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Topological K-theory of affine Hecke algebras

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Let $\mathcal{H}(\mathcal{R}, q)$ be an affine Hecke algebra with a positive parameter function q. We are interested in the topological K-theory of its C^*-completion $C_r^*(\mathcal{R}, q)$. We prove that $K_*(C_r^*(\mathcal{R}, q))$ does not depend on the parameter q, solving a long-standing conjecture of Higson and Plymen. For this we use representation-theoretic methods, in particular elliptic representations of Weyl groups and Hecke algebras.

Thus, for the computation of these K-groups it suffices to work out the case $q = 1$. These algebras are considerably simpler than for $q \neq 1$, just crossed products of commutative algebras with finite Weyl groups. We explicitly determine $K_*(C_r^*(\mathcal{R}, q))$ for all classical root data $\mathcal{R}$. This will be useful for analyzing the K-theory of the reduced C^*-algebra of any classical p-adic group.

For the computations in the case $q = 1$, we study the more general situation of a finite group Γ acting on a smooth manifold M. We develop a method to calculate the K-theory of the crossed product $C(M) \rtimes \Gamma$. In contrast to the equivariant Chern character of Baum and Connes, our method can also detect torsion elements in these K-groups.

Introduction

Affine Hecke algebras can be realized in two completely different ways. On the one hand, they are deformations of group algebras of affine Weyl groups, and on the other hand they appear as subalgebras of group algebras of reductive p-adic groups. Via the second interpretation, affine Hecke algebras (AHAs) have proven

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very useful in the representation theory of such groups. This use is in no small part due to their explicit construction in terms of root data, which makes them amenable to concrete calculations.

This paper is motivated by our desire to understand and compute the (topological) K-theory of the reduced C^*-algebra $C_r^*(G)$ of a reductive p-adic group G. This is clearly related to the representation theory of G. For instance, when G is semisimple, every discrete series G-representation gives rise to a one-dimensional direct summand in the K-theory of $C_r^*(G)$.

The problem can be transferred to AHAs in the following way. By the Bernstein decomposition, the Hecke algebra of G can be written as a countable direct sum of two-sided ideals:

$$\mathcal{H}(G) = \bigoplus\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}.$$

Borel [1976] and Iwahori and Matsumoto [1965] have shown that one particular summand, say $\mathcal{H}(G)^{\mathrm{IM}}$, is Morita equivalent to an AHA, say $\mathcal{H}(\mathcal{R}, q)^{\mathrm{IM}}$. It is expected that all other summands $\mathcal{H}(G)^{\mathfrak{s}}$ are also Morita equivalent to AHAs, or to closely related algebras. Indeed, this has been proven in many cases; see [Aubert et al. 2017a, §2.4] for an overview.

The reduced C^*-algebra of G is a completion of $\mathcal{H}(G)$, and it admits an analogous Bernstein decomposition

$$C_r^*(G) = \bigoplus\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} C_r^*(G)^{\mathfrak{s}},$$

where $C_r^*(G)^{\mathfrak{s}}$ is the closure of $\mathcal{H}(G)^{\mathfrak{s}}$ in $C_r^*(G)$. By [Bushnell et al. 2011], the Morita equivalence $\mathcal{H}(G)^{\mathrm{IM}} \sim_M \mathcal{H}(\mathcal{R}, q)$ extends to a Morita equivalence between $C_r^*(G)^{\mathrm{IM}}$ and the natural C^*-completion of $\mathcal{H}(\mathcal{R}, q)^{\mathrm{IM}}$. Again it can be expected that every summand $C_r^*(G)^{\mathfrak{s}}$ is Morita equivalent to the C^*-completion $C_r^*(\mathcal{R}, q)^{\mathfrak{s}}$ of some AHA $\mathcal{H}(\mathcal{R}, q)^{\mathfrak{s}}$. However, this is currently not yet proven in several cases where the Morita equivalence is known on the algebraic level. We will return to this issue in a subsequent paper. Assuming it for the moment, we get

$$K_*(C_r^*(G)) \cong \bigoplus\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} K_*(C_r^*(\mathcal{R}, q)^{\mathfrak{s}}).$$

The left-hand side figures in the Baum–Connes conjecture for reductive p-adic groups [Baum et al. 1994]. For applications to the Baum–Connes conjecture for algebraic groups over local fields, it would be useful to understand $K_*(C_r^*(G))$ better, in particular its torsion subgroup. Namely, from the work of Kasparov [1988] it is known that for many groups G the Baum–Connes assembly map is injective, and that its image is a direct summand of $K_*(C_r^*(G))$. There exist methods [Solleveld 2009, §3.4] which enable one to prove that the assembly map becomes an isomorphism after tensoring its domain and range by $\mathbb{Q}$, but which say little about the torsion elements in the K-groups. If one knew in advance that $K_*(C_r^*(G))$ is torsion-free, then one could prove instances of the Baum–Connes conjecture with such methods.

To construct an affine Hecke algebra, we use a root datum $\mathcal{R}$ in a lattice X. These give a Weyl group $W = W(\mathcal{R})$ and an extended affine Weyl group $W^e = X \rtimes W$. As parameters we take a tuple of nonzero complex numbers $q = (q_i)_i$. The AHA $\mathcal{H}(\mathcal{R}, q)$ is a deformation of the group algebra $\mathbb{C}[W^e]$, in the following sense: as a vector space it is $\mathbb{C}[W^e]$, with a multiplication rule depending algebraically on q, such that $\mathcal{H}(\mathcal{R}, 1) = \mathbb{C}[W^e]$. See Section 1C for the precise definition. To get a nice C^*-completion $C_r^*(\mathcal{R}, q)$, we must assume that q is positive, that is, $q_i \in \mathbb{R}_{>0}$ for all i. For $q = 1$ the C^*-completion can be described easily:

$$C_r^*(\mathcal{R}, 1) = C_r^*(W^e) = C(T_{\mathrm{un}}) \rtimes W,$$

where $T_{\mathrm{un}} = \mathrm{Hom}_{\mathbb{Z}}(X, S^1)$ is a compact torus.

All AHAs obtained from reductive p-adic groups G have rather special parameters: there are $n_i \in \mathbb{Z}_{\geq 0}$ such that $q_i = p^{n_i/2}$, where p is the characteristic of the local nonarchimedean field underlying G. Thus the realization of AHAs via root data admits more parameters than the realization as subalgebras of $\mathcal{H}(G)$. In particular the algebras $\mathcal{H}(\mathcal{R}, q)$ admit continuous parameter deformations, whereas the AHAs from reductive p-adic groups do not, since the prime powers $p^{n/2}$ are discrete in $\mathbb{R}_{>0}$.

In fact, for fixed $\mathcal{R}$ the family $C_r^*(\mathcal{R}, q)$, with varying positive q, form a continuous field of C^*-algebras. For a given $q \neq 1$ we have the half-line of parameters $q^\epsilon = (q_i^\epsilon)_i$ with $\epsilon \in \mathbb{R}_{\geq 0}$. It is known from [Solleveld 2012a, Theorem 4.4.2] that there exists a family of C^*-homomorphisms

$$\zeta_\epsilon : C_r^*(\mathcal{R}, q^\epsilon) \to C_r^*(\mathcal{R}, q), \quad \epsilon \geq 0,$$

such that ζ_ϵ is an isomorphism for all $\epsilon > 0$ and depends continuously on $\epsilon \in \mathbb{R}_{\geq 0}$. Via a general deformation principle, this yields a canonical homomorphism

$$K_*(C_r^*(W^e)) = K_*(C_r^*(\mathcal{R}, q^0)) \to K_*(C_r^*(\mathcal{R}, q)). \tag{0.1}$$

Loosely speaking, the construction goes as follows. Take a projection p_0 (or a unitary u_0) in a matrix algebra $M_n(C_r^*(W^e)) = M_n(C_r^*(\mathcal{R}, q^0))$. For $\epsilon > 0$ small, we can apply holomorphic functional calculus to p_0 to produce a new projection $p_\epsilon \in M_n(C_r^*(\mathcal{R}, q^\epsilon))$ (or a unitary u_ϵ). Then (0.1) sends $[p_0] \in K_0(C_r^*(\mathcal{R}, q^0))$ to the image of p_ϵ, and $u_0 \in K_1(C_r^*(\mathcal{R}, q^0))$ to the image of u_ϵ, under the isomorphism $M_n(C_r^*(\mathcal{R}, q^\epsilon)) \cong M_n(C_r^*(\mathcal{R}, q))$.

Actually, more is true: by [Solleveld 2012a, Lemma 5.1.2] the map $K_*(\zeta_0)$ equals (0.1). Furthermore, by [Solleveld 2012a, Theorem 5.1.4] ζ_0 induces an isomorphism

$$K_*(C_r^*(\mathcal{R}, q^0)) \otimes_{\mathbb{Z}} \mathbb{C} \to K_*(C_r^*(\mathcal{R}, q)) \otimes_{\mathbb{Z}} \mathbb{C}.$$

In view of the aforementioned relation with the Baum–Connes conjecture for p-adic groups, we also want to understand the torsion parts of these K-groups. We will prove:

Theorem 1 (see Theorem 2.11). *The map (0.1) is a canonical isomorphism*

$$K_*(C_r^*(\mathcal{R},1)) \to K_*(C_r^*(\mathcal{R},q)).$$

This theorem was conjectured first by Higson and Plymen (see [Plymen 1993, 6.4] and [Baum et al. 1994, 6.21]), at least when all parameters q_i are equal. It is similar to the Connes–Kasparov conjecture for Lie groups; see [Baum et al. 1994, Sections 4–6] for more background. Independently, Opdam [2004, Section 1.0.1] conjectured Theorem 1 for unequal parameters.

Unfortunately it is unclear how Theorem 1 could be proven by purely noncommutative geometric means. The search for an appropriate technique was a major drive behind the author's Ph.D. project (2002–2006), and partial results appeared already in his Ph.D. thesis [Solleveld 2007]. At that time, we hoped to derive representation consequences from a K-theoretic proof of Theorem 1. But so far, such a proof remains elusive.

In the meantime, substantial progress has been made in the representation theory of Hecke algebras; see in particular [Opdam and Solleveld 2010; Ciubotaru and Opdam 2015; Ciubotaru et al. 2014]. This enables us to turn things around (compared to 2004); now we can use representation theory to study the K-theory of $C_r^*(\mathcal{R},q)$.

Given an algebra or group A, let $\mathrm{Mod}_f(A)$ be the category of finite length A-modules, and let $R_{\mathbb{Z}}(A)$ be the Grothendieck group thereof. We deduce Theorem 1 from the following:

Theorem 2 (see Theorem 1.52). *The map*

$$\mathrm{Mod}_f(C_r^*(\mathcal{R},q)) \to \mathrm{Mod}(C_r^*(W^e)) : \pi \mapsto \pi \circ \zeta_0$$

induces $\mathbb{Z}$-linear bijections

$$R_{\mathbb{Z}}(C_r^*(\mathcal{R},q)) \to R_{\mathbb{Z}}(C_r^*(W^e)), \qquad R_{\mathbb{Z}}(\mathcal{H}(\mathcal{R},q)) \to R_{\mathbb{Z}}(W^e).$$

A substantial part of the proof of Theorem 2 boils down to representations of the finite Weyl group W. Following [Reeder 2001], we study the group $\overline{R_{\mathbb{Z}}}(W)$ of elliptic representations, that is, $R_{\mathbb{Z}}(W)$ modulo the subgroup spanned by all representations induced from proper parabolic subgroups of W. First we show that $\overline{R_{\mathbb{Z}}}(W)$ is always torsion-free (Theorem 1.12). Then we compare it with the analogous group of elliptic representations of $\mathcal{H}(\mathcal{R},q)$, which leads to Theorem 2.

Having established the general framework, we set out to compute $K_*(C_r^*(\mathcal{R},q))$ explicitly, for some root data $\mathcal{R}$ associated to well-known groups. By Theorem 1, we only have to consider one q for each $\mathcal{R}$. In most examples, the easiest is to take $q = 1$. Then we must determine

$$K_r(C_r^*(\mathcal{R}, 1)) = K_*(C(T_{\text{un}}) \rtimes W) \cong K_W^*(T_{\text{un}}),$$

where the right-hand side denotes the W-equivariant K-theory of the compact Hausdorff space T_{un}. Let $T_{\text{un}} /\!/ W$ be the extended quotient. Of course, the equivariant Chern character from [Baum and Connes 1988] gives a natural isomorphism

$$K_W^*(T_{\text{un}}) \otimes_{\mathbb{Z}} \mathbb{C} \to H^*(T_{\text{un}} /\!/ W; \mathbb{C}).$$

But this does not suffice for our purposes, because we are particularly interested in the torsion subgroup of $K_W^*(T_{\text{un}})$. Remarkably, that appears to be quite difficult to determine, already for cyclic groups acting on tori [Langer and Lück 2012]. Using equivariant cohomology, we develop a technique to facilitate the computation of $K_*(C(\Sigma) \rtimes W)$ for any finite group W acting smoothly on a manifold Σ. With extra conditions it can be made more explicit:

Theorem 3 (see Theorem 2.45). *Suppose that every isotropy group W_t $(t \in \Sigma)$ is a Weyl group, and that $H^*(\Sigma /\!/ W; \mathbb{Z})$ is torsion-free. Then*

$$K_*(C(\Sigma) \rtimes W) \cong H^*(\Sigma /\!/ W; \mathbb{Z}).$$

We note that $H^*(\Sigma /\!/ W; \mathbb{Z})$ can be computed relatively easily. Theorem 3 can be applied to all classical root data, and to some others as well. Let us summarize the outcome of our computations.

Theorem 4. *Let $\mathcal{R}$ be a root datum of type* GL_n, SL_n, PGL_n, SO_n, Sp_{2n} *or G_2. Let q be any positive parameter function for $\mathcal{R}$. Then $K_*(C_r^*(\mathcal{R}, q))$ is a free abelian group, whose rank is given explicitly in Section 3.*

Whether or not torsion elements can pop up in $K_*(C_r^*(\mathcal{R}, q))$ for other root data remains to be seen. In view of our results it does not seem very likely, but we do not have a general principle to rule it out.

1. Representation theory

1A. ***Weyl groups.*** In this first subsection we show that the representation ring $R_{\mathbb{Z}}(W)$ of any finite Weyl group W is the direct sum of two parts: the subgroup spanned by representations induced from proper parabolic subgroups, and an elliptic part $\overline{R_{\mathbb{Z}}}(W)$. We exhibit a $\mathbb{Z}$-basis of $\overline{R_{\mathbb{Z}}}(W)$ in terms of the Springer correspondence. These results rely mainly on case-by-case considerations in complex simple groups.

Let $\mathfrak{a}$ be a finite-dimensional real vector space and let $\mathfrak{a}^*$ be its dual. Let $Y \subset \mathfrak{a}$ be a lattice and $X = \mathrm{Hom}_{\mathbb{Z}}(Y, \mathbb{Z}) \subset \mathfrak{a}^*$ the dual lattice. Let

$$\mathcal{R} = (X, R, Y, R^\vee, \Delta) \tag{1.1}$$

be a based root datum. Thus, R is a reduced root system in X, $R^\vee \subset Y$ is the dual root system, Δ is a basis of R and the set of positive roots is denoted R^+. Furthermore, we are given a bijection $R \to R^\vee$, $\alpha \mapsto \alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and such that the corresponding reflections $s_\alpha : X \to X$ and $s_\alpha^\vee : Y \to Y$ stabilize R and $R^\vee$, respectively. We do not assume that R spans $\mathfrak{a}^*$. The reflections s_α generate the Weyl group $W = W(R)$ of R, and $S_\Delta := \{s_\alpha : \alpha \in \Delta\}$ is the collection of simple reflections.

For a set of simple roots $P \subset \Delta$ we let R_P be the root system they generate, and we let $W_P = W(R_P)$ be the corresponding parabolic subgroup of W.

Let $R_\mathbb{Z}(W)$ be the Grothendieck group of the category of finite-dimensional complex W-representations, and write $R_\mathbb{C}(W) = \mathbb{C} \otimes_\mathbb{Z} R_\mathbb{Z}(W)$. For any $P \subset \Delta$ the induction functor $\mathrm{ind}_{W_P}^W$ gives linear maps $R_\mathbb{Z}(W_P) \to R_\mathbb{Z}(W)$ and $R_\mathbb{C}(W_P) \to R_\mathbb{C}(W)$. In this subsection we are mainly interested in the abelian group of "elliptic W-representations"

$$\overline{R_\mathbb{Z}}(W) = R_\mathbb{Z}(W) \Big/ \sum_{P \subsetneq \Delta} \mathrm{ind}_{W_P}^W (R_\mathbb{Z}(W_P)). \tag{1.2}$$

In the literature [Reeder 2001; Ciubotaru et al. 2014], one more often encounters the vector space

$$\overline{R_\mathbb{C}}(W) = R_\mathbb{C}(W) \Big/ \sum_{P \subsetneq \Delta} \mathrm{ind}_{W_P}^W (R_\mathbb{C}(W_P)).$$

Recall that an element $w \in W$ is called elliptic if it fixes only the zero element of $\mathrm{Span}_\mathbb{R}(R)$, or equivalently if it does not belong to any proper parabolic subgroup of W. It was shown in [Reeder 2001, Proposition 2.2.2] that $\overline{R_\mathbb{C}}(W)$ is naturally isomorphic to the space of all class functions on W supported on elliptic elements. In particular, $\dim_\mathbb{C} \overline{R_\mathbb{C}}(W)$ is the number of elliptic conjugacy classes in W.

In [Ciubotaru et al. 2014], $\overline{R_\mathbb{Z}}(W)$ is defined as the subgroup of $\overline{R_\mathbb{C}}(W)$ generated by the W-representations. So in that work it is by definition a lattice. If $\overline{R_\mathbb{Z}}(W)$ (in our sense) is torsion-free, then it can be identified with the subgroup of $\overline{R_\mathbb{C}}(W)$ to which it is naturally mapped. For our purposes it will be essential to stick to the definition (1.2) and to use some results from [Ciubotaru et al. 2014]. Therefore we want to prove that (1.2) is always a torsion-free group.

In the analysis we will make ample use of Springer's construction of representations of Weyl groups, and of Reeder's results [2001]. Let G be a connected reductive complex group with a maximal torus T such that $R \cong R(G, T)$ and $W \cong W(G, T)$. For $u \in G$ let $\mathcal{B}^u = \mathcal{B}^u_G$ be the complex variety of Borel subgroups of G containing u. The group $Z_G(u)$ acts on $\mathcal{B}^u$ by conjugation, and that induces an action of $A_G(u) := \pi_0(Z_G(u)/Z(G))$ on the cohomology of $\mathcal{B}^u$. For a pair (u, ρ) with $u \in G$ unipotent and $\rho \in \mathrm{Irr}(A_G(u))$ we define

$$\begin{aligned} H(u,\rho) &= \mathrm{Hom}_{A_G(u)}(\rho, H^*(\mathcal{B}^u;\mathbb{C})), \\ \pi(u,\rho) &= \mathrm{Hom}_{A_G(u)}(\rho, H^{\mathrm{top}}(\mathcal{B}^u;\mathbb{C})), \end{aligned} \tag{1.3}$$

where top indicates the highest dimension in which the cohomology is nonzero, namely the dimension of $\mathcal{B}^u$ as a real variety. Let us call ρ geometric if $\pi(u,\rho) \neq 0$. Springer [1978] proved that

- $W \times A_G(u)$ acts naturally on $H^i(\mathcal{B}^u;\mathbb{C})$ for each $i \in \mathbb{Z}_{\geq 0}$,
- $\pi(u,\rho)$ is an irreducible W-representation whenever it is nonzero,
- this gives a bijection between $\mathrm{Irr}(W)$ and the G-conjugacy classes of pairs (u,ρ) with $u \in G$ unipotent and $\rho \in \mathrm{Irr}(A_G(u))$ geometric.

It follows from a result of Borho and MacPherson [1981] that the W-representations $H(u,\rho)$, parametrized by the same data (u,ρ), also form a basis of $R_{\mathbb{Z}}(W)$; see [Reeder 2001, Lemma 3.3.1]. Moreover, $\pi(u,\rho)$ appears with multiplicity one in $H(u,\rho)$.

Example 1.4.

- Type A. Only the n-cycles in $W = S_n$ are elliptic, and they form one conjugacy class. The only quasidistinguished unipotent class in $\mathrm{GL}_n(\mathbb{C})$ is the regular unipotent class. Then $A_{\mathrm{GL}_n(\mathbb{C})}(u_{\mathrm{reg}}) = 1$ for every regular unipotent element u_{reg} and $H(u_{\mathrm{reg}}, \mathrm{triv}) = H^0(\mathcal{B}^{u_{\mathrm{reg}}};\mathbb{C})$ is the sign representation of S_n (with our convention for the Springer correspondence).
- Types B and C. The elliptic classes in $W(B_n) = W(C_n) \cong S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ are parametrized by partitions of n. We will write them down explicitly as $\sigma(\varnothing,\lambda)$ with $\lambda \vdash n$ in (3.26).
- Type D. The elliptic classes in $W(D_n) = S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n_{\mathrm{ev}}$ are precisely the elliptic classes of $W(B_n)$ that are contained in $W(D_n)$. They can be parametrized by partitions $\lambda \vdash n$ such that λ has an even number of terms.
- Type G_2. There are three elliptic classes in $W(G_2) = D_6$: the rotations of order two, of order three and of order six. The quasidistinguished unipotent classes in $G_2(\mathbb{C})$ are the regular and the subregular class.

  We have $A_G(u_{\mathrm{reg}}) = 1$ and $H(u_{\mathrm{reg}}, \mathrm{triv}) = \pi(u_{\mathrm{reg}}, \mathrm{triv})$ is the sign representation of D_6. For u subregular $A_G(u) \cong S_3$, and the sign representation of $A_G(u)$ is not geometric. For ρ the two-dimensional irreducible representation of $A_G(u)$, $\pi(u,\rho) = H(u,\rho)$ is the character of $W(G_2)$ which is 1 on the reflections for long roots and -1 on the reflections for short roots. Furthermore $\pi(u,\mathrm{triv})$ is the standard two-dimensional representation of D_6 and $H(u,\mathrm{triv})$ is the direct sum of $\pi(u,\mathrm{triv})$ and the sign representation.

For a subset $P \subset \Delta$ let G_P be the standard Levi subgroup of G generated by T and the root subgroups for roots $\alpha \in R_P$. The irreducible representations of $W_P = W(G_P, T)$ are parametrized by G_P-conjugacy classes of pairs (u_P, ρ_P) with

$u_P \in G_P$ unipotent and $\rho_P \in \mathrm{Irr}(A_{G_P}(u_P))$ geometric, and the W_P-representations $H_P(u_P, \rho_P)$ form another basis of $R_{\mathbb{Z}}(W_P)$.

Recall from [Reeder 2001, §3.2] that $A_{G_P}(u_P)$ can be regarded as a subgroup of $A_G(u_P)$. By [Kato 1983, Proposition 2.5 and 6.2],

$$\mathrm{ind}_{W_P}^{W}(H^i(\mathcal{B}_{G_P}^{u_P};\mathbb{C})) \cong H^i(\mathcal{B}^{u_P};\mathbb{C}) \quad \text{as } W \times A_G(u_P)\text{-representations.} \tag{1.5}$$

It follows that for any (u_P, ρ_P) as above there are natural isomorphisms

$$\begin{aligned}\mathrm{ind}_{W_P}^{W}(H_P(u_P,\rho_P)) &\cong \mathrm{Hom}_{A_{G_P}(u_P)}(\rho_P, H^*(\mathcal{B}^{u_P};\mathbb{C})) \\ &\cong \bigoplus_{\rho\in\mathrm{Irr}(A_G(u_P))} \mathrm{Hom}_{A_{G_P}(u_P)}(\rho_P,\rho)\otimes H(u_P,\rho). \end{aligned} \tag{1.6}$$

For a unipotent conjugacy class $\mathcal{C} \subset G$ and $P \subset \Delta$, let $R_{\mathbb{Z}}(W_P, \mathcal{C})$ be the subgroup of $R_{\mathbb{Z}}(W_P)$ generated by the $H_P(u_P, \rho_P)$ with $u_P \in G_P \cap \mathcal{C}$ and $\rho_P \in \mathrm{Irr}(A_{G_P}(u_P))$. (Notice that $G_P \cap \mathcal{C}$ can consist of zero, one or more conjugacy classes.) In view of (1.6) we can define

$$\overline{R_{\mathbb{Z}}}(W,\mathcal{C}) = R_{\mathbb{Z}}(W,\mathcal{C}) \Big/ \sum_{P\subsetneq\Delta} \mathrm{ind}_{W_P}^{W}(R_{\mathbb{Z}}(W_P,\mathcal{C})).$$

We obtain a decomposition as in [Reeder 2001, §3.3]:

$$\overline{R_{\mathbb{Z}}}(W) = \bigoplus_{\mathcal{C}} \overline{R_{\mathbb{Z}}}(W,\mathcal{C}). \tag{1.7}$$

Following [Reeder 2001], we also define elliptic representation theories for the component groups $A_G(u)$. For $u, u_P \in \mathcal{C}$ the groups $A_G(u)$ and $A_G(u_P)$ are isomorphic. In general the isomorphism is not natural, but it is canonical up to inner automorphisms. This gives a natural isomorphism $R_{\mathbb{Z}}(A_G(u)) \cong R_{\mathbb{Z}}(A_G(u_P))$, which enables us to write

$$\overline{R_{\mathbb{Z}}}(A_G(u)) = R_{\mathbb{Z}}(A_G(u)) \Big/ \sum_{P\subsetneq\Delta,\, u_P\in\mathcal{C}\cap G_P} \mathrm{ind}_{A_{G_P}(u_P)}^{A_G(u_P)}\big(R_{\mathbb{Z}}(A_{G_P}(u_P))\big). \tag{1.8}$$

For any $u_P, u'_P \in \mathcal{C} \cap G_P$ there is a natural isomorphism

$$\mathrm{ind}_{A_{G_P}(u_P)}^{A_G(u_P)}\big(R_{\mathbb{Z}}(A_{G_P}(u_P))\big) \cong \mathrm{ind}_{A_{G_P}(u'_P)}^{A_G(u'_P)}\big(R_{\mathbb{Z}}(A_{G_P}(u'_P))\big),$$

so on the right-hand side of (1.8) it actually suffices to use only one u_P whenever $\mathcal{C} \cap G_P$ is nonempty.

Let $R_{\mathbb{Z}}^{\circ}(A_G(u))$ be the subgroup of $R_{\mathbb{Z}}(A_G(u))$ generated by the geometric irreducible $A_G(u)$-representations. By [Reeder 2001, §10],

$$\mathrm{ind}_{A_G(u_P)}^{A_G(u)}\big(R_{\mathbb{Z}}^{\circ}(A_{G_P}(u_P))\big) \subset R_{\mathbb{Z}}^{\circ}(A_G(u)).$$

Using this we can define

$$\overline{R_{\mathbb{Z}}^{\circ}}(A_G(u)) = R_{\mathbb{Z}}^{\circ}(A_G(u)) \Big/ \sum_{P \subsetneq \Delta,\, u_P \in \mathcal{C} \cap G_P} \mathrm{ind}_{A_{G_P}(u_P)}^{A_G(u_P)} \big(R_{\mathbb{Z}}^{\circ}(A_{G_P}(u_P)) \big).$$

It follows from (1.6) that every $\rho_P \in \mathrm{Irr}(A_{G_P}(u_P))$ which appears in ρ is itself geometric. Hence the inclusions $R_{\mathbb{Z}}^{\circ}(A_{G_P}(u_P)) \to R_{\mathbb{Z}}(A_{G_P}(u_P))$ induce an injection

$$\overline{R_{\mathbb{Z}}^{\circ}}(A_G(u)) \to \overline{R_{\mathbb{Z}}}(A_G(u)). \tag{1.9}$$

By [Reeder 2001, Proposition 3.4.1] the maps $\rho_P \mapsto \mathrm{Hom}_{A_{G_P}(u_P)}(\rho_P, H^*(\mathcal{B}^{u_P}; \mathbb{C}))$ for $P \subset \Delta$ induce a $\mathbb{Z}$-linear bijection

$$\overline{R_{\mathbb{Z}}^{\circ}}(A_G(u)) \to \overline{R_{\mathbb{Z}}}(W, \mathcal{C}). \tag{1.10}$$

(In [Reeder 2001] these groups are by definition subsets of complex vector spaces. But with the above definitions Reeder's proof still applies.) From (1.7), (1.10) and (1.9) we obtain an injection

$$\overline{R_{\mathbb{Z}}}(W) \to \bigoplus_u \overline{R_{\mathbb{Z}}}(A_G(u)), \tag{1.11}$$

where u runs over a set of representatives for the unipotent classes of G.

Theorem 1.12. *The group of elliptic representations $\overline{R_{\mathbb{Z}}}(W)$ is torsion-free.*

Proof. If W is a product of irreducible Weyl groups W_i, then it follows readily from (1.2) that

$$\overline{R_{\mathbb{Z}}}(W) = \bigotimes_i \overline{R_{\mathbb{Z}}}(W_i).$$

Hence we may assume that $W = W(R)$ is irreducible. By (1.11) it suffices to show that each $\overline{R_{\mathbb{Z}}}(A_G(u))$ is torsion-free. If u is distinguished, then $\mathcal{C} \cap G_P = \varnothing$ for all $P \subsetneq \Delta$, and $\overline{R_{\mathbb{Z}}}(A_G(u)) = R_{\mathbb{Z}}(A_G(u))$. That is certainly torsion-free, so we do not have to consider distinguished unipotent u anymore.

For root systems of type A and of exceptional type, the tables of component groups in [Carter 1985, §13.1] show that $A_G(u)$ is isomorphic to S_n with $n \leq 5$. Moreover, S_4 and S_5 only occur when u is distinguished. For $A_G(u) \cong S_2$ and for $A_G(u) \cong S_3$ one checks directly that $\overline{R_{\mathbb{Z}}}(A_G(u))$ is torsion-free, by listing all subgroups of $A_G(u)$ and all irreducible representations thereof.

That leaves the root systems of type B, C and D. As group of type B_n we take $G = \mathrm{SO}_{2n+1}(\mathbb{C})$. By the Bala–Carter classification, the unipotent classes $\mathcal{C}$ in G are parametrized by pairs of partitions (α, β) such that $2|\alpha| + |\beta| = 2n + 1$ and β has only odd parts, all distinct. A typical $u \in \mathcal{C}$ is distinguished in the standard Levi subgroup

$$G_\alpha := \mathrm{GL}_{\alpha_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{\alpha_d}(\mathbb{C}) \times \mathrm{SO}_{|\beta|}(\mathbb{C}).$$

The part of u in $\mathrm{SO}_{|\beta|}$ depends only on β; it has Jordan blocks of sizes $\beta_1, \beta_2, \dots$. Let α' be a partition consisting of a subset of the terms of α, say

$$\alpha' = (n)^{m'_n} (n-1)^{m'_{n-1}} \cdots (1)^{m'_1}. \tag{1.13}$$

Let α'' be a partition of $|\alpha| - |\alpha'|$ obtained from the remaining terms of α by repeatedly replacing some α_i, α_j by $\alpha_i + \alpha_j$. All the standard Levi subgroups of G containing this u are of the form $G_{\alpha''}$. The GL-factors of $G_{\alpha''}$ do not contribute to $A_{G_{\alpha''}}(u)$. The part u' of u in $\mathrm{SO}_{2(n-|\alpha''|)+1}(\mathbb{C})$ is parametrized by (α', β) and the quotient of $Z_{\mathrm{SO}_{2(n-|\alpha''|)+1}(\mathbb{C})}(u')$ by its unipotent radical is isomorphic to

$$\prod_{i \text{ even}} \mathrm{Sp}_{2m'_i}(\mathbb{C}) \times \prod_{i \text{ odd, not in } \beta} O_{2m'_i}(\mathbb{C}) \times S\Big(\prod_{i \text{ odd, in } \beta} O_{2m'_i+1}(\mathbb{C}) \Big), \tag{1.14}$$

where the S indicates that we take the subgroup of elements of determinant 1. From this one can deduce the component group:

$$A_{G_{\alpha''}}(u) \cong A_{\mathrm{Sp}_{2(n-|\alpha''|)}(\mathbb{C})}(u') \cong \prod_{i \text{ odd, not in } \beta,\, m'_i > 0} \mathbb{Z}/2\mathbb{Z} \times S\Big(\prod_{i \text{ in } \beta} \mathbb{Z}/2\mathbb{Z} \Big). \tag{1.15}$$

We see that if

- α has an even term,
- or α has an odd term with multiplicity > 1,
- or α has an odd term which also appears in β,

then there is a standard Levi subgroup $G_{\alpha''} \subsetneq G$ with $A_{G_{\alpha''}}(u) \cong A_G(u)$, namely with α'' just that one term of α. In these cases $\overline{R_{\mathbb{Z}}}(A_G(u)) = 0$.

Suppose now that α has only distinct odd terms, and that none of those appears in β. Then (1.15) becomes

$$A_G(u) \cong \prod_{i \text{ in } \alpha} \mathbb{Z}/2\mathbb{Z} \times A \quad \text{where } A = S\Big(\prod_{i \text{ in } \beta} \mathbb{Z}/2\mathbb{Z} \Big).$$

We get

$$\begin{aligned} &\sum_{P \subsetneq \Delta,\, u_P \in \mathcal{C} \cap G_P} \operatorname{ind}_{A_{G_P}(u_P)}^{A_G(u_P)} \big(R_{\mathbb{Z}}(A_{G_P}(u_P)) \big) \\ &\qquad \cong \sum_{j \text{ in } \alpha} \operatorname{ind}_{A_{G_{\alpha-(j)}}(u)}^{A_G(u)} R_{\mathbb{Z}} \Big(\prod_{i \text{ in } \alpha - (j)} \mathbb{Z}/2\mathbb{Z} \times A \Big) \\ &\qquad \cong \sum_{j \text{ in } \alpha} \operatorname{ind}_{\{1\}}^{\mathbb{Z}/2\mathbb{Z}} R_{\mathbb{Z}}(\{1\}) \otimes_{\mathbb{Z}} R_{\mathbb{Z}} \Big(\prod_{i \text{ in } \alpha - (j)} \mathbb{Z}/2\mathbb{Z} \Big) \otimes_{\mathbb{Z}} R_{\mathbb{Z}}(A). \end{aligned} \tag{1.16}$$

We conclude that $\overline{R_{\mathbb{Z}}}(A_G(u)) = R_{\mathbb{Z}}(A)$.

So $\overline{R_{\mathbb{Z}}}(A_G(u))$ is torsion free for all unipotent $u \in \mathrm{SO}_{2n+1}(\mathbb{C})$, which settles the case B_n. The root systems of types C_n and D_n can be handled in a completely analogous way, using the explicit descriptions in [Carter 1985, §13.1]. □

For every $w \in W$ there exists (more or less by definition) a unique parabolic subgroup $\widetilde{W} \subset W$ such that w is an elliptic element of $\widetilde{W}$. Let $\mathcal{C}(W)$ be the set of conjugacy classes of W. For $P \subset \Delta$ let $\mathcal{C}_P(W)$ be the subset consisting of those conjugacy classes that contain an elliptic element of W_P. Let $\mathcal{P}(\Delta)/W$ be a set of representatives for the W-association classes of subsets of Δ. Since every parabolic subgroup is conjugate to a standard one, for every conjugacy class $\mathcal{C}$ in W there exists a unique $P \in \mathcal{P}(\Delta)/W$ such that $\mathcal{C} \in \mathcal{C}_P(W)$.

Recall from [Reeder 2001, §3.3] that a unipotent element $u \in G$ is called quasi-distinguished if there exists a semisimple $t \in Z_G(u)$ such that tu is not contained in any proper Levi subgroup of G.

Proposition 1.17. *For every $P \in \mathcal{P}(\Delta)/W$ there exists an injection from $\mathcal{C}_P(W)$ to the set of G_P-conjugacy classes of pairs (u_P, ρ_P) with $u_P \in G_P$ quasidistinguished unipotent and $\rho_P \in \mathrm{Irr}(A_{G_P}(u_P))$ geometric, denoted $w \mapsto (u_{P,w}, \rho_{P,w})$, such that*:

(a) $\{H(u_w, \rho_w) : w \in \mathcal{C}_\Delta(W)\}$ *is a $\mathbb{Z}$-basis of $\overline{R_{\mathbb{Z}}}(W)$.*

(b) *The set*

$$\left\{\mathrm{ind}_{W_P}^{W}(H_P(u_{P,w}, \rho_{P,w})) : P \in \mathcal{P}(\Delta)/W,\, w \in \mathcal{C}_P(W)\right\}$$

is a $\mathbb{Z}$-basis of $R_{\mathbb{Z}}(W)$.

Proof. (a) By [Reeder 2001, Proposition 2.2.2] the rank of $\overline{R_{\mathbb{Z}}}(W)$ is the number of elliptic conjugacy classes of W. With Theorem 1.12 we find $\overline{R_{\mathbb{Z}}}(W) \cong \mathbb{Z}^{|\mathcal{C}_\Delta(W)|}$. By (1.11) and (1.10), $\overline{R_{\mathbb{Z}}}(W)$ has a basis consisting of representations of the form $H(u, \rho)$ with $\rho \in \mathrm{Irr}(A_G(u))$ geometric. By [Reeder 2001, Proposition 3.4.1] we need only quasidistinguished unipotent u. We choose such a set of pairs (u, ρ), and we parametrize it in an arbitrary way by $\mathcal{C}_\Delta(W)$.

(b) We prove this by induction on $|\Delta|$. For $|\Delta| = 0$ the statement is trivial.

Suppose now that $|\Delta| \geq 1$ and $\alpha \in \Delta$. By the induction hypothesis we can find maps $w \mapsto (u_P, \rho_P)$ such that the set

$$\left\{\mathrm{ind}_{W_P}^{W_{\Delta\setminus\{\alpha\}}}(H_P(u_{P,w}, \rho_{P,w})) : P \in \mathcal{P}(\Delta \setminus \{\alpha\})/W_{\Delta\setminus\{\alpha\}},\, w \in \mathcal{C}_P(W_{\Delta\setminus\{\alpha\}})\right\}$$

is a $\mathbb{Z}$-basis of $R_{\mathbb{Z}}(W_{\Delta\setminus\{\alpha\}})$. By means of any setwise splitting of $N_G(T) \to W$ we can arrange that $(u_{P,w}, \rho_{P,w})$ and $(u_{P',w'}, \rho_{P',w'})$ are G-conjugate whenever (P, w) and (P', w') are W-associate. Then (P, w) and (P', w') give rise to the same W-representation. Consequently,

$$\left\{\mathrm{ind}_{W_P}^{W}(H_P(u_{P,w}, \rho_{P,w})) : P \in \mathcal{P}(\Delta)/W,\, P \neq \Delta,\, w \in \mathcal{C}_P(W)\right\}$$

is well-defined and has $|\mathcal{C}(W) \setminus \mathcal{C}_\Delta(W)|$ elements. By the induction hypothesis it spans $\sum_{P \subsetneq \Delta} \mathrm{ind}_{W_P}^W (R_\mathbb{Z}(W_P))$, so it forms a $\mathbb{Z}$-basis thereof. Combine this with (1.2) and part (a). □

1B. ***Graded Hecke algebras.*** We consider the Grothendieck group $R_\mathbb{Z}(\mathbb{H})$ of finite length modules of a graded Hecke algebra $\mathbb{H}$ with parameters k. We show that it is the direct sum of the subgroup spanned by modules induced from proper parabolic subalgebras and an elliptic part $\overline{R_\mathbb{Z}}(\mathbb{H})$. We prove that $\overline{R_\mathbb{Z}}(\mathbb{H})$ is isomorphic to the elliptic part of the representation ring of the Weyl group associated to $\mathbb{H}$. By Section 1A, $\overline{R_\mathbb{Z}}(\mathbb{H})$ is free abelian and does not depend on the parameters k. The main ingredients are the author's work [Solleveld 2010] on the periodic cyclic homology of graded Hecke algebras, and the study of discrete series representations by Ciubotaru, Opdam and Trapa [Ciubotaru and Opdam 2017; Ciubotaru et al. 2014].

Graded Hecke algebras are also known as degenerate (affine) Hecke algebras. They were introduced in [Lusztig 1989]. In the notation from (1.1) we call

$$\widetilde{\mathcal{R}} = (\mathfrak{a}^*, R, \mathfrak{a}, R^\vee, \Delta) \tag{1.18}$$

a degenerate root datum. We pick complex numbers k_α for $\alpha \in \Delta$, such that $k_\alpha = k_\beta$ if α and β are in the same W-orbit. We put $\mathfrak{t} = \mathbb{C} \otimes_\mathbb{R} \mathfrak{a}$.

The graded Hecke algebra associated to these data is the complex vector space

$$\mathbb{H} = \mathbb{H}(\widetilde{\mathcal{R}}, k) = \mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[W],$$

with multiplication defined by the following rules:

- $\mathbb{C}[W]$ and $\mathcal{O}(\mathfrak{t})$ are canonically embedded as subalgebras;
- for $\xi \in \mathfrak{t}^*$ and $s_\alpha \in S_\Delta$ we have the cross relation

$$\xi \cdot s_\alpha - s_\alpha \cdot s_\alpha(\xi) = k_\alpha \langle \xi, \alpha^\vee \rangle. \tag{1.19}$$

Notice that $\mathbb{H}(\widetilde{\mathcal{R}}, 0) = \mathcal{O}(\mathfrak{t}) \rtimes W$.

Multiplication with any $\epsilon \in \mathbb{C}^\times$ defines a bijection $\mathfrak{t}^* \to \mathfrak{t}^*$, which clearly extends to an algebra automorphism of $\mathcal{O}(\mathfrak{t}) = S(\mathfrak{t}^*)$. From the cross relation (1.19) we see that it extends even further, to an algebra isomorphism

$$\mathbb{H}(\widetilde{\mathcal{R}}, \epsilon k) \to \mathbb{H}(\widetilde{\mathcal{R}}, k) \tag{1.20}$$

which is the identity on $\mathbb{C}[W]$. For $\epsilon = 0$ this map is well-defined, but obviously not bijective.

For a set of simple roots $P \subset \Delta$ we write

$$\begin{aligned} R_P &= \mathbb{Q}P \cap R, & R_P^\vee &= \mathbb{Q}R_P^\vee \cap R^\vee, \\ \mathfrak{a}_P &= \mathbb{R}P^\vee, & \mathfrak{a}^P &= (\mathfrak{a}_P^*)^\perp, \\ \mathfrak{a}_P^* &= \mathbb{R}P, & \mathfrak{a}^{P*} &= (\mathfrak{a}_P)^\perp, \\ \widetilde{\mathcal{R}}_P &= (\mathfrak{a}_P^*, R_P, \mathfrak{a}_P, R_P^\vee, P), & \widetilde{\mathcal{R}}^P &= (\mathfrak{a}^*, R_P, \mathfrak{a}, R_P^\vee, P). \end{aligned} \tag{1.21}$$

Let k_P be the restriction of k to R_P. We call

$$\mathbb{H}^P = \mathbb{H}(\widetilde{\mathcal{R}}^P, k_P)$$

a parabolic subalgebra of $\mathbb{H}$. It contains $\mathbb{H}_P = \mathbb{H}(\widetilde{\mathcal{R}}_P, k_P)$ as a direct summand.

The centre of $\mathbb{H}(\widetilde{\mathcal{R}}, k)$ is $\mathcal{O}(\mathfrak{t})^W = \mathcal{O}(\mathfrak{t}/W)$ [Lusztig 1989, Proposition 4.5]. Hence the central character of an irreducible $\mathbb{H}(\widetilde{\mathcal{R}}, k)$-representation is an element of $\mathfrak{t}/W$.

Let (π, V) be an $\mathbb{H}(\widetilde{\mathcal{R}}, k)$-representation. We say that $\lambda \in \mathfrak{t}$ is an $\mathcal{O}(\mathfrak{t})$-weight of V (or of π) if

$$\{v \in V : \pi(\xi)v = \lambda(\xi)v \text{ for all } \xi \in \mathfrak{t}^*\}$$

is nonzero. Let $\mathrm{Wt}(V) \subset \mathfrak{t}$ be the set of $\mathcal{O}(t)$-weights of V.

Temperedness of a representation is defined via its $\mathcal{O}(\mathfrak{t})$-weights. We write

$$\begin{aligned} \mathfrak{a}^+ &= \{\mu \in \mathfrak{a} : \langle \alpha, \mu \rangle \geq 0 \ \forall \alpha \in \Delta\}, \\ \mathfrak{a}^{*+} &:= \{x \in \mathfrak{a}^* : \langle x, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in \Delta\}, \\ \mathfrak{a}^- &= \{\lambda \in \mathfrak{a} : \langle x, \lambda \rangle \leq 0 \ \forall x \in \mathfrak{a}^{*+}\} = \left\{\textstyle\sum_{\alpha \in \Delta} \lambda_\alpha \alpha^\vee : \lambda_\alpha \leq 0\right\}. \end{aligned}$$

The interior $\mathfrak{a}^{--}$ of $\mathfrak{a}^-$ equals $\left\{\sum_{\alpha \in \Delta} \lambda_\alpha \alpha^\vee : \lambda_\alpha < 0\right\}$ if Δ spans $\mathfrak{a}^*$, and is empty otherwise.

We regard $\mathfrak{t} = \mathfrak{a} \oplus i\mathfrak{a}$ as the polar decomposition of $\mathfrak{t}$, with associated real part map $\Re : \mathfrak{t} \to \mathfrak{a}$. By definition, a finite-dimensional $\mathbb{H}(\widetilde{\mathcal{R}}, k)$-module (π, V) is tempered $\Re(\mathrm{Wt}(V)) \subset \mathfrak{a}^-$. More restrictively, we say that (π, V) belongs to the discrete series if $\Re(\mathrm{Wt}(V)) \subset \mathfrak{a}^{--}$.

We are interested in the restriction map

$$\mathrm{r} : \mathrm{Mod}(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to \mathrm{Mod}(\mathbb{C}[W]), \quad V \mapsto V|_W.$$

We can also regard it as the composition of representations with the algebra homomorphism (1.20) for $\epsilon = 0$, then its image consists of $\mathcal{O}(\mathfrak{t}) \rtimes W$-representations on which $\mathcal{O}(\mathfrak{t})$ acts via $0 \in \mathfrak{t}$.

Let $\mathrm{Irr}_0(\mathbb{H})$ be the set of irreducible tempered $\mathbb{H}(\widetilde{\mathcal{R}}, k)$-modules with central character in $\mathfrak{a}/W$. It is known from [Solleveld 2010, Theorem 6.5] that, for real-valued k, r induces a bijection

$$\mathrm{r}_{\mathbb{C}} : \mathbb{C}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to R_{\mathbb{C}}(W). \tag{1.22}$$

Using work of Lusztig, Ciubotaru [2008, Corollary 3.6] showed that, for parameters of "geometric" type,

$$\mathrm{r}_{\mathbb{Z}} : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to R_{\mathbb{Z}}(W) \text{ is bijective.} \tag{1.23}$$

We will generalize this to arbitrary real parameters. (Parameters k of geometric type need not be real-valued, but via (1.20) they can be reduced to that.)

We recall some notions from [Ciubotaru and Opdam 2015]. Let $R_{\mathbb{Z}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ be the Grothendieck group of (the category of) finite-dimensional $\mathbb{H}(\widetilde{\mathcal{R}}, k)$-modules. For any parabolic subalgebra $\mathbb{H}^P = \mathbb{H}(\widetilde{\mathcal{R}}^P, k_P)$, the induction functor $\mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}$ induces a map $R_{\mathbb{Z}}(\mathbb{H}^P) \to R_{\mathbb{Z}}(\mathbb{H})$. If the $\mathcal{O}(\mathfrak{t})$-weights of $V \in \mathrm{Mod}(\mathbb{H}^P)$ are contained in some $U \subset \mathfrak{t}$, then by [Barbasch and Moy 1993, Theorem 6.4], the $\mathcal{O}(\mathfrak{t})$-weights of $\mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}} V$ are contained in $W^P U$, where W^P is the set of shortest length representatives of W/W_P. This implies that $\mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}$ preserves temperedness [Barbasch and Moy 1993, Corollary 6.5] and central characters. In particular, it induces a map

$$\mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}} : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}^P) \to \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}). \tag{1.24}$$

Many arguments in this section make use of the group of "elliptic $\mathbb{H}$-representations"

$$\overline{R_{\mathbb{Z}}}(\mathbb{H}) = R_{\mathbb{Z}}(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \Big/ \textstyle\sum_{P \subsetneq \Delta} \mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}(R_{\mathbb{Z}}(\mathbb{H}^P)). \tag{1.25}$$

Since $\mathbb{H}(\widetilde{\mathcal{R}}, k) = \mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[W]$ as vector spaces,

$$\mathrm{r} \circ \mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}} = \mathrm{ind}_{W_P}^{W} \circ \mathrm{r}^P, \tag{1.26}$$

where r^P denotes the analogue of r for $\mathbb{H}^P$. Hence r induces a $\mathbb{Z}$-linear map

$$\bar{\mathrm{r}} : \overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to \overline{R_{\mathbb{Z}}}(W). \tag{1.27}$$

Proposition 1.28. *The map (1.27) is surjective, and its kernel is the torsion subgroup of* $\overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$.

Proof. By Theorem 1.12, $\overline{R_{\mathbb{Z}}}(W)$ is torsion-free, so it can be identified with its image in $\overline{R_{\mathbb{C}}}(W)$. This means that our definition of $\overline{R_{\mathbb{Z}}}(W)$ agrees with that in [Ciubotaru et al. 2014]. Likewise, in [Ciubotaru et al. 2014] the subgroup $\overline{R'_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ of $\overline{R_{\mathbb{C}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ generated by the actual representations is considered. In other words, $\overline{R'_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ is defined as the quotient of $\overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ by its torsion subgroup.

By [Ciubotaru et al. 2014, Proposition 5.6] the map

$$\bar{\mathrm{r}} : \overline{R'_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to \overline{R_{\mathbb{Z}}}(W) \tag{1.29}$$

is bijective, except possibly when R has type F_4 and k is not a generic parameter. However, in view of the more recent work [Ciubotaru and Opdam 2017, §3.2], the limit argument given (for types B_n and G_2) in [Ciubotaru et al. 2014, §5.1] also applies to F_4. Thus (1.29) is bijective for all $\widetilde{\mathcal{R}}$ and all real-valued parameters k. □

Lemma 1.30. *Let k be real-valued. The canonical map*

$$\mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to \overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$$

is surjective.

Proof. It was noted in [Opdam and Solleveld 2013, Lemma 6.3] (in the context of affine Hecke algebras) that every element of $\overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ can be represented by a tempered virtual representation. Consider any irreducible tempered $\mathbb{H}$-representation π. By [Solleveld 2012b, Proposition 8.2] there exists a $P \subset \Delta$, a discrete series representation δ of $\mathbb{H}_P$ and an element $\nu \in i\mathfrak{a}^P$ such that π is a direct summand of

$$\pi(P, \delta, \nu) = \mathrm{ind}_{\mathbb{H}_P \otimes \mathcal{O}(\mathfrak{t}^P)}^{\mathbb{H}} (\delta \otimes \mathbb{C}_\nu).$$

By [Solleveld 2012b, Proposition 8.3] the reducibility of $\pi(P, \delta, \nu)$ is determined by intertwining operators $\pi(w, P, \delta, \nu)$ for elements $w \in W$ that stabilize (P, δ, ν). Suppose that $\nu \neq 0$. Then W_ν is a proper parabolic subgroup of W, so the stabilizer of (P, δ, ν) is contained in W_Q for some $P \subset Q \subsetneq \Delta$. In that case, $\pi = \mathrm{ind}_{\mathbb{H}^Q}^{\mathbb{H}}(\pi^Q)$ for some irreducible representation π^Q of $\mathbb{H}^Q$, so π becomes zero in $\overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$.

Therefore we need only $\mathbb{Z}$-linear combinations of summands of $\pi(\delta, P, 0)$ (with varying P, δ) to surject to $\overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$. Since k is real, discrete series representations of $\mathbb{H}_P$ have central characters in $\mathfrak{a}_P / W_P$ [Slooten 2006, Lemma 2.13]. It follows that $\pi(P, \delta, 0)$ and all its constituents (among which is π) admit a central character in $\mathfrak{a}/W$. □

Theorem 1.31. *Let k be real-valued. The restriction-to-W maps*

$$\begin{aligned} \mathrm{r}_{\mathbb{Z}} &: \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to R_{\mathbb{Z}}(W), \\ \bar{\mathrm{r}} &: \overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to \overline{R_{\mathbb{Z}}}(W) \end{aligned}$$

are bijective.

Proof. We show this by induction on the semisimple rank of $\widetilde{\mathcal{R}}$ (i.e., the rank of R). Suppose first that the semisimple rank is zero. Then $W = 1$ and $\mathbb{H} = \mathcal{O}(\mathfrak{t})$. For $\lambda \in \mathfrak{t}$ the character

$$\mathrm{ev}_\lambda : f \mapsto f(\lambda)$$

is a tempered $\mathcal{O}(\mathfrak{t})$-representation if and only if $\Re(\lambda) = 0$. If λ is at the same time a real central character (i.e., $\lambda \in \mathfrak{a}$), then $\lambda = 0$. Hence $\mathrm{Irr}_0(\mathbb{H})$ consists just of ev_0. It is mapped to the trivial W-representation by r, so the theorem holds in this case.

Now let $\widetilde{\mathcal{R}}$ be of positive semisimple rank. It is a direct sum of degenerate root data with R irreducible or R empty, and $\mathbb{H}(\widetilde{\mathcal{R}}, k)$ decomposes accordingly. As we already know the result when R is empty, it remains to establish the case where R is irreducible.

Any proper parabolic subalgebra $\mathbb{H}^P \subset \mathbb{H}$ has smaller semisimple rank, so by the induction hypothesis

$$\mathrm{r}^P : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}^P) \to \mathbb{Z}\,\mathrm{Irr}_0(W_P) \text{ is bijective.} \tag{1.32}$$

Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \sum_{P \subsetneq \Delta} \mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}(\mathbb{Z}\ \mathrm{Irr}_0(\mathbb{H}^P)) \arrow[r] \arrow[d] & \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}) \arrow[r] \arrow[d] & \overline{R_{\mathbb{Z}}}(\mathbb{H}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \sum_{P \subsetneq \Delta} \mathrm{ind}_{W_P}^{W}(R_{\mathbb{Z}}(W_P)) \arrow[r] & R_{\mathbb{Z}}(W) \arrow[r] & \overline{R_{\mathbb{Z}}}(W) \arrow[r] & 0 \end{tikzcd} \tag{1.33}$$

The second row is exact by definition. By (1.32) and (1.26) the left vertical arrow is bijective and by Proposition 1.28 the right vertical arrow is surjective. Together with Lemma 1.30 these imply that the middle vertical arrow is surjective. By (1.22) both $\mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H})$ and $R_{\mathbb{Z}}(W)$ are free abelian groups of the same rank $|\mathrm{Irr}(W)| = |\mathrm{Irr}_0(\mathbb{H})|$, so the middle vertical arrow is in fact bijective.

The results so far imply that the kernel of $\mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}) \to \overline{R_{\mathbb{Z}}}(W)$ is precisely

$$\textstyle\sum_{P \subsetneq \Delta} \mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}(\mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}^P)).$$

The latter group is already killed in $\overline{R_{\mathbb{Z}}}(\mathbb{H})$, so $\overline{R_{\mathbb{Z}}}(\mathbb{H}) \to \overline{R_{\mathbb{Z}}}(W)$ is injective as well. We conclude that (1.33) is a bijection between two short exact sequences. $\square$

We will need Theorem 1.31 for somewhat more general algebras. Let Γ be a finite group acting on $\widetilde{\mathcal{R}}$; it acts $\mathbb{R}$-linearly on $\mathfrak{a}$, and the dual action on $\mathfrak{a}^*$ stabilizes R and Δ. We assume that $k_{\gamma(\alpha)} = k_\alpha$ for all $\alpha \in R$, $\gamma \in \Gamma$. Then Γ acts on $\mathbb{H}(\widetilde{\mathcal{R}}, k)$ by the algebra automorphisms satisfying

$$\gamma(\xi N_w) = \gamma(\xi) N_{\gamma w \gamma^{-1}}, \quad \gamma \in \Gamma,\ \xi \in \mathfrak{a}^*,\ w \in W.$$

Let $\natural : \Gamma^2 \to \mathbb{C}^\times$ be a 2-cocycle and let $\mathbb{C}[\Gamma, \natural]$ be the twisted group algebra. We recall that it has a standard basis $\{N_\gamma : \gamma \in \Gamma\}$ and multiplication rules

$$N_\gamma N_{\gamma'} = \natural(\gamma, \gamma') N_{\gamma\gamma'}, \quad \gamma, \gamma' \in \Gamma.$$

We can endow the vector space $\mathbb{H}(\widetilde{\mathcal{R}}, k) \otimes \mathbb{C}[\Gamma, \natural]$ with the algebra structure such that

- $\mathbb{H}(\widetilde{\mathcal{R}}, k)$ and $\mathbb{C}[\Gamma, \natural]$ are embedded as subalgebras,
- $N_\gamma h N_\gamma^{-1} = \gamma(h)$ for $\gamma \in \Gamma,\ h \in \mathbb{H}(\widetilde{\mathcal{R}}, k)$.

We denote this algebra by $\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural]$ and call it a twisted graded Hecke algebra. If $\natural$ is trivial, then it reduces to the crossed product $\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \Gamma$. All our previous notions for graded Hecke algebras admit natural generalizations to this setting.

Notice that $W\Gamma$ is a group with W as normal subgroup and Γ as quotient. The 2-cocycle $\natural$ can be lifted to $(W\Gamma)^2 \to \Gamma^2 \to (\mathbb{C}^\times)^2$, and that yields a twisted group algebra $\mathbb{C}[W\Gamma, \natural]$ in $\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural]$. It is worthwhile to note the case $k = 0$:

$$\mathbb{H}(\widetilde{\mathcal{R}}, 0) \rtimes \mathbb{C}[\Gamma, \natural] = \mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \natural]). \tag{1.34}$$

We consider the restriction map

$$\mathrm{r} : \mathrm{Mod}(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural]) \to \mathrm{Mod}(\mathbb{C}[W\Gamma, \natural]). \tag{1.35}$$

Every $\mathbb{C}[W\Gamma, \natural]$-module can be extended in a unique way to an $\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \natural])$-module on which $\mathcal{O}(\mathfrak{t})$ acts via evaluation at $0 \in \mathfrak{t}$, so the right-hand side of (1.35) can be considered as a subcategory of $\mathrm{Mod}(\mathbb{H}(\widetilde{\mathcal{R}}, 0) \rtimes \mathbb{C}[\Gamma, \natural])$.

Proposition 1.36. *Let $k : R / W\Gamma \to \mathbb{R}$ be a parameter function and let $\natural : \Gamma^2 \to \mathbb{C}^\times$ be a 2-cocycle. The map (1.35) induces a bijection*

$$\mathrm{r}_{\mathbb{Z}} : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural]) \to R_{\mathbb{Z}}(\mathbb{C}[W\Gamma, \natural]).$$

Proof. Let $\widetilde{\Gamma} \to \Gamma$ be a finite central extension such that $\natural$ becomes trivial in $H^2(\widetilde{\Gamma}, \mathbb{C}^\times)$. Such a group always exists: one can take the Schur extension from [Curtis and Reiner 1962, Theorem 53.7]. Then there exists a central idempotent $p_\natural \in \mathbb{C}[\ker(\widetilde{\Gamma} \to \Gamma)]$ such that

$$\mathbb{C}[\Gamma, \natural] \cong p_\natural \mathbb{C}[\widetilde{\Gamma}]. \tag{1.37}$$

The map $\mathrm{r}_{\mathbb{Z}}$ becomes

$$\mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes p_\natural \mathbb{C}[\widetilde{\Gamma}]) \to R_{\mathbb{Z}}(p_\natural \mathbb{C}[W\widetilde{\Gamma}]). \tag{1.38}$$

Since $p_\natural \mathbb{C}[\widetilde{\Gamma}]$ is a direct summand of $\mathbb{C}[\widetilde{\Gamma}]$, (1.38) is just a part of

$$\mathrm{r}_{\mathbb{Z}} : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \widetilde{\Gamma}) \to R_{\mathbb{Z}}(W \rtimes \widetilde{\Gamma}).$$

Hence it suffices to prove the proposition when $\natural$ is trivial, which we assume from now on. By [Solleveld 2010, Theorem 6.5(c)],

$$\mathrm{r}_{\mathbb{C}} : \mathbb{C}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \Gamma) \to R_{\mathbb{C}}(W\Gamma) \tag{1.39}$$

is a $\mathbb{C}$-linear bijection. So at least

$$\mathrm{r}_{\mathbb{Z}} : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \Gamma) \to R_{\mathbb{Z}}(W\Gamma) \tag{1.40}$$

is injective and has image of finite index in $R_{\mathbb{Z}}(W\Gamma)$.

Given $(\pi, V) \in \mathrm{Irr}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$, let Γ_π be the stabilizer in Γ of the isomorphism class of π. For every $\gamma \in \Gamma_\pi$ we can find $I^\gamma \in \mathrm{Aut}_{\mathbb{C}}(V)$ such that

$$I^\gamma \circ \pi(N_\gamma h N_\gamma^{-1}) = \pi(h) \circ I^\gamma \quad \text{for all } h \in \mathbb{H}(\widetilde{\mathcal{R}}, k).$$

By Schur's lemma there exists a 2-cocycle $\natural_\pi : \Gamma_\pi^2 \to \mathbb{C}^\times$ such that

$$I^{\gamma\gamma'} = \natural_\pi(\gamma, \gamma') I^\gamma I^{\gamma'} \quad \text{for all } \gamma, \gamma' \in \Gamma.$$

Let $(\tau, M) \in \mathrm{Irr}(\mathbb{C}[\Gamma_\pi, \natural_\pi])$. Then $M \otimes V$ becomes an irreducible $\mathbb{H} \rtimes \Gamma_\pi$-module. Clifford theory (see, e.g., [Ram and Ramagge 2003, Appendix], [Curtis and Reiner 1962, §51] or [Solleveld 2012b, Appendix]) tells us that $\mathrm{ind}_{\mathbb{H}\rtimes\Gamma_\pi}^{\mathbb{H}\rtimes\Gamma}(M \otimes V)$ is an irreducible $\mathbb{H} \rtimes \Gamma$-module. Moreover, this construction provides a bijection

$$\mathrm{Irr}(\mathbb{H} \rtimes \Gamma) \to \{(\pi, M) : \pi \in \mathrm{Irr}(\mathbb{H})/\Gamma, M \in \mathrm{Irr}(\mathbb{C}[\Gamma_\pi, \natural_\pi])\}.$$

We note that

$$\mathrm{r}\big(\mathrm{ind}_{\mathbb{H}\rtimes\Gamma_\pi}^{\mathbb{H}\rtimes\Gamma}(M \otimes V)\big) = \mathrm{ind}_{W\rtimes\Gamma_\pi}^{W\rtimes\Gamma}(M \otimes \mathrm{r}(V)). \tag{1.41}$$

Similarly, Clifford theory provides a bijection

$$\mathrm{Irr}(W \rtimes \Gamma) \to \{(\tau, N) : \tau \in \mathrm{Irr}(W)/\Gamma,\ N \in \mathrm{Irr}(\mathbb{C}[\Gamma_\tau, \natural_\tau])\}.$$

Since W is a Weyl group, the 2-cocycle $\natural_\tau$ is always trivial [Aubert et al. 2017c, Proposition 4.3]. With (1.41) it follows that $\natural_\pi$ is also trivial for all $\pi \in \mathrm{Irr}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$.

Consider any $\mathrm{ind}_{W\rtimes\Gamma_\tau}^{W\rtimes\Gamma}(N \otimes V_\tau) \in \mathrm{Irr}(W \rtimes \Gamma)$. Theorem 1.31 guarantees the existence of unique $m_\pi \in \mathbb{Z}$ such that $V_\tau = \sum_{(\pi,V)\in\mathrm{Irr}_0(\mathbb{H})} m_\pi \mathrm{r}(V)$. By the uniqueness, $\Gamma_\pi \supset \Gamma_\tau$ whenever $m_\pi \neq 0$. Hence $N \otimes V$ is a well-defined $\mathbb{H} \rtimes \Gamma_\pi$-module (it may be reducible though), and

$$\begin{aligned}\mathrm{ind}_{W\rtimes\Gamma_\tau}^{W\rtimes\Gamma}(N \otimes V_\tau) &= \mathrm{ind}_{W\rtimes\Gamma_\tau}^{W\rtimes\Gamma}\big(N \otimes \textstyle\sum_{(\pi,V)\in\mathrm{Irr}_0(\mathbb{H})} m_\pi \mathrm{r}(V)\big) \\ &= \mathrm{r}\big(\textstyle\sum_{(\pi,V)\in\mathrm{Irr}_0(\mathbb{H})} m_\pi\, \mathrm{ind}_{\mathbb{H}\rtimes\Gamma_\pi}^{\mathbb{H}\rtimes\Gamma}(N \otimes V)\big).\end{aligned}$$

This proves that (1.40) is also surjective. $\square$

1C. ***Affine Hecke algebras.*** Let $\mathcal{H}$ be an affine Hecke algebra with positive parameters q. We compare its Grothendieck group of finite length modules $R_\mathbb{Z}(\mathcal{H})$ with the analogous group for the parameters $q = 1$. By some of the main results of [Solleveld 2012a], the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_\mathbb{Z} R_\mathbb{Z}(\mathcal{H})$ is canonically isomorphic to its analogue for $q = 1$. We show that this is already an isomorphism for $R_\mathbb{Z}(\mathbb{H})$, without tensoring by $\mathbb{Q}$. This follows from the results of the previous paragraph, in combination with the standard reduction from affine Hecke algebras to graded Hecke algebras [Lusztig 1989].

As before, let $\mathcal{R} = (X, R, Y, R^\vee, \Delta)$ be a based root datum. We have the affine Weyl group $W^{\mathrm{aff}} = \mathbb{Z}R \rtimes W$ and the extended (affine) Weyl group $W^e = X \rtimes W$. Both can be considered as groups of affine transformations of $\mathfrak{a}^*$. We denote the translation corresponding to $x \in X$ by t_x. As is well-known, W^{aff} is a Coxeter group, and the basis Δ of R gives rise to a set S^{aff} of simple (affine) reflections.

More explicitly, let $\Delta_M^\vee$ be the set of maximal elements of $R^\vee$, with respect to the dominance ordering coming from Δ. Then

$$S^{\mathrm{aff}} = S_\Delta \cup \{t_\alpha s_\alpha : \alpha^\vee \in \Delta_M^\vee\}.$$

The length function ℓ of the Coxeter system $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ extends naturally to W^e. The elements of length zero form a subgroup $\Omega \subset W^e$ and $W^e = W^{\mathrm{aff}} \rtimes \Omega$.

A complex parameter function for $\mathcal{R}$ is a map $q : S^{\mathrm{aff}} \to \mathbb{C}^\times$ such that $q(s) = q(s')$ if s and s' are conjugate in W^e. This extends naturally to a map $q : W^e \to \mathbb{C}^\times$ which is 1 on Ω and satisfies

$$q(ww') = q(w)q(w') \quad \text{if} \quad \ell(ww') = \ell(w) + \ell(w').$$

Equivalently (see [Lusztig 1989, §3.1]), one can define q as a W-invariant function

$$q : R \cup \{2\alpha : \alpha^\vee \in 2Y\} \to \mathbb{C}^\times. \tag{1.42}$$

We speak of equal parameters if $q(s) = q(s')$ for all $s, s' \in S^{\mathrm{aff}}$ and of positive parameters if $q(s) \in \mathbb{R}_{>0}$ for all $s \in S^{\mathrm{aff}}$. We fix a square root $q^{1/2} : S^{\mathrm{aff}} \to \mathbb{C}^\times$.

The affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ is the unique associative complex algebra with basis $\{N_w : w \in W^e\}$ and multiplication rules

$$\begin{aligned} N_w N_{w'} &= N_{ww'} && \text{if} \quad \ell(ww') = \ell(w) + \ell(w'), \\ (N_s - q(s)^{1/2})(N_s + q(s)^{-1/2}) &= 0 && \text{if} \quad s \in S^{\mathrm{aff}}. \end{aligned} \tag{1.43}$$

In the literature one also finds this algebra defined in terms of the elements $q(s)^{1/2} N_s$, in which case the multiplication can be described without square roots. This explains why $q^{1/2}$ does not appear in the notation $\mathcal{H}(\mathcal{R}, q)$. For $q = 1$, (1.43) just reflects the defining relations of W^e, so $\mathcal{H}(\mathcal{R}, 1) = \mathbb{C}[W^e]$.

The set of dominant elements in X is

$$X^+ = \{x \in X : \langle x, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in \Delta\}.$$

The subset $\{N_{t_x} : x \in X^+\} \subset \mathcal{H}(\mathcal{R}, q)$ is closed under multiplication, and isomorphic to X^+ as a semigroup. For any $x \in X$ we put

$$\theta_x = N_{t_{x_1}} N_{t_{x_2}}^{-1}, \quad \text{where } x_1, x_2 \in X^+ \text{ and } x = x_1 - x_2.$$

This does not depend on the choice of x_1 and x_2, so $\theta_x \in \mathcal{H}(\mathcal{R}, q)^\times$ is well-defined. The Bernstein presentation of $\mathcal{H}(\mathcal{R}, q)$ [Lusztig 1989, §3] says that:

- $\{\theta_x : x \in X\}$ forms a $\mathbb{C}$-basis of a subalgebra of $\mathcal{H}(\mathcal{R}, q)$ isomorphic to $\mathbb{C}[X] \cong \mathcal{O}(T)$, which we identify with $\mathcal{O}(T)$.
- $\mathcal{H}(W, q) := \mathbb{C}\{N_w : w \in W\}$ is a finite-dimensional subalgebra of $\mathcal{H}(\mathcal{R}, q)$ (known as the Iwahori–Hecke algebra of W).

- The multiplication map $\mathcal{O}(T) \otimes \mathcal{H}(W, q) \to \mathcal{H}(\mathcal{R}, q)$ is a $\mathbb{C}$-linear bijection.
- There are explicit cross relations between $\mathcal{H}(W, q)$ and $\mathcal{O}(T)$, deformations of the standard action of W on $\mathcal{O}(T)$.

To define parabolic subalgebras of affine Hecke algebras, we associate some objects to any $P \subset \Delta$:

$$\begin{aligned} X_P &= X/(X \cap (P^\vee)^\perp), & X^P &= X/(X \cap \mathbb{Q}P), \\ Y_P &= Y \cap \mathbb{Q}P^\vee, & Y^P &= Y \cap P^\perp, \\ T_P &= \mathrm{Hom}_{\mathbb{Z}}(X_P, \mathbb{C}^\times), & T^P &= \mathrm{Hom}_{\mathbb{Z}}(X^P, \mathbb{C}^\times), \\ \mathcal{R}_P &= (X_P, R_P, Y_P, R_P^\vee, P), & \mathcal{R}^P &= (X, R_P, Y, R_P^\vee, P), \\ \mathcal{H}_P &= \mathcal{H}(\mathcal{R}_P, q_P), & \mathcal{H}^P &= \mathcal{H}(\mathcal{R}^P, q^P). \end{aligned}$$

Here q_P and q^P are derived from q via (1.42). Both $\mathcal{H}_P$ and $\mathcal{H}^P$ are called parabolic subalgebras of $\mathcal{H}$. One can regard $\mathcal{H}_P$ as a "semisimple" quotient of $\mathcal{H}^P$.

Any $t \in T^P$ and any $u \in T^P \cap T_P$ give rise to algebra automorphisms

$$\begin{aligned} \psi_u &: \mathcal{H}_P \to \mathcal{H}_P, \quad \theta_{x_P} N_w \mapsto u(x_P)\theta_{x_P} N_w, \\ \psi_t &: \mathcal{H}^P \to \mathcal{H}^P, \quad \theta_x N_w \mapsto t(x)\theta_x N_w. \end{aligned} \tag{1.44}$$

Let Γ be a finite group acting on $\mathcal{R}$, i.e., it acts $\mathbb{Z}$-linearly on X and preserves R and Δ. We also assume that Γ acts on T by affine transformations, whose linear part comes from the action on X. Thus Γ acts on $\mathcal{O}(T) \cong \mathbb{C}[X]$ by

$$\gamma(\theta_x) = z_\gamma(x)\theta_{\gamma x} \tag{1.45}$$

for some $z_\gamma \in T$. Since this is a group action, we must have $z_\gamma \in T^W$.

We suppose throughout that $q^{1/2}$ is Γ-invariant, so that $\gamma \in \Gamma$ acts on $\mathcal{H}(\mathcal{R}, q)$ by the algebra automorphism

$$\sum_{w \in W,\, x \in X} c_{x,w}\theta_x N_w \mapsto \sum_{w \in W,\, x \in X} c_{x,w} z_\gamma(x)\theta_{\gamma(x)} N_{\gamma w \gamma^{-1}}. \tag{1.46}$$

We can build the crossed product algebra

$$\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma. \tag{1.47}$$

In [Solleveld 2012a] we considered a slightly less general action of Γ on $\mathcal{H}(\mathcal{R}, q)$, where the elements $z_\gamma \in T^W$ from (1.45) were all equal to 1. But the relevant results from [Solleveld 2012a] do not rely on Γ fixing the unit element of T, so they are also valid for the actions as in (1.46). In this paper we will tacitly use some results from [Solleveld 2012a] in the generality of (1.46). We note that nontrivial $z_\gamma \in T^W$ are sometimes needed to describe Hecke algebras coming from p-adic groups, for example [Roche 2002, §4].

We can also endow the group Γ with a 2-cocycle $\natural : \Gamma^2 \to \mathbb{C}^\times$. Then the vector space $\mathcal{H}(\mathcal{R}, q) \otimes \mathbb{C}[\Gamma, \natural]$ obtains a multiplication such that $\mathcal{H}(\mathcal{R}, q)$ and $\mathbb{C}[\Gamma, \natural]$ are subalgebras and

$$N_\gamma h N_\gamma^{-1} = \gamma(h) \quad \text{for all } \gamma \in \Gamma,\ h \in \mathcal{H}(\mathcal{R}, q).$$

We denote this by $\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]$ and call it a twisted affine Hecke algebra. Such twists seem necessary to describe algebras appearing in the representation theory of nonsplit p-adic groups; see, e.g., [Aubert et al. 2017b, Example 5.5]. For reference we record the case $q = 1$:

$$\mathcal{H}(\mathcal{R}, 1) \rtimes \mathbb{C}[\Gamma, \natural] = \mathcal{O}(T) \rtimes \mathbb{C}[W\Gamma, \natural]. \tag{1.48}$$

The representation theory of (twisted) affine Hecke algebras is closely related to that of (twisted) graded Hecke algebras, as first shown by Lusztig [1989]. Since $\mathcal{H}(\mathcal{R}, q)$ is of finite rank as a module over its commutative subalgebra $\mathcal{O}(T)$, all irreducible $\mathcal{H}(\mathcal{R}, q)$-modules have finite dimension. The set of $\mathcal{O}(T)$-weights of an $\mathcal{H}(\mathcal{R}, q)$-module V is denoted by $\mathrm{Wt}(V)$.

The vector space $\mathfrak{t} = \mathfrak{a} \oplus i\mathfrak{a}$ can now be interpreted as the Lie algebra of the complex torus $T = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$. The latter has a polar decomposition $T = T_{\mathrm{rs}} \times T_{\mathrm{un}}$, where $T_{\mathrm{rs}} = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{R}_{>0})$ and T_{un} is the unique maximal compact subgroup of T. The polar decomposition of an element $t \in T$ is written as $t = |t|\ (t\,|t|^{-1})$.

We write $T^- = \exp(\mathfrak{a}^-) \subset T_{\mathrm{rs}}$ and $T^{--} = \exp(\mathfrak{a}^{--}) \subset T_{\mathrm{rs}}$. We say that a module V for $\mathcal{H}(\mathcal{R}, q)$ (or for $\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]$) is tempered if $|\mathrm{Wt}(V)| \subset T^-$, and that it is discrete series if $|\mathrm{Wt}(V)| \subset T^{--}$. (The latter is only possible if R spans $\mathfrak{a}$, for otherwise $\mathfrak{a}^{--}$ and T^{--} are empty.)

By the Bernstein presentation, the centre of $\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]$ contains $\mathcal{O}(T)^{W\Gamma}$. For any $W\Gamma$-invariant subset $U \subset T$, let

$$\mathrm{Mod}_{f,U}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural])$$

be the category of finite-dimensional $\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]$-modules whose $\mathcal{O}(T)^{W\Gamma}$-weights all lie in $U/W\Gamma$. We denote the Grothendieck group of this category by $R_{\mathbb{Z},U}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural])$.

The centre of $\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural]$ contains $\mathcal{O}(\mathfrak{t})^{W\Gamma}$. For any $W\Gamma$-invariant subset $V \subset \mathfrak{t}$ we define $\mathrm{Mod}_{f,V}(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural])$ analogously.

Fix $u \in T_{\mathrm{un}}$. To $\mathcal{R}$ and u we can associate some new objects. First we define the root system

$$R_u = \{\alpha \in R : s_\alpha(u) = u\},$$

and we let Δ_u be the unique basis of R_u contained in R^+. Then

$$(W\Gamma)_u = W(R_u) \rtimes \Gamma'_u, \qquad \Gamma'_u = \{w \in W\Gamma : w(u) = u,\ w(\Delta_u) = \Delta_u\}.$$

Now we can define the based root data

$$\mathcal{R}_u = (X, R_u, Y, R_u^\vee, \Delta_u) \quad \text{and} \quad \widetilde{\mathcal{R}_u} = (\mathfrak{a}^*, R_u, \mathfrak{a}, R_u^\vee, \Delta_u).$$

We define a parameter function $k_u : R_u \to \mathbb{R}$ for $\widetilde{\mathcal{R}_u}$ by

$$2k_{u,\alpha} = \log(q(s_\alpha)) + \alpha(u) \log(q(t_\alpha s_\alpha)).$$

Let $\natural_u : (\Gamma'_u)^2 \to \mathbb{C}^\times$ be the restriction to $\natural$. With a slight variation on Lusztig's reduction theorems [Lusztig 1989, §8–9], one can prove:

Theorem 1.49. *Let $q : W^e \to \mathbb{R}_{>0}$ be a positive parameter function. The categories*

$$\mathrm{Mod}_{f,W\Gamma u T_{\mathrm{rs}}}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]) \quad \textit{and} \quad \mathrm{Mod}_{f,\mathfrak{a}}(\mathbb{H}(\widetilde{\mathcal{R}_u}, k_u) \rtimes \mathbb{C}[\Gamma'_u, \natural_u])$$

are equivalent. The equivalence respects parabolic induction, temperedness and discrete series.

Proof. Let $\widetilde{\Gamma}$ and the central idempotent $p_\natural$ be as in (1.38). Then

$$\begin{aligned} \mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural] &= p_\natural(\mathcal{H}(\mathcal{R}, q) \rtimes \widetilde{\Gamma}), \\ \mathbb{H}(\widetilde{\mathcal{R}_u}, k_u) \rtimes \mathbb{C}[\Gamma'_u, \natural_u] &= p_\natural(\mathbb{H}(\widetilde{\mathcal{R}_u}, k_u) \rtimes \widetilde{\Gamma}'_u). \end{aligned} \tag{1.50}$$

By [Solleveld 2012a, Corollary 2.15] the theorem holds for $\mathcal{H}(\mathcal{R}, q) \rtimes \widetilde{\Gamma}$ and $\mathbb{H}(\widetilde{\mathcal{R}_u}, k_u) \rtimes \widetilde{\Gamma}'_u$. The claimed properties of this equivalence were checked in detail in [Aubert et al. 2016, §2.1].

This is based on a comparison of localizations of these algebras, as in [Lusztig 1989]. The comparison maps [Solleveld 2012a, Theorems 2.1.2 and 2.1.4] are the identity on $\mathbb{C}[\widetilde{\Gamma}'_u \cap \widetilde{\Gamma}]$, so they preserve $p_\natural$. Hence we can restrict the result from [Solleveld 2012a] to the direct summands (1.50). □

From Theorem 1.49 and (1.35) (and (1.34) and (1.48) for the bottom line) we construct a diagram

$$\begin{array}{ccc} \mathrm{Mod}_{f,W\Gamma u T_{\mathrm{rs}}}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]) & \xrightarrow{\sim} & \mathrm{Mod}_{f,\mathfrak{a}}(\mathbb{H}(\widetilde{\mathcal{R}_u}, k_u) \rtimes \mathbb{C}[\Gamma'_u, \natural_u]) \\ \downarrow{\scriptstyle \mathrm{r}_u} & & \downarrow{\scriptstyle \mathrm{r}} \\ \mathrm{Mod}_{f,W\Gamma u}(\mathcal{H}(\mathcal{R}, 1) \rtimes \mathbb{C}[\Gamma, \natural]) & \xleftarrow{\sim} & \mathrm{Mod}_{f,0}(\mathbb{H}(\widetilde{\mathcal{R}_u}, 0) \rtimes \mathbb{C}[\Gamma'_u, \natural_u]) \\ \| & & \| \\ \mathrm{Mod}_{f,W\Gamma u}(\mathcal{O}(T) \rtimes \mathbb{C}[W\Gamma, \natural]) & \xleftarrow{\sim} & \mathrm{Mod}_{f,0}(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[(W\Gamma)_u, \natural_u]) \end{array}$$

where r_u is the unique map that makes the diagram commutative. Using the technique in the proof of Theorem 1.49, we can immediately extend all relevant results in [Solleveld 2012a] from $\mathcal{H}(\mathcal{R}, q) \rtimes \widetilde{\Gamma}$ to twisted affine Hecke algebras. In view of this, we will freely use results from [Solleveld 2012a] in that generality.

As shown in [Solleveld 2012a, §2.3], there exists a unique system of $\mathbb{Z}$-linear maps

$$\zeta^\vee : R_{\mathbb{Z}}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]) \to R_{\mathbb{Z}}(\mathcal{H}(\mathcal{R}, 1) \rtimes \mathbb{C}[\Gamma, \natural]) \tag{1.51}$$

(for all possible $\mathcal{R}, q, \Gamma$) such that

- $\zeta^\vee(\pi) = \mathrm{r}_u(\pi)$ for tempered representations in $\mathrm{Mod}_{f, W\Gamma u T_{\mathrm{rs}}}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural])$,
- $\zeta^\vee$ commutes with parabolic induction,
- $\zeta^\vee$ respects the formation of standard modules for the Langlands classification, in the sense of [Solleveld 2012a, Corollary 2.2.5].

Theorem 1.52. *The map (1.51) is bijective for every positive parameter function q.*

Proof. Proposition 1.36 and Theorem 1.49 imply that (1.51) gives a bijection

$$R_{\mathbb{Z},\mathrm{temp}, W\Gamma u T_{\mathrm{rs}}}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]) \to R_{\mathbb{Z},\mathrm{temp}, W\Gamma u}(\mathcal{H}(\mathcal{R}, 1) \rtimes \mathbb{C}[\Gamma, \natural]), \tag{1.53}$$

where the subscripts “temp” indicate that we formed these Grothendieck groups by starting with tempered modules only. Any tempered $\mathcal{O}(T) \rtimes \mathbb{C}[W\Gamma, \natural]$-module only has $\mathcal{O}(T)$-weights in T_{un}, so on the right-hand side of (1.53) we may just as well replace $W\Gamma u$ by $W\Gamma u T_{\mathrm{rs}}$. Thus (1.51) restricts to a bijection between subgroups generated by tempered modules on both sides.

In [Solleveld 2012a, Corollary 2.3.2] it was shown that (1.51) becomes a $\mathbb{Q}$-linear bijection upon tensoring both sides with $\mathbb{Q}$. The second half of the proof of that result (see [Solleveld 2012a, §3.4]) extends the statement from the tempered to the general case. It says essentially that whatever happens in the space $\mathrm{Irr}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural])$ can be detected and understood already by looking at tempered representations. From that, the bijectivity in the tempered case and the multiplicity one property of the Langlands classification (every standard module has a unique irreducible quotient, appearing with multiplicity one [Solleveld 2012a, Theorem 2.2.4]), we obtain the bijectivity of (1.51) in general. □

2. Topological K-theory

2A. *The C^*-completion of an affine Hecke algebra.* In this paragraph we recall the structure of C^*-algebras associated to affine Hecke algebras. These deep results mainly stem from [Opdam 2004; Delorme and Opdam 2008; 2011].

Recall that q is a positive parameter function for $\mathcal{R}$. We define a *-operation and a trace on $\mathcal{H}(\mathcal{R}, q)$ by

$$\Big(\sum\nolimits_{w \in W^e} c_w N_w\Big)^* = \sum\nolimits_{w \in W^e} \overline{c_w} N_{w^{-1}}, \qquad \tau\Big(\sum\nolimits_{w \in W^e} c_w N_w\Big) = c_e.$$

Since $q(s_\alpha) > 0$, * preserves the relations (1.43) and defines an anti-involution of $\mathcal{H}(\mathcal{R}, q)$. The set $\{N_w : w \in W^e\}$ is an orthonormal basis of $\mathcal{H}(\mathcal{R}, q)$ for the inner product

$$\langle h_1, h_2 \rangle = \tau(h_1^* h_2).$$

This gives $\mathcal{H}(\mathcal{R}, q)$ the structure of a Hilbert algebra. The Hilbert space completion $L^2(\mathcal{R})$ of $\mathcal{H}(\mathcal{R}, q)$ is a module over $\mathcal{H}(\mathcal{R}, q)$, via left multiplication. Moreover, every $h \in \mathcal{H}(\mathcal{R}, q)$ acts as a bounded linear operator [Opdam 2004, Lemma 2.3]. The reduced C^*-algebra of $\mathcal{H}(\mathcal{R}, q)$ [Opdam 2004, §2.4], denoted by $C_r^*(\mathcal{R}, q)$, is defined as the closure of $\mathcal{H}(\mathcal{R}, q)$ in the algebra of bounded linear operators on $L^2(\mathcal{R})$.

As in (1.47), we can extend this to a C^*-algebra $C_r^*(\mathcal{R}, q) \rtimes \Gamma$, provided that q is Γ-invariant. We will not bother about twisted group algebras $\mathbb{C}[\Gamma, \natural]$ in this section, for with the technique from (1.50) it is easy to generalize our results to that setting, and in the context of C^*-algebras, crossed products with groups look much more natural.

Let us recall some background about $C_r^*(\mathcal{R}, q) \rtimes \Gamma$, mainly from [Opdam 2004; Solleveld 2012a]. It follows from [Delorme and Opdam 2008, Corollary 5.7] that it is a finite type I C^*-algebra and that $\mathrm{Irr}(C_r^*(\mathcal{R}, q))$ is precisely the tempered part of $\mathrm{Irr}(\mathcal{H}(\mathcal{R}, q))$. The structure of $C_r^*(\mathcal{R}, q) \rtimes \Gamma$ is described in terms of parabolically induced representations. As induction data we use triples (P, δ, t), where

- $P \subset \Delta$,
- δ is an irreducible discrete series representation of $\mathcal{H}_P$,
- $t \in T^P$.

We regard two triples (P, δ, t) and (P', δ', t') as equivalent if $P = P'$, $t = t'$ and $\delta \cong \delta'$. Notice that $\mathcal{H}_P$ comes from a semisimple root datum, so it can have discrete series representations. For every $t \in T^P$ there exists a surjection $\phi_t : \mathcal{H}^P \to \mathcal{H}_P$, which combines the projection $X \to X_P$ with evaluation at t. To such a triple (P, δ, t) we associate the $\mathcal{H} \rtimes \Gamma$-representation

$$\pi^\Gamma(P, \delta, t) = \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H} \rtimes \Gamma}(\delta \circ \phi_t).$$

(When $\Gamma = 1$, we often suppress it from these and similar notations.) For any $t \in T_{\mathrm{un}}^P = T^P \cap T_{\mathrm{un}}$ these representations extend continuously to the respective C^*-completions of the involved algebras. Let Ξ_{un} be the set of triples (P, δ, t) as above, such that moreover $t \in T_{\mathrm{un}}$. Considering P and δ as discrete variables, we regard Ξ_{un} as a disjoint union of finitely many compact real tori (of different dimensions).

Let $\mathcal{V}_\Xi^\Gamma$ be the vector bundle over Ξ_{un} whose fibre at (P, δ, t) is the vector space underlying $\pi^\Gamma(P, \delta, t)$. That vector space is independent of t, so the vector bundle is trivial. Let $\mathrm{End}(\mathcal{V}_\Xi^\Gamma)$ be the algebra bundle with fibres $\mathrm{End}_{\mathbb{C}}(\pi^\Gamma(P, \delta, t))$. Every element of $C_r^*(\mathcal{R}, q) \rtimes \Gamma$ naturally defines a continuous section of $\mathrm{End}(\mathcal{V}_\Xi^\Gamma)$.

There exists a finite groupoid $\mathcal{G}$ which acts on $\mathrm{End}(\mathcal{V}_\Xi^\Gamma)$. It is made from elements of $W \rtimes \Gamma$ and of $K_P := T_P \cap T^P$. More precisely, its base space is the power set of Δ, and for $P, Q \subseteq \Delta$ the collection of arrows from P to Q is

$$\mathcal{G}_{PQ} = \{(g, u) : g \in \Gamma \ltimes W, u \in K_P, g(P) = Q\}. \tag{2.1}$$

Whenever it is defined, the multiplication in $\mathcal{G}$ is

$$(g', u') \cdot (g, u) = (g'g, g^{-1}(u')u).$$

In particular, writing $W\Gamma(P, P) = \{w \in W\Gamma : w(P) = P\}$, we have the group

$$\mathcal{G}_{PP} = W\Gamma(P, P) \rtimes K_P. \tag{2.2}$$

Usually we will write elements of $\mathcal{G}$ simply as gu. For $\gamma \in \Gamma W$ with $\gamma(P) = Q \subset \Delta$, there are algebra isomorphisms

$$\begin{aligned} \psi_\gamma : \mathcal{H}_P \to \mathcal{H}_Q, \quad & \theta_{x_P} N_w \mapsto \theta_{\gamma(x_P)} N_{\gamma w \gamma^{-1}}, \\ \psi_\gamma : \mathcal{H}^P \to \mathcal{H}^Q, \quad & \theta_x N_w \mapsto \theta_{\gamma x} N_{\gamma w \gamma^{-1}}. \end{aligned} \tag{2.3}$$

The groupoid $\mathcal{G}$ acts from the left on Ξ_{un} by

$$(g, u) \cdot (P, \delta, t) := (g(P), \delta \circ \psi_u^{-1} \circ \psi_g^{-1}, g(ut)), \tag{2.4}$$

the action being defined if and only if $g(P) \subset \Delta$. Suppose that $g(P) = Q \subset \Delta$ and $\delta' \cong \delta \circ \psi_u^{-1} \circ \psi_g^{-1}$. By [Opdam 2004, Theorem 4.33] and [Solleveld 2012a, Theorem 3.1.5], there exists an intertwining operator

$$\pi^\Gamma(gu, P, \delta, t) \in \mathrm{Hom}_{\mathcal{H}(\mathcal{R},q)\rtimes\Gamma}\big(\pi^\Gamma(P, \delta, t), \pi^\Gamma(Q, \delta', g(ut))\big) \tag{2.5}$$

which depends algebraically on $t \in T^P_{\mathrm{un}}$. Then the action of $\mathcal{G}$ on the continuous sections $C(\Xi_{\mathrm{un}}; \mathrm{End}(\mathcal{V}_\Xi^\Gamma))$ is given by

$$(g\cdot f)(\xi) = \pi^\Gamma(g, g^{-1}\xi) f(g^{-1}\xi) \pi^\Gamma(g, g^{-1}\xi)^{-1}, \quad g \in \mathcal{G}_{PQ},\ \xi = (Q, \delta', t'). \tag{2.6}$$

Theorem 2.7 [Delorme and Opdam 2008, Corollary 5.7; Solleveld 2012a, Theorem 3.2.2]. *There exists a canonical isomorphism of C^*-algebras*

$$C_r^*(\mathcal{R}, q) \rtimes \Gamma \xrightarrow{\sim} C(\Xi_{\mathrm{un}}; \mathrm{End}(\mathcal{V}_\Xi^\Gamma))^{\mathcal{G}}.$$

For $q = 1$ this simplifies to the well-known isomorphism

$$C_r^*(\mathcal{R}, 1) \rtimes \Gamma = C(T_{\mathrm{un}}) \rtimes W\Gamma \xrightarrow{\sim} C(T_{\mathrm{un}}; \mathrm{End}_{\mathbb{C}}(\mathbb{C}[W\Gamma]))^{W\Gamma}. \tag{2.8}$$

Let $\mathcal{G}_{P,\delta}$ be the setwise stabilizer of $(P, \delta, T^P_{\mathrm{un}})$ in the group $\mathcal{G}_{PP}$. Let $(P, \delta)/\mathcal{G}$ be a set of representatives for the action of $\mathcal{G}$ on pairs (P, δ) obtained from (2.4). Theorem 2.7 can be rephrased as an isomorphism

$$C_r^*(\mathcal{R}, q) \rtimes \Gamma \xrightarrow{\sim} \bigoplus_{(P,\delta)/\mathcal{G}} C\big(T^P_{\mathrm{un}}; \mathrm{End}_{\mathbb{C}}(\pi^\Gamma(P, \delta, t))\big)^{\mathcal{G}_{P,\delta}}. \tag{2.9}$$

Let us discuss the representation theory of $C_r^*(\mathcal{R}, q) \rtimes \Gamma$ (i.e., the tempered unitary representations of $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$)) in more detail. Our approach, following Harish-Chandra and Opdam, starts with the discrete series of a parabolic subalgebra $\mathcal{H}(\mathcal{R}_P, q_P) = \mathcal{H}_P$. It is known from [Opdam 2004, Lemma 3.31] that the central character of any (irreducible) discrete series representation δ of $\mathcal{H}_P$ (a W_P-orbit in T_P) has a very specific property: it must consist of *residual points* in T_P, with respect to $(\mathcal{R}_P, q_P)$.

For $t \in T_P$ we write

$$\begin{aligned} R_P^z(t) &= \{\alpha \in R_P : \alpha(t) \in \{1, -1\}\}, \\ R_P^p(t) &= \big\{\alpha \in R_P : \alpha(t) \in \{q(s_\alpha)^{1/2} q(s_\alpha t_\alpha)^{1/2}, -q(s_\alpha)^{1/2} q(s_\alpha t_\alpha)^{-1/2}\}\big\}. \end{aligned}$$

(We remark that there is only one irreducible root datum for which $q(s_\alpha t_\alpha)$ need not be equal to $q(s_\alpha)$, namely with $R = B_n$.) By definition $t \in T_P$ is residual if

$$|R_P^p(t)| - |R_P^z(t)| = \dim_{\mathbb{C}}(T_P) = |P|.$$

Residuality depends in a subtle way on the parameters q. For instance, when $q = 1$ and $X_P \neq 0$, there are no residual points. Residual points have been classified in [Heckman and Opdam 1997]. It turns out that all the coordinates of a residual point t are monomials in the parameters $q(s)^{\pm 1/2}$, $s \in S^{\mathrm{aff}}$. Thus we can write $t = t(q^{1/2})$.

Let $\mathcal{Q}(\mathcal{R})$ be the space of all maps $q : S^{\mathrm{aff}} \to \mathbb{R}_{>0}$ such that $q(s) = q(s')$ if s and s' are conjugate in $X \rtimes W\Gamma$. Given $t = t(q^{1/2})$, there is a Zariski-open subset of the real variety $\mathcal{Q}(\mathcal{R})$ on which $t(q^{1/2})$ defines a residual point. For this reason we call the map

$$\mathcal{Q}(\mathcal{R}) \to T : q \mapsto t(q^{1/2})$$

a generic residual point. We say that a parameter function $q \in \mathcal{Q}(\mathcal{R})$ is generic if all generic residual points for parabolic subalgebras $\mathcal{H}_P$ of $\mathcal{H}$ are actually residual points for that q.

When there is only one free parameter in q, for instance when R is of type A, D or E, then every positive parameter function $q \neq 1$ is generic. On the other hand, when R contains root systems of type B, C, F or G, then usually no equal parameter function ($q(s) = q(s')$ for all $s, s' \in S^{\mathrm{aff}}$) is generic.

The discrete series representations of $\mathcal{H}(\mathcal{R}_P, q_P)$ were classified in [Opdam and Solleveld 2010], at least when R is irreducible and q_P generic. Later the classification was extended to the nongeneric cases, along with an actual construction of the representations, in [Ciubotaru and Opdam 2017]. Using these papers, it is in principle always possible to find a set of representatives for the action of $\mathcal{G}$ on the pairs (P, δ) as in (2.9).

Now we describe a single direct summand $C\big(T^P_{\mathrm{un}}; \mathrm{End}_{\mathbb{C}}(\pi^\Gamma(P,\delta,t))\big)^{\mathcal{G}_{P,\delta}}$ of (2.9) more explicitly. Fix $t \in T^P_{\mathrm{un}}$ and let $\mathcal{G}_\xi$ be the isotropy group of $\xi = (P,\delta,t)$ in $\mathcal{G}$. The intertwining operators $\pi^\Gamma(g,\xi)$, $g \in \mathcal{G}_\xi$ make $\pi^\Gamma(\xi)$ into a projective $\mathcal{G}_\xi$-representation. Decompose it as

$$\pi^\Gamma(\xi) = \bigoplus\nolimits_\rho \mathbb{C}^{m_\rho} \otimes V_\rho,$$

where (ρ, V_ρ) runs through the set of (equivalence classes of) irreducible projective $\mathcal{G}_\xi$-representations. From (2.6) we see that the evaluation at t of any element of $C\big(T^P_{\mathrm{un}}; \mathrm{End}_{\mathbb{C}}(\pi^\Gamma(P,\delta,t))\big)^{\mathcal{G}_{P,\delta}}$ lies in

$$\mathrm{End}_{\mathcal{G}_\xi}(\pi^\Gamma(\xi)) \cong \bigoplus\nolimits_\rho \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{m_\rho}).$$

The action of $\mathcal{G}_\xi$ on $\pi^\Gamma(P,\delta,t)$ can be analyzed further with the theory of R-groups from [Delorme and Opdam 2011]. In that paper there is no group Γ, but with the intertwining operators as in [Solleveld 2012a, Theorem 3.1.5] the extension to the case with Γ is straightforward. By [Delorme and Opdam 2011, Propositions 4.5 and 4.7] there exists a root system R_ξ on which $\mathcal{G}_\xi$ acts, and an R-group $\mathfrak{R}_\xi = \mathrm{Stab}_{\mathcal{G}_\xi}(R_\xi \cap R_P^+)$, such that

$$\mathcal{G}_\xi = W(R_\xi) \rtimes \mathfrak{R}_\xi. \tag{2.10}$$

By [Delorme and Opdam 2011, Theorem 4.3(iv)] the intertwining operator $\pi^\Gamma(g,\xi)$ is a scalar multiple of the identity if $g \in W(R_\xi)$. Hence,

$$\mathrm{End}_{\mathcal{G}_\xi}(\pi^\Gamma(\xi)) = \mathrm{End}_{\mathfrak{R}_\xi}(\pi^\Gamma(\xi)).$$

Moreover, the operators

$$\pi^\Gamma(r,\xi) \in \mathrm{End}_{\mathbb{C}}(\pi^\Gamma(\xi)), \quad r \in \mathfrak{R}_\xi,$$

are linearly independent by [Delorme and Opdam 2011, Theorem 5.4]. To classify all irreducible representations of $C\big(T^P_{\mathrm{un}}; \mathrm{End}_{\mathbb{C}}(\pi^\Gamma(P,\delta,t))\big)^{\mathcal{G}_{P,\delta}}$, it remains to determine (2.10) and to study $\pi^\Gamma(\xi)$ as a projective $\mathfrak{R}_\xi$-representation, for all $\xi = (P,\delta,t)$. In all cases that we will encounter in this paper, $\mathfrak{R}_\xi$ is abelian and $\pi^\Gamma(\xi)$ is actually a linear $\mathfrak{R}_\xi$-representation. Together with Theorem 1.52 this enables us to determine $\mathrm{Irr}(C_r^*(\mathcal{R},q) \rtimes \Gamma)$ in those cases.

2B. ***K-theory and equivariant cohomology.*** The computation of the topological K-theory of $C_r^*(\mathcal{R},q) \rtimes \Gamma$ is the main goal of this paper. It follows from (2.9), especially the compactness of T^P_{un}, that the abelian group

$$K_*(C_r^*(\mathcal{R},q) \rtimes \Gamma) = K_0(C_r^*(\mathcal{R},q) \rtimes \Gamma) \oplus K_1(C_r^*(\mathcal{R},q) \rtimes \Gamma)$$

is finitely generated; see [Solleveld 2012a, Lemma 5.1.3] and its proof. By [Solleveld 2012a, Theorem 5.1.4], which relies on the study of the representation theory and of parameter deformations of affine Hecke algebras in [Solleveld 2012a], the group $\mathbb{Q} \otimes_{\mathbb{Z}} K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$ does not depend on the parameters q. Combining this with the conclusions from Section 1C, we will deduce that also $K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$ itself is independent of q.

Next we use equivariant cohomology and the equivariant Chern character to express $K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$ in terms of the cohomology of a sheaf on a CW-complex. This is inspired by the equivariant Chern characters with values in Bredon cohomology developed in [Słomińska 1976; Lück and Oliver 2001]. Our version also applies to certain noncommutative algebras, and provides more information about the torsion elements than [Słomińska 1976; Lück and Oliver 2001].

In [Solleveld 2012a, Theorem 4.4.2] an injective homomorphism of C^*-algebras

$$\zeta_0 : C_r^*(\mathcal{R}, 1) \rtimes \Gamma \to C_r^*(\mathcal{R}, q) \rtimes \Gamma$$

was constructed, with the property

$$\pi \circ \zeta_0 \cong \zeta^\vee(\pi) \quad \text{for all } \pi \in \mathrm{Mod}_f(C_r^*(\mathcal{R}, q) \rtimes \Gamma).$$

Theorem 2.11. *The map $K_*(\zeta_0) : K_*(C_r^*(\mathcal{R}, 1) \rtimes \Gamma) \to K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$ is an isomorphism.*

Proof. Let $u \in T_{\mathrm{un}}$. Then (1.53) says that $\zeta^\vee$ provides a bijection between the Grothendieck group of finite length $C_r^*(\mathcal{R}, q) \rtimes \Gamma$-modules with $Z(\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma)$-character in $W\Gamma u T_{rs}$ and the analogous group for $C_r^*(X \rtimes W) \rtimes \Gamma$. For tempered modules $\zeta^\vee$ agrees with the map ζ^* from [Solleveld 2012a, §2.3].

These C^*-completions have the same irreducible representations as the respective Schwartz completions of these algebras (see [Opdam 2004, §6] or [Solleveld 2012a, §3.2]), namely the irreducible tempered representations of the underlying affine Hecke algebras. That follows from the comparison of Theorem 2.7 with its analogue for Schwartz completions [Solleveld 2012a, Theorem 3.2.2]. With these translation steps we see that part (c) of [Solleveld 2012a, Lemma 5.1.5] holds. Then [Solleveld 2012a, Lemma 5.1.5] tells us that also its part (a) holds, which is the statement of the theorem. □

When we want to compute $K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$, we can use Theorem 2.11 to replace q by 1, then apply it another time to replace 1 by any positive parameter function q' we like. We will do the actual computation either when $q = 1$ or when q is generic among all possible parameter functions.

In Section 3 we will encounter many root data $\mathcal{R}$ which are a product of root data $\mathcal{R}_1$ and $\mathcal{R}_2$. If Γ_i is a group acting on $\mathcal{R}_i$ in the usual way, then $\Gamma := \Gamma_1 \times \Gamma_2$ acts on $\mathcal{R}$. In this case $C_r^*(\mathcal{R}, q) \rtimes \Gamma$ is defined as an algebra of bounded linear operators on

$$L^2(\mathcal{R}) \otimes \mathbb{C}[\Gamma] = L^2(\mathcal{R}_1) \otimes \mathbb{C}[\Gamma_1] \otimes L^2(\mathcal{R}_2) \otimes \mathbb{C}[\Gamma_2].$$

It is the closure of the algebraic tensor product of the algebras $C_r^*(\mathcal{R}_1, q_1) \rtimes \Gamma_1$ and $C_r^*(\mathcal{R}_2, q_2) \rtimes \Gamma_2$ in $B(L^2(\mathcal{R}) \otimes \mathbb{C}[\Gamma])$, which means that

$$C_r^*(\mathcal{R}, q) \rtimes \Gamma = C_r^*(\mathcal{R}_1, q_1) \rtimes \Gamma_1 \otimes_{\min} C_r^*(\mathcal{R}_2, q_2) \rtimes \Gamma_2, \tag{2.12}$$

the minimal tensor product of C^*-algebras. These C^*-algebras are separable and of type I, so the paper [Schochet 1982] applies to them. The Künneth theorem [Schochet 1982] says that there exists a natural $\mathbb{Z}/2\mathbb{Z}$-graded short exact sequence

$$\begin{aligned} 0 \to K_*(C_r^*(\mathcal{R}_1, q_1) \rtimes \Gamma_1) \otimes_{\mathbb{Z}} K_*(C_r^*(\mathcal{R}_2, q_2) \rtimes \Gamma_2) \to K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma) \\ \to \mathrm{Tor}_{\mathbb{Z}}\big(K_*(C_r^*(\mathcal{R}_1, q_1) \rtimes \Gamma_1), K_*(C_r^*(\mathcal{R}_2, q_2) \rtimes \Gamma_2)\big) \to 0. \end{aligned} \tag{2.13}$$

In particular, this becomes an isomorphism

$$K_*(C_r^*(\mathcal{R}_1, q_1) \rtimes \Gamma_1) \otimes_{\mathbb{Z}} K_*(C_r^*(\mathcal{R}_2, q_2) \rtimes \Gamma_2) \xrightarrow{\sim} K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$$

if $K_*(C_r^*(\mathcal{R}_i, q_i) \rtimes \Gamma_i)$ has no torsion for $i = 1, 2$. With (2.13) we can often reduce the computation of K-groups to the case where R is irreducible.

By (2.8) and the Green–Julg Theorem [Julg 1981],

$$K_*(C_r^*(\mathcal{R}, 1) \rtimes \Gamma) = K_*(C(T_{\mathrm{un}}) \rtimes W\Gamma) \cong K_*^{W\Gamma}(C(T_{\mathrm{un}})).$$

Moreover, by the equivariant Serre–Swan theorem [Phillips 1987, Theorem 2.3.1],

$$K_*^{W\Gamma}(C(T_{\mathrm{un}})) \cong K^*_{W\Gamma}(T_{\mathrm{un}}). \tag{2.14}$$

Together with Theorem 2.7 we get

$$K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma) \cong K^*_{W\Gamma}(T_{\mathrm{un}}). \tag{2.15}$$

The right-hand side in (2.14) and (2.15) is just Atiyah's $W\Gamma$-equivariant K-theory of the compact Hausdorff space T_{un}. Let $T_{\mathrm{un}} /\!/ W\Gamma$ be the extended quotient (see also Section 2C). We recall from [Baum and Connes 1988, Theorem 1.19] that the equivariant Chern character gives a natural isomorphism

$$K^*_{W\Gamma}(T_{\mathrm{un}}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} H^*(T_{\mathrm{un}} /\!/ W\Gamma; \mathbb{C}). \tag{2.16}$$

(Here H^* could be many cohomology theories; in this paper we stick to Čech cohomology.) With (2.14) we find a canonical isomorphism

$$K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^*(T_{\mathrm{un}} /\!/ W\Gamma; \mathbb{C}). \tag{2.17}$$

In (2.16) it is essential to use complex coefficients, so this does not tell us much about the torsion in $K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$. To study the torsion elements better, we will compare the topological K-theory of relevant C^*-algebras with a suitable version of equivariant cohomology from [Bredon 1967]. Let Σ be a countable, locally finite and finite dimensional G-CW-complex, where G is a finite group. Assume that all cells are oriented and that the action of G preserves these orientations.

We define a category $\mathcal{K}$ whose objects are the finite subcomplexes of Σ. The morphisms from K to K' are the maps $K \to K' : x \to gx$ for $g \in G$ such that $gK \subset K'$. Now a local coefficient system on Σ is a covariant functor from $\mathcal{K}$ to the category of abelian groups, and the group $C^q(\Sigma; \mathfrak{L})$ of q-cochains is the set of all functions f on the q-cells of Σ with the property that $f(\tau) \in \mathfrak{L}(\tau)$ for all τ. Furthermore, we define a coboundary map $\mathrm{d} : C^q(\Sigma; \mathfrak{L}) \to C^{q+1}(\Sigma; \mathfrak{L})$ by

$$(\mathrm{d}f)(\sigma) = \sum_{\tau \in \Sigma^{(q)}} [\tau : \sigma] \mathfrak{L}(\tau \to \sigma) f(\tau), \tag{2.18}$$

where the sum runs over the set $\Sigma^{(q)}$ of all q-cells and the incidence number $[\tau : \sigma]$ is the degree of the attaching map from $\partial\sigma$ (the boundary of σ in the standard topological sense) to $\tau/\partial\tau$. The group G acts naturally on this complex by cochain maps so, for any $K \subset \Sigma$, $(C^*(K; \mathfrak{L})^G, \mathrm{d})$ is a differential complex. We define the equivariant cohomology of K with coefficients in $\mathfrak{L}$ as

$$H^q_G(K; \mathfrak{L}) := H^q(C^*(K; \mathfrak{L})^G, \mathrm{d}). \tag{2.19}$$

More generally, for $K' \subset K$, $C^*(K, K'; \mathfrak{L})$ is the kernel of the restriction map $C^*(K; \mathfrak{L}) \to C^*(K'; \mathfrak{L})$ and

$$H^q_G(K, K'; \mathfrak{L}) = H^q(C^*(K, K'; \mathfrak{L})^G, \mathrm{d}). \tag{2.20}$$

By construction there exists a local coefficient system $\mathfrak{L}^G$ (more or less consisting of the G-invariant elements of $\mathfrak{L}$) on the CW-complex Σ/G such that the differential complexes $(C^*(K, K'; \mathfrak{L})^G, \mathrm{d})$ and $(C^*(K/G, K'/G; \mathfrak{L}^G), \mathrm{d})$ are isomorphic. Notice that $\mathfrak{L}^G$ defines a sheaf over Σ/G (with the cells as cover), such that

$$H^q_G(K, K'; \mathfrak{L}) \cong \check{H}^q(K/G, K'/G; \mathfrak{L}^G). \tag{2.21}$$

Let Σ^p be the p-skeleton of Σ. We capture all the above things in a spectral sequence $(E_r^{p,q})_{r \geq 1}$, degenerating already for $r \geq 2$, as follows:

$$E_1^{p,q} = H_G^{p+q}(\Sigma^p, \Sigma^{p-1}; \mathfrak{L}) = \begin{cases} C^p(\Sigma; \mathfrak{L})^G & \text{if } q = 0, \\ 0 & \text{if } q > 0, \end{cases} \tag{2.22}$$

$$E_2^{p,q} = \begin{cases} H_G^p(\Sigma; \mathfrak{L}) & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases} \tag{2.23}$$

The differential d_1^E is the composition

$$E_1^{p,q} \to C^{p+q}(\Sigma^p; \mathfrak{L})^G \to E_1^{p+1,q} \tag{2.24}$$

of the maps induced by the inclusion $(\Sigma^p, \varnothing) \to (\Sigma^p, \Sigma^{p-1})$ and the coboundary d.

We are mostly interested in this cohomology theory for a particular coefficient system, which we now define. Consider the Fréchet algebra

$$B = C(\Sigma; M_N(\mathbb{C})) = M_N(C(\Sigma)). \tag{2.25}$$

(It is a C^*-algebra if Σ is compact.) We assume that we have $u_g \in B^\times$ such that

$$gb(x) = u_g(x) b(g^{-1}x) u_g^{-1}(x) \tag{2.26}$$

defines an action of G on B. Then the invariants B^G constitute a Fréchet subalgebra of B. Notice that by (2.6) and (2.9) the C^*-completion of an affine Hecke algebra is a direct sum of algebras of this form.

To associate a local coefficient system to B^G. we first assume that K is connected. In that case we let

$$G_K := \{g \in G : gx = x \ \forall x \in K\} \tag{2.27}$$

be the isotropy group of K and we define $\mathfrak{L}_u(K)$ to be the free abelian group on the (equivalence classes of) irreducible projective G_K-representations contained in $(\pi_x, \mathbb{C}^N)$, where $\pi_x(g) = u_g(x)$ for $g \in G_K$, $x \in K$. By the continuity of the u_g we get the same group for any $x \in K$. If K is not connected, then we let $\{K_i\}_i$ be its connected components, and we define

$$\mathfrak{L}_u(K) = \textstyle\prod_i \mathfrak{L}_u(K_i). \tag{2.28}$$

Suppose that $gK \subset K'$ and that ρ is a projective G_K-representation. Then we define a projective $G_{K'}$-representation by

$$\mathfrak{L}_u(g : K \to K')\rho(g') = \rho(g^{-1}g'g), \quad g' \in G_{K'}. \tag{2.29}$$

If $h \in G$ gives the same map from K to K' as g then $h^{-1}g \in G_K$ and

$$\mathfrak{L}_u(h : K \to K')\rho(g') = \rho(h^{-1}g'h) = \rho(h^{-1}g)\rho(g^{-1}g'g)\rho(g^{-1}h), \tag{2.30}$$

so $\mathfrak{L}_u(h : K \to K')\rho$ is isomorphic to $\mathfrak{L}_u(g : K \to K')\rho$ as a projective representation. This makes $\mathfrak{L}_u$ into a functor. We can regard $\mathfrak{L}_u$ as a sheaf on Σ, where a section s is continuous on U if and only if $s(K)|_{G_{K'}} = s(K')$ for every inclusion $K \subset K' \subset U$.

Example 2.31. Suppose that $u_g(x) = 1$ for all $x \in \Sigma$, $g \in G$. Then $\mathfrak{L}_u$ and $\mathfrak{L}_u^G$ are the constant sheaves $\mathbb{Z}$ over Σ and Σ/G, respectively, and

$$H^*_G(\Sigma; \mathfrak{L}_u) \cong \check{H}^*(\Sigma/G; \mathbb{Z}) \tag{2.32}$$

is the ordinary cellular cohomology of Σ/G. Furthermore,

$$K_*(B^G) \cong K_*(C(\Sigma/G; M_N(\mathbb{C}))) = K_*(C(\Sigma/G)),$$

which is isomorphic to $\check{H}^*(\Sigma/G; \mathbb{Z})$ modulo torsion.

It turns out that a relation like (2.16), between $K_*(B^G)$ and the Čech cohomology $H^*(\Sigma/G;\mathfrak{L}_u^G)$, is valid in the generality of the algebras B^G from (2.25) and (2.26). Notice that we do not require Σ to be compact; we consider the K-theory of B^G as a Fréchet algebra. The skeleton of the CW-complex Σ gives rise to the following filtration:

$$\begin{aligned} K_*(B^G) &= K_*^0(B^G) \supset K_*^1(B^G) \supset \cdots \supset K_*^{\dim\Sigma}(B^G) \supset K_*^{1+\dim\Sigma}(B^G) = 0, \\ K_*^p(B^G) &:= \mathrm{im}\big(K_*(C_0(\Sigma/\Sigma^{p-1}; M_N(\mathbb{C}))^G) \to K_*(C(\Sigma; M_N(\mathbb{C}))^G)\big). \end{aligned} \tag{2.33}$$

Theorem 2.34. *The graded group associated with the filtration (2.33) is isomorphic to $\check{H}^*(\Sigma/G;\mathfrak{L}_u^G)$. In particular, there is an (unnatural) isomorphism*

$$K_*(B^G)\otimes\mathbb{Q} \cong \check{H}^*(\Sigma/G;\mathfrak{L}_u^G\otimes\mathbb{Q}) \tag{2.35}$$

and

$$K_*(B^G) \cong \check{H}^*(\Sigma/G;\mathfrak{L}_u^G)$$

if the right-hand side is torsion free.

Proof. For $p, r \geq 0$ we set $K(p, p+r) = K_*(C_0(\Sigma^{p+r-1}/\Sigma^{p-1}; M_N(\mathbb{C}))^G)$. When $p' \geq p$ and $p'+r' \geq p+r$, the map

$$(\Sigma^{p+r-1}, \Sigma^{p-1}) \to (\Sigma^{p'+r'-1}, \Sigma^{p'-1})$$

induces a group homomorphism $K(p', p'+r') \to K(p, p+r)$. For any $s \geq 0$ the sequence

$$(\Sigma^{p+r-1}, \Sigma^{p-1}) \to (\Sigma^{p+r+s-1}, \Sigma^{p-1}) \to (\Sigma^{p+r+s-1}, \Sigma^{p+r-1}) \tag{2.36}$$

gives rise to a connecting homomorphism $K(p, p+r) \to K(p+r, p+r+s)$. Using [Cartan and Eilenberg 1956, Section XV.7] we construct a spectral sequence $(F_r^p)_{r\geq 1}$ with terms

$$\begin{aligned} F_1^p &= K(p, p+1)/K(p,p) = K_*(C_0(\Sigma^p/\Sigma^{p-1}; M_N(\mathbb{C}))^G), \\ F_\infty^p &= K(p,\infty)/K(p+1,\infty) = K_*^p(B^G)/K_*^{p+1}(B^G). \end{aligned} \tag{2.37}$$

The entire setting is $\mathbb{Z}/2\mathbb{Z}$-graded by the K-degree. We put

$$K^q(p, p+r) = K_{p+q}(C_0(\Sigma^{p+r-1}/\Sigma^{p-1}; M_N(\mathbb{C}))^G)$$

and we refine (2.37) to

$$\begin{aligned} F_1^{p,q} &= K_{p+q}(C_0(\Sigma^p/\Sigma^{p-1}; M_N(\mathbb{C}))^G), \\ F_\infty^{p,q} &= K_{p+q}^p(B^G)/K_{p+q}^{p+1}(B^G). \end{aligned} \tag{2.38}$$

By the definition of a G-CW-complex, the pointwise stabilizer of a p-cell σ is equal to its setwise stabilizer in G. Consequently,

$$C_0(\Sigma^p/\Sigma^{p-1}; M_N(\mathbb{C}))^G \cong \prod_{\sigma\in\Sigma^{(p)}/G} C_0(\mathbb{R}^p)\otimes M_N(\mathbb{C})^{G_\sigma}$$

and $F_1^{p,1}=0$. From Bott periodicity and the definition of $\mathfrak{L}_u$ in (2.28) we see that

$$F_1^{p,0}\cong \prod_{\sigma\in\Sigma^{(p)}/G}\mathfrak{L}_u(\sigma)\cong\Big(\prod_{\sigma\in\Sigma^{(p)}}\mathfrak{L}_u(\sigma)\Big)^G.$$

Now replace $\mathfrak{L}$ in (2.22) by $\mathfrak{L}_u$ and sum over all q to obtain E_r^p. If we compare the result with $F_1^p=F_1^{p,0}\oplus F_1^{p,1}$, we see that $E_1^p\cong F_1^p$. So we get a diagram

$$\begin{array}{ccc} F_1^{p,q} & \xrightarrow{\quad d_1^F\quad} & F_1^{p+1,q} \\ \| \wr & & \| \wr \\ \prod_{n\in\mathbb{Z}} E_1^{p,q+2n} & \xrightarrow{d_1^E} & \prod_{n\in\mathbb{Z}} E_1^{p+1,q+2n} \end{array} \tag{2.39}$$

The differential d_1^F for F_1^* is induced from the construction of a mapping cone of a Puppe sequence in the category of C^*-algebras, coming from (2.36). This is the noncommutative counterpart of the construction of the differential in cellular cohomology, so by naturality d_1^F corresponds to d_1^E under the above isomorphism. Therefore, the spectral sequences E_r^p and F_r^p are isomorphic, and in particular F_r^p degenerates for $r\geq 2$. Now the isomorphism (2.35) follows from (2.21).

If $\check{H}^*(\Sigma/G;\mathfrak{L}_u^G)$ is torsion free, then every term $E_\infty^p\cong F_\infty^p$ must be torsion free. Hence in this case both $K_*(B^G)$ and $\check{H}^*(\Sigma/G;\mathfrak{L}_u^G)$ are free abelian groups, of the same rank. □

Theorem 2.34 allows us to reduce the computations of $K_*(C_r^*(\mathcal{R},q))$ to Čech cohomology, where a lot of tools are available. For several root data it is easiest to look at the case $q=1$, for which we will develop more machinery in the next subsection. For some other root data (in particular of type PGL_n) it is more convenient to study $K_*(C_r^*(\mathcal{R},q))$ with $q\neq 1$, for then there are fewer possibilities for torsion elements, compared to $q=1$. In those cases we need the full force of Theorem 2.34.

2C. ***Crossed products.*** In the special case of crossed products the technique from Theorem 2.34 can be improved. A crucial role will be played by the extended quotient, whose definition we recall now. Let G be a finite group G acting on a topological space Σ. We define

$$\widetilde{\Sigma}=\{(g,t)\in G\times T_{\mathrm{un}}: g(t)=t\},$$

a closed subset of the topological space $G \times \Sigma$. The group G acts on $\widetilde{\Sigma}$ by

$$g(g', t) = (gg'g^{-1}, g(t)).$$

The (geometric) extended quotient of Σ by G is defined as

$$\Sigma /\!/ G = \widetilde{\Sigma}/G. \tag{2.40}$$

It decomposes as

$$\Sigma /\!/ G = \bigsqcup_{g \in \mathrm{cc}(G)} \Sigma^w / Z_G(g), \tag{2.41}$$

where $\mathrm{cc}(G)$ denotes a set of representatives for the conjugacy classes in G.

We will develop a method that allows one to pass from the G-equivariant K-theory of Σ to the integral cohomology of $\Sigma /\!/ G$. However, it does not work automatically; we require that the cohomology is torsion-free and that all G-isotropy groups of points of Σ are Weyl groups (and it uses some of our earlier results on the representation rings of Weyl groups).

From now on we assume that Σ is a smooth manifold (possibly with boundary) on which G acts smoothly. According to [Illman 1978] Σ also admits the structure of a countable, locally finite, finite-dimensional G-simplicial complex. The crossed product $C(\Sigma) \rtimes G$ fits in the framework of (2.25) and (2.26) by the isomorphisms

$$C(\Sigma) \rtimes G \cong C(\Sigma; \mathrm{End}_{\mathbb{C}}(\mathbb{C}[G]))^G = B^G. \tag{2.42}$$

In this case $u_g(x)$ is right multiplication by g^{-1} and π_x is the direct sum of $[G : G_x]$ copies of the regular representation of G_x. It is not hard to see that $\mathfrak{L}_u^G \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to the direct image of the constant sheaf $\mathbb{C}$ on $\widetilde{\Sigma}$, under the canonical map $pr : \widetilde{\Sigma}/G \to \Sigma/G$. Since pr is finite-to-one, there are no topological complications, and we get an isomorphism

$$H_G^*(\Sigma; \mathfrak{L}_u \otimes \mathbb{C}) \cong \check{H}^*(\Sigma/G; \mathfrak{L}_u^G \otimes_{\mathbb{Z}} \mathbb{C}) \cong \check{H}^*(\widetilde{\Sigma}/G; \mathbb{C}). \tag{2.43}$$

From this one can recover (2.16). Unfortunately, this approach does not automatically lead to an isomorphism between $\check{H}^*(\Sigma/G; \mathfrak{L}_u^G)$ and $\check{H}^*(\widetilde{\Sigma}/G; \mathbb{Z})$, for $\mathfrak{L}_u^G$ need not be isomorphic to the direct image of the constant sheaf $\mathbb{Z}$ under pr.

Sometimes this can be approached better via a dual homology theory. Let $C_q(\Sigma; \mathfrak{L}_u)$ be the subgroup of $C^q(\Sigma; \mathfrak{L}_u)$ consisting of functions supported on finitely many q-cells. The graded $\mathbb{Z}$-module $C_*(\Sigma; \mathfrak{L}_u)$ admits a G-equivariant boundary map, which in the notation of (2.18) can be written as

$$\partial : C^{q+1}(\Sigma; \mathfrak{L}_u) \to C^q(\Sigma; \mathfrak{L}_u), \quad (\partial f)(\tau) = \sum_{\sigma \in \Sigma^{(q+1)}} [\tau : \sigma] \, \mathrm{ind}_{G_\sigma}^{G_\tau}(f(\sigma)).$$

This is a natural perfect pairing on each $\mathfrak{L}_u(\sigma) \cong R_{\mathbb{Z}}(G_\sigma)$, since G_σ is a finite group. With that one sees that the differential complex $(C^*(\Sigma; \mathfrak{L}_u), \mathrm{d})$ is isomorphic to

$\mathrm{Hom}_{\mathbb{Z}}((C^*(\Sigma; \mathfrak{L}_u), \partial), \mathbb{Z})$. This persists to the G-invariants:

$$(C^*(\Sigma/G; \mathfrak{L}_u^G), \mathrm{d}) \cong \mathrm{Hom}_{\mathbb{Z}}((C^*(\Sigma/G; \mathfrak{L}_u^G), \partial), \mathbb{Z}). \tag{2.44}$$

Suppose now that Σ is a manifold on which the finite group G acts smoothly. For $t \in \Sigma$ the isotropy G_t acts $\mathbb{R}$-linearly on the tangent space $T_t(\Sigma)$. We say that G_t is a Weyl group if it is the Weyl group of some root system in $T_t(\Sigma)$.

Theorem 2.45. *Let G be a finite group acting smoothly on a manifold Σ.*

(a) *Suppose that G_t is a Weyl group for all $t \in \Sigma$. Then*

$$H_i(C_*(\Sigma/G; \mathfrak{L}_u^G), \partial) \cong H_i(\Sigma//G; \mathbb{Z}) \quad \text{for all } i \in \mathbb{Z}_{\geq 0}.$$

(b) *Suppose that the conclusion of part (a) holds, and that $H^*(\Sigma//G; \mathbb{Z})$ is torsion-free. Then*

$$K_*(C(\Sigma) \rtimes G) \cong H^*(\Sigma//G; \mathbb{Z}).$$

Proof. (a) For every subgroup $H \subset G$ the set of fixpoints Σ^H is a submanifold of Σ [Baum and Connes 1988, Lemma 4.1]. It follows that for every $g \in \mathrm{cc}(G)$ and every connected component Σ_i^g of Σ^g, the map $t \mapsto G_t$ is constant on an open dense subset of Σ_i^g. Pick a point t_i in this dense subset of Σ_i^g and write $G_{t_i} = W_i$. By assumption W_i is a Weyl group and $G_t \supset W_i$ for all $t \in \Sigma_i^g$.

For a cell τ and $t \in \tau \setminus \partial\tau$ we have $G_\tau = G_t$. Using Proposition 1.36 we define, for $t \in \Sigma_i^g$, $t \in \tau \setminus \partial\tau$,

$$s(g, t) = s(g, \tau) = \mathrm{ind}_{W_i}^{G_\tau}(H(u_g, \rho_g)). \tag{2.46}$$

We may and will assume that $s(g, h\tau) = h \cdot s(g, \tau)$ for all $h \in Z_G(g)$. This extends uniquely to a G-equivariant map $\Sigma^g \to \bigcup_{\tau \subset \Sigma^g} R_{\mathbb{Z}}(G_\tau)$, and hence defines an element $s(g) \in C^*(\Sigma; \mathfrak{L}_u^G)$. Thus $s(g)$ is nonzero at $Gt \in \Sigma/G$ if and only if $Gt \cap \Sigma^g$ is nonempty.

The $s(g)$ with $g \in \mathrm{cc}(G)$ yield precisely one representation for each element of the extended quotient

$$\Sigma//G = \bigsqcup_{g \in \mathrm{cc}(G)} \Sigma^g / Z_G(g).$$

So for every $t \in \Sigma$ we get exactly $|\mathrm{cc}(G_t)| = |\mathrm{Irr}(G_t)|$ representations $s(g, t)$. By Proposition 1.17 the $s(g, t)$ with $g \in \mathrm{cc}(G)$ and $t \in G\Sigma^g$ form a $\mathbb{Z}$-basis of the representation ring of the Weyl group G_t. This also shows that for $t \in \tau \setminus \partial\tau$, the set

$$\left\{h \cdot \mathrm{ind}_{W_i}^{G_\sigma}(H(u_g, \rho_g)) : \sigma \subset \Sigma^g,\ h \in G_\tau \setminus G,\ h\sigma = \tau\right\} \tag{2.47}$$

is linearly independent in $R_{\mathbb{Z}}(G_t) = R_{\mathbb{Z}}(G_\tau)$.

Let $\tau \otimes s(g, \tau)$ with $\tau \subset \Sigma^g$ be the terms of which $s(g)$ is made. Then (2.46) entails that the span of the $\tau \otimes s(g, \tau)$ forms a subchain complex $C(g, \Sigma)$ of $(C_*(\Sigma/G; \mathfrak{L}_u^G), \partial)$ and (2.47) implies that $C(g, \Sigma)$ is isomorphic to the cellular homology complex $C_*(\Sigma^g / Z_G(g); \mathbb{Z})$. Since the $s(g,t)$ form a basis of $R_{\mathbb{Z}}(G_t)$ for every $t \in \Sigma$,

$$C_*(\Sigma/G; \mathfrak{L}_u^G) = \bigoplus_{g \in \mathrm{cc}(G)} C(g, \Sigma).$$

The claim about the homology of $(C_*(\Sigma/G; \mathfrak{L}_u^G), \partial)$ follows.

(b) In the absence of torsion, the universal coefficient theorem says that the dual of the homology of a different complex is naturally isomorphic to the cohomology of the dual complex. This gives the horizontal isomorphisms in the following commutative diagram:

$$\begin{tikzcd} H^*(\Sigma/\!/G; \mathbb{Z}) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbb{Z}}(H_*(\Sigma/\!/G; \mathbb{Z}), \mathbb{Z}) \arrow[d] \\ H^*(C^*(\Sigma/G; \mathfrak{L}_u^G), \mathrm{d}) \arrow[u] \arrow[r, "\sim"] & \mathrm{Hom}_{\mathbb{Z}}(H_*(C_*(\Sigma/G; \mathfrak{L}_u^G), \partial), \mathbb{Z}) \end{tikzcd} \tag{2.48}$$

By assumption the right vertical arrow is an isomorphism. We define the left vertical arrow to be the isomorphism such that the diagram becomes commutative. The lower left corner of (2.48) is $H^*_G(\Sigma; \mathfrak{L})$, which by Theorem 2.34 is isomorphic to $K_*(C(\Sigma) \rtimes G)$. □

Let us return to the case of $C(T_{\mathrm{un}}) \rtimes W = C_r^*(W^e)$, where T_{un}, W and W^e come from a root datum $\mathcal{R}$. Then W acts by algebraic group automorphisms on the compact torus T_{un}.

Corollary 2.49. *Let $\mathcal{R}$ be the root datum of a reductive algebraic group with simply connected derived group, and assume that $H^*(T_{\mathrm{un}} /\!/ W; \mathbb{Z})$ is torsion-free. Then for any positive parameter function q,*

$$K_*(C_r^*(\mathcal{R}, q)) \cong H^*(T_{\mathrm{un}} /\!/ W; \mathbb{Z}).$$

Proof. Let $\mathcal{R}$ be the root datum of $(\mathcal{G}(\mathbb{C}), T)$. By Steinberg's connectedness theorem [Steinberg 1968], the group $Z_{\mathcal{G}(\mathbb{C})}(t)$ is connected for every $t \in T$. Hence $W_t = W(Z_{\mathcal{G}(\mathbb{C})}(t), T)$ is always a Weyl group. Now Theorem 2.45 says that

$$H^*(T_{\mathrm{un}} /\!/ W; \mathbb{Z}) \cong K_*(C(T_{\mathrm{un}}) \rtimes W) = K_*(C_r^*(\mathcal{R}, 1)).$$

Apply Theorem 2.11 to the right-hand side. □

In fact Corollary 2.49 also applies to some other root data, for example those of type SO_{2n+1}.

3. Examples

In this section, we compute the topological K-theory of the C^*-Hecke algebras $C_r^*(\mathcal{R}, q)$ associated to common root data $\mathcal{R}$. As discussed after Theorem 2.11, it suffices to do so for $q = 1$ or for generic parameter functions. For $q = 1$ we apply Theorem 2.45, when that is possible.

Our approach for $q \neq 1$ involves the following steps.

(1) Explicitly write down the root datum and the associated Weyl groups.

From (2.9) we get a canonical decomposition

$$C_r^*(\mathcal{R}, q) \rtimes \Gamma = \bigoplus_P C_r^*(\mathcal{R}, q)_P \rtimes \Gamma_P, \tag{3.1}$$

where P runs over a set of representatives for the action of $\mathcal{G}$ on the power set of Δ and Γ_P is the setwise stabilizer of P in Γ.

(2) List a good set of representatives P.

For every chosen P we do the following:

(3) Determine the root datum $\mathcal{R}_P$ and the residual points.

(4) Determine the discrete series of $\mathcal{H}(\mathcal{R}_P, q_P)$, and all the relevant intertwining operators.

(5) Describe $C_r^*(\mathcal{R}, q)_P \rtimes \Gamma_P$ and its space of irreducible representations.

(6) Calculate $K_*(C_r^*(\mathcal{R}, q)_P \rtimes \Gamma_P)$.

Often the final step can be reduced to commutative C^*-algebras. When this is not possible, we transfer the problem to sheaf cohomology via Theorem 2.34.

3A. ***Type* $\mathbf{GL}_n$.** The easiest root data to study are those associated with the reductive group GL_n. The right way to do this was shown by Plymen. From [Plymen 1987, Lemma 5.3] we know that the topological K-groups of these affine Hecke algebras are free abelian, of a finite rank which is explicitly given. Strictly speaking, we do not really need to study this root datum, as we could just refer to Plymen's results. Nevertheless, since many other examples rely on this case, we include an analysis.

From now on many things will be parametrized by partitions and permutations, so let us agree on some notations. We write partitions in decreasing order and abbreviate $(x)^3 = (x, x, x)$. A typical partition looks like

$$\mu = (\mu_1, \mu_2, \dots, \mu_d) = (n)^{m_n} \cdots (2)^{m_2} (1)^{m_1}, \tag{3.2}$$

where some of the multiplicities m_i may be 0. By $\mu \vdash n$ we mean that the weight of μ is

$$|\mu| = \mu_1 + \cdots + \mu_d = n.$$

The number of different μ_i (i.e., the number of blocks in the diagram of μ) is denoted by $b(\mu)$ and the dual partition (obtained by reflecting the diagram of μ)

by $\mu^\vee$. Sometimes we abbreviate

$$\begin{aligned} \gcd(\mu) &= \gcd(\mu_1, \dots, \mu_d), \\ \mu! &= \mu_1!\mu_2! \cdots \mu_d!. \end{aligned} \tag{3.3}$$

With a such partition μ of n we associate the permutation

$$\sigma(\mu) = (1\,2\,\cdots\,\mu_1)(\mu_1+1\,\cdots\,\mu_1+\mu_2)\cdots(n+1-\mu_d\,\cdots\,n) \in S_n.$$

As is well-known, this gives a bijection between partitions of n and conjugacy classes in the symmetric group S_n. The centralizer $Z_{S_n}(\sigma(\mu))$ is generated by the cycles

$$((\mu_1+\cdots+\mu_i+1)(\mu_1+\cdots+\mu_i+2)\cdots(\mu_1+\cdots+\mu_i+\mu_{i+1}))$$

and the “permutations of cycles of equal length” — for example, if $\mu_1 = \mu_2$,

$$(1\,\mu_1+1)(2\,\mu_1+2)\cdots(\mu_1\,2\mu_1). \tag{3.4}$$

Using the second presentation of μ, this means that

$$Z_{S_n}(\sigma(\mu)) \cong \prod_{l=1}^{n} (\mathbb{Z}/l\mathbb{Z})^{m_l} \rtimes S_{m_l}.$$

Let us recall the definition of $\mathcal{R}(\mathrm{GL}_n)$ and the associated groups. Below Q and $Q^\vee$ are the root and coroot lattices.

$$\begin{gathered}
X = \mathbb{Z}^n, \qquad Q = \{x \in X : x_1+\cdots+x_n = 0\}, \\
Y = \mathbb{Z}^n, \qquad Q^\vee = \{y \in Y : y_1+\cdots+y_n = 0\}, \\
T = (\mathbb{C}^\times)^n, \qquad t = (t(e_1), \dots, t(e_n)) = (t_1, \dots, t_n), \\
R = \{e_i - e_j \in X : i \neq j\}, \qquad \alpha_0 = e_1 - e_n, \\
R^\vee = \{e_i - e_j \in Y : i \neq j\}, \qquad \alpha_0^\vee = e_1 - e_n, \\
s_i = s_{\alpha_i} = s_{e_i - e_{i+1}}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{\alpha_1} s_{\alpha_0} t_{-\alpha_1} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x \rangle \alpha_0, \\
W = \langle s_1, \cdots, s_{n-1} \mid s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e : |i-j| > 1 \rangle \cong S_n, \\
S^{\mathrm{aff}} = \{s_0, s_1, \dots, s_{n-1}\}, \\
W^{\mathrm{aff}} = \langle s_0, W_0 \mid s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^3 = (s_0 s_{n-1})^3 = e \text{ if } 2 \leq i \leq n-2 \rangle, \\
W^e = W^{\mathrm{aff}} \rtimes \Omega, \qquad \Omega = \langle t_{e_1}(1\,2\,\cdots\,n) \rangle \cong \mathbb{Z}.
\end{gathered}$$

Because all roots of R are conjugate, s_0 is conjugate to any $s_i \in S^{\mathrm{aff}}$. Hence for any label function we have $q(s_0) = q(s_i) := q$. Every point of T is W-conjugate to one of the form $t = ((t_1)^{\mu_1}(t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d}) \in T$ and

$$W_t = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_d}. \tag{3.5}$$

• **Case $q = 1$.**

By (2.17) and (2.41) we have

$$K_*(C_r^*(W^e)) \otimes \mathbb{C} \cong \check{H}^*(\widetilde{T_{\mathrm{un}}}/S_n; \mathbb{C}) \cong \bigoplus_{\mu \vdash n} \check{H}^*\big(T_{\mathrm{un}}^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)); \mathbb{C}\big). \tag{3.6}$$

Therefore, we want to determine $T_{\mathrm{un}}^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$. If μ is as in (3.2) then

$$\begin{aligned} T^{\sigma(\mu)} &= \{(t_1)^{\mu_1}(t_{\mu_1+1})^{\mu_2}\cdots(t_n)^{\mu_d} \in T\}, \\ T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)) &\cong (\mathbb{C}^\times)^{m_n}/S_{m_n} \times \cdots \times (\mathbb{C}^\times)^{m_1}/S_{m_1}, \end{aligned} \tag{3.7}$$

where S_{m_l} acts on $(\mathbb{C}^\times)^{m_l}$ by permuting the coordinates. To handle this space we use the following nice, elementary result, a proof of which can be found for example in [Plymen 1987, Lemma 5.1].

Lemma 3.8. *For any $m \in \mathbb{N}$ there is an isomorphism of algebraic varieties*

$$(\mathbb{C}^\times)^m/S_m \cong \mathbb{C}^{m-1} \times \mathbb{C}^\times.$$

Consequently, $T_{\mathrm{un}}^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$ has the homotopy type of $(S^1)^{b(\mu)}$. In particular, its integral cohomology is torsion-free, so Corollary 2.49 is applicable. It says that (3.6) can be refined to

$$K_*(C_r^*(W^e)) \cong \bigoplus_{\mu \vdash n} \check{H}^*((S^1)^{b(\mu)}; \mathbb{Z}) \cong \bigoplus_{\mu \vdash n} \mathbb{Z}^{2^{b(\mu)}}. \tag{3.9}$$

• **Generic, equal parameter case $q \neq 1$.**

Inequivalent subsets of Δ are parametrized by partitions μ of n. For the typical partition (3.2) we have

$$\begin{aligned}
P_\mu &= \Delta \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{n-\mu_d}\}, \\
R_{P_\mu} &\cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \cong R^\vee_{P_\mu}, \\
X^{P_\mu} &\cong \mathbb{Z}(e_1 + \cdots + e_{\mu_1})/\mu_1 + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n)/\mu_d, \\
X_{P_\mu} &\cong (\mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n))^{m_n} + \cdots + (\mathbb{Z}^2/\mathbb{Z}(e_1 + e_2))^{m_2}, \\
Y^{P_\mu} &= \mathbb{Z}(e_1 + \cdots + e_{\mu_1}) + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n), \\
Y_{P_\mu} &= \{y \in \mathbb{Z}^n : y_1 + \cdots + y_{\mu_1} = \cdots = y_{n+1-\mu_d} + \cdots + y_n = 0\}, \\
T^{P_\mu} &= \{(t_1)^{\mu_1} \cdots (t_n)^{\mu_d} \in T\}, \\
T_{P_\mu} &= \{t \in T : t_1 t_2 \cdots t_{\mu_1} = \cdots = t_{n+1-\mu_d} \cdots t_n = 1\}, \\
K_{P_\mu} &= \{t \in T^{P_\mu} : t_1^{\mu_1} = \cdots = t_n^{\mu_d} = 1\},
\end{aligned}$$

$$W_{P_\mu} \cong (S_n)^{m_n} \times \cdots \times (S_2)^{m_2}, \qquad W(P_\mu, P_\mu) \cong S_{m_n} \times \cdots \times S_{m_2} \times S_{m_1},$$

$$\mathcal{G}_{P_\mu P_\mu} = K_{P_\mu} \rtimes W(P_\mu, P_\mu), \qquad Z_{S_n}(\sigma(\mu)) = W(P_\mu, P_\mu) \ltimes \textstyle\prod_{l=1}^n (\mathbb{Z}/l\mathbb{Z})^{m_l}.$$

The W_{P_μ}-orbits of residual points for $\mathcal{H}_{P_\mu}$ are parametrized by

$$K_{P_\mu}\big((q^{(\mu_1-1)/2}, q^{(\mu_1-3)/2}, \dots, q^{(1-\mu_1)/2}) \cdots (q^{(\mu_d-1)/2}, q^{(\mu_d-3)/2}, \dots, q^{(1-\mu_d)/2})\big).$$

This set is obviously in bijection with K_{P_μ}, and indeed the intertwiners $\pi(k)$ with $k \in K_{P_\mu}$ act on it by multiplication. From the classification of the discrete series we know that here every residual point carries precisely one discrete series representation, namely a twist of a Steinberg representation. The quickest way to see this is with the Kazhdan–Lusztig classification of irreducible representations of affine Hecke algebras with equal parameters, in particular [Kazhdan and Lusztig 1987, Theorems 7.12 and 8.13]. This implies

$$\bigcup_\delta (P_\mu, \delta, T^{P_\mu})/K_{P_\mu} \cong T^{P_\mu},$$
$$\bigcup_\delta (P_\mu, \delta, T^{P_\mu})/\mathcal{G}_{P_\mu P_\mu} \cong T^{P_\mu}/W(P_\mu, P_\mu) = T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)).$$

If a point $\xi = (P_\mu, \delta, t)$ has a nontrivial stabilizer $\mathcal{G}_\xi$, then by the above this stabilizer is contained in $W(P_\mu, P_\mu) \cong \prod_{l=1}^n S_{m_l}$. It is easily seen that this isotropy group is actually a Weyl group, and that it equals the group $W(R_\xi)$ from (2.10). In other words, all R-groups are trivial for this root datum and $q \neq 1$, and all intertwining operators $\pi(g, \xi)$ from a representation $\pi(\xi)$ to itself are scalar multiples of the identity. So the action of $\mathcal{W}_{P_\mu P_\mu}$ on

$$C\Big(\bigsqcup_\delta T_{\mathrm{un}}^{P_\mu}; M_{n!/\mu!}(\mathbb{C})\Big) \tag{3.10}$$

is essentially only on $\bigsqcup_\delta T_u^{P_\mu}$ and the conjugation part doesn't really matter. In particular, we deduce that

$$C_r^*(\mathcal{R}, q) \cong \bigoplus_{\mu \vdash n} M_{n!/\mu!}\Big(C\Big(\bigsqcup_\delta T_{\mathrm{un}}^{P_\mu}\Big)\Big) \cong \bigoplus_{\mu \vdash n} M_{n!/\mu!}\big(T_{\mathrm{un}}^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))\big). \tag{3.11}$$

In particular, $C_r^*(\mathcal{R}, q)$ is Morita-equivalent with the commutative C^*-algebra of continuous functions on $T_{\mathrm{un}}/\!/S_n$. Similar results were obtained by completely different methods in [Mischenko 1982].

We remark that $\mathrm{Irr}(C_r^*(\mathcal{R}, q))$ has a clear relation with the elliptic representation theory of symmetric groups. Every δ is essentially a Steinberg representation, so

$$\zeta^\vee(\delta \circ \phi_t) \in \mathrm{Mod}(\mathcal{O}(T) \rtimes Z_{S_n}(\sigma(\mu)))$$

is given by the $\mathcal{O}(T)$-character t and the sign representation of the Weyl group $Z_{S_n}(\sigma(\mu))_t$. Moreover, the group $Z_{S_n}(\sigma(\mu))_t$ can be identified with $R(\xi)$, where $\xi = (P_\mu, \delta, t)$. Then $\zeta^\vee(\pi(\xi)) = \mathrm{ind}_{W(R_\xi)}^{(S_n)_t}(\mathrm{sign})$ as $(S_n)_t$-representations, and this is exactly a member of the basis $R_{\mathbb{Z}}((S_n)_t)$ exhibited in Proposition 1.17(b).

Using the analysis from the case $q = 1$, it follows that

$$
\begin{aligned}
K_*(C_r^*(\mathcal{R}, q)) &\cong \bigoplus_{\mu \vdash n} K^*\big(T_{\mathrm{un}}^{\sigma(\mu)} / Z_{S_n}(\sigma(\mu))\big) \\
&\cong \bigoplus_{\mu \vdash n} K^*((S^1)^{b(\mu)}) \cong \bigoplus_{\mu \vdash n} \mathbb{Z}^{2^{b(\mu)}}. \qquad (3.12)
\end{aligned}
$$

Recall that the even cohomology of $(S^1)^b$ has the same dimension as its odd cohomology, unless $b = 0$. The same holds for K-theory, and $b(\mu) = 0$ does not occur because $b(\mu)$ counts the number of different terms in a partition of $n \geq 1$. So we can refine (3.12) to

$$
K_0(C_r^*(\mathcal{R}, q)) = \bigoplus_{\mu \vdash n} \mathbb{Z}^{2^{b(\mu)-1}}, \qquad K_1(C_r^*(\mathcal{R}, q)) = \bigoplus_{\mu \vdash n} \mathbb{Z}^{2^{b(\mu)-1}}. \qquad (3.13)
$$

3B. *Type* $\mathbf{SL}_n$. The affine Hecke algebra associated to a root datum of type SL_n describes the category of Iwahori-spherical representations of $\mathrm{PGL}_n(\mathbb{Q}_p)$. Since that is a subcategory of the Iwahori-spherical representations of $\mathrm{GL}_n(\mathbb{Q}_p)$, it can be expected this affine Hecke algebra behaves very similarly to those in the previous subsection. Indeed, we will see that the calculations of the K-theory are essentially the same as in Section 3A.

The root datum $\mathcal{R}(\mathrm{SL}_n)$ is given by:

$$
\begin{gathered}
X = \mathbb{Z}^n / \mathbb{Z}(e_1 + \cdots e_n) \cong Q + ((e_1 + \cdots + e_n)/n - e_n), \\
Q = \{x \in \mathbb{Z}^n : x_1 + \cdots + x_n = 0\}, \\
Y = Q^\vee = \{y \in \mathbb{Z}^n : y_1 + \cdots + y_n = 0\}, \\
T = \{t \in (\mathbb{C}^\times)^n : t_1 \cdots t_n = 1\}, \qquad t = (t(e_1), \ldots, t(e_n)) = (t_1, \ldots, t_n), \\
R = \{e_i - e_j \in X : i \neq j\}, \qquad \alpha_0 = e_1 - e_n, \\
R^\vee = \{e_i - e_j \in Y : i \neq j\}, \qquad \alpha_0 = e_1 - e_n, \\
s_i = s_{\alpha_i} = s_{e_i - e_{i+1}}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{\alpha_1} s_{\alpha_0} t_{-\alpha_1} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x \rangle \alpha_0, \\
W = \big\langle s_1, \cdots, s_{n-1} \mid s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e \text{ if } |i-j| > 1 \big\rangle \cong S_n, \\
S^{\mathrm{aff}} = \{s_0, s_1, \ldots, s_{n-1}\}, \\
W^{\mathrm{aff}} = \big\langle s_0, W_0 \mid s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^3 = (s_0 s_{n-1})^3 = e \text{ if } 2 \leq i \leq n-2 \big\rangle, \\
W^e = W^{\mathrm{aff}} \rtimes \Omega, \qquad \Omega = \langle t_{e_1 - (e_1 + \cdots e_n)/n} (12 \cdots n) \rangle \cong \mathbb{Z}/n\mathbb{Z}.
\end{gathered}
$$

Because all roots are conjugate, s_0 is conjugate to any $s_i \in S^{\mathrm{aff}}$, and for any label function $q(s_0) = q(s_i) = q$. The W-stabilizer of $((t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d})$ is isomorphic to $S_{\mu_1} \times \cdots \times S_{\mu_d}$. Generically, there are $n!\, n$ residual points, and they all satisfy $t(\alpha_i) = q$ or $t(\alpha_i) = q^{-1}$ for $1 \leq i < n$. These residual points form n conjugacy classes, unless $q = 1$.

• **Group case** $q = 1$**.**

In view of (2.17) and (2.41), we want to determine $T_{\text{un}}^{\sigma(\mu)} / Z_{S_n}(\sigma(\mu))$, where μ is any partition of n. Write it as in (3.2); then

$$
\begin{aligned}
T^{\sigma(\mu)} &= \{(t_1)^{\mu_1}(t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d} \in T\} \\
&\cong \{(t_1)^{\mu_1}(t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d} \in (\mathbb{C}^\times)^n\}/\mathbb{C}^\times \\
&\qquad\qquad\qquad \times \{(e^{2\pi i k/n})^n : 0 \le k < \gcd(\mu)\}, \\
T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)) &\cong \big((\mathbb{C}^\times)^{m_n}/S_{m_n} \times \cdots \times (\mathbb{C}^\times)^{m_1}/S_{m_1}\big)/\mathbb{C}^\times \\
&\qquad\qquad\qquad \times \{(e^{2\pi i k/n})^n : 0 \le k < \gcd(\mu)\},
\end{aligned}
$$

where $\mathbb{C}^\times$ acts diagonally. By Lemma 3.8, each factor $(\mathbb{C}^\times)^{m_i}/S_{m_i}$ is homotopy equivalent to a circle. The induced action of $S^1 \subset \mathbb{C}^\times$ on this direct product of circles identifies with a direct product of rotations. Hence, $T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$ is homotopy equivalent with $\mathbb{T}^{b(\mu)-1} \times \{\gcd(\mu) \text{ points}\}$, and the extended quotient $T/\!/W$ has torsion-free cohomology. By Corollary 2.49,

$$
K_*(C_r^*(W^e)) \cong \mathbb{Z}^{d(n)}, \qquad d(n) := \sum_{\mu \vdash n} \gcd(\mu) 2^{b(\mu)-1}. \tag{3.14}
$$

• **Generic, equal parameter case** $q \neq 1$**.**

Inequivalent subsets of Δ are parametrized by partitions μ of n. For the typical partition (3.2) we put

$$
\begin{aligned}
P_\mu &= \Delta \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{n-\mu_d}\}, \\
R_{P_\mu} &\cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \cong R^\vee_{P_\mu}, \\
X^{P_\mu} &\cong \big(\mathbb{Z}(e_1+\cdots+e_{\mu_1})/\mu_1+\cdots+\mathbb{Z}(e_{n+1-\mu_d}+\cdots+e_n)/\mu_d\big)/\mathbb{Z}(e_1+\cdots+e_n)/g, \\
X_{P_\mu} &\cong (\mathbb{Z}^n/\mathbb{Z}(e_1+\cdots+e_n))^{m_n} + \cdots + (\mathbb{Z}^2/\mathbb{Z}(e_1+e_2))^{m_2}, \\
Y^{P_\mu} &= \{y \in \mathbb{Z}(e_1+\cdots+e_{\mu_1}) + \cdots + \mathbb{Z}(e_{n+1-\mu_d}+\cdots+e_n) : y_1+\cdots+y_n = 0\}, \\
Y_{P_\mu} &= \{y \in Y : y_1+\cdots+y_{\mu_1} = \cdots = y_{n+1-\mu_d}+\cdots+y_n = 0\}, \\
T^{P_\mu} &= \{(t_1)^{\mu_1} \cdots (t_n)^{\mu_d} \in T : t_1^{\mu_1/g} \cdots t_n^{\mu_d/g} = 1\}, \quad g = \gcd(\mu), \\
T_{P_\mu} &= \{t \in T : t_1 t_2 \cdots t_{\mu_1} = \cdots = t_{n+1-\mu_d} \cdots t_n = 1\}, \\
K_{P_\mu} &= \{t \in T^{P_\mu} : t_1^{\mu_1} = \cdots = t_n^{\mu_d} = 1\}, \\
W_{P_\mu} &\cong (S_n)^{m_n} \times \cdots \times (S_2)^{m_2}, \qquad W(P_\mu, P_\mu) \cong S_{m_n} \times \cdots \times S_{m_2} \times S_{m_1}, \\
\mathcal{G}_{P_\mu P_\mu} &= K_{P_\mu} \rtimes W(P_\mu, P_\mu), \qquad Z_{S_n}(\sigma(\mu)) = W(P_\mu, P_\mu) \ltimes \textstyle\prod_{l=1}^n (\mathbb{Z}/l\mathbb{Z})^{m_l}.
\end{aligned}
$$

Theorem 3.15. *For $q \neq 1$ the C^*-algebra $C_r^*(\mathcal{R}(\mathrm{SL}_n), q)$ is Morita equivalent with the commutative algebra of continuous functions on $T_{\text{un}} /\!/ W$.*

Its K-theory is given by

$$K_0(C_r^*(\mathcal{R},q)) = \bigoplus_{\mu\vdash n,\, b(\mu)>1} \mathbb{Z}^{\gcd(\mu)2^{b(\mu)-2}} \oplus \bigoplus_{\mu\vdash n,\, b(\mu)=1} \mathbb{Z}^{\gcd(\mu)},$$
$$K_1(C_r^*(\mathcal{R},q)) = \bigoplus_{\mu\vdash n,\, b(\mu)>1} \mathbb{Z}^{\gcd(\mu)2^{b(\mu)-2}}.$$

Proof. The W_{P_μ}-orbits of residual points for $\mathcal{H}_{P_\mu}$ are represented by the points

$$\big((q^{(\mu_1-1)/2}, q^{(\mu_1-3)/2}, \dots, q^{(1-\mu_1)/2}) \cdots (q^{(\mu_d-1)/2}, q^{(\mu_d-3)/2}, \dots, q^{(1-\mu_d)/2})\big) \cdot \big((e^{2\pi i k_1/\mu_1})^{\mu_1} \cdots (e^{2\pi i k_d/\mu_d})^{\mu_d}\big), \quad 0 \le k_i < \mu_i. \tag{3.16}$$

These points are in bijection with $K_{P_\mu} \times \mathbb{Z}/\gcd(\mu)\mathbb{Z}$. Also $T^{\sigma(\mu)}$ consists of exactly $\gcd(\mu)$ components, one of which is T^{P_μ}. Just as in the type GL_n case, this leads to

$$\bigcup_\delta (P_\mu, \delta, T^{P_\mu})/K_{P_\mu} \cong T^{P_\mu} \times \mathbb{Z}/\gcd(\mu)\mathbb{Z} \cong T^{\sigma(\mu)},$$
$$\bigcup_\delta (P_\mu, \delta, T^{P_\mu})/\mathcal{W}_{P_\mu P_\mu} \cong T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)),$$
$$C_r^*(\mathcal{R},q) \cong \bigoplus_{\mu\vdash n} M_{n!/\mu!}\Big(C\Big(\bigsqcup_\delta T_u^{P_\mu}\Big)\Big) \cong \bigoplus_{\mu\vdash n} M_{n!/\mu!}\big(T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))\big).$$

The extended quotient $T_{\mathrm{un}}/\!/W$ is $\bigsqcup_{\mu\vdash n} T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$, which gives the desired Morita equivalence. It follows that

$$K_*(C_r^*(\mathcal{R},q)) \cong \bigoplus_{\mu\vdash n} K^*\big(T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))\big) \cong \bigoplus_{\mu\vdash n} K^*((S^1)^{b(\mu)-1})^{\gcd(\mu)}. \tag{3.17}$$

This a free abelian group of rank $d(n) = \sum_{\mu\vdash n} \gcd(\mu)2^{b(\mu)-1}$ with $b(\mu)$ as on page 431. Since the even K-theory of $(S^1)^b$ has the same rank as the odd K-theory unless $b=0$, (3.17) leads to K_0 and K_1 as claimed. □

3C. *Type* PGL_n. The root datum for the algebraic group PGL_n gives rise to

$$X = Q = \{x \in \mathbb{Z}^n : x_1 + \dots + x_n = 0\},$$
$$Q^\vee = \{y \in \mathbb{Z}^n : y_1 + \dots + y_n = 0\},$$
$$Y = \mathbb{Z}^n/\mathbb{Z}(e_1 + \dots + e_n) \cong Q^\vee + ((e_1 + \dots + e_n)/n - e_1),$$
$$T = (\mathbb{C}^\times)^n/\mathbb{C}^\times, \qquad t = (t_1, \dots, t_n) = (t(e_1), \dots, t(e_n)),$$
$$R = \{e_i - e_j \in X : i \neq j\}, \qquad \alpha_0 = e_1 - e_n,$$
$$R^\vee = \{e_i - e_j \in Y : i \neq j\}, \qquad \alpha_0 = e_1 - e_n,$$
$$s_i = s_{\alpha_i} = s_{e_i - e_{i+1}}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x\rangle \alpha_0,$$

$$W = \langle s_1, \dots, s_{n-1} \mid s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e \text{ if } |i-j| > 1 \rangle \cong S_n,$$
$$S^{\mathrm{aff}} = \{s_0, s_1, \dots, s_{n-1}\}, \qquad \Omega = \{e\},$$
$$W^e = W^{\mathrm{aff}} = \langle s_0, W_0 \mid s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^3 = (s_0 s_{n-1})^3 = e \text{ if } 2 \le i \le n-2 \rangle.$$

For $n > 2$, s_0 is conjugate to s_1 in W^{aff}, for $n = 2$ it is not. So for $n > 2$ there is only one parameter $q = q(s_i)$, $0 \le i \le n-1$, whereas for $n = 2$, q_0 may differ from q_1. In particular, for $n = 2$ the equal parameter function $q(s_0) = q(s_1)$ is not generic. Nevertheless, we only consider equal parameter functions in this subsection, explicit computations for the other parameter functions on $\mathcal{R}(\mathrm{PGL}_2)$ can be found in [Solleveld 2007, §6.1].

For $q \neq 1$, there are $n!$ residual points. They form one W-orbit, and a typical residual point is

$$(q^{(1-n)/2}, q^{(3-n)/2}, \dots, q^{(n-1)/2}).$$

To determine the isotropy group of points of T we have to be careful. In general the W-stabilizer of

$$((t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d}) \in T$$

is isomorphic to

$$S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_d} \subset W.$$

However, in some special cases the diagonal action of $\mathbb{C}^\times$ on $(\mathbb{C}^\times)^n$ gives rise to extra stabilizing elements. Let r be a divisor of n, $k \in (\mathbb{Z}/r\mathbb{Z})^\times$ and $\lambda = (\lambda_1, \dots, \lambda_l)$ a partition of n/r. The isotropy group of

$$(t_1)^{\lambda_1} (e^{2\pi i k/r} t_1)^{\lambda_1} \cdots (e^{-2\pi i k/r} t_1)^{\lambda_1} (t_{r\lambda_1+1})^{\lambda_2} \cdots (e^{-2\pi i k/r} t_{r\lambda_1+1})^{\lambda_2} \cdots (e^{-2\pi i k/r} t_n)^{\lambda_l}$$

is isomorphic to

$$S^r_{\lambda_1} \times S^r_{\lambda_2} \times \cdots \times S^r_{\lambda_l} \rtimes \mathbb{Z}/r\mathbb{Z}. \tag{3.18}$$

Explicitly, the subgroup $\mathbb{Z}/r\mathbb{Z}$ is generated by

$$(1\, \lambda_1{+}1\; 2\lambda_1{+}1 \cdots (r{-}1)\lambda_1{+}1)(2\, \lambda_1{+}2\; 2\lambda_1{+}2 \cdots (r{-}1)\lambda_1{+}2) \cdots (\lambda_1\, 2\lambda_1 \cdots r\lambda_1)$$
$$\cdots (n{+}1{-}r\lambda_d\; n{+}1{+}(1{-}r)\lambda_d \cdots n{+}1{+}(r{-}1)\lambda_d)(n{+}(1{-}r)\lambda_d\; n{+}(2{-}r)\lambda_d \cdots n),$$

and it acts on every factor $S^r_{\lambda_j}$ in (3.18) by cyclic permutations.

- **Case $q = 1$.**

As we noted before, we have to analyze $T_{\mathrm{un}}^{\sigma(\mu)} / Z_{S_n}(\sigma(\mu))$. For the typical partition μ we have

$$T^{\sigma(\mu)} = \{(t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d}\} / \mathbb{C}^\times \times \{t : t(e_j) = e^{2\pi i jk/g},\ 0 \le k < g\}, \tag{3.19}$$

which is the disjoint union of $g = \gcd(\mu)$ complex tori of dimension $m_n + m_{n-1} + \cdots + m_1 - 1$. We obtain

$$T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)) \cong \big((\mathbb{C}^\times)^{m_n}/S_{m_n} \times \cdots \times (\mathbb{C}^\times)^{m_1}/S_{m_1}\big)/\mathbb{C}^\times \\ \times \{t : t(e_j) = e^{2\pi i jk/g},\ 0 \leq k < g\}. \quad (3.20)$$

Remarkably enough, these sets are diffeomorphic to the corresponding sets for $\mathcal{R}(\mathrm{SL}_n)$. We take advantage of this by reusing our deduction that (3.20) is homotopy equivalent with $(S^1)^{b(\mu)-1} \times \{\gcd(\mu) \text{ points}\}$. With (2.17) we conclude that $K_*(C_r^*(W^e)) \otimes_{\mathbb{Z}} \mathbb{C}$ has dimension $d(n) = \sum_{\mu \vdash n} \gcd(\mu) 2^{b(\mu)-1}$.

• **Equal parameter case $q \neq 1$.**

This is noticeably different from the generic cases for $\mathcal{R}(\mathrm{GL}_n)$ and $\mathcal{R}(A_{n-1}^\vee)$, because $C_r^*(\mathcal{R}(A_{n-1}, q))$ is not Morita equivalent to a commutative C^*-algebra. Of course the inequivalent subsets of Δ are still parametrized by partitions μ of n:

$$P_\mu = \Delta \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{n-\mu_d}\},$$
$$R_{P_\mu} \cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \cong R_{P_\mu}^\vee,$$
$$X^{P_\mu} \cong \{x \in \mathbb{Z}(e_1 + \cdots + e_{\mu_1})/\mu_1 + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n)/\mu_d : \\ x_1 + \cdots + x_n = 0\},$$
$$X_{P_\mu} \cong \{x \in \mathbb{Z}^{\mu_1}/\mathbb{Z}(e_1 + \cdots + e_{\mu_1}) + \cdots + \mathbb{Z}^{\mu_d}/\mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n) : \\ x_1 + \cdots + x_n \in g\mathbb{Z}/g\mathbb{Z}\},$$
$$Y^{P_\mu} \cong \mathbb{Z}(e_1 + \cdots + e_{\mu_1}) + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n)/\mathbb{Z}(e_1 + \cdots + e_n),$$
$$Y_{P_\mu} \cong \{y : y_1 + \cdots + y_{\mu_1} = \cdots = y_{n+1-\mu_d} + \cdots + y_n = 0\}/\mathbb{Z}(e_1 + \cdots e_n),$$
$$T^{P_\mu} = \{(t_1)^{\mu_1} \cdots (t_n)^{\mu_d}\}/\mathbb{C}^\times,$$
$$T_{P_\mu} = \{t : t_1 t_2 \cdots t_{\mu_1} = \cdots = t_{n+1-\mu_d} \cdots t_n = 1\}/\{z \in \mathbb{C} : z^g = 1\},$$
$$K_{P_\mu} = \{(t_1)^{\mu_1} \cdots (t_n)^{\mu_d} : t_1^{\mu_1} = \cdots = t_n^{\mu_d} = 1\}/\{z \in \mathbb{C} : z^g = 1\},$$
$$W_{P_\mu} \cong S_n^{m_n} \times S_{n-1}^{m_{n-1}} \times \cdots \times S_2^{m_2}, \qquad W(P_\mu, P_\mu) \cong S_{m_n} \times \cdots \times S_{m_2} \times S_{m_1}.$$

We note that

$$T^{\sigma(\mu)} = T^{P_\mu} \times \{t : t(e_j) = e^{2\pi i jk/g},\ 0 \leq k < g\}.$$

The W_{P_μ}-orbits of residual points for $\mathcal{H}_{P_\mu}$ are represented by the points of

$$K_{P_\mu}(q^{(\mu_1-1)/2}, q^{(\mu_1-3)/2}, \ldots, q^{(1-\mu_1)/2}, q^{(\mu_2-1)/2}, \ldots, q^{(\mu_d-1)/2}, \ldots, q^{(1-\mu_d)/2}).$$

Hence, the intertwiners $\pi(k)$ with $k \in K_{P_\mu}$ permute the set of discrete series representations of $\mathcal{H}_{P_\mu}$ faithfully, and

$$\bigsqcup_\delta (P_\mu, \delta, T^{P_\mu})/K_{P_\mu} \cong T^{P_\mu} = (T^{\sigma(\mu)})^\circ.$$

Just before (3.10) we saw that the intertwiners for $\mathcal{R}(\mathrm{GL}_n)$, $q \neq 1$, have the property

$$w(t) = t \quad \Rightarrow \quad \pi(w, P_\mu, \delta, t) = 1.$$

This implies that in our present setting we can have $w(t)=t$ and $\pi(w, P_\mu, \delta, t) \neq 1$ only if $w(t)=t$ does not hold without taking the action of $\mathbb{C}^\times$ into account.

Let us classify such $w \in W(P_\mu, P_\mu)$ and $t \in T^{P_\mu}$ up to conjugacy. For a divisor r of $g^\vee := \gcd(\mu^\vee)$ we have the partition

$$\mu^{1/r} := (nr)^{m_n/r} \cdots (2r)^{m_2/r} (r)^{m_1/r}.$$

Notice that

$$b(\mu^{1/r}) = b(\mu) = b(\mu^\vee).$$

There exists a $\sigma \in S_n$ which is conjugate to $\sigma(\mu^{1/r})$ and satisfies $\sigma^r = \sigma(\mu)$. We construct a particular such σ as follows. If $r = g^\vee$ then (starting from the left) replace every block

$$(d{+}1\; d{+}2\; \cdots\; d{+}m)(d{+}1{+}m\; \cdots\; d{+}2m) \cdots (d{+}(g^\vee{-}1)m\; \cdots\; d{+}g^\vee m)$$

of $\sigma(\mu)$ by

$$(d{+}1\; d{+}1{+}m \cdots d{+}1{+}(g^\vee{-}1)m\; 2\; d{+}2{+}m \cdots d{+}2{+}(g^\vee{-}1)m\; d{+}3 \cdots d{+}g^\vee m).$$

We denote the resulting element by $\sigma(\mu)^{1/g^\vee}$, and for general $r \mid g^\vee$ we define

$$\sigma(\mu)^{1/r} := \big(\sigma(\mu)^{1/g^\vee}\big)^{g^\vee/r}.$$

Consider the cosets of subtori

$$T^{P_\mu}_{r,k} := \big(T^{\sigma(\mu)^{1/r}}\big)^\circ \big((1)^{g^\vee \mu_1/r} (e^{2\pi i k/r})^{g^\vee \mu_{1+g^\vee/r}/r} \cdots (e^{-2\pi i k/r})^{g^\vee \mu_d/r}\big), \quad k \in \mathbb{Z}.$$

If $\gcd(k, r) = 1$, then the generic points of $T^{P_\mu}_{r,k}$ have $W(P_\mu, P_\mu)$-stabilizer

$$\langle W_{P_\mu}, \sigma(\mu)^{1/r} \rangle \cap W(P_\mu, P_\mu) \cong \mathbb{Z}/r\mathbb{Z}.$$

Note that for $r' \mid g^\vee$,

$$T^{P_\mu}_{r',k} \subset T^{P_\mu}_{r,k} \quad \text{if } r \mid r'. \tag{3.21}$$

If a point $t \in T^{P_\mu}_{r,k}$ does not lie on any $T^{P_\mu}_{r',k'}$ with $r' > r$, then its $W(P_\mu, P_\mu)$-stabilizer may still be larger than $\mathbb{Z}/r\mathbb{Z}$. However, it is always of the form

$$S^r_{\lambda_1} \times \cdots \times S^r_{\lambda_l} \rtimes \mathbb{Z}/r\mathbb{Z}.$$

Here the product of symmetric groups is $W(R_\xi)$ from (2.10), and $\mathfrak{R}_\xi = \mathbb{Z}/r\mathbb{Z}$. With [Delorme and Opdam 2011] it follows that the intertwiners $\pi(w, P_\mu, \delta, t)$ are scalar for $w \in S^r_{\lambda_1} \times \cdots \times S^r_{\lambda_l}$ and nonscalar for $w \in (\mathbb{Z}/r\mathbb{Z}) \setminus \{e\}$. Because $\mathbb{Z}/r\mathbb{Z}$ is cyclic this implies that $\pi(P_\mu, \delta, t)$ is the direct sum of exactly r inequivalent irreducible representations.

Different choices of $\sigma(\mu)^{1/r}$ or of $k \in (\mathbb{Z}/r\mathbb{Z})^\times$ lead to conjugate subvarieties of T^{P_μ}, so we have a complete description of $\mathrm{Irr}(C^*_r(\mathcal{R}, q)_{P_\mu})$. To calculate the

K-theory of this algebra we use Theorem 2.34, which says that (at least modulo torsion) it is isomorphic to

$$H^*_{W(P_\mu,P_\mu)}\big(T_u^{P_\mu};\mathcal{L}_u\big) \cong \check{H}^*\big(T^{P_\mu}/W(P_\mu,P_\mu);\mathcal{L}_u^{W(P_\mu,P_\mu)}\big).$$

We can endow $T_u^{P_\mu}$ with the structure of a finite $W(P_\mu, P_\mu)$-CW-complex, such that every $T_{u,r,k}^{P_\mu}$ is a subcomplex. The local coefficient system $\mathcal{L}_u$ is not very complicated: $\mathcal{L}_u(B) \cong \mathbb{Z}^r$ if and only if $B \setminus \partial B$ consists of generic points in a conjugate of $T_{u,r,k}^{P_\mu}$. In suitable coordinates the maps $\mathcal{L}_u(B \to B')$ are all of the form

$$\mathbb{Z}^r \to \mathbb{Z}^{r/d} : (x_1,\dots,x_r) \to (x_1+x_2+\dots+x_d,\dots,x_{1+r-d}+\dots+x_r).$$

Hence, the associated sheaf is the direct sum of several subsheaves $\mathfrak{F}_r^\mu$, one for each divisor r of $\gcd(\mu^\vee)$. The support of $\mathfrak{F}_r^\mu$ is

$$W(P_\mu,P_\mu)T_{u,r,1}^{P_\mu}/W(P_\mu,P_\mu) \cong T_u^{P_{\mu^{1/r}}}/Z_{S_n}(\sigma(\mu^{1/r})),$$

and on that space it has constant stalk $\mathbb{Z}^{\phi(r)}$. Here ϕ is the Euler ϕ-function, i.e.,

$$\phi(r) = \#\{m \in \mathbb{Z} : 0 \le m < r,\ \gcd(m,r)=1\} = \#(\mathbb{Z}/r\mathbb{Z})^\times.$$

This is the rank of $\mathfrak{F}_r^\mu$, because in every point of $T_{u,r,1}$ we have r irreducible representations, but the ones corresponding to numbers that are not coprime to r are already accounted for by the sheaves $\mathfrak{F}_{r'}^\mu$ with $r' \mid r$. We calculate

$$\begin{aligned}
&\check{H}^*\big(T_{\mathrm{un}}^{P_\mu}/W(P_\mu,P_\mu);\mathcal{L}_u^{W(P_\mu,P_\mu)}\big)\\
&\qquad\cong \bigoplus_{r|\gcd(\mu^\vee)} \check{H}^*\big(T_{\mathrm{un}}^{P_\mu}/W(P_\mu,P_\mu);\mathfrak{F}_r^\mu\big)\\
&\qquad\cong \bigoplus_{r|\gcd(\mu^\vee)} \check{H}^*\big(T_{\mathrm{un}}^{P_{\mu^{1/r}}}/Z_{S_n}(\sigma(\mu^{1/r}));\mathbb{Z}^{\phi(r)}\big)\\
&\qquad\cong \bigoplus_{r|\gcd(\mu^\vee)} \check{H}^*\big((S^1)^{b(\mu^{1/r})-1};\mathbb{Z}^{\phi(r)}\big) \cong \bigoplus_{r|\gcd(\mu^\vee)} \mathbb{Z}^{\phi(r)2^{b(\mu^{1/r})-1}}\\
&\qquad= \bigoplus_{r|\gcd(\mu^\vee)} \mathbb{Z}^{\phi(r)2^{b(\mu^\vee)-1}} = \mathbb{Z}^{\gcd(\mu^\vee)2^{b(\mu^\vee)-1}}.
\end{aligned}\tag{3.22}$$

Now Theorem 2.34 says that $K_*(C_r^*(\mathcal{R},q)_{P_\mu})$ is also a free abelian group of rank $\gcd(\mu^\vee)2^{b(\mu^\vee)-1}$. Summing over partitions μ of n we find that $K_*(C_r^*(\mathcal{R},q))$ is a free abelian group of rank

$$\sum_{\mu\vdash n}\gcd(\mu^\vee)2^{b(\mu^\vee)-1} = \sum_{\mu\vdash n}\gcd(\mu)2^{b(\mu)-1}.$$

From Theorem 2.11 and the case $q = 1$ we see that these K-groups can also be obtained as the K-theory of a disjoint union of compact tori, with $\gcd(\mu)$ tori of dimension $b(\mu)-1$. This allows us to immediately determine K_0 and K_1 separately as well:

$$\begin{aligned} K_0(C_r^*(\mathcal{R}, q)) &= \bigoplus_{\mu \vdash n, b(\mu)>1} \mathbb{Z}^{\gcd(\mu) 2^{b(\mu)-2}} \oplus \bigoplus_{\mu \vdash n, b(\mu)=1} \mathbb{Z}^{\gcd(\mu)}, \\ K_1(C_r^*(\mathcal{R}, q)) &= \bigoplus_{\mu \vdash n, b(\mu)>1} \mathbb{Z}^{\gcd(\mu) 2^{b(\mu)-2}}. \end{aligned} \tag{3.23}$$

3D. *Type* $\mathbf{SO_{2n+1}}$. The root systems of type B_n are more complicated than those of type A_n because there are roots of different lengths. This implies that the associated root data allow label functions which have three independent parameters. Detailed information about the representations of type B_n affine Hecke algebras is available from [Slooten 2003].

Consider the root datum for the special orthogonal group SO_{2n+1}:

$$\begin{gathered} X = Q = \mathbb{Z}^n, \\ Y = \mathbb{Z}^n, \qquad Q^\vee = \{y \in Y : y_1 + \cdots + y_n \text{ even}\}, \\ T = (\mathbb{C}^\times)^n, \qquad t = (t_1, \ldots, t_n) = (t(e_1), \ldots, t(e_n)), \\ R = \{x \in X : \|x\| = 1 \text{ or } \|x\| = \sqrt{2}\}, \qquad \alpha_0 = e_1, \\ R^\vee = \{x \in X : \|x\| = 2 \text{ or } \|x\| = \sqrt{2}\}, \qquad \alpha_0^\vee = 2e_1, \\ \Delta = \{\alpha_i = e_i - e_{i+1} : i = 1, \ldots, n-1\} \cup \{\alpha_n = e_n\}, \\ s_i = s_{\alpha_i}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x \rangle \alpha_0, \\ W = \langle s_1, \ldots, s_n \mid s_j^2 = (s_i s_j)^2 = (s_i s_{i+1})^3 = (s_{n-1} s_n)^4 = e : i \le n-2,\ |i-j| > 1 \rangle, \\ S^{\mathrm{aff}} = \{s_0, s_1, \ldots, s_{n-1}, s_n\}, \qquad \Omega = \{e\}, \\ W^e = W^{\mathrm{aff}} = \langle W, s_0 \mid s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^4 = e : i \ge 2 \rangle. \end{gathered}$$

For a generic parameter function, we have different parameters $q_0 = q(s_0)$, $q_1 = q(s_i)$ for $1 \le i < n$ and $q_2 = q(s_n)$.

The finite reflection group $W = W(B_n)$ is naturally isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. Let $\mu \vdash n$ and consider a point

$$t = \big((t_1^{\pm})^{\mu_1} \cdots (t_{n-\mu_{d-1}-\mu_d}^{\pm})^{\mu_{d-2}} (1)^{\mu_{d-1}} (-1)^{\mu_d} \big) \in T, \tag{3.24}$$

where $(t_1^{\pm})^{\mu_1}$ means that μ_1 coordinates are equal to t_1 or t_1^{-1}, while the other $n - \mu_1$ coordinates of t are different. The stabilizer W_t of t is isomorphic to

$$S_{\mu_1} \times \cdots \times S_{\mu_{d-2}} \times W(B_{\mu_{d-1}}) \times W(B_{\mu_d}). \tag{3.25}$$

Notice that this is a Weyl group, generated by the reflections it contains.

• **Case $q_0 = q_1 = q_2 = 1$.**

In view of (2.17) we want to determine the extended quotient $\widetilde{T_{\rm un}} / W$. Therefore, we recall the explicit classification of conjugacy classes in W in terms of bipartitions, which be found (for example) in [Carter 1972]. We already know that the quotient of W by the normal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ of sign changes is isomorphic to S_n, and that conjugacy classes in S_n are parametrized by partitions of n. So we wonder what the different conjugacy classes in $(\mathbb{Z}/2\mathbb{Z})^n \sigma(\mu)$ are, for $\mu \vdash n$. To handle this we introduce some notation, assuming $|\mu| + |\lambda| = n$ and $|\mu| + |\lambda| + |\rho| = n'$:

$$
\begin{aligned}
\epsilon_I &= \textstyle\prod_{i \in I} s_{e_i}, \qquad I \subset \{1, \dots, n\},\\
I_\lambda &= \{1, 1+\lambda_1, 1+\lambda_1+\lambda_2, \dots\}, \qquad \lambda = (\lambda_1, \lambda_2, \lambda_3, \dots),\\
\sigma'(\lambda) &= \epsilon_{I_\lambda} \sigma(\lambda) \in W(B_{|\lambda|}),\\
\sigma(\mu, \lambda) &= \sigma(\mu)\,(m \to m - |\lambda| \bmod n)\,\sigma'(\lambda)\,(m \to m + |\lambda| \bmod n),\\
\sigma(\mu, \lambda, \rho) &= \sigma(\mu, \lambda)\,(m \to m - |\rho| \bmod n')\,\sigma'(\rho)\,(m \to m + |\rho| \bmod n').
\end{aligned}
\tag{3.26}
$$

Let $I \subset \{1, \dots, m\}$ and $J \subset \{m+1, \dots, 2m\}$. It is easily verified that $\epsilon_I (1\,2\,\cdots\,m)$ is conjugate to $\mu_J (m{+}1\; m{+}2\,\cdots\,2m)$ if and only if $|I|+|J|$ is even. Therefore the conjugacy classes in $W(B_n)$ are parametrized by ordered pairs of partitions of total weight n. Explicitly (μ, λ) corresponds to $\sigma(\mu, \lambda)$ as in (3.26). The set $T^{\sigma(\mu,\lambda)}$ and the group $Z_{W_0(B_n)}(\sigma(\mu, \lambda))$ are both the direct product of the corresponding objects for the blocks of μ and λ, i.e., for the parts $(m, m, \dots, m)$. The centralizer of $\sigma((m)^k)$ in $W(B_{km})$ is generated by $(1\,2\,\cdots\,m)$, $\epsilon_{\{1,2,\dots,m\}}$ and the transpositions of cycles

$$
(am+1\; am+m+1)(am+2\; am+m+2)\cdots(am+m\; am+2m), \tag{3.27}
$$

where $0 \le a \le k-2$. It follows that

$$
\begin{aligned}
Z_{W(B_{km})}(\sigma((m)^k)) &\cong W(B_k) \ltimes (\mathbb{Z}/m\mathbb{Z})^k,\\
((\mathbb{C}^\times)^{km})^{\sigma((m)^k)} &= \{((t_1)^m (t_{m+1})^m \cdots (t_{km+1-m})^m) : t_i \in \mathbb{C}^\times\},\\
((S^1)^{km})^{\sigma((m)^k)} / Z_{W_0(B_{km})}(\sigma((m)^k)) &\cong (S^1)^k / W(B_k) \cong [-1, 1]^k / S_k.
\end{aligned}
\tag{3.28}
$$

Now consider the following element of $W(B_{km})$:

$$
\sigma'((m)^k) = \epsilon_{\{1, m+1, \dots, km+1-m\}} (1\,2\,\cdots\,m)(m{+}1\,\cdots\,2m)\cdots(km{+}1{-}m\,\cdots\,km).
$$

It has only 2^k fixpoints, namely

$$
((\pm 1)^m (\pm 1)^m \cdots (\pm 1)^m). \tag{3.29}
$$

The centralizer of $\sigma'((m)^k)$ is generated by $\epsilon_{\{1\}}(1\,2\,\cdots\,m)$, $\epsilon_{\{1,2,\dots,m\}}$ and the elements (3.27). The latter two generate a subgroup isomorphic to $W(B_k)$, which fits in a short exact sequence

$$1 \to W(B_k) \to Z_{W(B_{mk})}(\sigma'((m)^k)) \to (\mathbb{Z}/m\mathbb{Z})^k \to 1, \tag{3.30}$$

where the first factor $\mathbb{Z}/m\mathbb{Z}$ is generated by the image of $\epsilon_{\{1\}}(1\,2\,\cdots\,m)$. We find

$$((S^1)^{km})^{\sigma'((m)^k)}/Z_{W(B_{mk})}(\sigma'((m)^k)) \cong \{(1)^{am}(-1)^{(k-a)m} : 0 \leq a \leq k\}. \tag{3.31}$$

Now we can see what $T_{\mathrm{un}}^{\sigma(\mu,\lambda)}/Z_W(\sigma(\mu,\lambda))$ looks like. Its number of components $N(\lambda)$ depends only on λ, and all these components are mutually homeomorphic contractible orbifolds, the shape and dimension being determined by μ. More precisely, for every block of μ of width k we get a factor $[-1,1]^k/S_k$, and for every block of λ of width l we must multiply the number of components by $l+1$. Alternatively, we can obtain the same space (modulo the action of W) as

$$\begin{aligned} &T_{\mathrm{un}}^{\sigma(\mu,\lambda)}/Z_{W(B_n)}(\sigma(\mu,\lambda)) \\ &\qquad = \bigsqcup_{\lambda_1\cup\lambda_2=\lambda} T_{\mathrm{un},c}^{\sigma(\mu,\lambda_1,\lambda_2)}/Z_{W(B_n)}(\sigma(\mu,\lambda_1,\lambda_2)) \\ &\qquad = \bigsqcup_{\lambda_1\cup\lambda_2=\lambda} ((S^1)^{|\mu|})^{\sigma(\mu)}/Z_{W(B_{|\mu|})}(\sigma(\mu))\,(-1)^{|\lambda_1|}\,(1)^{|\lambda_2|} \\ &\qquad = ([-1,1]^{|\mu|})^{\sigma(\mu)}/Z_{S_{|\mu|}}(\sigma(\mu)) \times \bigsqcup_{\lambda_1\cup\lambda_2=\lambda} (-1)^{|\lambda_1|}\,(1)^{|\lambda_2|}, \end{aligned} \tag{3.32}$$

where the subscript c means that we take only the connected component containing the point $((1)^{|\mu|}(-1)^{|\lambda_1|}(1)^{|\lambda_2|})$.

In effect we parametrized the components of the extended quotient $\widetilde{T_{\mathrm{un}}}/W$ by ordered triples of partitions $(\mu,\lambda_1,\lambda_2)$ of total weight n, and every such component is contractible. In combination with (3.25) this shows that the conditions of Theorem 2.45 are fulfilled.

Denote the number of ordered k-tuples of partitions of total weight n by $\mathcal{P}(k,n)$. Now Theorem 2.45 says that

$$K_*(C_r^*(W^e)) = \check{H}^*(\widetilde{T_{\mathrm{un}}}/W;\mathbb{Z}) = \check{H}^0(\widetilde{T_{\mathrm{un}}}/W;\mathbb{Z}) \cong \mathbb{Z}^{\mathcal{P}(3,n)}. \tag{3.33}$$

- **Generic case.**

The inequivalent subsets of Δ are parametrized by partitions μ of weight at most n:

$$\begin{aligned} P_\mu &= \Delta \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{|\mu|}\}, \\ R_{P_\mu} &\cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \times B_{n-|\mu|}, \\ R_{P_\mu}^\vee &\cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \times C_{n-|\mu|}, \\ X^{P_\mu} &\cong \mathbb{Z}(e_1+\cdots+e_{\mu_1})/\mu_1 + \cdots + \mathbb{Z}(e_{|\mu|+1-\mu_d}+\cdots+e_{|\mu|})/\mu_d, \end{aligned}$$

$$
\begin{aligned}
X_{P_\mu} &\cong (\mathbb{Z}^n/\mathbb{Z}(e_1+\cdots+e_n))^{m_n}+\cdots+(\mathbb{Z}^2/\mathbb{Z}(e_1+e_2))^{m_2}+\mathbb{Z}^{n-|\mu|},\\
Y^{P_\mu} &= \mathbb{Z}(e_1+\cdots+e_{\mu_1})+\cdots+\mathbb{Z}(e_{|\mu|+1-\mu_d}+\cdots+e_{|\mu|}),\\
Y_{P_\mu} &= \{y\in\mathbb{Z}^n : y_1+\cdots+y_{\mu_1}=\cdots=y_{|\mu|+1-\mu_d}+\cdots+y_{|\mu|}=0\},\\
T^{P_\mu} &= \{(t_1)^{\mu_1}(t_{\mu+1})^{\mu_2}\cdots(t_{|\mu|})^{\mu_d}(1)^{n-|\mu|} : t_i\in\mathbb{C}^\times\},\\
T_{P_\mu} &= \{t\in(\mathbb{C}^\times)^n : t_1\cdots t_{\mu_1}=t_{\mu_1+1}\cdots t_{\mu_1+\mu_2}=\cdots=t_{|\mu|+1-\mu_d}\cdots t_{|\mu|}=1\},\\
K_{P_\mu} &= \{t\in T^{P_\mu} : t_1^{\mu_1}=\cdots=t_{|\mu|}^{\mu_d}=1\},\\
W_{P_\mu} &\cong S_n^{m_n}\times\cdots\times S_2^{m_2}\times W(B_{n-|\mu|}),\\
W(P_\mu,P_\mu) &\cong W(B_{m_n})\times\cdots\times W(B_{m_2})\times W(B_{m_1}).
\end{aligned}
$$

We see that $\mathcal{R}_{P_\mu}$ is the product of various root data of type SL_m and one factor $\mathcal{R}(\mathrm{SO}_{2(n-|\mu|)+1})$. Hence $\mathcal{H}_{P_\mu}$ is the tensor product of a type A part and a type B part. From our study of $\mathcal{R}(\mathrm{SL}_m)$, we recall that the discrete series representations of the type A part of $\mathcal{H}_{P_\mu}$ are in bijection with K_{P_μ}. From [Heckman and Opdam 1997, Proposition 4.3] and [Opdam 2004, Appendix A.2] we know that the residual points for $\mathcal{R}(\mathrm{SO}_{2(n-|\mu|)+1}, q)$ are parametrized by ordered pairs (λ_1,λ_2) of total weight $n-|\mu|$. The unitary part of such a residual point is in the component we indicated in (3.32). Let $\mathrm{RP}(\mathcal{R},q)$ denote the collection of residual points for the pair $(\mathcal{R},q)$. The above gives canonical bijections

$$
\begin{aligned}
&\bigsqcup_{t\in\mathrm{RP}(\mathcal{R}_{P_\mu},q_{P_\mu})} tT_{\mathrm{un}}^{P_\mu}/\mathcal{W}_{P_\mu P_\mu}\\
&\qquad\cong \bigsqcup_{t\in\mathrm{RP}(\mathcal{R}(\mathrm{SO}_{2(n-|\mu|)+1},q))} tT_{\mathrm{un}}^{P_\mu}/W(P_\mu,P_\mu)\\
&\qquad\cong T_{\mathrm{un}}^{P_\mu}/Z_{W_0(B_{|\mu|})}(\sigma(\mu))\times\bigsqcup_{(\lambda_1,\lambda_2):|\lambda_1|+|\lambda_2|=n}(-1)^{|\lambda_1|}(1)^{|\lambda_2|}.
\end{aligned}
\tag{3.34}
$$

Theorem 3.35. (a) *For generic q, $C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1}),q)$ is Morita equivalent with the commutative C^*-algebra of continuous functions on* (3.34).

(b) $K_1(C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1}),q))=0$ *and* $K_0(C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1},q)))$ *is a free abelian group of rank* $\mathcal{P}(3,n)$.

Proof. (a) First we note that (3.34) can be identified with the extended quotient $\widetilde{T_{\mathrm{un}}}/W$ described in (3.32) and the subsequent lines.

Fix any $u\in T_{\mathrm{un}}$. The fibre over u of the projection

$$p:\widetilde{T_{\mathrm{un}}}/W\to T_{\mathrm{un}}/W$$

is in bijection with the set of conjugacy classes of W. By Clifford theory, $|p^{-1}(Wu)|$ is also the number of inequivalent irreducible representations of $C(T_{\mathrm{un}})\rtimes W$ with central character Wu. Equivalently, $|p^{-1}(Wu)|$ is the number of inequivalent tempered irreducible representations of $\mathcal{O}(T)\rtimes W$ with central character Wu. By

Theorem 1.52 the latter equals the number of inequivalent irreducible tempered $\mathcal{H}(\mathcal{R}, q)$-representations with central character in WuT_{rs}.

By Theorem 2.7 every point of (3.34) is the $Z(C_r^*(\mathcal{R}, q))$-character of at least one irreducible $C_r^*(\mathcal{R}, q)$-representation. The projection p' from (3.34) to T/W corresponds to restriction from $Z(C_r^*(\mathcal{R}, q)) \cong C(\Xi_{\mathrm{un}}/\mathcal{G})$ to $Z(\mathcal{H}(\mathcal{R}, q)) \cong \mathcal{O}(T/W)$.

Suppose that a point of $p'^{-1}(WuT_{\mathrm{rs}})$ would carry more than one inequivalent irreducible $C_r^*(\mathcal{R}, q)$-representation. Then the inverse image of WuT_{rs} under

$$\mathrm{Irr}(C_r^*(\mathcal{R}, q)) = \mathrm{Irr}_{\mathrm{temp}}(\mathcal{H}(\mathcal{R}, q)) \to T/W$$

would have more than $|p^{-1}(u)|$ elements. This would contradict what we concluded above, using Theorem 1.52. Thus every $\pi(P_\mu, \delta, t)$ with $(P_\mu, \delta, t) \in \Xi_{\mathrm{un}}$ is irreducible and (3.34) is exactly the space $\mathrm{Irr}(C_r^*(\mathcal{R}, q))$.

When we compare this with Theorem 2.7 and (2.6), we see that all intertwining operators $\pi(g, P_\mu, \delta, t)$ with $g(P_\mu, \delta, t) = (P_\mu, \delta, t)$ must be scalar. Recall from (2.9) that every indecomposable direct summand of $C_r^*(\mathcal{R}, q)$ is of the form

$$C\big(T_{\mathrm{un}}^{P_\mu}; \mathrm{End}_{\mathbb{C}}(\pi(P_\mu, \delta, t))\big)^{\mathcal{G}_{P\mu,\delta}}. \tag{3.36}$$

From (3.31) we know that the space $T_{\mathrm{un}}^{P_\mu}/\mathcal{G}_{P\mu,\delta}$ is a direct product of factors $(S^1)^k/W(B_k) \cong [-1, 1]/S_k$. We note that

$$\{(z_1, z_2, \dots, z_k) \in (S^1)^k : \Im(z_i) \geq 0, \Re(z_1) \geq \Re(z_2) \geq \cdots \geq \Re(z_k)\}$$

is a closed, connected fundamental domain for action of $W(B_k)$ on $(S^1)^k$. With this it is easy to find a closed fundamental domain $D_{P_\mu,\delta}$ for the action of $\mathcal{G}_{P_\mu,\delta}$ on $T_\delta^{P_\mu}$, such that $D_{P_\mu,\delta}$ is homeomorphic to $T_\delta^{P_\mu}/\mathcal{G}_{P_\mu,\delta}$. Then restriction from $T_{\mathrm{un}}^{P_\mu}$ to $D_{P_\mu,\delta}$ gives a monomorphism of C^*-algebras from (3.36) to

$$C(D_{P_\mu,\delta}; \mathrm{End}_{\mathbb{C}}(\pi(P_\mu, \delta, t))) = C(D_{P_\mu,\delta}) \otimes \mathrm{End}_{\mathbb{C}}(\pi(P_\mu, \delta, t)).$$

It is surjective because the intertwining operators $\pi(g, P_\mu, \delta, t)$, $g \in \mathcal{G}_{P_\mu,\delta}$, from (2.5) depend continuously on $t \in T_{\mathrm{un}}^{P_\mu}$ and are scalar multiples of the identity whenever they map a representation to itself. Hence $C_r^*(\mathcal{R}, q)$ is Morita equivalent with $\bigoplus_{(P_\mu,\delta)/\mathcal{G}} C(D_{P_\mu,\delta})$, as required.

(b) By the Serre–Swan theorem, $K_*(C_r^*(\mathcal{R}, q))$ is the topological K-theory of the underlying space (3.34). Since every connected component of this space is contractible, $K_1(C_r^*(\mathcal{R}, q)) = 0$ and $K_0(C_r^*(\mathcal{R}, q))$ is a free abelian group whose rank equals the number of connected components of (3.34). In the lines following (3.32) we showed that that number is $\mathcal{P}(3, n)$. By Theorem 2.11 these K-groups are independent of the parameters q. $\square$

3E. *Type* $\mathbf{Sp_{2n}}$. The root datum for the symplectic group Sp_{2n} is dual to that for SO_{2n+1}. Concretely, $\mathcal{R}(\mathrm{Sp}_{2n})$ is given by

$$
\begin{gathered}
X=\{y\in Y: y_1+\cdots+y_n \text{ even}\}, \qquad Q=\mathbb{Z}^n, \qquad Y=Q^\vee=\mathbb{Z}^n,\\
T=(\mathbb{C}^\times)^n, \qquad t=(t_1,\dots,t_n)=(t(e_1),\dots,t(e_n)),\\
R=\{x\in X:\|x\|=2 \text{ or } \|x\|=\sqrt{2}\}, \qquad \alpha_0=e_1+e_2,\\
R^\vee=\{x\in X:\|x\|=1 \text{ or } \|x\|=\sqrt{2}\}, \qquad \alpha_0^\vee=e_1+e_2,\\
\Delta=\{\alpha_i=e_i-e_{i+1}: i=1,\dots,n-1\}\cup\{\alpha_n=2e_n\},\\
s_i=s_{\alpha_i}, \qquad s_0=t_{\alpha_0}s_{\alpha_0}=t_{e_1}s_{\alpha_0}t_{-e_1}: x\to x+\langle\alpha_0^\vee,x\rangle\alpha_0,\\
W=\langle s_1,\dots,s_n \mid s_j^2=(s_is_j)^2=(s_is_{i+1})^3=(s_{n-1}s_n)^4=e: i\le n-2,\ |i-j|>1\rangle,\\
S^{\mathrm{aff}}=\{s_0,s_1,\dots,s_{n-1},s_n\}, \qquad \Omega=\{e,t_{e_1}s_{2e_1}\},\\
W^{\mathrm{aff}}=\langle W,s_0 \mid s_0^2=(s_0s_i)^2=(s_0s_2)^3=e: i\ne 2\rangle, \qquad W^e=W^{\mathrm{aff}}\rtimes\Omega.
\end{gathered}
$$

For a generic parameter function we have two independent parameters $q_1=q(s_1)$ and $q_2=q(s_n)$.

The groups X, W and W^e are exactly the same as for $\mathcal{R}(\mathrm{SO}_{2n+1})$. Everything that we said in Section 3D about the stabilizers in W of points of T obviously is valid here as well. In particular, for $q=1$ the algebra $\mathcal{H}(\mathcal{R}(\mathrm{Sp}_{2n}),1)$ is identical to $\mathcal{H}(\mathcal{R}(\mathrm{SO}_{2n+1}),1)$, and the entire analysis of the K-theory of its C^*-completion can be found in the previous paragraph.

For all other q we can use Theorem 2.11. Thus, we get

$$
\begin{aligned}
K_*(C_r^*(\mathcal{R}(\mathrm{Sp}_{2n}),q)) &\cong K_*(C_r^*(\mathcal{R}(\mathrm{Sp}_{2n}),1))\\
&= K_*(C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1}),1)) \cong K_*(C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1}),q)).
\end{aligned}
$$

The last group is the one we actually computed, for generic parameters. Let us phrase the results explicitly:

$$
K_0(C_r^*(\mathcal{R}(\mathrm{Sp}_{2n}),q))\cong\mathbb{Z}^{\mathcal{P}(3,n)}, \qquad K_1(C_r^*(\mathcal{R}(\mathrm{Sp}_{2n}),q))=0. \tag{3.37}
$$

3F. *Type* $\mathbf{SO_{2n}}$. The root datum for the even special orthogonal group SO_{2n} has groups contained in those for the root datum of type SO_{2n+1}:

$$
\begin{gathered}
X=\mathbb{Z}^n, \qquad Q=\{y\in Y: y_1+\cdots+y_n \text{ even}\},\\
Y=\mathbb{Z}^n, \qquad Q^\vee=\{y\in Y: y_1+\cdots+y_n \text{ even}\},\\
T=(\mathbb{C}^\times)^n, \qquad t=(t_1,\dots,t_n)=(t(e_1),\dots,t(e_n)),\\
R=\{x\in X:\|x\|=\sqrt{2}\}, \qquad \alpha_0=e_1+e_2,\\
R^\vee=\{x\in X:\|x\|=\sqrt{2}\}, \qquad \alpha_0^\vee=e_1+e_2,\\
\Delta=\{\alpha_i=e_i-e_{i+1}: i=1,\dots,n-1\}\cup\{\alpha_n=e_{n-1}+e_n\},
\end{gathered}
$$

$$s_i = s_{\alpha_i}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{e_1} s_{\alpha_0} t_{-e_1} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x \rangle \alpha_0,$$
$$W = \langle s_1, \dots, s_n \mid s_j^2 = (s_i s_j)^2 = (s_i s_{i+1})^3 = (s_{n-2} s_n)^4 = e : i \le n-2,\ |i-j| > 1 \rangle,$$
$$S^{\mathrm{aff}} = \{s_0, s_1, \dots, s_{n-1}, s_n\}, \qquad \Omega = \{e, t_{e_1} s_{e_1} s_{e_n}\},$$
$$W^{\mathrm{aff}} = \langle W, s_0 \mid s_0^2 = (s_0 s_i)^2 = (s_0 s_2)^3 = e : i \neq 2 \rangle \subsetneq W^e.$$

When $n > 2$, all the simple affine reflections are conjugate in W^e, and

$$q(s_i) = q, \quad i = 0, 1, \dots, n,$$

for every parameter function. For $n = 2$ the root system $R \cong A_1 \times A_1$ is reducible, there is an additional simple affine reflection and there are more possible parameter functions. For $n = 1$, $\mathcal{R}(\mathrm{SO}_2)$ is the root datum of a one-dimensional torus, in particular $W = 1$.

The based root datum $\mathcal{R}(\mathrm{SO}_{2n})$ has one nontrivial automorphism, which exchanges the roots α_{n-1} and α_n. It is easily seen that

$$W^e(\mathrm{SO}_{2n}) \rtimes \mathrm{Aut}(\mathcal{R}(\mathrm{SO}_{2n})) \cong W^e(\mathrm{SO}_{2n+1}).$$

With Theorem 2.11 we conclude that, for every equal parameter function q,

$$\begin{aligned} K_*\big(C_r^*(\mathcal{R}(\mathrm{SO}_{2n}), q) \rtimes \mathrm{Aut}(\mathcal{R}(\mathrm{SO}_{2n}))\big) \\ &\cong K_*\big(W^e(\mathrm{SO}_{2n}) \rtimes \mathrm{Aut}(\mathcal{R}(\mathrm{SO}_{2n}))\big) \\ &= K_*\big(C_r^*(W^e(\mathrm{SO}_{2n+1}))\big) \cong K_*\big(C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1}), q)\big). \end{aligned} \tag{3.38}$$

Unfortunately, no such shortcut is available for $K_*(C_r^*(\mathcal{R}(\mathrm{SO}_{2n}), q))$. Therefore we will just compute $K_*(W^e(\mathrm{SO}_{2n}))$ by hand, in several steps:

- We determine the extended quotient $T_{\mathrm{un}} /\!/ W(D_n)$ and its cohomology.
- We analyze the (elliptic) representations of the $W(D_n)$-isotropy groups of points of T.
- We relate the second bullet to the sheaf $\mathfrak{L}_u^{W(D_n)}$ on $T_{\mathrm{un}} / W(D_n)$.
- Then we are finally in the right position to apply Theorem 2.34.

The finite reflection group $W(D_n)$ is naturally isomorphic to the index two subgroup of $W(B_n) = W(C_n)$ consisting of those elements that involve an even number of sign changes. In other words, let $(\mathbb{Z}/2\mathbb{Z})^n_{\mathrm{ev}}$ be the kernel of the summation map $(\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}$. Then

$$W(D_n) = (\mathbb{Z}/2\mathbb{Z})^n_{\mathrm{ev}} \rtimes S_n.$$

The conjugacy classes in $W(D_n)$ are similar to but slightly different from those in $W(B_n)$. We rephrase Young's parametrization in the notations from (3.26). For every bipartition (μ, λ) of n where λ has an even number of parts, $\sigma(\mu, \lambda)$ represents one class in $W(D_n)$. Suppose now that $\mu \vdash n$ has only even terms, and define

$$\sigma''(\mu) = \sigma(\mu)\epsilon_{\{n-1,n\}} = \epsilon_{\{n\}}^{-1}\sigma(\mu)\epsilon_{\{n\}}. \tag{3.39}$$

Then $\sigma''(\mu)$ represents a class of $W(D_n)$ different from the above. The $\sigma(\mu, \lambda)$ and the $\sigma''(\mu)$ form a set of representatives for all conjugacy classes of $W(D_n)$.

In the representation theory of classical groups, some almost direct products of root data of type D arise [Goldberg 1994; Heiermann 2011]. Therefore it will be useful to investigate a more general situation, as in the Appendix. Fix $n_1, \dots, n_d$ with $n_1 + \cdots + n_d = n$ and consider the root datum

$$\mathcal{R}'_{\vec{n}} = \mathcal{R}(\mathrm{SO}_{2n_1}) \times \cdots \times \mathcal{R}(\mathrm{SO}_{2n_d}).$$

Let $W'_{\vec{n}} = W(D_{\vec{n}}) \rtimes \Gamma$ be as in (A.1), so $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^d_{\mathrm{ev}}$. The conjugacy classes for $W'_{\vec{n}}$ are a mixture of those for $W(D_n)$ and for $W(B_{\vec{n}})$. Let us analyze them and the extended quotient $T_{\mathrm{un}} /\!/ W'_{\vec{n}}$ together.

Recall that for $w \in W(B_{\vec{n}})$, the groups T^w_{un} and $Z_{W(B_{\vec{n}})}(w)$ were already computed in Section 3D; see in particular (3.28), (3.29) and (3.30). We say that $\vec{\mu} \vdash \vec{n}$ if $\vec{\mu}$ is a d-tuple of partitions $(\mu^{(1)}, \dots, \mu^{(d)})$ with $|\mu^{(i)}| = n_i$, and that $(\vec{\mu}, \vec{\lambda}) \vdash \vec{n}$ if $\vec{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ such that $|\mu^{(i)}| + |\lambda^{(i)}|$. To these we can associate $\sigma(\vec{\mu})$ and $\sigma(\vec{\mu}, \vec{\lambda})$, as products of (3.26) over the indices i.

- Consider $\sigma(\vec{\mu}, \vec{\lambda})$, where $\vec{\lambda}$ is nonempty and has an even number of terms. Notice that $Z_{W(B_{\vec{n}})}(\sigma(\vec{\mu}, \vec{\lambda}))$ contains an element not in $W(D_n)$ which fixes $T^{\sigma(\vec{\mu},\vec{\lambda})}$ pointwise, namely a single factor $\epsilon_{\{a_1\}}(a_1 \cdots a_m)$ of $\vec{\lambda}$. Hence the $W(B_{\vec{n}})$-conjugacy class of $\sigma(\vec{\mu}, \vec{\lambda})$ is precisely the $W'_{\vec{n}}$-conjugacy class of $\sigma(\vec{\mu}, \vec{\lambda})$. Furthermore,

$$T_{\mathrm{un}}^{\sigma(\vec{\mu},\vec{\lambda})} / Z_{W'_{\vec{n}}}(\sigma(\vec{\mu}, \vec{\lambda})) = T_{\mathrm{un}}^{\sigma(\vec{\mu},\vec{\lambda})} / Z_{W(B_{\vec{n}})}(\sigma(\vec{\mu}, \vec{\lambda})),$$

  and as described in (3.32), this is a disjoint union of contractible spaces. The number of components is given explicitly in terms of $\vec{\lambda}$.

- Suppose that $\vec{\mu} \vdash \vec{n}$ and that all terms of $\vec{\mu}$ are even. Then the $W(B_{\vec{n}})$-conjugacy class of $\sigma(\vec{\mu})$ splits into two $W'_{\vec{n}}$-conjugacy classes, the other one represented by

$$\sigma''(\vec{\mu}) = \sigma(\vec{\mu})\epsilon_{\{n-1,n\}}.$$

  Both $Z_{W(B_{\vec{n}})}(\sigma(\vec{\mu}))$ and

$$Z_{W(B_{\vec{n}})}(\sigma''(\vec{\mu})) = \epsilon_{\{n\}}^{-1} Z_{W(B_{\vec{n}})}(\sigma(\vec{\mu}))\epsilon_{\{n\}}$$

  are contained in $W'_{\vec{n}}$. Let m_l be the multiplicity of l in $\vec{\mu}$. By (3.32),

$$T_{\mathrm{un}}^{\sigma''(\vec{\mu})} / Z_{W'_{\vec{n}}}(\sigma''(\vec{\mu})) \cong T_{\mathrm{un}}^{\sigma(\vec{\mu})} / Z_{W'_{\vec{n}}}(\sigma(\vec{\mu})) \cong \prod_{l=1}^{n} [-1, 1]^{m_l} / S_{m_l},$$

  which is a contractible space.

- Let $\mu \vdash n$ be a partition with at least one odd term. Again, the $W(B_{\vec{n}})$-conjugacy class of $\sigma(\vec{\mu})$ is precisely the $W'_{\vec{n}}$-conjugacy class of $\sigma(\vec{\mu})$. Now
$$Z_{W'_{\vec{n}}}(\sigma(\vec{\mu})) \subsetneq Z_{W(B_{\vec{n}})}(\sigma(\vec{\mu})),$$
and this really makes a difference. From (3.28) we deduce
$$T_{\mathrm{un}}^{\sigma(\vec{\mu})} / Z_{W'_{\vec{n}}}(\sigma(\vec{\mu})) \cong \prod_{l=1}^{n} (S^1)^{m_l} \Big/ \Big(\prod_{l=1}^{n} W(B_{m_l}) \cap W(D_n) \Big). \tag{3.40}$$
The group $\prod_{l=1}^{n} W(B_{m_l}) \cap W(D_n)$ equals $\big(\prod_{l=1}^{n} (\mathbb{Z}/2\mathbb{Z})^{m_l}\big)_+ \rtimes \prod_{l=1}^{m_l} S_{m_l}$, where the subscript + means that the total number of sign changes for odd l must be even. The quotient map
$$\prod_{l \text{ odd}} (S^1)^{m_l} \Big/ \Big(\prod_{l \text{ odd}} (\mathbb{Z}/2\mathbb{Z})^{m_l} \Big)_+ \to \prod_{l \text{ odd}} (S^1)^{m_l} / (\mathbb{Z}/2\mathbb{Z})^{m_l} \cong \prod_{l \text{ odd}} [-1, 1]^{m_l} \tag{3.41}$$
is a two-fold cover which ramifies precisely at the boundary of the unit cube $\prod_{l \text{ odd}} [-1, 1]^{m_l}$. Therefore the left-hand side of (3.41) is homeomorphic to the unit sphere of dimension $m_1 + m_3 + m_5 + \cdots$. This entails that (3.40) is homeomorphic to
$$\prod_{l \text{ even}} ([-1, 1]^{m_l} / S_{m_l}) \times S^{m_1 + m_3 + \cdots} \Big/ \prod_{l \text{ odd}} S_{m_l}. \tag{3.42}$$
This space is contractible unless $m_l = 1$ for all odd l; then it is homotopic to $S^{m_1 + m_3 + \cdots}$.

The extended quotient $T_{\mathrm{un}} /\!/ W'_{\vec{n}}$ is the disjoint union of the spaces $T_{\mathrm{un}}^{w} / Z_{W'_{\vec{n}}}(w)$, as w runs over representatives for the conjugacy classes of $W'_{\vec{n}}$. Since we covered all conjugacy classes for $W(B_{\vec{n}})$ intersecting $W'_{\vec{n}}$, we have a complete description of conjugacy classes for the latter group. From the above calculations we immediately get the cohomology of the extended quotient.

Lemma 3.43. *The abelian group $\check{H}^*(T_{\mathrm{un}} /\!/ W'_{\vec{n}})$ is torsion-free.*

In the case $\vec{n} = n$, $W'_{\vec{n}} = W(D_n)$, we can describe the cohomology of $T_{\mathrm{un}} /\!/ W(D_n)$ explicitly. The rank of the odd cohomology is the number of partitions $\mu \vdash n$ such that every odd term appears with multiplicity one, and there is an odd number of odd terms.

The rank of the even cohomology of $T_{\mathrm{un}} /\!/ W(D_n)$ is the sum of four contributions:

- $\prod_i (k_i + 1)$, *for every bipartition (μ, λ) of n with $\lambda = (n)^{k_n} \cdots (1)^{k_1}$ such that $\sum_i k_i$ is positive and even;*
- *two times the number of partitions of n with only even terms;*

- *the number of partitions of n with at least one odd term*;
- *the number of partitions of n such that every odd term appears only once*, *and the number of odd terms is positive and even.*

Every point of $T \cong (\mathbb{C}^\times)^n$ is $W(B_{\vec{n}})$-conjugate to one of the form

$$t = (t^{(1)}, \dots, t^{(d)}), \qquad t^{(i)} = \big((t_1)^{\mu_1^{(i)}} \cdots (t_{n_i - m_1^{(i)} - m_2^{(i)}})^{\mu_{d_i}} (1)^{m_1^{(i)}} (-1)^{m_2^{(i)}}\big) \in (\mathbb{C}^\times)^{n_i}.$$

The isotropy group of t in $W'_{\vec{n}}$ is

$$\begin{aligned}(W'_{\vec{n}})_t &= \Big(\prod_{i=1}^d S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}} \times W(B_{m_1^{(i)}}) \times W(B_{m_2^{(i)}})\Big) \cap W(D_n) \\ &= \Big(\prod_{i=1}^d S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}}\Big) \times \Big(\prod_{i=1}^d W\big(B_{m_1^{(i)}}\big) \times W\big(B_{m_2^{(i)}}\big)\Big) \\ &\qquad\qquad \cap W\big(D_{m_1^{(1)} + \cdots + m_2^{(d)}}\big).\end{aligned} \tag{3.44}$$

We note that $(W'_{\vec{n}})_t$ is generated by the reflections it contains if t has no coordinates 1 or -1. Otherwise the reflection subgroup of $W(D_n)_t$ is

$$(W'_{\vec{n}})_t^\circ := \prod_{i=1}^d S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}} \times W\big(D_{m_1^{(i)}}\big) \times W\big(D_{m_2^{(i)}}\big),$$

where $W(D_0) = W(D_1) = 1$. In that case,

$$(W'_{\vec{n}})_t = \Big(\prod_{i=1}^d S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}}\Big) \times W'_{\vec{m}}, \tag{3.45}$$

where $\vec{m}$ consists of those terms $m_1^{(i)}, m_2^{(i)}$ which are nonzero. The group $W'_{\vec{m}}$ is a particular instance of the almost Weyl groups studied in the Appendix. Thus $(W'_{\vec{n}})_t$ is an example of the groups considered in Lemma A.9, and we may use that result.

Proposition 3.46. *For any positive parameter function q, $K_*(C_r^*(\mathcal{R}'_{\vec{n}}, q))$ is a free abelian group, isomorphic to $H^*(T_{\mathrm{un}} /\!/ W'_{\vec{n}}; \mathbb{Z})$.*

In particular, for $\vec{n} = n$, $\mathcal{R}'_{\vec{n}} = \mathcal{R}(\mathrm{SO}_{2n})$, $W'_{\vec{n}} = W(D_n)$, the free abelian group

$$K_*(C_r^*(\mathcal{R}(\mathrm{SO}_{2n}), q)) \cong H^*(T_{\mathrm{un}} /\!/ W(D_n); \mathbb{Z})$$

has even and odd ranks as given in Lemma 3.43.

Proof. By Theorem 1.52 it suffices to prove this when $q = 1$.

We adapt the notations from (3.32) to the present setting. Let $(\vec{\mu}, \vec{\lambda}_1, \vec{\lambda}_2)$ be a d-tuple of tripartitions of $n_1, \dots, n_d$, respectively, and such that $\vec{\lambda}_1 \cup \vec{\lambda}_2$ has an even number of terms. As in (3.45) we write

$$W_{\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2} := \Big(\prod_{i=1}^{d} S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}} \times W\big(B_{|\lambda_1^{(i)}|}\big) \times W\big(B_{|\lambda_2^{(i)}|}\big) \Big) \cap W(D_n)$$
$$= \Big(\prod_{i=1}^{d} S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}} \Big) \times W'_{\vec{m}},$$

where $\vec{m}$ consists of the nonzero terms among the $|\lambda_1^{(i)}|, |\lambda_2^{(i)}|$. The group $W_{\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2}$ is the full stabilizer of some point of T_{un}, and of the form considered in Lemma A.9. We note that $\sigma(\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2)$ is an elliptic element of $W_{\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2}$.

For every $t \in T_{\mathrm{un},c}^{\sigma(\mu,\lambda_1,\lambda_2)}$ we have $(W'_{\vec{n}})_t \supset W_{\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2}$. Using Lemma A.9 we define

$$s(\sigma(\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2),t) = \mathrm{ind}_{W_{\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2}}^{(W'_{\vec{n}})_t} H(u_{\sigma(\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2)}, \rho_{\sigma(\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2)}). \tag{3.47}$$

Suppose that $\vec{\mu} \vdash \vec{n}$ and that $\vec{\mu}$ has only even terms. Then $\sigma''(\vec{\mu}) = \epsilon_{\{n-1,n\}}\sigma(\vec{\mu})$ is conjugate to $\sigma(\vec{\mu})$ in $W(B_{\vec{n}})$ but not in $W'_{\vec{n}}$. The element $\sigma''(\vec{\mu})$ is elliptic in $\epsilon_{\{n\}}\big(\prod_{i=1}^{d} S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}}\big)\epsilon_{\{n\}}$, and for every $t \in T^{\sigma''(\vec{\mu})}$ we have

$$(W'_{\vec{n}})_t \supset \epsilon_{\{n\}} \Big(\prod_{i=1}^{d} S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}} \Big) \epsilon_{\{n\}}.$$

For such t we define

$$s(\sigma''(\vec{\mu}),t) = \mathrm{ind}_{\epsilon_{\{n\}}(\prod_{i=1}^{d} S_{\mu_1^{(i)}} \times\cdots\times S_{\mu_{d_i}^{(i)}})\epsilon_{\{n\}}}^{(W'_{\vec{n}})_t} H(u_{\sigma''(\vec{\mu})}, \rho_{\sigma''(\vec{\mu})}). \tag{3.48}$$

As discussed before Lemma 3.43, every conjugacy class of $W'_{\vec{n}}$ appears precisely once in (3.47) and (3.48) together.

With this information and Lemma A.9 available, the same argument as in the proof of Theorem 2.45(a) works in the present setting, and shows that the conclusion of Theorem 2.45(a) is fulfilled. Then we apply Theorem 2.45(b). □

3G. *Type G_2.* As basis for the root lattice X of type G_2, we take the two simple roots. We coordinatize the dual lattice Y so that the pairing between X and Y becomes the standard pairing on $\mathbb{Z}^2$. Explicitly, $\mathcal{R}(G_2)$ becomes

$$X = Q = \mathbb{Z}^2, \qquad Y = Q^\vee = \mathbb{Z}^2,$$
$$T = (\mathbb{C}^\times)^2, \qquad t = (t(e_1), t(e_2)) = (t_1, t_2),$$
$$R^+ = \{e_1, e_2, e_1+e_2, 2e_1+e_2, 3e_1+e_2, 3e_1+2e_2\}, \qquad R = R^+ \cup -R^+,$$
$$R^{\vee,+} = \{2e_1-3e_2, 2e_1-e_2, 3e_2-e_1, e_1, e_1-e_2, e_2\}, \qquad R^\vee = R^{\vee,+} \cup -R^{\vee,+},$$
$$\Delta = \{e_1, e_2\}, \qquad \alpha_0^\vee = e_1, \qquad \alpha_0 = 2e_1+e_2,$$

$$s_1 = s_{e_1}, \qquad s_2 = s_{e_2}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{e_1} s_{\alpha_0} t_{-e_1} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x \rangle \alpha_0,$$
$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^6 = e \rangle \cong D_6,$$
$$S^{\mathrm{aff}} = \{s_0, s_1, s_2\}, \qquad \Omega = \{e\},$$
$$W^e = W^{\mathrm{aff}} = \langle s_0, W_0 \mid s_0^2 = (s_0 s_2)^2 = (s_0 s_1)^3 = e \rangle.$$

A generic parameter function q for $\mathcal{R}(G_2)$ has two independent parameters $q_1 = q(s_1)$ and $q_2 = q(s_2)$.

The group $W \cong D_6$ has six conjugacy classes: the identity, reflections associated to short roots, reflections associated to long roots, the rotation of order two, rotations of order three and rotations of order six. Representatives are e, s_1, s_2, $\rho_\pi = (s_1 s_2)^3$, $\rho_{2\pi/3} = (s_1 s_2)^2$ and $\rho_{\pi/6} = s_1 s_2$. We determine the connected components of the extended quotient $T_{\mathrm{un}} /\!/ W$:

w	T^w	$Z_{D_6}(w)$	$T_{\mathrm{un}}^w / Z_{D_6}(w)$
e	T	D_6	$(S^1)^2/D_6 \cong$ solid triangle
s_1	$\{(1, t_2) : t_2 \in \mathbb{C}^\times\}$	$\langle s_1, s_{3e_1+2e_2} \rangle$	$S^1/\langle s_{3e_1+2e_2} \rangle \cong [-1, 1]$
s_2	$\{(t_1, 1) : t_1 \in \mathbb{C}^\times\}$	$\langle s_2, s_{2e_1+e_2} \rangle$	$S^1/\langle s_{2e_1+e_2} \rangle \cong [-1, 1]$
ρ_π	$\{(a, b) : a, b \in \{\pm 1\}\}$	D_6	2 points
$\rho_{2\pi/3}$	$\{(1, 1), (\zeta_3, 1), (\zeta_3^2, 1)\}$	$C_6 = \langle \rho_{\pi/3} \rangle$	2 points
$\rho_{\pi/3}$	$\{(1, 1)\}$	$C_6 = \langle \rho_{\pi/3} \rangle$	1 point

Here ζ_3 is a primitive third root of unity. We see that every connected component of $T_{\mathrm{un}} /\!/ W$ is contractible, and that its cohomology is zero in positive degrees and $\mathbb{Z}^8$ in degree zero.

The root datum $\mathcal{R}(G_2)$ is simply connected, so W_t is a Weyl group for every $t \in T$. This can also be checked directly: For $t \in T$ with $W_t = \{e\}$ or W_t generated by one reflection it is true. For all $t \in T$ not of that form, W_t contains a nontrivial rotation. All rotations (or their inverses) appear in the above table, along with their fixpoints. We list the isotropy groups of those points:

$$W_{(1,1)} = D_6,$$
$$W_{(\zeta_3,1)} = W_{(\zeta_3^2,1)} = \langle s_2, \rho_{2\pi/3} \rangle \cong S_3,$$
$$W_{(-1,-1)} \cong W_{(-1,1)} \cong W_{(1,-1)} = \langle s_1, s_{3e_1+2e_2} \rangle \cong S_2 \times S_2.$$

We have checked all the conditions of Theorem 2.45. By Corollary 2.49, for every positive parameter function q,

$$\begin{aligned} K_0(C_r^*(\mathcal{R}(G_2), q)) &\cong \mathbb{Z}^8, \\ K_1(C_r^*(\mathcal{R}(G_2), q)) &= 0. \end{aligned} \tag{3.49}$$

Appendix: Some almost Weyl groups

We study some finite groups which are almost Weyl groups. Such groups can arise as the component groups of unipotent elements of classical complex groups, and they play a role in the affine Hecke algebras associated to general Bernstein components for classical p-adic groups [Goldberg 1994; Heiermann 2011]. The results from this appendix are only needed in Section 3F.

Fix $n_1, n_2, \dots, n_d \in \mathbb{Z}_{\geq 1}$ with $n_1 + \dots + n_d = n$ and consider

$$W'_{\vec{n}} := (W(B_{n_1}) \times \dots \times W(B_{n_d})) \cap W(D_n).$$

We use the convention that $W(D_1)$ is the trivial group. The group $W'_{\vec{n}}$ acts on the root system

$$D_{\vec{n}} := D_{n_1} \times \dots \times D_{n_d}.$$

Let $\Delta_{\vec{n}}$ be the standard basis of $D_{\vec{n}}$ and let Γ be the stabilizer of $\Delta_{\vec{n}}$ in $W'_{\vec{n}}$. Since $W(D_{\vec{n}})$ acts simply transitively on the collection of bases of $D_{\vec{n}}$,

$$W'_{\vec{n}} = W(D_{\vec{n}}) \rtimes \Gamma. \tag{A.1}$$

Explicitly, the group $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^{d-1}$ is generated by the elements $\epsilon^{(k)}\epsilon^{(k+1)}$ for $k = 1, \dots, d-1$, where $\epsilon^{(k)} = s_{e_{n_k}}$ is the reflection associated to the short simple root of B_{n_k}.

The Springer correspondence was extended to groups of this kind in [Kato 1983; Aubert et al. 2017c]. Let T be the diagonal torus of the connected complex group

$$G^\circ = \mathrm{SO}_{2n_1}(\mathbb{C}) \times \dots \times \mathrm{SO}_{2n_d}(\mathbb{C}). \tag{A.2}$$

Then $W'_{\vec{n}}$ acts naturally on T and we recover $W(D_{\vec{n}})$ as the Weyl group of (G°, T). The Lie algebra of T can be identified with the defining representation of

$$W(B_{\vec{n}}) := W(B_{n_1}) \times \dots \times W(B_{n_d}). \tag{A.3}$$

Since Γ consists of diagram automorphisms of $D_{\vec{n}}$, we can build the reductive group

$$G = G^\circ \rtimes \Gamma. \tag{A.4}$$

Then $W'_{\vec{n}}$ becomes the "Weyl" group of this disconnected group:

$$W'_{\vec{n}} = W(G, T) := N_G(T)/T.$$

For $u \in G^\circ$ unipotent let $\mathcal{B}^u = \mathcal{B}^u_{G^\circ}$ be the variety of Borel subgroups of G° containing u. The group $Z_G(u)$ acts naturally on $\mathcal{B}^u \times \Gamma$, and that induces an action of $A_G(u) = \pi_0(Z_G(u)/Z(G))$ on $H^i(\mathcal{B}^u; \mathbb{C}) \otimes \mathbb{C}[\Gamma]$. For $\rho' \in \mathrm{Irr}(A_G(u))$ we form the $W'_{\vec n}$-representations

$$H(u,\rho') = H_{A_G(u)}(\rho, H^*(\mathcal{B}^u;\mathbb{C})\otimes\mathbb{C}[\Gamma]),$$
$$\pi(u,\rho') = H_{A_G(u)}(\rho, H^{\mathrm{top}}(\mathcal{B}^u;\mathbb{C})\otimes\mathbb{C}[\Gamma]).$$

We call ρ' geometric if $\pi(u,\rho)\neq 0$. Then [Aubert et al. 2017c, Theorem 4.4] says that $\pi(u,\rho')\in\mathrm{Irr}(W'_{\vec n})$ and that this yields a bijection between $\mathrm{Irr}(W'_{\vec n})$ and the G-conjugacy classes of pairs (u,ρ') with $u\in G^\circ$ unipotent and $\rho'\in\mathrm{Irr}(A_G(u))$ geometric.

The $W'_{\vec n}$-representations $H'(u,\rho')$, with (u,ρ') as above, form another $\mathbb{Z}$-basis of $R_{\mathbb{Z}}(W'_{\vec n})$. Indeed, this can be shown in the same way as for Weyl groups in [Reeder 2001, Lemma 3.3.1]; the input from [Borho and MacPherson 1981] holds for W' by [Aubert et al. 2017c, Lemma 4.5].

For $P\subset\Delta_{\vec n}$ we define the standard parabolic subgroup

$$W'_P := \langle s_\alpha : \alpha\in P\rangle \rtimes \mathrm{Stab}_\Gamma(P).$$

As usual, a parabolic subgroup of $W'_{\vec n}$ is a conjugate of some W'_P. Let P_A be the standard basis of the union of the type A root subsystems of R_P and let P_B be the standard basis of the union of the type B root subsystems of $\mathbb{Q}R_P\cap B_{\vec n}$. (So P_B need not be contained in P.) It is easily seen that

$$W'_P = W_{P_A}\times W_{P_B}\cap W(D_n) = W_{P_A}\times W'_{\vec n_P}, \tag{A.5}$$

where $\vec n_P$ consists of the numbers $|P_B\cap B_{n_i}|$ which are nonzero.

All the above notions for $W'_{\vec n}$ have natural analogues for W'_P, which we indicate by an additional subscript P. In particular, [Kato 1983, Proposition 6.2] entails that, as in (1.5) and (1.6),

$$\mathrm{ind}_{W'_P}^{W'_{\vec n}}(H_{W'_P}(u_P,\rho'_P)) \cong \mathrm{Hom}_{A_{G_P}(u_P)}(\rho_P, H^*(\mathcal{B}^{u_P};\mathbb{C})\otimes\mathbb{C}[\Gamma]).$$

Lemma A.6. *The parabolic subgroups of $W'_{\vec n}$ are precisely the isotropy groups of the points of Lie(T).*

Proof. Considering the standard representation of $W(B_{\vec n})$ on $\mathrm{Lie}(T)$, we see that for any $y\in\mathrm{Lie}(T)$ the isotropy group $(W'_{\vec n})_y$ is $W(B_{\vec n})$-conjugate to $W(B_{\vec n})_Q\cap W(D_{\vec n})$, where $W(B_{\vec n})_Q$ is a standard parabolic subgroup of $W(B_{\vec n})$. From (A.5) we see that the group $W(B_{\vec n})_Q\cap W(D_{\vec n})$ equals W'_P for $R_P = R_Q\cap D_{\vec n}$. Hence every isotropy group $(W'_{\vec n})_y$ is $W(B_{\vec n})$-conjugate to some standard parabolic subgroup of $W'_{\vec n}$. Since the diagram automorphisms $\epsilon^{(k)}$ stabilize the collection of parabolic subgroups of $W'_{\vec n}$ and $W(B_{\vec n})$ is generated by $W(D_{\vec n})$ and the $\epsilon^{(k)}$, we conclude that $(W'_{\vec n})_y$ is $W'_{\vec n}$-conjugate to a parabolic subgroup of $W'_{\vec n}$. □

With Lemma A.6 we can define ellipticity in two equivalent ways. An element of $W'_{\vec n}$ is elliptic if it is not contained in a proper parabolic subgroup, or equivalently, if it fixes a nonzero element of $\mathrm{Lie}(T)$. With these notions we can develop the elliptic representation theory of $W'_{\vec{n}}$, exactly as in [Reeder 2001] and as in Section 1A. In particular, (1.11) remains valid.

Lemma A.7. *The group of elliptic representations $\overline{R_{\mathbb{Z}}}(W'_{\vec{n}})$ is torsion-free.*

Proof. We follow the proof of Theorem 1.12, with the group G° from (A.2). Every Levi subgroup of G° can be described by a d-tuple of partitions $\vec{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(d)})$. The standard Levi subgroup associated to $\vec{\alpha}$ is

$$G^\circ_{\vec{\alpha}} = \prod_{i=1}^{d} \mathrm{SO}_{2n_i}(\mathbb{C})_{\alpha^{(i)}} = \prod_{i=1}^{d} \mathrm{GL}_{\alpha_1^{(i)}}(\mathbb{C}) \times \dots \times \mathrm{GL}_{\alpha_{d_i}^{(i)}}(\mathbb{C}) \times \mathrm{SO}_{2(n_i - |\alpha^{(i)}|)}(\mathbb{C}).$$

(We note that sometimes several $P \subset \Delta$ are associated to one $\vec{\alpha}$, as already for $\mathrm{SO}_{2n}(\mathbb{C})$.) We mimic (A.4) by putting

$$\begin{aligned} G_{\vec{\alpha}} &= G^\circ_{\vec{\alpha}} \rtimes \big\langle \epsilon^{(i)}\epsilon^{(j)} : |\alpha^{(i)}| < n_i \text{ and } |\alpha^{(j)}| < n_j \big\rangle \\ &= \Big(\prod_{i=1}^{d} \mathrm{GL}_{\alpha_1^{(i)}}(\mathbb{C}) \times \dots \times \mathrm{GL}_{\alpha_{d_i}^{(i)}}(\mathbb{C}) \Big) \times S\Big(\prod_{i=1}^{d} O_{2(n_i - |\alpha^{(i)}|)}(\mathbb{C}) \Big). \end{aligned}$$

Then $W(G_{\vec{\alpha}}, T) \cong W'_P$ for $P \subset \Delta$ corresponding to $\vec{\alpha}$.

The Bala–Carter classification says that the unipotent classes in G° can be parametrized by d-tuples of bipartitions $(\vec{\alpha}, \vec{\beta})$ such that $2|\alpha^{(i)}| + |\beta^{(i)}| = 2n_i$, $\beta^{(i)}$ has only odd parts and all parts of $\beta^{(i)}$ are distinct. A typical u in this conjugacy class is distinguished in the standard Levi subgroup $G^\circ_{\vec{\alpha}}$.

Like in (1.13) and (1.14), let $G_{\vec{\alpha}''}$ be a standard Levi subgroup containing u. Then $u = u''u'$ with u' in a product of groups $\mathrm{GL}_{n_k}(\mathbb{C})$ and

$$u' \in S\Big(\prod_{i=1}^{d} O_{2(n_i - |\alpha''^{(i)}|)}(\mathbb{C}) \Big) =: H.$$

The GL-factors and u'' do not contribute to $A_{G_{\vec{\alpha}''}}(u)$.

In the upcoming calculations we omit the case that $\vec{\beta}$ is empty; that case is a bit different but can be handled in the same way.

With [Carter 1972, §13.1], we find that the quotient of $Z_H(u')$ by its unipotent radical is

$$\prod_{i=1}^{d} \prod_{j \text{ even}} \mathrm{Sp}_{2m_j'^{(i)}}(\mathbb{C}) \times \prod_{i=1}^{d} \prod_{j \text{ odd, not in } \beta^{(i)}} O_{2m_j'^{(i)}}(\mathbb{C}) \times S\Big(\prod_{i=1}^{d} \prod_{j \text{ odd, in } \beta^{(i)}} O_{2m_j'^{(i)}+1}(\mathbb{C}) \Big).$$

The component groups become

$$A_{G_{\vec{\alpha}''}}(u) \cong A_H(u') \cong \Big(\prod_{i=1}^{d} \prod_{j \text{ odd, in } \alpha'^{(i)}, \text{ not in } \beta^{(i)}} \mathbb{Z}/2\mathbb{Z} \Big) \times S\Big(\prod_{i=1}^{d} \prod_{j \text{ odd, in } \beta^{(i)}} \mathbb{Z}/2\mathbb{Z} \Big).$$

In the same way as after (1.15), we see that $\overline{R_{\mathbb{Z}}}(A_G(u)) = 0$ unless each $\alpha^{(i)}$ has only distinct odd terms, none of them appearing in $\beta^{(i)}$. For such $(\vec{\alpha}, \vec{\beta})$, the maximal reductive quotient of $Z_G(u)$ simplifies to

$$\Big(\prod_{i=1}^{d} \prod_{j \text{ odd, in } \alpha^{(i)}} O_2(\mathbb{C})\Big) \times S\Big(\prod_{i=1}^{d} \prod_{j \text{ odd, in } \beta^{(i)}} O_1(\mathbb{C})\Big) \tag{A.8}$$

and the component group becomes

$$A_G(u) = \prod_{i=1}^{d} \prod_{j \text{ odd, in } \alpha^{(i)}} \mathbb{Z}/2\mathbb{Z} \times A \quad \text{with} \quad A = S\Big(\prod_{i=1}^{d} \prod_{j \text{ odd, in } \beta^{(i)}} \mathbb{Z}/2\mathbb{Z}\Big).$$

Just as in (1.16), we can calculate that $\overline{R_{\mathbb{Z}}}(A_G(u)) \cong R_{\mathbb{Z}}(A)$. □

With Lemmas A.6 and A.7 at hand the proof of Proposition 1.17 also becomes valid for $W'_{\vec{n}}$. Let us formulate this somewhat more generally. Let W' be a finite group which is a direct product of a Weyl group and a number of groups of the form $W'_{\vec{n}}$. Let G' be the corresponding direct product of the groups called G in (1.3) and (A.4). We denote the basis of the root system R' underlying W' by Δ', and the standard parabolic subgroup associated to $P \subset \Delta$ by W'_P.

Lemma A.9. *For every $w \in \mathcal{C}_P(W')$, there exists a pair $(u_{P,w}, \rho'_{P,w})$ such that*

- *$u_{P,w}$ is quasidistinguished unipotent in ${G'}_P$,*
- *$\rho'_{P,w} \in \mathrm{Irr}(A_{{G'}_P}(u_{P,w}))$ is geometric,*
- *the set*

$$\big\{ \mathrm{ind}_{W'_P}^{W'_{\vec{n}}} (H_P(u_{P,w}, \rho'_{P,w})) : P \in \mathcal{P}(\Delta_{\vec{n}})/W'_{\vec{n}},\ w \in \mathcal{C}_{P,\mathrm{ell}}(W'_{\vec{n}}) \big\}$$

forms a $\mathbb{Z}$-basis of $R_{\mathbb{Z}}(W')$.

Proof. Let $(W'_i)_i$ be the indecomposable factors of W', with root systems R'_i. For every $P \subset \Delta'$,

$$W'_P = \prod_i W'_{P \cap R'_i} \quad \text{and} \quad R_{\mathbb{Z}}(W'_P) = \bigotimes_i R_{\mathbb{Z}}(W'_{P \cap R'_i}).$$

Thus we reduce to the case of a single W'_i. If W'_i is an irreducible Weyl group, then Proposition 1.17 applies immediately, so we may assume that $W'_i = W'_{\vec{n}}$.

Let $u \in G$ be unipotent and assume that $\overline{R_{\mathbb{Z}}}(A_G(u)) \neq 0$. From the proof of Lemma A.7, we see that a maximal reductive subgroup of $Z_G(u)$ is of the form (A.8). For each (i, j) with j in $\alpha^{(i)}$, we pick an element $t_{i,j} \in \mathrm{SO}_2(\mathbb{C}) \setminus \big\{\pm\big(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\big)\big\}$, all different. This gives a semisimple element

$$t := \prod_{i=1}^{d} \Big(\prod_{j \text{ in } \alpha^{(i)}} t_{i,j} \times \prod_{j \text{ in } \beta^{(i)}} 1 \Big) \in Z_G(u)^{\circ}.$$

Furthermore, t does not lie in any proper Levi subgroup of G° containing u, so tu does lie in any proper Levi subgroup of G°. Thus, u is quasidistinguished in G.

Knowing this and Lemma A.7, the proof of Proposition 1.17 goes through. □

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