# ANNALS OF K-THEORY

vol. 3 no. 3 2018



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A JOURNAL OF THE K-THEORY FOUNDATION

# ANNALS OF K-THEORY

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Annals of K-Theory is a journal of the K-Theory Foundation (ktheoryfoundation.org). The K-Theory Foundation acknowledges the precious support of Foundation Compositio Mathematica, whose help has been instrumental in the launch of the Annals of K-Theory.

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Annals of K-Theory (ISSN 2379-1681 electronic, 2379-1683 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

AKT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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**nonprofit scientific publishing**

http://msp.org/

ANNALS OF K-THEORY
Vol. 3, No. 3, 2018
dx.doi.org/10.2140/akt.2018.3.369

# Triple linkage

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We study the condition on a field that any triple of (bilinear) Pfister forms of a given dimension are linked. This is a strengthening of the condition of linkage investigated by Elman and Lam, which asks the same for pairs of Pfister forms. In characteristic different from two this condition for triples of 2-fold Pfister forms is related to the Hasse number.

## 1. Introduction

Milnor's seminal article [1969/1970] on $K$-theory of fields had an enormous impact on quadratic form theory. In a series of articles Elman and Lam explored the correspondence between Pfister forms and symbols (canonical generators) in the $K$-theory modulo 2 of a field. The notion of *linkage* for Pfister forms was introduced in [Elman and Lam 1972a]. With the definition from [Elman and Lam 1972a, Section 4] one can consider linkage of a finite number of Pfister forms. However, the study of linkage has mostly been limited to pairs of Pfister forms. Initially, this study was restricted to fields of characteristic different from 2, where quadratic forms are characterised by their associated (symmetric bilinear) polar forms.

When trying to extend notions and statements to cover the case of characteristic 2, one has to choose between quadratic forms or symmetric bilinear forms. In this article we work mainly in the setup of Milnor $K$-theory over a field of arbitrary characteristic. We study linkage of symbols in the Milnor $K$-groups modulo 2, or equivalently, of symmetric bilinear Pfister forms. In particular, we study the condition that a certain Milnor $K$-group modulo 2 has *triple linkage*, i.e., that any three symbols have a common linkage. This condition turns out to have stronger consequences than usual linkage, in particular on the vanishing of higher $K$-groups. In the last section we focus on fields of characteristic different from 2 and relate the condition of triple linkage to quadratic forms and the Hasse number $\tilde{u}$ (the $u$-invariant if the field is nonreal).

For a recent study of triple linkage of quadratic Pfister forms covering fields of characteristic 2, we refer the reader to [Chapman et al. 2018].

*MSC2010:* 11E04, 11E81, 19D45.
*Keywords:* field, Milnor $K$-theory, symbol, linkage, bilinear Pfister form, quadratic form, Hasse number, $u$-invariant.

## 2. Symbols and Pfister forms

We refer to [Elman et al. 2008] for standard results used from quadratic form theory. Let $E$ be a field. By a *form over $E$* we mean a pair $(V, b)$ of a finite-dimensional $E$-vector space $V$ and a nondegenerate symmetric bilinear form $b$ on $V$. We use an equality sign to indicate that two forms are isometric.

Let $n$ always denote a nonnegative integer. Given $a_1, \ldots, a_n \in E^\times$ we denote the bilinear $n$-fold Pfister form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ over $E$ by $\langle\langle a_1, \ldots, a_n\rangle\rangle$. In the sequel we refer to bilinear Pfister forms simply as *Pfister forms*. Given a Pfister form $\pi$, the orthogonal complement of the subform $\langle 1\rangle$ in $\pi$ is called the *pure part of $\pi$*.

**Theorem 2.1** (Elman–Lam). *Let $r \in \mathbb{N}$. Let $\rho$ be an anisotropic $r$-fold Pfister form over $E$ and let $\rho'$ denote its pure part. Let $\pi$ be a Pfister form over $E$ such that $\pi \otimes \rho$ is anisotropic and let $c_1 \in E^\times$ be such that $-c_1$ is represented by $\pi \otimes \rho'$. Then there exist $c_2, \ldots, c_r \in E^\times$ such that $\pi \otimes \rho = \pi \otimes \langle\langle c_1, \ldots, c_r\rangle\rangle$.*

*Proof.* See [Elman et al. 2008, Proposition 6.15] or [Elman and Lam 1972a, Theorem 2.6]. □

We denote by $\mathsf{k}_n E$ the $n$-th Milnor $K$-group of $E$ modulo 2; this is the abelian group generated by symbols $\{a_1, \ldots, a_n\}$, with $a_1, \ldots, a_n \in E^\times$, subject to the defining relations that the map $(E^\times)^n \to \mathsf{k}_n E$ given by $(a_1, \ldots, a_n) \mapsto \{a_1, \ldots, a_n\}$ is multilinear and further that $\{a_1, \ldots, a_n\} = 0$ whenever $a_i \in E^{\times 2}$ for some $i \leqslant n$ or $a_i + a_{i+1} = 1$ for some $i < n$. The direct sum $\bigoplus_{n\in\mathbb{N}} \mathsf{k}_n E$ is a graded ring with the multiplication induced by concatenation of symbols.

We recall some results from [Elman et al. 2008] on the relation of symbols and Pfister forms, which for fields of characteristic different from 2 go back to [Elman and Lam 1972a]. We begin with the one-to-one correspondence between symbols and Pfister forms.

**Theorem 2.2** (Elman–Lam). *For $a_1 \ldots, a_n, b_1, \ldots, b_n \in E^\times$, we have*

$$\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_n\} \quad \textit{if and only if} \quad \langle\langle a_1, \ldots, a_n\rangle\rangle = \langle\langle b_1, \ldots, b_n\rangle\rangle.$$

*Proof.* See [Elman et al. 2008, Theorem 6.20]. □

We denote by $\Sigma E^2$ the subgroup of $E^\times$ consisting of the nonzero sums of squares in $E$. Recall that the field $E$ is *real* if $-1 \notin \Sigma E^2$, *nonreal* otherwise. For $m \in \mathbb{N}$ we denote by $\mathsf{D}_E(m)$ the subset of $\Sigma E^2$ consisting of the elements that are sums of $m$ squares in $E$.

**Corollary 2.3.** *For $a \in E^\times$ and the symbol $\tau = \{-1, \ldots, -1\}$ in $\mathsf{k}_n E$, we have:*

(a) *$a \in \mathsf{D}_E(2^n)$ if and only if $\tau \cdot \{a\} = 0$ in $\mathsf{k}_{n+1} E$.*

(b) *$a \in \mathsf{D}_E(2^n - 1)$ if and only if $\tau = \{-a, a_2, \ldots, a_n\}$ for certain $a_2, \ldots, a_n \in E^\times$.*

*Proof.* This follows from Theorem 2.1 and Theorem 2.2. □

**Lemma 2.4.** *Let $\tau$ and $\tau'$ be symbols in $\mathsf{k}_n E$ and let $a, a' \in E^\times$ be such that $\tau \cdot \{a\} = \tau' \cdot \{a'\}$. Then $\tau \cdot \{a\} = \tau \cdot \{c\} = \tau' \cdot \{c\} = \tau' \cdot \{a'\}$ for some $c \in E^\times$.*

*Proof.* This follows from [Elman et al. 2008, Corollary 6.16 and Theorem 6.20]. □

## 3. Linkage

Assume from now on that $n \geqslant 2$. Given two symbols $\sigma_1, \sigma_2 \in \mathsf{k}_n E$, the sum $\sigma_1 + \sigma_2 \in \mathsf{k}_n E$ is equal to a symbol if and only if there exists a symbol $\sigma' \in \mathsf{k}_{n-1} E$ and $b_1, b_2 \in E^\times$ such that $\sigma_i = \sigma' \cdot \{b_i\}$ for $i = 1, 2$ (see [Elman and Lam 1972a, Lemma 5.4]); in this case, we say that $\sigma_1$ and $\sigma_2$ are *linked*.

We say that $\mathsf{k}_n E$ *is linked* if any two symbols in $\mathsf{k}_n E$ are linked (which in the terminology of [Elman and Lam 1973] corresponds to saying that $\mathsf{I}^n E$ is linked.) Obviously, if $\mathsf{k}_n E$ is linked, then so is $\mathsf{k}_m E$ for any integer $m \geqslant n$.

The following statement was obtained in [Elman and Lam 1973, Corollary 2.8 and Corollary 2.9]. For convenience of the reader we include a compact proof, whose first lines follow [Elman and Lam 1972b, Section 3, Example 3]. The statement should be compared with Theorem 5.1.

**Theorem 3.1** (Elman–Lam). *Assume that $\mathsf{k}_n E$ is linked. Then $\Sigma E^2 = \mathsf{D}_E(2^{n+1})$ and $\mathsf{k}_{n+2} E = \{-1, \dots, -1\} \cdot \mathsf{k}_1 E$. In particular, if $E$ is nonreal then $\mathsf{k}_{n+2} E = 0$.*

*Proof.* Consider an arbitrary symbol $\tau \in \mathsf{k}_{n-2} E$ and $a_1, a_2, b_1, b_2 \in E^\times$. Since $\mathsf{k}_n E$ is linked and by Lemma 2.4, there exist $c_1, c_2 \in E^\times$ and a symbol $\sigma \in \mathsf{k}_{n-1} E$ such that $\tau \cdot \{a_i, b_i\} = \tau \cdot \{a_i, c_i\} = \sigma \cdot \{c_i\}$ for $i = 1, 2$. It follows that

$$\tau \cdot \{a_1, c_1, c_2\} = \sigma \cdot \{c_1, c_2\} = \tau \cdot \{a_2, c_1, c_2\},$$

whereby

$$\tau \cdot \{a_1, b_1, a_2, b_2\} = \tau \cdot \{a_1, c_1, a_2, c_2\} = \tau \cdot \{a_2, c_1, a_2, c_2\} = \{-1, c_1\} \cdot \tau \cdot \{a_2, b_2\}.$$

This argument shows that for any $a_1, \dots, a_{n+2} \in E^\times$, there exists $c \in E^\times$ such that

$$\{a_1, \dots, a_{n+2}\} = \{a_2, \dots, a_{n+1}, -1, c\}.$$

Applying this rule $n+1$ times, we conclude that every symbol in $\mathsf{k}_{n+2} E$ is of the form $\{-1, \dots, -1, c\}$ with $c \in E^\times$. Hence $\mathsf{k}_{n+2} E = \{-1, \dots, -1\} \cdot \mathsf{k}_1 E$.

Moreover, if $a_1, \dots, a_{n+2} \in E^\times$ are such that $a_2 \in \mathsf{D}_E(2)$, then $\{-1, a_2\} = 0$ and we obtain from the above rule that $\{a_1, \dots, a_{n+2}\} = 0$. Thus we have $\{a\} \cdot \mathsf{k}_{n+1} E = 0$ for any $a \in \mathsf{D}_E(2)$.

Consider an element $c \in \mathsf{D}_E(2^{n+1} + 1)$. We write $c = a + b$ with $a \in \mathsf{D}_E(2)$ and $b \in \mathsf{D}_E(2^{n+1} - 1)$. In $\mathsf{k}_{n+1} E$ we obtain that $\{-1, \dots, -1\} = \{-b\} \cdot \tau$ for a symbol $\tau$ in $\mathsf{k}_n E$, by Corollary 2.3. Since $c-b=a$ we have $\{-b,c\}=\{a,bc\}$. As $\{a\}\cdot\mathsf{k}_{n+1}E=0$, we obtain in $\mathsf{k}_{n+2}E$ that

$$\{-1,\ldots,-1,c\}=\{-b,c\}\cdot\tau=\{a,bc\}\cdot\tau=0,$$

which shows that $c\in\mathsf{D}_E(2^{n+1})$. This argument shows that $\Sigma E^2=\mathsf{D}_E(2^{n+1})$.

Assume finally that $E$ is nonreal. If $-1\in E^{\times 2}$ then $\{-1\}=\{1\}=0$ in $\mathsf{k}_1 E$. If $\mathsf{char}(E)\neq 2$ then $E^\times=\Sigma E^2=\mathsf{D}_E(2^{n+1})$. Hence, in any case we obtain that $\mathsf{k}_{n+2}E=\{-1,\ldots,-1\}\cdot\mathsf{k}_1 E=0$. □

If $E$ is nonreal and $\mathsf{k}_n E$ is linked with $n\geqslant 2$, then $\mathsf{k}_{n+2}E$ vanishes by Theorem 3.1, but we may have that $\mathsf{k}_{n+1}E\neq 0$, as the following well-known example shows.

**Example 3.2.** For $E=\mathbb{C}((t_1))\cdots((t_{n+1}))$, $\mathsf{k}_n E$ is linked and $\mathsf{k}_{n+1}E\simeq\mathbb{Z}/2\mathbb{Z}$.

## 4. The linkage pairing

We are going to investigate an operation on linked symbols. Let $n\geqslant 2$. To any pair of linked symbols in $\mathsf{k}_n E$ we associate a symbol in $\mathsf{k}_{n+1}E$.

**Proposition 4.1.** *Let $\sigma_1,\sigma_2\in\mathsf{k}_n E$ be two linked symbols. There is a unique symbol $\rho\in\mathsf{k}_{n+1}E$ such that, for any symbol $\tau\in\mathsf{k}_{n-1}E$ and any $a_1,a_2\in E^\times$ with $\sigma_i=\tau\cdot\{a_i\}$ for $i=1,2$, we have $\rho=\tau\cdot\{a_1,a_2\}$.*

*Proof.* By the hypothesis there exist a symbol $\tau\in\mathsf{k}_{n-1}E$ and elements $a_1,a_2\in E^\times$ with $\sigma_i=\tau\cdot\{a_i\}$ for $i=1,2$. Suppose we have another symbol $\tau'\in\mathsf{k}_{n-1}E$ and $a_1',a_2'\in E^\times$ with $\sigma_i=\tau'\cdot\{a_i'\}$ for $i=1,2$. By Lemma 2.4 there exist $c_1,c_2\in E^\times$ such that $\tau\cdot\{c_i\}=\sigma_i=\tau'\cdot\{c_i\}$ holds for $i=1,2$. We obtain that

$$\tau\cdot\{a_1,a_2\}=\tau\cdot\{c_1,c_2\}=\tau'\cdot\{c_1,c_2\}=\tau'\cdot\{a_1',a_2'\}.\qquad\square$$

**Corollary 4.2.** *Suppose $a,b,c\in E^\times$ and let $\tau$ be a symbol in $\mathsf{k}_{n-1}E$ such that $\tau\cdot\{-1,a\}=0$. Assume there exist a symbol $\rho$ in $\mathsf{k}_n E$ and $x,y,z\in E^\times$ such that $\tau\cdot\{a,b\}=\rho\cdot\{x\}$, $\tau\cdot\{a,c\}=\rho\cdot\{y\}$ and $\tau\cdot\{b,c\}=\rho\cdot\{z\}$. Then $\tau\cdot\{a,b,c\}=0$.*

*Proof.* By Proposition 4.1 we obtain that $\rho\cdot\{x,z\}=\tau\cdot\{a,b,c\}=\rho\cdot\{y,z\}$, whereby $\rho\cdot\{xy,z\}=0$. We have $\tau\cdot\{b,-bc\}=\tau\cdot\{b,c\}=\rho\cdot\{z\}$. Since $\tau\cdot\{-1,a\}=0$ we further have $\tau\cdot\{a,-bc\}=\tau\cdot\{a,bc\}=\rho\cdot\{xy\}$. We conclude with Proposition 4.1 that $\tau\cdot\{a,b,c\}=\tau\cdot\{a,b,-bc\}=\rho\cdot\{xy,z\}=0$. □

**Corollary 4.3.** *Assume that $\mathsf{k}_n E$ is linked. We obtain a surjective pairing*

$$\langle\cdot,\cdot\rangle:\mathsf{k}_n E\times\mathsf{k}_n E\to\mathsf{k}_{n+1}E$$

*by letting $\langle\tau\cdot\{a_1\},\tau\cdot\{a_2\}\rangle=\tau\cdot\{a_1,a_2\}$ for any symbol $\tau\in\mathsf{k}_{n-1}E$ and any $a_1,a_2\in E^\times$.*

*Proof.* Let $\sigma_1, \sigma_2 \in \mathsf{k}_n E$ be given. As $\mathsf{k}_n E$ is linked, there exist $a_1, a_2 \in E^\times$ and a symbol $\tau \in \mathsf{k}_{n-1} E$ such that $\sigma_i = \tau \cdot \{a_i\}$ for $i = 1, 2$. By Proposition 4.1 the symbol $\rho = \tau \cdot \{a_1, a_2\} \in \mathsf{k}_{n+1} E$ only depends on $\sigma_1$ and $\sigma_2$ but not on the choice of $\tau$ and $a_1, a_2 \in E^\times$. Hence, the pairing is well-defined. As $\mathsf{k}_n E$ is linked, so is $\mathsf{k}_{n+1} E$, and it follows that the pairing is surjective. $\square$

If $\mathsf{k}_n E$ is linked then we call the pairing in Corollary 4.3 the *linkage pairing on* $\mathsf{k}_n E$.

**Theorem 4.4.** *Assume that $\mathsf{k}_n E$ is linked. Then the following are equivalent*:

(i) *The linkage pairing on $\mathsf{k}_n E$ is bilinear.*

(ii) $\Sigma E^2 = \mathsf{D}_E(2^n)$ *and* $\mathsf{k}_{n+1} E = \{-1, \dots, -1\} \cdot \mathsf{k}_1 E$.

(iii) *Either $\mathsf{k}_{n+1} E = 0$, or $E$ is real and the rule $c \mapsto \{-1, \dots, -1, c\}$ determines an isomorphism $E^\times / \Sigma E^2 \to \mathsf{k}_{n+1} E$.*

*Proof.* As a consequence of the definition of the linkage pairing

$$\langle \cdot , \cdot \rangle : \mathsf{k}_n E \times \mathsf{k}_n E \to \mathsf{k}_{n+1} E,$$

we have for any $\rho, \rho' \in \mathsf{k}_n E$ that $\langle \rho, \rho' \rangle = \rho \cdot \{d\}$ for some $d \in E^\times$.

(i)⇒(ii): Consider an arbitrary symbol $\tau \in \mathsf{k}_{n-2} E$ and $a, b, c \in E^\times$. Set $\sigma_1 = \{a, b\}$, $\sigma_2 = \{a, c\}$, $\sigma_3 = \{b, c\}$ and $\sigma_4 = \{-ab, c\}$. We obtain that $\langle \tau \cdot \sigma_i, \tau \cdot \sigma_1 \rangle = \tau \cdot \{a, b, c\}$ for $i = 2, 3, 4$. Assuming that the pairing is bilinear, we get that

$$\langle \tau \cdot (\sigma_2 + \sigma_3 + \sigma_4), \tau \cdot \sigma_1 \rangle = \tau \cdot \{a, b, c\}.$$

Since $\sigma_2 + \sigma_3 + \sigma_4 = \{-1, c\}$, we conclude that

$$\tau \cdot \{a, b, c\} = \langle \tau \cdot \{-1, c\}, \tau \cdot \{a, b\} \rangle = \tau \cdot \{-1, c, d\}$$

for some $d \in E^\times$. This argument shows that, for any $c_1, \dots, c_n, c_{n+1} \in E^\times$, there exists $d \in E^\times$ such that $\{c_1, \dots, c_n, c_{n+1}\} = \{-1, c_1, \dots, c_{n-1}, d\}$ in $\mathsf{k}_{n+1} E$. Using this rule $n$ times we obtain for any $c_1, \dots, c_n, c_{n+1} \in E^\times$ that

$$\{c_1, \dots, c_{n+1}\} = \{-1, \dots, -1, c_1, d\} = \{-1, \dots, -1, d'\}$$

for some $d, d' \in E^\times$. This shows that $\mathsf{k}_{n+1} E = \{-1, \dots, -1\} \cdot \mathsf{k}_1 E$ and that $\{a\} \cdot \mathsf{k}_n E = 0$ for any $a \in \mathsf{D}_E(2^{n-1})$.

Consider now an element $c \in \mathsf{D}_E(2^n + 1)$. We choose $a, b \in \mathsf{D}_E(2^{n-1})$ such that $c - a - b$ is a square in $E$. Then we have $\{-a, -b, c\} = 0$ in $\mathsf{k}_3 E$, whereby $\{a, -b, c\} = \{-1, -b, c\}$. For the symbol $\epsilon = \{-1, \dots, -1\}$ in $\mathsf{k}_{n-2} E$ we obtain that $\epsilon \cdot \{-1, b\} = 0$ and conclude that

$$\epsilon \cdot \{-1, -1, c\} = \epsilon \cdot \{-1, -b, c\} = \epsilon \cdot \{a, -b, c\} = 0$$

for $a \in \mathsf{D}_E(2^{n-1})$. Hence $\{-1, \dots, -1, c\} = 0$ in $\mathsf{k}_{n+1}E$, whereby $c \in \mathsf{D}_E(2^n)$. This shows that $\Sigma E^2 = \mathsf{D}_E(2^n)$.

(ii)⇒(iii): This implication is obvious.

(iii)⇒(i): Let $\varepsilon = \{-1, \dots, -1\}$ in $\mathsf{k}_{n-1}E$. For any symbol $\tau \in \mathsf{k}_{n-1}E$ and any $a, b \in E^\times$ we have $\tau \cdot \{a\} \cdot \tau \cdot \{b\} = \varepsilon \cdot \tau \cdot \{a, b\} = \varepsilon \cdot \langle \tau \cdot \{a\}, \tau \cdot \{b\}\rangle$. Since $\mathsf{k}_n E$ is linked, this means that

$$\rho \cdot \rho' = \varepsilon \cdot \langle \rho, \rho' \rangle \text{ in } \mathsf{k}_{2n}E \text{ for any } \rho, \rho' \in \mathsf{k}_n E.$$

Hence, the pairing $\mathsf{k}_n E \times \mathsf{k}_n E \to \mathsf{k}_{2n}E$, $(\rho, \rho') \mapsto \varepsilon \cdot \langle \rho, \rho' \rangle$ is bilinear. On the other hand, (iii) implies that $\mathsf{k}_{n+1}E \to \mathsf{k}_{2n}E$, $\xi \mapsto \varepsilon \cdot \xi$ is an isomorphism. Therefore the pairing $\langle \cdot, \cdot \rangle$ is bilinear. □

## 5. Triple linkage

Let $n \geqslant 2$. We say $\mathsf{k}_n E$ has *triple linkage* if for any three symbols $\sigma_1, \sigma_2, \sigma_3 \in \mathsf{k}_n E$ there exist a symbol $\tau \in \mathsf{k}_{n-1}E$ and $a_1, a_2, a_3 \in E^\times$ such that $\sigma_i = \tau \cdot \{a_i\}$ for $i = 1, 2, 3$. Note that this implies that $\mathsf{k}_n E$ is linked.

**Theorem 5.1.** *Assume that $\mathsf{k}_n E$ has triple linkage. Then $\Sigma E^2 = \mathsf{D}_E(2^n)$ and $\mathsf{k}_{n+1}E = \{-1, \dots, -1\} \cdot \mathsf{k}_1 E$. In particular, if $E$ is nonreal then $\mathsf{k}_{n+1}E = 0$.*

*Proof.* Let $\langle \cdot, \cdot \rangle : \mathsf{k}_n E \times \mathsf{k}_n E \to \mathsf{k}_{n+1}E$ be the linkage pairing. Consider three symbols $\sigma_1, \sigma_2, \sigma_3 \in \mathsf{k}_n E$. By the hypothesis there exist a symbol $\tau \in \mathsf{k}_{n-1}E$ and $a_1, a_2, a_3 \in E^\times$ such that $\sigma_i = \tau \cdot \{a_i\}$ for $i = 1, 2, 3$. As $\sigma_1 + \sigma_2 = \tau \cdot \{a_1 a_2\}$ we obtain that

$$\langle \sigma_1 + \sigma_2, \sigma_3 \rangle = \tau \cdot \{a_1 a_2, a_3\} = \tau \cdot \{a_1, a_3\} + \tau \cdot \{a_2, a_3\} = \langle \sigma_1, \sigma_3 \rangle + \langle \sigma_2, \sigma_3 \rangle.$$

Hence the linkage pairing is bilinear and Theorem 4.4 yields the statement. □

**Question 5.2.** If $\mathsf{k}_n E$ has triple linkage, do then any finite number of symbols in $\mathsf{k}_n E$ have a common linkage (by a symbol in $\mathsf{k}_{n-1}E$)?

Triple linkage holds for $\mathsf{k}_n E$ if $E$ is a $\mathcal{C}_n$-field, in the terms of Tsen–Lang theory (see [Pfister 1995, Chapter 5]). This is a direct consequence of the next statement. For $n = 1$ and $|\mathcal{S}| = 3$ the statement corresponds to [Sivatski 2014, Proposition 9].

**Proposition 5.3.** *Assume that there exists a finite system $\mathcal{S}$ of nonzero symbols in $\mathsf{k}_n E$ that do not have a common linkage. Then there exists an anisotropic system of $|\mathcal{S}| - 1$ quadratic forms in $|\mathcal{S}| \cdot 3 \cdot 2^{n-2}$ variables over $E$.*

*Proof.* Let $m \in \mathbb{N}$ be as large as possible such that there exist $a_1, \dots, a_m \in E^\times$ for which the symbol $\{a_1, \dots, a_m\}$ factors every element of $\mathcal{S}$. By the hypothesis, $m \leqslant n - 2$. We set $\pi = \langle\langle a_1, \dots, a_m \rangle\rangle$. Using Theorem 2.1 and the one-to-one correspondence between Pfister forms and symbols, we choose for $\sigma \in \mathcal{S}$ an $(n-m)$-fold Pfister form $\rho_\sigma$ over $E$ such that $\sigma$ corresponds to the $n$-fold Pfister form $\pi \otimes \rho_\sigma$ over $E$, and we denote by $\rho'_\sigma$ the pure part of $\rho_\sigma$. Note that $\dim(\pi \otimes \rho'_\sigma) = 2^n - 2^m$ for any $\sigma \in \mathcal{S}$. By Theorem 2.1 and by the maximality of $m$, there exists no element $c \in E^\times$ such that $-c$ is represented by all the forms $\pi \otimes \rho'_\sigma$ with $\sigma \in \mathcal{S}$. We fix $\sigma_0 \in \mathcal{S}$ and set $\mathcal{S}' = \mathcal{S} \smallsetminus \{\sigma_0\}$. Considering each of the forms $\pi \otimes \rho'_\sigma$ for $\sigma \in \mathcal{S}$ with its own variables, we obtain an anisotropic system of quadratic forms $(\pi \otimes \rho'_{\sigma_0} - \pi \otimes \rho'_\sigma)_{\sigma \in \mathcal{S}'}$ in $|\mathcal{S}| \cdot (2^n - 2^m)$ variables over $E$. If $m < n-2$ we may substitute zero for some of these variables. So in any case we obtain an anisotropic system of $|\mathcal{S}| - 1$ quadratic forms over $E$ in exactly $|\mathcal{S}| \cdot 3 \cdot 2^{n-2}$ variables. □

Let $\tilde{u}(E)$ denote the *Hasse number of $E$*, which is defined as the supremum in $\mathbb{N} \cup \{\infty\}$ on the dimension of anisotropic totally indefinite quadratic forms over $E$; see [Pfister 1995, Chapter 8, Section 3]. The study of this invariant was initiated in [Elman et al. 1973], and the notation was introduced in [Elman 1977]. The definition of the Hasse number captures one of several possibilities to study bounds on the dimension of anisotropic quadratic forms in a meaningful way without restriction to nonreal fields. The results below have their main interest in the case where $E$ is nonreal, and in this case $\tilde{u}(E)$ is the usual $u$-invariant; see [Pfister 1995, Chapter 8; Elman et al. 2008, Chapter VI].

**Corollary 5.4.** *If $\tilde{u}(E(t)) \leqslant 2^{n+1}$ then $\mathsf{k}_n E$ has triple linkage.*

*Proof.* Suppose first that $E$ is real and $\tilde{u}(E(t)) < \infty$. It follows by [Elman et al. 1973, Theorem I] that $E$ is hereditarily euclidean. Hence $\mathsf{k}_n E \simeq \mathbb{Z}/2\mathbb{Z}$, whereby $\mathsf{k}_n E$ has triple linkage.

Assume now that $E$ is nonreal and $\tilde{u}(E(t)) \leqslant 2^{n+1} < 9 \cdot 2^{n-2}$. By the Amer–Brumer theorem [Pfister 1995, Chapter 9, Proposition 1.10], it follows that every pair of quadratic forms in $9 \cdot 2^{n-2}$ variables over $E$ is isotropic. Hence $\mathsf{k}_n E$ has triple linkage, by Proposition 5.3. □

The next example shows that the converse to the statement in Corollary 5.4 does not hold.

**Example 5.5.** Let $E_0$ be a quadratically closed field of characteristic not 2 having a finite field extension of even degree. (One can for example take $E_0$ as the quadratic closure of $\mathbb{Q}$: any polynomial over $\mathbb{Q}$ having as Galois group a dihedral group of order $2m$ for an odd positive integer $m$ will have as splitting field over $E_0$ an extension of order $2m$. See also the discussion of finite extensions of quadratically closed fields in [Lam 2005, Chapter VII, §7].) It follows from this choice of $E_0$ that $\tilde{u}(E_0) = 1$ while $\tilde{u}(F_0) \geqslant 2$ for some finite separable extension $F_0/E_0$. Thus $\mathsf{k}_1 F_0 \neq 0$. Consider the fields of iterated power series in $n$ variables

$$E = E_0((u_1)) \cdots ((u_n)) \quad \text{and} \quad F = F_0((u_1)) \cdots ((u_n)).$$

We obtain $\tilde{u}(E)=2^n$ and $\mathsf{k}_{n+1}F\neq 0$. Since $F/E$ is a finite separable extension, $F$ is the residue field of a discrete valuation on $E(t)$. We conclude that $\mathsf{k}_{n+2}E(t)\neq 0$. After translation to Pfister forms via [Elman et al. 2008, Theorem 6.20], the Arason–Pfister Hauptsatz [Elman et al. 2008, Theorem 6.18] yields $\tilde{u}(E(t))\geqslant 2^{n+2}$.

Note that $\{u_1,\dots,u_n\}$ is the only nonzero symbol in $\mathsf{k}_nE$. In particular $\mathsf{k}_nE$ has triple linkage. Finally, $\mathsf{k}_nF$ does not have triple linkage, for $\mathsf{k}_{n+1}F\neq 0$.

The following reformulates and enhances [Elman et al. 1973, Theorem G]. The notation $\mathsf{I}^nE$ refers to the $n$-th power of the fundamental ideal $\mathsf{I}E$ in the Witt ring (of symmetric bilinear forms) of $E$; recall that $\mathsf{I}^nE$ is additively generated by the classes of the $n$-fold Pfister forms over $E$.

**Theorem 5.6** (Elman–Lam–Prestel). *Assume that* $\mathsf{char}(E)\neq 2$. *The following are equivalent*:

(i) $\tilde{u}(E)\leqslant 4$.

(ii) *$E$ is linked and $\mathsf{I}^3E$ is torsion-free.*

(iii) *$E$ is linked and the linkage pairing $\mathsf{k}_2E\times\mathsf{k}_2E\to\mathsf{k}_3E$ is bilinear.*

(iv) $\Sigma E^2=\mathsf{D}_E(4)$ *and* $\mathsf{k}_3E=\{-1,-1\}\cdot\mathsf{k}_1E$.

(v) *$\mathsf{k}_2E$ is linked and either $\mathsf{k}_3E=0$, or $E$ is real and $c\mapsto\{-1,-1,c\}$ defines an isomorphism $E^\times/\Sigma E^2\to\mathsf{k}_3E$.*

*Proof.* Conditions (iii)–(v) are equivalent by Theorem 4.4. The equivalence of (i) and (ii) is shown in [Elman et al. 1973, Theorem G] and in [Elman 1977, Theorem 4.7]. By [Elman and Lam 1973, Corollary 2.9], (ii) implies that $\mathsf{I}^3E=4\cdot\mathsf{I}E$. Therefore, the equivalence of (ii) and (iv) follows using Theorem 2.2. □

**Corollary 5.7.** *Assume that* $\mathsf{char}(E)\neq 2$ *and that $\mathsf{k}_2E$ has triple linkage. Then* $\tilde{u}(E)\leqslant 4$.

*Proof.* This follows from Theorem 5.1 together with Theorem 5.6. □

In the case where $E$ is nonreal, one can show the converse of Corollary 5.7 by using the following statement, which is a direct consequence of a deep result of Peyre [1995, Proposition 6.1] combined with an observation by Sivatski [2014, Corollary 11]. This was pointed out to the author of the present article by Adam Chapman and David Leep.

**Proposition 5.8** (Peyre–Sivatski). *Assume that* $\mathsf{char}(E)\neq 2$. *Let $\mathcal{H}$ be a subgroup of $\mathsf{k}_2E$ with $|\mathcal{H}|\leqslant 8$. Assume that every element of $\mathcal{H}$ is a symbol and that $\mathsf{I}^3E=0$. Then there exists $a\in E^\times$ such that for every $\sigma\in\mathcal{H}$ one has $\sigma=\{a,b_\sigma\}$ for some $b_\sigma\in E^\times$.*

*Proof.* By the hypothesis, every $\sigma\in\mathcal{H}$ corresponds to an $E$-quaternion algebra. Let $\sigma_1,\sigma_2,\sigma_3\in\mathsf{k}_2E$ be three symbols that generate $\mathcal{H}$ and let $Q_1$, $Q_2$, $Q_3$ denote the corresponding quaternion algebras. Since $\mathsf{I}^3 E = 0$ we have $H^3(E, \mathbb{Z}/\mathbb{Q}(2)) = 0$. It follows by [Peyre 1995, Proposition 6.1] that there exists a field extension $F/E$ with $[F : E] = 2m$ for an odd integer $m$ and such that $(Q_i)_F$ is split for $i = 1, 2, 3$. Then, by [Sivatski 2014, Corollary 11], $Q_1$, $Q_2$, $Q_3$ have a common slot $a \in E^\times$. It follows that any $\sigma \in \mathcal{H}$ is of the form $\sigma = \{a, b_\sigma\}$ with $b_\sigma \in E^\times$. □

In [Sivatski 2014], Sivatski seems to be unaware that he provides a proof of Proposition 5.8. In [Sivatski 2014, Corollary 12] he comes to a closely related conclusion, but at the end of his article he asks whether fields of cohomological 2-dimension 2 satisfy the conclusion stated in Proposition 5.8.

**Corollary 5.9.** *Assume that $E$ is nonreal with* $\mathsf{char}(E) \neq 2$. *Then $\tilde{u}(E) \leqslant 4$ if and only if $\mathsf{k}_2 E$ has triple linkage.*

*Proof.* One of the implications is Corollary 5.7, while the converse follows from Proposition 5.8. □

The hypothesis in Proposition 5.8 on $E$ can be weakened. Instead of assuming $\mathsf{I}^3 E = 0$, which requires $E$ to be nonreal, it is sufficient to assume that $\mathsf{I}^3 E$ is torsion-free and that $E$ has the so-called *ED-property* about field orderings introduced in [Prestel and Ware 1979]. This can be proven by using algebras with involution and skew-hermitian forms over quaternion algebras. In this way Proposition 5.8 is recovered with a different proof, which in particular is independent of the algebraic geometry behind Peyre's result [1995, Proposition 6.1].

Since $\tilde{u}(E) < \infty$ implies that $E$ satisfies the *ED*-property, the generalisation carries over to Corollary 5.9 and makes the condition that $E$ is nonreal superfluous: if $\mathsf{char}(E) \neq 2$, then $\tilde{u}(E) \leqslant 4$ if and only if $\mathsf{k}_2 E$ has triple linkage.

The author is planning to give details on this argument in a future article.

## Acknowledgments

This work was supported by the FWO Odysseus Programme (project *Explicit Methods in Quadratic Form Theory*), funded by the Fonds Wetenschappelijk Onderzoek – Vlaanderen. I further wish to express my gratitude to Andrew Dolphin, Parul Gupta, David Leep and Anne Quéguiner-Mathieu for inspiring discussions around this topic. I thank Adam Chapman and David Leep in particular for pointing out Proposition 5.8. Finally, I am grateful to the referee for helping to improve the readability of the article.

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Received 14 Sep 2015. Revised 12 Oct 2017. Accepted 31 Oct 2017.

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ANNALS OF K-THEORY
Vol. 3, No. 3, 2018
dx.doi.org/10.2140/akt.2018.3.379

# $\mathbb{A}^1$-equivalence of zero cycles on surfaces, II

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Using recent developments in the theory of mixed motives, we prove that the log Bloch conjecture holds for an open smooth complex surface if the Bloch conjecture holds for its compactification. This verifies the log Bloch conjecture for all $\mathbb{Q}$-homology planes and for open smooth surfaces which are not of log general type.

## 1. Introduction

Throughout this paper, we work with varieties over the complex numbers.

**1A.** ***Statement of the main theorem.*** Let $U$ be a smooth quasiprojective algebraic variety. Let

$$a : h_0(U)^0 \to \mathrm{Alb}(U)$$

be the Albanese morphism from the zeroth Suslin homology of degree zero to the Albanese variety of $U$, and let $T(U) := \ker(a)$ be the Albanese kernel. When $U$ is projective, $h_0(U)$ reduces to the Chow group of zero cycles $\mathrm{CH}_0(U)$. Indeed, we get the classical Albanese map.

In dimension one, the Albanese morphism is well-understood by the classical work of Abel and Jacobi in the projective case, and by Rosenlicht in the open case.

**Theorem 1.1** (Abel–Jacobi; Rosenlicht [1952; 1954]). *When* $\dim U = 1$, *the Albanese morphism is an isomorphism.*

The higher-dimensional analogue of Theorem 1.1 is much more subtle, although the torsion part of the Albanese morphism is known.

**Theorem 1.2** [Rojtman 1980; Spieß and Szamuely 2003]. *In arbitrary dimension, the Albanese morphism induces an isomorphism on torsion subgroups.*

In this paper, we study the two-dimensional case. In one direction, the log Mumford theorem says that the Albanese morphism fails to be injective as long as $p_g(U) \neq 0$.

Yin was supported by the grant ERC-2012-AdG-320368-MCSK.
*MSC2010:* 14C15, 14C25, 14F42.
*Keywords:* Bloch's conjecture, open algebraic surfaces, $\mathbb{Q}$-homology planes, Suslin homology, mixed motives.

**Theorem 1.3** [Mumford 1968; Zhu 2018]. *Let $U$ be a smooth algebraic surface with $p_g(U) \neq 0$. Then $T(U)$ is infinite-dimensional.*

In the other direction, we expect the following conjecture. When $U$ is projective, it is famously known as the Bloch conjecture [1980].

**Conjecture 1.4** (log Bloch conjecture). Let $U$ be a smooth algebraic surface with $p_g(U) = 0$. Then

$$T(U) = 0.$$

Using recent developments in the theory of mixed motives [Ayoub and Barbieri-Viale 2009; Ayoub 2011; Barbieri-Viale and Kahn 2016; Ayoub 2017], we prove the following theorem.

**Theorem 1.5.** *Let $(X, D)$ be a log smooth projective surface pair with interior $U$. If $p_g(U) = 0$, in particular, $p_g(X) = 0$ as well, then the log Bloch conjecture holds for $U$ if and only if it holds for $X$.*

Since the Bloch conjecture holds for any smooth projective surface $X$ with $\kappa(X) \leq 1$ [Bloch et al. 1976], our main theorem yields the following corollary.

**Corollary 1.6.** *The log Bloch conjecture holds for $U$ if $\kappa(X) \leq 1$.* □

Since $\kappa(X) \leq \kappa(U)$, Corollary 1.6 generalizes the result of Bloch, Kas, and Lieberman [Bloch et al. 1976] to open surfaces of $\kappa(U) \leq 1$. It also covers the second author's previous result [Zhu 2018] on the log Bloch conjecture for $\kappa(U) = -\infty$.

Further, we may apply Theorem 1.5 to the case where $X$ is of general type and the Bloch conjecture is true. The Bloch conjecture holds in a great number of cases; see [Bauer et al. 2011; Pedrini and Weibel 2016; Voisin 2014] for recent developments.

During the preparation of this paper, Binda and Krishna [2018] proved more general results in the context of Chow groups with modulus using cycle-theoretic methods.

**1B.** ***Applications of Theorem 1.5 and Corollary 1.6.*** The birational geometry of open surfaces is developed by Kawamata [1979], while it is almost impossible to hope for a complete classification even for $\kappa(U) \leq 1$. We would like to focus on three special classes of surfaces whose geometry is extremely complicated.

**Example 1.7** ($\kappa(U) = -\infty$: **log del Pezzo surfaces**). Let $U$ be the smooth locus of a singular del Pezzo surface of Picard number one with at worst quotient singularities. In general, such singular del Pezzos form an unbounded family. Partial classifications are obtained in [Keel and McKernan 1999] with more than sixty exceptional collections. A difficult theorem of [Keel and McKernan 1999] states that $U$ is log rationally connected. In particular, it implies the log Bloch conjecture for $U$ [Zhu 2018, Proposition 4.3].

Since Theorem 1.5 and Corollary 1.6 do not depend on Keel and McKernan's result, we give a new proof of the following result.

**Corollary 1.8.** *With the notation as above, we have $h_0(U) = \mathbb{Z}$.*

**Example 1.9** ($\kappa(U) = 0$: **log Enriques surfaces**). A projective normal surface $Y$ is said to be a *log Enriques surface* if

(1) $Y$ has at worst quotient singularities;

(2) $NK_Y \sim \mathcal{O}_Y$ for some positive integer $N$;

(3) $\dim H^1(Y, \mathcal{O}_Y) = 0$.

Since $K_Y$ is $\mathbb{Q}$-Cartier, we define the *index* $I$ of $Y$ to be the smallest positive integer such that $IK_Y \sim \mathcal{O}_Y$. By [Kawamata 1979; Tsunoda 1983; Zhang 1991], the index is bounded by 66, while classically (when $Y$ is smooth projective) it is bounded by 6.

**Corollary 1.10.** *Let $U$ be the smooth locus of a log Enriques surface of index $\geq 2$ defined as above. Then $h_0(U) = \mathbb{Z}$.*

Log Enriques surfaces are partially classified in [Zhang 1991; 1993; Kudryavtsev 2002; 2004]. There are more than 1000 examples of log Enriques surfaces with $\delta$-invariant 2 [Kudryavtsev 2002].

*Proof of Corollaries 1.8 and 1.10.* Let $(X, D)$ be a minimal log resolution of $U$. By Corollary 1.6, the log Bloch conjecture holds in both cases. It suffices to show $q(U) = 0$. Since $D$ is the exceptional set of the resolution of quotient singularities, we have $q(U) = q(X)$. Now the del Pezzo case follows from [Zhang 1989, Lemma 1.1(3)] and the Enriques case from [Zhang 1991, Lemma 1.2]. □

**Example 1.11.** **$\mathbb{Q}$-homology planes** A smooth surface $U$ is a *$\mathbb{Q}$-homology plane* if $H^i(U, \mathbb{Q}) = H^i(\mathbb{A}^2, \mathbb{Q})$ for any $i$. A $\mathbb{Q}$-homology plane can have log Kodaira dimension $-\infty$, 0, 1, or 2. Ramanujam [1971] constructed the first homology plane of log general type which is topologically contractible. They are classified for log Kodaira dimension $\leq 1$, but there is no thorough classification of $\mathbb{Q}$-homology planes of log general type [Miyanishi 2001, Section 3.4].

As all $\mathbb{Q}$-homology planes are rational [Gurjar and Pradeep 1999], Corollary 1.6 implies the following:

**Corollary 1.12.** *Let $U$ be a $\mathbb{Q}$-homology plane. Then the log Bloch conjecture holds, that is, $h_0(U) = \mathbb{Z}$.* □

The Bloch conjecture for fake projective planes remains unknown.

**1C.** ***Ideas of proof.*** The proof of our main theorem has two major ingredients. One is the work in [Ayoub and Barbieri-Viale 2009; Ayoub 2011; Barbieri-Viale and Kahn 2016] on the derived category of 1-motives, especially the construction of a derived Albanese functor. The use is twofold: first, it gives a motivic interpretation of the Albanese morphism, allowing us to apply results from the theory of mixed motives. Second, it provides a way to eliminate "easy" pieces of the motive of $U$ (essentially 1-motives) while keeping track of the homological realization.

The other ingredient is the famous conservativity conjecture; see [Ayoub 2017]. Regarded as a key conjecture in the study of motives, it notably says that a geometric motive is trivial if and only if its homological realization is trivial. By truncating the motive of $U$ using the derived Albanese functor, we eventually arrive at a motive which has trivial homological realization and whose motivic homology controls the Albanese kernel $T(U)$. Therefore, the conservativity conjecture implies the log Bloch conjecture for $U$. Part of our main theorem then follows from a special case of the conservativity conjecture proven by Wildeshaus [2015].

**1D.** ***Notation.*** A *log pair* $(X, D)$ means a variety $X$ with a reduced Weil divisor $D$. We say that $(X, D)$ is *log smooth* if $X$ is smooth and $D$ is a simple normal crossing divisor on $X$. A log pair is projective if the ambient variety is projective.

Given any smooth quasiprojective variety $U$, by the resolution of singularities, we may choose a log smooth projective compactification $(X, D)$ with interior $U$. We use $\kappa(X, D)$ to denote the *log Kodaira dimension*. We define the *log geometric genus* $p_g(X, D) := \dim H^0(\Omega_X^{\dim X}(\log D))$ and the *log irregularity* $q(X, D) := \dim H^0(\Omega_X^1(\log D))$. Since they do not depend on the compactification, we may write $\kappa(U)$, $p_g(U)$, and $q(U)$ as well.

## 2. Preliminaries

By Theorem 1.2, it suffices to consider the Albanese morphism with $\mathbb{Q}$-coefficients. From now on, all (co)homology, cycle groups, and motives are taken with $\mathbb{Q}$-coefficients.

**2A.** ***Mixed motives and conservativity.*** We refer to [Voevodsky et al. 2000; Mazza et al. 2006] for Voevodsky's theory of mixed motives. With $\mathbb{Q}$-coefficients, the categories of mixed motives in the Nisnevich and étale topologies are equivalent.

Let $\mathrm{DM}_{\mathrm{gm}}$ denote the triangulated category of geometric motives, and let $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ denote the triangulated category of effective geometric motives. We follow the homological convention. The unit object of $\mathrm{DM}_{\mathrm{gm}}$ is denoted by $\mathbb{Q}(0)$, or simply $\mathbb{Q}$, and the Tate object $\mathbb{Q}(1)$. Given an object $M \in \mathrm{DM}_{\mathrm{gm}}$, its dual object $\mathcal{H}om_{\mathrm{DM}_{\mathrm{gm}}}(M, \mathbb{Q})$ is denoted by $M^\vee$. The motive of a smooth variety $Y$ is denoted by $M(Y) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$.

The $i$-th motivic homology of $M \in \mathrm{DM}_{\mathrm{gm}}$ is defined to be

$$h_i(M) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}[i], M).$$

For $M = M(Y)$, this recovers the $i$-th Suslin homology $h_i(Y) = h_i(M(Y))$.

Further, we refer to [Huber 2000] for the Hodge realization functor

$$R^H : \mathrm{DM}_{\mathrm{gm}} \to D^b(\mathrm{MHS}).$$

Here we use the covariant version of $R^H$. Composing with the forgetful functor $D^b(\mathrm{MHS}) \to D^b(\mathbb{Q})$, we obtain the Betti realization

$$R^B : \mathrm{DM}_{\mathrm{gm}} \to D^b(\mathbb{Q}).$$

Recall the statement of the conservativity conjecture.

**Conjecture 2.1** (see [Ayoub 2017, Conjecture 2.1]). The Betti realization functor $R^B$ is conservative. In other words, a morphism $f : M \to N$ in $\mathrm{DM}_{\mathrm{gm}}$ is an isomorphism if and only if $R^B(f) : R^B(M) \to R^B(N)$ is an isomorphism.

Using consequences of the standard conjecture D for abelian varieties [André and Kahn 2002], Kimura–O'Sullivan finiteness [Kimura 2005], and Bondarko's weight structures [2009; 2010], Wildeshaus proved the following special case of the conservativity conjecture.

**Theorem 2.2** [Wildeshaus 2015, Theorem 1.12]. *Let* $\mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}} \subset \mathrm{DM}_{\mathrm{gm}}$ *denote the smallest triangulated subcategory containing the motives of smooth curves and closed under direct summands, tensor products, and duality. Then the restriction of* $R^B$ *to* $\mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$ *is conservative.*

With the notion of $\mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$, we now state our main theorem extending Theorem 1.5.

**Theorem 2.3.** *Let* $(X, D)$ *be a log smooth projective surface pair with interior* $U$. *Then the following four conditions are equivalent*:

(1) $T(U) = 0$;

(2) $T(X) = 0$;

(3) $M(U) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$;

(4) $M(X) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$.

**2B.** ***Derived category of* 1*-motives.*** We mainly follow the book [Barbieri-Viale and Kahn 2016]. Let $\mathcal{M}_1$ denote Deligne's category of 1-motives [Deligne 1974] with $\mathbb{Q}$-coefficients. By [Orgogozo 2004, Théorème 3.4.1], the bounded derived category $D^b(\mathcal{M}_1)$ can be naturally identified with the thick triangulated subcategory of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ generated by the motives of smooth curves, denoted by $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. The identification is compatible with realizations [Vologodsky 2012]. For simplicity we always make this identification.

One of the main results of [Barbieri-Viale and Kahn 2016] is the construction of a *derived Albanese functor*.

**Theorem 2.4** [Barbieri-Viale and Kahn 2016, Corollary 6.2.2]. *The inclusion*

$$d_{\le 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \hookrightarrow \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$$

*admits a left adjoint*

$$L\,\mathrm{Alb} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to d_{\le 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,.$$

We list a number of results and facts about the functor $L\,\mathrm{Alb}$ which will be used in the proof of our main theorem. To begin with, when $Y$ is a smooth variety, we write $L\,\mathrm{Alb}(Y) = L\,\mathrm{Alb}(M(Y))$. Then the natural morphism $M(Y) \to L\,\mathrm{Alb}(Y)$ induces a morphism in motivic homology

$$h_0(Y) \to h_0(L\,\mathrm{Alb}(Y)). \tag{2.5}$$

By [Barbieri-Viale and Kahn 2016, Lemma 13.4.2], we have

$$h_0(L\,\mathrm{Alb}(Y))^0 = \mathrm{Alb}(Y) \otimes \mathbb{Q},$$

and the degree zero part of (2.5) coincides with the Albanese morphism.

The next statement concerns the Hodge realization of $L\,\mathrm{Alb}(M)$ for $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. Recall that a mixed Hodge structure $H$ is *effective* if the $(i, j)$-th part of the weight-graded piece $\mathrm{Gr}^W_{i+j} H$ vanishes unless $i, j \le 0$. Given an effective mixed Hodge structure $H$, let $H_{\le 1}$ denote the maximal quotient of $H$ of weights $\ge -2$ and of types $(0, 0)$, $(0, -1)$, $(-1, 0)$, and $(-1, -1)$.

**Theorem 2.6** [Barbieri-Viale and Kahn 2016, Theorem 15.3.1]. *For $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, the morphism $M \to L\,\mathrm{Alb}(M)$ induces isomorphisms*

$$H_i(R^H(M))_{\le 1} \xrightarrow{\sim} H_i\big(R^H(L\,\mathrm{Alb}(M))\big).$$

The theorem above applies to $L\,\mathrm{Alb}(Y)$ and also to the Borel–Moore variant of $L\,\mathrm{Alb}(Y)$. Let $M^c(Y) \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ denote the motive of $Y$ with compact support. Note that by [Voevodsky et al. 2000, Chapter 5, Theorem 4.3.7], there is an isomorphism

$$M^c(Y) \simeq M(Y)^\vee(\dim Y)[2\dim Y]. \tag{2.7}$$

We write $L\,\mathrm{Alb}^c(Y) = L\,\mathrm{Alb}(M^c(Y))$.

**Corollary 2.8** [Barbieri-Viale and Kahn 2016, Corollary 15.3.2]. *By Theorem 2.6, we have*

$$H_i\big(R^H(L\,\mathrm{Alb}(Y))\big) = \begin{cases} H_0(Y,\mathbb{Q}), & i=0,\\ H_1(Y,\mathbb{Q}), & i=1,\\ H_2(Y,\mathbb{Q})_{\le 1}, & i=2,\\ 0, & i<0 \text{ or } i>2,\end{cases}$$

$$H_i\big(R^H(L\,\mathrm{Alb}^c(Y))\big) = \begin{cases} H_0^{\mathrm{BM}}(Y,\mathbb{Q}), & i=0,\\ H_1^{\mathrm{BM}}(Y,\mathbb{Q}), & i=1,\\ H_i^{\mathrm{BM}}(Y,\mathbb{Q})_{\le 1}, & 2\le i\le \dim Y+1,\\ 0, & i<0 \text{ or } i>\dim Y+1.\end{cases}$$

Finally, we recall the fact that $\mathcal{M}_1$ is of cohomological dimension one [Orgogozo 2004, Proposition 3.2.4]. Hence, all elements in $D^b(\mathcal{M}_1)$ can be represented by complexes with zero differentials. In particular, we have

$$L\,\mathrm{Alb}(Y) \simeq \bigoplus_{i=0}^{2} L_i\,\mathrm{Alb}(Y)[i] \quad\text{and}\quad L\,\mathrm{Alb}^c(Y) \simeq \bigoplus_{i=0}^{\dim Y+1} L_i\,\mathrm{Alb}^c(Y)[i], \tag{2.9}$$

with $L_i\,\mathrm{Alb}(Y)$, $L_i\,\mathrm{Alb}^c(Y) \in \mathcal{M}_1$; see [Barbieri-Viale and Kahn 2016, Corollary 9.2.3 and Proposition 10.6.2]. When $\dim Y = 1$, this gives the "Chow–Künneth" decomposition of $M(Y)$ [Barbieri-Viale and Kahn 2016, Corollary 11.1.1]

$$M(Y) \simeq L\,\mathrm{Alb}(Y) \simeq \bigoplus_{i=0}^{2} L_i\,\mathrm{Alb}(Y)[i]. \tag{2.10}$$

## 3. Proof of the main theorem

In this section we prove our main theorem, that is, Theorem 2.3.

**3A.** ***Proof of (1) ⇒ (2) ⇒ (3) ⇔ (4).*** For (1) ⇒ (2), consider a partial compactification $U \subset Y \subset X$ such that $C = Y \setminus U$ is a smooth curve. By induction, it suffices to show that $T(U) = 0$ implies $T(Y) = 0$.

Recall the Gysin distinguished triangle [Voevodsky et al. 2000, Chapter 5, Proposition 3.5.4]

$$M(U) \to M(Y) \to M(C)(1)[2] \to M(U)[1]. \tag{3.1}$$

By applying the functor $L\,\mathrm{Alb}$, we find a morphism of distinguished triangles

$$\begin{array}{ccccccc} M(U) & \longrightarrow & M(Y) & \longrightarrow & M(C)(1)[2] & \longrightarrow & M(U)[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L\,\mathrm{Alb}(U) & \longrightarrow & L\,\mathrm{Alb}(Y) & \longrightarrow & \mathbb{Q}(1)[2] & \longrightarrow & L\,\mathrm{Alb}(U)[1] \end{array} \tag{3.2}$$

Here we used the fact that $L\,\mathrm{Alb}(M(C)(1)) \simeq \mathbb{Q}(1)$ [Barbieri-Viale and Kahn 2016, Proposition 8.2.3]. Moreover, the morphism

$$M(C)(1) \to L\,\mathrm{Alb}(M(C)(1)) \simeq \mathbb{Q}(1)$$

coincides with the projection in (2.10),

$$M(C) \to L_0\,\mathrm{Alb}(C) \simeq \mathbb{Q},$$

twisted by $\mathbb{Q}(1)$.

Now we apply motivic homology to the distinguished triangles in (3.2). Since $h_0(U) \to h_0(Y)$ is surjective [Zhu 2018, Lemma 4.2] and

$$h_0(\mathbb{Q}(1)[2]) = \mathrm{CH}_{-1}(\mathrm{pt}) = 0,$$

we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
h_1(M(C)(1)[2]) \arrow[r] \arrow[d] & h_0(U) \arrow[r] \arrow[d] & h_0(Y) \arrow[r] \arrow[d] & 0 \arrow[d] \\
h_1(\mathbb{Q}(1)[2]) \arrow[r] & h_0(L\,\mathrm{Alb}(U)) \arrow[r] & h_0(L\,\mathrm{Alb}(Y)) \arrow[r] & 0
\end{tikzcd}$$

The first vertical arrow is surjective since it comes from a projection. The middle vertical arrows are given by the Albanese morphisms of $U$ and $Y$. Our assumption $T(U) = 0$ says that the second vertical arrow is injective. Then, by the five lemma, the third vertical arrow is also injective, and hence $T(Y) = 0$.

The implication (2) ⇒ (4) is essentially due to Guletskiĭ and Pedrini [2003, Theorem 7]. The precise statement can be found in [Kahn et al. 2007, Corollary. 4.9], where it is shown that $T(X) = 0$ is equivalent to the vanishing of the transcendental part in the Chow–Künneth decomposition of $M(X)$, and that the remaining parts belong to $\mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$. Further, by the distinguished triangle (3.1) and the fact that $M(C) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$, we see that $M(U) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$ if and only if $M(Y) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$. The equivalence (3) ⇔ (4) then follows by induction. □

**3B.** ***Proof of (3) ⇒ (1).*** We define the motive $M'(U) \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ by the distinguished triangle

$$M'(U) \to M(U) \to L\,\mathrm{Alb}(U) \to M'(U)[1]. \tag{3.3}$$

Our assumption $p_g(U) = 0$ says that $H_2(U, \mathbb{Q}) = H_2(U, \mathbb{Q})_{\le 1}$. Then, by Theorem 2.6 and Corollary 2.8, we have

$$H_i\bigl(R^B(M'(U))\bigr) = \begin{cases} H_3(U, \mathbb{Q}), & i = 3, \\ H_4(U, \mathbb{Q}), & i = 4, \\ 0, & i < 3 \text{ or } i > 4. \end{cases}$$

Now consider the motive $M'(U)^\vee(2)[4]$. By the duality

$$H_i^{\mathrm{BM}}(U, \mathbb{Q}) = H_{4-i}(U, \mathbb{Q})^\vee(2),$$

we have

$$H_i\big(R^B(M'(U)^\vee(2)[4])\big) = \begin{cases} H_0^{\mathrm{BM}}(U,\mathbb{Q}), & i=0,\\ H_1^{\mathrm{BM}}(U,\mathbb{Q}), & i=1,\\ 0, & i<0 \text{ or } i>1.\end{cases} \tag{3.4}$$

There is a dual distinguished triangle to (3.3),

$$L\,\mathrm{Alb}(U)^\vee(2)[4] \to M(U)^\vee(2)[4] \to M'(U)^\vee(2)[4] \to L\,\mathrm{Alb}(U)^\vee(2)[5].$$

By (2.7), we have $M(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. Since $L\,\mathrm{Alb}(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ by Cartier duality [Barbieri-Viale and Kahn 2016, Proposition 4.5.1], we also have $M'(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. This allows us to apply the functor $L\,\mathrm{Alb}$ to $M'(U)^\vee(2)[4]$. By Theorem 2.6 and Corollary 2.8, the morphism

$$M'(U)^\vee(2)[4] \to L\,\mathrm{Alb}(M'(U)^\vee(2)[4]) \tag{3.5}$$

induces an isomorphism

$$R^B(M'(U)^\vee(2)[4]) \xrightarrow{\sim} R^B\big(L\,\mathrm{Alb}(M'(U)^\vee(2)[4])\big).$$

We are ready to apply conservativity. Our assumption $M(U) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$ implies $M(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$. Furthermore, since $d_{\le 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \subset \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$, we have $L\,\mathrm{Alb}(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$ and thus $M'(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$. Then, according to Theorem 2.2, the morphism (3.5) is itself an isomorphism.

We thus obtain from (3.3) a distinguished triangle

$$\begin{aligned} L\,\mathrm{Alb}(M'(U)^\vee(2)[4])^\vee(2)[4] &\to M(U) \to L\,\mathrm{Alb}(U)\\ &\to L\,\mathrm{Alb}(M'(U)^\vee(2)[4])^\vee(2)[5]. \end{aligned} \tag{3.6}$$

Taking motivic homology, we have an exact sequence

$$h_0\big(L\,\mathrm{Alb}(M'(U)^\vee(2)[4])^\vee(2)[4]\big) \to h_0(U) \to h_0(L\,\mathrm{Alb}(U)),$$

where the second arrow is given by the Albanese morphism of $U$. Hence, to prove $T(U)=0$, it suffices to show that

$$h_0\big(L\,\mathrm{Alb}(M'(U)^\vee(2)[4])^\vee(2)[4]\big) = 0.$$

By [Deligne 1974, Construction 10.1.3], the Hodge realization gives a full embedding $\mathcal{M}_1 \subset \mathrm{MHS}$. A comparison of realizations yields the isomorphisms

$$
\begin{aligned}
L\,\mathrm{Alb}(M'(U)^\vee(2)[4]) &\simeq \bigoplus_{i=0}^{1} L_i\,\mathrm{Alb}(M'(U)^\vee(2)[4])[i] \\
&\simeq \bigoplus_{i=0}^{1} L_i\,\mathrm{Alb}(M(U)^\vee(2)[4])[i] \\
&\simeq \bigoplus_{i=0}^{1} L_i\,\mathrm{Alb}^c(U)[i]. \qquad (3.7)
\end{aligned}
$$

More precisely, the first isomorphism is a consequence of (2.9) and (3.4). The last two isomorphisms follow from the duality (2.7), Corollary 2.8, and (3.4). Alternatively, one may also deduce (3.7) from Theorem 2.2 since all motives involved belong to $\mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$.

We compute

$$
\begin{aligned}
&h_0\big(L\,\mathrm{Alb}(M'(U)^\vee(2)[4])^\vee(2)[4]\big) \\
&\quad = h_0\Bigg(\bigoplus_{i=0}^{1}(L_i\,\mathrm{Alb}^c(U)[i])^\vee(2)[4]\Bigg) \\
&\quad = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}\Bigg(\mathbb{Q}, \bigoplus_{i=0}^{1}(L_i\,\mathrm{Alb}^c(U)[i])^\vee(2)[4]\Bigg) \\
&\quad = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}\Bigg(\bigoplus_{i=0}^{1} L_i\,\mathrm{Alb}^c(U)[i], \mathbb{Q}(2)[4]\Bigg) \\
&\quad = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(L_0\,\mathrm{Alb}^c(U), \mathbb{Q}(2)[4]) \oplus \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(L_1\,\mathrm{Alb}^c(U), \mathbb{Q}(2)[3]).
\end{aligned}
$$

By [Barbieri-Viale and Kahn 2016, Proposition 10.6.2], we have

$$
L_0\,\mathrm{Alb}^c(U) \simeq \begin{cases} \mathbb{Q} & \text{if } U \text{ is projective,} \\ 0 & \text{if not.} \end{cases}
$$

Since

$$
\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}, \mathbb{Q}(2)[4]) = \mathrm{CH}_{-2}(\mathrm{pt}) = 0,
$$

we find in both cases $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(L_0\,\mathrm{Alb}^c(U), \mathbb{Q}(2)[4]) = 0$.

Further, by [Barbieri-Viale and Kahn 2016, Corollary 12.11.2], the 1-motive $L_1\,\mathrm{Alb}^c(U)$ is represented by a two-term complex in degrees 0 and 1

$$
\mathbb{Q}^{\oplus r} \to A \otimes \mathbb{Q},
$$

where $A$ is an abelian variety and $r = \#\{\text{connected components of } D\} - 1$. In other words, there is an extension of 1-motives

$$
0 \to (A \otimes \mathbb{Q})[-1] \to L_1\,\mathrm{Alb}^c(U) \to \mathbb{Q}^{\oplus r} \to 0, \qquad (3.8)
$$

which yields an exact sequence

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}, \mathbb{Q}(2)[3])^{\oplus r} \to \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(L_1 \operatorname{Alb}^c(U), \mathbb{Q}(2)[3]) \\ \to \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}((A \otimes \mathbb{Q})[-1], \mathbb{Q}(2)[3]).$$

Since

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}, \mathbb{Q}(2)[3]) = \mathrm{CH}_{-2}(\mathrm{pt}, 1) = 0,$$

it suffices to show that $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}((A \otimes \mathbb{Q})[-1], \mathbb{Q}(2)[3]) = 0$.

We may assume $A$ to be the Albanese variety of a smooth projective surface $S$. If $\dim A > 2$, the surface $S$ is obtained by taking a sequence of general hyperplane sections of $A$. Then we have $\operatorname{Alb}(S) \simeq \operatorname{Alb}(A) \simeq A$ by the Lefschetz hyperplane theorem. Recall the Chow–Künneth decomposition of $M(S)$ [Murre 1990, Theorem 3]:

$$M(S) \simeq \bigoplus_{i=0}^{4} M_i(S)[i].$$

We have $M_{4-i}(S) \simeq M_i(S)^\vee(2)$ and $M_1(S) \simeq (A \otimes \mathbb{Q})[-1]$. Hence

$$\begin{aligned} \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}((A \otimes \mathbb{Q})[-1], \mathbb{Q}(2)[3]) &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(M_1(S), \mathbb{Q}(2)[3]) \\ &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}, M_3(S)[3]) \\ &= \mathrm{CH}_0(M_3(S)[3]) \\ &= 0, \end{aligned} \tag{3.9}$$

where the last equality follows again from [Murre 1990, Theorem 3]. The proof of Theorem 2.3 is now complete. □

**3C. *"Chow–Künneth" decomposition.*** Our proof of Theorem 2.3 also leads to the following consequence.

**Corollary 3.10.** *Assume one of the equivalent conditions in Theorem 2.3. Then $M(U)$ admits a "Chow–Künneth" decomposition*

$$M(U) \simeq \bigoplus_{i=0}^{2} L_i \operatorname{Alb}(U)[i] \oplus \bigoplus_{i=3}^{4} L_{4-i} \operatorname{Alb}^c(U)^\vee(2)[i].$$

*In particular, it is Kimura–O'Sullivan finite.*

*Proof.* Consider the distinguished triangle (3.6) obtained under the assumption $M(U) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$. By (2.9) and (3.7), there are isomorphisms

$$L \operatorname{Alb}(U) \simeq \bigoplus_{i=0}^{2} L_i \operatorname{Alb}(U)[i] \quad \text{and} \quad L \operatorname{Alb}(M'(U)^\vee(2)[4]) \simeq \bigoplus_{i=0}^{1} L_i \operatorname{Alb}^c(U)[i].$$

Hence (3.6) induces a distinguished triangle

$$\bigoplus_{i=3}^{4} L_{4-i}\,\mathrm{Alb}^{c}(U)^{\vee}(2)[i] \to M(U) \to \bigoplus_{i=0}^{2} L_i\,\mathrm{Alb}(U)[i] \\ \to \bigoplus_{i=3}^{4} L_{4-i}\,\mathrm{Alb}^{c}(U)^{\vee}(2)[i+1].$$

For the distinguished triangle to split, it suffices to show that

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}\Bigl(\bigoplus_{i=0}^{2} L_i\,\mathrm{Alb}(U)[i], \bigoplus_{i=3}^{4}(L_{4-i}\,\mathrm{Alb}^{c}(U))^{\vee}(2)[i+1]\Bigr) = 0.$$

The left-hand side consists of six direct summands, all of which can be computed explicitly. To keep the paper short we only do the most complicated one, that is,

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(L_1\,\mathrm{Alb}(U)[1], L_1\,\mathrm{Alb}^{c}(U)^{\vee}(2)[4]). \tag{3.11}$$

By [Barbieri-Viale and Kahn 2016, Corollary 9.2.3], the 1-motive $L_1\,\mathrm{Alb}(U)$ is represented by the two-term complex in degrees 0 and 1

$$0 \to \mathrm{Alb}(U) \otimes \mathbb{Q}.$$

Since the abelian part of the semiabelian variety $\mathrm{Alb}(U)$ is $\mathrm{Alb}(X)$, this gives an extension of 1-motives

$$0 \to (\mathbb{G}_m \otimes \mathbb{Q})^{\oplus s}[-1] \to L_1\,\mathrm{Alb}(U) \to (\mathrm{Alb}(X) \otimes \mathbb{Q})[-1] \to 0. \tag{3.12}$$

We have $(\mathbb{G}_m \otimes \mathbb{Q})[-1] \simeq \mathbb{Q}(1)$ and $(\mathrm{Alb}(X) \otimes \mathbb{Q})[-1] \simeq M_1(X)$.

Combining (3.8) and (3.12), we see that (3.11) sits in the middle of several extensions involving the following four terms:

(1) $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(M_1(X)[1], M_3(S)[4])$;

(2) $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(M_1(X)[1], \mathbb{Q}(2)[4])$;

(3) $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}(1)[1], M_3(S)[4])$;

(4) $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}(1)[1], \mathbb{Q}(2)[4])$.

The vanishing of the second term is shown in (3.9) (with $X$ replaced by $S$). The vanishing of the three other terms follows from the fact that given two Chow motives $M$ and $M'$, we have $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(M, M'[i]) = 0$ for all $i > 0$. This in turn follows from [Voevodsky et al. 2000, Chapter 5, Corollary 4.2.6] and the cancellation theorem [Voevodsky et al. 2000, Chapter 5, Theorem 4.3.1]. Hence (3.11) vanishes.

Finally, by [Mazza 2004, Remark 5.11], all elements in $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ are Kimura–O'Sullivan finite. The last statement follows since Kimura–O'Sullivan finiteness is closed under direct sums and tensor products. □

On the other hand, there exist motives of smooth surfaces which are not Kimura–O'Sullivan finite [Mazza 2004, Theorem 5.18].

## Acknowledgments

We would like to thank Joseph Ayoub for explaining his results. We thank Qile Chen and Javier Fresán for helpful discussions, and the referee for comments and suggestions. This work was initiated during the AMS Summer Institute in Algebraic Geometry at the University of Utah, 2015. The authors would like to thank the Summer Institute for its hospitality and inspiring environment.

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Received 28 Mar 2016. Revised 28 Nov 2017. Accepted 14 Dec 2017.

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ANNALS OF K-THEORY
Vol. 3, No. 3, 2018
dx.doi.org/10.2140/akt.2018.3.395

# Topological K-theory of affine Hecke algebras

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Let $\mathcal{H}(\mathcal{R}, q)$ be an affine Hecke algebra with a positive parameter function $q$. We are interested in the topological K-theory of its $C^*$-completion $C_r^*(\mathcal{R}, q)$. We prove that $K_*(C_r^*(\mathcal{R}, q))$ does not depend on the parameter $q$, solving a long-standing conjecture of Higson and Plymen. For this we use representation-theoretic methods, in particular elliptic representations of Weyl groups and Hecke algebras.

Thus, for the computation of these K-groups it suffices to work out the case $q = 1$. These algebras are considerably simpler than for $q \neq 1$, just crossed products of commutative algebras with finite Weyl groups. We explicitly determine $K_*(C_r^*(\mathcal{R}, q))$ for all classical root data $\mathcal{R}$. This will be useful for analyzing the K-theory of the reduced $C^*$-algebra of any classical $p$-adic group.

For the computations in the case $q = 1$, we study the more general situation of a finite group $\Gamma$ acting on a smooth manifold $M$. We develop a method to calculate the K-theory of the crossed product $C(M) \rtimes \Gamma$. In contrast to the equivariant Chern character of Baum and Connes, our method can also detect torsion elements in these K-groups.



## Introduction

Affine Hecke algebras can be realized in two completely different ways. On the one hand, they are deformations of group algebras of affine Weyl groups, and on the other hand they appear as subalgebras of group algebras of reductive $p$-adic groups. Via the second interpretation, affine Hecke algebras (AHAs) have proven

The author is supported by a NWO Vidi grant "A Hecke algebra approach to the local Langlands correspondence" (nr. 639.032.528).
*MSC2010:* primary 20C08, 46L80; secondary 19L47.
*Keywords:* topological K-theory, affine Hecke algebra, Weyl group, crossed product algebra.

very useful in the representation theory of such groups. This use is in no small part due to their explicit construction in terms of root data, which makes them amenable to concrete calculations.

This paper is motivated by our desire to understand and compute the (topological) K-theory of the reduced $C^*$-algebra $C_r^*(G)$ of a reductive $p$-adic group $G$. This is clearly related to the representation theory of $G$. For instance, when $G$ is semisimple, every discrete series $G$-representation gives rise to a one-dimensional direct summand in the K-theory of $C_r^*(G)$.

The problem can be transferred to AHAs in the following way. By the Bernstein decomposition, the Hecke algebra of $G$ can be written as a countable direct sum of two-sided ideals:

$$\mathcal{H}(G) = \bigoplus\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}}.$$

Borel [1976] and Iwahori and Matsumoto [1965] have shown that one particular summand, say $\mathcal{H}(G)^{\mathrm{IM}}$, is Morita equivalent to an AHA, say $\mathcal{H}(\mathcal{R}, q)^{\mathrm{IM}}$. It is expected that all other summands $\mathcal{H}(G)^{\mathfrak{s}}$ are also Morita equivalent to AHAs, or to closely related algebras. Indeed, this has been proven in many cases; see [Aubert et al. 2017a, §2.4] for an overview.

The reduced $C^*$-algebra of $G$ is a completion of $\mathcal{H}(G)$, and it admits an analogous Bernstein decomposition

$$C_r^*(G) = \bigoplus\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} C_r^*(G)^{\mathfrak{s}},$$

where $C_r^*(G)^{\mathfrak{s}}$ is the closure of $\mathcal{H}(G)^{\mathfrak{s}}$ in $C_r^*(G)$. By [Bushnell et al. 2011], the Morita equivalence $\mathcal{H}(G)^{\mathrm{IM}} \sim_M \mathcal{H}(\mathcal{R}, q)$ extends to a Morita equivalence between $C_r^*(G)^{\mathrm{IM}}$ and the natural $C^*$-completion of $\mathcal{H}(\mathcal{R}, q)^{\mathrm{IM}}$. Again it can be expected that every summand $C_r^*(G)^{\mathfrak{s}}$ is Morita equivalent to the $C^*$-completion $C_r^*(\mathcal{R}, q)^{\mathfrak{s}}$ of some AHA $\mathcal{H}(\mathcal{R}, q)^{\mathfrak{s}}$. However, this is currently not yet proven in several cases where the Morita equivalence is known on the algebraic level. We will return to this issue in a subsequent paper. Assuming it for the moment, we get

$$K_*(C_r^*(G)) \cong \bigoplus\nolimits_{\mathfrak{s} \in \mathfrak{B}(G)} K_*(C_r^*(\mathcal{R}, q)^{\mathfrak{s}}).$$

The left-hand side figures in the Baum–Connes conjecture for reductive $p$-adic groups [Baum et al. 1994]. For applications to the Baum–Connes conjecture for algebraic groups over local fields, it would be useful to understand $K_*(C_r^*(G))$ better, in particular its torsion subgroup. Namely, from the work of Kasparov [1988] it is known that for many groups $G$ the Baum–Connes assembly map is injective, and that its image is a direct summand of $K_*(C_r^*(G))$. There exist methods [Solleveld 2009, §3.4] which enable one to prove that the assembly map becomes an isomorphism after tensoring its domain and range by $\mathbb{Q}$, but which say little about the torsion elements in the K-groups. If one knew in advance that $K_*(C_r^*(G))$ is torsion-free, then one could prove instances of the Baum–Connes conjecture with such methods.

To construct an affine Hecke algebra, we use a root datum $\mathcal{R}$ in a lattice $X$. These give a Weyl group $W = W(\mathcal{R})$ and an extended affine Weyl group $W^e = X \rtimes W$. As parameters we take a tuple of nonzero complex numbers $q = (q_i)_i$. The AHA $\mathcal{H}(\mathcal{R}, q)$ is a deformation of the group algebra $\mathbb{C}[W^e]$, in the following sense: as a vector space it is $\mathbb{C}[W^e]$, with a multiplication rule depending algebraically on $q$, such that $\mathcal{H}(\mathcal{R}, 1) = \mathbb{C}[W^e]$. See Section 1C for the precise definition. To get a nice $C^*$-completion $C_r^*(\mathcal{R}, q)$, we must assume that $q$ is positive, that is, $q_i \in \mathbb{R}_{>0}$ for all $i$. For $q = 1$ the $C^*$-completion can be described easily:

$$C_r^*(\mathcal{R}, 1) = C_r^*(W^e) = C(T_{\mathrm{un}}) \rtimes W,$$

where $T_{\mathrm{un}} = \mathrm{Hom}_{\mathbb{Z}}(X, S^1)$ is a compact torus.

All AHAs obtained from reductive $p$-adic groups $G$ have rather special parameters: there are $n_i \in \mathbb{Z}_{\geq 0}$ such that $q_i = p^{n_i/2}$, where $p$ is the characteristic of the local nonarchimedean field underlying $G$. Thus the realization of AHAs via root data admits more parameters than the realization as subalgebras of $\mathcal{H}(G)$. In particular the algebras $\mathcal{H}(\mathcal{R}, q)$ admit continuous parameter deformations, whereas the AHAs from reductive $p$-adic groups do not, since the prime powers $p^{n/2}$ are discrete in $\mathbb{R}_{>0}$.

In fact, for fixed $\mathcal{R}$ the family $C_r^*(\mathcal{R}, q)$, with varying positive $q$, form a continuous field of $C^*$-algebras. For a given $q \neq 1$ we have the half-line of parameters $q^\epsilon = (q_i^\epsilon)_i$ with $\epsilon \in \mathbb{R}_{\geq 0}$. It is known from [Solleveld 2012a, Theorem 4.4.2] that there exists a family of $C^*$-homomorphisms

$$\zeta_\epsilon : C_r^*(\mathcal{R}, q^\epsilon) \to C_r^*(\mathcal{R}, q), \quad \epsilon \geq 0,$$

such that $\zeta_\epsilon$ is an isomorphism for all $\epsilon > 0$ and depends continuously on $\epsilon \in \mathbb{R}_{\geq 0}$. Via a general deformation principle, this yields a canonical homomorphism

$$K_*(C_r^*(W^e)) = K_*(C_r^*(\mathcal{R}, q^0)) \to K_*(C_r^*(\mathcal{R}, q)). \tag{0.1}$$

Loosely speaking, the construction goes as follows. Take a projection $p_0$ (or a unitary $u_0$) in a matrix algebra $M_n(C_r^*(W^e)) = M_n(C_r^*(\mathcal{R}, q^0))$. For $\epsilon > 0$ small, we can apply holomorphic functional calculus to $p_0$ to produce a new projection $p_\epsilon \in M_n(C_r^*(\mathcal{R}, q^\epsilon))$ (or a unitary $u_\epsilon$). Then (0.1) sends $[p_0] \in K_0(C_r^*(\mathcal{R}, q^0))$ to the image of $p_\epsilon$, and $u_0 \in K_1(C_r^*(\mathcal{R}, q^0))$ to the image of $u_\epsilon$, under the isomorphism $M_n(C_r^*(\mathcal{R}, q^\epsilon)) \cong M_n(C_r^*(\mathcal{R}, q))$.

Actually, more is true: by [Solleveld 2012a, Lemma 5.1.2] the map $K_*(\zeta_0)$ equals (0.1). Furthermore, by [Solleveld 2012a, Theorem 5.1.4] $\zeta_0$ induces an isomorphism

$$K_*(C_r^*(\mathcal{R}, q^0)) \otimes_{\mathbb{Z}} \mathbb{C} \to K_*(C_r^*(\mathcal{R}, q)) \otimes_{\mathbb{Z}} \mathbb{C}.$$

In view of the aforementioned relation with the Baum–Connes conjecture for $p$-adic groups, we also want to understand the torsion parts of these K-groups. We will prove:

**Theorem 1** (see Theorem 2.11). *The map (0.1) is a canonical isomorphism*

$$K_*(C_r^*(\mathcal{R}, 1)) \to K_*(C_r^*(\mathcal{R}, q)).$$

This theorem was conjectured first by Higson and Plymen (see [Plymen 1993, 6.4] and [Baum et al. 1994, 6.21]), at least when all parameters $q_i$ are equal. It is similar to the Connes–Kasparov conjecture for Lie groups; see [Baum et al. 1994, Sections 4–6] for more background. Independently, Opdam [2004, Section 1.0.1] conjectured Theorem 1 for unequal parameters.

Unfortunately it is unclear how Theorem 1 could be proven by purely noncommutative geometric means. The search for an appropriate technique was a major drive behind the author's Ph.D. project (2002–2006), and partial results appeared already in his Ph.D. thesis [Solleveld 2007]. At that time, we hoped to derive representation consequences from a K-theoretic proof of Theorem 1. But so far, such a proof remains elusive.

In the meantime, substantial progress has been made in the representation theory of Hecke algebras; see in particular [Opdam and Solleveld 2010; Ciubotaru and Opdam 2015; Ciubotaru et al. 2014]. This enables us to turn things around (compared to 2004); now we can use representation theory to study the K-theory of $C_r^*(\mathcal{R}, q)$.

Given an algebra or group $A$, let $\mathrm{Mod}_f(A)$ be the category of finite length $A$-modules, and let $R_{\mathbb{Z}}(A)$ be the Grothendieck group thereof. We deduce Theorem 1 from the following:

**Theorem 2** (see Theorem 1.52). *The map*

$$\mathrm{Mod}_f(C_r^*(\mathcal{R}, q)) \to \mathrm{Mod}(C_r^*(W^e)) : \pi \mapsto \pi \circ \zeta_0$$

*induces $\mathbb{Z}$-linear bijections*

$$R_{\mathbb{Z}}(C_r^*(\mathcal{R}, q)) \to R_{\mathbb{Z}}(C_r^*(W^e)), \qquad R_{\mathbb{Z}}(\mathcal{H}(\mathcal{R}, q)) \to R_{\mathbb{Z}}(W^e).$$

A substantial part of the proof of Theorem 2 boils down to representations of the finite Weyl group $W$. Following [Reeder 2001], we study the group $\overline{R_{\mathbb{Z}}}(W)$ of elliptic representations, that is, $R_{\mathbb{Z}}(W)$ modulo the subgroup spanned by all representations induced from proper parabolic subgroups of $W$. First we show that $\overline{R_{\mathbb{Z}}}(W)$ is always torsion-free (Theorem 1.12). Then we compare it with the analogous group of elliptic representations of $\mathcal{H}(\mathcal{R}, q)$, which leads to Theorem 2.

Having established the general framework, we set out to compute $K_*(C_r^*(\mathcal{R}, q))$ explicitly, for some root data $\mathcal{R}$ associated to well-known groups. By Theorem 1, we only have to consider one $q$ for each $\mathcal{R}$. In most examples, the easiest is to take $q = 1$. Then we must determine

$$K_r(C_r^*(\mathcal{R}, 1)) = K_*(C(T_{\mathrm{un}}) \rtimes W) \cong K_W^*(T_{\mathrm{un}}),$$

where the right-hand side denotes the $W$-equivariant K-theory of the compact Hausdorff space $T_{\mathrm{un}}$. Let $T_{\mathrm{un}} /\!/ W$ be the extended quotient. Of course, the equivariant Chern character from [Baum and Connes 1988] gives a natural isomorphism

$$K_W^*(T_{\mathrm{un}}) \otimes_{\mathbb{Z}} \mathbb{C} \to H^*(T_{\mathrm{un}} /\!/ W; \mathbb{C}).$$

But this does not suffice for our purposes, because we are particularly interested in the torsion subgroup of $K_W^*(T_{\mathrm{un}})$. Remarkably, that appears to be quite difficult to determine, already for cyclic groups acting on tori [Langer and Lück 2012]. Using equivariant cohomology, we develop a technique to facilitate the computation of $K_*(C(\Sigma) \rtimes W)$ for any finite group $W$ acting smoothly on a manifold $\Sigma$. With extra conditions it can be made more explicit:

**Theorem 3** (see Theorem 2.45). *Suppose that every isotropy group $W_t$ $(t \in \Sigma)$ is a Weyl group, and that $H^*(\Sigma /\!/ W; \mathbb{Z})$ is torsion-free. Then*

$$K_*(C(\Sigma) \rtimes W) \cong H^*(\Sigma /\!/ W; \mathbb{Z}).$$

We note that $H^*(\Sigma /\!/ W; \mathbb{Z})$ can be computed relatively easily. Theorem 3 can be applied to all classical root data, and to some others as well. Let us summarize the outcome of our computations.

**Theorem 4.** *Let $\mathcal{R}$ be a root datum of type $\mathrm{GL}_n$, $\mathrm{SL}_n$, $\mathrm{PGL}_n$, $\mathrm{SO}_n$, $\mathrm{Sp}_{2n}$ or $G_2$. Let $q$ be any positive parameter function for $\mathcal{R}$. Then $K_*(C_r^*(\mathcal{R}, q))$ is a free abelian group, whose rank is given explicitly in Section 3.*

Whether or not torsion elements can pop up in $K_*(C_r^*(\mathcal{R}, q))$ for other root data remains to be seen. In view of our results it does not seem very likely, but we do not have a general principle to rule it out.

## 1. Representation theory

**1A. *Weyl groups.*** In this first subsection we show that the representation ring $R_{\mathbb{Z}}(W)$ of any finite Weyl group $W$ is the direct sum of two parts: the subgroup spanned by representations induced from proper parabolic subgroups, and an elliptic part $\overline{R_{\mathbb{Z}}}(W)$. We exhibit a $\mathbb{Z}$-basis of $\overline{R_{\mathbb{Z}}}(W)$ in terms of the Springer correspondence. These results rely mainly on case-by-case considerations in complex simple groups.

Let $\mathfrak{a}$ be a finite-dimensional real vector space and let $\mathfrak{a}^*$ be its dual. Let $Y \subset \mathfrak{a}$ be a lattice and $X = \mathrm{Hom}_{\mathbb{Z}}(Y, \mathbb{Z}) \subset \mathfrak{a}^*$ the dual lattice. Let

$$\mathcal{R} = (X, R, Y, R^\vee, \Delta) \tag{1.1}$$

be a based root datum. Thus, $R$ is a reduced root system in $X$, $R^\vee \subset Y$ is the dual root system, $\Delta$ is a basis of $R$ and the set of positive roots is denoted $R^+$. Furthermore, we are given a bijection $R \to R^\vee$, $\alpha \mapsto \alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and such that the corresponding reflections $s_\alpha : X \to X$ and $s_\alpha^\vee : Y \to Y$ stabilize $R$ and $R^\vee$, respectively. We do not assume that $R$ spans $\mathfrak{a}^*$. The reflections $s_\alpha$ generate the Weyl group $W = W(R)$ of $R$, and $S_\Delta := \{s_\alpha : \alpha \in \Delta\}$ is the collection of simple reflections.

For a set of simple roots $P \subset \Delta$ we let $R_P$ be the root system they generate, and we let $W_P = W(R_P)$ be the corresponding parabolic subgroup of $W$.

Let $R_{\mathbb{Z}}(W)$ be the Grothendieck group of the category of finite-dimensional complex $W$-representations, and write $R_{\mathbb{C}}(W) = \mathbb{C} \otimes_{\mathbb{Z}} R_{\mathbb{Z}}(W)$. For any $P \subset \Delta$ the induction functor $\mathrm{ind}_{W_P}^W$ gives linear maps $R_{\mathbb{Z}}(W_P) \to R_{\mathbb{Z}}(W)$ and $R_{\mathbb{C}}(W_P) \to R_{\mathbb{C}}(W)$. In this subsection we are mainly interested in the abelian group of "elliptic $W$-representations"

$$\overline{R_{\mathbb{Z}}}(W) = R_{\mathbb{Z}}(W) \Big/ \sum_{P \subsetneq \Delta} \mathrm{ind}_{W_P}^W (R_{\mathbb{Z}}(W_P)). \tag{1.2}$$

In the literature [Reeder 2001; Ciubotaru et al. 2014], one more often encounters the vector space

$$\overline{R_{\mathbb{C}}}(W) = R_{\mathbb{C}}(W) \Big/ \sum_{P \subsetneq \Delta} \mathrm{ind}_{W_P}^W (R_{\mathbb{C}}(W_P)).$$

Recall that an element $w \in W$ is called elliptic if it fixes only the zero element of $\mathrm{Span}_{\mathbb{R}}(R)$, or equivalently if it does not belong to any proper parabolic subgroup of $W$. It was shown in [Reeder 2001, Proposition 2.2.2] that $\overline{R_{\mathbb{C}}}(W)$ is naturally isomorphic to the space of all class functions on $W$ supported on elliptic elements. In particular, $\dim_{\mathbb{C}} \overline{R_{\mathbb{C}}}(W)$ is the number of elliptic conjugacy classes in $W$.

In [Ciubotaru et al. 2014], $\overline{R_{\mathbb{Z}}}(W)$ is defined as the subgroup of $\overline{R_{\mathbb{C}}}(W)$ generated by the $W$-representations. So in that work it is by definition a lattice. If $\overline{R_{\mathbb{Z}}}(W)$ (in our sense) is torsion-free, then it can be identified with the subgroup of $\overline{R_{\mathbb{C}}}(W)$ to which it is naturally mapped. For our purposes it will be essential to stick to the definition (1.2) and to use some results from [Ciubotaru et al. 2014]. Therefore we want to prove that (1.2) is always a torsion-free group.

In the analysis we will make ample use of Springer's construction of representations of Weyl groups, and of Reeder's results [2001]. Let $G$ be a connected reductive complex group with a maximal torus $T$ such that $R \cong R(G, T)$ and $W \cong W(G, T)$. For $u \in G$ let $\mathcal{B}^u = \mathcal{B}^u_G$ be the complex variety of Borel subgroups of $G$ containing $u$. The group $Z_G(u)$ acts on $\mathcal{B}^u$ by conjugation, and that induces an action of $A_G(u) := \pi_0(Z_G(u)/Z(G))$ on the cohomology of $\mathcal{B}^u$. For a pair $(u, \rho)$ with $u \in G$ unipotent and $\rho \in \mathrm{Irr}(A_G(u))$ we define

$$
\begin{aligned}
H(u,\rho) &= \mathrm{Hom}_{A_G(u)}(\rho, H^*(\mathcal{B}^u;\mathbb{C})),\\
\pi(u,\rho) &= \mathrm{Hom}_{A_G(u)}(\rho, H^{\mathrm{top}}(\mathcal{B}^u;\mathbb{C})),
\end{aligned}
\tag{1.3}
$$

where top indicates the highest dimension in which the cohomology is nonzero, namely the dimension of $\mathcal{B}^u$ as a real variety. Let us call $\rho$ geometric if $\pi(u,\rho)\neq 0$. Springer [1978] proved that

- $W \times A_G(u)$ acts naturally on $H^i(\mathcal{B}^u;\mathbb{C})$ for each $i \in \mathbb{Z}_{\geq 0}$,
- $\pi(u,\rho)$ is an irreducible $W$-representation whenever it is nonzero,
- this gives a bijection between $\mathrm{Irr}(W)$ and the $G$-conjugacy classes of pairs $(u,\rho)$ with $u\in G$ unipotent and $\rho \in \mathrm{Irr}(A_G(u))$ geometric.

It follows from a result of Borho and MacPherson [1981] that the $W$-representations $H(u,\rho)$, parametrized by the same data $(u,\rho)$, also form a basis of $R_{\mathbb{Z}}(W)$; see [Reeder 2001, Lemma 3.3.1]. Moreover, $\pi(u,\rho)$ appears with multiplicity one in $H(u,\rho)$.

**Example 1.4.**

- Type $A$. Only the $n$-cycles in $W=S_n$ are elliptic, and they form one conjugacy class. The only quasidistinguished unipotent class in $\mathrm{GL}_n(\mathbb{C})$ is the regular unipotent class. Then $A_{\mathrm{GL}_n(\mathbb{C})}(u_{\mathrm{reg}})=1$ for every regular unipotent element $u_{\mathrm{reg}}$ and $H(u_{\mathrm{reg}},\mathrm{triv}) = H^0(\mathcal{B}^{u_{\mathrm{reg}}};\mathbb{C})$ is the sign representation of $S_n$ (with our convention for the Springer correspondence).
- Types $B$ and $C$. The elliptic classes in $W(B_n)=W(C_n)\cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ are parametrized by partitions of $n$. We will write them down explicitly as $\sigma(\varnothing,\lambda)$ with $\lambda \vdash n$ in (3.26).
- Type $D$. The elliptic classes in $W(D_n)=S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n_{\mathrm{ev}}$ are precisely the elliptic classes of $W(B_n)$ that are contained in $W(D_n)$. They can be parametrized by partitions $\lambda \vdash n$ such that $\lambda$ has an even number of terms.
- Type $G_2$. There are three elliptic classes in $W(G_2)=D_6$: the rotations of order two, of order three and of order six. The quasidistinguished unipotent classes in $G_2(\mathbb{C})$ are the regular and the subregular class.

  We have $A_G(u_{\mathrm{reg}})=1$ and $H(u_{\mathrm{reg}},\mathrm{triv})=\pi(u_{\mathrm{reg}},\mathrm{triv})$ is the sign representation of $D_6$. For $u$ subregular $A_G(u)\cong S_3$, and the sign representation of $A_G(u)$ is not geometric. For $\rho$ the two-dimensional irreducible representation of $A_G(u)$, $\pi(u,\rho)=H(u,\rho)$ is the character of $W(G_2)$ which is 1 on the reflections for long roots and $-1$ on the reflections for short roots. Furthermore $\pi(u,\mathrm{triv})$ is the standard two-dimensional representation of $D_6$ and $H(u,\mathrm{triv})$ is the direct sum of $\pi(u,\mathrm{triv})$ and the sign representation.

For a subset $P\subset\Delta$ let $G_P$ be the standard Levi subgroup of $G$ generated by $T$ and the root subgroups for roots $\alpha\in R_P$. The irreducible representations of $W_P=W(G_P,T)$ are parametrized by $G_P$-conjugacy classes of pairs $(u_P,\rho_P)$ with

$u_P \in G_P$ unipotent and $\rho_P \in \mathrm{Irr}(A_{G_P}(u_P))$ geometric, and the $W_P$-representations $H_P(u_P, \rho_P)$ form another basis of $R_{\mathbb{Z}}(W_P)$.

Recall from [Reeder 2001, §3.2] that $A_{G_P}(u_P)$ can be regarded as a subgroup of $A_G(u_P)$. By [Kato 1983, Proposition 2.5 and 6.2],

$$\mathrm{ind}_{W_P}^{W}(H^i(\mathcal{B}_{G_P}^{u_P}; \mathbb{C})) \cong H^i(\mathcal{B}^{u_P}; \mathbb{C}) \quad \text{as } W \times A_G(u_P)\text{-representations.} \tag{1.5}$$

It follows that for any $(u_P, \rho_P)$ as above there are natural isomorphisms

$$\begin{aligned} \mathrm{ind}_{W_P}^{W}(H_P(u_P, \rho_P)) &\cong \mathrm{Hom}_{A_{G_P}(u_P)}(\rho_P, H^*(\mathcal{B}^{u_P}; \mathbb{C})) \\ &\cong \bigoplus_{\rho \in \mathrm{Irr}(A_G(u_P))} \mathrm{Hom}_{A_{G_P}(u_P)}(\rho_P, \rho) \otimes H(u_P, \rho). \end{aligned} \tag{1.6}$$

For a unipotent conjugacy class $\mathcal{C} \subset G$ and $P \subset \Delta$, let $R_{\mathbb{Z}}(W_P, \mathcal{C})$ be the subgroup of $R_{\mathbb{Z}}(W_P)$ generated by the $H_P(u_P, \rho_P)$ with $u_P \in G_P \cap \mathcal{C}$ and $\rho_P \in \mathrm{Irr}(A_{G_P}(u_P))$. (Notice that $G_P \cap \mathcal{C}$ can consist of zero, one or more conjugacy classes.) In view of (1.6) we can define

$$\overline{R_{\mathbb{Z}}}(W, \mathcal{C}) = R_{\mathbb{Z}}(W, \mathcal{C}) \Big/ \sum_{P \subsetneq \Delta} \mathrm{ind}_{W_P}^{W}(R_{\mathbb{Z}}(W_P, \mathcal{C})).$$

We obtain a decomposition as in [Reeder 2001, §3.3]:

$$\overline{R_{\mathbb{Z}}}(W) = \bigoplus_{\mathcal{C}} \overline{R_{\mathbb{Z}}}(W, \mathcal{C}). \tag{1.7}$$

Following [Reeder 2001], we also define elliptic representation theories for the component groups $A_G(u)$. For $u, u_P \in \mathcal{C}$ the groups $A_G(u)$ and $A_G(u_P)$ are isomorphic. In general the isomorphism is not natural, but it is canonical up to inner automorphisms. This gives a natural isomorphism $R_{\mathbb{Z}}(A_G(u)) \cong R_{\mathbb{Z}}(A_G(u_P))$, which enables us to write

$$\overline{R_{\mathbb{Z}}}(A_G(u)) = R_{\mathbb{Z}}(A_G(u)) \Big/ \sum_{P \subsetneq \Delta, u_P \in \mathcal{C} \cap G_P} \mathrm{ind}_{A_{G_P}(u_P)}^{A_G(u_P)}\big(R_{\mathbb{Z}}(A_{G_P}(u_P))\big). \tag{1.8}$$

For any $u_P, u'_P \in \mathcal{C} \cap G_P$ there is a natural isomorphism

$$\mathrm{ind}_{A_{G_P}(u_P)}^{A_G(u_P)}\big(R_{\mathbb{Z}}(A_{G_P}(u_P))\big) \cong \mathrm{ind}_{A_{G_P}(u'_P)}^{A_G(u'_P)}\big(R_{\mathbb{Z}}(A_{G_P}(u'_P))\big),$$

so on the right-hand side of (1.8) it actually suffices to use only one $u_P$ whenever $\mathcal{C} \cap G_P$ is nonempty.

Let $R_{\mathbb{Z}}^{\circ}(A_G(u))$ be the subgroup of $R_{\mathbb{Z}}(A_G(u))$ generated by the geometric irreducible $A_G(u)$-representations. By [Reeder 2001, §10],

$$\mathrm{ind}_{A_G(u_P)}^{A_G(u)}\big(R_{\mathbb{Z}}^{\circ}(A_{G_P}(u_P))\big) \subset R_{\mathbb{Z}}^{\circ}(A_G(u)).$$

Using this we can define

$$\overline{R^\circ_{\mathbb{Z}}}(A_G(u)) = R^\circ_{\mathbb{Z}}(A_G(u)) \Big/ \sum_{P \subsetneq \Delta,\, u_P \in \mathcal{C} \cap G_P} \mathrm{ind}_{A_{G_P}(u_P)}^{A_G(u_P)} \big( R^\circ_{\mathbb{Z}}(A_{G_P}(u_P)) \big).$$

It follows from (1.6) that every $\rho_P \in \mathrm{Irr}(A_{G_P}(u_P))$ which appears in $\rho$ is itself geometric. Hence the inclusions $R^\circ_{\mathbb{Z}}(A_{G_P}(u_P)) \to R_{\mathbb{Z}}(A_{G_P}(u_P))$ induce an injection

$$\overline{R^\circ_{\mathbb{Z}}}(A_G(u)) \to \overline{R_{\mathbb{Z}}}(A_G(u)). \tag{1.9}$$

By [Reeder 2001, Proposition 3.4.1] the maps $\rho_P \mapsto \mathrm{Hom}_{A_{G_P}(u_P)}(\rho_P, H^*(\mathcal{B}^{u_P}; \mathbb{C}))$ for $P \subset \Delta$ induce a $\mathbb{Z}$-linear bijection

$$\overline{R^\circ_{\mathbb{Z}}}(A_G(u)) \to \overline{R_{\mathbb{Z}}}(W, \mathcal{C}). \tag{1.10}$$

(In [Reeder 2001] these groups are by definition subsets of complex vector spaces. But with the above definitions Reeder's proof still applies.) From (1.7), (1.10) and (1.9) we obtain an injection

$$\overline{R_{\mathbb{Z}}}(W) \to \bigoplus_u \overline{R_{\mathbb{Z}}}(A_G(u)), \tag{1.11}$$

where $u$ runs over a set of representatives for the unipotent classes of $G$.

**Theorem 1.12.** *The group of elliptic representations $\overline{R_{\mathbb{Z}}}(W)$ is torsion-free.*

*Proof.* If $W$ is a product of irreducible Weyl groups $W_i$, then it follows readily from (1.2) that

$$\overline{R_{\mathbb{Z}}}(W) = \bigotimes_i \overline{R_{\mathbb{Z}}}(W_i).$$

Hence we may assume that $W = W(R)$ is irreducible. By (1.11) it suffices to show that each $\overline{R_{\mathbb{Z}}}(A_G(u))$ is torsion-free. If $u$ is distinguished, then $\mathcal{C} \cap G_P = \varnothing$ for all $P \subsetneq \Delta$, and $\overline{R_{\mathbb{Z}}}(A_G(u)) = R_{\mathbb{Z}}(A_G(u))$. That is certainly torsion-free, so we do not have to consider distinguished unipotent $u$ anymore.

For root systems of type $A$ and of exceptional type, the tables of component groups in [Carter 1985, §13.1] show that $A_G(u)$ is isomorphic to $S_n$ with $n \leq 5$. Moreover, $S_4$ and $S_5$ only occur when $u$ is distinguished. For $A_G(u) \cong S_2$ and for $A_G(u) \cong S_3$ one checks directly that $\overline{R_{\mathbb{Z}}}(A_G(u))$ is torsion-free, by listing all subgroups of $A_G(u)$ and all irreducible representations thereof.

That leaves the root systems of type $B$, $C$ and $D$. As group of type $B_n$ we take $G = \mathrm{SO}_{2n+1}(\mathbb{C})$. By the Bala–Carter classification, the unipotent classes $\mathcal{C}$ in $G$ are parametrized by pairs of partitions $(\alpha, \beta)$ such that $2|\alpha| + |\beta| = 2n+1$ and $\beta$ has only odd parts, all distinct. A typical $u \in \mathcal{C}$ is distinguished in the standard Levi subgroup

$$G_\alpha := \mathrm{GL}_{\alpha_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{\alpha_d}(\mathbb{C}) \times \mathrm{SO}_{|\beta|}(\mathbb{C}).$$

The part of $u$ in $\mathrm{SO}_{|\beta|}$ depends only on $\beta$; it has Jordan blocks of sizes $\beta_1, \beta_2, \dots$. Let $\alpha'$ be a partition consisting of a subset of the terms of $\alpha$, say

$$\alpha' = (n)^{m'_n} (n-1)^{m'_{n-1}} \cdots (1)^{m'_1}. \tag{1.13}$$

Let $\alpha''$ be a partition of $|\alpha| - |\alpha'|$ obtained from the remaining terms of $\alpha$ by repeatedly replacing some $\alpha_i, \alpha_j$ by $\alpha_i + \alpha_j$. All the standard Levi subgroups of $G$ containing this $u$ are of the form $G_{\alpha''}$. The GL-factors of $G_{\alpha''}$ do not contribute to $A_{G_{\alpha''}}(u)$. The part $u'$ of $u$ in $\mathrm{SO}_{2(n-|\alpha''|)+1}(\mathbb{C})$ is parametrized by $(\alpha', \beta)$ and the quotient of $Z_{\mathrm{SO}_{2(n-|\alpha''|)+1}(\mathbb{C})}(u')$ by its unipotent radical is isomorphic to

$$\prod_{i \text{ even}} \mathrm{Sp}_{2m'_i}(\mathbb{C}) \times \prod_{i \text{ odd, not in } \beta} O_{2m'_i}(\mathbb{C}) \times S\Big( \prod_{i \text{ odd, in } \beta} O_{2m'_i+1}(\mathbb{C}) \Big), \tag{1.14}$$

where the $S$ indicates that we take the subgroup of elements of determinant 1. From this one can deduce the component group:

$$A_{G_{\alpha''}}(u) \cong A_{\mathrm{Sp}_{2(n-|\alpha''|)}(\mathbb{C})}(u') \cong \prod_{i \text{ odd, not in } \beta,\, m'_i > 0} \mathbb{Z}/2\mathbb{Z} \times S\Big( \prod_{i \text{ in } \beta} \mathbb{Z}/2\mathbb{Z} \Big). \tag{1.15}$$

We see that if

- $\alpha$ has an even term,
- or $\alpha$ has an odd term with multiplicity $> 1$,
- or $\alpha$ has an odd term which also appears in $\beta$,

then there is a standard Levi subgroup $G_{\alpha''} \subsetneq G$ with $A_{G_{\alpha''}}(u) \cong A_G(u)$, namely with $\alpha''$ just that one term of $\alpha$. In these cases $\overline{R_{\mathbb{Z}}}(A_G(u)) = 0$.

Suppose now that $\alpha$ has only distinct odd terms, and that none of those appears in $\beta$. Then (1.15) becomes

$$A_G(u) \cong \prod_{i \text{ in } \alpha} \mathbb{Z}/2\mathbb{Z} \times A \quad \text{where } A = S\Big( \prod_{i \text{ in } \beta} \mathbb{Z}/2\mathbb{Z} \Big).$$

We get

$$\begin{aligned} &\sum_{P \subsetneq \Delta,\, u_P \in \mathcal{C} \cap G_P} \mathrm{ind}_{A_{G_P}(u_P)}^{A_G(u_P)} \big( R_{\mathbb{Z}}(A_{G_P}(u_P)) \big) \\ &\qquad \cong \sum_{j \text{ in } \alpha} \mathrm{ind}_{A_{G_{\alpha-(j)}}(u)}^{A_G(u)} R_{\mathbb{Z}} \Big( \prod_{i \text{ in } \alpha-(j)} \mathbb{Z}/2\mathbb{Z} \times A \Big) \\ &\qquad \cong \sum_{j \text{ in } \alpha} \mathrm{ind}_{\{1\}}^{\mathbb{Z}/2\mathbb{Z}} R_{\mathbb{Z}}(\{1\}) \otimes_{\mathbb{Z}} R_{\mathbb{Z}} \Big( \prod_{i \text{ in } \alpha-(j)} \mathbb{Z}/2\mathbb{Z} \Big) \otimes_{\mathbb{Z}} R_{\mathbb{Z}}(A). \end{aligned} \tag{1.16}$$

We conclude that $\overline{R_{\mathbb{Z}}}(A_G(u)) = R_{\mathbb{Z}}(A)$.

So $\overline{R_{\mathbb{Z}}}(A_G(u))$ is torsion free for all unipotent $u \in \mathrm{SO}_{2n+1}(\mathbb{C})$, which settles the case $B_n$. The root systems of types $C_n$ and $D_n$ can be handled in a completely analogous way, using the explicit descriptions in [Carter 1985, §13.1]. $\square$

For every $w \in W$ there exists (more or less by definition) a unique parabolic subgroup $\widetilde{W} \subset W$ such that $w$ is an elliptic element of $\widetilde{W}$. Let $\mathcal{C}(W)$ be the set of conjugacy classes of $W$. For $P \subset \Delta$ let $\mathcal{C}_P(W)$ be the subset consisting of those conjugacy classes that contain an elliptic element of $W_P$. Let $\mathcal{P}(\Delta)/W$ be a set of representatives for the $W$-association classes of subsets of $\Delta$. Since every parabolic subgroup is conjugate to a standard one, for every conjugacy class $\mathcal{C}$ in $W$ there exists a unique $P \in \mathcal{P}(\Delta)/W$ such that $\mathcal{C} \in \mathcal{C}_P(W)$.

Recall from [Reeder 2001, §3.3] that a unipotent element $u \in G$ is called quasidistinguished if there exists a semisimple $t \in Z_G(u)$ such that $tu$ is not contained in any proper Levi subgroup of $G$.

**Proposition 1.17.** *For every $P \in \mathcal{P}(\Delta)/W$ there exists an injection from $\mathcal{C}_P(W)$ to the set of $G_P$-conjugacy classes of pairs $(u_P, \rho_P)$ with $u_P \in G_P$ quasidistinguished unipotent and $\rho_P \in \mathrm{Irr}(A_{G_P}(u_P))$ geometric, denoted $w \mapsto (u_{P,w}, \rho_{P,w})$, such that*:

(a) $\{H(u_w, \rho_w) : w \in \mathcal{C}_\Delta(W)\}$ *is a $\mathbb{Z}$-basis of $\overline{R_{\mathbb{Z}}}(W)$.*

(b) *The set*

$$\left\{\mathrm{ind}_{W_P}^{W}(H_P(u_{P,w}, \rho_{P,w})) : P \in \mathcal{P}(\Delta)/W,\, w \in \mathcal{C}_P(W)\right\}$$

*is a $\mathbb{Z}$-basis of $R_{\mathbb{Z}}(W)$.*

*Proof.* (a) By [Reeder 2001, Proposition 2.2.2] the rank of $\overline{R_{\mathbb{Z}}}(W)$ is the number of elliptic conjugacy classes of $W$. With Theorem 1.12 we find $\overline{R_{\mathbb{Z}}}(W) \cong \mathbb{Z}^{|\mathcal{C}_\Delta(W)|}$. By (1.11) and (1.10), $\overline{R_{\mathbb{Z}}}(W)$ has a basis consisting of representations of the form $H(u, \rho)$ with $\rho \in \mathrm{Irr}(A_G(u))$ geometric. By [Reeder 2001, Proposition 3.4.1] we need only quasidistinguished unipotent $u$. We choose such a set of pairs $(u, \rho)$, and we parametrize it in an arbitrary way by $\mathcal{C}_\Delta(W)$.

(b) We prove this by induction on $|\Delta|$. For $|\Delta| = 0$ the statement is trivial.

Suppose now that $|\Delta| \geq 1$ and $\alpha \in \Delta$. By the induction hypothesis we can find maps $w \mapsto (u_P, \rho_P)$ such that the set

$$\left\{\mathrm{ind}_{W_P}^{W_{\Delta\setminus\{\alpha\}}}(H_P(u_{P,w}, \rho_{P,w})) : P \in \mathcal{P}(\Delta \setminus \{\alpha\})/W_{\Delta\setminus\{\alpha\}},\, w \in \mathcal{C}_P(W_{\Delta\setminus\{\alpha\}})\right\}$$

is a $\mathbb{Z}$-basis of $R_{\mathbb{Z}}(W_{\Delta\setminus\{\alpha\}})$. By means of any setwise splitting of $N_G(T) \to W$ we can arrange that $(u_{P,w}, \rho_{P,w})$ and $(u_{P',w'}, \rho_{P',w'})$ are $G$-conjugate whenever $(P, w)$ and $(P', w')$ are $W$-associate. Then $(P, w)$ and $(P', w')$ give rise to the same $W$-representation. Consequently,

$$\left\{\mathrm{ind}_{W_P}^{W}(H_P(u_{P,w}, \rho_{P,w})) : P \in \mathcal{P}(\Delta)/W,\, P \neq \Delta,\, w \in \mathcal{C}_P(W)\right\}$$

is well-defined and has $|\mathcal{C}(W) \setminus \mathcal{C}_\Delta(W)|$ elements. By the induction hypothesis it spans $\sum_{P \subsetneq \Delta} \mathrm{ind}_{W_P}^W (R_{\mathbb{Z}}(W_P))$, so it forms a $\mathbb{Z}$-basis thereof. Combine this with (1.2) and part (a). $\square$

**1B. *Graded Hecke algebras.*** We consider the Grothendieck group $R_{\mathbb{Z}}(\mathbb{H})$ of finite length modules of a graded Hecke algebra $\mathbb{H}$ with parameters $k$. We show that it is the direct sum of the subgroup spanned by modules induced from proper parabolic subalgebras and an elliptic part $\overline{R_{\mathbb{Z}}}(\mathbb{H})$. We prove that $\overline{R_{\mathbb{Z}}}(\mathbb{H})$ is isomorphic to the elliptic part of the representation ring of the Weyl group associated to $\mathbb{H}$. By Section 1A, $\overline{R_{\mathbb{Z}}}(\mathbb{H})$ is free abelian and does not depend on the parameters $k$. The main ingredients are the author's work [Solleveld 2010] on the periodic cyclic homology of graded Hecke algebras, and the study of discrete series representations by Ciubotaru, Opdam and Trapa [Ciubotaru and Opdam 2017; Ciubotaru et al. 2014].

Graded Hecke algebras are also known as degenerate (affine) Hecke algebras. They were introduced in [Lusztig 1989]. In the notation from (1.1) we call

$$\widetilde{\mathcal{R}} = (\mathfrak{a}^*, R, \mathfrak{a}, R^\vee, \Delta) \tag{1.18}$$

a degenerate root datum. We pick complex numbers $k_\alpha$ for $\alpha \in \Delta$, such that $k_\alpha = k_\beta$ if $\alpha$ and $\beta$ are in the same $W$-orbit. We put $\mathfrak{t} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{a}$.

The graded Hecke algebra associated to these data is the complex vector space

$$\mathbb{H} = \mathbb{H}(\widetilde{\mathcal{R}}, k) = \mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[W],$$

with multiplication defined by the following rules:

- $\mathbb{C}[W]$ and $\mathcal{O}(\mathfrak{t})$ are canonically embedded as subalgebras;
- for $\xi \in \mathfrak{t}^*$ and $s_\alpha \in S_\Delta$ we have the cross relation

$$\xi \cdot s_\alpha - s_\alpha \cdot s_\alpha(\xi) = k_\alpha \langle \xi, \alpha^\vee \rangle. \tag{1.19}$$

Notice that $\mathbb{H}(\widetilde{\mathcal{R}}, 0) = \mathcal{O}(\mathfrak{t}) \rtimes W$.

Multiplication with any $\epsilon \in \mathbb{C}^\times$ defines a bijection $\mathfrak{t}^* \to \mathfrak{t}^*$, which clearly extends to an algebra automorphism of $\mathcal{O}(\mathfrak{t}) = S(\mathfrak{t}^*)$. From the cross relation (1.19) we see that it extends even further, to an algebra isomorphism

$$\mathbb{H}(\widetilde{\mathcal{R}}, \epsilon k) \to \mathbb{H}(\widetilde{\mathcal{R}}, k) \tag{1.20}$$

which is the identity on $\mathbb{C}[W]$. For $\epsilon = 0$ this map is well-defined, but obviously not bijective.

For a set of simple roots $P \subset \Delta$ we write

$$
\begin{aligned}
R_P &= \mathbb{Q}P \cap R, & R_P^\vee &= \mathbb{Q}R_P^\vee \cap R^\vee,\\
\mathfrak{a}_P &= \mathbb{R}P^\vee, & \mathfrak{a}^P &= (\mathfrak{a}_P^*)^\perp,\\
\mathfrak{a}_P^* &= \mathbb{R}P, & \mathfrak{a}^{P*} &= (\mathfrak{a}_P)^\perp,\\
\widetilde{\mathcal{R}}_P &= (\mathfrak{a}_P^*, R_P, \mathfrak{a}_P, R_P^\vee, P), & \widetilde{\mathcal{R}}^P &= (\mathfrak{a}^*, R_P, \mathfrak{a}, R_P^\vee, P).
\end{aligned}
\tag{1.21}
$$

Let $k_P$ be the restriction of $k$ to $R_P$. We call

$$\mathbb{H}^P = \mathbb{H}(\widetilde{\mathcal{R}}^P, k_P)$$

a parabolic subalgebra of $\mathbb{H}$. It contains $\mathbb{H}_P = \mathbb{H}(\widetilde{\mathcal{R}}_P, k_P)$ as a direct summand.

The centre of $\mathbb{H}(\widetilde{\mathcal{R}}, k)$ is $\mathcal{O}(\mathfrak{t})^W = \mathcal{O}(\mathfrak{t}/W)$ [Lusztig 1989, Proposition 4.5]. Hence the central character of an irreducible $\mathbb{H}(\widetilde{\mathcal{R}}, k)$-representation is an element of $\mathfrak{t}/W$.

Let $(\pi, V)$ be an $\mathbb{H}(\widetilde{\mathcal{R}}, k)$-representation. We say that $\lambda \in \mathfrak{t}$ is an $\mathcal{O}(\mathfrak{t})$-weight of $V$ (or of $\pi$) if

$$\{v \in V : \pi(\xi)v = \lambda(\xi)v \text{ for all } \xi \in \mathfrak{t}^*\}$$

is nonzero. Let $\mathrm{Wt}(V) \subset \mathfrak{t}$ be the set of $\mathcal{O}(t)$-weights of $V$.

Temperedness of a representation is defined via its $\mathcal{O}(\mathfrak{t})$-weights. We write

$$
\begin{aligned}
\mathfrak{a}^+ &= \{\mu \in \mathfrak{a} : \langle \alpha, \mu \rangle \geq 0 \ \forall \alpha \in \Delta\},\\
\mathfrak{a}^{*+} &:= \{x \in \mathfrak{a}^* : \langle x, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in \Delta\},\\
\mathfrak{a}^- &= \{\lambda \in \mathfrak{a} : \langle x, \lambda \rangle \leq 0 \ \forall x \in \mathfrak{a}^{*+}\} = \left\{\textstyle\sum_{\alpha \in \Delta} \lambda_\alpha \alpha^\vee : \lambda_\alpha \leq 0\right\}.
\end{aligned}
$$

The interior $\mathfrak{a}^{--}$ of $\mathfrak{a}^-$ equals $\left\{\sum_{\alpha \in \Delta} \lambda_\alpha \alpha^\vee : \lambda_\alpha < 0\right\}$ if $\Delta$ spans $\mathfrak{a}^*$, and is empty otherwise.

We regard $\mathfrak{t} = \mathfrak{a} \oplus i\mathfrak{a}$ as the polar decomposition of $\mathfrak{t}$, with associated real part map $\Re : \mathfrak{t} \to \mathfrak{a}$. By definition, a finite-dimensional $\mathbb{H}(\widetilde{\mathcal{R}}, k)$-module $(\pi, V)$ is tempered $\Re(\mathrm{Wt}(V)) \subset \mathfrak{a}^-$. More restrictively, we say that $(\pi, V)$ belongs to the discrete series if $\Re(\mathrm{Wt}(V)) \subset \mathfrak{a}^{--}$.

We are interested in the restriction map

$$\mathrm{r} : \mathrm{Mod}(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to \mathrm{Mod}(\mathbb{C}[W]), \quad V \mapsto V|_W.$$

We can also regard it as the composition of representations with the algebra homomorphism (1.20) for $\epsilon = 0$, then its image consists of $\mathcal{O}(\mathfrak{t}) \rtimes W$-representations on which $\mathcal{O}(\mathfrak{t})$ acts via $0 \in \mathfrak{t}$.

Let $\mathrm{Irr}_0(\mathbb{H})$ be the set of irreducible tempered $\mathbb{H}(\widetilde{\mathcal{R}}, k)$-modules with central character in $\mathfrak{a}/W$. It is known from [Solleveld 2010, Theorem 6.5] that, for real-valued $k$, r induces a bijection

$$\mathrm{r}_\mathbb{C} : \mathbb{C}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to R_\mathbb{C}(W). \tag{1.22}$$

Using work of Lusztig, Ciubotaru [2008, Corollary 3.6] showed that, for parameters of "geometric" type,

$$\mathrm{r}_{\mathbb{Z}} : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to R_{\mathbb{Z}}(W) \text{ is bijective.} \tag{1.23}$$

We will generalize this to arbitrary real parameters. (Parameters $k$ of geometric type need not be real-valued, but via (1.20) they can be reduced to that.)

We recall some notions from [Ciubotaru and Opdam 2015]. Let $R_{\mathbb{Z}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ be the Grothendieck group of (the category of) finite-dimensional $\mathbb{H}(\widetilde{\mathcal{R}}, k)$-modules. For any parabolic subalgebra $\mathbb{H}^P = \mathbb{H}(\widetilde{\mathcal{R}}^P, k_P)$, the induction functor $\mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}$ induces a map $R_{\mathbb{Z}}(\mathbb{H}^P) \to R_{\mathbb{Z}}(\mathbb{H})$. If the $\mathcal{O}(\mathfrak{t})$-weights of $V \in \mathrm{Mod}(\mathbb{H}^P)$ are contained in some $U \subset \mathfrak{t}$, then by [Barbasch and Moy 1993, Theorem 6.4], the $\mathcal{O}(\mathfrak{t})$-weights of $\mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}} V$ are contained in $W^P U$, where $W^P$ is the set of shortest length representatives of $W/W_P$. This implies that $\mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}$ preserves temperedness [Barbasch and Moy 1993, Corollary 6.5] and central characters. In particular, it induces a map

$$\mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}} : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}^P) \to \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}). \tag{1.24}$$

Many arguments in this section make use of the group of "elliptic $\mathbb{H}$-representations"

$$\overline{R_{\mathbb{Z}}}(\mathbb{H}) = R_{\mathbb{Z}}(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \Big/ \textstyle\sum_{P \subsetneq \Delta} \mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}(R_{\mathbb{Z}}(\mathbb{H}^P)). \tag{1.25}$$

Since $\mathbb{H}(\widetilde{\mathcal{R}}, k) = \mathcal{O}(\mathfrak{t}) \otimes \mathbb{C}[W]$ as vector spaces,

$$\mathrm{r} \circ \mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}} = \mathrm{ind}_{W_P}^{W} \circ \mathrm{r}^P, \tag{1.26}$$

where $\mathrm{r}^P$ denotes the analogue of r for $\mathbb{H}^P$. Hence r induces a $\mathbb{Z}$-linear map

$$\bar{\mathrm{r}} : \overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to \overline{R_{\mathbb{Z}}}(W). \tag{1.27}$$

**Proposition 1.28.** *The map (1.27) is surjective, and its kernel is the torsion subgroup of $\overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$.*

*Proof.* By Theorem 1.12, $\overline{R_{\mathbb{Z}}}(W)$ is torsion-free, so it can be identified with its image in $\overline{R_{\mathbb{C}}}(W)$. This means that our definition of $\overline{R_{\mathbb{Z}}}(W)$ agrees with that in [Ciubotaru et al. 2014]. Likewise, in [Ciubotaru et al. 2014] the subgroup $\overline{R'_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ of $\overline{R_{\mathbb{C}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ generated by the actual representations is considered. In other words, $\overline{R'_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ is defined as the quotient of $\overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ by its torsion subgroup.

By [Ciubotaru et al. 2014, Proposition 5.6] the map

$$\bar{\mathrm{r}} : \overline{R'_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to \overline{R_{\mathbb{Z}}}(W) \tag{1.29}$$

is bijective, except possibly when $R$ has type $F_4$ and $k$ is not a generic parameter. However, in view of the more recent work [Ciubotaru and Opdam 2017, §3.2], the limit argument given (for types $B_n$ and $G_2$) in [Ciubotaru et al. 2014, §5.1] also applies to $F_4$. Thus (1.29) is bijective for all $\widetilde{\mathcal{R}}$ and all real-valued parameters $k$. □

**Lemma 1.30.** *Let $k$ be real-valued. The canonical map*

$$\mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k)) \to \overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$$

*is surjective.*

*Proof.* It was noted in [Opdam and Solleveld 2013, Lemma 6.3] (in the context of affine Hecke algebras) that every element of $\overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$ can be represented by a tempered virtual representation. Consider any irreducible tempered $\mathbb{H}$-representation $\pi$. By [Solleveld 2012b, Proposition 8.2] there exists a $P \subset \Delta$, a discrete series representation $\delta$ of $\mathbb{H}_P$ and an element $\nu \in i\mathfrak{a}^P$ such that $\pi$ is a direct summand of

$$\pi(P, \delta, \nu) = \mathrm{ind}_{\mathbb{H}_P \otimes \mathcal{O}(\mathfrak{t}^P)}^{\mathbb{H}} (\delta \otimes \mathbb{C}_\nu).$$

By [Solleveld 2012b, Proposition 8.3] the reducibility of $\pi(P, \delta, \nu)$ is determined by intertwining operators $\pi(w, P, \delta, \nu)$ for elements $w \in W$ that stabilize $(P, \delta, \nu)$. Suppose that $\nu \neq 0$. Then $W_\nu$ is a proper parabolic subgroup of $W$, so the stabilizer of $(P, \delta, \nu)$ is contained in $W_Q$ for some $P \subset Q \subsetneq \Delta$. In that case, $\pi = \mathrm{ind}_{\mathbb{H}^Q}^{\mathbb{H}}(\pi^Q)$ for some irreducible representation $\pi^Q$ of $\mathbb{H}^Q$, so $\pi$ becomes zero in $\overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$.

Therefore we need only $\mathbb{Z}$-linear combinations of summands of $\pi(\delta, P, 0)$ (with varying $P, \delta$) to surject to $\overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$. Since $k$ is real, discrete series representations of $\mathbb{H}_P$ have central characters in $\mathfrak{a}_P / W_P$ [Slooten 2006, Lemma 2.13]. It follows that $\pi(P, \delta, 0)$ and all its constituents (among which is $\pi$) admit a central character in $\mathfrak{a}/W$. □

**Theorem 1.31.** *Let $k$ be real-valued. The restriction-to-$W$ maps*

$$\begin{array}{rcccl} \mathrm{r}_{\mathbb{Z}} & : & \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k)) & \to & R_{\mathbb{Z}}(W), \\ \bar{\mathrm{r}} & : & \overline{R_{\mathbb{Z}}}(\mathbb{H}(\widetilde{\mathcal{R}}, k)) & \to & \overline{R_{\mathbb{Z}}}(W) \end{array}$$

*are bijective.*

*Proof.* We show this by induction on the semisimple rank of $\widetilde{\mathcal{R}}$ (i.e., the rank of $R$). Suppose first that the semisimple rank is zero. Then $W = 1$ and $\mathbb{H} = \mathcal{O}(\mathfrak{t})$. For $\lambda \in \mathfrak{t}$ the character

$$\mathrm{ev}_\lambda : f \mapsto f(\lambda)$$

is a tempered $\mathcal{O}(\mathfrak{t})$-representation if and only if $\Re(\lambda) = 0$. If $\lambda$ is at the same time a real central character (i.e., $\lambda \in \mathfrak{a}$), then $\lambda = 0$. Hence $\mathrm{Irr}_0(\mathbb{H})$ consists just of $\mathrm{ev}_0$. It is mapped to the trivial $W$-representation by r, so the theorem holds in this case.

Now let $\widetilde{\mathcal{R}}$ be of positive semisimple rank. It is a direct sum of degenerate root data with $R$ irreducible or $R$ empty, and $\mathbb{H}(\widetilde{\mathcal{R}}, k)$ decomposes accordingly. As we already know the result when $R$ is empty, it remains to establish the case where $R$ is irreducible.

Any proper parabolic subalgebra $\mathbb{H}^P \subset \mathbb{H}$ has smaller semisimple rank, so by the induction hypothesis

$$\mathrm{r}^P : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}^P) \to \mathbb{Z}\,\mathrm{Irr}_0(W_P) \text{ is bijective.} \tag{1.32}$$

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sum_{P \subsetneq \Delta} \mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}(\mathbb{Z}\ \mathrm{Irr}_0(\mathbb{H}^P)) & \longrightarrow & \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}) & \longrightarrow & \overline{R_{\mathbb{Z}}}(\mathbb{H}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \sum_{P \subsetneq \Delta} \mathrm{ind}_{W_P}^{W}(R_{\mathbb{Z}}(W_P)) & \longrightarrow & R_{\mathbb{Z}}(W) & \longrightarrow & \overline{R_{\mathbb{Z}}}(W) & \longrightarrow & 0 \end{array} \tag{1.33}$$

The second row is exact by definition. By (1.32) and (1.26) the left vertical arrow is bijective and by Proposition 1.28 the right vertical arrow is surjective. Together with Lemma 1.30 these imply that the middle vertical arrow is surjective. By (1.22) both $\mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H})$ and $R_{\mathbb{Z}}(W)$ are free abelian groups of the same rank $|\mathrm{Irr}(W)| = |\mathrm{Irr}_0(\mathbb{H})|$, so the middle vertical arrow is in fact bijective.

The results so far imply that the kernel of $\mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}) \to \overline{R_{\mathbb{Z}}}(W)$ is precisely

$$\textstyle\sum_{P \subsetneq \Delta} \mathrm{ind}_{\mathbb{H}^P}^{\mathbb{H}}(\mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}^P)).$$

The latter group is already killed in $\overline{R_{\mathbb{Z}}}(\mathbb{H})$, so $\overline{R_{\mathbb{Z}}}(\mathbb{H}) \to \overline{R_{\mathbb{Z}}}(W)$ is injective as well. We conclude that (1.33) is a bijection between two short exact sequences. $\square$

We will need Theorem 1.31 for somewhat more general algebras. Let $\Gamma$ be a finite group acting on $\widetilde{\mathcal{R}}$; it acts $\mathbb{R}$-linearly on $\mathfrak{a}$, and the dual action on $\mathfrak{a}^*$ stabilizes $R$ and $\Delta$. We assume that $k_{\gamma(\alpha)} = k_\alpha$ for all $\alpha \in R$, $\gamma \in \Gamma$. Then $\Gamma$ acts on $\mathbb{H}(\widetilde{\mathcal{R}}, k)$ by the algebra automorphisms satisfying

$$\gamma(\xi N_w) = \gamma(\xi) N_{\gamma w \gamma^{-1}}, \quad \gamma \in \Gamma,\ \xi \in \mathfrak{a}^*,\ w \in W.$$

Let $\natural : \Gamma^2 \to \mathbb{C}^\times$ be a 2-cocycle and let $\mathbb{C}[\Gamma, \natural]$ be the twisted group algebra. We recall that it has a standard basis $\{N_\gamma : \gamma \in \Gamma\}$ and multiplication rules

$$N_\gamma N_{\gamma'} = \natural(\gamma, \gamma') N_{\gamma\gamma'}, \quad \gamma, \gamma' \in \Gamma.$$

We can endow the vector space $\mathbb{H}(\widetilde{\mathcal{R}}, k) \otimes \mathbb{C}[\Gamma, \natural]$ with the algebra structure such that

- $\mathbb{H}(\widetilde{\mathcal{R}}, k)$ and $\mathbb{C}[\Gamma, \natural]$ are embedded as subalgebras,
- $N_\gamma h N_\gamma^{-1} = \gamma(h)$ for $\gamma \in \Gamma,\ h \in \mathbb{H}(\widetilde{\mathcal{R}}, k)$.

We denote this algebra by $\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural]$ and call it a twisted graded Hecke algebra. If $\natural$ is trivial, then it reduces to the crossed product $\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \Gamma$. All our previous notions for graded Hecke algebras admit natural generalizations to this setting.

Notice that $W\Gamma$ is a group with $W$ as normal subgroup and $\Gamma$ as quotient. The 2-cocycle $\natural$ can be lifted to $(W\Gamma)^2 \to \Gamma^2 \to (\mathbb{C}^\times)^2$, and that yields a twisted group algebra $\mathbb{C}[W\Gamma, \natural]$ in $\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural]$. It is worthwhile to note the case $k = 0$:

$$\mathbb{H}(\widetilde{\mathcal{R}}, 0) \rtimes \mathbb{C}[\Gamma, \natural] = \mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \natural]). \tag{1.34}$$

We consider the restriction map

$$\mathrm{r} : \mathrm{Mod}(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural]) \to \mathrm{Mod}(\mathbb{C}[W\Gamma, \natural]). \tag{1.35}$$

Every $\mathbb{C}[W\Gamma, \natural]$-module can be extended in a unique way to an $\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[W\Gamma, \natural])$-module on which $\mathcal{O}(\mathfrak{t})$ acts via evaluation at $0 \in \mathfrak{t}$, so the right-hand side of (1.35) can be considered as a subcategory of $\mathrm{Mod}(\mathbb{H}(\widetilde{\mathcal{R}}, 0) \rtimes \mathbb{C}[\Gamma, \natural])$.

**Proposition 1.36.** *Let $k : R / W\Gamma \to \mathbb{R}$ be a parameter function and let $\natural : \Gamma^2 \to \mathbb{C}^\times$ be a 2-cocycle. The map (1.35) induces a bijection*

$$\mathrm{r}_\mathbb{Z} : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural]) \to R_\mathbb{Z}(\mathbb{C}[W\Gamma, \natural]).$$

*Proof.* Let $\widetilde{\Gamma} \to \Gamma$ be a finite central extension such that $\natural$ becomes trivial in $H^2(\widetilde{\Gamma}, \mathbb{C}^\times)$. Such a group always exists: one can take the Schur extension from [Curtis and Reiner 1962, Theorem 53.7]. Then there exists a central idempotent $p_\natural \in \mathbb{C}[\ker(\widetilde{\Gamma} \to \Gamma)]$ such that

$$\mathbb{C}[\Gamma, \natural] \cong p_\natural \mathbb{C}[\widetilde{\Gamma}]. \tag{1.37}$$

The map $\mathrm{r}_\mathbb{Z}$ becomes

$$\mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes p_\natural \mathbb{C}[\widetilde{\Gamma}]) \to R_\mathbb{Z}(p_\natural \mathbb{C}[W\widetilde{\Gamma}]). \tag{1.38}$$

Since $p_\natural \mathbb{C}[\widetilde{\Gamma}]$ is a direct summand of $\mathbb{C}[\widetilde{\Gamma}]$, (1.38) is just a part of

$$\mathrm{r}_\mathbb{Z} : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \widetilde{\Gamma}) \to R_\mathbb{Z}(W \rtimes \widetilde{\Gamma}).$$

Hence it suffices to prove the proposition when $\natural$ is trivial, which we assume from now on. By [Solleveld 2010, Theorem 6.5(c)],

$$\mathrm{r}_\mathbb{C} : \mathbb{C}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \Gamma) \to R_\mathbb{C}(W\Gamma) \tag{1.39}$$

is a $\mathbb{C}$-linear bijection. So at least

$$\mathrm{r}_\mathbb{Z} : \mathbb{Z}\,\mathrm{Irr}_0(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \Gamma) \to R_\mathbb{Z}(W\Gamma) \tag{1.40}$$

is injective and has image of finite index in $R_\mathbb{Z}(W\Gamma)$.

Given $(\pi, V) \in \mathrm{Irr}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$, let $\Gamma_\pi$ be the stabilizer in $\Gamma$ of the isomorphism class of $\pi$. For every $\gamma \in \Gamma_\pi$ we can find $I^\gamma \in \mathrm{Aut}_\mathbb{C}(V)$ such that

$$I^\gamma \circ \pi(N_\gamma h N_\gamma^{-1}) = \pi(h) \circ I^\gamma \quad \text{for all } h \in \mathbb{H}(\widetilde{\mathcal{R}}, k).$$

By Schur's lemma there exists a 2-cocycle $\natural_\pi : \Gamma_\pi^2 \to \mathbb{C}^\times$ such that

$$I^{\gamma\gamma'} = \natural_\pi(\gamma, \gamma') I^\gamma I^{\gamma'} \quad \text{for all } \gamma, \gamma' \in \Gamma.$$

Let $(\tau, M) \in \mathrm{Irr}(\mathbb{C}[\Gamma_\pi, \natural_\pi])$. Then $M \otimes V$ becomes an irreducible $\mathbb{H} \rtimes \Gamma_\pi$-module. Clifford theory (see, e.g., [Ram and Ramagge 2003, Appendix], [Curtis and Reiner 1962, §51] or [Solleveld 2012b, Appendix]) tells us that $\mathrm{ind}_{\mathbb{H}\rtimes\Gamma_\pi}^{\mathbb{H}\rtimes\Gamma}(M \otimes V)$ is an irreducible $\mathbb{H} \rtimes \Gamma$-module. Moreover, this construction provides a bijection

$$\mathrm{Irr}(\mathbb{H} \rtimes \Gamma) \to \{(\pi, M) : \pi \in \mathrm{Irr}(\mathbb{H})/\Gamma, M \in \mathrm{Irr}(\mathbb{C}[\Gamma_\pi, \natural_\pi])\}.$$

We note that

$$\mathrm{r}\big(\mathrm{ind}_{\mathbb{H}\rtimes\Gamma_\pi}^{\mathbb{H}\rtimes\Gamma}(M \otimes V)\big) = \mathrm{ind}_{W\rtimes\Gamma_\pi}^{W\rtimes\Gamma}(M \otimes \mathrm{r}(V)). \tag{1.41}$$

Similarly, Clifford theory provides a bijection

$$\mathrm{Irr}(W \rtimes \Gamma) \to \{(\tau, N) : \tau \in \mathrm{Irr}(W)/\Gamma,\ N \in \mathrm{Irr}(\mathbb{C}[\Gamma_\tau, \natural_\tau])\}.$$

Since $W$ is a Weyl group, the 2-cocycle $\natural_\tau$ is always trivial [Aubert et al. 2017c, Proposition 4.3]. With (1.41) it follows that $\natural_\pi$ is also trivial for all $\pi \in \mathrm{Irr}(\mathbb{H}(\widetilde{\mathcal{R}}, k))$.

Consider any $\mathrm{ind}_{W\rtimes\Gamma_\tau}^{W\rtimes\Gamma}(N \otimes V_\tau) \in \mathrm{Irr}(W \rtimes \Gamma)$. Theorem 1.31 guarantees the existence of unique $m_\pi \in \mathbb{Z}$ such that $V_\tau = \sum_{(\pi,V)\in\mathrm{Irr}_0(\mathbb{H})} m_\pi \mathrm{r}(V)$. By the uniqueness, $\Gamma_\pi \supset \Gamma_\tau$ whenever $m_\pi \neq 0$. Hence $N \otimes V$ is a well-defined $\mathbb{H} \rtimes \Gamma_\pi$-module (it may be reducible though), and

$$\begin{aligned}\mathrm{ind}_{W\rtimes\Gamma_\tau}^{W\rtimes\Gamma}(N \otimes V_\tau) &= \mathrm{ind}_{W\rtimes\Gamma_\tau}^{W\rtimes\Gamma}\big(N \otimes \textstyle\sum_{(\pi,V)\in\mathrm{Irr}_0(\mathbb{H})} m_\pi \mathrm{r}(V)\big) \\ &= \mathrm{r}\big(\textstyle\sum_{(\pi,V)\in\mathrm{Irr}_0(\mathbb{H})} m_\pi \,\mathrm{ind}_{\mathbb{H}\rtimes\Gamma_\pi}^{\mathbb{H}\rtimes\Gamma}(N \otimes V)\big).\end{aligned}$$

This proves that (1.40) is also surjective. □

**1C. *Affine Hecke algebras.*** Let $\mathcal{H}$ be an affine Hecke algebra with positive parameters $q$. We compare its Grothendieck group of finite length modules $R_\mathbb{Z}(\mathcal{H})$ with the analogous group for the parameters $q = 1$. By some of the main results of [Solleveld 2012a], the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_\mathbb{Z} R_\mathbb{Z}(\mathcal{H})$ is canonically isomorphic to its analogue for $q = 1$. We show that this is already an isomorphism for $R_\mathbb{Z}(\mathbb{H})$, without tensoring by $\mathbb{Q}$. This follows from the results of the previous paragraph, in combination with the standard reduction from affine Hecke algebras to graded Hecke algebras [Lusztig 1989].

As before, let $\mathcal{R} = (X, R, Y, R^\vee, \Delta)$ be a based root datum. We have the affine Weyl group $W^{\mathrm{aff}} = \mathbb{Z}R \rtimes W$ and the extended (affine) Weyl group $W^e = X \rtimes W$. Both can be considered as groups of affine transformations of $\mathfrak{a}^*$. We denote the translation corresponding to $x \in X$ by $t_x$. As is well-known, $W^{\mathrm{aff}}$ is a Coxeter group, and the basis $\Delta$ of $R$ gives rise to a set $S^{\mathrm{aff}}$ of simple (affine) reflections.

More explicitly, let $\Delta_M^\vee$ be the set of maximal elements of $R^\vee$, with respect to the dominance ordering coming from $\Delta$. Then

$$S^{\mathrm{aff}} = S_\Delta \cup \{t_\alpha s_\alpha : \alpha^\vee \in \Delta_M^\vee\}.$$

The length function $\ell$ of the Coxeter system $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ extends naturally to $W^e$. The elements of length zero form a subgroup $\Omega \subset W^e$ and $W^e = W^{\mathrm{aff}} \rtimes \Omega$.

A complex parameter function for $\mathcal{R}$ is a map $q : S^{\mathrm{aff}} \to \mathbb{C}^\times$ such that $q(s) = q(s')$ if $s$ and $s'$ are conjugate in $W^e$. This extends naturally to a map $q : W^e \to \mathbb{C}^\times$ which is 1 on $\Omega$ and satisfies

$$q(ww') = q(w)q(w') \quad \text{if} \quad \ell(ww') = \ell(w) + \ell(w').$$

Equivalently (see [Lusztig 1989, §3.1]), one can define $q$ as a $W$-invariant function

$$q : R \cup \{2\alpha : \alpha^\vee \in 2Y\} \to \mathbb{C}^\times. \tag{1.42}$$

We speak of equal parameters if $q(s) = q(s')$ for all $s, s' \in S^{\mathrm{aff}}$ and of positive parameters if $q(s) \in \mathbb{R}_{>0}$ for all $s \in S^{\mathrm{aff}}$. We fix a square root $q^{1/2} : S^{\mathrm{aff}} \to \mathbb{C}^\times$.

The affine Hecke algebra $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ is the unique associative complex algebra with basis $\{N_w : w \in W^e\}$ and multiplication rules

$$\begin{aligned} N_w N_{w'} &= N_{ww'} && \text{if} \quad \ell(ww') = \ell(w) + \ell(w'), \\ (N_s - q(s)^{1/2})(N_s + q(s)^{-1/2}) &= 0 && \text{if} \quad s \in S^{\mathrm{aff}}. \end{aligned} \tag{1.43}$$

In the literature one also finds this algebra defined in terms of the elements $q(s)^{1/2} N_s$, in which case the multiplication can be described without square roots. This explains why $q^{1/2}$ does not appear in the notation $\mathcal{H}(\mathcal{R}, q)$. For $q = 1$, (1.43) just reflects the defining relations of $W^e$, so $\mathcal{H}(\mathcal{R}, 1) = \mathbb{C}[W^e]$.

The set of dominant elements in $X$ is

$$X^+ = \{x \in X : \langle x, \alpha^\vee \rangle \geq 0 \ \forall \alpha \in \Delta\}.$$

The subset $\{N_{t_x} : x \in X^+\} \subset \mathcal{H}(\mathcal{R}, q)$ is closed under multiplication, and isomorphic to $X^+$ as a semigroup. For any $x \in X$ we put

$$\theta_x = N_{t_{x_1}} N_{t_{x_2}}^{-1}, \quad \text{where } x_1, x_2 \in X^+ \text{ and } x = x_1 - x_2.$$

This does not depend on the choice of $x_1$ and $x_2$, so $\theta_x \in \mathcal{H}(\mathcal{R}, q)^\times$ is well-defined. The Bernstein presentation of $\mathcal{H}(\mathcal{R}, q)$ [Lusztig 1989, §3] says that:

- $\{\theta_x : x \in X\}$ forms a $\mathbb{C}$-basis of a subalgebra of $\mathcal{H}(\mathcal{R}, q)$ isomorphic to $\mathbb{C}[X] \cong \mathcal{O}(T)$, which we identify with $\mathcal{O}(T)$.
- $\mathcal{H}(W, q) := \mathbb{C}\{N_w : w \in W\}$ is a finite-dimensional subalgebra of $\mathcal{H}(\mathcal{R}, q)$ (known as the Iwahori–Hecke algebra of $W$).

- The multiplication map $\mathcal{O}(T) \otimes \mathcal{H}(W, q) \to \mathcal{H}(\mathcal{R}, q)$ is a $\mathbb{C}$-linear bijection.
- There are explicit cross relations between $\mathcal{H}(W, q)$ and $\mathcal{O}(T)$, deformations of the standard action of $W$ on $\mathcal{O}(T)$.

To define parabolic subalgebras of affine Hecke algebras, we associate some objects to any $P \subset \Delta$:

$$\begin{aligned} X_P &= X/(X \cap (P^\vee)^\perp), & X^P &= X/(X \cap \mathbb{Q}P), \\ Y_P &= Y \cap \mathbb{Q}P^\vee, & Y^P &= Y \cap P^\perp, \\ T_P &= \mathrm{Hom}_{\mathbb{Z}}(X_P, \mathbb{C}^\times), & T^P &= \mathrm{Hom}_{\mathbb{Z}}(X^P, \mathbb{C}^\times), \\ \mathcal{R}_P &= (X_P, R_P, Y_P, R_P^\vee, P), & \mathcal{R}^P &= (X, R_P, Y, R_P^\vee, P), \\ \mathcal{H}_P &= \mathcal{H}(\mathcal{R}_P, q_P), & \mathcal{H}^P &= \mathcal{H}(\mathcal{R}^P, q^P). \end{aligned}$$

Here $q_P$ and $q^P$ are derived from $q$ via (1.42). Both $\mathcal{H}_P$ and $\mathcal{H}^P$ are called parabolic subalgebras of $\mathcal{H}$. One can regard $\mathcal{H}_P$ as a "semisimple" quotient of $\mathcal{H}^P$.

Any $t \in T^P$ and any $u \in T^P \cap T_P$ give rise to algebra automorphisms

$$\begin{aligned} \psi_u : \mathcal{H}_P \to \mathcal{H}_P, &\quad \theta_{x_P} N_w \mapsto u(x_P)\theta_{x_P} N_w, \\ \psi_t : \mathcal{H}^P \to \mathcal{H}^P, &\quad \theta_x N_w \mapsto t(x)\theta_x N_w. \end{aligned} \tag{1.44}$$

Let $\Gamma$ be a finite group acting on $\mathcal{R}$, i.e., it acts $\mathbb{Z}$-linearly on $X$ and preserves $R$ and $\Delta$. We also assume that $\Gamma$ acts on $T$ by affine transformations, whose linear part comes from the action on $X$. Thus $\Gamma$ acts on $\mathcal{O}(T) \cong \mathbb{C}[X]$ by

$$\gamma(\theta_x) = z_\gamma(x)\theta_{\gamma x} \tag{1.45}$$

for some $z_\gamma \in T$. Since this is a group action, we must have $z_\gamma \in T^W$.

We suppose throughout that $q^{1/2}$ is $\Gamma$-invariant, so that $\gamma \in \Gamma$ acts on $\mathcal{H}(\mathcal{R}, q)$ by the algebra automorphism

$$\sum_{w \in W,\, x \in X} c_{x,w}\theta_x N_w \;\mapsto\; \sum_{w \in W,\, x \in X} c_{x,w} z_\gamma(x)\theta_{\gamma(x)} N_{\gamma w \gamma^{-1}}. \tag{1.46}$$

We can build the crossed product algebra

$$\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma. \tag{1.47}$$

In [Solleveld 2012a] we considered a slightly less general action of $\Gamma$ on $\mathcal{H}(\mathcal{R}, q)$, where the elements $z_\gamma \in T^W$ from (1.45) were all equal to 1. But the relevant results from [Solleveld 2012a] do not rely on $\Gamma$ fixing the unit element of $T$, so they are also valid for the actions as in (1.46). In this paper we will tacitly use some results from [Solleveld 2012a] in the generality of (1.46). We note that nontrivial $z_\gamma \in T^W$ are sometimes needed to describe Hecke algebras coming from $p$-adic groups, for example [Roche 2002, §4].

We can also endow the group $\Gamma$ with a 2-cocycle $\natural : \Gamma^2 \to \mathbb{C}^\times$. Then the vector space $\mathcal{H}(\mathcal{R}, q) \otimes \mathbb{C}[\Gamma, \natural]$ obtains a multiplication such that $\mathcal{H}(\mathcal{R}, q)$ and $\mathbb{C}[\Gamma, \natural]$ are subalgebras and

$$N_\gamma h N_\gamma^{-1} = \gamma(h) \quad \text{for all } \gamma \in \Gamma,\ h \in \mathcal{H}(\mathcal{R}, q).$$

We denote this by $\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]$ and call it a twisted affine Hecke algebra. Such twists seem necessary to describe algebras appearing in the representation theory of nonsplit $p$-adic groups; see, e.g., [Aubert et al. 2017b, Example 5.5]. For reference we record the case $q = 1$:

$$\mathcal{H}(\mathcal{R}, 1) \rtimes \mathbb{C}[\Gamma, \natural] = \mathcal{O}(T) \rtimes \mathbb{C}[W\Gamma, \natural]. \tag{1.48}$$

The representation theory of (twisted) affine Hecke algebras is closely related to that of (twisted) graded Hecke algebras, as first shown by Lusztig [1989]. Since $\mathcal{H}(\mathcal{R}, q)$ is of finite rank as a module over its commutative subalgebra $\mathcal{O}(T)$, all irreducible $\mathcal{H}(\mathcal{R}, q)$-modules have finite dimension. The set of $\mathcal{O}(T)$-weights of an $\mathcal{H}(\mathcal{R}, q)$-module $V$ is denoted by $\mathrm{Wt}(V)$.

The vector space $\mathfrak{t} = \mathfrak{a} \oplus i\mathfrak{a}$ can now be interpreted as the Lie algebra of the complex torus $T = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$. The latter has a polar decomposition $T = T_{\mathrm{rs}} \times T_{\mathrm{un}}$, where $T_{\mathrm{rs}} = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{R}_{>0})$ and $T_{\mathrm{un}}$ is the unique maximal compact subgroup of $T$. The polar decomposition of an element $t \in T$ is written as $t = |t|\ (t\,|t|^{-1})$.

We write $T^- = \exp(\mathfrak{a}^-) \subset T_{\mathrm{rs}}$ and $T^{--} = \exp(\mathfrak{a}^{--}) \subset T_{\mathrm{rs}}$. We say that a module $V$ for $\mathcal{H}(\mathcal{R}, q)$ (or for $\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]$) is tempered if $|\mathrm{Wt}(V)| \subset T^-$, and that it is discrete series if $|\mathrm{Wt}(V)| \subset T^{--}$. (The latter is only possible if $R$ spans $\mathfrak{a}$, for otherwise $\mathfrak{a}^{--}$ and $T^{--}$ are empty.)

By the Bernstein presentation, the centre of $\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]$ contains $\mathcal{O}(T)^{W\Gamma}$. For any $W\Gamma$-invariant subset $U \subset T$, let

$$\mathrm{Mod}_{f,U}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural])$$

be the category of finite-dimensional $\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]$-modules whose $\mathcal{O}(T)^{W\Gamma}$-weights all lie in $U/W\Gamma$. We denote the Grothendieck group of this category by $R_{\mathbb{Z},U}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural])$.

The centre of $\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural]$ contains $\mathcal{O}(\mathfrak{t})^{W\Gamma}$. For any $W\Gamma$-invariant subset $V \subset \mathfrak{t}$ we define $\mathrm{Mod}_{f,V}(\mathbb{H}(\widetilde{\mathcal{R}}, k) \rtimes \mathbb{C}[\Gamma, \natural])$ analogously.

Fix $u \in T_{\mathrm{un}}$. To $\mathcal{R}$ and $u$ we can associate some new objects. First we define the root system

$$R_u = \{\alpha \in R : s_\alpha(u) = u\},$$

and we let $\Delta_u$ be the unique basis of $R_u$ contained in $R^+$. Then

$$(W\Gamma)_u = W(R_u) \rtimes \Gamma'_u, \qquad \Gamma'_u = \{w \in W\Gamma : w(u) = u,\ w(\Delta_u) = \Delta_u\}.$$

Now we can define the based root data

$$\mathcal{R}_u = (X, R_u, Y, R_u^\vee, \Delta_u) \quad \text{and} \quad \widetilde{\mathcal{R}_u} = (\mathfrak{a}^*, R_u, \mathfrak{a}, R_u^\vee, \Delta_u).$$

We define a parameter function $k_u : R_u \to \mathbb{R}$ for $\widetilde{\mathcal{R}_u}$ by

$$2k_{u,\alpha} = \log(q(s_\alpha)) + \alpha(u)\log(q(t_\alpha s_\alpha)).$$

Let $\natural_u : (\Gamma'_u)^2 \to \mathbb{C}^\times$ be the restriction to $\natural$. With a slight variation on Lusztig's reduction theorems [Lusztig 1989, §8–9], one can prove:

**Theorem 1.49.** *Let $q : W^e \to \mathbb{R}_{>0}$ be a positive parameter function. The categories*

$$\mathrm{Mod}_{f,W\Gamma u T_{\mathrm{rs}}}(\mathcal{H}(\mathcal{R},q) \rtimes \mathbb{C}[\Gamma,\natural]) \quad \textit{and} \quad \mathrm{Mod}_{f,\mathfrak{a}}(\mathbb{H}(\widetilde{\mathcal{R}_u},k_u) \rtimes \mathbb{C}[\Gamma'_u,\natural_u])$$

*are equivalent. The equivalence respects parabolic induction, temperedness and discrete series.*

*Proof.* Let $\widetilde{\Gamma}$ and the central idempotent $p_\natural$ be as in (1.38). Then

$$\begin{aligned} \mathcal{H}(\mathcal{R},q) \rtimes \mathbb{C}[\Gamma,\natural] &= p_\natural(\mathcal{H}(\mathcal{R},q) \rtimes \widetilde{\Gamma}), \\ \mathbb{H}(\widetilde{\mathcal{R}_u},k_u) \rtimes \mathbb{C}[\Gamma'_u,\natural_u] &= p_\natural(\mathbb{H}(\widetilde{\mathcal{R}_u},k_u) \rtimes \widetilde{\Gamma}'_u). \end{aligned} \tag{1.50}$$

By [Solleveld 2012a, Corollary 2.15] the theorem holds for $\mathcal{H}(\mathcal{R},q) \rtimes \widetilde{\Gamma}$ and $\mathbb{H}(\widetilde{\mathcal{R}_u},k_u) \rtimes \widetilde{\Gamma}'_u$. The claimed properties of this equivalence were checked in detail in [Aubert et al. 2016, §2.1].

This is based on a comparison of localizations of these algebras, as in [Lusztig 1989]. The comparison maps [Solleveld 2012a, Theorems 2.1.2 and 2.1.4] are the identity on $\mathbb{C}[\widetilde{\Gamma}'_u \cap \widetilde{\Gamma}]$, so they preserve $p_\natural$. Hence we can restrict the result from [Solleveld 2012a] to the direct summands (1.50). $\square$

From Theorem 1.49 and (1.35) (and (1.34) and (1.48) for the bottom line) we construct a diagram

$$\begin{array}{ccc} \mathrm{Mod}_{f,W\Gamma u T_{\mathrm{rs}}}(\mathcal{H}(\mathcal{R},q) \rtimes \mathbb{C}[\Gamma,\natural]) & \xrightarrow{\sim} & \mathrm{Mod}_{f,\mathfrak{a}}(\mathbb{H}(\widetilde{\mathcal{R}_u},k_u) \rtimes \mathbb{C}[\Gamma'_u,\natural_u]) \\ \downarrow \mathrm{r}_u & & \downarrow \mathrm{r} \\ \mathrm{Mod}_{f,W\Gamma u}(\mathcal{H}(\mathcal{R},1) \rtimes \mathbb{C}[\Gamma,\natural]) & \xleftarrow{\sim} & \mathrm{Mod}_{f,0}(\mathbb{H}(\widetilde{\mathcal{R}_u},0) \rtimes \mathbb{C}[\Gamma'_u,\natural_u]) \\ \| & & \| \\ \mathrm{Mod}_{f,W\Gamma u}(\mathcal{O}(T) \rtimes \mathbb{C}[W\Gamma,\natural]) & \xleftarrow{\sim} & \mathrm{Mod}_{f,0}(\mathcal{O}(\mathfrak{t}) \rtimes \mathbb{C}[(W\Gamma)_u,\natural_u]) \end{array}$$

where $\mathrm{r}_u$ is the unique map that makes the diagram commutative. Using the technique in the proof of Theorem 1.49, we can immediately extend all relevant results in [Solleveld 2012a] from $\mathcal{H}(\mathcal{R},q) \rtimes \widetilde{\Gamma}$ to twisted affine Hecke algebras. In view of this, we will freely use results from [Solleveld 2012a] in that generality.

As shown in [Solleveld 2012a, §2.3], there exists a unique system of $\mathbb{Z}$-linear maps

$$\zeta^\vee : R_{\mathbb{Z}}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]) \to R_{\mathbb{Z}}(\mathcal{H}(\mathcal{R}, 1) \rtimes \mathbb{C}[\Gamma, \natural]) \tag{1.51}$$

(for all possible $\mathcal{R}$, $q$, $\Gamma$) such that

- $\zeta^\vee(\pi) = \mathrm{r}_u(\pi)$ for tempered representations in $\mathrm{Mod}_{f, W\Gamma u T_{\mathrm{rs}}}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural])$,
- $\zeta^\vee$ commutes with parabolic induction,
- $\zeta^\vee$ respects the formation of standard modules for the Langlands classification, in the sense of [Solleveld 2012a, Corollary 2.2.5].

**Theorem 1.52.** *The map (1.51) is bijective for every positive parameter function $q$.*

*Proof.* Proposition 1.36 and Theorem 1.49 imply that (1.51) gives a bijection

$$R_{\mathbb{Z},\mathrm{temp}, W\Gamma u T_{\mathrm{rs}}}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural]) \to R_{\mathbb{Z},\mathrm{temp}, W\Gamma u}(\mathcal{H}(\mathcal{R}, 1) \rtimes \mathbb{C}[\Gamma, \natural]), \tag{1.53}$$

where the subscripts “temp” indicate that we formed these Grothendieck groups by starting with tempered modules only. Any tempered $\mathcal{O}(T) \rtimes \mathbb{C}[W\Gamma, \natural]$-module only has $\mathcal{O}(T)$-weights in $T_{\mathrm{un}}$, so on the right-hand side of (1.53) we may just as well replace $W\Gamma u$ by $W\Gamma u T_{\mathrm{rs}}$. Thus (1.51) restricts to a bijection between subgroups generated by tempered modules on both sides.

In [Solleveld 2012a, Corollary 2.3.2] it was shown that (1.51) becomes a $\mathbb{Q}$-linear bijection upon tensoring both sides with $\mathbb{Q}$. The second half of the proof of that result (see [Solleveld 2012a, §3.4]) extends the statement from the tempered to the general case. It says essentially that whatever happens in the space $\mathrm{Irr}(\mathcal{H}(\mathcal{R}, q) \rtimes \mathbb{C}[\Gamma, \natural])$ can be detected and understood already by looking at tempered representations. From that, the bijectivity in the tempered case and the multiplicity one property of the Langlands classification (every standard module has a unique irreducible quotient, appearing with multiplicity one [Solleveld 2012a, Theorem 2.2.4]), we obtain the bijectivity of (1.51) in general. □

## 2. Topological K-theory

**2A.** ***The $C^*$-completion of an affine Hecke algebra.*** In this paragraph we recall the structure of $C^*$-algebras associated to affine Hecke algebras. These deep results mainly stem from [Opdam 2004; Delorme and Opdam 2008; 2011].

Recall that $q$ is a positive parameter function for $\mathcal{R}$. We define a $*$-operation and a trace on $\mathcal{H}(\mathcal{R}, q)$ by

$$\Big(\sum\nolimits_{w \in W^e} c_w N_w\Big)^* = \sum\nolimits_{w \in W^e} \overline{c_w} N_{w^{-1}}, \qquad \tau\Big(\sum\nolimits_{w \in W^e} c_w N_w\Big) = c_e.$$

Since $q(s_\alpha) > 0$, $*$ preserves the relations (1.43) and defines an anti-involution of $\mathcal{H}(\mathcal{R}, q)$. The set $\{N_w : w \in W^e\}$ is an orthonormal basis of $\mathcal{H}(\mathcal{R}, q)$ for the

inner product
$$\langle h_1, h_2 \rangle = \tau(h_1^* h_2).$$

This gives $\mathcal{H}(\mathcal{R}, q)$ the structure of a Hilbert algebra. The Hilbert space completion $L^2(\mathcal{R})$ of $\mathcal{H}(\mathcal{R}, q)$ is a module over $\mathcal{H}(\mathcal{R}, q)$, via left multiplication. Moreover, every $h \in \mathcal{H}(\mathcal{R}, q)$ acts as a bounded linear operator [Opdam 2004, Lemma 2.3]. The reduced $C^*$-algebra of $\mathcal{H}(\mathcal{R}, q)$ [Opdam 2004, §2.4], denoted by $C_r^*(\mathcal{R}, q)$, is defined as the closure of $\mathcal{H}(\mathcal{R}, q)$ in the algebra of bounded linear operators on $L^2(\mathcal{R})$.

As in (1.47), we can extend this to a $C^*$-algebra $C_r^*(\mathcal{R}, q) \rtimes \Gamma$, provided that $q$ is $\Gamma$-invariant. We will not bother about twisted group algebras $\mathbb{C}[\Gamma, \natural]$ in this section, for with the technique from (1.50) it is easy to generalize our results to that setting, and in the context of $C^*$-algebras, crossed products with groups look much more natural.

Let us recall some background about $C_r^*(\mathcal{R}, q) \rtimes \Gamma$, mainly from [Opdam 2004; Solleveld 2012a]. It follows from [Delorme and Opdam 2008, Corollary 5.7] that it is a finite type I $C^*$-algebra and that $\mathrm{Irr}(C_r^*(\mathcal{R}, q))$ is precisely the tempered part of $\mathrm{Irr}(\mathcal{H}(\mathcal{R}, q))$. The structure of $C_r^*(\mathcal{R}, q) \rtimes \Gamma$ is described in terms of parabolically induced representations. As induction data we use triples $(P, \delta, t)$, where

- $P \subset \Delta$,
- $\delta$ is an irreducible discrete series representation of $\mathcal{H}_P$,
- $t \in T^P$.

We regard two triples $(P, \delta, t)$ and $(P', \delta', t')$ as equivalent if $P = P'$, $t = t'$ and $\delta \cong \delta'$. Notice that $\mathcal{H}_P$ comes from a semisimple root datum, so it can have discrete series representations. For every $t \in T^P$ there exists a surjection $\phi_t : \mathcal{H}^P \to \mathcal{H}_P$, which combines the projection $X \to X_P$ with evaluation at $t$. To such a triple $(P, \delta, t)$ we associate the $\mathcal{H} \rtimes \Gamma$-representation
$$\pi^\Gamma(P, \delta, t) = \mathrm{ind}_{\mathcal{H}^P}^{\mathcal{H} \rtimes \Gamma}(\delta \circ \phi_t).$$

(When $\Gamma = 1$, we often suppress it from these and similar notations.) For any $t \in T^P_{\mathrm{un}} = T^P \cap T_{\mathrm{un}}$ these representations extend continuously to the respective $C^*$-completions of the involved algebras. Let $\Xi_{\mathrm{un}}$ be the set of triples $(P, \delta, t)$ as above, such that moreover $t \in T_{\mathrm{un}}$. Considering $P$ and $\delta$ as discrete variables, we regard $\Xi_{\mathrm{un}}$ as a disjoint union of finitely many compact real tori (of different dimensions).

Let $\mathcal{V}_\Xi^\Gamma$ be the vector bundle over $\Xi_{\mathrm{un}}$ whose fibre at $(P, \delta, t)$ is the vector space underlying $\pi^\Gamma(P, \delta, t)$. That vector space is independent of $t$, so the vector bundle is trivial. Let $\mathrm{End}(\mathcal{V}_\Xi^\Gamma)$ be the algebra bundle with fibres $\mathrm{End}_{\mathbb{C}}(\pi^\Gamma(P, \delta, t))$. Every element of $C_r^*(\mathcal{R}, q) \rtimes \Gamma$ naturally defines a continuous section of $\mathrm{End}(\mathcal{V}_\Xi^\Gamma)$.

There exists a finite groupoid $\mathcal{G}$ which acts on $\mathrm{End}(\mathcal{V}_\Xi^\Gamma)$. It is made from elements of $W \rtimes \Gamma$ and of $K_P := T_P \cap T^P$. More precisely, its base space is the power set of $\Delta$, and for $P, Q \subseteq \Delta$ the collection of arrows from $P$ to $Q$ is

$$\mathcal{G}_{PQ} = \{(g, u) : g \in \Gamma \ltimes W, u \in K_P, g(P) = Q\}. \tag{2.1}$$

Whenever it is defined, the multiplication in $\mathcal{G}$ is

$$(g', u') \cdot (g, u) = (g'g, g^{-1}(u')u).$$

In particular, writing $W\Gamma(P, P) = \{w \in W\Gamma : w(P) = P\}$, we have the group

$$\mathcal{G}_{PP} = W\Gamma(P, P) \rtimes K_P. \tag{2.2}$$

Usually we will write elements of $\mathcal{G}$ simply as $gu$. For $\gamma \in \Gamma W$ with $\gamma(P) = Q \subset \Delta$, there are algebra isomorphisms

$$\begin{aligned} \psi_\gamma : \mathcal{H}_P \to \mathcal{H}_Q, \quad & \theta_{x_P} N_w \mapsto \theta_{\gamma(x_P)} N_{\gamma w \gamma^{-1}}, \\ \psi_\gamma : \mathcal{H}^P \to \mathcal{H}^Q, \quad & \theta_x N_w \mapsto \theta_{\gamma x} N_{\gamma w \gamma^{-1}}. \end{aligned} \tag{2.3}$$

The groupoid $\mathcal{G}$ acts from the left on $\Xi_{\mathrm{un}}$ by

$$(g, u) \cdot (P, \delta, t) := (g(P), \delta \circ \psi_u^{-1} \circ \psi_g^{-1}, g(ut)), \tag{2.4}$$

the action being defined if and only if $g(P) \subset \Delta$. Suppose that $g(P) = Q \subset \Delta$ and $\delta' \cong \delta \circ \psi_u^{-1} \circ \psi_g^{-1}$. By [Opdam 2004, Theorem 4.33] and [Solleveld 2012a, Theorem 3.1.5], there exists an intertwining operator

$$\pi^\Gamma(gu, P, \delta, t) \in \mathrm{Hom}_{\mathcal{H}(\mathcal{R},q)\rtimes\Gamma}\big(\pi^\Gamma(P, \delta, t), \pi^\Gamma(Q, \delta', g(ut))\big) \tag{2.5}$$

which depends algebraically on $t \in T^P_{\mathrm{un}}$. Then the action of $\mathcal{G}$ on the continuous sections $C(\Xi_{\mathrm{un}}; \mathrm{End}(\mathcal{V}_\Xi^\Gamma))$ is given by

$$(g\cdot f)(\xi) = \pi^\Gamma(g, g^{-1}\xi) f(g^{-1}\xi) \pi^\Gamma(g, g^{-1}\xi)^{-1}, \quad g \in \mathcal{G}_{PQ},\ \xi = (Q, \delta', t'). \tag{2.6}$$

**Theorem 2.7** [Delorme and Opdam 2008, Corollary 5.7; Solleveld 2012a, Theorem 3.2.2]. *There exists a canonical isomorphism of $C^*$-algebras*

$$C_r^*(\mathcal{R}, q) \rtimes \Gamma \xrightarrow{\sim} C(\Xi_{\mathrm{un}}; \mathrm{End}(\mathcal{V}_\Xi^\Gamma))^{\mathcal{G}}.$$

For $q = 1$ this simplifies to the well-known isomorphism

$$C_r^*(\mathcal{R}, 1) \rtimes \Gamma = C(T_{\mathrm{un}}) \rtimes W\Gamma \xrightarrow{\sim} C(T_{\mathrm{un}}; \mathrm{End}_{\mathbb{C}}(\mathbb{C}[W\Gamma]))^{W\Gamma}. \tag{2.8}$$

Let $\mathcal{G}_{P,\delta}$ be the setwise stabilizer of $(P, \delta, T^P_{\mathrm{un}})$ in the group $\mathcal{G}_{PP}$. Let $(P, \delta)/\mathcal{G}$ be a set of representatives for the action of $\mathcal{G}$ on pairs $(P, \delta)$ obtained from (2.4). Theorem 2.7 can be rephrased as an isomorphism

$$C_r^*(\mathcal{R}, q) \rtimes \Gamma \xrightarrow{\sim} \bigoplus_{(P,\delta)/\mathcal{G}} C\big(T^P_{\mathrm{un}}; \mathrm{End}_{\mathbb{C}}(\pi^\Gamma(P, \delta, t))\big)^{\mathcal{G}_{P,\delta}}. \tag{2.9}$$

Let us discuss the representation theory of $C_r^*(\mathcal{R}, q) \rtimes \Gamma$ (i.e., the tempered unitary representations of $\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma$)) in more detail. Our approach, following Harish-Chandra and Opdam, starts with the discrete series of a parabolic subalgebra $\mathcal{H}(\mathcal{R}_P, q_P) = \mathcal{H}_P$. It is known from [Opdam 2004, Lemma 3.31] that the central character of any (irreducible) discrete series representation $\delta$ of $\mathcal{H}_P$ (a $W_P$-orbit in $T_P$) has a very specific property: it must consist of *residual points* in $T_P$, with respect to $(\mathcal{R}_P, q_P)$.

For $t \in T_P$ we write

$$R_P^z(t) = \{\alpha \in R_P : \alpha(t) \in \{1, -1\}\},$$
$$R_P^p(t) = \big\{\alpha \in R_P : \alpha(t) \in \{q(s_\alpha)^{1/2} q(s_\alpha t_\alpha)^{1/2}, -q(s_\alpha)^{1/2} q(s_\alpha t_\alpha)^{-1/2}\}\big\}.$$

(We remark that there is only one irreducible root datum for which $q(s_\alpha t_\alpha)$ need not be equal to $q(s_\alpha)$, namely with $R = B_n$.) By definition $t \in T_P$ is residual if

$$|R_P^p(t)| - |R_P^z(t)| = \dim_{\mathbb{C}}(T_P) = |P|.$$

Residuality depends in a subtle way on the parameters $q$. For instance, when $q = 1$ and $X_P \neq 0$, there are no residual points. Residual points have been classified in [Heckman and Opdam 1997]. It turns out that all the coordinates of a residual point $t$ are monomials in the parameters $q(s)^{\pm 1/2}$, $s \in S^{\mathrm{aff}}$. Thus we can write $t = t(q^{1/2})$.

Let $\mathcal{Q}(\mathcal{R})$ be the space of all maps $q : S^{\mathrm{aff}} \to \mathbb{R}_{>0}$ such that $q(s) = q(s')$ if $s$ and $s'$ are conjugate in $X \rtimes W\Gamma$. Given $t = t(q^{1/2})$, there is a Zariski-open subset of the real variety $\mathcal{Q}(\mathcal{R})$ on which $t(q^{1/2})$ defines a residual point. For this reason we call the map

$$\mathcal{Q}(\mathcal{R}) \to T : q \mapsto t(q^{1/2})$$

a generic residual point. We say that a parameter function $q \in \mathcal{Q}(\mathcal{R})$ is generic if all generic residual points for parabolic subalgebras $\mathcal{H}_P$ of $\mathcal{H}$ are actually residual points for that $q$.

When there is only one free parameter in $q$, for instance when $R$ is of type $A$, $D$ or $E$, then every positive parameter function $q \neq 1$ is generic. On the other hand, when $R$ contains root systems of type $B$, $C$, $F$ or $G$, then usually no equal parameter function ($q(s) = q(s')$ for all $s, s' \in S^{\mathrm{aff}}$) is generic.

The discrete series representations of $\mathcal{H}(\mathcal{R}_P, q_P)$ were classified in [Opdam and Solleveld 2010], at least when $R$ is irreducible and $q_P$ generic. Later the classification was extended to the nongeneric cases, along with an actual construction of the representations, in [Ciubotaru and Opdam 2017]. Using these papers, it is in principle always possible to find a set of representatives for the action of $\mathcal{G}$ on the pairs $(P, \delta)$ as in (2.9).

Now we describe a single direct summand $C\big(T^P_{\mathrm{un}}; \mathrm{End}_{\mathbb{C}}(\pi^\Gamma(P,\delta,t))\big)^{\mathcal{G}_{P,\delta}}$ of (2.9) more explicitly. Fix $t \in T^P_{\mathrm{un}}$ and let $\mathcal{G}_\xi$ be the isotropy group of $\xi = (P,\delta,t)$ in $\mathcal{G}$. The intertwining operators $\pi^\Gamma(g,\xi)$, $g \in \mathcal{G}_\xi$ make $\pi^\Gamma(\xi)$ into a projective $\mathcal{G}_\xi$-representation. Decompose it as

$$\pi^\Gamma(\xi) = \bigoplus\nolimits_\rho \mathbb{C}^{m_\rho} \otimes V_\rho,$$

where $(\rho, V_\rho)$ runs through the set of (equivalence classes of) irreducible projective $\mathcal{G}_\xi$-representations. From (2.6) we see that the evaluation at $t$ of any element of $C\big(T^P_{\mathrm{un}}; \mathrm{End}_{\mathbb{C}}(\pi^\Gamma(P,\delta,t))\big)^{\mathcal{G}_{P,\delta}}$ lies in

$$\mathrm{End}_{\mathcal{G}_\xi}(\pi^\Gamma(\xi)) \cong \bigoplus\nolimits_\rho \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{m_\rho}).$$

The action of $\mathcal{G}_\xi$ on $\pi^\Gamma(P,\delta,t)$ can be analyzed further with the theory of R-groups from [Delorme and Opdam 2011]. In that paper there is no group $\Gamma$, but with the intertwining operators as in [Solleveld 2012a, Theorem 3.1.5] the extension to the case with $\Gamma$ is straightforward. By [Delorme and Opdam 2011, Propositions 4.5 and 4.7] there exists a root system $R_\xi$ on which $\mathcal{G}_\xi$ acts, and an R-group $\mathfrak{R}_\xi = \mathrm{Stab}_{\mathcal{G}_\xi}(R_\xi \cap R_P^+)$, such that

$$\mathcal{G}_\xi = W(R_\xi) \rtimes \mathfrak{R}_\xi. \tag{2.10}$$

By [Delorme and Opdam 2011, Theorem 4.3(iv)] the intertwining operator $\pi^\Gamma(g,\xi)$ is a scalar multiple of the identity if $g \in W(R_\xi)$. Hence,

$$\mathrm{End}_{\mathcal{G}_\xi}(\pi^\Gamma(\xi)) = \mathrm{End}_{\mathfrak{R}_\xi}(\pi^\Gamma(\xi)).$$

Moreover, the operators

$$\pi^\Gamma(r,\xi) \in \mathrm{End}_{\mathbb{C}}(\pi^\Gamma(\xi)), \quad r \in \mathfrak{R}_\xi,$$

are linearly independent by [Delorme and Opdam 2011, Theorem 5.4]. To classify all irreducible representations of $C\big(T^P_{\mathrm{un}}; \mathrm{End}_{\mathbb{C}}(\pi^\Gamma(P,\delta,t))\big)^{\mathcal{G}_{P,\delta}}$, it remains to determine (2.10) and to study $\pi^\Gamma(\xi)$ as a projective $\mathfrak{R}_\xi$-representation, for all $\xi = (P,\delta,t)$. In all cases that we will encounter in this paper, $\mathfrak{R}_\xi$ is abelian and $\pi^\Gamma(\xi)$ is actually a linear $\mathfrak{R}_\xi$-representation. Together with Theorem 1.52 this enables us to determine $\mathrm{Irr}(C_r^*(\mathcal{R},q) \rtimes \Gamma)$ in those cases.

**2B. *K-theory and equivariant cohomology.*** The computation of the topological K-theory of $C_r^*(\mathcal{R},q) \rtimes \Gamma$ is the main goal of this paper. It follows from (2.9), especially the compactness of $T^P_{\mathrm{un}}$, that the abelian group

$$K_*(C_r^*(\mathcal{R},q) \rtimes \Gamma) = K_0(C_r^*(\mathcal{R},q) \rtimes \Gamma) \oplus K_1(C_r^*(\mathcal{R},q) \rtimes \Gamma)$$

is finitely generated; see [Solleveld 2012a, Lemma 5.1.3] and its proof. By [Solleveld 2012a, Theorem 5.1.4], which relies on the study of the representation theory and of parameter deformations of affine Hecke algebras in [Solleveld 2012a], the group $\mathbb{Q} \otimes_{\mathbb{Z}} K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$ does not depend on the parameters $q$. Combining this with the conclusions from Section 1C, we will deduce that also $K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$ itself is independent of $q$.

Next we use equivariant cohomology and the equivariant Chern character to express $K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$ in terms of the cohomology of a sheaf on a CW-complex. This is inspired by the equivariant Chern characters with values in Bredon cohomology developed in [Słomińska 1976; Lück and Oliver 2001]. Our version also applies to certain noncommutative algebras, and provides more information about the torsion elements than [Słomińska 1976; Lück and Oliver 2001].

In [Solleveld 2012a, Theorem 4.4.2] an injective homomorphism of $C^*$-algebras

$$\zeta_0 : C_r^*(\mathcal{R}, 1) \rtimes \Gamma \to C_r^*(\mathcal{R}, q) \rtimes \Gamma$$

was constructed, with the property

$$\pi \circ \zeta_0 \cong \zeta^\vee(\pi) \quad \text{for all } \pi \in \mathrm{Mod}_f(C_r^*(\mathcal{R}, q) \rtimes \Gamma).$$

**Theorem 2.11.** *The map $K_*(\zeta_0) : K_*(C_r^*(\mathcal{R}, 1) \rtimes \Gamma) \to K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$ is an isomorphism.*

*Proof.* Let $u \in T_{\mathrm{un}}$. Then (1.53) says that $\zeta^\vee$ provides a bijection between the Grothendieck group of finite length $C_r^*(\mathcal{R}, q) \rtimes \Gamma$-modules with $Z(\mathcal{H}(\mathcal{R}, q) \rtimes \Gamma)$-character in $W\Gamma u T_{rs}$ and the analogous group for $C_r^*(X \rtimes W) \rtimes \Gamma$. For tempered modules $\zeta^\vee$ agrees with the map $\zeta^*$ from [Solleveld 2012a, §2.3].

These $C^*$-completions have the same irreducible representations as the respective Schwartz completions of these algebras (see [Opdam 2004, §6] or [Solleveld 2012a, §3.2]), namely the irreducible tempered representations of the underlying affine Hecke algebras. That follows from the comparison of Theorem 2.7 with its analogue for Schwartz completions [Solleveld 2012a, Theorem 3.2.2]. With these translation steps we see that part (c) of [Solleveld 2012a, Lemma 5.1.5] holds. Then [Solleveld 2012a, Lemma 5.1.5] tells us that also its part (a) holds, which is the statement of the theorem. □

When we want to compute $K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$, we can use Theorem 2.11 to replace $q$ by 1, then apply it another time to replace 1 by any positive parameter function $q'$ we like. We will do the actual computation either when $q = 1$ or when $q$ is generic among all possible parameter functions.

In Section 3 we will encounter many root data $\mathcal{R}$ which are a product of root data $\mathcal{R}_1$ and $\mathcal{R}_2$. If $\Gamma_i$ is a group acting on $\mathcal{R}_i$ in the usual way, then $\Gamma := \Gamma_1 \times \Gamma_2$ acts on $\mathcal{R}$. In this case $C_r^*(\mathcal{R}, q) \rtimes \Gamma$ is defined as an algebra of bounded linear operators on

$$L^2(\mathcal{R}) \otimes \mathbb{C}[\Gamma] = L^2(\mathcal{R}_1) \otimes \mathbb{C}[\Gamma_1] \otimes L^2(\mathcal{R}_2) \otimes \mathbb{C}[\Gamma_2].$$

It is the closure of the algebraic tensor product of the algebras $C_r^*(\mathcal{R}_1, q_1) \rtimes \Gamma_1$ and $C_r^*(\mathcal{R}_2, q_2) \rtimes \Gamma_2$ in $B(L^2(\mathcal{R}) \otimes \mathbb{C}[\Gamma])$, which means that

$$C_r^*(\mathcal{R}, q) \rtimes \Gamma = C_r^*(\mathcal{R}_1, q_1) \rtimes \Gamma_1 \otimes_{\min} C_r^*(\mathcal{R}_2, q_2) \rtimes \Gamma_2, \tag{2.12}$$

the minimal tensor product of $C^*$-algebras. These $C^*$-algebras are separable and of type I, so the paper [Schochet 1982] applies to them. The Künneth theorem [Schochet 1982] says that there exists a natural $\mathbb{Z}/2\mathbb{Z}$-graded short exact sequence

$$\begin{aligned} 0 \to K_*(C_r^*(\mathcal{R}_1, q_1) \rtimes \Gamma_1) \otimes_{\mathbb{Z}} K_*(C_r^*(\mathcal{R}_2, q_2) \rtimes \Gamma_2) \to K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma) \\ \to \mathrm{Tor}_{\mathbb{Z}}\big(K_*(C_r^*(\mathcal{R}_1, q_1) \rtimes \Gamma_1), K_*(C_r^*(\mathcal{R}_2, q_2) \rtimes \Gamma_2)\big) \to 0. \end{aligned} \tag{2.13}$$

In particular, this becomes an isomorphism

$$K_*(C_r^*(\mathcal{R}_1, q_1) \rtimes \Gamma_1) \otimes_{\mathbb{Z}} K_*(C_r^*(\mathcal{R}_2, q_2) \rtimes \Gamma_2) \xrightarrow{\sim} K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$$

if $K_*(C_r^*(\mathcal{R}_i, q_i) \rtimes \Gamma_i)$ has no torsion for $i = 1, 2$. With (2.13) we can often reduce the computation of K-groups to the case where $R$ is irreducible.

By (2.8) and the Green–Julg Theorem [Julg 1981],

$$K_*(C_r^*(\mathcal{R}, 1) \rtimes \Gamma) = K_*(C(T_{\mathrm{un}}) \rtimes W\Gamma) \cong K_*^{W\Gamma}(C(T_{\mathrm{un}})).$$

Moreover, by the equivariant Serre–Swan theorem [Phillips 1987, Theorem 2.3.1],

$$K_*^{W\Gamma}(C(T_{\mathrm{un}})) \cong K^*_{W\Gamma}(T_{\mathrm{un}}). \tag{2.14}$$

Together with Theorem 2.7 we get

$$K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma) \cong K^*_{W\Gamma}(T_{\mathrm{un}}). \tag{2.15}$$

The right-hand side in (2.14) and (2.15) is just Atiyah's $W\Gamma$-equivariant K-theory of the compact Hausdorff space $T_{\mathrm{un}}$. Let $T_{\mathrm{un}} /\!/ W\Gamma$ be the extended quotient (see also Section 2C). We recall from [Baum and Connes 1988, Theorem 1.19] that the equivariant Chern character gives a natural isomorphism

$$K^*_{W\Gamma}(T_{\mathrm{un}}) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} H^*(T_{\mathrm{un}} /\!/ W\Gamma; \mathbb{C}). \tag{2.16}$$

(Here $H^*$ could be many cohomology theories; in this paper we stick to Čech cohomology.) With (2.14) we find a canonical isomorphism

$$K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^*(T_{\mathrm{un}} /\!/ W\Gamma; \mathbb{C}). \tag{2.17}$$

In (2.16) it is essential to use complex coefficients, so this does not tell us much about the torsion in $K_*(C_r^*(\mathcal{R}, q) \rtimes \Gamma)$. To study the torsion elements better, we will compare the topological K-theory of relevant $C^*$-algebras with a suitable version of equivariant cohomology from [Bredon 1967]. Let $\Sigma$ be a countable, locally finite and finite dimensional $G$-CW-complex, where $G$ is a finite group. Assume that all cells are oriented and that the action of $G$ preserves these orientations.

We define a category $\mathcal{K}$ whose objects are the finite subcomplexes of $\Sigma$. The morphisms from $K$ to $K'$ are the maps $K \to K' : x \to gx$ for $g \in G$ such that $gK \subset K'$. Now a local coefficient system on $\Sigma$ is a covariant functor from $\mathcal{K}$ to the category of abelian groups, and the group $C^q(\Sigma; \mathfrak{L})$ of $q$-cochains is the set of all functions $f$ on the $q$-cells of $\Sigma$ with the property that $f(\tau) \in \mathfrak{L}(\tau)$ for all $\tau$. Furthermore, we define a coboundary map $\mathrm{d} : C^q(\Sigma; \mathfrak{L}) \to C^{q+1}(\Sigma; \mathfrak{L})$ by

$$(\mathrm{d}f)(\sigma) = \sum_{\tau \in \Sigma^{(q)}} [\tau : \sigma] \mathfrak{L}(\tau \to \sigma) f(\tau), \tag{2.18}$$

where the sum runs over the set $\Sigma^{(q)}$ of all $q$-cells and the incidence number $[\tau : \sigma]$ is the degree of the attaching map from $\partial\sigma$ (the boundary of $\sigma$ in the standard topological sense) to $\tau/\partial\tau$. The group $G$ acts naturally on this complex by cochain maps so, for any $K \subset \Sigma$, $(C^*(K; \mathfrak{L})^G, \mathrm{d})$ is a differential complex. We define the equivariant cohomology of $K$ with coefficients in $\mathfrak{L}$ as

$$H^q_G(K; \mathfrak{L}) := H^q(C^*(K; \mathfrak{L})^G, \mathrm{d}). \tag{2.19}$$

More generally, for $K' \subset K$, $C^*(K, K'; \mathfrak{L})$ is the kernel of the restriction map $C^*(K; \mathfrak{L}) \to C^*(K'; \mathfrak{L})$ and

$$H^q_G(K, K'; \mathfrak{L}) = H^q(C^*(K, K'; \mathfrak{L})^G, \mathrm{d}). \tag{2.20}$$

By construction there exists a local coefficient system $\mathfrak{L}^G$ (more or less consisting of the $G$-invariant elements of $\mathfrak{L}$) on the CW-complex $\Sigma/G$ such that the differential complexes $(C^*(K, K'; \mathfrak{L})^G, \mathrm{d})$ and $(C^*(K/G, K'/G; \mathfrak{L}^G), \mathrm{d})$ are isomorphic. Notice that $\mathfrak{L}^G$ defines a sheaf over $\Sigma/G$ (with the cells as cover), such that

$$H^q_G(K, K'; \mathfrak{L}) \cong \check{H}^q(K/G, K'/G; \mathfrak{L}^G). \tag{2.21}$$

Let $\Sigma^p$ be the $p$-skeleton of $\Sigma$. We capture all the above things in a spectral sequence $(E_r^{p,q})_{r \geq 1}$, degenerating already for $r \geq 2$, as follows:

$$E_1^{p,q} = H_G^{p+q}(\Sigma^p, \Sigma^{p-1}; \mathfrak{L}) = \begin{cases} C^p(\Sigma; \mathfrak{L})^G & \text{if } q = 0, \\ 0 & \text{if } q > 0, \end{cases} \tag{2.22}$$

$$E_2^{p,q} = \begin{cases} H_G^p(\Sigma; \mathfrak{L}) & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases} \tag{2.23}$$

The differential $d_1^E$ is the composition

$$E_1^{p,q} \to C^{p+q}(\Sigma^p; \mathfrak{L})^G \to E_1^{p+1,q} \tag{2.24}$$

of the maps induced by the inclusion $(\Sigma^p, \varnothing) \to (\Sigma^p, \Sigma^{p-1})$ and the coboundary d.

We are mostly interested in this cohomology theory for a particular coefficient system, which we now define. Consider the Fréchet algebra

$$B = C(\Sigma; M_N(\mathbb{C})) = M_N(C(\Sigma)). \tag{2.25}$$

(It is a $C^*$-algebra if $\Sigma$ is compact.) We assume that we have $u_g \in B^\times$ such that

$$gb(x) = u_g(x)b(g^{-1}x)u_g^{-1}(x) \tag{2.26}$$

defines an action of $G$ on $B$. Then the invariants $B^G$ constitute a Fréchet subalgebra of $B$. Notice that by (2.6) and (2.9) the $C^*$-completion of an affine Hecke algebra is a direct sum of algebras of this form.

To associate a local coefficient system to $B^G$. we first assume that $K$ is connected. In that case we let

$$G_K := \{g \in G : gx = x \ \forall x \in K\} \tag{2.27}$$

be the isotropy group of $K$ and we define $\mathfrak{L}_u(K)$ to be the free abelian group on the (equivalence classes of) irreducible projective $G_K$-representations contained in $(\pi_x, \mathbb{C}^N)$, where $\pi_x(g) = u_g(x)$ for $g \in G_K, x \in K$. By the continuity of the $u_g$ we get the same group for any $x \in K$. If $K$ is not connected, then we let $\{K_i\}_i$ be its connected components, and we define

$$\mathfrak{L}_u(K) = \textstyle\prod_i \mathfrak{L}_u(K_i). \tag{2.28}$$

Suppose that $gK \subset K'$ and that $\rho$ is a projective $G_K$-representation. Then we define a projective $G_{K'}$-representation by

$$\mathfrak{L}_u(g : K \to K')\rho(g') = \rho(g^{-1}g'g), \quad g' \in G_{K'}. \tag{2.29}$$

If $h \in G$ gives the same map from $K$ to $K'$ as $g$ then $h^{-1}g \in G_K$ and

$$\mathfrak{L}_u(h : K \to K')\rho(g') = \rho(h^{-1}g'h) = \rho(h^{-1}g)\rho(g^{-1}g'g)\rho(g^{-1}h), \tag{2.30}$$

so $\mathfrak{L}_u(h : K \to K')\rho$ is isomorphic to $\mathfrak{L}_u(g : K \to K')\rho$ as a projective representation. This makes $\mathfrak{L}_u$ into a functor. We can regard $\mathfrak{L}_u$ as a sheaf on $\Sigma$, where a section $s$ is continuous on $U$ if and only if $s(K)|_{G_{K'}} = s(K')$ for every inclusion $K \subset K' \subset U$.

**Example 2.31.** Suppose that $u_g(x) = 1$ for all $x \in \Sigma$, $g \in G$. Then $\mathfrak{L}_u$ and $\mathfrak{L}_u^G$ are the constant sheaves $\mathbb{Z}$ over $\Sigma$ and $\Sigma/G$, respectively, and

$$H_G^*(\Sigma; \mathfrak{L}_u) \cong \check{H}^*(\Sigma/G; \mathbb{Z}) \tag{2.32}$$

is the ordinary cellular cohomology of $\Sigma/G$. Furthermore,

$$K_*(B^G) \cong K_*(C(\Sigma/G; M_N(\mathbb{C}))) = K_*(C(\Sigma/G)),$$

which is isomorphic to $\check{H}^*(\Sigma/G; \mathbb{Z})$ modulo torsion.

It turns out that a relation like (2.16), between $K_*(B^G)$ and the Čech cohomology $H^*(\Sigma/G;\, \mathfrak{L}_u^G)$, is valid in the generality of the algebras $B^G$ from (2.25) and (2.26). Notice that we do not require $\Sigma$ to be compact; we consider the K-theory of $B^G$ as a Fréchet algebra. The skeleton of the CW-complex $\Sigma$ gives rise to the following filtration:

$$\begin{aligned} K_*(B^G) &= K_*^0(B^G) \supset K_*^1(B^G) \supset \cdots \supset K_*^{\dim \Sigma}(B^G) \supset K_*^{1+\dim \Sigma}(B^G) = 0, \\ K_*^p(B^G) &:= \mathrm{im}\big(K_*(C_0(\Sigma/\Sigma^{p-1};\, M_N(\mathbb{C}))^G) \to K_*(C(\Sigma;\, M_N(\mathbb{C}))^G)\big). \end{aligned} \tag{2.33}$$

**Theorem 2.34.** *The graded group associated with the filtration (2.33) is isomorphic to $\check H^*(\Sigma/G;\, \mathfrak{L}_u^G)$. In particular, there is an (unnatural) isomorphism*

$$K_*(B^G) \otimes \mathbb{Q} \cong \check H^*(\Sigma/G;\, \mathfrak{L}_u^G \otimes \mathbb{Q}) \tag{2.35}$$

*and*

$$K_*(B^G) \cong \check H^*(\Sigma/G;\, \mathfrak{L}_u^G)$$

*if the right-hand side is torsion free.*

*Proof.* For $p, r \geq 0$ we set $K(p, p+r) = K_*(C_0(\Sigma^{p+r-1}/\Sigma^{p-1};\, M_N(\mathbb{C}))^G)$. When $p' \geq p$ and $p' + r' \geq p + r$, the map

$$(\Sigma^{p+r-1}, \Sigma^{p-1}) \to (\Sigma^{p'+r'-1}, \Sigma^{p'-1})$$

induces a group homomorphism $K(p', p'+r') \to K(p, p+r)$. For any $s \geq 0$ the sequence

$$(\Sigma^{p+r-1}, \Sigma^{p-1}) \to (\Sigma^{p+r+s-1}, \Sigma^{p-1}) \to (\Sigma^{p+r+s-1}, \Sigma^{p+r-1}) \tag{2.36}$$

gives rise to a connecting homomorphism $K(p, p+r) \to K(p+r, p+r+s)$. Using [Cartan and Eilenberg 1956, Section XV.7] we construct a spectral sequence $(F_r^p)_{r\geq 1}$ with terms

$$\begin{aligned} F_1^p &= K(p, p+1)/K(p, p) = K_*(C_0(\Sigma^p/\Sigma^{p-1};\, M_N(\mathbb{C}))^G), \\ F_\infty^p &= K(p, \infty)/K(p+1, \infty) = K_*^p(B^G)/K_*^{p+1}(B^G). \end{aligned} \tag{2.37}$$

The entire setting is $\mathbb{Z}/2\mathbb{Z}$-graded by the K-degree. We put

$$K^q(p, p+r) = K_{p+q}(C_0(\Sigma^{p+r-1}/\Sigma^{p-1};\, M_N(\mathbb{C}))^G)$$

and we refine (2.37) to

$$\begin{aligned} F_1^{p,q} &= K_{p+q}(C_0(\Sigma^p/\Sigma^{p-1};\, M_N(\mathbb{C}))^G), \\ F_\infty^{p,q} &= K_{p+q}^p(B^G)/K_{p+q}^{p+1}(B^G). \end{aligned} \tag{2.38}$$

By the definition of a $G$-CW-complex, the pointwise stabilizer of a $p$-cell $\sigma$ is equal to its setwise stabilizer in $G$. Consequently,

$$C_0(\Sigma^p/\Sigma^{p-1}; M_N(\mathbb{C}))^G \cong \prod_{\sigma\in\Sigma^{(p)}/G} C_0(\mathbb{R}^p)\otimes M_N(\mathbb{C})^{G_\sigma}$$

and $F_1^{p,1}=0$. From Bott periodicity and the definition of $\mathfrak{L}_u$ in (2.28) we see that

$$F_1^{p,0}\cong \prod_{\sigma\in\Sigma^{(p)}/G}\mathfrak{L}_u(\sigma)\cong\Big(\prod_{\sigma\in\Sigma^{(p)}}\mathfrak{L}_u(\sigma)\Big)^G.$$

Now replace $\mathfrak{L}$ in (2.22) by $\mathfrak{L}_u$ and sum over all $q$ to obtain $E_r^p$. If we compare the result with $F_1^p=F_1^{p,0}\oplus F_1^{p,1}$, we see that $E_1^p\cong F_1^p$. So we get a diagram

$$\begin{array}{ccc} F_1^{p,q} & \xrightarrow{\quad d_1^F\quad} & F_1^{p+1,q} \\ \| \wr & & \| \wr \\ \prod_{n\in\mathbb{Z}} E_1^{p,q+2n} & \xrightarrow{d_1^E} & \prod_{n\in\mathbb{Z}} E_1^{p+1,q+2n} \end{array} \tag{2.39}$$

The differential $d_1^F$ for $F_1^*$ is induced from the construction of a mapping cone of a Puppe sequence in the category of $C^*$-algebras, coming from (2.36). This is the noncommutative counterpart of the construction of the differential in cellular cohomology, so by naturality $d_1^F$ corresponds to $d_1^E$ under the above isomorphism. Therefore, the spectral sequences $E_r^p$ and $F_r^p$ are isomorphic, and in particular $F_r^p$ degenerates for $r\geq 2$. Now the isomorphism (2.35) follows from (2.21).

If $\check{H}^*(\Sigma/G;\mathfrak{L}_u^G)$ is torsion free, then every term $E_\infty^p\cong F_\infty^p$ must be torsion free. Hence in this case both $K_*(B^G)$ and $\check{H}^*(\Sigma/G;\mathfrak{L}_u^G)$ are free abelian groups, of the same rank. □

Theorem 2.34 allows us to reduce the computations of $K_*(C_r^*(\mathcal{R},q))$ to Čech cohomology, where a lot of tools are available. For several root data it is easiest to look at the case $q=1$, for which we will develop more machinery in the next subsection. For some other root data (in particular of type $\mathrm{PGL}_n$) it is more convenient to study $K_*(C_r^*(\mathcal{R},q))$ with $q\neq 1$, for then there are fewer possibilities for torsion elements, compared to $q=1$. In those cases we need the full force of Theorem 2.34.

**2C.** ***Crossed products.*** In the special case of crossed products the technique from Theorem 2.34 can be improved. A crucial role will be played by the extended quotient, whose definition we recall now. Let $G$ be a finite group $G$ acting on a topological space $\Sigma$. We define

$$\widetilde{\Sigma}=\{(g,t)\in G\times T_{\mathrm{un}}: g(t)=t\},$$

a closed subset of the topological space $G \times \Sigma$. The group $G$ acts on $\widetilde{\Sigma}$ by

$$g(g', t) = (gg'g^{-1}, g(t)).$$

The (geometric) extended quotient of $\Sigma$ by $G$ is defined as

$$\Sigma//G = \widetilde{\Sigma}/G. \tag{2.40}$$

It decomposes as

$$\Sigma//G = \bigsqcup_{g \in \mathrm{cc}(G)} \Sigma^w / Z_G(g), \tag{2.41}$$

where $\mathrm{cc}(G)$ denotes a set of representatives for the conjugacy classes in $G$.

We will develop a method that allows one to pass from the $G$-equivariant K-theory of $\Sigma$ to the integral cohomology of $\Sigma//G$. However, it does not work automatically; we require that the cohomology is torsion-free and that all $G$-isotropy groups of points of $\Sigma$ are Weyl groups (and it uses some of our earlier results on the representation rings of Weyl groups).

From now on we assume that $\Sigma$ is a smooth manifold (possibly with boundary) on which $G$ acts smoothly. According to [Illman 1978] $\Sigma$ also admits the structure of a countable, locally finite, finite-dimensional $G$-simplicial complex. The crossed product $C(\Sigma) \rtimes G$ fits in the framework of (2.25) and (2.26) by the isomorphisms

$$C(\Sigma) \rtimes G \cong C(\Sigma; \mathrm{End}_{\mathbb{C}}(\mathbb{C}[G]))^G = B^G. \tag{2.42}$$

In this case $u_g(x)$ is right multiplication by $g^{-1}$ and $\pi_x$ is the direct sum of $[G : G_x]$ copies of the regular representation of $G_x$. It is not hard to see that $\mathfrak{L}_u^G \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic to the direct image of the constant sheaf $\mathbb{C}$ on $\widetilde{\Sigma}$, under the canonical map $pr : \widetilde{\Sigma}/G \to \Sigma/G$. Since $pr$ is finite-to-one, there are no topological complications, and we get an isomorphism

$$H_G^*(\Sigma; \mathfrak{L}_u \otimes \mathbb{C}) \cong \check{H}^*(\Sigma/G; \mathfrak{L}_u^G \otimes_{\mathbb{Z}} \mathbb{C}) \cong \check{H}^*(\widetilde{\Sigma}/G; \mathbb{C}). \tag{2.43}$$

From this one can recover (2.16). Unfortunately, this approach does not automatically lead to an isomorphism between $\check{H}^*(\Sigma/G; \mathfrak{L}_u^G)$ and $\check{H}^*(\widetilde{\Sigma}/G; \mathbb{Z})$, for $\mathfrak{L}_u^G$ need not be isomorphic to the direct image of the constant sheaf $\mathbb{Z}$ under $pr$.

Sometimes this can be approached better via a dual homology theory. Let $C_q(\Sigma; \mathfrak{L}_u)$ be the subgroup of $C^q(\Sigma; \mathfrak{L}_u)$ consisting of functions supported on finitely many $q$-cells. The graded $\mathbb{Z}$-module $C_*(\Sigma; \mathfrak{L}_u)$ admits a $G$-equivariant boundary map, which in the notation of (2.18) can be written as

$$\partial : C^{q+1}(\Sigma; \mathfrak{L}_u) \to C^q(\Sigma; \mathfrak{L}_u), \quad (\partial f)(\tau) = \sum_{\sigma \in \Sigma^{(q+1)}} [\tau : \sigma] \, \mathrm{ind}_{G_\sigma}^{G_\tau}(f(\sigma)).$$

This is a natural perfect pairing on each $\mathfrak{L}_u(\sigma) \cong R_{\mathbb{Z}}(G_\sigma)$, since $G_\sigma$ is a finite group. With that one sees that the differential complex $(C^*(\Sigma; \mathfrak{L}_u), \mathrm{d})$ is isomorphic to

$\mathrm{Hom}_{\mathbb{Z}}((C^*(\Sigma; \mathfrak{L}_u), \partial), \mathbb{Z})$. This persists to the $G$-invariants:

$$(C^*(\Sigma/G; \mathfrak{L}_u^G), \mathrm{d}) \cong \mathrm{Hom}_{\mathbb{Z}}((C^*(\Sigma/G; \mathfrak{L}_u^G), \partial), \mathbb{Z}). \tag{2.44}$$

Suppose now that $\Sigma$ is a manifold on which the finite group $G$ acts smoothly. For $t \in \Sigma$ the isotropy $G_t$ acts $\mathbb{R}$-linearly on the tangent space $T_t(\Sigma)$. We say that $G_t$ is a Weyl group if it is the Weyl group of some root system in $T_t(\Sigma)$.

**Theorem 2.45.** *Let $G$ be a finite group acting smoothly on a manifold $\Sigma$.*

(a) *Suppose that $G_t$ is a Weyl group for all $t \in \Sigma$. Then*

$$H_i(C_*(\Sigma/G; \mathfrak{L}_u^G), \partial) \cong H_i(\Sigma/\!/G; \mathbb{Z}) \quad \text{for all } i \in \mathbb{Z}_{\geq 0}.$$

(b) *Suppose that the conclusion of part (a) holds, and that $H^*(\Sigma/\!/G; \mathbb{Z})$ is torsion-free. Then*

$$K_*(C(\Sigma) \rtimes G) \cong H^*(\Sigma/\!/G; \mathbb{Z}).$$

*Proof.* (a) For every subgroup $H \subset G$ the set of fixpoints $\Sigma^H$ is a submanifold of $\Sigma$ [Baum and Connes 1988, Lemma 4.1]. It follows that for every $g \in \mathrm{cc}(G)$ and every connected component $\Sigma_i^g$ of $\Sigma^g$, the map $t \mapsto G_t$ is constant on an open dense subset of $\Sigma_i^g$. Pick a point $t_i$ in this dense subset of $\Sigma_i^g$ and write $G_{t_i} = W_i$. By assumption $W_i$ is a Weyl group and $G_t \supset W_i$ for all $t \in \Sigma_i^g$.

For a cell $\tau$ and $t \in \tau \setminus \partial\tau$ we have $G_\tau = G_t$. Using Proposition 1.36 we define, for $t \in \Sigma_i^g$, $t \in \tau \setminus \partial\tau$,

$$s(g, t) = s(g, \tau) = \mathrm{ind}_{W_i}^{G_\tau}(H(u_g, \rho_g)). \tag{2.46}$$

We may and will assume that $s(g, h\tau) = h \cdot s(g, \tau)$ for all $h \in Z_G(g)$. This extends uniquely to a $G$-equivariant map $\Sigma^g \to \bigcup_{\tau \subset \Sigma^g} R_{\mathbb{Z}}(G_\tau)$, and hence defines an element $s(g) \in C^*(\Sigma; \mathfrak{L}_u^G)$. Thus $s(g)$ is nonzero at $Gt \in \Sigma/G$ if and only if $Gt \cap \Sigma^g$ is nonempty.

The $s(g)$ with $g \in \mathrm{cc}(G)$ yield precisely one representation for each element of the extended quotient

$$\Sigma/\!/G = \bigsqcup_{g \in \mathrm{cc}(G)} \Sigma^g / Z_G(g).$$

So for every $t \in \Sigma$ we get exactly $|\mathrm{cc}(G_t)| = |\mathrm{Irr}(G_t)|$ representations $s(g, t)$. By Proposition 1.17 the $s(g, t)$ with $g \in \mathrm{cc}(G)$ and $t \in G\Sigma^g$ form a $\mathbb{Z}$-basis of the representation ring of the Weyl group $G_t$. This also shows that for $t \in \tau \setminus \partial\tau$, the set

$$\big\{h \cdot \mathrm{ind}_{W_i}^{G_\sigma}(H(u_g, \rho_g)) : \sigma \subset \Sigma^g,\ h \in G_\tau \setminus G,\ h\sigma = \tau\big\} \tag{2.47}$$

is linearly independent in $R_{\mathbb{Z}}(G_t) = R_{\mathbb{Z}}(G_\tau)$.

Let $\tau \otimes s(g, \tau)$ with $\tau \subset \Sigma^g$ be the terms of which $s(g)$ is made. Then (2.46) entails that the span of the $\tau \otimes s(g, \tau)$ forms a subchain complex $C(g, \Sigma)$ of $(C_*(\Sigma/G; \mathfrak{L}_u^G), \partial)$ and (2.47) implies that $C(g, \Sigma)$ is isomorphic to the cellular homology complex $C_*(\Sigma^g / Z_G(g); \mathbb{Z})$. Since the $s(g,t)$ form a basis of $R_{\mathbb{Z}}(G_t)$ for every $t \in \Sigma$,

$$C_*(\Sigma/G; \mathfrak{L}_u^G) = \bigoplus_{g \in \mathrm{cc}(G)} C(g, \Sigma).$$

The claim about the homology of $(C_*(\Sigma/G; \mathfrak{L}_u^G), \partial)$ follows.

(b) In the absence of torsion, the universal coefficient theorem says that the dual of the homology of a different complex is naturally isomorphic to the cohomology of the dual complex. This gives the horizontal isomorphisms in the following commutative diagram:

$$\begin{array}{ccc} H^*(\Sigma /\!/ G; \mathbb{Z}) & \xrightarrow{\;\sim\;} & \mathrm{Hom}_{\mathbb{Z}}(H_*(\Sigma /\!/ G; \mathbb{Z}), \mathbb{Z}) \\ \uparrow & & \downarrow \\ H^*(C^*(\Sigma/G; \mathfrak{L}_u^G), \mathrm{d}) & \xrightarrow{\;\sim\;} & \mathrm{Hom}_{\mathbb{Z}}(H_*(C_*(\Sigma/G; \mathfrak{L}_u^G), \partial), \mathbb{Z}) \end{array} \tag{2.48}$$

By assumption the right vertical arrow is an isomorphism. We define the left vertical arrow to be the isomorphism such that the diagram becomes commutative. The lower left corner of (2.48) is $H_G^*(\Sigma; \mathfrak{L})$, which by Theorem 2.34 is isomorphic to $K_*(C(\Sigma) \rtimes G)$. □

Let us return to the case of $C(T_{\mathrm{un}}) \rtimes W = C_r^*(W^e)$, where $T_{\mathrm{un}}$, $W$ and $W^e$ come from a root datum $\mathcal{R}$. Then $W$ acts by algebraic group automorphisms on the compact torus $T_{\mathrm{un}}$.

**Corollary 2.49.** *Let $\mathcal{R}$ be the root datum of a reductive algebraic group with simply connected derived group, and assume that $H^*(T_{\mathrm{un}} /\!/ W; \mathbb{Z})$ is torsion-free. Then for any positive parameter function $q$,*

$$K_*(C_r^*(\mathcal{R}, q)) \cong H^*(T_{\mathrm{un}} /\!/ W; \mathbb{Z}).$$

*Proof.* Let $\mathcal{R}$ be the root datum of $(\mathcal{G}(\mathbb{C}), T)$. By Steinberg's connectedness theorem [Steinberg 1968], the group $Z_{\mathcal{G}(\mathbb{C})}(t)$ is connected for every $t \in T$. Hence $W_t = W(Z_{\mathcal{G}(\mathbb{C})}(t), T)$ is always a Weyl group. Now Theorem 2.45 says that

$$H^*(T_{\mathrm{un}} /\!/ W; \mathbb{Z}) \cong K_*(C(T_{\mathrm{un}}) \rtimes W) = K_*(C_r^*(\mathcal{R}, 1)).$$

Apply Theorem 2.11 to the right-hand side. □

In fact Corollary 2.49 also applies to some other root data, for example those of type $\mathrm{SO}_{2n+1}$.

## 3. Examples

In this section, we compute the topological K-theory of the $C^*$-Hecke algebras $C_r^*(\mathcal{R}, q)$ associated to common root data $\mathcal{R}$. As discussed after Theorem 2.11, it suffices to do so for $q = 1$ or for generic parameter functions. For $q = 1$ we apply Theorem 2.45, when that is possible.

Our approach for $q \neq 1$ involves the following steps.

(1) Explicitly write down the root datum and the associated Weyl groups.

From (2.9) we get a canonical decomposition

$$C_r^*(\mathcal{R}, q) \rtimes \Gamma = \bigoplus_P C_r^*(\mathcal{R}, q)_P \rtimes \Gamma_P, \tag{3.1}$$

where $P$ runs over a set of representatives for the action of $\mathcal{G}$ on the power set of $\Delta$ and $\Gamma_P$ is the setwise stabilizer of $P$ in $\Gamma$.

(2) List a good set of representatives $P$.

For every chosen $P$ we do the following:

(3) Determine the root datum $\mathcal{R}_P$ and the residual points.

(4) Determine the discrete series of $\mathcal{H}(\mathcal{R}_P, q_P)$, and all the relevant intertwining operators.

(5) Describe $C_r^*(\mathcal{R}, q)_P \rtimes \Gamma_P$ and its space of irreducible representations.

(6) Calculate $K_*(C_r^*(\mathcal{R}, q)_P \rtimes \Gamma_P)$.

Often the final step can be reduced to commutative $C^*$-algebras. When this is not possible, we transfer the problem to sheaf cohomology via Theorem 2.34.

**3A. *Type* $\mathbf{GL}_n$.** The easiest root data to study are those associated with the reductive group $\mathrm{GL}_n$. The right way to do this was shown by Plymen. From [Plymen 1987, Lemma 5.3] we know that the topological K-groups of these affine Hecke algebras are free abelian, of a finite rank which is explicitly given. Strictly speaking, we do not really need to study this root datum, as we could just refer to Plymen's results. Nevertheless, since many other examples rely on this case, we include an analysis.

From now on many things will be parametrized by partitions and permutations, so let us agree on some notations. We write partitions in decreasing order and abbreviate $(x)^3 = (x, x, x)$. A typical partition looks like

$$\mu = (\mu_1, \mu_2, \dots, \mu_d) = (n)^{m_n} \cdots (2)^{m_2}(1)^{m_1}, \tag{3.2}$$

where some of the multiplicities $m_i$ may be 0. By $\mu \vdash n$ we mean that the weight of $\mu$ is

$$|\mu| = \mu_1 + \cdots + \mu_d = n.$$

The number of different $\mu_i$ (i.e., the number of blocks in the diagram of $\mu$) is denoted by $b(\mu)$ and the dual partition (obtained by reflecting the diagram of $\mu$)

by $\mu^\vee$. Sometimes we abbreviate

$$\begin{aligned} \gcd(\mu) &= \gcd(\mu_1, \dots, \mu_d), \\ \mu! &= \mu_1!\mu_2! \cdots \mu_d!. \end{aligned} \tag{3.3}$$

With a such partition $\mu$ of $n$ we associate the permutation

$$\sigma(\mu) = (1\,2\, \cdots\, \mu_1)(\mu_1 + 1\, \cdots\, \mu_1 + \mu_2) \cdots (n + 1 - \mu_d\, \cdots\, n) \in S_n.$$

As is well-known, this gives a bijection between partitions of $n$ and conjugacy classes in the symmetric group $S_n$. The centralizer $Z_{S_n}(\sigma(\mu))$ is generated by the cycles

$$((\mu_1 + \cdots + \mu_i + 1)(\mu_1 + \cdots + \mu_i + 2) \cdots (\mu_1 + \cdots + \mu_i + \mu_{i+1}))$$

and the "permutations of cycles of equal length" — for example, if $\mu_1 = \mu_2$,

$$(1\, \mu_1 + 1)(2\, \mu_1 + 2) \cdots (\mu_1\, 2\mu_1). \tag{3.4}$$

Using the second presentation of $\mu$, this means that

$$Z_{S_n}(\sigma(\mu)) \cong \prod_{l=1}^{n} (\mathbb{Z}/l\mathbb{Z})^{m_l} \rtimes S_{m_l}.$$

Let us recall the definition of $\mathcal{R}(\mathrm{GL}_n)$ and the associated groups. Below $Q$ and $Q^\vee$ are the root and coroot lattices.

$$\begin{aligned}
X &= \mathbb{Z}^n, & Q &= \{x \in X : x_1 + \cdots + x_n = 0\}, \\
Y &= \mathbb{Z}^n, & Q^\vee &= \{y \in Y : y_1 + \cdots + y_n = 0\}, \\
T &= (\mathbb{C}^\times)^n, & t &= (t(e_1), \dots, t(e_n)) = (t_1, \dots, t_n),
\end{aligned}$$
$$R = \{e_i - e_j \in X : i \neq j\}, \qquad \alpha_0 = e_1 - e_n,$$
$$R^\vee = \{e_i - e_j \in Y : i \neq j\}, \qquad \alpha_0^\vee = e_1 - e_n,$$
$$s_i = s_{\alpha_i} = s_{e_i - e_{i+1}}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{\alpha_1} s_{\alpha_0} t_{-\alpha_1} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x \rangle \alpha_0,$$
$$W = \langle s_1, \cdots, s_{n-1} \mid s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e : |i - j| > 1 \rangle \cong S_n,$$
$$S^{\mathrm{aff}} = \{s_0, s_1, \dots, s_{n-1}\},$$
$$W^{\mathrm{aff}} = \langle s_0, W_0 \mid s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^3 = (s_0 s_{n-1})^3 = e \text{ if } 2 \leq i \leq n-2 \rangle,$$
$$W^e = W^{\mathrm{aff}} \rtimes \Omega, \qquad \Omega = \langle t_{e_1}(1\,2\, \cdots\, n) \rangle \cong \mathbb{Z}.$$

Because all roots of $R$ are conjugate, $s_0$ is conjugate to any $s_i \in S^{\mathrm{aff}}$. Hence for any label function we have $q(s_0) = q(s_i) := q$. Every point of $T$ is $W$-conjugate to one of the form $t = ((t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d}) \in T$ and

$$W_t = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_d}. \tag{3.5}$$

• **Case $q = 1$.**

By (2.17) and (2.41) we have

$$K_*(C_r^*(W^e)) \otimes \mathbb{C} \cong \check{H}^*(\widetilde{T_{\mathrm{un}}}/S_n; \mathbb{C}) \cong \bigoplus_{\mu \vdash n} \check{H}^*\big(T_{\mathrm{un}}^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)); \mathbb{C}\big). \tag{3.6}$$

Therefore, we want to determine $T_{\mathrm{un}}^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$. If $\mu$ is as in (3.2) then

$$\begin{aligned} T^{\sigma(\mu)} &= \{(t_1)^{\mu_1}(t_{\mu_1+1})^{\mu_2}\cdots(t_n)^{\mu_d} \in T\}, \\ T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)) &\cong (\mathbb{C}^\times)^{m_n}/S_{m_n} \times \cdots \times (\mathbb{C}^\times)^{m_1}/S_{m_1}, \end{aligned} \tag{3.7}$$

where $S_{m_l}$ acts on $(\mathbb{C}^\times)^{m_l}$ by permuting the coordinates. To handle this space we use the following nice, elementary result, a proof of which can be found for example in [Plymen 1987, Lemma 5.1].

**Lemma 3.8.** *For any $m \in \mathbb{N}$ there is an isomorphism of algebraic varieties*

$$(\mathbb{C}^\times)^m/S_m \cong \mathbb{C}^{m-1} \times \mathbb{C}^\times.$$

Consequently, $T_{\mathrm{un}}^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$ has the homotopy type of $(S^1)^{b(\mu)}$. In particular, its integral cohomology is torsion-free, so Corollary 2.49 is applicable. It says that (3.6) can be refined to

$$K_*(C_r^*(W^e)) \cong \bigoplus_{\mu \vdash n} \check{H}^*((S^1)^{b(\mu)}; \mathbb{Z}) \cong \bigoplus_{\mu \vdash n} \mathbb{Z}^{2^{b(\mu)}}. \tag{3.9}$$

• **Generic, equal parameter case $q \neq 1$.**

Inequivalent subsets of $\Delta$ are parametrized by partitions $\mu$ of $n$. For the typical partition (3.2) we have

$$\begin{aligned} P_\mu &= \Delta \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{n-\mu_d}\}, \\ R_{P_\mu} &\cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \cong R^\vee_{P_\mu}, \\ X^{P_\mu} &\cong \mathbb{Z}(e_1 + \cdots + e_{\mu_1})/\mu_1 + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n)/\mu_d, \\ X_{P_\mu} &\cong (\mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n))^{m_n} + \cdots + (\mathbb{Z}^2/\mathbb{Z}(e_1 + e_2))^{m_2}, \\ Y^{P_\mu} &= \mathbb{Z}(e_1 + \cdots + e_{\mu_1}) + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n), \\ Y_{P_\mu} &= \{y \in \mathbb{Z}^n : y_1 + \cdots + y_{\mu_1} = \cdots = y_{n+1-\mu_d} + \cdots + y_n = 0\}, \\ T^{P_\mu} &= \{(t_1)^{\mu_1} \cdots (t_n)^{\mu_d} \in T\}, \\ T_{P_\mu} &= \{t \in T : t_1 t_2 \cdots t_{\mu_1} = \cdots = t_{n+1-\mu_d} \cdots t_n = 1\}, \\ K_{P_\mu} &= \{t \in T^{P_\mu} : t_1^{\mu_1} = \cdots = t_n^{\mu_d} = 1\}, \end{aligned}$$

$$\begin{aligned} W_{P_\mu} &\cong (S_n)^{m_n} \times \cdots \times (S_2)^{m_2}, & W(P_\mu, P_\mu) &\cong S_{m_n} \times \cdots \times S_{m_2} \times S_{m_1}, \\ \mathcal{G}_{P_\mu P_\mu} &= K_{P_\mu} \rtimes W(P_\mu, P_\mu), & Z_{S_n}(\sigma(\mu)) &= W(P_\mu, P_\mu) \ltimes \textstyle\prod_{l=1}^n (\mathbb{Z}/l\mathbb{Z})^{m_l}. \end{aligned}$$

The $W_{P_\mu}$-orbits of residual points for $\mathcal{H}_{P_\mu}$ are parametrized by

$$K_{P_\mu}\big((q^{(\mu_1-1)/2}, q^{(\mu_1-3)/2}, \dots, q^{(1-\mu_1)/2}) \cdots (q^{(\mu_d-1)/2}, q^{(\mu_d-3)/2}, \dots, q^{(1-\mu_d)/2})\big).$$

This set is obviously in bijection with $K_{P_\mu}$, and indeed the intertwiners $\pi(k)$ with $k \in K_{P_\mu}$ act on it by multiplication. From the classification of the discrete series we know that here every residual point carries precisely one discrete series representation, namely a twist of a Steinberg representation. The quickest way to see this is with the Kazhdan–Lusztig classification of irreducible representations of affine Hecke algebras with equal parameters, in particular [Kazhdan and Lusztig 1987, Theorems 7.12 and 8.13]. This implies

$$\bigcup_\delta (P_\mu, \delta, T^{P_\mu})/K_{P_\mu} \cong T^{P_\mu},$$

$$\bigcup_\delta (P_\mu, \delta, T^{P_\mu})/\mathcal{G}_{P_\mu P_\mu} \cong T^{P_\mu}/W(P_\mu, P_\mu) = T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)).$$

If a point $\xi = (P_\mu, \delta, t)$ has a nontrivial stabilizer $\mathcal{G}_\xi$, then by the above this stabilizer is contained in $W(P_\mu, P_\mu) \cong \prod_{l=1}^n S_{m_l}$. It is easily seen that this isotropy group is actually a Weyl group, and that it equals the group $W(R_\xi)$ from (2.10). In other words, all R-groups are trivial for this root datum and $q \neq 1$, and all intertwining operators $\pi(g, \xi)$ from a representation $\pi(\xi)$ to itself are scalar multiples of the identity. So the action of $\mathcal{W}_{P_\mu P_\mu}$ on

$$C\Big(\bigsqcup_\delta T_{\mathrm{un}}^{P_\mu}; M_{n!/\mu!}(\mathbb{C})\Big) \tag{3.10}$$

is essentially only on $\bigsqcup_\delta T_u^{P_\mu}$ and the conjugation part doesn't really matter. In particular, we deduce that

$$C_r^*(\mathcal{R}, q) \cong \bigoplus_{\mu \vdash n} M_{n!/\mu!}\Big(C\Big(\bigsqcup_\delta T_{\mathrm{un}}^{P_\mu}\Big)\Big) \cong \bigoplus_{\mu \vdash n} M_{n!/\mu!}\big(T_{\mathrm{un}}^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))\big). \tag{3.11}$$

In particular, $C_r^*(\mathcal{R}, q)$ is Morita-equivalent with the commutative $C^*$-algebra of continuous functions on $T_{\mathrm{un}}//S_n$. Similar results were obtained by completely different methods in [Mischenko 1982].

We remark that $\mathrm{Irr}(C_r^*(\mathcal{R}, q))$ has a clear relation with the elliptic representation theory of symmetric groups. Every $\delta$ is essentially a Steinberg representation, so

$$\zeta^\vee(\delta \circ \phi_t) \in \mathrm{Mod}(\mathcal{O}(T) \rtimes Z_{S_n}(\sigma(\mu)))$$

is given by the $\mathcal{O}(T)$-character $t$ and the sign representation of the Weyl group $Z_{S_n}(\sigma(\mu))_t$. Moreover, the group $Z_{S_n}(\sigma(\mu))_t$ can be identified with $R(\xi)$, where $\xi = (P_\mu, \delta, t)$. Then $\zeta^\vee(\pi(\xi)) = \mathrm{ind}_{W(R_\xi)}^{(S_n)_t}(\text{sign})$ as $(S_n)_t$-representations, and this is exactly a member of the basis $R_{\mathbb{Z}}((S_n)_t)$ exhibited in Proposition 1.17(b).

Using the analysis from the case $q = 1$, it follows that

$$K_*(C_r^*(\mathcal{R}, q)) \cong \bigoplus_{\mu \vdash n} K^*\big(T_{\mathrm{un}}^{\sigma(\mu)} / Z_{S_n}(\sigma(\mu))\big) \cong \bigoplus_{\mu \vdash n} K^*((S^1)^{b(\mu)}) \cong \bigoplus_{\mu \vdash n} \mathbb{Z}^{2^{b(\mu)}}. \tag{3.12}$$

Recall that the even cohomology of $(S^1)^b$ has the same dimension as its odd cohomology, unless $b = 0$. The same holds for K-theory, and $b(\mu) = 0$ does not occur because $b(\mu)$ counts the number of different terms in a partition of $n \geq 1$. So we can refine (3.12) to

$$K_0(C_r^*(\mathcal{R}, q)) = \bigoplus_{\mu \vdash n} \mathbb{Z}^{2^{b(\mu)-1}}, \qquad K_1(C_r^*(\mathcal{R}, q)) = \bigoplus_{\mu \vdash n} \mathbb{Z}^{2^{b(\mu)-1}}. \tag{3.13}$$

**3B.** ***Type*** $\mathbf{SL}_n$**.** The affine Hecke algebra associated to a root datum of type $\mathrm{SL}_n$ describes the category of Iwahori-spherical representations of $\mathrm{PGL}_n(\mathbb{Q}_p)$. Since that is a subcategory of the Iwahori-spherical representations of $\mathrm{GL}_n(\mathbb{Q}_p)$, it can be expected this affine Hecke algebra behaves very similarly to those in the previous subsection. Indeed, we will see that the calculations of the K-theory are essentially the same as in Section 3A.

The root datum $\mathcal{R}(\mathrm{SL}_n)$ is given by:

$$\begin{gathered}
X = \mathbb{Z}^n / \mathbb{Z}(e_1 + \cdots e_n) \cong Q + ((e_1 + \cdots + e_n)/n - e_n), \\
Q = \{x \in \mathbb{Z}^n : x_1 + \cdots + x_n = 0\}, \\
Y = Q^\vee = \{y \in \mathbb{Z}^n : y_1 + \cdots + y_n = 0\}, \\
T = \{t \in (\mathbb{C}^\times)^n : t_1 \cdots t_n = 1\}, \qquad t = (t(e_1), \ldots, t(e_n)) = (t_1, \ldots, t_n), \\
R = \{e_i - e_j \in X : i \neq j\}, \qquad \alpha_0 = e_1 - e_n, \\
R^\vee = \{e_i - e_j \in Y : i \neq j\}, \qquad \alpha_0 = e_1 - e_n, \\
s_i = s_{\alpha_i} = s_{e_i - e_{i+1}}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{\alpha_1} s_{\alpha_0} t_{-\alpha_1} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x \rangle \alpha_0, \\
W = \big\langle s_1, \cdots, s_{n-1} \mid s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e \text{ if } |i - j| > 1 \big\rangle \cong S_n, \\
S^{\mathrm{aff}} = \{s_0, s_1, \ldots, s_{n-1}\}, \\
W^{\mathrm{aff}} = \big\langle s_0, W_0 \mid s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^3 = (s_0 s_{n-1})^3 = e \text{ if } 2 \leq i \leq n-2 \big\rangle, \\
W^e = W^{\mathrm{aff}} \rtimes \Omega, \qquad \Omega = \langle t_{e_1 - (e_1 + \cdots e_n)/n} (12 \cdots n) \rangle \cong \mathbb{Z}/n\mathbb{Z}.
\end{gathered}$$

Because all roots are conjugate, $s_0$ is conjugate to any $s_i \in S^{\mathrm{aff}}$, and for any label function $q(s_0) = q(s_i) = q$. The $W$-stabilizer of $((t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d})$ is isomorphic to $S_{\mu_1} \times \cdots \times S_{\mu_d}$. Generically, there are $n!\, n$ residual points, and they all satisfy $t(\alpha_i) = q$ or $t(\alpha_i) = q^{-1}$ for $1 \leq i < n$. These residual points form $n$ conjugacy classes, unless $q = 1$.

• **Group case $q = 1$.**

In view of (2.17) and (2.41), we want to determine $T_{\mathrm{un}}^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$, where $\mu$ is any partition of $n$. Write it as in (3.2); then

$$\begin{aligned} T^{\sigma(\mu)} &= \{(t_1)^{\mu_1}(t_{\mu_1+1})^{\mu_2}\cdots(t_n)^{\mu_d} \in T\} \\ &\cong \{(t_1)^{\mu_1}(t_{\mu_1+1})^{\mu_2}\cdots(t_n)^{\mu_d} \in (\mathbb{C}^\times)^n\}/\mathbb{C}^\times \\ &\qquad\qquad \times \{(e^{2\pi ik/n})^n : 0 \le k < \gcd(\mu)\}, \\ T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)) &\cong \big((\mathbb{C}^\times)^{m_n}/S_{m_n} \times \cdots \times (\mathbb{C}^\times)^{m_1}/S_{m_1}\big)/\mathbb{C}^\times \\ &\qquad\qquad \times \{(e^{2\pi ik/n})^n : 0 \le k < \gcd(\mu)\}, \end{aligned}$$

where $\mathbb{C}^\times$ acts diagonally. By Lemma 3.8, each factor $(\mathbb{C}^\times)^{m_i}/S_{m_i}$ is homotopy equivalent to a circle. The induced action of $S^1 \subset \mathbb{C}^\times$ on this direct product of circles identifies with a direct product of rotations. Hence, $T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$ is homotopy equivalent with $\mathbb{T}^{b(\mu)-1} \times \{\gcd(\mu) \text{ points}\}$, and the extended quotient $T/\!/W$ has torsion-free cohomology. By Corollary 2.49,

$$K_*(C_r^*(W^e)) \cong \mathbb{Z}^{d(n)}, \qquad d(n) := \sum_{\mu \vdash n} \gcd(\mu) 2^{b(\mu)-1}. \tag{3.14}$$

• **Generic, equal parameter case $q \neq 1$.**

Inequivalent subsets of $\Delta$ are parametrized by partitions $\mu$ of $n$. For the typical partition (3.2) we put

$$\begin{aligned}
&P_\mu = \Delta \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{n-\mu_d}\}, \\
&R_{P_\mu} \cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \cong R^\vee_{P_\mu}, \\
&X^{P_\mu} \cong \big(\mathbb{Z}(e_1+\cdots+e_{\mu_1})/\mu_1+\cdots+\mathbb{Z}(e_{n+1-\mu_d}+\cdots+e_n)/\mu_d\big)/\mathbb{Z}(e_1+\cdots+e_n)/g, \\
&X_{P_\mu} \cong (\mathbb{Z}^n/\mathbb{Z}(e_1+\cdots+e_n))^{m_n} + \cdots + (\mathbb{Z}^2/\mathbb{Z}(e_1+e_2))^{m_2}, \\
&Y^{P_\mu} = \{y \in \mathbb{Z}(e_1+\cdots+e_{\mu_1}) + \cdots + \mathbb{Z}(e_{n+1-\mu_d}+\cdots+e_n) : y_1+\cdots+y_n = 0\}, \\
&Y_{P_\mu} = \{y \in Y : y_1+\cdots+y_{\mu_1} = \cdots = y_{n+1-\mu_d}+\cdots+y_n = 0\}, \\
&T^{P_\mu} = \{(t_1)^{\mu_1}\cdots(t_n)^{\mu_d} \in T : t_1^{\mu_1/g}\cdots t_n^{\mu_d/g} = 1\}, \quad g = \gcd(\mu), \\
&T_{P_\mu} = \{t \in T : t_1t_2\cdots t_{\mu_1} = \cdots = t_{n+1-\mu_d}\cdots t_n = 1\}, \\
&K_{P_\mu} = \{t \in T^{P_\mu} : t_1^{\mu_1} = \cdots = t_n^{\mu_d} = 1\}, \\
&W_{P_\mu} \cong (S_n)^{m_n} \times \cdots \times (S_2)^{m_2}, \qquad W(P_\mu, P_\mu) \cong S_{m_n} \times \cdots \times S_{m_2} \times S_{m_1}, \\
&\mathcal{G}_{P_\mu P_\mu} = K_{P_\mu} \rtimes W(P_\mu, P_\mu), \qquad Z_{S_n}(\sigma(\mu)) = W(P_\mu, P_\mu) \ltimes \textstyle\prod_{l=1}^n (\mathbb{Z}/l\mathbb{Z})^{m_l}.
\end{aligned}$$

**Theorem 3.15.** *For $q \neq 1$ the $C^*$-algebra $C_r^*(\mathcal{R}(\mathrm{SL}_n), q)$ is Morita equivalent with the commutative algebra of continuous functions on $T_{\mathrm{un}}/\!/W$.*

*Its K-theory is given by*

$$K_0(C_r^*(\mathcal{R},q)) = \bigoplus_{\mu\vdash n,\, b(\mu)>1} \mathbb{Z}^{\gcd(\mu)2^{b(\mu)-2}} \oplus \bigoplus_{\mu\vdash n,\, b(\mu)=1} \mathbb{Z}^{\gcd(\mu)},$$
$$K_1(C_r^*(\mathcal{R},q)) = \bigoplus_{\mu\vdash n,\, b(\mu)>1} \mathbb{Z}^{\gcd(\mu)2^{b(\mu)-2}}.$$

*Proof.* The $W_{P_\mu}$-orbits of residual points for $\mathcal{H}_{P_\mu}$ are represented by the points

$$\big((q^{(\mu_1-1)/2}, q^{(\mu_1-3)/2}, \dots, q^{(1-\mu_1)/2}) \cdots (q^{(\mu_d-1)/2}, q^{(\mu_d-3)/2}, \dots, q^{(1-\mu_d)/2})\big) \cdot \big((e^{2\pi i k_1/\mu_1})^{\mu_1} \cdots (e^{2\pi i k_d/\mu_d})^{\mu_d}\big), \quad 0 \le k_i < \mu_i. \tag{3.16}$$

These points are in bijection with $K_{P_\mu} \times \mathbb{Z}/\gcd(\mu)\mathbb{Z}$. Also $T^{\sigma(\mu)}$ consists of exactly $\gcd(\mu)$ components, one of which is $T^{P_\mu}$. Just as in the type $\mathrm{GL}_n$ case, this leads to

$$\bigcup_\delta (P_\mu, \delta, T^{P_\mu})/K_{P_\mu} \cong T^{P_\mu} \times \mathbb{Z}/\gcd(\mu)\mathbb{Z} \cong T^{\sigma(\mu)},$$
$$\bigcup_\delta (P_\mu, \delta, T^{P_\mu})/\mathcal{W}_{P_\mu P_\mu} \cong T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)),$$
$$C_r^*(\mathcal{R},q) \cong \bigoplus_{\mu\vdash n} M_{n!/\mu!}\Big(C\Big(\bigsqcup_\delta T_u^{P_\mu}\Big)\Big) \cong \bigoplus_{\mu\vdash n} M_{n!/\mu!}\big(T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))\big).$$

The extended quotient $T_{\mathrm{un}}/\!/W$ is $\bigsqcup_{\mu\vdash n} T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$, which gives the desired Morita equivalence. It follows that

$$K_*(C_r^*(\mathcal{R},q)) \cong \bigoplus_{\mu\vdash n} K^*\big(T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))\big) \cong \bigoplus_{\mu\vdash n} K^*((S^1)^{b(\mu)-1})^{\gcd(\mu)}. \tag{3.17}$$

This a free abelian group of rank $d(n) = \sum_{\mu\vdash n} \gcd(\mu)2^{b(\mu)-1}$ with $b(\mu)$ as on page 431. Since the even K-theory of $(S^1)^b$ has the same rank as the odd K-theory unless $b=0$, (3.17) leads to $K_0$ and $K_1$ as claimed. □

**3C. *Type* $\mathbf{PGL}_n$.** The root datum for the algebraic group $\mathrm{PGL}_n$ gives rise to

$$X = Q = \{x \in \mathbb{Z}^n : x_1 + \cdots + x_n = 0\},$$
$$Q^\vee = \{y \in \mathbb{Z}^n : y_1 + \cdots + y_n = 0\},$$
$$Y = \mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n) \cong Q^\vee + ((e_1 + \cdots + e_n)/n - e_1),$$
$$T = (\mathbb{C}^\times)^n/\mathbb{C}^\times, \qquad t = (t_1, \dots, t_n) = (t(e_1), \dots, t(e_n)),$$
$$R = \{e_i - e_j \in X : i \ne j\}, \qquad \alpha_0 = e_1 - e_n,$$
$$R^\vee = \{e_i - e_j \in Y : i \ne j\}, \qquad \alpha_0 = e_1 - e_n,$$
$$s_i = s_{\alpha_i} = s_{e_i - e_{i+1}}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x\rangle \alpha_0,$$

$$W = \langle s_1, \dots, s_{n-1} \mid s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e \text{ if } |i-j| > 1 \rangle \cong S_n,$$
$$S^{\text{aff}} = \{s_0, s_1, \dots, s_{n-1}\}, \qquad \Omega = \{e\},$$
$$W^e = W^{\text{aff}} = \langle s_0, W_0 \mid s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^3 = (s_0 s_{n-1})^3 = e \text{ if } 2 \le i \le n-2 \rangle.$$

For $n > 2$, $s_0$ is conjugate to $s_1$ in $W^{\text{aff}}$, for $n = 2$ it is not. So for $n > 2$ there is only one parameter $q = q(s_i)$, $0 \le i \le n-1$, whereas for $n = 2$, $q_0$ may differ from $q_1$. In particular, for $n = 2$ the equal parameter function $q(s_0) = q(s_1)$ is not generic. Nevertheless, we only consider equal parameter functions in this subsection, explicit computations for the other parameter functions on $\mathcal{R}(\text{PGL}_2)$ can be found in [Solleveld 2007, §6.1].

For $q \neq 1$, there are $n!$ residual points. They form one $W$-orbit, and a typical residual point is

$$(q^{(1-n)/2}, q^{(3-n)/2}, \dots, q^{(n-1)/2}).$$

To determine the isotropy group of points of $T$ we have to be careful. In general the $W$-stabilizer of

$$((t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d}) \in T$$

is isomorphic to

$$S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_d} \subset W.$$

However, in some special cases the diagonal action of $\mathbb{C}^\times$ on $(\mathbb{C}^\times)^n$ gives rise to extra stabilizing elements. Let $r$ be a divisor of $n$, $k \in (\mathbb{Z}/r\mathbb{Z})^\times$ and $\lambda = (\lambda_1, \dots, \lambda_l)$ a partition of $n/r$. The isotropy group of

$$(t_1)^{\lambda_1} (e^{2\pi i k/r} t_1)^{\lambda_1} \cdots (e^{-2\pi i k/r} t_1)^{\lambda_1} (t_{r\lambda_1+1})^{\lambda_2} \cdots (e^{-2\pi i k/r} t_{r\lambda_1+1})^{\lambda_2} \cdots (e^{-2\pi i k/r} t_n)^{\lambda_l}$$

is isomorphic to

$$S_{\lambda_1}^r \times S_{\lambda_2}^r \times \cdots \times S_{\lambda_l}^r \rtimes \mathbb{Z}/r\mathbb{Z}. \tag{3.18}$$

Explicitly, the subgroup $\mathbb{Z}/r\mathbb{Z}$ is generated by

$$\begin{aligned}(1\, \lambda_1{+}1\; 2\lambda_1{+}1 \cdots (r{-}1)\lambda_1{+}1)(2\, \lambda_1{+}2\; 2\lambda_1{+}2 \cdots (r{-}1)\lambda_1{+}2) \cdots (\lambda_1\, 2\lambda_1 \cdots r\lambda_1)\\ \cdots (n{+}1{-}r\lambda_d\; n{+}1{+}(1{-}r)\lambda_d \cdots n{+}1{+}(r{-}1)\lambda_d)(n{+}(1{-}r)\lambda_d\; n{+}(2{-}r)\lambda_d \cdots n),\end{aligned}$$

and it acts on every factor $S_{\lambda_j}^r$ in (3.18) by cyclic permutations.

- **Case $q = 1$.**

As we noted before, we have to analyze $T_{\text{un}}^{\sigma(\mu)} / Z_{S_n}(\sigma(\mu))$. For the typical partition $\mu$ we have

$$T^{\sigma(\mu)} = \{(t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d}\} / \mathbb{C}^\times \times \{t : t(e_j) = e^{2\pi i j k/g},\ 0 \le k < g\}, \tag{3.19}$$

which is the disjoint union of $g = \gcd(\mu)$ complex tori of dimension $m_n + m_{n-1} + \cdots + m_1 - 1$. We obtain

$$T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)) \cong \big((\mathbb{C}^\times)^{m_n}/S_{m_n} \times \cdots \times (\mathbb{C}^\times)^{m_1}/S_{m_1}\big)/\mathbb{C}^\times \\ \times \{t : t(e_j) = e^{2\pi ijk/g},\ 0 \le k < g\}. \quad (3.20)$$

Remarkably enough, these sets are diffeomorphic to the corresponding sets for $\mathcal{R}(\mathrm{SL}_n)$. We take advantage of this by reusing our deduction that (3.20) is homotopy equivalent with $(S^1)^{b(\mu)-1} \times \{\gcd(\mu) \text{ points}\}$. With (2.17) we conclude that $K_*(C_r^*(W^e)) \otimes_{\mathbb{Z}} \mathbb{C}$ has dimension $d(n) = \sum_{\mu\vdash n} \gcd(\mu) 2^{b(\mu)-1}$.

• **Equal parameter case** $q \neq 1$**.**

This is noticeably different from the generic cases for $\mathcal{R}(\mathrm{GL}_n)$ and $\mathcal{R}(A_{n-1}^\vee)$, because $C_r^*(\mathcal{R}(A_{n-1}, q))$ is not Morita equivalent to a commutative $C^*$-algebra. Of course the inequivalent subsets of $\Delta$ are still parametrized by partitions $\mu$ of $n$:

$$P_\mu = \Delta \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{n-\mu_d}\},$$
$$R_{P_\mu} \cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \cong R_{P_\mu}^\vee,$$
$$X^{P_\mu} \cong \{x \in \mathbb{Z}(e_1 + \cdots + e_{\mu_1})/\mu_1 + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n)/\mu_d : \\ x_1 + \cdots + x_n = 0\},$$
$$X_{P_\mu} \cong \{x \in \mathbb{Z}^{\mu_1}/\mathbb{Z}(e_1 + \cdots + e_{\mu_1}) + \cdots + \mathbb{Z}^{\mu_d}/\mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n) : \\ x_1 + \cdots + x_n \in g\mathbb{Z}/g\mathbb{Z}\},$$
$$Y^{P_\mu} \cong \mathbb{Z}(e_1 + \cdots + e_{\mu_1}) + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n)/\mathbb{Z}(e_1 + \cdots + e_n),$$
$$Y_{P_\mu} \cong \{y : y_1 + \cdots + y_{\mu_1} = \cdots = y_{n+1-\mu_d} + \cdots + y_n = 0\}/\mathbb{Z}(e_1 + \cdots e_n),$$
$$T^{P_\mu} = \{(t_1)^{\mu_1} \cdots (t_n)^{\mu_d}\}/\mathbb{C}^\times,$$
$$T_{P_\mu} = \{t : t_1 t_2 \cdots t_{\mu_1} = \cdots = t_{n+1-\mu_d} \cdots t_n = 1\}/\{z \in \mathbb{C} : z^g = 1\},$$
$$K_{P_\mu} = \{(t_1)^{\mu_1} \cdots (t_n)^{\mu_d} : t_1^{\mu_1} = \cdots = t_n^{\mu_d} = 1\}/\{z \in \mathbb{C} : z^g = 1\},$$
$$W_{P_\mu} \cong S_n^{m_n} \times S_{n-1}^{m_{n-1}} \times \cdots \times S_2^{m_2}, \qquad W(P_\mu, P_\mu) \cong S_{m_n} \times \cdots \times S_{m_2} \times S_{m_1}.$$

We note that

$$T^{\sigma(\mu)} = T^{P_\mu} \times \{t : t(e_j) = e^{2\pi ijk/g},\ 0 \le k < g\}.$$

The $W_{P_\mu}$-orbits of residual points for $\mathcal{H}_{P_\mu}$ are represented by the points of

$$K_{P_\mu}(q^{(\mu_1-1)/2}, q^{(\mu_1-3)/2}, \ldots, q^{(1-\mu_1)/2}, q^{(\mu_2-1)/2}, \ldots, q^{(\mu_d-1)/2}, \ldots, q^{(1-\mu_d)/2}).$$

Hence, the intertwiners $\pi(k)$ with $k \in K_{P_\mu}$ permute the set of discrete series representations of $\mathcal{H}_{P_\mu}$ faithfully, and

$$\bigsqcup_\delta (P_\mu, \delta, T^{P_\mu})/K_{P_\mu} \cong T^{P_\mu} = (T^{\sigma(\mu)})^\circ.$$

Just before (3.10) we saw that the intertwiners for $\mathcal{R}(\mathrm{GL}_n)$, $q \neq 1$, have the property

$$w(t) = t \quad \Rightarrow \quad \pi(w, P_\mu, \delta, t) = 1.$$

This implies that in our present setting we can have $w(t)=t$ and $\pi(w, P_\mu, \delta, t) \neq 1$ only if $w(t)=t$ does not hold without taking the action of $\mathbb{C}^\times$ into account.

Let us classify such $w \in W(P_\mu, P_\mu)$ and $t \in T^{P_\mu}$ up to conjugacy. For a divisor $r$ of $g^\vee := \gcd(\mu^\vee)$ we have the partition

$$\mu^{1/r} := (nr)^{m_n/r} \cdots (2r)^{m_2/r} (r)^{m_1/r}.$$

Notice that

$$b(\mu^{1/r}) = b(\mu) = b(\mu^\vee).$$

There exists a $\sigma \in S_n$ which is conjugate to $\sigma(\mu^{1/r})$ and satisfies $\sigma^r = \sigma(\mu)$. We construct a particular such $\sigma$ as follows. If $r = g^\vee$ then (starting from the left) replace every block

$$(d{+}1\; d{+}2 \;\cdots\; d{+}m)(d{+}1{+}m \;\cdots\; d{+}2m)\cdots(d{+}(g^\vee{-}1)m \;\cdots\; d{+}g^\vee m)$$

of $\sigma(\mu)$ by

$$(d{+}1\; d{+}1{+}m \cdots d{+}1{+}(g^\vee{-}1)m\; 2\; d{+}2{+}m \cdots d{+}2{+}(g^\vee{-}1)m\; d{+}3 \cdots d{+}g^\vee m).$$

We denote the resulting element by $\sigma(\mu)^{1/g^\vee}$, and for general $r \mid g^\vee$ we define

$$\sigma(\mu)^{1/r} := \big(\sigma(\mu)^{1/g^\vee}\big)^{g^\vee/r}.$$

Consider the cosets of subtori

$$T^{P_\mu}_{r,k} := \big(T^{\sigma(\mu)^{1/r}}\big)^\circ \big((1)^{g^\vee \mu_1/r} (e^{2\pi i k/r})^{g^\vee \mu_{1+g^\vee/r}/r} \cdots (e^{-2\pi i k/r})^{g^\vee \mu_d/r}\big), \quad k \in \mathbb{Z}.$$

If $\gcd(k, r) = 1$, then the generic points of $T^{P_\mu}_{r,k}$ have $W(P_\mu, P_\mu)$-stabilizer

$$\langle W_{P_\mu}, \sigma(\mu)^{1/r}\rangle \cap W(P_\mu, P_\mu) \cong \mathbb{Z}/r\mathbb{Z}.$$

Note that for $r' \mid g^\vee$,

$$T^{P_\mu}_{r',k} \subset T^{P_\mu}_{r,k} \quad \text{if } r \mid r'. \tag{3.21}$$

If a point $t \in T^{P_\mu}_{r,k}$ does not lie on any $T^{P_\mu}_{r',k'}$ with $r' > r$, then its $W(P_\mu, P_\mu)$-stabilizer may still be larger than $\mathbb{Z}/r\mathbb{Z}$. However, it is always of the form

$$S^r_{\lambda_1} \times \cdots \times S^r_{\lambda_l} \rtimes \mathbb{Z}/r\mathbb{Z}.$$

Here the product of symmetric groups is $W(R_\xi)$ from (2.10), and $\mathfrak{R}_\xi = \mathbb{Z}/r\mathbb{Z}$. With [Delorme and Opdam 2011] it follows that the intertwiners $\pi(w, P_\mu, \delta, t)$ are scalar for $w \in S^r_{\lambda_1} \times \cdots \times S^r_{\lambda_l}$ and nonscalar for $w \in (\mathbb{Z}/r\mathbb{Z}) \setminus \{e\}$. Because $\mathbb{Z}/r\mathbb{Z}$ is cyclic this implies that $\pi(P_\mu, \delta, t)$ is the direct sum of exactly $r$ inequivalent irreducible representations.

Different choices of $\sigma(\mu)^{1/r}$ or of $k \in (\mathbb{Z}/r\mathbb{Z})^\times$ lead to conjugate subvarieties of $T^{P_\mu}$, so we have a complete description of $\mathrm{Irr}(C_r^*(\mathcal{R}, q)_{P_\mu})$. To calculate the

K-theory of this algebra we use Theorem 2.34, which says that (at least modulo torsion) it is isomorphic to

$$H^*_{W(P_\mu,P_\mu)}\big(T_u^{P_\mu};\mathcal{L}_u\big)\cong \check{H}^*\big(T^{P_\mu}/W(P_\mu,P_\mu);\mathcal{L}_u^{W(P_\mu,P_\mu)}\big).$$

We can endow $T_u^{P_\mu}$ with the structure of a finite $W(P_\mu,P_\mu)$-CW-complex, such that every $T_{u,r,k}^{P_\mu}$ is a subcomplex. The local coefficient system $\mathcal{L}_u$ is not very complicated: $\mathcal{L}_u(B)\cong\mathbb{Z}^r$ if and only if $B\setminus\partial B$ consists of generic points in a conjugate of $T_{u,r,k}^{P_\mu}$. In suitable coordinates the maps $\mathcal{L}_u(B\to B')$ are all of the form

$$\mathbb{Z}^r\to\mathbb{Z}^{r/d}:(x_1,\dots,x_r)\to(x_1+x_2+\dots+x_d,\dots,x_{1+r-d}+\dots+x_r).$$

Hence, the associated sheaf is the direct sum of several subsheaves $\mathfrak{F}_r^\mu$, one for each divisor $r$ of $\gcd(\mu^\vee)$. The support of $\mathfrak{F}_r^\mu$ is

$$W(P_\mu,P_\mu)T_{u,r,1}^{P_\mu}/W(P_\mu,P_\mu)\cong T_u^{P_{\mu^{1/r}}}/Z_{S_n}(\sigma(\mu^{1/r})),$$

and on that space it has constant stalk $\mathbb{Z}^{\phi(r)}$. Here $\phi$ is the Euler $\phi$-function, i.e.,

$$\phi(r)=\#\{m\in\mathbb{Z}:0\le m<r,\ \gcd(m,r)=1\}=\#(\mathbb{Z}/r\mathbb{Z})^\times.$$

This is the rank of $\mathfrak{F}_r^\mu$, because in every point of $T_{u,r,1}$ we have $r$ irreducible representations, but the ones corresponding to numbers that are not coprime to $r$ are already accounted for by the sheaves $\mathfrak{F}_{r'}^\mu$ with $r'\mid r$. We calculate

$$\begin{aligned}
&\check{H}^*\big(T_{\text{un}}^{P_\mu}/W(P_\mu,P_\mu);\mathcal{L}_u^{W(P_\mu,P_\mu)}\big)\\
&\qquad\cong\bigoplus_{r\mid\gcd(\mu^\vee)}\check{H}^*\big(T_{\text{un}}^{P_\mu}/W(P_\mu,P_\mu);\mathfrak{F}_r^\mu\big)\\
&\qquad\cong\bigoplus_{r\mid\gcd(\mu^\vee)}\check{H}^*\big(T_{\text{un}}^{P_{\mu^{1/r}}}/Z_{S_n}(\sigma(\mu^{1/r}));\mathbb{Z}^{\phi(r)}\big)\\
&\qquad\cong\bigoplus_{r\mid\gcd(\mu^\vee)}\check{H}^*\big((S^1)^{b(\mu^{1/r})-1};\mathbb{Z}^{\phi(r)}\big)\cong\bigoplus_{r\mid\gcd(\mu^\vee)}\mathbb{Z}^{\phi(r)2^{b(\mu^{1/r})-1}}\\
&\qquad=\bigoplus_{r\mid\gcd(\mu^\vee)}\mathbb{Z}^{\phi(r)2^{b(\mu^\vee)-1}}=\mathbb{Z}^{\gcd(\mu^\vee)2^{b(\mu^\vee)-1}}.
\end{aligned}\tag{3.22}$$

Now Theorem 2.34 says that $K_*(C_r^*(\mathcal{R},q)_{P_\mu})$ is also a free abelian group of rank $\gcd(\mu^\vee)2^{b(\mu^\vee)-1}$. Summing over partitions $\mu$ of $n$ we find that $K_*(C_r^*(\mathcal{R},q))$ is a free abelian group of rank

$$\sum_{\mu\vdash n}\gcd(\mu^\vee)2^{b(\mu^\vee)-1}=\sum_{\mu\vdash n}\gcd(\mu)2^{b(\mu)-1}.$$

From Theorem 2.11 and the case $q = 1$ we see that these K-groups can also be obtained as the K-theory of a disjoint union of compact tori, with $\gcd(\mu)$ tori of dimension $b(\mu)-1$. This allows us to immediately determine $K_0$ and $K_1$ separately as well:

$$
\begin{aligned}
K_0(C_r^*(\mathcal{R}, q)) &= \bigoplus_{\mu\vdash n, b(\mu)>1} \mathbb{Z}^{\gcd(\mu)2^{b(\mu)-2}} \oplus \bigoplus_{\mu\vdash n, b(\mu)=1} \mathbb{Z}^{\gcd(\mu)}, \\
K_1(C_r^*(\mathcal{R}, q)) &= \bigoplus_{\mu\vdash n, b(\mu)>1} \mathbb{Z}^{\gcd(\mu)2^{b(\mu)-2}}.
\end{aligned}
\tag{3.23}
$$

**3D. *Type* $\mathbf{SO_{2n+1}}$.** The root systems of type $B_n$ are more complicated than those of type $A_n$ because there are roots of different lengths. This implies that the associated root data allow label functions which have three independent parameters. Detailed information about the representations of type $B_n$ affine Hecke algebras is available from [Slooten 2003].

Consider the root datum for the special orthogonal group $SO_{2n+1}$:

$$
\begin{gathered}
X = Q = \mathbb{Z}^n, \\
Y = \mathbb{Z}^n, \qquad Q^\vee = \{y \in Y : y_1 + \cdots + y_n \text{ even}\}, \\
T = (\mathbb{C}^\times)^n, \qquad t = (t_1, \ldots, t_n) = (t(e_1), \ldots, t(e_n)), \\
R = \{x \in X : \|x\| = 1 \text{ or } \|x\| = \sqrt{2}\}, \qquad \alpha_0 = e_1, \\
R^\vee = \{x \in X : \|x\| = 2 \text{ or } \|x\| = \sqrt{2}\}, \qquad \alpha_0^\vee = 2e_1, \\
\Delta = \{\alpha_i = e_i - e_{i+1} : i = 1, \ldots, n-1\} \cup \{\alpha_n = e_n\}, \\
s_i = s_{\alpha_i}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x\rangle \alpha_0, \\
W = \langle s_1, \ldots, s_n \mid s_j^2 = (s_i s_j)^2 = (s_i s_{i+1})^3 = (s_{n-1}s_n)^4 = e : i \le n-2,\ |i-j| > 1\rangle, \\
S^{\text{aff}} = \{s_0, s_1, \ldots, s_{n-1}, s_n\}, \qquad \Omega = \{e\}, \\
W^e = W^{\text{aff}} = \langle W, s_0 \mid s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^4 = e : i \ge 2\rangle.
\end{gathered}
$$

For a generic parameter function, we have different parameters $q_0 = q(s_0)$, $q_1 = q(s_i)$ for $1 \le i < n$ and $q_2 = q(s_n)$.

The finite reflection group $W = W(B_n)$ is naturally isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. Let $\mu \vdash n$ and consider a point

$$
t = \big((t_1^\pm)^{\mu_1} \cdots (t_{n-\mu_{d-1}-\mu_d}^\pm)^{\mu_{d-2}} (1)^{\mu_{d-1}} (-1)^{\mu_d}\big) \in T, \tag{3.24}
$$

where $(t_1^\pm)^{\mu_1}$ means that $\mu_1$ coordinates are equal to $t_1$ or $t_1^{-1}$, while the other $n-\mu_1$ coordinates of $t$ are different. The stabilizer $W_t$ of $t$ is isomorphic to

$$
S_{\mu_1} \times \cdots \times S_{\mu_{d-2}} \times W(B_{\mu_{d-1}}) \times W(B_{\mu_d}). \tag{3.25}
$$

Notice that this is a Weyl group, generated by the reflections it contains.

• **Case $q_0 = q_1 = q_2 = 1$.**

In view of (2.17) we want to determine the extended quotient $\widetilde{T_{\mathrm{un}}}/W$. Therefore, we recall the explicit classification of conjugacy classes in $W$ in terms of bipartitions, which be found (for example) in [Carter 1972]. We already know that the quotient of $W$ by the normal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ of sign changes is isomorphic to $S_n$, and that conjugacy classes in $S_n$ are parametrized by partitions of $n$. So we wonder what the different conjugacy classes in $(\mathbb{Z}/2\mathbb{Z})^n \sigma(\mu)$ are, for $\mu \vdash n$. To handle this we introduce some notation, assuming $|\mu| + |\lambda| = n$ and $|\mu| + |\lambda| + |\rho| = n'$:

$$
\begin{aligned}
\epsilon_I &= \textstyle\prod_{i \in I} s_{e_i}, \qquad I \subset \{1, \dots, n\},\\
I_\lambda &= \{1, 1+\lambda_1, 1+\lambda_1+\lambda_2, \dots\}, \qquad \lambda = (\lambda_1, \lambda_2, \lambda_3, \dots),\\
\sigma'(\lambda) &= \epsilon_{I_\lambda} \sigma(\lambda) \in W(B_{|\lambda|}),\\
\sigma(\mu, \lambda) &= \sigma(\mu)\, (m \to m - |\lambda| \bmod n)\, \sigma'(\lambda)\, (m \to m + |\lambda| \bmod n),\\
\sigma(\mu, \lambda, \rho) &= \sigma(\mu, \lambda)\, (m \to m - |\rho| \bmod n')\, \sigma'(\rho)\, (m \to m + |\rho| \bmod n').
\end{aligned} \tag{3.26}
$$

Let $I \subset \{1, \dots, m\}$ and $J \subset \{m+1, \dots, 2m\}$. It is easily verified that $\epsilon_I (1\,2\,\cdots\,m)$ is conjugate to $\mu_J (m+1\; m+2\,\cdots\,2m)$ if and only if $|I|+|J|$ is even. Therefore the conjugacy classes in $W(B_n)$ are parametrized by ordered pairs of partitions of total weight $n$. Explicitly $(\mu, \lambda)$ corresponds to $\sigma(\mu, \lambda)$ as in (3.26). The set $T^{\sigma(\mu,\lambda)}$ and the group $Z_{W_0(B_n)}(\sigma(\mu, \lambda))$ are both the direct product of the corresponding objects for the blocks of $\mu$ and $\lambda$, i.e., for the parts $(m, m, \dots, m)$. The centralizer of $\sigma((m)^k)$ in $W(B_{km})$ is generated by $(1\,2\,\cdots\,m)$, $\epsilon_{\{1,2,\dots,m\}}$ and the transpositions of cycles

$$(am+1\; am+m+1)(am+2\; am+m+2)\cdots(am+m\; am+2m), \tag{3.27}$$

where $0 \le a \le k-2$. It follows that

$$
\begin{aligned}
Z_{W(B_{km})}(\sigma((m)^k)) &\cong W(B_k) \ltimes (\mathbb{Z}/m\mathbb{Z})^k,\\
((\mathbb{C}^\times)^{km})^{\sigma((m)^k)} &= \{((t_1)^m (t_{m+1})^m \cdots (t_{km+1-m})^m) : t_i \in \mathbb{C}^\times\},\\
((S^1)^{km})^{\sigma((m)^k)} / Z_{W_0(B_{km})}(\sigma((m)^k)) &\cong (S^1)^k / W(B_k) \cong [-1, 1]^k / S_k.
\end{aligned} \tag{3.28}
$$

Now consider the following element of $W(B_{km})$:

$$\sigma'((m)^k) = \epsilon_{\{1, m+1, \dots, km+1-m\}} (1\,2\,\cdots\,m)(m+1\,\cdots\,2m)\cdots(km+1-m\,\cdots\,km).$$

It has only $2^k$ fixpoints, namely

$$((\pm 1)^m (\pm 1)^m \cdots (\pm 1)^m). \tag{3.29}$$

The centralizer of $\sigma'((m)^k)$ is generated by $\epsilon_{\{1\}}(1\,2\,\cdots\,m)$, $\epsilon_{\{1,2,\dots,m\}}$ and the elements (3.27). The latter two generate a subgroup isomorphic to $W(B_k)$, which fits in a short exact sequence

$$1 \to W(B_k) \to Z_{W(B_{mk})}(\sigma'((m)^k)) \to (\mathbb{Z}/m\mathbb{Z})^k \to 1, \tag{3.30}$$

where the first factor $\mathbb{Z}/m\mathbb{Z}$ is generated by the image of $\epsilon_{\{1\}}(1\,2\,\cdots\,m)$. We find

$$((S^1)^{km})^{\sigma'((m)^k)}/Z_{W(B_{mk})}(\sigma'((m)^k)) \cong \{(1)^{am}(-1)^{(k-a)m} : 0 \leq a \leq k\}. \tag{3.31}$$

Now we can see what $T_{\mathrm{un}}^{\sigma(\mu,\lambda)}/Z_W(\sigma(\mu,\lambda))$ looks like. Its number of components $N(\lambda)$ depends only on $\lambda$, and all these components are mutually homeomorphic contractible orbifolds, the shape and dimension being determined by $\mu$. More precisely, for every block of $\mu$ of width $k$ we get a factor $[-1, 1]^k/S_k$, and for every block of $\lambda$ of width $l$ we must multiply the number of components by $l+1$. Alternatively, we can obtain the same space (modulo the action of $W$) as

$$\begin{aligned}
T_{\mathrm{un}}^{\sigma(\mu,\lambda)}&/Z_{W(B_n)}(\sigma(\mu,\lambda)) \\
&= \bigsqcup_{\lambda_1\cup\lambda_2=\lambda} T_{\mathrm{un},c}^{\sigma(\mu,\lambda_1,\lambda_2)}/Z_{W(B_n)}(\sigma(\mu,\lambda_1,\lambda_2)) \\
&= \bigsqcup_{\lambda_1\cup\lambda_2=\lambda} ((S^1)^{|\mu|})^{\sigma(\mu)}/Z_{W(B_{|\mu|})}(\sigma(\mu))\,(-1)^{|\lambda_1|}\,(1)^{|\lambda_2|} \\
&= ([-1,1]^{|\mu|})^{\sigma(\mu)}/Z_{S_{|\mu|}}(\sigma(\mu)) \times \bigsqcup_{\lambda_1\cup\lambda_2=\lambda} (-1)^{|\lambda_1|}\,(1)^{|\lambda_2|},
\end{aligned} \tag{3.32}$$

where the subscript $c$ means that we take only the connected component containing the point $((1)^{|\mu|}(-1)^{|\lambda_1|}(1)^{|\lambda_2|})$.

In effect we parametrized the components of the extended quotient $\widetilde{T_{\mathrm{un}}}/W$ by ordered triples of partitions $(\mu, \lambda_1, \lambda_2)$ of total weight $n$, and every such component is contractible. In combination with (3.25) this shows that the conditions of Theorem 2.45 are fulfilled.

Denote the number of ordered $k$-tuples of partitions of total weight $n$ by $\mathcal{P}(k, n)$. Now Theorem 2.45 says that

$$K_*(C_r^*(W^e)) = \check{H}^*(\widetilde{T_{\mathrm{un}}}/W; \mathbb{Z}) = \check{H}^0(\widetilde{T_{\mathrm{un}}}/W; \mathbb{Z}) \cong \mathbb{Z}^{\mathcal{P}(3,n)}. \tag{3.33}$$

• **Generic case.**

The inequivalent subsets of $\Delta$ are parametrized by partitions $\mu$ of weight at most $n$:

$$\begin{aligned}
P_\mu &= \Delta \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{|\mu|}\}, \\
R_{P_\mu} &\cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \times B_{n-|\mu|}, \\
R_{P_\mu}^\vee &\cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \times C_{n-|\mu|}, \\
X^{P_\mu} &\cong \mathbb{Z}(e_1 + \cdots + e_{\mu_1})/\mu_1 + \cdots + \mathbb{Z}(e_{|\mu|+1-\mu_d} + \cdots + e_{|\mu|})/\mu_d,
\end{aligned}$$

$$\begin{aligned}
X_{P_\mu} &\cong (\mathbb{Z}^n/\mathbb{Z}(e_1+\cdots+e_n))^{m_n}+\cdots+(\mathbb{Z}^2/\mathbb{Z}(e_1+e_2))^{m_2}+\mathbb{Z}^{n-|\mu|},\\
Y^{P_\mu} &= \mathbb{Z}(e_1+\cdots+e_{\mu_1})+\cdots+\mathbb{Z}(e_{|\mu|+1-\mu_d}+\cdots+e_{|\mu|}),\\
Y_{P_\mu} &= \{y\in\mathbb{Z}^n : y_1+\cdots+y_{\mu_1}=\cdots=y_{|\mu|+1-\mu_d}+\cdots+y_{|\mu|}=0\},\\
T^{P_\mu} &= \{(t_1)^{\mu_1}(t_{\mu+1})^{\mu_2}\cdots(t_{|\mu|})^{\mu_d}(1)^{n-|\mu|} : t_i\in\mathbb{C}^\times\},\\
T_{P_\mu} &= \{t\in(\mathbb{C}^\times)^n : t_1\cdots t_{\mu_1}=t_{\mu_1+1}\cdots t_{\mu_1+\mu_2}=\cdots=t_{|\mu|+1-\mu_d}\cdots t_{|\mu|}=1\},\\
K_{P_\mu} &= \{t\in T^{P_\mu} : t_1^{\mu_1}=\cdots=t_{|\mu|}^{\mu_d}=1\},\\
W_{P_\mu} &\cong S_n^{m_n}\times\cdots\times S_2^{m_2}\times W(B_{n-|\mu|}),\\
W(P_\mu,P_\mu) &\cong W(B_{m_n})\times\cdots\times W(B_{m_2})\times W(B_{m_1}).
\end{aligned}$$

We see that $\mathcal{R}_{P_\mu}$ is the product of various root data of type $\mathrm{SL}_m$ and one factor $\mathcal{R}(\mathrm{SO}_{2(n-|\mu|)+1})$. Hence $\mathcal{H}_{P_\mu}$ is the tensor product of a type $A$ part and a type $B$ part. From our study of $\mathcal{R}(\mathrm{SL}_m)$, we recall that the discrete series representations of the type $A$ part of $\mathcal{H}_{P_\mu}$ are in bijection with $K_{P_\mu}$. From [Heckman and Opdam 1997, Proposition 4.3] and [Opdam 2004, Appendix A.2] we know that the residual points for $\mathcal{R}(\mathrm{SO}_{2(n-|\mu|)+1}, q)$ are parametrized by ordered pairs $(\lambda_1,\lambda_2)$ of total weight $n-|\mu|$. The unitary part of such a residual point is in the component we indicated in (3.32). Let $\mathrm{RP}(\mathcal{R}, q)$ denote the collection of residual points for the pair $(\mathcal{R}, q)$. The above gives canonical bijections

$$\begin{aligned}
&\bigsqcup_{t\in\mathrm{RP}(\mathcal{R}_{P_\mu},q_{P_\mu})} tT_{\mathrm{un}}^{P_\mu}/\mathcal{W}_{P_\mu P_\mu}\\
&\qquad\cong \bigsqcup_{t\in\mathrm{RP}(\mathcal{R}(\mathrm{SO}_{2(n-|\mu|)+1},q))} tT_{\mathrm{un}}^{P_\mu}/W(P_\mu,P_\mu)\\
&\qquad\cong T_{\mathrm{un}}^{P_\mu}/Z_{W_0(B_{|\mu|})}(\sigma(\mu))\times \bigsqcup_{(\lambda_1,\lambda_2):|\lambda_1|+|\lambda_2|=n}(-1)^{|\lambda_1|}(1)^{|\lambda_2|}. \qquad (3.34)
\end{aligned}$$

**Theorem 3.35.** (a) *For generic $q$, $C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1}), q)$ is Morita equivalent with the commutative $C^*$-algebra of continuous functions on (3.34).*

(b) *$K_1(C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1}), q)) = 0$ and $K_0(C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1}, q)))$ is a free abelian group of rank $\mathcal{P}(3, n)$.*

*Proof.* (a) First we note that (3.34) can be identified with the extended quotient $\widetilde{T_{\mathrm{un}}}/W$ described in (3.32) and the subsequent lines.

Fix any $u\in T_{\mathrm{un}}$. The fibre over $u$ of the projection

$$p:\widetilde{T_{\mathrm{un}}}/W\to T_{\mathrm{un}}/W$$

is in bijection with the set of conjugacy classes of $W$. By Clifford theory, $|p^{-1}(Wu)|$ is also the number of inequivalent irreducible representations of $C(T_{\mathrm{un}})\rtimes W$ with central character $Wu$. Equivalently, $|p^{-1}(Wu)|$ is the number of inequivalent tempered irreducible representations of $\mathcal{O}(T)\rtimes W$ with central character $Wu$. By

Theorem 1.52 the latter equals the number of inequivalent irreducible tempered $\mathcal{H}(\mathcal{R}, q)$-representations with central character in $WuT_{\mathrm{rs}}$.

By Theorem 2.7 every point of (3.34) is the $Z(C_r^*(\mathcal{R}, q))$-character of at least one irreducible $C_r^*(\mathcal{R}, q)$-representation. The projection $p'$ from (3.34) to $T/W$ corresponds to restriction from $Z(C_r^*(\mathcal{R}, q)) \cong C(\Xi_{\mathrm{un}}/\mathcal{G})$ to $Z(\mathcal{H}(\mathcal{R}, q)) \cong \mathcal{O}(T/W)$.

Suppose that a point of $p'^{-1}(WuT_{\mathrm{rs}})$ would carry more than one inequivalent irreducible $C_r^*(\mathcal{R}, q)$-representation. Then the inverse image of $WuT_{\mathrm{rs}}$ under

$$\mathrm{Irr}(C_r^*(\mathcal{R}, q)) = \mathrm{Irr}_{\mathrm{temp}}(\mathcal{H}(\mathcal{R}, q)) \to T/W$$

would have more than $|p^{-1}(u)|$ elements. This would contradict what we concluded above, using Theorem 1.52. Thus every $\pi(P_\mu, \delta, t)$ with $(P_\mu, \delta, t) \in \Xi_{\mathrm{un}}$ is irreducible and (3.34) is exactly the space $\mathrm{Irr}(C_r^*(\mathcal{R}, q))$.

When we compare this with Theorem 2.7 and (2.6), we see that all intertwining operators $\pi(g, P_\mu, \delta, t)$ with $g(P_\mu, \delta, t) = (P_\mu, \delta, t)$ must be scalar. Recall from (2.9) that every indecomposable direct summand of $C_r^*(\mathcal{R}, q)$ is of the form

$$C\big(T_{\mathrm{un}}^{P_\mu}; \mathrm{End}_{\mathbb{C}}(\pi(P_\mu, \delta, t))\big)^{\mathcal{G}_{P\mu,\delta}}. \tag{3.36}$$

From (3.31) we know that the space $T_{\mathrm{un}}^{P_\mu}/\mathcal{G}_{P\mu,\delta}$ is a direct product of factors $(S^1)^k/W(B_k) \cong [-1, 1]/S_k$. We note that

$$\{(z_1, z_2, \dots, z_k) \in (S^1)^k : \Im(z_i) \geq 0, \Re(z_1) \geq \Re(z_2) \geq \cdots \geq \Re(z_k)\}$$

is a closed, connected fundamental domain for action of $W(B_k)$ on $(S^1)^k$. With this it is easy to find a closed fundamental domain $D_{P_\mu,\delta}$ for the action of $\mathcal{G}_{P_\mu,\delta}$ on $T_\delta^{P_\mu}$, such that $D_{P_\mu,\delta}$ is homeomorphic to $T_\delta^{P_\mu}/\mathcal{G}_{P_\mu,\delta}$. Then restriction from $T_{\mathrm{un}}^{P_\mu}$ to $D_{P_\mu,\delta}$ gives a monomorphism of $C^*$-algebras from (3.36) to

$$C(D_{P_\mu,\delta}; \mathrm{End}_{\mathbb{C}}(\pi(P_\mu, \delta, t))) = C(D_{P_\mu,\delta}) \otimes \mathrm{End}_{\mathbb{C}}(\pi(P_\mu, \delta, t)).$$

It is surjective because the intertwining operators $\pi(g, P_\mu, \delta, t)$, $g \in \mathcal{G}_{P_\mu,\delta}$, from (2.5) depend continuously on $t \in T_{\mathrm{un}}^{P_\mu}$ and are scalar multiples of the identity whenever they map a representation to itself. Hence $C_r^*(\mathcal{R}, q)$ is Morita equivalent with $\bigoplus_{(P_\mu,\delta)/\mathcal{G}} C(D_{P_\mu,\delta})$, as required.

(b) By the Serre–Swan theorem, $K_*(C_r^*(\mathcal{R}, q))$ is the topological K-theory of the underlying space (3.34). Since every connected component of this space is contractible, $K_1(C_r^*(\mathcal{R}, q)) = 0$ and $K_0(C_r^*(\mathcal{R}, q))$ is a free abelian group whose rank equals the number of connected components of (3.34). In the lines following (3.32) we showed that that number is $\mathcal{P}(3, n)$. By Theorem 2.11 these K-groups are independent of the parameters $q$. □

**3E.** ***Type*** $\mathbf{Sp_{2n}}$**.** The root datum for the symplectic group $\mathrm{Sp}_{2n}$ is dual to that for $\mathrm{SO}_{2n+1}$. Concretely, $\mathcal{R}(\mathrm{Sp}_{2n})$ is given by

$$
\begin{gathered}
X = \{y \in Y : y_1 + \cdots + y_n \text{ even}\}, \qquad Q = \mathbb{Z}^n, \qquad Y = Q^\vee = \mathbb{Z}^n, \\
T = (\mathbb{C}^\times)^n, \qquad t = (t_1, \dots, t_n) = (t(e_1), \dots, t(e_n)), \\
R = \{x \in X : \|x\| = 2 \text{ or } \|x\| = \sqrt{2}\}, \qquad \alpha_0 = e_1 + e_2, \\
R^\vee = \{x \in X : \|x\| = 1 \text{ or } \|x\| = \sqrt{2}\}, \qquad \alpha_0^\vee = e_1 + e_2, \\
\Delta = \{\alpha_i = e_i - e_{i+1} : i = 1, \dots, n-1\} \cup \{\alpha_n = 2e_n\}, \\
s_i = s_{\alpha_i}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{e_1} s_{\alpha_0} t_{-e_1} : x \to x + \langle \alpha_0^\vee, x \rangle \alpha_0, \\
W = \langle s_1, \dots, s_n \mid s_j^2 = (s_i s_j)^2 = (s_i s_{i+1})^3 = (s_{n-1} s_n)^4 = e : i \le n-2, |i-j| > 1 \rangle, \\
S^{\mathrm{aff}} = \{s_0, s_1, \dots, s_{n-1}, s_n\}, \qquad \Omega = \{e, t_{e_1} s_{2e_1}\}, \\
W^{\mathrm{aff}} = \langle W, s_0 \mid s_0^2 = (s_0 s_i)^2 = (s_0 s_2)^3 = e : i \neq 2 \rangle, \qquad W^e = W^{\mathrm{aff}} \rtimes \Omega.
\end{gathered}
$$

For a generic parameter function we have two independent parameters $q_1 = q(s_1)$ and $q_2 = q(s_n)$.

The groups $X$, $W$ and $W^e$ are exactly the same as for $\mathcal{R}(\mathrm{SO}_{2n+1})$. Everything that we said in Section 3D about the stabilizers in $W$ of points of $T$ obviously is valid here as well. In particular, for $q = 1$ the algebra $\mathcal{H}(\mathcal{R}(\mathrm{Sp}_{2n}), 1)$ is identical to $\mathcal{H}(\mathcal{R}(\mathrm{SO}_{2n+1}), 1)$, and the entire analysis of the K-theory of its $C^*$-completion can be found in the previous paragraph.

For all other $q$ we can use Theorem 2.11. Thus, we get

$$
\begin{aligned}
K_*(C_r^*(\mathcal{R}(\mathrm{Sp}_{2n}), q)) &\cong K_*(C_r^*(\mathcal{R}(\mathrm{Sp}_{2n}), 1)) \\
&= K_*(C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1}), 1)) \cong K_*(C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1}), q)).
\end{aligned}
$$

The last group is the one we actually computed, for generic parameters. Let us phrase the results explicitly:

$$
K_0(C_r^*(\mathcal{R}(\mathrm{Sp}_{2n}), q)) \cong \mathbb{Z}^{\mathcal{P}(3,n)}, \qquad K_1(C_r^*(\mathcal{R}(\mathrm{Sp}_{2n}), q)) = 0. \tag{3.37}
$$

**3F.** ***Type*** $\mathbf{SO_{2n}}$**.** The root datum for the even special orthogonal group $\mathrm{SO}_{2n}$ has groups contained in those for the root datum of type $\mathrm{SO}_{2n+1}$:

$$
\begin{gathered}
X = \mathbb{Z}^n, \qquad Q = \{y \in Y : y_1 + \cdots + y_n \text{ even}\}, \\
Y = \mathbb{Z}^n, \qquad Q^\vee = \{y \in Y : y_1 + \cdots + y_n \text{ even}\}, \\
T = (\mathbb{C}^\times)^n, \qquad t = (t_1, \dots, t_n) = (t(e_1), \dots, t(e_n)), \\
R = \{x \in X : \|x\| = \sqrt{2}\}, \qquad \alpha_0 = e_1 + e_2, \\
R^\vee = \{x \in X : \|x\| = \sqrt{2}\}, \qquad \alpha_0^\vee = e_1 + e_2, \\
\Delta = \{\alpha_i = e_i - e_{i+1} : i = 1, \dots, n-1\} \cup \{\alpha_n = e_{n-1} + e_n\},
\end{gathered}
$$

$$s_i = s_{\alpha_i}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{e_1} s_{\alpha_0} t_{-e_1} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x \rangle \alpha_0,$$
$$W = \langle s_1, \dots, s_n \mid s_j^2 = (s_i s_j)^2 = (s_i s_{i+1})^3 = (s_{n-2} s_n)^4 = e : i \leq n-2,\ |i-j| > 1 \rangle,$$
$$S^{\mathrm{aff}} = \{s_0, s_1, \dots, s_{n-1}, s_n\}, \qquad \Omega = \{e, t_{e_1} s_{e_1} s_{e_n}\},$$
$$W^{\mathrm{aff}} = \langle W, s_0 \mid s_0^2 = (s_0 s_i)^2 = (s_0 s_2)^3 = e : i \neq 2 \rangle \subsetneq W^e.$$

When $n > 2$, all the simple affine reflections are conjugate in $W^e$, and

$$q(s_i) = q, \quad i = 0, 1, \dots, n,$$

for every parameter function. For $n = 2$ the root system $R \cong A_1 \times A_1$ is reducible, there is an additional simple affine reflection and there are more possible parameter functions. For $n = 1$, $\mathcal{R}(\mathrm{SO}_2)$ is the root datum of a one-dimensional torus, in particular $W = 1$.

The based root datum $\mathcal{R}(\mathrm{SO}_{2n})$ has one nontrivial automorphism, which exchanges the roots $\alpha_{n-1}$ and $\alpha_n$. It is easily seen that

$$W^e(\mathrm{SO}_{2n}) \rtimes \mathrm{Aut}(\mathcal{R}(\mathrm{SO}_{2n})) \cong W^e(\mathrm{SO}_{2n+1}).$$

With Theorem 2.11 we conclude that, for every equal parameter function $q$,

$$\begin{aligned} K_*\big(C_r^*(\mathcal{R}(\mathrm{SO}_{2n}), q) \rtimes \mathrm{Aut}(\mathcal{R}(\mathrm{SO}_{2n}))\big) \\ &\cong K_*\big(W^e(\mathrm{SO}_{2n}) \rtimes \mathrm{Aut}(\mathcal{R}(\mathrm{SO}_{2n}))\big) \\ &= K_*\big(C_r^*(W^e(\mathrm{SO}_{2n+1}))\big) \cong K_*\big(C_r^*(\mathcal{R}(\mathrm{SO}_{2n+1}), q)\big). \end{aligned} \tag{3.38}$$

Unfortunately, no such shortcut is available for $K_*(C_r^*(\mathcal{R}(\mathrm{SO}_{2n}), q))$. Therefore we will just compute $K_*(W^e(\mathrm{SO}_{2n}))$ by hand, in several steps:

- We determine the extended quotient $T_{\mathrm{un}} /\!/ W(D_n)$ and its cohomology.
- We analyze the (elliptic) representations of the $W(D_n)$-isotropy groups of points of $T$.
- We relate the second bullet to the sheaf $\mathfrak{L}_u^{W(D_n)}$ on $T_{\mathrm{un}} / W(D_n)$.
- Then we are finally in the right position to apply Theorem 2.34.

The finite reflection group $W(D_n)$ is naturally isomorphic to the index two subgroup of $W(B_n) = W(C_n)$ consisting of those elements that involve an even number of sign changes. In other words, let $(\mathbb{Z}/2\mathbb{Z})^n_{\mathrm{ev}}$ be the kernel of the summation map $(\mathbb{Z}/2\mathbb{Z})^n \to \mathbb{Z}/2\mathbb{Z}$. Then

$$W(D_n) = (\mathbb{Z}/2\mathbb{Z})^n_{\mathrm{ev}} \rtimes S_n.$$

The conjugacy classes in $W(D_n)$ are similar to but slightly different from those in $W(B_n)$. We rephrase Young's parametrization in the notations from (3.26). For every bipartition $(\mu, \lambda)$ of $n$ where $\lambda$ has an even number of parts, $\sigma(\mu, \lambda)$ represents one class in $W(D_n)$. Suppose now that $\mu \vdash n$ has only even terms, and define

$$\sigma''(\mu) = \sigma(\mu)\epsilon_{\{n-1,n\}} = \epsilon_{\{n\}}^{-1}\sigma(\mu)\epsilon_{\{n\}}. \tag{3.39}$$

Then $\sigma''(\mu)$ represents a class of $W(D_n)$ different from the above. The $\sigma(\mu,\lambda)$ and the $\sigma''(\mu)$ form a set of representatives for all conjugacy classes of $W(D_n)$.

In the representation theory of classical groups, some almost direct products of root data of type $D$ arise [Goldberg 1994; Heiermann 2011]. Therefore it will be useful to investigate a more general situation, as in the Appendix. Fix $n_1, \dots, n_d$ with $n_1 + \dots + n_d = n$ and consider the root datum

$$\mathcal{R}'_{\vec{n}} = \mathcal{R}(\mathrm{SO}_{2n_1}) \times \dots \times \mathcal{R}(\mathrm{SO}_{2n_d}).$$

Let $W'_{\vec{n}} = W(D_{\vec{n}}) \rtimes \Gamma$ be as in (A.1), so $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^d_{\mathrm{ev}}$. The conjugacy classes for $W'_{\vec{n}}$ are a mixture of those for $W(D_n)$ and for $W(B_{\vec{n}})$. Let us analyze them and the extended quotient $T_{\mathrm{un}} /\!/ W'_{\vec{n}}$ together.

Recall that for $w \in W(B_{\vec{n}})$, the groups $T^w_{\mathrm{un}}$ and $Z_{W(B_{\vec{n}})}(w)$ were already computed in Section 3D; see in particular (3.28), (3.29) and (3.30). We say that $\vec{\mu} \vdash \vec{n}$ if $\vec{\mu}$ is a $d$-tuple of partitions $(\mu^{(1)}, \dots, \mu^{(d)})$ with $|\mu^{(i)}| = n_i$, and that $(\vec{\mu}, \vec{\lambda}) \vdash \vec{n}$ if $\vec{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ such that $|\mu^{(i)}| + |\lambda^{(i)}|$. To these we can associate $\sigma(\vec{\mu})$ and $\sigma(\vec{\mu}, \vec{\lambda})$, as products of (3.26) over the indices $i$.

- Consider $\sigma(\vec{\mu}, \vec{\lambda})$, where $\vec{\lambda}$ is nonempty and has an even number of terms. Notice that $Z_{W(B_{\vec{n}})}(\sigma(\vec{\mu}, \vec{\lambda}))$ contains an element not in $W(D_n)$ which fixes $T^{\sigma(\vec{\mu},\vec{\lambda})}$ pointwise, namely a single factor $\epsilon_{\{a_1\}}(a_1 \cdots a_m)$ of $\vec{\lambda}$. Hence the $W(B_{\vec{n}})$-conjugacy class of $\sigma(\vec{\mu}, \vec{\lambda})$ is precisely the $W'_{\vec{n}}$-conjugacy class of $\sigma(\vec{\mu}, \vec{\lambda})$. Furthermore,
$$T_{\mathrm{un}}^{\sigma(\vec{\mu},\vec{\lambda})} / Z_{W'_{\vec{n}}}(\sigma(\vec{\mu}, \vec{\lambda})) = T_{\mathrm{un}}^{\sigma(\vec{\mu},\vec{\lambda})} / Z_{W(B_{\vec{n}})}(\sigma(\vec{\mu}, \vec{\lambda})),$$
and as described in (3.32), this is a disjoint union of contractible spaces. The number of components is given explicitly in terms of $\vec{\lambda}$.
- Suppose that $\vec{\mu} \vdash \vec{n}$ and that all terms of $\vec{\mu}$ are even. Then the $W(B_{\vec{n}})$-conjugacy class of $\sigma(\vec{\mu})$ splits into two $W'_{\vec{n}}$-conjugacy classes, the other one represented by
$$\sigma''(\vec{\mu}) = \sigma(\vec{\mu})\epsilon_{\{n-1,n\}}.$$
Both $Z_{W(B_{\vec{n}})}(\sigma(\vec{\mu}))$ and
$$Z_{W(B_{\vec{n}})}(\sigma''(\vec{\mu})) = \epsilon_{\{n\}}^{-1} Z_{W(B_{\vec{n}})}(\sigma(\vec{\mu}))\epsilon_{\{n\}}$$
are contained in $W'_{\vec{n}}$. Let $m_l$ be the multiplicity of $l$ in $\vec{\mu}$. By (3.32),
$$T_{\mathrm{un}}^{\sigma''(\vec{\mu})} / Z_{W'_{\vec{n}}}(\sigma''(\vec{\mu})) \cong T_{\mathrm{un}}^{\sigma(\vec{\mu})} / Z_{W'_{\vec{n}}}(\sigma(\vec{\mu})) \cong \prod_{l=1}^{n} [-1, 1]^{m_l} / S_{m_l},$$
which is a contractible space.

- Let $\mu \vdash n$ be a partition with at least one odd term. Again, the $W(B_{\vec n})$-conjugacy class of $\sigma(\vec\mu)$ is precisely the $W'_{\vec n}$-conjugacy class of $\sigma(\vec\mu)$. Now
$$Z_{W'_{\vec n}}(\sigma(\vec\mu)) \subsetneq Z_{W(B_{\vec n})}(\sigma(\vec\mu)),$$
and this really makes a difference. From (3.28) we deduce
$$T_{\mathrm{un}}^{\sigma(\vec\mu)}/Z_{W'_{\vec n}}(\sigma(\vec\mu)) \cong \prod_{l=1}^{n}(S^1)^{m_l} \Big/ \Big( \prod_{l=1}^{n} W(B_{m_l}) \cap W(D_n) \Big). \tag{3.40}$$
The group $\prod_{l=1}^{n} W(B_{m_l}) \cap W(D_n)$ equals $\big(\prod_{l=1}^{n}(\mathbb{Z}/2\mathbb{Z})^{m_l}\big)_+ \rtimes \prod_{l=1}^{m_l} S_{m_l}$, where the subscript + means that the total number of sign changes for odd $l$ must be even. The quotient map
$$\prod_{l \text{ odd}}(S^1)^{m_l} \Big/ \Big( \prod_{l \text{ odd}}(\mathbb{Z}/2\mathbb{Z})^{m_l} \Big)_+ \to \prod_{l \text{ odd}}(S^1)^{m_l}/(\mathbb{Z}/2\mathbb{Z})^{m_l} \cong \prod_{l \text{ odd}}[-1,1]^{m_l} \tag{3.41}$$
is a two-fold cover which ramifies precisely at the boundary of the unit cube $\prod_{l \text{ odd}}[-1,1]^{m_l}$. Therefore the left-hand side of (3.41) is homeomorphic to the unit sphere of dimension $m_1 + m_3 + m_5 + \cdots$. This entails that (3.40) is homeomorphic to
$$\prod_{l \text{ even}}([-1,1]^{m_l}/S_{m_l}) \times S^{m_1+m_3+\cdots} \Big/ \prod_{l \text{ odd}} S_{m_l}. \tag{3.42}$$
This space is contractible unless $m_l = 1$ for all odd $l$; then it is homotopic to $S^{m_1+m_3+\cdots}$.

The extended quotient $T_{\mathrm{un}} /\!/ W'_{\vec n}$ is the disjoint union of the spaces $T_{\mathrm{un}}^{w}/Z_{W'_{\vec n}}(w)$, as $w$ runs over representatives for the conjugacy classes of $W'_{\vec n}$. Since we covered all conjugacy classes for $W(B_{\vec n})$ intersecting $W'_{\vec n}$, we have a complete description of conjugacy classes for the latter group. From the above calculations we immediately get the cohomology of the extended quotient.

**Lemma 3.43.** *The abelian group $\check H^*(T_{\mathrm{un}} /\!/ W'_{\vec n})$ is torsion-free.*

*In the case $\vec n = n$, $W'_{\vec n} = W(D_n)$, we can describe the cohomology of $T_{\mathrm{un}} /\!/ W(D_n)$ explicitly. The rank of the odd cohomology is the number of partitions $\mu \vdash n$ such that every odd term appears with multiplicity one, and there is an odd number of odd terms.*

*The rank of the even cohomology of $T_{\mathrm{un}} /\!/ W(D_n)$ is the sum of four contributions:*

- $\prod_i (k_i + 1)$, *for every bipartition $(\mu, \lambda)$ of $n$ with $\lambda = (n)^{k_n} \cdots (1)^{k_1}$ such that $\sum_i k_i$ is positive and even;*
- *two times the number of partitions of $n$ with only even terms;*

- *the number of partitions of n with at least one odd term*;
- *the number of partitions of n such that every odd term appears only once, and the number of odd terms is positive and even.*

Every point of $T \cong (\mathbb{C}^\times)^n$ is $W(B_{\vec{n}})$-conjugate to one of the form

$$t = (t^{(1)}, \ldots, t^{(d)}), \qquad t^{(i)} = \big((t_1)^{\mu_1^{(i)}} \cdots (t_{n_i - m_1^{(i)} - m_2^{(i)}})^{\mu_{d_i}} (1)^{m_1^{(i)}} (-1)^{m_2^{(i)}}\big) \in (\mathbb{C}^\times)^{n_i}.$$

The isotropy group of $t$ in $W'_{\vec{n}}$ is

$$\begin{aligned}(W'_{\vec{n}})_t &= \left(\prod_{i=1}^d S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}} \times W(B_{m_1^{(i)}}) \times W(B_{m_2^{(i)}})\right) \cap W(D_n) \\ &= \left(\prod_{i=1}^d S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}}\right) \times \left(\prod_{i=1}^d W\big(B_{m_1^{(i)}}\big) \times W\big(B_{m_2^{(i)}}\big)\right) \\ &\qquad \cap W\big(D_{m_1^{(1)} + \cdots + m_2^{(d)}}\big). \end{aligned} \tag{3.44}$$

We note that $(W'_{\vec{n}})_t$ is generated by the reflections it contains if $t$ has no coordinates 1 or $-1$. Otherwise the reflection subgroup of $W(D_n)_t$ is

$$(W'_{\vec{n}})_t^\circ := \prod_{i=1}^d S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}} \times W\big(D_{m_1^{(i)}}\big) \times W\big(D_{m_2^{(i)}}\big),$$

where $W(D_0) = W(D_1) = 1$. In that case,

$$(W'_{\vec{n}})_t = \left(\prod_{i=1}^d S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}}\right) \times W'_{\vec{m}}, \tag{3.45}$$

where $\vec{m}$ consists of those terms $m_1^{(i)}, m_2^{(i)}$ which are nonzero. The group $W'_{\vec{m}}$ is a particular instance of the almost Weyl groups studied in the Appendix. Thus $(W'_{\vec{n}})_t$ is an example of the groups considered in Lemma A.9, and we may use that result.

**Proposition 3.46.** *For any positive parameter function $q$, $K_*(C_r^*(\mathcal{R}'_{\vec{n}}, q))$ is a free abelian group, isomorphic to $H^*(T_{\mathrm{un}} /\!/ W'_{\vec{n}}; \mathbb{Z})$.*

*In particular, for $\vec{n} = n$, $\mathcal{R}'_{\vec{n}} = \mathcal{R}(\mathrm{SO}_{2n})$, $W'_{\vec{n}} = W(D_n)$, the free abelian group*

$$K_*(C_r^*(\mathcal{R}(\mathrm{SO}_{2n}), q)) \cong H^*(T_{\mathrm{un}} /\!/ W(D_n); \mathbb{Z})$$

*has even and odd ranks as given in Lemma 3.43.*

*Proof.* By Theorem 1.52 it suffices to prove this when $q = 1$.

We adapt the notations from (3.32) to the present setting. Let $(\vec{\mu}, \vec{\lambda}_1, \vec{\lambda}_2)$ be a $d$-tuple of tripartitions of $n_1, \ldots, n_d$, respectively, and such that $\vec{\lambda}_1 \cup \vec{\lambda}_2$ has an even number of terms. As in (3.45) we write

$$W_{\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2} := \Big( \prod_{i=1}^{d} S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}} \times W\big(B_{|\lambda_1^{(i)}|}\big) \times W\big(B_{|\lambda_2^{(i)}|}\big) \Big) \cap W(D_n)$$
$$= \Big( \prod_{i=1}^{d} S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}} \Big) \times W'_{\vec{m}},$$

where $\vec{m}$ consists of the nonzero terms among the $|\lambda_1^{(i)}|, |\lambda_2^{(i)}|$. The group $W_{\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2}$ is the full stabilizer of some point of $T_{\mathrm{un}}$, and of the form considered in Lemma A.9. We note that $\sigma(\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2)$ is an elliptic element of $W_{\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2}$.

For every $t \in T_{\mathrm{un},c}^{\sigma(\mu,\lambda_1,\lambda_2)}$ we have $(W'_{\vec{n}})_t \supset W_{\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2}$. Using Lemma A.9 we define

$$s(\sigma(\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2), t) = \mathrm{ind}_{W_{\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2}}^{(W'_{\vec{n}})_t} H(u_{\sigma(\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2)}, \rho_{\sigma(\vec{\mu},\vec{\lambda}_1,\vec{\lambda}_2)}). \tag{3.47}$$

Suppose that $\vec{\mu} \vdash \vec{n}$ and that $\vec{\mu}$ has only even terms. Then $\sigma''(\vec{\mu}) = \epsilon_{\{n-1,n\}}\sigma(\vec{\mu})$ is conjugate to $\sigma(\vec{\mu})$ in $W(B_{\vec{n}})$ but not in $W'_{\vec{n}}$. The element $\sigma''(\vec{\mu})$ is elliptic in $\epsilon_{\{n\}}\big(\prod_{i=1}^{d} S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}}\big)\epsilon_{\{n\}}$, and for every $t \in T^{\sigma''(\vec{\mu})}$ we have

$$(W'_{\vec{n}})_t \supset \epsilon_{\{n\}} \Big( \prod_{i=1}^{d} S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}} \Big) \epsilon_{\{n\}}.$$

For such $t$ we define

$$s(\sigma''(\vec{\mu}), t) = \mathrm{ind}_{\epsilon_{\{n\}}(\prod_{i=1}^{d} S_{\mu_1^{(i)}} \times \cdots \times S_{\mu_{d_i}^{(i)}})\epsilon_{\{n\}}}^{(W'_{\vec{n}})_t} H(u_{\sigma''(\vec{\mu})}, \rho_{\sigma''(\vec{\mu})}). \tag{3.48}$$

As discussed before Lemma 3.43, every conjugacy class of $W'_{\vec{n}}$ appears precisely once in (3.47) and (3.48) together.

With this information and Lemma A.9 available, the same argument as in the proof of Theorem 2.45(a) works in the present setting, and shows that the conclusion of Theorem 2.45(a) is fulfilled. Then we apply Theorem 2.45(b). □

**3G. *Type $G_2$.*** As basis for the root lattice $X$ of type $G_2$, we take the two simple roots. We coordinatize the dual lattice $Y$ so that the pairing between $X$ and $Y$ becomes the standard pairing on $\mathbb{Z}^2$. Explicitly, $\mathcal{R}(G_2)$ becomes

$$X = Q = \mathbb{Z}^2, \qquad Y = Q^\vee = \mathbb{Z}^2,$$
$$T = (\mathbb{C}^\times)^2, \qquad t = (t(e_1), t(e_2)) = (t_1, t_2),$$
$$R^+ = \{e_1, e_2, e_1+e_2, 2e_1+e_2, 3e_1+e_2, 3e_1+2e_2\}, \qquad R = R^+ \cup -R^+,$$
$$R^{\vee,+} = \{2e_1-3e_2, 2e_1-e_2, 3e_2-e_1, e_1, e_1-e_2, e_2\}, \qquad R^\vee = R^{\vee,+} \cup -R^{\vee,+},$$
$$\Delta = \{e_1, e_2\}, \qquad \alpha_0^\vee = e_1, \qquad \alpha_0 = 2e_1+e_2,$$

$$s_1 = s_{e_1}, \qquad s_2 = s_{e_2}, \qquad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{e_1} s_{\alpha_0} t_{-e_1} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x \rangle \alpha_0,$$
$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^6 = e \rangle \cong D_6,$$
$$S^{\text{aff}} = \{s_0, s_1, s_2\}, \qquad \Omega = \{e\},$$
$$W^e = W^{\text{aff}} = \langle s_0, W_0 \mid s_0^2 = (s_0 s_2)^2 = (s_0 s_1)^3 = e \rangle.$$

A generic parameter function $q$ for $\mathcal{R}(G_2)$ has two independent parameters $q_1 = q(s_1)$ and $q_2 = q(s_2)$.

The group $W \cong D_6$ has six conjugacy classes: the identity, reflections associated to short roots, reflections associated to long roots, the rotation of order two, rotations of order three and rotations of order six. Representatives are $e$, $s_1$, $s_2$, $\rho_\pi = (s_1 s_2)^3$, $\rho_{2\pi/3} = (s_1 s_2)^2$ and $\rho_{\pi/6} = s_1 s_2$. We determine the connected components of the extended quotient $T_{\text{un}} /\!/ W$:

| $w$ | $T^w$ | $Z_{D_6}(w)$ | $T^w_{\text{un}}/Z_{D_6}(w)$ |
|---|---|---|---|
| $e$ | $T$ | $D_6$ | $(S^1)^2/D_6 \cong$ solid triangle |
| $s_1$ | $\{(1, t_2) : t_2 \in \mathbb{C}^\times\}$ | $\langle s_1, s_{3e_1+2e_2} \rangle$ | $S^1/\langle s_{3e_1+2e_2} \rangle \cong [-1, 1]$ |
| $s_2$ | $\{(t_1, 1) : t_1 \in \mathbb{C}^\times\}$ | $\langle s_2, s_{2e_1+e_2} \rangle$ | $S^1/\langle s_{2e_1+e_2} \rangle \cong [-1, 1]$ |
| $\rho_\pi$ | $\{(a, b) : a, b \in \{\pm 1\}\}$ | $D_6$ | 2 points |
| $\rho_{2\pi/3}$ | $\{(1, 1), (\zeta_3, 1), (\zeta_3^2, 1)\}$ | $C_6 = \langle \rho_{\pi/3} \rangle$ | 2 points |
| $\rho_{\pi/3}$ | $\{(1, 1)\}$ | $C_6 = \langle \rho_{\pi/3} \rangle$ | 1 point |

Here $\zeta_3$ is a primitive third root of unity. We see that every connected component of $T_{\text{un}} /\!/ W$ is contractible, and that its cohomology is zero in positive degrees and $\mathbb{Z}^8$ in degree zero.

The root datum $\mathcal{R}(G_2)$ is simply connected, so $W_t$ is a Weyl group for every $t \in T$. This can also be checked directly: For $t \in T$ with $W_t = \{e\}$ or $W_t$ generated by one reflection it is true. For all $t \in T$ not of that form, $W_t$ contains a nontrivial rotation. All rotations (or their inverses) appear in the above table, along with their fixpoints. We list the isotropy groups of those points:

$$W_{(1,1)} = D_6,$$
$$W_{(\zeta_3,1)} = W_{(\zeta_3^2,1)} = \langle s_2, \rho_{2\pi/3} \rangle \cong S_3,$$
$$W_{(-1,-1)} \cong W_{(-1,1)} \cong W_{(1,-1)} = \langle s_1, s_{3e_1+2e_2} \rangle \cong S_2 \times S_2.$$

We have checked all the conditions of Theorem 2.45. By Corollary 2.49, for every positive parameter function $q$,

$$K_0(C_r^*(\mathcal{R}(G_2), q)) \cong \mathbb{Z}^8, \qquad K_1(C_r^*(\mathcal{R}(G_2), q)) = 0. \tag{3.49}$$

## Appendix: Some almost Weyl groups

We study some finite groups which are almost Weyl groups. Such groups can arise as the component groups of unipotent elements of classical complex groups, and they play a role in the affine Hecke algebras associated to general Bernstein components for classical $p$-adic groups [Goldberg 1994; Heiermann 2011]. The results from this appendix are only needed in Section 3F.

Fix $n_1, n_2, \ldots, n_d \in \mathbb{Z}_{\geq 1}$ with $n_1 + \cdots + n_d = n$ and consider

$$W'_{\vec{n}} := (W(B_{n_1}) \times \cdots \times W(B_{n_d})) \cap W(D_n).$$

We use the convention that $W(D_1)$ is the trivial group. The group $W'_{\vec{n}}$ acts on the root system

$$D_{\vec{n}} := D_{n_1} \times \cdots \times D_{n_d}.$$

Let $\Delta_{\vec{n}}$ be the standard basis of $D_{\vec{n}}$ and let $\Gamma$ be the stabilizer of $\Delta_{\vec{n}}$ in $W'_{\vec{n}}$. Since $W(D_{\vec{n}})$ acts simply transitively on the collection of bases of $D_{\vec{n}}$,

$$W'_{\vec{n}} = W(D_{\vec{n}}) \rtimes \Gamma. \tag{A.1}$$

Explicitly, the group $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^{d-1}$ is generated by the elements $\epsilon^{(k)}\epsilon^{(k+1)}$ for $k = 1, \ldots, d-1$, where $\epsilon^{(k)} = s_{e_{n_k}}$ is the reflection associated to the short simple root of $B_{n_k}$.

The Springer correspondence was extended to groups of this kind in [Kato 1983; Aubert et al. 2017c]. Let $T$ be the diagonal torus of the connected complex group

$$G^\circ = \mathrm{SO}_{2n_1}(\mathbb{C}) \times \cdots \times \mathrm{SO}_{2n_d}(\mathbb{C}). \tag{A.2}$$

Then $W'_{\vec{n}}$ acts naturally on $T$ and we recover $W(D_{\vec{n}})$ as the Weyl group of $(G^\circ, T)$. The Lie algebra of $T$ can be identified with the defining representation of

$$W(B_{\vec{n}}) := W(B_{n_1}) \times \cdots \times W(B_{n_d}). \tag{A.3}$$

Since $\Gamma$ consists of diagram automorphisms of $D_{\vec{n}}$, we can build the reductive group

$$G = G^\circ \rtimes \Gamma. \tag{A.4}$$

Then $W'_{\vec{n}}$ becomes the "Weyl" group of this disconnected group:

$$W'_{\vec{n}} = W(G, T) := N_G(T)/T.$$

For $u \in G^\circ$ unipotent let $\mathcal{B}^u = \mathcal{B}^u_{G^\circ}$ be the variety of Borel subgroups of $G^\circ$ containing $u$. The group $Z_G(u)$ acts naturally on $\mathcal{B}^u \times \Gamma$, and that induces an action of $A_G(u) = \pi_0(Z_G(u)/Z(G))$ on $H^i(\mathcal{B}^u; \mathbb{C}) \otimes \mathbb{C}[\Gamma]$. For $\rho' \in \mathrm{Irr}(A_G(u))$ we form the $W'_{\vec n}$-representations

$$H(u,\rho') = H_{A_G(u)}(\rho, H^*(\mathcal{B}^u;\mathbb{C})\otimes\mathbb{C}[\Gamma]),$$
$$\pi(u,\rho') = H_{A_G(u)}(\rho, H^{\mathrm{top}}(\mathcal{B}^u;\mathbb{C})\otimes\mathbb{C}[\Gamma]).$$

We call $\rho'$ geometric if $\pi(u,\rho)\neq 0$. Then [Aubert et al. 2017c, Theorem 4.4] says that $\pi(u,\rho')\in\mathrm{Irr}(W'_{\vec n})$ and that this yields a bijection between $\mathrm{Irr}(W'_{\vec n})$ and the $G$-conjugacy classes of pairs $(u,\rho')$ with $u\in G^\circ$ unipotent and $\rho'\in\mathrm{Irr}(A_G(u))$ geometric.

The $W'_{\vec n}$-representations $H'(u,\rho')$, with $(u,\rho')$ as above, form another $\mathbb{Z}$-basis of $R_{\mathbb{Z}}(W'_{\vec n})$. Indeed, this can be shown in the same way as for Weyl groups in [Reeder 2001, Lemma 3.3.1]; the input from [Borho and MacPherson 1981] holds for $W'$ by [Aubert et al. 2017c, Lemma 4.5].

For $P\subset\Delta_{\vec n}$ we define the standard parabolic subgroup

$$W'_P := \langle s_\alpha : \alpha\in P\rangle \rtimes \mathrm{Stab}_\Gamma(P).$$

As usual, a parabolic subgroup of $W'_{\vec n}$ is a conjugate of some $W'_P$. Let $P_A$ be the standard basis of the union of the type $A$ root subsystems of $R_P$ and let $P_B$ be the standard basis of the union of the type $B$ root subsystems of $\mathbb{Q}R_P\cap B_{\vec n}$. (So $P_B$ need not be contained in $P$.) It is easily seen that

$$W'_P = W_{P_A}\times W_{P_B}\cap W(D_n) = W_{P_A}\times W'_{\vec n_P}, \tag{A.5}$$

where $\vec n_P$ consists of the numbers $|P_B\cap B_{n_i}|$ which are nonzero.

All the above notions for $W'_{\vec n}$ have natural analogues for $W'_P$, which we indicate by an additional subscript $P$. In particular, [Kato 1983, Proposition 6.2] entails that, as in (1.5) and (1.6),

$$\mathrm{ind}_{W'_P}^{W'_{\vec n}}(H_{W'_P}(u_P,\rho'_P)) \cong \mathrm{Hom}_{A_{G_P}(u_P)}(\rho_P, H^*(\mathcal{B}^{u_P};\mathbb{C})\otimes\mathbb{C}[\Gamma]).$$

**Lemma A.6.** *The parabolic subgroups of $W'_{\vec n}$ are precisely the isotropy groups of the points of Lie$(T)$.*

*Proof.* Considering the standard representation of $W(B_{\vec n})$ on $\mathrm{Lie}(T)$, we see that for any $y\in\mathrm{Lie}(T)$ the isotropy group $(W'_{\vec n})_y$ is $W(B_{\vec n})$-conjugate to $W(B_{\vec n})_Q\cap W(D_{\vec n})$, where $W(B_{\vec n})_Q$ is a standard parabolic subgroup of $W(B_{\vec n})$. From (A.5) we see that the group $W(B_{\vec n})_Q\cap W(D_{\vec n})$ equals $W'_P$ for $R_P = R_Q\cap D_{\vec n}$. Hence every isotropy group $(W'_{\vec n})_y$ is $W(B_{\vec n})$-conjugate to some standard parabolic subgroup of $W'_{\vec n}$. Since the diagram automorphisms $\epsilon^{(k)}$ stabilize the collection of parabolic subgroups of $W'_{\vec n}$ and $W(B_{\vec n})$ is generated by $W(D_{\vec n})$ and the $\epsilon^{(k)}$, we conclude that $(W'_{\vec n})_y$ is $W'_{\vec n}$-conjugate to a parabolic subgroup of $W'_{\vec n}$. □

With Lemma A.6 we can define ellipticity in two equivalent ways. An element of $W'_{\vec n}$ is elliptic if it is not contained in a proper parabolic subgroup, or equivalently, if it fixes a nonzero element of $\mathrm{Lie}(T)$. With these notions we can develop the elliptic representation theory of $W'_{\vec{n}}$, exactly as in [Reeder 2001] and as in Section 1A. In particular, (1.11) remains valid.

**Lemma A.7.** *The group of elliptic representations $\overline{R_{\mathbb{Z}}}(W'_{\vec{n}})$ is torsion-free.*

*Proof.* We follow the proof of Theorem 1.12, with the group $G^\circ$ from (A.2). Every Levi subgroup of $G^\circ$ can be described by a $d$-tuple of partitions $\vec{\alpha} = (\alpha^{(1)}, \dots, \alpha^{(d)})$. The standard Levi subgroup associated to $\vec{\alpha}$ is

$$G^\circ_{\vec{\alpha}} = \prod_{i=1}^d \mathrm{SO}_{2n_i}(\mathbb{C})_{\alpha^{(i)}} = \prod_{i=1}^d \mathrm{GL}_{\alpha_1^{(i)}}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{\alpha_{d_i}^{(i)}}(\mathbb{C}) \times \mathrm{SO}_{2(n_i - |\alpha^{(i)}|)}(\mathbb{C}).$$

(We note that sometimes several $P \subset \Delta$ are associated to one $\vec{\alpha}$, as already for $\mathrm{SO}_{2n}(\mathbb{C})$.) We mimic (A.4) by putting

$$\begin{aligned} G_{\vec{\alpha}} &= G^\circ_{\vec{\alpha}} \rtimes \langle \epsilon^{(i)}\epsilon^{(j)} : |\alpha^{(i)}| < n_i \text{ and } |\alpha^{(j)}| < n_j \rangle \\ &= \Big( \prod_{i=1}^d \mathrm{GL}_{\alpha_1^{(i)}}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{\alpha_{d_i}^{(i)}}(\mathbb{C}) \Big) \times S\Big( \prod_{i=1}^d O_{2(n_i - |\alpha^{(i)}|)}(\mathbb{C}) \Big). \end{aligned}$$

Then $W(G_{\vec{\alpha}}, T) \cong W'_P$ for $P \subset \Delta$ corresponding to $\vec{\alpha}$.

The Bala–Carter classification says that the unipotent classes in $G^\circ$ can be parametrized by $d$-tuples of bipartitions $(\vec{\alpha}, \vec{\beta})$ such that $2|\alpha^{(i)}| + |\beta^{(i)}| = 2n_i$, $\beta^{(i)}$ has only odd parts and all parts of $\beta^{(i)}$ are distinct. A typical $u$ in this conjugacy class is distinguished in the standard Levi subgroup $G^\circ_{\vec{\alpha}}$.

Like in (1.13) and (1.14), let $G_{\vec{\alpha}''}$ be a standard Levi subgroup containing $u$. Then $u = u''u'$ with $u'$ in a product of groups $\mathrm{GL}_{n_k}(\mathbb{C})$ and

$$u' \in S\Big( \prod_{i=1}^d O_{2(n_i - |\alpha''^{(i)}|)}(\mathbb{C}) \Big) =: H.$$

The GL-factors and $u''$ do not contribute to $A_{G_{\vec{\alpha}''}}(u)$.

In the upcoming calculations we omit the case that $\vec{\beta}$ is empty; that case is a bit different but can be handled in the same way.

With [Carter 1972, §13.1], we find that the quotient of $Z_H(u')$ by its unipotent radical is

$$\prod_{i=1}^d \prod_{j \text{ even}} \mathrm{Sp}_{2m_j'^{(i)}}(\mathbb{C}) \times \prod_{i=1}^d \prod_{j \text{ odd, not in } \beta^{(i)}} O_{2m_j'^{(i)}}(\mathbb{C}) \times S\Big( \prod_{i=1}^d \prod_{j \text{ odd, in } \beta^{(i)}} O_{2m_j'^{(i)}+1}(\mathbb{C}) \Big).$$

The component groups become

$$A_{G_{\vec{\alpha}''}}(u) \cong A_H(u') \cong \Big( \prod_{i=1}^d \prod_{j \text{ odd, in } \alpha'^{(i)}, \text{ not in } \beta^{(i)}} \mathbb{Z}/2\mathbb{Z} \Big) \times S\Big( \prod_{i=1}^d \prod_{j \text{ odd, in } \beta^{(i)}} \mathbb{Z}/2\mathbb{Z} \Big).$$

In the same way as after (1.15), we see that $\overline{R_{\mathbb{Z}}}(A_G(u)) = 0$ unless each $\alpha^{(i)}$ has only distinct odd terms, none of them appearing in $\beta^{(i)}$. For such $(\vec{\alpha}, \vec{\beta})$, the maximal reductive quotient of $Z_G(u)$ simplifies to

$$\Big(\prod_{i=1}^{d} \prod_{j \text{ odd, in } \alpha^{(i)}} O_2(\mathbb{C})\Big) \times S\Big(\prod_{i=1}^{d} \prod_{j \text{ odd, in } \beta^{(i)}} O_1(\mathbb{C})\Big) \tag{A.8}$$

and the component group becomes

$$A_G(u) = \prod_{i=1}^{d} \prod_{j \text{ odd, in } \alpha^{(i)}} \mathbb{Z}/2\mathbb{Z} \times A \quad \text{with} \quad A = S\Big(\prod_{i=1}^{d} \prod_{j \text{ odd, in } \beta^{(i)}} \mathbb{Z}/2\mathbb{Z}\Big).$$

Just as in (1.16), we can calculate that $\overline{R_{\mathbb{Z}}}(A_G(u)) \cong R_{\mathbb{Z}}(A)$. □

With Lemmas A.6 and A.7 at hand the proof of Proposition 1.17 also becomes valid for $W'_{\vec{n}}$. Let us formulate this somewhat more generally. Let $W'$ be a finite group which is a direct product of a Weyl group and a number of groups of the form $W'_{\vec{n}}$. Let $G'$ be the corresponding direct product of the groups called $G$ in (1.3) and (A.4). We denote the basis of the root system $R'$ underlying $W'$ by $\Delta'$, and the standard parabolic subgroup associated to $P \subset \Delta$ by $W'_P$.

**Lemma A.9.** *For every $w \in \mathcal{C}_P(W')$, there exists a pair $(u_{P,w}, \rho'_{P,w})$ such that*

- *$u_{P,w}$ is quasidistinguished unipotent in $G'_P$,*
- *$\rho'_{P,w} \in \mathrm{Irr}(A_{G'_P}(u_{P,w}))$ is geometric,*
- *the set*

$$\big\{ \mathrm{ind}_{W'_P}^{W'_{\vec{n}}}(H_P(u_{P,w}, \rho'_{P,w})) : P \in \mathcal{P}(\Delta_{\vec{n}})/W'_{\vec{n}},\ w \in \mathcal{C}_{P,\mathrm{ell}}(W'_{\vec{n}}) \big\}$$

*forms a $\mathbb{Z}$-basis of $R_{\mathbb{Z}}(W')$.*

*Proof.* Let $(W'_i)_i$ be the indecomposable factors of $W'$, with root systems $R'_i$. For every $P \subset \Delta'$,

$$W'_P = \prod_i W'_{P \cap R'_i} \quad \text{and} \quad R_{\mathbb{Z}}(W'_P) = \bigotimes_i R_{\mathbb{Z}}(W'_{P \cap R'_i}).$$

Thus we reduce to the case of a single $W'_i$. If $W'_i$ is an irreducible Weyl group, then Proposition 1.17 applies immediately, so we may assume that $W'_i = W'_{\vec{n}}$.

Let $u \in G$ be unipotent and assume that $\overline{R_{\mathbb{Z}}}(A_G(u)) \neq 0$. From the proof of Lemma A.7, we see that a maximal reductive subgroup of $Z_G(u)$ is of the form (A.8). For each $(i, j)$ with $j$ in $\alpha^{(i)}$, we pick an element $t_{i,j} \in \mathrm{SO}_2(\mathbb{C}) \setminus \big\{\pm\big(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\big)\big\}$, all different. This gives a semisimple element

$$t := \prod_{i=1}^{d} \Big( \prod_{j \text{ in } \alpha^{(i)}} t_{i,j} \times \prod_{j \text{ in } \beta^{(i)}} 1 \Big) \in Z_G(u)^{\circ}.$$

Furthermore, $t$ does not lie in any proper Levi subgroup of $G^\circ$ containing $u$, so $tu$ does lie in any proper Levi subgroup of $G^\circ$. Thus, $u$ is quasidistinguished in $G$.

Knowing this and Lemma A.7, the proof of Proposition 1.17 goes through. □

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Received 6 Nov 2016. Revised 18 Sep 2017. Accepted 19 Oct 2017.

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ANNALS OF K-THEORY
Vol. 3, No. 3, 2018
dx.doi.org/10.2140/akt.2018.3.461

# On a localization formula of epsilon factors via microlocal geometry

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Given a lisse $l$-adic sheaf $\mathcal{G}$ on a smooth proper variety $X$ and a lisse sheaf $\mathcal{F}$ on an open dense $U$ in $X$, Kato and Saito conjectured a localization formula for the global $l$-adic epsilon factor $\varepsilon_l(X, \mathcal{F} \otimes \mathcal{G})$ in terms of the global epsilon factor of $\mathcal{F}$ and a certain intersection number associated to $\det(\mathcal{G})$ and the Swan class of $\mathcal{F}$. In this article, we prove an analog of this conjecture for global de Rham epsilon factors in the classical setting of $\mathcal{D}_X$-modules on smooth projective varieties over a field of characteristic zero.

## 1. Introduction

Let $X$ denote a smooth proper variety of dimension $d$ over a finite field $F$ of characteristic $p$, and let $\mathcal{G}$ be a smooth étale $\mathbb{Q}_l$ (or $\mathbb{F}_l$) sheaf. Then, one has the usual global $l$-adic epsilon factor

$$\varepsilon_l(X, \mathcal{F}) := \prod_{q=0}^{2d} \det(-\sigma : \mathrm{H}^q_c(U_{\bar{F}}, \mathcal{F}))^{(-1)^{q+1}},$$

where $\sigma \in \mathrm{Gal}(\bar{F}/F)$ is the geometric Frobenius. In this setting, Kato and Saito conjectured the following "localization" formula for the epsilon factor of the tensor product:

**Conjecture** [Kato and Saito 2008, Conjecture 4.3.11]. Let $\mathcal{F}$ be a constructible sheaf on $X$, and $\mathcal{G}$ a smooth sheaf on $X$. Then one has

$$\varepsilon_l(X, \mathcal{F} \otimes \mathcal{G}) = \varepsilon_l(X, \mathcal{F})^{r_{\mathcal{G}}} \cdot \langle \det(\mathcal{G}), \mathrm{CC}(\mathcal{F}) \rangle.$$

Here $r_{\mathcal{G}}$ denotes the rank of $\mathcal{G}$, and $\langle -, - \rangle$ denotes a pairing defined using the class field theory which we do not recall here.

Abe is supported by Grant-in-Aid for Young Scientists (A) 16H05993. Patel would like to acknowledge support from the National Science Foundation award DMS-1502296. Finally, both authors would like to thank the referee for reading the manuscript very carefully.
*MSC2010:* 14C35, 14F10, 19M05.
*Keywords:* K-theory, epsilon factors.

When $X$ is a proper smooth variety over a field $k$ of characteristic 0, the second author constructed the de Rham epsilon factor formalism in [Patel 2012]. More precisely, let $\mathrm{K}(\mathcal{D}_X)$ denote the K-theory spectrum of coherent $\mathcal{D}_X$-modules, and $\mathrm{K}(T^*X)$ denote the K-theory spectrum of coherent sheaves. Then he constructed a map of spectra

$$\varepsilon : \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(T^*X).$$

At the level of Grothendieck groups, given a holonomic module $\mathcal{F}$, the class $[\varepsilon(\mathcal{F})] \in \mathrm{K}_0(T^*X)$ is the class $[\mathrm{gr}^F(\mathcal{F})]$, where $F$ is a good filtration of $\mathcal{F}$. It is well-known that the class is independent of the choice of good filtration. The composition of $\varepsilon$ with the pull-back by a certain twist of the zero-section followed by the push-forward $R\Gamma : \mathrm{K}(X) \to \mathrm{K}(k)$ is homotopic to the de Rham cohomology map $R\Gamma_{\mathrm{dR}}$ (see Lemma 2.7.5). In particular, passing to Grothendieck groups, we may proceed via $\varepsilon$ in order to compute the Euler–Poincaré characteristic. Moreover, an automorphism $f$ of $\mathcal{F}$ determines an element in $\pi_1\mathrm{K}(\mathcal{D}_X)$ whose image under the morphism $R\Gamma_{\mathrm{dR}}$ gives an element of $\pi_1\mathrm{K}(k) \cong k^\times$. The latter is precisely the determinant of the induced automorphism on the de Rham cohomology of $\mathcal{F}$. In [Patel 2012], a "microlocalized" version of $\varepsilon$ was also constructed, which allows one to pass to the K-theory of holonomic $\mathcal{D}_X$-modules and construct a morphism of spectra

$$\mathrm{CC} : \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{K}^{(d)}(X, -).$$

Here $\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X)$ is the K-theory spectrum of holonomic $\mathcal{D}_X$-modules, and $\mathrm{K}^{(d)}(X, -)$ is part of Levine's homotopy coniveau tower. We do not recall the definition here, but only note that $\pi_0(\mathrm{K}^{(d)}(X, -)) = \mathrm{CH}_0(X)$ and, at the level of $\pi_0$, CC associates to the class of a holonomic $\mathcal{D}_X$-module the zero cycle given by pulling back its characteristic cycle by the zero section. Our main result is the following analog of the Kato–Saito localization formula in the de Rham and K-theoretic setting. Below, we let $\mathrm{K}_X(\mathcal{D}_X)$ denote the K-theory spectrum of $\mathcal{D}_X$-modules with singular support contained in the zero section. Since any such $\mathcal{D}_X$-module is just a flat connection, one has a natural morphism $\mathrm{K}_X(\mathcal{D}_X) \xrightarrow{\mathrm{for}^\nabla} \mathrm{K}(X)$ given by forgetting the connection.

**Theorem 4.1.1.** *Let $d$ be the dimension of $X$. The following diagram commutes up to homotopy*:

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "\mathrm{for}^\nabla \wedge \mathrm{CC}"'] & \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[d, "R\Gamma_{\mathrm{dR}}"] \\
\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X, -) \arrow[r, "\langle -, - \rangle^{\mathrm{K}}_{(d,-)}"'] & \mathrm{K}(k)
\end{tikzcd}$$

The pairing $\langle -, - \rangle^{\mathrm{K}}_{(d,-)}$ is an analog in our setting of the pairing appearing in the conjecture above. The usual dictionary between connective spectra and Picard groupoids allows one to get formulas for determinants of endomorphisms, and, in particular, by taking $\pi_1$ of the commutative diagram above we get an equality of actual numbers analogous to that in the conjecture above. We refer the reader to Theorem 4.3.1 for a precise statement. We note that this particular consequence can be shown with a much simpler argument as described in the proof of Theorem 4.3.1. On the other hand, by the same method, we also obtain similar formulas in the setting of correspondences (and not just endomorphisms). In particular, suppose we are given an automorphism $\varphi$ of $X$. Then a correspondence of a $\mathcal{D}_X$-module $\mathcal{F}$ is an isomorphism $\mathcal{F} \to \varphi_*\mathcal{F}$. Given a correspondence, it induces an automorphism on the cohomology $R\Gamma_{\mathrm{dR}}(X, \mathcal{F})$, and we may again consider the determinant of this automorphism. We also obtain a localization formula in this setting.

We note that, after most of this paper was written, the original conjecture of Kato and Saito was proven, with some modification of the definition of characteristic cycles and following recent developments in ramification theory for $l$-adic sheaves, in [Umezaki et al. 2018]. However, following the philosophy of Beilinson [2007], we believe that the K-theoretic method gives a different perspective on localization formulas for epsilon factors. In principle, proving the formula at the level of K-theory spectra should also give formulas in higher K-theory. At the level of $\mathrm{K}_0$ one gets formulas for the Euler characteristic, and at the level of $\mathrm{K}_1$, for determinants. It would be interesting to see the consequences at the level of $\mathrm{K}_2$ (or higher K-groups).

Let us explain the structure of the paper. We begin with collecting some materials from K-theory used in this paper. In particular, we recall some basic properties of Levine's homotopy coniveau tower. In Section 3, we define the pairing $\langle -,-\rangle^{\mathrm{K}}_{(d,-)}$, and prove a key vanishing lemma (Lemma 3.7.1). This allows us to compute the pairing in the setting of correspondences. We formulate and prove the localization formula in the last section. The localization formula as an equality of values is especially easy to prove when we are given actual automorphism of modules. We conclude the paper by providing an elementary proof of this simple case.

## 2. Background

In this article, we shall make use of K-theory spectra and their associated Picard groupoids. However, our applications will mostly use these constructions in a formal manner. We briefly recall the required concepts and constructions for ease of exposition.

**2.1.** ***Spectra.*** In the following, we fix a symmetric monoidal category of spectra and denote it by $\mathcal{S}$. For example, one could take for $\mathcal{S}$ Lurie's $(\infty, 1)$-category of spectra or the category of symmetric spectra. We only make use of this category in a formal manner. Moreover, our results on traces only depend on the associated homotopy category (which are all known to be equivalent for the various models for spectra). Recall that $\mathcal{S}$ is a proper simplicial model category. In particular, one has functorial fibrant-cofibrant replacements. In the following, we assume all our spectra are fibrant-cofibrant. We denote by $\wedge$ the monoidal structure in $\mathcal{S}$.

The homotopy category of $\mathcal{S}$ is denoted by $\mathrm{Ho}(\mathcal{S})$. By definition, this is the localization of $\mathcal{S}$ with respect to the weak equivalences. A weak equivalence of spectra $P \to Q$ can be inverted as a morphism in the homotopy category. However, in general such a morphism cannot be inverted as a morphism of spectra. To remedy this situation, one can use the more general notion of a homotopy morphism of spectra. A *homotopy morphism* $P \to Q$ consists of a contractible simplicial set $K$ and a genuine morphism of spectra $f : K \wedge P \to Q$. We refer to $K$ as the base of the homotopy morphism, and by abuse of notation we denote the homotopy morphism simply by $f : P \to Q$. Given two homotopy morphisms $f, g$ with bases $K_f$, $K_g$, an *identification of $f$ and $g$* is a homotopy morphism $h$ with base $K_h$ together with morphisms $K_f \to K_h \leftarrow K_g$ such that $f, g$ are the respective pull-backs of $h$. One can define the composition of two homotopy morphisms $f : P \to Q$ and $g : Q \to R$ as the composition $K_g \wedge K_f \wedge P \to K_g \wedge Q \to R$. A homotopy morphism from a sphere spectrum to a given spectrum $P$ will be referred to as a homotopy point of $P$. If $f$ and $g$ are identified, then they induce the same maps on homotopy groups. A weak equivalence between fibrant-cofibrant spectra can be canonically inverted as a homotopy morphism. We refer to [Patel 2012, Section 2.1] or [Beilinson 2007, Section 1.4, Example (ii)] for the details. We note that in the following the language of homotopy morphisms is not necessary, since, for our purposes, we could work directly in the homotopy category. However, it is a convenient notion for constructions at the level of actual spectra (rather than the homotopy category).

**2.2.** K-***theory spectra.*** Let $\mathcal{E}$ be a small exact category. Then Quillen's K-theory construction gives a functor from the category of small exact categories to the category of spectra. If $F_1 : \mathcal{E}_1 \to \mathcal{E}_2$ and $F_2 : \mathcal{E}_2 \to \mathcal{E}_3$ are exact functors, then one has

$$\mathrm{K}(F_2) \circ \mathrm{K}(F_1) = \mathrm{K}(F_2 \circ F_1).$$

More generally, a natural isomorphism of functors induces a homotopy equivalence of the corresponding morphisms of K-theory spectra. By taking a large enough Grothendieck universe, we may assume all our categories are small.

More generally, Waldhausen associates to any category with cofibrations and weak equivalences a corresponding K-theory spectrum. Moreover, an exact functor between Waldhausen categories induces a morphism between the corresponding spectra. In this article, we are mostly interested in complicial bi-Waldhausen categories and complicial exact functors; we refer the reader to [Thomason and Trobaugh 1990] for details. If $\mathcal{E}$ is an exact category, then $\mathrm{C}^{\mathrm{b}}(\mathcal{E})$ is a complicial bi-Waldhausen category with weak equivalences. A fundamental result of Thomason, Trobaugh, Waldhausen, and Gillet [Thomason and Trobaugh 1990] shows that the inclusion of $\mathcal{E}$ into $\mathrm{C}^{\mathrm{b}}(\mathcal{E})$ as degree zero morphisms induces a canonical weak equivalence of spectra $\mathrm{K}(\mathcal{E}) \to \mathrm{K}(\mathrm{C}^{\mathrm{b}}(\mathcal{E}))$. Here the right side is the Waldhausen K-theory spectrum associated to $\mathrm{C}^{\mathrm{b}}(\mathcal{E})$. This allows us to canonically identify various Quillen and Waldhausen K-theory spectra. In the following, we always assume all our spectra to be fibrant-cofibrant. In particular, the machinery from the previous section allows us to invert various weak equivalences canonically as homotopy morphisms.

Given a Waldhausen category $\mathcal{A}$, we denote by $\mathcal{A}^{\mathrm{tri}}$ the associated homotopy category given by inverting the weak equivalences; note that this is a triangulated category. If $F : \mathcal{A} \to \mathcal{B}$ is a complicial exact functor between two complicial bi-Waldhausen categories such that the induced map on homotopy categories is an equivalence of categories, then the induced map on K-theory spectra is a weak equivalence. We often consider derived functors which are a priori only defined on $\mathcal{A}^{\mathrm{tri}}$. Usually, these can be lifted to functors on certain full complicial bi-Waldhausen subcategories $\mathcal{C} \subset \mathcal{A}$ such that the inclusion induces an equivalence on the associated triangulated categories. Using the formalism of homotopy morphisms, we can lift the derived functor to a morphism of K-theory spectra. A typical application is the following: Let $X$ be a proper scheme over $k$, and let $\mathrm{K}(X)$ be the K-theory spectrum of perfect complexes on $X$. Since $X$ is proper, we can define $R\Gamma : \mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{perf}}(k)$. The above approach allows us to lift this to a homotopy morphism $R\Gamma : \mathrm{K}(X) \to \mathrm{K}(k)$, where $\mathrm{K}(X)$ is the K-theory spectrum of the category of perfect complexes on $X$ and similarly for $\mathrm{K}(k)$. First, we may consider the (full) complicial bi-Waldhausen subcategory of flasque perfect complexes. On this subcategory, $R\Gamma$ is represented by $\Gamma$. Furthermore, the properness assumption implies that $\Gamma$ preserves perfectness. We refer to [Thomason and Trobaugh 1990] for more details.

**Remark 2.2.1.** Let $X$ be a smooth projective variety over a field $k$, and $\mathcal{D}_X$ the sheaf of differential operators on $X$. Let $\mathrm{K}(\mathcal{D}_X)$ denote the K-theory spectrum of complexes of coherent-$\mathcal{D}_X$-modules. Then, via the above procedure, the $\mathcal{D}_X$-module push-forward induces a homotopy morphism $R\Gamma_{\mathrm{dR}} : \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(k)$. For example, one can take the usual locally free resolution by the de Rham complex and restrict to flasque complexes.

**2.3.** ***Picard groupoids, determinants, and traces.*** We recall some basic facts about Picard groupoids and determinants which will be useful in the following. We refer to the beautiful article [Deligne 1987] for the basic theory of Picard groupoids and determinants.

A *Picard groupoid* $\mathcal{P}$ is a symmetric monoidal category in which every object is invertible, which satisfies natural commutativity and associativity constraints. We refer the reader to [Patel 2012, Section 5.2] for a discussion of the definition. In the following, we always assume that our Picard groupoids come with a fixed unit $\mathbb{1}$. In order to avoid confusion, we denote by $+$ the monoidal structure in a Picard groupoid. The following will be one of our main examples of a Picard groupoid.

**Example 2.3.1.** Let $\mathrm{Pic}^{\mathbb{Z}}(X)$ denote the category whose objects are pairs $(\mathcal{L},\alpha)$, where $\mathcal{L}$ is a line bundle on $X$, and $\alpha : X \to \mathbb{Z}$ is a continuous function. We define $\mathrm{Hom}((\mathcal{L},\alpha),(\mathcal{L}',\alpha'))$ to be the set of isomorphisms $\mathcal{L}\to\mathcal{L}'$ if $\alpha=\alpha'$ and the empty set if $\alpha\neq\alpha'$. The monoidal structure is given by setting

$$(\mathcal{L},\alpha)+(\mathcal{L}',\alpha') := (\mathcal{L}\otimes\mathcal{L}',\alpha+\alpha').$$

The commutativity constraint

$$c_{\mathcal{L},\mathcal{L}'} : (\mathcal{L},\alpha)+(\mathcal{L}',\alpha') \cong (\mathcal{L}',\alpha')+(\mathcal{L},\alpha)$$

is given (locally) by sending $l_x\otimes l'_x$ to $(-1)^{\alpha(x)\cdot\alpha'(x)}(l'_x\otimes l_x)$.

Given a vector bundle $V$ on $X$, one can associate to it an object $\det(V)\in\mathrm{Pic}^{\mathbb{Z}}(X)$, where $\alpha(x)$ is taken to be the rank of $V$ at $x$. This construction gives rise to a *determinant* functor

$$\det : \mathrm{Vect}(X)^{\mathrm{iso}} \to \mathrm{Pic}^{\mathbb{Z}}(X).$$

Here $\mathrm{Vect}(X)^{\mathrm{iso}}$ denotes the category whose objects are vector bundles on $X$, and morphisms are isomorphisms of vector bundles. We do not recall the definition of a determinant functor and refer to [Deligne 1987] for details. We only note here that there are natural isomorphisms

$$\det(x\oplus y)\cong\det(x)+\det(y)$$

which are compatible with commutativity constraints. In fact, one can define the notion of a $\mathcal{P}$-valued determinant functor for any exact category $\mathcal{E}$ or even derived categories of exact categories; see [Knudsen 2002]. Moreover, one can extend the determinant functor det above to the category of coherent sheaves or even derived category of perfect complexes on $X$ [Knudsen and Mumford 1976; Knudsen 2002].

One can associate natural homotopy groups to a Picard groupoid. By definition, $\pi_0(\mathcal{P})$ is the group of isomorphism classes of objects in $\mathcal{P}$ and $\pi_1(\mathcal{P}) := \mathrm{End}_{\mathcal{P}}(\mathbb{1})$. We note that if $\mathrm{L}\in\mathcal{P}$, then there is a canonical isomorphism

$$\mathrm{End}_{\mathcal{P}}(\mathrm{L})\to\pi_1(\mathcal{P})$$

defined as follows. If $f:\mathrm{L}\to\mathrm{L}$ is an endomorphism, then it induces an endomorphism

$$f\otimes\mathrm{Id} : \mathrm{L}\otimes\mathrm{L}^{-1}\to\mathrm{L}\otimes\mathrm{L}^{-1},$$

and composing this with the natural isomorphisms $\mathbb{I} \to \mathrm{L} \otimes \mathrm{L}^{-1}$ and $\mathrm{L} \otimes \mathrm{L}^{-1} \to \mathbb{I}$ gives an element of $\mathrm{End}_{\mathcal{P}}(\mathbb{I})$. We call this the trace of $f$, denoted $\mathrm{Tr}(f \mid \mathrm{L}) \in \pi_1(\mathcal{P})$. The following example explains this terminology.

**Example 2.3.2.** For a field $k$, $\pi_1(\mathrm{Pic}^{\mathbb{Z}}(k)) = k^\times$. An automorphism $f : V \to V$ of a finite dimensional vector space over $k$ gives a map

$$\det(f) : (\det(V), \dim(V)) \to (\det(V), \dim(V))$$

in $\mathrm{Pic}^{\mathbb{Z}}(k)$. One can check that $\mathrm{Tr}(\det(f) \mid \det(V)) \in k^\times$ is the usual determinant of $f$.

The following lemma is immediate, and only recorded here for future use

**Lemma 2.3.3.** *Let $\mathcal{P}$ be a Picard groupoid and* $\mathrm{L} \in \mathcal{P}$.

(1) *If* $\mathrm{Id} : \mathrm{L} \to \mathrm{L}$ *is the identity*, *then* $\mathrm{Tr}(\mathrm{Id} \mid \mathrm{L}) = \mathrm{Id} \in \mathrm{End}_{\mathcal{P}}(\mathbb{I})$.

(2) *If* $f, g : \mathrm{L} \to \mathrm{L}$ *are two automorphisms*, *then*

$$\mathrm{Tr}(f \circ g \mid \mathrm{L}) = \mathrm{Tr}(f \mid \mathrm{L}) \circ \mathrm{Tr}(g \mid \mathrm{L}).$$

**2.4. *Picard groupoids and spectra.*** Let Pic be the category of Picard groupoids. We let Ho(Pic) denote the homotopy category of Picard groupoids. This is by definition the category of Picard groupoids localized at equivalences of Picard groupoids. It is well-known that the category of Picard groupoids identifies homotopically with the category of spectra [Patel 2012, §5] with homotopy groups concentrated in degrees 0 and 1. In particular, there are natural adjoint functors $\Pi : \mathcal{S}^{\geq 0} \to \mathrm{Pic}$ and $\mathrm{B} : \mathrm{Pic} \to \mathcal{S}^{\geq 0}$ which induce an equivalence on the associated homotopy categories when restricted to spectra with only nonvanishing homotopy groups in degree 0 or 1. Here B takes a Picard groupoid to its usual classifying space, $\Pi$ is the fundamental groupoid associated to a connective spectrum, and $\mathcal{S}^{\geq 0}$ denotes the category of spectra with nonvanishing homotopy groups only in nonnegative degrees.

This construction allows one to view the Picard groupoid associated to K-theory as a universal determinant functor. Let $\mathcal{E}$ be an exact category and $C^{\mathrm{b}}(\mathcal{E})$ the corresponding Waldhausen category of bounded chain complexes in $\mathcal{E}$. The homotopy point construction gives rise to a natural universal determinant functor

$$\det : (\mathrm{D}^{\mathrm{b}}(\mathcal{E}), \mathrm{qis}) \to \Pi(\mathrm{K}(C^{\mathrm{b}}(\mathcal{E}))).$$

In the following, we are mostly interested in applying this construction to the K-theory spectrum of a scheme. In particular, let $\mathrm{K}(X)$ denote the K-theory spectrum of vector bundles (or coherent sheaves or perfect complexes) on a smooth scheme $X$. In that case, there is a natural map

$$\mathrm{Det} : \Pi(\mathrm{K}(X)) \to \mathrm{Pic}^{\mathbb{Z}}(X).$$

Moreover, the usual determinant functor $\det : (\mathrm{D}^b(X), \mathrm{qis}) \to \mathrm{Pic}^{\mathbb{Z}}(X)$ is compatible with the previous two. In particular, the following diagram is commutative:

$$\begin{tikzcd}
(\mathrm{D}^{\mathrm{b}}(X), \mathrm{qis}) \arrow[r] \arrow[dr] & \Pi(\mathrm{K}(X)) \arrow[d] \\
& \mathrm{Pic}^{\mathbb{Z}}(X)
\end{tikzcd}$$

We note that an explicit construction of a model for the Picard groupoid $\Pi(\mathrm{K}(X))$ can be given by Deligne's virtual categories [1987].

**2.5. *Distributive functors.*** In the following, we are interested in certain pairings of Picard groupoids. Given two Picard groupoids $\mathcal{P}$ and $\mathcal{P}'$, let $\mathcal{P} \times \mathcal{P}'$ denote the product groupoid. Note that we consider this as a mere groupoid (and not a Picard groupoid). A *distributive functor* is a functor

$$\langle -, - \rangle : \mathcal{P} \times \mathcal{P}' \to \mathcal{P}''$$

which satisfies some natural "bilinearity" or "distributive" conditions. We refer to [Deligne 1987, 4.11] for the precise definitions. The definition, in particular, implies that for fixed $\mathrm{L} \in \mathcal{P}$ and $\mathrm{L}' \in \mathcal{P}$, the induced functors $\langle \mathrm{L}, - \rangle$ and $\langle -, \mathrm{L}' \rangle$ are morphisms of Picard groupoids. These morphisms are natural in L and L$'$, respectively. Moreover, one also has natural isomorphisms

$$\langle \mathrm{L}_1 + \mathrm{L}_2, \mathrm{L}' \rangle \cong \langle \mathrm{L}_1, \mathrm{L}' \rangle + \langle \mathrm{L}_2, \mathrm{L}' \rangle \quad \text{and} \quad \langle \mathrm{L}, \mathrm{L}'_1 + \mathrm{L}'_2 \rangle \cong \langle \mathrm{L}, \mathrm{L}'_1 \rangle + \langle \mathrm{L}, \mathrm{L}'_1 \rangle.$$

We refer to such a distributive functor simply as a pairing of Picard groupoids. The following is one of our main examples of a pairing.

**Example 2.5.1.** Let $X$ be an integral scheme over $k$. The tensor product $\otimes$ of line bundles induces a distributive functor:

$$(- \otimes -) : \mathrm{Pic}^{\mathbb{Z}}(X) \times \mathrm{Pic}^{\mathbb{Z}}(X) \to \mathrm{Pic}^{\mathbb{Z}}(X).$$

Explicitly, it sends $(\mathcal{L}, \alpha) \otimes (\mathcal{L}', \alpha') := (\mathcal{L}^{\otimes \alpha'} \otimes \mathcal{L}'^{\otimes \alpha}, \alpha\alpha')$. Note that for vector bundles $G$ and $G'$, one has $\det(G \otimes G') \cong \det(G) \otimes \det(G')$ in $\mathrm{Pic}^{\mathbb{Z}}(X)$.

**Lemma 2.5.2.** *Let $\langle -, - \rangle : \mathcal{P} \times \mathcal{P}' \to \mathcal{P}''$ be a distributive functor. Given morphisms $f : \mathrm{L} \to \mathrm{L} \in \mathcal{P}$ and $g : \mathrm{L}' \to \mathrm{L}' \in \mathcal{P}'$, let $\langle \mathrm{L}, \mathrm{L}' \rangle(f, g) := \mathrm{Tr}(\langle f, g \rangle \mid \langle \mathrm{L}, \mathrm{L}' \rangle) \in \pi_1(\mathcal{P}'')$.*

(i) *If $f$ is the identity, then $\langle \mathrm{L}, \mathrm{L}' \rangle(\mathrm{Id}, g)$ is the image of* $\mathrm{Tr}(g \mid \mathrm{L}')$ *under the induced map $\pi_1 \langle \mathrm{L}, - \rangle : \pi_1(\mathcal{P}') \to \pi_1(\mathcal{P}'')$.*

(ii) *If $f_i : \mathrm{L}_i \to \mathrm{L}_i \in \mathcal{P}$ $(i = 1, 2)$, then*

$$\langle \mathrm{L}_1 + \mathrm{L}_2, \mathrm{L}' \rangle(f_1 + f_2, g) = \langle \mathrm{L}_1, \mathrm{L}' \rangle(f_1, g) \langle \mathrm{L}_2, \mathrm{L}' \rangle(f_2, g).$$

Any abelian group can be considered as a Picard groupoid. We will sometimes be interested in a pairing of a Picard groupoid $\mathcal{P}$ and an abelian group $G$ with values in a Picard groupoid $\mathcal{P}''$. By definition, this means that for each $g \in G$, we have a morphism of Picard groupoids $F_g : \mathcal{P} \to \mathcal{P}''$ such that $F_e = \mathbb{I}$ (where $e \in G$ is the identity), and there are natural isomorphisms $F_{g+h} \cong F_g + F_h$. Note that $\mathrm{Hom}(\mathcal{P}, \mathcal{P}'')$ is also a Picard groupoid, and such a pairing can be interpreted as a morphism of groupoids

$$G \to \mathrm{Hom}(\mathcal{P}, \mathcal{P}'').$$

The following is our central example of such a pairing.

**Example 2.5.3.** Let $f : X \to \mathrm{Spec}(k)$ denote a smooth proper scheme over a field $k$, and $\mathrm{Z}_0(X)$ denote the abelian group of 0-cycles on $X$. Then we have a pairing

$$\langle -, - \rangle : \mathrm{Pic}^{\mathbb{Z}}(X) \times \mathrm{Z}_0(X) \to \mathrm{Pic}^{\mathbb{Z}}(k)$$

defined as follows. If $i_Z : Z \subset X$ is a closed integral subscheme of dimension 0, then we set

$$\langle (\mathcal{L}, \alpha), [Z] \rangle := \big(\det(\pi_{Z,*}\mathcal{O}_Z)^{\otimes \alpha} \otimes N(i_Z^*\mathcal{L}), \mathrm{Tr}(\alpha)\big).$$

Here $\pi_Z : Z \to \mathrm{Spec}(k)$ is the structure map, $N$ is the norm functor on line bundles, and Tr is the trace map. Note $\mathrm{Tr}(\alpha)$ is just given by $n\alpha$, where $n$ is the degree of $k(Z)$ over $k$. We refer the reader to [Deligne 1987, §7] for the details. This defines the pairing for all effective cycles, and then we extend by linearity.

Finally, we note that pairings of spectra induce pairings of Picard groupoids. We refer to [Schwede 2012, Chapter I, Section 5.1] for details on the notion of bilinear pairings of spectra. Here we only note that a bilinear pairing of spectra $\mathrm{K}_1$ and $\mathrm{K}_2$ with values in $\mathrm{K}_3$ is equivalent to giving a morphism of spectra

$$\mathrm{K}_1 \wedge \mathrm{K}_2 \to \mathrm{K}_3.$$

Furthermore, a biexact functor of exact categories (or Waldhausen categories) induces a bilinear pairing of the corresponding K-theory spectra [Thomason and Trobaugh 1990, 3.15]. Moreover, under the equivalence of categories between Picard groupoids and spectra, a bilinear map gives rise to a pairing of the associated Picard groupoids. In particular, the usual tensor product of vector bundles induces a pairing of spectra $\mathrm{K}(X) \wedge \mathrm{K}(X) \to \mathrm{K}(X)$ and, therefore, a pairing $\Pi(\mathrm{K}(X)) \times \Pi(\mathrm{K}(X)) \to \Pi(\mathrm{K}(X))$. Moreover, this pairing is compatible with the one from Example 2.5.1 under $\mathrm{Det} : \Pi(\mathrm{K}(X)) \to \mathrm{Pic}^{\mathbb{Z}}(X)$. In the following, we sometimes use the notation

$$\mathcal{P} \wedge \mathcal{P}' \to \mathcal{P}''$$

to mean a pairing $\mathcal{P} \times \mathcal{P}' \to \mathcal{P}''$. We note that this should only be thought of as formal notation, and the Picard groupoid $\mathcal{P} \wedge \mathcal{P}'$ has not been defined.

**Remark 2.5.4.** We can take the fundamental Picard groupoid associated to $\mathrm{B}\mathcal{P}\wedge\mathrm{B}\mathcal{P}'$ as the definition of $\mathcal{P}\wedge\mathcal{P}'$. Moreover, there is an equivalence between pairings $\mathcal{P}\times\mathcal{P}'\to\mathcal{P}''$ and morphisms of Picard groupoids $\mathcal{P}\wedge\mathcal{P}'\to\mathcal{P}''$. However, we shall not need this in what follows. Note that for strictly commutative Picard groupoids this construction is described in [SGA $4_3$ 1973, Exposé XVIII].

A homotopy equivalence between two morphisms of spectra induces a monoidal natural transformation of the corresponding morphisms of Picard groupoids. In particular, it is compatible with the monoidal structures. A homotopy equivalence between bilinear pairing of spectra induces a natural transformation between the corresponding distributive functor, which is a monoidal natural transformation when restricted to each variable. We refer to such a natural transformation as an equivalence of distributive functors. In the following, a diagram of Picard groupoids

$$\begin{array}{ccc} \mathcal{P}_1\wedge\mathcal{P}_2 & \xrightarrow{F} & \mathcal{P}_3 \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G'} \\ \mathcal{P}'_1\wedge\mathcal{P}'_2 & \xrightarrow{F'} & \mathcal{P}'_3 \end{array}$$

with horizontal maps distributive functors, right vertical map a morphism of Picard groupoids, and left vertical map a functor which is a morphism of Picard groupoids in each variable, is said to be commutative if the induced distributive functors

$$G'\circ F,\ \ F'\circ G:\mathcal{P}_1\wedge\mathcal{P}_2\to\mathcal{P}_3$$

are equivalent. A homotopy commutative square of spectra

$$\begin{array}{ccc} K_1\wedge K_2 & \longrightarrow & K_3 \\ \downarrow & & \downarrow \\ K'_1\wedge K'_2 & \longrightarrow & K'_3 \end{array}$$

induces a commutative diagram at the level of Picard groupoids.

**2.6.** ***Levine's homotopy coniveau tower.*** In this subsection, $X$ will be a smooth scheme of finite type over a field $k$. Moreover, $\mathrm{K}(X)$ will denote the K-theory spectrum of coherent sheaves on $X$. We recall the construction and some basic properties of Levine's homotopy coniveau tower associated to the K-theory of schemes which shall be used in the following. We refer to [Levine 2006; 2008] for details.

Let $\Delta^n:=\mathrm{Spec}\big(k[t_0,\dots,t_r]/\big(\sum_j t_j-1\big)\big)$ denote the usual $n$-simplex. These form a cosimplicial scheme. A face of $\Delta^n$ is a closed subscheme defined by equations of the form $t_{i_1}=\cdots=t_{i_s}=0$. Then one defines

$$\mathrm{K}^{(q)}(X,p):=\underrightarrow{\mathrm{holim}}_W\,\mathrm{K}_W(X\times\Delta^p),$$

where the homotopy limit is taken over closed subschemes $W \subset X \times \Delta^p$ such that

$$\mathrm{codim}_{X\times F}(W \cap (X \times F)) \geq q$$

for all faces $F \subset \Delta^p$. We set $\mathrm{K}^{(q)}(X) := \mathrm{K}^{(q)}(X, 0)$. The spectra $\mathrm{K}^{(q)}(X, p)$ form a simplicial spectrum, and we let $\mathrm{K}^{(q)}(X, -)$ denote the corresponding total spectrum [Levine 2006, §1.5]. Moreover, one has a tower of spectra

$$\cdots \to \mathrm{K}^{(q)}(X, -) \to \mathrm{K}^{(q-1)}(X, -) \to \cdots \to \mathrm{K}^{(0)}(X, -).$$

This tower of spectra is referred to as the homotopy coniveau tower. It satisfies the following properties proved by Levine:

(1) Given a morphism of smooth schemes $F : X \to Y$ there is a natural pull-back morphism on the corresponding coniveau towers [Levine 2008, Theorem 4.1.1].

(2) There are natural augmentation maps $\eta_q : \mathrm{K}^{(q)}(X) \to \mathrm{K}^{(q)}(X, -)$. Moreover, the composition

$$\eta : \mathrm{K}(X) \to \mathrm{K}^{(0)}(X) \to \mathrm{K}^{(0)}(X, -)$$

is a weak equivalence.

(3) The cofibers $\mathrm{K}^{(p/p+1)}(X, -)$ of the homotopy coniveau tower are naturally weak equivalent to Bloch's higher Chow groups cycles complex [Levine 2008, Theorem 6.4.2]. In particular, there is a functorial (with respect to pull-backs) isomorphism $\mathrm{CH}^d(X) \to \pi_0(\mathrm{K}^{(d)}(X, -))$ if $d = \dim(X)$, since $\pi_0 \mathrm{K}^{(d+1)}(X, -)$ is 0 for reasons of dimension.

(4) Finally, we note that the tensor product induces natural (functorial) pairings:

$$\mathrm{K}^{(d)}(X, -) \wedge \mathrm{K}^{(d')}(X, -) \to \mathrm{K}^{(d+d')}(X, -).$$

**Remark 2.6.1.** The existence of a pairing as in (4) is a deep theorem and relies on Levine's moving lemma for the homotopy coniveau tower. However, we shall only use the result in the case where $d' = 0$. In that case, $\eta : \mathrm{K}(X) \to \mathrm{K}^{(0)}(X, -)$ is a weak equivalence, and the induced pairing

$$\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X, -) \to \mathrm{K}^{(d)}(X, -)$$

is simply given by tensor product. In particular, no "moving" is required.

**2.7.** ***Microlocalization map of K-theory of $\mathcal{D}_X$-modules.*** In this paragraph, $X$ will denote a smooth projective variety over a field $k$ of characteristic zero. Below we recall the construction of a microlocalization map for K-theory spectra of $\mathcal{D}_X$-modules. We refer to [Patel 2012] for details.

Let $\mathrm{K}(\mathcal{D}_X)$ denote the K-theory spectrum of the abelian category of coherent $\mathcal{D}_X$-modules, and similarly let $\mathrm{K}_S(\mathcal{D}_X)$ denote the K-theory spectrum of the abelian category of coherent $\mathcal{D}_X$-modules with singular support contained in $S \subset T^*X$. Recall that any $\mathcal{D}_X$ module $\mathcal{M}$ has a good filtration $\mathcal{F}^\bullet$ such that the associated graded gives rise to a coherent $\mathcal{O}_{T^*X}$-module. This construction gives rise to a well-defined (i.e., independent of the choice of filtration) map $\mathrm{K}_0(\mathcal{D}_X) \to \mathrm{K}_0(T^*X)$. One has an analogous statement in the setting of supports. The following theorem extends this construction to the setting of higher K-theory. Below, let $\mathrm{KF}(\mathcal{D}_X)$ denote the K-theory spectrum of the exact category whose objects are pairs $(\mathcal{M}, \mathcal{F})$, where $(\mathcal{M}, \mathcal{F})$ is a coherent $\mathcal{D}_X$-module and $\mathcal{F}$ is a good filtration. We can similarly define $\mathrm{KF}_S(\mathcal{D}_X)$. There is a natural map $\mathrm{gr}^{\mathrm{F}}_S : \mathrm{KF}_S(\mathcal{D}_X) \to \mathrm{K}_S(T^*X)$ induced by sending a pair $(\mathcal{M}, \mathcal{F})$ to $\mathrm{gr}^{\mathcal{F}}(\mathcal{M})$. One also has a natural map $\mathrm{ff} : \mathrm{KF}_S(\mathcal{D}_X) \to \mathrm{K}_S(\mathcal{D}_X)$ given by simply forgetting the filtration.

**Theorem 2.7.1** [Patel 2012]. *Let $X$ be as above. There is a natural* (*in $S$*) *microlocalization morphism of K-theory spectra*:

$$\mathrm{gr}_S : \mathrm{K}_S(\mathcal{D}_X) \to \mathrm{K}_S(T^*X).$$

*In particular*, *these are compatible with respect to the inclusions $S \subset S'$. Moreover*, *by construction*, $\mathrm{gr}_S \circ \mathrm{ff}$ *is homotopic to* $\mathrm{gr}^{\mathrm{F}}_S$.

Let $\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X)$ denote the K-theory spectrum of the abelian category of holonomic $\mathcal{D}_X$-modules. The preceding theorem immediately gives the following corollary by passing to homotopy colimits.

**Corollary 2.7.2.** *With notation as above*, *one has a morphism of spectra*:

$$\varepsilon : \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{K}^{(d)}(T^*X).$$

*Proof.* By definition, we may view the category of holonomic $\mathcal{D}_X$-modules as a direct limit of the categories of the full subcategories of $\mathcal{D}_X$-modules with singular support in a fixed codimension $d$ subset $S \subset T^*X$. Since K-theory commutes with direct limits, we may write $\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X)$ as the colimit of the corresponding spectra $\mathrm{K}_S(\mathcal{D}_X)$. The result now follows from the previous theorem by taking limits. $\square$

**Remark 2.7.3.** Note that there is a natural map $\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \to \mathrm{K}(\mathcal{D}_X)$. Moreover, by the compatibility of $\mathrm{gr}_S$, one has a natural commutative diagram:

$$\begin{tikzcd}
\mathrm{K}(\mathcal{D}_X) \arrow[r, "\mathrm{gr}"] & \mathrm{K}(T^*X) \\
\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[u] \arrow[r, "\varepsilon"] & \mathrm{K}^{(d)}(T^*X) \arrow[u]
\end{tikzcd}$$

Let $f : X \to \mathrm{Spec}(k)$ denote the structure map, $\pi : T^*X \to X$ the projection map, and $\sigma : X \to T^*X$ the zero section. Then $f$ and $\sigma$ induce morphisms of K-theory spectra (Section 2.2) $\mathrm{K}(X) \xrightarrow{f_*} \mathrm{K}(k)$ and $\mathrm{K}(T^*X) \xrightarrow{\sigma^*} \mathrm{K}(X)$. The canonical bundle $\omega_X$ induces a natural morphism

$$\mathrm{K}(X) \xrightarrow{(-\otimes\omega_X)} \mathrm{K}(X).$$

We define the twisted pull-back $\sigma^+$ as the composition

$$\mathrm{K}(T^*X) \xrightarrow{\sigma^*} \mathrm{K}(X) \xrightarrow{(-\otimes\omega_X)} \mathrm{K}(X).$$

These give rise to a morphism $f_* \circ \sigma^+ \circ \mathrm{gr} : \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(k)$. On the other hand, the $\mathcal{D}_X$-module push-forward induces a morphism of K-theory spectra $R\Gamma_{\mathrm{dR}} : \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(k)$ (Remark 2.2.1). The next lemma is a restatement of the following remark in terms of K-theory.

**Remark 2.7.4.** Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module. Then there is a natural isomorphism $Rf_*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{E}) \cong Rf_+(\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{E})$. We refer to [Laumon 1983, 6.5] for the details.

**Lemma 2.7.5.** *The morphisms $R\Gamma_{\mathrm{dR}}$ and $f_* \circ \sigma^+ \circ \mathrm{gr}$ are homotopic.*

*Proof.* First note that the composition $\mathrm{K}(X) \to \mathrm{K}(\mathcal{D}_X) \to \mathrm{K}(T^*X) \to \mathrm{K}(X)$ is homotopic to the identity. Here the first map is the natural map induced by $\mathcal{D}_X \otimes_{\mathcal{O}_X} (-)$, which is a weak equivalence by a theorem of Quillen [Patel 2012]. Thus, one is reduced to showing the diagram

$$\begin{tikzcd}
\mathrm{K}(X) \arrow[r, "(-\otimes\omega_X)"] \arrow[d, "(\mathcal{D}_X\otimes -)"'] & \mathrm{K}(X) \arrow[d, "f_*"] \\
\mathrm{K}(\mathcal{D}_X) \arrow[r, "R\Gamma_{\mathrm{dR}}"] & \mathrm{K}(k).
\end{tikzcd}$$

is commutative. This follows from Remark 2.7.4. □

**Remark 2.7.6.** We think of $R\Gamma_{\mathrm{dR}}$ as the global de Rham epsilon factor. Recall that at the level of $\mathcal{D}_X$-modules, up to a shift, the $\mathcal{D}_X$-module push-forward computes de Rham cohomology of the corresponding $\mathcal{D}_X$-modules.

**2.8.** ***Global epsilon factors and tensor products.*** We record an elementary lemma computing the global epsilon factor of a tensor product. Below, we denote by $\pi^* : \mathrm{K}_X(\mathcal{D}_X) \to \mathrm{K}(T^*X)$ the morphism given by pulling back a flat connection under the projection map $\pi : T^*X \to X$. The following remark will be useful in the proof of the lemma, and, in fact, the lemma itself is the K-theoretic version of the remark.

**Remark 2.8.1.** Let $\mathcal{M}$ be a flat connection on $X$, and $\mathcal{N}$ a filtered $\mathcal{D}_X$-module. Note that $\mathcal{M}$ has a canonical good filtration given by taking the whole module in degrees greater than or equal to 0 and 0 in negative degrees. Then $\mathrm{gr}(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})$ is isomorphic to $\pi^*(\mathcal{M}) \otimes_{\mathcal{O}_{T^*X}} \mathrm{gr}(\mathcal{N})$.

**Lemma 2.8.2.** *The following diagram is commutative*:

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}(\mathcal{D}_X) \arrow[r, "(-\otimes-)"] \arrow[d, "\pi^* \wedge \mathrm{gr}"'] & \mathrm{K}(\mathcal{D}_X) \arrow[d, "\mathrm{gr}"] \\
\mathrm{K}(T^*X) \wedge \mathrm{K}(T^*X) \arrow[r, "(-\otimes-)"] & \mathrm{K}(T^*X)
\end{tikzcd}$$

*Proof.* Consider the following diagram:

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}(X) \arrow[r, "\star"] \arrow[d, "\mathrm{Id} \wedge (\mathcal{D}_X \otimes -)"', "\sim"] & \mathrm{KF}_X(\mathcal{D}_X) \wedge \mathrm{KF}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[dl, bend right, "\mathrm{ff} \wedge \mathrm{ff}"'] & \mathrm{KF}(\mathcal{D}_X) \arrow[d, "\mathrm{ff}"] \\
\mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}(\mathcal{D}_X) \arrow[rr, "\otimes"'] \arrow[d, "\pi^* \wedge \mathrm{gr}"'] & & \mathrm{K}(\mathcal{D}_X) \arrow[d, "\mathrm{gr}"] \\
\mathrm{K}(T^*X) \wedge \mathrm{K}(T^*X) \arrow[rr, "\otimes"] & & \mathrm{K}(T^*X)
\end{tikzcd}$$

Here, $\star$ is defined as follows: There is a natural map $\mathrm{K}_X(\mathcal{D}_X) \to \mathrm{KF}_X(\mathcal{D}_X)$ induced by giving a flat connection $\mathcal{M}$ the trivial filtration (i.e., it is $\mathcal{M}$ in degree greater than or equal to 0 and 0 in negative degrees). Similarly, there is a natural map $\mathrm{K}(X) \to \mathrm{KF}(\mathcal{D}_X)$ induced by $(\mathcal{D}_X \otimes -)$ and taking the filtration induced by the usual filtration by order on $\mathcal{D}_X$. The map $\star$ is defined by taking $\wedge$ of these two maps.

Our goal is to show that the lower square is commutative. The upper square is commutative. Since the first left vertical map is a weak equivalence, it suffices to check the commutativity for the outer square. The composition of the top horizontal and right vertical maps is given by sending a bundle with connection $\mathcal{M}$ and an induced $\mathcal{D}_X$-module $\mathcal{D}_X \otimes \mathcal{N}$ to the associated graded of the tensor product. By Remark 2.8.1 above, this composition is homotopic to

$$(-\otimes-) \circ (\pi^* \wedge (\mathrm{gr} \circ (\mathcal{D}_X \otimes -))). \qquad \square$$

## 3. Comparison of traces for various pairings of Picard groupoids

In this subsection, we recall the construction of some pairings on K-theory spectra at the level of Levine's homotopy coniveau tower, and make some computations of traces of tensor products in this setting. In this section, let $X$ be a smooth and connected scheme of finite type over $k$.

**3.1.** ***Pairings on K-theory with supports.*** Given a closed subset $Z \subset X$, there is a natural pairing of K-theory spectra

$$\mathrm{K}(X) \wedge \mathrm{K}_Z(X) \xrightarrow{\otimes} \mathrm{K}_Z(X)$$

induced by the tensor product [Thomason and Trobaugh 1990, 3.15]. Since $X$ is smooth, Quillen's localization theorem implies that the natural map $\mathrm{K}_Z(X) \to \mathrm{G}(Z)$ is a weak equivalence. Here $\mathrm{G}(Z)$ is the K-theory spectrum of coherent sheaves. If $i_Z : Z \hookrightarrow X$ is a reduced closed subscheme of dimension 0, and hence regular, then we may identify $\mathrm{G}(Z)$ with the K-theory spectrum $\mathrm{K}(Z)$ of locally free sheaves. Below we shall make this assumption on $Z$. Moreover, one also has a natural pairing

$$\mathrm{K}(X) \wedge \mathrm{K}(Z) \xrightarrow{i_Z^* \wedge \mathrm{Id}} \mathrm{K}(Z) \wedge \mathrm{K}(Z) \to \mathrm{K}(Z).$$

We record the following standard lemma for future use.

**Lemma 3.1.1.** *The following diagram is commutative*:

$$\begin{tikzcd}
\mathrm{K}(X) \wedge \mathrm{K}_Z(X) \arrow[r] & \mathrm{K}_Z(X) \\
\mathrm{K}(X) \wedge \mathrm{K}(Z) \arrow[r] \arrow[u, "\mathrm{Id}\wedge i_{Z,*}"] & \mathrm{K}(Z) \arrow[u]
\end{tikzcd}$$

*Proof.* This is a special case of the projection formula [Thomason and Trobaugh 1990, Proposition 3.17]. □

One has a natural norm map (given by the push-forward):

$$N : \mathrm{K}(Z) \to \mathrm{K}(k).$$

Similarly, one has the usual push-forward $\pi_* : \mathrm{K}_Z(X) \to \mathrm{K}(k)$. Composing the pairings above with these push-forward maps give rise to pairings

$$\langle -, - \rangle^{\mathrm{K}} : \mathrm{K}(X) \wedge \mathrm{K}_Z(X) \to \mathrm{K}(k) \quad \text{and} \quad \langle -, - \rangle^{\mathrm{K}} : \mathrm{K}(X) \wedge \mathrm{K}(Z) \to \mathrm{K}(k).$$

By the previous lemma these two pairings are identified via the natural weak equivalence $i_{Z,*} : \mathrm{K}(Z) \to \mathrm{K}_Z(X)$. Therefore, in the following we use the two interchangeably and use the same notation to denote the two pairings.

Since the pairings above are compatible with respect to inclusions $Z' \subset Z$, we may pass to homotopy limits and deduce a pairing

$$\langle -, - \rangle^{\mathrm{K}}_{(d)} : \mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X) \to \mathrm{K}(k).$$

Note this pairing is simply the composition

$$\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X) \xrightarrow{(-\otimes-)} \mathrm{K}^{(d)}(X) \xrightarrow{\pi_*} \mathrm{K}(k).$$

Here we define $\pi_*$ as the one induced by taking homotopy limits of the maps $\pi_* : \mathrm{K}_Z(X) \to \mathrm{K}(k)$.

**Remark 3.1.2.** We may also define $\pi_*$ by taking homotopy limits of the compositions $\mathrm{K}_Z(X) \to \mathrm{K}(Z) \xrightarrow{N} \mathrm{K}(k)$. The two constructions are homotopic.

**3.2.** ***Pairings on the homotopy coniveau tower.*** We now explain how the constructions of the previous paragraph lift to Levine's homotopy coniveau tower. We may consider the composition

$$\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X) \to \mathrm{K}^{(0)}(X,-) \wedge \mathrm{K}^{(d)}(X,-) \to \mathrm{K}^{(d)}(X,-),$$

where we refer to (2) in Section 2.6 for the first map and (4) for the last map. Moreover, we have $\pi_*^{(d)} : \mathrm{K}^{(d)}(X,-) \to \mathrm{K}^{(0)}(X,-) \xrightarrow{\sim} \mathrm{K}(X) \xrightarrow{f_*} \mathrm{K}(k)$, where the isomorphism is induced by inverting $\eta$ of (2). Composing with the map above, it gives rise to a pairing

$$\langle -,-\rangle^{\mathrm{K}}_{(d,-)} : \mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \to \mathrm{K}(k).$$

This pairing is compatible with the pairing constructed in the previous paragraph. Namely, we have the following lemma.

**Lemma 3.2.1.** *The following diagram is commutative*:

$$\begin{tikzcd}
\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X) \arrow[r, "\langle -,-\rangle^{\mathrm{K}}_{(d)}"] \arrow[d, "\mathrm{Id}\wedge\eta_d"'] & \mathrm{K}(k) \arrow[d, "\mathrm{Id}"] \\
\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \arrow[r, "\langle -,-\rangle^{\mathrm{K}}_{(d,-)}"] & \mathrm{K}(k)
\end{tikzcd}$$

*Proof.* First, recall that the tensor product is compatible with the augmentation. Therefore, one is reduced to showing that

$$\mathrm{K}^{(d)}(X) \xrightarrow{\eta_d} \mathrm{K}^{(d)}(X,-) \xrightarrow{\pi_*^{(d)}} \mathrm{K}(k)$$

is homotopic to $\mathrm{K}^{(d)}(X) \xrightarrow{\pi_*} \mathrm{K}(k)$. The latter is evident from the construction of $\pi_*^{(d)}$. $\square$

**Remark 3.2.2.** One also has a product $\mathrm{K}(T^*X) \wedge \mathrm{K}^{(d)}(T^*X,-) \to \mathrm{K}^{(d)}(T^*X,-)$. By functoriality, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{K}(T^*X) \wedge \mathrm{K}^{(d)}(T^*X,-) \arrow[r] \arrow[d, "\sigma^*\wedge\sigma^*"'] & \mathrm{K}^{(d)}(T^*X,-) \arrow[d, "\sigma^*"] \\
\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \arrow[r] & \mathrm{K}^{(d)}(X,-)
\end{tikzcd}$$

**3.3.** ***Pairings on Picard groupoids.*** Recall that pairings on spectra give rise to pairings on the corresponding fundamental Picard groupoids (Section 2.5). In particular, the pairing $\langle -,-\rangle^{\mathrm{K}}_{(d)}$ induces a pairing

$$\langle -,-\rangle^{\Pi}_{(d)} : \Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X)) \to \Pi(\mathrm{K}(k)).$$

By definition, it is defined as the composition

$$\Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X)) \to \Pi(\mathrm{K}^{(d)}(X)) \to \Pi(\mathrm{K}(k)),$$

where the first map is induced by the tensor product and the second by $\pi_*$. On the other hand, one has the following pairing which is a variant of Example 2.5.1:

$$\langle -,-\rangle : \mathrm{Pic}^{\mathbb{Z}}(X) \wedge \mathrm{Pic}^{\mathbb{Z}}(Z) \to \mathrm{Pic}^{\mathbb{Z}}(Z).$$

Explicitly, this pairing sends $(\mathcal{L},\alpha) \in \mathrm{Pic}^{\mathbb{Z}}(X)$ and $(\mathcal{M},\beta) \in \mathrm{Pic}^{\mathbb{Z}}(Z)$ to the element $(\mathcal{L}|_Z^{\beta} \otimes \mathcal{M}^{\alpha}, \alpha|_Z\beta)$. Recall that we have the universal determinant map $\mathrm{Det} : \Pi(\mathrm{K}(X)) \to \mathrm{Pic}^{\mathbb{Z}}(X)$, and similarly for $Z$. As before (see Section 2.5), this gives rise to a commutative diagram:

$$\begin{array}{ccc} \Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}(Z)) & \longrightarrow & \Pi(\mathrm{K}(Z)) \\ \downarrow & & \downarrow \\ \mathrm{Pic}^{\mathbb{Z}}(X) \wedge \mathrm{Pic}^{\mathbb{Z}}(Z) & \longrightarrow & \mathrm{Pic}^{\mathbb{Z}}(Z) \end{array}$$

The push-forward induces a norm map $N : \mathrm{Pic}^{\mathbb{Z}}(Z) \to \mathrm{Pic}^{\mathbb{Z}}(k)$. In particular, one has a natural pairing

$$N \circ \langle -,-\rangle : \mathrm{Pic}^{\mathbb{Z}}(X) \wedge \mathrm{Pic}^{\mathbb{Z}}(Z) \to \mathrm{Pic}^{\mathbb{Z}}(k).$$

By abuse of notation, we also denote this pairing by $\langle -,-\rangle$. Explicitly, this pairing sends $(\mathcal{L},\alpha) \in \mathrm{Pic}^{\mathbb{Z}}(X)$ and $(\mathcal{M},\beta) \in \mathrm{Pic}^{\mathbb{Z}}(Z)$ to the element

$$\big(\det(\pi_{Z,*}\mathcal{O}_Z)^{\otimes(\alpha|_Z\beta)} \otimes N(\mathcal{L}|_Z^{\beta} \otimes \mathcal{M}^{\alpha}), \mathrm{Tr}(\alpha|_Z\beta)\big),$$

where $\pi_Z : Z \to \mathrm{Spec}(k)$ is the natural structure map (see Example 2.5.3). The previous remarks show that the following diagram commutes:

$$\begin{array}{ccc} \Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}(Z)) & \xrightarrow{\langle -,-\rangle^{\Pi}} & \Pi(\mathrm{K}(k)) \\ \downarrow & & \downarrow \\ \mathrm{Pic}^{\mathbb{Z}}(X) \wedge \mathrm{Pic}^{\mathbb{Z}}(Z) & \xrightarrow{\langle -,-\rangle} & \mathrm{Pic}^{\mathbb{Z}}(k) \end{array}$$

**Remark 3.3.1.** Recall that the map $\mathrm{Det} : \Pi(\mathrm{K}(k)) \to \mathrm{Pic}^{\mathbb{Z}}(k)$ is an isomorphism of Picard groupoids. In the following, we make this identification in our resulting pairings.

**3.4.** ***Picard groupoid pairings coming from coniveau tower.*** We now descend the pairings $\langle -,-\rangle^{\mathrm{K}}_{(d,-)}$ to the level of Picard groupoids. In particular, taking fundamental groupoids gives a pairing

$$\langle -,-\rangle^{\Pi}_{(d,-)} : \Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X,-)) \to \mathrm{Pic}^{\mathbb{Z}}(k).$$

Combining everything gives rise to the following commutative diagrams, which we record as a lemma for future use.

**Lemma 3.4.1.** *The following diagrams commute* (*up to natural transformations*):

$$\begin{array}{ccc} \Pi(\mathrm{K}(X))\wedge\Pi(\mathrm{K}(Z)) & \longrightarrow & \mathrm{Pic}^{\mathbb{Z}}(k) \\ \downarrow & & \downarrow \\ \Pi(\mathrm{K}(X))\wedge\Pi(\mathrm{K}^{(d)}(X)) & \longrightarrow & \mathrm{Pic}^{\mathbb{Z}}(k) \end{array} \qquad \begin{array}{ccc} \Pi(\mathrm{K}(X))\wedge\Pi(\mathrm{K}^{(d)}(X)) & \longrightarrow & \mathrm{Pic}^{\mathbb{Z}}(k) \\ \downarrow & & \downarrow \\ \Pi(\mathrm{K}(X))\wedge\Pi(\mathrm{K}^{(d)}(X,-)) & \longrightarrow & \mathrm{Pic}^{\mathbb{Z}}(k) \end{array}$$

*Proof.* The commutativity of the first diagram follows from the remarks in Section 3.1 and that of the second follows from Lemma 3.2.1. □

**3.5.** ***Compatibility of various traces of endomorphisms.*** We explain how the constructions of the previous subsection pass to traces in the presence of endomorphisms. Given endomorphisms $g$ of $\mathcal{G} \in \mathrm{Pic}^{\mathbb{Z}}(X)$ and $f$ of $\mathcal{F} \in \mathrm{Pic}^{\mathbb{Z}}(Z)$, we have an induced endomorphism $g \otimes f$ of $\langle \mathcal{G}, \mathcal{F}\rangle \in \mathrm{Pic}^{\mathbb{Z}}(k)$. Therefore, we have an element $\mathrm{Tr}(g\otimes f) \in \mathrm{Pic}^{\mathbb{Z}}(k) = k^{\times}$. We denote the latter trace by $\langle \mathcal{G}, \mathcal{F}\rangle(g, f)$. Similarly, given endomorphisms $g$ of $\mathcal{G} \in \Pi(\mathrm{K}(X))$, $f$ of $\mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X))$, and $f'$ of $\mathcal{F}' \in \Pi(K^{(d)}(X,-))$, we can define the traces $\langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d)}(g, f)$ and $\langle \mathcal{G}, \mathcal{F}'\rangle^{\Pi}_{(d,-)}(g, f')$ in $k^{\times}$. Note that a pair $(\mathcal{F}, f)$ of an object and an endomorphism in $\Pi(\mathrm{K}^{(d)}(X))$ can also be considered as an object and endomorphism of $\Pi(\mathrm{K}^{(d)}(X,-))$ simply by considering its image under the natural map $\Pi(\mathrm{K}^{(d)}(X)) \to \Pi(\mathrm{K}^{(d)}(X,-))$. We record the following corollary of the previous result for future reference.

**Corollary 3.5.1.** *Let $\mathcal{G} \in \Pi(\mathrm{K}(X))$ and $\mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X))$, and let $g$ and $f$ denote endomorphisms of $\mathcal{G}$ and $\mathcal{F}$, respectively. Then one has*

$$\langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d,-)}(g, f) = \langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d)}(g, f).$$

*Proof.* This is a direct consequence of Lemma 3.4.1. □

By definition, $\pi_i(\Pi(\mathrm{K}^{(d)}(X))) = \lim \pi_i(\mathrm{K}(Z))$ for $i \leq 1$, where the direct limit is over closed reduced subschemes $Z$ of dimension zero. Therefore, for any object $\mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X))$ and endomorphism $f : \mathcal{F} \to \mathcal{F}$, we can choose a $Z$ such that the pair $(\mathcal{F}, f)$ lifts to $\Pi(\mathrm{K}(Z))$. In particular, there is a pair $(\mathcal{F}_Z, f_Z)$ consisting of an object and an endomorphism in $\Pi(\mathrm{K}(Z))$, and an isomorphism $h$ of the image of this pair in $\Pi(\mathrm{K}^{(d)}(X))$ (under the natural map $\Pi(\mathrm{K}(Z)) \to \Pi(\mathrm{K}^{(d)}(X))$) with the pair $(\mathcal{F}, f)$. In this setting, we have the following lemma:

**Lemma 3.5.2.** *With notation as above*, $\langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d)}(g, f) = \langle \mathcal{G}, \mathcal{F}_Z\rangle^{\Pi}(g, f_Z)$. *Moreover*, *we have an equality* $\langle \mathcal{G}, \mathcal{F}_Z\rangle^{\Pi}(g, f_Z) = \langle \mathrm{Det}(\mathcal{G}), \mathrm{Det}(\mathcal{F}_Z)\rangle(g, f_Z)$.

*Proof.* The first statement follows from commutativity of the second diagram in Lemma 3.4.1 after passing to Picard groupoids and traces. The second similarly follows from the remarks in Section 3.3. □

**3.6.** ***Formula for traces of tensor products of endomorphisms.*** In this subsection, we prove an elementary formula for traces of tensor products of endomorphisms. In Section 3.9, we shall prove a similar formula in the more general setting of correspondences. The formula presented in this paragraph will be an easy corollary of that more general formula. However, we present the simpler version here since the proof has some features of the more general situation and might be useful in understanding the more complicated version presented later.

Suppose we are given $\mathcal{G} \in \Pi(\mathrm{K}(X))$ and $\mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X))$ with endomorphisms $f : \mathcal{F} \to \mathcal{F}$ and $g : \mathcal{G} \to \mathcal{G}$ as before. For any element of $\mathcal{G} \in \Pi(\mathrm{K}(X))$, we let $r_{\mathcal{G}}$ (the rank of $\mathcal{G}$) denote its image by the canonical homomorphism $\pi_0(\mathrm{Pic}^{\mathbb{Z}}(X)) \to \mathbb{Z}$. In this setting, one has the following standard formula at the level of traces.

**Proposition 3.6.1.** *With notation as above*,

$$\langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d)}(g, f) = \mathrm{Tr}(\pi_*(f) \mid \pi_*(\mathcal{F}))^{r_{\mathcal{G}}} \times \langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d)}(g, \mathrm{Id}).$$

*Here*, $\pi_*(f)$ *and* $\pi_*(\mathcal{F})$ *are the images under the natural map*

$$\Pi(\mathrm{K}^{(d)}(X)) \xrightarrow{\pi_*} \mathrm{Pic}^{\mathbb{Z}}(k).$$

*Proof.* By Lemma 3.5.2, we are reduced to showing that

$$\langle \mathcal{G}, \mathcal{F}\rangle(g, f) = \mathrm{Tr}\big(\det(\pi_{Z,*}(f)) \mid \det(\pi_{Z,*}(\mathcal{F}))\big)^{r_{\mathcal{G}}} \times \langle \mathcal{G}, \mathcal{F}\rangle(g, \mathrm{Id}),$$

where $\mathcal{G} \in \mathrm{Pic}^{\mathbb{Z}}(X)$ and $\mathcal{F} \in \mathrm{Pic}^{\mathbb{Z}}(Z)$. Here $i_Z : Z \hookrightarrow X$ is a closed subscheme of dimension zero and $\pi_Z : Z \to \mathrm{Spec}(k)$ is the structure map. Since

$$\langle \mathcal{G}, \mathcal{F}\rangle(g, f) = \langle \mathcal{G}, \mathcal{F}\rangle(g, \mathrm{Id}) \times \langle \mathcal{G}, \mathcal{F}\rangle(\mathrm{Id}, f)$$

by Lemma 2.3.3, we are reduced to showing that

$$\langle \mathcal{G}, \mathcal{F}\rangle(\mathrm{Id}, f) = \mathrm{Tr}\big(\det(\pi_{Z,*}(f)) \mid \det(\pi_{Z,*}(\mathcal{F}))\big)^{r_{\mathcal{G}}}. \qquad (\star)$$

We may assume $Z$ is a closed integral point such that the degree of $k(Z)$ over $k$ is $n$, and denote by $N_{k(Z)/k}$ the field norm map. The map

$$\rho := \pi_1\langle \mathcal{G}|_Z, -\rangle : \pi_1\mathrm{Pic}^{\mathbb{Z}}(Z) \to \pi_1\mathrm{Pic}^{\mathbb{Z}}(Z)$$

is nothing but the map sending $\alpha \in \pi_1\mathrm{Pic}^{\mathbb{Z}}(Z) \cong k(Z)^\times$ to $\alpha^{r_{\mathcal{G}}}$. Recall the notation of Section 3.3. We have

$$\begin{aligned}\langle \mathcal{G}, \mathcal{F}\rangle(\mathrm{Id}, f) &= \mathrm{Tr}\big(N\langle \mathrm{Id}, f\rangle \mid \langle \mathcal{G}, \mathcal{F}\rangle\big) = N_{k(Z)/k}\mathrm{Tr}\big(\langle \mathrm{Id}, f\rangle \mid \langle \mathcal{G}, \mathcal{F}\rangle\big)\\ &= N_{k(Z)/k}\big(\rho(\mathrm{Tr}(f \mid \mathcal{F}))\big) = N_{k(Z)/k}(\mathrm{Tr}(f \mid \mathcal{F})^{r_{\mathcal{G}}}),\end{aligned}$$

where the third equality holds by Lemma 2.5.2, and we get the equality ($\star$). □

**3.7.** ***A key vanishing lemma.*** We would like a formula similar to that of the last subsection for $\langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d,-)}(g, f)$, where $\mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X, -))$. If the pair $(\mathcal{F}, f)$ can be lifted to $\Pi(\mathrm{K}^{(d)}(X))$, then we would get such a formula as a consequence of the previous proposition. Unfortunately, while we may lift any such object $\mathcal{F}$ to $\Pi(\mathrm{K}^{(d)}(X))$, it is not always possible to lift the endomorphism $f$. However, we shall see that the desired formula (in the more general setting of correspondences) is an easy consequence of the following lemma.

**Lemma 3.7.1.** *Let $X$ be a smooth projective variety of dimension $d$ and $W$ a closed subscheme of codimension $> 0$. The map $\pi_1\mathrm{K}^{(d)}(X, -) \to \pi_1\mathrm{K}^{(d)}(X, -)$ induced by $\otimes\mathcal{O}_W$ is trivial.*

*Proof of Lemma 3.7.1.* First, we remark that [Levine 2006, Theorem 2.6.2] holds when $X$ is projective. Below, we follow the notation of [loc. cit.]. Using that theorem for $\mathcal{C} = \{W\}$ and $e = 0$, we get a weak equivalence $\mathrm{K}^{(d)}(X, -)_{\mathcal{C},e} \xrightarrow{\sim} \mathrm{K}^{(d)}(X, -)$. Now, the map $\otimes\mathcal{O}_W$ factors through $\mathrm{K}^{(d+1)}(X, -)_{\mathcal{C},e} \to \mathrm{K}^{(d)}(X, -)_{\mathcal{C},e}$, and we have the commutative diagram

$$\begin{tikzcd}
\mathrm{K}^{(d)}(X,-)_{\mathcal{C},e} \arrow[r, "\otimes\mathcal{O}_W"] \arrow[d] & \mathrm{K}^{(d+1)}(X,-)_{\mathcal{C},e} \arrow[r] \arrow[d] & \mathrm{K}^{(d)}(X,-)_{\mathcal{C},e} \arrow[d] \\
\mathrm{K}^{(d)}(X,-) \arrow[rr, bend right, "\otimes\mathcal{O}_W"'] & \mathrm{K}^{(d+1)}(X,-) \arrow[r] & \mathrm{K}^{(d)}(X,-)
\end{tikzcd}$$

For $n \in \{d, d+1\}$, consider the spectral sequences

$$\begin{tikzcd}
(E_1^{p,q})^{(n)}_{\mathcal{C},e} = \pi_{-q}\mathrm{K}^{(n)}(X, -p)_{\mathcal{C},e} \arrow[r, Rightarrow] \arrow[d] & \pi_{-p-q}\mathrm{K}^{(n)}(X,-)_{\mathcal{C},e} \arrow[d, "\sim"] \\
(E_1^{p,q})^{(n)} = \pi_{-q}\mathrm{K}^{(n)}(X, -p) \arrow[r, Rightarrow] & \pi_{-p-q}\mathrm{K}^{(n)}(X,-)
\end{tikzcd}$$

By dimension reasons, we have $\mathrm{K}^{(d+1)}(X, 0)_{\mathcal{C},e} = \mathrm{K}^{(d+1)}(X, 0) = \{*\}$, which implies $(E_1^{0,q})^{(d+1)}_{\mathcal{C},e} = (E_1^{0,q})^{(d+1)} = 0$ for any $q$. Thus,

$$(E_2^{-1,0})^{(d+1)}_{\mathcal{C},e} \cong (E_2^{-1,0})^{(d+1)} \cong \pi_1\mathrm{K}^{(d+1)}(X, -).$$

Now, we have the following big commutative diagram:

$$\begin{tikzcd}
\pi_1\mathrm{K}^{(d)}(X,-)_{\mathcal{C},e} \arrow[rr, "\otimes\mathcal{O}_W"] \arrow[d, two heads] & & \pi_1\mathrm{K}^{(d+1)}(X,-)_{\mathcal{C},e} \arrow[r] \arrow[d, dash, "\sim"] & \pi_1\mathrm{K}^{(d+1)}(X,-) \arrow[r] \arrow[d, dash, "\sim"] & \pi_1\mathrm{K}^{(d)}(X,-) \\
(E_\infty^{-1,0})^{(d)}_{\mathcal{C},e} \arrow[r, dash, "\sim"] & (E_2^{-1,0})^{(d)}_{\mathcal{C},e} \arrow[r, "\otimes\mathcal{O}_W"'] & (E_2^{-1,0})^{(d+1)}_{\mathcal{C},e} \arrow[r] & (E_2^{-1,0})^{(d+1)} & \\
 & K \arrow[u, two heads] \arrow[r] & (E_1^{-1,0})^{(d+1)}_{\mathcal{C},e} \arrow[u, two heads] & &
\end{tikzcd}$$

where $K := \mathrm{Ker}\big((E_1^{-1,0})^{(d)}_{\mathcal{C},e} \to (E_1^{0,0})^{(d)}_{\mathcal{C},e}\big)$. Take $\alpha \in \pi_1\mathrm{K}^{(d)}(X,-) \cong \pi_1\mathrm{K}^{(d)}(X,-)_{\mathcal{C},e}$. Our goal is to show that the image of $\alpha$ by the composition of the homomorphisms of the first row is trivial. By the diagram above, there exists

$$\tilde{\alpha} \in K \subset (E_1^{-1,0})^{(d)}_{\mathcal{C},e} = \pi_0\mathrm{K}^{(d)}(X, 1)_{\mathcal{C},e}$$

such that $\alpha \otimes \mathcal{O}_W$ coincides with $\tilde{\alpha} \otimes \mathcal{O}_W$ in $\pi_1\mathrm{K}^{(d+1)}(X,-)_{\mathcal{C},e}$. It suffices to show that the image of $\tilde{\alpha} \otimes \mathcal{O}_W$ in $(E_2^{-1,0})^{(d+1)}$ is 0.

There exists a closed subscheme $Z \subset X \times \Delta^1$ belonging to $\mathcal{S}^{(d)}_{X,\mathcal{C},e}(1)$ (in particular, dimension 1) such that $\tilde{\alpha}$ can be lifted to $\mathrm{K}_Z(X, 1)$, which we denote by $\tilde{\alpha}'$. We have $\tilde{\alpha}' \otimes \mathcal{O}_W \in \pi_0\mathrm{K}_{Z\cap\mathrm{pr}^{-1}(W)}(X \times \Delta^1)$ (where $\mathrm{pr} : X \times \Delta^1 \to X$ is the projection). Since $Z \in \mathcal{S}^{(d)}_{X,\mathcal{C},e}(1)$, the intersection $Z \cap \mathrm{pr}^{-1}(W)$ is 0-dimensional. By definition of $\mathcal{S}^{(d)}_{X,\mathcal{C},e}(1)$, note that $Z \cap \mathrm{pr}^{-1}(W) \subset X \times (\Delta^1 \setminus \{0, 1\})$. The canonical coordinates of $\Delta^2$ are denoted by $t_1, t_2$. Take a closed point $(w, s) \in X \times (\Delta^1 \setminus \{0, 1\})$. Let

$$H_{(w,s)} := \{w\} \times \{t_1 + st_2 - s = 0\} \subset X \times \Delta^2,$$

namely the closed subscheme in $\{w\} \times \Delta^2$ which is the line connecting $(s, 0)$ and $(0, 1)$. We have the morphism $\rho_{(w,s)} : H_{(w,s)} \to \{(w, s)\} \hookrightarrow X \times \Delta^1$. Now, put

$$\beta := \bigoplus_{(w,s)\in Z\cap\mathrm{pr}^{-1}(W)} \rho^*_{(w,s)}(\tilde{\alpha}'\otimes\mathcal{O}_W) \in \pi_0\mathrm{K}_H(X\times\Delta^2) \quad \text{for } H := \bigcup_{(w,s)\in Z\cap\mathrm{pr}^{-1}(W)} H_{(w,s)}.$$

By construction, this gives a homotopy between $\tilde{\alpha}' \otimes \mathcal{O}_W$ and 0. Indeed, let $f_1 : X \times \Delta^1 \hookrightarrow X \times \Delta^2$ be the map defined by $t_2 = 1$, $f_3$ by $t_1 = 0$, and $f_2$ by $t_1 + t_2 = 1$ sending 0 and 1 to $(0, 1)$ and $(1, 0)$, respectively. The homotopy $\beta$ defines

$$f_1^*(\beta) + f_2^*(\beta) \sim f_3^*(\beta).$$

On the other hand, $f_1^*(\beta) = \tilde{\alpha}' \otimes \mathcal{O}_W$ and $f_2^*(\beta) = f_3^*(\beta)$, and thus $\tilde{\alpha}' \otimes \mathcal{O}_W$ is homotopic to 0. $\square$

**3.8.** ***An elementary projection formula.*** In this subsection, we recall an elementary projection formula which will be used in the next subsection. Let $\varphi : X \to X$ be an endomorphism. In this setting, we have the following elementary projective formula.

**Lemma 3.8.1.** *The following diagram is commutative*:

$$\begin{tikzcd} \mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \arrow[r, "\mathrm{Id}\wedge\varphi_*"] \arrow[d, "\varphi^*\wedge\mathrm{Id}"'] & \mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \arrow[d, "(-\otimes-)"] \\ \mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \arrow[r, "\varphi_*\circ(-\otimes-)"] & \mathrm{K}^{(d)}(X,-) \end{tikzcd}$$

*Proof.* By definition of $(-\otimes-)$, we are reduced to showing the corresponding statement for each level of the simplicial spectrum corresponding to $\mathrm{K}^{(d)}(X,-)$. In that case, it follows directly from Thomason's projection formula for K-theory spectra [Thomason and Trobaugh 1990]. □

It follows that we have a commutative diagram at the level of Picard groupoids:

$$\begin{tikzcd} \Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X,-)) \arrow[r, "\mathrm{Id}\wedge\varphi_*"] \arrow[d, "\varphi^*\wedge\mathrm{Id}"'] & \Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X,-)) \arrow[d, "(-\otimes-)"] \\ \Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X,-)) \arrow[r, "\varphi_*\circ(-\otimes-)"] & \Pi(\mathrm{K}^{(d)}(X,-)) \end{tikzcd}$$

In particular, for $\mathcal{G} \in \Pi(\mathrm{K}^{(d)}(X,-))$ and $\mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X,-))$ we have a natural isomorphism

$$\mathrm{proj}_{\mathcal{G},\mathcal{F}} : \varphi_*(\varphi^*(\mathcal{G}) \otimes \mathcal{F}) \to \mathcal{G} \otimes \varphi_*\mathcal{F}.$$

**3.9.** ***Formula for traces of tensor products of correspondences.*** We now prove a formula for the traces of tensor products of correspondences. Let $\varphi : X \to X$ be an endomorphism. Then one has an induced map $\varphi_* : \mathrm{K}^{(d)}(X,-) \to \mathrm{K}^{(d)}(X,-)$. Moreover, we also have the push-forward map

$$\pi_*^{(d)} : \mathrm{K}^{(d)}(X,-) \to \mathrm{K}(k).$$

Note that $\pi_*^{(d)} \circ \varphi_*$ is homotopic to $(\pi_*^{(d)} \circ \varphi)_* = \pi_*^{(d)}$. Below, we use the same notation to denote the corresponding induced morphisms on the associated Picard groupoids.

**Definition 3.9.1.** Let $\mathcal{F} \in \Pi\mathrm{K}^{(d)}(X,-)$. A *right correspondence* on $\mathcal{F}$ is a morphism $\Phi_{\mathcal{F}} : \mathcal{F} \to \varphi_*\mathcal{F}$ in $\Pi\mathrm{K}^{(d)}(X,-)$, and a *left correspondence* is a morphism $\Psi_{\mathcal{F}} : \varphi^*\mathcal{F} \to \mathcal{F}$ in $\Pi\mathrm{K}^{(d)}(X,-)$. If no confusion can arise, we abbreviate right or left correspondence simply by correspondence.

Let $(\mathcal{F}, \Phi_{\mathcal{F}})$ be an object in $\Pi\mathrm{K}^{(d)}(X, -)$ endowed with a left correspondence. Then we have morphisms

$$\pi_*^{(d)}(\Phi_{\mathcal{F}}) : \pi_*^{(d)}(\mathcal{F}) \to \pi_*^{(d)}(\varphi_*\mathcal{F}) \cong \pi_*^{(d)}(\mathcal{F})$$

in $\Pi\mathrm{K}(k)$. Suppose now we are given $\mathcal{G} \in \Pi\mathrm{K}(X)$ and a left correspondence $\Psi_{\mathcal{G}} : \varphi^*\mathcal{G} \to \mathcal{G}$. Then $\mathcal{F} \otimes \mathcal{G} \in \Pi\mathrm{K}^{(d)}(X, -)$ is endowed with a correspondence as follows:

$$\Psi_{\mathcal{G}} \otimes \Phi_{\mathcal{F}} : \mathcal{G} \otimes \mathcal{F} \xrightarrow{\mathrm{Id}\otimes\Phi_{\mathcal{F}}} \mathcal{G} \otimes \varphi_*\mathcal{F} \xleftarrow{\mathrm{proj}_{\mathcal{G},\mathcal{F}}} \varphi_*(\varphi^*(\mathcal{G}) \otimes \mathcal{F}) \xrightarrow{\varphi_*(\Psi_{\mathcal{G}}\otimes\mathrm{Id})} \varphi_*(\mathcal{G} \otimes \mathcal{F}).$$

In the following, we sometimes denote the trace $\mathrm{Tr}(\pi_*^{(d)}(\Psi_{\mathcal{G}} \otimes \Phi_{\mathcal{F}}))$, which is an element of $k^\times$, by $\langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d,-)}(\Psi_{\mathcal{G}}, \Phi_{\mathcal{F}})$. When $\varphi = \mathrm{Id}$, this notation is compatible with the one in Section 3.4.

**Proposition 3.9.2.** *Let $X$ be* projective *and $\varphi : X \to X$ be an endomorphism. Suppose $\mathcal{G} \in \Pi\mathrm{K}(X)$, $\mathcal{F} \in \Pi\mathrm{K}^{(d)}(X, -)$, and both are endowed with correspondences $\Psi_{\mathcal{G}} : \varphi^*\mathcal{G} \to \mathcal{G}$ and $\Phi_{\mathcal{F}} : \mathcal{F} \to \varphi_*\mathcal{F}$. Assume given another correspondence $\Phi'_{\mathcal{F}} : \mathcal{F} \to \varphi_*\mathcal{F}$. Then one has the formula*

$$\langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d,-)}(\Psi_{\mathcal{G}}, \Phi_{\mathcal{F}}) \times \mathrm{Tr}(\pi_*^{(d)}(\Phi_{\mathcal{F}}))^{-r_{\mathcal{G}}} = \langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d,-)}(\Psi_{\mathcal{G}}, \Phi'_{\mathcal{F}}) \times \mathrm{Tr}(\pi_*^{(d)}(\Phi'_{\mathcal{F}}))^{-r_{\mathcal{G}}},$$

*where $r_{\mathcal{G}}$ is the generic rank of $\mathcal{G}$, i.e., the image of $\mathcal{G}$ by the canonical map $\pi_0(\mathrm{Pic}^{\mathbb{Z}}(X)) \to \mathbb{Z}$.*

*Proof.* We may write $\mathcal{G} = [\mathcal{O}_X^{\oplus r_{\mathcal{G}}}] + \mathcal{G}_0 \in \Pi(\mathrm{K}(X))$. Note that $r_{\mathcal{G}_0} = 0$. Now, $\mathcal{O}_X$ comes equipped with a canonical correspondence $\mathrm{can} : \varphi^*\mathcal{O}_X \to \mathcal{O}_X$, and therefore $\mathcal{O}_X^{r_G}$ also comes equipped with a canonical correspondence (also denoted by can). Using this, we define a correspondence $\Psi_{\mathcal{G}_0}$ on $\mathcal{G}_0$ so that $\Psi_{\mathcal{G}} = \mathrm{can} + \Psi_{\mathcal{G}_0}$. Since $\langle -, -\rangle^{\Pi}_{(d,-)}$ is distributive we have

$$\mathrm{Tr}(\pi_*^{(d)}(\Psi_{\mathcal{G}} \otimes \Phi_{\mathcal{F}})) = \mathrm{Tr}(\pi_*^{(d)}(\mathrm{can} \otimes \Phi_{\mathcal{F}})) \times \mathrm{Tr}(\pi_*^{(d)}(\Psi_{\mathcal{G}_0} \otimes \Phi_{\mathcal{F}})).$$

Since $\mathrm{Tr}(\pi_*^{(d)}(\mathrm{can} \otimes \Phi_{\mathcal{F}})) = \mathrm{Tr}(\pi_*^{(d)}(\Phi_{\mathcal{F}}))^{r_{\mathcal{G}}}$, we are reduced to showing that $\langle \mathcal{G}_0, \mathcal{F}\rangle^{\Pi}_{(d,-)}(\Psi_{\mathcal{G}_0}, \Phi_{\mathcal{F}}) = \langle \mathcal{G}_0, \mathcal{F}\rangle^{\Pi}_{(d,-)}(\Psi_{\mathcal{G}_0}, \Phi'_{\mathcal{F}})$. The result follows if we show that the two ways of composing the following maps are homotopic:

$$\langle \mathcal{G}_0, \mathcal{F}\rangle^{\Pi}_{(d,-)} \underset{\langle \mathrm{Id}, \Phi'_{\mathcal{F}}\rangle}{\overset{\langle \mathrm{Id}, \Phi_{\mathcal{F}}\rangle}{\rightrightarrows}} \langle \mathcal{G}_0, \varphi_*\mathcal{F}\rangle^{\Pi}_{(d,-)} \xrightarrow{\mathrm{proj}} \langle \varphi^*\mathcal{G}_0, \mathcal{F}\rangle^{\Pi}_{(d,-)} \xrightarrow{\langle \Psi_{\mathcal{G}_0}, \mathrm{Id}\rangle} \langle \mathcal{G}_0, \mathcal{F}\rangle^{\Pi}_{(d,-)}.$$

To check this, we only need to show that the first two maps, namely $\langle \mathrm{Id}, \Phi_{\mathcal{F}}\rangle$ and $\langle \mathrm{Id}, \Phi'_{\mathcal{F}}\rangle$, are homotopic. Recall that for any sheaf $\mathcal{L}$ of generic rank $r$, there exists a coherent sheaf $\mathcal{L}'$ the codimension of whose support is $\geq 1$ and $[\mathcal{L}] = [\mathcal{O}_X^{\oplus r}] + [\mathcal{L}']$ in $\mathrm{K}_0(X)$; see [Fulton 1998, Example 15.1.5]. This implies that, since $\mathcal{G}_0$ has rank zero, there exists $C \in \Pi\mathrm{K}^{(1)}(X)$ such that $C \cong \mathcal{G}_0$. Then the two maps above are isomorphic to

$$\langle C, \mathcal{F}\rangle^{\Pi}_{(d,-)} \underset{\langle \mathrm{Id}, \Phi'_{\mathcal{F}}\rangle}{\overset{\langle \mathrm{Id}, \Phi_{\mathcal{F}}\rangle}{\rightrightarrows}} \langle C, \varphi_*\mathcal{F}\rangle^{\Pi}_{(d,-)}.$$

It is enough to show that the path $\Phi'^{-1}_{\mathcal{F}} \circ \Phi_{\mathcal{F}} \in \pi_1\mathrm{K}^{(d)}(X, -)$ tensored with $C$ is homotopic to the identity. In particular, we just need to show that it maps to the identity when viewed as an element of $\pi_1\mathrm{K}^{(d)}(X, -)$. But this is precisely the content of Lemma 3.7.1. □

**Corollary 3.9.3.** *Suppose $f : \mathcal{F} \to \mathcal{F} \in \Pi(\mathrm{K}^{(d)}(X, -))$ and $g : \mathcal{G} \to \mathcal{G} \in \Pi(\mathrm{K}(X))$. Then*

$$\langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d,-)}(g, f) = \mathrm{Tr}\big(\pi^{(d)}_*(f) \mid \pi^{(d)}_*(\mathcal{F})\big)^{r_{\mathcal{G}}} \times \langle \mathcal{G}, \mathcal{F}\rangle^{\Pi}_{(d,-)}(g, \mathrm{Id}).$$

*Here, $\pi^{(d)}_*(f)$ and $\pi^{(d)}_*(\mathcal{F})$ are the images under the natural map*

$$\Pi(\mathrm{K}^{(d)}(X, -)) \xrightarrow{\pi_*} \mathrm{Pic}^{\mathbb{Z}}(k).$$

*Proof.* Note that if $\varphi = \mathrm{Id} : X \to X$, then a correspondence on $\mathcal{F}$ just amounts to giving an endomorphism of $\mathcal{F}$, and likewise a correspondence on $\mathcal{G}$. The corollary follows by taking $\Phi_{\mathcal{F}} = f$, $\Psi_{\mathcal{G}} = g$, and $\Phi'_{\mathcal{F}} = \mathrm{id}$ in the previous proposition. □

## 4. Localization formula for holonomic $\mathcal{D}_X$-modules

We now prove our main results on the global epsilon factors of tensor products of holonomic $\mathcal{D}_X$-modules and flat connections. In the following, $\pi : X \to \mathrm{Spec}(k)$ is a smooth projective variety over a field of characteristic zero.

**4.1.** ***The main theorem.*** Let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$-module. We set

$$\varepsilon_{\mathrm{dR}}(X, \mathcal{F}) := \det(R\Gamma_{\mathrm{dR}}(X, \mathcal{F})) \in \mathrm{Pic}^{\mathbb{Z}}(k).$$

We consider the following variant of the microlocalization map of Corollary 2.7.2:

$$\mathrm{CC}^{\mathrm{K}} : \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \xrightarrow{\varepsilon} \mathrm{K}^{(d)}(T^*X) \to \mathrm{K}^{(d)}(T^*X, -),$$

where the second map is the natural augmentation map. Recall that we have defined a twisted pull-back map

$$\sigma^+ : \mathrm{K}(T^*X) \to \mathrm{K}(X).$$

In an analogous manner we can define the twisted pull-back

$$\sigma^+ : \mathrm{K}^{(d)}(T^*X, -) \xrightarrow{\sigma^*} \mathrm{K}^{(d)}(X, -) \xrightarrow{\otimes\omega_X} \mathrm{K}^{(d)}(X, -).$$

We set $\mathrm{CC} := \sigma^+ \circ \mathrm{CC}^{\mathrm{K}}$, and let $\mathrm{for}^{\nabla} : \mathrm{K}_X(\mathcal{D}_X) \to \mathrm{K}(X)$ denote the morphism induced by forgetting the $\mathcal{D}_X$ module structure. Recall that this is well-defined since any $\mathcal{D}_X$-module with singular support in the zero section is coherent as an $\mathcal{O}_X$-module. The following is the main result of this section.

**Theorem 4.1.1.** *The following diagram commutes up to homotopy equivalence*:

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "\mathrm{for}^{\nabla} \wedge \mathrm{CC}"'] & \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[d, "R\Gamma_{\mathrm{dR}}"] \\
\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \arrow[r, "\langle -,- \rangle^{\mathrm{K}}_{(d,-)}"] & \mathrm{K}(k)
\end{tikzcd}$$

*Proof.* We only need to show that the following two diagrams commute:

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "\mathrm{for}^{\nabla} \wedge \mathrm{CC}"'] & \mathrm{K}(\mathcal{D}_X) \arrow[d, "\sigma^+ \circ \mathrm{gr}"] \\
\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \arrow[r, "\otimes"] & \mathrm{K}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{K}(\mathcal{D}_X) \arrow[r, "\sigma^+ \circ \mathrm{gr}"] \arrow[d, "R\Gamma_{\mathrm{dR}}"'] & \mathrm{K}(X) \arrow[dl, "f_*"] \\
\mathrm{K}(k) &
\end{tikzcd}$$

The commutativity of the right-hand diagram follows from Lemma 2.7.5. Therefore, it is enough to verify that the diagram on the left is commutative. The bottom horizontal in this diagram is by definition the composition

$$\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \xrightarrow{\otimes} \mathrm{K}^{(d)}(X,-) \to \mathrm{K}^{(0)}(X,-) \leftarrow \mathrm{K}(X).$$

Since $\sigma^+$ commutes with $\otimes$ and augmentation, we are reduced to showing that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "\mathrm{for}^{\nabla} \wedge \mathrm{CC}^{\mathrm{K}}"'] & \mathrm{K}(\mathcal{D}_X) \arrow[d, "\sigma^* \circ \mathrm{gr}"] \\
\mathrm{K}(X) \wedge \mathrm{K}^{(d)}(X,-) \arrow[r, "\otimes"] & \mathrm{K}(X)
\end{tikzcd}$$

Note that $\mathrm{for}^{\nabla}$ is homotopic to $\sigma^* \circ \pi^*$ (see Lemma 2.8.2 for the definition of $\pi^*$). Therefore, by Lemma 3.2.1 and Remark 3.2.2, we are reduced to showing that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "\pi^* \wedge \varepsilon"'] & \mathrm{K}(\mathcal{D}_X) \arrow[d, "\mathrm{gr}"] \\
\mathrm{K}(T^*X) \wedge \mathrm{K}^{(d)}(T^*X) \arrow[r, "\otimes"] & \mathrm{K}(T^*X)
\end{tikzcd}$$

By definition, this commutative diagram factors as

$$\begin{tikzcd}
\mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X) \arrow[r] \arrow[d, "\pi^* \wedge \varepsilon"'] & \mathrm{K}_X(\mathcal{D}_X) \wedge \mathrm{K}(\mathcal{D}_X) \arrow[r, "\otimes"] \arrow[d, "\pi^* \wedge \mathrm{gr}"'] & \mathrm{K}(\mathcal{D}_X) \arrow[d, "\mathrm{gr}"] \\
\mathrm{K}(T^*X) \wedge \mathrm{K}^{(d)}(T^*X) \arrow[r] & \mathrm{K}(T^*X) \wedge \mathrm{K}(T^*X) \arrow[r, "\otimes"] & \mathrm{K}(T^*X)
\end{tikzcd}$$

The left square in this diagram commutes by Remark 2.7.3, and the right square commutes by Lemma 2.8.2. □

The theorem has a direct consequence for the pairing $\langle -,-\rangle^{\Pi}_{(d,-)}$. Namely, let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$-modules, and $\mathcal{G}$ a vector bundle with connection. Then, forgetting the connection, $\mathcal{G}$ gives rise to a natural object $\det(\mathcal{G}) \in \Pi(\mathrm{K}(X))$. On the other hand, $\mathcal{F}$ gives rise to an object of the Picard groupoid associated to $\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X)$, and therefore, an object of $\Pi(\mathrm{K}^{(d)}(X,-))$ via the morphism CC. We denote the corresponding object by $\mathrm{CC}(\mathcal{F}) \in \Pi(\mathrm{K}^{(d)}(X,-))$. Applying the pairing

$$\langle -,-\rangle_{(d)} : \Pi(\mathrm{K}(X)) \wedge \mathrm{K}^{(d)}(X,-) \to \mathrm{Pic}^{\mathbb{Z}}(k)$$

to $\det(\mathcal{G})$ and $\mathrm{CC}(\mathcal{F})$ gives rise to an object $\langle \det(\mathcal{G}), \mathrm{CC}(\mathcal{F})\rangle \in \mathrm{Pic}^{\mathbb{Z}}(k)$. An isomorphism of $\mathcal{D}_X$-modules $g : \mathcal{G} \to \mathcal{G}'$ induces an isomorphism $g : \det(\mathcal{G}) \to \det(\mathcal{G}')$, and an isomorphism $f : \mathcal{F} \to \mathcal{F}'$ induces an isomorphism $f : \mathrm{CC}(\mathcal{F}) \to \mathrm{CC}(\mathcal{F}')$. Therefore we have an isomorphism $f \otimes g : \langle \det(\mathcal{G}), \mathrm{CC}(\mathcal{F})\rangle \to \langle \det(\mathcal{G}'), \mathrm{CC}(\mathcal{F}')\rangle$. Similarly, we get an isomorphism $\varepsilon(g \otimes f) : \varepsilon_{\mathrm{dR}}(X, \mathcal{G} \otimes \mathcal{F}) \to \varepsilon_{\mathrm{dR}}(X, \mathcal{G}' \otimes \mathcal{F}')$.

**Corollary 4.1.2.** *One has a natural* (*in $f$ and $g$ as above*) *isomorphism in* $\mathrm{Pic}^{\mathbb{Z}}(k)$:

$$\varepsilon_{\mathrm{dR}}(X, \mathcal{G} \otimes \mathcal{F}) \cong \langle \det(\mathcal{G}), \mathrm{CC}(\mathcal{F})\rangle.$$

*Proof.* The theorem gives rise to the following commutative diagram:

$$\begin{tikzcd}
\Pi(\mathrm{K}_X(\mathcal{D}_X) \wedge \Pi(\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X)) \arrow[r] \arrow[d] & \Pi(\mathrm{K}(\mathcal{D}_X)) \arrow[d] \\
\Pi(\mathrm{K}(X)) \wedge \Pi(\mathrm{K}^{(d)}(X,-)) \arrow[r] & \mathrm{Pic}^{\mathbb{Z}}(k)
\end{tikzcd}$$

Recall, $\mathcal{G}$ gives rise to a homotopy point of $\mathrm{K}_X(\mathcal{D}_X)$, and therefore an object, also denoted by $\mathcal{G}$, in $\Pi(\mathrm{K}_X(\mathcal{D}_X))$. Likewise, $\mathcal{F}$ gives a homotopy point of $\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X)$ and therefore an object $\mathcal{F}$ in $\Pi(\mathrm{K}_{\mathrm{hol}}(\mathcal{D}_X))$. By construction, the composition of the top arrow and right vertical is naturally isomorphic to $\varepsilon_{\mathrm{dR}}(X, \mathcal{G} \otimes \mathcal{F})$. The image of $\mathcal{G}$ in $\Pi(\mathrm{K}(X))$ is by definition $\det(\mathcal{G})$ and similarly the image of $\mathcal{F}$ in $\Pi(\mathrm{K}^{(d)}(X,-))$ is $\mathrm{CC}(\mathcal{F})$. Therefore, the commutativity of the diagram above gives rise to the desired natural isomorphism. $\square$

We now apply the previous corollary to compute traces of correspondences and endomorphisms. Let $\mathcal{F}$ denote a holonomic $\mathcal{D}_X$-module and $\mathcal{G}$ a flat connection as above, and fix an automorphism $\varphi : X \to X$.

**Definition 4.1.3.** A *correspondence* $\Phi_{\mathcal{F}}$ on $\mathcal{F}$ is an isomorphism $\Phi_{\mathcal{F}} : \mathcal{F} \to \varphi_*\mathcal{F}$ of $\mathcal{D}_X$-modules. Since $\varphi$ is assumed to be an automorphism, this is equivalent to giving an isomorphism $\Psi_{\mathcal{F}} : \varphi^*\mathcal{F} \to \mathcal{F}$.

We fix correspondences $\Phi_{\mathcal{F}}$ and $\Psi_{\mathcal{G}}$ on $\mathcal{F}$ and $\mathcal{G}$. Note that if $\varphi = \mathrm{id}$ is the identity, then a correspondence is simply an automorphism. Moreover, just as in Section 3.9, one has an induced correspondence

$$\Phi_{\mathcal{F}} \otimes \Psi_{\mathcal{G}} : \mathcal{F} \otimes \mathcal{G} \to \varphi_*(\mathcal{F} \otimes \mathcal{G}).$$

It follows that one has an induced quasi-isomorphism:

$$R\Gamma(\Phi_{\mathcal{F}} \otimes \Psi_{\mathcal{G}}) : R\Gamma_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}) \to R\Gamma_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}).$$

We let $\varepsilon_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}; \Phi_{\mathcal{F}} \otimes \Psi_{\mathcal{G}}) := \mathrm{Tr}\big(\Phi_{\mathcal{F}} \otimes \Psi_{\mathcal{G}} \mid \det(R\Gamma_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}))\big) \in k^{\times}$. If $\varphi$ is the identity, we have simply automorphisms $f : \mathcal{F} \to \mathcal{F}$ and $g : \mathcal{G} \to \mathcal{G}$ (as $\mathcal{D}_X$-modules). In this case we denote the corresponding epsilon factor by $\varepsilon_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}; f \otimes g) := \mathrm{Tr}\big(f \otimes g \mid \det(R\Gamma_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}))\big) \in k^{\times}$. In the following, we fix a lift $\mathrm{SS}(\mathcal{F}) \in \mathrm{Z}_0(X)$ of $[\mathrm{CC}(\mathcal{F})] \in \mathrm{CH}_0(X)$. Moreover, we fix an object (as in Section 3.9), also denoted by $\mathrm{SS}(\mathcal{F})$, of $\Pi(\mathrm{K}^{(d)}(X))$ whose image in $\Pi(\mathrm{K}^{(d)}(X, -))$ is isomorphic to $\mathrm{CC}(\mathcal{F})$. Since $\mathcal{F}$ is equipped with a correspondence, we have $\varphi_*(\mathrm{CC}(\mathcal{F})) = \mathrm{CC}(\mathcal{F})$ in $\mathrm{CH}_0(X) \cong \pi_0\mathrm{K}^{(d)}(X, -)$. This enables us to take a path $\alpha : \mathrm{CC}(\mathcal{F}) \to \varphi_*(\mathrm{CC}(\mathcal{F}))$, and we normalize so that $\mathrm{Tr}(\pi_*^{(d)}(\alpha)) = 1$. By Proposition 3.9.2, this data allows us to define $\langle \mathcal{G}_0, \mathrm{CC}(\mathcal{F})\rangle(\Psi_{\mathcal{G}}, \alpha)^{\Pi}_{(d,-)} \in k^{\times}$.

**Theorem 4.1.4.** *With notation as above*:

(i) *One has*

$$\varepsilon_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}; \Phi_{\mathcal{F}} \otimes \Psi_{\mathcal{G}}) = \varepsilon_{\mathrm{dR}}(X, \mathcal{F}; \Phi_{\mathcal{F}})^{r_{\mathcal{G}}} \times \langle \mathcal{G}, \mathrm{CC}(\mathcal{F})\rangle(\Psi_{\mathcal{G}}, \alpha)^{\Pi}_{(d,-)}.$$

(ii) *In the setting of endomorphisms* (*i.e.*, $\varphi = \mathrm{id}$), *we have*

$$\varepsilon_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}; f \otimes g) = \varepsilon_{\mathrm{dR}}(X, \mathcal{F}; f)^{r_{\mathcal{G}}} \times \langle \det(\mathcal{G}), \mathrm{SS}(\mathcal{F})\rangle(g).$$

*Proof.* The first equality follows directly from Theorem 4.1.1 and Proposition 3.9.2. For the second statement, we note that in the setting of endomorphisms one has, by Corollary 3.9.3,

$$\varepsilon_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}; f \otimes g) = \varepsilon_{\mathrm{dR}}(X, \mathcal{F}; f)^{r_{\mathcal{G}}} \times \langle \det(\mathcal{G}), \mathrm{SS}(\mathcal{F})\rangle^{\Pi}_{(d,-)}(g, \mathrm{Id}).$$

On the other hand, the latter is

$$\langle \det(\mathcal{G}), \mathrm{SS}(\mathcal{F})\rangle^{\Pi}_{(d,-)}(g, \mathrm{Id}) = \langle \det(\mathcal{G}), \mathrm{SS}(\mathcal{F})\rangle^{\Pi}_{(d)}(g, \mathrm{Id}) = \langle \det(\mathcal{G}), \mathrm{SS}(\mathcal{F})\rangle(g)$$

by Corollary 3.5.1. □

**Remark 4.1.5.** We note that $\mathrm{CC}(\mathcal{F}) \in \mathrm{CH}^d(X)$ is precisely the pull-back of the characteristic cycle of $\mathcal{F}$ under the zero section $\sigma^* : \mathrm{CH}^d(T^*X) \to \mathrm{CH}^d(X)$.

**4.2.** ***A formula for the local pairing.*** Let $\mathcal{G} \in \Pi\mathrm{K}(X)$, and $\Psi_{\mathcal{G}} : \varphi^*\mathcal{G} \to \mathcal{G}$ be a correspondence in $\Pi\mathrm{K}(X)$. Assume given a cycle $z \in \mathrm{CH}^d(X)$ such that $z = \varphi_*(z)$. We take an object $\mathcal{O}_Z \in \Pi\mathrm{K}^{(d)}(X, -)$ which corresponds to $z$ via the isomorphism $\pi_0\mathrm{K}^{(d)}(X, -) \cong \mathrm{CH}_0(X)$, and take a correspondence $P : \mathcal{O}_Z \to \varphi_*\mathcal{O}_Z$, normalized so that the trace of the action of $P$ on the cohomology is 1 as well. Since $z = \varphi_*(z)$, such a correspondence must exist (though it may not be unique).

In this setting, we have seen in the proof of Proposition 3.9.2 that

$$\langle \mathcal{G}, \mathcal{O}_Z \rangle^{\Pi}_{(d,-)}(\Psi_{\mathcal{G}}, P) \in k^{\times}$$

is independent of the choice of $\mathcal{O}_Z$ and $P$. When $z$ is represented by $z_0 \in \mathrm{Z}^d(X)$ such that $\varphi_*(z_0) = z_0$, we may take $P$ such that the description of the pair is especially simple. For simplicity, we assume that $z_0$ is an effective cycle. In the general case, we can proceed by writing it as a difference of two effective cycles. In this case, let $W$ be the underlying reduced scheme of $z_0$ in $X$. Note that $W$ is a smooth scheme of dimension 0. Since, by assumption, $\varphi_*(z_0) = z_0$, there exists an endomorphism $\varphi_W$ of $W$ such that

$$\begin{tikzcd} W \arrow[r, "\varphi_W"] \arrow[d, hook, "i"'] & W \arrow[d, hook, "i"] \\ X \arrow[r, "\varphi"] & X \end{tikzcd}$$

is commutative. Since $z_0$ is an effective cycle, we may write $z_0 = \sum_{w \in |W|} n_w \cdot [w]$, where $n_w > 0$. We set $\mathcal{O}_{z_0} := \bigoplus_{w \in |W|} \mathcal{O}_w^{\oplus n_w}$. The endomorphism $\varphi_W$ yields a correspondence $P : \mathcal{O}_{z_0} \to \varphi_* \mathcal{O}_{z_0}$. We can pull back the correspondence $\varphi^* A \to A$ by $i$, and get a correspondence $i^*\Psi : \varphi_W^*(i^* A) \to i^* A$. One can check that

$$\langle A, z \rangle(\Psi, P) = \mathrm{Tr}(R\Gamma(i^*\Psi)).$$

**4.3.** ***Elementary proof of localization formula for endomorphisms.*** In this section, we give an elementary proof of the main theorem when the correspondence is merely an automorphism. While the proof below is elementary, it doesn't seem to generalize to the setting of correspondences (unlike the K-theoretic approach of the previous sections). We only give an outline of the proof below, and leave the details to the reader.

We begin by recalling the statement for the reader's convenience. Let $X$ denote a smooth projective variety over an algebraically closed field $k$ of characteristic zero. Let $\mathcal{G}$ denote a flat connection on $X$, and $\mathcal{F}$ a holonomic $\mathcal{D}_X$-module. Let $f$ denote a $\mathcal{D}_X$-module automorphism of $\mathcal{F}$, and $g$ a $\mathcal{D}_X$-module automorphism of $\mathcal{G}$. Given a cycle $\mathrm{S}(\mathcal{F}) \in \mathrm{CH}_0(X)$ representing the pull-back (by the zero section) of the characteristic cycle of $\mathcal{F}$, we have defined the trace $\langle \det(\mathcal{G}), \mathrm{S}(\mathcal{F}) \rangle(g) \in k^{\times}$. Note that $\mathrm{S}(\mathcal{F}) = [\mathrm{CC}(\mathcal{F})]$ using the previous notation.

**Theorem 4.3.1.** *With notation as above*:

$$\varepsilon_{\mathrm{dR}}(X, \mathcal{F} \otimes \mathcal{G}, f \otimes g) = \varepsilon_{\mathrm{dR}}(X, \mathcal{F}, f)^{r_{\mathcal{G}}} \times \langle \det(\mathcal{G})(g), \mathrm{S}(\mathcal{F}) \rangle.$$

*Proof.* Suppose that $0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}$ is a finite filtration of $\mathcal{F}$ and that $f$ is an endomorphism which preserves this filtration. Since both sides of the formula are compatible with exact sequences (i.e., are multiplicative), we are reduced to showing the validity of the given formula for $\mathcal{F}$ replaced by $\mathrm{gr}_i(\mathcal{F})$ with the morphism induced by $f$. A similar assertion holds for $\mathcal{G}$. In particular, we can assume that $\mathcal{F}$ is a simple holonomic $\mathcal{D}_X$-module. Then $f$ is given by multiplication by a scalar. A similar assertion holds for $\mathcal{G}$ and $g$. Suppose $f=\alpha\in k^\times$ and $g=\beta\in k^\times$. Then the left-hand side of the formula is given by $(\alpha\beta)^{\chi(\mathcal{F}\otimes\mathcal{G})}$. The right-hand side is given by $\alpha^{\chi(\mathcal{F})r_{\mathcal{G}}}\beta^{r_{\mathcal{G}}\chi(\mathcal{F})}$. Therefore, we are reduced to showing that $\chi(\mathcal{F}\otimes\mathcal{G})=\chi(\mathcal{F})r_{\mathcal{G}}$. This follows from a direct computation or by the Dubson–Kashiwara formula once one notes that the associated graded (with respect to a good filtration) commutes with the tensor product since $\mathcal{G}$ is $\mathcal{O}_X$-coherent (see Remark 2.8.1). □

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Received 3 Feb 2017. Revised 16 May 2017. Accepted 23 Jul 2017.

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ANNALS OF K-THEORY
Vol. 3, No. 3, 2018
dx.doi.org/10.2140/akt.2018.3.491

# Poincaré duality and Langlands duality for extended affine Weyl groups

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In this paper we construct an equivariant Poincaré duality between dual tori equipped with finite group actions. We use this to demonstrate that Langlands duality induces a rational isomorphism between the group $C^*$-algebras of extended affine Weyl groups at the level of $K$-theory.



## Introduction

Let $T$ be a compact torus and let $W$ be a finite group acting on $T$ with fixed point. We construct a $W$-equivariant degree-0 Poincaré duality between $C(T)$ and $C(T^\vee)$, where $T^\vee$ denotes the dual torus equipped with the dual action of $W$.

Moreover we show that there is a nonequivariant Poincaré duality between the crossed product algebras $C(T) \rtimes W$ and $C(T^\vee) \rtimes W$. Indeed we provide a general mechanism to descend equivariant Poincaré duality to Poincaré duality for crossed products. As far as we are aware this does not appear elsewhere in the literature.

In the case when $W$ is trivial, our degree-0 duality is connected to the Baum–Connes assembly map in the following way: Let $T$ be a compact torus (equipped with the structure of a Lie group), and let $X^*(T)$, $X_*(T)$ be the groups of characters and cocharacters respectively. By definition the dual torus $T^\vee$ is the torus such that $X^*(T^\vee) = X_*(T)$ and $X_*(T^\vee) = X^*(T)$. Whence the Pontryagin dual of $X_*(T)$

Wright was partially supported by EPSRC grant EP/J015806/1.
*MSC2010:* 46L80.
*Keywords:* Poincaré duality, Langlands duality, extended affine Weyl groups.

is the torus $T^\vee$. The Baum–Connes assembly map for $X_*(T)$ gives a degree-0 isomorphism

$$K_*(T) \xrightarrow{\cong} K_*(C^*X_*(T)) \cong K^*(T^\vee).$$

This isomorphism agrees with our Poincaré duality, though this is not immediate from the definition of the two maps, see Section 4D.

For an isometric action of a group $W$ on a closed Riemannian manifold $M^n$, Kasparov's [1988] Poincaré duality, by contrast with our Poincaré duality, provides an isomorphism from $KK_W(C(M), \mathbb{C})$ to $KK_W(\mathbb{C}, C_\tau(M))$, where $C_\tau(M)$ denotes the algebra of continuous sections of the Clifford bundle for the cotangent bundle $\tau$ of $M$. See also [Echterhoff et al. 2008]. If the action is trivial and $M$ is a spin manifold, then the twisting by the Clifford algebra simply induces a dimension shift so Kasparov's Poincaré duality has degree $n$ modulo 2. In the case where $M$ is a torus and $W$ is trivial, this is connected to our Poincaré duality via the Dirac-dual-Dirac method, which addresses the dimension shift. In the equivariant case the group acts nontrivially on the Clifford bundle, so the appearance of this bundle no longer simply gives a dimension shift. Indeed, for example, letting $\mathbb{Z}/2\mathbb{Z}$ act by complex conjugation on the 1-dimensional torus $U(1)$, then $KK_W(\mathbb{C}, C_\tau(U(1)))$ is $\mathbb{Z}^3$ in dimension 0 and 0 in dimension 1, which agrees with the *unshifted* $K$-theory group $KK_W(\mathbb{C}, C(U(1)))$.

In this paper, in order to describe the $KK$-cycles defining our Poincaré dualities explicitly, we have given direct proofs of the relevant properties of these cycles and their pairings. As remarked by the referee, it is in principle possible to obtain these elements by combining Kasparov's Poincaré duality elements with the Dirac and dual-Dirac cycles. Providing full details of this reduction to the known results is in itself somewhat complicated and we have opted to give the direct, self-contained argument.

As an application of our Poincaré duality we consider the case where $T$ is the maximal torus in a compact connected semisimple Lie group and $W$ is the Weyl group. The dual torus is then the maximal torus in the Langlands dual Lie group. In general there is no $W$-equivariant homeomorphism between the two tori, hence *a priori* one would not expect them to have the same equivariant $K$-theory. However our Poincaré duality gives a canonical pairing between these two equivariant $K$-theory groups, and hence ignoring torsion these groups are isomorphic. Moreover our Poincaré duality also provides a canonical pairing between the $K$-theory of the extended affine Weyl groups of the original Lie group and its Langlands dual. This again yields an isomorphism up to torsion in $K$-theory, although these discrete groups are not typically isomorphic. In [Niblo et al. 2016] we explore this phenomenon in further detail and give detailed computations of these $K$-theory groups in a number of cases.

The connection between $T$-duality and Langlands duality has been studied by Daenzer and van Erp [2014], who showed that Langlands duality for complex reductive Lie groups can be implemented by $T$-dualization for groups whose simple factors are of type A, D or E. This was generalised by Bunke and Nikolaus [2015]. The study of $T$-duality in these papers, involves examining the Lie group viewed as a principal bundle of tori via the action of the maximal torus on the group. Here by contrast we study the Weyl group action on the maximal torus, instead of the maximal torus action on the Lie group. In both cases there is a natural duality arising from Langlands duality of root systems and the possible unification of these two perspectives would provide an interesting future project.

## 1. Statement of results

Let $W$ be a finite group acting isometrically with a global fixed point on a flat Riemannian torus $T$, and let $\mathfrak{t}$ denote the universal cover of $T$. The notation reflects the observation that $T$ can be equipped with the structure of an abelian Lie group with identity at the fixed point, and $\mathfrak{t}$ is then its Lie algebra which inherits a linear isometric action of $W$. Denote by $\Gamma$ the lattice in $\mathfrak{t}$ mapping to the identity in $T$, or equivalently the fundamental group of $T$. This inherits an action of $W$ from $\mathfrak{t}$.

Now let $T^\vee$ be the dual torus of $T$, that is, the group of characters of $\Gamma$. We similarly denote by $\mathfrak{t}^*$ the Lie algebra of $T^\vee$ (which is the dual space of $\mathfrak{t}$) and denote by $\Gamma^\vee$ the fundamental group of $T^\vee$. The action of $W$ on $T$ induces dual actions on $T^\vee$, $\mathfrak{t}^*$ and $\Gamma^\vee$.

Let $\mathcal{P} \in KK_W(\mathbb{C}, C(T) \mathbin{\widehat{\otimes}} C(T^\vee))$ denote the class of the Poincaré line bundle. To construct our Poincaré duality we will, in Section 3B, define an element $\mathcal{Q} \in KK_W(C(T^\vee) \mathbin{\widehat{\otimes}} C(T), \mathbb{C})$ given by a triple $(L^2(\mathfrak{t}) \mathbin{\widehat{\otimes}} \mathcal{S}, \rho, Q_0)$, for which $\mathcal{P}, \mathcal{Q}$ is a Poincaré duality pair. The operator is

$$Q_0 = \frac{\partial}{\partial y^j} \otimes \varepsilon^j - 2\pi i y^j \otimes \boldsymbol{e}_j,$$

where $\{\varepsilon^j, \boldsymbol{e}_j : j = 1, \ldots, n\}$ denotes a suitable basis for $\mathfrak{t}^* \times \mathfrak{t}$ acting on a space of spinors $\mathcal{S}$. The representation $\rho$ is defined by

$$\rho\biggl(\sum_{\gamma\in\Gamma} a_\gamma e^{2\pi i\langle\eta,\gamma\rangle} \otimes f\biggr)\xi \otimes s = \sum_{\gamma\in\Gamma} a_\gamma \gamma \cdot (\tilde{f}\xi) \otimes s,$$

where $\gamma$ acts by translation on $L^2(\mathfrak{t})$, $\tilde{f}$ denotes the lift of $f$ to a periodic function on $\mathfrak{t}$ and $\eta$ denotes a variable in $\mathfrak{t}^*$.

**Theorem 1.1.** *Let $T$ be a torus with flat Riemannian metric and $T^\vee$ its dual. Suppose that $W$ is a finite group acting isometrically on $T$ with a global fixed point. The elements $\mathcal{P}$, $\mathcal{Q}$ define a $W$-equivariant Poincaré duality in KK-theory* *from $C(T)$ to $C(T^\vee)$ and there is a "descended" nonequivariant Poincaré duality from $C_0(\mathfrak{t}) \rtimes (\Gamma \rtimes W)$ to $C_0(\mathfrak{t}^*) \rtimes (\Gamma^\vee \rtimes W)$. This is natural in the sense that there is a commutative diagram*

$$\begin{array}{ccc} KK^*_W(C(T),\mathbb{C}) & \xrightarrow{\cong} & KK^*_W(\mathbb{C}, C(T^\vee)). \\ \downarrow{\cong} & & {\cong}\downarrow \\ KK^*(C_0(\mathfrak{t}) \rtimes (\Gamma \rtimes W), \mathbb{C}) & \xrightarrow{\cong} & KK^*(\mathbb{C}, C_0(\mathfrak{t}^*) \rtimes (\Gamma^\vee \rtimes W)) \end{array}$$

*where*

- *the top and bottom maps are induced by the Poincaré dualities,*
- *the left-hand map is the composition of the $W$-equivariant Morita equivalence $C(T) \sim C_0(\mathfrak{t}) \rtimes \Gamma$ with the dual Green–Julg isomorphism in $K$-homology,*
- *the right-hand map is its dual, i.e., the composition of the Morita equivalence $C(T^\vee) \sim C_0(\mathfrak{t}^*) \rtimes \Gamma^\vee$ with the Green–Julg isomorphism in $K$-theory.*

The vertical maps factor through $KK(C(T) \rtimes W, \mathbb{C})$ and $KK(\mathbb{C}, C(T^\vee) \rtimes W)$ on the left and right, respectively, and these may be identified (by Fourier–Pontryagin duality) with the groups $KK^*(C^*(\Gamma^\vee \rtimes W), \mathbb{C})$ and $KK^*(\mathbb{C}, C^*(\Gamma \rtimes W))$ respectively.

**Theorem 1.2.** *Let $T$ be a torus with flat Riemannian metric and $T^\vee$ its dual. Suppose that $W$ is a finite group acting isometrically on $T$ with a global fixed point. The Poincaré duality from $C(T)$ to $C(T^\vee)$ descends to give a nonequivariant Poincaré duality as follows*:

$$\begin{array}{ccc} KK^*_W(C(T),\mathbb{C}) & \xrightarrow{\cong} & KK^*_W(\mathbb{C}, C(T^\vee)). \\ \downarrow{\cong} & & {\cong}\downarrow \\ KK^*(C^*(\Gamma^\vee \rtimes W), \mathbb{C}) & \xrightarrow{\cong} & KK^*(\mathbb{C}, C^*(\Gamma \rtimes W)) \end{array}$$

*where*

- *the top and bottom maps are induced by the Poincaré dualities*,
- *the left-hand map is the composition of the $W$-equivariant Fourier–Pontryagin duality $C(T) \cong C^*(\Gamma^\vee)$ with the dual Green–Julg isomorphism in $K$-homology*,
- *the right-hand map is its dual*, *i.e.*, *the composition of the $W$-equivariant Fourier–Pontryagin duality $C(T^\vee) \cong C^*(\Gamma)$ with the Green–Julg isomorphism in $K$-theory.*

In Section 4D we turn to the question of the relationship between the Baum–Connes assembly map and our Poincaré duality. In particular, we show that the following diagram of isomorphisms commutes.

$$\begin{tikzcd}
KK^*_{\Gamma \rtimes W}(C_0(\mathfrak{t}), \mathbb{C}) \arrow[r, "\text{Baum–Connes}"] \arrow[d, "\text{dual Green–Julg}"] & KK^*(\mathbb{C}, C^*(\Gamma \rtimes W)) \arrow[d, "\text{Morita equivalence}"] \\
KK^*(C_0(\mathfrak{t}) \rtimes (\Gamma \rtimes W), \mathbb{C}) \arrow[r, "\text{Poincaré duality}"] & KK^*(\mathbb{C}, C_0(\mathfrak{t}^*) \rtimes (\Gamma^\vee \rtimes W))
\end{tikzcd}$$

Given the definitions of the maps this is, in some sense surprising since both the Baum–Connes and the dual Green–Julg maps factor through the descent map, which has target $KK(\mathbb{C}, C_0(\mathfrak{t}^*) \rtimes (\Gamma^\vee \rtimes W) \otimes C^*(\Gamma \rtimes W))$. The corresponding square with this latter group in the top left corner as illustrated in Section 4D, does not commute.

A case of particular interest is provided by the action of a Weyl group $W$ on a torus, provided by a root datum $(X^*, R, X_*, R^\vee)$. Let $W'_a = X_* \rtimes W$ be the corresponding extended affine Weyl group. The Langlands dual root system $(X_*, R^\vee, X^*, R)$ gives rise to a dual extended affine Weyl group $(W'_a)^\vee = X^* \rtimes W$, which is not usually isomorphic to $W'_a$. However the Poincaré duality in Theorem 1.2 provides an isomorphism between $K^*(C^*((W'_a)^\vee))$ and $K_*(C^*(W'_a))$.

The Langlands duality between $W'_a$ and $(W'_a)^\vee$ is further amplified by the following theorem.

**Corollary 1.3.** *Let $G$ be a compact connected semisimple Lie group and $G^\vee$ its Langlands dual, with $W'_a$, $(W'_a)^\vee$ the corresponding extended affine Weyl groups. Then there is a rational isomorphism*

$$K_*(C^*((W'_a)^\vee)) \cong K_*(C^*(W'_a)).$$

In particular we obtain a duality between affine and extended affine Weyl groups of the following form:

**Corollary 1.4.** *Let $W'_a$ be the extended affine Weyl group of $G$, and let $W_a$, $W_a^\vee$ be the affine Weyl groups of $G$ and its Langlands dual $G^\vee$. If $G$ is of adjoint type then rationally*

$$K_*(C^*(W_a^\vee)) \cong K_*(C^*(W'_a)).$$

*If additionally $G$ is of type $A_n$, $D_n$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$ then rationally*

$$K_*(C^*(W_a)) \cong K_*(C^*(W'_a)).$$

Recall that the extended affine Weyl group $W'_a$ is an extension of $W_a$ by the cyclic group $\pi_1(G)$ so the content of Corollary 1.4 is that, surprisingly, this particular extension does not change the $K$-theory.

In a companion paper [Niblo et al. 2016] we explore this phenomenon in further detail and give detailed computations of these $K$-theory groups in a number of cases.

## 2. Background

**2A. *Real Langlands duality.*** Recall that a connected complex reductive Lie group $\boldsymbol{H}$ is classified by its root datum. That is a 4-tuple $(X^*, R, X_*, R^\vee)$ where $X^*$ is the lattice of characters on a maximal torus in $\boldsymbol{H}$, and $X_*$ is the lattice of cocharacters, or equivalently the fundamental group of the maximal torus. The set of roots $R \subset X^*$ is in bijection with the reflections in the Weyl group $W$ and in bijection with the set of coroots $R^\vee \subset X_*$. Root data classify connected complex reductive Lie groups, in the sense that two such groups are isomorphic if and only if their root data are isomorphic (in the obvious sense). The Langlands dual of $\boldsymbol{H}$, denoted $\boldsymbol{H}^\vee$ is then the unique connected complex reductive Lie group associated to the dual root datum $(X_*, R^\vee, X^*, R)$. See [Bourbaki 2002; 2005] for details.

One of the key motivations of this paper is that for extended affine Weyl groups the Baum–Connes correspondence should be thought of as an equivariant duality between maximal tori in a compact connected semisimple Lie group and its real Langlands dual. As in the complex case these are classified by their root data, and we can define the (real) Langlands dual by dualising the root datum as before. Since the real case is not as well known we recall the relationship with the complex case.

For a Lie group $G$, the *complexification* $G_{\mathbb{C}}$ is a complex Lie group together with a morphism from $G$, satisfying the universal property that for any morphism of $G$ into a complex Lie group $\boldsymbol{L}$ there is a unique factorisation through $G_{\mathbb{C}}$.

For $T$ a maximal torus in $G$, the complexification $\boldsymbol{S} := T_{\mathbb{C}}$ of $T$ is a maximal torus in $\boldsymbol{H} := G_{\mathbb{C}}$, and so the dual torus $\boldsymbol{S}^\vee$ is well-defined in the dual group $\boldsymbol{H}^\vee$. Then $T^\vee$ is defined to be the maximal compact subgroup of $\boldsymbol{S}^\vee$, and satisfies the condition

$$(T^\vee)_{\mathbb{C}} = \boldsymbol{S}^\vee.$$

The groups $X^*$, $X_*$ in the root datum are again the groups of characters and cocharacters of the torus $T$ respectively. Dually $X_*$, $X^*$ are the groups of characters and cocharacters on the dual torus $T^\vee$, giving the $T$-duality equation

$$X^*(T^\vee) = X_*(T). \tag{2.1}$$

The torus $T^\vee$ is given explicitly by $T^\vee = \mathrm{Hom}(X_*(T), \boldsymbol{U})$. The *Langlands dual* of $G$, denoted $G^\vee$, is defined to be a maximal compact subgroup of $\boldsymbol{H}^\vee$ containing the torus $T^\vee$.

The process of passing to a maximal compact subgroup is inverse to complexification in the sense that complexifying $G^\vee$ recovers $\boldsymbol{H}^\vee$.

**2A1.** *A table of Langlands dual groups.* Given a compact connected semisimple Lie group $G$, the product $|\pi_1(G)| \cdot |\mathcal{Z}(G)|$ is unchanged by Langlands duality, i.e., it agrees with the product $|\pi_1(G^\vee)| \cdot |\mathcal{Z}(G^\vee)|$. This product is equal to the *connection index*, denoted $f$, (see [Bourbaki 2005, Chapter IX, p. 320]), which is defined in [Bourbaki 2002, Chapter VI, p. 240]. The connection indices are listed in [Bourbaki 2002, Chapter VI, Plates I–X, p. 265–292].

The following is a table of Langlands duals and connection indices for compact connected semisimple groups:

| $G$ | $G^\vee$ | $f$ |
|---|---|---|
| $A_n = \mathrm{SU}_{n+1}$ | $\mathrm{PSU}_{n+1}$ | $n+1$ |
| $B_n = \mathrm{SO}_{2n+1}$ | $\mathrm{Sp}_{2n}$ | 2 |
| $C_n = \mathrm{Sp}_{2n}$ | $\mathrm{SO}_{2n+1}$ | 2 |
| $D_n = \mathrm{SO}_{2n}$ | $\mathrm{SO}_{2n}$ | 4 |
| $E_6$ | $E_6$ | 3 |
| $E_7$ | $E_7$ | 2 |
| $E_8$ | $E_8$ | 1 |
| $F_4$ | $F_4$ | 1 |
| $G_2$ | $G_2$ | 1 |

In this table, the simply connected form of $E_6$ (resp. $E_7$) dualises to the adjoint form of $E_6$ (resp. $E_7$).

The Lie group $G$ and its dual $G^\vee$ admit a common Weyl group

$$W = N(T)/T = N(T^\vee)/T^\vee.$$

The $T$-duality Equation (2.1) identifies the action of the Weyl group of $T$ on $X_*(T)$ with the dual action of the Weyl group of $T^\vee$ on $X^*(T^\vee)$.

**Remark 2.2.** In general, $T$ and $T^\vee$ are *not* isomorphic as $W$-spaces. For example, let $G = \mathrm{SU}_3$ and take $T = \{(z_1, z_2, z_3) : z_j \in \boldsymbol{U}, z_1z_2z_3 = 1\}$. Then in homogeneous coordinates we have $T^\vee = \{(z_1 : z_2 : z_3) : z_j \in \boldsymbol{U}, z_1z_2z_3 = 1\}$. The Weyl group $W$ is the symmetric group $S_3$ which acts by permuting coordinates in both cases. Note that the torus $T$ admits three $W$-fixed points whereas the unique $W$-fixed point in $T^\vee$ is the identity $(1 : 1 : 1) \in T^\vee$, hence $T$ and $T^\vee$ are not $W$-equivariantly isomorphic.

The *nodal group* of the torus $T$ is defined to be $\Gamma(T) := \ker(\exp : \mathfrak{t} \to T)$ and differentiating the action of the Weyl group $W$ we obtain a linear action of $W$ on the Lie algebra $\mathfrak{t}$ which restricts to an action on the nodal group $\Gamma(T)$. Indeed there is a $W$-equivariant isomorphism $X_*(T) \cong \Gamma(T)$.

We remark that by definition $T^\vee$ is the Pontryagin dual of the nodal group $\Gamma(T)$. Moreover the natural action of $W$ on $T^\vee$ is the dual of the action on $\Gamma(T)$. Hence we have the following:

**Lemma 2.3.** *Let $\widehat{\Gamma}$ denote the Pontryagin dual of $\Gamma = \Gamma(T)$. Then we have a $W$-equivariant isomorphism*

$$\widehat{\Gamma} \cong T^{\vee}$$

*and hence an isomorphism of $W$-$C^*$-algebras*

$$C^*(\Gamma) \cong C(T^{\vee}).$$

**2B.** ***Affine and extended affine Weyl groups.*** In this section we will give the definitions of the affine and extended affine Weyl groups of a compact connected semisimple Lie group. As noted in the introduction these are semidirect products of lattices in the Lie algebra $\mathfrak{t}$ of a maximal torus $T$ by the Weyl group $W$. The affine Weyl group $W_a$ is a Coxeter group while the extended affine Weyl group contains $W_a$ as a finite index normal subgroup and the quotient is the fundamental group of the Lie group $G$.

Let $p : \widetilde{G} \to G$ denote the universal cover and let $\widetilde{T}$ be the preimage of $T$ which is a maximal torus in $\widetilde{G}$. We consider the following commutative diagram:

$$\begin{tikzcd} & \Gamma(\widetilde{T}) \arrow[r] \arrow[d, "\iota"] & \mathfrak{t} \arrow[r] \arrow[d, "\mathrm{id}"] & \widetilde{T} \arrow[r] \arrow[d, "p|_{\widetilde{T}}"] & 0 \\ 0 \arrow[r] & \Gamma(T) \arrow[r] & \mathfrak{t} \arrow[r] & T & \end{tikzcd}$$

By the snake lemma the sequence

$$\begin{array}{ccccccc} \ker(\mathrm{id}) & \longrightarrow & \ker(p|_{\widetilde{T}}) & \longrightarrow & \operatorname{coker}(\iota) & \longrightarrow & \operatorname{coker}(\mathrm{id}) \\ \| & & \| & & \| & & \| \\ 0 & & \pi_1(G) & & \Gamma(T)/\Gamma(\widetilde{T}) & & 0 \end{array}$$

is exact, hence $\Gamma(T)/\Gamma(\widetilde{T})$ is isomorphic to $\pi_1(G)$. We thus have a map from $\Gamma(T)$ onto $\pi_1(G)$. The kernel of this map (more commonly denoted $N(G, T)$) is the nodal lattice $\Gamma(\widetilde{T})$ for the torus $\widetilde{T}$ and we have:

**Definition 2.4.** The *affine Weyl group of $G$* is

$$W_a(G) = \Gamma(\widetilde{T}) \rtimes W$$

and the *extended affine Weyl group of $G$* is

$$W_a'(G) = \Gamma(T) \rtimes W,$$

where $W$ denotes the Weyl group of $G$.

The following is now immediate:

**Lemma 2.5.** *Let $\widetilde{G}$ denote the universal cover of $G$ and let $\widetilde{T}$ denote a maximal torus in $\widetilde{G}$. Then we have*

$$W_a(G) = W'_a(\widetilde{G}) = W_a(\widetilde{G}).$$

We remark that the extended affine Weyl group $W'_a(G)$ is a split extension of $W_a(G)$ by $\pi_1(G)$.

## 3. Equivariant Poincaré duality between $C(T)$ and $C(T^\vee)$

We begin by recalling the general framework of Poincaré duality in *KK*-theory. For $\mathfrak{G}$-$C^*$-algebras $A$, $B$ a Poincaré duality is given by elements $\mathfrak{a} \in KK_{\mathfrak{G}}(B \mathbin{\widehat{\otimes}} A, \mathbb{C})$ and $\mathfrak{b} \in KK_{\mathfrak{G}}(\mathbb{C}, A \mathbin{\widehat{\otimes}} B)$ with the property that

$$\mathfrak{b} \otimes_A \mathfrak{a} = 1_B \in KK_{\mathfrak{G}}(B, B), \quad \mathfrak{b} \otimes_B \mathfrak{a} = 1_A \in KK_{\mathfrak{G}}(A, A). \tag{3.1}$$

These then yield isomorphisms between the $K$-homology of $A$ and the $K$-theory of $B$ (and vice versa) given by

$$\mathfrak{x} \mapsto \mathfrak{b} \otimes_A \mathfrak{x} \in KK_{\mathfrak{G}}(\mathbb{C}, B) \quad \text{for } \mathfrak{x} \in KK(A, \mathbb{C}),$$
$$\mathfrak{y} \mapsto \mathfrak{y} \otimes_B \mathfrak{a} \in KK_{\mathfrak{G}}(A, \mathbb{C}) \quad \text{for } \mathfrak{y} \in KK(\mathbb{C}, B).$$

Throughout this section $T$ will denote a torus with flat Riemannian metric, $T^\vee$ its dual torus and $W$ a finite group acting by isometries on $T$ (and dually on $T^\vee$). We will construct elements

$$\mathcal{Q} \in KK_W(C(T^\vee) \mathbin{\widehat{\otimes}} C(T), \mathbb{C}) \quad \text{and} \quad \mathcal{P} \in KK_W(\mathbb{C}, C(T) \mathbin{\widehat{\otimes}} C(T^\vee))$$

satisfying (3.1).

**3A.** ***The Poincaré line bundle.*** Recall that the Poincaré line bundle over $T \times T^\vee$ is the bundle with total space given by the quotient of $\mathfrak{t} \times T^\vee \times \mathbb{C}$ under the action of $\Gamma$ defined by $\gamma(x, \chi, z) = (\gamma + x, \chi, \chi(\gamma)z)$. The projection onto the base $T \times T^\vee$ maps the $\Gamma$ orbit of $(x, \chi, z)$ to the $\Gamma$ orbit of $(x, \chi)$. Here we are identifying elements of $T^\vee$ with characters on $\Gamma$. We note that the bundle is $W$-equivariant with respect to the diagonal action of $W$ on $\mathfrak{t} \times T^\vee$, hence it defines an element in $W$-equivariant K-theory allowing it to play the role of the element $\mathcal{P}$ in our Poincaré duality.

To place this in the language of *KK*-theory we consider sections of this bundle, which are given by functions $\sigma : \mathfrak{t} \times T^\vee \to \mathbb{C}$ such that $\sigma(\gamma + x, \chi) = \chi(\gamma)\sigma(x, \chi)$. They naturally form a module over $C(T \times T^\vee)$ and given two such sections we define $\langle \sigma_1, \sigma_2 \rangle = \overline{\sigma}_1 \sigma_2$. We note that this is a $\Gamma$ periodic function in the first variable, hence the inner product takes values in $C(T \times T^\vee)$, giving the space of sections the structure of a Hilbert module.

We will now give an alternative construction of this Hilbert module. Let $C_c(\mathfrak{t})$ denote the space of continuous compactly supported functions on $\mathfrak{t}$ and equip this with a $C(T)\otimes C(T^\vee)$-valued inner product defined by

$$\langle \phi_1, \phi_2\rangle(x,\eta) = \sum_{\alpha,\beta\in\Gamma} \overline{\phi_1(x-\alpha)}\phi_2(x-\beta)e^{2\pi i\langle \eta,\beta-\alpha\rangle}.$$

We remark that the support condition ensures that this is a finite sum, and that it is easy to check that $\langle\phi_1,\phi_2\rangle(x,\eta)$ is $\Gamma$-periodic in $x$ and $\Gamma^\vee$-periodic in $\eta$.

The space $C_c(\mathfrak{t})$ has a $C(T)\otimes\mathbb{C}[\Gamma]$-module structure

$$(\phi\cdot(f\otimes[\gamma])) = \phi(x+\gamma)\tilde{f}(x),$$

where we view the function $f$ in $C(T)$ as a $\Gamma$-periodic function $\tilde{f}$ on $\mathfrak{t}$.

Completing $C_c(\mathfrak{t})$ with respect to the inner product norm, the module structure extends by continuity to give $\overline{C_c(\mathfrak{t})}$ the structure of a $C(T)\widehat{\otimes}C^*(\Gamma)\cong C(T)\widehat{\otimes}C(T^\vee)$ Hilbert module. We denote this Hilbert module by $\mathcal{E}$ and give this the trivial grading.

The group $W$ acts on $\mathfrak{t}$ and hence on $C_c(\mathfrak{t})$ by $(w\cdot\phi)(x)=\phi(w^{-1}x)$. We have

$$\big(w\cdot(\phi\cdot(f\otimes[\gamma]))\big)(x) = \phi(w^{-1}x+\gamma)\tilde{f}(w^{-1}x) = \big((w\cdot\phi)\cdot(w\cdot f\otimes[w\gamma])\big)(x)$$

so the action is compatible with the module structure. Now for the inner product we have

$$\begin{aligned}\langle w\cdot\phi_1, w\cdot\phi_2\rangle(x,\eta) &= \sum_{\alpha,\beta\in\Gamma}\overline{(w\cdot\phi_1)(x-\alpha)}(w\cdot\phi_2)(x-\beta)e^{2\pi i\langle\eta,\beta-\alpha\rangle}\\ &= \sum_{\alpha,\beta\in\Gamma}\overline{\phi_1(w^{-1}x-w^{-1}\alpha)}\phi_2(w^{-1}x-w^{-1}\beta)e^{2\pi i\langle\eta,\beta-\alpha\rangle}\\ &= \sum_{\alpha',\beta'\in\Gamma}\overline{\phi_1(w^{-1}x-\alpha')}\phi_2(w^{-1}x-\beta')e^{2\pi i\langle\eta,w(\beta'-\alpha')\rangle}\\ &= \sum_{\alpha',\beta'\in\Gamma}\overline{\phi_1(w^{-1}x-\alpha')}\phi_2(w^{-1}x-\beta')e^{2\pi i\langle w^{-1}\eta,\beta'-\alpha'\rangle}\\ &= (w\cdot\langle\phi_1,\phi_2\rangle)(x,\eta).\end{aligned}$$

Hence $\mathcal{E}$ is a $W$-equivariant Hilbert module.

The identification of the module $\mathcal{E}$ with the sections of the Poincaré line bundle is given by the following analogue of the Fourier transform. For each element $\phi\in C_c(\mathfrak{t})$ set

$$\sigma(x,\chi) = \sum_{\gamma\in\Gamma}\phi(x-\gamma)\chi(\gamma).$$

It is routine to verify that $\sigma(x+\delta,\chi) = \chi(\delta)\sigma(x,\chi)$ hence $\sigma$ is a section of the Poincaré line bundle, and that the $W$ action on $C_c(\mathfrak{t})$ corresponds precisely to the $W$ action on the bundle.

**Theorem 3.2.** *The triple* $(\mathcal{E}, 1, 0)$, *where* $1$ *denotes the identity representation of* $\mathbb{C}$ *on* $\mathcal{E}$, *is a W-equivariant Kasparov triple defining an element* $\mathcal{P}$ *in*

$$KK_W\big(\mathbb{C}, C(T)\,\widehat{\otimes}\, C(T^\vee)\big).$$

We remark that there is a connection with Fourier–Mukai duality. We recall that Fourier–Mukai duality is given by the map

$$\mathfrak{x} \mapsto p_{2*}(p_1^*\mathfrak{x} \otimes \mathcal{P}),$$

where $p_1, p_2$ are the projections of $T \times T^\vee$ onto the first and second factors. From the point of view of $K$-theory the subtlety is to interpret the wrong-way map $p_{2*}$. This should give a map from the $W$-equivariant $K$-theory of $T \times T^\vee$ to the $W$ equivariant $K$-theory of $T^\vee$, but to make this well defined we must twist by the Clifford algebra $\mathcal{C}\ell(\mathfrak{t})$. Specifically we can take

$$p_{2*} := [D] \otimes 1_{C(T^\vee)} \in KK_W\big(C(T \times T^\vee) \otimes \mathcal{C}\ell(\mathfrak{t}), C(T^\vee)\big),$$

where $[D]$ is the Dirac class in $KK_W(C(T) \otimes \mathcal{C}\ell(\mathfrak{t}), \mathbb{C})$. The Fourier–Mukai map is then given by taking the Kasparov product with the element $p_1^*\mathcal{P} i^* p_{2*} = \mathcal{P} p_1^* i^* p_{2*}$ where $i$ is the diagonal inclusion of $T \times T^\vee$ into $(T \times T^\vee)^2$. We note that $p_1^* i^* p_{2*}$ is the tensor product of Kasparov's Poincaré duality element for $T$ (given by its Dirac element) with the identity on $C(T^\vee)$.

**3B.** ***Construction of the element*** $\mathcal{Q}$ ***in*** $KK_W(C(T^\vee)\,\widehat{\otimes}\, C(T), \mathbb{C})$**.** We consider the differential operator $Q_0$ on $\mathfrak{t}$ with coefficients in the Clifford algebra $\mathcal{C}\ell(\mathfrak{t} \times \mathfrak{t}^*)$ defined using Einstein summation convention by

$$Q_0 = \frac{\partial}{\partial y^j} \otimes \varepsilon^j - 2\pi i y^j \otimes \boldsymbol{e}_j.$$

Here $\left\{\boldsymbol{e}_j = \frac{\partial}{\partial y^j}\right\}$ is an orthonormal basis for $\mathfrak{t}$, $\{\varepsilon^j\}$ denotes the dual basis of $\mathfrak{t}^*$ and we regard these as generators of the Clifford algebra $\mathcal{C}\ell(\mathfrak{t} \times \mathfrak{t}^*)$.

We view $Q_0$ as an unbounded operator on the Hilbert space $L^2(\mathfrak{t})\,\widehat{\otimes}\,\mathcal{S}$, where $\mathcal{S}$ denotes the space of spinors $\mathcal{S} = \mathcal{C}\ell(\mathfrak{t} \times \mathfrak{t}^*)P$ with $P$ the projection $\prod_j \frac{1}{2}(1 - i\boldsymbol{e}_j\varepsilon^j)$ in the Clifford algebra. (The space $\mathcal{S}$ is naturally equipped with a representation of $\mathcal{C}\ell(\mathfrak{t} \times \mathfrak{t}^*)$ by left multiplication inducing the action of $Q_0$.)

The subtlety in constructing an element in equivariant $KK$-theory is the need to ensure that $P$ is $W$-invariant with respect to the diagonal action of $W$ on $\mathfrak{t} \times \mathfrak{t}^*$ and hence that the action of $W$ on $\mathcal{C}\ell(\mathfrak{t} \times \mathfrak{t}^*)$ restricts to a representation on $\mathcal{S}$. The corner algebra $P\mathcal{C}\ell(\mathfrak{t}\times\mathfrak{t}^*)P$ is $\mathbb{C}P$, which we identify with $\mathbb{C}$, and this gives $\mathcal{S}$ a canonical inner product given by $\langle aP, bP\rangle = Pa^*bP$.

As a simple example consider the 1-dimensional case. Here $\mathcal{C}\ell(\mathfrak{t}\times\mathfrak{t}^*) = M_2(\mathbb{C})$ and the two generators are $\boldsymbol{e}_1 = \left(\begin{smallmatrix} 0 & i \\ i & 0\end{smallmatrix}\right)$ and $\varepsilon^1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$. The projection $P$ is therefore $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0\end{smallmatrix}\right)$ so $\mathcal{S}$ is the space of matrices of the form $\left(\begin{smallmatrix} * & 0 \\ * & 0\end{smallmatrix}\right)$ and the operator is

$$Q_0 = \begin{pmatrix} 0 & -\frac{\partial}{\partial y^1} + 2\pi y^1 \\ \frac{\partial}{\partial y^1} + 2\pi y^1 & 0 \end{pmatrix}.$$

The off-diagonal elements are of course the 1-dimensional annihilation and creation operators.

For the general case we must now construct a representation of $C(T^\vee)\mathbin{\widehat{\otimes}} C(T)$ on $L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}$. It suffices to define commuting representations of $C(T^\vee)\mathbin{\widehat{\otimes}} 1$ and $1\mathbin{\widehat{\otimes}} C(T)$. The representation of $C(T)$ is the usual pointwise multiplication on $L^2(\mathfrak{t})$ viewing elements of $C(T)$ as $\Gamma$-periodic functions on $\mathfrak{t}$. The representation of $C(T^\vee)$ involves the action of $\Gamma$ on $\mathfrak{t}$.

For $a$ an affine isometry of $\mathfrak{t}$, let $L_a$ be the operator on $L^2(\mathfrak{t})$ induced by the action of $a$ on $\mathfrak{t}$:

$$(L_a\xi)(y) = \xi(a^{-1}\cdot y).$$

For the function $\eta\mapsto e^{2\pi i\langle\eta,\gamma\rangle}$ in $C(T^\vee)$ we define

$$\rho(e^{2\pi i\langle\eta,\gamma\rangle}) = L_\gamma\otimes 1_{\mathcal{S}}.$$

Identifying $C(T^\vee)$ with $C^*(\Gamma)$ and identifying $L^2(\mathfrak{t})$ with $\ell^2(\Gamma)\otimes L^2(X)$, where $X$ is a fundamental domain for the action of $\Gamma$, the representation of the algebra is given by the left regular representation on $\ell^2(\Gamma)$.

Consider the commutators of $Q_0$ with the representation $\rho$. For $f\in C(T)$, the operator $\rho(f)$ commutes exactly with the second term $2\pi i y^j\otimes\boldsymbol{e}_j$ in $Q_0$, while, for $f$ smooth, the commutator of $\rho(f)$ with $\frac{\partial}{\partial y^j}\otimes\varepsilon^j$ is given by the bounded operator $\frac{\partial f}{\partial y^j}\otimes\varepsilon^j$. Now for the function $\eta\mapsto e^{2\pi i\langle\eta,\gamma\rangle}$ in $C(T^\vee)$ we have $\rho(e^{2\pi i\langle\eta,\gamma\rangle}) = L_\gamma\otimes 1_{\mathcal{S}}$. This commutes exactly with the differential term of the operator, while

$$L_\gamma(2\pi i y^j)L_\gamma^* = 2\pi i(y^j - \gamma^j)$$

hence the commutator $[L_\gamma\otimes 1_{\mathcal{S}}, 2\pi i y^j\otimes\boldsymbol{e}_j]$ is again bounded.

We have verified that $Q_0$ commutes with the representation $\rho$ modulo bounded operators, on a dense subalgebra of $C(T^\vee)\mathbin{\widehat{\otimes}} C(T)$. Thus to show that the triple

$$(L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}, \rho, Q_0)$$

is an unbounded Kasparov triple it remains to prove the following:

**Theorem 3.3.** *The operator $Q_0$ has compact resolvent. It has a 1-dimensional kernel with even grading.*

*Proof.* In the following argument we will *not* use summation convention. We consider the following operators on $L^2(\mathfrak{t}) \mathbin{\widehat{\otimes}} \mathcal{S}$:

$$p_j = \frac{\partial}{\partial y^j} \otimes \varepsilon^j, \qquad x_j = -2\pi i y^j \otimes \boldsymbol{e}_j,$$
$$q_j = \tfrac{1}{2}(1 + 1 \otimes i\boldsymbol{e}_j \varepsilon^j), \quad A_j = \frac{1}{2\sqrt{\pi}}(p_j + x_j).$$

Since $A_j$ anticommutes with $1 \otimes i\boldsymbol{e}_j\varepsilon^j$ we have $q_j A_j = A_j(1 - q_j)$, hence we can think of $A_j$ as an off-diagonal matrix with respect to $q_j$. We write $A_j$ as $a_j + a_j^*$, where $a_j = q_j A_j = A_j(1 - q_j)$ and hence $a_j^* = A_j q_j = (1 - q_j)A_j$. We think of $a_j^*$ and $a_j$ as creation and annihilation operators respectively and we define a number operator $N_j = a_j^* a_j$. The involution $i\varepsilon^j$ intertwines $q_j$ with $1 - q_j$. We define $A_j'$, $N_j'$ to be the conjugates of $A_j$, $N_j$ respectively by $i\varepsilon^j$. Note that

$$A_j' = \frac{1}{2\sqrt{\pi}}(p_j - x_j)$$

and hence

$$A_j^2 = (A_j')^2 + 2\frac{1}{4\pi}[x_j, p_j] = (A_j')^2 + 1 \otimes i\boldsymbol{e}_j\varepsilon^j.$$

We have

$$N_j' = A_j'(1 - q_j)A_j' = q_j(A_j')^2.$$

Thus

$$a_j a_j^* = q_j A_j^2 q_j = q_j A_j^2 = q_j(A_j')^2 + q_j(1 \otimes i\boldsymbol{e}_j\varepsilon^j) = N_j' + q_j.$$

Hence the spectrum of $a_j a_j^*$ (viewed as an operator on the range of $q_j$) is the spectrum of $N_j'$ shifted by 1. However $N_j'$ is conjugate to $N_j = a_j^* a_j$ so we conclude that

$$\mathrm{Sp}(a_j a_j^*) = \mathrm{Sp}(a_j^* a_j) + 1.$$

But $\mathrm{Sp}(a_j a_j^*) \setminus \{0\} = \mathrm{Sp}(a_j^* a_j) \setminus \{0\}$ so we conclude that the spectrum is

$$\mathrm{Sp}(a_j^* a_j) = \{0, 1, 2, \dots\} \quad \text{while} \quad \mathrm{Sp}(a_j a_j^*) = \{1, 2, \dots\}.$$

Now since the operators $A_j$ pairwise gradedly commute we have

$$Q_0^2 = 4\pi \sum_j A_j^2 = 4\pi \sum_j a_j^* a_j + a_j a_j^*$$

and noting that the summands commute we see that $Q_0^2$ has discrete spectrum. To show that $(1 + Q_0^2)^{-1}$ is compact, it remains to verify that $\ker Q_0$ is finite dimensional (and hence that all eigenspaces are finite dimensional). We have

$$\ker Q_0 = \ker Q_0^2 = \bigcap_j \ker A_j^2 = \bigcap_j \ker A_j.$$

Multiplying the differential equation $(p_j+x_j)f=0$ by $-\exp(\pi(y^j)^2\otimes i\varepsilon^j\boldsymbol{e}_j)\varepsilon^j$ we see that the kernel of $A_j$ is the space of solutions of the differential equation

$$\frac{\partial}{\partial y^j}\big(\exp(\pi(y^j)^2\otimes i\varepsilon^j\boldsymbol{e}_j)f\big)=0$$

whence for $f$ in the kernel we have

$$f(y^1,\dots,y^n)=\exp(-\pi(y^j)^2\otimes i\varepsilon^j\boldsymbol{e}_j)f(y^1,\dots,y^{j-1},0,y^{j+1},\dots,y^n).$$

Since the solutions must be square integrable the values of $f$ must lie in the $+1$ eigenspace of the involution $i\varepsilon^j\boldsymbol{e}_j$, that is, the range of the projection $1-q_j$. On this subspace the operator $\exp(-\pi(y^j)^2\otimes i\varepsilon^j\boldsymbol{e}_j)$ reduces to $e^{-\pi(y^j)^2}(1-q_j)$. Since the kernel of $Q_0$ is the intersection of the kernels of the operators $A_j$ an element of the kernel must have the form

$$f(y)=e^{-\pi|y|^2}\prod_j(1-q_j)f(0)$$

so the kernel is 1-dimensional. Indeed the product $\prod_j(1-q_j)$ is the projection $P$ used to define the space of spinors $\mathcal{S}=\mathcal{C}\ell(\mathfrak{t}\times\mathfrak{t}^*)P$, and hence $\prod_j(1-q_j)f(0)$ lies in the 1-dimensional space $P\mathcal{S}=P\mathcal{C}\ell(\mathfrak{t}\times\mathfrak{t}^*)P$ which has even grading. $\square$

We have shown that $(L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S},\rho,Q_0)$ defines an unbounded Kasparov triple. It remains to establish $W$-equivariance.

Let $V$ be a finite dimensional vector space and equip $V\otimes V^*$ with the natural diagonal action of $\mathrm{GL}(V)$. If $V$ is equipped with a nondegenerate symmetric bilinear form $g$ then we can form the Clifford algebra $\mathcal{C}\ell(V)$ and dually $\mathcal{C}\ell(V^*)$. The subgroup $O(g)$ of $\mathrm{GL}(V)$, consisting of those elements preserving $g$, acts diagonally on $\mathcal{C}\ell(V)\mathbin{\widehat{\otimes}}\mathcal{C}\ell(V^*)$, which we identify with $\mathcal{C}\ell(V\times V^*)$.

We say that an element $a$ of $\mathcal{C}\ell(V\times V^*)$ is *symmetric* if there exists a $g$-orthonormal[1] basis $\{\boldsymbol{e}_j : j=1,\dots,n\}$ with dual basis $\{\varepsilon^j : j=1,\dots,n\}$ such that $a$ can be written as $p(\boldsymbol{e}_1\varepsilon^1,\dots,\boldsymbol{e}_n\varepsilon^n)$ where $p(x_1,\dots,x_n)$ is a symmetric polynomial.

**Proposition 3.4.** *For any basis $\{\boldsymbol{e}_j\}$ of $V$ with dual basis $\{\varepsilon^j\}$ for $V^*$, the Einstein sum $\boldsymbol{e}_j\otimes\varepsilon^j$ in $V\otimes V^*$ is $\mathrm{GL}(V)$-invariant.*

*Suppose moreover that $V$ is equipped with a nondegenerate symmetric bilinear form $g$ and that the underlying field has characteristic zero. Then every symmetric element of $\mathcal{C}\ell(V)\mathbin{\widehat{\otimes}}\mathcal{C}\ell(V^*)\cong\mathcal{C}\ell(V\times V^*)$ is $O(g)$-invariant.*

*Proof.* Identifying $V\otimes V^*$ with endomorphisms of $V$ in the natural way, the action of $\mathrm{GL}(V)$ is the action by conjugation and $\boldsymbol{e}_j\otimes\varepsilon^j$ is the identity, which is invariant under conjugation.

[1]We say that $\{\boldsymbol{e}_j\}$ is $g$-orthonormal if $g_{jk}=\pm\delta_{jk}$ for each $j,k$.

For the second part, over a field of characteristic zero the symmetric polynomials are generated by power sum symmetric polynomials $p(x_1, \dots, x_n) = x_1^k + \cdots + x_n^k$, so it suffices to consider

$$\begin{aligned} p(\boldsymbol{e}_1\varepsilon^1, \dots, \boldsymbol{e}_n\varepsilon^n) &= (\boldsymbol{e}_1\varepsilon^1)^k + \cdots + (\boldsymbol{e}_n\varepsilon^n)^k \\ &= (-1)^{k(k-1)/2}\big((\boldsymbol{e}_1)^k(\varepsilon^1)^k + \cdots + (\boldsymbol{e}_n)^k(\varepsilon^n)^k\big). \end{aligned}$$

When $k$ is even, writing $(\boldsymbol{e}_j)^k = (\boldsymbol{e}_j^2)^{k/2} = (g_{jj})^{k/2}$ and similarly $(\varepsilon^j)^k = (g^{jj})^{k/2}$, we see that each term $(\boldsymbol{e}_j)^k(\varepsilon^j)^k$ is 1 since $g_{jj} = g^{jj} = \pm 1$ for an orthonormal basis. Thus $p(\boldsymbol{e}_1\varepsilon^1, \dots, \boldsymbol{e}_n\varepsilon^n) = n(-1)^{k(k-1)/2}$, which is invariant.

Similarly when $k$ is odd we get $(\boldsymbol{e}_j)^k(\varepsilon^j)^k = \boldsymbol{e}_j\varepsilon^j$ so

$$p(\boldsymbol{e}_1\varepsilon^1, \dots, \boldsymbol{e}_n\varepsilon^n) = (-1)^{k(k-1)/2}(\boldsymbol{e}_1\varepsilon^1 + \cdots + \boldsymbol{e}_n\varepsilon^n).$$

As the sum $\boldsymbol{e}_j \otimes \varepsilon^j$ in $V \otimes V^*$ is invariant under $\mathrm{GL}(V)$, it is in particular invariant under $O(g)$, and hence the sum $\boldsymbol{e}_j\varepsilon^j$ is $O(g)$-invariant in the Clifford algebra. □

Returning to our construction, the projection $P$ is a symmetric element of the Clifford algebra and hence is $W$-invariant by Proposition 3.4. It follows that $\mathcal{S}$ carries a representation of $W$. The space $L^2(\mathfrak{t})$ also carries a representation of $W$ given by the action of $W$ on $\mathfrak{t}$ and we equip $L^2(\mathfrak{t}) \mathbin{\widehat{\otimes}} \mathcal{S}$ with the diagonal action of $W$.

To verify that the representation $\rho$ is $W$-equivariant it suffices to consider the representations of $C(T)$ and $C(T^\vee)$ separately. As the exponential map $\mathfrak{t} \to T$ is $W$-equivariant it is clear that the representation of $C(T)$ on $L^2(\mathfrak{t})$ by pointwise multiplication is $W$-equivariant.

For $e^{2\pi i\langle \eta, \gamma\rangle} \in C(T^\vee)$ we have $w \cdot (e^{2\pi i\langle \eta, \gamma\rangle}) = e^{2\pi i\langle w^{-1}\cdot\eta, \gamma\rangle} = e^{2\pi i\langle \eta, w\cdot\gamma\rangle}$ thus $\rho(w \cdot (e^{2\pi i\langle \eta, \gamma\rangle})) = L_{w\cdot\gamma} \otimes 1_{\mathcal{S}} = L_w L_\gamma L_{w^{-1}} \otimes 1_{\mathcal{S}}$. Thus the representation of $C(T^\vee)$ is also $W$-equivariant.

It remains to check that the operator $Q_0$ is $W$-equivariant. By definition

$$Q_0 = \frac{\partial}{\partial y^j} \otimes \varepsilon^j - 2\pi i y^j \otimes \boldsymbol{e}_j.$$

Now by Proposition 3.4 $\frac{\partial}{\partial y^j} \otimes \varepsilon^j = \boldsymbol{e}_j \otimes \varepsilon^j$ is a $\mathrm{GL}(\mathfrak{t})$-invariant element of $\mathfrak{t} \otimes \mathfrak{t}^*$ and so in particular it is $W$-invariant. Writing $y^j = \langle \varepsilon^j, y\rangle$ the $W$-invariance of the second term again follows from invariance of $\boldsymbol{e}_j \otimes \varepsilon^j$.

Hence we conclude the following:

**Theorem 3.5.** *The triple $(L^2(\mathfrak{t}) \mathbin{\widehat{\otimes}} \mathcal{S}, \rho, Q_0)$ constructed above defines an element $\mathcal{Q}$ of $KK_W(C(T^\vee) \mathbin{\widehat{\otimes}} C(T), \mathbb{C})$.*

**3C.** ***The Kasparov product*** $\mathcal{P} \otimes_{C(T^\vee)} \mathcal{Q}$**.** We will compute the Kasparov product of the Poincaré line bundle $\mathcal{P} \in KK_W(\mathbb{C}, C(T) \mathbin{\widehat{\otimes}} C(T^\vee))$ with our inverse $\mathcal{Q} \in KK_W(C(T^\vee) \mathbin{\widehat{\otimes}} C(T), \mathbb{C})$, where the product is taken over $C(T^\vee)$ (not $C(T) \mathbin{\widehat{\otimes}} C(T^\vee)$).

Recall that $\mathcal{P}$ is given by the Kasparov triple $(\mathcal{E}, 1, 0)$, where $\mathcal{E}$ is the completion of $C_c(\mathfrak{t})$ with the inner product

$$\langle \phi_1, \phi_2 \rangle (x, \eta) = \sum_{\alpha,\beta\in\Gamma} \overline{\phi_1(x-\alpha)}\phi_2(x-\beta)e^{2\pi i\langle \eta, \beta-\alpha\rangle}$$

in $C(T)\mathbin{\widehat{\otimes}} C(T^\vee)$. As above, $\mathcal{Q}$ is given by the triple $(L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}, \rho, Q_0)$.

To form the Kasparov product we must take that tensor product of $\mathcal{E}$ with $L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}$ over $C(T^\vee)$ and as the operator in the first triple is zero, the operator required for the Kasparov product can be any connection for $Q_0$.

We note that the representation $\rho$ is the identity on $\mathcal{S}$ and hence

$$\mathcal{E}\mathbin{\widehat{\otimes}}_{C(T^\vee)} (L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}) = (\mathcal{E}\mathbin{\widehat{\otimes}}_{C(T^\vee)} L^2(\mathfrak{t}))\mathbin{\widehat{\otimes}}\mathcal{S}.$$

Thus we can focus on identifying the tensor product $\mathcal{E}\mathbin{\widehat{\otimes}}_{C(T^\vee)} L^2(\mathfrak{t})$. By abuse of notation we will also let $\rho$ denote the representation of $C(T)\mathbin{\widehat{\otimes}} C(T^\vee)$ on $L^2(\mathfrak{t})$.

As we are taking the tensor product over $C(T^\vee)$, not over $C(T)\mathbin{\widehat{\otimes}} C(T^\vee)$, we are forming the Hilbert module

$$(\mathcal{E}\mathbin{\widehat{\otimes}} C(T))\mathbin{\widehat{\otimes}}_{C(T)\widehat{\otimes}C(T^\vee)\widehat{\otimes}C(T)} (C(T)\mathbin{\widehat{\otimes}} L^2(\mathfrak{t})),$$

however, since the algebra $C(T)$ is unital, it suffices to consider elementary tensors of the form $(\phi\otimes 1)\otimes(1\otimes\xi)$. Where there is no risk of confusion we will abbreviate these as $\phi\otimes\xi$.

Let $\phi_1, \phi_2 \in C_c(\mathfrak{t})$ and let $\xi_1, \xi_2$ be elements of $L^2(\mathfrak{t})$. Then

$$\langle \phi_1\otimes\xi_1, \phi_2\otimes\xi_2\rangle = \big\langle 1\otimes\xi_1, (1\otimes\rho)(\langle\phi_1,\phi_2\rangle\otimes 1)(1\otimes\xi_2)\big\rangle.$$

The operator $(1\otimes\rho)(\langle\phi_1,\phi_2\rangle\otimes 1)$ corresponds to a field of operators

$$\begin{aligned}(1\otimes\rho)(\langle\phi_1,\phi_2\rangle\otimes 1)(x) &= \sum_{\alpha,\beta\in\Gamma} \overline{\phi_1(x-\alpha)}\phi_2(x-\beta)\otimes\rho(e^{2\pi i\langle\eta,\beta-\alpha\rangle}\otimes 1)\\ &= \sum_{\alpha,\beta\in\Gamma} \overline{\phi_1(x-\alpha)}\phi_2(x-\beta)\otimes L_\alpha^* L_\beta\end{aligned}$$

and so

$$\begin{aligned}\langle \phi_1\otimes\xi_1, \phi_2\otimes\xi_2\rangle(x) &= \sum_{\alpha,\beta\in\Gamma} \overline{\phi_1(x-\alpha)}\phi_2(x-\beta)\langle L_\alpha\xi_1, L_\beta\xi_2\rangle\\ &= \Big\langle \sum_{\alpha\in\Gamma}\phi_1(x-\alpha)L_\alpha\xi_1, \sum_{\beta\in\Gamma}\phi_2(x-\beta)L_\beta\xi_2\Big\rangle.\end{aligned}$$

We note that $x\mapsto \sum_{\alpha\in\Gamma}\phi_1(x-\alpha)L_\alpha\xi_1$ is a continuous $\Gamma$-equivariant (and hence bounded) function from $\mathfrak{t}$ to $L^2(\mathfrak{t})$. Let $C(\mathfrak{t}, L^2(\mathfrak{t}))^\Gamma$ denote the space of such functions equipped with the $C(T)$-module structure of pointwise multiplication in the first variable and give it the pointwise inner product $\langle g_1, g_2\rangle(x) = \langle g_1(x), g_2(x)\rangle$.

We remark that equivariance implies this inner product is a $\Gamma$-periodic function on $\mathfrak{t}$.

The above calculation shows that $\mathcal{E}\,\widehat{\otimes}_{C(T^\vee)}\,L^2(\mathfrak{t})$ is mapped isometrically into $C(\mathfrak{t}, L^2(\mathfrak{t}))^\Gamma$ via the map

$$\phi \otimes \xi \mapsto \sum_{\alpha\in\Gamma} \phi(x-\alpha) L_\alpha \xi.$$

Moreover this map is surjective. To see this, note that if $\phi$ is supported inside a single fundamental domain then for $x$ in that fundamental domain we obtain the function $\phi(x)\xi$. This is extended by equivariance to a function on $\mathfrak{t}$, and using a partition of unity one can approximate an arbitrary element of $C(\mathfrak{t}, L^2(\mathfrak{t}))^\Gamma$ by sums of functions of this form.

We now remark that $C(\mathfrak{t}, L^2(\mathfrak{t}))^\Gamma$ is in fact isomorphic to the Hilbert module $C(T, L^2(\mathfrak{t}))$ via a change of variables. Given $g \in C(\mathfrak{t}, L^2(\mathfrak{t}))^\Gamma$, let $\tilde{h}(x) = L_{-x} g(x)$. The $\Gamma$-equivariance of $g$ ensures that $g(\gamma + x) = L_\gamma g(x)$, whence

$$\tilde{h}(\gamma + x) = L_{-x-\gamma} g(\gamma + x) = L_{-x-\gamma} L_\gamma g(x) = L_{-x} g(x) = \tilde{h}(x).$$

As $\tilde{h}$ is a $\Gamma$-periodic function from $\mathfrak{t}$ to $L^2(\mathfrak{t})$, we identify it via the exponential map with the continuous function $h$ from $T$ to $L^2(\mathfrak{t})$ such that $\tilde{h}(x) = h(\exp(x))$. Hence $g \mapsto h$ defines the isomorphism $C(\mathfrak{t}, L^2(\mathfrak{t}))^\Gamma \cong C(T, L^2(\mathfrak{t}))$.

We now state the following theorem.

**Theorem 3.6.** *There is an isomorphism from the Hilbert module $\mathcal{E}\widehat{\otimes}_{C(T^\vee)}(L^2(\mathfrak{t})\widehat{\otimes}\mathcal{S})$ to $C(T, L^2(\mathfrak{t})\,\widehat{\otimes}\,\mathcal{S})$ given by the map*

$$\phi \otimes (\xi \otimes s) \mapsto \sum_{\alpha\in\Gamma} \phi(x-\alpha) L_{\alpha-x} \xi \otimes s.$$

*The representation of $C(T)$ on $L^2(\mathfrak{t})$ induces a representation, $\sigma$, of $C(T)$ on $C(T, L^2(\mathfrak{t})\,\widehat{\otimes}\,\mathcal{S})$ defined by*

$$[\sigma(f)h](\exp(x), y) = f(\exp(x+y)) h(\exp(x), y).$$

*Here the notation $h(\exp(x), y)$ denotes the value at the point $y \in \mathfrak{t}$ of $h(\exp(x)) \in L^2(\mathfrak{t})\,\widehat{\otimes}\,\mathcal{S}$.*

*Proof.* We recall that $\mathcal{E}\,\widehat{\otimes}_{C(T^\vee)}\,(L^2(\mathfrak{t})\widehat{\otimes}\mathcal{S})$ is isomorphic to $(\mathcal{E}\,\widehat{\otimes}_{C(T^\vee)}\,L^2(\mathfrak{t}))\,\widehat{\otimes}\,\mathcal{S}$ and we have established that $\mathcal{E}\,\widehat{\otimes}_{C(T^\vee)}\,L^2(\mathfrak{t}) \cong C(T, L^2(\mathfrak{t}))$. This provides the claimed isomorphism.

It remains to identify the representation. Given $f \in C(T)$ let $\tilde{f}(x) = f(\exp(x))$ denote the corresponding periodic function on $\mathfrak{t}$. By definition the representation of $C(T)$ on $\mathcal{E}\,\widehat{\otimes}_{C(T^\vee)}\,(L^2(\mathfrak{t})\,\widehat{\otimes}\,\mathcal{S})$ takes $\phi \otimes \xi \otimes s$ to $\phi \otimes \tilde{f}\xi \otimes s$. This is mapped under the isomorphism to the $\Gamma$-periodic function on $\mathfrak{t}$ whose value at $x$ is

$$\sum_{\alpha\in\Gamma}\phi(x-\alpha)L_{\alpha-x}(\tilde{f}\xi)\otimes s\in L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}.$$

Evaluating this element of $L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}$ at a point $y\in\mathfrak{t}$ we have

$$\sum_{\alpha\in\Gamma}\phi(x-\alpha)\tilde{f}(x-\alpha+y)\xi(x-\alpha+y)\otimes s=\tilde{f}(x+y)\sum_{\alpha\in\Gamma}\phi(x-\alpha)[L_{\alpha-x}\xi](y)\otimes s$$

by $\Gamma$-periodicity of $\tilde{f}$. Thus $\sigma(f)$ pointwise multiplies the image of $\phi\otimes\xi\otimes s$ in $C(T,L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S})$ by $\tilde{f}(x+y)=f(\exp(x+y))$ as claimed. □

We now define an operator $\boldsymbol{Q}$ on $C(T,L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S})$ by

$$(\boldsymbol{Q}h)(\exp(x))=Q_0(h(\exp(x)))$$

for $h\in C(T,L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S})$.

**Theorem 3.7.** *The unbounded operator $\boldsymbol{Q}$ is a connection for $Q_0$ in the sense that the bounded operator $\boldsymbol{F}=\boldsymbol{Q}(1+\boldsymbol{Q}^2)^{-\frac{1}{2}}$ is a connection for $F_0=Q_0(1+Q_0^2)^{-\frac{1}{2}}$, after making the identification of Hilbert modules as in Theorem 3.6.*

*Proof.* Let $Q_x=(L_x\otimes 1_{\mathcal{S}})Q_0(L_{-x}\otimes 1_{\mathcal{S}})$ and correspondingly define

$$F_x=Q_x(1+Q_x^2)^{-\frac{1}{2}}=(L_x\otimes 1_{\mathcal{S}})F_0(L_{-x}\otimes 1_{\mathcal{S}}).$$

The commutators $[L_x\otimes 1_{\mathcal{S}},Q_0]$ are bounded (the argument is the same as for $[L_\gamma\otimes 1_{\mathcal{S}},Q_0]$ in Section 3B). It follows (in the spirit of Baaj and Julg [1983]) that the commutators $[L_x\otimes 1_{\mathcal{S}},F_0]$ are compact. Thus $F_x-F_0$ is a compact operator for all $x\in\mathfrak{t}$.

To show that $\boldsymbol{F}$ is a connection for $F_0$ we must show that for $\phi\in\mathcal{E}$, the diagram

$$\begin{tikzcd}
L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S} \arrow[r, "F_0"] \arrow[d, "\phi\otimes"'] & L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S} \arrow[d, "\phi\otimes"'] \\
\mathcal{E}\otimes L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S} \arrow[d, "\cong"'] & \mathcal{E}\otimes L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S} \arrow[d, "\cong"'] \\
C(T,L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}) \arrow[r, "\boldsymbol{F}"'] & C(T,L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S})
\end{tikzcd}$$

commutes modulo compact operators.

Following the diagram around the right-hand side we have

$$\xi\otimes s\mapsto\sum_{\alpha\in\Gamma}\phi(x-\alpha)(L_{\alpha-x}\otimes 1_{\mathcal{S}})F_0(\xi\otimes s)$$

while following the left-hand side we have

$$\boldsymbol{F}\left[\sum_{\alpha\in\Gamma}\phi(x-\alpha)(L_{\alpha-x}\otimes 1_{\mathcal{S}})(\xi\otimes s)\right]=\sum_{\alpha\in\Gamma}\phi(x-\alpha)F_0(L_{\alpha-x}\otimes 1_{\mathcal{S}})(\xi\otimes s).$$

As $[F_0, L_{\alpha-x}\otimes 1_{\mathcal{S}}]$ is a compact operator for each $x$ and the sum is finite for each $x$, the difference between the two paths around the diagram is a function from $T$ to compact operators on $L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}$. It is thus a compact operator from the Hilbert space $L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}$ to the Hilbert module $C(T, L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S})$ as required. □

**Theorem 3.8.** *The Kasparov product $\mathcal{P}\otimes_{C(T^\vee)}\mathcal{Q}$ is the identity $1_{C(T)}$ in*

$$KK_W(C(T), C(T)).$$

*Proof.* We define a homotopy of representations of $C(T)$ on $C(T, L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S})$ by

$$[\sigma_\lambda(f)h](\exp(x), y)=f(\exp(x+\lambda y))h(\exp(x), y)$$

and note that $\sigma_1=\sigma$ while $\sigma_0$ is simply the representation of $C(T)$ on $C(T, L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S})$ by pointwise multiplication of functions on $T$. It is easy to see that these representations are $W$-equivariant.

Let $f$ be a smooth function on $T$ and let $h\in C(T, L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S})$. Let $\tilde{f}(x)=f(\exp(x))$ and let $\tilde{h}(x, y)=h(\exp(x), y)$. Then

$$\begin{aligned}
&\big([\boldsymbol{Q}, \sigma_\lambda(f)]h\big)(\exp(x), y)\\
&\quad=\left[\frac{\partial}{\partial y^j}(\varepsilon^j\tilde{f}(x+\lambda y)\tilde{h}(x, y))-2\pi i y^j\boldsymbol{e}_j\tilde{f}(x+\lambda y)\tilde{h}(x, y)\right]\\
&\qquad\qquad-\left[\tilde{f}(x+\lambda y)\frac{\partial}{\partial y^j}(\varepsilon^j\tilde{h}(x, y))-\tilde{f}(x+\lambda y)2\pi i y^j\boldsymbol{e}_j\tilde{h}(x, y)\right]\\
&\quad=\frac{\partial}{\partial y^j}(\tilde{f}(x+\lambda y))(\varepsilon^j\tilde{h}(x, y)).
\end{aligned}$$

For each $\lambda$ the operator $\boldsymbol{Q}$ thus commutes with the representation $\sigma_\lambda$ modulo bounded operators on a dense subalgebra of $C(T)$. Hence for each $\lambda$

$$\big(C(T, L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}), \sigma_\lambda, \boldsymbol{Q}\big)$$

defines an unbounded Kasparov triple.

This is true in particular for $\lambda=1$ and thus $(C(T, L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}), \sigma, \boldsymbol{Q})$ is a Kasparov triple so, as the operator in the triple $\mathcal{P}$ is zero while $\boldsymbol{Q}$ is a connection for $Q_0$, it follows that $\mathcal{P}\otimes_{C(T^\vee)}\mathcal{Q}=(C(T, L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}), \sigma, \boldsymbol{Q})$ in $KK_W(C(T), C(T))$.

Now applying the homotopy we have $\mathcal{P}\otimes_{C(T^\vee)}\mathcal{Q}=(C(T, L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S}), \sigma_0, \boldsymbol{Q})$. Since $\sigma_0$ commutes exactly with the operator $\boldsymbol{Q}$, the representation $\sigma_0$ respects the direct sum decomposition of $C(T, L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S})$ as $C(T, \ker(Q_0))\oplus C(T, \ker(Q_0)^\perp)$.

The operator $\boldsymbol{Q}$ is invertible on the second summand (and commutes with the representation) and hence the corresponding Kasparov triple

$$\big(C(T,\ker(Q_0)^{\perp}),\sigma_0|_{C(T,\ker(Q_0)^{\perp})},\boldsymbol{Q}|_{C(T,\ker(Q_0)^{\perp})}\big)$$

is zero in *KK*-theory.

We thus conclude that $\mathcal{P}\otimes_{C(T^{\vee})}\mathcal{Q}=(C(T,\ker(Q_0)),\sigma_0|_{C(T,\ker(Q_0))},0)$. Since $\ker Q_0$ is 1-dimensional (Theorem 3.3), the module $C(T,\ker(Q_0))$ is isomorphic to $C(T)$ and the restriction of $\sigma_0$ to this is the identity representation of $C(T)$ on itself. Thus $\mathcal{P}\otimes_{C(T^{\vee})}\mathcal{Q}=(C(T),1,0)=1_{C(T)}$. □

**3D.** ***The Kasparov product*** $\mathcal{P}\otimes_{C(T)}\mathcal{Q}$**.** We begin by considering the dual picture, which exchanges the roles of $T$ and $T^{\vee}$. There exist elements

$$\mathcal{Q}^{\vee}\in KK_W(C(T)\mathbin{\widehat{\otimes}}C(T^{\vee}),\mathbb{C})\quad\text{and}\quad\mathcal{P}^{\vee}\in KK_W(\mathbb{C},C(T^{\vee})\mathbin{\widehat{\otimes}}C(T))$$

for which the result of the previous section implies $\mathcal{P}^{\vee}\otimes_{C(T)}\mathcal{Q}^{\vee}=1_{C(T^{\vee})}$ in $KK_W(C(T^{\vee}),C(T^{\vee}))$.

We will show that there is an isomorphism

$$\theta:C(T^{\vee})\mathbin{\widehat{\otimes}}C(T)\to C(T)\mathbin{\widehat{\otimes}}C(T^{\vee})$$

such that $\mathcal{Q}=\theta^*\mathcal{Q}^{\vee}$ and $\mathcal{P}=\theta_*^{-1}\mathcal{P}^{\vee}$. This will imply that

$$\mathcal{P}\otimes_{C(T)}\mathcal{Q}=\mathcal{P}^{\vee}\otimes_{C(T)}\mathcal{Q}^{\vee}=1_{C(T^{\vee})}\text{ in }KK_W(C(T^{\vee}),C(T^{\vee}))$$

and hence will complete the proof of the Poincaré duality between $C(T)$ and $C(T^{\vee})$.

We recall that $\mathcal{Q}$ is represented by the (unbounded) Kasparov triple

$$(L^2(\mathfrak{t})\mathbin{\widehat{\otimes}}\mathcal{S},\rho,Q_0),$$

where $\mathcal{S}=\mathcal{C}\ell(\mathfrak{t}\times\mathfrak{t}^*)P$, for $P$ the projection $P=\prod_j\frac{1}{2}(1-i\boldsymbol{e}_j\varepsilon^j)$ and

$$Q_0=\frac{\partial}{\partial y^j}\otimes\varepsilon^j-2\pi i y^j\otimes\boldsymbol{e}_j.$$

For $\gamma\in\Gamma$, $\chi\in\Gamma^{\vee}$ and correspondingly $e^{2\pi i\langle\eta,\gamma\rangle}$ in $C(T^{\vee})$, $e^{2\pi i\langle\chi,x\rangle}$ in $C(T)$, the representation $\rho$ of $C(T^{\vee})\mathbin{\widehat{\otimes}}C(T)$ is defined by

$$\rho(e^{2\pi i\langle\eta,\gamma\rangle})(\xi\otimes s)=L_\gamma\xi\otimes s,\quad\text{and}\quad\rho(e^{2\pi i\langle\chi,x\rangle})(\xi\otimes s)=e^{2\pi i\langle\chi,x\rangle}\xi\otimes s.$$

By definition $\mathcal{Q}^{\vee}$ is represented by the triple $(L^2(\mathfrak{t}^*)\mathbin{\widehat{\otimes}}\mathcal{S}^{\vee},\rho^{\vee},Q_0^{\vee})$, where $\mathcal{S}^{\vee}=\mathcal{C}\ell(\mathfrak{t}^*\times\mathfrak{t})P^{\vee}$, for $P^{\vee}$ the projection $P^{\vee}=\prod_j\frac{1}{2}(1-i\varepsilon^j\boldsymbol{e}_j)$ and

$$Q_0^{\vee}=\frac{\partial}{\partial\eta_j}\otimes\boldsymbol{e}_j-2\pi i\eta_j\otimes\varepsilon^j.$$

For $\gamma \in \Gamma$, $\chi \in \Gamma^\vee$ and correspondingly $e^{2\pi i\langle \eta,\gamma\rangle}$ in $C(T^\vee)$, $e^{2\pi i\langle \chi,x\rangle}$ in $C(T)$, the representation $\rho^\vee$ of $C(T)\,\widehat{\otimes}\,C(T^\vee)$ is now defined by

$$\rho^\vee(e^{2\pi i\langle \chi,x\rangle})(\xi^\vee \otimes s^\vee) = L^\vee_\chi \xi^\vee \otimes s^\vee,$$
$$\rho^\vee(e^{2\pi i\langle \eta,\gamma\rangle})(\xi^\vee \otimes s^\vee) = e^{2\pi i\langle \eta,\gamma\rangle} \xi^\vee \otimes s^\vee.$$

Here $L^\vee_\chi$ denotes the translation action of $\chi \in \Gamma^\vee$ on $L^2(\mathfrak{t}^*)$.

In our notation, $\varepsilon^j$ is again an orthonormal basis for $\mathfrak{t}^*$ and $\boldsymbol{e}_j$ is an orthonormal basis for $\mathfrak{t}$. We can canonically identify $\mathcal{C}\ell(\mathfrak{t} \times \mathfrak{t}^*)$ with $\mathcal{C}\ell(\mathfrak{t}^* \times \mathfrak{t})$, and hence think of both $\mathcal{S}$ and $\mathcal{S}^\vee$ as subspaces of this algebra.

We can identify $L^2(\mathfrak{t})$ with $L^2(\mathfrak{t}^*)$ via the Fourier transform: let $\mathcal{F}: L^2(\mathfrak{t}) \to L^2(\mathfrak{t}^*)$ denote the Fourier transform isomorphism

$$[\mathcal{F}\xi](\eta) = \int_{\mathfrak{t}} \xi(y) e^{2\pi i\langle \eta,y\rangle}\, dy.$$

It is easy to see that this is $W$-equivariant.

To identify $\mathcal{S}$ with $\mathcal{S}^\vee$, let $u \in \mathcal{C}\ell(\mathfrak{t} \times \mathfrak{t}^*)$ be defined by $u = \varepsilon^1\varepsilon^2 \cdots \varepsilon^n$ when $n = \dim(\mathfrak{t})$ is even and $u = e_1 e_2 \cdots e_n$ when $n$ is odd.

**Lemma 3.9.** *Conjugation by $u$ defines a $W$-equivariant unitary isomorphism $\mathcal{U}$ : $\mathcal{S} \to \mathcal{S}^\vee$. For $a \in \mathcal{C}\ell(\mathfrak{t} \times \mathfrak{t}^*)$ (viewed as an operator on $\mathcal{S}$ by Clifford multiplication) $\mathcal{U}a\mathcal{U}^*$ is Clifford multiplication by $uau^*$ on $\mathcal{S}^\vee$ and in particular $\mathcal{U}\boldsymbol{e}_j\mathcal{U}^* = \boldsymbol{e}_j$, while $\mathcal{U}\varepsilon^j\mathcal{U}^* = -\varepsilon^j$.*

*Proof.* We first note that $u$ respectively commutes and anticommutes with $\boldsymbol{e}_j$, $\varepsilon^j$ (there being respectively an even or odd number of terms in $u$ which anticommute with $\boldsymbol{e}_j$, $\varepsilon^j$). It follows that $uPu^* = P^\vee$, hence conjugation by $u$ maps $\mathcal{S}$ to $\mathcal{S}^\vee$.

Denoting by $\pi : \mathbb{C}P \to \mathbb{C}$ the identification of $\mathbb{C}P$ with $\mathbb{C}$, the inner product on $\mathcal{S}$ is given by $\langle s_1, s_2\rangle = \pi(s_1^* s_2)$ while the inner product on $\mathcal{S}^\vee$ is given by $\langle s_1^\vee, s_2^\vee\rangle = \pi(u^*(s_1^\vee)^* s_2^\vee u)$. Thus

$$\langle usu^*, s^\vee\rangle = \pi(u^*(usu^*)^* s^\vee u) = \pi(s^* u^* s^\vee u) = \langle s, u^* s^\vee u\rangle$$

so $\mathcal{U}^*$ is conjugation by $u^*$ which inverts $\mathcal{U}$ establishing that $\mathcal{U}$ is unitary.

We now check that $\mathcal{U}$ is $W$-equivariant. In the case that $\mathfrak{t}$ is even-dimensional, we note that identifying $\mathcal{C}\ell(\mathfrak{t}^*)$ with the exterior algebra of $\mathfrak{t}^*$ (as a $W$-vector space), $u$ corresponds to the volume form on $\mathfrak{t}^*$ so $w \cdot u = \det(w)u$. Similarly in the odd dimensional case $u$ corresponds to the volume form on $\mathfrak{t}$ and again the action of $w$ on $u$ is multiplication by the determinant. Thus

$$w \cdot \mathcal{U}(s) = w \cdot (usu^*) = (w \cdot u)(w \cdot s)(w \cdot u^*) = \det(w)^2\, u(w \cdot s)u^* = \mathcal{U}(w \cdot s)$$

since $\det(w) = \pm 1$.

Finally for $s^\vee \in \mathcal{S}^\vee$ and $a \in \mathcal{C}\ell(\mathfrak{t} \times \mathfrak{t}^*)$ we have

$$\mathcal{U}a\mathcal{U}^* s^\vee = \mathcal{U}(au^* s^\vee u) = uau^* s^\vee$$

and hence $\mathcal{U}\boldsymbol{e}_j\mathcal{U}^* = u\boldsymbol{e}_j u^* = \boldsymbol{e}_j$, $\mathcal{U}\varepsilon^j\mathcal{U}^* = u\varepsilon^j u^* = -\varepsilon^j$. □

Since $\mathcal{F}\otimes\mathcal{U}$ is a $W$-equivariant unitary isomorphism from $L^2(\mathfrak{t})\widehat{\otimes}\mathcal{S}$ to $L^2(\mathfrak{t}^*)\widehat{\otimes}\mathcal{S}^\vee$, the triple $(L^2(\mathfrak{t})\,\widehat{\otimes}\,\mathcal{S}, \rho, Q_0)$ representing $\mathcal{Q}$ is isomorphic to the Kasparov triple

$$\big(L^2(\mathfrak{t}^*)\,\widehat{\otimes}\,\mathcal{S}^\vee, (\mathcal{F}\otimes\mathcal{U})\rho(\mathcal{F}^*\otimes\mathcal{U}^*), (\mathcal{F}\otimes u)Q_0(\mathcal{F}^*\otimes\mathcal{U}^*)\big).$$

**Theorem 3.10.** *Let $\theta : C(T^\vee)\,\widehat{\otimes}\,C(T) \to C(T)\,\widehat{\otimes}\,C(T^\vee)$ be defined by*

$$\theta(g\otimes f) = f\otimes(g\circ\zeta),$$

*where $\zeta$ is the involution on $T^\vee$ defined by $\zeta(\exp(\eta)) = \exp(-\eta)$. Then $\mathcal{Q} = \theta^*\mathcal{Q}^\vee$ in $KK_W(C(T^\vee)\,\widehat{\otimes}\,C(T), \mathbb{C})$.*

*Proof.* We will show that $\rho^\vee\circ\theta = (\mathcal{F}\otimes\mathcal{U})\rho(\mathcal{F}^*\otimes\mathcal{U}^*)$ and $(\mathcal{F}\otimes u)Q_0(\mathcal{F}^*\otimes\mathcal{U}^*) = Q_0^\vee$. We begin with the operator.

The operator $Q_0$ is given by

$$\frac{\partial}{\partial y^j}\otimes\varepsilon^j - 2\pi i y^j\otimes\boldsymbol{e}_j.$$

Conjugating the operator $\frac{\partial}{\partial y^j}$ by the Fourier transform we obtain the multiplication by $2\pi i\eta_j$, while conjugating $-2\pi i y^j$ by the Fourier transform we obtain the multiplication by $-2\pi i\big(\frac{i}{2\pi}\frac{\partial}{\partial\eta_j}\big) = \frac{\partial}{\partial\eta_j}$. Conjugation by $\mathcal{U}$ negates $\varepsilon^j$ and preserves $\boldsymbol{e}_j$ hence

$$(\mathcal{F}\otimes u)Q_0(\mathcal{F}^*\otimes\mathcal{U}^*) = 2\pi i\eta_j\otimes(-\varepsilon^j) + \frac{\partial}{\partial\eta_j}\otimes\boldsymbol{e}_j = Q_0^\vee.$$

For the representation, $\rho(e^{2\pi i\langle\chi,x\rangle})$ is multiplication by $e^{2\pi i\langle\chi,x\rangle}$ on $L^2(\mathfrak{t})$ (with the identity on $\mathcal{S}$) and conjugating by the Fourier transform we get the translation $L_\chi^\vee$, hence $(\mathcal{F}\otimes\mathcal{U})\rho(e^{2\pi i\langle\chi,x\rangle})(\mathcal{F}^*\otimes\mathcal{U}^*) = \rho^\vee(e^{2\pi i\langle\chi,x\rangle})$. On the other hand $\rho(e^{2\pi i\langle\eta,\gamma\rangle})$ is the translation $L_\gamma$ and Fourier transforming we get the multiplication by $e^{-2\pi i\langle\eta,\gamma\rangle}$. Thus $(\mathcal{F}\otimes\mathcal{U})\rho(e^{2\pi i\langle\eta,\gamma\rangle})(\mathcal{F}^*\otimes\mathcal{U}^*) = \rho^\vee(e^{2\pi i\langle-\eta,\gamma\rangle})$.

We conclude that $(\mathcal{F}\otimes\mathcal{U})\rho(\mathcal{F}^*\otimes\mathcal{U}^*) = \rho^\vee\circ\theta$ as required. □

**Theorem 3.11.** *The Kasparov product $\mathcal{P}\otimes_{C(T)}\mathcal{Q}$ is $1_{C(T^\vee)}$ in the Kasparov group $KK_W(C(T^\vee), C(T^\vee))$.*

*Proof.* We have $\mathcal{P}\otimes_{C(T^\vee)}\mathcal{Q} = 1_{C(T)}$ in $KK_W(C(T), C(T))$ by Theorem 3.8 while $\mathcal{P}^\vee\otimes_{C(T)}\mathcal{Q}^\vee = 1_{C(T^\vee)}$ in $KK_W(C(T^\vee), C(T^\vee))$ by Theorem 3.8 for the dual group.

By Theorem 3.10 we have $\mathcal{Q}^\vee = (\theta^{-1})^*\mathcal{Q}$, whence

$$1_{C(T^\vee)} = \mathcal{P}^\vee\otimes_{C(T)}\mathcal{Q}^\vee = (\theta^{-1})_*\mathcal{P}^\vee\otimes_{C(T)}\mathcal{Q}.$$

Let $\mathcal{P}' = (\theta^{-1})_* \mathcal{P}^\vee$ in $KK_W(\mathbb{C}, C(T) \mathbin{\widehat{\otimes}} C(T^\vee))$. Then

$$\mathcal{P} = \mathcal{P} \otimes_{C(T^\vee)} 1_{C(T^\vee)} = \mathcal{P} \otimes_{C(T^\vee)} (\mathcal{P}' \otimes_{C(T)} \mathcal{Q}).$$

By definition $\mathcal{P}' \otimes_{C(T)} \mathcal{Q} = (\mathcal{P}' \otimes 1_{C(T^\vee)}) \otimes_{C(T) \otimes C(T^\vee)} \mathcal{Q}$ and hence

$$\mathcal{P} = (\mathcal{P} \otimes \mathcal{P}') \otimes_{C(T^\vee) \widehat{\otimes} C(T)} \mathcal{Q}$$

by associativity of the Kasparov product. Here $\mathcal{P} \otimes \mathcal{P}'$ is the "external" product and lives in $KK_W(\mathbb{C}, C(T) \mathbin{\widehat{\otimes}} C(T) \mathbin{\widehat{\otimes}} C(T^\vee) \mathbin{\widehat{\otimes}} C(T^\vee))$, with $\mathcal{P}$ appearing in the first and last factors, and $\mathcal{P}'$ in the second and third. The product with $\mathcal{Q}$ is over the second and last factors. Similarly

$$\mathcal{P}' = \mathcal{P}' \otimes_{C(T)} (\mathcal{P} \otimes_{C(T^\vee)} \mathcal{Q}) = (\mathcal{P}' \otimes \mathcal{P}) \otimes_{C(T) \widehat{\otimes} C(T^\vee)} \mathcal{Q},$$

where $\mathcal{P}'$ now appears as the first and last factors and the product with $\mathcal{Q}$ is over the first and third factors. Up to reordering terms of the tensor product,

$$(\mathcal{P} \otimes \mathcal{P}') \otimes_{C(T^\vee) \widehat{\otimes} C(T)} \mathcal{Q} = (\mathcal{P}' \otimes \mathcal{P}) \otimes_{C(T) \widehat{\otimes} C(T^\vee)} \mathcal{Q}.$$

Thus (by commutativity of the external product) $\mathcal{P} = \mathcal{P}' = (\theta^{-1})_* \mathcal{P}^\vee$ and hence $\mathcal{P} \otimes_{C(T)} \mathcal{Q} = 1_{C(T^\vee)}$. □

**Corollary 3.12.** *The elements*

$$\mathcal{Q} \in KK_W(C(T^\vee) \mathbin{\widehat{\otimes}} C(T), \mathbb{C}) \quad \textit{and} \quad \mathcal{P} \in KK_W(\mathbb{C}, C(T) \mathbin{\widehat{\otimes}} C(T^\vee))$$

*exhibit a $W$-equivariant Poincaré duality between the algebras $C(T)$ and $C(T^\vee)$.*

## 4. Poincaré duality between $C_0(\mathfrak{t}) \rtimes (\Gamma \rtimes W)$ and $C_0(\mathfrak{t}^*) \rtimes (\Gamma^\vee \rtimes W)$

**4A.** ***Descent of Poincaré duality.*** For $W$ a group, a Poincaré duality between two $W$-$C^*$-algebras $A$, $B$ induces a natural family of isomorphisms

$$KK_W(A \mathbin{\widehat{\otimes}} D_1, D_2) \cong KK_W(D_1, B \mathbin{\widehat{\otimes}} D_2)$$

for $W$-$C^*$-algebras $D_1$, $D_2$. In other words the functor $A \widehat{\otimes}$ is left-adjoint to $B \widehat{\otimes}$ on the $KK_W$ category when there is a Poincaré duality from $A$ to $B$. (The symmetry of Poincaré dualities means that $B \widehat{\otimes}$ is also left-adjoint to $A \widehat{\otimes}$ ). The element in $KK_W(\mathbb{C}, A \mathbin{\widehat{\otimes}} B)$ defining the Poincaré duality is precisely the unit of the adjunction, while the counit is given by the element in $KK_W(B \mathbin{\widehat{\otimes}} A, \mathbb{C})$. This categorical view of Poincaré duality appears in [Echterhoff et al. 2008; Emerson 2011; Emerson and Meyer 2010].

Now let $D_1$, $D_2$ be $C^*$-algebras (without $W$-action). Let $\tau$ denote the trivial-action functor from $KK$ to $KK_W$, i.e., $\tau D_1$, $\tau D_2$ are $W$-$C^*$-algebras with trivial action of $W$. In the case that $W$ is a *finite* group, a Poincaré duality yields isomorphisms

$$\begin{aligned} KK(A \rtimes W \mathbin{\widehat{\otimes}} D_1, D_2) &\cong KK_W(A \mathbin{\widehat{\otimes}} \tau D_1, \tau D_2) \\ &\cong KK_W(\tau D_1, B \mathbin{\widehat{\otimes}} \tau D_2) \\ &\cong KK(D_1, B \rtimes W \mathbin{\widehat{\otimes}} D_2). \end{aligned}$$

The first and last isomorphisms are the dual Green–Julg and Green–Julg isomorphisms respectively, and in categorical terms these amount to the fact that the $\tau$ functor is (right and left) adjoint to the descent functor $\rtimes W$, see [Meyer 2008]. We denote the unit and counit by $\alpha$ and $\beta$, for the left-adjunction from $\tau$ to $\rtimes W$, and by $\widehat{\alpha}$ and $\widehat{\beta}$ for the left adjunction from $\rtimes W$ to $\tau$.

Since this is natural $A \rtimes W \widehat{\otimes}$ is left-adjoint to $B \rtimes W \widehat{\otimes}$, hence there must exist a unit and a counit providing this descended Poincaré duality. We will identify these elements explicitly.

**Theorem 4.1.** *Let $\mathfrak{a} \in KK_W(B \mathbin{\widehat{\otimes}} A, \mathbb{C})$ and $\mathfrak{b} \in KK_W(\mathbb{C}, A \mathbin{\widehat{\otimes}} B)$ define a $W$-equivariant Poincaré duality between $W$-$C^*$-algebras $A$, $B$, with $W$ finite. Then*

$$\begin{aligned} \widetilde{\mathfrak{a}} &= \mathfrak{Tr}(\mathfrak{a} \rtimes W)\beta_{\mathbb{C}} \in KK(B \rtimes W \ \widehat{\otimes}\ A \rtimes W, \mathbb{C}), \\ \widetilde{\mathfrak{b}} &= \alpha_{\mathbb{C}}(\mathfrak{b} \rtimes W)\Delta \in KK(\mathbb{C}, A \rtimes W \ \widehat{\otimes}\ B \rtimes W) \end{aligned}$$

*define a Poincaré duality such that the following diagram commutes:*

$$\begin{tikzcd} KK^*_W(A, \mathbb{C}) \arrow[r, "\mathfrak{b}\widehat{\otimes}_A -"] \arrow[d, "\text{dual Green–Julg} \cong"'] & KK^*_W(\mathbb{C}, B) \arrow[d, "\text{Green–Julg} \cong"] \\ KK^*(A \rtimes W, \mathbb{C}) \arrow[r, "\widetilde{\mathfrak{b}}\widehat{\otimes}_{A \rtimes W} -"] & KK^*(\mathbb{C}, B \rtimes W) \end{tikzcd}$$

Here $\Delta \in KK\big((A \mathbin{\widehat{\otimes}} B) \rtimes W,\ A \rtimes W \ \widehat{\otimes}\ B \rtimes W\big)$ is given by the diagonal inclusion of $W$ into $W \times W$ and $\mathfrak{Tr} \in KK\big(B \rtimes W \ \widehat{\otimes}\ A \rtimes W,\ (B \mathbin{\widehat{\otimes}} A) \rtimes W\big)$ is dual to this: We define a positive linear map $\mathrm{Tr} : (B \mathbin{\widehat{\otimes}} A) \rtimes (W \times W) \to (B \mathbin{\widehat{\otimes}} A) \rtimes W$ by

$$\mathrm{Tr} : (a \otimes b)[w_1, w_2] \mapsto \begin{cases} (a \otimes b)[w_1] & \text{if } w_1 = w_2, \\ 0 & \text{otherwise.} \end{cases}$$

This is a $(B \mathbin{\widehat{\otimes}} A) \rtimes W$-module map and we equip the algebra $(B \mathbin{\widehat{\otimes}} A) \rtimes (W \times W)$ with inner product in $(B \mathbin{\widehat{\otimes}} A) \rtimes W$ defined by

$$\big\langle (b \otimes a)[w_1, w_2], (b' \otimes a')[w_1', w_2'] \big\rangle_{\mathrm{Tr}} = \mathrm{Tr}\big([w_1^{-1}, w_2^{-1}](b^* \otimes a^*)(b' \otimes a')[w_1', w_2']\big).$$

The completion of this as a Hilbert module, equipped with the left multiplication representation of $(B \mathbin{\widehat{\otimes}} A) \rtimes (W \times W)$ provides the required element

$$\mathfrak{Tr} \in KK\big((B \mathbin{\widehat{\otimes}} A) \rtimes (W \times W), (B \mathbin{\widehat{\otimes}} A) \rtimes W\big).$$

To identify the unit and counit $\widetilde{\mathfrak{b}}$ and $\widetilde{\mathfrak{a}}$ one proceeds as follows. The unit $\widetilde{\mathfrak{b}}$ is the image of the identity $1_{A\rtimes W}$ under the isomorphism $KK(A\rtimes W, A\rtimes W) \cong KK(\mathbb{C}, B\rtimes W \mathbin{\widehat{\otimes}} A\rtimes W)$. This is the composition of the dual Green–Julg, equivariant Poincaré duality, and Green–Julg maps. The first two yield the Poincaré dual of the unit $\widehat{\alpha}_A$. One must then descend this and pair with the unit $\alpha_{\mathbb{C}}$. Hence $\widetilde{\mathfrak{b}} = \alpha_{\mathbb{C}}(b(\widehat{\alpha}_A \otimes 1_B))\rtimes W = \alpha_{\mathbb{C}}(b\rtimes W)((\widehat{\alpha}_A \otimes 1_B)\rtimes W)$ by naturality of descent. It is not hard to identify $(\widehat{\alpha}_A \otimes 1_B)\rtimes W$ as the element $\Delta$.

Similarly the counit $\widetilde{\mathfrak{a}}$ is the image of the identity $1_{B\rtimes W}$ under the isomorphism $KK(B\rtimes W, B\rtimes W) \cong KK(A\rtimes W \mathbin{\widehat{\otimes}} B\rtimes W, \mathbb{C})$. This is the composition of the Green–Julg, equivariant Poincaré duality, and dual Green–Julg maps, hence $\widetilde{\mathfrak{a}}$ is obtained by taking the Poincaré dual of the counit $\beta_B$, descending, and applying the counit $\widehat{\beta}_{\mathbb{C}}$. We have $\widetilde{\mathfrak{a}} = ((\widehat{\beta}_B \otimes 1_A)a)\rtimes W)\widehat{\beta}_{\mathbb{C}} = ((\widehat{\beta}_B \otimes 1_A)\rtimes W)(a\rtimes W)\widehat{\beta}_{\mathbb{C}}$. A change of variables identifies $(\widehat{\beta}_B \otimes 1_A)\rtimes W$ with $\mathfrak{Tr}$.

**Remark 4.2.** Given a Kasparov triple $(\mathcal{E}, 1, D)$ representing $\mathfrak{b}$ we can describe explicitly a triple $(\widetilde{\mathcal{E}}, \widetilde{\alpha}_{\mathbb{C}}, D\otimes 1)$ for $\widetilde{\mathfrak{b}}$.

The module $\widetilde{\mathcal{E}}$ is given by descending $\mathcal{E}$ and inflating the action of $W$ to $W \times W$. Explicitly $\widetilde{\mathcal{E}}$ is the completion of $\mathcal{E} \mathbin{\widehat{\otimes}} \mathbb{C}[W \times W]$ with respect to the inner product

$$\left\langle \xi \otimes [w_1, w_2], \xi' \otimes [w_1', w_2'] \right\rangle = (w_1^{-1}, w_2^{-1}) \cdot \langle \xi, \xi' \rangle [w_1^{-1} w_1', w_2^{-1} w_2'].$$

The operator is simply $D \otimes 1$ on $\widetilde{\mathcal{E}}$.

The representation $\widetilde{\alpha}_{\mathbb{C}}$ of $\mathbb{C}$ on $\widetilde{\mathcal{E}}$ takes 1 to the projection corresponding to the trivial representation of $W$, where $W$ acts diagonally on $\widetilde{\mathcal{E}}$ — the unit $\alpha_{\mathbb{C}} \in KK(\mathbb{C}, (\tau\mathbb{C})\rtimes W)$ is given by inclusion of $\mathbb{C}$ as the trivial representation in $\mathbb{C}[W] = (\tau\mathbb{C})\rtimes W$.

**4B.** ***Proof of Theorem 1.1.*** The theorem follows from Theorem 4.1 by consideration of the following diagram:

$$\begin{tikzcd}
KK^*_W(C(T), \mathbb{C}) \arrow[r, "\mathcal{P}\widehat{\otimes}_{C(T)} -"] \arrow[d, "\cong"', "\text{Morita}"] & KK^*_W(\mathbb{C}, C(T^\vee)) \arrow[d, "\cong"', "\text{Morita}"] \\
KK^*_W(C_0(\mathfrak{t}) \rtimes \Gamma, \mathbb{C}) \arrow[r, "\mathfrak{b}\widehat{\otimes}_{C_0(\mathfrak{t})\rtimes\Gamma} -"] \arrow[d, "\cong"', "\text{dual Green–Julg}"] & KK^*_W(\mathbb{C}, C_0(\mathfrak{t}^*) \rtimes \Gamma^\vee) \arrow[d, "\cong"', "\text{Green–Julg}"] \\
KK^*(C_0(\mathfrak{t}) \rtimes (\Gamma \rtimes W), \mathbb{C}) \arrow[r, "\widetilde{\mathfrak{b}}\widehat{\otimes}_{C_0(\mathfrak{t})\rtimes(\Gamma\rtimes W)} -"] & KK^*(\mathbb{C}, C_0(\mathfrak{t}^*) \rtimes (\Gamma^\vee \rtimes W))
\end{tikzcd}$$

Composition of $\mathcal{P}$ with the Morita equivalences and of $\mathcal{Q}$ with the inverse Morita equivalences, yields a $W$-equivariant Poincaré duality between $C_0(\mathfrak{t}) \rtimes \Gamma$ and $C_0(\mathfrak{t}^*) \rtimes \Gamma^\vee$ inducing the middle arrow.

To determine the element $\mathfrak{b}$ explicitly, recall that $\mathcal{P}$ is given by the Hilbert module of functions $\sigma : \mathfrak{t} \times \mathfrak{t}^* \to \mathbb{C}$ which are $\Gamma^\vee$ periodic in the second variable and satisfy

$$\sigma(\gamma+x,\eta)=e^{2\pi i\langle\eta,\gamma\rangle}\sigma(x,\eta).$$

This module is equipped with the inner product

$$\langle\sigma_1,\sigma_2\rangle(x,\eta)=\overline{\sigma_1(x,\eta)}\sigma_2(x,\eta).$$

The Morita equivalence from $C(T)$ to $C_0(\mathfrak{t})\rtimes\Gamma$ is given by the completion of $C_c(\mathfrak{t})$ with respect to the inner product

$$\langle\phi_1,\phi_2\rangle=\sum_{\gamma\in\Gamma}\overline{\phi_1}\,(\gamma\cdot\phi_2)[\gamma],$$

and similarly for $C(T^\vee)$.

It follows that $\mathfrak{b}$ is given by the Hilbert module completion of $C_c(\mathfrak{t}\times\mathfrak{t}^*)$ with respect to the inner product

$$\langle\theta_1,\theta_2\rangle=\sum_{(\gamma,\chi)\in\Gamma\times\Gamma^\vee}\overline{\theta}_1((\gamma,\chi)\cdot\theta_2)e^{2\pi i\langle\eta,\gamma\rangle}[(\gamma,\chi)].$$

Applying Theorem 4.1 yields the bottom arrow. Here we identify

$$(C_0(\mathfrak{t})\rtimes\Gamma)\rtimes W\quad\text{with}\quad C_0(\mathfrak{t})\rtimes(\Gamma\rtimes W)$$

and

$$(C_0(\mathfrak{t}^*)\rtimes\Gamma^\vee)\rtimes W)\quad\text{with}\quad C_0(\mathfrak{t}^*)\rtimes(\Gamma^\vee\rtimes W).$$

As noted in Remark 4.2 the element $\widetilde{\mathfrak{b}}$ has Hilbert module obtained by descending the module and inflating the $W$ action to $W\times W$.

In conclusion we obtain the module by completing $C_c(\mathfrak{t}\times\mathfrak{t}^*)\rtimes(W\times W)$ with respect to the inner product

$$\langle\theta[w_1,w_2],\theta'[w_1',w_2']\rangle=(w_1,w_2)^{-1}\cdot\langle\theta,\theta'\rangle[w_1^{-1}w_1',w_2^{-1}w_2'],$$

where $\langle\theta,\theta'\rangle$ is the inner product on $C_c(\mathfrak{t}\times\mathfrak{t}^*)$ defined above which is equipped with the representation of $\mathbb{C}$ given by the trivial projection in $\mathbb{C}[W]$, where $W$ acts diagonally on all factors.

**4C.** ***Proof of Theorem 1.2.*** The result follows from Theorem 4.1 by the consideration of the following diagram:

$$\begin{tikzcd}
KK_W^*(C(T),\mathbb{C}) \arrow[r, "\mathcal{P}\widehat{\otimes}_{C(T)}-"] \arrow[d, "\cong", "\text{Fourier–Pontryagin}"'] & KK_W^*(\mathbb{C},C(T^\vee)) \arrow[d, "\cong"', "\text{Fourier–Pontryagin}"] \\
KK_W^*(C^*(\Gamma^\vee),\mathbb{C}) \arrow[r, "\mathfrak{b}\widehat{\otimes}_{C^*(\Gamma^\vee)}-"] \arrow[d, "\cong", "\text{dual Green–Julg}"'] & KK_W^*(\mathbb{C},C^*(\Gamma)) \arrow[d, "\cong"', "\text{Green–Julg}"] \\
KK^*(C^*(\Gamma^\vee\rtimes W),\mathbb{C}) \arrow[r, "\widetilde{\mathfrak{b}}\widehat{\otimes}_{C^*(\Gamma^\vee\rtimes W)}-"] & KK^*(\mathbb{C},C^*(\Gamma\rtimes W))
\end{tikzcd}$$

Composition of $\mathcal{P}$ and of $\mathcal{Q}$ with the Fourier–Pontryagin isomorphisms yields a $W$-equivariant Poincaré duality between $C^*(\Gamma^\vee)$ and $C^*(\Gamma)$ inducing the middle arrow. Applying Theorem 4.1 yields the bottom arrow. Here we identify $C^*(\Gamma^\vee) \rtimes W$ with $C^*(\Gamma^\vee \rtimes W)$ and $C^*(\Gamma) \rtimes W$ with $C^*(\Gamma \rtimes W)$.

**4D.** ***The connection with the Baum–Connes assembly map.*** The Poincaré duality isomorphism appearing in Theorem 1.1

$$KK^*\big(C_0(\mathfrak{t}) \rtimes (\Gamma \rtimes W), \mathbb{C}\big) \to KK^*\big(\mathbb{C}, C_0(\mathfrak{t}^*) \rtimes (\Gamma^\vee \rtimes W)\big)$$

can be identified with the Baum–Connes assembly map for the group $\Gamma \rtimes W$ in a sense made explicit by the following diagram. Note that while we have suppressed the indices, these are degree-0 maps of $\mathbb{Z}_2$-graded groups.

$$\begin{tikzcd}
KK_{\Gamma\rtimes W}(C_0(\mathfrak{t}), \mathbb{C}) \arrow[r, "\text{Baum–Connes}"] \arrow[d, "\text{dual Green–Julg}"] \arrow[dd, bend right] & KK(\mathbb{C}, C^*(\Gamma\rtimes W)) \arrow[d, "\text{Morita equivalence}"] \\
KK(C_0(\mathfrak{t})\rtimes(\Gamma\rtimes W), \mathbb{C}) \arrow[r, "\text{Poincaré duality}"] & KK(\mathbb{C}, C_0(\mathfrak{t}^*)\rtimes(\Gamma^\vee\rtimes W)) \\
KK(C_0(\mathfrak{t})\rtimes(\Gamma\rtimes W), C^*(\Gamma\rtimes W)) \arrow[r, "\text{P. d.}"] \arrow[u, "\times\widehat{\beta}_{\mathbb{C}}"] & KK(\mathbb{C}, C_0(\mathfrak{t}^*)\rtimes(\Gamma^\vee\rtimes W)\otimes C^*(\Gamma\rtimes W)) \arrow[u, "\times\widehat{\beta}_{\mathbb{C}}"']
\end{tikzcd}$$

The curved arrow is the descent map. Note that since $\Gamma \rtimes W$ is amenable, the full and reduced $C^*$-algebras agree. The counit $\widehat{\beta}_{\mathbb{C}} \in KK((\tau\mathbb{C})\rtimes\Gamma\rtimes W), \mathbb{C}) = KK(C^*(\Gamma\rtimes W), \mathbb{C})$ is given by the trivial representation of the group $\Gamma \rtimes W$. This element has the effect of collapsing the coefficients $C^*(\Gamma \rtimes W)$.

The upper and lower Poincaré dualities in the diagram are both provided by Theorem 1.1, in the lower case with the coefficients $C^*(\Gamma \rtimes W)$, and the element inducing the map from $K$-homology to $K$-theory is described in detail in Section 4B.

Clearly the lower square commutes by associativity of the Kasparov product, while the left-hand triangle commutes by definition. Therefore, to show that the Baum–Connes assembly map corresponds to the upper Poincaré duality it suffices to show that the outer pentagon is commutative.

By definition the assembly map is the composition of descent with a Kasparov product. We denote by $\boldsymbol{A}_{C_0(\mathfrak{t})}$ the relevant element of $KK(\mathbb{C}, C_0(\mathfrak{t}) \rtimes \Gamma \rtimes W)$, which is given by the Hilbert module obtained by completing $C_c(\mathfrak{t})$ with respect to the inner product

$$\langle f, f'\rangle = \sum_{\gamma, w} \bar{f}((\gamma w) \cdot f)[\gamma w].$$

We thus have the following diagram, where the bottom arrow is our Poincaré duality:

$$\begin{tikzcd}
KK_{\Gamma\rtimes W}(C_0(\mathfrak{t}),\mathbb{C}) \arrow[r, "\text{Baum–Connes}"] \arrow[dd, "\text{descent}"'] \arrow[dd, phantom, shift left=8ex, "\circlearrowleft"] & KK(\mathbb{C}, C^*(\Gamma\rtimes W)) \arrow[d, "\text{Morita equivalence}"] \\
 & KK(\mathbb{C}, C_0(\mathfrak{t}^*)\rtimes(\Gamma^\vee\rtimes W)) \\
KK(C_0(\mathfrak{t})\rtimes(\Gamma\rtimes W), C^*(\Gamma\rtimes W)) \arrow[r] \arrow[uur, "A_{C_0(\mathfrak{t})}\times"] & KK(\mathbb{C}, C_0(\mathfrak{t}^*)\rtimes(\Gamma^\vee\rtimes W)\otimes C^*(\Gamma\rtimes W)) \arrow[u, "\times\widehat{\beta}_{\mathbb{C}}"'] \arrow[u, phantom, shift left=12ex, "\not\circlearrowleft"]
\end{tikzcd}$$

The upper triangle commutes by definition of the assembly map, however, it should be noted that the lower quadrilateral does not commute: the two directions around the quadrilateral collapse different algebras. It is thus not entirely obvious that the outer pentagon itself commutes. However we will show that the quadrilateral does commute on the image of the descent map so that the outer pentagon commutes as required.

We start with a Kasparov cycle $(H,\rho,T)\in KK_{\Gamma\rtimes W}(C_0(\mathfrak{t}),\mathbb{C})$. Note that since the action of $\Gamma\rtimes W$ on $\mathfrak{t}$ is proper we may, without loss of generality, take $T$ to be exactly invariant and of finite propagation. Now we descend to get $(\mathcal{E},\hat{\rho},T\otimes 1)$, where $\mathcal{E}=H\mathbin{\widehat{\otimes}}C^*\Gamma\rtimes W$ and $\hat{\rho}$ is a representation defined by

$$\hat{\rho}(f[g])=\rho(f)\pi(g)\otimes[g],$$

($\pi$ denotes the representation of $\Gamma\rtimes W$ on $H$).

Applying our Poincaré duality, given by the completion of $C_{\mathrm{c}}(\mathfrak{t}\times\mathfrak{t}^*)\rtimes(\Gamma\times\Gamma^\vee)$ described in Section 4B, along with the representation of $\mathbb{C}$ given by the trivial representation of $W$, we obtain a Kasparov triple as follows:

Let $H_c=\rho(C_{\mathrm{c}}(\mathfrak{t}))H$. The module in our triple is the completion of

$$H_c\mathbin{\widehat{\otimes}}C_{\mathrm{c}}(\mathfrak{t}^*)\mathbin{\widehat{\otimes}}\mathbb{C}[(\Gamma\rtimes W)\times W]$$

with respect to the inner product

$$\big\langle \xi\otimes f[(g,w)],\xi'\otimes f'[(g',w')]\big\rangle \\ =\sum_{\delta\in\Gamma}\sum_{\chi\in\Gamma^\vee}\langle\xi,\delta\cdot\xi'\rangle[g^{-1}\delta g']\big\langle f[w],[\chi]e^{2\pi i\langle\eta,\delta\rangle}f'[w']\big\rangle,$$

where the last inner product in the formula is taken in the algebra $C_0(\mathfrak{t}^*)\rtimes(\Gamma^\vee\rtimes W)$ viewed as a module over itself. The representation of $\mathbb{C}$ is once again given by the trivial projection in $\mathbb{C}[W]$, where $W$ acts diagonally on $H_c$, $C_{\mathrm{c}}(\mathfrak{t}^*)$, $\Gamma\rtimes W$ and $W$ itself. The operator is given by $T$ on $H_c$ and by the identity on the other factors.

This is a well-defined adjointable operator as we took $T$ to be exactly invariant under the action of $\Gamma \rtimes W$ and of finite propagation.

Applying the element $\widehat{\beta_{\mathbb{C}}}$ reduces this to a module over $C_0(\mathfrak{t}^*) \rtimes (\Gamma^\vee \rtimes W)$, where the inner product is

$$\big\langle \xi \otimes f[(g, w)], \xi' \otimes f'[(g', w')]\big\rangle = \sum_{\delta \in \Gamma} \sum_{\chi \in \Gamma^\vee} \langle \xi, \delta \cdot \xi' \rangle \big\langle f[w], [\chi] e^{2\pi i \langle \eta, \delta \rangle} f'[w'] \big\rangle.$$

Note that as this no longer depends on $g$ and $g'$, vectors of the form $\xi \otimes f[(g_1, w)]$ and $\xi \otimes f[(g_2, w)]$ are identified. Thus the module, which we will denote $\mathcal{E}_1$, is really a completion of $H_c \mathbin{\widehat{\otimes}} C_c(\mathfrak{t}^*) \mathbin{\widehat{\otimes}} \mathbb{C}[W]$. Once again the representation is provided by the trivial representation of $W$, and we denote the corresponding projection on $\mathcal{E}_1$ by $p_W$. The operator on $\mathcal{E}_1$ is given by $T \otimes 1 \otimes 1$.

We now trace the other route around the diagram. As before, starting with a Kasparov triple $(H, \rho, T)$ we obtain the descended triple $(\mathcal{E}, \hat{\rho}, T \otimes 1)$. We next apply the element $\boldsymbol{A}_{C_0(\mathfrak{t})}$ which is given by the completion of $C_c(\mathfrak{t})$ described earlier in this section. We obtain the completion of $H_c \mathbin{\widehat{\otimes}} \mathbb{C}[\Gamma \rtimes W]$ with respect to the inner product

$$\langle \xi[g], \xi'[g'] \rangle = \sum_{h \in \Gamma \rtimes W} \langle \xi, h \cdot \xi' \rangle [g^{-1} h g'].$$

The representation of $\mathbb{C}$ is given by the identity while the operator, once again, is given by $T$ on $H_c$ and the identity on the other factor.

The Hilbert module realising the descended Morita equivalence is given by completing the module $C_c(\mathfrak{t}^*) \rtimes W$ with respect to the inner product

$$\langle f[w], f'[w'] \rangle = \sum_{\chi \in \Gamma^\vee} [w^{-1}] \bar{f} (\chi \cdot f') [\chi w']$$

in $C_0(\mathfrak{t}^*) \rtimes W$.

The representation of $C^*(\Gamma \rtimes W)$ on this module is given by the representation of $\Gamma \rtimes W$, where $((\gamma w') \cdot f[w])(\eta) = e^{2\pi i \langle \eta, \gamma \rangle} (w' \cdot f)(\eta)[w'w]$. Hence, applying the Morita equivalence we obtain a Kasparov triple where the module is the completion, which we denote by $\mathcal{E}_2$, of $H_c \mathbin{\widehat{\otimes}} C_0(\mathfrak{t}^*) \mathbin{\widehat{\otimes}} \mathbb{C}[W]$ with respect to the inner product

$$\big\langle \xi \otimes f[w], \xi' \otimes f'[w'] \big\rangle = \sum_{\delta \in \Gamma} \sum_{u \in W} \sum_{\chi \in \Gamma^\vee} \langle \xi, (\delta u) \cdot \xi' \rangle \big\langle f[w], e^{2\pi i \langle \eta, \delta \rangle} [\chi u] f'[w'] \big\rangle,$$

the representation of $\mathbb{C}$ is given by the identity and the operator is given by $T$ on $H_c$ and the identity elsewhere.

To identify this triple with the Kasparov element obtained via the first route, we note that the module $\mathcal{E}_2$ is isomorphic to the range of the projection $p_W$ on $\mathcal{E}_1$.

Indeed,

$$\big\langle p_w(\xi\otimes f[w]),\, p_w(\xi'\otimes f'[w'])\big\rangle_{\mathcal{E}_1}=\frac{1}{|W|}\big\langle \xi\otimes f[w],\, \xi'\otimes f'[w']\big\rangle_{\mathcal{E}_2}.$$

This completes the proof.

## 5. Langlands duality and *K*-theory

In this section we will consider the $K$-theory of the affine and extended affine Weyl groups of a compact connected semisimple Lie group.

As remarked in the introduction an extended affine Weyl group and its Langlands dual $(W_a')^\vee$ need not be isomorphic. For example the extended affine Weyl groups of $\mathrm{PSU}_3$ and its Langlands dual $\mathrm{SU}_3$ are nonisomorphic. However their group $C^*$-algebras have the same $K$-theory, see [Niblo et al. 2016].

In this section we will show that this is not a coincidence, indeed passing to the Langlands dual always rationally preserves the $K$-theory for the extended affine Weyl groups. In particular, where the extended affine Weyl group of the dual of $G$ agrees with the affine Weyl group of $G$ (as for $\mathrm{PSU}_3$) the $K$-theory for the affine and extended affine Weyl groups of $G$ agrees up to rational isomorphism.

**Corollary 1.3.** *Let $G$ be a compact connected semisimple Lie group and $G^\vee$ its Langlands dual, with $W_a'$, $(W_a')^\vee$ the corresponding extended affine Weyl groups. Then there is a rational isomorphism*

$$K_*(C^*((W_a')^\vee))\cong K_*(C^*(W_a')).$$

*Proof.* The proof combines the universal coefficient theorem with our Poincaré duality as follows.

We start by writing $W_a'=\Gamma\rtimes W$ and $(W_a')^\vee=\Gamma^\vee\rtimes W$. By the Green–Julg theorem and Fourier–Pontryagin duality,

$$K_*(C^*(W_a')^\vee)\cong K_*^W(C^*(\Gamma^\vee)\cong K_*^W(C(T))=K_W^*(T). \tag{5.1}$$

Applying the universal coefficient theorem, we have the exact sequence

$$0\to \mathrm{Ext}^1_{\mathbb{Z}}(K_W^{*-1}(T),\mathbb{Z})\to K_*^W(T)\to \mathrm{Hom}(K_W^*(T),\mathbb{Z})\to 0.$$

In particular the torsion-free part of $K_*^W(T)$ agrees with the torsion-free part of $K_W^*(T)$ therefore rationally we have

$$K_W^*(T)\cong K_*^W(T). \tag{5.2}$$

As in Theorem 1.2, we can identify $K_*^W(T)=K_W^*(C(T))$ with $K^*(C^*(W_a')^\vee)$. The theorem now follows by applying our Poincaré duality from Theorem 1.2 to obtain

$$K^*(C^*(W_a')^\vee)\cong K_*(C^*(W_a')).$$

□

In a subsequent paper, [Niblo et al. 2016] we construct the admissible duals for the extended affine Weyl groups of all Lie groups of type $A_n$, exhibiting these spaces as varieties which decompose as a union of spaces indexed by the partitions of $n+1$. Furthermore we show that the rational isomorphism given above is induced by a homotopy equivalence between the varieties which respects the decomposition. The special case of $\mathrm{SU}(n)$ itself was considered by Solleveld [2007].

For the affine Weyl groups we have the following:

**Corollary 1.4.** *Let $W_a'$ be the extended affine Weyl group of $G$, and let $W_a$, $W_a^\vee$ be the affine Weyl groups of $G$ and its Langlands dual $G^\vee$. If $G$ is of adjoint type then rationally*

$$K_*(C^*(W_a^\vee)) \cong K_*(C^*(W_a')).$$

*If additionally $G$ is of type $A_n$, $D_n$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$ then rationally*

$$K_*(C^*(W_a)) \cong K_*(C^*(W_a')).$$

*Proof.* If $G$ is a compact connected semisimple Lie group of adjoint type then its Langlands dual $G^\vee$ is simply connected, so $(W_a')^\vee = W_a^\vee$.

In the case that $G$ is additionally of type $A_n$, $D_n$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$, the group $G^\vee$ is the universal cover of $G$ and hence $W_a = W_a^\vee$. □

## Acknowledgements

We would like to thank Maarten Solleveld for his helpful comments on the first version of this paper. We would also like to thank the editor Jonathan Rosenberg and the referee, whose careful reading of our paper led to the examination, in Section 4D, of the relationship between our Poincaré duality and the Baum–Connes conjecture. We are also grateful to the referee for suggesting the simplified proof of Theorem 4.1 presented here.

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Received 23 Feb 2017. Revised 10 Nov 2017. Accepted 26 Nov 2017.

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ANNALS OF K-THEORY
Vol. 3, No. 3, 2018
dx.doi.org/10.2140/akt.2018.3.523

# Geometric obstructions for Fredholm boundary conditions for manifolds with corners

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For every connected manifold with corners there is a homology theory called conormal homology, defined in terms of faces and orientation of their conormal bundle and whose cycles correspond geometrically to corner cycles. Its Euler characteristic (over the rationals, dimension of the total even space minus the dimension of the total odd space), $\chi_{\text{cn}} := \chi_0 - \chi_1$, is given by the alternating sum of the number of (open) faces of a given codimension.

The main result of the present paper is that for a compact connected manifold with corners $X$, given as a finite product of manifolds with corners of codimension less or equal to three, we have that:

1) If $X$ satisfies the Fredholm perturbation property (every elliptic pseudodifferential b-operator on $X$ can be perturbed by a b-regularizing operator so it becomes Fredholm) then the even Euler corner character of $X$ vanishes, i.e., $\chi_0(X) = 0$.

2) If the even periodic conormal homology group vanishes, i.e., $H_0^{\text{pcn}}(X) = 0$, then $X$ satisfies the stably homotopic Fredholm perturbation property (i.e., every elliptic pseudodifferential b-operator on $X$ satisfies the same named property up to stable homotopy among elliptic operators).

3) If $H_0^{\text{pcn}}(X)$ is torsion free and if the even Euler corner character of $X$ vanishes, i.e., $\chi_0(X) = 0$, then $X$ satisfies the stably homotopic Fredholm perturbation property. For example, for every finite product of manifolds with corners of codimension at most two the conormal homology groups are torsion free.

The main theorem behind the above result is the explicit computation in terms of conormal homology of the K-theory groups of the algebra $\mathcal{K}_b(X)$ of b-compact operators for $X$ as above. Our computation unifies the known cases of codimension zero (smooth manifolds) and of codimension one (smooth manifolds with boundary).

Both authors were partially supported by ANR-14-CE25-0012-01. An important part of this work started during a stay of the first author at the Max Planck Institut for Mathematics at Bonn. This author wishes to express his gratitude to this institution for the excellent working conditions.

*MSC2010:* 19K56, 58H05.

*Keywords:* Index theory, K-theory, Manifolds with corners, Lie groupoids.

## 1. Introduction

On a smooth compact manifold, ellipticity of (classical) pseudodifferential operators is equivalent to Fredholmness, and the vanishing of the Fredholm index of an elliptic pseudodifferential operator is equivalent to its invertibility after perturbation by a regularizing operator. In the case of a smooth manifold with boundary, not every elliptic operator is Fredholm and it has been known since Atiyah and Bott that there exist obstructions to the existence of local boundary conditions in order to upgrade an elliptic operator into a Fredholm boundary value problem. Nonetheless, if one moves to nonlocal boundary conditions, obstructions disappear: for instance, not every elliptic pseudodifferential b-operator is Fredholm but it can be perturbed with a regularizing operator to become Fredholm. This nontrivial fact, which goes back to Atiyah, Patodi and Singer [Atiyah et al. 1975], can also be obtained from the vanishing of a boundary analytic index (see [Melrose and Piazza 1997a; 1997b; Monthubert and Nistor 2012], and below). In fact, in this case the boundary analytic index takes values in the $K_0$-theory group of the algebra of regularizing operators and this K-theory group is easily seen to vanish. It is known that obstructions to the existence of perturbations of elliptic operators into Fredholm ones reappear in the case of manifolds with corners of arbitrary codimension [Bunke 2009; Nazaikinskii et al. 2009] (this includes for instance many useful domains in Euclidean spaces). In this paper we will show that the global topology and geometry of the corners and the way the corners form cycles contribute in a fundamental way to a primary obstruction to Fredholm boundary conditions. As we will see, the answer passes by the computation of some K-theory groups. We explain now with more detail the problem and the content of this paper.

Using K-theoretical tools for solving index problems was the main asset in the series of papers by Atiyah and Singer [1968a; 1968b] in which they introduce and prove several index formulas for smooth compact manifolds. In the case of manifolds with boundary, K-theory is still useful to understand the vanishing of the obstruction to the existence of perturbations of elliptic operators into Fredholm ones (even if K-theory is not essential in the computation of analytical indices [Atiyah et al. 1975]), and a fortiori to understand this obstruction in the case of families of manifolds with boundary [Melrose and Piazza 1997a; 1997b; Melrose and Rochon 2006]. For manifolds with corners, Bunke [2009] has delivered for Dirac type operators a complete study of the obstruction, which lives in the homology of a complex associated with the faces of the manifold. As we shall see later in the present work, this homology also appears as the $E^2$-term of the spectral sequence computing the K-group that contains the obstruction we define for general elliptic b-pseudodifferential operators. Nazaikinskii, Savin and Sternin [Nazaikinskii et al. 2008b; 2009] also use K-theory to express the obstruction for their pseudodifferential calculus on manifolds with corners and stratified spaces.

Let us briefly recall the framework in which we are going to work. The algebra of pseudodifferential operators $\Psi_{\mathrm{b}}^{*}(X)$ associated to any manifold with corners $X$ is defined in [Melrose and Piazza 1992]: it generalizes the case of manifolds with boundary treated in [Melrose 1993] (see also [Hörmander 1985, Section 18.3]). The elements in this algebra are called b-pseudodifferential operators,[1] the subscript b identifies these operators as obtained by "microlocalization" of the Lie algebra of $C^\infty$ vector fields on $X$ tangent to the boundary. This Lie algebra of vector fields can be explicitly obtained as sections of the so called b-tangent bundle ${}^{\mathrm{b}}TX$ (compressed tangent bundle that we will recall below). The b-pseudodifferential calculus has the classic and expected properties. In particular there is a principal symbol map

$$\sigma_{\mathrm{b}} : \Psi_{\mathrm{b}}^{m}(X) \to S^{[m]}({}^{\mathrm{b}}T^{*}X).$$

Ellipticity has the usual meaning, namely invertibility of the principal symbol. Moreover (see the discussion below and Theorem 2.15 in [Melrose and Piazza 1992]), an operator is elliptic if and only[2] if it has a quasiinverse modulo $\Psi_{\mathrm{b}}^{-\infty}(X)$. Now, $\Psi_{\mathrm{b}}^{-\infty}(X)$ contains compact operators, but also noncompact ones (as soon as $\partial X \neq \varnothing$), and compactness is characterized there by the vanishing of a suitable indicial map [loc. cit., p. 8]. Elliptic b-pseudodifferential operators, being invertible modulo compact operators (and hence Fredholm),[3] are usually said to be *fully* elliptic.

[1] To simplify we discuss only the case of scalar operators, the passage to operators acting on sections of vector bundles is done in the classic way.

[2] Notice that this remark implies that to an elliptic b-pseudodifferential operator one can associate an "index" in the algebraic K-theory group $K_0(\Psi_{\mathrm{b}}^{-\infty}(X))$ (the classic construction of quasiinverses).

[3] See page 8 in [Melrose and Piazza 1992] for a characterization of Fredholm operators in terms of an indicial map or [Loya 2005, Theorem 2.3] for the proof that Fredholm $\Longleftrightarrow$ fully elliptic.

Now, by the properties of the b-calculus, $\Psi_{\mathrm{b}}^0(X)$ is included in the algebra of bounded operators on $L^2(X)$, where the $L^2$ structure is provided by some b-metric in the interior of $X$. We denote by $\mathcal{K}_{\mathrm{b}}(X)$ the norm completion of the subalgebra $\Psi_{\mathrm{b}}^{-\infty}(X)$. This $C^*$-algebra fits in a short exact sequence of $C^*$-algebras of the form

$$0 \longrightarrow \mathcal{K}(X) \xrightarrow{i_0} \mathcal{K}_{\mathrm{b}}(X) \xrightarrow{r} \mathcal{K}_{\mathrm{b}}(\partial X) \longrightarrow 0, \tag{1.1}$$

where $\mathcal{K}(X)$ is the algebra of compact operators in $L^2(X)$. In order to study Fredholm problems and analytic index problems one has to understand the K-theory of the above short exact sequence.

To better explain how these K-theory groups enter into the study of Fredholm perturbation properties and in order to enounce our first main results we need to settle some definitions.

***Analytic and Boundary analytic Index morphism.*** Given an elliptic b-pseudodifferential $D$, the classic construction of parametrices adapts to give a K-theory-valued index in $K_0(\mathcal{K}_{\mathrm{b}}(X))$ that only depends on its principal symbol class $[\sigma_{\mathrm{b}}(D)] \in K^0_{\mathrm{top}}({}^{\mathrm{b}}T^*X)$. In more precise terms, the short exact sequence

$$0 \longrightarrow \mathcal{K}_{\mathrm{b}}(X) \longrightarrow \overline{\Psi_{\mathrm{b}}^0(X)} \xrightarrow{\sigma_{\mathrm{b}}} C({}^{\mathrm{b}}S^*X) \longrightarrow 0 \tag{1.2}$$

gives rise to a K-theory index morphism $K_1(C({}^{\mathrm{b}}S^*X)) \to K_0(\mathcal{K}_{\mathrm{b}}(X))$ that factors in a canonical way by an index morphism

$$K^0_{\mathrm{top}}({}^{\mathrm{b}}T^*X) \xrightarrow{\mathrm{Ind}_X^a} K_0(\mathcal{K}_{\mathrm{b}}(X)) \tag{1.3}$$

called *the Analytic Index morphism of X*. By composing the Analytic index with the morphism induced by the restriction to the boundary we have a morphism

$$K^0_{\mathrm{top}}({}^{\mathrm{b}}T^*X) \xrightarrow{\mathrm{Ind}_X^\partial} K_0(\mathcal{K}_{\mathrm{b}}(\partial X)) \tag{1.4}$$

called *the Boundary analytic index morphism of X*. In fact

$$r : K_0(\mathcal{K}_{\mathrm{b}}(X)) \to K_0(\mathcal{K}_{\mathrm{b}}(\partial X))$$

is an isomorphism if $\partial X \neq \varnothing$, Proposition 5.40, and so the two indices above are essentially the same. In other words we completely understand the six term short exact sequence in K-theory associated to the sequence (1.1). Notice that in particular there is no contribution of the Fredholm index in the $K_0$-analytic index.

To state the next theorem we need to define the Fredholm perturbation property and its stably homotopic version.

**Definition 1.5.** Let $D \in \Psi_{\mathrm{b}}^m(X)$ be elliptic. We say that $D$ satisfies

- the *Fredholm perturbation property* ($\mathcal{FP}$) if there is $R \in \Psi_{\mathrm{b}}^{-\infty}(X)$ such that $D + R$ is fully elliptic,
- the *stably homotopic Fredholm perturbation property* ($\mathcal{HFP}$) if there is a fully elliptic operator $D'$ with $[\sigma_{\mathrm{b}}(D')] = [\sigma_{\mathrm{b}}(D)] \in K^0_{\mathrm{top}}({}^{\mathrm{b}}T^*X)$.

We also say that $X$ satisfies the *Fredholm perturbation property* or the *stably homotopic Fredholm perturbation property* if any elliptic b-operator on $X$ satisfies ($\mathcal{FP}$) or ($\mathcal{HFP}$) respectively.

Property ($\mathcal{FP}$) is of course stronger than property ($\mathcal{HFP}$). Nistor [2003] characterized ($\mathcal{FP}$) in terms of the vanishing of an index in some particular cases. Nazaikinskii, Savin and Sternin [2008b] characterized ($\mathcal{HFP}$) for arbitrary manifolds with corners using an index map associated with their dual manifold construction. We now rephrase the result of [Nazaikinskii et al. 2008b] and we give a new proof in terms of deformation groupoids.

**Theorem 1.6.** *Let $D$ be an elliptic b-pseudodifferential operator on a compact manifold with corners $X$. Then $D$ satisfies ($\mathcal{HFP}$) if and only if $\mathrm{Ind}^{\partial}_X([\sigma_{\mathrm{b}}(D)]) = 0$. In particular if $D$ satisfies ($\mathcal{FP}$) then its boundary analytic index vanishes.*

The above results fit exactly with the K-theory vs index theory program of Atiyah and Singer, and in that sense it is not completely unexpected. Now, in order to give a full characterization of the Fredholm perturbation property one is first led to compute or understand the K-theory groups for the algebras (1.1) preferably in terms of the geometry and topology of the manifold with corners. As it happens, the only previously known cases are

- the K-theory of the compact operators $\mathcal{K}(X)$, giving $K_0(\mathcal{K}(X)) = \mathbb{Z}$ and $K_1(\mathcal{K}(X)) = 0$, which is of course essential for classic index theory purposes;
- the K-theory of $\mathcal{K}_{\mathrm{b}}(X)$ for a smooth manifold with boundary, giving
$$K_0(\mathcal{K}_{\mathrm{b}}(X)) = 0 \quad \text{and} \quad K_1(\mathcal{K}_{\mathrm{b}}(X)) = \mathbb{Z}^{1-p}$$
with $p$ the number of boundary components, which has the nontrivial consequence that any elliptic b-operator on a manifold with boundary can be endowed with Fredholm boundary conditions.

***Computation of the K-theory groups in terms of corner cycles.*** In this paper we explicitly compute the above K-theory groups for any finite product of manifolds with corners of codimension $\leq 3$ in terms of corner cycles (explanation below). Our computations and results are based on a geometric interpretation of the algebras of b-pseudodifferential operators in terms of Lie groupoids. We explain and recall the basic facts on groupoids and the b-pseudodifferential calculus in the first two sections. Besides being extremely useful to compute K-theory groups, the groupoid approach we propose turns out to be very powerful for computing index morphisms and relating several indices. Indeed, the relation between the different indices for manifolds with corners was only partially understood for some examples. Let us explain this in detail. Let $X$ be a manifold with corners. Let $F_p = F_p(X)$ be the set of faces (connected, without boundary) of $X$ of codimension $p$. To compute $K_*(\mathcal{K}_b(X))$, we use an increasing filtration of $X$ given by the open subspaces:

$$X_p = \bigcup_{k\leq p,\ f\in F_k} f. \tag{1.7}$$

We have $X_0 = \overset{\circ}{X}$ and $X_d = X$. We extend if necessary the filtration over $\mathbb{Z}$ by setting $X_k = \varnothing$ if $k < 0$ and $X_k = X$ if $k > d$. The $C^*$-algebra of $\mathcal{K}_b(X)$ inherits an increasing filtration by $C^*$-ideals (for full details see Section 5):

$$\mathcal{K}(L^2(\overset{\circ}{X})) = A_0 \subset A_1 \subset \cdots A_d = A = \mathcal{K}_b(X). \tag{1.8}$$

The spectral sequence $(E^*_{*,*}(\mathcal{K}_b(X)), d^*_{*,*})$ associated with this filtration can be used, in principle, to have a better understanding of these K-theory groups. This filtration was already considered by Melrose and Nistor [1998] and their main theorem is the expression of the first differential [loc. cit., Theorem 9]. In trying to figure out an expression for the differentials of this spectral sequence in all degrees, we found a differential $\mathbb{Z}$-module $(\mathcal{C}(X), \delta^{\mathrm{pcn}})$ constructed in a very simple way out of the set of open connected faces of the given manifold with (embedded) corners $X$. Roughly speaking, the $\mathbb{Z}$-module $\mathcal{C}(X)$ is generated by open connected faces provided with a coorientation (that is, an orientation of their conormal bundles in $X$), while the differential map $\delta^{\mathrm{pcn}}$ associates to a given cooriented face of codimension $p$, the sum of cooriented faces of codimension $p-2k-1$, $k\geq 0$, containing it in their closures. This gives a well defined differential module for two reasons. The first one is that once a labeling of the boundary hyperfaces is chosen, the coorientation of a given face induces coorientations of the faces containing it in their closures, proving that the module map $\delta^{\mathrm{pcn}}$ is well defined. The second one is that the jumps by $2k+1$, $k\geq 0$, in the codimension guarantee the relation $\delta^{\mathrm{pcn}}\circ\delta^{\mathrm{pcn}} = 0$. We call the homology of $(\mathcal{C}(X), \delta^{\mathrm{pcn}})$ *periodic conormal homology*, and denote it by $H^{\mathrm{pcn}}(X)$. Note that it is $\mathbb{Z}_2$-graded by sorting faces by even and odd codimension.

Actually, the differential $\delta^{\mathrm{pcn}}$ retracts onto the simpler differential map $\delta$ where one stops at $-1$ in the codimension, that is, $\delta$ maps a given cooriented face of codimension $p$ to the sum of cooriented faces of codimension $p-1$ containing it in their closures. We call *conormal homology* and denote it by $H^{\mathrm{cn}}(X)$ the homology of $(\mathcal{C}(X), \delta)$: this is a $\mathbb{Z}$-graded homology and the resulting $\mathbb{Z}_2$-graduation coincides with the periodic conormal groups. For full details about conormal homology see Section 4 and the Appendix. It is worthwhile to note that the conormal complex $(\mathcal{C}(X), \delta)$ already appears as the complex of (connected) faces in [Bunke

2009]. The complex considered by Bunke is made of mutually compatible oriented faces and the consistency of these orientations induces orientations of the conormal spaces of the faces. There is then an obvious isomorphism between both complexes. Recall that Bunke proved that the obstruction for the existence of a *boundary taming* of a Dirac type operator on $X$ is given by an explicit class in this homology, which also implicitly appears in the work of Melrose and Nistor [1998], through the quasiisomorphism that we prove here (Corollary 5.35). It is thus all but a surprise that conormal homology emerges from the computation of $K_*(\mathcal{K}_b(X))$ and we conclude this paragraph by recording that there is a natural isomorphism

$$H_p^{\mathrm{cn}}(X) \simeq E^2_{p,0}(\mathcal{K}_b(X)). \tag{1.9}$$

Our main K-theory computation can now be stated:

**Theorem 5.43.** *Let $X = \Pi_i X_i$ be a finite product of manifolds with corners of codimension less or equal to three. There are natural isomorphisms*

$$\begin{aligned} H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} &\xrightarrow[\cong]{\phi_X} K_0(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}, \\ H_1^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} &\xrightarrow[\cong]{\phi_X} K_1(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned} \tag{1.10}$$

*In the case where $X$ contains a factor of codimension at most two or $X$ is of codimension three, the result holds even without tensoring by $\mathbb{Q}$.*

We insist on the fact that (periodic) conormal homology groups are easily computable, because the underlying chain complexes as well as the differential maps are obtained from elementary and explicit ingredients. To continue let us introduce the Corner characters.

**Definition 1.11** (corner characters). Let $X$ be a manifold with corners. We define the *even conormal character of $X$* as the finite sum

$$\chi_0(X) = \dim_{\mathbb{Q}} H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{1.12}$$

Similarly, we define the *odd conormal character of $X$* as the finite sum

$$\chi_1(X) = \dim_{\mathbb{Q}} H_1^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{1.13}$$

We can consider as well

$$\chi(X) = \chi_0(X) - \chi_1(X), \tag{1.14}$$

then by definition

$$\chi(X) = 1 - \#F_1 + \#F_2 - \cdots + (-1)^d \#F_d. \tag{1.15}$$

We refer to the integer $\chi(X)$ as the Euler corner character of $X$.

In particular one can rewrite Theorem 5.43 to have, for $X$ as stated,

$$K_0(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{\chi_0(X)}, \quad K_1(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{\chi_1(X)}, \tag{1.16}$$

and in terms of the corner character,

$$\chi(X) = \operatorname{rank}(K_0(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}) - \operatorname{rank}(K_1(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}). \tag{1.17}$$

Or, in the case where $X$ is a finite product of manifolds with corners of codimension at most 2, we even have

$$K_0(\mathcal{K}_b(X)) \simeq \mathbb{Z}^{\chi_0(X)}, \quad \text{and} \quad K_1(\mathcal{K}_b(X)) \simeq \mathbb{Z}^{\chi_1(X)} \tag{1.18}$$

and also $\chi_{\mathrm{cn}}(X) = \operatorname{rank}\big(K_0(\mathcal{K}_b(X))\big) - \operatorname{rank}\big(K_1(\mathcal{K}_b(X))\big)$.

We can finally state the following primary obstruction to the Fredholm perturbation theorem in terms of corner characters and corner cycles:

**Theorem 6.9.** *Let $X$ be a compact manifold with corners of codimension greater than or equal to one. If $X$ is a finite product of manifolds with corners of codimension less than or equal to three we have that*:

(1) *If $X$ satisfies the Fredholm perturbation property then the even Euler corner character of $X$ vanishes*, *i.e.*, $\chi_0(X) = 0$.

(2) *If the even periodic conormal homology group vanishes*, *i.e.*, $H_0^{\mathrm{pcn}}(X) = 0$, *then $X$ satisfies the stably homotopic Fredholm perturbation property.*

(3) *If $H_0^{\mathrm{pcn}}(X)$ is torsion free and if the even Euler corner character of $X$ vanishes*, *i.e.*, $\chi_0(X) = 0$, *then $X$ satisfies the stably homotopic Fredholm perturbation property.*

We believe that the results above hold beyond the case of finite products of manifolds with corners of codimension $\leq 3$. On one side conormal homology can be defined and computed in all generality and, in all examples we have, the isomorphisms above still hold. The problem in general is to compute beyond the third differential of the naturally associated spectral sequence for the K-theory groups for manifolds with corners of codimension greater or equal to four. This is technically a very hard task and explicit interesting examples (not products) become rare. In fact, for any codimension, the corresponding spectral sequence in periodic conormal homology collapses at the second page as shown in the Appendix. We believe it does collapse as well for K-theory because of the results above. Another problem is related with the possible torsion of the conormal homology groups. Indeed, we prove in Theorem 4.22 that for a finite product of manifolds with corners of codimension at most two these groups are torsion free and that the odd group for a three codimensional manifold with corners is torsion free as well. We think that in general these groups are torsion free but the combinatorics become very hard and one needs a good way to deal with all these data. We will discuss all these topics elsewhere.

Partial results in the direction of this paper were obtained by several authors; we have already mentioned the seminal works of Melrose and Nistor [1998] and of Nazaikinskii et al. [2008a; 2008b]. In particular Melrose and Nistor start the computation of the K-groups of the algebra of zero order b-operators and some particular cases of boundary analytic index morphisms as defined here (together with some topological formulas for them). Also, Nistor [2003] solves the Fredholm perturbation problem for manifolds with corners of the form a canonical simplex times a smooth manifold. Let us mention also the work of Monthubert and Nistor [2012] in which they construct a classifying space associated to a manifold with corners whose K-theory can be in principle used to compute the analytic index above. We were very much inspired by all these works. Bunke [2009] focuses on the obstruction of Fredholm perturbations for Dirac operators on manifolds with corners, for which he gives a precise answer in terms of conormal homology, while we focus on the *receptacle* for these obstructions: our results are then less precise for a given operator, but address generic b-pseudodifferential operators.

The theorems above show the importance and interest in computing the boundary analytic and the Fredholm indices associated to a manifold with corners and if possible, in a unified and in a topological and geometrical way. Using K-theory as above, for the case of a smooth compact connected manifold, the computation we are mentioning is none other than the Atiyah and Singer [1968a] index theorem. As we mentioned already, for manifolds with boundary, Atiyah, Patodi and Singer gave a formula for the Fredholm index of a Dirac type operator. In fact, with the groupoid approach to index theory, several authors have contributed to the now realizable idea that one can actually use these tools to have a nice K-theoretical framework and to actually compute more general index theorems as in the classic smooth case. For example, in our common work with Monthubert, [Carrillo Rouse et al. 2014], we give a topological formula for the Fredholm index morphism for manifolds with boundary that will allow us in a sequel paper to compare with the APS formula and obtain geometric information on the eta invariant. In the second paper of this series we will generalize our results of [Carrillo Rouse et al. 2014] for general manifolds with corners by giving explicit topological index computations for the indices appearing above.

## 2. Melrose b-calculus for manifolds with corners via groupoids

**2A.** ***Preliminaries on groupoids, K-theory $C^*$-algebras and pseudodifferential calculus.*** All the material in this section is well known and by now classic for the people working in groupoid $C^*$-algebras, K-theory and index theory. For more details and references see [Debord and Lescure 2010; Nistor et al. 1999; Monthubert and Pierrot 1997; Lescure et al. 2017; Hilsum and Skandalis 1983; Renault 1980; Anantharaman-Delaroche and Renault 2000].

*Groupoids.* Let us start with the definition.

**Definition 2.1.** A *groupoid* consists of the following data: two sets $\mathscr{G}$ and $\mathscr{G}^{(0)}$, and maps

(1) $s, r : \mathscr{G} \to \mathscr{G}^{(0)}$ called the source and range (or target) maps,

(2) $m : \mathscr{G}^{(2)} \to \mathscr{G}$ called the product map, where

$$\mathscr{G}^{(2)} = \{(\gamma, \eta) \in \mathscr{G} \times \mathscr{G} : s(\gamma) = r(\eta)\},$$

such that there exist two maps, $u : \mathscr{G}^{(0)} \to \mathscr{G}$ (the unit map) and $i : \mathscr{G} \to \mathscr{G}$ (the inverse map), which, if we denote $m(\gamma, \eta) = \gamma \cdot \eta$, $u(x) = x$ and $i(\gamma) = \gamma^{-1}$, satisfy the following properties:

(i) $r(\gamma \cdot \eta) = r(\gamma)$ and $s(\gamma \cdot \eta) = s(\eta)$.

(ii) $\gamma \cdot (\eta \cdot \delta) = (\gamma \cdot \eta) \cdot \delta$ for all $\gamma, \eta, \delta \in \mathscr{G}$ when this is possible.

(iii) $\gamma \cdot x = \gamma$ and $x \cdot \eta = \eta$ for all $\gamma, \eta \in \mathscr{G}$ with $s(\gamma) = x$ and $r(\eta) = x$.

(iv) $\gamma \cdot \gamma^{-1} = u(r(\gamma))$ and $\gamma^{-1} \cdot \gamma = u(s(\gamma))$ for all $\gamma \in \mathscr{G}$.

Generally, we denote a groupoid by $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$. A morphism $f$ from a groupoid $\mathscr{H} \rightrightarrows \mathscr{H}^{(0)}$ to a groupoid $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ is given by a map $f$ from $\mathscr{G}$ to $\mathscr{H}$ which preserves the groupoid structure, i.e., $f$ commutes with the source, target, unit, and inverse maps, and respects the groupoid product in the sense that $f(h_1 \cdot h_2) = f(h_1) \cdot f(h_2)$ for any $(h_1, h_2) \in \mathscr{H}^{(2)}$.

For $A, B$ subsets of $\mathscr{G}^{(0)}$ we use the notation $\mathscr{G}_A^B$ for the subset

$$\{\gamma \in \mathscr{G} : s(\gamma) \in A,\ r(\gamma) \in B\}.$$

We will also need the following definition:

**Definition 2.2** (saturated subgroupoids). Let $\mathscr{G} \rightrightarrows M$ be a groupoid.

(1) A subset $A \subset M$ of the units is said to be saturated by $\mathscr{G}$ (or only saturated if the context is clear enough) if it is a union of orbits of $\mathscr{G}$.

(2) A subgroupoid

$$\begin{tikzcd} \mathscr{G}_1 \arrow[r, hook, "\subset"] \arrow[d, shift right, "r"'] \arrow[d, shift left, "s"] & \mathscr{G} \arrow[d, shift right, "r"'] \arrow[d, shift left, "s"] \\ M_1 \arrow[r, hook, "\subset"] & M \end{tikzcd} \tag{2.3}$$

is a saturated subgroupoid if its set of units $M_1 \subset M$ is saturated by $\mathscr{G}$.

A groupoid can be endowed with a structure of topological space, or manifold, for instance. In the case when $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are smooth manifolds, and $s, r, m, u$ are smooth maps (with $s$ and $r$ submersions), then $\mathcal{G}$ is called a Lie groupoid. In the case of manifolds with boundary, or with corners, this notion can be generalized to that of continuous families groupoids (see [Paterson 1999]) or as Lie groupoids if one considers the category of smooth manifolds with corners.

*$C^*$-algebras.* To any Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ one has several $C^*$-algebra completions for the $^*$-convolution algebra $C_c^\infty(\mathcal{G})$. Since in this paper all the groupoids considered are amenable, we will be denoting by $C^*(\mathcal{G})$ the maximal and hence reduced $C^*$-algebra of $\mathcal{G}$. From now on, all groupoids will be assumed amenable.

In the sequel we will use the following two results which hold in the generality of locally compact groupoids equipped with Haar systems.

(1) Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two locally compact groupoids equipped with Haar systems. Then for the locally compact groupoid $\mathcal{G}_1 \times \mathcal{G}_2$ we have

$$C^*(\mathcal{G}_1 \times \mathcal{G}_2) \cong C^*(\mathcal{G}_1) \otimes C^*(\mathcal{G}_2). \tag{2.4}$$

(2) Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ a locally compact groupoid with Haar system $\mu$. Let $U \subset \mathcal{G}^{(0)}$ be a saturated open subset. Then $F := \mathcal{G}^{(0)} \setminus U$ is a closed saturated subset. The Haar system $\mu$ can be restricted to the restriction groupoids $\mathcal{G}_U := \mathcal{G}_U^U \rightrightarrows U$ and $\mathcal{G}_F := \mathcal{G}_F^F \rightrightarrows F$, and we have the following short exact sequence of $C^*$-algebras:

$$0 \longrightarrow C^*(\mathcal{G}_U) \xrightarrow{\ i\ } C^*(\mathcal{G}) \xrightarrow{\ r\ } C^*(\mathcal{G}_F) \longrightarrow 0, \tag{2.5}$$

where $i : C_c(\mathcal{G}_U) \to C_c(\mathcal{G})$ is the extension of functions by zero and $r : C_c(\mathcal{G}) \to C_c(\mathcal{G}_F)$ is the restriction of functions.

*K-theory.* We will be considering the K-theory groups of the $C^*$-algebra of a groupoid. For space purposes we will be denoting these groups by

$$K^*(\mathcal{G}) := K_*(C^*(\mathcal{G})). \tag{2.6}$$

We will use the classic properties of the K-theory functor, mainly its homotopy invariance and the six term exact sequence associated to a short exact sequence. Whenever the groupoid in question is a space $X$ we will use the notation

$$K^*_{\mathrm{top}}(X) := K_*(C_0(X)). \tag{2.7}$$

to indicate that in this case this group is indeed isomorphic to the topological K-theory group.

*ΨDO calculus for groupoids.* A pseudodifferential operator on a Lie groupoid (or more generally a continuous family groupoid) $\mathcal{G}$ is a family of pseudodifferential operators on the fibers of $\mathcal{G}$ (which are smooth manifolds without boundary), the family being equivariant under the natural action of $\mathcal{G}$.

Compactly supported pseudodifferential operators form an algebra, denoted by $\Psi^\infty(\mathcal{G})$. The algebra of order-0 pseudodifferential operators can be completed into a $C^*$-algebra, $\overline{\Psi^0}(\mathcal{G})$. There exists a symbol map, $\sigma$, whose kernel is $C^*(\mathcal{G})$. This gives rise to the following exact sequence:

$$0 \to C^*(\mathcal{G}) \to \overline{\Psi^0}(\mathcal{G}) \to C_0(S^*(\mathcal{G})) \to 0,$$

where $S^*(\mathcal{G})$ is the cosphere bundle of the Lie algebroid of $\mathcal{G}$.

In the general context of index theory on groupoids, there is an analytic index which can be defined in two ways. The first way, which is classical, is to consider the boundary map of the 6-terms exact sequence in K-theory induced by the short exact sequence above:

$$\mathrm{ind}_a : K_1(C_0(S^*(\mathcal{G}))) \to K_0(C^*(\mathcal{G})).$$

Actually, an alternative is to define it through the tangent groupoid of Connes, which was originally defined for the groupoid of a smooth manifold and later extended to the case of continuous family groupoids [Monthubert and Pierrot 1997; Lauter et al. 2000]. The tangent groupoid of a Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is the groupoid

$$\mathcal{G}^{\tan} = A(\mathcal{G}) \bigsqcup \mathcal{G} \times (0,1] \rightrightarrows \mathcal{G}^{(0)} \times [0,1],$$

where $A(\mathcal{G}) = T_{\mathcal{G}^{(0)}}\mathcal{G}/T\mathcal{G}^{(0)}$ is the Lie algebroid of $\mathcal{G}$. The groupoid $\mathcal{G}^{\tan}$ has a smooth structure given by *the deformation to the normal cone* construction, see for example [Carrillo Rouse 2008] for a survey.

Using the evaluation maps, one has two K-theory morphisms,

$$e_0 : K_0(C^*(\mathcal{G}^{\tan})) \to K^0(A^*\mathcal{G}),$$

which is an isomorphism (since $K_*(C^*(\mathcal{G} \times (0,1])) = 0$), and

$$e_1 : K_*(C^0(\mathcal{G}^{\tan})) \to K_0(C^*(\mathcal{G})).$$

The analytic index can be defined as

$$\mathrm{ind}_a = e_1 \circ e_0^{-1} : K^0(A^*\mathcal{G}) \to K_0(C^*(\mathcal{G})),$$

modulo the surjection $K_1(C_0(S^*(\mathcal{G})) \to K^0(A^*\mathcal{G})$.

See [Monthubert and Pierrot 1997; Nistor et al. 1999; Monthubert 2003; Lauter et al. 2000; Vassout 2006] for a detailed presentation of pseudodifferential calculus on groupoids.

**2B.** ***Melrose b-calculus for manifolds with corners via the b-groupoid.*** We start by defining the manifolds with corners we will be using in the entire paper.

A manifold with corners is a Hausdorff space covered by compatible coordinate charts with coordinate functions modeled in the spaces

$$\mathbb{R}^n_k := [0, +\infty)^k \times \mathbb{R}^{n-k}$$

for fixed $n$ and possibly variable $k$.

**Definition 2.8.** A *manifold with embedded corners* $X$ is a Hausdorff topological space endowed with a subalgebra $C^\infty(X) \in C^0(X)$ satisfying the following conditions:

(1) There is a smooth manifold $\tilde{X}$ and a map $\iota : X \to \tilde{X}$ such that

$$\iota^*(C^\infty(\tilde{X})) = C^\infty(X).$$

(2) There is a finite family of functions $\rho_i \in C^\infty(\tilde{X})$, called the defining functions of the hyperfaces, such that

$$\iota(X) = \bigcap_{i \in I} \{\rho_i \geq 0\}.$$

(3) For any $J \subset I$,

$$d_x\rho_i(x) \text{ are linearly independent in } T^*_x\tilde{X} \text{ for all } x \in F_J := \textstyle\bigcap_{i\in J}\{\rho_i = 0\}.$$

**Terminology.** In this paper we will only be considering manifolds with embedded corners. We will refer to them simply as manifolds with corners. We will also assume our manifolds to be connected. More general manifold with corners deserve attention but as we will see in further papers it will be more simple to consider them as stratified pseudomanifolds and desingularize them as manifolds with embedded corners with an iterated fibration structure.

Given a compact manifold with corners $X$, Melrose[4] [1993] constructed the algebra $\Psi^*_{\mathrm{b}}(X)$ of b-pseudodifferential operators. The elements in this algebra are called b-pseudodifferential operators; the subscript b identifies these operators as obtained by "microlocalization" of the Lie algebra of $C^\infty$ vector fields on $X$ tangent to the boundary. This Lie algebra of vector fields can be explicitly obtained as sections of the so called b-tangent bundle ${}^{\mathrm{b}}TX$ (the compressed tangent bundle that will appear below as the Lie algebroid of an explicit Lie groupoid). The b-pseudodifferential calculus developed by Melrose has the classic and expected properties. In particular there is a principal symbol map

$$\sigma_{\mathrm{b}} : \Psi^m_{\mathrm{b}}(X) \to S^{[m]}({}^{\mathrm{b}}T^*X).$$

[4]For full details in the case with corners see the paper of Melrose and Piazza [1992].

Ellipticity has the usual meaning, namely invertibility of the principal symbol. Moreover (see the discussion below Theorem 2.15 in [Melrose and Piazza 1992]), an operator is elliptic if and only[5] if it has a quasiinverse modulo $\Psi_b^{-\infty}(X)$. Now, the operators in $\Psi_b^{-\infty}(X)$ are not all compact (unless the topological boundary $\partial X = \varnothing$) but they contain a subalgebra consisting of compact operators (those for which certain indicial map is zero [loc. cit., p. 8]). Hence, among elliptic b-pseudodifferential operators one has those for which the quasiinverse is actually modulo compact operators and hence Fredholm (again, see [loc. cit., p. 8] for a characterization of Fredholm operators in terms of an indicial map). These b-elliptic operators are called fully elliptic operators.

Now, as for every 0-order b-pseudodifferential operator [loc. cit., (2.16)], the operators in $\Psi_b^{-\infty}(X)$ extend to bounded operators on $L^2(X)$ and hence if we consider its completion as bounded operators one obtains an algebra denoted in this paper by $\mathcal{K}_b(X)$ that fits in a short exact sequence of $C^*$-algebras of the form

$$0 \longrightarrow \mathcal{K}(X) \xrightarrow{i_0} \mathcal{K}_b(X) \xrightarrow{r} \mathcal{K}_b(\partial X) \longrightarrow 0, \tag{2.9}$$

where $\mathcal{K}(X)$ is the algebra of compact operators in $L^2(X)$.

Let $X$ be a compact manifold with embedded corners, so by definition we are assuming there is a smooth compact manifold (of the same dimension) $\tilde{X}$ with $X \subset \tilde{X}$ and $\rho_1, \dots, \rho_n$ defining functions of the faces. Monthubert [2003] constructed a Lie groupoid (called the puff groupoid) associated to any decoupage $(\tilde{X}, (\rho_i))$; it has the following expression as a Lie subgroupoid of $\tilde{X} \times \tilde{X} \times \mathbb{R}^k$:

$$G(\tilde{X}, (\rho_i)) = \left\{(x, y, \lambda_1, \dots, \lambda_n) \in \tilde{X} \times \tilde{X} \times \mathbb{R}^n : \rho_i(x) = e^{\lambda_i} \rho_i(y)\right\}. \tag{2.10}$$

The puff groupoid is not s-connected; we denote by $G_c(\tilde{X}, (\rho_i))$ its s-connected component.

**Definition 2.11** (the b-groupoid). The b-groupoid $\Gamma_b(X)$ of $X$ is by definition the restriction to $X$ of the s-connected puff groupoid (2.10) considered above, that is

$$\Gamma_b(X) := G_c(\tilde{X}, (\rho_i))|_X \rightrightarrows X. \tag{2.12}$$

The b-groupoid was introduced by B. Monthubert in order to give a groupoid description for the Melrose's algebra of b-pseudodifferential operators. We summarize below the main properties we will be using of this groupoid:

**Theorem 2.13** [Monthubert 2003]. *Let $X$ be a manifold with corners, as above. We have that*:

(1) $\Gamma_b(X)$ *is a $C^{0,\infty}$-amenable groupoid.*

[5] Notice that this remark implies that to a b-pseudodifferential operator one can associate an "index" in the algebraic K-theory group $K_0(\Psi_b^{-\infty}(X))$ (the classic construction of quasiinverses).

(2) *$X$ has Lie algebroid $A(\Gamma_{\mathrm{b}}(X)) = {}^{\mathrm{b}}TX$, the b-tangent bundle of Melrose.*

(3) *The $C^*$-algebra of $X$ (reduced or maximal is the same since we have amenability) coincides with the algebra of b-compact operators. The canonical isomorphism*

$$C^*(\Gamma_{\mathrm{b}}(X)) \cong \mathcal{K}_{\mathrm{b}}(X) \tag{2.14}$$

*is given as usual by the Schwartz kernel theorem.*

(4) *The pseudodifferential calculus of $\Gamma_{\mathrm{b}}(X)$ coincides with compactly supported b-calculus of Melrose.*

**Remark 2.15.** To simplify the exposition, in the present paper we only discuss the case of scalar operators. The case of operators acting on sections of vector bundles is treated classically by considering bundles of homomorphisms.

## 3. Boundary analytic and Fredholm indices for manifolds with corners: relations and Fredholm perturbation characterization

We will now introduce the several index morphisms we will be using, mainly the analytic and the Fredholm index. In all this section, $X$ denotes a compact and connected manifold with embedded corners.

**3A.** ***Analytic and boundary analytic index morphisms.*** Any elliptic b-pseudodifferential $D$ has an analytical index $\mathrm{Ind}_{\mathrm{an}}(D)$ given by

$$\mathrm{Ind}_{\mathrm{an}}(D) = I([\sigma_{\mathrm{b}}(D)]_1) \in K_0(\mathcal{K}_{\mathrm{b}}(X)),$$

where $I$ is the connecting homomorphism in K-theory of the exact sequence

$$0 \longrightarrow \mathcal{K}_{\mathrm{b}}(X) \longrightarrow \overline{\Psi^0_{\mathrm{b}}(X)} \xrightarrow{\sigma_{\mathrm{b}}} C({}^{\mathrm{b}}S^*X) \longrightarrow 0, \tag{3.1}$$

and $[\sigma_{\mathrm{b}}(D)]_1$ is the class in $K_1(C({}^{\mathrm{b}}S^*X))$ of the principal symbol $\sigma_{\mathrm{b}}(D)$ of $D$.

Alternatively, we can express $\mathrm{Ind}(D)$ using the adiabatic deformation groupoid of $\Gamma_{\mathrm{b}}(X)$ and the class in $K_0$ of the same symbol, namely

$$[\sigma_{\mathrm{b}}(D)] = \delta([\sigma_{\mathrm{b}}(D)]_1) \in K_0(C_0({}^{\mathrm{b}}T^*X)), \tag{3.2}$$

where $\delta$ is the connecting homomorphism of the exact sequence relating the vector and sphere bundles:

$$0 \longrightarrow C_0({}^{\mathrm{b}}T^*X) \longrightarrow C_0({}^{\mathrm{b}}B^*X) \longrightarrow C({}^{\mathrm{b}}S^*X) \longrightarrow 0. \tag{3.3}$$

Indeed, consider the exact sequence

$$0 \longrightarrow C^*(\Gamma_{\mathrm{b}}(X) \times (0,1]) \longrightarrow C^*(\Gamma^{\tan}_{\mathrm{b}}(X)) \xrightarrow{r_0} C^*({}^{\mathrm{b}}TX) \cong C_0({}^{\mathrm{b}}T^*X) \longrightarrow 0, \tag{3.4}$$

in which the ideal is K-contractible and set

$$\mathrm{Ind}^a_X = r_1 \circ r_0^{-1} : K^0_{\mathrm{top}}({}^{\mathrm{b}}T^*X) \to K_0(\mathcal{K}_{\mathrm{b}}(X)), \tag{3.5}$$

where $r_1 : K_0\big(C^*(\Gamma^{\tan}_{\mathrm{b}}(X))\big) \to K_0\big(C^*(\Gamma_{\mathrm{b}}(X))\big)$ is induced by the restriction morphism to $t = 1$. Applying a mapping cone argument to the exact sequence (3.1) gives a commutative diagram

$$\begin{tikzcd} K_1(C({}^{\mathrm{b}}S^*X)) \arrow[rr, "I"] \arrow[dr, "\delta"'] & & K_0(\mathcal{K}_{\mathrm{b}}(X)) \\ & K^0_{\mathrm{top}}({}^{\mathrm{b}}T^*X) \arrow[ur, "\mathrm{Ind}^a_X"'] & \end{tikzcd} \tag{3.6}$$

Therefore we get, as announced:

$$\mathrm{Ind}_{\mathrm{an}}(D) = \mathrm{Ind}^a_X([\sigma_{\mathrm{b}}(D)]). \tag{3.7}$$

The map $\mathrm{Ind}^a_X$ will be called the *analytic index morphism* of $X$. A closely related homomorphism is the *boundary analytic index morphism*, in which the restriction to $X \times \{1\}$ is replaced by the one to $\partial X \times \{1\}$, that is, we set

$$\mathrm{Ind}^\partial_X = r_\partial \circ r_0^{-1} : K_0(C_0({}^{\mathrm{b}}T^*X)) \to K_0(C^*(\Gamma_{\mathrm{b}}(X)|_{\partial X})), \tag{3.8}$$

where $r_\partial$ is induced by the homomorphism $C^*(\Gamma^{\tan}_{\mathrm{b}}(X)) \to C^*(\Gamma_{\mathrm{b}}(X))|_{\partial X}$. We have of course

$$\mathrm{Ind}^\partial_X = r_{1,\partial} \circ \mathrm{Ind}^a_X \tag{3.9}$$

if $r_{1,\partial}$ denotes the map induced by the homomorphism $C^*(\Gamma_{\mathrm{b}}(X)) \to C^*(\Gamma_{\mathrm{b}}(X)|_{\partial X})$.

**3B. *Fredholm index morphism.*** In general, elliptic b-operators on $X$ are not Fredholm. Indeed, to construct an inverse of a b-operator modulo compact terms, we have to invert not only the principal symbol, but also all the family of boundary symbols. One way to summarize this situation is to introduce the algebra of full, or joint, symbols. Let $\mathcal{H}$ be the set of closed boundary hyperfaces of $X$, and set

$$\mathcal{A}_{\mathcal{F}} = \Big\{ \big(a, (q_H)_{H\in\mathcal{H}}\big) \in C^\infty({}^{\mathrm{b}}S^*X) \times \prod_{H\in\mathcal{H}} \Psi^0(\Gamma_{\mathrm{b}}(X)|_H) : \\ \forall H \in \mathcal{H},\ a|_H = \sigma_{\mathrm{b}}(q_H) \Big\}. \tag{3.10}$$

The full symbol map:

$$\sigma_F : \Psi^0(\Gamma_{\mathrm{b}}(X)) \ni P \mapsto \big(\sigma_{\mathrm{b}}(P), (P|_H)_{H\in\mathcal{H}}\big) \in \mathcal{A}_{\mathcal{F}} \tag{3.11}$$

extends to the $C^*$-closures of the algebras and the assertion about the invertibility modulo compact operators amounts to the exactness of the sequence [Lauter et al.

2000]:

$$0 \longrightarrow \mathcal{K}(X) \longrightarrow \overline{\Psi^0(\Gamma_{\mathrm{b}}(X))} \xrightarrow{\sigma_F} \overline{\mathcal{A}_{\mathcal{F}}} \longrightarrow 0 \tag{3.12}$$

**Definition 3.13** (full ellipticity). An operator $D \in \Psi^0(\Gamma_{\mathrm{b}}(X))$ is said to be fully elliptic if $\sigma_F(D)$ is invertible.

We then recall the following result of Loya [2005] (the statement also appears in [Melrose and Piazza 1992]). Remember that b-Sobolev spaces $H^s_{\mathrm{b}}(X)$ are defined using b-metrics and b-operators map $H^m_{\mathrm{b}}(X)$ to $H^{s-m}_{\mathrm{b}}(X)$ continuously for every $s$.

**Theorem 3.14** [Loya 2005, Theorem 2.3]. *An operator $D \in \Psi^0_{\mathrm{b}}(X)$ is fully elliptic if and only if it is Fredholm on $H^s_{\mathrm{b}}(X)$ for some $s$ (and then for any $s$, with Fredholm index independent of $s$).*

For a given fully elliptic operator $D$, we denote by $\mathrm{Ind}_{\mathrm{Fred}}(D)$ its Fredholm index. We are going to express this number in terms of K-theory and clarify the relationship between the analytical index and full ellipticity on $X$ using deformation groupoids. Let us start with the tangent groupoid

$$\Gamma_{\mathrm{b}}(X)^{\tan} := (G_c(\tilde{X}, (\rho_i))^{\tan})|_{X\times[0,1]} = T_{\mathrm{b}}X \sqcup (\Gamma_{\mathrm{b}}(X)\times(0,1]) \rightrightarrows X\times[0,1]. \tag{3.15}$$

Now we introduce the two following saturated subspaces of $X \times [0, 1]$:

$$X_{\mathcal{F}} := X\times[0,1]\setminus\partial X\times\{1\} \quad \text{and} \quad X_\partial := X_{\mathcal{F}}\setminus\mathring{X}\times(0,1] = X\cup\partial X\times[0,1). \tag{3.16}$$

The *Fredholm b-groupoid* and the *noncommutative tangent space* of $X$ are defined by

$$\Gamma_{\mathrm{b}}(X)^{\mathrm{Fred}} := \Gamma_{\mathrm{b}}(X)^{\tan}|_{X_{\mathcal{F}}} \quad \text{and} \quad T_{\mathrm{nc}}X := \Gamma_{\mathrm{b}}(X)^{\mathrm{Fred}}|_{X_\partial} \tag{3.17}$$

respectively. They are obviously KK-equivalent, as one sees using the exact sequence

$$0 \longrightarrow C^*(\mathring{X}\times\mathring{X}\times(0,1]) \longrightarrow C^*(\Gamma_{\mathrm{b}}(X)^{\mathrm{Fred}}) \xrightarrow{r_F} C^*(T_{\mathrm{nc}}X) \longrightarrow 0 \tag{3.18}$$

whose ideal is K-contractible. We then define the *Fredholm index morphism* by

$$\mathrm{Ind}^X_F = (r_1)_* \circ (r_F)_*^{-1} : K^0(T_{\mathrm{nc}}X) \to K^0(\mathring{X}\times\mathring{X}) \simeq \mathbb{Z}. \tag{3.19}$$

Following [Debord et al. 2015, Definition 10.4], we denote by $\mathrm{FE}(X)$ the group of order-0 fully elliptic operators modulo stable homotopy. Then the vocabulary above is justified by:

**Proposition 3.20.** *There exists a group isomorphism*

$$\sigma_{\mathrm{nc}} : \mathrm{FE}(X) \to K_0(C^*(\mathcal{T}_{\mathrm{nc}}X)) \tag{3.21}$$

*such that*

$$\begin{aligned} r_0([\sigma_{\mathrm{nc}}(D)]) &= [\sigma_{\mathrm{b}}(D)] \in K_0(C_0({}^{\mathrm{b}}T^*X)), \\ \mathrm{Ind}_F^X([\sigma_{\mathrm{nc}}(D)]) &= \mathrm{Ind}_{\mathrm{Fred}}(D), \end{aligned} \tag{3.22}$$

*where $r_0$ comes from the natural restriction map $C^*(\mathcal{T}_{\mathrm{nc}}X) \to C_0({}^{\mathrm{b}}T^*X)$.*

This is proved by the method leading to [Savin 2005, Theorem 4; Debord et al. 2015, Theorem 10.6] in exactly the same way. Also, this homotopy classification appears in [Nazaikinskii et al. 2008b], in which the K-homology of a suitable *dual manifold* is used instead of the K-theory of the noncommutative tangent space. Previous related results appeared in [Lauter et al. 2000] for differential operators and using different algebras to classify their symbols.

The construction of the various index maps above is summarized into the commutative diagram:

$$\begin{tikzcd}
& K^0(\Gamma_{\mathrm{b}}^F) \arrow[rr, "i_F"] \arrow[dl, "e_1"'] & & K^0(\Gamma_{\mathrm{b}}^{\tan}) \arrow[rr, "r_\partial"] \arrow[dl, "e_1"'] & & K^0(\Gamma_{\mathrm{b}}|_\partial) \arrow[dl, "\mathrm{id}"'] \arrow[dd] \\
K^0(\mathring{X} \times \mathring{X}) \arrow[rr, "i_0"] & & K^0(\Gamma_{\mathrm{b}}) \arrow[rr, "r_{\mathrm{b}}"] & & K^0(\Gamma_{\mathrm{b}}|_\partial) \arrow[dd] & \\
& K^1(\Gamma_{\mathrm{b}}|_\partial) \arrow[uu, dotted] \arrow[dl, dotted, "\mathrm{id}"'] & & K^1(\Gamma_{\mathrm{b}}^{\tan}) \arrow[ll, dotted, "r_\partial"] \arrow[dl, dotted, "e_1"'] & & K^1(\Gamma_{\mathrm{b}}^F) \arrow[ll, dotted] \arrow[dl, "e_1"'] \\
K^1(\Gamma_{\mathrm{b}}|_\partial) \arrow[uu, "\partial_1"] & & K^1(\Gamma_{\mathrm{b}}) \arrow[ll, "r_{\mathrm{b}}"] & & K^1(\mathring{X} \times \mathring{X}) \arrow[ll, "i_0"] &
\end{tikzcd}$$

**3C. *Fredholm perturbation property.*** We are ready to define the Fredholm perturbation property [Nistor 2003] and its stably homotopic version.

**Definition 3.23.** Let $D \in \Psi_{\mathrm{b}}^m(X)$ be elliptic. We say that $D$ satisfies:

- the *Fredholm perturbation property* ($\mathcal{FP}$) if there is $R \in \Psi_{\mathrm{b}}^{-\infty}(X)$ such that $D + R$ is fully elliptic.
- the *stably homotopic Fredholm perturbation property* ($\mathcal{HFP}$) if there is a fully elliptic operator $D'$ with $[\sigma_{\mathrm{b}}(D')] = [\sigma_{\mathrm{b}}(D)] \in K_0(C^*({}^{\mathrm{b}}TX))$.

We also say that $X$ satisfies the (*stably homotopic*) *Fredholm perturbation property* if any elliptic b-operator on $X$ satisfies (($\mathcal{H}$)$\mathcal{FP}$).

Property ($\mathcal{FP}$) is of course stronger than property ($\mathcal{HFP}$). Nistor [2003] characterized ($\mathcal{FP}$) in terms of the vanishing of an index in some particular cases. Nazaikinskii et al. [2008b] characterized ($\mathcal{HFP}$) for arbitrary manifolds with corners using an index map associated with their dual manifold construction. We now rephrase the result of [Nazaikinskii et al. 2008b] in terms of deformation groupoids.

**Theorem 3.24.** *Let $D$ be an elliptic b-pseudodifferential operator on a compact manifold with corners $X$. Then $D$ satisfies $(\mathcal{HFP})$ if and only if* $\mathrm{Ind}_\partial([\sigma_{\mathrm{b}}(D)]) = 0$. *In particular if $D$ satisfies $(\mathcal{FP})$ then its analytic indicial index vanishes.*

*Proof.* Note that the Fredholm and the tangent groupoids are related by the exact sequence

$$0 \longrightarrow C^*(\Gamma_{\mathrm{b}}^{\mathrm{Fred}}(X)) \xrightarrow{i_F} C^*(\Gamma_{\mathrm{b}}^{\tan}(X)) \xrightarrow{r_\partial} C^*(\Gamma_{\mathrm{b}}(X)_{\partial X}) \longrightarrow 0. \tag{3.25}$$

Then Proposition 3.20, together with this exact sequence and the commutative diagram:

$$\begin{array}{ccc} K_0(C^*(\Gamma_{\mathrm{b}}^F)) & \xrightarrow[r_F]{\simeq} & K_0(C^*(T_{\mathrm{nc}}X)) \\ {\scriptstyle i_F}\big\downarrow & & \big\downarrow{\scriptstyle r_0} \\ K_0(C^*(\Gamma_{\mathrm{b}}^{\tan})) & \xrightarrow[r_0]{\simeq} & K_0(C^*({}^{\mathrm{b}}TX)) \end{array} \tag{3.26}$$

yields the result. □

Loosely speaking, this theorem tells us that the K-theory of $\Gamma_{\mathrm{b}}(X)_{\partial X}$, or equivalently the one of $\Gamma_{\mathrm{b}}(X)$ as we shall see later, is the receptacle for the obstruction to Fredholmness of elliptic symbols in the b-calculus. This is why we now focus on the understanding of these K-theory groups. If the result is well known in codimension less or equal to 1, the general case is far from understood. Meanwhile, we will also clarify the equivalent role played by $\Gamma_{\mathrm{b}}(X)$ and $\Gamma_{\mathrm{b}}(X)_{\partial X}$.

## 4. The conormal homology of a manifold with corners

In all this section, $X$ is a manifold with embedded corners of codimension $d$, whose connected hyperfaces $H_1, \dots, H_N$ are provided with defining functions $r_1, \dots, r_N$.

**4A.** ***Definition of the homology.*** The one form $e_j = dr_j$ trivializes the conormal bundle of $H_j$ for any $1 \le j \le N$. By convention, $p$-tuples of integers $I = (i_1, \dots, i_p) \in \mathbb{N}^p$ are always labeled so that $1 \le i_1 < \cdots < i_p \le N$. Let $I$ be a $p$-tuple, set

$$H_I = r_I^{-1}(\{0\}) = H_{i_1} \cap \cdots \cap H_{i_p}, \tag{4.1}$$

and denote by $c(I)$ the set of open connected faces of codimension $p$ included in $H_I$. Also, we denote by $e_I$ the exterior product

$$e_I = e_{i_1} \cdot e_{i_2} \cdot \cdots \cdot e_{i_p}. \tag{4.2}$$

Let $f$ be a face of codimension $p$ and $I$ the $p$-tuple such that $f \in c(I)$. The conormal bundle $N(f)$ of $f$ has a global basis given by the sections $e_j$, $j \in I$, and its orientations are identified with $\pm e_I$. For any integer $0 \le p \le d$, we denote by $C_p(X)$ the free $\mathbb{Z}$-module generated by

$$\{f \otimes \varepsilon : f \in F_p,\ \varepsilon \text{ is an orientation of } N(f)\}. \tag{4.3}$$

Let $f \in F_p$, $\varepsilon_f$ an orientation of $N(f)$ and $g \in F_{p-1}$ such that $f \subset \bar{g}$. The face $f$ is characterized in $\bar{g}$ by the vanishing of a defining function $r_{i_{(g,f)}}$. Then the contraction $e_{i_{(g,f)}} \lrcorner\, \varepsilon_f$ is an orientation of $N(g)$. Recall that the contraction $\lrcorner$ is defined by

$$e_i \lrcorner\, e_I = \begin{cases} 0 & \text{if } i \notin I, \\ (-1)^{j-1} e_{I \setminus \{i\}} & \text{if } i \text{ is the } j\text{-th coordinate of } I. \end{cases} \tag{4.4}$$

We then define $\delta_p : C_p(X) \to C_{p-1}(X)$ by

$$\delta_p(f \otimes \varepsilon_f) = \sum_{\substack{g \in F_{p-1}, \\ f \subset \bar{g}}} g \otimes e_{i_{(g,f)}} \lrcorner\, \varepsilon_f. \tag{4.5}$$

It is not hard to check directly that $(C_*(X), \delta_*)$ is a differential complex. Actually, $\delta_*$ is the component of degree $-1$ of another natural differential map $\delta^{\mathrm{pcn}} = \sum_{k \ge 0} \delta^{2k+1}$, which eventually produces a quasiisomorphic differential complex. Details are provided in the Appendix.

We define the *conormal homology* of $X$ as the homology of $(C_*(X), \delta_*)$, and we write

$$H_p^{\mathrm{cn}}(X) := H_p(C_*(X), \delta_*). \tag{4.6}$$

This homology was first considered in [Bunke 2009], in a slightly different but equivalent way. Also, the graduation of the conormal homology into even and odd degree, called here *periodic conormal homology*, will be used and we denote

$$H_0^{\mathrm{pcn}}(X) = \oplus_{p \ge 0} H_{2p}^{\mathrm{cn}}(X) \quad \text{and} \quad H_1^{\mathrm{pcn}}(X) = \oplus_{p \ge 0} H_{2p+1}^{\mathrm{cn}}(X). \tag{4.7}$$

**4B.** ***Examples.*** The determination of the groups $H_*^{\mathrm{cn}}(X)$ is completely elementary in all concrete cases. In the following examples, it is understood that faces $f$ arise with the orientation given by $e_I$ if $f \in c(I)$.

**Example 4.8.**

- Assume that $X$ has no boundary. Then $H_0^{\mathrm{pcn}}(X) = H_0^{\mathrm{cn}}(X) \simeq \mathbb{Z}$, $H_1^{\mathrm{pcn}}(X) = 0$.
- Assume that $X$ has a boundary with $n$ connected components. Then $H_0^{\mathrm{pcn}}(X) = 0$ and $H_1^{\mathrm{pcn}}(X) = H_1^{\mathrm{cn}}(X) \simeq \mathbb{Z}^{n-1}$. More precisely, if we set $F_1 = \{f_1, \dots, f_n\}$ then $\{f_1 - f_2, f_2 - f_3, \dots, f_{n-1} - f_n\}$ provides a basis of $\ker \delta_1$.
- Assume that $X$ has codimension 2 and that $\partial X$ is connected. Then $H_0^{\mathrm{pcn}}(X) = H_2^{\mathrm{cn}}(X) = \ker \delta_2 \simeq \mathbb{Z}^k$, where all nonnegative integers $k$ can arise. For instance, consider the unit closed ball $B$ in $\mathbb{R}^3$, cut $k+1$ small disjoint disks out of its

boundary and glue two copies of such spaces along the pairs of cut out disks. We get a space $X$ satisfying the statement: the boundaries $s_0, \dots, s_k$ of the original disks provide a basis of $F_2$ and the family $s_0 - s_j$, $1 \le j \le k$ a basis of $\ker \delta_2$. Finally, $[0, +\infty)^2$ provides an example with $k = 0$.

- Consider the cube $X = [0, 1]^3$.
  (1) We have $H_0^{\mathrm{pcn}}(X) = 0$ and $H_1^{\mathrm{pcn}}(X) = H_3^{\mathrm{cn}}(X) \simeq \mathbb{Z}$.
  (2) Remove a small open cube into the interior of $X$ and call the new space $Y$. Then
  $$H_0^{\mathrm{pcn}}(Y) = 0 \quad \text{and} \quad H_1^{\mathrm{pcn}}(Y) = H_3^{\mathrm{cn}}(Y) \oplus H_1^{\mathrm{cn}}(Y) \simeq \mathbb{Z}^2 \oplus \mathbb{Z}.$$
  (3) Remove a small open ball from the interior of $X$ and call the new space $Z$. Then
  $$H_0^{\mathrm{pcn}}(Z) = 0 \quad \text{and} \quad H_1^{\mathrm{pcn}}(Y) = H_3^{\mathrm{cn}}(Y) \oplus H_1^{\mathrm{cn}}(Y) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

**4C.** ***Long exact sequence in conormal homology.*** We define a filtration of $X$ by open submanifolds with corners by setting:

$$X_m = \bigcup_{f \in F_k,\ k \le m} f, \quad 0 \le m \le d. \tag{4.9}$$

This leads to differential complexes $(C_*(X_m), \delta)$ for $0 \le m \le d$. We can also filter the differential complex $(C_*(X), \delta)$ by the codimension of faces:

$$F_m(C_*(X)) = \bigoplus_{k=0}^{m} C_k(X). \tag{4.10}$$

There is an obvious identification $C_*(X_m) \simeq F_m(C_*(X))$ and we thus consider $(C_*(X_m), \delta)$ as a subcomplex of $(C_*(X), \delta)$, with quotient complex denoted by $(C_*(X, X_m), \delta)$. The quotient module is also naturally embedded in $C_*(X)$:

$$C_*(X, X_m) = C_*(X)/C_*(X_m) \simeq \bigoplus_{k=m+1}^{d} C_k(X) \subset C_*(X). \tag{4.11}$$

The embedding, denoted by $\rho$, is a section of the quotient map. The short exact sequence:

$$0 \longrightarrow C_*(X_m) \longrightarrow C_*(X) \longrightarrow C_*(X, X_m) \longrightarrow 0 \tag{4.12}$$

induces a long exact sequence in conormal homology:

$$\cdots \xrightarrow{\partial_{p+1}} H_p^{\mathrm{cn}}(X_m) \longrightarrow H_p^{\mathrm{cn}}(X) \longrightarrow H_p^{\mathrm{cn}}(X, X_m) \xrightarrow{\partial_p} H_{p-1}^{\mathrm{cn}}(X_m) \longrightarrow \cdots \tag{4.13}$$

and we need to make precise the connecting homomorphism.

**Proposition 4.14.** *Let* $[c] \in H_p^{\mathrm{cn}}(X, X_m)$. *Then*

$$\partial_p[c] = [\delta(\rho(c))]. \tag{4.15}$$

*Proof.* Since $c$ is by assumption a cycle in $(C_*(X, X_m), \delta)$, the chain $\rho(c)$ has a boundary made of faces contained in $X_m$. The result follows. $\square$

**Remarks 4.16.** • We can replace $X$ by $X_l$ and quotient the exact sequence (4.12) by $C_*(X_q)$ for any integers $l, m, q$ such that $0 \le q \le m \le l \le d$. This leads to long exact sequences:

$$\cdots \xrightarrow{\partial} H_p^{\mathrm{cn}}(X_m, X_q) \longrightarrow H_p^{\mathrm{cn}}(X_l, X_q) \longrightarrow H_p^{\mathrm{pcn}}(X_l, X_m) \xrightarrow{\partial} H_{p-1}^{\mathrm{pcn}}(X_m, X_q) \longrightarrow \cdots \tag{4.17}$$

whose connecting homomorphisms are again given by (4.15).

• If we split the conormal homology into even and odd periodic groups, then the long exact sequence (4.13) becomes a six-term exact sequence:

$$\begin{tikzcd} H_0^{\mathrm{pcn}}(X_m) \arrow[r] & H_0^{\mathrm{pcn}}(X) \arrow[r] & H_0^{\mathrm{pcn}}(X, X_m) \arrow[d, "\partial^0"] \\ H_1^{\mathrm{pcn}}(X, X_m) \arrow[u, "\partial^1"] & H_1^{\mathrm{pcn}}(X) \arrow[l] & H_1^{\mathrm{pcn}}(X_m) \arrow[l] \end{tikzcd} \tag{4.18}$$

where $\partial^0, \partial^1$ are given by the direct sum in even/odd degrees of the maps $\partial_*$ of Proposition 4.14.

• We can replace $X_m$ in the exact sequence (4.12) by an open saturated submanifold $U \subset X_m$, that is, an open subset of $X$ consisting of a union of faces. This gives in the same way a subcomplex $(C_*(U), \delta)$ of $(C_*(X), \delta)$ and a section $\rho : C_*(X, U) \to C_*(X)$ allowing us to state Proposition 4.14 verbatim. More generally, if $U$ is any open submanifold of $X$ and $\widetilde{U}$ denotes the smallest open saturated submanifold containing $U$, then any face $f$ of $U$ is contained in a unique face $\tilde{f}$ of $X$ and an orientation of $N(f)$ determines an orientation of $N(\tilde{f})$. This gives rise to a quasiisomorphism $C_*(U) \to C_*(\widetilde{U})$.

Finally, assume that $d \ge 1$. Since $X$ is connected, the map $\delta_1 : C_1(X) \to C_0(X)$ is surjective, which implies by Proposition 4.14 the surjectivity of the connecting homomorphism $\partial^1 : H_1^{\mathrm{pcn}}(X, X_0) \to H_0^{\mathrm{pcn}}(X_0)$. This fact and $H_1^{\mathrm{pcn}}(X_0) = 0$ gives, using (4.18), the useful corollary:

**Corollary 4.19.** *For any connected manifold with corners $X$ of codimension $d \ge 1$ the canonical morphism $H_0^{\mathrm{pcn}}(X) \to H_0^{\mathrm{pcn}}(X, X_0)$ is an isomorphism.*

**4D.** ***Torsion free in low codimensions.*** Here we will show that up to codimension 2 the conormal homology groups (and later on the K-theory groups) are free abelian groups.

**Lemma 4.20.** *Let $X$ be of arbitrary codimension and assume that $\partial X$ has $l$ connected components. Then $H_1^{\mathrm{cn}}(X) \simeq \mathbb{Z}^{l-1}$.*

*Proof.* For any face $f$, denote by $cc(f)$ the connected component of $\partial X$ containing $f$. It is obvious that $\ker \delta_1$ is generated by the differences $f - g$, where $f, g$ run through $F_1$. Let $f, g \in F_1$ such that $cc(f) = cc(g)$. Then there exist $f_0, \dots, f_l \in F_1$ such that $f = f_0$, $g = f_l$ and $\overline{f_i} \cap \overline{f_{i+1}} \neq \varnothing$ for any $i$. Therefore for any $i$, there exists $f_{i,i+1} \in F_2$ such that $\delta_2(f_{i,i+1}) = f_i - f_{i+1}$, hence $f - g = \delta_2\big(\sum f_{i,i+1}\big)$ is a boundary in conormal homology.

Now assume that $cc(f) \neq cc(g)$. By the previous discussion, we also have $[f-g] = [f'-g'] \in H_1^{\mathrm{cn}}(X)$ for any $f', g' \in F_1$ such that $f' \subset cc(f)$ and $g' \subset cc(g)$. Therefore, pick up one hyperface in each connected component of $\partial X$, call them $f_1, \dots, f_l$, and set $\alpha_i = [f_1 - f_i] \in H_1^{\mathrm{cn}}(X)$ for $i \in \{2, \dots, l\}$. It is obvious that $(\alpha_i)_{2 \le i \le l}$ generates $H_1^{\mathrm{cn}}(X)$. So, consider integers $x_2, \dots, x_l$ such that

$$\sum_{i=2}^{l} x_i \alpha_i = 0.$$

In other words, there exists $x \in C_2(X)$ such that

$$\left(\sum_{i=2}^{l} x_i\right) f_1 - \sum_{i=2}^{l} x_i f_i = \delta_2(x). \tag{4.21}$$

For any $p \ge 1$ and $2 \le j \le l$ denote by $\pi_j : C_p(X) \to C_p(X)$ the map defined by $\pi_j(h) = h$ if $h \subset cc(f_j)$ and $\pi_j(h) = 0$ otherwise. All the $\pi_i$ commute with $\delta_*$, hence (4.21) gives

$$\text{for all } 2 \le j \le l, \quad x_j f_j = \delta_2(\pi_j(x)).$$

Since $\delta_1(f_j) = \mathring{X} \neq 0$, we conclude $x_j = 0$ for all $j$. □

**Theorem 4.22.** *Assume that $X$ is connected and has codimension $d \le 2$. Then $H_*^{\mathrm{pcn}}(X)$ is a free abelian group.*

*Proof.* This is essentially a compilation of previous examples and computations. The first two cases in Section 4B give the result for $d = 0$ and $d = 1$. If $X$ is of codimension 2, then the third case in Section 4B says that $H_0^{\mathrm{pcn}}(X)$ is free. In codimension 2 again, we have $H_1^{\mathrm{pcn}}(X) = H_1^{\mathrm{cn}}(X)$, hence we are done by Lemma 4.20. □

**Remark 4.23.** If $\mathrm{codim}(X) = 3$, then $H_1^{\mathrm{pcn}}(X) = H_1^{\mathrm{cn}}(X) \oplus H_3^{\mathrm{cn}}(X)$. Since $H_3^{\mathrm{cn}}(X) = \ker \delta_3$, Lemma 4.20 also gives that $H_1^{\mathrm{pcn}}(X)$ is free. The combinatorics needed to prove that $H_2^{\mathrm{cn}}(X)$ — and therefore $H_0^{\mathrm{pcn}}(X)$ — is free are much more involved. The torsion of conormal homology for manifolds of arbitrary codimension will be studied somewhere else.

**4E. *Künneth formula for conormal homology.*** Taking advantage of the previous paragraph, we consider a product $X = X_1 \times X_2$ of two manifolds with corners, one of them being of codimension $\leq 2$. It is understood that the defining functions used for $X$ are obtained by pulling back the ones used for $X_1$ and $X_2$. The tensor product $(\widehat{C}_*, \hat{\delta})$ of the conormal complexes of $X_1$ and $X_2$ is given by

$$\widehat{C}_p = \bigoplus_{s+t=p} C_s(X_1) \otimes C_t(X_2) \quad \text{and} \quad \hat{\delta}(x \otimes y) = \delta(x) \otimes y + (-1)^t x \otimes \delta(y), \tag{4.24}$$

where $x \in C_t(X_1)$ in the second formula. We have an isomorphism of differential complexes:

$$(\widehat{C}_*, \hat{\delta}) \simeq (C_*(X), \delta). \tag{4.25}$$

It is given by the map

$$\Psi_p : \widehat{C}_p = \bigoplus_{s+t=p} C_s(X_1) \otimes C_t(X_2) \to C_p(X) \tag{4.26}$$

defined by

$$(f \otimes \varepsilon_f) \otimes (g \otimes \varepsilon_g) \mapsto (f \times g) \otimes \varepsilon_f \cdot \varepsilon_g, \tag{4.27}$$

where we did not distinguish differential forms on $X_j$ and their pull-back to $X$ via the canonical projections and $\cdot$ denotes again the exterior product. Since $H_*^{\mathrm{cn}}(X_j)$ is torsion free for $j = 1$ or $2$ by assumption, we get by Künneth Theorem:

$$H_p(\widehat{C}_*, \hat{\delta}) = \bigoplus_{s+t=p} H_s^{\mathrm{cn}}(X_1) \otimes H_t^{\mathrm{cn}}(X_2). \tag{4.28}$$

Therefore:

**Proposition 4.29** (Künneth formula). *Assume that $X = X_1 \times X_2$ with one factor at least of codimension $\leq 2$. Then we have*:

$$H_0^{\mathrm{pcn}}(X) \simeq H_0^{\mathrm{pcn}}(X_1) \otimes H_0^{\mathrm{pcn}}(X_2) \oplus H_1^{\mathrm{pcn}}(X_1) \otimes H_1^{\mathrm{pcn}}(X_2), \tag{4.30}$$

$$H_1^{\mathrm{pcn}}(X) \simeq H_0^{\mathrm{pcn}}(X_1) \otimes H_1^{\mathrm{pcn}}(X_2) \oplus H_1^{\mathrm{pcn}}(X_1) \otimes H_0^{\mathrm{pcn}}(X_2). \tag{4.31}$$

The following straightforward corollary will be useful later on:

**Corollary 4.32.** *If $X = \Pi_i X_i$ is a finite product of manifolds with corners $X_i$ with* $\mathrm{codim}(X_i) \leq 2$, *then the groups $H_*^{\mathrm{pcn}}(X)$ are torsion free.*

The exact same arguments as above work to show that the Künneth formula holds in full generality for conormal homology with rational coefficients, i.e., for $H_*^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. We state the proposition as we will use it later:

**Proposition 4.33** (Künneth formula with rational coefficients). *For $X = X_1 \times X_2$ we have*:

$$H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \left(H_0^{\mathrm{pcn}}(X_1) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \otimes \left(H_0^{\mathrm{pcn}}(X_2) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \oplus \left(H_1^{\mathrm{pcn}}(X_1) \otimes_{\mathbb{Z}} \mathbb{Q}\right) \otimes \left(H_1^{\mathrm{pcn}}(X_2) \otimes_{\mathbb{Z}} \mathbb{Q}\right), \tag{4.34}$$

$$H_1^{\mathrm{pcn}}(X)\otimes_{\mathbb{Z}}\mathbb{Q}\simeq\big(H_0^{\mathrm{pcn}}(X_1)\otimes_{\mathbb{Z}}\mathbb{Q}\big)\otimes\big(H_1^{\mathrm{pcn}}(X_2)\otimes_{\mathbb{Z}}\mathbb{Q}\big)\\ \oplus\big(H_1^{\mathrm{pcn}}(X_1)\otimes_{\mathbb{Z}}\mathbb{Q}\big)\otimes\big(H_0^{\mathrm{pcn}}(X_2)\otimes_{\mathbb{Z}}\mathbb{Q}\big). \tag{4.35}$$

## 5. The computation of $K_*(\mathcal{K}_{\mathrm{b}}(X))$

We keep all the notation and conventions of Section 4. In particular, the defining functions induce a trivialization of the conormal bundle of any face $f$,

$$N(f)\simeq f\times E_f, \tag{5.1}$$

in which the $p$-dimensional real vector space $E_f$ inherits a basis $b_f=(e_i)_{i\in I}$, where $I$ is characterized by $f\in c(I)$. These data induce an isomorphism

$$\Gamma_{\mathrm{b}}(X)|_f\simeq C^*(\mathcal{C}(f)\times E_f), \tag{5.2}$$

where $\mathcal{C}(f)$ denotes the pair groupoid over $f$, as well as a linear isomorphism $\varphi_f:\mathbb{R}^p\to E_f$.

Also, the filtration (4.9) gives rise to the following filtration of the $C^*$-algebra $\mathcal{K}_{\mathrm{b}}(X)=C^*(\Gamma_{\mathrm{b}}(X))$ by ideals:

$$\mathcal{K}(L^2(\mathring{X}))=A_0\subset A_1\subset\cdots\subset A_d=A=\mathcal{K}_{\mathrm{b}}(X), \tag{5.3}$$

with $A_m=C^*(\Gamma_{\mathrm{b}}(X)|_{X_m})$ for any $0\le m\le d$. The isomorphisms (5.2) induce

$$A_m/A_{m-1}\simeq C^*(\Gamma_{\mathrm{b}}(X)|_{X_m\setminus X_{m-1}})\simeq\bigoplus_{f\in F_m}C^*(\mathcal{C}(f)\times E_f). \tag{5.4}$$

**5A.** ***The first differential of the spectral sequence for*** $K_*(A)$**.** The K-theory spectral sequence $(E^r_{*,*},d^r_{*,*})_{r\ge1}$ associated with (5.3) [Schochet 1981; Kono and Tamaki 2006] converges to

$$E^\infty_{p,q}=K_{p+q}(A_p)/K_{p+q}(A_{p-1}). \tag{5.5}$$

Here we have set $K_n(A)=K_0(A\otimes C_0(\mathbb{R}^n))$ for any $C^*$-algebra $A$. By construction, all the terms $E^r_{p,2q+1}$ vanish, and by Bott periodicity, $E^r_{p,2q}\simeq E^r_{p,0}$. Also, all the differentials $d^{2r}_{p,q}$ vanish. By definition

$$d^1_{p,q}:E^1_{p,q}=K_{p+q}(A_p/A_{p-1})\to E^1_{p-1,q}=K_{p+q-1}(A_{p-1}/A_{p-2}) \tag{5.6}$$

is the connecting homomorphism of the short exact sequence

$$0\to A_{p-1}/A_{p-2}\to A_p/A_{p-2}\to A_p/A_{p-1}\to 0. \tag{5.7}$$

By (5.4), we get isomorphisms:

$$E^1_{p,q}\simeq\bigoplus_{f\in F_p}K_{p+q}\big(C^*(\mathcal{C}(f)\times E_f)\big). \tag{5.8}$$

Since the real vector space $E_f$ has dimension $p$, the groups $E^1_{p,q}$ vanish for odd $q$ and for even $q$, we have after applying Bott periodicity, $E^1_{p,q} \simeq \mathbb{Z}^{\# F_p}$.

Melrose and Nistor [1998, Theorem 9] already achieved the computation of $d^1_{*,*}$. In order to relate the terms $E^2_{*,*}$ with the elementary defined conormal homology, we reproduce their computation in a slightly different way. Our approach is based on the next two lemmas.

**Lemma 5.9.** *Let $\mathbb{R}_+ \rtimes \mathbb{R}$ be the groupoid of the action of $\mathbb{R}$ onto $\mathbb{R}_+$ given by*

$$t.\lambda = te^{\lambda}, \quad t \in \mathbb{R}_+,\ \lambda \in \mathbb{R}. \tag{5.10}$$

*The element $\alpha \in \mathrm{KK}_1(C^*(\mathbb{R}), C^*(\mathbb{R}^*_+))$ associated with the exact sequence*

$$0 \to C^*(\mathcal{C}(\mathbb{R}^*_+)) \to C^*(\mathbb{R}_+ \rtimes \mathbb{R}) \to C^*(\mathbb{R}) \to 0 \tag{5.11}$$

*is a* KK*-equivalence.*

*Proof.* By the Thom–Connes isomorphism, the $C^*$-algebras $C^*(\mathbb{R}_+ \rtimes \mathbb{R})$ and $C^*(\mathbb{R}_+ \times \mathbb{R})$ are KK-equivalent. The latter being K-contractible, the result follows. □

**Lemma 5.12.** *Let $\mathbb{R}_+ \rtimes_i \mathbb{R}^p$ be the groupoid given by the action of the $i$-th coordinate of $\mathbb{R}^p$ on $\mathbb{R}_+$ by (5.10). Let $\alpha_{i,p} \in \mathrm{KK}_1(C^*(\mathbb{R}^p), C^*(\mathbb{R}^{p-1}))$ be the* KK*-element induced by the exact sequence*

$$0 \to C^*(\mathcal{C}(\mathbb{R}^*_+) \times \mathbb{R}^{p-1}) \to C^*(\mathbb{R}_+ \rtimes_i \mathbb{R}^p) \to C^*(\mathbb{R}^p) \to 0. \tag{5.13}$$

*Then for all $1 \le i \le p$ we have*

$$\alpha_{i,p} = (-1)^{i-1}\alpha_{1,p} \quad \textit{and} \quad \alpha_{1,p} = \sigma_{C^*(\mathbb{R}^{p-1})}(\alpha), \tag{5.14}$$

*where $\sigma_D : K_*(A, B) \to K_*(A \otimes D, B \otimes D)$ denotes the Kasparov suspension map.*

*Proof.* Let $\tau$ be a permutation of $\{1, 2, \dots, p\}$ and $i \in \{1, \dots, p\}$. We denote in the same way the corresponding automorphisms of $\mathbb{R}^p$ and $C^*(\mathbb{R}^p)$. We have a groupoid isomorphism

$$\widetilde{\tau} : \mathbb{R}_+ \rtimes_i \mathbb{R}^p \xrightarrow{\simeq} \mathbb{R}_+ \rtimes_{\tau(i)} \mathbb{R}^p$$

and if we denote by $\tau_i$ the automorphism of $\mathbb{R}^{p-1}$ obtained by removing the $i$-th factor in the domain of $\tau$ and the $\tau(i)$-th factor in the range of $\tau$, we get a commutative diagram of exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & C^*(\mathcal{C}(\mathbb{R}^*_+) \times \mathbb{R}^{p-1})) \arrow[r] \arrow[d, "\tau_i"] & C^*(\mathbb{R}_+ \rtimes_i \mathbb{R}^p) \arrow[r] \arrow[d, "\widetilde{\tau}"] & C^*(\mathbb{R}^p) \arrow[r] \arrow[d, "\tau"] & 0 \\ 0 \arrow[r] & C^*(\mathcal{C}(\mathbb{R}^*_+) \times \mathbb{R}^{p-1})) \arrow[r] & C^*(\mathbb{R}_+ \rtimes_{\tau(i)} \mathbb{R}^p) \arrow[r] & C^*(\mathbb{R}^p) \arrow[r] & 0 \end{tikzcd} \tag{5.15}$$

It follows that

$$\alpha_{\tau(i),p} = [\tau^{-1}] \otimes \alpha_{i,p} \otimes [\tau_i] \in \mathrm{KK}_1\big(C^*(\mathbb{R}^p), \mathcal{K} \otimes C^*(\mathbb{R}^{p-1})\big). \tag{5.16}$$

Taking $\tau = (1, i)$, we get $\tau = \tau^{-1}$ and $\tau_i = \mathrm{id}$, so that $\alpha_{i,p} = [\tau] \otimes \alpha_{1,p}$. Moreover, observe that for any $j$,

$$[(j-1, j)] = 1_{j-2} \otimes [f] \otimes 1_{p-j} \in \mathrm{KK}(C^*(\mathbb{R}^p), C^*(\mathbb{R}^p)), \tag{5.17}$$

where $[f] = -1 \in \mathrm{KK}(C^*(\mathbb{R}^2), C^*(\mathbb{R}^2))$ is the class of the flip automorphism and we have used the identification

$$C^*(\mathbb{R}^p) = C^*(\mathbb{R}^{j-2}) \otimes C^*(\mathbb{R}^2) \otimes C^*(\mathbb{R}^{p-j}).$$

Using

$$(1, i) = (1, 2).(2, 3).\ldots.(i-1, i)$$

now gives $[\tau] = (-1)^{i-1}$. Factorizing $C^*(\mathbb{R}^{p-1})$ on the right in the sequence (5.13) for $i = 1$ gives the assertion $\alpha_{1,p} = \sigma_{C^*(\mathbb{R}^{p-1})}(\alpha)$. □

Using the canonical isomorphism $\mathrm{KK}_1(C^*(\mathbb{R}), C^*(\mathbb{R}_+^*)) \simeq \mathrm{KK}_1(C_0(\mathbb{R}), \mathbb{C})$, we can define a generator $\beta$ of $K_1(C_0(\mathbb{R}))$ by

$$\beta \otimes \alpha = +1. \tag{5.18}$$

For any $f \in F_p$ we then obtain a generator $\beta_f$ of $K_p(C_0(E_f))$ by

$$\beta_f = (\varphi_f)_*(\beta^p) \in K_p(C_0(E_f)), \tag{5.19}$$

where $\beta^p$ is the external product:

$$\beta^p = \beta \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \beta \in K_p(C_0(\mathbb{R}^p)). \tag{5.20}$$

Picking up rank one projectors $p_f$ in $C^*(\mathcal{C}(f))$, we get a basis of the free $\mathbb{Z}$-module $E^1_{p,0}$:

$$(p_f \otimes \beta_f)_{f \in F_p}. \tag{5.21}$$

Bases of $E^1_{p,q}$ for all even $q$ are deduced from the previous one by applying Bott periodicity.

Now consider faces $f \in F_p$ and $g \in F_{p-1}$ such that $f \subset \partial \bar{g}$. The $p$ and $p-1$ tuples $I$, $J$ such that $f \in c(I)$ and $g \in c(J)$ differ by exactly one index, say the $j$-th, and we define

$$\sigma(f, g) = (-1)^{j-1}. \tag{5.22}$$

Introduce the exact sequence

$$0 \to C^*(\mathcal{C}(f \times \mathbb{R}_+^*) \times E_g) \to C^*(\mathcal{C}(f) \times (\mathbb{R}_+ \rtimes_j E_f)) \to C^*(\mathcal{C}(f) \times E_f) \to 0, \tag{5.23}$$

where $\mathbb{R}_+ \rtimes_j E_f$ denotes the transformation groupoid where the $j$-th coordinate (only) of $E_f$ acts on $\mathbb{R}_+$ by (5.10) again. We denote by

$$\partial_{f,g} : K_p(C^*(\mathcal{C}(f) \times E_f)) \to K_{p-1}(C^*(\mathcal{C}(g) \times E_g))$$

the connecting homomorphism associated with (5.23), followed by the unique KK-equivalence

$$C^*(\mathcal{C}(f \times \mathbb{R}_+^*)) \to C^*(\mathcal{C}(g)) \tag{5.24}$$

provided by any tubular neighborhood of $f$ into $g$.

**Proposition 5.25.** *With the notation above, we get*

$$\partial_{f,g}(p_f \otimes \beta_f) = \sigma(f, g).p_g \otimes \beta_g. \tag{5.26}$$

*Proof.* Identify $E_f \simeq \mathbb{R}^p$ and $E_g \simeq \mathbb{R}^{p-1}$ using $b_f, b_g$ and apply Lemmas 5.12 and 5.9. □

We can now achieve the determination of $d^1_{*,*}$.

**Theorem 5.27.** *We have*

$$\text{for all } f \in F_p, \quad d^1_{p,0}(p_f \otimes \beta_f) = \sum_{\substack{g \in F_{p-1} \\ f \subset \partial \bar{g}}} \sigma(f, g) p_g \otimes \beta_g. \tag{5.28}$$

*Proof.* For $p = 0$, we have $F_{p-1} = \varnothing$ and $d^1_{p,0} = 0$; the result follows. For $p \geq 1$, we recall that

$$d^1_{p,0} : \bigoplus_{f \in F_p} K_p(C^*(\mathcal{C}(f) \times E_f)) \to \bigoplus_{g \in F_{p-1}} K_{p-1}(C^*(\mathcal{C}(g) \times E_g)). \tag{5.29}$$

is the connecting homomorphism in K-theory of the exact sequence (5.7). We obviously have

$$d^1_{p,0}(p_f \otimes \beta_f) = \sum_{g \in F_{p-1}} \partial_g(p_f \otimes \beta_f), \tag{5.30}$$

where $\partial_g$ is the connecting homomorphism in K-theory of the exact sequence

$$0 \to C^*(\Gamma_{\mathrm{b}}(X)|_g) \to C^*(\Gamma_{\mathrm{b}}(X)|_{g \cup f}) \to C^*(\Gamma_{\mathrm{b}}(X)|_f) \to 0. \tag{5.31}$$

If $f \not\subset \partial \bar{g}$ then the sequence splits and $\partial_g(p_f \otimes \beta_f) = 0$. Let $g \in F_{p-1}$ be such that $f \subset \partial \bar{g}$. Let $\mathcal{U}$ be an open neighborhood of $f$ in $X$ such that there exists a diffeomorphism

$$\mathcal{U}_g := \mathcal{U} \cap g \to f \times (0, +\infty), \quad x \mapsto (\phi(x), r_{i_g}(x)), \tag{5.32}$$

where $r_{i_g}$ is the defining function of $f$ in $\bar{g}$. This yields a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C^*(\Gamma_{\mathrm{b}}(X)|_{\mathcal{U}_g}) \arrow[r] \arrow[d, hook, "\iota"'] & C^*(\Gamma_{\mathrm{b}}(X)|_{\mathcal{U}_g \cup f}) \arrow[r] \arrow[d, hook] & C^*(\Gamma_{\mathrm{b}}(X)|_f) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C^*(\Gamma_{\mathrm{b}}(X)|_g) \arrow[r] & C^*(\Gamma_{\mathrm{b}}(X)|_{g \cup f}) \arrow[r] & C^*(\Gamma_{\mathrm{b}}(X)|_f) \arrow[r] & 0 \end{tikzcd} \tag{5.33}$$

whose upper sequence coincides with (5.23) using (5.32). This implies

$$\partial_g = \partial_{f,g}. \tag{5.34}$$

The result follows by Proposition 5.25. □

The map $d^1_{p,q}$, $q$ even, is deduced from $d^1_{p,0}$ by Bott periodicity. We are ready to relate the $E^2$ terms with conormal homology.

**Corollary 5.35.** *For every $p \in \{1, \dots, d\}$ there are isomorphisms*

$$\phi^i_{p,1} : H^{\mathrm{pcn}}_{\mathrm{i}}(X_p, X_{p-1}) \to K_i(A_p/A_{p-1}), \quad \{0,1\} \ni i \equiv p \bmod 2, \tag{5.36}$$

*such that the following diagram commutes*

$$\begin{tikzcd} H^{\mathrm{pcn}}_{\mathrm{i}}(X_p, X_{p-1}) \arrow[r, "\phi^i_{p,1}"] \arrow[d, "\partial"'] & K_p(A_p/A_{p-1}) \arrow[d, "d^1_{p,0}"] \\ H^{\mathrm{pcn}}_{1-\mathrm{i}}(X_{p-1}, X_{p-2}) \arrow[r, "\phi^{1-i}_{p-1,1}"] & K_{p-1}(A_{p-1}/A_{p-2}) \end{tikzcd} \tag{5.37}$$

*where $\partial$ stands for the connecting morphism in conormal homology.*

*Proof.* If $i \equiv p \mod 2$ then

$$H^{\mathrm{pcn}}_{\mathrm{i}}(X_p, X_{p-1}) = C_p(X) \quad \text{and} \quad \partial = \delta_p : C_p(X) \to C_{p-1}(X).$$

We define (5.36) by $\phi^i_{p,1}(f \otimes \varepsilon_f) = p_f \otimes \beta_f$ and Theorem 5.27 gives the commutativity of (5.37). □

In other words, the map $f \otimes \varepsilon_f \mapsto p_f \otimes \beta_f$ induces a isomorphism

$$H^{\mathrm{cn}}_p(X) \simeq E^2_{p,0}. \tag{5.38}$$

It would be very interesting to compute the higher differentials $d^{2r+1}_{p,0}$.

**5B.** ***The final computation for $K_*(\mathcal{K}_{\mathrm{b}}(X))$ in terms of conormal homology.*** Before getting to the explicit computations and to the analytic corollaries in term of these, let us give a simple but interesting result. It is about the full understanding of the six term exact sequence in K-theory of the fundamental sequence

$$0 \longrightarrow \mathcal{K}(X) \xrightarrow{i_0} \mathcal{K}_{\mathrm{b}}(X) \xrightarrow{r} \mathcal{K}_{\mathrm{b}}(\partial X) \longrightarrow 0. \tag{5.39}$$

**Proposition 5.40.** *For a connected manifold with corners $X$ of codimension greater or equal to one the induced morphism by $r$ in $K_0$, $r : K_0(\mathcal{K}_b(X)) \to K_0(\mathcal{K}_b(\partial X))$, is an isomorphism. Equivalently*:

(1) *The morphism $i_F : K_0(\mathcal{K}) \cong \mathbb{Z} \to K_0(\mathcal{K}_b(X))$ is the zero morphism.*

(2) *The connecting morphism $K_1(\mathcal{K}_b(\partial X)) \to K_0(\mathcal{K}) \cong \mathbb{Z}$ is surjective.*

*Proof.* Let $X$ be a connected manifold with corners of codimension $d$. With the notations of the last section, the sequence (5.39) correspond to the canonical sequence

$$0 \longrightarrow A_0 \longrightarrow A_d \longrightarrow A_d/A_0 \longrightarrow 0.$$

We will prove that the connecting morphism $K_1(A_d/A_0) \to K_0(A_0) \cong \mathbb{Z}$ is surjective. The proof will proceed by induction, the case $d = 1$ immediately satisfies this property. So let us assume that the connecting morphism $K_1(A_{d-1}/A_0) \to K_0(A_0)$ associated to the short exact sequence

$$0 \longrightarrow A_0 \longrightarrow A_{d-1} \longrightarrow A_{d-1}/A_0 \longrightarrow 0.$$

is surjective. Consider now the following commutative diagram of short exact sequences

$$\begin{CD}
0 @>>> 0 @>>> A_d/A_{d-1} @>>> A_d/A_{d-1} @>>> 0\\
@. @AAA @AAA @AAA @.\\
0 @>>> A_0 @>>> A_d @>>> A_d/A_0 @>>> 0\\
@. @AAA @AAA @AAA @.\\
0 @>>> A_0 @>>> A_{d-1} @>>> A_{d-1}/A_0 @>>> 0
\end{CD} \tag{5.41}$$

By applying the six-term short exact sequence in K-theory to it we obtain that the following diagram is commutative, where $\partial_d$ and $\partial_{d-1}$ are the connecting morphisms associated to the middle and to the bottom rows respectively:

$$\begin{tikzcd}
K_(A_d/A_0) \arrow[dr, "\partial_d"] & \\
K_1(A_{d-1}/A_0) \arrow[u] \arrow[r, "\partial_{d-1}"'] & K_0(A_0)
\end{tikzcd}$$

Hence, by the inductive hypothesis, we obtain that $\partial_d$ is surjective. $\square$

**Remark 5.42.** Roughly speaking, the previous proposition tells us that the analytical index of a fully elliptic element carries no information about its Fredholm index, this information being completely contained in some element of $K_1(\mathcal{K}_b(\partial X))$.

We have next our main K-theoretical computation:

**Theorem 5.43.** *Let $X = \Pi_i X_i$ be a finite product of manifolds with corners of codimension less or equal to three. There are natural isomorphisms*

$$\begin{aligned} H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} &\xrightarrow[\cong]{\phi_X} K_0(\mathcal{K}_b(X))) \otimes_{\mathbb{Z}} \mathbb{Q}, \\ H_1^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} &\xrightarrow[\cong]{\phi_X} K_1(\mathcal{K}_b(X)) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned} \tag{5.44}$$

*In the case where $X$ contains a factor of codimension at most two or $X$ is of codimension three, the result holds even without tensoring by $\mathbb{Q}$.*

*Proof. 1A.* $\operatorname{codim}(X) = 0$: The only face of codimension 0 is $\mathring{X}$ (we are always assuming $X$ to be connected). The isomorphism

$$H_0^{\mathrm{cn}}(X_0) \xrightarrow{\phi_0} K_0(A_0)$$

is simply given by sending $\mathring{X}$ to the rank one projector $p_{\mathring{X}}$ chosen in Section 5A.

*1B.* $\operatorname{codim}(X) = 1$: Consider the canonical short exact sequence

$$0 \longrightarrow A_0 \longrightarrow A_1 \longrightarrow A_1/A_0 \longrightarrow 0.$$

That gives, since $d^1_{1,0}$ is surjective, the following exact sequence in K-theory:

$$0 \longrightarrow K_1(A_1) \longrightarrow K_1(A_1/A_0) \xrightarrow{d^1_{1,0}} K_0(A_0) \longrightarrow 0,$$

from which $K_1(A_1) \cong \ker d^1_{1,0}$ and $K_0(A_1) = 0$ (since $K_0(A_1/A_0) = 0$ by a direct computation for K-theory or for conormal homology). By Theorem 5.27 and Corollary 5.35, we have the following commutative diagram:

$$\begin{tikzcd} K_1(A_1/A_0) \arrow[r, "d^1_{1,0}"] & K_0(A_0) \\ H_1^{\mathrm{pcn}}(X_1 \setminus X_0) \arrow[u, "\phi_{1,0}", "\cong"'] \arrow[r, "\delta_1"'] & H_0^{\mathrm{pcn}}(X_0) \arrow[u, "\phi_0", "\cong"'] \end{tikzcd}$$

Then there is a unique natural isomorphism

$$H_1^{\mathrm{pcn}}(X_1) \xrightarrow{\phi_1} K_1(A_1),$$

fitting the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & K_1(A_1) \arrow[r] & K_1(A_1/A_0) \arrow[r, "d^1_{1,0}"] & K_0(A_0) \arrow[r] & 0 \\ 0 \arrow[r] & H_1^{\mathrm{pcn}}(X_1) \arrow[u, "\phi_1", "\cong"'] \arrow[r] & H_1^{\mathrm{pcn}}(X_1 \setminus X_0) \arrow[u, "\phi_{1,0}", "\cong"'] \arrow[r, "\partial_{1,0}"] & H_0^{\mathrm{pcn}}(X_0) \arrow[u, "\phi_0", "\cong"'] \arrow[r] & 0 \end{tikzcd}$$

*1C.* $\operatorname{codim}(X)=2$: We first prove that we have natural isomorphisms

$$H_*^{\mathrm{cn}}(X_l, X_m) \xrightarrow[\cong]{\phi_{l,m}} K_*(A_l/A_m) \tag{5.45}$$

for every $0 \leq m \leq l$ with $l-m=2$ and for every manifold with corners (of any codimension). Indeed, this case can be treated very similarly to the above one. Suppose $l$ is even, the odd case is treated in the same way by exchanging $K_0$ by $K_1$ and $H_0$ by $H_1$. By comparing the long exact sequences in conormal homology we have that there exist unique natural isomorphisms $\phi^0_{l,l-2}$ and $\phi^1_{l,l-2}$ making the following diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & K_0(A_l/A_{l-2}) \arrow[r] & K_0(A_l/A_{l-1}) \arrow[r, "d^1_{l,0}"] & K_1(A_{l-1}/A_{l-2}) \arrow[r] & K_1(A_l/A_{l-2}) \arrow[r] & 0 \\
0 \arrow[r] & H_0^{\mathrm{pcn}}(X_l \backslash X_{l-2}) \arrow[r] \arrow[u, "\phi^0_{l,l-2}", "\cong"'] & H_0^{\mathrm{pcn}}(X_l \backslash X_{l-1}) \arrow[r, "\partial_{l,0}"] \arrow[u, "\phi_{l,l-1}", "\cong"'] & H_1^{\mathrm{pcn}}(X_{l-1} \backslash X_{l-2}) \arrow[r] \arrow[u, "\phi_{l-1,l-2}", "\cong"'] & H_1^{\mathrm{pcn}}(X_l \backslash X_{l-2}) \arrow[r] \arrow[u, "\phi^1_{l,l-2}", "\cong"'] & 0
\end{tikzcd}$$

since the diagram in the middle is commutative again by Corollary 5.35.

Let us now pass to the case when $\operatorname{codim}(X)=2$. Consider the short exact sequence:

$$0 \longrightarrow A_0 \longrightarrow A_2 \longrightarrow A_2/A_0 \longrightarrow 0. \tag{5.46}$$

We compare its associated six term short exact sequence in K-theory with the one in conormal homology to get

$$\begin{tikzcd}
& \mathbb{Z} \arrow[rr] & & K^0(A_2) \arrow[rr] & & K_0(A_2/A_0) \arrow[dd] \\
H_0(X_0) \arrow[ur, "\phi_0", "\cong"'] \arrow[rr] & & H_0^{\mathrm{pcn}}(X_2) \arrow[rr] \arrow[ur, dotted, "?_2"] & & H_0^{\mathrm{pcn}}(X_2, X_0) \arrow[ur, "\phi_{2,0}", "\cong"'] \arrow[dd] & \\
& K_1(A_2/A_0) \arrow[uu, dotted] & & K_1(A_2) \arrow[ll, dotted] & & 0 \arrow[ll, dotted] \\
H_1^{\mathrm{pcn}}(X_2, X_0) \arrow[uu] \arrow[ur, dotted, "\phi_{2,0}", "\cong"'] & & H_1^{\mathrm{pcn}}(X_2) \arrow[ll] \arrow[ur, dotted, "?_1"] & & 0 \arrow[ll] \arrow[ur] &
\end{tikzcd} \tag{5.47}$$

where we need now to define isomorphisms $?_1$ and $?_2$. In fact if we can define morphisms such that the diagrams are commutative then by a simple five lemma argument they would be isomorphisms. The first thing to check is that

$$
\begin{tikzcd}
K_1(A_2/A_0) \arrow[r, "d_{2,0}"] & K_0(A_0) \cong \mathbb{Z} \\
H_1^{\mathrm{pcn}}(X_2, X_0) \arrow[r, "\partial_{2,0}"'] \arrow[u, "\phi_{2,0}", "\cong"'] & H_0^{\mathrm{pcn}}(X_0) \arrow[u, "\cong", "\phi_0"']
\end{tikzcd}
\tag{5.48}
$$

is commutative. Indeed, this can be seen by considering the following commutative diagram of short exact sequences:

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & A_2/A_1 \arrow[r] & A_2/A_1 \arrow[r] & 0 \\
0 \arrow[r] & A_0 \arrow[r] \arrow[u] & A_2 \arrow[r] \arrow[u] & A_2/A_0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A_0 \arrow[r] \arrow[u] & A_1 \arrow[r] \arrow[u] & A_1/A_0 \arrow[r] \arrow[u] & 0
\end{tikzcd}
\tag{5.49}
$$

Applying the associated diagram between the short exact sequences that gives that the connecting morphism for the middle row, $K_1(A_2/A_0) \xrightarrow{d_{2,0}} K_0(A_0)$, is given by a (any) splitting of $K_1(A_1/A_0) \to K_1(A_2/A_0)$ (both modules are free $\mathbb{Z}$-modules by Theorem 4.22) followed by the connecting morphism associated to the exact sequence on the bottom of the above diagram. By definition of $\phi_{2,0}$ in (5.45) above and by Corollary 5.35 we have that these two last morphisms are compatible with the analogs in the respective conormal homologies. Since the connecting morphism $\partial_{2,0}$ in conormal homology is obtained in this way as well, we conclude that (5.48) is commutative. We are ready to define $?_1$ and $?_2$. For the first one, $?_1$, there is a unique isomorphism $\phi_2^1$ fitting the following commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & K_1(A_2) \arrow[r] & K_1(A_2/A_0) \\
0 \arrow[r] & H_1^{\mathrm{pcn}}(X_2) \arrow[r] \arrow[u, "\phi_2^1", "\cong"'] & H_1^{\mathrm{pcn}}(X_2, X_0) \arrow[u, "\phi_{2,0}^1", "\cong"']
\end{tikzcd}
$$

and given by restriction of $\phi_{2,0}^1$ to the image of $H_1^{\mathrm{pcn}}(X_2) \to H_1^{\mathrm{pcn}}(X_2, X_0)$. Now, for defining $?_2$ we have by Proposition 5.40 a unique isomorphism $\phi_2^0$ fitting the following diagram

$$
\begin{tikzcd}
K_0(A_2) \arrow[r, "\cong"] & K_0(A_2/A_0) \\
H_0^{\mathrm{pcn}}(X_2) \arrow[r, "\cong"'] \arrow[u, "\phi_2^0", "\cong"'] & H_0^{\mathrm{pcn}}(X_2, X_0) \arrow[u, "\phi_{2,0}^0", "\cong"']
\end{tikzcd}
$$

*1D.* $\operatorname{codim}(X) = 3$: Consider the short exact sequence

$$0 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_3/A_2 \longrightarrow 0.$$

We compare its associated six term short exact sequence in K-theory with the one in conormal homology to get

$$\begin{tikzcd}
& K_0(A_2) \arrow[rr] && K_0(A_3) \arrow[rr] && 0 \arrow[dd] \\
H_0^{\mathrm{pcn}}(X_2) \arrow[ur, "\phi_2", "\cong"'] \arrow[rr] && H_0^{\mathrm{pcn}}(X_3) \arrow[ur, dotted, "?_2"] \arrow[rr] && 0 \arrow[ur] \arrow[dd] & \\
& K_1(A_3/A_2) \arrow[uu, dotted] && K_1(A_3) \arrow[ll, dotted] && K_1(A_2) \arrow[ll, dotted] \\
H_1^{\mathrm{pcn}}(X_3, X_2) \arrow[uu] \arrow[ur, dotted, "\phi_{3,2}", "\cong"'] && H_1^{\mathrm{pcn}}(X_3) \arrow[ll] \arrow[ur, dotted, "?_1"] && H_1^{\mathrm{pcn}}(X_2) \arrow[ll] \arrow[ur, "\phi_2", "\cong"'] &
\end{tikzcd} \tag{5.50}$$

where we need now to define isomorphisms $?_1$ and $?_2$. Again, if we can define morphisms such that the diagrams are commutative then by a simple five lemma argument they would be isomorphisms. Let us first check that the diagram

$$\begin{tikzcd}
K_1(A_3/A_2) \arrow[r, "\partial"] & K_0(A_2) \\
H_1^{\mathrm{pcn}}(X_3, X_2) \arrow[u, "\phi_{3,2}", "\cong"'] \arrow[r, "\partial"'] & H_0^{\mathrm{pcn}}(X_2) \arrow[u, "\cong", "\phi_2"']
\end{tikzcd} \tag{5.51}$$

is commutative. For this consider the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & A_3/A_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2/A_1 \arrow[r] & A_3/A_1 \arrow[r] & A_3/A_2 \arrow[r] & 0
\end{tikzcd} \tag{5.52}$$

It implies that the connecting morphism $K_1(A_3/A_2) \xrightarrow{\partial} K_0(A_2)$ followed by the morphism $K_0(A_2) \to K_0(A_2/A_1)$ coincides with the connecting morphism $K_1(A_3/A_2) \xrightarrow{\partial} K_0(A_2/A_1)$. Now, the two latter morphisms are compatible with the analogs in conormal homology via the isomorphisms described above, and the morphism $K_0(A_2) \to K_0(A_2/A_1)$ is injective (since $K_0(A_1) = 0$); hence the commutativity of diagram (5.51) above follows. From diagram (5.50), by passage to the quotient, there is unique isomorphism $\phi_3^0$ (the one filling $?_2$ in the above diagram) such that

$$\begin{tikzcd} K_0(A_2) \arrow[r] & K_0(A_3) \arrow[r] & 0 \\ H_0^{\mathrm{pcn}}(X_2) \arrow[r] \arrow[u, "\phi_2^0", "\cong"'] & H_0^{\mathrm{pcn}}(X_3) \arrow[r] \arrow[u, "\phi_3^0", "\cong"'] & 0 \end{tikzcd}$$

is commutative. Finally, for defining $?_1$, it is now enough to choose splittings for the map

$$0 \to H_1^{\mathrm{pcn}}(X_2) \to H_1^{\mathrm{pcn}}(X_3),$$

which is possible since $H_1^{\mathrm{pcn}}(X_3)$ is free (see Theorem 4.22 and the remark below it) and for the map

$$K_1(A_3) \to \operatorname{im} j \to 0,$$

where $j$ is the canonical morphism $j : K_1(A_3) \to K_1(A_3/A_2)$ (remember all the groups $K_*(A_p/A_{p-1})$ are torsion free).

*1E. If $X = \Pi_i X_i$ is a finite product with* $\operatorname{codim}(X_i) \leq 3$ *and with at least one factor of codimension at most 2*: In this case the result would follow, by all the points above, if both periodic conormal homology and K-theory satisfy the Künneth formula. Since the algebras $\mathcal{K}_b(X)$ are nuclear because the groupoids $\Gamma_b(X)$ are amenable, we have the Künneth formula in K-theory for these kind of algebras. Now, for conormal homology we verified the Künneth formula in Proposition 4.29.

*1F. If $X = \Pi_i X_i$ is a finite product with* $\operatorname{codim}(X_i) \leq 3$, *for all $i$*: In this case the result holds rationally by the same arguments as above by using Proposition 4.33.

□

## 6. Fredholm perturbation properties and Euler conormal characters

The previous results yield a criterion for property $(\mathcal{HFP})$ in terms of the Euler characteristic for conormal homology. To fit with the assumptions of Theorem 5.43, we consider a manifold with corners $X$ of codimension $d$, which is given by the cartesian product of manifolds with corners of codimension at most 3.

**Definition 6.1** (corner characters). Let $X$ be a manifold with corners. We define the even conormal character of $X$ as

$$\chi_0(X) = \dim_{\mathbb{Q}} H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{6.2}$$

Similarly, we define the odd conormal character of $X$ as

$$\chi_1(X) = \dim_{\mathbb{Q}} H_1^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{6.3}$$

We can consider as well

$$\chi(X)=\chi_0(X)-\chi_1(X), \tag{6.4}$$

then we have (by the rank nullity theorem)

$$\chi(X)=1-\#F_1+\#F_2-\cdots+(-1)^d\#F_d. \tag{6.5}$$

We refer to the integer $\chi(X)$ as the Euler corner character of $X$. These numbers are clearly invariant under the natural notion of isomorphism of manifolds with corners. Their computation is elementary in any concrete situation.

In particular one can rewrite Theorem 5.43 to have, for $X$ as in the theorem statement,

$$K_0(\mathcal{K}_b(X))\otimes_{\mathbb{Z}}\mathbb{Q}\cong\mathbb{Q}^{\chi_0(X)},\quad K_1(\mathcal{K}_b(X))\otimes_{\mathbb{Z}}\mathbb{Q}\cong\mathbb{Q}^{\chi_1(X)} \tag{6.6}$$

and, in terms of the corner character,

$$\chi(X)=\operatorname{rank}(K_0(\mathcal{K}_b(X))\otimes_{\mathbb{Z}}\mathbb{Q})-\operatorname{rank}(K_1(\mathcal{K}_b(X))\otimes_{\mathbb{Z}}\mathbb{Q}). \tag{6.7}$$

In the case where $X$ is a finite product of manifolds with corners of codimension at most 2 we even have

$$K_0(\mathcal{K}_b(X))\simeq\mathbb{Z}^{\chi_0(X)}\quad\text{and}\quad K_1(\mathcal{K}_b(X))\simeq\mathbb{Z}^{\chi_1(X)} \tag{6.8}$$

and also $\chi_{cn}(X)=\operatorname{rank}(K_0(\mathcal{K}_b(X)))-\operatorname{rank}(K_1(\mathcal{K}_b(X)))$.

We end with the characterization of property $(\mathcal{HFP})$ in terms of conormal characteristics.

**Theorem 6.9.** *Let $X$ be a compact connected manifold with corners of codimension greater than or equal to one. If $X$ is a finite product of manifolds with corners of codimension less than or equal to three we have that*:

1. *If $X$ satisfies the Fredholm perturbation property then the even Euler corner character of $X$ vanishes, i.e., $\chi_0(X)=0$.*
2. *If the even periodic conormal homology group vanishes, i.e., $H_0^{\mathrm{pcn}}(X)=0$, then $X$ satisfies the stably homotopic Fredholm perturbation property.*
3. *If $H_0^{\mathrm{pcn}}(X)$ is torsion free and if the even Euler corner character of $X$ vanishes, i.e., $\chi_0(X)=0$ then $X$ satisfies the stably homotopic Fredholm perturbation property.*

*Proof.* (1) Suppose $\chi_0(X)\neq 0$ then $K_0(\mathcal{K}_b(X))\otimes_{\mathbb{Z}}\mathbb{Q}\cong\mathbb{Q}^{\chi_0(X)}$ is not the zero group. By Theorem 3.24 it is enough to prove that the rationalized analytic indicial index morphism

$$\operatorname{Ind}_a: K^0_{\mathrm{top}}({}^bT^*X)\otimes_{\mathbb{Z}}\mathbb{Q}\to K_0(\mathcal{K}_b(X))\otimes_{\mathbb{Z}}\mathbb{Q}$$

is not the zero morphism. Monthubert and Nistor [2012, Theorems 12 and 13 and Proposition 7] construct a manifold with corners $Y$ and a closed embedding of manifolds with embedded corners $X \xrightarrow{i} Y$ to obtain a commutative diagram

$$\begin{tikzcd} K^0_{\mathrm{top}}({}^{\mathrm{b}}T^*X) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, "\mathrm{Ind}_a"] \arrow[d, "i!"'] & K_0(\mathcal{K}_{\mathrm{b}}(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[d, "i_*", "\cong"'] \\ K^0_{\mathrm{top}}({}^{\mathrm{b}}T^*Y) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, "\cong", "\mathrm{Ind}_a"'] & K_0(\mathcal{K}_{\mathrm{b}}(Y)) \otimes_{\mathbb{Z}} \mathbb{Q} \end{tikzcd} \tag{6.10}$$

They call such a $Y$ a classifying space of $X$. For our purposes it would be then enough to show that the morphism

$$i! : K^0_{\mathrm{top}}({}^{\mathrm{b}}T^*X) \otimes_{\mathbb{Z}} \mathbb{Q} \to K^0_{\mathrm{top}}({}^{\mathrm{b}}T^*Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is not the zero morphism. But now we are at the topological K-theory level (with compact supports) where classic topological arguments apply to get that the morphism above is not the zero morphism. Indeed, to construct $i!$ one uses a tubular neighborhood (which exist in this setting, see for example Douady [1961/1962]); the first step is then a Thom isomorphism followed by a morphism induced by a classic extension by zero. This is summarized in [Monthubert and Nistor 2012, Proposition 5]. The conclusion follows.

(2) If $H_0^{\mathrm{pcn}}(X) = 0$ then $H_0^{\mathrm{pcn}}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and the result follows from Theorems 5.43 and 3.24.

(3) In this case $K_0(\mathcal{K}_{\mathrm{b}}(X)) \cong \mathbb{Z}^{\chi_0(X)}$ by Theorem 5.43 and the arguments applied in the last two points identically apply to get the result (the results of Monthubert and Nistor cited above hold over $\mathbb{Z}$). □

## Appendix: more on conormal homology

We reproduce the discussion leading to the definition of the conormal differential in a slightly more general way. We keep the same notations. Let $f \in F_p$, $\varepsilon_f$ an orientation of $N(f)$ and $g \in F_{p-k}$ such that $f \subset \bar{g}$. The face $f$ is characterized in $\bar{g}$ by the vanishing of $k$ defining functions and we denote by $(g, f)$ the corresponding $k$-tuple of their indices. Then the contraction $\varepsilon_g := e_{(g,f)} \lrcorner\, \varepsilon_f$ is an orientation of $N(g)$. Recall that

$$e_J \lrcorner \cdot = e_{j_1} \lrcorner (\cdots \lrcorner (e_{j_k} \lrcorner \cdot) \cdots). \tag{A.1}$$

For any integers $0 \le k \le p$, we define $\delta_p^k : C_p(X) \to C_{p-k}(X)$ by

$$\delta_p^k(f \otimes \varepsilon_f) = \sum_{\substack{g \in F_{p-k},\\ f \subset \bar{g}}} g \otimes e_{(g,f)} \lrcorner\, \varepsilon_f. \tag{A.2}$$

We get a homomorphism $\delta^{\mathrm{pcn}} : \mathcal{C}(X) \to \mathcal{C}(X)$ of degree 1 with respect to the $\mathbb{Z}_2$-grading by setting

$$\delta_i^{\mathrm{pcn}} = \sum_{\substack{k\geq 0,\\ p\equiv i \bmod 2}} \delta_p^{2k+1}, \quad i = 0, 1. \tag{A.3}$$

**Proposition A.4.** *The map $\delta^{\mathrm{pcn}}$ is a differential, that is $\delta^{\mathrm{pcn}} \circ \delta^{\mathrm{pcn}} = 0$.*

*Proof.* Let $f \in F_p(X)$ and $\varepsilon$ be an orientation of $N(f)$. We have

$$\delta^{\mathrm{pcn}}(\delta^{\mathrm{pcn}}(f \otimes \varepsilon)) = \sum_{\substack{g,h \text{ s.t. } \bar{h}\supset\bar{g}\supset f\\ (g,f),(h,g) \text{ are odd}}} \big(h \otimes e_{(h,g)} \lrcorner (e_{(g,f)} \lrcorner \varepsilon)\big). \tag{A.5}$$

Let $g, h$ be such that they contribute a term in the sum above and denote by $I, J, K$ the tuples labeling the defining functions of $f, g, h$ respectively. Then set

$$J' = I \setminus (h, g). \tag{A.6}$$

By definition of manifolds with (embedded) corners, $H_{J'}$ is not empty and there exists a unique face $g' \in c(J')$ with $f \subset \overline{g'}$. This face $g' = \iota(g, h, f)$ satisfies the following properties:

- $f \subset \overline{g'} \subset \bar{h}$,
- $(g', f) = (h, g)$ and $(h, g') = (g, f)$ are odd,
- $\iota(g', h, f) = g$.

Finally, note that $\#(g, f) \neq \#(h, g)$, otherwise we would have $(h, f) = (h, g) + (g, f)$ even. This implies in particular that $g \neq g'$. These observations allow us to reorganize the sum (A.5) as follows:

$$\delta^{\mathrm{pcn}}(\delta^{\mathrm{pcn}}(f \otimes \varepsilon)) = \sum_{\substack{g,h \text{ s.t. } \bar{h}\supset\bar{g}\supset f\\ \#(g,f)<\#(h,g) \text{ odd}}} \big(h \otimes (e_{(h,g)} \lrcorner (e_{(g,f)} \lrcorner \varepsilon + e_{(h,g')} \lrcorner (e_{(g',f)} \lrcorner \varepsilon)))\big).$$

Now

$$e_{(h,g)} \lrcorner (e_{(g,f)} \lrcorner \varepsilon) + e_{(h,g')} \lrcorner (e_{(g',f)} \lrcorner \varepsilon) = e_{(h,g)} \lrcorner (e_{(g,f)} \lrcorner \varepsilon) + e_{(g,f)} \lrcorner (e_{(h,g)} \lrcorner \varepsilon) = 0$$

since $\#(g, f)$ and $\#(h, g)$ are odd. $\square$

Proposition A.4 implies $\delta_{p-1}^1 \circ \delta_p^1 = 0$ for any $p$. Since $\delta_*^1 = \delta_*$, this proves the claim of Section 4A. Moreover:

**Proposition A.7.** *The identity map $(\mathcal{C}_*(X), \delta^1) \to (\mathcal{C}_*(X), \delta)$ induces an isomorphism between the $\mathbb{Z}_2$-graded homology groups.*

**Lemma A.8.** *The following equality hold for any $k \geq 0$:*

$$\delta^{2k+1} = \delta^{2k} \circ \delta^1 = \delta^1 \circ \delta^{2k}. \tag{A.9}$$

*Proof of Lemma A.8.* Let $f$ be a codimension-$p$ face and $\varepsilon$ an orientation of $N(f)$. Let $I$ be the $p$-tuple defining $f$. Then $g$ is a face such that $f \subset \bar{g}$ if and only if $g$ is a connected component of $H_J$ for some $J \subset I$. Since the definition of $\delta(f)$ only involves faces $g$ with $f \subset \bar{g}$, it is no restriction to remove the connected component of $H_J$ disjoint from $f$ for any $J \subset I$, or equivalently to assume that such $H_J$ are connected. It follows that the faces appearing in the definition of $\delta(f)$ are in one-to-one correspondence with the tuples $J \subset I$ so they can be indexed by them and eventually omitted in the sum defining $\delta^*(f)$. It follows that, $\varepsilon_I$ denoting an orientation of $N(f)$,

$$\begin{aligned}\delta^{2k} \circ \delta^1(\varepsilon_I) &= \sum_{|J|=2k} \sum_{1\le i\le N} e_J \lrcorner e_i \lrcorner \varepsilon_I \\ &= \sum_{|J|=2k+1} \sum_{l=1}^{2k+1} e_{j_1} \lrcorner \cdots \widehat{e_{j_l}} \lrcorner \cdots \lrcorner e_{j_{2k+1}} \lrcorner e_{j_l} \lrcorner \varepsilon_I \\ &= \sum_{|J|=2k+1} \sum_{l=1}^{2k+1} (-1)^{l-1} e_J \lrcorner \varepsilon_I = \sum_{|J|=2k+1} e_J \lrcorner \varepsilon_I = \delta^{2k+1}(\varepsilon_I).\end{aligned}$$

The equality $\delta^{2k+1} = \delta^1 \circ \delta^{2k}$ is obtained in the same way. □

*Proof of Proposition A.7.* Let us set $N = \sum_{k\ge0} \delta^{2k}$ and $h = \mathrm{id} + N$. Using the lemma, we get:

$$\delta^{\mathrm{pcn}} = \delta^1 \circ h = h \circ \delta^1. \tag{A.10}$$

Since $N$ is nilpotent, the map $h$ is invertible with inverse given by the finite sum

$$h^{-1} = \sum_{j\ge0} (-1)^j N^j.$$

This proves that $\delta^1(x) = 0$ if and only if $\delta^{\mathrm{pcn}}(x) = 0$ and that $x = \delta^1(y)$ if and only if $x = \delta^{\mathrm{pcn}}(y')$ for some $y$, $y'$ as well. The proposition follows. □

The differential $\delta^1$ is of course much simpler to handle than $\delta^{\mathrm{pcn}}$.

## Acknowledgements

We would like to thank the referee for his/her careful comments and remarks and, in particular, for pointing out some important references that helped us to understand better our own work and its place in the index theory literature.

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Received 6 Jul 2017. Revised 16 Feb 2018. Accepted 7 Mar 2018.

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ANNALS OF K-THEORY
Vol. 3, No. 3, 2018
dx.doi.org/10.2140/akt.2018.3.565

# Positive scalar curvature and low-degree group homology

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Let $\Gamma$ be a discrete group. Assuming rational injectivity of the Baum–Connes assembly map, we provide new lower bounds on the rank of the positive scalar curvature bordism group and the relative group in Stolz' positive scalar curvature sequence for $\mathrm{B}\Gamma$. The lower bounds are formulated in terms of the part of degree up to 2 in the group homology of $\Gamma$ with coefficients in the $\mathbb{C}\Gamma$-module generated by finite order elements. Our results use and extend work of Botvinnik and Gilkey which treated the case of finite groups. Further crucial ingredients are a real counterpart to the delocalized equivariant Chern character and Matthey's work on explicitly inverting this Chern character in low homological degrees.

## 1. Introduction

There exists a natural comparison mapping between the positive scalar curvature (psc) sequence of Stolz (top row) to the analytic surgery sequence of Higson and Roe (bottom row):

$$\begin{CD}
\Omega_n^{\mathrm{spin}}(\mathrm{B}\Gamma) @>>> \mathrm{R}_n^{\mathrm{spin}}(\mathrm{B}\Gamma) @>{\partial}>> \mathrm{P}_{n-1}^{\mathrm{spin}}(\mathrm{B}\Gamma) @>>> \Omega_{n-1}^{\mathrm{spin}}(\mathrm{B}\Gamma) @>>> \mathrm{R}_{n-1}^{\mathrm{spin}}(\mathrm{B}\Gamma)\\
@VV{\beta}V @VV{\alpha}V @VV{\rho}V @VV{\beta}V @VV{\alpha}V\\
\mathrm{KO}_n(\mathrm{B}\Gamma) @>{\nu}>> \mathrm{KO}_n(\mathrm{C}_\mathrm{r}^*\Gamma) @>{\partial}>> \mathrm{S}_{n-1}^{\mathbb{R}}(\Gamma) @>>> \mathrm{KO}_{n-1}(\mathrm{B}\Gamma) @>{\nu}>> \mathrm{KO}_{n-1}(\mathrm{C}_\mathrm{r}^*\Gamma)
\end{CD} \tag{1.1}$$

This diagram was first established by Piazza and Schick [2014, Theorem 1.28] for complex K-theory and $n$ even. It was extended by Xie and Yu [2014, Theorem B] and by the second-named author [2016b, Theorem 3.1.13] to cover all dimensions and the real case. See also Zenobi [2017, Remark 6.2].

We briefly explain the constituents in the diagram above. Start with Stolz' psc sequence. The group $\Omega_n^{\mathrm{spin}}(\mathrm{B}\Gamma)$ is the singular *spin bordism group* of the classifying space $\mathrm{B}\Gamma$. That is, it consists of bordism classes of pairs $(M, \phi)$, where $M$ is a closed spin manifold of dimension $n$ and $\phi : M \to \mathrm{B}\Gamma$ a continuous map. The *psc spin bordism group* $\mathrm{P}_n^{\mathrm{spin}}(\mathrm{B}\Gamma)$ consists of bordism classes of $(M, \phi, g)$,

*MSC2010:* primary 58D27, 58J22; secondary 19K33, 19L10, 19L47, 55N91.
*Keywords:* positive scalar curvature, secondary index theory, $\rho$-invariant, equivariant Chern character, group homology.

where $(M, \phi)$ is as before and $g \in \mathcal{R}^+(M)$ is a metric of psc. Here we require bordisms to have metrics of positive scalar curvature with product structure near the boundary. *Stolz' relative group* $\mathrm{R}^{\mathrm{spin}}_{n+1}(\mathrm{B}\Gamma)$ consists of bordism classes of triples $(W, \phi, g)$, where $W$ is a compact spin manifold of dimension $(n+1)$, $\phi : W \to \mathrm{B}\Gamma$ a continuous map, and $g \in \mathcal{R}^+(\partial W)$ a metric of psc on the boundary. The horizontal arrows in Stolz' sequence are the evident forgetful maps.

The (real version of the) *analytic surgery sequence* of Higson and Roe consists of the *real* K-*homology* of $\mathrm{B}\Gamma$, the topological K-*theory of the reduced group* $\mathrm{C}^*$-*algebra* of $\Gamma$, and the *analytic structure group* of $\Gamma$. We denote the latter by $\mathrm{S}^{\mathbb{R}}_*(\Gamma)$. It is defined in such a way that it fits into a long exact sequence together with the real *Novikov assembly map* $\nu : \mathrm{KO}_*(\mathrm{B}\Gamma) \to \mathrm{KO}_*(\mathrm{C}^*_{\mathrm{r}}\Gamma)$. We also use their complex counterparts, which we denote by $\mathrm{K}_*(\mathrm{B}\Gamma)$, $\mathrm{K}_*(\mathrm{C}^*_{\mathrm{r}}\Gamma)$, and $\mathrm{S}^{\mathbb{C}}_*(\Gamma)$.

The groups $\mathrm{P}^{\mathrm{spin}}_{n-1}(\mathrm{B}\Gamma)$ and $\mathrm{R}^{\mathrm{spin}}_{n}(\mathrm{B}\Gamma)$ classify psc metrics up to bordism and concordance, respectively, on spin manifolds with fundamental group $\Gamma$. For the latter see [Rosenberg and Stolz 2001, Theorem 5.1]. Alas, at present there are no tools known that allow a computation of these groups (not even in simple special cases). However, the comparison (1.1) allows us to obtain lower bounds on these groups using the index-theoretic information contained in the sequence of Higson and Roe. To that end, it is important to know something about the size of the image of the *relative index map* $\alpha : \mathrm{R}^{\mathrm{spin}}_{n}(\mathrm{B}\Gamma) \to \mathrm{KO}_n(\mathrm{C}^*_{\mathrm{r}}\Gamma)$ and the *higher* $\rho$-*invariant* $\rho : \mathrm{P}^{\mathrm{spin}}_{n-1}(\mathrm{B}\Gamma) \to \mathrm{S}^{\mathbb{R}}_{n-1}(\Gamma)$.

The first case where something can be said is the class of finite groups. Indeed, let $H$ be a finite group. Let $\mathrm{R}(H)$ denote its complex representation ring. Let $\mathrm{R}^q_0(H)$ be the subgroup generated by those representations $\rho$ of virtual dimension 0 such that its character $\chi_\rho$ satisfies $\chi_\rho(\gamma^{-1}) = (-1)^q \chi_\rho(\gamma)$ for all $\gamma \in H$. Botvinnik and Gilkey [1995, Theorem 2.1] showed that the rank of the positive scalar curvature bordism group $\mathrm{P}^{\mathrm{spin}}_{2q+4k-1}(\mathrm{B}H)$ is bounded below by the rank of $\mathrm{R}^q_0(H)$, where $k \geq 1$, $q \in \{0, 1\}$ with $4k + 2q \geq 6$. They used relative $\eta$-invariants. These are numerical invariants that are related to the higher $\rho$-invariant via certain trace maps; see for instance [Higson and Roe 2010]. In fact, Botvinnik and Gilkey's result [1995] implies that both

$$\rho : \mathrm{P}^{\mathrm{spin}}_{n-1}(\mathrm{B}H) \to \mathrm{S}^{\mathbb{R}}_{n-1}(H) \quad \text{and} \quad \alpha : \mathrm{R}^{\mathrm{spin}}_{n}(\mathrm{B}H) \to \mathrm{KO}_n(\mathrm{C}^*_{\mathrm{r}} H)$$

are *rationally surjective* for $n \geq 6$ (we explain this in Section 4). Moreover, recently Weinberger and Yu [2015] and Xie and Yu [2017] gave lower bounds for a large class of infinite groups based on the number of torsion elements with pairwise different orders. We also refer to [Piazza and Schick 2007] for lower bounds on the positive scalar bordism group based on the $L^2$-$\rho$-invariant.

The results mentioned above only yield information for $n$ even. Using product formulas one can obtain further ad hoc examples of nontrivial relative indices and

$\rho$-invariants by taking certain products; see [Zeidler 2016a, Corollary 6.10; Zenobi 2017, Corollary 5.24]. For instance, taking products with the circle allows one to also produce examples for $n$ odd.

In the main results of this paper, we give new systematic lower bounds for all $n \geq 7$ on the image of the relative index and the $\rho$-invariant based on the part of degree up to 2 of a certain group homology. The result of Botvinnik and Gilkey [1995] yields the 0-dimensional part. Then the idea is that degrees 1 and 2 can be obtained from this by taking products with circles and oriented surfaces. We use the *Baum–Connes assembly map* $\mu : \mathrm{K}^\Gamma_*(\underline{\mathrm{E}}\Gamma) \to \mathrm{K}_*(\mathrm{C}^*_\mathrm{r}\Gamma)$, the *delocalized Chern character* of Baum and Connes [1988], and — most centrally — its explicit partial inverse in degrees up to 2 due to Matthey [2004].

To state our results, we start with some preparations. Let $\Gamma$ be a discrete group and denote by $\Gamma_{\mathrm{fin}}$ the set of elements of finite order of $\Gamma$. Let $\mathrm{F}\Gamma$ be the set of all finitely supported functions $\Gamma_{\mathrm{fin}} \to \mathbb{C}$. Letting $\Gamma$ act by conjugation on $\Gamma_{\mathrm{fin}}$ turns $\mathrm{F}\Gamma$ into a $\mathbb{C}\Gamma$-module. The delocalized equivariant Chern character yields an isomorphism

$$\underline{\mathrm{ch}}_\Gamma : \mathrm{K}^\Gamma_p(\underline{\mathrm{E}}\Gamma) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{k \in \mathbb{Z}} \mathrm{H}_{p+2k}(\Gamma; \mathrm{F}\Gamma). \tag{1.2}$$

It was first introduced by Baum and Connes [1988] but we will instead work with the "handicrafted Chern character" of Matthey [2004]. Matthey [2004] also constructed maps

$$\beta^{(\mathrm{t})}_p : \mathrm{H}_p(\Gamma; \mathrm{F}\Gamma) \to \mathrm{K}^\Gamma_p(\underline{\mathrm{E}}\Gamma) \otimes \mathbb{C}$$

for $p \in \{0, 1, 2\}$ which are right-inverse to the delocalized Chern character. Moreover, he defined explicit maps

$$\beta^{(\mathrm{a})}_p : \mathrm{H}_p(\Gamma; \mathrm{F}\Gamma) \to \mathrm{K}_p(\mathrm{C}^*_\mathrm{r}\Gamma)$$

which satisfy $\beta^{(\mathrm{a})}_p = (\mu \otimes \mathbb{C}) \circ \beta^{(\mathrm{t})}_p$ for $p \in \{0, 1, 2\}$. They thereby describe the Baum–Connes assembly map explicitly in low homological degrees.

To use these maps for our purposes, we need to adapt the above to real K-homology. To that end, for $q \in \{0, 1\}$, let

$$\mathrm{F}^q\Gamma = \{f \in \mathrm{F}\Gamma \mid f(\gamma) = (-1)^q f(\gamma^{-1})\ \forall \gamma \in \Gamma_{\mathrm{fin}}\}.$$

Then $\mathrm{F}\Gamma = \mathrm{F}^0\Gamma \oplus \mathrm{F}^1\Gamma$ as $\mathbb{C}\Gamma$-modules. We can now state our main result and its corollaries.

**Theorem 1.3.** *For each $p \in \{0, 1, 2\}$, $q \in \{0, 1\}$ and $k \geq 1$ with $4k + 2q \geq 6$, there exists a linear map*

$$\beta^{(\mathrm{psc})}_{p,q,k} : \mathrm{H}_p(\Gamma; \mathrm{F}^q\Gamma) \to \mathrm{R}^{\mathrm{spin}}_{p+2q+4k}(\mathrm{B}\Gamma) \otimes \mathbb{C}$$

*such that the following diagram commutes*:

$$\begin{tikzcd}
\mathrm{H}_p(\Gamma;\mathrm{F}^q\Gamma) \arrow[r, hook] \arrow[d, "\beta^{(\mathrm{psc})}_{p,q,k}"] & \mathrm{H}_p(\Gamma;\mathrm{F}\Gamma) \arrow[dd, "\beta^{(\mathrm{a})}_p"] \\
\mathrm{R}^{\mathrm{spin}}_{p+2q+4k}(\mathrm{B}\Gamma)\otimes\mathbb{C} \arrow[d, "\alpha\otimes\mathbb{C}"] & \\
\mathrm{KO}_{p+2q}(\mathrm{C}^*_{\mathrm{r}}\Gamma)\otimes\mathbb{C} \arrow[r, "\mathrm{c}\otimes\mathbb{C}"] & \mathrm{K}_p(\mathrm{C}^*_{\mathrm{r}}\Gamma)\otimes\mathbb{C}
\end{tikzcd}$$

Here $\mathrm{c}:\mathrm{KO}_*(\mathrm{C}^*_{\mathrm{r}}\Gamma)\to\mathrm{K}_*(\mathrm{C}^*_{\mathrm{r}}\Gamma)$ is the complexification map from real to complex K-theory. We implicitly use that complex K-theory is 2-periodic and real K-theory is rationally 4-periodic.

**Remark 1.4.** We do not claim that our maps $\beta^{(\mathrm{psc})}_{p,q,k}$ are canonical (unlike the original maps of Matthey). Indeed, their construction depends on choosing preimages under the surjective map $\alpha\otimes\mathbb{Q}:\mathrm{R}^{\mathrm{spin}}_*(\mathrm{B}H)\otimes\mathbb{Q}\to\mathrm{KO}_*(\mathrm{C}^*_{\mathrm{r}}H)\otimes\mathbb{Q}$ for each finite cyclic group $H$. However, after fixing these choices it is in principle possible to trace through the construction to obtain explicit formulas for $\beta^{(\mathrm{psc})}_{p,q,k}$ similarly as in Matthey's work.

In any case, the existence of $\beta^{(\mathrm{psc})}_{p,q,k}$ implies lower bounds and surjectivity results:

**Corollary 1.5.** *Let $n\geq 7$. If the rational Baum–Connes assembly map $\mu\otimes\mathbb{Q}$ is injective, then the rank of $\mathrm{R}^{\mathrm{spin}}_n(\mathrm{B}\Gamma)$ is at least the dimension of*

$$\begin{cases}\mathrm{H}_0(\Gamma;\mathrm{F}^0\Gamma)\oplus\mathrm{H}_2(\Gamma;\mathrm{F}^1\Gamma), & n\equiv 0 \mod 4,\\ \mathrm{H}_1(\Gamma;\mathrm{F}^0\Gamma), & n\equiv 1 \mod 4,\\ \mathrm{H}_0(\Gamma;\mathrm{F}^1\Gamma)\oplus\mathrm{H}_2(\Gamma;\mathrm{F}^0\Gamma), & n\equiv 2 \mod 4,\\ \mathrm{H}_1(\Gamma;\mathrm{F}^1\Gamma), & n\equiv 3 \mod 4.\end{cases}$$

**Corollary 1.6.** *Let $n\geq 7$. If the rational Baum–Connes assembly map $\mu\otimes\mathbb{Q}$ is injective, then the rank of $\mathrm{P}^{\mathrm{spin}}_{n-1}(\mathrm{B}\Gamma)$ is at least the dimension of*

$$\begin{cases}\mathrm{H}_0(\Gamma;\mathrm{F}^0_0\Gamma)\oplus\mathrm{H}_2(\Gamma;\mathrm{F}^1\Gamma), & n\equiv 0 \mod 4,\\ \mathrm{H}_1(\Gamma;\mathrm{F}^0_0\Gamma), & n\equiv 1 \mod 4,\\ \mathrm{H}_0(\Gamma;\mathrm{F}^1\Gamma)\oplus\mathrm{H}_2(\Gamma;\mathrm{F}^0_0\Gamma), & n\equiv 2 \mod 4,\\ \mathrm{H}_1(\Gamma;\mathrm{F}^1\Gamma), & n\equiv 3 \mod 4,\end{cases}$$

*where* $\mathrm{F}^0_0=\{f\in\mathrm{F}^0\mid f(1)=0\}$.

In comparison, Botvinnik and Gilkey [1995, Theorem 0.1] imply that for a finite group $H$ and $n\geq 6$ even, the rank of $\mathrm{P}^{\mathrm{spin}}_{n-1}(\mathrm{B}H)$ is bounded below by the dimension of $\mathrm{H}_0(H;\mathrm{F}^0_0H)$ if $n\equiv 0 \mod 4$ or $\mathrm{H}_0(H;\mathrm{F}^1H)$ if $n\equiv 2 \mod 4$.

**Example 1.7.** We describe an explicit example illustrating the nontrivial content of Corollaries 1.5 and 1.6. Let $\Sigma_g$ denote the oriented surface of genus $g \geq 1$. Let $n$ be a positive integer. Consider the group $\Gamma = \pi_1(\Sigma_g) \times \mathbb{Z}/n\mathbb{Z}$. Then the Baum–Connes assembly map for $\Gamma$ is an isomorphism.[1] So our results are applicable. Next we explicitly compute the homology groups that appear in the corollaries for this example. Start with the group homology of $\Gamma$ with trivial coefficients $\mathbb{C}$. The group homology of $\mathbb{Z}/n\mathbb{Z}$ is torsion in all positive degrees. Hence the Künneth theorem implies that $\mathrm{H}_*(\Gamma; \mathbb{C}) \cong \mathrm{H}_*(\Sigma_g; \mathbb{C})$. Thus the homology of $\Gamma$ is $\mathrm{H}_p(\Gamma; \mathbb{C}) \cong \mathbb{C}$ for $p \in \{0, 2\}$, $\mathrm{H}_1(\Gamma; \mathbb{C}) \cong \mathbb{C}^{2g}$, and zero in degrees greater than 2. To proceed, observe that any finite order element of $\Gamma$ is of the form $(1, t^k)$, where 1 denotes the neutral element of $\pi_1(\Sigma_g)$ and $t$ the generator of $\mathbb{Z}/n\mathbb{Z}$. The action by conjugation is trivial on these elements. We deduce that $\mathrm{F}\Gamma$ is isomorphic to the trivial $\mathbb{C}\Gamma$-module $\mathbb{C}^n$. By counting dimensions, we see that $\mathrm{F}^0\Gamma \cong \mathbb{C}^{\lfloor n/2 \rfloor + 1}$, $\mathrm{F}^0_0\Gamma \cong \mathbb{C}^{\lfloor n/2 \rfloor}$, $\mathrm{F}^1\Gamma \cong \mathbb{C}^{\lceil n/2 \rceil - 1}$ as trivial $\mathbb{C}\Gamma$-modules. Together with the computation of $\mathrm{H}_*(\Gamma; \mathbb{C})$, we deduce

$$
\begin{aligned}
&\mathrm{H}_0(\Gamma; \mathrm{F}^0\Gamma) \cong \mathbb{C}^{\lfloor n/2 \rfloor + 1}, && \mathrm{H}_0(\Gamma; \mathrm{F}^0_0\Gamma) \cong \mathbb{C}^{\lfloor n/2 \rfloor}, && \mathrm{H}_0(\Gamma; \mathrm{F}^1\Gamma) \cong \mathbb{C}^{\lceil n/2 \rceil - 1}, \\
&\mathrm{H}_1(\Gamma; \mathrm{F}^0\Gamma) \cong \mathbb{C}^{2g(\lfloor n/2 \rfloor + 1)}, && \mathrm{H}_1(\Gamma; \mathrm{F}^0_0\Gamma) \cong \mathbb{C}^{2g\lfloor n/2 \rfloor}, && \mathrm{H}_1(\Gamma; \mathrm{F}^1\Gamma) \cong \mathbb{C}^{2g(\lceil n/2 \rceil - 1)}, \\
&\mathrm{H}_2(\Gamma; \mathrm{F}^0\Gamma) \cong \mathbb{C}^{\lfloor n/2 \rfloor + 1}, && \mathrm{H}_2(\Gamma; \mathrm{F}^0_0\Gamma) \cong \mathbb{C}^{\lfloor n/2 \rfloor}, && \mathrm{H}_2(\Gamma; \mathrm{F}^1\Gamma) \cong \mathbb{C}^{\lceil n/2 \rceil - 1}.
\end{aligned}
$$

This shows that for $n \geq 3$ all homology groups which appear in the conclusion of Corollaries 1.5 and 1.6 are nontrivial.

**Corollary 1.8.** *Let $n \geq 7$. Let the rational homological dimension of $\Gamma$ be at most 2. Then, if the rational Baum–Connes assembly map $\mu \otimes \mathbb{Q}$ is surjective, the rational relative index map*

$$\alpha \otimes \mathbb{Q} : \mathrm{R}^{\mathrm{spin}}_n(\mathrm{B}\Gamma) \otimes \mathbb{Q} \twoheadrightarrow \mathrm{KO}_n(\mathrm{C}^*_{\mathrm{r}}\Gamma) \otimes \mathbb{Q}$$

*is surjective.*

*If $\mu \otimes \mathbb{Q}$ is an isomorphism, then the rational higher $\rho$-invariant*

$$\rho \otimes \mathbb{Q} : \mathrm{P}^{\mathrm{spin}}_{n-1}(\mathrm{B}\Gamma) \otimes \mathbb{Q} \twoheadrightarrow \mathrm{S}^{\mathbb{R}}_{n-1}(\Gamma) \otimes \mathbb{Q}$$

*is also surjective.*

## 2. The delocalized equivariant Pontryagin character

In this section, we exhibit the *delocalized equivariant Pontryagin character*, which is the real counterpart to the delocalized equivariant Chern character. It is obtained from the delocalized Chern character by precomposing it with complexification.

[1] This follows readily from [Higson and Kasparov 2001] because $\pi_1(\Sigma_g) \times \mathbb{Z}/n\mathbb{Z}$ is a-T-menable. However, we should note that the case of surface groups goes back to the original article of Baum and Connes [2000].

Start with some preparations. The rationalized equivariant real K-homology $\mathrm{KO}_*^\bullet \otimes \mathbb{Q}$ is 4-periodic. Indeed, it is a module over

$$\mathrm{KO}_*(\mathrm{pt}) \otimes \mathbb{Q} \cong \mathbb{Q}[\alpha, \beta, \beta^{-1}]/\langle \alpha^2 - 4\beta \rangle$$

with $\alpha \in \mathrm{KO}_4(\mathrm{pt})$, $\beta \in \mathrm{KO}_8(\mathrm{pt})$, and module multiplication with $\alpha/2$ implements the 4-periodicity. We will implicitly use this 4-periodicity whenever convenient.

The complexification $\mathrm{c} : \mathrm{KO}_*(\mathrm{pt}) \to \mathrm{K}_*(\mathrm{pt}) \cong \mathbb{Z}[\xi, \xi^{-1}]$ satisfies $\mathrm{c}(\alpha) = 2\xi^2$ and $\mathrm{c}(\beta) = \xi^4$, where $\xi \in \mathrm{K}_2(\mathrm{pt})$.

Complex K-homology rationally decomposes into two copies of real K-homology:

**Proposition 2.1.** *Complexification yields an isomorphism of proper equivariant homology theories*:

$$\mathbf{c} := \mathrm{c} + \xi^{-1}\,\mathrm{c} : (\mathrm{KO}_*^\bullet \oplus \mathrm{KO}_{*+2}^\bullet) \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{K}_*^\bullet \otimes \mathbb{Q}. \tag{2.2}$$

The decomposition (2.2) can be used to decompose the equivariant delocalized Chern character and thereby obtain the *delocalized Pontryagin character*:

**Proposition 2.3.** *The equivariant delocalized Chern character composed with complexification yields an isomorphism*

$$\underline{\mathrm{ph}}_\Gamma := \underline{\mathrm{ch}}_\Gamma \circ \mathbf{c} : \mathrm{KO}_p^\Gamma(\underline{\mathrm{E}}\Gamma) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_{k \in \mathbb{Z}} \mathrm{H}_{p+4k}(\Gamma; \mathrm{F}^0\Gamma) \oplus \mathrm{H}_{p+2+4k}(\Gamma; \mathrm{F}^1\Gamma).$$

Because Matthey's maps are right-inverse to the delocalized Chern character, Proposition 2.3 immediately implies that they decompose as follows:

**Corollary 2.4.** *Using the identification* (2.2), *Matthey's inverse maps* [2004]

$$\beta_p^{(\mathrm{t})} : \mathrm{H}_p(\Gamma; \mathrm{F}\Gamma) \to \mathrm{K}_p^\Gamma(\underline{\mathrm{E}}\Gamma) \otimes \mathbb{C}, \quad p \in \{0, 1, 2\},$$

*restrict to maps*

$$\beta_{p,q}^{(\mathrm{t})} : \mathrm{H}_p(\Gamma; \mathrm{F}^q\Gamma) \to \mathrm{KO}_{p+2q}^\Gamma(\underline{\mathrm{E}}\Gamma) \otimes \mathbb{C}, \quad p \in \{0, 1, 2\},\ q \in \{0, 1\}.$$

Analogously to the complex case we write $\beta_{p,q}^{(\mathrm{a})} := \mu \otimes \mathbb{C} \circ \beta_{p,q}^{(\mathrm{t})}$. By abuse of notation $\mu : \mathrm{KO}_{p+2q}^\Gamma(\underline{\mathrm{E}}\Gamma) \to \mathrm{KO}_{p+2q}(\mathrm{C}_\mathrm{r}^*\Gamma)$ denotes the real version of the Baum–Connes assembly map here.

The proofs of Propositions 2.1 and 2.3 can essentially be reduced to the case of finite groups. Thus we need some facts about equivariant K-homology of a finite group and to fix some notation. For more details on the following we refer for example to [Bruner and Greenlees 2010, Chapter 2].

Let $\mathrm{R}(H)$ denote the complex representation ring and $\mathrm{RO}(H)$ its real counterpart. Let $\mathrm{c} : \mathrm{RO}(H) \to \mathrm{R}(H)$ be complexification and $\mathrm{r} : \mathrm{R}(H) \to \mathrm{RO}(H)$ be realification. Let $\tau : \mathrm{R}(H) \to \mathrm{R}(H)$ be the map induced by complex conjugation.

Then $\mathrm{c} \circ \mathrm{r} = 1 + \tau$ and $\mathrm{r} \circ \mathrm{c} = 2$. Let $\mathrm{R}(H)/(1+\tau)$ be a shorthand for the quotient group $\mathrm{R}(H)/(1+\tau)(\mathrm{R}(H))$. Then the equivariant K-homology of a point satisfies

$$\mathrm{KO}_i^H(\mathrm{pt}) \otimes \mathbb{Q} \cong \begin{cases} \mathrm{RO}(H) \otimes \mathbb{Q}, & i \equiv 0 \mod 4, \\ \mathrm{R}(H)/(1+\tau) \otimes \mathbb{Q}, & i \equiv 2 \mod 4, \\ 0, & \text{otherwise}, \end{cases} \tag{2.5}$$

$$\mathrm{K}_i^H(\mathrm{pt}) \otimes \mathbb{Q} \cong \begin{cases} \mathrm{R}(H) \otimes \mathbb{Q}, & i \equiv 0 \mod 2, \\ 0, & \text{otherwise}. \end{cases} \tag{2.6}$$

Complexification $\mathrm{c} : \mathrm{KO}_i^H(\mathrm{pt}) \to \mathrm{K}_i^H(\mathrm{pt})$ is given by complexification of representations for $i \equiv 0 \mod 4$, and by the map $1 - \tau : \mathrm{R}(H)/(1+\tau) \to \mathrm{R}(H)$ for $i \equiv 2 \mod 4$.

We are now ready to prove the propositions of this section.

*Proof of Proposition 2.1*. To show that $\mathbf{c}$ is an isomorphism of proper equivariant homology theories, it suffices to show that

$$\mathbf{c} = \mathrm{c} + \xi^{-1}\,\mathrm{c} : (\mathrm{KO}_i^H(\mathrm{pt}) \oplus \mathrm{KO}_{i+2}^H(\mathrm{pt})) \otimes \mathbb{Q} \to \mathrm{K}_i^H(\mathrm{pt}) \otimes \mathbb{Q} \tag{2.7}$$

is an isomorphism for every finite group $H$ and every $i \in \mathbb{Z}$. The map

$$(\mathrm{KO}_i^\bullet \oplus \mathrm{KO}_{i+2}^\bullet) \otimes \mathbb{Q} \to (\mathrm{KO}_{i+2}^\bullet \oplus \mathrm{KO}_{i+4}^\bullet) \otimes \mathbb{Q}, \quad x \oplus y \mapsto y \oplus (\alpha/2) \cdot x$$

defines a 2-periodicity on the left-hand side which corresponds to Bott periodicity after applying $\mathbf{c} = \mathrm{c} + \xi^{-1}\,\mathrm{c}$. Hence it suffices to check (2.7) for $i \in \{0, 1\}$. For $i = 1$, both sides of (2.7) are zero by (2.5) and (2.6). It remains to check $i = 0$. In this case $\mathbf{c}$ corresponds to the map

$$(\mathrm{RO}(H) \oplus \mathrm{R}(H)/(1+\tau)) \otimes \mathbb{Q} \to \mathrm{R}(H) \otimes \mathbb{Q}, \quad x \oplus [y] \mapsto \mathrm{c}(x) + y - \tau(y).$$

This is an isomorphism because the map

$$\mathrm{R}(H) \otimes \mathbb{Q} \to (\mathrm{RO}(H) \oplus \mathrm{R}(H)/(1+\tau)) \otimes \mathbb{Q}, \quad z \mapsto \tfrac{1}{2}(\mathrm{r}(z) \oplus [z])$$

is an explicit inverse. □

*Proof of Proposition 2.3*. Start with the (nonequivariant) Pontryagin character isomorphism

$$\mathrm{ph} = \mathrm{ch} \circ \mathrm{c} : \mathrm{KO}_p(X) \otimes \mathbb{Q} \xrightarrow{\cong} \bigoplus_{k \in \mathbb{Z}} \mathrm{H}_{p+4k}(X; \mathbb{Q}). \tag{2.8}$$

It is, by definition, the Chern character applied after complexification. In particular, the proposition holds if $\Gamma$ is torsion-free.

Next we deal with the case where $\Gamma = H = \mathbb{Z}/n\mathbb{Z}$ is a finite cyclic group. Then the Chern character yields an isomorphism $\underline{\mathrm{ch}}_H : \mathrm{R}(H) = \mathrm{K}_0^H(\mathrm{pt}) \to \mathrm{H}_0(H; \mathrm{F}H)$ as all the other homology groups vanish. We have $\mathrm{H}_0(H; \mathrm{F}H) = \mathrm{F}H = \mathbb{C}^H$ and $\underline{\mathrm{ch}}_H$ associates to a representation $[\rho] \in \mathrm{R}(H)$ its character $\chi_\rho$. As the character of any real representation is symmetric and the character of an element in the image of $(1-\tau):\mathrm{R}(H)\to\mathrm{R}(H)$ is antisymmetric, we conclude that

$$\underline{\mathrm{ch}}_\Gamma\big(\mathrm{c}(\mathrm{KO}^H_{2q}(\mathrm{pt}))\big)=\mathrm{H}_0(H;\mathrm{F}^q H). \tag{2.9}$$

Next, let $G$ be some group and consider $\Gamma:=G\times\mathbb{Z}/n\mathbb{Z}$ and $y\times z\in\mathrm{KO}^\Gamma_{p+2q}(\underline{\mathrm{E}}\Gamma)$ with $y\in\mathrm{KO}_p(\mathrm{B}G)\cong\mathrm{KO}^G_p(\mathrm{E}G)$ and $z\in\mathrm{KO}^{\mathbb{Z}/n\mathbb{Z}}_{2q}(\mathrm{pt})$. There is a natural isomorphism

$$\mathrm{K}^{G\times\mathbb{Z}/n\mathbb{Z}}_p(\underline{\mathrm{E}}G)\cong\mathrm{K}^G_p(\underline{\mathrm{E}}G)\otimes\mathrm{R}(\mathbb{Z}/n\mathbb{Z})\cong\mathrm{K}^G_p(\underline{\mathrm{E}}G)\otimes\mathrm{K}^{\mathbb{Z}/n\mathbb{Z}}_0(\mathrm{pt})$$

and a commutative diagram

$$\begin{tikzcd}
\mathrm{K}_p(\mathrm{B}G)\otimes\mathrm{K}^{\mathbb{Z}/n\mathbb{Z}}_0(\mathrm{pt})\otimes\mathbb{C} \arrow[r, "\times"] \arrow[d, "\mathrm{ch}\otimes\underline{\mathrm{ch}}_{\mathbb{Z}/n\mathbb{Z}}"] & \mathrm{K}^{G\times\mathbb{Z}/n\mathbb{Z}}_p(\underline{\mathrm{E}}G)\otimes\mathbb{C} \arrow[d, "\underline{\mathrm{ch}}_\Gamma"] \\
\bigoplus_{k\in\mathbb{Z}}\mathrm{H}_{p+2k}(G;\mathbb{C})\otimes\mathrm{H}_0(\mathbb{Z}/n\mathbb{Z};\mathrm{F}\mathbb{Z}/n\mathbb{Z}) \arrow[r, "\times"] & \bigoplus_{k\in\mathbb{Z}}\mathrm{H}_{p+2k}(\Gamma;\mathrm{F}\Gamma)
\end{tikzcd}$$

In view of (2.2), (2.8), and (2.9) this diagram restricts to

$$\begin{tikzcd}
\mathrm{KO}_p(\mathrm{B}G)\otimes\mathrm{KO}^{\mathbb{Z}/n\mathbb{Z}}_{2q}(\mathrm{pt})\otimes\mathbb{C} \arrow[r, "\times"] \arrow[d, "\mathrm{ph}\otimes\underline{\mathrm{ch}}_{\mathbb{Z}/n\mathbb{Z}}\circ\mathrm{c}"] & \mathrm{KO}^{G\times\mathbb{Z}/n\mathbb{Z}}_{p+2q}(\underline{\mathrm{E}}G)\otimes\mathbb{C} \arrow[d, "\underline{\mathrm{ch}}_\Gamma\circ\mathrm{c}"] \\
\bigoplus_{k\in\mathbb{Z}}\mathrm{H}_{p+4k}(G;\mathbb{C})\otimes\mathrm{H}_0(\mathbb{Z}/n\mathbb{Z};\mathrm{F}^q\mathbb{Z}/n\mathbb{Z}) \arrow[r, "\times"] & \bigoplus_{k\in\mathbb{Z}}\mathrm{H}_{p+4k}(\Gamma;\mathrm{F}^q\Gamma)
\end{tikzcd}$$

We conclude

$$\underline{\mathrm{ch}}_\Gamma(\mathrm{c}(y\times z))\in\bigoplus_{k\in\mathbb{Z}}\mathrm{H}_{p+4k}(\Gamma;\mathrm{F}^q\Gamma). \tag{2.10}$$

Now let $\Gamma$ be general. The equivariant K-homology $\mathrm{K}^\Gamma_p(\underline{\mathrm{E}}\Gamma)$ is generated by elements of the form $\varphi_*(y\times z)$ with $G\subseteq\Gamma$ some subgroup, $y\in\mathrm{K}_p(\mathrm{B}G)$, $z\in\mathrm{K}^{\mathbb{Z}/n\mathbb{Z}}_0(\mathrm{pt})$ and $\varphi:G\times\mathbb{Z}/n\mathbb{Z}\to\Gamma$ some group homomorphism. This follows from [Matthey 2004, Theorem 1.3 and Section 7]. Using (2.2), this implies that $\mathrm{KO}^\Gamma_i(\underline{\mathrm{E}}\Gamma)$ is generated by elements of the form $\varphi_*(y\times z)$ with $y\in\mathrm{KO}_p(\mathrm{B}G)$, $z\in\mathrm{KO}^{\mathbb{Z}/n\mathbb{Z}}_{2q}(\mathrm{pt})$ such that $i=p+2q$. Thus (2.10) implies

$$(\underline{\mathrm{ch}}_\Gamma\circ\mathrm{c})(\mathrm{KO}^\Gamma_p(\underline{\mathrm{E}}\Gamma))\subseteq\bigoplus_{k\in\mathbb{Z}}\mathrm{H}_{p+4k}(\Gamma;\mathrm{F}^0\Gamma)\oplus\mathrm{H}_{p+2+4k}(\Gamma;\mathrm{F}^1\Gamma). \tag{2.11}$$

Now the proposition follows by combining (2.2) and (2.11) and using the fact that $\underline{\mathrm{ch}}_\Gamma$ is an isomorphism. □

## 3. Matthey's maps

In this section, we exhibit the real versions of Matthey's maps from Corollary 2.4 more explicitly. We start with a brief summary of the material in [Matthey 2004] that leads to his definition of

$$\beta_p^{(\mathrm{t})} : \mathrm{H}_p(\Gamma; \mathrm{F}(\Gamma)) \to \mathrm{K}_p^{\Gamma}(\underline{\mathrm{E}}\Gamma) \otimes \mathbb{C}, \quad p \in \{0, 1, 2\}.$$

Let $G^{(0)} := 1$, $G^{(1)} := \mathbb{Z}$ and $G^{(2),g} := \Gamma_g := \pi_1(\Sigma_g)$, where $\Sigma_g$ is the oriented surface of genus $g \geq 1$. To simplify the notation, we let $G^{(2)}$ stand for $G^{(2),g}$ for some $g$. Moreover, let $G_n^{(p)} := G^{(p)} \times \mathbb{Z}/n\mathbb{Z}$.

There is an isomorphism $\mathrm{H}_p(\Gamma; \mathrm{F}\Gamma) \cong \bigoplus_C \mathrm{H}_p(\mathrm{B}Z_C; \mathbb{C})$. Here the direct sum runs over all conjugacy classes of finite order elements and $Z_C$ denotes the centralizer of some element in the conjugacy class $C$. An element $x \in \mathrm{H}_p(\Gamma; \mathrm{F}\Gamma)$ is called *homogeneous* if it is contained in one of the direct summands. For each $p \in \{0, 1, 2\}$ and $n \geq 0$, there is a certain fundamental class $[G_n^{(p)}] \in \mathrm{H}_p(G_n^{(p)}; \mathrm{F}G_n^{(p)})$. These have the property that for $p \in \{0, 1, 2\}$ any homogeneous element $x \in \mathrm{H}_p(\Gamma; \mathrm{F}\Gamma)$ can be written as $x = \phi_*[G_n^{(p)}]$ for some $G_n^{(p)}$ and some group homomorphism $\phi : G_n^{(p)} \to \Gamma$.

Moreover, there is a certain K-homological fundamental class

$$[G_n^{(p)}]_{\mathrm{K}} \in \mathrm{K}_p^{G_n^{(p)}}(\underline{\mathrm{E}}G_n^{(p)}) \otimes \mathbb{C}.$$

Setting $\phi_*[G_n^{(p)}] \mapsto \phi_*[G_n^{(p)}]_{\mathrm{K}}$ defines a map $\mathrm{H}_p(\Gamma; \mathrm{F}(\Gamma)) \to \mathrm{K}_p^{\Gamma}(\underline{\mathrm{E}}\Gamma) \otimes \mathbb{C}$ that is right-inverse to the equivariant Chern character. This is Matthey's definition of $\beta_p^{(\mathrm{t})}$.

To describe how this map decomposes in the real case, we need to recall the definition of the K-homological fundamental class $[G_n^{(p)}]_{\mathrm{K}}$. We start with the fundamental classes $[G^{(p)}]_{\mathrm{K}} \in \mathrm{K}_p(\underline{\mathrm{E}}G^{(p)})$, which are defined as

$$\begin{aligned}
&[G^{(0)}]_{\mathrm{K}} := 1 \in \mathrm{K}_0(\underline{\mathrm{E}}G^{(0)}) = \mathrm{K}_0(\mathrm{pt}),\\
&[G^{(1)}]_{\mathrm{K}} := [\mathrm{S}^1]_{\mathrm{K}} \in \mathrm{K}_1(\mathrm{S}^1) \cong \mathrm{K}_1^{\mathbb{Z}}(\underline{\mathrm{E}}\mathbb{Z}),\\
&[G^{(2)}]_{\mathrm{K}} := [\Sigma_g]_{\mathrm{K}} \in \mathrm{K}_2(\Sigma_g) \cong \mathrm{K}_2^{\Gamma_g}(\underline{\mathrm{E}}\Gamma_g).
\end{aligned}$$

That is, $[G^{(p)}]_{\mathrm{K}}$ is the K-homological fundamental class of the point, the circle or an oriented surface of positive genus. Observe that we may take $\underline{\mathrm{E}}G_n^{(p)} = \underline{\mathrm{E}}G^{(p)}$ by letting $\mathbb{Z}/n\mathbb{Z}$ act trivially. Then we set

$$[G_n^{(p)}]_{\mathrm{K}} := \sum_{l=0}^{n-1} \big([G^{(p)}]_{\mathrm{K}} \times [\omega_n^l]\big) \otimes \omega_n^{-l} \in \mathrm{K}_p^{G_n^{(p)}}(\underline{\mathrm{E}}G_n^{(p)}) \otimes \mathbb{C},$$

where $\omega_n := \mathrm{e}^{2\pi\mathrm{i}/n}$ is the primitive $n$-th root of unity and $[\omega_n^l] \in \mathrm{R}(\mathbb{Z}/n\mathbb{Z}) = \mathrm{K}_0^{\mathbb{Z}/n\mathbb{Z}}(\mathrm{pt})$ the corresponding representation.

To obtain the real counterparts to this, we first observe how $[\omega_n^l] \in \mathrm{K}_0^{\mathbb{Z}/n\mathbb{Z}}(\mathrm{pt})$ decomposes under the isomorphism from Proposition 2.1, (2.5) and (2.6). Indeed,

$$\begin{aligned}\mathrm{K}_0^{\mathbb{Z}/n\mathbb{Z}}(\mathrm{pt}) \otimes \mathbb{Q} &\cong \big(\mathrm{KO}_0^{\mathbb{Z}/n\mathbb{Z}}(\mathrm{pt}) \oplus \mathrm{KO}_2^{\mathbb{Z}/n\mathbb{Z}}(\mathrm{pt})\big) \otimes \mathbb{Q} \\ &\cong (\mathrm{RO}(\mathbb{Z}/n\mathbb{Z}) \oplus \mathrm{R}(\mathbb{Z}/n\mathbb{Z})/(1+\tau)) \otimes \mathbb{Q}.\end{aligned}$$

Given an element $x \in \mathrm{R}(Z/n\mathbb{Z}) \otimes \mathbb{Q}$, we write $\Re x := \mathrm{r}(x)/2 \in \mathrm{RO}(\mathbb{Z}/n\mathbb{Z}) \otimes \mathbb{Q}$ and $\Im x \in \mathrm{R}(\mathbb{Z}/n\mathbb{Z})/(1+\tau) = \mathrm{KO}_2^{\mathbb{Z}/n\mathbb{Z}}(\mathrm{pt})$ for the class represented by $x/2$. Then we have $x = \mathrm{c}(\Re x + \Im x)$. We define

$$\begin{aligned}[G_n^{(p)}]^0_{\mathrm{KO}} &:= \sum_{l=0}^{n-1} ([G^{(p)}]_{\mathrm{KO}} \times \Re[\omega_n^l]) \otimes \omega_n^{-l} \in \mathrm{KO}_p^{G_n^{(p)}}(\underline{\mathrm{E}}G_n^{(p)}) \otimes \mathbb{C}, \\ [G_n^{(p)}]^1_{\mathrm{KO}} &:= \sum_{l=0}^{n-1} ([G^{(p)}]_{\mathrm{KO}} \times \Im[\omega_n^l]) \otimes \omega_n^{-l} \in \mathrm{KO}_{p+2}^{G_n^{(p)}}(\underline{\mathrm{E}}G_n^{(p)}) \otimes \mathbb{C}.\end{aligned} \tag{3.1}$$

Here $[G^{(p)}]_{\mathrm{KO}}$ denotes the KO-fundamental class of the point, the circle or a surface, respectively. We find that $[G_n^{(p)}]_{\mathrm{K}} = [G_n^{(p)}]^0_{\mathrm{KO}} \oplus [G_n^{(p)}]^1_{\mathrm{KO}}$ under the isomorphism from Proposition 2.1. The homological fundamental class also decomposes as $[G_n^{(p)}] = [G_n^{(p)}]^0 \oplus [G_n^{(p)}]^1$ according to

$$\mathrm{H}_p(G_n^{(p)}; \mathrm{F}G_n^{(p)}) = \mathrm{H}_p(G_n^{(p)}; \mathrm{F}^0 G_n^{(p)}) \oplus \mathrm{H}_p(G_n^{(p)}; \mathrm{F}^1 G_n^{(p)}).$$

From this discussion we deduce:

**Proposition 3.2.** *The real versions of Matthey's maps from Corollary 2.4 are given by*

$$\beta^{(\mathrm{t})}_{p,q} : \mathrm{H}_p(\Gamma; \mathrm{F}^q(\Gamma)) \to \mathrm{KO}^\Gamma_{p+2q}(\underline{\mathrm{E}}\Gamma) \otimes \mathbb{C}, \quad \phi_*[G_n^{(p)}]^q \mapsto \phi_*[G_n^{(p)}]^q_{\mathrm{KO}}.$$

**Remark 3.3.** The element $[G_n^{(p)}]^0_{\mathrm{KO}}$ can be rewritten as

$$[G_n^{(p)}]^0_{\mathrm{KO}} = \sum_{l=0}^{\lfloor n/2 \rfloor} ([G^{(p)}]_{\mathrm{KO}} \times \Re[\omega_n^l]) \otimes 2\cos(2\pi l/n).$$

A similar equation involving the sine function holds for $[G_n^{(p)}]^1_{\mathrm{KO}}$. Thus it would be possible to restrict to real coefficients. However, we shall not use this any further, and continue to keep complex coefficients everywhere.

## 4. Secondary index classes of psc metrics for finite groups

The proposition below is essentially due to Botvinnik and Gilkey [1995], albeit formulated in a slightly different way. In the proof, we briefly explain for the convenience of the reader how its statement can be deduced from the result in the literature:

**Proposition 4.1.** *Suppose that $H$ is a finite group and $n \geq 6$. Then the $\rho$-invariant $\rho : \mathrm{P}^{\mathrm{spin}}_{n-1}(\mathrm{B}H) \to \mathrm{S}^{\mathbb{R}}_{n-1}(H)$ is rationally surjective.*

*Proof.* We only need to consider $n$ to be even because the analytic structure group of a finite group rationally vanishes in odd degrees. Let $n = 4k + 2q$ with $k \geq 1$ and $q \in \{0, 1\}$. Each (virtual) unitary representation $\pi$ of $H$ induces a trace functional $\mathrm{tr}_\pi : \mathrm{K}_0(\mathbb{C}H) \to \mathbb{Z}$. If $\pi$ is of virtual dimension 0, then $\mathrm{tr}_\pi$ extends to a functional $\eta_\pi : \mathrm{S}_1^{\mathbb{C}}(H) \to \mathbb{R}$ on the complex version of the analytic structure group; see [Higson and Roe 2010]. By the construction of $\eta_\pi$, the composition $\eta_\pi \circ \rho : \mathrm{P}^{\mathrm{spin}}_{4k+2q-1}(\mathrm{B}H) \to \mathbb{R}$ recovers the relative $\eta$-invariant used in [Botvinnik and Gilkey 1995].

Since finite groups satisfy the Baum–Connes conjecture, the Higson–Roe sequence rationally becomes a short exact sequence:

$$0 \to \mathbb{Q} \to \mathrm{K}_0(\mathbb{C}H) \otimes \mathbb{Q} \to \mathrm{S}_1^{\mathbb{C}}(H) \otimes \mathbb{Q} \to 0. \tag{4.2}$$

Let $\mathrm{R}_0(H)$ denote the space of virtual unitary representations of dimension 0. The pairing $\mathrm{R}(H) \otimes \mathbb{Q} \times \mathrm{K}_0(\mathbb{C}H) \otimes \mathbb{Q} \to \mathbb{Q}$, $(\pi, x) \mapsto \mathrm{tr}_\pi(x)$ is nondegenerate. We conclude from this and (4.2) that the pairing

$$\mathrm{R}_0(H) \otimes \mathbb{R} \times \mathrm{S}_1^{\mathbb{C}}(H) \otimes \mathbb{R} \to \mathbb{R}, \quad (\pi, x) \mapsto \eta_\pi(x) \tag{4.3}$$

is also nondegenerate. The complex analytic structure group admits a decomposition $\mathrm{S}_1^{\mathbb{C}}(H) \otimes \mathbb{Q} \cong (\mathrm{S}_1^{\mathbb{R}}(H) \oplus \mathrm{S}_3^{\mathbb{R}}(H)) \otimes \mathbb{Q}$ analogous to Proposition 2.1. Applying this to (4.3) yields a nondegenerate pairing

$$\mathrm{R}_0^q(H) \otimes \mathbb{R} \times \mathrm{S}^{\mathbb{R}}_{2q-1}(H) \otimes \mathbb{R} \to \mathbb{R}, \quad (\pi, x) \mapsto \eta_\pi(x) \tag{4.4}$$

for each $q \in \{0, 1\}$.

Finally, let $4k + 2q \geq 6$. Then [Botvinnik and Gilkey 1995, Theorem 2.1] implies that the composition

$$\mathrm{P}^{\mathrm{spin}}_{4k+2q-1}(\mathrm{B}H) \otimes \mathbb{R} \xrightarrow{\rho \otimes \mathbb{R}} \mathrm{S}^{\mathbb{R}}_{2q-1}(H) \otimes \mathbb{R} \xrightarrow{\bigoplus_i \eta_{\pi_i}} \mathbb{R}^{\dim \mathrm{R}_0^q(H)}$$

is surjective, where $(\pi_i)_i$ is a basis of $\mathrm{R}_0^q(H) \otimes \mathbb{R}$. Since the pairing (4.4) is nondegenerate, the latter map in this composition is an isomorphism. Thus $\rho \otimes \mathbb{R}$ must be surjective as well. □

**Corollary 4.5.** *Suppose that $H$ is a finite group and $n \geq 6$. Then the relative index map $\alpha : \mathrm{R}^{\mathrm{spin}}_n(\mathrm{B}H) \to \mathrm{KO}_n(\mathrm{C}^*_{\mathrm{r}}H)$ is rationally surjective.*

*Proof.* Again we only need to consider $n$ to be even and let $n = 4k + 2q \geq 6$. For a finite group $H$, the groups $\Omega^{\mathrm{spin}}_l(\mathrm{B}H)$ and $\mathrm{KO}_l(\mathrm{B}H)$ are torsion for $l \not\equiv 0$ mod 4. Moreover, $\beta \otimes \mathbb{Q} : \Omega^{\mathrm{spin}}_{4k}(\mathrm{B}H) \otimes \mathbb{Q} \to \mathrm{KO}_0(\mathrm{B}H) \otimes \mathbb{Q}$ is surjective because $\mathrm{KO}_0(\mathrm{B}H) \otimes \mathbb{Q} \cong \mathrm{KO}_0(\mathrm{pt}) \otimes \mathbb{Q}$ is generated by the class represented by any product of Kummer surfaces. By Proposition 4.1, the $\rho$-invariant

$$\rho \otimes \mathbb{Q} : \mathrm{P}^{\mathrm{spin}}_{4k+2q-1}(\mathrm{B}H) \otimes \mathbb{Q} \to \mathrm{S}^{\mathbb{R}}_{2q-1}(H) \otimes \mathbb{Q}$$

is also surjective. Thus we have a diagram of exact sequences

$$\begin{tikzcd}
\Omega^{\mathrm{spin}}_{4k+2q}(\mathrm{B}H)\otimes\mathbb{Q} \arrow[r] \arrow[d, two heads, "\beta\otimes\mathbb{Q}"] & \mathrm{R}^{\mathrm{spin}}_{4k+2q}(\mathrm{B}H)\otimes\mathbb{Q} \arrow[r] \arrow[d, "\alpha\otimes\mathbb{Q}"] & \mathrm{P}^{\mathrm{spin}}_{4k+2q-1}(\mathrm{B}H)\otimes\mathbb{Q} \arrow[r] \arrow[d, two heads, "\rho\otimes\mathbb{Q}"] & 0 \arrow[d, hook] \\
\mathrm{KO}_{2q}(\mathrm{B}H)\otimes\mathbb{Q} \arrow[r] & \mathrm{KO}_{2q}(\mathbb{R}H)\otimes\mathbb{Q} \arrow[r] & \mathrm{S}^{\mathbb{R}}_{2q-1}(H)\otimes\mathbb{Q} \arrow[r] & 0
\end{tikzcd}$$

The four lemma implies that $\alpha\otimes\mathbb{Q}$ must be surjective as well. □

## 5. Proof of main results

Our main result, Theorem 1.3, follows immediately from Proposition 3.2 and the following lemma.

**Lemma 5.1.** *For each $n\in\mathbb{N}$, $p\in\{0,1,2\}$, $q\in\{0,1\}$ and $k\geq 1$ with $4k+2q\geq 6$, there exists $[G_n^{(p)}]^q_{\mathrm{psc}}\in\mathrm{R}^{\mathrm{spin}}_{p+2q+4k}(\mathrm{B}G_n^{(p)})\otimes\mathbb{C}$ with*

$$\alpha([G_n^{(p)}]^q_{\mathrm{psc}})=\mu([G_n^{(p)}]^q_{\mathrm{KO}})\in\mathrm{KO}_{2q}(\mathrm{C}^*_{\mathrm{r}}G_n^{(p)})\otimes\mathbb{C}.$$

*Proof.* Let

$$x^q_{n,l}:=\begin{cases}\Re[\omega_n^l], & q=0\\ \Im[\omega_n^l], & q=1\end{cases}\in\mathrm{KO}^{\mathbb{Z}/n\mathbb{Z}}_{2q}(\mathrm{pt}).$$

By Corollary 4.5, we can choose an element $y^{q,k}_{n,l}\in\mathrm{R}^{\mathrm{spin}}_{2q+4k}(\mathrm{B}\mathbb{Z}/n\mathbb{Z})\otimes\mathbb{Q}$ such that $\alpha(y^{q,k}_{n,l})=\mu(x^q_{n,l})\in\mathrm{KO}_{2q}(\mathrm{C}^*_{\mathrm{r}}(\mathbb{Z}/n\mathbb{Z}))\otimes\mathbb{Q}$. Now let $[G^{(0)}]_\Omega:=[\mathrm{pt}]\in\Omega^{\mathrm{spin}}_0(\mathrm{pt})$, $[G^{(1)}]_\Omega:=[\mathrm{S}^1]\in\Omega^{\mathrm{spin}}_1(\mathrm{B}\mathbb{Z})$ and $[G^{(2),g}]_\Omega:=[\Sigma_g]\in\Omega^{\mathrm{spin}}_2(\mathrm{B}\Gamma_g)\otimes\mathbb{Q}$. Note that for the latter we need to choose one from the $2^{2g}$ different spin structures on the oriented surface. However, rationally the element $[\Sigma_g]$ is independent of this choice. Taking direct products yields a map $\Omega^{\mathrm{spin}}_l(X)\otimes\mathrm{R}^{\mathrm{spin}}_m(Y)\xrightarrow{\times}\mathrm{R}^{\mathrm{spin}}_{l+m}(X\times Y)$. Using this, we let

$$[G_n^{(p)}]^q_{\mathrm{psc}}:=\sum_{l=0}^{n-1}([G^{(p)}]_\Omega\times y^{q,k}_{n,l})\otimes\omega_n^{-l}\in\mathrm{R}^{\mathrm{spin}}_{p+2q+4k}(\mathrm{B}G_n^{(p)})\otimes\mathbb{C}.$$ □

*Proof of Theorem 1.3.* We have the diagram

$$\begin{tikzcd}
 & \mathrm{H}_p(\Gamma;\mathrm{F}^q\Gamma) \arrow[r, hook] \arrow[d, dashed, "\beta^{(\mathrm{psc})}_{p,q,k}"] \arrow[ddl, bend right, "\beta^{(\mathrm{t})}_{p,q}"'] & \mathrm{H}_p(\Gamma;\mathrm{F}\Gamma) \arrow[dd, "\beta^{(\mathrm{a})}_p"] \\
 & \mathrm{R}^{\mathrm{spin}}_{p+2q+4k}(\mathrm{B}\Gamma)\otimes\mathbb{C} \arrow[d, "\alpha\otimes\mathbb{C}"] & \\
\mathrm{KO}^{\Gamma}_{p+2q}(\underline{\mathrm{E}}\Gamma)\otimes\mathbb{C} \arrow[r, "\mu\otimes\mathbb{C}"] & \mathrm{KO}_{p+2q}(\mathrm{C}^*_{\mathrm{r}}\Gamma)\otimes\mathbb{C} \arrow[r, "\mathrm{c}\otimes\mathbb{C}"] & \mathrm{K}_p(\mathrm{C}^*_{\mathrm{r}}\Gamma)\otimes\mathbb{C}
\end{tikzcd}$$

where by construction of $\beta^{(\mathrm{t})}_{p,q}$ the outer paths commute. To prove the existence of $\beta^{(\mathrm{psc})}_{p,q,k}$, it suffices to show that the image of $\mu \otimes \mathbb{C} \circ \beta^{(\mathrm{t})}_{p,q}$ is contained in the image of $\alpha \otimes \mathbb{C}$. Proposition 3.2 implies that the image of $\beta^{(\mathrm{t})}_{p,q}$ is generated by elements of the form $\phi_*[G_n^{(p)}]^q_{\mathrm{KO}}$, where $[G_n^{(p)}]^q_{\mathrm{KO}}$ is defined in (3.1) and $\phi : G_n^{(p)} \to \Gamma$ is a group homomorphism. Thus it suffices to show that the elements $\mu(\phi_*[G_n^{(p)}]^q_{\mathrm{KO}})$ are contained in the image of $\alpha \otimes \mathbb{C}$. Indeed, Lemma 5.1 states that $\mu([G_n^{(p)}]^q_{\mathrm{KO}})$ admits a lift to $\mathrm{R}^{\mathrm{spin}}_{p+2q+4k}(\mathrm{B}G_n^{(p)})$. Therefore, by functoriality, we conclude that $\mu(\phi_*[G_n^{(p)}]^q_{\mathrm{KO}})$ admits a lift to $\mathrm{R}^{\mathrm{spin}}_{p+2q+4k}(\mathrm{B}\Gamma)$. □

*Proof of Corollary 1.5.* If $\mu\otimes\mathbb{Q}$ is injective, then $\beta^{(\mathrm{a})}_p = \mu\otimes\mathbb{C}\circ\beta^{(\mathrm{t})}_p$ maps $\mathrm{H}_p(\Gamma;\mathrm{F}\Gamma)$ injectively into $\mathrm{K}_p(\mathrm{C}^*_{\mathrm{r}}\Gamma)\otimes\mathbb{C}$. Thus the diagram in Theorem 1.3 implies that for fixed $n$ the following map must be injective:

$$\sum_{p+2q+4k=n} \beta^{(\mathrm{psc})}_{p,q,k} : \bigoplus_{p+2q\in n+4\mathbb{Z}} \mathrm{H}_p(\Gamma;\mathrm{F}^q\Gamma) \to \mathrm{R}^{\mathrm{spin}}_n(\mathrm{B}\Gamma)\otimes\mathbb{C}. \tag{5.2}$$

Here $p$, $q$, $k$ range over $\{0, 1, 2\}$, $\{0, 1\}$, $\mathbb{Z}$, respectively. Unpacking this yields the table in the statement of Corollary 1.5. □

*Proof of Corollary 1.6.* The image in $\mathrm{K}^\Gamma_p(\underline{\mathrm{E}}\Gamma)\otimes\mathbb{C}$ of the restriction of $\beta^{(\mathrm{t})}_p$ to $\mathrm{H}_p(\Gamma;\mathrm{F}^0_0\Gamma\oplus\mathrm{F}^1\Gamma)$ intersects trivially with the image of $\mathrm{K}_p(\mathrm{B}\Gamma)\otimes\mathbb{C}\hookrightarrow\mathrm{K}^\Gamma_p(\underline{\mathrm{E}}\Gamma)\otimes\mathbb{C}$. This follows from the decomposition of the handicrafted Chern character based on the Shapiro isomorphism; see [Matthey 2004, Theorem 1.4]. Thus the injectivity of (5.2) together with a diagram chase involving (1.1) implies that the following map must be injective as well:

$$\sum_{p+2q+4k=n} \partial\circ\beta^{(\mathrm{psc})}_{p,q,k} : \bigoplus_{p+2q\in n+4\mathbb{Z}} \mathrm{H}_p(\Gamma;\mathrm{F}^q_0\Gamma) \to \mathrm{P}^{\mathrm{spin}}_{n-1}(\mathrm{B}\Gamma)\otimes\mathbb{C}.$$

Here we use the convention $\mathrm{F}^1_0\Gamma := \mathrm{F}^1\Gamma$. □

*Proof of Corollary 1.8.* If the rational homological dimension of $\Gamma$ is at most 2, then the map

$$\sum_{p+2q\in n+4\mathbb{Z}} \beta^{(\mathrm{t})}_{p,q} : \bigoplus_{p+2q\in n+4\mathbb{Z}} \mathrm{H}_p(\Gamma;\mathrm{F}\Gamma^q) \to \mathrm{KO}^\Gamma_n(\underline{\mathrm{E}}\Gamma)\otimes\mathbb{C}$$

is the inverse to the Chern character. In particular, it is surjective. If the rational Baum–Connes assembly map $\mu\otimes\mathbb{Q}$ is also surjective, then this implies that the following is surjective too:

$$\sum_{p+2q\in n+4\mathbb{Z}} \beta^{(\mathrm{a})}_{p,q} = \mu\otimes\mathbb{C}\circ\sum_{p+2q\in n+4\mathbb{Z}} \beta^{(\mathrm{t})}_{p,q} : \bigoplus_{p+2q\in n+4\mathbb{Z}} \mathrm{H}_p(\Gamma;\mathrm{F}\Gamma^q) \to \mathrm{KO}_n(\mathrm{C}^*_{\mathrm{r}}\Gamma)\otimes\mathbb{C}.$$

Theorem 1.3 implies that for $n\geq 7$, the image of $\beta^{(\mathrm{a})}_{p,q}$ is contained in the image of $\alpha\otimes\mathbb{C}$, which proves surjectivity of $\alpha\otimes\mathbb{C}$ and thus of $\alpha\otimes\mathbb{Q}$.

If $\mu \otimes \mathbb{Q}$ is injective, then $\nu \otimes \mathbb{Q}$ is injective and by exactness the boundary map $\partial \otimes \mathbb{Q} : \mathrm{KO}_n(\mathrm{C}^*_\mathrm{r}\Gamma) \otimes \mathbb{Q} \to \mathrm{S}^{\mathbb{R}}_{n-1}(\Gamma) \otimes \mathbb{Q}$ is surjective. Hence the surjectivity statement for $\rho \otimes \mathbb{Q}$ if $\mu \otimes \mathbb{Q}$ is an isomorphism follows from surjectivity of $\alpha \otimes \mathbb{Q}$ and commutativity of the diagram (1.1). □

## Acknowledgements

Bárcenas thanks the support of PAPIIT-UNAM grants IA 100117 "Topología Equivariante y Teoría de Índice", as well as CONACYT-SEP Foundational Research grant "Geometría No Conmutativa, Aplicación de Baum–Connes y Topología Algebraica". Zeidler gratefully acknowledges the hospitality of the Centro de Ciencias Matemáticas at UNAM Morelia, where this project was initiated. Both authors thank Johannes Ebert for valuable discussions.

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Received 3 Oct 2017. Revised 21 Dec 2017. Accepted 6 Jan 2018.

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2018 vol. 3 no. 3