Friedlander and Sulin [1997] proved it for functors with finite dimensional values, but their proof actually works in the general case; see also [Krause 2013].) In particular, $\mathcal{P}_\Bbbk$ has nice properties similar to those of modules over a finite dimensional algebra. We shall use the following ones in the sequel.

(1) Simple functors are homogeneous functors, and their values are finite dimensional vector spaces. A functor has a finite composition series if and only if it has finite dimensional values; such functors are called *finite*. Finally, every functor is the union of its finite subfunctors.

(2) Arbitrary direct sums of injective functors are injective, and every functor can be embedded into a direct sum of finite injectives.

(3) Any nonzero strict polynomial functor has a nonzero socle, a nonzero head and a finite Loewy length.

2B. ***Frobenius twists.*** let $\Bbbk$ be a field of positive characteristic p. For all $r \geq 0$, we denote by $I^{(r)}$ the r-th Frobenius twist functor. The functor $I^{(0)} = I = S^1 = \Gamma^1 = \Lambda^1$ is the identity functor. More generally, for all $r \geq 0$ the functor $I^{(r)}$ is the unique simple additive functor of degree p^r (up to isomorphism).

Notation 2.1. We use the traditional notation $F^{(r)} = F \circ I^{(r)}$. We also denote by $F \otimes G^{(r)}$ the tensor product of F and $G^{(r)}$, i.e., Frobenius twists have a priority higher than tensor products in our notations.

The effect on Ext^* of precomposition by Frobenius twist is now well understood in all degrees [Touzé 2013a; Chałupnik 2015]. In particular, in degrees $i = 0$ or $i = 1$, the $\Bbbk$-linear morphism

$$\mathrm{Ext}^i_{\mathcal{P}_\Bbbk}(F,G) \to \mathrm{Ext}^i_{\mathcal{P}_\Bbbk}(F^{(r)}, G^{(r)})$$

induced by precomposition by $I^{(r)}$ is an isomorphism. This description of the effect of precomposition by Frobenius twists in degrees 0 and 1 can be proved by very elementary means; see, e.g., [Breen et al. 2016, Appendix A]. We will not need to know about higher degrees, except in the proof of Proposition 7.3.

2C. ***Elementary facts regarding simple functors.*** Traditionally, simple polynomial $\mathrm{GL}_n(\Bbbk)$-modules are classified by examining the action of a maximal torus on $\mathrm{GL}_n(\Bbbk)$-modules, that is, using the concept of highest weights; see, e.g., [Martin 1993, Chapter 1]. In the sequel of the article, we shall use the following consequences of this classification.

(1) Isomorphism classes of simple functors are in bijective correspondence with partitions. For each partition $\lambda = (\lambda_1, \dots, \lambda_k)$ we fix a simple functor L_λ in the corresponding isomorphism class. Then L_λ is homogeneous of degree $\sum \lambda_i$. We call λ the *highest weight* of L_λ. Indeed, by evaluating on $\Bbbk^n$, we obtain a simple polynomial module $L_\lambda(\Bbbk^n)$ with highest weight λ. For example, the only simple functor of degree 0 is $L_{(0)} = \Bbbk$.

(2) Simple functors are self-dual. To be more specific, each simple functor L is isomorphic to its dual $L^\sharp$, defined by $L^\sharp(V) := L(V^*)^*$.

(3) Simple functors have endomorphism rings of dimension 1.

(4) For all partitions λ and μ and all $r \ge 0$, $L_{\lambda+p^r\mu}$ is a composition factor of $L_\lambda \otimes L_\mu^{(r)}$.

Remark 2.2. Actually, one needs the fact that $\Bbbk$ is algebraically closed to obtain easily (by Schur's lemma) that the endomorphism ring of a simple functor has dimension one. When $\Bbbk$ is not algebraically closed, this can be proved using the fact that Schur algebras are quasihereditary; see, e.g., [Martin 1993, Chapter 3].

2D. ***Bifunctors and sum-diagonal adjunction.*** We will need strict polynomial functors with several variables for intermediate computations, as well as in the study of tensor products of Steinberg type in Section 5. Definitions and basic properties of strict polynomial functors extend without problem to the case of functors with several variables, and we refer to [Suslin et al. 1997, Section 2], [Touzé 2010, Section 2] or [Touzé 2017b, Section 3] for details. We recall here the main features of the theory in the context of bifunctors, and leave to the reader the obvious formulas with three variables or more.

Given two nonnegative integers d_1 and d_2, we denote by $\mathcal{P}_{d_1,d_2,\Bbbk}$ the category of homogeneous strict polynomial bifunctors of bidegree (d_1, d_2) (with possibly infinite dimensional values). Typical examples of objects of this category are the *bifunctors of separable type*, which are the bifunctors of the form

$$F \boxtimes G : (V, W) \mapsto F(V) \otimes G(W),$$

where F and G are homogeneous strict polynomial functors of degree d_1 and d_2, respectively. Just as in the one variable case, evaluating bifunctors on a pair of vector spaces $(\Bbbk^n, \Bbbk^m)$ yields a functor

$$\mathcal{P}_{d_1,d_2,\Bbbk} \to S(n, d_1) \otimes S(m, d_2)\text{-Mod},$$

where $S(m, d_1)$ and $S(m, d_2)$ are Schur algebras (which are finite dimensional). Moreover, this functor is an equivalence of categories if $n \geq d_1$ and $m \geq d_2$. In particular $\mathcal{P}_{d_1,d_2,\Bbbk}$ satisfies the three properties mentioned at the end of Section 2A. We have a Künneth morphism

$$\mathrm{Ext}^*_{\mathcal{P}_{d_1,\Bbbk}}(F_1, G_1) \otimes \mathrm{Ext}^*_{\mathcal{P}_{d_2,\Bbbk}}(F_2, G_2) \xrightarrow{\kappa} \mathrm{Ext}^*_{\mathcal{P}_{d_1,d_2,\Bbbk}}(F_1 \boxtimes F_2, G_1 \boxtimes G_2),$$

which is an isomorphism if the quadruple (F_1, G_1, F_2, G_2) satisfies the following condition.

Condition 2.3 (Künneth condition). In the quadruple (F_1, G_1, F_2, G_2), F_1 and F_2 are both finite functors, or F_1 and G_1 are both finite functors.

We also denote by $\mathcal{P}_{d,\Bbbk}(2)$ the category of homogeneous strict polynomial bifunctors of total degree d, and by $\mathcal{P}_{\Bbbk}(2)$ the category of strict polynomial functors of bounded degree, with possibly infinite dimensional values. We have decompositions

$$\mathcal{P}_{\Bbbk}(2) = \bigoplus_{d \geq 0} \mathcal{P}_{d,\Bbbk}(2), \qquad \mathcal{P}_{d,\Bbbk}(2) = \bigoplus_{d_1+d_2=d} \mathcal{P}_{d_1,d_2,\Bbbk}.$$

In particular, each bifunctor B decomposes uniquely as a direct sum $B = \bigoplus B^{(d_1,d_2)}$, where $B^{(d_1,d_2)}$ is a homogeneous strict polynomial bifunctor of bidegree (d_1, d_2). We shall refer to $B^{(d_1,d_2)}$ as the *homogeneous component of bidegree* (d_1, d_2) *of* B. A typical example of (degree d homogeneous) bifunctor is the bifunctor

$$F_{\boxplus} : (V, W) \mapsto F(V \oplus W),$$

where F is a (degree d homogeneous) strict polynomial functor of degree d. Conversely, from a (degree d homogeneous) bifunctor B of total degree d one can construct a (degree d homogeneous) strict polynomial functor with one variable by *diagonal evaluation*:

$$B_{\Delta} : V \mapsto B(V, V).$$

These two constructions are exact and adjoint to each other on both sides. Hence we have graded isomorphisms

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(B, F_{\boxplus}) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(B_{\Delta}, F),$$
$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F_{\boxplus}, B) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F, B_{\Delta}).$$

These two isomorphisms were first used in the context of strict polynomial functors in [Franjou et al. 1999]. In this article, they will be the key tool for Theorem 3.6. As in [Franjou et al. 1999], we will often use them when B is of separable type $B = G \boxtimes H$, and hence when $B_{\Delta} = G \otimes H$.

2E. ***The internal tensor product.*** The category $\mathcal{P}_{d,\Bbbk}$ is endowed with a closed symmetric monoidal structure. We denote this internal tensor product by $\underline{\otimes}$, and by $\underline{\mathrm{Hom}}$ the associated internal hom. We refer the reader to [Krause 2013] for a presentation of this internal tensor product. We study the internal tensor product of simple functors in Section 6. For this purpose, we will use the following facts.

(1) If F is a functor, we denote by F^V the parametrized functor

$$W \mapsto F(\mathrm{Hom}_{\Bbbk}(V, W)).$$

Then the internal Hom is the functor given by

$$\underline{\mathrm{Hom}}(F, G)(V) = \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F^V, G).$$

(2) The study of internal tensor products can be reduced to the study of internal Hom by using the isomorphism natural with respect to F, G:

$$(F \underline{\otimes} G)^\sharp \simeq \underline{\mathrm{Hom}}(F, G^\sharp).$$

Here $^\sharp$ is the duality defined by $F^\sharp(V) = F(V^*)^*$, where * is the $\Bbbk$-linear duality of vector spaces.

Remark 2.4. Schur algebras do not have a Hopf algebra structure in general. (Indeed, Schur algebras have finite global dimension, and a Hopf algebra structure would make them self-injective in addition, and hence semisimple.) Thus the internal tensor product on $\mathcal{P}_{d,\Bbbk}$ is an example of a monoidal product which does not come from a Hopf algebra structure.

2F. ***Connection with representations of symmetric groups.*** The Schur functors relate strict polynomial functors to representations of the symmetric groups $\mathfrak{S}_d$. We will use these Schur functors in Sections 6 and 8. Let d be a positive integer. Consider the right action of the symmetric group $\mathfrak{S}_d$ on $\otimes^d$ given by permuting the factors of the tensor product. The Schur functor is the functor

$$f_d := \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(\otimes^d, -) : \mathcal{P}_{d,\Bbbk} \to \Bbbk\mathfrak{S}_d\text{-Mod}.$$

Since $\otimes^d$ is projective, the Schur functor f_d is exact. It has adjoints on both sides. To be more specific, the left adjoint ℓ_d is defined by $\ell_d(M) = (\otimes^d) \otimes_{\mathfrak{S}_d} M$, while the right adjoint r_d is defined by $r_d(M) = ((\otimes^d) \otimes M)^{\mathfrak{S}_d}$. The unit and counit of adjunction induce natural isomorphisms

$$M \xrightarrow{\simeq} f_d(\ell_d(M)), \qquad f_d(r_d(M)) \xrightarrow{\simeq} M.$$

In particular, the Schur functor f_d is a quotient functor.

3. Exts in low degrees between tensor products

3A. *Definition of $i(F, r)$ and $p(F, r)$.* For all tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ of nonnegative integers, we let

$$\Gamma^\lambda := \Gamma^{\lambda_1} \otimes \cdots \otimes \Gamma^{\lambda_n} \quad \text{and} \quad S^\lambda := S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_n}.$$

Let $\mathcal{T}$ denote the set of all tuples of nonnegative integers. Then the family $(\Gamma^\lambda)_{\lambda \in \mathcal{T}}$ forms a projective generator of $\mathcal{P}_{\Bbbk}$, while the family $(S^\lambda)_{\lambda\in\mathcal{T}}$ forms an injective cogenerator of $\mathcal{P}_{\Bbbk}$.

Definition 3.1. Let r be a nonnegative integer. A tuple of nonnegative integers $\lambda = (\lambda_1, \dots, \lambda_n)$ is *p^r-bounded* if $\lambda_k < p^r$ for all k. A *basic p^r-bounded projective* (resp. *injective*) is a functor of the form Γ^λ (resp. S^λ), where λ is p^r-bounded. A strict polynomial functor F is *left p^r-bounded* if it is a quotient of a direct sum of basic p^r-bounded projectives. Similarly, F is *right p^r-bounded* if it embeds in a product of basic p^r-bounded injectives.

Remark 3.2. If $r = 0$, the tuple $(0, \dots, 0)$ is the only p^r-bounded tuple. Since $\Gamma^0 = S^0 = \Bbbk$, a functor is p^0-bounded if and only if it is constant.

The following lemma collects elementary facts on p^r-bounded functors.

Lemma 3.3. (1) *The following statements are equivalent*:

(i) *F is right p^r-bounded*,

(ii) $\mathrm{Soc}(F)$ *is right p^r-bounded*,

(iii) *F embeds into a direct sum of basic p^r-bounded injectives.*

(2) *The following statements are equivalent*:

(i′) *F is left p^r-bounded*,

(ii′) $\mathrm{Head}(F)$ *is left p^r-bounded*,

(iii′) *F is the union of finite left p^r-bounded subfunctors.*

Proof. (1) It is clear that (iii)⇒(i)⇒(ii). If (ii) holds, then each simple summand of $\mathrm{Soc}(F)$ embeds into a basic p^r-bounded injective. Thus $\mathrm{Soc}(F)$ embeds into a direct sum of basic p^r-bounded injectives J. Since J is injective, the monomorphism $\mathrm{Soc}(F) \hookrightarrow J$ extends to a map $\phi : F \to J$. But $\mathrm{Soc}(\ker\phi) \subset \ker\phi \cap \mathrm{Soc}(F) = 0$ so ϕ is injective. This proves (iii).

(2) It is clear that (i′)⇒(ii′). The proof of (ii′)⇒(i′) is dual to the one of (ii)⇒(iii). Let us prove (i′)⇔(iii′). If F is left p^r-bounded, there is a map $\pi : \bigoplus_{\lambda\in A} \Gamma^\lambda \twoheadrightarrow F$. Thus F is the union of the images of the $\pi\left(\bigoplus_{\lambda\in B} \Gamma^\lambda\right)$, where B is a finite subset of A. Conversely, if F is the union of a family of finite left p^r-bounded functors F_α, then F is a quotient of $\bigoplus F_\alpha$. Hence F is left p^r-bounded. □

Definition 3.4. Let r be a nonnegative integer, and let F be a strict polynomial functor.

(1) We define $p(F, r) \in [0, +\infty]$ as the supremum of all the integers $n \geq 0$ such that F admits a projective resolution P in which the first n objects $P_0, \dots, P_{n-1}$ are left p^r-bounded.

(2) We define $i(F, r) \in [0, +\infty]$ as the supremum of all the integers $n \geq 0$ such that F admits an injective resolution J in which the first n objects $J^0, \dots, J^{n-1}$ are right p^r-bounded.

Remark 3.5. (i) By definition $p(F, r) > 0$ if and only if F is left p^r-bounded, and $i(F, r) > 0$ if and only if F is right p^r-bounded.

(ii) If $p^r > \deg F$, then all projectives or injectives appearing in any resolution of F are p^r-bounded, so $p(F, r) = i(F, r) = +\infty$. In particular, if F is constant, it is homogeneous of degree 0 and $i(F, r) = p(F, r) = +\infty$ for all $r \geq 0$.

(iii) In the definition, $p(F, r)$ and $i(F, r)$ belong to $[0, +\infty]$. However, the category $\mathcal{P}_{d,\Bbbk}$ has finite global dimension $\mathrm{gldim}(d, \Bbbk)$, which is explicitly computed in [Totaro 1997]. If F is homogeneous of degree d, then $p(F, r)$ and $i(F, r)$ actually belong to $[0, \dots, \mathrm{gldim}(d, \Bbbk)] \cup \{+\infty\}$.

3B. ***Application to the connectedness of cup products.*** The tensor product on $\mathcal{P}_\Bbbk$, $\otimes : \mathcal{P}_\Bbbk \times \mathcal{P}_\Bbbk \to \mathcal{P}_\Bbbk$, induces a cup product on extension groups in the usual way; see, e.g., [Benson 1998, Section 3.2]. The purpose of this section is to prove the following result.

Theorem 3.6. *Let (F, G, X, Y) be a quadruple of homogeneous strict polynomial functors satisfying the Künneth condition* (Condition 2.3), *and let $r \geq 0$. The cup product induces a graded injective map*

$$\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F, G) \otimes \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(X^{(r)}, Y^{(r)}) \hookrightarrow \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F \otimes X^{(r)}, G \otimes Y^{(r)}).$$

Moreover, *this graded injective map is an isomorphism in degree k in the following situations*:

(1) *when* $\deg F < \deg G$, *and* $k < i(G, r)$;

(2) *when* $\deg F > \deg G$, *and* $k < p(F, r)$;

(3) *when* $\deg F = \deg G$, *and* $k < p(F, r) + i(G, r)$.

Remark 3.7. If $\deg F \neq \deg G$ then the domain of the cup product is zero, as there is no nonzero Ext between homogeneous functors of different degrees. Thus, in cases (1) and (2), Theorem 3.6 merely says that the codomain of the cup product is zero in low degrees.

The remainder of Section 3 is devoted to the proof of Theorem 3.6. Observe that we have a factorization of cup products

$$\begin{tikzcd}
\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F,G)\otimes \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(X^{(r)},Y^{(r)}) \arrow[r, "\cup"] \arrow[d, "\kappa"] & \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F\otimes X^{(r)}, G\otimes Y^{(r)}) \\
\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F\boxtimes X^{(r)}, G\boxtimes Y^{(r)}) \arrow[ur, "-_\Delta"'] &
\end{tikzcd}$$

In particular, Theorem 3.6 is a consequence of the following slightly more general statement, in which the Künneth condition is removed.

Theorem 3.8. *Let F, G, X, Y be homogeneous functors, and let $r \geq 0$. Diagonal evaluation induces a graded injective map*

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F\boxtimes X^{(r)}, G\boxtimes Y^{(r)}) \hookrightarrow \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F\otimes X^{(r)}, G\otimes Y^{(r)}).$$

Moreover, this graded injective map is an isomorphism in degree k in the situations listed in Theorem 3.6.

The proof of Theorem 3.8 relies on a series of lemmas. The proofs of these lemmas are all based upon the sum-diagonal adjunction technique recalled in Section 2D.

Lemma 3.9. *Let F, G, F', G' be homogeneous functors satisfying $\deg F = \deg G$ and $\deg F' = \deg G'$. Diagonal evaluation yields an injective map*

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F\boxtimes F', G\boxtimes G') \hookrightarrow \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F\otimes F', G\otimes G')$$

whose cokernel is isomorphic to the following direct sum, indexed by the tuples of nonnegative integers (d_1, d_2, e_1, e_2) such that $d_2 > 0$ and $e_1 > 0$:

$$\bigoplus_{\substack{0<d_2,e_1\\ 0\leq d_1,e_2}} \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}\big(F\boxtimes F', (G_{\boxplus})^{(d_1,d_2)}\otimes (G'_{\boxplus})^{(e_1,e_2)}\big).$$

This cokernel is also isomorphic to the direct sum

$$\bigoplus_{\substack{0<d_2,e_1\\ 0\leq d_1,e_2}} \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}\big((F_{\boxplus})^{(d_1,d_2)}\otimes (F'_{\boxplus})^{(e_1,e_2)}, G\boxtimes G'\big).$$

Proof. We recall the proof of injectivity from [Touzé 2010] and prove the first description of the cokernel. The proof of the second description is similar. The map given by diagonal evaluation is equal to the composite of the map

$$\eta_* : \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F\boxtimes F', G\boxtimes G') \to \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F\boxtimes F', (G\otimes G')_{\boxplus})$$

induced by the canonical map $\eta : G\boxtimes G' \to (G\otimes G')_{\boxplus}$, together with the adjunction isomorphism

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F\boxtimes F', (G\otimes G')_{\boxplus}) \simeq \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F\otimes F', G\otimes G').$$

Thus, to prove Lemma 3.9, it suffices to prove that η_* is injective and to identify its cokernel. But

$$(G\otimes G')_{\boxplus} = G_{\boxplus}\otimes G'_{\boxplus},$$

and there is a decomposition

$$(G\otimes G')_{\boxplus} = G\boxtimes G' \oplus \bigoplus_{d_2>0 \text{ or } e_1>0} (G_{\boxplus})^{(d_1,d_2)}\otimes (G'_{\boxplus})^{(e_1,e_2)}.$$

The map η identifies with the inclusion of $G\boxtimes G'$ into the right-hand side, and since the decomposition is a direct sum, it follows that η_* admits a section, and the cokernel of η_* is isomorphic to

$$\bigoplus_{d_2>0 \text{ or } e_1>0} \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F\boxtimes F',\, (G_{\boxplus})^{(d_1,d_2)}\otimes (G'_{\boxplus})^{(e_1,e_2)}).$$

This is almost the description of the cokernel given in Lemma 3.9, but the summation index is different. Since there are no nonzero extensions between homogeneous bifunctors of different degrees, all the terms in the direct sum are zero, except the ones satisfying $d_1+e_1=\deg F$ and $d_2+e_2=\deg F'$. Since $d_1+d_2=\deg G=\deg F$, the nonzero terms in the direct sum satisfy $e_1=d_2$. Thus we can replace the summation index "$d_2>0$ or $e_1>0$" by "$e_1>0$ and $d_2>0$" and we are done. □

The proof of the next lemma is omitted since it is very similar to the proof of Lemma 3.9.

Lemma 3.10. *Let F, F', G, G' be homogeneous functors. If $\deg F>\deg G$, then $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F\otimes F', G\otimes G')$ is isomorphic to the following direct sum, indexed by the tuples of nonnegative integers (d_1,d_2,e_1,e_2) such that $e_1>0$:*

$$\bigoplus_{\substack{0<e_1\\ 0\le d_1,d_2,e_2}} \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(F\boxtimes F',\, (G_{\boxplus})^{(d_1,d_2)}\otimes (G'_{\boxplus})^{(e_1,e_2)}).$$

If $\deg F<\deg G$, then it is isomorphic to

$$\bigoplus_{\substack{0<e_1\\ 0\le d_1,d_2,e_2}} \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}((F_{\boxplus})^{(d_1,d_2)}\otimes (F'_{\boxplus})^{(e_1,e_2)},\, G\boxtimes G').$$

The next two vanishing lemmas are analogues of the key vanishing result (i.e., Pirashvili's vanishing) of [Friedlander and Suslin 1997, Theorem 2.13].

Lemma 3.11. *Let F and G be homogeneous functors with $\deg G>0$ and let λ be a p^r-bounded tuple. Then*

$$\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F\otimes G^{(r)},\, S^{\lambda}) = 0 = \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(\Gamma^{\lambda},\, F\otimes G^{(r)}).$$

Proof. We prove the first equality. The proof of the second one is similar. We will use the fact that for all homogeneous G of positive degree and for all p^r-bounded tuples ν,

$$\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(G^{(r)}, S^{\nu}) = 0. \tag{$*$}$$

This is proved when G has finite dimensional values in [Touzé 2012, Lemma 2.3], and it holds for an arbitrary G because any functor is the colimit of its finite subfunctors. (Alternatively, one could also prove this vanishing by sum-diagonal adjunction.) To reduce the equality of Lemma 3.11 to formula $(*)$, we proceed as follows. First, sum-diagonal adjunction yields an isomorphism:

$$\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F \otimes G^{(r)}, S^{\lambda}) \simeq \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F \boxtimes G^{(r)}, (S^{\lambda})_{\boxplus}).$$

We observe that $(S^{\lambda})_{\boxplus}$ decomposes as a direct sum of tensor products of the form $S^{\mu} \boxtimes S^{\nu}$ such that μ and ν are p^r-bounded. Thus Lemma 3.11 will be proved if we can prove that $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(F \boxtimes G^{(r)}, S^{\mu} \boxtimes S^{\nu})$ is zero when μ and ν are p^r-bounded. So let $\phi : F \boxtimes G^{(r)} \to S^{\mu} \boxtimes S^{\nu}$ be a morphism. By freezing the first variable of the bifunctors, we obtain for all V a morphism of functors

$$\phi_V : F(V) \otimes G^{(r)}(-) \to S^{\mu}(V) \otimes S^{\nu}(-).$$

By formula $(*)$, ϕ_V must be zero for all V. In particular, ϕ must be zero. □

Lemma 3.12. *Let r be a positive integer, let J be a be a right p^r-bounded injective functor, let P be a left p^r-bounded projective functor, let Z be a homogeneous functor and let B and C be two homogeneous bifunctors. If* $\deg C = (e_1, e_2)$ *with* $e_1 > 0$, *and $C^{(r)}$ denotes the bifunctor* $(V, W) \mapsto C(V^{(r)}, W^{(r)})$, *then*

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(B \otimes C^{(r)}, J \boxtimes Z) = 0 = \mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(P \boxtimes Z, B \otimes C^{(r)}).$$

Proof. We prove the first equality. The proof of the second one is similar. If J_Z is an injective resolution of the functor Z, then $J \boxtimes J_Z$ is an injective resolution of the bifunctor $J \boxtimes Z$. Thus, it is sufficient to do the proof in degree zero (i.e., for Hom) and when Z is injective, the general case will follow by taking resolutions. So let us take a morphism of bifunctors $\phi : B \otimes C^{(r)} \to J \boxtimes Z$. Then by freezing the first variable of the bifunctors, we obtain for all V a morphism of functors:

$$\phi_V : B(V, -) \otimes C^{(r)}(V, -) \to J(V) \otimes Z(-).$$

But by Lemma 3.11, ϕ_V is zero for all V. In particular, ϕ must be zero. □

Proof of Theorem 3.8. By Lemma 3.9, diagonal evaluation yields an injective morphism on the Ext-level (if $\deg F \neq \deg G$ or $\deg X \neq \deg Y$, the source of the cup product morphism is zero for degree reasons, so that injectivity is trivial). Hence, it remains to prove the cancellation in low degrees of the cokernel, described in Lemmas 3.9 and 3.10.

Assume that $\deg F \geq \deg G$. Take a finite resolution of F of the form:

$$0 \to \bar{F} \to F_{p(F,r)-1} \to \cdots \to F_0 \to F \to 0$$

where the functors F_k with $k < p(F,r)$ are left p^r-bounded projective functors. Take B and C as in Lemma 3.12. By using long exact sequences, we obtain that for all $k \in \mathbb{Z}$ (with the convention that Ext are zero in nonpositive degrees):

$$\operatorname{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}(F \boxtimes X^{(r)}, B \otimes C^{(r)}) \simeq \operatorname{Ext}^{*-p(F,r)}_{\mathcal{P}_{\Bbbk}(2)}(\bar{F} \boxtimes X^{(r)}, B \otimes C^{(r)}). \qquad (*)$$

In particular the Ext on the left-hand side is $(p(F,r)-1)$-connected, i.e., zero in degrees $* < p(F,r)$. By Lemmas 3.9 and 3.10, the case where $B = (F_{\boxplus})^{(d_1,d_2)}$ and $C^{(r)} = (Y^{(r)}_{\boxplus})^{(e_1,e_2)}$ with $e_1 > 0$ implies that the cup product induces an isomorphism in degrees less than $p(F,r)$. A similar argument shows that the cup product is an isomorphism in degrees less than $i(G,r)$ if $\deg F \leq \deg G$. Assume now that $\deg F = \deg G$. By Lemma 3.9 and isomorphism $(*)$, the statement of Theorem 3.8 is equivalent to showing that

$$\operatorname{Ext}^*_{\mathcal{P}_{\Bbbk}(2)}\big(\bar{F} \boxtimes X^{(r)}, (G_{\boxplus})^{(d_1,d_2)} \otimes (Y^{(r)}_{\boxplus})^{(e_1,e_2)}\big)$$

is $(i(G,r)-1)$-connected for $d_2 > 0$ and $e_1 > 0$. By Lemma 3.9 again, this is equivalent to showing that the cup product

$$\operatorname{Ext}^*_{\mathcal{P}_{\Bbbk}}(\bar{F}, G) \otimes \operatorname{Ext}^*_{\mathcal{P}_{\Bbbk}}(X^{(r)}, Y^{(r)}) \to \operatorname{Ext}^*_{\mathcal{P}_{\Bbbk}}(\bar{F} \otimes X^{(r)}, G \otimes Y^{(r)})$$

is an isomorphism in degrees less than $i(G,r)$. But we have already proved that the latter holds, since $\deg \bar{F} \leq \deg G$. □

4. An equivalent definition of $p(F,r)$ and $i(F,r)$

The next proposition gives an equivalent definition of $p(F,r)$ and $i(F,r)$. While the proof of Theorem 3.6 really relies on Definition 3.1, this new definition is useful for applying Theorem 3.6 in concrete situations. In particular, the translation of Theorem 3.6 in low degrees given in Corollary 4.4 will be used in Sections 5 and 6.

Recall that a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is *p^r-restricted* (for some nonnegative integer r) if $\lambda_n < p^r$ and for $i < n$, $\lambda_i - \lambda_{i+1} < p^r$. By convention, the partition (0) is p^r-restricted for all $r \geq 0$. Using euclidean division, one sees that any partition λ can be written in a unique way as a sum $\lambda = \lambda^0 + p^r\lambda^1$, where λ^0 is p^r-restricted. A simple indexed by a p^r-restricted partition will be loosely called a p^r-restricted simple. The proof of Proposition 4.1 relies on two classical fundamental results on simple polynomial representations in positive characteristic. We state them here with references to the literature, but we prove in Appendix B that both of them can actually be derived from Theorem 3.6.

(1) Steinberg's tensor product theorem [Jantzen 2003, II.3.17]. *If* λ *is* p^r*-restricted and* μ *is an arbitrary partition, then* $L_\lambda \otimes L_\mu^{(r)}$ *is isomorphic to* $L_{\lambda+p^r\mu}$.

(2) Clausen and James' theorem [Clausen 1980; James 1980]. *A partition* λ *of* d *is* p*-restricted if and only if* $\mathrm{Hom}_{\mathcal{P}}(L_\lambda, \otimes^d) = \mathrm{Hom}_{\mathcal{P}}(\otimes^d, L_\lambda)$ *is nonzero.*

Proposition 4.1. *Let* r *be a nonnegative integer, and let* F *be a functor.*

(1) *The integer* $p(F, r)$ *is the supremum of all* $n \geq 0$ *such that* F *admits a projective resolution* P, *in which the first* n *objects* $P_0, \dots, P_{n-1}$ *are direct sums of projective covers of* p^r*-restricted simples.*

(2) *The integer* $i(F, r)$ *is the supremum of all* $n \geq 0$ *such that* F *admits an injective resolution* J, *in which the first* n *objects* $J^0, \dots, J^{n-1}$ *are direct sums of injective envelopes of* p^r*-restricted simples.*

Proof. We restrict ourselves to proving the second statement; the proof of the first one is similar. Let us denote by J_μ the injective envelope of L_μ. We have to prove that

(i) for all p^r-restricted partitions μ, there is a p^r-bounded tuple λ such that J_μ is a direct summand of S^λ, and

(ii) for all p^r-bounded tuples λ, the indecomposable direct summands of S^λ are all isomorphic to J_μ with μ p^r-restricted.

Write $\mu = \sum_{i=0}^n p^i \mu^i$ for p-restricted partitions μ^i. By Steinberg's tensor product theorem, L_μ is isomorphic to

$$L_{\mu^0}^{(0)} \otimes \cdots \otimes L_{\mu^n}^{(n)}.$$

By Clausen and James' theorem, L_μ is a subfunctor of $(I^{(0)})^{\otimes|\mu^0|} \otimes \cdots \otimes (I^{(n)})^{\otimes|\mu^n|}$. Since $I^{(i)} \subset S^{p^i}$, we obtain that L_μ is a subfunctor of $\bigotimes_{0\leq i\leq n}(S^{p^i})^{\otimes|\mu^i|}$. If μ is p^r-restricted then $n < r$, so L_μ (hence also J_μ) is a subfunctor of S^λ with λ p^r-bounded. This proves (i). Let λ be a p^r-bounded tuple, and let μ be a partition such that μ is not p^r-restricted. By Steinberg's tensor product theorem, L_μ is isomorphic to $L_{\mu'} \otimes L_{\mu''}^{(r)}$ for a p^r-restricted partition μ' and a nonzero partition μ''. So by Lemma 3.11, $\mathrm{Hom}_{\mathcal{P}}(L_\mu, S^\lambda)$ is zero, and hence J_μ is not a direct summand of S^λ. This proves (ii). □

Corollary 4.2. *For all* F, $i(F, r) > 0$ *if and only if* $\mathrm{Soc}(F)$ *is a direct sum of* p^r*-restricted simples. Likewise,* $p(F, r) > 0$ *if and only if* $\mathrm{Head}(F)$ *is a direct sum of* p^r*-restricted simples.*

If $(d_0, \dots, d_k)$ is a tuple of nonnegative integers, we let

$$T^{(d_0,\dots,d_k)} = (\otimes^{d_0}) \otimes (\otimes^{d_1})^{(1)} \otimes \cdots \otimes (\otimes^{d_k})^{(k)}.$$

Corollary 4.3. *Let* L *be a simple functor. The following are equivalent:*

(i) $p(L,r)>0$,

(ii) $i(L,r)>0$,

(iii) *L is p^r-restricted*,

(iv) *there exists a tuple $(d_0,\dots,d_{r-1})$ such that L is a quotient of $T^{(d_0,\dots,d_{r-1})}$, and*

(v) *there exists a tuple $(d_0,\dots,d_{r-1})$ such that L embeds into $T^{(d_0,\dots,d_{r-1})}$.*

Proof. We have (i)$\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) by Corollary 4.2, (iv) $\Leftrightarrow$ (v) by Kuhn duality (both L and $T^{(d_0,\dots,d_{r-1})}$ are self-dual) and (iii) $\Rightarrow$ (iv) by Steinberg's tensor product theorem and Clausen and James' theorem (as used in the proof of Proposition 4.1). Finally, the functor $T^{(d_0,\dots,d_{r-1})}$ is a quotient of $(\Gamma^1)^{\otimes d_0}\otimes\cdots\otimes(\Gamma^{p^{r-1}})^{\otimes d_r}$, so that $p(T^{(d_0,\dots,d_{r-1})},r)>0$. Hence (iv)$\Rightarrow$(iii) by Corollary 4.2. ☐

The next corollary is a translation of Theorem 3.6 in low degrees in terms of p^r-restricted weights. It will be used in Sections 5 and 6.

Corollary 4.4. *Let (F,G,X,Y) be a quadruple of homogeneous strict polynomial functors satisfying the Künneth condition* (Condition 2.3), *and let $r\geq 0$. Precomposing by $I^{(r)}$ and taking cup products induces injective morphisms*

$$\mathrm{Hom}_{\mathcal{P}_\Bbbk}(F,G)\otimes\mathrm{Hom}_{\mathcal{P}_\Bbbk}(X,Y)\hookrightarrow\mathrm{Hom}_{\mathcal{P}_\Bbbk}(F\otimes X^{(r)},G\otimes Y^{(r)}),\tag{4.5}$$

$$\begin{array}{c}\mathrm{Hom}_{\mathcal{P}_\Bbbk}(F,G)\otimes\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(X,Y)\\ \oplus\ \mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(F,G)\otimes\mathrm{Hom}_{\mathcal{P}_\Bbbk}(X,Y)\end{array}\hookrightarrow\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(F\otimes X^{(r)},G\otimes Y^{(r)}).\tag{4.6}$$

If one of the conditions (C1) *or* (C2) *below holds, morphism* (4.5) *is an isomorphism. If both* (C1) *and* (C2) *hold, then morphism* (4.6) *is also an isomorphism.*

(C1) $\deg F\leq\deg G$ *and* $\mathrm{Head}(G)$ *is a direct sum of p^r-restricted simples.*

(C2) $\deg F\geq\deg G$ *and* $\mathrm{Soc}(F)$ *is a direct sum of p^r-restricted simples.*

Proof. Recall from Section 2B that precomposing by $I^{(r)}$ yields a $\Bbbk$-linear isomorphism on the level of Hom and Ext^1. Thus the statement of Corollary 4.4 is equivalent to the statement where X and Y are replaced by $X^{(r)}$ and $Y^{(r)}$ at the source of the cup product maps. By Corollary 4.2, (C1) means that $i(G,r)>0$, and (C2) that $i(F,r)>0$. Thus Corollary 4.2 follows directly from Theorem 3.6. ☐

Remark 4.7. In Sections 5 and 6 we will use Corollary 4.4 for quite general functors F and G. However, this result is already interesting in the very special case where F and G are p^r-restricted simples. Indeed the isomorphism given by Corollary 4.4 is then a stronger form, albeit valid only for stable polynomial representations of GL_n, of formulas of Donkin [1982] and Andersen [1984]; see also [Jantzen 2003, II.10.16, II.10.17]. For example, Corollary 4.4 implies that if $\lambda\neq\lambda'$ are partitions of d and $G=\mathrm{GL}_n$ with n big enough, then the number of L_μ

in the socle of the tensor product $\mathrm{Ext}^1_{G_r}(L_\lambda, L_{\lambda'})^{(-r)} \otimes L'_\mu$ is zero, unless $\mu = \mu'$, in which case it equals the dimension of $\mathrm{Ext}^1_G(L_\lambda, L_{\lambda'})$.

5. Tensor products of Steinberg type

Recall that a simple functor L is a composition factor of an arbitrary functor F if L is a subquotient of F. Equivalently, if $0 = F^{-1} \subset F^0 \subset \cdots \subset F$ is an exhaustive filtration of F whose successive quotients are semisimple (e.g., the Loewy filtration of F), then L appears as a direct summand in one of these successive quotients.

Definition 5.1. A tensor product of Steinberg type is a strict polynomial functor isomorphic to a tensor product $F \otimes G^{(r)}$, where r is a nonnegative integer and F is a functor whose composition factors are all p^r-restricted.

The purpose of the present section is to explore the structure of these tensor products of Steinberg type. Note that by Steinberg's tensor product theorem (applied to the tensor product of the socle filtration of F by the socle filtration of $G^{(r)}$), all composition factors of $F \otimes G^{(r)}$ are of the form $L_\lambda \otimes L_\mu^{(r)}$, with L_λ a composition factor of F and L_μ a composition factor of G. This observation motivates the following definition.

Definition 5.2. Let e, d, r be nonnegative integers. We let $\mathcal{S}t(d, e, r)$ be the full subcategory of $\mathcal{P}_{d+ep^r,\Bbbk}$ supported by the strict polynomial functors whose composition factors are all of the form $L_\lambda \otimes L_\mu^{(r)}$ for p^r-restricted partition λ of d and partitions μ of e.

Lemma 5.3. *The category $\mathcal{S}t(d, e, r)$ contains all the tensor products of Steinberg type $F \otimes G^{(r)}$, where F is homogeneous of degree d and G is homogeneous of degree e. Moreover, it is localizing and colocalizing, i.e., it is closed under sums, products, subobjects, quotients and extensions.*

Proof. Everything is straightforward from the definition of $\mathcal{S}t(d, e, r)$ except maybe that $\mathcal{S}t(d, e, r)$ is closed under arbitrary products. Let L be a composition factor of a product $\prod X_\alpha$. Then there is a nonzero map $P_L \to \prod X_\alpha$, where P_L denotes the projective cover of L. Thus there is an α such that $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(P_L, X_\alpha) \neq 0$, so that $L = L_\lambda \otimes L_\mu^{(r)}$ with λ p^r-restricted. $\square$

The next lemma makes critical use of Corollary 4.4.

Lemma 5.4. *In the category $\mathcal{S}t(d, e, r)$, any object X has a presentation $P_1 \to P_0 \to X \to 0$ in which the P_i are direct sums of tensor products of Steinberg type $F \otimes G^{(r)}$ with F and G finite. Similarly, X has a copresentation $0 \to X \to Q^0 \to Q^1$ in which the Q^i are products of such tensor products.*

Proof. It suffices to prove that all the objects of $\mathcal{S}t(d, e, r)$ are quotients of direct sums of tensor products of Steinberg type with values in finite dimensional spaces (then using the duality $\sharp$, they will also embed into products of such functors). Let X be an object of $\mathcal{S}t(d,e,r)$, and let X^i denote the i-th term of the socle filtration of X. Assume that X^{i-1} is a quotient of P^{i-1}, where P^{i-1} has the required form. Then X^i/X^{i-1} is a direct sum of $L_\lambda \otimes L_\mu^{(r)}$, and each of these functors is a homomorphic quotient of $P_\lambda \otimes P_\mu^{(r)}$, where P_μ and P_λ are projective functors, and P_λ is left p^r-bounded. Using Corollary 4.4 and the projectivity of P_μ and P_λ, we obtain $\mathrm{Ext}^1_{\mathcal{P}_{\Bbbk}}(P_\lambda \otimes P_\mu^{(r)}, X^{i-1}) = 0$. Hence the map $P_\lambda \otimes P_\mu^{(r)} \to X^i/X^{i-1}$ lifts to $f : P_\lambda \otimes P_\mu^{(r)} \to X^i$. The functor P_λ has a unique largest quotient P'_λ whose composition factors are p^r-restricted. Let K_λ be the kernel of the quotient map $P_\lambda \twoheadrightarrow P'_\lambda$. By Corollary 4.4, $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(K_\lambda \otimes P_\mu^{(r)}, X^i) = 0$. Therefore, f induces a map $P'_\lambda \otimes P_\mu^{(r)} \to X^i$. In particular, if we define $P^i := P^{i-1} \oplus \bigoplus P'_\lambda \otimes P_\mu^{(r)}$, then P^i is a direct sum of tensor products of Steinberg type with values in finite dimensional vector spaces, and X^i is a quotient of P^i. Since homogeneous strict polynomial functors have finite socle filtrations, this proves the lemma. $\square$

We will prove that the categories $\mathcal{S}t(d,e,r)$ have an alternative description in terms of bifunctors. To be more specific, we denote by

$$\Phi : \mathcal{P}_{d,e,\Bbbk}(2) \to \mathcal{P}_{d+p^r e,\Bbbk}$$

the functor such that $\Phi(B)(V) = B(V, V^{(r)})$. We observe that Φ is exact, but it is not an equivalence of categories. For example, if $d = p^r$ and $e = 1$, the bifunctor $I^{(r)} \boxtimes I$ is simple, while its image $\otimes^{2(r)}$ is not ($\Lambda^{2(r)}$ is a proper subfunctor). However, Φ behaves better if we suitably restrict its source and target categories.

Definition 5.5. Let e, d, r be nonnegative integers. We denote by $\mathcal{S}t'(d,e,r)$ the full subcategory of $\mathcal{P}_{d,e,\Bbbk}(2)$ supported by the strict polynomial bifunctors whose composition factors are all of the form $L_\lambda \boxtimes L_\mu$, where λ is a p^r-restricted partition of weight d and μ is a partition of weight e.

We have the following analogues of Lemmas 5.3 and 5.4.

Lemma 5.6. *The subcategory $\mathcal{S}t'(d,e,r)$ contains all the separable functors $F \boxtimes G$, where F is homogeneous of degree d with p^r-restricted composition factors and G is homogeneous of degree e. Moreover, $\mathcal{S}t'(d,e,r)$ is closed under sums, products, subobjects, quotients and extensions.*

Lemma 5.7. *In the category $\mathcal{S}t'(d,e,r)$, any object B has a presentation $P_1 \to P_0 \to X \to 0$ in which the P_i are direct sums of tensor products of separable type $F \boxtimes G$, where F and G are finite and the composition factors of F are p^r-restricted. Similarly, B has a copresentation $0 \to X \to Q^0 \to Q^1$ in which the Q^i are products of such tensor products.*

We can now state the central theorem of this section.

Theorem 5.8. *The functor Φ restricts to an equivalence of categories*:

$$\Phi : \mathcal{S}t'(d,e,r) \xrightarrow{\simeq} \mathcal{S}t(d,e,r).$$

Proof. We first prove that Φ is fully faithful. Let $\mathcal{T}$ be the full subcategory of $\mathcal{S}t'(d,e,r)$ supported by the bifunctors of separable type $F \boxtimes G$ with F and G finite. By Lemma 5.7 and exactness of Φ, it suffices to prove that the restriction of Φ to $\mathcal{T}$ is fully faithful. This follows from Corollary 4.4. To prove that Φ is essentially surjective, we consider the functor $\Psi : \mathcal{P}_{d+ep^r,\Bbbk} \to \mathcal{P}_{d,e,\Bbbk}(2)$ which sends a functor F to the bifunctor

$$(\Psi F)(V,W) = \mathrm{Hom}_{\mathcal{P}_{d+p^r e,\Bbbk}}(\Gamma^{d,V} \otimes (\Gamma^{e,W})^{(r)}, F).$$

If $F \otimes G^{(r)}$ is a tensor product of Steinberg type, by Corollary 4.2 F is right p^r-bounded, so that Corollary 4.4 and [Friedlander and Suslin 1997, Theorem 2.10] yield isomorphisms of strict polynomial functors:

$$\begin{aligned}(\Phi\Psi(F \otimes G^{(r)}))(V) &\simeq \mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(\Gamma^{d,V}, F) \otimes \mathrm{Hom}_{\mathcal{P}_{e,\Bbbk}}(\Gamma^{e,V^{(r)}}, G)\\ &\simeq F(V) \otimes G(V^{(r)}).\end{aligned}$$

Thus $\Phi \circ \Psi$ is the identity functor on the tensor products of Steinberg type. By Lemma 5.4, all the functors of $\mathcal{S}t(d,e,r)$ are kernels of products of tensor products of Steinberg type. Thus by left exactness of $\Phi \circ \Psi$, the restriction of $\Phi \circ \Psi$ to the whole category $\mathcal{S}t(d,e,r)$ is isomorphic to the identity functor. Hence Φ is essentially surjective (and Ψ is the inverse of Φ). $\square$

Theorem 5.8 generalizes the Steinberg tensor product theorem. Indeed, external tensor products $L_\lambda \boxtimes L_\mu$ of simple functors are simple bifunctors, so that Theorem 5.8 and the stability of $\mathcal{S}t(d,e,r)$ by subobjects imply that the functors $\Phi(L_\lambda \boxtimes L_\mu) = L_\lambda \otimes L_\mu^{(r)}$ are simple. More generally, Theorem 5.8 can be used to convert any question about the structure of the tensor products of Steinberg type (socle length, submodule lattices, or even Ext^1 issues) into similar questions about the structure of bifunctors of separable type which are much easier to study. To illustrate this, we give new properties of tensor products of Steinberg type, obtained by translating some general properties of representations of tensor products of finite dimensional algebras given in Appendix A (recall that the category $\mathcal{P}_{d,e,\Bbbk}(2)$ is equivalent to the category of $S(d,d) \otimes S(e,e)$-modules).

Remark 5.9. In the following corollaries, we do not assume that F and G are homogeneous. In each case, the proof reduces easily to the homogeneous case by additivity of tensor products. We also observe that each of these corollaries is a stronger statement than the classical Steinberg tensor product theorem.

Corollary 5.10 (socle series). *If the composition factors of F are p^r-restricted, then for all G, the socle filtration of $F \otimes G^{(r)}$ is the tensor product of the socle filtration of F by the socle filtration of G, precomposed by $I^{(r)}$.*

Corollary 5.11 (subfunctors). *Assume that the composition factors of F are p^r-restricted. Let G be any functor. Assume that F or G is multiplicity free. Then the subfunctors $S \subset F \otimes G^{(r)}$ are of the form*

$$S = \sum_{\alpha} F_\alpha \otimes G_\alpha^{(r)}$$

for some families of subfunctors $F_\alpha \subset F$ and $G_\alpha \subset G$.

Corollary 5.12 (diagrams). *Assume that F and G are multiplicity-free and the composition factors of F are p^r-restricted. Then the diagram associated to $F \otimes G^{(r)}$ as defined in [Alperin 1980] has the functors $L_\lambda \otimes L_\mu^{(r)}$ as vertices, where L_λ is a composition factor of F and L_μ is a composition factor of G, and there is an edge $L_\lambda \otimes L_\mu^{(r)} \to L'_\lambda \otimes {L'_\mu}^{(r)}$ if and only if one of the following two conditions holds:*

(i) *$L_\lambda = L'_\lambda$ and there is an edge $L_\mu \to L'_\mu$ in the diagram of G,*

(ii) *$L_\mu = L'_\mu$ and there is an edge $L_\lambda \to L'_\lambda$ in the diagram of F.*

The next statement follows from Proposition A.8. It uses the fact that all simple strict polynomial functors satisfy $\mathrm{Ext}^1_{\mathcal{P}_{\Bbbk}}(L, L) = 0$, which follows from the fact that the Schur algebras are quasihereditary.

Corollary 5.13 (tensor products on the left). *Let λ be a p^r-restricted partition. Let $L_\lambda \otimes \mathcal{P}_{\Bbbk}^{(r)}$ denote the full subcategory of $\mathcal{P}_{\Bbbk}$ whose objects are the functors isomorphic to tensor products of the form $L_\lambda \otimes F^{(r)}$. Then*

(1) *the subcategory $L_\lambda \otimes \mathcal{P}_{\Bbbk}^{(r)}$ is localizing and colocalizing,*

(2) *precomposing by $I^{(r)}$ and tensoring by L_λ yields an equivalence of categories $\mathcal{P}_{\Bbbk} \simeq L_\lambda \otimes \mathcal{P}_{\Bbbk}^{(r)}$.*

6. Application to internal tensor products

The purpose of this section is to study the internal tensor product of simple functors. In particular, Theorem 6.2 plays a role for internal tensor products similar to the role of the Steinberg theorem for ordinary tensor products.

6A. ***Internal tensor products of simple functors.*** Let F_1 and G_1 be two homogeneous functors of degree d, and F_2 and G_2 homogeneous functors of degree e. The internal tensor product is equipped with a coproduct

$$(F_1 \otimes F_2) \underline{\otimes} (G_1 \otimes G_2) \to (F_1 \underline{\otimes} G_1) \otimes (F_2 \underline{\otimes} G_2).$$

To be more specific, this coproduct coincides on the standard projectives with the following composite (where the first map is the canonical inclusion and the second one is the canonical projection):

$$\begin{aligned}(\Gamma^{d,T}\otimes\Gamma^{e,U})\,\underline{\otimes}\,(\Gamma^{d,V}\otimes\Gamma^{e,W}) &\hookrightarrow (\Gamma^{d+e,T\oplus U}\,\underline{\otimes}\,\Gamma^{d+e,V\oplus W})\\ &=\Gamma^{d+e,(T\oplus U)\otimes(V\oplus W)}\\ &\twoheadrightarrow \Gamma^{d,T\otimes V}\otimes\Gamma^{e,U\otimes W}\\ &=(\Gamma^{d,T}\,\underline{\otimes}\,\Gamma^{d,V})\otimes(\Gamma^{e,U}\,\underline{\otimes}\,\Gamma^{e,W}).\end{aligned}$$

The following proposition is a consequence of Corollary 4.4.

Proposition 6.1. *Let F, G, X, Y be homogeneous strict polynomial functors, and let $r\geq 0$. If $\deg F<\deg G$ and G is left p^r-bounded, or if $\deg F>\deg G$ and F is left p^r-bounded, then*

$$(F\otimes X^{(r)})\,\underline{\otimes}\,(G\otimes Y^{(r)})=0.$$

If $\deg F=\deg G$ and F or G is left p^r-bounded, then the coproduct induces an isomorphism

$$(F\otimes X^{(r)})\,\underline{\otimes}\,(G\otimes Y^{(r)})\simeq(F\,\underline{\otimes}\,G)\otimes(X\,\underline{\otimes}\,Y)^{(r)}.$$

Proof. In this proof, we assume that $\deg F\geq\deg G$ and F is p^r-bounded (the proof is similar if $\deg F\leq\deg G$ and G is p^r-bounded). Since the internal tensor product is right exact and commutes with arbitrary direct sums, it suffices to prove Proposition 6.1 when G and Y are finite.

Since F is left p^r-bounded, the parametrized functor F^V also is. Hence, if $\deg F>\deg G$, Corollary 4.4 implies that

$$\underline{\mathrm{Hom}}(F\otimes X^{(r)},G^{\sharp}\otimes(Y^{\sharp})^{(r)})=0. \tag{$*$}$$

Since G and Y are finite, $G^{\sharp}\otimes(Y^{\sharp})^{(r)}$ is isomorphic to $(G\otimes Y^{(r)})^{\sharp}$. Hence the equality $(*)$ can be reinterpreted as

$$((F\otimes X^{(r)})\,\underline{\otimes}\,(G\otimes Y^{(r)}))^{\sharp}=0.$$

This proves the asserted cancellation. Assume now that $\deg F=\deg G$. Then by Corollary 4.4 the cup product induces an isomorphism

$$\underline{\mathrm{Hom}}(F,G)\otimes\underline{\mathrm{Hom}}(X,Y)^{(r)}\simeq\underline{\mathrm{Hom}}(F\otimes X^{(r)},Y\otimes Y^{(r)}). \tag{$**$}$$

But the coproduct is dual to the cup product; that is, for all functors F, G, H and K there is a commutative diagram in which the horizontal isomorphisms are the canonical isomorphisms recalled in Section 2E:

$$\begin{tikzcd}
(F \underline{\otimes} H)^\sharp \otimes (G \underline{\otimes} K)^\sharp \arrow[r, "\simeq"] \arrow[d, "\text{can}"] & \underline{\mathrm{Hom}}(F, H^\sharp) \otimes \underline{\mathrm{Hom}}(G, K^\sharp) \arrow[d, "\cup"] \\
((F \underline{\otimes} H) \otimes (G \underline{\otimes} K))^\sharp \arrow[d, "\text{coproduct}^\sharp"] & \underline{\mathrm{Hom}}(F \otimes H, G^\sharp \otimes K^\sharp) \arrow[d, "\text{can}"] \\
((F \otimes H) \underline{\otimes} (G \otimes K))^\sharp \arrow[r, "\simeq"] & \underline{\mathrm{Hom}}(F \otimes H, (G \otimes K)^\sharp)
\end{tikzcd}$$

If the functors G and K are finite, so is $G \underline{\otimes} K$ and the canonical maps denoted "can" in the diagram above are isomorphisms. Thus, the isomorphism of Proposition 6.1 can be deduced from the diagram above with $H = X^{(r)}$ and $K = Y^{(r)}$, and from the isomorphism $(**)$. □

The following theorem reduces the study of internal tensor products of simple functors to the case of p-restricted simple functors. In other terms, it plays the same role for internal tensor products as the classical Steinberg tensor product theorem does for ordinary tensor products.

Theorem 6.2. *Let $\lambda^0, \dots, \lambda^r$ and $\mu^0, \dots, \mu^s$ be p-restricted partitions, and let $\lambda = \sum p^i \lambda^i$ and $\mu = \sum p^i \mu^i$.*

(1) *If $r = s$ and μ^i and λ^i have the same weight for all i, then $L_\lambda \underline{\otimes} L_\mu$ is nonzero and there is an isomorphism*

$$L_\lambda \underline{\otimes} L_\mu \simeq (L_{\lambda^0} \underline{\otimes} L_{\mu^0}) \otimes (L_{\lambda^1} \underline{\otimes} L_{\mu^1})^{(1)} \otimes \cdots \otimes (L_{\lambda^r} \underline{\otimes} L_{\mu^r})^{(r)}.$$

(2) *Otherwise, $L_\lambda \underline{\otimes} L_\mu$ is zero.*

Proof. The classical Steinberg tensor product theorem shows that

$$L_\lambda = L_{\lambda^0} \otimes \cdots \otimes L_{\lambda^r}^{(r)} \quad \text{and} \quad L_\mu = L_{\mu^0} \otimes \cdots \otimes L_{\mu^s}^{(s)},$$

where the L_{λ^i} and the L_{μ^j} are p-restricted, hence right p-bounded by Corollary 4.2. Thus the result follows by applying Proposition 6.1. □

6B. ***The case of p-restricted simple functors.*** To investigate internal tensor products of p-restricted simple functors, we rely on the Schur functor.

Lemma 6.3. *For all strict polynomial functors F, there are isomorphisms of functors, natural with respect to F:*

$$F \underline{\otimes} \otimes^d \simeq \underline{\mathrm{Hom}}(\otimes^d, F) \simeq \otimes^d \otimes f_d(F).$$

Moreover, if we consider the action of $\mathfrak{S}_d$ on the left-hand side induced by the left action of $\mathfrak{S}_d$ on $\otimes^d$, the action on the middle term induced by the right action of $\mathfrak{S}_d$ on $\otimes^d$, and the diagonal action of $\mathfrak{S}_d$ on the right-hand side, then these isomorphisms are $\mathfrak{S}_d$-equivariant.

Proof. We have isomorphisms of strict polynomial functors, natural with respect to V and F:

$$\underline{\mathrm{Hom}}(\Gamma^{d,V}, F) \simeq F_V \simeq F \underline{\otimes} \Gamma^d_V. \tag{$*$}$$

Take $V = \Bbbk^d$ and let the torus $(\mathbb{G}_m)^{\times d}$ act on $\Bbbk^d$ by $(\lambda_1, \dots, \lambda_d) \cdot (x_1, \dots, x_d) = (\lambda_1 x_1, \dots, \lambda_d x_d)$. Then the summand of weight $(1, \dots, 1)$ of the right-hand side of isomorphism $(*)$ is $F \underline{\otimes} \otimes^d$, and it is isomorphic to the summand of weight $(1, \dots, 1)$ of the left-hand side, which is $\underline{\mathrm{Hom}}(\otimes^d, F)$. Moreover, $\underline{\mathrm{Hom}}(\otimes^d, F)$ is isomorphic to the functor $U \mapsto \mathrm{Hom}_{\mathcal{P}_\Bbbk}((\otimes^d)^U, F)$. Since $(\otimes^d)^U$ is isomorphic to $(U^*)^{\otimes d} \otimes \otimes^d$, we get an isomorphism of strict polynomial functors with variable U:

$$\underline{\mathrm{Hom}}(\otimes^d, F) \simeq \mathrm{Hom}_{\mathcal{P}_\Bbbk}((U^*)^{\otimes d} \otimes \otimes^d, F) \simeq U^{\otimes d} \otimes f_d(F).$$

Finally, one easily checks that these explicit constructions of the isomorphisms of Lemma 6.3 yield $\mathfrak{S}_d$-equivariant isomorphisms. $\square$

Proposition 6.4. *For all functors F, G, there is an isomorphism of $\Bbbk\mathfrak{S}_d$-modules*

$$f_d(F \underline{\otimes} G) \simeq f_d(F) \otimes f_d(G),$$

where the tensor product on the right is the Kronecker product of $f_d(F)$ and $f_d(G)$ (i.e., $\mathfrak{S}_d$ acts diagonally).

Proof. Lemma 6.3 yields a chain of isomorphisms:

$$F \underline{\otimes} (G \underline{\otimes} \otimes^d) \simeq F \underline{\otimes} (\otimes^d \otimes f_d(G)) \simeq (F \underline{\otimes} \otimes^d) \otimes f_d(G) \simeq \otimes^d \otimes f_d(F) \otimes f_d(G).$$

Thus the evaluation of $F \underline{\otimes} (G \underline{\otimes} \otimes^d)$ at $\Bbbk$ is isomorphic to $f_d(F) \otimes f_d(G)$. On the other hand, $F \underline{\otimes} (G \underline{\otimes} \otimes^d)$ is isomorphic to $(F \underline{\otimes} G) \underline{\otimes} \otimes^d$ and Lemma 6.3 shows that the evaluation of the latter at $\Bbbk$ is isomorphic to $f_d(F \otimes G)$. $\square$

The following corollary shows that in the first case of Theorem 6.2, the internal tensor product is always nonzero.

Corollary 6.5. *Let L and L' be two p-restricted simples. Then $L \underline{\otimes} L'$ is nonzero.*

Proof. By Clausen and James' theorem, $f_d(L)$ and $f_d(L')$ are nonzero. Hence, by Proposition 6.4, $f_d(L \underline{\otimes} L')$ is nonzero. Thus $L \underline{\otimes} L'$ is nonzero. $\square$

Given two p-restricted simples L and L', a natural question is to determine if the analogue of Theorem B.12 holds, i.e., if the nonzero functor $L \underline{\otimes} L'$ is simple. In fact, Bessenrodt and Kleshchev [2000] have proved that the Kronecker product of two simple representations of symmetric groups is almost never simple. In particular, Proposition 6.4 has the following consequence in odd characteristic.

Corollary 6.6. *Assume that p is odd. Let L and L' be two p-restricted simples such that $f_d(L)$ and $f_d(L')$ both have dimension at least two. Then $L \underline{\otimes} L'$ is not simple.*

Proof. Since the right adjoint of f_d satisfies $f_d \circ r_d = \mathrm{Id}$, $f_d(L)$ sends simple functors either to simple $\Bbbk\mathfrak{S}_d$-modules or to zero. But $f_d(L \underline{\otimes} L') \simeq f_d(L) \otimes f_d(L')$ is a Kronecker product of two simple $\Bbbk\mathfrak{S}_d$-modules, so is not simple by [Bessenrodt and Kleshchev 2000, Theorem 2]. Thus $L \underline{\otimes} L'$ cannot be simple. □

Remark 6.7. Corollary 6.6 uses [Bessenrodt and Kleshchev 2000, Theorem 2], which is a nontrivial result on symmetric groups. It would be interesting to find a more elementary proof of Corollary 6.6, in the spirit of the proof of Theorem B.12.

To solve completely (in odd characteristic) the problem of knowing if an internal tensor product $L \underline{\otimes} L'$ can be simple, it remains to study the case where $f_d(L')$ has dimension 1. The remainder of the section is devoted to this study. In our discussion below, we show in Corollary 6.10 that when $f_d(L')$ has dimension one, $L \underline{\otimes} L'$ may sometimes be simple and sometimes not, and in Corollary 6.9 we show that it suffices to study the case $L' = Q^d$. The latter case is studied in [Reischuk 2016], where the simplicity of $L \underline{\otimes} Q^d$ is shown to be equivalent to $p(L, 1) > 1$.

There are two $\Bbbk\mathfrak{S}_d$-modules of dimension 1, namely the signature module $\Bbbk^{\mathrm{alt}}$ and the trivial module $\Bbbk$. The signature module is the image by the Schur functor of $\Lambda^d = L_{(1,\dots,1)}$. Since $\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(\otimes^d, S^d)$ has dimension 1 and since S^d is a quotient of $\otimes^d$, the head of S^d is a p-restricted simple functor. This functor is known under the name of truncated symmetric powers, and we denote it by Q^d as in [Breen et al. 2016]. Then $f_d(Q^d)$ is the trivial $\Bbbk\mathfrak{S}_d$-module. Thus, to solve completely (in odd characteristic) the problem of knowing if an internal tensor product $L \underline{\otimes} L'$ can be simple, it remains to investigate the internal tensor products $L \underline{\otimes} Q^d$ and $L \underline{\otimes} \Lambda^d$ for p-restricted simples L.

Proposition 6.8. *Let F be a homogeneous functor of degree d. Consider the right action of $\mathfrak{S}_d$ on $\otimes^d$ given by permuting the factors of the tensor product. If $p \neq 2$ then*

$$F \underline{\otimes} \Lambda^d \simeq (\otimes^d) \otimes_{\mathfrak{S}_d} (\Bbbk^{\mathrm{alt}} \otimes f_d(F)).$$

If $\mathrm{Head}(F)$ *is a direct sum of p-restricted simples* (*and p arbitrary*), *then*

$$F \underline{\otimes} Q^d \simeq (\otimes^d) \otimes_{\mathfrak{S}_d} f_d(F).$$

Proof. Lemma 6.3 yields an $\mathfrak{S}_d$-equivariant isomorphism $F \underline{\otimes} \otimes^d \simeq \otimes^d \otimes f_d(F)$. Taking the coinvariants under the signed action of $\mathfrak{S}_d$ and using right exactness of internal tensor products, we obtain the first isomorphism. For the second, let R^d be the radical of S^d. Since $f_d(S^d) = f_d(Q^d)$ and the Schur functor is exact, we have $f_d(R^d) = 0$. Hence, by Lemma 6.3, $R^d \underline{\otimes} \otimes^d$ is zero. But if P is left p-bounded projective, it is a direct summand in a direct sum of copies of $\otimes^d$, and hence $R^d \underline{\otimes} P$ is zero. Now F is left p-bounded by Corollary 4.2, so $R^d \underline{\otimes} F = F \underline{\otimes} R^d = 0$. By right exactness of tensor products we thus obtain an isomorphism $F \underline{\otimes} S^d \simeq F \underline{\otimes} Q^d$. Then the computation of $F \underline{\otimes} S^d$ is done in the same fashion as that of $F \underline{\otimes} \Lambda^d$. □

If M is a simple $\mathfrak{S}_d$-module, then $M \otimes \Bbbk^{\mathrm{alt}}$ is also simple. Let L_μ be the simple p-restricted functor such that $f_d(L_\mu) = M$. We denote by $m(\mu)$ the p-restricted partition such that $f_d(L_{m(\mu)}) = M \otimes \Bbbk^{\mathrm{alt}}$. The involution $\mu \mapsto m(\mu)$ (or rather $\mu' \mapsto m(\mu')$ where μ' stands for the conjugate partition of μ) is known as the Mullineux correspondence [Martin 1993, Section 4.2], and its combinatorial description has been proved by Ford and Kleshchev [1997]; see also the work of Brundan and Kujawa [2003] for a more recent and different proof. Proposition 6.8 has the following consequence.

Corollary 6.9. *Let μ be a p-restricted partition. Then*

$$L_\mu \underline{\otimes} \Lambda^d \simeq L_{m(\mu)} \underline{\otimes} Q^d.$$

As another consequence of Proposition 6.8, we obtain that the internal tensor product of two simple functors may sometimes be simple and sometimes not. The problem of knowing exactly for which p-restricted partitions μ the functor $L_\mu \underline{\otimes} \Lambda^d$ is simple is studied in [Reischuk 2016].

Corollary 6.10. *Assume that p is odd. Then $Q^d \underline{\otimes} \Lambda^d$ is isomorphic to Λ^d, and $\Lambda^d \underline{\otimes} \Lambda^d$ is isomorphic to S^d.*

7. Estimates for $p(F, r)$ and $i(F, r)$

7A. *Basic properties of $p(F, r)$ and $i(F, r)$.* Let r be a positive integer. We introduce the following two homogeneous functors of degree d, where $T^{(d_0,\dots,d_k)} = (\otimes^{d_0}) \otimes (\otimes^{d_1})^{(1)} \otimes \cdots \otimes (\otimes^{d_k})^{(k)}$ as in Corollary 4.3:

$$L(d, r) = \bigoplus_{\substack{\lambda \text{ not } p^r\text{-restricted} \\ \text{and } |\lambda| = d}} L_\lambda, \qquad T(d, r) = \bigoplus_{\substack{\sum_{0\le i<r} p^i d_i < d \\ \sum_{0\le i} p^i d_i = d}} T^{(d_0,\dots,d_k)}.$$

These functors are defined so that they contain all the simples of degree d, or all the twisted tensor powers of degree d, which have at least one factor precomposed by $I^{(s)}$ with $s \ge r$. Hence they are nonzero if and only if $d \ge p^r$. By Corollary 4.3, $L(d, r)$ is a quotient of $T(d, r)$. Since these two functors are self-dual, it follows that $L(d, r)$ is also a subfunctor of $T(d, r)$.

Proposition 7.1. *Let F be a homogeneous functor of degree d, and let $G(d, r)$ be equal to either $L(d, r)$ or $T(d, r)$. Then $p(F, r)$ is the lowest (possibly $+\infty$) degree k such that the vector space $\mathrm{Ext}^k_{\mathcal{P}_\Bbbk}(F, G(d, r))$ is nonzero, and $i(F, r)$ is the lowest k such that $\mathrm{Ext}^k_{\mathcal{P}_\Bbbk}(G(d, r), F)$ is nonzero.*

Proof. Let P be a degree d homogeneous p^r-bounded projective. Then Theorem 3.6 implies that $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(P, G(d, r))$ is zero. Take a projective resolution Q of F whose first $p(F, r)$-terms (i.e., up to index $p(F, r) - 1$) are left p^r-bounded projectives, and let K be the kernel of $Q_{p(F,r)-1} \to Q_{p(F,r)-2}$. By definition of $p(F,r)$, K is not p^r-bounded. By Corollary 4.2, this means that there exists a nonzero map $K \to L(d,r)$, and hence also a nonzero map $K \to T(d,r)$. By dimension shifting,

$$\mathrm{Ext}^i_{\mathcal{P}_\Bbbk}(F, G(d,r)) \simeq \begin{cases} 0 & \text{if } i < p(F,r), \\ \mathrm{Hom}_{\mathcal{P}_\Bbbk}(K, G(d,r)) \neq 0 & \text{if } i = p(F,r). \end{cases}$$

The proof for $i(F,r)$ is similar. □

Since $T(d,r)$ is a self-dual functor, $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(T(d,r), F^\sharp)$ is always isomorphic to $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F, T(d,r))$. Thus we obtain the following corollary.

Corollary 7.2. *For all functors F, we have $i(F^\sharp, r) = p(F,r)$.*

We now indicate how $i(F,r)$ behaves with respect to some usual operations on strict polynomial functors. There are similar statements for $p(F,r)$ which can be deduced from the formula $p(F,r) = i(F^\sharp, r)$ or by repeating the proofs with projective resolutions. We leave this to the reader.

Proposition 7.3. *Let F and G be two functors. The following hold*:

(a) $i(F_V, r) = i(F,r)$.

(b) $i(F,r) = i(F^{(s)}, r+s)$.

(c) $i(F \otimes G, r) = \min\{i(F,r), i(G,r)\}$.

(d) $i(F \oplus G, r) = \min\{i(F,r), i(G,r)\}$.

(e) $i(F,r) \geq \min\{i(S,r) : S \text{ is finite and } S \subset F\}$.

Proof. Statement (d) is straightforward from the characterization of $i(F,r)$ in terms of Ext^* provided by Proposition 7.1, and implies that for the remaining statements, we can assume that F and G are homogeneous. We let $d := \deg F$ and $g := \deg G$. Statement (e) follows from the interpretation of $i(F,r)$ given in Proposition 7.1 and the fact that $\mathrm{Ext}^*(T(d,r), -)$ commutes with directed colimits. To prove (a), observe that F is a direct summand in F^V so that $i(F,r) \geq i(F_V, r)$. Moreover, if J is a standard p^r-bounded injective then J_V is a direct sum of standard p^r-bounded injectives. Hence if Q is an injective resolution of F whose first $i(F,r)$ terms are left p^r-bounded injectives, then Q_V is an injective resolution of F_V whose first $i(F,r)$ terms are left p^r-bounded injectives, so that $i(F_V,r) \geq i(F,r)$.

To prove (b), we use the isomorphisms

$$\mathrm{Ext}^*_{\mathcal{P},\Bbbk}(T(d,r+s), F^{(s)}) \simeq \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(T(d,r)^{(s)}, F^{(s)}) \simeq \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(T(d,r), F_{E_s}).$$

The first isomorphism is induced by the inclusion $T(d,r)^{(s)} \subset T(d,r+s)$; the cokernel of this split inclusion is easily seen to be zero by using the sum diagonal adjunction. The second isomorphism is proved in [Touzé 2013a; Chałupnik 2015].

In this formula F_{E_s} is a nonnegatively graded functor, and the degree on the right-hand side is the total degree. The graded functor F_{E_s} equals $F_{\Bbbk^{p^s}}$ in an ungraded way, so that the lowest nonzero degree k on the right-hand side of the isomorphism is greater or equal to $i(F_{\Bbbk^{p^s}}, r) = i(F, r)$. Hence $i(F^{(s)}, r+s) \geq i(F, r)$. Conversely, the degree zero component of F_{E_s} is isomorphic to F, so that the lowest nonzero degree k on the right-hand side of the isomorphism is lower or equal to $i(F, r)$, and hence $i(F^{(s)}, r+s) \leq i(F, r)$.

It remains to prove (c). Assume for example that $i(F, r) \leq i(G, r)$. If Q and Q' are injective resolutions of F and G, respectively, whose first $i(F, r)$ terms are p^r-bounded, then $Q \otimes Q'$ is an injective resolution of $F \otimes G$ whose first $i(F, r)$ terms are p^r-bounded, and hence $i(F \otimes G, r) \geq i(F, r)$. Conversely, let x be a nonzero extension in $\mathrm{Ext}^{i(F,r)}(T(d, r), F)$. Let L be a simple subfunctor of G. As L is a quotient of a functor $T^{(d_0,\dots,d_\ell)}$ for some tuple $(d_0, \dots, d_\ell)$ by Corollary 4.3, there is a nonzero map $f : T^{(d_0,\dots,d_\ell)} \to G$. Since cup products are injective (by Theorem 3.6 with $r = 0$ or by Lemma 3.9), $x \cup f$ is a nonzero element of $\mathrm{Ext}^{i(F,r)}(T(d, r) \otimes T^{(d_0,\dots,d_\ell)}, F \otimes G)$. But $T(d, r) \otimes T^{(d_0,\dots,d_\ell)}$ is a direct summand of $T(d+g, r)$, so that $i(F \otimes G, r) \leq i(F, r)$. □

7B. *A few examples.*

Proposition 7.4. *Let r be a nonnegative integer. The following hold*:

(1) *If* $\deg F < p^r$, *then* $i(F, r) = +\infty$.

(2) *If* $d \geq p^r$, *then* $i(S^d, r) = 0$.

(3) *If* $d \geq p^r$, *then* $i(\Lambda^d, r) = p^r - 1$.

(4) *If* $d \geq p^r$, *then* $i(\Gamma^d, r) = 2(p^r - 1)$.

Proof. The first statement follows from the fact that when $d < p^r$, all basic injectives of degree d are p^r-bounded. If $d \geq p^r$, the multiplication of the symmetric algebra and the natural inclusion $I^{(r)} \hookrightarrow S^{p^r}$ induce a nonzero map $\otimes^{d-p^r} \otimes I^{(r)} \to S^d$. Hence, by Proposition 7.1, $i(S^d, r) = 0$. Let us prove that $i(\Lambda^d, p^r) = p^r - 1$. The homogeneous part of degree d of the reduced bar construction of the symmetric algebra S provides an injective resolution of Λ^d whose first $p^r - 1$ terms are basic p^r-bounded injectives; see, e.g., [Totaro 1997]. Thus $i(\Lambda^d, r) \geq p^r - 1$. Conversely, using sum-diagonal adjunction one obtains that $\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(\otimes^{d-p^r} \otimes I^{(r)}, \Lambda^d)$ is isomorphic to the tensor product

$$\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(\otimes^{d-p^r}, \Lambda^{d-p^r}) \otimes \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(I^{(r)}, \Lambda^{p^r}).$$

The factor on the left-hand side of the tensor product is concentrated in degree zero (as $\otimes^{d-p^r}$ is projective) and one-dimensional by [Friedlander and Suslin 1997, Corollary 2.12], and by [Friedlander and Suslin 1997, (4.5.1), p. 251], the factor on the right-hand side of the tensor product is one-dimensional and concentrated in degree p^r-1. Now $\otimes^{d-p^r}\otimes I^{(r)}$ is a direct summand in $T(d,r)$, so that $i(\Lambda^d,r)\leq p^r-1$ by Proposition 7.1. A similar argument applies to (4): the homogeneous part of degree d of the twofold reduced bar construction of the symmetric algebra yields an injective resolution whose first $2(p^r-1)$ terms are basic p^r-bounded injectives, and on the other hand, one can compute that $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(\otimes^{d-p^r}\otimes I^{(r)},\Gamma^d)$ is one-dimensional and concentrated in degree $2(p^r-1)$. □

Let us denote by S_λ the Schur functor associated to a partition λ and by W_λ the Weyl functor associated to λ. These are finite homogeneous strict polynomial functors, whose degree is the weight of the partition λ, and we have $W_\lambda=S_\lambda^\sharp$. They generalize the functors S^d, Λ^d and Γ^d. Indeed,

$$W_{(d,0,0,\dots)}=\Gamma^d,\qquad S_{(d,0,0,\dots)}=S^d,\qquad S_{(1,\dots,1)}=W_{(1,\dots,1)}=\Lambda^d.$$

The S_λ are the costandard, and the W_λ the standard, objects of the highest weight category structure of $\mathcal{P}_{\Bbbk}$. In particular $\mathrm{Soc}(S_\lambda)=L_\lambda=\mathrm{Head}(W_\lambda)$. We refer the reader to [Touzé 2013b, Section 6.1.1] or [Krause 2017] for more details and references on these functors. The following lemma may be useful for computations.

Lemma 7.5. *Let λ be a partition and let λ' be the dual partition. For all tuples $(d_0,\dots,d_k)$ there is a graded isomorphism* (*where* Ext *is understood as zero in negative degrees*):

$$\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(T^{(d_0,\dots,d_k)},S_\lambda)\simeq\mathrm{Ext}^{*+s}_{\mathcal{P}_{\Bbbk}}(T^{(d_0,\dots,d_k)},W_{\lambda'}),$$

where $s=d_1(p-1)+d_2(p^2-1)+\cdots+d_k(p^k-1)$.

Proof. We use Ringel duality Θ, which is an autoequivalence of $\mathbf{D}(\mathcal{P}_{d,\Bbbk})$. See, e.g., [Touzé 2013b, Section 3; Chałupnik 2008, Section 2]. We have $\Theta S_\lambda=W_{\lambda'}$ and $\Theta T^{(d_0,\dots,d_k)}=T^{(d_0,\dots,d_k)}[-s]$, so that the lemma follows from interpreting morphisms of degree s in the derived category as extensions of degree s. □

Proposition 7.6. *Let λ be a partition and λ' the dual partition. Then we have*

$$i(S_\lambda,r)+p^r-1\leq i(W_{\lambda'},r).$$

Assume, moreover, that $\lambda=\lambda^0+p\lambda^1+\cdots+p^k\lambda^k$, where each λ^k is a p-restricted partition of d_k, and $k\geq r$. Then

$$i(W_{\lambda'},r)\leq\sum_{i=1}^{k}d_i(p^i-1).$$

Proof. We use the isomorphism of Lemma 7.5. If $T^{(d_0,\dots,d_r)}$ is a direct summand of $T(d,r)$, then the associated shift s is always greater than or equal to p^r-1. This proves the first inequality. As regards the second inequality, we have $L_\lambda=L_{\lambda^0}\otimes\cdots\otimes L_{\lambda^k}^{(k)}$ by the Steinberg tensor product theorem. By Clausen and James' theorem, L_λ is then a quotient of $T^{(d_0,\dots,d_k)}$. Thus we get a nonzero element in $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(T^{(d_0,\dots,d_k)}, S_\lambda)$ by composing the quotient map $T^{(d_0,\dots,d_k)} \to L_\lambda$ with the inclusion $L_\lambda \subset S_\lambda$. Therefore, by Lemma 7.5, there is a nonzero extension of degree $\sum_{i=1}^{k} d_i(p^i-1)$ between $T^{(d_0,\dots,d_k)}$ (hence $T(d,r)$) and $W_{\lambda'}$. □

We finish this section by computing the integers $i(F,r)$ when F is any Schur or Weyl functor of degree 4 in characteristic $p=2$. The result is already known for S^4, Λ^4 and Γ^4 by Proposition 7.4. For the three remaining partitions, the computation relies on the following short exact sequences.

Lemma 7.7. *Let $\Bbbk$ be a field of characteristic $p=2$. There are short exact sequences*

(1) $0 \to \Lambda^4 \to \Lambda^3 \otimes \Lambda^1 \to S_{(2,1,1)} \to 0$,

(2) $0 \to S_{(3,1)} \to S^3 \otimes S^1 \to S^4 \to 0$,

(3) $0 \to S_{(2,2)} \to S^2 \otimes S^2 \to S_{(3,1)} \oplus S^4 \to 0$.

Proof. The first two sequences are the standard presentation and copresentation of Schur functors and are valid over an arbitrary ring [Akin et al. 1982]. Only the last one is specific to the characteristic 2 case and needs to be proved. As proved in [Akin et al. 1982], the Schur functor $S_{(2,2)}$ has copresentation given by

$$0 \to S_{(2,2)} \to S^2 \otimes S^2 \xrightarrow{(\phi,\mathrm{mult})} S^3 \otimes S^1 \oplus S^4, \qquad (*)$$

where mult denotes the map induced by the multiplication for the symmetric algebra and ϕ is defined as the composite map

$$S^2 \otimes S^2 \xrightarrow{S^2\otimes\Delta} S^2 \otimes \otimes^2 \xrightarrow{\mathrm{mult}\otimes S^1} S^3 \otimes S^1,$$

for Δ induced by the comultiplication of the symmetric algebra. Since the field has characteristic 2, there is a surjective map $\pi : S^2 \to \Lambda^2$, and ϕ factors in a unique way as $\phi = \psi \circ (S^2 \otimes \pi)$. Now the composite $\mathrm{mult} \circ \psi : S^2 \otimes \Lambda^2 \to S^4$ is zero, so that the image of ψ is contained in $S_{(3,1)}$. Thus the copresentation $(*)$ induces a copresentation

$$0 \to S_{(2,2)} \to S^2 \otimes S^2 \to S_{(3,1)} \oplus S^4.$$

The last map on the right is surjective for Euler characteristic reasons (the dimensions being independent of the characteristic, one can do the computation in characteristic zero, where $S^2 \otimes S^2$ is isomorphic to $S_{(2,2)} \oplus S_{(3,1)} \oplus S^4$ by the Pieri rule). □

The extension groups between $\otimes^2 \otimes I^{(1)}$, $I^{(1)} \otimes I^{(1)}$ or $I^{(2)}$ on the one hand, and tensor products of symmetric or exterior powers on the other, are easy to compute. Now one can completely compute the extension groups between $T(4,r)$ and the Schur functors simply by inspecting the (not very) long exact $\mathrm{Ext}^*_{\mathcal{P}_{\Bbbk}}(T(d,r),-)$-sequences associated to the short exact sequences of Lemma 7.7. One can then obtain the corresponding computations with Weyl functors by Lemma 7.5. We record the resulting computations of $i(F, r)$ in the following proposition. Since $p^3 = 2^3 > 4 = d$, only the cases $r = 1$ and $r = 2$ are interesting.

Proposition 7.8. *Let $\Bbbk$ be a field of characteristic 2. The following computations hold.*

F	Γ^4	$W_{(3,1)}$	$W_{(2,2)}$	$W_{(2,1,1)}$	Λ^4	$S_{(2,1,1)}$	$S_{(2,2)}$	$S_{(3,1)}$	S^4
$i(F,1)$	2	2	2	1	1	1	0	0	0
$i(F,2)$	6	5	4	4	2	2	1	1	0

Remark 7.9. One sees in this example that the integers $i(F, r)$ are increasing with respect to the dominance order for Weyl functors, and decreasing with respect to the dominance order for Schur functors. It would be quite interesting to know if this is the shadow of some general phenomenon.

8. Application to symmetric groups

Lemma 8.1. *The Schur functor sends p-bounded projectives and injectives to projective and injective $\Bbbk\mathfrak{S}_d$-modules, respectively. Moreover, if F is a p-bounded projective or if G is a p-bounded injective, then the Schur functor induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F, G) \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{S}_d}(f_d(F), f_d(G)).$$

Proof. The left adjoint of f_d sends $\Bbbk\mathfrak{S}_d$ to $\otimes^d$. Thus $f_d(\otimes^d) = f_d(\ell_d(\Bbbk\mathfrak{S}_d)) \simeq \Bbbk\mathfrak{S}_d$ is projective. Moreover, the map induced by f_d,

$$\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(\otimes^d, G) \to \mathrm{Hom}_{\mathfrak{S}_d}(\Bbbk\mathfrak{S}_d, f_d(G)),$$

is an isomorphism because it identifies with the adjunction isomorphism for (ℓ_d, f_d). This proves Lemma 8.1 for the p-bounded projective $\otimes^d$. If $i < p$ then Γ^i is a direct summand of $\otimes^i$ (the retract of the canonical inclusion $\Gamma^i \hookrightarrow \otimes^i$ is the natural transformation which sends $v_1 \otimes \cdots \otimes v_i$ to $\sum_{\sigma\in\mathfrak{S}_i} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)}$.). Thus p-bounded projectives are direct summands of direct sums of copies of $\otimes^d$. As f_d commutes with arbitrary direct sums, this implies that Lemma 8.1 holds for all p-bounded projectives. The proof for p-bounded injectives is similar, using the right adjoint r_d. □

The next theorem generalizes many theorems in [Kleshchev and Nakano 2001]. In particular, Theorem 8.2 does not require any restriction on the characteristic, and works for all F and all G. As regards concrete computations, the explicit bounds for $i(G, 1)$ for Weyl functors G given in Section 7B yield connectedness bounds which are at least as good as those given in [Kleshchev and Nakano 2001]. However, we have not investigated estimates for $i(G, 1)$ when G is simple. Hence, unlike

[Kleshchev and Nakano 2001], we don't have concrete connectedness estimates for simple functors.

Theorem 8.2. *Let F and G be homogeneous strict polynomial functors of degree d. The map induced by the Schur functor*

$$\mathrm{Ext}^k_{\mathcal{P}_{d,\Bbbk}}(F, G) \to \mathrm{Ext}^k_{\Bbbk\mathfrak{S}_d}(f_d(F), f_d(G))$$

is an isomorphism in degrees $k < p(F, 1) + i(G, 1) - 1$, and it is injective in degree $k = p(F, 1) + i(G, 1) - 1$.

Proof. Assume that there is a short exact sequence $0 \to H \to J \to H' \to 0$, where J is a p-bounded injective. The Schur functor induces a morphism from the induced $\mathrm{Ext}^*_{\mathcal{P}_{d,\Bbbk}}(F, -)$-long exact sequence to the induced $\mathrm{Ext}^*_{\mathfrak{S}_d}(f_d(F), f_d(-))$-long exact sequence. Using Lemma 8.1 together with the five lemma, we see that the Schur functor is k-connected for the pair (F, H) (i.e., an isomorphism in Ext-degree $< k$ and injective in Ext-degree k) if and only if it is $(k-1)$-connected for the pair (F, H'). Using this argument, we reduce the proof of Theorem 8.2 to the case where $i(G, 1) = 0$. By a similar argument applied to the contravariant variable of Ext, we reduce the proof further to the case where $i(F, 1) = 1$. In the latter case, F is a quotient of a p-bounded projective P and we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(P, G) \arrow[r, "f_d", "\simeq"'] & \mathrm{Hom}_{\mathfrak{S}_d}(f_d(P), f_d(G)) \\
\mathrm{Hom}_{\mathcal{P}_{d,\Bbbk}}(F, G) \arrow[r, "f_d"] \arrow[u, hook] & \mathrm{Hom}_{\mathfrak{S}_d}(f_d(F), f_d(G)) \arrow[u, hook]
\end{tikzcd}$$

which proves that for the pair (F, G) the Schur functor is indeed $p(F, 1) - 1$ connected (i.e., injective in degree zero). □

The following examples show that the bounds in Theorem 8.2 are optimal.

Example 8.3. Let Q^p be the socle of Γ^p. Then Q^p is the simple functor with highest weight $(p-1, 1)$. In particular, $i(Q^p, 1) \geq 1$ by Corollary 4.2. Since Γ^p is the middle term of a nonsplit extension

$$0 \to Q^p \to \Gamma^p \to I^{(1)} \to 0,$$

we have $\mathrm{Ext}^1_{\mathcal{P}_\Bbbk}(I^{(1)}, Q^p) \neq 0$, which proves that $i(Q^p, 1) \leq 1$ by Proposition 7.1. Thus $i(Q^p, 1) = 1$. We claim that the following map is not an isomorphism:

$$\mathrm{Ext}^{i(Q^p,1)-1}_{\mathcal{P}_{p,\Bbbk}}(\Gamma^p, Q^p) \to \mathrm{Ext}^{i(Q^p,1)-1}_{\Bbbk\mathfrak{S}_p}(f_p(Q^p), f_p(\Gamma^p)).$$

Indeed, the domain $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(\Gamma^p, Q^p)$ is zero as $\mathrm{Head}(\Gamma^p) = I^{(1)} \neq Q^p$. But $f_p(\Gamma^p) = f_p(Q^p) = \Bbbk$. Thus the codomain $\mathrm{Hom}_{\Bbbk\mathfrak{S}_p}(f_p(\Gamma^p), f_p(Q^p))$ has dimension one.

Example 8.4. Let F be a homogeneous functor of degree d. By Proposition 7.1, $\mathrm{Ext}^{i(F,1)}_{\mathcal{P}_{\Bbbk,d}}(T(d,1), F)$ is nonzero. On the other hand $f_d(T(d,1))=0$ by Corollary 4.4, so that the following map is not injective:

$$\mathrm{Ext}^{i(F,1)}_{\mathcal{P}_{d,\Bbbk}}(T(d,1), F) \to \mathrm{Ext}^{i(F,1)}_{\Bbbk\mathfrak{S}_d}(f_d(T(d,1)), f_d(F)).$$

Appendix A. Representations of tensor product algebras

This appendix collects some results about representations of tensor product algebras. All these results are standard (except maybe Proposition A.6), but they are scattered in the literature and not always stated under the form that we want to use.

In the remainder of the section, we fix two finite dimensional algebras A and B over a ground field $\Bbbk$. We assume furthermore that $\Bbbk$ is a splitting field for these two algebras; that is, the endomorphism rings of simple modules have dimension one over $\Bbbk$. (This hypothesis is satisfied for quasihereditary algebras, and of course for all algebras if $\Bbbk$ is algebraically closed).

If M is an A-module and N a B-module, we denote by $M\boxtimes N$ their tensor product, viewed as an $A\otimes B$-module. The tensor product yields a Künneth morphism

$$\mathrm{Ext}^*_A(M, M')\otimes \mathrm{Ext}^*_B(N, N') \xrightarrow{\kappa} \mathrm{Ext}^*_{A\otimes B}(M\boxtimes N, M'\boxtimes N').$$

Proposition A.1. *The Künneth morphism κ is an isomorphism if M and M' have finite dimension or if M and N have finite dimension.*

Proof. If M has finite dimension, then it has a projective resolution by finite dimensional projective A-modules. Thus, it suffices to prove the result in degree zero (i.e., for Hom), the general result follows formally by taking resolutions. Using semi-exactness and additivity of Hom and $\boxtimes$ with respect to their first variable, one reduces furthermore to the case $M = A$. If M' has finite dimension, the Künneth morphism in degree zero identifies with the map

$$M'\otimes \mathrm{Hom}_B(N, N') \to \mathrm{Hom}_B(N, M'\otimes N'),$$

which is an isomorphism since M' has finite dimension. If N has finite dimension, one can also reduce to the case $N = B$, and in the latter case it is clear that κ is an isomorphism. □

Proposition A.2. *Up to isomorphism, the simple $A\otimes B$-modules are the tensor products $L_1\boxtimes L_2$ where L_1 is a simple module over A and L_2 a simple module over B. Moreover, two such simple modules $L_1\boxtimes L_2$ and $L_1'\boxtimes L_2'$ are isomorphic if and only if $L_1\simeq L_1'$ and $L_2\simeq L_2'$.*

Proof. The fact that $L_1\boxtimes L_2$ is simple if L_1 and L_2 are simple follows from the density theorem [Curtis and Reiner 1981, (3.27)]. If two such tensor products $L_1\boxtimes L_2$ and $L_1'\boxtimes L_2'$ are isomorphic, then $L_1\simeq L_1'$ and $L_2\simeq L_2'$ because

$\mathrm{Hom}_{A\otimes B}(L_1 \boxtimes L_2, L_1' \boxtimes L_2')$ is isomorphic to $\mathrm{Hom}_A(L_1, L_1') \otimes \mathrm{Hom}_B(L_2, L_2')$. It remains to prove that any simple $A\otimes B$-module is of the form $L_1 \boxtimes L_2$. The Jacobson radical $J(A\otimes B)$ of $A\otimes B$ contains $J(A)\otimes B + A\otimes J(B)$, since the latter is a nilpotent ideal [Curtis and Reiner 1981, (5.15)]. Thus we have a surjective morphism

$$\pi : A/J(A)\otimes B/J(B) \twoheadrightarrow A\otimes B/J(A\otimes B).$$

Since the quotient $C/J(C)$ of a $\Bbbk$-algebra C is a semisimple ring [Curtis and Reiner 1981, (5.19)] with the same simple modules as C, it follows form the Wedderburn theorem and dimension counting that π is an isomorphism and that all simple $A\otimes B$-modules have the form $L_1 \boxtimes L_2$. □

Lemma A.3. *For all modules M and N*, $\mathrm{Soc}(M)\boxtimes \mathrm{Soc}(N) = \mathrm{Soc}(M\boxtimes N)$.

Proof. By Proposition A.2, $\mathrm{Soc}(M)\boxtimes\mathrm{Soc}(N)$ is a semisimple submodule of $M\boxtimes N$. Moreover, for all simple modules $L_1\boxtimes L_2$, we have

$$\begin{aligned}\mathrm{Hom}_{A\otimes B}(L_1\boxtimes L_2, M\boxtimes N) &= \mathrm{Hom}_A(L_1, M)\otimes \mathrm{Hom}_B(L_2, N)\\ &= \mathrm{Hom}_A(L_1, \mathrm{Soc}(M))\otimes \mathrm{Hom}_B(L_2, \mathrm{Soc}(N))\\ &= \mathrm{Hom}_{A\otimes B}(L_1\boxtimes L_2, \mathrm{Soc}(M)\boxtimes \mathrm{Soc}(N)).\end{aligned}$$

Consequently, all simple submodules of $M\boxtimes N$ are submodules of $\mathrm{Soc}(M)\boxtimes\mathrm{Soc}(N)$. This proves the lemma. □

Lemma A.4. *For all modules M and N*, $\mathrm{Head}(M\boxtimes N) = \mathrm{Head}(M)\boxtimes \mathrm{Head}(N)$.

Proof. If M and N have finite dimension, the proof is dual to the proof of Lemma A.3. By additivity of $\boxtimes$ with respect to both variables, the result is then true when M and N are arbitrary projectives. In general, let P and Q be projective covers of $\mathrm{Head}(M)$ and $\mathrm{Head}(N)$, respectively. One has quotient maps

$$P\boxtimes Q \twoheadrightarrow M\boxtimes N \twoheadrightarrow \mathrm{Head}(M)\boxtimes \mathrm{Head}(N),$$

and the result follows by taking heads of these modules. □

Lemma A.3 can be applied iteratively to identify the socle filtration of $M\boxtimes N$. We index socle filtrations of modules so that the (-1)-th term is zero and the zeroth term is the socle of the modules.

Proposition A.5. *For all modules M and N, the socle filtration of $M\boxtimes N$ is the tensor product of the socle filtration of M with the socle filtration of N.*

Proof. Let M^i, N^i and $(M\boxtimes N)^i$ be the terms of the socle filtrations of M, N and $M\boxtimes N$, and let $F^n := \sum_{i+j=n} M^i\boxtimes N^j$. We prove $F^n = (M\boxtimes N)^n$ by induction on n. We have $F^0 = M^0\boxtimes N^0 = (M\boxtimes N)^0$ by Lemma A.3. Assume that $F^n = (M \boxtimes N)^n$. Let ι be the canonical inclusion

$$\bigoplus_{i+j=n+1} (M^i/M^{i-1}) \boxtimes (N^j/N^{j-1}) = F^{n+1}/F^n \hookrightarrow (M \boxtimes N)/F^n.$$

Let ϕ denote the canonical inclusion

$$(M \boxtimes N)/F^n \hookrightarrow \bigoplus_{i+j=n+1} (M/M^{j-1}) \boxtimes (N/N^{j-1}).$$

The composite $\phi \circ \iota$ is the direct sum of the canonical inclusions

$$(M^{i+1}/M^i) \boxtimes (N^{j+1}/N^j) \hookrightarrow (M/M^i) \boxtimes (N/N^j).$$

Thus, it follows from Lemma A.3 that $\phi \circ \iota$ maps the semisimple module F^{n+1}/F^n isomorphically onto the socle of the target of ϕ. In particular, the inclusion ι is in fact an isomorphism. $\square$

Recall that a finite module is multiplicity free if it has a composition series whose composition factors are pairwise nonisomorphic.

Proposition A.6. *Assume that one of the modules M or N is multiplicity free. Then for all submodules $S \subset M \boxtimes N$, there are submodules U_α of M and submodules V_α of N such that $S = \sum U_\alpha \boxtimes V_\alpha$.*

Proof. Since any module over a finite dimensional algebra is the sum of its finite submodules, it suffices to prove Proposition A.6 when all modules have finite dimension. Assume for example that M is multiplicity free, and fix a submodule $S \subset M \boxtimes N$.

Let T be a submodule of S such that $T/\mathrm{Rad}(T) \simeq L_1 \boxtimes L_2$ is simple. There is a submodule $U \subset M$ such that $\mathrm{Head}(U) \simeq L_1$. We claim that $T \subset U \boxtimes N$. Indeed, since M is multiplicity free, $L_1 \boxtimes L_2$ is not a composition factor of $(M/U) \boxtimes N$. Since $\mathrm{Head}(T) = L_1 \boxtimes L_2$, no nontrivial homomorphic image of T can be contained in $(M/U) \boxtimes N$. Thus $T \subset U \boxtimes N$.

We now construct a strictly decreasing sequence of modules $V_0 = N \supset V_1 \supset \cdots \supset V_n$ such that $U \boxtimes V_n = T$. Assume that V_i is constructed such that $T \subset U \boxtimes V_i$. If the inclusion is an equality then the construction is finished. Otherwise, the canonical map $\phi : \mathrm{Head}(T) \to \mathrm{Head}(U \boxtimes V_i)$ is not surjective. By Lemma A.4, $\mathrm{Head}(U \boxtimes V_i) = \mathrm{Head}(U) \boxtimes \mathrm{Head}(V_i)$ and by using the Künneth formula, we see that all submodules of $\mathrm{Head}(U \boxtimes V_i)$ are of the form $\mathrm{Head}(U) \boxtimes W$, where W is a submodule of $\mathrm{Head}(V_i)$. In particular, $\mathrm{Im}\,\phi$ is of the form $\mathrm{Head}(U) \boxtimes W_\phi$. The inverse image of $\mathrm{Im}\,\phi$ by the quotient map

$$\pi \boxtimes \pi_i : U \boxtimes V_i \twoheadrightarrow \mathrm{Head}(U) \boxtimes \mathrm{Head}(V_i)$$

is $\mathrm{Rad}(U) \boxtimes V_i + U \boxtimes \pi_i^{-1}(W_\phi)$. This is a submodule of $U \boxtimes V_i$ which contains T.

But $\mathrm{Head}(T) \simeq L_1 \boxtimes L_2$ is not a composition factor of

$$\mathrm{Rad}(U) \boxtimes (V_i/\pi_i^{-1}(W_\phi)) \simeq \frac{\mathrm{Rad}(U) \boxtimes V_i + U \boxtimes \pi_i^{-1}(W_\phi)}{U \boxtimes \pi_i^{-1}(W_\phi)}.$$

Thus T is actually a submodule of $U \boxtimes \pi_i^{-1}(W_\phi)$. We define $V_{i+1} := \pi_i^{-1}(W_\phi)$. Since ϕ is not surjective, V_{i+1} is a strict submodule of V_i and $T \subset U \boxtimes V_{i+1}$. Since $V_0 = N$ has finite dimension, we cannot indefinitely repeat this construction and decrease the dimension of the submodules V_i. Hence there must be an integer n such that $U \boxtimes V_n = T$.

We have proved so far that all submodules of $T \subset S$ with simple head are of the form $U \boxtimes V$ for some submodules $U \subset M$ and $V \subset N$. But for each composition factor L_α of S we can find a T_α with $T_\alpha/\mathrm{Rad}(T_\alpha) = L_\alpha$. Then $S = \sum T_\alpha = \sum U_\alpha \boxtimes V_\alpha$ and we are done. □

The submodule lattice of multiplicity free modules can be described in terms of certain oriented diagrams [Alperin 1980]. To be more specific, the diagram $D(M)$ associated to a module M has the composition factors of M as vertices, and there is an edge $L \to L'$ if and only if there is a submodule $U \subset M$ such that $\mathrm{Head}(U) \simeq L$ and L' is a homomorphic image of $\mathrm{Rad}(U)$ (such a module U is unique [Alperin 1980, Lemma 4]). The following proposition describes the diagrams of tensor products $M \boxtimes N$.

Proposition A.7. *The tensor product $M \boxtimes N$ is multiplicity free if and only if both M and N are multiplicity free. If this happens, then the vertices of $D(M \boxtimes N)$ are the tensor products $L_1 \boxtimes L_2$, where L_1 is a composition factor of M and L_2 a composition factor of N. Moreover, there is an edge $L_1 \boxtimes L_2 \to L_1' \boxtimes L_2'$ if and only if either $L_1 = L_1'$ and there is an edge $L_2 \to L_2'$ in $D(N)$ or if $L_2 = L_2'$ and there is an edge $L_1 \to L_1'$ in $D(M)$.*

Proof. We only prove the statement about the edges of $D(M \boxtimes N)$. Let $L_1 \boxtimes L_2$ be a composition factor of $M \boxtimes N$ and let $U \subset M$ such that $\mathrm{Head}(U) = L_1$ and $V \subset N$ such that $\mathrm{Head}(V) = L_2$. Then by Lemma A.4, $\mathrm{Head}(U \boxtimes V) = L_1 \boxtimes L_2$. Thus there is an edge $L_1 \boxtimes L_2 \to L_1' \boxtimes L_2'$ if and only if $L_1' \boxtimes L_2'$ is a homomorphic image of $\mathrm{Rad}(U) \boxtimes V + U \boxtimes \mathrm{Rad}(V)$, that is, if and only if $\mathrm{Hom}_{A\boxtimes B}(\mathrm{Rad}(U) \boxtimes V, L_1' \boxtimes L_2') \neq 0$ or $\mathrm{Hom}_{A\boxtimes B}(U \boxtimes \mathrm{Rad}(V), L_1' \boxtimes L_2') \neq 0$. By the Künneth formula, the first condition is equivalent to the fact that L_1' is a homomorphic image of $\mathrm{Rad}(U)$ and that $L_2' = L_2$ while the second one is equivalent to the fact that L_2' is a homomorphic image of $\mathrm{Rad}(V)$ and that $L_1' = L_1$. □

Proposition A.8. *Let L be a simple A-module satisfying* $\mathrm{Ext}^1_A(L, L) = 0$, *let $\mathcal{C}$ be a localizing and colocalizing subcategory of B-Mod, and let $L \boxtimes \mathcal{C}$ denote the full subcategory of $A \otimes B$-Mod whose objects are isomorphic to tensor products of the form $L \boxtimes M$, where M is an object of $\mathcal{C}$. Then*

(i) *$L \boxtimes \mathcal{C}$ is a localizing and colocalizing subcategory of $A \otimes B$-Mod,*

(ii) *tensor product by L induces an equivalence of categories $\mathcal{C} \simeq L \boxtimes \mathcal{C}$.*

Proof. The second statement follows from the Künneth formula and the fact that $\mathrm{End}_A(L) = \Bbbk$. Let us prove (i). The stability of $L \boxtimes \mathcal{C}$ by arbitrary direct sums is obvious, and since L is finite dimensional the canonical morphism $L \boxtimes \prod M_i \to \prod L \boxtimes M_i$ is an isomorphism, which proves the stability by direct products. If $S \subset L \boxtimes N$ then $S = \sum U_\alpha \boxtimes V_\alpha$ by Proposition A.6. But the only nonzero submodule of L is L itself, so that $S = \sum L \boxtimes V_\alpha \simeq L \boxtimes \left(\sum V_\alpha\right)$ is an object of $L \boxtimes \mathcal{C}$. The stability by quotients follows from the stability by subobjects. Finally, since $\mathrm{Ext}^1_A(L, L) = 0$ and $\mathrm{End}_A(L) = \Bbbk$, the Künneth formula shows that $\mathrm{Ext}^1_{A\otimes B}(L \boxtimes N, L \boxtimes N')$ is isomorphic to $\mathrm{Ext}^1_B(N, N')$. Thus, all extensions of $L \boxtimes N$ by $L \boxtimes N'$ are of the form $L \boxtimes E$, where E is an extension of N by N'. Hence $L \boxtimes \mathcal{C}$ is stable by extensions. □

Appendix B. On theorems of Steinberg and Clausen–James

In this appendix, we give new proofs of Steinberg's tensor product theorem for GL_n and Clausen and James' theorem, based on Theorem 3.6.

Lemma B.1. *A strict polynomial functor is simple if and only if it is self-dual and its endomorphism ring has dimension one.*

Proof. The condition is necessary by facts (2) and (3) from Section 2C. We prove it is sufficient. Let L be a simple subfunctor of F. The composite

$$F \simeq F^\sharp \twoheadrightarrow L^\sharp \simeq L \hookrightarrow F$$

is a nonzero endomorphism of F. Since the endomorphism ring of F has dimension one, this morphism must be a nonzero multiple of the identity, and hence an isomorphism. Thus one must have $L = F$. □

Proposition B.2 (weak Steinberg theorem). *Let $r \geq 0$, let L_1 be a left p^r-bounded simple functor, and let L_2 be any simple functor. Then $L_1 \otimes L_2^{(r)}$ is simple.*

Proof. Self-duality of L_1, L_2 and $I^{(r)}$ and general properties of duality imply that $L_1 \otimes L_2^{(r)}$ is self-dual. Moreover, since L_1 is left p^r-bounded, Theorem 3.6 yields an isomorphism:

$$\mathrm{End}_{\mathcal{P}_\Bbbk}(L_1 \otimes L_2^{(r)}) \simeq \mathrm{End}_{\mathcal{P}_\Bbbk}(L_1) \otimes \mathrm{End}_{\mathcal{P}_\Bbbk}(L_2) \simeq \Bbbk \otimes \Bbbk = \Bbbk.$$

Hence, $L_1 \otimes L_2^{(r)}$ is simple by Lemma B.1. □

Our next task is to prove that the p-restricted simple functors are left p-bounded. Our proof will use the following proposition, which extends the classification of additive strict polynomial functors proved in [Touzé 2017b].

Proposition B.3. *Let $F \in \mathcal{P}_{d_0,d_1,\dots,d_n,\Bbbk}$ be a strict polynomial functor with $1+n$ variables, such that F is nonzero and additive with respect to each of the last n variables. Let G be the strict polynomial functor defined by $G(V) = F(V, \Bbbk, \dots, \Bbbk)$. Then the d_is, $1 \le i \le n$, are powers of p, i.e., $d_i = p^{r_i}$ and there is an isomorphism*

$$F \simeq G \boxtimes I^{(r_1)} \boxtimes \cdots \boxtimes I^{(r_n)}.$$

Proof. By induction, we can reduce ourselves to proving that $d_n = p^{r_n}$ and that F is isomorphic to $\bar{F} \boxtimes I^{(r_n)}$, where $\bar{F}(V_0, \dots, V_{n-1}) := F(V_0, \dots, V_{n-1}, \Bbbk)$. The functors with $n+1$ variables of the form $P \boxtimes \Gamma^{\mu}$, where P is a projective functor with n variables, homogeneous of multidegree $(d_0, \dots, d_{n-1})$, and $\mu = (\mu_1, \dots, \mu_k)$ is a tuple with $\sum \mu_i = d_n$, form a projective generator of $\mathcal{P}_{d_0,d_1,\dots,d_n,\Bbbk}$. Thus F is a quotient of a direct sum $\bigoplus P_i \boxtimes \Gamma^{\mu^i}$.

Observe that if μ has more than one nonzero coefficient, then there are no nonzero morphisms from a functor of the form $P \boxtimes \Gamma^{\mu}$ to F. Indeed, for some n-tuple $\overline{V} = (V_0, \dots, V_{n-1})$, such a nonzero morphism would induce a nonzero morphism of strict polynomial functors from $P(\overline{V}) \otimes \Gamma^{\mu}(-)$ to the additive functor $F(\overline{V}, -)$. This would contradict [Friedlander and Suslin 1997, Theorem 2.13].

In particular, F is in fact a quotient of $\bigoplus P_i \boxtimes \Gamma^{d_n} = P \boxtimes \Gamma^{d_n}$ with $P = \bigoplus P_i$. Moreover, the following composite is zero, where $\phi = P \boxtimes \mathrm{mult}$, with "mult" referring to the multiplication of the divided power algebra:

$$\bigoplus_{k=1}^{d_n} P \boxtimes (\Gamma^k \otimes \Gamma^{d_n - k}) \xrightarrow{\phi} P \boxtimes \Gamma^{d_n} \to F.$$

Hence F is a quotient of $P \boxtimes (\operatorname{Coker} \phi)$. But $\operatorname{Coker} \phi$ is nonzero if and only if $d_n = p^{r_n}$ for some r_n, and in this case it is equal to $I^{(r_n)}$. Thus $d_n = p^{r_n}$, and we have a surjective map $\psi : P \boxtimes I^{(r_n)} \twoheadrightarrow F$. By replacing the last variable by $\Bbbk$, we obtain a surjective map $\bar{\psi} : P \twoheadrightarrow \bar{F}$. We then take a projective functor with n variables Q and a map $\chi : Q \to P$ whose image is $\operatorname{Ker} \bar{\psi}$. Then using additivity with respect with the last variable, one sees that we have a right exact sequence:

$$Q \boxtimes I^{(r_n)} \xrightarrow{\chi \boxtimes I^{(r)}} P \boxtimes I^{(r_n)} \xrightarrow{\psi} F \to 0.$$

This implies that F is isomorphic to $\bar{F} \boxtimes I^{(r_n)}$. □

Corollary B.4. *If L is a simple functor, there exists nonnegative integers $d_0, \dots, d_r$ such that L is a quotient of the functor $T^{(d_0,\dots,d_r)} = \bigotimes_{0 \le i \le r} (\otimes^{d_i})^{(i)}$.*

Proof. If L has degree zero, then L is the constant functor $\Bbbk$. Hence it is a quotient of $T^{(0)} = \otimes^0 = \Bbbk$. Assume L is not constant. Then there exists a positive integer n, the Eilenberg–Mac Lane degree of L, such that the functor with n variables

$$L_{\boxplus_n} : (V_1, \dots, V_n) \mapsto L(V_1 \oplus \cdots \oplus V_n)$$

contains a nonzero homogeneous direct summand F which is additive with respect to each of its variables (see, e.g., [Touzé 2017b, Section 2] for more details on Eilenberg–Mac Lane degrees for strict polynomial functors). By Proposition B.3, F is of the form

$$F = G \boxtimes I^{(r_1)} \boxtimes \cdots \boxtimes I^{(r_n)}$$

where G is a homogeneous functor of degree zero, i.e., a constant functor. In particular, F (hence also $L_{\boxplus_n}$) contains $I^{(r_1)} \boxtimes \cdots \boxtimes I^{(r_n)}$ as a direct summand. Thus we have

$$0 \neq \mathrm{Hom}_{\mathcal{P}_{\Bbbk}(n)}(I^{(r_1)} \boxtimes \cdots \boxtimes I^{(r_n)}, L_{\boxplus_n}) \simeq \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(I^{(r_1)} \otimes \cdots \otimes I^{(r_n)}, L).$$

Since L is simple, any nonzero morphism with target L is surjective. Thus the inequality above proves that L is a quotient of $I^{(r_1)} \otimes \cdots \otimes I^{(r_n)}$. By reordering the factors of this tensor product (and using that $(I^{(k)})^{\otimes d_k} = (\otimes^{d_k})^{(k)}$), we obtain the result. □

We now consider two assertions, indexed by a nonnegative integer k.

- $A(k)$ If L is a p-restricted functor of degree d with $d \leq k$, then L is a quotient of $\otimes^d$.
- $B(k)$ Let d be a nonnegative integer and let T be a homogeneous functor of positive degree e. If $d + pe \leq k + 1$, then no p-restricted simple functor occurs as a composition factor of the tensor product $\otimes^d \otimes T^{(1)}$.

Lemma B.5. *Assertion $A(0)$ is true.*

Proof. If L is a simple functor of degree 0, then L is the constant functor $\Bbbk$. Hence it is a quotient of $\otimes^0 = \Bbbk$. □

Lemma B.6. *If $A(k)$ is true, then $B(k)$ is true.*

Proof. The functor $\otimes^d \otimes T^{(1)}$ admits a filtration whose successive quotients are direct sums of functors of the form $L_\lambda \otimes T^{(1)}$, where L_λ is a simple functor of degree d. Thus, it suffices to prove that these tensor products $L_\lambda \otimes T^{(1)}$ have no p-restricted composition factors. Let us write $\lambda = \alpha + p\beta$, where α is a p-restricted partition and β is a partition. Since $|\alpha| \leq d \leq k$, assertion $A(k)$ implies that the simple functor L_α is left p-bounded. By the weak Steinberg theorem of Proposition B.2, $L_\lambda \simeq L_\alpha \otimes L_\beta^{(1)}$ is simple, and it is isomorphic to L_λ by elementary highest weight theory (see item (4) in Section 2C). Hence,

$$L_\lambda \otimes T^{(1)} \simeq L_\alpha \otimes (L_\beta \otimes T)^{(1)}.$$

The functor $(L_\beta \otimes T)^{(1)}$ has composition factors of the form $(L_\gamma)^{(1)}$ with $\gamma \neq (0)$. Since L_α is left p-bounded, Proposition B.2 implies that the composition factors of $L_\lambda \otimes T^{(1)}$ have the form $L_\alpha \otimes L_\gamma^{(1)} = L_{\alpha + p\gamma}$, and hence are not p-restricted. □

Lemma B.7. *If $A(k)$ and $B(k)$ are true, then $A(k+1)$ is true.*

Proof. By elementary highest weight theory (see (4) in Section 2C), L_λ is a composition factor of

$$L_{\lambda^0} \otimes L_{\lambda^1}^{(1)} \otimes \cdots \otimes L_{\lambda^r}^{(r)}.$$

Thus it suffices to prove that the latter is a simple functor. Since $A(k)$ is true, it remains to prove that a p-restricted functor L of degree $k+1$ is necessarily a quotient of $\otimes^{k+1}$. By Corollary B.4, there exists a tuple of nonnegative integers $(d_0, \dots, d_r)$ such that L is a quotient of a tensor product of the form $T^{(d_0,\dots,d_r)}$. But assertion $B(k)$ says that such tensor products have no p-restricted composition factor except maybe if $(d_0, \dots, d_r) = (k+1, 0, \dots, 0)$. □

Lemmas B.5, B.6 and B.7 imply that $A(k)$ is true for all $k \geq 0$. We are now ready to prove:

Theorem B.8 (Steinberg's tensor product theorem). *Let $\lambda^0, \dots, \lambda^r$ be p-restricted partitions, and let $\lambda = \sum_{i=0}^r p^i \lambda^i$. There is an isomorphism*

$$L_\lambda \simeq L_{\lambda^0} \otimes L_{\lambda^1}^{(1)} \otimes \cdots \otimes L_{\lambda^r}^{(r)}.$$

Proof. Since $A(k)$ is true for all $k \geq 0$, p-restricted simples are quotients of tensor powers $\otimes^d$. Moreover $(\otimes^d)^{(i)} = (I^{(i)})^{\otimes d}$ is a quotient of $(\Gamma^{p^i})^{\otimes d}$. Thus, for all $k \leq r$, $\bigotimes_{i<k} L_{\lambda^i}^{(i)}$ is left p^k-bounded. An induction on k using Proposition B.2 shows that each tensor product $\bigotimes_{i \leq k} L_{\lambda^i}^{(i)}$ is simple. □

Remark B.9. If $\lambda = (\lambda_1, \dots, \lambda_k)$, then $L_\lambda(\Bbbk^n)$ is a simple polynomial $\mathrm{GL}_n(\Bbbk)$-module if $n \geq k$ and is zero if $n < k$ (This follows from the properties of the deflating Schur functor $d_{N,n}$ given in [Martin 1993, pp. 109–110], and the fact that $\mathrm{ev}_{\Bbbk^n} = d_{N,n} \circ \mathrm{ev}_{\Bbbk^N}$.) Thus, Theorem B.8 actually implies the Steinberg tensor product theorem for polynomial representations of $\mathrm{GL}_n(\Bbbk)$, for all values of n (and in particular without requiring that the representations are stable). Finally, all simple rational representations of $\mathrm{GL}_n(\Bbbk)$ can be obtained by tensoring simple polynomial representations of $\mathrm{GL}_n(\Bbbk)$ by a power of the determinant representation. Thus, Theorem B.8 implies the classical Steinberg tensor product theorem as in [Jantzen 2003, II.3.17].

Theorem B.10 (Clausen and James' theorem). *A simple functor L is p-restricted if and only if $\mathrm{Hom}_{\mathcal{P}_\Bbbk}(L, \otimes^d) = \mathrm{Hom}_{\mathcal{P}_\Bbbk}(\otimes^d, L)$ is nonzero.*

Proof. Property $A(k)$ gives the "only if" part. Conversely, assume that the highest weight λ of L is not p-restricted. Using euclidean division, we write $\lambda = \lambda^0 + p\lambda^1$ with λ^0 p-restricted and λ^1 nonzero. Thus $L \simeq L_{\lambda^0} \otimes L_{\lambda^1}^{(1)}$ by Steinberg's tensor product theorem. By property $A(k)$, L_{λ^0} is left and right p-bounded, so that by Theorem 3.6, $\mathrm{Hom}_{\mathcal{P}}(L, \otimes^d) = \mathrm{Hom}_{\mathcal{P}}(\otimes^d, L) = 0$. □

Remark B.11. There already exists a functorial proof of Steinberg's tensor product theorem in the literature [Kuhn 2002, Theorem 7.11]. However, the proof given in this appendix is quite different from that in [Kuhn 2002]. Let us stress two differences. First, the proof in [Kuhn 2002] uses finite fields, while the size of the ground field plays no role in our proof. Second, to obtain a concrete form of [Kuhn 2002, Theorem 7.11], one needs to know the classification of simple representations of symmetric groups. On the contrary, our proof does not use any knowledge of representations of symmetric groups. Better still, our reasoning also proves Clausen and James' theorem, so we can actually use our approach to derive the classification of simple representations of symmetric groups from the classification of simple representations of GL_n.

Steinberg's tensor product theorem tells us that if L_λ is simple and p-restricted and L_μ is simple, then $L_\lambda \otimes L_\mu^{(1)}$ is simple. The following statement completes the picture regarding tensor products of simple objects.

Theorem B.12. *Let L and L' be both simple and p-restricted. Then $L \otimes L'$ is not simple, unless one of the two is the constant functor $\Bbbk$.*

The remainder of the section is devoted to the proof of Theorem B.12.

Lemma B.13. *Let d be a positive integer, and let L be a simple quotient of $\otimes^d$. The following injection induced by the tensor product is not surjective:*

$$\mathrm{Hom}_{\mathcal{P}_\Bbbk}(\otimes^d, L) \otimes \mathrm{Hom}_{\mathcal{P}_\Bbbk}(I, I) \hookrightarrow \mathrm{Hom}_{\mathcal{P}_\Bbbk}(\otimes^{d+1}, L \otimes I).$$

Proof. Fix a vector space V equipped with an isomorphism $\Bbbk^d \oplus \Bbbk \simeq V$. Let $\iota_1 : \Bbbk \hookrightarrow V$, $\iota_2 : \Bbbk^d \hookrightarrow V$, $\pi_1 : V \to \Bbbk$ and $\pi_2 : V \to \Bbbk^d$ be the associated canonical maps. Since $\mathrm{End}_{\mathcal{P}_\Bbbk}(I) \simeq \Bbbk$, any nonzero map ϕ in the image of the injection of Lemma B.13 is of the form $\phi = f \otimes \mathrm{Id}$ for a nonzero f. Thus the following composite is nonzero (it equals the map induced by f):

$$(\Bbbk^d)^{\otimes d} \otimes \Bbbk \xrightarrow{(\iota_2)^{\otimes d} \otimes \iota_1} V^{\otimes d+1} \xrightarrow{\phi} L(V) \otimes V \xrightarrow{L(\pi_2)\otimes \pi_1} L(\Bbbk^d) \otimes \Bbbk. \qquad (*)$$

For all morphisms $f : \otimes^d \to L$, we define a morphism $\psi_f : \otimes^{d+1} \to L \otimes I$ by $\psi_f(x_1 \otimes \cdots \otimes x_{d+1}) = f(x_2 \otimes \cdots \otimes x_{d+1}) \otimes x_1$. If f is nonzero, then ψ_f is nonzero, while for $\phi = \psi_f$ the composite $(*)$ is zero. In particular ψ_f is not in the image of the inclusion. $\square$

Lemma B.14. *Let d be a positive integer, and let L be a simple quotient of $\otimes^d$. Let $L^{(d-1,1)}$ be the homogeneous summand of bidegree $(d-1, 1)$ of the bifunctor $(V, W) \mapsto L(V \oplus W)$. There is an isomorphism $L^{(d-1,1)} \simeq F_L \boxtimes I$, where F_L is a* nonzero *homogeneous functor of degree $d - 1$.*

Proof. Proposition B.3 provides an isomorphism $L^{(d-1,1)} \simeq F_L \boxtimes I$. We have to prove that F_L is nonzero. By using the sum-diagonal adjunction and the Künneth formula, we obtain that $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(\otimes^{d+1}, L \otimes I)$ is isomorphic to

$$\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(\otimes^d, L) \otimes \mathrm{End}_{\mathcal{P}_{\Bbbk}}(I) \oplus \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(\otimes^d, F_L \otimes I) \otimes \mathrm{End}_{\mathcal{P}_{\Bbbk}}(I).$$

For dimension reasons, Lemma B.13 implies that $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(\otimes^d, F_L \otimes I)$ is nonzero. Hence F_L is nonzero. □

Proof of Theorem B.12. We will show that the dimension of $\mathrm{End}_{\mathcal{P}_{\Bbbk}}(L \otimes L')$ is not one. To this purpose, we use the sum-diagonal adjunction and the Künneth formula. We obtain that the vector space $\mathrm{End}_{\mathcal{P}_{\Bbbk}}(L \otimes L')$ contains

$$\mathrm{End}_{\mathcal{P}_{\Bbbk}}(L) \otimes \mathrm{End}_{\mathcal{P}_{\Bbbk}}(L') \oplus \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(L, F_L \otimes I) \otimes \mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(L', F_{L'} \otimes I)$$

as a direct summand, with F_L and $F_{L'}$ defined as in Lemma B.14. Again using the sum-diagonal adjunction and the Künneth formula, we get that $\mathrm{Hom}_{\mathcal{P}_{\Bbbk}}(L, F_L \otimes I)$ contains $\mathrm{End}_{\mathcal{P}_{\Bbbk}}(F_L) \otimes \mathrm{End}_{\mathcal{P}_{\Bbbk}}(I)$ as a direct summand (and similarly for L'). But Lemma B.14 asserts that F_L and $F_{L'}$ are nonzero, so that the dimension of the corresponding endomorphism spaces is at least one. So, the dimension of $\mathrm{End}_{\mathcal{P}_{\Bbbk}}(L \otimes L')$ is at least two. □

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Stable $\mathbb{A}^1$-connectivity over Dedekind schemes

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We show that $\mathbb{A}^1$-localization decreases the stable connectivity by at most one over a Dedekind scheme with infinite residue fields. For the proof, we establish a version of Gabber's geometric presentation lemma over a henselian discrete valuation ring with infinite residue field.

Introduction

Background. Morel [2005] formulated the following property on a scheme S, called the *stable $\mathbb{A}^1$-connectivity property*:

> *The $\mathbb{A}^1$-localization of a connected spectrum on the smooth Nisnevich site over S is still connected.*

Here, the notion of connectivity refers to the associated Nisnevich homotopy sheaves or equivalently to the connectivity of the Nisnevich stalks. Further, he proved this property for $S = \mathrm{Spec}(k)$, where k is a field. This celebrated result is known as the *stable $\mathbb{A}^1$-connectivity theorem* [Morel 2005, Theorem 6.1.8] and has diverse implications. Most of the content from Morel's monograph [2012] is based on this result, such as the unstable $\mathbb{A}^1$-connectivity theorem and its implication, the Hurewicz theorem in $\mathbb{A}^1$-homotopy theory [Morel 2012, Theorem 6.37]. This leads to a computation of the 0-line of the stable homotopy groups of motivic spheres as the Milnor–Witt K-theory $K_*^{\mathrm{MW}}(S)$ of the base S [Morel 2012, Corollary 6.43]. More immediately, the $\mathbb{A}^1$-connectivity theorem implies the vanishing of the negative lines which is analogous to the vanishing of the negative stable homotopy groups of the sphere in topology.

Morel [2005, Conjecture 2] conjectured that the stable $\mathbb{A}^1$-connectivity property holds over every regular base. However, Ayoub [2006] constructed a counterexample to this conjecture (see Remark 4.4 below).

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Aim and results. In this paper, we want to replace Morel's stable $\mathbb{A}^1$-connectivity property by the following weaker property on a base scheme S of Krull-dimension d which is consistent with Ayoub's counterexample:

> *The $\mathbb{A}^1$-localization of a d-connected spectrum on the smooth Nisnevich site over S is still connected.*

We refer to this property as the *shifted stable $\mathbb{A}^1$-connectivity property*. In other words, S has this property if $\mathbb{A}^1$-localization lowers the connectivity by at most the dimension of S. Question 4.12 below asks whether every regular base scheme has this shifted stable $\mathbb{A}^1$-connectivity property. Morel's stable connectivity theorem is a positive answer in the case $d = 0$. In the main theorem of this paper, we give a positive answer in the one-dimensional case, assuming infinite residue fields (see also Theorem 4.16):

Theorem A. *A Dedekind scheme with only infinite residue fields has the shifted stable $\mathbb{A}^1$-connectivity property: if E is an i-connected spectrum, then its $\mathbb{A}^1$-localization $L^{\mathbb{A}^1}E$ is $(i-1)$-connected.*

Examples for such base schemes are algebraic curves over infinite fields in geometric settings, or $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{nr}})$ for $\mathbb{Z}_p^{\mathrm{nr}}/\mathbb{Z}_p$ the maximal unramified extension in more arithmetic settings.

Morel's proof of the $\mathbb{A}^1$-connectivity theorem needs a strong geometric input referred to as *Gabber's geometric presentation lemma* and written up in [Colliot-Thélène et al. 1997, Theorem 3.1.1]. These authors show how Gabber's presentation result leads to universal exactness of certain Cousin complexes. In particular, they derive the Bloch–Ogus theorem and the Gersten conjecture for algebraic K-theory for smooth varieties over a field, as first proved by Quillen [1973, Theorem 5.11]. In Section 2, we prove a version of this presentation result over a henselian discrete valuation ring with infinite residue fields (compare Theorem 2.1):

Theorem B. *Let $\mathfrak{o}$ be a henselian discrete valuation ring with infinite residue field and let σ denote the closed point of $S = \mathrm{Spec}(\mathfrak{o})$. Let X be a smooth S-scheme of finite type and let $Z \hookrightarrow X$ be a proper closed subscheme. Let z be a point in Z. If z lies in the special fibre Z_σ, suppose that $Z_\sigma \neq X_\sigma$. Then, Nisnevich-locally around z, there exists a smooth $\mathfrak{o}$-scheme V of finite type and a cartesian square*

$$\begin{array}{ccc} X \setminus Z & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{A}^1_V \setminus p(Z) & \longrightarrow & \mathbb{A}^1_V \end{array}$$

such that p is étale, the restriction $p|_Z : Z \hookrightarrow \mathbb{A}^1_V$ is a closed subscheme and Z is finite over V. In particular, this square is a Nisnevich-distinguished square.

The proof is based on [Colliot-Thélène et al. 1997, Theorem 3.1.1] combined with a Noether normalization over a Dedekind base; cf. [Kai 2015, Theorem 4.6].

Apart from this geometric input to the proof of Theorem A, we need a second key ingredient of a more homotopical kind: In Section 3, we examine a vanishing result for the nonsheafified homotopy classes of the $\mathbb{A}^1$-localization of a connected spectrum. This is a slight generalization of the argument in [Morel 2005, Lemma 4.3.1] to arbitrary noetherian base schemes of finite Krull-dimension. As a byproduct, we obtain that the S^1- and the $\mathbb{P}^1$-homotopy t-structure over any base scheme is left complete, i.e., a presheaf of spectra is recovered as the homotopy limit over its Postnikov truncations (see Corollaries 3.6 and 3.8).

1. Preliminaries

In this paper, our base scheme S is always a noetherian scheme of finite Krull-dimension. Let Sm_S be the category of smooth schemes of finite type over S. The category Sm_S is essentially small and sometimes we choose a small skeleton implicitly without mentioning. Let $\mathrm{sPre}_+(S)$ be the category of pointed simplicial presheaves on Sm_S. We mostly ignore S in the notation. For an object $U \in \mathrm{Sm}_S$, let U_+ denote the presheaf $\hom_{\mathrm{Sm}_S}(-, U)$ considered as a discrete simplicial set with an additional disjoint basepoint. Whenever we speak of a category having all limits and colimits we actually mean that it has all *small* limits and all *small* colimits.

Model structures. In contrast to the foundational address [Morel and Voevodsky 1999] of $\mathbb{A}^1$-homotopy theory, we use projective analogues of the unstable model structures and obtain the (pointed) *objectwise*, *Nisnevich-local* and $\mathbb{A}^1$*-Nisnevich-local* model structure (see [Dundas et al. 2003, Section 2]). Throughout the whole text, let L^{ob}, L^s and $L^{\mathbb{A}^1}$ denote fixed (pointed) objectwise, Nisnevich-local and $\mathbb{A}^1$-Nisnevich-local fibrant replacement functors, respectively. Given a symbol $\tau \in \{\mathrm{ob}, s, \mathbb{A}^1\}$, a nonnegative integer n and $F \in \mathrm{sPre}_+$, define the *n-th τ-homotopy sheaf* $\pi_n^\tau(F)$ of F as the Nisnevich sheafification of the *n-th τ-homotopy presheaf*

$$[(-)_+ \wedge S^n, L^\tau F].$$

Here, the brackets denote (pointed) objectwise homotopy classes. Notice that $\pi_n^{\mathrm{ob}}(F) \cong \pi_n^s(F)$. Whenever the objectwise model structure is considered, we omit the symbol $^{\mathrm{ob}}$ from the notation.

Let $\mathrm{Spt}_{S^1}(S)$ be the category of (nonsymmetric) S^1-spectra on the category $\mathrm{sPre}_+(S)$ [Hovey 2001, Definition 1.1]. The functor $(-)_0$ sending an S^1-spectrum to its zeroth level and the S^1-suspension spectrum functor $\Sigma^\infty_{S^1}$ fit into an adjunction $\Sigma^\infty_{S^1} : \mathrm{sPre}_+ \rightleftarrows \mathrm{Spt}_{S^1} : (-)_0$. For an integer $n \geq 0$, there is also an adjunction $[-n] : \mathrm{Spt}_{S^1} \rightleftarrows \mathrm{Spt}_{S^1} : [n]$ of shift functors defined for an integer n

by $E[n]_m := E_{n+m}$ whenever $n+m \geq 0$ and $E[n]_m := *$ otherwise. Following the general procedure of [Hovey 2001], we equip the category $\mathrm{Spt}_{S^1}(S)$ with stable model structures (see Hovey's Definition 3.3) having homotopy categories $\mathcal{SH}^{\mathrm{ob}}_{S^1}(S)$, $\mathcal{SH}^{s}_{S^1}(S)$ and $\mathcal{SH}^{\mathbb{A}^1}_{S^1}(S)$. The two above-mentioned adjunctions turn into Quillen adjunctions. Each of these stable homotopy categories is a triangulated category with distinguished triangles given by the homotopy cofibre sequences [Hovey 1999, Proposition 7.1.6]. In fact, by choosing symmetric spectra as a more elaborate model, these homotopy categories carry the structure of a closed symmetric monoidal category with a compatible triangulation in the sense of [May 2001, Definition 4.1]. For details of this construction we refer to [Hovey 2001; Jardine 2000; Ayoub 2007]. There are functors

$$- \wedge \Sigma^{\infty}_{S^1}(-) : \mathrm{Spt}_{S^1} \times \mathrm{sPre}_+ \to \mathrm{Spt}_{S^1},$$
$$\underline{\mathrm{hom}}(\Sigma^{\infty}_{S^1}(-), -) : \mathrm{sPre}_+^{\mathrm{op}} \times \mathrm{Spt}_{S^1} \to \mathrm{Spt}_{S^1},$$

defined in the obvious way. For a cofibrant $F \in \mathrm{sPre}_+$, they fit into a Quillen adjunction

$$- \wedge \Sigma^{\infty}_{S^1} F : \mathrm{Spt}_{S^1} \rightleftarrows \mathrm{Spt}_{S^1} : \underline{\mathrm{hom}}(\Sigma^{\infty}_{S^1} F, -)$$

whose derived adjunction models the monoidal structure from before.

Let Ω_{S^1} denote the functor $\underline{\mathrm{hom}}(\Sigma^{\infty}_{S^1} S^1, -)$. We mention that a concrete fibrant replacement functor for the stable τ-model structure on Spt_{S^1} is given by

$$\Theta^{\tau}_{S^1} E = \mathrm{colim}\big((L^{\tau} E) \to \Omega_{S^1}(L^{\tau} E)[1] \to (\Omega_{S^1})^2(L^{\tau} E)[2] \to \cdots\big), \tag{1.1}$$

where $\tau \in \{\mathrm{ob}, s, \mathbb{A}^1\}$ and where the application of L^{τ} to a spectrum is levelwise [Hovey 2001, Theorem 4.12]. We write $\Omega^{\infty}_{S^1} : \mathrm{Spt}_{S^1} \to \mathrm{sPre}_+$ for the composition of this fibrant replacement functor with $(-)_0$.

As for the unstable structures, we define the *n-th stable τ-homotopy sheaf* $\pi^{\tau}_n(E)$ of a spectrum $E \in \mathrm{Spt}_{S^1}$ as the Nisnevich sheafification of the *n-th stable τ-homotopy presheaf*

$$[\Sigma^{\infty}_{S^1}(-_+)[n], L^{\tau} E].$$

Here, the brackets denote the morphism sets of $\mathcal{SH}^{\mathrm{ob}}_{S^1}$. Since it will be evident from the context if the unstable or the stable homotopy sheaf is considered, we do not introduce an extra decoration.

We use the following explicit model for $L^{\mathbb{A}^1}$ in the stable context introduced in [Morel 2004, Lemma 4.2.4].

Lemma 1.2 (Morel). *Let S be an arbitrary base scheme. For each integer $k \geq 0$, we set $L^k(E) := \underline{\mathrm{hom}}(F^{\wedge k}, L^s(E))$ with $F := \Sigma^{\infty}_{S^1} C[-1]$, where C is a cofibrant replacement of the cofibre of the morphism*

$$S^0 \xrightarrow{0,1} \mathbb{A}^1$$

in sPre_+. *Then the functor* $L^\infty : \mathrm{Spt}_{S^1} \to \mathrm{Spt}_{S^1}$ *defined by*

$$L^\infty(E) := \operatorname*{hocolim}_{k\to\infty} L^k(E)$$

is a fibrant replacement functor for the stable $\mathbb{A}^1$*-Nisnevich-local model.*

Remark 1.3. Likewise, the spectrum F from the above Lemma 1.2 may be defined by the distinguished triangle

$$F \longrightarrow \Sigma^\infty_{S^1} S^0 \xrightarrow{0,1} \Sigma^\infty_{S^1}\mathbb{A}^1.$$

Let $k \geq 1$ be an integer. After rotation and smashing with the spectrum $F^{\wedge(k-1)}$, the above triangle becomes $\Sigma^\infty_{S^1}\mathbb{A}^1 \wedge F^{\wedge(k-1)}[-1] \to F^{\wedge k} \to F^{\wedge(k-1)}$. Applying $\underline{\mathrm{hom}}(-, L^s(E))$ yields the distinguished triangle

$$L^{k-1}(E) \to L^k(E) \to \underline{\mathrm{hom}}(\Sigma^\infty_{S^1}\mathbb{A}^1, L^{k-1}(E)[1]).$$

Here $L^{k-1}(E)[1] \simeq L^{k-1}(E[1])$ holds by definition and homotopy-exactness of L^s.

Base change. We briefly recall the construction of base change functors in $\mathbb{A}^1$-homotopy theory. For details, see the monograph [Ayoub 2007] and [Hu 2001].

Let $f : R \to S$ be a morphism between noetherian schemes of finite Krull-dimension. There is an adjunction

$$f^* : \mathrm{sPre}_+(S) \rightleftarrows \mathrm{sPre}_+(R) : f_*$$

where the *direct image* functor f_* is defined by $(f_*G)(-) := G(- \times_S R)$ and where the left adjoint *inverse image* is determined by $f^*(U_+) := (U \times_S R)_+$ for $U \in \mathrm{Sm}_S$. The functor f^* is strong symmetric monoidal with respect to the smash product and there is a natural isomorphism $f_* \underline{\mathrm{hom}}_+(f^*F, G) \cong \underline{\mathrm{hom}}_+(F, f_*G)$ [Fausk et al. 2003, (3.4)]. If the morphism $f : R \to S$ is smooth and of finite type, the inverse image functor has a left adjoint

$$f_\sharp : \mathrm{sPre}_+(R) \rightleftarrows \mathrm{sPre}_+(S) : f^*$$

determined by $f_\sharp(V_+) = f_\sharp(V_+ \to R) := (V \sqcup S \to R \sqcup S \to S) = f_\sharp^{\text{unpointed}}(V)_+$. We emphasize that throughout the whole text, the functor $f_\sharp$ is considered in this pointed sense: it does not only postcompose with f but also quotients out the base-point along f. In the case of a smooth f of finite type, the inverse image is given by $f^*F = F \wedge R_+$, and one has a *projection formula* $f_\sharp(G \wedge f^*F) \cong f_\sharp G \wedge F$ (see, e.g., [Hoyois 2017, Section 5.1]) and a natural isomorphism

$$f^* \underline{\mathrm{hom}}_+(A, B) \cong \underline{\mathrm{hom}}_+(f^*A, f^*B) \tag{1.4}$$

by [Fausk et al. 2003, Proposition 4.1]. Note that, since S is noetherian, any open immersion $R \hookrightarrow S$ is smooth and of finite type.

The adjunction (f^*, f_*) is a Quillen adjunction for the objectwise, the Nisnevich-local and the $\mathbb{A}^1$-Nisnevich-local model structures. If $f : R \to S$ is smooth and of finite type, then the adjunction $(f_\sharp, f^*)$ is a Quillen adjunction for the objectwise, the Nisnevich-local and the $\mathbb{A}^1$-Nisnevich-local model structures as well and f^* preserves all weak equivalences; see [Ayoub 2007, Theorem 4.5.10].

Remark 1.5. For the projective versions of the model structures, it is easy to see that f^* and $f_\sharp$ preserve the generating cofibrations and hence all cofibrations. By the same reason, the objectwise acyclic cofibrations are preserved, so $(f_\sharp, f^*)$ and (f^*, f_*) are Quillen adjunctions for the objectwise structures. In order to see that the right adjoints f_* and f^* preserve fibrations for the Nisnevich-local and the $\mathbb{A}^1$-Nisnevich-local model structure, it suffices to show that they preserve fibrations between fibrant objects [Dugger 2001, Corollary A.2]. As the right adjoints preserve objectwise fibrations, it suffices to show that they preserve fibrant objects. The fibrant objects of a Bousfield localization may be detected by a particular set J' of acyclic cofibrations [Hirschhorn 2003, Lemma 3.3.11]. It remains to be shown that the left adjoints preserve these acyclic cofibrations in J', which is straightforward; see [Dundas et al. 2003, Definition 2.14].

In particular, Remark 1.5 implies the following lemma.

Lemma 1.6. *Suppose $f : R \to S$ is a smooth morphism of finite type. For each $F \in \mathrm{sPre}_+(S)$, there are canonical (objectwise) weak equivalences*

$$L^s(f^*F) \sim f^*(L^sF) \quad \textit{and} \quad L^{\mathbb{A}^1}(f^*F) \sim f^*(L^{\mathbb{A}^1}F)$$

in $\mathrm{sPre}_+(R)$.

The spectrum $S_s^h := \mathrm{Spec}(\mathcal{O}_{S,s}^h)$ of a henselian local ring of a point $s \in S$ is usually not of finite type over S. Hence, Lemma 1.6 does not apply directly to the canonical morphism $\mathfrak{s} : S_s^h \to S$. Instead, we treat S_s^h as a cofiltered limit of the diagram $\underline{D}$ given by the affine Nisnevich neighbourhoods of s in S and invoke the following lemma.

Lemma 1.7. *Let $d : D \to S$ be a noetherian S-scheme of finite Krull-dimension. Suppose d is the limit of a cofiltered diagram $\underline{D} : \mathcal{I} \to \mathrm{Sm}_S$ with affine transition morphisms, where $d_i : D_i \to S$ denotes the structure morphism of each $D_i := \underline{D}(i)$. Assume that each D_i is quasiseparated. Let $V \to D$ be an element of Sm_D. Then the following statements hold:*

(1) *There is a cofinal functor $\mathcal{I}_V \to \mathcal{I}$, a cofiltered diagram $\underline{V} : \mathcal{I}_V \to \mathrm{Sm}_S$ with affine transition morphisms and a natural transformation $\underline{V} \to \underline{D}|_{\mathcal{I}_V}$ inducing $V \to D$ on the limit over $\mathcal{I}_V$ in Sch_S.*

(2) *For $V \to \underline{V}$ as in (1) and for each $F \in \mathrm{sPre}_+(S)$, the morphism of diagrams*

$\Gamma(\underline{V}, d^*F) \to \Gamma(V, d^*F)$ *induces a canonical natural isomorphism*

$$\Gamma(V, d^*F) \cong \operatorname*{colim}_{i \in \mathcal{I}_V} \Gamma(V_i, d_i^* F).$$

(3) *For* $V \to \underline{V}$ *as in* (1) *and for each* $F \in \mathrm{sPre}_+(S)$, *there is a canonical natural isomorphism of pointed* (*objectwise*) *homotopy classes*

$$[V_+, d^*F] \cong \operatorname*{colim}_{i \in \mathcal{I}_V} [V_{i+}, d_i^* F].$$

(4) *In* (1), *open embeddings*, *étale morphisms*, *smooth morphisms and Nisnevich-distinguished squares in* Sm_D *can be approximated by their sectionwise counterparts in* Sm_S.

(5) *For each* $F \in \mathrm{sPre}_+(S)$, *there are canonical* (*objectwise*) *weak equivalences*

$$L^s(d^*F) \sim d^*(L^sF) \quad \textit{and} \quad L^{\mathbb{A}^1}(d^*F) \sim d^*(L^{\mathbb{A}^1}F)$$

in $\mathrm{sPre}_+(D)$.

Proof. (1) This follows from [EGA IV$_3$ 1966, Theorem 8.8.2, Proposition 17.7.8]. In fact, we may (and always will) even assume that $\mathcal{I}_V = \mathcal{I} \downarrow i_0$ for a suitable object $i_0 \in \mathcal{I}$ and $\underline{V} = V_{i_0} \times_{D_{i_0}} \underline{D}|_{\mathcal{I}\downarrow i_0}$ for a suitable smooth morphism $V_{i_0} \to D_{i_0}$.

(2) We may assume F to be simplicially discrete, i.e., a presheaf. As we may write F as the colimit over representable presheaves, and pullback- as well as section-functors preserve colimits in the category of presheaves, we may assume that F is representable by a suitable object $U \to S$ in Sm_S. Then $d^*F = U \times_S D$ and $d_i^*F = U \times_S D_i$, and (2) follows from (1) and [EGA IV$_3$ 1966, Theorem 8.8.2].

(3) Let us first observe that d^* preserves objectwise fibrant objects. Indeed, this holds for the d_i^* by Remark 1.5. Taking sections and applying (2), it suffices to observe that a filtered colimit of fibrant simplicial sets is again fibrant. The assertion of (3) now follows by taking homotopies with respect to the functorial standard cylinder $(-) \times \Delta^1$.

(4) Let $f : V' \to V$ be an open embedding (resp. an étale or smooth morphism) in Sm_D. We apply (1) first to the structural map $V \to D$ of the target and then to $V' \to V$ itself. We get approximations $\underline{V}, \underline{V}' : \mathcal{I}_f \to \mathrm{Sm}_S$ and a natural transformation $\underline{f} : \underline{V}' \to \underline{V}$ inducing f after taking limits. By [EGA IV$_3$ 1966, Proposition 8.6.3] (resp. [EGA IV$_3$ 1966, Proposition 17.7.8]) we may assume that $\underline{f}$ is sectionwise an open embedding (resp. an étale or smooth morphism). As in (1), we may assume $\underline{f} = f_{i_0} \times_{D_{i_0}} \underline{D}|_{\mathcal{I}\downarrow i_0}$.

Let f be étale and $j : U \hookrightarrow V$ an open immersion inducing a Nisnevich-distinguished square. Choose an approximation $\underline{f} = f_{i_0} \times_{D_{i_0}} \underline{D}|_{\mathcal{I}\downarrow i_0}$ of f as above. By possibly enlarging i_0, we can find an approximation $\underline{j} = j_{i_0} \times_{D_{i_0}} \underline{D}|_{\mathcal{I}\downarrow i_0}$ of j

by open immersions. We get a levelwise pullback square

$$\begin{tikzcd} \underline{U} \times_{\underline{V}} \underline{V}' \arrow[r] \arrow[d] & \underline{V}' \arrow[d, "\underline{f}"] \\ \underline{U} \arrow[r, "\underline{j}"] & \underline{V} \end{tikzcd}$$

In particular, the sectionwise definition of $\underline{f}^*(\underline{Z}) \to \underline{Z} := \underline{V} \setminus \underline{U}$ (sectionwise with the reduced structure) gives a well defined approximation of $f^*(Z) \to Z := V \setminus U$. By [EGA IV$_3$ 1966, Corollary 8.8.2.4] we may even assume that this approximation is sectionwise an isomorphism, i.e., the above square of approximations is sectionwise a Nisnevich-distinguished square.

(5) Note that the first assertion is equivalent to d^* preserving Nisnevich-local fibrant objects and that the second assertion is equivalent to d^* preserving $\mathbb{A}^1$-Nisnevich-local fibrant objects. Let $F \in \mathrm{sPre}_+$ be Nisnevich-local fibrant. We have to show that d^*F sends Nisnevich-distinguished squares to homotopy pullback squares of simplicial sets. Let Q be a Nisnevich-distinguished square in Sm_D. By (4), Q may be approximated by a diagram $\underline{Q}$ of Nisnevich-distinguished squares. By (2), we have $(d^*F)(Q) \cong \mathrm{colim}(d_i^*F)(Q_i)$. Again, as the d_i^* admit Quillen left adjoints for the Nisnevich-local model, it suffices to show that a filtered colimit of homotopy pullback squares of simplicial sets is again a homotopy pullback square. This, in turn, follows from the fact that those colimits preserve categorical pullback squares, fibrations and weak equivalences of simplicial sets. This shows $L^s(d^*F) \sim d^*(L^sF)$.

For the second assertion it suffices to show that d^* preserves $\mathbb{A}^1$-invariant simplicial presheaves. This is the case for the d_i^* as they admit left adjoints $d_{i,\sharp}$. The assertion follows directly from (2). □

We need the following glueing property. Let S be a base scheme of finite Krull dimension and $i : Z \hookrightarrow S$ a closed subscheme with complementary open immersion $j : U \hookrightarrow S$. For a pointed simplicial presheaf $F \in \mathrm{sPre}_+(S)$, there is a homotopy cofibre sequence

$$j_\sharp j^* F \to F \to i_* L_{\mathbb{A}^1} i^* F \tag{1.8}$$

for the *pointed* $\mathbb{A}^1$-Nisnevich-local model structure. This fact follows (e.g., by [Hoyois 2017, Section 5.1]) from the unpointed analogue due to Morel and Voevodsky [1999, Theorem 3.2.21] (see also [Ayoub 2006, Theorem 4.5.36]).

For a morphism $f : R \to S$ of noetherian schemes of finite Krull dimension, there is also an adjunction

$$f^* : \mathrm{Spt}_{S^1}(S) \rightleftarrows \mathrm{Spt}_{S^1}(R) : f_*$$

on the level of spectra where one defines $f^*(E)_n := f^*(E_n)$ and $f_*(D)_n := f_*(D_n)$ with obvious structure maps. If the morphism $f : R \to S$ is smooth and of finite type, there is an adjunction

$$f_\sharp : \mathrm{Spt}_{S^1}(R) \rightleftarrows \mathrm{Spt}_{S^1}(S) : f^*$$

with $f_\sharp(D)_n := f_\sharp(D_n)$ and structure maps given by the projection formula. The adjunction (f^*, f_*) is a Quillen adjunction for the stable objectwise, the stable Nisnevich-local and the stable $\mathbb{A}^1$-Nisnevich-local model structures, respectively. If $f : R \to S$ is smooth and of finite type, then the adjunction $(f_\sharp, f^*)$ is a Quillen adjunction for these stable model structures as well and f^* preserves all stable weak equivalences; see [Ayoub 2007, Theorem 4.5.23].

We have the following analogue of Lemma 1.6 and Lemma 1.7 in the stable setting.

Lemma 1.9. *Let $\underline{D}$ and $V \to D$ be as in Lemma 1.7. Let $f : R \to S$ be either smooth of finite type or the canonical map $\lim \underline{D} \to S$. Let $E \in \mathrm{Spt}_{S^1}(S)$ be a spectrum. Then the following statements hold*:

(1) $L^{\mathrm{ob}}(f^*E) \sim f^*(L^{\mathrm{ob}}E)$, $L^s(f^*E) \sim f^*(L^sE)$ *and* $L^{\mathbb{A}^1}(f^*E) \sim f^*(L^{\mathbb{A}^1}E)$.

(2) *With the notation of Lemma 1.7(1), there is a canonical natural isomorphism of pointed (stable objectwise) homotopy classes*

$$[\Sigma^\infty_{S^1}(V_+), d^*E] \cong \operatorname*{colim}_{i\in\mathcal{I}_V} [\Sigma^\infty_{S^1}(V_{i+}), d_i^*E].$$

Proof. The assertions follow from (1.4) and the explicit form of a fibrant replacement (1.1) using Lemma 1.6 and Lemma 1.7. □

Corollary 1.10. *Let $E \in \mathrm{Spt}_{S^1}(S)$ be a spectrum. Then the following statements are equivalent*:

(1) *The homotopy sheaf $\pi_0(E)$ is trivial.*

(2) *For all schemes $V \in \mathrm{Sm}_S$ with structure morphism $p : V \to S$ and all points $v \in V$ with canonical morphism $\mathfrak{v} : V^h_v := \mathrm{Spec}(\mathcal{O}^h_{V,v}) \to V$, the homotopy sheaf $\pi_0(\mathfrak{v}^*p^*E)$ is trivial.*

(3) *For all points $s \in S$ with canonical morphism $\mathfrak{s} : S^h_s \to S$, the homotopy sheaf $\pi_0(\mathfrak{s}^*E)$ is trivial.*

Proof. First suppose (2) holds. We want to show (1), i.e., we have to show that the Nisnevich stalk at (V, v) of the sheaf $\pi_0(E)$ is trivial for all such (V, v). By (2) of the previous lemma, we get

$$\pi_0(E)_{(V,v)} = \operatorname*{colim}_{f:(W,w)\to(V,v)} [\Sigma^\infty_{S^1}(W_+), f^*p^*E] \cong [\Sigma^\infty_{S^1}(V^h_{v,+}), \mathfrak{v}^*p^*E],$$

where the colimit runs over the Nisnevich neighbourhoods of (V, v). The identity $\mathrm{id}_{(V_v^h, v)}$ is cofinal in the Nisnevich neighbourhoods of (V_v^h, v), so we obtain $\Sigma^\infty_{S^1}(V^h_{v,+}), \mathfrak{v}^* p^* E] = \pi_0(\mathfrak{v}^* p^* E)_{(V_v^h, v)}$, which is trivial by assumption.

For the implication (1) $\Rightarrow$ (2), suppose that $\pi_0(E) = 0$. Let $V_i \to V$ be the diagram given by the affine Nisnevich neighbourhoods of (V, v). In particular, we have $\lim_i V_i \cong V_v^h$. By Lemma 1.7(4), every object in $\mathrm{Sm}_{V_v^h}$ has the form $W_v^h := W \times_V V_v^h$ for a suitable $W \in \mathrm{Sm}_V$. Take a point $w \in W_v^h$ and let w_0 be its image in W. Again by Lemma 1.7(4), we find a diagram of étale maps $W_j \to W$ such that $(W_v^h)_w^h \cong \lim_j (W_j)_v^h \cong \lim_j \lim_i W_j \times_V V_i$. Using Lemma 1.9(2), we compute

$$\begin{aligned}\pi_0(\mathfrak{v}^* p^* E)_{(W_v^h, w)} &\cong \mathrm{colim}_j\, [\Sigma^\infty_{S^1}((W_j)^h_{v,+}), \mathfrak{v}^* p^* E] \\ &\cong \mathrm{colim}_j\, \mathrm{colim}_i\, [\Sigma^\infty_{S^1}((W_j \times_V V_i)_+), p^* E].\end{aligned}$$

The pro-object $\{W_j \times_V V_i\}_{i,j}$ in Sm_S induces a Nisnevich point α of $\mathrm{Sh}(\mathrm{Sm}_S)$. Note that this point may not correspond to the henselian scheme X_x^h for some $X \in \mathrm{Sm}_S$ and $x \in X$ but to a subextension of the strict henselization $W^{sh}_{\bar{w}_0}/W^h_{w_0}$. The assumption $\pi_0(E) = 0$ now implies

$$\mathrm{colim}_j\, \mathrm{colim}_i\, [\Sigma^\infty_{S^1}((W_j \times_V V_i)_+), p^* E] \cong \alpha(\pi_0(E)) = 0.$$

As a special case we get the implication (3) $\Rightarrow$ (2). Finally, the reverse implication (2) $\Rightarrow$ (3) is trivial. $\square$

$\mathbb{P}^1$-spectra. In this subsection, we briefly recall a model for the $\mathbb{P}^1$-stable *motivic homotopy category*. As an underlying category of this model structure, we use $(\mathbb{G}_m, S^1)$-bispectra, i.e., the category $\mathrm{Spt}_{\mathbb{G}_m}(\mathrm{Spt}_{S^1}(S))$ of $\mathbb{G}_m$-spectra with entries in Spt_{S^1}; see [Hovey 2001, Definition 1.1]. Here, by abuse of notation, $\mathbb{G}_m$ denotes the S^1-suspension spectrum of a cofibrant replacement of the pointed object $(\mathbb{G}_m, 1)$ of $\mathrm{sPre}_+(S)$; see [Morel 2004, Remark 5.1.10]. Again, by an abuse of notation, we abbreviate this category by $\mathrm{Spt}_{\mathbb{P}^1}(S)$ and call its objects $\mathbb{P}^1$-*spectra*. Similarly to the passage from sPre_+ to S^1-spectra, the zeroth entry of a $\mathbb{P}^1$-spectrum and the $\mathbb{G}_m$-suspension spectrum functor fit into an adjunction

$$\Sigma^\infty_{\mathbb{G}_m} : \mathrm{Spt}_{S^1} \rightleftarrows \mathrm{Spt}_{\mathbb{P}^1} : (-)_0. \tag{1.11}$$

For $q \geq 0$, there is also an adjunction $\langle -q \rangle : \mathrm{Spt}_{\mathbb{P}^1} \rightleftarrows \mathrm{Spt}_{\mathbb{P}^1} : \langle q \rangle$ of shift functors defined for an integer q by $E\langle q \rangle_m := E_{q+m}$ whenever $q + m \geq 0$ and $E\langle q \rangle_m := *$ otherwise. Again by the general procedure of [Hovey 2001], we equip $\mathrm{Spt}_{\mathbb{P}^1}$ with the stable model structure induced via (1.11) by the stable $\mathbb{A}^1$-Nisnevich-local structure on Spt_{S^1} [Hovey 2001, Definition 3.3]. Its homotopy category $\mathcal{SH}(S)$ is the ($\mathbb{P}^1$-stable) *motivic homotopy category*. The two above-mentioned adjunctions turn into Quillen adjunctions for these structures, respectively.

The motivic homotopy category $\mathcal{SH}$ is a triangulated category with distinguished triangles again given by the homotopy cofibre sequences. Note that here the triangulated shift is again induced by the simplicial shift [1] and not by the $\mathbb{G}_m$-shift $\langle 1\rangle$.

Finally, let us mention a concrete fibrant replacement functor for the above model structure on $\mathrm{Spt}_{\mathbb{P}^1}$. This is completely analogous to the S^1-stabilization process from sPre_+ to Spt_{S^1}. Let $E \in \mathrm{Spt}_{\mathbb{P}^1}$. By [Hovey 2001, Theorem 4.12], we may use the functor

$$\Theta_{\mathbb{G}_m} E = \operatorname{colim}(E \to \Omega_{\mathbb{G}_m} E\langle 1\rangle \to (\Omega_{\mathbb{G}_m})^2 E\langle 2\rangle \to \cdots)$$

if each level of E is already a fibrant spectrum in Spt_{S^1}. Otherwise we can first apply the stable $\mathbb{A}^1$-Nisnevich-local fibrant replacement functor $\Theta^{\mathbb{A}^1}_{S^1}$ levelwise. We write $\Omega^\infty_{\mathbb{G}_m} : \mathrm{Spt}_{\mathbb{P}^1} \to \mathrm{Spt}_{S^1}$ for the composition of this fibrant replacement functor with $(-)_0$ from (1.11).

Preliminaries on t-structures. We briefly recall the definition of a homological t-structure and basic properties. Details can be found in [Gelfand and Manin 2003].

Definition 1.12. A *(homological) t-structure* on a triangulated category $\mathcal{D}$ is a pair of full subcategories $\mathcal{D}_{\leq 0}$ and $\mathcal{D}_{\geq 0}$ which are closed under isomorphisms in $\mathcal{D}$ such that the following axioms hold, where for an integer n, one sets $\mathcal{D}_{\geq n} := \mathcal{D}_{\geq 0}[n]$ and $\mathcal{D}_{\leq n} := \mathcal{D}_{\leq 0}[n]$.

(1) For all $X \in \mathcal{D}_{\geq 0}$ and all $Y \in \mathcal{D}_{\leq -1}$ we have $\hom_{\mathcal{D}}(X, Y) = 0$.

(2) $\mathcal{D}_{\geq 0}$ is closed under [1] (i.e., $\mathcal{D}_{\geq 1} \subseteq \mathcal{D}_{\geq 0}$) and dually $\mathcal{D}_{\leq -1} \subseteq \mathcal{D}_{\leq 0}$.

(3) For all $Y \in \mathcal{D}$ there exists a distinguished $X \to Y \to Z \to X[1]$ with $X \in \mathcal{D}_{\geq 0}$ and $Z \in \mathcal{D}_{\leq -1}$.

Set $\mathcal{D}_{=n} := \mathcal{D}_{\geq n} \cap \mathcal{D}_{\leq n}$ and call $\mathcal{D}_{=0}$ the *heart* of the t-structure. A t-structure is called *nondegenerate* if $\bigcap_{n\geq 0} \mathcal{D}_{\geq n} = \{0\}$ and $\bigcap_{n\leq 0} \mathcal{D}_{\leq n} = \{0\}$. A t-structure is called *left complete* if for all $X \in \mathcal{D}$ the canonical morphism

$$X \to \operatorname*{holim}_{n\to\infty} X_{\leq n}$$

is an isomorphism. Dually, a t-structure is called *right complete* if for all $X \in \mathcal{D}$ the canonical morphism $\operatorname*{hocolim}_{n\to-\infty} X_{\geq n} \to X$ is an isomorphism.

Remark 1.13. The adjunctions

$$\begin{aligned} \text{inclusion} : \mathcal{D}_{\geq n} &\rightleftarrows \mathcal{D} \quad : \tau_{\geq n} \\ \tau_{\leq n} : \quad \mathcal{D} &\rightleftarrows \mathcal{D}_{\leq n} : \text{inclusion} \end{aligned}$$

turn $\mathcal{D}_{\geq n}$ into a coreflective and $\mathcal{D}_{\leq n}$ into a reflective subcategory of $\mathcal{D}$. The counit of the first adjunction is denoted by $(-)_{\geq n} : \mathcal{D} \to \mathcal{D}$ and called the *n-skeleton*. The unit of the second adjunction is denoted by $(-)_{\leq n}$ and called the *n-coskeleton*. The skeleton and the coskeleton induce a distinguished triangle

$$X_{\geq n} \to X \to X_{\leq n-1} \to (X_{\geq n})[1].$$

Remark 1.14. Let $\mathcal{D}$ be a triangulated category obtained from the homotopy category of a stable model category together with a t-structure. If the t-structure is left complete, then $\bigcap_{n\geq 0} \mathcal{D}_{\geq n} = \{0\}$, which can be seen as follows. Take $X \in \bigcap_{n\geq 0} \mathcal{D}_{\geq n}$ and suppose that $X \to \operatorname{holim} X_{\leq n}$ is an isomorphism. The homotopy limit of the diagram

$$\begin{tikzcd}
{} \arrow[d, dashed] & {} \arrow[d, dashed] & {} \arrow[d, dashed] \\
X_{\geq n+1} \arrow[r, "\cong"] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq n} \arrow[d] \\
X_{\geq n} \arrow[r, "\cong"] & X \arrow[r] & X_{\leq n-1}
\end{tikzcd}$$

of triangles is the triangle $\operatorname{holim} X_{\geq n} \to X \to \operatorname{holim} X_{\leq n}$. Since the homotopy limit of weak equivalences is a weak equivalence, the first morphism $\operatorname{holim} X_{\geq n} \to X$ of this triangle is an isomorphism. This implies $\operatorname{holim} X_{\leq n} \cong 0$ and hence $X \cong 0$. In the same way, right completeness implies $\bigcap_{n\leq 0} \mathcal{D}_{\leq n} = \{0\}$.

For the converse, consider [Lurie 2017, Proposition 1.2.1.19]: Suppose that $\mathcal{D}_{\geq 0}$ is stable under countable homotopy products. Then $\bigcap_{n\geq 0} \mathcal{D}_{\geq n} = \{0\}$ implies left completeness. Dually, if $\mathcal{D}_{\leq 0}$ is stable under countable homotopy coproducts, the relation $\bigcap_{n\leq 0} \mathcal{D}_{\leq n} = \{0\}$ implies right completeness.

Proposition 1.15 [Ayoub 2007, Proposition 2.1.70]. *Let $\mathcal{D}$ be a triangulated category with coproducts and let $\mathcal{S}$ be a set of compact objects of $\mathcal{D}$. Define*

- *$\mathcal{D}_{\leq -1}$ as the full subcategory of those Y of $\mathcal{D}$ with* $\hom_{\mathcal{D}}(S[n], Y) = 0$ *for all $n \geq 0$ and all $S \in \mathcal{S}$,*
- *$\mathcal{D}_{\geq 0}$ as the full subcategory of those X of $\mathcal{D}$ with* $\hom_{\mathcal{D}}(X, Y) = 0$ *for all $Y \in \mathcal{D}_{\leq -1}$.*

The pair $\mathcal{D}_{\leq 0} = \mathcal{D}_{\leq -1}[1]$ and $\mathcal{D}_{\geq 0}$ forms a t-structure. The category $\mathcal{D}_{\geq 0}$ is the full subcategory of $\mathcal{D}$ generated under extensions, (small) sums and cones from $\mathcal{S}$ and in particular $\mathcal{S} \subseteq \mathcal{D}_{\geq 0}$. Moreover, the truncation functor $\tau_{\leq -1}$ is given by $\tau_{\leq -1}(X) := \operatorname{hocolim}_{k\to\infty} \Phi^k(X)$, where $\Phi(X)$ is defined as the cone

$$\coprod_{\substack{\mathrm{Hom}(S[n],X)\\ S\in\mathcal{S},\, n\geq 0}} S[n] \to X \to \Phi(X).$$

Remark 1.16. Let $\mathcal{D}$ be a triangulated category obtained from the homotopy category of a stable model category and let $\mathcal{S}$ be a set of compact objects of $\mathcal{D}$. The t-structure obtained from the previous Proposition 1.15 satisfies the property that $\mathcal{D}_{\leq 0}$ is stable under countable homotopy coproducts. If $\mathcal{D}$ has an underlying cofibrantly generated model category and $\mathcal{S}$ equals (up to shifts) the set of cofibres of the generating cofibrations, then $\bigcap_{n\leq 0}\mathcal{D}_{\leq n}=\{0\}$ by [Hovey 1999, Theorem 7.3.1] and $\mathcal{D}$ is right complete by Remark 1.14. It is however usually a nontrivial issue to show left completeness of a t-structure obtained from Proposition 1.15.

Canonical t-structures on S^1-spectra. In this subsection we recall some basic properties of canonical t-structures on S^1-spectra arising in $\mathbb{A}^1$-homotopy theory.

Definition 1.17. Consider the set $\mathcal{S}:=\{\Sigma^{\infty}_{S^1}U_+ \mid U\in \mathrm{Sm}_S\}$. The *objectwise t-structure* (resp. *Nisnevich-local t-structure*, $\mathbb{A}^1$*-Nisnevich-local t-structure*) on $\mathcal{SH}^{\mathrm{ob}}_{S^1}$ (resp. $\mathcal{SH}^{s}_{S^1}$, $\mathcal{SH}^{\mathbb{A}^1}_{S^1}$) is obtained by applying Proposition 1.15 to the triangulated category $\mathcal{SH}^{\mathrm{ob}}_{S^1}$ (resp. $\mathcal{SH}^{s}_{S^1}$, $\mathcal{SH}^{\mathbb{A}^1}_{S^1}$) and to $\mathcal{S}$.

Remark 1.18. In [Morel 2005] the Nisnevich-local t-structure on $\mathcal{SH}^{s}_{S^1}$ is called the *standard t-structure*. In [Morel 2004, Section 4.3] the $\mathbb{A}^1$-Nisnevich-local t-structure on $\mathcal{SH}^{\mathbb{A}^1}_{S^1}$ is called the *homotopy t-structure* (on S^1-spectra).

Remark 1.19. By definition we have

$$\mathcal{SH}^{\mathrm{ob}}_{S^1\,\leq -1}=\big\{Y\in\mathcal{SH}^{\mathrm{ob}}_{S^1}\mid Y \text{ has trivial homotopy presheaves } [\Sigma^{\infty}_{S^1}(-_+)[i],Y] \text{ for all } i\geq 0\big\}.$$

Applying the classical [Margolis 1983, Proposition 3.6] objectwise, we get

$$\mathcal{SH}^{\mathrm{ob}}_{S^1\,\geq 0}=\big\{X\in\mathcal{SH}^{\mathrm{ob}}_{S^1}\mid X \text{ has trivial homotopy presheaves } [\Sigma^{\infty}_{S^1}(-_+)[i],X] \text{ for all } i\leq -1\big\}.$$

The objectwise t-structure is clearly nondegenerate as there are no nonzero spectra without nontrivial homotopy presheaves. The objectwise t-structure is right complete by Remark 1.16 and left complete by Remark 1.14 as $\mathcal{SH}^{\mathrm{ob}}_{S^1\,\geq 0}$ is stable under countable homotopy products.

Remark 1.20. Again, by definition we have

$$\mathcal{SH}^{s}_{S^1\,\leq -1}=\big\{Y\in\mathcal{SH}^{s}_{S^1}\mid Y \text{ has trivial homotopy presheaves } [\Sigma^{\infty}_{S^1}(-_+)[i],L^sY] \text{ for all } i\geq 0\big\}$$

and using on Nisnevich stalks the classical result [Margolis 1983, Proposition 3.6], we get

$$\mathcal{SH}^{s}_{S^1\,\geq 0}=\{X\in\mathcal{SH}^{s}_{S^1}\mid X \text{ has trivial homotopy sheaves } \pi^s_iX \text{ for all } i\leq -1\},$$
$$\mathcal{SH}^{s}_{S^1\,\leq -1}=\{Y\in\mathcal{SH}^{s}_{S^1}\mid Y \text{ has trivial homotopy sheaves } \pi^s_iY \text{ for all } i\geq 0\}.$$

The Nisnevich-local t-structure is clearly nondegenerate as all nonzero spectra have at least one nontrivial homotopy sheaf. The Nisnevich-local t-structure is right complete by Remark 1.16 and left complete by, e.g., [Spitzweck 2014, Lemma 4.4].

Remark 1.21. A Nisnevich-local fibrant replacement functor L^s respects only the truncation from above, i.e., if E is in $\mathcal{SH}^{\mathrm{ob}}_{S^1\le -1}$, then the spectrum $L^s E$ is in $\mathcal{SH}^{s}_{S^1\le -1}$. The analogous statement is not true for the positive part $\mathcal{SH}^{\mathrm{ob}}_{S^1\ge 0}$, which can be seen as follows: By Hilbert's Theorem 90, there is an isomorphism $\mathrm{Pic}(X) \cong H^1_{\mathrm{Nis}}(X, \mathbb{G}_{\mathrm{m}})$. The Eilenberg–Mac Lane spectrum $H\mathbb{G}_{\mathrm{m}}$ is in the heart of the objectwise t-structure but

$$H^1_{\mathrm{Nis}}(X, \mathbb{G}_{\mathrm{m}}) = [\Sigma^\infty_{S^1} X_+, L^s H\mathbb{G}_{\mathrm{m}}[1]] = [\Sigma^\infty_{S^1} X_+[-1], L^s H\mathbb{G}_{\mathrm{m}}]$$

and certainly there are schemes X with nontrivial Picard group.

Remark 1.22. By definition and Remark 1.20, one has

$$\begin{aligned} \mathcal{SH}^{\mathbb{A}^1}_{S^1\le -1} &= \bigl\{Y \in \mathcal{SH}^{\mathbb{A}^1}_{S^1} \mid Y \text{ has trivial homotopy presheaves} \\ &\qquad\qquad [\Sigma^\infty_{S^1}(-_+)[i], L^{\mathbb{A}^1} Y] \text{ for all } i \ge 0\bigr\} \\ &= \{Y \in \mathcal{SH}^{\mathbb{A}^1}_{S^1} \mid Y \text{ has trivial homotopy sheaves } \pi_i^{\mathbb{A}^1} Y \text{ for all } i \ge 0\}. \end{aligned}$$

The $\mathbb{A}^1$-Nisnevich-local t-structure is right complete by Remark 1.16 and we have $\bigcap_{n\le 0} \mathcal{SH}^{\mathbb{A}^1}_{S^1\le n} = \{0\}$. It will be shown in Corollary 3.6 that the $\mathbb{A}^1$-Nisnevich-local t-structure is left complete and hence nondegenerate.

Definition 1.23. We define

$$\mathcal{SH}^{\mathbb{A}^1,\pi}_{S^1\ge 0} := \{X \in \mathcal{SH}^{\mathbb{A}^1}_{S^1} \mid X \text{ has trivial homotopy sheaves } \pi_i^{\mathbb{A}^1} X \text{ for all } i \le -1\}.$$

Remark 1.24. The full subcategory $\mathcal{SH}^{\mathbb{A}^1,\pi}_{S^1\ge 0}$ in $\mathcal{SH}^{\mathbb{A}^1}_{S^1}$ is closed under homotopy colimits and extensions. There is an inclusion $\mathcal{SH}^{\mathbb{A}^1,\pi}_{S^1\ge 0} \subseteq \mathcal{SH}^{\mathbb{A}^1}_{S^1\ge 0}$ due to [Spitzweck 2014, Lemmas 4.1 and 4.3]. Conversely, the other implication $\mathcal{SH}^{\mathbb{A}^1}_{S^1\ge 0} \subseteq \mathcal{SH}^{\mathbb{A}^1,\pi}_{S^1\ge 0}$ holds if and only if $L^{\mathbb{A}^1}\Sigma^\infty_{S^1} U_+ \in \mathcal{SH}^{\mathbb{A}^1,\pi}_{S^1\ge 0}$ for all $U \in \mathrm{Sm}_S$. Unfortunately, there are schemes S such that these two equivalent conditions do not hold (see Remark 4.4). However, they hold true over the spectrum of a field S [Morel 2005, Theorem 6.1.8] and we have $\mathcal{SH}^{\mathbb{A}^1}_{S^1\ge 0} = \mathcal{SH}^{\mathbb{A}^1,\pi}_{S^1\ge 0}$ in that case.

Homotopy t-structures on $\mathbb{P}^1$-spectra. In this subsection we recall the homotopy t-structure on the motivic homotopy category $\mathcal{SH}$. We remind the reader that $\langle q\rangle$ denotes the $\mathbb{G}_m$-shift operation.

Definition 1.25. The *homotopy t-structure* on $\mathcal{SH}$ is the t-structure obtained by applying Proposition 1.15 to the triangulated category $\mathcal{SH}$ and the set

$$\mathcal{S} = \{\Sigma^\infty_{\mathbb{P}^1}(U_+)\langle q\rangle \mid U \in \mathrm{Sm}_S \text{ and } q \in \mathbb{Z}\}.$$

Remark 1.26. We use the name "homotopy t-structure" in order to agree with the terminology of [Morel 2004; 2005].

Remark 1.27. Unravelling the definitions, one identifies

$$\begin{aligned}\mathcal{SH}_{\le -1} &= \{Y \in \mathcal{SH} \mid \Omega^{\infty}_{\mathbb{G}_m}(Y\langle q\rangle) \in \mathcal{SH}^{\mathbb{A}^1}_{S^1 \le -1} \text{ for all } q \in \mathbb{Z}\} \\ &= \{Y \in \mathcal{SH} \mid (\operatorname{colim}_k \Omega^{k}_{\mathbb{G}_m} Y_{k+q}) \in \mathcal{SH}^{\mathbb{A}^1}_{S^1 \le -1} \text{ for all } q \in \mathbb{Z}\}.\end{aligned}$$

In particular,

$$\Omega^{\infty}_{\mathbb{G}_m}(\mathcal{SH}_{\le -1}) \subseteq \mathcal{SH}^{\mathbb{A}^1}_{S^1 \le -1} \quad \text{and} \quad \Sigma^{\infty}_{\mathbb{G}_m}(\mathcal{SH}^{\mathbb{A}^1}_{S^1 \ge 0}) \subseteq \mathcal{SH}_{\ge 0},$$

using [Ayoub 2006, Lemma 2.1.16] for the latter.

Remark 1.28. Over a field, using [Morel 2004, Lemma 4.3.11] and the equality $\mathcal{SH}^{\mathbb{A}^1}_{S^1 \ge 0} = \mathcal{SH}^{\mathbb{A}^1,\pi}_{S^1 \ge 0}$ from Remark 1.24, we can also identify

$$\mathcal{SH}_{\ge 0} = \{X \in \mathcal{SH} \mid \Omega^{\infty}_{\mathbb{G}_m}(X\langle q\rangle) \in \mathcal{SH}^{\mathbb{A}^1}_{S^1 \ge 0} \text{ for all } q \in \mathbb{Z}\};$$

cf. [Morel 2004, Section 5.2]. In particular, we have $\Omega^{\infty}_{\mathbb{G}_m}(\mathcal{SH}_{\ge 0}) \subseteq \mathcal{SH}^{\mathbb{A}^1}_{S^1 \ge 0}$ in this case.

Remark 1.29. The homotopy t-structure on the motivic homotopy category is right complete by Remark 1.16 and we have $\bigcap_{n \le 0} \mathcal{SH}_{\le n} = \{0\}$. It will be shown in Corollary 3.8 that the homotopy t-structure on the motivic homotopy category is also left complete and hence nondegenerate.

2. Gabber presentations over henselian discrete valuation rings

Throughout this section, fix a henselian discrete valuation ring $\mathfrak{o}$ with maximal ideal $\mathfrak{m} \trianglelefteq \mathfrak{o}$, local uniformizer $\pi \in \mathfrak{m}$, residue field $\mathbb{F} = \mathfrak{o}/\mathfrak{m}$ and field of fractions k. Assume that $\mathbb{F}$ is an infinite field. Let S be the spectrum of $\mathfrak{o}$ and denote by σ the closed point of S and by η the generic point of S. We want to prove the following version of Gabber's geometric presentation lemma over $\mathfrak{o}$.

Theorem 2.1. *Let $\mathfrak{o}$ be a henselian discrete valuation ring with infinite residue field. Let $X/\mathfrak{o}$ be a smooth $\mathfrak{o}$-scheme of finite type and let $Z \hookrightarrow X$ be a proper closed subscheme. Let z be a point in Z. If z lies in the special fibre, suppose that $Z_\sigma \ne X_\sigma$. Then, Nisnevich-locally around z, there exists a smooth $\mathfrak{o}$-scheme V of finite type and a cartesian square*

$$\begin{tikzcd} X \setminus Z \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \mathbb{A}^1_V \setminus p(Z) \arrow[r] & \mathbb{A}^1_V \end{tikzcd}$$

such that p is étale, the restriction $p|_Z : Z \hookrightarrow \mathbb{A}^1_V$ is a closed subscheme and Z is finite over V. In particular, this square is a Nisnevich-distinguished square and therefore, the induced canonical morphism $X/(X \setminus Z) \to \mathbb{A}^1_V/(\mathbb{A}^1_V \setminus p(Z))$ is an isomorphism of Nisnevich sheaves.

Remark 2.2. The essential case of Theorem 2.1 is that of an effective Cartier-divisor $Z \hookrightarrow X$ (see the proof of Theorem 2.4 below). Earlier results for relative effective Cartier-divisors $Z \hookrightarrow X$ over discrete valuation rings are [Gillet and Levine 1987, Lemma 1] and [Dutta 1995, Theorem 3.4]. These results are even Zariski local and do not assume infinite residue fields. However, they do not include an analogue for the crucial finiteness claim of Z/V.

The map p in Theorem 2.1 will be provided by a careful choice of suitable linear projections. Before we give a short outline of the proof, let us first recall some basic facts about linear projections.

Linear projections. Denote by $\mathbb{A}_{x_1/x_0,\dots,x_N/x_0,S} = \mathbb{A}_{\underline{x}/x_0,S}$ the affine N-space $\mathbb{A}^N_S$ with coordinates $x_1/x_0, \dots, x_N/x_0$ and by $\mathbb{P}_{x_0:\dots:x_N,S} = \mathbb{P}_{\underline{x},S}$ the projective N-space of $\mathbb{P}^N_S$ with homogeneous coordinates $x_0 : \cdots : x_N$. We get the standard open embedding $\mathbb{A}_{\underline{x}/x_0,S} \hookrightarrow \mathbb{P}_{\underline{x},S}$. By abuse of notation, we identify $x_1/x_0, \dots, x_N/x_0$ with $x_1, \dots, x_N$ and write just $\mathbb{A}_{\underline{x},S}$ for $\mathbb{A}_{\underline{x}/x_0,S}$ (and similarly for other coordinates).

Let $\mathbb{A}^\vee_{x_1,\dots,x_N,S} = \mathbb{A}_{x_1^*,\dots,x_N^*,S}$ be the dual affine space, i.e., for any $\mathfrak{o}$-algebra A, $\mathbb{A}^\vee_{x_1,\dots,x_N,S}(A)$ is the free A-module generated by the coordinate functions $x_1, \dots, x_N$ of $\mathbb{A}_{\underline{x},S}$. Dually, we can view $\mathbb{A}_{\underline{x},S}(A)$ as the free A-module generated by the dual coordinate functions $x_1^*, \dots, x_N^*$. To be more precise, take r copies of $\mathbb{A}^\vee_{\underline{x},S}$ and denote the j-th copy by $\mathbb{A}^{\vee,(j)}_{\underline{x},S}$ with coordinate functions $x^*_{i,j} := x^*_i$ for $1 \le i \le N$. Mapping $t_j \mapsto \sum_i x_i \otimes x^*_{i,j}$ defines the dual pairing $\langle -,- \rangle : \mathbb{A}_{\underline{x},S} \times_S \mathbb{A}^{\vee,(j)}_{\underline{x},S} \to \mathbb{A}_{t_j,S}$. Via this pairing, each A-point u of $\mathbb{A}^{\vee,(j)}_{\underline{x},S}$ induces a linear A-morphism (in abuse of notation also denoted by) u defined as the composition

$$\mathbb{A}_{\underline{x},A} \xrightarrow{\mathrm{id}\times u} \mathbb{A}_{\underline{x},A} \times_A \mathbb{A}^{\vee,(j)}_{\underline{x},A} \xrightarrow{\langle -,- \rangle} \mathbb{A}_{t_j,A} \qquad \text{via} \qquad t_j \mapsto \textstyle\sum_i \langle x_i^*, u \rangle x_i .$$

This map is precisely the linear form given by the A-point u seen as the corresponding linear combination of the x_i in $\mathbb{A}^{\vee,(j)}_{\underline{x},S}(A) = \bigoplus_i Ax_i$. We define

$$\mathfrak{E}_r := \mathbb{A}^{\vee,(1)}_{\underline{x},S} \times_S \cdots \times_S \mathbb{A}^{\vee,(r)}_{\underline{x},S}$$

and look at it as the space of linear projections $\mathbb{A}_{\underline{x},S} \to \mathbb{A}_{t_1,\dots,t_r,S}$. Indeed, each A-point $\underline{u}$ of $\mathfrak{E}_r$ induces a linear A-morphism

$$\underline{u} : \mathbb{A}_{\underline{x},A} \longrightarrow \mathbb{A}_{t_1,\dots,t_r,A} \qquad \text{via} \qquad t_j \mapsto \textstyle\sum_i \langle x_i^*, u_j \rangle x_i .$$

Mapping $t_0 \mapsto x_0$, this extends to a rational map

$$\underline{u} : \mathbb{P}_{\underline{x},A} \dashrightarrow \mathbb{P}_{t_0:\dots:t_r,A}$$

with locus of indeterminacy $L_{\underline{u}} := V_+(x_0, u_1, \ldots, u_r) \subseteq H_\infty$, where u_j is the linear form in the coordinates $x_1, \ldots, x_N$ corresponding to the j-th component of $\underline{u}$ and $H_\infty \subset \mathbb{P}_{\underline{x},A}$ is the hyperplane at infinity $V_+(x_0)$.

Assume $Y \hookrightarrow \mathbb{A}_{\underline{x},A}$ is a (reduced) closed subscheme with (reduced) projective closure $\overline{Y} \hookrightarrow \mathbb{P}_{\underline{x},A}$ such that $\overline{Y} \cap L_{\underline{u}} = \varnothing$. Then $\underline{u}$ induces regular maps

$$p_{\underline{u}} : Y \longrightarrow \mathbb{A}_{t_1,\ldots,t_r,A} \quad \text{and} \quad \bar{p}_{\underline{u}} : \overline{Y} \longrightarrow \mathbb{P}_{t_0:\cdots:t_r,A}$$

satisfying $p_{\underline{u}} = \bar{p}_{\underline{u}} \times_{\mathbb{P}_{\underline{t},A}} \mathbb{A}_{\underline{t},A}$. Observe that [Shafarevich 1994, Theorem I.5.3.7] remains true in our setting:

Lemma 2.3. *For any $\underline{u} \in \mathfrak{E}_r(A)$ and any closed $Y \hookrightarrow \mathbb{A}_{\underline{x},S}$ with $\overline{Y} \cap L_{\underline{u}} = \varnothing$, the linear projections $p_{\underline{u}}$ and $\bar{p}_{\underline{u}}$ are finite maps.*

Proof. It suffices to show that $\bar{p}_{\underline{u}}$ is finite. As a map between projective schemes over S, $\bar{p}_{\underline{u}}$ itself is projective. It remains to show that $\bar{p}_{\underline{u}}$ is quasifinite. Let $\bar{\sigma}$ be a geometric point of S over σ, and likewise $\bar{\eta}$ over η. By [Shafarevich 1994, Theorem I.5.3.7], $\bar{\sigma}^* \bar{p}_{\underline{u}}$ and $\bar{\eta}^* \bar{p}_{\underline{u}}$ are finite. It follows that $\bar{p}_{\underline{u}}$ is finite on the special fibre $\sigma^* \bar{p}_{\underline{u}}$ and the geometric fibre $\eta^* \bar{p}_{\underline{u}}$, hence quasifinite. ☐

Outline of the proof of Theorem 2.1. The proof of Theorem 2.1 principally follows the proof of Gabber's geometric presentation lemma over fields in [Colliot-Thélène et al. 1997]. The crucial part of Theorem 2.1 turns out to be the finiteness claim for Z/V. We make the Ansatz $p = p_{\underline{u}}$ for a closed embedding $i : X \hookrightarrow \mathbb{A}_{x_1,\ldots,x_N,S}$ and a linear projection $\underline{u} \in \mathfrak{E}_n(\mathfrak{o})$, for n the relative dimension of X/S. Using Lemma 2.3, one can show that the property

$$\textit{the induced map } p_{(u_1,\ldots,u_{n-1})}|_Z : Z \to \mathbb{A}_{t_1,\ldots,t_{n-1},S} \textit{ is finite}$$

is open in our space of linear projections $\mathfrak{E}_n$ (see the proof of Lemma 2.11, below). If $W \hookrightarrow \mathfrak{E}_n$ is the corresponding open locus, we first need to make sure that the special fibre W_σ is nonempty — because the residue field $\mathbb{F}$ is infinite and $\mathfrak{E}_n$ isomorphic to an affine space, $W(\mathfrak{o})$ is automatically nonempty in this case. We will see in the proof of Lemma 2.11 that W_σ is nonempty, as soon as X over S is fibrewise dense inside its closure in $\mathbb{P}_{x_0:\cdots:x_N,S}$. In fact, we just need this closure to be dense on the special fibre, but this obviously is equivalent to fibrewise density. Therefore, special care needs to be taken about the choice of our initial closed embedding $i : X \hookrightarrow \mathbb{A}_{\underline{x},S}$. In Proposition 2.6 we will provide a closed embedding of this type, but the price to pay is that we need to replace (X, z) by a suitable Nisnevich neighbourhood.

Let $F = F_{\underline{u}}$ denote the set of preimages under $p_{(u_1,\ldots,u_{n-1})}$ of $p_{(u_1,\ldots,u_{n-1})}(z)$ in Z. The next goal is to find $\underline{u}$ in $W(\mathfrak{o})$ such that $p_{\underline{u}}$ is étale around each point of F and the restriction of $p_{\underline{u}}$ to F is universally injective. This again corresponds to nonempty open conditions in (the special fibre of) $\mathfrak{E}_n$ (see the proof of Lemma 2.12). Shrinking W accordingly, we may assume that this is the case for all $\underline{u} \in W(\mathfrak{o})$ (see Proposition 2.9).

Making use of the finiteness of $p_{(u_1,\dots,u_{n-1})}|_Z$, we get an open neighbourhood $V = V_{\underline{u}}$ of $p_{(u_1,\dots,u_{n-1})}(z)$ in $\mathbb{A}_{t_1,\dots,t_{n-1},S}$ such that $p_{\underline{u}}$ is étale at all points in the intersection $Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V)$ and the restriction of $p_{\underline{u}}$ induces a closed embedding $Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V) \hookrightarrow \mathbb{A}^1_V$ (see Lemma 2.13). Finally, replacing (X, z) by a suitable Zariski-neighbourhood, we establish in Lemma 2.14 the remaining properties claimed in Theorem 2.1.

Summing up, we will in fact show slightly more, proving the following version of [Colliot-Thélène et al. 1997, Theorem 3.2.2] over $\mathfrak{o}$.

Theorem 2.4. *Let $X = \operatorname{Spec}(A)/S$ be a smooth affine S-scheme of finite type, fibrewise of pure dimension n and let $Z = \operatorname{Spec}(B) \hookrightarrow X$ be a proper closed subscheme. Let z be a point in Z. If z lies in the special fibre, suppose that $Z_\sigma \neq X_\sigma$. Then, Nisnevich-locally around z, there exists a closed embedding $X \hookrightarrow \mathbb{A}^N_S$ and a Zariski-open subset $W \subseteq \mathfrak{E}_n$ with $W(\mathfrak{o}) \neq \varnothing$ such that the following holds:*

For all $\underline{u} \in W(\mathfrak{o})$ with linear projections

$$p_{\underline{u}} = p_{(u_1,\dots,u_{n-1})} \times_S p_{u_n} : X \to \mathbb{A}^n_S = \mathbb{A}^{n-1}_S \times_S \mathbb{A}^1_S,$$

there are Zariski-open neighbourhoods $V \subseteq \mathbb{A}^{n-1}_S$ containing $p_{(u_1,\dots,u_{n-1})}(z)$ and $U \subseteq p^{-1}_{(u_1,\dots,u_{n-1})}(V)$ containing z satisfying.

(1) *$p_{(u_1,\dots,u_{n-1})}|_Z : Z \to \mathbb{A}^{n-1}_S$ is finite,*

(2) *$Z \cap U = Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V)$,*

(3) *$p_{\underline{u}}|_U : U \to \mathbb{A}^n_S$ is étale and restricts to a closed embedding $Z \cap U \hookrightarrow \mathbb{A}^1_V$ and*

(4) *$p_{\underline{u}}^{-1}(p_{\underline{u}}(Z \cap U)) \cap U = Z \cap U$.*

The proof of Theorem 2.4 follows the proof in [Colliot-Thélène et al. 1997] and the outline sketched above.

Proof. Clearly, we may assume that both X and Z are connected. Next, observe that the case of z lying in the generic fibre X_η of X/S is already covered by [Colliot-Thélène et al. 1997, Theorem 3.2.2]. Thus, we may assume that z lies in the special fibre X_σ of X/S. Finally, observe that we may enlarge Z. In particular, picking any element f in the kernel of $A \twoheadrightarrow B$ with $f \neq 0$ in the special fibre $A \otimes_{\mathfrak{o}} \mathbb{F}$, we may assume $B = A/f$, i.e., $Z = V(f)$.

We follow the outline of the proof sketched above: Up to a refinement by a suitable Nisnevich neighbourhood $(X', z') \to (X, z)$, Proposition 2.6 provides a closed embedding $i_0 : X' \hookrightarrow \mathbb{A}_{x_1,\dots,x_N,S}$ such that $Z' = Z \times_X X'$ is fibrewise dense in its Zariski-closure $\overline{Z}'$ in $\mathbb{P}_{x_0:\dots:x_N,S}$. Replacing our base-point z by a specialization to a closed point z_0 in the image of $X' \to X$, we can assume that z is closed itself (see Reduction 2.8). Further, we replace (X, z) by (X', z'), i.e., we assume

$X' = X$. Composing the closed embedding $i_0 : X \hookrightarrow \mathbb{A}_{\underline{x},S}$ with a linear projection $\mathbb{A}_{\underline{x},S} \to \mathbb{A}_{t_1,\dots,t_n,S}$, corresponding to an $\mathfrak{o}$-point $\underline{u}$ of the space of linear projections $\mathfrak{E}_n$, we get maps

$$\begin{tikzcd} X \arrow[r, "p_{\underline{u}}"] \arrow[dr, "p_{(u_1,\dots,u_{n-1})}"'] & \mathbb{A}_{t_1,\dots,t_n,S} \arrow[d, "\mathrm{pr}"] \\ & \mathbb{A}_{t_1,\dots,t_{n-1},S} \end{tikzcd}$$

Here n is the dimension of X. Proposition 2.9 provides an open W in our space of linear projections $\mathfrak{E}_n$ with $W(\mathfrak{o})$ nonempty, and for each linear projection $\underline{u}$ in $W(\mathfrak{o})$ the restriction $p_{(u_1,\dots,u_{n-1})}|_Z$ is finite (i.e., part (1) in Theorem 2.4), $p_{\underline{u}}$ is étale around

$$F = F_{\underline{u}} = p^{-1}_{(u_1,\dots,u_{n-1})}(p_{(u_1,\dots,u_{n-1})}(z)) \cap Z$$

and $p_{\underline{u}}|_F : F \to p_{\underline{u}}(F)$ is universally injective.

Fix any such $\underline{u}$ in $W(\mathfrak{o})$. In Lemma 2.13, we will replace $\mathbb{A}_{t_1,\dots,t_{n-1},S}$ by a Zariski-neighbourhood $V = V_{\underline{u}}$ of $p_{(u_1,\dots,u_{n-1})}(z)$ such that $p_{\underline{u}}$ is étale around every point of $Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V)$ and such that the induced restriction

$$p_{\underline{u}}|_{Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V)} : Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V) \hookrightarrow \mathbb{A}_{t_n,V}$$

is a closed embedding. In Lemma 2.14, we will shrink $p^{-1}_{(u_1,\dots,u_{n-1})}(V)$ to a Zariski-neighbourhood U_1 of z satisfying the analogue of (4) in Theorem 2.4, i.e.,

$$p_{\underline{u}}^{-1}(p_{\underline{u}}(Z \cap U_1)) \cap U_1 = Z \cap U_1,$$

without changing $Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V) = Z \cap U_1$. In particular, $p_{\underline{u}}$ restricts to a closed embedding $Z \cap U_1 \to \mathbb{A}_{t_n,V}$. Since $p_{\underline{u}}$ is étale already around every point of $Z \cap U_1$, we may shrink U_1 a bit more (by intersecting it with the open étale locus of $p_{\underline{u}}$) to get the desired Zariski-neighbourhood $U = U_{\underline{u}}$ of (X, z) satisfying (2), (3) and (4) in Theorem 2.4. $\square$

Towards the finiteness part. The key part in the proof of Theorem 2.4 is the finiteness assertion (1). By Lemma 2.3, we need to find a closed embedding $i_0 : X \hookrightarrow \mathbb{A}_{\underline{x},S}$ and an $\mathfrak{o}$-point $\underline{u} \in \mathfrak{E}_n(\mathfrak{o})$ such that the closure $\bar{Z}$ of Z in $\mathbb{P}_{\underline{x},S}$ intersects $L_{(u_1,\dots,u_{n-1})}$ trivially. Unfortunately, it is not enough to require that the fibrewise closure of Z in $\mathbb{P}_{\underline{x},S}$ misses $L_{(u_1,\dots,u_{n-1})}$. Indeed, $\bar{Z}$ might not be the fibrewise projective closure of Z over S — the special fibre $\bar{Z}_\sigma$ might be strictly larger than the projective closure of Z_σ, as seen in the next example.

Example 2.5. Let $A = \mathfrak{o}[x_1]$ and $B = \mathfrak{o}[x_1]/(\pi x_1^2 + x_1 + 1)$. Then $\mathrm{Spec}(B) \subset \mathbb{P}_{x_0:x_1,S}$ is fibrewise closed but at least one solution of $\pi x_1^2 + x_1 + 1$ in k^{alg} specializes to ∞ in $\mathbb{P}_{x_0:x_1,\mathbb{F}}$, i.e., $\mathrm{Spec}(B) \subset \mathbb{P}_{x_0:x_1,S}$ is not closed.

To avoid these difficulties, we need to make a careful choice for the embedding $i_0 : X \hookrightarrow \mathbb{A}_{\underline{x},S}$.

Proposition 2.6. *Nisnevich-locally around z, there exists a closed embedding*

$$i_0 : X \hookrightarrow \mathbb{A}_{\underline{x},S}$$

with Z fibrewise dense over S inside its closure $\bar{Z}$ in $\mathbb{P}_{\underline{x},S}$.

Proof. We need to adapt [Kai 2015, Theorem 4.6] to our situation. Since z lies in the special fibre and $\mathfrak{o}$ is henselian, [loc. cit.] gives us an affine Nisnevich neighbourhood $(Z', z') \to (Z, z)$ and a closed embedding $\bar{i}_0 : Z' \hookrightarrow \mathbb{A}_{x_1,\dots,x_m,S}$, such that Z' is fibrewise dense over S inside its closure $\bar{Z}'$ in $\mathbb{P}_{x_0:\dots:x_m,S}$. Since both Z and Z' are affine, the underlying étale morphism $Z' \to Z$ is standard smooth, i.e., $Z' = \operatorname{Spec}(B')$ with $B' = B[t_1, \dots, t_s]/(\bar{g}_1, \dots, \bar{g}_s)$ and invertible Jacobi-determinant $\det(\{\partial \bar{g}_i/\partial t_j\}_{i,j}) \in B'^{\times}$.

We want to extend (Z', z') to a Nisnevich neighbourhood (X', z') of (X, z). Since the Jacobi-determinant $\det(\{\partial \bar{g}_i/\partial t_j\}_{i,j})$ is invertible in B', it is nontrivial in $B' \otimes k(z')$. Choose a lift $g_i \in A[\underline{t}]$ for each $\bar{g}_i$ and set $A' := A[\underline{t}]/(g_1, \dots, g_s)$ and $X' = \operatorname{Spec}(A')$. By construction, $B' = A' \otimes_A B$, so z' induces a point (also denoted by) z' in X'. Since $\det(\{\partial g_i/\partial t_j\}_{i,j}) \equiv \det(\{\partial \bar{g}_i/\partial t_j\}_{i,j}) \neq 0$ in $A' \otimes k(z')$, the Jacobi-determinant $\det(\{\partial g_i/\partial t_j\}_{i,j})$ is invertible around z' in X'. By shrinking X' without changing Z' (since the Jacobi-determinant is invertible on the latter), we may assume that $(Z', z') \to (Z, z)$ extends to an affine Nisnevich neighbourhood $(X', z') \to (X, z)$. Further, lifting the images of x_i in B' to A', $\bar{i}_0$ extends to a map $i_0' : X' \to \mathbb{A}_{x_1,\dots,x_n,S}$.

Unfortunately, there is no reason for i_0' to be a closed embedding. To repair this, choose a closed embedding $X' \hookrightarrow \mathbb{A}_{y_1,\dots,y_r,S}$ over S, i.e., generators a_j of A' as an $\mathfrak{o}$-algebra. Recall that we assumed $B = A/f$ for f in A nonzero in the special fibre $A \otimes_{\mathfrak{o}} \mathbb{F}$ (see the proof of Theorem 2.4). Writing $\underline{a}^{\underline{j}} = a_1^{j_1} \cdot \dots \cdot a_r^{j_r}$, any element of the ideal $f \cdot A'$ is of the form $\sum_{\underline{j}} \lambda_{\underline{j}} \cdot f \underline{a}^{\underline{j}}$, where $\lambda_{\underline{j}} \in \mathfrak{o}$ and $\underline{j}$ runs over a finite subset of $\mathbb{N}^r$. Mapping $y^{(\underline{j})} \mapsto f \underline{a}^{\underline{j}}$, we get a map $X' \to \mathbb{A}_{\{y^{(\underline{j})} \mid \underline{j} \in \mathbb{N}^r\},S}$ into a copy of the infinite affine space over S. Together with the map $i_0' : X' \to \mathbb{A}_{x_1,\dots,x_n,S}$, we get a *closed* embedding

$$i_\infty : X' \hookrightarrow \mathbb{A}_{x_1,\dots,x_m,S} \times_S \mathbb{A}_{\{y^{(\underline{j})} \mid \underline{j} \in \mathbb{N}^r\},S} \cong \mathbb{A}_S^\infty$$

into the fibre product. Indeed, $i_0' : \mathfrak{o}[x_1, \dots, x_m] \to A'$ is surjective modulo f and $\mathfrak{o}[y^{(\underline{j})} \mid \underline{j} \in \mathbb{N}^r] \to A'$ has image $\mathfrak{o}[f \cdot A']$ by construction. By Lemma 2.7, i_∞ induces $i_0 : X' \hookrightarrow \mathbb{A}_{x_1,\dots,x_m,S} \times_S \mathbb{A}_{y^{(\underline{j}_1)},\dots,y^{(\underline{j}_l)},S}$, still a closed embedding for suitable $\underline{j}_1, \dots, \underline{j}_l \in \mathbb{N}^r$. Setting $x_{m+s} := y^{(\underline{j}_s)}$ and $N := m+l$, we have constructed a closed embedding $i_0 : X' \hookrightarrow \mathbb{A}_{x_1,\dots,x_N,S}$ such that $i_0|_{Z'}$ factors over $\bar{i}_0 : Z' \hookrightarrow \mathbb{A}_{x_1,\dots,x_m,S} = V(x_{m+1}, \dots, x_N) \subseteq \mathbb{A}_{x_1,\dots,x_N,S}$. In particular, the closure of Z' in $\mathbb{P}_{x_0:\dots:x_N,S}$ is just

$\bar{Z}'$ inside the linear subspace $V_+(x_{m+1}, \dots, x_N) \subseteq \mathbb{P}_{x_0:\dots:x_N,S}$, so Z' is fibrewise dense over S inside this closure. $\square$

Lemma 2.7. *Let C be an $\mathfrak{o}$-algebra of finite type. Let ι: $\mathrm{Spec}(C) \hookrightarrow \mathbb{A}_{t_1,t_2,\dots,S} = \mathbb{A}^\infty_S$ be a closed embedding and let $\mathrm{pr}_{\leq N} : \mathbb{A}_{t_1,t_2,\dots,S} \to \mathbb{A}_{t_1,\dots,t_N,S} = \mathbb{A}^N_S$ be the canonical projection. Then $\mathrm{pr}_{\leq N} \circ \iota$ is a closed embedding for $N \gg 0$.*

Proof. Suppose that C is generated as an $\mathfrak{o}$-algebra by $c_1, \dots, c_r \in C$. Since the corresponding map on algebras $\mathfrak{o}[t_1, t_2, \dots] \twoheadrightarrow C$ is surjective, we can find polynomials $f_i \in \mathfrak{o}[t_1, t_2, \dots]$ mapping to c_i. Pick $N \gg 0$ such that all the f_i lie inside $\mathfrak{o}[t_1, \dots, t_N]$. Then ι restricted to $\mathfrak{o}[t_1, \dots, t_N]$ is still surjective, hence the claim. $\square$

Choosing linear projections. In the next step, we want to find the Zariski-open subset $W \subseteq \mathfrak{C}_n$ parametrizing the linear projections $p_{\underline{u}}$ in Theorem 2.4. To do so, let us first make one further reduction.

Reduction 2.8. By Proposition 2.6, there is a Nisnevich neighbourhood $(X', z') \to (X, z)$ and a closed embedding $X' \hookrightarrow \mathbb{A}_{\underline{x},S}$ such that $Z' = Z \times_X X'$ is fibrewise dense in its Zariski-closure $\bar{Z}'$ in $\mathbb{P}_{\underline{x},S}$. Let z_0 be a specialization of z in the image of $X' \to X$. We can find a point z'_0 in Z' such that $k(z'_0) = k(z_0)$, i.e., $(X', z'_0) \to (X, z_0)$ is a Nisnevich neighbourhood, too. The Nisnevich localization $(X', z') \to (X, z)$ will be the only non-Zariski-localization in the proof of Theorem 2.4. Thus we may assume that z is a closed point in the following. Further, from now on we may identify $X' = X$.

The Zariski-open subset $W \subseteq \mathfrak{C}_n$ in Theorem 2.4 will be provided in the following proposition.

Proposition 2.9. *Let $X = \mathrm{Spec}(A)/S$ be a connected smooth affine S-scheme of finite type, fibrewise of pure dimension n, f an element in A which is nonzero in $A \otimes_{\mathfrak{o}} \mathbb{F}$ and $Z = \mathrm{Spec}(B = A/f) \hookrightarrow X$ the closed embedding. Let z be a closed point in the special fibre of Z. Suppose there is a closed embedding $i_0 : X \hookrightarrow \mathbb{A}_{\underline{x},S}$ such that Z is fibrewise dense over S inside its closure $\bar{Z}$ in $\mathbb{P}_{\underline{x},S}$.*

Then there is a Zariski-open subset $W \subseteq \mathfrak{C}_n$ with $W(\mathfrak{o}) \neq \varnothing$, such that for all $\underline{u} \in W(\mathfrak{o})$ the following hold:

(1) *$p_{(u_1,\dots,u_{n-1})}|_Z : Z \to \mathbb{A}_{t_1,\dots,t_{n-1},S}$ is finite,*

(2) *$p_{\underline{u}}$ is étale at all points of $F = p^{-1}_{(u_1,\dots,u_{n-1})}(p_{(u_1,\dots,u_{n-1})}(z)) \cap Z$ and*

(3) *$p_{\underline{u}}|_F : F \to p_{\underline{u}}(F)$ is radicial.*

Let us first fix the following notation.

Remark 2.10. For Y/S a smooth scheme, denote by $\mathrm{red} : Y(\mathfrak{o}) \to Y(\mathbb{F}) = Y_\sigma(\mathbb{F})$ the reduction map we get by precomposing with the closed point σ. Because $\mathfrak{o}$ is henselian and Y/S smooth, this reduction map is always surjective.

Proof of Proposition 2.9. We divide Proposition 2.9 into two parts: Lemma 2.11 will provide an open W_1 of $\mathfrak{E}_n$ such that $W_1(\mathfrak{o})$ is nonempty and every $\underline{u}$ in $W_1(\mathfrak{o})$ satisfies claim (1), while Lemma 2.12 will provide an open W_2 such that $W_2(\mathfrak{o})$ is nonempty and every $\underline{u}$ in $W_2(\mathfrak{o})$ satisfies claims (2) and (3) in Proposition 2.9. The intersection $W = W_1 \cap W_2$ has all the properties claimed by Proposition 2.9. For the nonemptiness of $W(\mathfrak{o})$, recall that the reduction map $W(\mathfrak{o}) \twoheadrightarrow W(\mathbb{F})$ is surjective and $W_1(\mathbb{F}) \cap W_2(\mathbb{F})$ is nonempty as the special fibre of $W_1 \cap W_2$ is a nonempty open subscheme of an affine space over the infinite field $\mathbb{F}$. $\square$

Lemma 2.11. *Under the assumptions of Proposition 2.9, there is a Zariski-open subset $W_1 \subseteq \mathfrak{E}_n$ with $W_1(\mathfrak{o}) \neq \varnothing$, such that for all $\underline{u} \in W_1(\mathfrak{o})$ the restriction $p_{(u_1,\dots,u_{n-1})}|_Z : Z \to \mathbb{A}_{t_1,\dots,t_{n-1},S}$ is finite.*

Lemma 2.12. *Under the assumptions of Proposition 2.9, there is a Zariski-open subset $W_2 \subseteq \mathfrak{E}_n$ with $W_2(\mathfrak{o}) \neq \varnothing$, such that $p_{\underline{u}}$ is étale at all points of F and $p_{\underline{u}}|_F : F \to p_{\underline{u}}(F)$ is radicial for all $\underline{u} \in W_2(\mathfrak{o})$.*

Proof of Lemma 2.11. This is just a version of the arguments leading to [Grayson 1978, Proposition 1.1]. Recall that the j-th factor $\mathbb{A}^{\vee,(j)}_{\underline{x},S}$ of $\mathfrak{E}_n$ is $\mathbb{A}_{x^*_{1,j},\dots,x^*_{N,j},S}$, i.e., $\mathfrak{E}_n = \mathbb{A}_{\{x^*_{i,j} | 1 \le i \le N, 1 \le j \le n\},S}$. Define

$$\mathbb{L} := V_+\Big(x_0, \sum_j x^*_{i,j} \otimes x_j \;\Big|\; 1 \le i < n\Big) \subseteq \mathfrak{E}_n \times_S H_\infty \quad \text{and} \quad \bar{Z}_\infty := \bar{Z} \cap H_\infty.$$

Here, $H_\infty = V_+(x_0) \subset \mathbb{P}_{\underline{x},S}$ is the hyperplane at infinity. By construction, $\mathbb{L} \to \mathfrak{E}_n$ has fibre $\mathbb{L}_{\underline{u}} = L_{(u_1,\dots,u_{n-1})}$ over $\underline{u} \in \mathfrak{E}_n(\mathfrak{o})$. Since the projection $\mathrm{pr} : \mathfrak{E}_n \times_S H_\infty \to \mathfrak{E}_n$ is projective, hence closed,

$$W_1 := \mathfrak{E}_n \setminus \mathrm{pr}(\mathbb{L} \cap (\mathfrak{E}_n \times_S \bar{Z}_\infty))$$

is open. Again by construction, for any $\underline{u} \in W_1(\mathfrak{o})$, $L_{(u_1,\dots,u_{n-1})} \cap \bar{Z} = \varnothing$, so $p_{(u_1,\dots,u_{n-1})}|_Z : Z \to \mathbb{A}_{t_1,\dots,t_{n-1},S}$ is finite by Lemma 2.3.

It remains to show that $W_1(\mathfrak{o}) \neq \varnothing$. The reduction map $\mathrm{red} : W_1(\mathfrak{o}) \twoheadrightarrow W_1(\mathbb{F})$ is surjective, so we have to show $W_1(\mathbb{F}) = W_{1,\sigma}(\mathbb{F}) \neq \varnothing$. The special fibre $W_{1,\sigma}$ equals $\mathfrak{E}_{n,\sigma} \setminus \mathrm{pr}(\mathbb{L}_\sigma \cap (\mathfrak{E}_{n,\sigma} \times_{\mathbb{F}} \bar{Z}_{\infty,\sigma}))$. Further, $Z_\sigma \subset \bar{Z}_\sigma$ is dense by assumption so $\bar{Z}_\sigma$ is the closure $\overline{Z_\sigma}$ of Z_σ inside $\mathbb{P}_{\underline{x},\mathbb{F}}$. It follows that $\bar{Z}_{\infty,\sigma} = \overline{Z_\sigma} \cap H_{\infty,\sigma}$, i.e., we are in the situation of [Grayson 1978, Proposition 1.1] and $W_{1,\sigma}(\mathbb{F}) \neq \varnothing$. $\square$

Lemma 2.12 can easily be derived from [Colliot-Thélène et al. 1997, Lemmas 3.4.1 and 3.4.2] applied over the special fibre.

Proof of Lemma 2.12. As a closed embedding of smooth S-schemes, $i_0 : X \hookrightarrow \mathbb{A}_{\underline{x},S}$ is regular. Let $I = (f_1, \dots, f_{N-n}) \trianglelefteq \mathfrak{o}[\underline{x}]$ be the ideal of i_0 for $f_1, \dots, f_{N-n}$ a regular sequence. Write $A = \mathcal{O}(X)$ over $\mathfrak{o}[\underline{t}]$ (via $p_{\underline{u}}$) as $\mathfrak{o}[\underline{t}][\underline{x}]/(f_i, u_j - t_j \mid i, j)$.

Then $p_{\underline{u}}$ is étale at a point $x \in X$ if it is standard smooth around x, i.e., if the Jacobi-determinant

$$\det\left(\left\{\frac{\partial f_i}{\partial x_s}\right\}_{i,s} \,\Big|\, \left\{\frac{\partial(u_j - t_j)}{\partial x_s}\right\}_{j,s}\right) = \det\left(\left\{\frac{\partial f_i}{\partial x_s}\right\}_{i,s} \,\Big|\, \left\{\frac{\partial(u_j)}{\partial x_s}\right\}_{j,s}\right)$$

is invertible in $\mathcal{O}_{X,x}$. We may write the latter determinant as $df_1 \wedge \cdots \wedge df_{N-n} \wedge du_1 \wedge \cdots \wedge du_n$ in $\Omega^N_{\mathfrak{o}[\underline{x}]/\mathfrak{o}} \otimes_{\mathfrak{o}[\underline{x}]} \mathcal{O}_{X,x}$. Since $X \hookrightarrow \mathbb{A}_{\underline{x},S}$ is a smooth pair, the conormal sequence

$$0 \to I/I^2 \otimes_A \mathcal{O}_{X,x} \to \Omega^1_{\mathfrak{o}[\underline{x}]/\mathfrak{o}} \otimes_{\mathfrak{o}[\underline{x}]} \mathcal{O}_{X,x} \to \Omega^1_{A/\mathfrak{o}} \otimes_A \mathcal{O}_{X,x} \to 0$$

is split exact and

$$\Omega^N_{\mathfrak{o}[\underline{x}]/\mathfrak{o}} \otimes_{\mathfrak{o}[\underline{x}]} \mathcal{O}_{X,x} = \bigwedge^{N-n}(I/I^2 \otimes_A \mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{X,x}} (\Omega^n_{A/\mathfrak{o}} \otimes_A \mathcal{O}_{X,x}).$$

Note that I/I^2 is free over A with basis given by the regular sequence $f_1, \dots, f_{N-n}$. In particular, $f_1 \wedge \cdots \wedge f_n$ is invertible in $\bigwedge^{N-n}(I/I^2 \otimes_A \mathcal{O}_{X,x}) = \mathcal{O}_{X,x}$ and

$$df_1 \wedge \cdots \wedge df_{N-n} \wedge du_1 \wedge \cdots \wedge du_n = (f_1 \wedge \cdots \wedge f_n) \otimes (d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_n)$$

is invertible if and only if $d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_n$ is invertible in $\Omega^n_{A/\mathfrak{o}} \otimes_A \mathcal{O}_{X,x} = \mathcal{O}_{X,x}$ for $\bar{u}_j$ the image of t_j under $\mathfrak{o}[\underline{t}] \to A$. By Nakayama's lemma, this is equivalent to $d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_n \neq 0$ in $\Omega^n_{X/S} \otimes_{\mathcal{O}_X} k(x)$. Suppose x is contained in F. Since z lies in the special fibre, so does x and $\Omega^n_{X/S} \otimes_{\mathcal{O}_X} k(x) = \Omega^n_{X_\sigma/\mathbb{F}} \otimes_{\mathcal{O}_{X_\sigma}} k(x)$. Summing up, $p_{\underline{u}}$ is étale at $x \in F$ if $d\bar{u}_1 \wedge \cdots \wedge d\bar{u}_n \neq 0$ in $\Omega^n_{X_\sigma/\mathbb{F}} \otimes_{\mathcal{O}_{X_\sigma}} k(x)$. Thus, we are in fact in the situation of [Colliot-Thélène et al. 1997, Lemma 3.4.1], i.e., we get a nonempty open subset $\overline{W}'_2 \subseteq \mathfrak{E}_{n,\sigma} \otimes_{\mathbb{F}} \mathbb{F}^{\mathrm{alg}}$ with $p_{\underline{u}}$ étale around F for all $\underline{u}$ with $\mathrm{red}(\underline{u}) \in \overline{W}'_2(\mathbb{F}^{\mathrm{alg}})$. Here, $\mathbb{F}^{\mathrm{alg}}/\mathbb{F}$ is an algebraic closure.

For the universal injectivity, observe that $p_{\underline{u}}|_F = p_{\mathrm{red}(\underline{u})}|_F$. Thus, we are in the situation of [Colliot-Thélène et al. 1997, Lemma 3.4.2], i.e., we get a nonempty open subset $\overline{W}''_2 \subseteq \mathfrak{E}_{n,\sigma} \otimes_{\mathbb{F}} \mathbb{F}^{\mathrm{alg}}$ with $p_{\underline{u}}|_{F \otimes_{\mathbb{F}} \mathbb{F}^{\mathrm{alg}}}$ (universally) injective for all $\underline{u}$ with $\mathrm{red}(\underline{u}) \in \overline{W}''_2(\mathbb{F}^{\mathrm{alg}})$.

Finally, by a standard descent argument (see the proof of [Colliot-Thélène et al. 1997, Lemma 3.4.3]) for the intersection $\overline{W}'_2 \cap \overline{W}''_2$, we get an open subset $\overline{W}_2 \subseteq \mathfrak{E}_{n,\sigma}$ with $\overline{W}_2(\mathbb{F}) \neq \varnothing$, such that $p_{\underline{u}}$ is étale at all points of F and $p_{\underline{u}}|_F$ is universally injective for all $\underline{u}$ with $\mathrm{red}(\underline{u}) \in \overline{W}_2(\mathbb{F})$. Let $W_2 \subseteq \mathfrak{E}_n$ be any open subset with special fibre $W_{2,\sigma} = \overline{W}_2$. Then $\underline{u} \in W_2(\mathfrak{o})$ if and only if $\mathrm{red}(\underline{u}) \in \overline{W}_2(\mathbb{F})$ and $W_2(\mathfrak{o}) \neq \varnothing$, since the reduction map is surjective. □

Choosing neighbourhoods. Fix a linear projection $p_{\underline{u}}$ for an $\mathfrak{o}$-point $\underline{u}$ in the open subset $W \subseteq \mathfrak{E}_n$ provided by Proposition 2.9. In the following, we construct the open neighbourhoods V and U in Theorem 2.4.

As in [Colliot-Thélène et al. 1997], we first secure $V \subseteq \mathbb{A}_{t_1,\dots,t_{n-1},S}$ in Lemma 2.13 and an open neighbourhood $z \in U_1 \subseteq p^{-1}_{(u_1,\dots,u_{n-1})}(V)$ covering Theorem 2.4 parts

(2) and (4) in Lemma 2.14. If we define U as the intersection of U_1 with the étale locus of $p_{\underline{u}}$, the pair V and U will finally satisfy claims (2), (3) and (4) of Theorem 2.4. The proofs can almost literally be transferred from [Colliot-Thélène et al. 1997].

Lemma 2.13 (cf. [Colliot-Thélène et al. 1997, Lemma 3.5.1]). *Under the assumptions of Proposition 2.9 and any choice of linear projection $\underline{u}$ in $W(\mathfrak{o})$, there is a Zariski-open neighbourhood $V \subseteq \mathbb{A}_{t_1,\dots,t_{n-1},S}$ of $p_{(u_1,\dots,u_{n-1})}(z)$ such that $p_{\underline{u}}$ is étale in a Zariski-open neighbourhood of $Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V)$ and restricts to a closed embedding $Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V) \hookrightarrow \mathbb{A}^1_V$.*

Proof. We will get V as $V_1 \cap V_2$, where $V_1 \subseteq \mathbb{A}_{t_1,\dots,t_{n-1},S}$ is an open neighbourhood of $p_{(u_1,\dots,u_{n-1})}(z)$ such that $p_{\underline{u}}$ is étale at all points of $Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V_1)$ and $V_2 \subseteq \mathbb{A}_{t_1,\dots,t_{n-1},S}$ is an open neighbourhood such that $p_{\underline{u}}$ restricts to a closed embedding $Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V_2) \to \mathbb{A}^1_{V_2}$.

Let $U' \subseteq X$ be the étale locus of $p_{\underline{u}}$. Since $\underline{u} \in W(\mathfrak{o})$, U' is an open neighbourhood of $F = p^{-1}_{(u_1,\dots,u_{n-1})}(p_{(u_1,\dots,u_{n-1})}(z)) \cap Z$ in X (Proposition 2.9). As $p_{(u_1,\dots,u_{n-1})}|_Z$ is finite, $p_{(u_1,\dots,u_{n-1})}(Z \setminus U')$ is closed and we set

$$V_1 := \mathbb{A}_{t_1,\dots,t_{n-1},S} \setminus p_{(u_1,\dots,u_{n-1})}(Z \setminus U').$$

By construction, V_1 is a Zariski-open neighbourhood of the image $p_{(u_1,\dots,u_{n-1})}(z)$ of z and moreover $Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V_1) \subseteq Z \cap U'$ is contained in the étale locus U' of $p_{\underline{u}}$.

To get the neighbourhood V_2, consider $p_{\underline{u}}|_Z : Z \to \mathbb{A}_{\underline{t},S} = \mathbb{A}_{t_n,\mathbb{A}_{t_1,\dots,t_{n-1},S}}$ as a family of maps over $\mathbb{A}_{t_1,\dots,t_{n-1},S}$. Since $Z/\mathbb{A}_{t_1,\dots,t_{n-1},S}$ is finite, the property "$p_{\underline{u}}|_Z$ is a closed embedding" is Zariski-open in the base $\mathbb{A}_{t_1,\dots,t_{n-1},S}$ by Nakayama's lemma. Thus we have to show that the fibre of this family

$$p_{\underline{u}}|_F : Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(p_{(u_1,\dots,u_{n-1})}(z)) = F \to \mathbb{A}_{t_n,p_{(u_1,\dots,u_{n-1})}(z)}$$

over $p_{(u_1,\dots,u_{n-1})}(z)$ is a closed embedding. But $p_{\underline{u}}(F) \subset \mathbb{A}_{t_n,p_{(u_1,\dots,u_{n-1})}(z)}$ is closed as a finite set of closed points and $p_{\underline{u}}|_F : F \to p_{\underline{u}}(F)$ is a closed embedding as by Proposition 2.9, it is radicial and $p_{\underline{u}}$ is étale and hence unramified at each point of F. □

Lemma 2.14 (cf. [Colliot-Thélène et al. 1997, Lemma 3.6.1]). *Under the assumptions of Proposition 2.9 and any choice of linear projection $\underline{u}$ in $W(\mathfrak{o})$, let*

$$Z' := Z \cap p^{-1}_{(u_1,\dots,u_{n-1})}(V) \quad \textit{and} \quad U_1 := p^{-1}_{(u_1,\dots,u_{n-1})}(V) \setminus (p^{-1}_{\underline{u}}(p_{\underline{u}}(Z')) \setminus Z').$$

Then $U_1 \subseteq p^{-1}_{(u_1,\dots,u_{n-1})}(V)$ is a Zariski-open neighbourhood of the point z and we have $Z \cap U_1 = Z'$ and $p^{-1}_{\underline{u}}(p_{\underline{u}}(Z')) \cap U_1 = Z'$.

Proof. By definition of U_1, z lies inside U_1, $Z \cap U_1 = Z'$ and $p^{-1}_{\underline{u}}(p_{\underline{u}}(Z')) \cap U_1 = Z'$. It remains to show that $U_1 \subseteq p^{-1}_{(u_1,\dots,u_{n-1})}(V)$ is open. By Lemma 2.13, $p_{\underline{u}}$ restricts to a closed embedding $Z' \to \mathbb{A}_{t_n,V}$, so $p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \subset p_{(u_1,\dots,u_{n-1})}^{-1}(V)$ is closed. We need to show that $Z' \subseteq p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))$ is open.

To this end, consider the étale locus U'' of

$$p_{\underline{u}}|_{p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))} : p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \to p_{\underline{u}}(Z').$$

By Lemma 2.13, $p_{\underline{u}}$ is étale at all points of Z'. Thus, the base change $p_{\underline{u}}|_{p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))}$ is still étale at all points of Z', that is, Z' is contained inside the open subset $U'' \subseteq p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))$. But $Z' \to p_{\underline{u}}(Z')$ is an isomorphism by Lemma 2.13, so both $Z'/p_{\underline{u}}(Z')$ and $U''/p_{\underline{u}}(Z')$ are étale and hence $Z' \subseteq U''$ is open. □

3. Objectwise stable $\mathbb{A}^1$-connectivity

In this section, we derive connectivity results for homotopy presheaves (i.e., "objectwise" connectivity results). These are used in the proof of our main theorem in the following section. Moreover, we show the left completeness of the $\mathbb{A}^1$-Nisnevich-local t-structure on S^1- and $\mathbb{P}^1$-spectra. Throughout this section, let S be an arbitrary noetherian scheme of finite dimension.

Results for S^1-spectra. We start with objectwise connectivity results for S^1-spectra.

Proposition 3.1. *Let $U \in \mathrm{Sm}_S$ be a scheme of dimension e. Then given E in $\mathcal{SH}^s_{S^1>i+e}(S)$, one has*

$$[\Sigma^\infty_{S^1}(U_+)[i], L^{\mathbb{A}^1}E] = 0,$$

where $L^{\mathbb{A}^1}$ is a fibrant replacement functor for the stable $\mathbb{A}^1$-Nisnevich-local model structure.

Remark 3.2. Proposition 3.1 gives a connectivity result for a U-section of the homotopy presheaf $[\Sigma^\infty_{S^1}(-_+)[i], L^{\mathbb{A}^1}E]$ with respect to the dimension of U. However, we are interested in a connectivity result depending only on the dimension of the base scheme S. The price we have to pay for this is to sheafify the homotopy presheaf, i.e., eventually we are interested in connectivity results for the Nisnevich stalks of the presheaf $[\Sigma^\infty_{S^1}(-_+)[i], L^{\mathbb{A}^1}E]$. Unfortunately we cannot apply Proposition 3.1 directly to the stalks as their dimension is unbounded.

Proof of Proposition 3.1. We work with the explicit model L^∞ of Lemma 1.2 as an $\mathbb{A}^1$-Nisnevich-local fibrant replacement functor $L^{\mathbb{A}^1}$. By homotopy-exactness of L^∞, we have to show that

$$\big[\Sigma^\infty_{S^1}(U_+), \operatorname*{hocolim}_{k\to\infty} L^k(E)\big] = 0$$

for $U \in \mathrm{Sm}_S$ of dimension e and $E \in \mathcal{SH}^s_{S^1>e}$. Since $\Sigma^\infty_{S^1}(U_+)$ is compact, every homotopy class in question is represented by some $\Sigma^\infty_{S^1}(U_+) \to L^k(E)$. Hence, it suffices to show, that for every $k \geq 0$,

$$[\Sigma^\infty_{S^1}(U_+), L^k E] = 0.$$

We argue by induction on $k \geq 0$ for all $U \in \mathrm{Sm}_S$ of dimension e and all spectra $E \in \mathcal{SH}^s_{S^1 > e}$ at once.

For $k = 0$ the statement follows directly from Lemma 3.3 below. Let $k \geq 1$. The distinguished triangle in Remark 1.3 induces the long exact sequence

$$\cdots \to [\Sigma^\infty_{S^1}(U_+), L^{(k-1)} E)] \to [\Sigma^\infty_{S^1}(U_+), L^k E] \\ \to [\Sigma^\infty_{S^1}(U_+ \wedge \mathbb{A}^1), L^{(k-1)} E[1]] \to \cdots .$$

The abelian group on the left-hand side vanishes by the induction hypothesis on k. In order to see the vanishing of the right-hand side, we observe that

$$U \sqcup \mathbb{A}^1 \cong U_+ \vee \mathbb{A}^1 \to (U \times \mathbb{A}^1) \sqcup \mathbb{A}^1 \cong U_+ \times \mathbb{A}^1 \to U_+ \wedge \mathbb{A}^1$$

and therefore $U_+ \to (U \times \mathbb{A}^1)_+ \to U_+ \wedge \mathbb{A}^1$ is a homotopy cofibre sequence in $\mathrm{sPre}_+(S)$. This yields a distinguished triangle after applying the left Quillen functor $\Sigma^\infty_{S^1}$. Consider the long exact sequence obtained by an application of $[-, L^{(k-1)} E[1]]$ to this triangle. It suffices to show the vanishing of both of the abelian groups $[\Sigma^\infty_{S^1}(U_+)[1], L^{(k-1)} E[1]]$ and $[\Sigma^\infty_{S^1}(U \times \mathbb{A}^1)_+, L^{(k-1)} E[1]]$. For the first group, this follows from the inductive hypothesis on k and likewise for the second, since the dimension of $U \times \mathbb{A}^1$ is $e+1$ and $E[1] \in \mathcal{SH}^s_{S^1 > e+1}$. □

Lemma 3.3. *Let $U \in \mathrm{Sm}_S$ be a scheme of dimension e. Then for $D \in \mathcal{SH}^s_{S^1 > i+e}(S)$, one has*

$$[\Sigma^\infty_{S^1}(U_+)[i], L^s D] = 0.$$

Proof. It suffices to show that $[\Sigma^\infty_{S^1}(U_+), L^s D] = 0$ for $D \in \mathcal{SH}^s_{S^1 > e}$. Indeed, $[\Sigma^\infty_{S^1}(U_+)[i], L^s D] \cong [\Sigma^\infty_{S^1}(U_+), L^s(D[-i])]$ as L^s is homotopy-exact. Recall that the Nisnevich-cohomological dimension is bounded by the Krull-dimension, i.e., for any sheaf G of abelian groups on Sm_S and $n > \dim(U)$, we have

$$[\Sigma^\infty_{S^1}(U_+), L^s HG[n]] = H^n_{\mathrm{Nis}}(U, G) = 0;$$

see, e.g., [Thomason and Trobaugh 1990, Lemma E.6(c)].

By the left completeness of the Nisnevich-local structure, there is a filtration

$$\begin{array}{ccccccc} 0 \simeq \operatorname*{holim}_{n\to\infty} L^s D_{\geq n} & \longrightarrow \cdots \longrightarrow & L^s D_{\geq e+2} & \longrightarrow & L^s D_{\geq e+1} = L^s D \\ & & \downarrow & & \downarrow \\ & \cdots & L^s H\pi_{e+2}(D)[e+2] & & L^s H\pi_{e+1}(D)[e+1] \end{array}$$

and a surjection $0 = [\Sigma^\infty_{S^1}(U_+), \mathrm{holim}_n\, L^s D_{\geq n}] \twoheadrightarrow \lim_n [\Sigma^\infty_{S^1}(U_+), L^s D_{\geq n}]$ by the Milnor-$\lim^1$-sequence. Hence, $\lim_n [\Sigma^\infty_{S^1}(U_+), L^s D_{\geq n}] = 0$. For $i \geq 1$, there is a long exact sequence

$$\cdots \to [\Sigma^\infty_{S^1}(U_+), L^s D_{\geq e+i+1}] \to [\Sigma^\infty_{S^1}(U_+), L^s D_{\geq e+i}] \\ \to [\Sigma^\infty_{S^1}(U_+), L^s H\pi_{e+i}(D)[e+i]] \to \cdots,$$

where the abelian group on the right-hand side is zero by the above-mentioned result on the Nisnevich-cohomological dimension. For this reason, the projection $\lim[\Sigma^\infty_{S^1}(U_+), L^s D_{\geq n}] \twoheadrightarrow [\Sigma^\infty_{S^1}(U_+), L^s D_{\geq e+1}]$ is surjective and therefore we get $[\Sigma^\infty_{S^1}(U_+), L^s D_{\geq e+1}] = 0$ as desired. ☐

Corollary 3.4. *Let $U \in \mathrm{Sm}_S$ be an S-pointed scheme of dimension e. Then for $D \in \mathcal{SH}^s_{S^1 > i+e}(S)$, one has*

$$[\Sigma^\infty_{S^1}(U)[i], L^s D] = 0.$$

Proof. The basepoint $s : S \to U$ is a splitting of the structure morphism $p : U \to S$. In particular, $\dim(U) \geq \dim(S)$. Consider the distinguished triangle

$$\Sigma^\infty_{S^1}(S_+) \to \Sigma^\infty_{S^1}(U_+) \to \Sigma^\infty_{S^1}(U) \to \Sigma^\infty_{S^1}(S_+)[1].$$

If $\dim(U) > \dim(S)$ then the assertion follows from the previous Lemma 3.3 applied to the entries $\Sigma^\infty_{S^1}(U_+)$ and $\Sigma^\infty_{S^1}(S_+)[1]$ of the triangle. Now we consider the case $\dim(U) = \dim(S)$. Because of the splitting $s : S \to U$, p is surjective. Since p is smooth of relative dimension zero, it follows that p is étale. Thus, the section s itself is étale. As it is also a closed immersion, the image of s is a component of U, i.e., $U \cong (U')_+$ for some $U' \in \mathrm{Sm}_S$ with $\dim(U') \leq \dim(U)$. The result then follows from the previous Lemma 3.3 applied to $\Sigma^\infty_{S^1}(U'_+)$. ☐

We get the following analogue as a corollary to Proposition 3.1.

Corollary 3.5. *Let $U \in \mathrm{Sm}_S$ be an S-pointed scheme of dimension e. Then for $E \in \mathcal{SH}^s_{S^1 > i+e}(S)$, one has*

$$[\Sigma^\infty_{S^1}(U)[i], L^{\mathbb{A}^1} E] = 0.$$

Proof. The proof is literally the same as that of Corollary 3.4 using Proposition 3.1 instead of Lemma 3.3. ☐

Corollary 3.6. *The $\mathbb{A}^1$-Nisnevich-local t-structure on $\mathcal{SH}^{\mathbb{A}^1}_{S^1}(S)$ is left complete and hence nondegenerate. In particular,*

$$L^{\mathbb{A}^1} \operatorname*{holim}_{n\to\infty}(E_{\leq n}) \sim \operatorname*{holim}_{n\to\infty} L^{\mathbb{A}^1}(E_{\leq n}).$$

Proof. First note that the truncation functors of the $\mathbb{A}^1$-Nisnevich-local t-structure are (after inclusion to $\mathcal{SH}^s_{S^1}$) given by $L^{\mathbb{A}^1}((-)_{\leq n})$ (see Proposition 1.15). Consider a spectrum $E \in \mathrm{Spt}_{S^1}(S)$. To see that $L^{\mathbb{A}^1} E \to \mathrm{holim}_n L^{\mathbb{A}^1}(E_{\leq n})$ is an isomorphism in $\mathcal{SH}^s_{S^1}$, we may equivalently show $\mathrm{holim}_n L^{\mathbb{A}^1}(E_{\geq n}) \simeq 0$, which is implied by the triviality of the group $\pi_i(\operatorname{holim}_n L^{\mathbb{A}^1}(E_{\geq n}))(U)$ for every integer i and every $U \in \mathrm{Sm}_S$. Equivalently, we show that $\pi_i \operatorname{holim}_n(L^{\mathbb{A}^1}(E_{\geq n})(U))$ is trivial. Proposition 3.1 yields $[\Sigma^\infty_{S^1} U_+[i], L^{\mathbb{A}^1}(E_{\geq n})] = 0$ for all integers $n > i + \dim(U)$. Hence, we obtain $\lim_n \pi_i(L^{\mathbb{A}^1}(E_{\geq n})(U)) = 0$. Using Milnor's $\lim^1$-sequence, it follows that the group $\pi_i \operatorname{holim}_n(L^{\mathbb{A}^1}(E_{\geq n})(U))$ is trivial. Indeed, the $\lim^1$-term is trivial as the occurring groups are eventually zero. $\square$

Results for* $\mathbb{P}^1$*-spectra. In this subsection, we show some analogous statements to those of the preceding section for $\mathbb{P}^1$-spectra. The results of this subsection are not needed for the rest of the paper but are of independent interest.

Proposition 3.7. *Let $U \in \mathrm{Sm}_S$ be a scheme of dimension e. For $E \in \mathcal{SH}_{>i+e}(S)$, one has*

$$[\Sigma^\infty_{\mathbb{P}^1}(U_+)[i]\langle q\rangle, E]_{\mathcal{SH}} = 0$$

for all $q \in \mathbb{Z}$.

Proof. Set $\mathcal{F} := [\Sigma^\infty_{\mathbb{P}^1}(U_+)[i]\langle q\rangle, -]_{\mathcal{SH}}$ for abbreviation. By the construction in Proposition 1.15, the class $\mathcal{SH}_{>i+e}$ is generated under extensions, (small) sums and cones from $\mathcal{S}[i+e+1]$. If E is obtained from an extension $E' \to E \to E''$ and $\mathcal{F}$ vanishes on E' and E'', then it also vanishes on E. If E is a (small) sum of objects E'_α on which $\mathcal{F}$ vanishes, we use the homotopy-compactness of $\Sigma^\infty_{\mathbb{P}^1}(U_+)[i]\langle q\rangle$ to conclude that $\mathcal{F}(E) = 0$. Suppose that E sits in a distinguished triangle $E' \to E'' \to E \to E'[1]$ and we know the vanishing of $\mathcal{F}$ on E'' and $E'[1]$. Then we know it on E. Summing up, it suffices to show that $\mathcal{F}(\mathcal{S}[n]) = 0$ for all $n \geq i+e+1$, i.e., $\left[\Sigma^\infty_{\mathbb{P}^1}(U_+)[i]\langle q\rangle, \Sigma^\infty_{\mathbb{P}^1}(V_+)[n]\langle q'\rangle\right]_{\mathcal{SH}} = 0$ for all $V \in \mathrm{Sm}_S$ and $q' \in \mathbb{Z}$. We compute

$$\begin{aligned}
&\left[\Sigma^\infty_{\mathbb{P}^1}(U_+)[i]\langle q\rangle,\ \Sigma^\infty_{\mathbb{P}^1}(V_+)[n]\langle q'\rangle\right]_{\mathcal{SH}} \\
&\qquad\cong \left[\Sigma^\infty_{\mathbb{P}^1}(U_+)[i-n], \Sigma^\infty_{\mathbb{P}^1}(V_+)\langle q'-q\rangle\right]_{\mathcal{SH}} \\
&\qquad\cong \left[\Sigma^\infty_{S^1}(U_+)[i-n], \Omega^\infty_{\mathbb{G}_m}(\Sigma^\infty_{\mathbb{G}_m}\Sigma^\infty_{S^1}(V_+)\langle q'-q\rangle)\right] \\
&\qquad\cong \left[\Sigma^\infty_{S^1}(U_+)[i-n], \operatorname{colim}_k \Omega^k_{\mathbb{G}_m} L^{\mathbb{A}^1}(\Sigma^\infty_{S^1}(V_+) \wedge \mathbb{G}_m^{\wedge(k+q'-q)})\right] \\
&\qquad\cong \operatorname{colim}_k \left[\Sigma^\infty_{S^1}(U_+)[i-n] \wedge \mathbb{G}_m^{\wedge k}, L^{\mathbb{A}^1}(\Sigma^\infty_{S^1}(V_+) \wedge \mathbb{G}_m^{\wedge(k+q'-q)})\right],
\end{aligned}$$

where the last isomorphism is due to compactness of $\Sigma^\infty_{S^1}(U_+)$. Now we use the fact that $\mathbb{A}^1$-Nisnevich-locally there is an equivalence $\mathbb{G}_m \sim \mathbb{P}^1[-1]$. Hence, it suffices to show that for all but finitely many $k \geq 0$ (and in particular, we may assume $k + q' - q \geq 0$), one has

$$\begin{aligned}
&\left[\Sigma^\infty_{S^1}(U_+)[i-n] \wedge \mathbb{G}_m^{\wedge k}, L^{\mathbb{A}^1}(\Sigma^\infty_{S^1}(V_+) \wedge \mathbb{G}_m^{\wedge(k+q'-q)})\right] \\
&\qquad\cong \left[\Sigma^\infty_{S^1}(U_+ \wedge (\mathbb{P}^1)^{\wedge k})[i-n-k], L^{\mathbb{A}^1}\Sigma^\infty_{S^1}(V_+ \wedge \mathbb{G}_m^{\wedge(k+q'-q)})\right] = 0.
\end{aligned}$$

By the same arguments as in the proof of Proposition 3.1, this is implied by the vanishing of the group $\big[\Sigma^\infty_{S^1}(U \times \mathbb{P}^k)_+[i-n-k], L^{\mathbb{A}^1}\Sigma^\infty_{S^1}(V_+ \wedge \mathbb{G}_m^{\wedge(k+q'-q)})\big]$. Since the spectrum $\Sigma^\infty_{S^1}(V_+ \wedge \mathbb{G}_m^{\wedge(k+q'-q)})$ is in $\mathcal{SH}^s_{S^1\geq 0}$, the result follows from Proposition 3.1 as the scheme $U \times \mathbb{P}^k$ has dimension $e+k$. □

Corollary 3.8. *Let S be a noetherian scheme of finite Krull-dimension. Then the homotopy t-structure on the motivic homotopy category $\mathcal{SH}(S)$ is left complete and hence nondegenerate.*

Proof. Let $E \in \mathcal{SH}$. We have to show that the canonical morphism $E \to \operatorname{holim} E_{\leq n}$ is an isomorphism in $\mathcal{SH}$. Equivalently, we may show that $\operatorname{holim} E_{\geq n} \simeq 0$. By [Hovey 1999, Theorem 7.3.1], this is implied by the vanishing of the homotopy classes

$$\big[\Sigma^\infty_{\mathbb{P}^1}(U_+)[i]\langle q\rangle, \operatorname*{holim}_{n\to\infty} E_{\geq n}\big]$$

in $\mathcal{SH}$ for all $U \in \mathrm{Sm}_S$ and all $i, q \in \mathbb{Z}$. Using Milnor's $\lim^1$-sequence as in Corollary 3.6, this, in turn, is implied by the following statement: for all $U \in \mathrm{Sm}_S$ and $i, q \in \mathbb{Z}$ there exists an integer n_0 with $\big[\Sigma^\infty_{\mathbb{P}^1}(U_+)[i]\langle q\rangle, E_{\geq n}\big] = 0$ for all $n \geq n_0$. Setting $n_0 := i + \dim(U)$, this is precisely the preceding Proposition 3.7. □

4. Stalkwise stable $\mathbb{A}^1$-connectivity

In this section, we derive our main connectivity result for homotopy sheaves (i.e., a "stalkwise" connectivity result). We formulate the *shifted stable $\mathbb{A}^1$-connectivity property* on the base scheme and show that this property holds for every Dedekind scheme with infinite residue fields.

Stable $\mathbb{A}^1$-connectivity. Let us recall the following property on a base scheme S introduced by Morel [2005, Definition 1].

Definition 4.1. A noetherian scheme S of dimension d has the *stable $\mathbb{A}^1$-connectivity property*, if for every integer i and every spectrum E in $\mathcal{SH}^s_{S^1\geq i}(S)$, the $\mathbb{A}^1$-localization $L^{\mathbb{A}^1}E$ is contained in $\mathcal{SH}^s_{S^1\geq i}(S)$.

Theorem 4.2 [Morel 2005, Theorem 6.1.8]. *If S is the spectrum of a field, then S has the stable $\mathbb{A}^1$-connectivity property.*

Corollary 4.3. *If S is the spectrum of a field, then $\mathcal{SH}^{\mathbb{A}^1}_{S^1\geq 0}(S) = \mathcal{SH}^{\mathbb{A}^1,\pi}_{S^1\geq 0}(S)$.*

Remark 4.4. Ayoub [2006] gave examples of base schemes that do not have the stable $\mathbb{A}^1$-connectivity property: Let S/k be a connected normal surface over k an algebraically closed field, regular away from one closed singular point s. Let $S' \to S$ be a resolution with exceptional divisor E and let E_{red} be the underlying reduced subscheme. Then by [op. cit., Corollary 3.3], S does not have the stable $\mathbb{A}^1$-connectivity property if $\mathrm{Pic}_{E_{\mathrm{red}}}$ is not $\mathbb{A}^1$-invariant. Here, $\mathrm{Pic}_{E_{\mathrm{red}}}$ is the Nisnevich sheafification of the presheaf $U \mapsto \mathrm{Pic}(U \times_s E_{\mathrm{red}})$ on $\mathrm{Sm}_{k(s)}$. A family of concrete examples for such a surface S (due to Barbieri-Viale) is given in the example in [op. cit., Section 3] as hypersurfaces of $\mathbb{P}^3_k$. Even worse, it follows from [op. cit., Lemma 1.3] that no $\mathbb{P}^n_k$ for $n \geq 3$ has the stable $\mathbb{A}^1$-connectivity property.

Towards stable $\mathbb{A}^1$-connectivity. We saw in Remark 4.4 that connectivity may drop for general base schemes. Thus, it is an interesting question if, for a given base scheme S, there is at least *some* uniform bound r for the loss of connectivity, i.e., for E an i-connected spectrum, the $\mathbb{A}^1$-localization $L^{\mathbb{A}^1}E$ is at least $(i-r)$-connected. In this subsection, we want to discuss a general recipe for finding such a bound, based on Morel's original work over a field.

Proposition 4.5. *Let S be a noetherian scheme of finite Krull-dimension and let $r \geq 0$ be an integer. Let $E \in \mathcal{SH}^s_{S^1 \geq i}(S)$ be a spectrum. Suppose for all $V \in \mathrm{Sm}_S$, all integers $k < i - r$ and all $f \in [\Sigma^\infty_{S^1} V_+, L^{\mathbb{A}^1}E[-k]]$, Nisnevich-locally in V, there exists a Zariski-open $W \hookrightarrow V$ such that*

(1) $f|_{\Sigma^\infty_{S^1} W_+} = 0$, *and*

(2) $\pi_0^{\mathbb{A}^1}(V/W) = 0$.

Then $L^{\mathbb{A}^1}E \in \mathcal{SH}^s_{S^1 \geq i-r}(S)$.

Proof. We may assume $k = 0$. We have to show that the sheaf $\pi_0^{\mathbb{A}^1}(E)$ is trivial. Take a connected scheme $V \in \mathrm{Sm}_S$ with structure morphism $p : V \to S$ and a point $v \in V$. It suffices to show that the Nisnevich stalk of $\pi_0^{\mathbb{A}^1}(E)$ at (V, v) is trivial. Let $f_{(V,v)}$ be a germ in this stalk. Possibly refining (V, v) Nisnevich-locally, we may assume that $f_{(V,v)}$ is induced by an element $f \in [\Sigma^\infty_{S^1} V_+, L^{\mathbb{A}^1}E]$. After a further Nisnevich refinement of (V, v), we find a Zariski-open $W \hookrightarrow V$ satisfying properties (1) and (2). The homotopy cofibre sequence $W_+ \to V_+ \to V/W$ induces a long exact sequence

$$\cdots \to [\Sigma^\infty_{S^1} V/W, L^{\mathbb{A}^1}E] \to [\Sigma^\infty_{S^1} V_+, L^{\mathbb{A}^1}E] \to [\Sigma^\infty_{S^1} W_+, L^{\mathbb{A}^1}E] \to \cdots.$$

Since the restriction of f to $\Sigma^\infty_{S^1} W_+$ is trivial by (1), f is the image of an element in the group $[\Sigma^\infty_{S^1} V/W, L^{\mathbb{A}^1}E]$, i.e., a morphism $g : V/W \to (L^{\mathbb{A}^1}E)_0$ in the (unstable) objectwise (pointed) homotopy category. We want to show the triviality of the germ $f_{(V,v)}$, so it is enough to show that $\pi_0(g)$ is trivial. As the adjunction $(\Sigma^\infty_{S^1}, (-)_0)$ is a Quillen-adjunction for the $\mathbb{A}^1$-Nisnevich-local model, $(L^{\mathbb{A}^1}E)_0$ is $\mathbb{A}^1$-Nisnevich-local. Therefore, the morphism g factors through $h : L^{\mathbb{A}^1}(V/W) \to (L^{\mathbb{A}^1}E)_0$ and it suffices to show that $\pi_0(h)$ is trivial, which follows from assumption (2). $\square$

Now let us discuss how to obtain assumptions (1) and (2) from the previous proposition. We start with assumption (2).

We first recall a well-known construction. The singular functor $\mathrm{Sing} : \mathrm{sPre}(S) \to \mathrm{sPre}(S)$ is given on U-sections by the diagonal of the bisimplicial set $\mathrm{Sing}(F)(U) = F_\bullet(\Delta^\bullet \times U)$, where $\Delta^\bullet$ denotes the standard cosimplicial object in Sm_S; see [Morel and Voevodsky 1999, Section 2.3.2] in the analogous situation for simplicial sheaves. An infinite alternating composition of a Nisnevich-local fibrant replacement functor L^s and Sing yields an $\mathbb{A}^1$-Nisnevich-local fibrant replacement functor $L^{\mathbb{A}^1}$ in the unstable setting. We refer to [loc. cit.] for details of this construction.

Note that for Lemma 4.6 and its Corollary 4.7, we work in the unstable setting.

Lemma 4.6. *Let $V \in \mathrm{Sm}_S$ be an irreducible scheme and $W \hookrightarrow V$ a nonempty open subscheme. Let $Z = (V \setminus W)_{\mathrm{red}}$ be the reduced complement. Suppose, moreover, that each point v of V admits a Nisnevich neighbourhood V' (with pullback W' and Z' to V') and an étale map $p : V' \to \mathbb{A}^1_Y$ in Sm_S with $Z' \to Y$ finite such that*

$$\begin{tikzcd} W' \arrow[r] \arrow[d] & V' \arrow[d, "p"] \\ \mathbb{A}^1_Y \setminus p(Z') \arrow[r] & \mathbb{A}^1_Y \end{tikzcd}$$

is a Nisnevich-distinguished square. Then $\pi_0(\mathrm{Sing}(a_{\mathrm{Nis}}(V/W)))$ is trivial for a_{Nis}, the Nisnevich sheafification.

Proof. We follow the proof of [Morel 2005, Lemma 6.1.4]. Since a simplicial presheaf F has the same 0-simplices as the simplicial presheaf $\mathrm{Sing}(F)$, there is an epimorphism $[(-)_+, F] \twoheadrightarrow [(-)_+, \mathrm{Sing}(F)]$ of presheaves. As Nisnevich sheafification preserves epimorphisms, we get a natural epimorphism $\pi_0(F) \twoheadrightarrow \pi_0(\mathrm{Sing}(F))$ of sheaves. Applying this to the discrete simplicial presheaf $F = a_{\mathrm{Nis}}(V/W)$ and precomposing with the epimorphism $V = a_{\mathrm{Nis}} V \twoheadrightarrow a_{\mathrm{Nis}}(V/W)$ of sheaves, we get a natural epimorphism of sheaves

$$V \twoheadrightarrow a_{\mathrm{Nis}}(V/W) = \pi_0(a_{\mathrm{Nis}}(V/W)) \twoheadrightarrow \pi_0(\mathrm{Sing}(a_{\mathrm{Nis}}(V/W))).$$

Hence, it suffices to show that for each point $v \in V$ there exists a Nisnevich neighbourhood V' such that $V' \to \pi_0(\mathrm{Sing}(a_{\mathrm{Nis}}(V'/W')))$ is zero where $W' := W \times_V V'$.

Let $v \in V$ be a point and choose the Nisnevich neighbourhood V' from the assumption of the proposition. In particular, $V'/W' \to \mathbb{A}^1_Y/(\mathbb{A}^1_Y \setminus Z')$ is an isomorphism of Nisnevich sheaves, so we may assume $V' = \mathbb{A}^1_Y$ with closed $Z' \hookrightarrow \mathbb{A}^1_Y$ and $Z' \to Y$ finite. By finiteness, the morphism $Z' \to \mathbb{A}^1_Y \hookrightarrow \mathbb{P}^1_Y$ is proper and hence a closed immersion. Therefore, we get a diagram

$$\begin{tikzcd} \mathbb{A}^1_Y \setminus Z' \arrow[r] \arrow[d] & \mathbb{A}^1_Y \arrow[r, "q"] \arrow[d, "j"] & a_{\mathrm{Nis}}\big(\mathbb{A}^1_Y/_{\mathbb{A}^1_Y \setminus Z'}\big) \arrow[d, "\cong"] \\ \mathbb{P}^1_Y \setminus Z' \arrow[r] & \mathbb{P}^1_Y \arrow[r, "q'"] & a_{\mathrm{Nis}}\big(\mathbb{P}^1_Y/_{\mathbb{P}^1_Y \setminus Z'}\big) \end{tikzcd}$$

where the right vertical morphism is an isomorphism of Nisnevich sheaves as the left-hand square is a Zariski- and therefore a Nisnevich-distinguished square.

There exists an elementary $\mathbb{A}^1$-homotopy $Y \times \mathbb{A}^1 \to \mathbb{P}^1_Y$ from the zero-section $s_0 : Y \to \mathbb{A}^1_Y \hookrightarrow \mathbb{P}^1_Y$ to the section $s_\infty : Y \to \mathbb{P}^1_Y$ at infinity, and the latter factorizes over $\mathbb{P}^1_Y \setminus Z'$. As by [Morel and Voevodsky 1999, Lemma 2.3.6] the functor Sing turns elementary $\mathbb{A}^1$-homotopies into objectwise homotopies, the maps $\mathrm{Sing}(s_0)$ and $\mathrm{Sing}(s_\infty)$ are identified in the objectwise homotopy category. The morphism $\mathrm{Sing}(Y) \xrightarrow{\sim} \mathrm{Sing}(\mathbb{A}^1_Y)$ is an objectwise weak equivalence by [Morel and Voevodsky 1999, Corollary 2.3.5]. Hence, the composition $\mathrm{Sing}(q' \circ j)$ is the constant map to the point in the objectwise homotopy category. It follows that the same is true for $\mathrm{Sing}(q)$, as desired. Note that the cited arguments of [Morel and Voevodsky 1999] are valid for the objectwise structure on simplicial presheaves. □

Using the epimorphism $\pi_0(\mathrm{Sing}(a_{\mathrm{Nis}}(V/W))) \twoheadrightarrow \pi_0^{\mathbb{A}^1}(V/W)$ of sheaves [Morel and Voevodsky 1999, Corollary 2.3.22], we get:

Corollary 4.7. *In the situation of Lemma 4.6, we have*

$$\pi_0^{\mathbb{A}^1}(V/W) = 0.$$

Remark 4.8. As explained in [Morel 2005, Remark 6.1.5], $\pi_0^{\mathbb{A}^1}(V/W) = 0$ might fail for arbitrary open subschemes $W \hookrightarrow V$. For example, let S be the spectrum of a local ring with closed point $i : \sigma \hookrightarrow S$ and open complement $j : W \hookrightarrow S$. Define $V := S$ and consider the $\mathbb{A}^1$-Nisnevich-local homotopy cofibre sequence $j_\sharp j^*(V/W) \to V/W \to i_* L^{\mathbb{A}^1} i^*(V/W)$ from (1.8). We have $j_\sharp j^*(S/W) \simeq *$ and therefore $L^{\mathbb{A}^1}(V/W) \simeq i_* L^{\mathbb{A}^1}(i^*(V/W))$. On the other hand, $i^*(S/W) \simeq i^*(S)/i^*(W) \simeq \sigma/\varnothing \simeq S^0_\sigma$, and hence $i_* L^{\mathbb{A}^1}(i^*(V/W))$ has nontrivial π_0.

Now we turn to assumption (1) of Proposition 4.5. For the special case that S is the spectrum of a field, this is an observation of Morel [2004, Lemma 3.3.6]. Please note that the extra claim $s^*(W) \neq \varnothing$ in the next lemma excludes the obvious obstacle to assumption (2) of Proposition 4.5 in our application (see Remark 4.8).

Lemma 4.9. *Let S be a noetherian scheme of finite Krull-dimension together with a codimension c point $s \in S$. Let $E \in \mathcal{SH}^s_{S^1 > c}(S)$ be a spectrum. Then for any $V \in \mathrm{Sm}_S$ with $s^*(V) \neq \varnothing$ and any $f \in [\Sigma^\infty_{S^1} V_+, L^{\mathbb{A}^1} E]$ there exists an open subscheme $W \hookrightarrow V$ with $f|_{\Sigma^\infty_{S^1} W_+} = 0$ and $s^*(W) \neq \varnothing$.*

Proof. Let $\eta_Z \in V$ be a generic point of an irreducible component Z of $s^*(V)$. In particular, the ring $\mathcal{O}_{V,\eta_Z}$ has dimension c. We write

$$\mathrm{j} : \mathfrak{U} := \mathrm{Spec}(\mathcal{O}_{V,\eta_Z}) \cong \lim_{j_i : U_i \hookrightarrow V} U_i \to V,$$

where the limit on the right-hand side is indexed by the diagram constituted by the open immersions $j_i : U_i \hookrightarrow V$ with U_i affine and $U_i \cap Z \neq \varnothing$.

Let $p : V \to S$ denote the structural morphism. We have

$$\begin{aligned}
&\operatorname*{colim}_{j_i : U_i \hookrightarrow V} [\Sigma^\infty_{S^1}(U_i \to S)_+, L^{\mathbb{A}^1} E]_{\mathrm{Spt}_{S^1}(S)} \\
&\cong \operatorname*{colim}_{j_i : U_i \hookrightarrow V} [\Sigma^\infty_{S^1} p_\sharp((U_i \to V)_+), L^{\mathbb{A}^1} E]_{\mathrm{Spt}_{S^1}(S)} \\
&\cong \operatorname*{colim}_{j_i : U_i \hookrightarrow V} [p_\sharp \Sigma^\infty_{S^1}(U_i \to V)_+, L^{\mathbb{A}^1} E]_{\mathrm{Spt}_{S^1}(S)} \\
&\cong \operatorname*{colim}_{j_i : U_i \hookrightarrow V} [\Sigma^\infty_{S^1}(U_i \to V)_+, p^*(L^{\mathbb{A}^1} E)]_{\mathrm{Spt}_{S^1}(V)} \\
&\cong \operatorname*{colim}_{j_i : U_i \hookrightarrow V} [\Sigma^\infty_{S^1}(U_i \to V)_+, L^{\mathbb{A}^1}(p^* E)]_{\mathrm{Spt}_{S^1}(V)} && \text{(by Lemma1.9(1))} \\
&\cong \operatorname*{colim}_{j_i : U_i \hookrightarrow V} [j_{i,\sharp} \Sigma^\infty_{S^1}(U_i \to U_i)_+, L^{\mathbb{A}^1}(p^* E)]_{\mathrm{Spt}_{S^1}(V)} && (j_{i,\sharp}((U_i \to U_i)_+) = (U_i \to V)_+) \\
&\cong \operatorname*{colim}_{j_i : U_i \hookrightarrow V} [\Sigma^\infty_{S^1}(U_i \to U_i)_+, j_i^* L^{\mathbb{A}^1}(p^* E)]_{\mathrm{Spt}_{S^1}(U_i)} \\
&\cong [\Sigma^\infty_{S^1}(\mathfrak{U} \to \mathfrak{U})_+, \mathrm{j}^* L^{\mathbb{A}^1}(p^* E)]_{\mathrm{Spt}_{S^1}(\mathfrak{U})} && \text{(by Lemma1.9(2))} \\
&\cong [\Sigma^\infty_{S^1}(\mathfrak{U} \to \mathfrak{U})_+, L^{\mathbb{A}^1}(\mathrm{j}^* p^* E)]_{\mathrm{Spt}_{S^1}(\mathfrak{U})} && \text{(by Lemma1.9(1))}.
\end{aligned}$$

Using the Quillen adjoint pair $(p_\sharp, p^*)$, we see that p^* preserves connectivity. By Lemma 1.9, the same is true for j^*, so $\mathrm{j}^*(p^* E)$ is contained in $\mathcal{SH}^s_{S^1>c}(\mathfrak{U})$. By Proposition 3.1, we get $[\Sigma^\infty_{S^1}\mathfrak{U}_+, \mathrm{j}^* p^* E] = 0$ as the scheme $\mathfrak{U}$ has dimension c.

The restrictions of $f \in [\Sigma^\infty_{S^1} V_+, L^{\mathbb{A}^1} E]$ induce an element of the set

$$\operatorname{colim}_{j_i} [\Sigma^\infty_{S^1} U_{i,+}, L^{\mathbb{A}^1} E] = 0$$

from the left-hand side of the chain of equations above. This means that there exists an open subscheme $W := U_i \hookrightarrow V$ with $W \cap Z \neq \varnothing$ and $f|_{\Sigma^\infty_{S^1} W_+} = 0$. Since $Z \subseteq s^*(V)$, we have $s^*(W) \neq \varnothing$. □

Finally, let us mention that for connectivity results we may restrict ourselves to local base schemes.

Lemma 4.10. *Let S be a noetherian scheme of finite Krull-dimension and let $r \geq 0$ be an integer. Let $E \in \mathcal{SH}^s_{S^1 \geq i}(S)$ be a spectrum. Suppose that for all points $s \in S$ and $\mathfrak{s} : S^h_s \to S$, we have that $L^{\mathbb{A}^1}\mathfrak{s}^* E \in \mathcal{SH}^s_{S^1 \geq i-r}(S^h_s)$. Then $L^{\mathbb{A}^1} E \in \mathcal{SH}^s_{S^1 \geq i-r}(S)$.*

Proof. After shifting, we can assume that $i = r$. We have to show that the sheaf $\pi_0^{\mathbb{A}^1}(E)$ is trivial. It follows from Corollary 1.10 that $\pi_0(L^{\mathbb{A}^1} E)$ is trivial if and only if $\pi_0(\mathfrak{s}^* L^{\mathbb{A}^1} E)$ is trivial for all $s \in S$. Hence the claim follows. □

Shifted stable $\mathbb{A}^1$-connectivity. In order to obtain a uniform bound for the loss of connectivity of a spectrum E, we may restrict ourselves to a local base scheme S by Lemma 4.10. We want to invoke Proposition 4.5. Given a V-section f of a homotopy presheaf of E, we have to search for an open subscheme $W \hookrightarrow V$

fulfilling assumptions (1) and (2) of Proposition 4.5, i.e., $f = 0$ when restricted to W, and V/W is $\mathbb{A}^1$-Nisnevich-locally connected. For the former condition, we want to apply Lemma 4.9. To avoid the obstacle to the latter condition coming from Remark 4.8, we need to take care that W has nonempty fibre over the closed point of S. This point has codimension c equal to the dimension d of S. Thus, again by Lemma 4.9, a natural candidate for a uniform bound on the loss of connectivity of E is $c = d$. This motivates the following definition and question.

Definition 4.11. A noetherian scheme S of dimension d has the *shifted stable $\mathbb{A}^1$-connectivity property*, if for every integer i and every spectrum E in $\mathcal{SH}^s_{S^1 \geq i}(S)$, the $\mathbb{A}^1$-localization $L^{\mathbb{A}^1}E$ is contained in $\mathcal{SH}^s_{S^1 \geq i-d}(S)$.

Question 4.12. Let S be a regular noetherian scheme of dimension d. Does S have the shifted stable $\mathbb{A}^1$-connectivity property?

Remark 4.13. Morel's connectivity theorem (Theorem 4.2 above) provides a positive answer in the case of S the spectrum of a field. In the case of S a Dedekind scheme with all residue fields infinite, we get a positive answer by Theorem 4.16 below. Unfortunately, we do not have a positive or negative answer for more general base schemes.

Remark 4.14. The example of Ayoub discussed in Remark 4.4 does not provide a negative answer to Question 4.12. In fact, Ayoub gave an example of a base scheme S (of dimension ≥ 2) and a spectrum E whose homotopy sheaves $\pi_i^{\mathbb{A}^1}(E)$ are not *strictly* $\mathbb{A}^1$-invariant. The latter property is a consequence of the *nonshifted* stable $\mathbb{A}^1$-connectivity property, i.e., the property that $\mathbb{A}^1$-localization does not lower the connectivity at all. However, the proof (see [Morel 2005, Theorem 6.2.7]) that the nonshifted stable $\mathbb{A}^1$-connectivity property implies strictly $\mathbb{A}^1$-invariant homotopy sheaves does not carry over from the nonshifted to the shifted stable connectivity property of the base.

At least, a positive answer to Question 4.12 would follow from $\mathbb{A}^1$-invariance of $\mathbb{A}^1$-homotopy sheaves $\pi_k^{\mathbb{A}^1}(E)$ as the following proposition shows. Note that the property of a Nisnevich sheaf to be strictly $\mathbb{A}^1$-invariant is a stronger property than just being $\mathbb{A}^1$-invariant (see Remark 4.14).

Proposition 4.15. *Let S be a noetherian scheme of dimension d. Let i be an integer and $E \in \mathcal{SH}^s_{\geq i}(S)$ be such that the sheaf $\pi_k^{\mathbb{A}^1}(E)$ is $\mathbb{A}^1$-invariant for all integers $k < i - d$. Then $L^{\mathbb{A}^1}E \in \mathcal{SH}^s_{S^1 \geq i-d}(S)$.*

Proof. First note that for any open immersion $j : S' \hookrightarrow S$ the functor j^* preserves $\mathbb{A}^1$-invariance of (simplicial) presheaves and $j^*\pi_0 \cong \pi_0 j^*$. In particular, our assumptions on the spectrum are stable under restriction to open subschemes of the base. Let E be a spectrum in $\mathcal{SH}^s_{S^1 > d}$ with $\mathbb{A}^1$-invariant homotopy sheaves $\pi_k^{\mathbb{A}^1}(E)$ in degrees $k \leq 0$. To prove Proposition 4.15, it is again enough to show that $\pi_0^{\mathbb{A}^1}(E)$

is trivial. We argue by induction on the dimension d of the base S. The case $d = 0$ is Theorem 4.2. Let $d > 0$. By Corollary 1.10, we may assume that S is local with closed point $i : \sigma \hookrightarrow S$. Take a connected scheme $V \in \mathrm{Sm}_S$ with structure morphism $p : V \to S$ and a point $v \in V$. It suffices to show that the Nisnevich stalk of $\pi_0^{\mathbb{A}^1}(E)$ at (V, v) is trivial. By the induction hypothesis, we may assume that v lies in the fibre over σ as the open complement $S \setminus \sigma$ has Krull-dimension strictly smaller than d. Moreover, we may assume that $i^*(V)$ is connected. Let $f_{(V,v)}$ be a germ in this stalk. We have to show that $f_{(V,v)}$ is trivial. After possibly refining (V, v) Nisnevich-locally, we may assume that $f_{(V,v)}$ is induced by an element $f \in [\Sigma^\infty_{S^1} V_+, L^{\mathbb{A}^1} E]$. By Lemma 4.9, there exists an open subscheme $W \hookrightarrow V$ with $f|_{\Sigma^\infty_{S^1} W_+} = 0$ and $i^*(W) \neq \varnothing$. Clearly, we may assume that $v \notin W$. The cofibre sequence $W_+ \to V_+ \to V/W$ induces an exact sequence

$$0 \to \tilde{\pi}_0(L^{\mathbb{A}^1} E)(V/W) \to \tilde{\pi}_0(L^{\mathbb{A}^1} E)(V) \to \tilde{\pi}_0(L^{\mathbb{A}^1} E)(W)$$

of homotopy sheaves. Here we write $\tilde{\pi}_0(L^{\mathbb{A}^1} E)(V/W)$ for $\mathrm{Hom}(V/W, \tilde{\pi}_0(L^{\mathbb{A}^1} E))$. Since the restriction of f to W is trivial, it suffices to show that $\tilde{\pi}_0(L^{\mathbb{A}^1} E)(V/W)$ is trivial. The $\mathbb{A}^1$-Nisnevich-local homotopy cofibre sequence

$$j_\sharp j^*(V/W) \to V/W \to i_* L^{\mathbb{A}^1} i^*(V/W)$$

from (1.8) induces a long exact sequence

$$\cdots \to [i_* L^{\mathbb{A}^1} i^*(V/W), \pi_0^{\mathbb{A}^1}(E)] \to [V/W, \pi_0^{\mathbb{A}^1}(E)] \to [j_\sharp j^*(V/W), \pi_0^{\mathbb{A}^1}(E)]$$

by the $\mathbb{A}^1$-Nisnevich-local fibrancy of $\pi_0^{\mathbb{A}^1}(E) = \pi_0(L^{\mathbb{A}^1} E)$. For the latter, note that a sheaf considered as a discrete simplicial presheaf is Nisnevich-locally fibrant. The right-hand side equals $[j^*(V/W), j^*\pi_0^{\mathbb{A}^1}(E)]$, and $j^*\pi_0^{\mathbb{A}^1}(E) \cong \pi_0^{\mathbb{A}^1}(j^*E)$ is trivial by induction. The triviality of the set on the left-hand side follows from the triviality of $\pi_0(i_* L^{\mathbb{A}^1} i^*(V/W))$. By [Spitzweck 2014, Proposition 4.2], the latter is zero if $\pi_0^{\mathbb{A}^1}(i^*(V/W)) = 0$. Since $i^*(V)$ is irreducible and $i^*(W)$ is nonempty, we conclude by [Morel 2005, Lemma 6.1.4]. □

The one-dimensional case. Using the Gabber presentation given by Theorem 2.4, we can give a positive answer to Question 4.12 for a Dedekind scheme S with infinite residue fields.

Theorem 4.16. *Let S be a Dedekind scheme and assume that all of its residue fields are infinite. Then S has the shifted stable $\mathbb{A}^1$-connectivity property: $E \in \mathcal{SH}^s_{S^1 \geq i}(S)$ implies $L^{\mathbb{A}^1} E \in \mathcal{SH}^s_{S^1 \geq i-1}(S)$.*

Proof. By Lemma 4.10, we may assume that S is henselian local of dimension ≤ 1 with infinite residue field and closed point σ. The case of dimension zero is covered by Theorem 4.2. Hence we may assume that S is the spectrum of a henselian discrete valuation ring. We want to apply Proposition 4.5. Consider an element $f \in [\Sigma^{\infty}_{S^1} V_+, L^{\mathbb{A}^1} E]$ for $V \in \mathrm{Sm}_S$. We may assume that $\sigma^*(V) \neq \varnothing$, since otherwise we argue as Morel in the proof of Theorem 4.2. By Lemma 4.9 applied to the closed point σ of S, we find an open subscheme $W \hookrightarrow V$ such that $f_{\Sigma^{\infty}_{S^1} W_+} = 0$ and $\sigma^*(W) \neq \varnothing$. Let $i : Z \hookrightarrow V$ be the reduced closed complement of W. In particular, $Z_\sigma \neq V_\sigma$. By Theorem 2.1, the conditions of Lemma 4.6 are fulfilled, and we get the second assumption $\pi_0^{\mathbb{A}^1}(V/W) = 0$ of Proposition 4.5 as well. □

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