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The A_∞-structure of the index map

Oliver Braunling, Michael Groechenig and Jesse Wolfson

Let F be a local field with residue field k. The classifying space of $\mathrm{GL}_n(F)$ comes canonically equipped with a map to the delooping of the K-theory space of k. Passing to loop spaces, such a map abstractly encodes a homotopy coherently associative map of A_∞-spaces $\mathrm{GL}_n(F) \to K_k$. Using a generalized Waldhausen construction, we construct an explicit model built for the A_∞-structure of this map, built from nested systems of lattices in F^n. More generally, we construct this model in the framework of Tate objects in exact categories, with finite dimensional vector spaces over local fields as a motivating example.

1. Introduction

Let F be a local field with residue field k, e.g., $F = \mathbb{Q}_p$ and $k = \mathbb{F}_p$, or $F = \mathbb{F}_p((t))$ and $k = \mathbb{F}_p$. Let $O \subset F$ be the ring of integers, $\mathfrak{m} \subset O$ the maximal ideal, and denote by $\mathrm{Tor}_{\mathfrak{m},f}(O)$ the category of finitely generated torsion O-modules. Let $S_\bullet$ denote Waldhausen's S-construction. For any finite dimensional vector space V over F, the authors constructed in [Braunling et al. 2018] an "index" map, i.e., a map of spaces

$$B\mathrm{GL}(V) \xrightarrow{\mathsf{Index}} |S_\bullet(\mathrm{Tor}_{\mathfrak{m},f}(O))^\times| \xrightarrow{\simeq} BK_k$$

from the classifying space of $\mathrm{GL}(V)$, a group which we shall always tacitly view as equipped with the discrete topology, to Waldhausen's delooping of the K-theory space of k.[1]

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[1] For F such that k is not a subfield of F, the existence of the map $|S_\bullet(\mathrm{Tor}_{\mathfrak{m},f}(O))^\times| \to BK_k$ relies on devissage.

To sketch the bigger picture, for an equicharacteristic local field F with residue field k, Quillen's localization sequence gives a boundary map

$$\Omega K_F \longrightarrow K_k, \tag{1.1}$$

where K_F is the algebraic K-theory of the category of finite dimensional F-vector spaces. On the other hand, by a general procedure a finite dimensional F-vector space V can be written as an ind-pro limit of finite dimensional k-vector spaces. The "index map" has the property that (the classifying space of) the group of automorphisms of V as such an ind-pro limit can also be mapped to the K-theory K_k of the residue field. Restricting to those automorphisms which genuinely come from F-vector space automorphisms, [Braunling et al. 2018] shows that, suitably restricted to a common source, this map agrees with the one coming from (1.1).

Let Vect_f denote the category of finite dimensional vector spaces. The index map encodes, after passing to loop spaces, a homotopy coherently associative map of loop spaces

$$\mathrm{GL}(V) \xrightarrow{\simeq} \Omega B\mathrm{GL}(V) \to \Omega|S_\bullet(\mathsf{Vect}_f(k))^\times| \xrightarrow{\simeq} K_k,$$

which in turn amounts to a coherent collection of homotopies

$$\mathsf{Index}(g_1) + \mathsf{Index}(g_2) \simeq \mathsf{Index}(g_1 g_2). \tag{1.2}$$

In applications, e.g., [Braunling et al. 2014], one would like to be able to manipulate these homotopies in detail. The goal of the present paper is to construct a map of reduced Segal spaces

$$B_\bullet \mathrm{GL}(V) \to K_{S_\bullet(\mathsf{Vect}_f(k))},$$

whose geometric realization is the index map.[2] Our main tool for this construction is a generalized Waldhausen construction, developed in Section 3A. Our model for this construction follows from an analogy with index theory. Given an invertible element $f \in F^\times$ such that $f \cdot O \subset O$, the linear map $O \xrightarrow{f} O$ has finite dimensional cokernel, and the assignment $f \mapsto O/f \cdot O$ extends to a map of spaces

$$\mathrm{GL}_1(F) \to K_k.$$

To extend this to a full map of simplicial spaces (and to handle the case where k is not a subfield of F, or when $\dim V > 1$), we employ the framework of Tate objects in an exact category C, as developed in [Braunling et al. 2016]. Tate objects provide a setting for working with "locally compact" objects modeled on C. For example, a finite dimensional vector space over $\mathbb{Q}_p$ is canonically a locally compact topological abelian group (with the p-adic topology), and also an elementary Tate object in the category $\mathrm{Ab}_{p,f}$ of finite abelian p-groups. A key advantage of

[2] Here $B_\bullet G$ denotes the bar construction (or nerve) of the group G. This is a reduced Segal space with $|B_\bullet G| \simeq BG$.

working with Tate objects is that the category $\mathsf{Tate}(\mathsf{C})$ of Tate objects in C is itself an exact category, and can be treated on the same footing as C (without requiring any topological constructions).

To define Tate objects, we rely on the notion of "admissible Ind-objects". Recall that an admissible Ind-object in C is a left exact presheaf $\widehat{X}$ of abelian groups on C such that $\widehat{X}$ can be written as the colimit of a filtering diagram $X : I \to \mathsf{C}$ in which all maps $X_i \to X_j$ are admissible monics. The category of admissible Ind-objects $\mathsf{Ind}^a(\mathsf{C})$ is a full subcategory of the category $\mathrm{Lex}(\mathsf{C})$ of all left exact presheaves of abelian groups, and it inherits an exact structure from $\mathrm{Lex}(\mathsf{C})$; see [Braunling et al. 2016, Section 3]. We define the category of admissible Pro-objects by $\mathsf{Pro}^a(\mathsf{C}) := \mathsf{Ind}^a(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$. Since $\mathsf{Pro}^a(\mathsf{C})$ is an exact category, we can consider the exact category $\mathsf{Ind}^a(\mathsf{Pro}^a(\mathsf{C}))$, and we define $\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})$ to be the smallest full subcategory of $\mathsf{Ind}^a(\mathsf{Pro}^a(\mathsf{C}))$ which contains $\mathsf{Ind}^a(\mathsf{C})$ and $\mathsf{Pro}^a(\mathsf{C})$ and is closed under extensions.

The key feature of Tate objects is that they have "lattices", i.e., admissible subobjects $L \subset V$ such that $L \in \mathsf{Pro}^a(\mathsf{C})$ and $V/L \in \mathsf{Ind}^a(\mathsf{C})$. For example, the ring of integers $\mathbb{Z}_p \subset \mathbb{Q}_p$ is canonically an object in $\mathsf{Pro}^a(\mathrm{Ab}_{p,f})$, and $\mathbb{Q}_p/\mathbb{Z}_p$ is a discrete abelian p-group, or equivalently, an object of $\mathsf{Ind}^a(\mathrm{Ab}_{p,f})$. In the above analogy with index theory, any Tate object V can play the role of F, any lattice $L \subset V$ the role of O, and any automorphism $g \in \mathrm{GL}(V)$ the role of $f \in F^\times$. Following this analogy, coherent homotopies as in (1.2) should correspond to choices of nested systems of lattices in V. In the present paper, we make this precise by using a generalized Waldhausen construction to exhibit, for a Tate object V in an idempotent complete exact category C, a map of reduced Segal objects

$$B_\bullet \mathrm{GL}(V) \to K_{S_\bullet(\mathsf{C})} \tag{1.3}$$

whose geometric realization is the index map. The present construction is independent of our approach in [Braunling et al. 2018]. In Section 3C, we exhibit a homotopy between the geometric realization of (1.3) and the "index map" of [Braunling et al. 2018].

2. Preliminaries

Throughout this paper we work in the ∞-categories of spaces and spectra. We take [Lurie 2009; 2017] as standard references for ∞-categories.

2A. ***Exact categories and Tate objects.*** We follow the notation of [Braunling et al. 2018] throughout. We consider *exact categories* C, i.e., additive categories equipped with a collection of distinguished kernel-cokernel pairs

$$X \hookrightarrow Y \twoheadrightarrow Z$$

called *exact sequences* which satisfy axioms modeled on the behavior of exact sequences of abelian groups or of projective modules. See [Bühler 2010] for an excellent exposition. An exact category C is *idempotent complete* if every idempotent splits, i.e., if for all $p : X \to X$ in C with $p^2 = p$, there exists an isomorphism $X \cong Y \oplus Z$ which takes p to $1_Y \oplus 0$. Fixing language, we refer to maps which arise as kernels of exact sequences as *admissible monics*, and those which arise as cokernels of exact sequences as *admissible epics*.

Given an exact category C, there are associated exact categories $\mathsf{Ind}^a(\mathsf{C})$ and $\mathsf{Pro}^a(\mathsf{C})$ of *admissible Ind-objects* and *admissible Pro-objects* and also exact categories $\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})$ and $\mathsf{Tate}(\mathsf{C})$ of *elementary Tate objects* and *Tate objects* in C. We quickly recall the definitions here, and refer the reader to [Braunling et al. 2016] for full details.

Denote by $\mathrm{Lex}(\mathsf{C})$ the abelian category of left exact presheaves of abelian groups on C. The Yoneda embedding allows us to view C as a fully exact subcategory of $\mathrm{Lex}(\mathsf{C})$ which is closed under extensions; see, e.g., [Keller 1990, Appendix A].

Definition 2.1. Let C be an exact category. An *admissible Ind-object* in C is an object $\widehat{X} \in \mathrm{Lex}(\mathsf{C})$ such that $\widehat{X}$ is the colimit (in $\mathrm{Lex}(\mathsf{C})$) of a filtering diagram $X : I \to \mathsf{C}$ in which all maps $X_i \to X_j$ are admissible monics in C. Define the category of admissible Ind-objects $\mathsf{Ind}^a(\mathsf{C})$ as a full subcategory of $\mathrm{Lex}(\mathsf{C})$. Define the category of *admissible Pro-objects* $\mathsf{Pro}^a(\mathsf{C})$ by $\mathsf{Pro}^a(\mathsf{C}) := \mathsf{Ind}^a(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$.

Following [Keller 1990, Appendix A], we show in [Braunling et al. 2016, Theorem 3.7] that $\mathsf{Ind}^a(\mathsf{C})$ is closed under extensions in $\mathrm{Lex}(\mathsf{C})$, and thus has a canonical structure as an exact category.

Remark 2.2. Unpacking the definitions, one can also realize $\mathsf{Pro}^a(\mathsf{C})$ as a localization of the category $\mathsf{Inv}^a(\mathsf{C})$ of cofiltering systems of admissible epimorphisms, where one localizes at all morphisms of diagrams which are invertible on a cofinal subdiagram. Equivalently, one localizes at all morphisms which become invertible under the evaluation map $\mathsf{Inv}^a(\mathsf{C}) \to \mathrm{Lex}(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$.

Definition 2.3. Let C be an exact category. Define the category of *elementary Tate objects* $\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})$ to be the smallest full subcategory of $\mathsf{Ind}^a(\mathsf{Pro}^a(\mathsf{C}))$ which contains $\mathsf{Ind}^a(\mathsf{C})$ and $\mathsf{Pro}^a(\mathsf{C})$ and which is closed under extensions. Define the category of *Tate objects* $\mathsf{Tate}(\mathsf{C})$ to be the idempotent completion of $\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})$.

By [Braunling et al. 2016, Theorem 5.6], the category of elementary Tate objects is well-defined, and thus inherits a canonical exact structure from $\mathsf{Ind}^a(\mathsf{Pro}^a(\mathsf{C}))$.

Example 2.4. Let $\mathrm{Ab}_{p,f}$ be the category of finitely generated abelian p-groups. There exists an exact functor

$$\mathsf{Vect}_f(\mathbb{Q}_p) \to \mathsf{Tate}^{\mathrm{el}}(\mathrm{Ab}_{p,f})$$

from the category of finite dimensional vector spaces over $\mathbb{Q}_p$ to the category of elementary Tate objects in $\mathrm{Ab}_{p,f}$.

For the present, we need the following.

Definition 2.5. Let V be an elementary Tate object in C.

(1) A *lattice* $L \hookrightarrow V$ is an admissible subobject, with $L \in \mathsf{Pro}^a_\kappa(\mathsf{C})$ and the cokernel $V/L \in \mathsf{Ind}^a_\kappa(\mathsf{C})$.

(2) The *Sato Grassmannian* $\mathrm{Gr}(V)$ is the partially ordered set of lattices in V, where $L_0 \le L_1$ if there exists a commuting triangle of admissible monics

$$\begin{tikzcd} L_0 \arrow[r, hook] \arrow[dr, hook] & L_1 \arrow[d, hook] \\ & V \end{tikzcd}$$

Lattices and the Sato Grassmannian play a key role in our study of Tate objects. We view (c) in the theorem below as the main result of [Braunling et al. 2016].

Theorem 2.6 [Braunling et al. 2016, Proposition 6.6, Theorem 6.7]. *Let* C *be an exact category.*

(a) *Every elementary Tate object in* C *has a lattice.*

(b) *The quotient of a lattice by a sublattice is an object of* C.

(c) *If* C *is idempotent complete, and* $L_0 \hookrightarrow V$ *and* $L_1 \hookrightarrow V$ *are two lattices in an elementary Tate object* V, *then there exists a lattice* $N \hookrightarrow V$ *with* $L_0, L_1 \le N$ *in* $\mathrm{Gr}(V)$. *Similarly,* L_0 *and* L_1 *have a common sublattice* $M \le L_0, L_1$.

2B. ***Algebraic K-theory.*** Following [Quillen 1973], one associates to every exact category C its K-theory space K_{C}. The space K_{C} is an infinite loop space which serves as a universal target for *additive* invariants of C. Waldhausen [1985] gave an alternate construction of K_{C}, and proved his fundamental "additivity theorem". Waldhausen's treatment of algebraic K-theory hinges on two simplicial exact categories, denoted by $S_\bullet(\mathsf{C})$, and $S^r_\bullet(f)$, where C is an exact category and $f : \mathsf{C} \to \mathsf{D}$ is an exact functor. The simplicial object $S_\bullet(\mathsf{C})$ associates to every finite nonempty totally ordered set $[k]$ the exact category $S_k(\mathsf{C})$, which consists of functors $[k] \to \mathsf{C}$, sending every arrow in $[k]$ to an admissible monic. Likewise, the simplicial object $S^r_\bullet(f)$ associates to $[k]$ the exact category $S_k(\mathsf{C})$ consisting of functors $[k] \to \mathsf{D}$, sending every arrow in $[k]$ to an admissible monic in D with cokernel in C. Given a category C, denote by $\mathsf{C}^\times$ the groupoid of all isomorphism in C. With this notation, Waldhausen's definition can be given as

$$K_{\mathsf{C}} := \Omega |S_\bullet(\mathsf{C})^\times|.$$

See [Braunling et al. 2018, Section 2] for a discussion of Waldhausen's approach to K-theory tailored to the present setting. As discussed there, the fundamental property of algebraic K-theory is the following "additivity theorem". The results of this paper and [Braunling et al. 2018] can be seen as consequences of the additivity theorem combined with Theorem 2.6.

Theorem 2.7 (Waldhausen's additivity theorem [Waldhausen 1985, Theorem 1.4.2, Proposition 1.3.2(4)]). *Let $F_1 \hookrightarrow F_2 \twoheadrightarrow F_3$ be an exact sequence of functors $\mathsf{C}_1 \to \mathsf{C}_2$. Then the map*

$$|S_\bullet F_2| : |S_\bullet(\mathsf{C}_1)^\times| \to |S_\bullet(\mathsf{C}_2)^\times|$$

is naturally homotopic to

$$|S_\bullet F_1 \oplus S_\bullet F_3| : |S_\bullet(\mathsf{C}_1)^\times| \to |S_\bullet(\mathsf{C}_2)^\times|.$$

Several equivalent reformulations exist. We need the following.

Definition 2.8 (Waldhausen). Let D be an exact category, and let C_1 and C_2 be full subcategories of D which are closed under extensions. Define $\mathcal{E}(\mathsf{C}_1, \mathsf{D}, \mathsf{C}_2)$ to be the full subcategory of $\mathcal{E}\,\mathsf{D}$ consisting of the exact sequences $X_1 \hookrightarrow Y \twoheadrightarrow X_2$ with $X_i \in \mathsf{C}_i$.

Note that, because C_1 and C_2 are closed under extensions in D, $\mathcal{E}(\mathsf{C}_1, \mathsf{D}, \mathsf{C}_2)$ is closed under extensions in $\mathcal{E}\,\mathsf{D}$; in particular, it is an exact category.

Theorem 2.9. *Let $\mathsf{A} \xrightarrow{i} \mathsf{B} \xrightarrow{p} \mathsf{C}$ be a composable pair of exact functors such that i is fully faithful and induces an equivalence with the full subcategory of B annihilated by p. Moreover, assume that p has a left adjoint*

$$s : \mathsf{C} \to \mathsf{B},$$

such that $ps \cong 1_{\mathsf{C}}$ and such that, for every object $Y \in \mathsf{B}$, the co-unit $sp(Y) \to Y$ is an admissible monic with cokernel in A. Then, the map

$$i \times s : K_{\mathsf{A}} \times K_{\mathsf{C}} \xrightarrow{\simeq} K_{\mathsf{B}}$$

is an equivalence of spaces.

While this theorem is, without doubt, well-known, we have chosen a less conventional statement which is convenient for our applications. Therefore, we now give a proof.

Proof. We have a well-defined map of spaces $i \times s : K_{\mathsf{A}} \times K_{\mathsf{C}} \to K_{\mathsf{B}}$. By the Whitehead lemma it suffices to show that it establishes an equivalence on all homotopy groups.

The admissible monic of functors

$$sp \hookrightarrow 1_{\mathsf{B}} : \mathsf{B} \to \mathsf{B},$$

given by the co-unit of the adjunction (p, s), extends to a short exact sequence

$$sp \hookrightarrow 1_{\mathsf{B}} \twoheadrightarrow f : \mathsf{B} \to \mathsf{B}\,.$$

By construction, $pf = 0$, and therefore f can be expressed as ir, where $r : \mathsf{B} \to \mathsf{A}$ is an exact functor. By the additivity theorem (Theorem 2.7), we have

$$\pi_i(K(ir) \oplus K(sp)) = \pi_i(K(1_{\mathsf{B}})).$$

Moreover, the relations $ps = 1_{\mathsf{C}}$ and $ri = 1_{\mathsf{A}}$ imply that we also have

$$\pi_i(K_{\mathsf{B}}) \cong \pi_i(K_{\mathsf{A}}) \times \pi_i(K_{\mathsf{C}}).$$

The Whitehead lemma concludes the proof. □

2C. *Segal objects.* Segal [1974] introduced a definition which, in the hands of May and Thomason [May and Thomason 1978; Thomason 1979], Rezk [2001], Lurie [2017] and many others, has become fundamental to the study of A_∞-objects (also known as E_1-objects or homotopy coherent associative monoids) in a homotopical setting.

Definition 2.10. Let C be an ∞-category with finite products. For each n, consider the collection of maps

$$\left\{[1] = \{0 < 1\} \xrightarrow{\cong} \{i-1 < i\} \subset [n]\right\}_{i=1}^{n}.$$

A *Segal object* in C is a simplicial object $X_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{C})$ such that, for $n \geq 2$, the map

$$X_n \to \underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{n}$$

induced by the above collection is an equivalence. A *reduced Segal object* $X_\bullet$ is a Segal object with $X_0 \simeq *$. Segal objects form a full subcategory of simplicial objects in C.

For a basic example, the bar construction associates to a group G a simplicial space $B_\bullet G$ with n-simplices the discrete space G^n. A standard exercise shows that $B_\bullet G$ is a reduced Segal space, and the Segal structure is just a rewriting of the group law. For a richer example, given an exact category C, we can consider the simplicial exact category $S_\bullet\mathsf{C}$ given by Waldhausen's $S_\bullet$-construction. Waldhausen's additivity theorem (Theorem 2.7) implies that the simplicial space $K_{S_\bullet\mathsf{C}}$ obtained by taking the K-theory space of each category of n-simplices is a reduced Segal object in the ∞-category of spaces. The Segal space structure encodes the homotopy coherent addition of elements in K_{C}.

2D. ***The index map.*** We now recall the index map. For $n \geq 0$, denote by $[n]$ the partially ordered set $\{0 < \cdots < n\}$ viewed as a category, and, for a category C, denote by $\mathrm{Fun}([n], \mathsf{C})$ the category of functors from $[n]$ to C.

Definition 2.11. Let C be an exact category. Define the *Sato complex* $\mathrm{Gr}^{\leq}_{\bullet}(\mathsf{C})$ to be the simplicial diagram of exact categories with

(1) n-simplices $\mathrm{Gr}^{\leq}_{n}(\mathsf{C})$ given by the full subcategory of $\mathrm{Fun}([n+1], \mathsf{Tate}^{\mathrm{el}}(\mathsf{C}))$ consisting of sequences of admissible monics

$$L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V$$

where, for all i, $L_i \hookrightarrow V$ is the inclusion of a lattice,

(2) face maps are given by the functors

$$d_i(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) := (L_0 \hookrightarrow \cdots \hookrightarrow L_{i-1} \hookrightarrow L_{i+1} \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V),$$

(3) and degeneracy maps are given by the functors

$$s_i(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) := (L_0 \hookrightarrow \cdots \hookrightarrow L_i \hookrightarrow L_i \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V).$$

The simplicial object $\mathrm{Gr}^{\leq}_{\bullet}(\mathsf{C})$ allows us to introduce the index map.

Definition 2.12. Let C be an exact category. The *categorical index map* is the span of simplicial maps

$$\mathsf{Tate}^{\mathrm{el}}(\mathsf{C}) \longleftarrow \mathrm{Gr}^{\leq}_{\bullet}(\mathsf{C}) \xrightarrow{\mathsf{Index}} S_{\bullet}(\mathsf{C}), \tag{2.13}$$

where the left-facing arrow is given on n-simplices by the assignment

$$(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) \mapsto V,$$

and Index is given on n-simplices by the assignment

$$(L_0 \hookrightarrow \cdots \hookrightarrow L_n \hookrightarrow V) \mapsto (L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0).$$

Recall the following.

Proposition 2.14 [Braunling et al. 2018, Proposition 3.3]. *Let C be an idempotent complete exact category. Then the map $\mathrm{Gr}^{\leq}_{\bullet}(\mathsf{C}) \to \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})$ of* (2.13) *induces an equivalence*

$$|\,\mathrm{Gr}^{\leq}_{\bullet}(\mathsf{C})^{\times}| \xrightarrow{\simeq} |\,\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^{\times}|. \tag{2.15}$$

Remark 2.16. The proposition follows from the fact that if C is idempotent complete, then the Sato Grassmannian $\mathrm{Gr}(V)$ of every elementary Tate object is a directed and codirected poset [Braunling et al. 2016, Theorem 6.7]. The nerve of this poset is therefore contractible, and the geometric realizations of these nerves are the fibres of the map (2.15).

Following the proposition, we obtain the K-theoretic index map by restricting the categorical index map (2.13) to the groupoids of all isomorphisms, geometrically realizing, and picking a homotopy inverse to (2.15) to obtain the map

$$\mathsf{Index} : |\,\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^{\times}| \xrightarrow{\simeq} |\,\mathsf{Gr}^{\leq}_{\bullet}(\mathsf{C})^{\times}| \to |S_{\bullet}(\mathsf{C})^{\times}| =: BK_{\mathsf{C}}. \tag{2.17}$$

Our goal is to construct an explicit map of Segal objects $B_{\bullet}\,\mathrm{Aut}(V) \to K_{S_{\bullet}(\mathsf{C})}$, for any elementary Tate object V, whose geometric realization is equivalent to the restriction of (2.17) along the map $|\ast /\!/ \mathrm{Aut}(V)| \to |\,\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^{\times}|$.[3]

3. The A_∞-structure of the index map

3A. ***A generalized Waldhausen construction.*** Let C be an exact category, and $f : \mathsf{C} \to \mathsf{D}$ an exact functor. Waldhausen's approach to algebraic K-theory [1985] hinges on the simplicial exact categories $S_{\bullet}(\mathsf{C})$ and $S^r_{\bullet}(f)$ recalled above. We now extend the functors

$$S_{\bullet}(\mathsf{C}),\ S^r_{\bullet}(f) : \Delta^{\mathrm{op}} \to \mathsf{Cat}_{ex}$$

from the ordinal category, i.e., the category of finite nonempty linearly ordered sets, to the category of filtered finite partially ordered sets. We refer to the resulting functors as the "generalized Waldhausen construction". In Section 3B we then use the generalized Waldhausen construction to give a treatment of the A_∞-structure of the index map.

Partially ordered sets and related structures. The current subsection contains several definitions of a combinatorial nature.

Definition 3.1. Let I be a partially ordered set. We denote by $\Gamma(I)$ the directed graph given by the set underlying I as set of vertices, and intervals $a < b$ as edges. We denote the set of directed edges of $\Gamma(I)$ by $E(I)$.

Example 3.2. For the ordinal [2] we obtain

$$\begin{tikzcd} \bullet \\ \bullet \arrow[u] \\ \bullet \arrow[u] \arrow[uu, bend right] \end{tikzcd}$$

for the oriented graph $\Gamma([2])$. While this graph is more traditionally drawn as the boundary of a 2-simplex, the present depiction is chosen to highlight the maximal tree.

[3] Here $\ast /\!/ G$ denotes the one object groupoid with automorphisms G, and the map $\ast /\!/ \mathrm{Aut}(V) \to \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^{\times}$ is given on objects by $\ast \mapsto V$ and is the identity map on automorphisms.

We work with finite, filtered, partially ordered sets with *basepoints* (which are chosen to be minimal elements).

Definition 3.3. A *based, finite, filtered, partially ordered set* is a pair $(I; x_0, \dots, x_k)$, where I is a *finite* partially ordered set with a final element, and $(x_0, \dots, x_k)$ is a tuple of *minimal elements* in I.[4] A morphism of based partially ordered sets is a map of pairs

$$(f, \sigma) : (I; x_0, \dots, x_k) \to (I'; y_0, \dots, y_m),$$

where $f : I \to I'$ is a map of partially ordered sets, $\sigma : [m] \to [k]$ is a map of finite ordinals, and $f(x_i) = y_{\sigma(i)}$. The category of based, finite, filtered, partially ordered sets is denoted by $\mathrm{poSet}_{\mathrm{f}}^{\mathrm{filt}}$.

The assumption of *finiteness* is crucial for the inductive proofs that are given later, but could eventually be relaxed.

Some arguments require choosing a maximal tree in $\Gamma(I)$ with good properties.

Definition 3.4. Let Γ be an oriented graph. A maximal tree $T \subset \Gamma$ is said to be *admissible* if for every pair of vertices (x, y), there exists a vertex z and unique oriented paths from x to z and from y to z within T.

The following examples help to clarify this definition.

Example 3.5. Consider the trees below:

The tree on the left is admissible, while the one on the right is not (there is no common vertex that receives an oriented path from the two upper vertices).

Example 3.6. Let I be a finite, filtered, partially ordered set. An admissible tree $T \subset \Gamma(I)$ always exists. Indeed, let $m \in I$ denote the final element. Then the tree T given by the union of all edges (x, m) for $x \in I$ is admissible.

The definition below introduces the concept of a *framing* of a based partially ordered set.

Definition 3.7. A *framed partially ordered set* is a triple $(I, E(T), x_0, \dots, x_k)$, where $E(T) \subset E(I)$ is the set of edges of an admissible maximal tree, and the pair $(I; x_0, \dots, x_k)$ is a based, finite, filtered, partially ordered set. The category

[4] It is important to note that the basepoints are not assumed to be pairwise distinct.

of framed, partially ordered sets $\mathrm{poSet}_{\mathrm{f}}^{\mathrm{fr,filt}}$ is the category with framed, partially ordered sets as objects, and morphisms

$$\phi : (I, E(T), x_0) \to (I', E(T'), x_0'),$$

where $\phi : I \to I'$ is a map of partially ordered sets, mapping the basepoints *bijectively* onto each other, and satisfying $\phi(T) \subset \phi(T')$. We denote by

$$\phi_\sharp : E(T) \to E(T')_+ = E(T) \cup \{\star\},$$

the map which sends $e \in E(T)$ either to its image $\phi(e) \in E(T')$, or, if $\phi(e)$ consists of a single point, to the basepoint $\star$.

Pairs of exact categories and diagrams. We define the generalized Waldhausen construction in the context of extension closed subcategories of exact categories.

Definition 3.8. We denote by Cat_{ex}^{pair} the 2-category of pairs of exact categories $\mathsf{C} \subset \mathsf{D}$ such that C is an *extension-closed* subcategory of D. Objects in this category are also referred to using the notation (D, C).

For every partially ordered set I we have an associated category. For notational convenience, we do not distinguish between these.

Definition 3.9. Let $(\mathsf{D}, \mathsf{C}) \in \mathsf{Cat}_{ex}^{pair}$ be a pair of exact categories. Let I be a partially ordered set. An *admissible I-diagram* in (D, C) is a functor $I \to \mathsf{D}$, sending each arrow in I to an admissible monic in D with cokernel an object of C. We denote the exact category of such functors by $\mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D})$.

The following example serves as a motivation for this definition.

Example 3.10. We observe that $\mathrm{Fun}_{\mathsf{C}}([n], \mathsf{D}) = S_n^r(\mathsf{C} \subset \mathsf{D})$ (see Section 2B).

In Definition 3.7 we introduced the concept of framed partially ordered sets. Recall the map $\phi_\sharp : E(T) \to E(T')_+$. By abuse of notation we also use the symbol $\phi_\sharp$ to denote the unique map of pointed sets

$$E(T)_+ \to E(T')_+.$$

Note that, for every object X in a pointed ∞-category C with finite coproducts, we have a natural functor

$$\coprod_{?} X : (\mathrm{Set}_*^{\mathrm{fin}})^{\mathrm{op}} \to \mathsf{C}.$$

An inductive argument allows us to establish the following lemma. The choice of a maximal tree $T \subset \Gamma(I)$ should be understood as analogous to choosing a basis for a vector space.

Lemma 3.11. *Let $(I; E(T), x_0, \dots, x_k)$ be a framed, partially ordered set. We denote by $T \subset \Gamma(I)$ an admissible maximal tree of $\Gamma(I)$. Then there exists an equivalence*

$$\phi(T) : K_{\mathsf{Fun}_{\mathsf{C}}(I,\mathsf{D})} \cong K_{\mathsf{D}} \times K_{\mathsf{C}}^{\times E(T)}.$$

Moreover, this equivalence can be seen as a natural equivalence of functors

$$K_{\mathrm{Fun}_{-}(-,-)} \simeq K_{-} \times K_{-}^{\times E(-)} : \mathsf{Cat}_{ex}^{pair} \times (\mathrm{poSet}_{\mathrm{f}}^{\mathrm{fr,filt}})^{\mathrm{op}} \to \mathsf{Spaces}.$$

Although the lemma is stated for a framed partially ordered set with basepoints $x_0, \dots, x_k$, we actually only need the zeroth basepoint x_0. An inspection of the proof below shows that all the other basepoints could be discarded.

Proof of Lemma 3.11. For every $e = (y_i \le y_{i+1}) \in E(T)$ we denote by X_e the quotient $F(y_{i+1})/F(y_i)$. We have an exact functor

$$\mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D}) \to \mathsf{D} \times \mathsf{C}^{E(T)},$$

which sends $F : I \to \mathsf{D}$ to $(F(x_0), (X_e)_{e \in E(T)})$. This map defines a natural transformation between the functors

$$\mathrm{Fun}_{-}(-,-),\ (-) \times (-)^{E(-)} : \mathsf{Cat}_{ex}^{pair} \times (\mathrm{poSet}_{\mathrm{f}}^{\mathrm{fr,filt}})^{\mathrm{op}} \to \mathsf{Cat}_{ex}.$$

Applying the functor $K_{-} : \mathsf{Cat}_{ex} \to \mathsf{Spaces}$, we obtain the natural transformation $\phi(T)$. It remains to show that $\phi(T)$ is an equivalence for each triple $(I, \mathsf{D}, \mathsf{C})$. We use induction on the cardinality of I to show this. As a warmup, we begin with the case that I is a totally ordered set. Without loss of generality we may identify it with $\{0 < \cdots < n\}$. Moreover, in the totally ordered case, there is only one possible choice for the framing (T, x_0). The induction is anchored to the case $n = 0$, i.e., the case of the singleton set, which is evidently true.

Assume that $\phi(T)$ has been shown to be an equivalence for totally ordered sets of cardinality $< n$. We denote by I' the framed partially ordered set defined by the subset $\{0 < \cdots < n-1\}$. The restriction functor $\mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D}) \to \mathrm{Fun}_{\mathsf{C}}(I', \mathsf{D})$ sits in a short exact sequence of exact categories

$$\mathsf{C} \hookrightarrow \mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D}) \twoheadrightarrow \mathrm{Fun}_{\mathsf{C}}(I', \mathsf{D}),$$

where we send $X \in \mathsf{C}$ to $(0 \hookrightarrow \cdots \hookrightarrow 0 \hookrightarrow X) \in \mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D})$. We also have a splitting, given by

$$\mathrm{Fun}_{\mathsf{C}}(I', \mathsf{D}) \to \mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D}),$$

which sends $(Y_0 \hookrightarrow \cdots \hookrightarrow Y_{n-1})$ to $(Y_0 \hookrightarrow \cdots \hookrightarrow Y_{n-1} \hookrightarrow Y_{n-1})$. By means of the additivity theorem (Theorem 2.9), we conclude

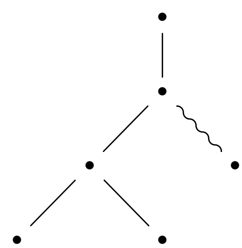
$$K_{\mathrm{Fun}_{\mathsf{C}}(I,\mathsf{D})} \cong K_{\mathrm{Fun}_{\mathsf{C}}(I',\mathsf{D})} \times K_{\mathsf{C}}.$$

Applying the inductive hypothesis to $\mathrm{Fun}_{\mathsf{C}}(I', \mathsf{D})$, we conclude the assertion for totally ordered sets.

The proof for general I also works by induction on the number of elements. If I is not totally ordered, but of cardinality $n+1$, we may decompose our framed partially ordered set

$$(I, T) = (I', T') \cup (I'', T''),$$

where I'' is totally ordered, $I' \cap I'' = \{\max I''\}$, and $x_0 \in I'$. Consider for example the graph



where edges belonging to I'' have been drawn as squiggly lines.

There exists a positive integer $1 \leq k \leq n$ such that $I'' \cong \{0 < \cdots < k\}$. The restriction functor from I-diagrams to I'-diagrams belongs to a short exact sequence of exact categories

$$\mathrm{Fun}_{\mathsf{C}}(I'' \setminus \{\max I''\}, \mathsf{C}) \hookrightarrow \mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D}) \twoheadrightarrow \mathrm{Fun}_{\mathsf{C}}(I', \mathsf{D}),$$

where the left-hand side is seen as the exact category of morphisms

$$(Y_0 \hookrightarrow Y_1 \hookrightarrow \cdots \hookrightarrow Y_{k-1}),$$

which extends to an I-diagram by sending the object Y_{k-1} to every vertex in I'. This short exact sequence is split by the functor

$$\mathrm{Fun}_{\mathsf{C}}(I', \mathsf{D}) \to \mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D}),$$

which extends an I'-diagram to an I-diagram, by sending each vertex y of I'' to the object $\max I'' \in I' \cap I''$ (with the identity morphisms as admissible epimorphisms between them). The additivity from Theorem 2.9 yields

$$K_{\mathrm{Fun}_{\mathsf{C}}(I,\mathsf{D})} \cong K_{\mathrm{Fun}_{\mathsf{C}}(I',\mathsf{D})} \times K_{S_k(\mathsf{C})}.$$

Using the induction hypothesis, we see that the first component is equivalent to $K_{\mathsf{D}} \times K_{\mathsf{C}}^{\times E(T')}$, and the second component to $K_{\mathsf{C}}^{\times E(T'')}$, proving the assertion. $\square$

The index space. Let $(I; x_0, \ldots, x_k)$ be a based, finite, filtered, partially ordered set (Definition 3.3). Together with a pair of exact categories $\mathsf{C} \subset \mathsf{D}$ such that C is extension-closed in D, we define the *index space*, which is the recipient of a map from $K_{\mathrm{Fun}_{\mathsf{C}}(I,\mathsf{D})}$. It can be thought of as measuring the difference between the basepoints.

Definition 3.12. (a) For a based, finite, filtered, partially ordered set $(I; x_0, \dots, x_k)$ we denote by I^Δ the partially ordered set obtained by identifying the basepoints. Cofunctoriality of $\mathrm{Fun}_{\mathsf{C}}(-, \mathsf{C})$ yields a forgetful functor

$$\mathrm{Fun}_{\mathsf{C}}(I^\Delta, \mathsf{D}) \to \mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D}).$$

(b) For an exact category D, let $\mathcal{K}_{\mathsf{D}}$ be the connective K-theory spectrum. We denote by $\mathbb{I}\mathrm{dx}_{\mathsf{C},I}\,\mathsf{D}$ the space underlying (i.e., Ω^∞ of) the cofibre of the morphism[5]

$$\mathcal{K}_{\mathrm{Fun}_{\mathsf{C}}(I^\Delta,\mathsf{D})} \to \mathcal{K}_{\mathrm{Fun}_{\mathsf{C}}(I,\mathsf{D})}\,.$$

By functoriality of cofibres, this gives rise to a functor

$$\mathbb{I}\mathrm{dx} : \mathsf{Cat}_{ex}^{pair} \times (\mathrm{poSet}_{\mathrm{f}}^{\mathrm{filt}})^{\mathrm{op}} \to \mathsf{Spaces}.$$

We refer to $\mathbb{I}\mathrm{dx}_{\mathsf{C},I}\,\mathsf{D}$ as the *index-space* of (D, C) relative to $(I; x_0, \dots, x_k)$.

(c) We refer to the map of spaces

$$|\,\mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D})^\times| \to K_{\mathrm{Fun}_C(I,D)} \to \mathbb{I}\mathrm{dx}_{C,I}(D)$$

as the *pre-index map* of the pair (D, C) relative to $(I; x_0, \dots, x_k)$.

The index space is to a large extent independent of I, as guaranteed by its functorial nature in Definition 3.12(b). We record this observation in the next two results. In Proposition 3.22 we further refine this statement.

Lemma 3.13. *Let* $\mathsf{C} \hookrightarrow \mathsf{D}$ *be an extension-closed exact subcategory of an exact category* D*. We consider an injective morphism of finite, based, filtered, partially ordered sets, in the sense of Definition 3.3,*

$$(I; x_0, \dots, x_k) \to (I'; y_0, \dots, y_k),$$

which induces a bijection of basepoints (i.e., on basepoints, it corresponds to the identity map $[k] \to [k]$*). Then the induced morphism of index spaces*

$$\mathbb{I}\mathrm{dx}_{\mathsf{C},I}\,\mathsf{D} \to \mathbb{I}\mathrm{dx}_{\mathsf{C},I'}\,\mathsf{D}$$

is an equivalence.

Proof. By virtue of Lemma 3.11, the choice of an admissible maximal tree T in I induces an equivalence of K-theory spaces

$$K_{\mathrm{Fun}_{\mathsf{C}}(I,\mathsf{D})} \cong K_{\mathsf{D}} \times K_{\mathsf{C}}^{\times E(T)}.$$

Recall from Definition 3.12 that I^Δ denotes the finite, based, filtered, partially ordered set obtained by identifying all basepoints. We can choose T in a way, such that its image T^Δ in I^Δ is also an admissible tree. For instance, we could

[5] The long exact sequence of homotopy groups implies that this cofibre is again a connective spectrum.

take the tree given by the edges (x, m), where $m = \max I$ and x runs through the elements of $I \setminus \{m\}$. We denote by e_i the (unique) edge of T which contains x_i. By construction, the edges e_i map to the same edge in T^Δ, and we denote this edge by e. We can apply the functoriality of Lemma 3.11 to obtain the commutative square of connective K-theory spectra

$$\begin{tikzcd} \mathcal{K}_{\mathrm{Fun}_{\mathsf{C}}(I^\Delta,\mathsf{D})} \arrow[r] \arrow[d, "\cong"'] & \mathcal{K}_{\mathrm{Fun}_{\mathsf{C}}(I,\mathsf{D})} \arrow[d, "\cong"] \\ \mathcal{K}_{\mathsf{D}} \oplus \mathcal{K}_{\mathsf{C}}^{\oplus E(T^\Delta)} \arrow[r, "\alpha"] & \mathcal{K}_{\mathsf{D}} \oplus \mathcal{K}_{\mathsf{C}}^{\oplus E(T)} \end{tikzcd}$$

where the morphism α is given by the identity $1_{\mathcal{K}_{\mathsf{C}}}$ for edges in $E(T) \setminus \{e_0, \dots, e_k\}$, and given by the diagonal map

$$\Delta_{\mathcal{K}_{\mathsf{C}}} : \mathcal{K}_{\mathsf{C}} \to \mathcal{K}_{\mathsf{C}}^{\oplus(k+1)}$$

for the component e. In particular, we see that $\mathrm{cofib}(\alpha) \cong \mathrm{cofib}(\Delta_{\mathcal{K}_{\mathsf{C}}})$.

The same analysis applies to I'. Because we can choose an admissible maximal tree T in I which extends to an admissible maximal tree T' in I', we see that $\mathrm{cofib}(\mathcal{K}_{\mathrm{Fun}_{\mathsf{C}}(I^\Delta,\mathsf{D})} \to \mathcal{K}_{\mathrm{Fun}_{\mathsf{C}}(I,\mathsf{D})})$ is equivalent to

$$\mathrm{cofib}(\Delta_{\mathcal{K}_{\mathsf{C}}} : \mathcal{K}_{\mathsf{C}} \to \mathcal{K}_{\mathsf{C}}^{k+1}) \cong \mathrm{cofib}(\mathcal{K}_{\mathrm{Fun}^\Delta_{\mathsf{C}}(I',\mathsf{D})} \to \mathcal{K}_{\mathrm{Fun}_{\mathsf{C}}(I',\mathsf{D})}).$$

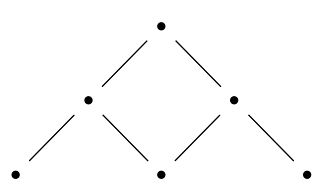
The restriction functor $\mathbb{I}\mathrm{dx}_{\mathsf{C},I}\,\mathsf{D} \to \mathbb{I}\mathrm{dx}_{\mathsf{C},I'}\,\mathsf{D}$ is defined independently of any choices. The admissible maximal trees T and T' only play a role in verifying that this map is an equivalence. We therefore see that we have a canonical equivalence between $\mathbb{I}\mathrm{dx}_{\mathsf{C},I}\,\mathsf{D}$ and $\mathbb{I}\mathrm{dx}_{\mathsf{C},I'}\,\mathsf{D}$. □

Definition 3.14. For every positive integer k we have an object

$$B[k] = (B[k]; b_0, \dots, b_k) \in \mathrm{poSet}_{\mathrm{f}}^{\mathrm{filt}},$$

given by the set of nonempty intervals in the ordinal $[k]$. An interval is understood to be a subset $J \subset [k]$ with the property that $x \le y \le z$ and $x, z \in J$ implies that $y \in J$. The basepoints $(b_i)_{i=0,\dots,k}$ are given by the singletons $\{i\}$.

We have drawn the filtered partially ordered set $B[2]$ below:



Definition 3.15. For an arbitrary $I = (I; x_0, \dots, x_k)$ in $\mathrm{poSet}_{\mathrm{f}}^{\mathrm{filt}}$, we denote by $I^B = (I^B; x_0, \dots, x_k)$ the based, finite, filtered, partially ordered set given by $I \cup B[k]$,

where we identify the basepoints $b_i = x_i$ and extend the inductive ordering of I to I^B by demanding $x \leq y$, for all $x \in B[k]$ and $y \in I \setminus \{x_0, \dots, x_k\}$. To summarize the previous construction, we obtain I^B from I by gluing on a copy of $B[k]$ to I, with all new elements being $\leq$ than elements in I. This process is functorial in I; we denote the resulting functor by

$$(-)^B : \mathrm{poSet}_{\mathrm{f}}^{\mathrm{filt}} \to \mathrm{poSet}_{\mathrm{f}}^{\mathrm{filt}}.$$

The inclusion $I \subset I^B$ gives rise to a natural transformation of functors

$$1_{\mathrm{poSet}_{\mathrm{f}}^{\mathrm{filt}}} \Rightarrow (-)^B.$$

The category $\mathrm{poSet}_{\mathrm{f}}^{\mathrm{filt}}$ satisfies the property that for two objects $(I; x_0, \dots, x_k)$ and $(I'; y_0, \dots, y_k)$ we can find an $(I'', z_1, \dots, z_k)$, containing subobjects isomorphic to I and I' (respecting basepoints). Combining this observation with the lemma proven above, we obtain a complete description of index spaces.

Corollary 3.16. *Let $(I; x_0, \dots, x_k)$ be a based, finite, filtered, partially ordered set with pairwise distinct basepoints. Then the index space of the pair (D, C) is equivalent to*

$$K_{S_k(\mathsf{C})} \cong K_{\mathsf{C}}^{\times k}.$$

This equivalence is functorial in the pair $\mathsf{C} \subset \mathsf{D}$, where C is extension-closed in D, and it is contravariantly functorial in the based filtered partially ordered set I. Moreover, if $M_\bullet$ is a simplicial object in $\mathrm{poSet}_{\mathrm{f}}^{\mathrm{filt}}$ such that, for every nonnegative integer k, M_k has $k+1$ pairwise distinct basepoints, then we have an equivalence of simplicial spaces

$$\mathbb{I}\mathrm{dx}_{\mathsf{C}, M_\bullet} \mathsf{D} \cong K_{S_\bullet(\mathsf{C})}.$$

Proof. Lemma 3.13 implies that we have a canonical equivalence

$$\mathbb{I}\mathrm{dx}_{\mathsf{C}, I} \mathsf{D} \cong \mathbb{I}\mathrm{dx}_{\mathsf{C}, I^B} \mathsf{D} \cong \mathbb{I}\mathrm{dx}_{\mathsf{C}, B[k]} \mathsf{D}.$$

To conclude the argument, we have to show that $\mathbb{I}\mathrm{dx}_{\mathsf{C}, B[k]} \mathsf{D} \cong K_{S_k(\mathsf{C})}$. This equivalence will be shown to be induced by the exact functor

$$S_k(\mathsf{C}) \to \mathrm{Fun}_{\mathsf{C}}(B[k], \mathsf{D}), \tag{3.17}$$

sending $(0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_k)$ to the functor F in $\mathrm{Fun}_{\mathsf{C}}(B[k], \mathsf{D})$, which maps the interval $[i, j]$ to the object X_j. We draw the resulting diagram for $k = 2$ to illustrate the idea behind the definition:

$$
\begin{tikzcd}[column sep=small]
 & & X_2 & & \\
 & X_1 \arrow[ur] & & X_2 \arrow[ul] & \\
0 \arrow[ur] & & X_1 \arrow[ul] \arrow[ur] & & X_2 \arrow[ul]
\end{tikzcd}
$$

Alluding to Lemma 3.11, one can prove with the help of the right choice of admissible maximal tree in $B[k]$ that the induced map of index spaces is indeed an equivalence. We choose to work with the naive admissible maximal tree T in $B[k]$, uniquely defined by the property that for every nonmaximal element there is a unique edge in T connecting it with the maximum. The image of T in $B[k]^\Delta$, i.e., the partially ordered set obtained by identifying the basepoints $b_0, \dots, b_k$ (see Definition 3.12), is also an admissible maximal tree. We can therefore apply Lemma 3.11 to analyze the map of spaces

$$K_{\mathrm{Func}_{\mathsf{C}}(B[k]^\Delta, \mathsf{D})} \to K_{\mathrm{Func}_{\mathsf{C}}(B[k], \mathsf{D})}.$$

Doing so, we obtain a commutative diagram of connective K-theory spectra (as in the proof of Lemma 3.13)

$$
\begin{tikzcd}
\mathcal{K}_{\mathrm{Func}_{\mathsf{C}}(B[k]^\Delta, \mathsf{D})} \arrow[r] \arrow[d, "\cong"'] & \mathcal{K}_{\mathrm{Func}_{\mathsf{C}}(B[k], \mathsf{D})} \arrow[d, "\cong"] \\
\mathcal{K}_{\mathsf{D}} \oplus \mathcal{K}_{\mathsf{C}}^{\oplus E(T^\Delta)} \arrow[r, "\alpha"] & \mathcal{K}_{\mathsf{D}} \oplus \mathcal{K}_{\mathsf{C}}^{\oplus E(T)}
\end{tikzcd}
\tag{3.18}
$$

where the morphism α agrees with the identity $1_{\mathcal{K}_{\mathsf{C}}}$ for edges in $E(T) \setminus \{e_0, \dots, e_k\}$, and with the diagonal map

$$\Delta_{\mathcal{K}_{\mathsf{C}}} : \mathcal{K}_{\mathsf{C}} \to \mathcal{K}_{\mathsf{C}}^{\oplus(k+1)}$$

for the component e. This is the same map arising in the proof of Lemma 3.13, and we have

$$\mathbb{I}\mathrm{dx}_{\mathsf{C}, B[k]}\, \mathsf{D} \cong \Omega^\infty \operatorname{cofib}\bigl(\mathcal{K}_{\mathsf{C}} \xrightarrow{\Delta_{\mathcal{K}_{\mathsf{C}}}} \mathcal{K}_{\mathsf{C}}^{\{b_0,\dots,b_k\}}\bigr) \cong K_{\mathsf{C}}^{\times k},$$

where the last equivalence is defined as the inverse to the composition

$$K_{\mathsf{C}}^{\times k} \xrightarrow{i} K_{\mathsf{C}}^{\{b_0,\dots,b_k\}} \to \Omega^\infty \operatorname{cofib}\bigl(\mathcal{K}_{\mathsf{C}} \xrightarrow{\Delta_{\mathcal{K}_{\mathsf{C}}}} \mathcal{K}_{\mathsf{C}}^{\{b_0,\dots,b_k\}}\bigr), \tag{3.19}$$

where the map i is the inclusion of $K_{\mathsf{C}}^{\times k}$ into $K_{\mathsf{C}}^{\{b_0,\dots,b_k\}}$, which misses the $K_{\mathsf{C}}^{\{b_0\}}$-factor. In particular, we see that i corresponds to the map of K-theory spaces induced by the functor $\mathsf{C}^{\times k} \to \mathsf{C}^{\{b_0,\dots,b_k\}}$ given by the inclusion of the last k factors.

Recall that we have $K_{S_k(\mathsf{C})} \cong K_{\mathsf{C}}^{\times k}$, with respect to the map induced by the exact functor

$$\mathsf{C}^{\times k} \to S_k(\mathsf{C}) \tag{3.20}$$

sending

$$(X_1, \dots, X_k) \mapsto (0 \hookrightarrow X_1 \hookrightarrow X_1 \oplus X_2 \hookrightarrow \cdots \hookrightarrow X_1 \oplus \cdots \oplus X_k).$$

Composing the functors

$$K_{\mathsf{C}}^{\times k} \to K_{S_k(\mathsf{C})} \to K_{\mathrm{Func}_{\mathsf{C}}(B[k],\mathsf{D})} \to \mathbb{I}\mathrm{dx}_{\mathsf{C},B[k]}\,\mathsf{D} \to K_{\mathsf{C}}^{\times k},$$

we obtain the identity, as can be checked on the level of exact categories: we have a commutative diagram of exact functors

$$\begin{tikzcd}
 & & \mathsf{D} \times \mathsf{C}^{\{b_1,\dots,b_k\}} \\
\mathsf{C}^{\times k} \arrow[r] \arrow[urr] & S_k(\mathsf{C}) \arrow[r] & \mathrm{Func}_{\mathsf{C}}(B[k],\mathsf{D}) \arrow[u]
\end{tikzcd}$$

where the right vertical functor sends

$$F \mapsto (F(b_0), F([1])/F(b_0), \dots, F([k])/F(b_{k-1})).$$

The composition of exact functors represented by the diagonal arrow is given on objects by

$$\begin{aligned}
(X_1, \dots, X_k) &\mapsto (0 \hookrightarrow X_1 \hookrightarrow X_1 \oplus X_2 \hookrightarrow \cdots X_1 \oplus \cdots \oplus X_k) \\
&\mapsto ([i,j] \mapsto X_1 \oplus \cdots \oplus X_j) \\
&\mapsto (0, X_1, X_2, \dots, X_k),
\end{aligned}$$

i.e., it is equivalent to the inclusion of the last k factors in $\mathsf{C}^{\times k+1}$. Applying K-theory, and juxtaposing with (3.18), we obtain a commutative diagram of spaces

$$\begin{tikzcd}
 & & K_{\mathsf{D}} \times K_{\mathsf{C}}^{\{b_0,\dots,b_k\}} \arrow[r] & \Omega^\infty \operatorname{cofib}\big(\mathcal{K}_{\mathsf{C}} \xrightarrow{\Delta_{\mathcal{K}_{\mathsf{C}}}} \mathcal{K}_{\mathsf{C}}^{\{b_0,\dots,b_k\}}\big) \\
K_{\mathsf{C}}^{\times k} \arrow[r] \arrow[urr] & K_{S_k(\mathsf{C})} \arrow[r] & K_{\mathrm{Func}_{\mathsf{C}}(B[k],\mathsf{D})} \arrow[u] \arrow[r] & \mathbb{I}\mathrm{dx}_{\mathsf{C},B[k]}\,\mathsf{D} \arrow[u, "\cong"']
\end{tikzcd}$$

As we observed in (3.19), the composition of the arrows on the top agrees with the equivalence $\mathbb{I}\mathrm{dx}_{\mathsf{C},B[k]}\,\mathsf{D} \cong K_{\mathsf{C}}^{\times k}$.

To conclude the argument it suffices to establish the last claim. The functoriality of the index space construction guarantees that $\mathbb{I}\mathrm{dx}_{\mathsf{C},M_\bullet}\,\mathsf{D}$ is a well-defined simplicial space. Since the construction $I \mapsto I^B$ is functorial, we obtain a well-defined simplicial object $M^B_\bullet$, which acts as a bridge between $\mathbb{I}\mathrm{dx}_{\mathsf{C},M_\bullet}\,\mathsf{D}$ and $\mathbb{I}\mathrm{dx}_{\mathsf{C},B[\bullet]}\,\mathsf{D}$, i.e., according to Lemma 3.13 we have equivalences

$$\mathbb{I}\mathrm{dx}_{\mathsf{C},M_\bullet}\,\mathsf{D} \cong \mathbb{I}\mathrm{dx}_{\mathsf{C},M^B_\bullet}\,\mathsf{D} \cong \mathbb{I}\mathrm{dx}_{\mathsf{C},B[\bullet]}\,\mathsf{D}\,.$$

It therefore suffices to show that $\mathbb{I}\mathrm{dx}_{\mathsf{C},B[\bullet]}\,\mathsf{D} \cong K_{S_\bullet(\mathsf{C})}$ as simplicial spaces. Since the map (3.17) is clearly a map of simplicial objects in exact categories, and a map of simplicial objects is an equivalence if it is a levelwise equivalence, we may conclude the proof. □

Rigidity of the pre-index map. We now record a consequence of Lemma 3.13, which we refer to as the *rigidity* of the pre-index map. In order to formulate the result, we have to introduce a localization of the category $\mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}}$.

Lemma 3.21. *Consider the class of morphisms W in the category* $\mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}}$ *which consists of maps* $\big(I \to I', [k] \xrightarrow{\phi} [k']\big)$ *such that* $\phi : [k] \to [k']$ *is an isomorphism. We denote by* $\mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}}[W^{-1}]$ *the ∞-category obtained by localization at W. This localization is canonically equivalent to the category Δ of finite nonempty ordinals, by means of the functor*

$$\mathrm{base} : \mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}} \to \Delta\,,$$

which sends the pair $(I, (x_0, \ldots, x_k))$ to $[k]$. The functor $B[\bullet] : \Delta \to \mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}}$ (Definition 3.14) *is an inverse equivalence*

$$\Delta \to \mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}}[W^{-1}].$$

Proof. Note that we have $\mathrm{base} \circ B[\bullet] \xrightarrow{\cong} \mathrm{id}_\Delta$.

The universal property of localization of ∞-categories implies that the functor base induces a functor

$$\widetilde{\mathrm{base}} : \mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}}[W^{-1}] \to \Delta\,.$$

In particular, we obtain a natural equivalence

$$\widetilde{\mathrm{base}} \circ B[\bullet] \xrightarrow{\cong} \mathrm{id}_\Delta\,.$$

Similarly, we recall from the proof of Corollary 3.16 that we have a natural transformation

$$\mathrm{id}_{\mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}}} \to (-)^B : \mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}} \to \mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}},$$

as well as $B[\bullet] \circ \mathrm{base} \to (-)^B$. Putting these two natural transformations together, we obtain a zigzag

$$\mathrm{id}_{\mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}}} \to (-)^B \leftarrow B[\bullet] \circ \mathrm{base},$$

which induces a natural equivalence of functors

$$\mathrm{id}_{\mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}}[W^{-1}]} \xrightarrow{\cong} B[\bullet] \circ \widetilde{\mathrm{base}}.$$

We conclude that the functors $B[\bullet]$ and $\widetilde{\text{base}}$ are mutually inverse equivalences of ∞-categories (in fact, this shows that the ∞-category $\text{poSet}_{\text{f}}^{\text{filt}}[W^{-1}]$ is equivalent to a category). $\square$

We use this localization to get the below porism from the proof of Corollary 3.16.

Proposition 3.22. *The functor* $\mathbb{I}\text{dx} : \mathsf{Cat}_{ex}^{pair} \times \text{poSet}_{\text{f}}^{\text{filt}^{\text{op}}} \to \mathsf{Spaces}$ *of Definition 3.12 descends along the localization* $\text{poSet}_{\text{f}}^{\text{filt}} \to \text{poSet}_{\text{f}}^{\text{filt}}[W^{-1}]$ *of Lemma 3.21. In particular, by virtue of the equivalence*

$$\text{poSet}_{\text{f}}^{\text{filt}}[W^{-1}] \cong \Delta,$$

we see that $\mathbb{I}\text{dx}$ *induces a functor*

$$\mathsf{Cat}_{ex}^{pair} \times \Delta^{\text{op}} \to \mathsf{Spaces}.$$

Remark 3.23. The above implies that the functor $\mathbb{I}\text{dx}$ gives rise to a simplicial object $\mathbb{I}\text{dx}_\bullet$ in the ∞-category of functors $\text{Fun}(\mathsf{Cat}_{ex}^{pair}, \mathsf{Spaces})$. Corollary 3.16 can be restated as

$$\mathbb{I}\text{dx}_{\mathsf{C},\bullet}\, \mathsf{D} \cong K_{S_\bullet(\mathsf{C})}.$$

Proof of Proposition 3.22. We have seen, in Lemma 3.13, that every inclusion $I \subset I'$ which restricts to a bijection on basepoints induces an equivalence of index spaces

$$\mathbb{I}\text{dx}_{\mathsf{C},I}\, \mathsf{D} \cong \mathbb{I}\text{dx}_{\mathsf{C},I'}\, \mathsf{D}\,.$$

As in the proof of Corollary 3.16 we observe that the zigzag of inclusions

$$I \subset I^B \supset B[\text{base}(I)]$$

yields a zigzag of equivalences of index spaces. In particular, we see that the functor $\mathbb{I}\text{dx}$ is equivalent to $\mathbb{I}\text{dx} \circ B[\bullet] \circ \text{base}$. In particular, it factors through the map $\text{base} : \text{poSet}_{\text{f}}^{\text{filt}} \to \Delta$. $\square$

In Section 3B we sketch a construction of index spaces for infinite filtered sets, using the rigidity property as main ingredient.

Three examples for the structure of the pre-index map. In order to shed some light on the abstract constructions introduced above, we take a look at a few concrete examples. This serves a purely expository purpose, and we only refer to the results of this paragraph to illustrate the theory. The first example is a simple lemma illustrating that the ostensible complexity of the definitions above can be avoided if $\mathsf{C} = \mathsf{D}$.

Example 3.24. Let C be an exact category. Then for every based, filtered, partially ordered set $(I; x_0, \ldots, x_k)$, the pre-index map

$$|\,\text{Fun}_{\mathsf{C}}(I, \mathsf{C})^{\times}| \to \mathbb{I}\text{dx}_{\mathsf{C},I}\, \mathsf{C} \cong K_{\mathsf{C}}^{\times k}$$

is equivalent to the map

$$F \mapsto (F(x_1) - F(x_0), \dots, F(x_k) - F(x_{k-1})),$$

where we view $F(x_i)$ as a point in the K-theory space K_{C} and we use the subtraction operation stemming from the infinite loop space structure of K-theory spaces (which is well-defined, up to a contractible space of choices).

This follows directly from the next example, by setting $\mathsf{D} = \mathsf{C}$ and using the fact that for every diagram $F \in \mathrm{Fun}_{\mathsf{C}}(I, \mathsf{C})$ the maps $F(m)/F(x_i) - F(m)/F(x_{i+1})$ and $F(x_{i+1}) - F(x_i)$ are naturally homotopic (this follows from the basic properties of algebraic K-theory).

Example 3.25. Let I be a based, finite, filtered, partially ordered set such that the k basepoints are pairwise distinct. We denote the unique maximal element of I by m. Then the pre-index map

$$|\mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D})^{\times}| \to K_{\mathsf{C}}^{\times k}$$

can be expressed as

$$(F(m)/F(x_0) - F(m)/F(x_1), \dots, F(m)/F(x_{k-1}) - F(m)/F(x_k)).$$

Proof. For the proof we recall the description of the index space $\mathbb{I}\mathrm{dx}_{\mathsf{C},I}$ given in terms of admissible trees (see the proof of Lemma 3.13). Let T be the admissible tree in $\Gamma(I)$, consisting precisely of the set of edges $\{e_x\}_{x \in I}$, where e_x connects the point x with the maximal element m. As observed in the proof of Lemma 3.13, the infinite loop space underlying $\mathbb{I}\mathrm{dx}_{\mathsf{C},I}\,\mathsf{D}$, is equivalent to the cofibre of the map of connective spectra

$$\mathcal{K}_{\mathsf{C}}^{E(T^{\Delta})} \xrightarrow{\alpha} \mathcal{K}_{\mathsf{C}}^{E(T)}.$$

In the homotopy category of spectra this morphism belongs to a distinguished triangle which can be written as a sum of two distinguished triangles: the first summand is given by

$$\mathcal{K}_{\mathsf{C}}^{E(T)\setminus\{e_{b_0},\dots,e_{b_k}\}} \to \mathcal{K}_{E}^{E(T)\setminus\{e_{b_0},\dots,e_{b_k}\}} \to 0 \to \Sigma\, \mathcal{K}_{\mathsf{C}}^{E(T)\setminus\{e_{b_0},\dots,e_{b_k}\}}$$

and corresponds to the edges in T which do not contain a base point. The second summand is

$$\mathcal{K}_{\mathsf{C}} \xrightarrow{\Delta} \mathcal{K}_{\mathsf{C}}^{k+1} \xrightarrow{\beta} \mathcal{K}_{\mathsf{C}}^{k} \to \Sigma\, \mathcal{K}_{\mathsf{C}},$$

where Δ denotes the diagonal inclusion, and β is given by

$$(x_0, \dots, x_k) \mapsto (x_0 - x_1, \dots, x_{m-1} - x_m).$$

The claim now follows from the definition of the exact functor

$$\mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D}) \to \mathsf{D} \times \mathsf{C}^{\times E(T)} \quad \text{as} \quad F \mapsto (F(b_0), (F(m)/F(x))_{x \in I\setminus\{m\}}),$$

where we use the identification $E(T) = I \setminus \{m\}$. ☐

Example 3.26. Let I be $B[2]$ with its three basepoints b_0, b_1, and b_2. It contains three copies of $B[1]$, indexed by the set of unordered pairs of distinct elements in $\{b_0, b_1, b_2\}$. We denote these inclusions by $\phi_{ij} : B[1] \to B[2]$. For every $F \in \mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D})$, we have a contractible space of homotopies

$$\phi_{01}^* F + \phi_{12}^* F \simeq \phi_{02}^* F$$

in $K_{\mathsf{C}} \cong K_{S_1(\mathsf{C})} \cong \mathbb{I}\mathrm{dx}_{\mathsf{C},B[1]}\,\mathsf{D}$.

Proof. We construct these homotopies as homotopies of loops in $K_{\mathsf{C}} \cong \Omega|K_{S_\bullet(\mathsf{C})}|$. By Corollary 3.16, for every simplicial object $M_\bullet$ in $\mathrm{poSet}_{\mathrm{f}}^{\mathrm{filt}}$ with $k+1$ basepoints in level k, we have a map of simplicial spaces

$$(\mathrm{Fun}_{\mathsf{C}}(M_\bullet, \mathsf{D}))^\times \to K_{S_\bullet(\mathsf{C})}.$$

We apply this observation to the degenerate simplicial object $M_\bullet$, which agrees with $B[k]$ for $k \leq 2$, and satisfies $M_k = B[2]$ for $k \geq 2$, with the last basepoint x_2 repeated $k-2$ times in M_k. In particular, a diagram F gives rise to a 2-simplex of the left-hand side

$$\begin{tikzcd} & \bullet \arrow[dr, dash, "\phi_{12}^* F"] & \\ \bullet \arrow[ur, dash, "\phi_{01}^* F"] \arrow[rr, dash, "\phi_{02}^* F"'] & F & \bullet \end{tikzcd}$$

with boundary faces $\phi_{01}^* F$, $\phi_{12}^* F$, and $\phi_{02}^* F$. Since $K_{S_0(\mathsf{C})} \cong 0$, every 1-simplex induces an element of $\Omega|K_{S_\bullet(\mathsf{C})}|$. The geometric realization of this triangle yields a contractible space of homotopies between the loops $\phi_{01}^* F \cdot \phi_{12}^* F$ and $\phi_{02}^* F$. $\square$

The existence of such a homotopy is not surprising. Indeed, passing to K_0, this statement amounts to the simple observation that we have the identity

$$\begin{aligned} F(x_{01})/F(x_0) - F(x_{01})/F(x_1) + F(x_{12})/F(x_1) - F(x_{12})/F(x_2) \\ = F(x_{02})/F(x_0) - F(x_{02})/F(x_2). \end{aligned}$$

The pre-index provides a natural contractible space of choices for this homotopy. We return to this at the end of this section.

3B. *The index map for Tate objects revisited.* We now apply the generalized Waldhausen construction to produce a simplicial map

$$N_\bullet \,\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times \to K_{S_\bullet(\mathsf{C})} \tag{3.27}$$

whose geometric realization is equivalent to the index map. For any elementary Tate object V, by precomposing (3.27) with the map

$$B_\bullet \operatorname{Aut}(V) \to N_\bullet \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$$

we obtain a map of reduced Segal objects in Spaces

$$B_\bullet \operatorname{Aut}(V) \to K_{S_\bullet(\mathsf{C})}$$

which encodes the A_∞-structure of the index map.

Let $\mathrm{poSet}^{\mathrm{filt}}$ denote the category of (possibly infinite) filtered posets I, together with a choice of basepoints $(x_0, \ldots, x_k) \in I^{[k]}$. Note that we do not impose the condition that the basepoints are minimal in I.

Definition 3.28. For $(I; x_0, \ldots, x_k) \in \mathrm{poSet}^{\mathrm{filt}}$, and $(\mathsf{D}, \mathsf{C}) \in \mathsf{Cat}_{ex}^{pair}$, we define:

(a) $\mathrm{Fun}_{\mathsf{C}}(I, \mathsf{D})$ is the exact category of functors $I \to \mathsf{D}$ such that $x \leq y$ in I is sent to an admissible monomorphism in C with cokernel in D.

(b) $\mathrm{Fun}^*_{\mathsf{C}}(I, \mathsf{D})$ as the colimit of exact categories $\varinjlim_{I'} \mathrm{Fun}_{\mathsf{C}}(I', \mathsf{D})$.

(c) $\mathbb{I}\mathrm{dx}_{\mathsf{C},I}\, \mathsf{D}$ as the colimit of spaces $\varinjlim_{I'} \mathbb{I}\mathrm{dx}_{\mathsf{C},I'}\, \mathsf{D}$.

Here I' ranges over the filtered category of finite based sets $(I'; x_0, \ldots, x_k)$ together with a map of based sets $(I'; x_0, \ldots, x_k) \to (I; x_0, \ldots, x_k)$ corresponding to $\mathrm{id}_{[k]}$.

Just as in the case of finite based sets, these constructions are sufficiently natural in the pair (D, C) and the based set I. This follows from Lurie's functoriality of (co)limits result [2009, Proposition 4.2.2.7], applied to the following setup: Let S be (the nerve of) the category $\mathrm{poSet}^{\mathrm{filt}}$, and $Y \to S$ the constant cartesian fibration with fibre given by the ∞-category $\mathrm{Fun}(\mathsf{Cat}_{ex}^{pair}, \mathsf{Spaces})$. Consider the diagram $K \to S$ given by (the nerve of) the category $\mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}} / \mathrm{poSet}^{\mathrm{filt}}$ together with the obvious functor to $\mathrm{poSet}^{\mathrm{filt}}$. The functor $\mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}} \to \mathrm{Fun}(\mathsf{Cat}_{ex}^{pair}, \mathsf{Spaces})$ of Definition 3.12 gives rise to a functor $K \to Y$ belonging to a commutative diagram

$$\begin{tikzcd} K \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$

According to [Lurie 2009, Proposition 4.2.2.7] there exists a functor

$$S = \mathrm{poSet}^{\mathrm{filt}} \xrightarrow{\mathbb{I}\mathrm{dx}} \mathrm{Fun}(\mathsf{Cat}_{ex}^{pair}, \mathsf{Spaces}),$$

such that for every $I \in \mathrm{poSet}^{\mathrm{filt}}$ we have an equivalence $\mathbb{I}\mathrm{dx}_I(D, \mathsf{C}) \cong \varinjlim_{I'/I} \mathbb{I}\mathrm{dx}_{\mathsf{C},I'}\, \mathsf{D}$, where $I' \in \mathrm{poSet}^{\mathrm{filt}}_{\mathrm{f}}$. We record these observations in the lemma below.

Lemma 3.29. *There exist functors*

$$\begin{aligned}\mathrm{Fun} : \mathrm{poSet}^{\mathrm{filt}^{\mathrm{op}}} \times \mathsf{Cat}_{ex}^{pair} &\to \mathsf{Cat}_{ex}^{pair},\\ \mathrm{Fun}^* : \mathrm{poSet}^{\mathrm{filt}^{\mathrm{op}}} \times \mathsf{Cat}_{ex}^{pair} &\to \mathsf{Cat}_{ex}^{pair},\\ \mathbb{I}\mathrm{dx} : \mathrm{poSet}^{\mathrm{filt}^{\mathrm{op}}} \times \mathsf{Cat}_{ex}^{pair} &\to \mathsf{Spaces},\end{aligned}$$

which are compatible with Definition 3.28. Moreover there are natural transformations

$$\mathrm{Fun}^\times \to (\mathrm{Fun}^*)^\times \to \mathbb{I}\mathrm{dx}$$

extending the canonical one for finite based sets.

Since the category we are taking the colimit over in Definition 3.28 is cofiltered, and for a morphism $I' \to I''$ (inducing the identity on base points) the induced map of index spaces

$$\mathbb{I}\mathrm{dx}_{\mathsf{C},I''}\,\mathsf{D} \to \mathbb{I}\mathrm{dx}_{\mathsf{C},I'}\,\mathsf{D}$$

is an equivalence by Lemma 3.13, we are taking an inverse limit over a cofiltered system of equivalences. Hence, we have a canonical equivalence of index spaces $\mathbb{I}\mathrm{dx}_{\mathsf{C},I}\,\mathsf{D} \cong \mathbb{I}\mathrm{dx}_{\mathsf{C},I'}\,\mathsf{D}$. This implies at once that the rigidity property (Proposition 3.22) holds as well for objects in $\mathrm{poSet}^{\mathrm{filt}}$.

Definition 3.30. Let $\mathrm{Gr}_\bullet(\mathsf{C})^\times$ denote the Grothendieck construction of the functor $\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times \to \mathsf{sSet}$, which sends $V \in \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$ to the simplicial set of (unordered) tuples of lattices in $\mathrm{Gr}(V)$, i.e., an n-simplex in $\mathrm{Gr}_\bullet(\mathsf{C})^\times$ is given by the data $(V; L_0, \dots, L_n)$, where $V \in \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$, and each L_i denotes a lattice in V.

We construct a morphism

$$\mathrm{Gr}_\bullet(\mathsf{C})^\times_\bullet \to K_{S_\bullet(\mathsf{C})}$$

which, informally stated, sends $(V; L_0, \dots, L_k)$ to $(\mathrm{Gr}(V); L_0, \dots, L_k) \in \mathrm{poSet}^{\mathrm{filt}}$, and then computes the index of the tautological diagram $\mathrm{Gr}(V) \to \mathsf{Pro}^a(\mathsf{C})$, which sends $L \in \mathrm{Gr}(V)$ to the corresponding Pro-object. To make this rigorous we begin with a technical observation.

Remark 3.31. The Grothendieck construction (for simplicial *sets*) turns a simplicial set $M_\bullet$ into a category $\widetilde{M}_\bullet \to \Delta^{\mathrm{op}}$ over the opposite category of finite nonempty ordinals. We have a canonical equivalence

$$M_\bullet \cong \varinjlim_{\widetilde{M}_\bullet/\Delta^{\mathrm{op}}} \{\bullet\},$$

where we take a fibrewise colimit (in the ∞-category of spaces [Lurie 2009, Section 4.3.1]) on the left-hand side over the constant, singleton-valued diagram indexed by $\widetilde{M}$.

We apply this remark to the simplicial set $\mathrm{Gr}_\bullet(V)$, where V is a Tate object, in order to define the following morphism.

Definition 3.32. For $V \in \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$, consider the canonical map

$$\{\mathrm{Fun}_{\mathsf{C}}(\mathrm{Gr}(V)^\times, \mathsf{Pro}^a(\mathsf{C}))^\times\}_{\widetilde{\mathrm{Gr}}_\bullet(V)} \to \mathbb{I}\mathrm{dx}_{\mathsf{C},\bullet}\, \mathsf{Pro}^a(\mathsf{C}) \cong K_{S_\bullet(\mathsf{C})}.$$

Precomposing it with the map

$$\mathrm{Gr}_\bullet(V) \to \mathrm{Fun}_{\mathsf{C}}(\mathrm{Gr}(V), \mathsf{D})^\times$$

which sends $(L_0, \dots, L_k) \in \mathrm{Gr}_k(V)$ to the tautological C-diagram $\mathrm{Gr}(V) \to \mathsf{Pro}^a(\mathsf{C})$ of the based set $(\mathrm{Gr}(V), L_0, \dots, L_k)$, we obtain a natural transformation of diagrams indexed by $\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$:

$$\{\mathrm{Gr}_\bullet(V)\}_{\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times} \to \{K_{S_\bullet(\mathsf{C})}\}.$$

By virtue of the universal property of colimits (since the right-hand side is a constant diagram), we obtain a morphism

$$\mathrm{Gr}_\bullet(\mathsf{C})^\times \to K_{S_\bullet(\mathsf{C})}.$$

3C. *Comparison.* It remains to verify compatibility of Definition 3.32 with the index map.

Proposition 3.33. *There exists a commutative diagram*

$$\begin{tikzcd} \mathrm{Gr}^{\leq}_\bullet(\mathsf{C})^\times \arrow[r] \arrow[dr, "\mathsf{Index}"'] & \mathrm{Gr}^\times_\bullet \arrow[d] \\ & K_{S_\bullet(\mathsf{C})} \end{tikzcd}$$

in the ∞-category of simplicial diagrams of spaces.

The proof rests on the following technical lemma.

Lemma 3.34. *Let $S \in \mathrm{poSet}^{\mathrm{filt}}$ be a based filtered set with basepoints $(x_0, \dots, x_n)$. We assume that*

(a) *we have $x_0 \leq \cdots \leq x_n$,*

(b) *for $s \in S$ we have that if $s \leq x_i$ for $i = 0, \dots, n$ then $s = y_j$ for some j with $0 \leq j \leq i$,*

(c) *there exists $y \in S$ such that $y \geq x_i$ for $i = 0, \dots, n$,*

(d) *there is a surjective morphism $S \xrightarrow{\phi} S'$ of based filtered sets, which contracts the elements $(x_0, \dots, x_n)$ to a single point $x \in S'$, and is an equivalence on $S \setminus \{x_0, \dots, x_n\}$.*

Then the functor $\phi^* : \mathrm{Fun}_{\mathsf{C}}(S', \mathsf{D}) \to \mathrm{Fun}_{\mathsf{C}}(S, \mathsf{D})$ is a left s-filtering embedding (in the sense of [Schlichting 2004, Definition 1.5]).

Proof. Let $S' \to S$ be the unique section to ϕ sending x to x_n. There is a natural transformation $\phi^* s^* \hookrightarrow \mathrm{id}$, which is objectwise an admissible monomorphism. Moreover, we have a natural isomorphism $s^*\phi^* \simeq (\phi \circ s)^* \simeq \mathrm{id}$. We therefore conclude that s^* is the left adjoint to ϕ^*, and that ϕ^* is fully faithful.

If we are given an admissible short exact sequence $X \hookrightarrow Y \twoheadrightarrow \phi(Z)$ with $Z \in \phi^*(\mathrm{Fun}_{\mathsf{C}}(S', \mathsf{D}))$ then we may apply the exact functor $\phi^* s^*$ to obtain a short exact sequence $\phi^* s^* X \hookrightarrow \phi^* s^* Y \twoheadrightarrow \phi(Z)$ in the essential image of ϕ. The natural transformation $\phi^* s^* \to \mathrm{id}$ yields that ϕ^* is left special.

It remains to show that ϕ^* is left special, by noting that every morphism $\phi(X) \to Z$ factors through an admissible monomorphism $\phi(X) \to \phi(Y) \hookrightarrow Z$. This is possible since one can define $Y = s^* Z$, and consider the admissible monomorphism $\phi^* s^*(Z) \hookrightarrow Z$. $\square$

Theorem 2.10 in [Schlichting 2004] implies the following.

Corollary 3.35. *For S and S' as in Lemma 3.34, there is a natural morphism*

$$K_{\mathrm{Fun}_{\mathsf{C}}(S,\mathsf{D})/\phi^* \mathrm{Fun}_{\mathsf{C}}(S',\mathsf{D})} \to \mathbb{I}\mathrm{dx}_{S,\mathsf{C}}\, \mathsf{D},$$

and in particular we have a commutative diagram of spaces

$$\begin{tikzcd}
\mathrm{Fun}_{\mathsf{C}}(S, \mathsf{D})^{\times} \arrow[r] \arrow[dr] & (\mathrm{Fun}_{\mathsf{C}}(S, \mathsf{D})/\phi^* \mathrm{Fun}_{\mathsf{C}}(S', \mathsf{D}))^{\times} \arrow[d] \\
 & \mathbb{I}\mathrm{dx}_{S,\mathsf{C}}\, \mathsf{D}
\end{tikzcd}$$

Proof of Proposition 3.33. By Definition 3.32, the composition

$$\mathrm{Gr}^{\le}_{\bullet}(\mathsf{C})^{\times}_{\bullet} \to \mathrm{Gr}_{\bullet}(\mathsf{C})^{\times}_{\bullet} \to K_{S_{\bullet}(\mathsf{C})}$$

is equivalent to the levelwise colimit of the map of constant diagrams

$$\{*\}_{\widetilde{\mathrm{Gr}^{\le}_{\bullet}(V)}/\Delta^{\mathrm{op}}} \to \{\mathrm{Fun}_{\mathsf{C}}(\mathrm{Gr}(V), \mathsf{Pro}^a(\mathsf{C}))^{\times}\}_{\widetilde{\mathrm{Gr}^{\le}_{\bullet}(V)}/\Delta^{\mathrm{op}}} \to \{\mathbb{I}\mathrm{dx}_{\mathsf{C},\mathrm{Gr}_{\bullet}}\, \mathsf{Pro}^a(\mathsf{C})\}_{\widetilde{\mathrm{Gr}^{\le}_{\bullet}(V)}/\Delta^{\mathrm{op}}},$$

where $*$ is sent to the canonical admissible diagram $\mathrm{Gr}(V) \to \mathsf{Pro}^a(\mathsf{C})$ sending $L \in \mathrm{Gr}(V)$ to the Pro-object L.

Next we introduce a variant of the construction S^B. Let $A[n]$ be the filtered poset $\{(x, y) \in [n] \times [n] \mid x \le y\}$, ordered lexicographically. It is clear that this defines a cosimplicial object in the category of filtered posets. For a based poset $(S; x_0, \ldots, x_n)$, we define S^A to be the pushout of posets

$$S^A = S \cup_{[n]} A[n]$$

along the map $[n] \to S$ given by $i \mapsto x_i$, and $[n] \to A[n]$ given by the diagonal. As basepoints we choose $a_i = (i, 0) \in A[n]$ for $0 \le i \le n$.

In the following we use the notation $L_0 \subset \cdots \subset L_k$ to denote an element in $\mathrm{Gr}_k^{\leq}(V)$. The tautological $\mathrm{Gr}(V)$-diagram extends to $\mathrm{Gr}(V)^A$, by sending the interval (x, y) to L_x. For the resulting $A[n]$-subdiagram, we have an admissible epimorphism in $\mathrm{Fun}_{\mathsf{C}}(A[n], \mathsf{Pro}^a(\mathsf{C}))$, to the admissible $A[n]$-diagram obtained by restricting the admissible $[n]$-diagram

$$0 \hookrightarrow L_1/L_0 \hookrightarrow \cdots \hookrightarrow L_n/L_0 \tag{3.36}$$

to the morphism of filtered posets $A[n] \to [n]$ given by the projection to the first component.

The kernel of the admissible epimorphism relating the two diagrams lies in $\mathrm{Fun}_{\mathsf{C}}(A([n])', \mathsf{Pro}^a(\mathsf{C}))$. By Corollary 3.35 the above colimit is therefore equivalent to the colimit of constant diagrams

$$\{*\}_{\widetilde{\mathrm{Gr}_\bullet^{\leq}(V)/\Delta^{\mathrm{op}}}} \to S_\bullet \mathsf{C}^\times \to \{\mathrm{Fun}_{\mathsf{C}}(A[\bullet], \mathsf{Pro}^a(\mathsf{C}))^\times\}_{\widetilde{\mathrm{Gr}_\bullet^{\leq}(V)/\Delta^{\mathrm{op}}}} \\ \to \{\mathbb{I}\mathrm{dx}_{\mathsf{C},A[\bullet]} \mathsf{Pro}^a(\mathsf{C})\}_{\widetilde{\mathrm{Gr}_\bullet^{\leq}(V)/\Delta^{\mathrm{op}}}}.$$

This shows that the resulting $A[n]$-subdiagram lies in the image of the functor

$$S_\bullet(\mathsf{C}) \to \mathrm{Fun}_{\mathsf{C}}(A[n], \mathsf{Pro}^a(\mathsf{C})).$$

Assuming this functor is compatible with the equivalence $\mathbb{I}\mathrm{dx}_{\mathsf{C},\bullet} \mathsf{Pro}^a(\mathsf{C}) \cong K_{S_\bullet(\mathsf{C})}$, we use the fact that the morphism

$$\mathrm{Gr}_\bullet^{\leq}(V)^\times \to K_{S_\bullet(\mathsf{C})}$$

factors through the canonical map $\mathrm{Gr}_\bullet^{\leq} \to S_\bullet(\mathsf{C})^\times$ to conclude the proof.

In order to establish the required compatibility, we denote by $T[n]$ the based filtered set, given by $n+1$ basepoints $x_0, \ldots, x_n$ and a unique maximal point m. There are natural maps $T[n] \to A[n]$ and $T[n] \to B[n]$. The commutative diagram

$$\begin{tikzcd} & \mathrm{Fun}_{\mathsf{C}}(B[\bullet], \mathsf{D}) \arrow[dr] & \\ S_\bullet \mathsf{C} \arrow[ur] \arrow[dr] & & \mathrm{Fun}_{\mathsf{C}}(T[\bullet], \mathsf{D}) \\ & \mathrm{Fun}_{\mathsf{C}}(A[\bullet], \mathsf{D}) \arrow[ur] & \end{tikzcd}$$

of exact categories commutes. It induces a commutative diagram

$$\begin{tikzcd} & \mathbb{I}\mathrm{dx}_{\mathsf{C},B[\bullet]} \mathsf{D} \arrow[dr, "\simeq"] & \\ S_\bullet \mathsf{C} \arrow[ur, "\simeq"] \arrow[dr] & & \mathbb{I}\mathrm{dx}_{\mathsf{C},T[\bullet]} \mathsf{D} \\ & \mathbb{I}\mathrm{dx}_{\mathsf{C},A[\bullet]} \mathsf{D} \arrow[ur, "\simeq"'] & \end{tikzcd}$$

of equivalences by virtue of rigidity (Proposition 3.22). □

Choose a representative V for every isomorphism class of elementary Tate objects, and select a lattice $L \in \mathrm{Gr}(V)$. This allows one to define a pseudosimplicial map of simplicial groupoids

$$N_\bullet \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times \simeq \coprod_{V \in \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})/\mathrm{iso}} B_\bullet \mathrm{Aut}(V) \xrightarrow{\mathcal{L}} \mathrm{Gr}_\bullet(\mathsf{C})^\times,$$

where we view $B_\bullet \mathrm{Aut}(V)$ as a discrete simplicial groupoid (i.e., having no nontrivial morphisms), and where $\mathcal{L}$ sends an n-simplex $(g_1, \dots, g_n) \in B_n \mathrm{Aut}(V)$ to $(L, g_1 L, \dots, g_n \cdots g_1 L)$. Note that this map is simplicial away from d_0, i.e., $d_i \mathcal{L} = \mathcal{L} d_i$ for $i > 0$, and $s_i \mathcal{L} = \mathcal{L} s_i$ for all i. The component at $\bar{g} := (g_1, \dots, g_n)$ of the natural isomorphism $\mathcal{L} d_0 \xrightarrow{\alpha} d_0 \mathcal{L}$ is given by

$$\alpha_{\bar{g}} = (g_1, g_2 g_1 g_2^{-1}, \dots, g_n \cdots g_1 g_2^{-1} \cdots g_n^{-1}) : \\ (L, g_2 L, \dots, g_n \cdots g_2 L) \to (g_1 L, g_2 g_1 L, \dots, g_n \cdots g_1 L).$$

One can check directly that $d_0 \alpha_{\bar{g}} \circ \alpha_{d_0 \bar{g}} = \alpha_{d_1 \bar{g}}$ as required for $(\mathcal{L}, \alpha)$ to give a pseudosimplicial map.

Postcomposing this map with $\mathrm{Gr}_\bullet(\mathsf{C})^\times \to K_{S_\bullet \mathsf{C}}$ of Definition 3.32 we obtain a morphism of Segal objects

$$N_\bullet \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times \to K_{S_\bullet \mathsf{C}}. \tag{3.37}$$

Theorem 3.38. *The map of A_∞-objects* $\mathrm{Aut}(V) \to K_{\mathsf{C}}$ *encoded by* (3.37) *agrees with the natural A_∞-structure obtained by applying Ω to the map $B\,\mathrm{Aut}(V) \to BK_{\mathsf{C}}$.*

Proof. We have a morphism of simplicial objects $B_\bullet \mathrm{Aut}(V) \to \mathrm{Gr}_\bullet(\mathsf{C})^\times \to K_{S_\bullet \mathsf{C}}$. We claim that the forgetful map $\mathrm{Gr}_\bullet(\mathsf{C})^\times \to \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$ is an equivalence after geometrically realizing. Indeed, by its definition as a Grothendieck construction, we have an equivalence of spaces

$$|\mathrm{Gr}_\bullet(\mathsf{C})^\times| \simeq \varinjlim_{\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times} |\mathrm{Gr}_\bullet(V)|,$$

where the colimit on the right-hand side is the colimit in the ∞-category of spaces of the functor

$$\mathrm{Gr}_\bullet(-) : \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times \to \mathsf{sSet}, \quad V \mapsto \mathrm{Gr}_\bullet(V).$$

Let $\{\bullet\}$ denote the constant diagram

$$\{\bullet\} : \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times \to \mathsf{sSet}, \quad V \mapsto \Delta^0$$

and consider the map to the constant diagram $\mathrm{Gr}_\bullet(-) \to \{\bullet\}$. After geometrically realizing, this gives a pointwise equivalence of diagrams; indeed, for any

$V \in \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$, the simplicial set $\mathrm{Gr}_\bullet(V)$ is 0-coskeletal, which implies that the map $\mathrm{Gr}_\bullet(V) \to \Delta^0$ is a trivial fibration. Therefore,

$$|\,\mathrm{Gr}_\bullet(\mathsf{C})^\times| \simeq \varinjlim_{\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times} |\,\mathrm{Gr}_\bullet(V)| \simeq \varinjlim_{\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times} \{\bullet\} \simeq |\,\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times|$$

as claimed.

We now show that the geometric realization of the map $\mathcal{L}$ is homotopy inverse to this map. Denote by $B^{\mathrm{css}}_\bullet \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$ the complete Segal space associated to the groupoid $\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$, i.e.,

$$B^{\mathrm{css}}_n \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times := \mathrm{Fun}([n], \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times)^\times.$$

Recall the adjunctions

$$p_j^* : \mathsf{sSet} \leftrightarrows \mathsf{ssSet} : \iota_j^*$$

for $j = 1, 2$ (see the Appendix). Observe that the inclusion of horizontal and vertical 0-simplices give canonical maps

$$p_j^* N_\bullet \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times \to B^{\mathrm{css}}_\bullet \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$$

for $j = 1, 2$. For $j = 1$, this is an equivalence of complete Segal spaces by [Joyal and Tierney 2007, Theorem 4.11] (it is the co-unit for the Quillen equivalence $p_1^* \dashv \iota_1^*$; see the Appendix). By Lemma A.3, these two inclusions become equivalent after applying the functor

$$t_! : \mathsf{ssSet} \to \mathsf{sSet}$$

(see again the Appendix). By [Joyal and Tierney 2007, Theorem 4.12], $t_!$ is a Quillen equivalence from the model structure for complete Segal spaces to the model structure for quasicategories. By Corollary A.4, we conclude that the two inclusions, viewed as a zigzag from $\mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$ to itself, are canonically equal to the identity.

The pseudosimplicial map $\mathcal{L}$ extends (along the inclusion of vertical 0-simplices $N_\bullet \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times \to B^{\mathrm{css}}_\bullet \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times$) to a pseudosimplicial map of simplicial groupoids

$$B^{\mathrm{css}}_\bullet \mathsf{Tate}^{\mathrm{el}}(\mathsf{C})^\times \xrightarrow{\mathcal{L}} \mathrm{Gr}_\bullet(\mathsf{C})^\times$$

where concretely, $\mathcal{L}$ is given on objects by the formula above. On morphisms, $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L}((g_1, \dots, g_n) \xrightarrow{(h_0,\dots,h_n)} (h_1 g_1 h_0^{-1}, \dots, h_n g_n h_{n-1}^{-1})) = (L, g_1 L, \dots, g_n \cdots g_1 L)\\ \xrightarrow{(1, h_1 g_1 h_0^{-1} g_1^{-1}, \dots, h_n (g_n \cdots g_1) h_0^{-1} (g_n \cdots g_1)^{-1})} (L, h_1 g_1 h_0^{-1} L, \dots, h_n g_n \cdots g_1 h_0^{-1} L).\end{aligned}$$

One can check that α as above defines a natural transformation $\alpha : d_0\mathcal{L} \to \mathcal{L}d_0$. By inspection, the composition

$$p_2^* N_\bullet \operatorname{\mathsf{Tate}}^{\mathrm{el}}(\mathsf{C})^\times \to B_\bullet^{\mathrm{css}} \operatorname{\mathsf{Tate}}^{\mathrm{el}}(\mathsf{C})^\times \xrightarrow{\mathcal{L}} \mathrm{Gr}_\bullet(\mathsf{C})^\times \to \operatorname{\mathsf{Tate}}^{\mathrm{el}}(\mathsf{C})^\times$$

is the identity. By the above, the maps

$$p_j^* N_\bullet \operatorname{\mathsf{Tate}}^{\mathrm{el}}(\mathsf{C})^\times \to B_\bullet^{\mathrm{css}} \operatorname{\mathsf{Tate}}^{\mathrm{el}}(\mathsf{C})^\times \xrightarrow{\mathcal{L}} \mathrm{Gr}_\bullet(\mathsf{C})^\times$$

are canonically equivalent for $j = 1, 2$; in particular, the map

$$\mathcal{L} : N_\bullet \operatorname{\mathsf{Tate}}^{\mathrm{el}}(\mathsf{C})^\times \to \mathrm{Gr}_\bullet(\mathsf{C})^\times$$

is canonically inverse to the equivalence

$$\mathrm{Gr}_\bullet(\mathsf{C})^\times \to \operatorname{\mathsf{Tate}}^{\mathrm{el}}(\mathsf{C})^\times$$

as claimed.

According to Proposition 3.33, the geometric realization of the chain of maps

$$N_\bullet \operatorname{Aut}(V) \xrightarrow{\mathcal{L}} \mathrm{Gr}_\bullet(\mathsf{C})^\times \to K_{S_\bullet(\mathsf{C})}$$

is therefore equivalent to the index map

$$B \operatorname{Aut}(V) \xrightarrow{\mathsf{Index}} BK_{\mathsf{C}}.$$

Theorem 5.2.6.15 of [Lurie 2017] implies that geometric realization induces an equivalence between the ∞-category of Segal objects $X_\bullet$ with X_0 contractible, and the ∞-category of connected pointed spaces. This shows that the A_∞-structure we defined above agrees with the one which naturally lives on the index map. □

Appendix

In this appendix, we recall basic facts about complete Segal spaces and groupoids.

Let C be a category. Let $B_\bullet^{\mathrm{css}}\mathsf{C}$ be the associated complete Segal space, i.e.,

$$B_n^{\mathrm{css}}\mathsf{C} = |\operatorname{Fun}([n], \mathsf{C})^\times|.$$

For definiteness of notation, we view a complete Segal space as a bisimplicial set, with the simplicial direction horizontal, and the spaces given by the columns, e.g.,

$$(B_\bullet^{\mathrm{css}}\mathsf{C})_{m,n} := N_n \operatorname{Fun}([m], \mathsf{C})^\times.$$

Recall the Quillen equivalence

$$t_! : \mathsf{ssSet} \leftrightarrows \mathsf{sSet} : t^!$$

of [Joyal and Tierney 2007, Section 2 and Theorem 4.12] from the Rezk model structure (for complete Segal spaces) on ssSet to the Joyal model structure (for quasicategories) on sSet. By definition,

$$t_!([m] \times [n]) := \Delta^m \times \Delta'[n],$$

where $\Delta'[n]$ denotes the nerve of the groupoid freely generated by the category $[n]$. In general, $t_!$ is the left Kan extension of $t_!$ along the Yoneda embedding, while $t^!$ is the functor

$$(t^! X)_{m,n} := \hom_{\mathsf{sSet}}(\Delta^m \times \Delta'[n], X).$$

Recall also the projections and inclusions

$$\iota_j : \Delta \to \Delta \times \Delta : p_j,$$

where Δ is the ordinal category and $j = 1, 2$. We denote the associated functors

$$p_j^* : \mathsf{sSet} \to \mathsf{ssSet} : \iota_j^*.$$

Then $p_j^* \dashv \iota_j^*$ for $j = 1, 2$. By [Joyal and Tierney 2007, Theorem 4.11], $p_1^* \dashv \iota_1^*$ is also a Quillen equivalence from the Rezk model structure (for complete Segal spaces) on ssSet to the Joyal model structure (for quasicategories) on sSet.

Lemma A.1. *For a category* C, *with nerve* $N\mathsf{C}$, *there is a natural isomorphism of bisimplicial sets*

$$B_\bullet^{\mathrm{css}}\mathsf{C} \cong t^! N\mathsf{C}.$$

Proof. By definition,

$$(B_\bullet^{\mathrm{css}}\mathsf{C})_{m,n} := N_n \operatorname{Fun}([m], \mathsf{C})^\times = \operatorname{ob} \operatorname{Fun}([m] \times \Delta'[n], \mathsf{C}).$$

Further, because the nerve preserves products and gives a fully faithful embedding of the category of categories into the category of simplicial sets, the right-hand side is naturally isomorphic to

$$\hom_{\mathsf{sSet}}(\Delta^m \times \Delta'[n], N\mathsf{C}) = (t^! N\mathsf{C}). \qquad \square$$

Lemma A.2. *For a category* C *with core* $\mathsf{C}^\times$, *there exist natural isomorphisms*

$$N\mathsf{C} \cong \iota_1^* t^! N\mathsf{C}, \qquad N\mathsf{C}^\times \cong \iota_2^* t^! N\mathsf{C}, \qquad N\mathsf{C}^\times \cong t_! p_2^* N\mathsf{C}^\times.$$

Proof. The first statement is immediate from the definitions, and in fact holds for any simplicial set X. For the second, by definition,

$$\begin{aligned}
(\iota_2^* t^! N\mathsf{C})_n &= \hom_{\mathsf{sSet}}(\Delta[0] \times \Delta'[n], N\mathsf{C}) \\
&\cong \hom_{\mathsf{sSet}}(\Delta'[n], N\mathsf{C}) \\
&\cong \hom_{\mathsf{sSet}}(\Delta'[n], N\mathsf{C}^\times) \\
&\cong \hom_{\mathsf{sSet}}(\Delta^n, N\mathsf{C}^\times) = N_n \mathsf{C}^\times.
\end{aligned}$$

The second claim follows from the first by the uniqueness of adjoints. Concretely, we restrict the adjunction
$$t_! p_2^* \dashv \iota_2^* t^!$$
to the full subcategories of (nerves of) groupoids in sSet and (Rezk nerves of) groupoids in ssSet. Then the above shows that after restricting to groupoids, $\iota_2^* t^! \cong 1$; therefore, the left adjoints, i.e., $t_! p_2^*$ and 1, are also isomorphic. □

Let $\varepsilon_t : t_! t^! \Rightarrow 1$ denote the co-unit of the adjunction $t_! \dashv t^!$. For a bisimplicial set $X_{\bullet,\bullet}$, let $\varepsilon_2 : p_2^* \iota_2^* X \hookrightarrow X$ denote the inclusion of horizontal 0-simplices, i.e., the co-unit of the adjunction $p_2^* \dashv \iota_2^*$.

Lemma A.3. *Let $\mathcal{G}$ be a groupoid. Then the compositions*
$$N\mathcal{G} \xrightarrow{\cong} t_! p_2^* N\mathcal{G} \xrightarrow{\cong} t_! p_2^* \iota_2^* t^! N\mathcal{G} \xrightarrow{t_! \varepsilon_2} t_! t^! N\mathcal{G} = t_! B_\bullet^{\mathrm{css}} \mathcal{G} \xrightarrow{\varepsilon_t} N\mathcal{G}$$
and
$$N\mathcal{G} \xrightarrow{\cong} t_! p_1^* N\mathcal{G} \xrightarrow{\cong} t_! p_1^* \iota_1^* t^! N\mathcal{G} \xrightarrow{t_! \varepsilon_1} t_! t^! N\mathcal{G} = t_! B_\bullet^{\mathrm{css}} \mathcal{G} \xrightarrow{\varepsilon_t} N\mathcal{G}$$
are the identity. In particular, the two maps
$$N\mathcal{G} \xrightarrow{\cong} t_! p_j^* N\mathcal{G} \xrightarrow{\cong} t_! p_j^* \iota_j^* t^! N\mathcal{G} \xrightarrow{t_! \varepsilon_j} t_! t^! N\mathcal{G}$$
for $j = 1, 2$ are canonically equivalent.

Proof. For the first, by the adjunction $t_! \dashv t^!$, it suffices to prove that
$$p_2^* N\mathcal{G} \xrightarrow{\cong} p_2^* \iota_2^* t^! N\mathcal{G} \xrightarrow{\varepsilon_{2,t^!}} t^! N\mathcal{G} \xrightarrow{1} t^! N\mathcal{G}$$
is the inclusion of horizontal 0-simplices. But this follows immediately from Lemma A.2. Similarly, for the second, it suffices to prove that
$$p_1^* N\mathcal{G} \xrightarrow{\cong} p_1^* \iota_1^* t^! N\mathcal{G} \xrightarrow{\varepsilon_!} t^! N\mathcal{G} \xrightarrow{1} t^! N\mathcal{G}$$
is the inclusion of vertical 0-simplices. But this follows by inspection. For the last claim, the two maps are each (strict) inverses of the weak equivalence ε_t; the claim follows. □

Corollary A.4. *Let $\mathcal{G}$ be a groupoid. Then $t_!$ takes the zigzag of weak equivalences*
$$p_2^* N\mathcal{G} \to t^! N\mathcal{G} \leftarrow p_1^* N\mathcal{G}$$
to the identity.

Proof. By Lemma A.3, $t_!$ applied to both maps gives ε_t^{-1}. This is equivalent to the identity via the map of spans

$$\begin{tikzcd} N\mathcal{G} \arrow[r, "\varepsilon_t^{-1}"] \arrow[d, "1"] & t_! t^! N\mathcal{G} \arrow[d, "\varepsilon_t"] & N\mathcal{G} \arrow[l, "\varepsilon_t^{-1}"'] \arrow[d, "1"] \\ N\mathcal{G} \arrow[r, "1"] & N\mathcal{G} & N\mathcal{G} \arrow[l, "1"'] \end{tikzcd}$$

and the result follows. □

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Localization C^*-algebras and K-theoretic duality

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Based on the localization algebras of Yu, and their subsequent analysis by Qiao and Roe, we give a new picture of KK-theory in terms of time-parametrized families of (locally) compact operators that asymptotically commute with appropriate representations.

1. Introduction

Let A be a unital C^*-algebra, unitally represented on a Hilbert space H. Assume that there is a continuous family $(q_t)_{t\in[0,\infty)}$ of compact projections on H that asymptotically commutes with A, meaning that $[q_t, a] \to 0$ as $t \to \infty$ for all $a \in A$. Note that if p is a projection in A, then the family $t \mapsto pq_t$ of compact operators gets close to being a projection, and is thus close to a projection that is uniquely defined up to homotopy; in particular, there is a well-defined K-theory class $[pq_t] \in K_0(K(H)) = \mathbb{Z}$. It is moreover not difficult to see that this idea can be bootstrapped up to define a homomorphism

$$[q_t] : K_0(A) \to \mathbb{Z}, \quad [p] \mapsto [pq_t]. \tag{1.1}$$

This suggests using such parametrized families $(q_t)_{t\in[0,\infty)}$ to define elements of K-homology.

Indeed, something like this has been done when $A = C(X)$ is commutative. In this case, the condition that $[q_t, a] \to 0$ is equivalent to the condition that the "propagation" of q_t (in the sense of [Roe 1993, Definition 4.5]) tends to zero, up to an arbitrarily good approximation. Motivated by considerations like the above, and by the heat kernel approach to the Atiyah–Singer index theorem, Yu [1997] described K-homology for simplicial complexes in terms of families with asymptotically vanishing propagation using his localization algebras. Subsequently, Qiao and Roe [2010] gave a new approach to this result of Yu that works for all compact (in fact, all proper) metric spaces.

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In this paper, we present a new picture of Kasparov's KK groups [Kasparov 1980b] based on asymptotically commuting families. Thanks to the relationship between asymptotically vanishing propagation and asymptotic commutation, our picture can be thought of as an extension of the results of Yu and Qiao–Roe from commutative to general (separable) C^*-algebras, and from K-homology to KK-theory. We think this gives an attractive picture of KK-theory. We also suspect that the ease with which the pairing in (1.1) is defined — note that unlike in the case of Paschke duality, there is no dimension shift, and unlike in the case of E-theory, there is no suspension — should be useful for future applications. Having said this, we should note that the picture of the pairing in (1.1) is overly simplified, as in general to get the whole KK group one needs to consider formal differences of such families of projections (q_t) in an appropriate sense.

We now give precise statements of our main results. For a C^*-algebra B, we denote by $C_u(T, B)$ the C^*-algebra of bounded and uniformly continuous functions from $T = [0, \infty)$ to B. Inspired by [Yu 1997; Qiao and Roe 2010], we define the localization algebra $\mathcal{C}_L(\pi)$ associated to a representation π of a separable C^*-algebra A on a separable Hilbert space to be the C^*-subalgebra of $C_u(T, L(H))$ consisting of all the functions f such that for all $a \in A$,

$$[f, \pi(a)] \in C_0(T, K(H)) \quad \text{and} \quad \pi(a)f \in C_u(T, K(H)).$$

Let us recall that a representation π is ample if it is nondegenerate, faithful and $\pi(A) \cap K(H) = \{0\}$. One verifies that the isomorphism class of $\mathcal{C}_L(\pi)$ does not depend on the choice of an ample representation π. In this case, we write $\mathcal{C}_L(A)$ in place of $\mathcal{C}_L(\pi)$ and view A as a C^*-subalgebra of $L(H)$. Note that if A is unital, then

$$\mathcal{C}_L(A) = \{f \in C_u(T, K(H)) : [f, a] \in C_0(T, K(H)),\ \forall a \in A\}.$$

In this paper we establish canonical isomorphisms $K^i(A) \cong K_i(\mathcal{C}_L(A))$, $i = 0, 1$, between the K-homology of A and the K-theory of the localization algebra $\mathcal{C}_L(A)$. More generally, we use results of [Thomsen 2001] to show that for separable C^*-algebras A, B and any absorbing representation $\pi : A \to L(H_B)$ on the standard infinite dimensional countably generated right Hilbert B-module H_B, there are canonical isomorphisms of groups

$$KK_i(A, B) \xrightarrow{\cong} K_i(\mathcal{C}_L(\pi)), \quad i = 0, 1, \tag{1.2}$$

where the localization C^*-algebra $\mathcal{C}_L(\pi)$ consists of those functions $f \in C_u(T, L(H_B))$ such that for all $a \in A$,

$$[f, \pi(a)] \in C_0(T, K(H_B)) \quad \text{and} \quad \pi(a)f \in C_u(T, K(H_B)).$$

The isomorphism in (1.2) is defined and proved by combining Paschke duality with a generalization of the techniques used by Roe and Qiao in the commutative case.

The paper is structured as follows. In Section 2, we discuss absorbing representations and give a version of Voiculescu's theorem appropriate to localization algebras. In Section 3, we define the various dual algebras and localization algebras that we use, and show that they do not depend on the choice of absorbing representation. In Section 4, we prove the isomorphism in (1.2). Finally, in Section 5, we construct maps $K_i(\mathcal{C}_L(\pi)) \to E_i(A, B)$ and show that they "invert" the isomorphism in (1.2) in the sense that the composition $KK_i(A, B) \to K_i(\mathcal{C}_L(\pi)) \to E_i(A, B)$ is the canonical natural transformation from KK-theory to E-theory.

2. Absorbing representations

Let A and B be separable C^*-algebras. If E and F are countably generated right Hilbert B-modules, we denote by $L(E, F)$ the C^*-algebra of bounded B-linear adjointable operators from E to F. The corresponding C^*-algebra of "compact" operators is denoted by $K(E, F)$ [Kasparov 1980a]. Set $L(E) = L(E, E)$ and $K(E) = K(E, E)$. Recall that H_B is the standard infinite dimensional countably generated right Hilbert B-module.

We shall use the notion of (unitally) absorbing $*$-representations $\pi : A \to L(H_B)$; see [Thomsen 2001].

Definition 2.1. (i) Suppose that A is a unital separable C^*-algebra. A unital representation $\pi : A \to L(H_B)$ is called *unitally absorbing* for the pair (A, B) if for any other unital representation $\sigma : A \to L(E)$, there is an isometry $v \in C_b(\mathbb{N}, L(E, H_B))$ such that $v\sigma(a) - \pi(a)v \in C_0(\mathbb{N}, K(E, H_B))$ for all $a \in A$.

(ii) Suppose that A is a separable C^*-algebra. We denote by $\widetilde{A}$ the unitalization of A, with the convention that $\widetilde{A} = A$ if A is already unital. A representation $\pi : A \to L(H_B)$ is called *absorbing* for the pair (A, B) if its unitalization $\widetilde{\pi} : \widetilde{A} \to L(H_B)$ is unitally absorbing for the pair $(\widetilde{A}, B)$.

Note that in Definition 2.1, if we denote the components of v by v_n, we have $v_n\sigma(a) - \pi(a)v_n \in K(E, H_B)$ and $\lim_{n\to\infty} \|v_n\sigma(a) - \pi(a)v_n\| = 0$ for all $a \in A$.

Theorem 2.2 [Voiculescu 1976]. *Any ample representation of a separable C^*-algebra on a separable infinite dimensional Hilbert space is absorbing.*

Theorem 2.3 [Kasparov 1980a]. *Let A be a unital separable C^*-algebra and let B be a σ-unital C^*-algebra. If either A or B are nuclear, then any unital ample representation $\pi : A \to L(H) \subset L(H_B)$ is absorbing for the pair (A, B).*

Theorem 2.4 [Thomsen 2001]. *For any separable C^*-algebras A and B there exist absorbing representations $\pi : A \to L(H_B)$.*

Given two $*$-representations $\pi_i : A \to L(E_i)$ we write that $\pi_1 \underset{v}{\precsim} \pi_2$ if there is an isometry $v \in C_u(T, L(E_1, E_2))$ such that

$$v\pi_1(a) - \pi_2(a)v \in C_0(T, K(E_1, E_2)).$$

If in addition $v \in C_u(T, L(E_1, E_2))$ is a unitary with the same property, then we write $\pi_1 \underset{v}{\approx} \pi_2$.

Let $w^\infty : E_1^\infty \to E_1 \oplus E_1^\infty$ be the unitary defined by

$$w^\infty(h_0, h_1, h_2, \dots) = h_0 \oplus (h_1, h_2, \dots).$$

Lemma 2.5 [Dadarlat and Eilers 2002, Lemma 2.16]. *Let $\pi_i : A \to L(E_i)$ for $i = 1, 2$ be two representations and let $v \in L(E_1^\infty, E_2)$ be an isometry such that $v\pi_1^\infty(a) - \pi_2(a)v \in K(E_1^\infty, E_2)$ for all $a \in A$. Then*

$$u = (1_{E_1} \oplus v)w^\infty v^* + (1_{E_2} - vv^*) \in L(E_2, E_1 \oplus E_2)$$

is a unitary operator such that $\pi_1(a) \oplus \pi_2(a) - u\pi_2(a)u^* \in K(E_1 \oplus E_2)$ for all $a \in A$ and moreover,

$$\|\pi_1(a) \oplus \pi_2(a) - u\pi_2(a)u^*\| \leq 6\|v\pi_1^\infty(a) - \pi_2(a)v\| + 4\|v\pi_1^\infty(a^*) - \pi_2(a^*)v\|.$$

Using this lemma, one gets the following strengthened variation of Voiculescu's theorem [1976]. This result appears in [Dadarlat and Eilers 2001] as Theorem 3.11, except that the uniform continuity of the isometry v and the unitary u were not addressed explicitly in the statement.

Theorem 2.6. *Let A, B be separable C^*-algebras and let $\pi_i : A \to L(E_i)$, $i = 1, 2$, be two representations where $E_i \cong H_B$. If π_2 is absorbing, then $\pi_1 \underset{v}{\precsim} \pi_2$ for some isometry $v \in C_u(T, L(E_1, E_2))$. If both π_1 and π_2 are absorbing, then $\pi_1 \underset{u}{\approx} \pi_2$ for some unitary $u \in C_u(T, L(E_1, E_2))$.*

Proof. As π_2 absorbs π_2^∞ there is an isometry $u = (u_n)_n \in C_b(\mathbb{N}, L(E_2^\infty, E_2))$ such that $u\pi_2^\infty(a) - \pi_2(a)u \in C_0(\mathbb{N}, K(E_2^\infty, E_2))$ for all $a \in A$. As π_2 absorbs π_1, there is a sequence of isometries $w_n \in L(E_1, E_2^\infty)$ with mutually orthogonal ranges such that $w_n\pi_1(a) - \pi_2^\infty(a)w_n \in K(E_1, E_2^\infty)$ and $\lim_{n\to\infty} \|w_n\pi_1(a) - \pi_2^\infty(a)w_n\| = 0$ for all $a \in A$. Then $v_n = u_n w_n \in L(E_1, E_2)$ is a sequence of isometries with orthogonal ranges such that the corresponding isometry $v \in C_b(\mathbb{N}, L(E_1, E_2))$ satisfies $v\pi_1(a) - \pi_2(a)v \in C_0(\mathbb{N}, K(E_1, E_2))$ for all $a \in A$. This follows from the identity

$$u_n w_n \pi_1(a) - \pi_2(a)u_n w_n = u_n(w_n\pi_1(a) - \pi_2^\infty(a)w_n) + (u_n\pi_2^\infty(a) - \pi_2(a)u_n)w_n.$$

Since $v_n^* v_m = 0$ for $n \neq m$, one observes that $\boldsymbol{v}(n+s) = (1-s)^{1/2}v_n + s^{1/2}v_{n+1}$, $0 \leq s \leq 1$, extends v to a uniformly continuous isometry $\boldsymbol{v} \in C_u(T, L(E_1, E_2))$ that satisfies $\pi_1 \underset{\boldsymbol{v}}{\precsim} \pi_2$.

For the second part of the statement, we note that by the first part $\pi_1^\infty \precsim_v \pi_2$. Thus, $v\pi_1^\infty(a) - \pi_2(a)v \in C_0(T, K(E_1^\infty, E_2))$ for all $a \in A$ where $v = (v_t)_{t\in T}$ is a uniformly continuous isometry with $v_t \in L(E_1^\infty, E_2)$. It follows by Lemma 2.5 that

$$u_t = (1_{E_1} \oplus v_t)w^\infty v_t^* + (1_{E_2} - v_t v_t^*)$$

is a uniformly continuous unitary such that $\pi_1 \oplus \pi_2 \approx_u \pi_2$. By symmetry we have that $\pi_1 \oplus \pi_2 \approx_u \pi_1$ and hence $\pi_1 \approx_u \pi_2$. □

3. Dual algebras

Let A and B be separable C^*-algebras and let $\pi : A \to L(H_B)$ be a $*$-representation.

Definition 3.1. The *localization algebra* $\mathcal{C}_L(\pi)$ associated to π is the C^*-subalgebra of $C_u(T, L(H_B))$ consisting of all functions f such that $[f, \pi(a)] \in C_0(T, K(H_B))$ and $\pi(a) f \in C_u(T, K(H_B))$ for all $a \in A$.

While $\mathcal{C}_L(\pi)$ is the central object of the paper, we also need to consider a series of pairs of C^*-algebras and ideals which will play a supporting role:

$$\mathcal{D}(\pi) = \{b \in L(H_B) : [b, \pi(a)] \in K(H_B),\ \forall a \in A\},$$
$$\mathcal{C}(\pi) = \{b \in L(H_B) : \pi(a)b \in K(H_B),\ \forall a \in A\},$$

and their parametrized versions,

$$\mathcal{D}_T(\pi) = \{f \in C_u(T, L(H_B)) : [f, \pi(a)] \in C_u(T, K(H_B)),\ \forall a \in A\} \cong C_u(T, \mathcal{D}(\pi)),$$
$$\mathcal{C}_T(\pi) = \{f \in C_u(T, L(H_B)) : \pi(a) f \in C_u(T, K(H_B)),\ \forall a \in A\} \cong C_u(T, \mathcal{C}(\pi)).$$

The evaluation map at 0 leads to the pair

$$\mathcal{D}_T^0(\pi) = \{f \in \mathcal{D}_T(\pi) : f(0) = 0\},$$
$$\mathcal{C}_T^0(\pi) = \{f \in \mathcal{C}_T(\pi) : f(0) = 0\}.$$

Finally, we view the localization algebra $\mathcal{C}_L(\pi)$ as an ideal of

$$\mathcal{D}_L(\pi) = \{f \in C_u(T, L(H_B)) : [f, \pi(a)] \in C_0(T, K(H_B)),\ \forall a \in A\},$$
$$\mathcal{C}_L(\pi) = \{f \in \mathcal{D}_L(\pi) : \pi(a) f \in C_u(T, K(H_B)),\ \forall a \in A\}.$$

In order to simplify some of the statements, it is useful to introduce the following notation: $A_1(\pi) = \mathcal{D}_T(\pi)$, $A_2(\pi) = \mathcal{C}_T(\pi)$, $A_3(\pi) = \mathcal{D}_T^0(\pi)$, $A_4(\pi) = \mathcal{C}_T^0(\pi)$, $A_5(\pi) = \mathcal{D}_L(\pi)$ and $A_6(\pi) = \mathcal{C}_L(\pi)$. We are going to see that the isomorphism classes of these C^*-algebras are independent of π, provided that π is an absorbing representation. We follow the presentation from [Higson and Roe 2000, Section 5.2], where analogous properties of $\mathcal{D}(\pi)$ and $\mathcal{C}(\pi)$ are established, except that we need to employ a strengthened version of Voiculescu's theorem, contained in Theorem 2.6 above.

Let $\pi_1, \pi_2 : A \to L(H_B)$ be two representations.

Lemma 3.2. *If $\pi_1 \underset{v}{\precsim} \pi_2$, then the equation $\Phi_v(f) = vfv^*$ defines a $*$-homomorphism*

$$\Phi_v : \mathcal{D}_T(\pi_1) \to \mathcal{D}_T(\pi_2)$$

with the property that $\Phi_v(A_j(\pi_1)) \subset A_j(\pi_2)$ for all $1 \le j \le 6$.

Proof. This follows from the identities

$$\begin{aligned}[][vfv^*, \pi_2(a)] &= v[f, \pi_1(a)]v^* + (v\pi_1(a) - \pi_2(a)v)fv^* \\ &\qquad\qquad - vf(v\pi_1(a^*) - \pi_2(a^*)v)^*, \\ \pi_2(a)vfv^* &= v\pi_1(a)fv^* - (v\pi_1(a) - \pi_2(a)v)fv^*. \end{aligned}$$ □

Corollary 3.3. *Let $\pi_1, \pi_2 : A \to L(H_B)$ be two absorbing representations. Then $A_j(\pi_1) \cong A_j(\pi_2)$ for all $1 \le j \le 6$.*

Proof. Theorem 2.6 yields a unitary $v \in C_u(T, L(H_B))$ such that $\pi_1 \underset{v}{\approx} \pi_2$. The corresponding maps $\Phi_v : A_j(\pi_1) \to A_j(\pi_2)$ are isomorphisms. □

Lemma 3.4. *Let $\pi_1, \pi_2 : A \to L(H_B)$ be two representations of A and suppose that v_1, v_2 are two isometries such that $\pi_1 \underset{v_i}{\precsim} \pi_2$, $i = 1, 2$. Then*

$$(\Phi_{v_1})_* = (\Phi_{v_2})_* : K_*(A_j(\pi_1)) \to K_*(A_j(\pi_2))$$

for all $1 \le j \le 6$.

Proof. The unitary

$$u = \begin{pmatrix} 1 - v_1v_1^* & v_1v_2^* \\ v_2v_1^* & 1 - v_2v_2^* \end{pmatrix} \in M_2(\mathcal{D}_L(\pi_2))$$

conjugates $\left(\begin{smallmatrix} \Phi_{v_1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ over $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \Phi_{v_1} \end{smallmatrix}\right)$. It follows that

$$(\Phi_{v_1})_* = (\Phi_{v_2})_* : K_*(\mathcal{D}_T(\pi_1)) \to K_*(\mathcal{D}_T(\pi_2)).$$

Similarly, one verifies that the equality $(\Phi_{v_1})_* = (\Phi_{v_2})_* : K_*(A_j(\pi_1)) \to K_*(A_j(\pi_2))$ holds for all $1 \le j \le 6$. □

Denote by π^∞ the direct sum $\pi^\infty = \bigoplus_{n=1}^\infty \pi : A \to L(H_B^\infty) = L\big(\bigoplus_{n=1}^\infty H_B\big)$.

Corollary 3.5. *If $\pi : A \to L(H_B)$ is an absorbing representation, then the inclusion $\mathcal{D}_T(\pi) \to \mathcal{D}_T(\pi^\infty)$, $f \mapsto (f, 0, 0, \ldots)$ induces isomorphisms on K-theory: $K_*(A_j(\pi)) \to K_*(A_j(\pi^\infty))$, for all $1 \le j \le 6$.*

Proof. We have $\pi \underset{v}{\precsim} \pi^\infty$, where $v \in C_u(T, L(H_B, H_B^\infty))$ is the constant isometry defined by $v(t)(h) = (h, 0, 0, \ldots)$ for any $t \in T$ and $h \in H_B$. The inclusion map from the statement coincides with Φ_v. On the other hand, $\pi \underset{u}{\approx} \pi^\infty$ since π is absorbing, and hence Φ_u is an isomorphism. We conclude the proof by noting that $(\Phi_v)_* = (\Phi_u)_*$ by Lemma 3.4. □

4. A duality isomorphism

Let A and B be separable C^*-algebras. We are going to show that when we fix an absorbing representation $\pi : A \to L(H_B)$ — the existence of such an absorbing representation is guaranteed by Theorem 2.4 — the K-theory of $\mathcal{C}_L(\pi)$ is canonically isomorphic to the KK-theory of the pair (A, B).

We start with a technical lemma that will be used several times later.

Lemma 4.1. *For any separable C^*-algebra $D \subset C_u(T, L(H_B))$, there is a positive contraction $x \in C_u(T, K(H_B))$ such that*

(a) $[x, d] \in C_0(T, K(H_B))$ *for all $d \in D$, and*

(b) $(1 - x)d \in C_0(T, K(H_B))$ *for all $d \in D \cap C_u(T, K(H_B))$.*

Proof. Our arguments will in fact show that the statement holds true in the more general situation where $L(H_B)$ is replaced by a C^*-algebra L and $K(H_B)$ is replaced by a two-sided closed ideal I of L. Let $\dot{D}$ denote the C^*-subalgebra of L generated by all images $d(t)$ as d ranges over D and t over T. This is separable, and contains $\dot{C} = \dot{D} \cap I$ as an ideal. Let $(x_n)_n$ be a positive contractive approximate unit for $\dot{C}$ which is quasicentral in $\dot{D}$. Choose countable dense subsets $(d_k)_{k=1}^{\infty}$ and $(c_k)_{k=1}^{\infty}$ of D and $D \cap C_u(T, I)$, respectively. As for each n, the subsets $\bigcup_{k=1}^{n} \{d_k(t) : t \in [0, n+1]\} \subseteq \dot{D}$ and $\bigcup_{k=1}^{n} \{c_k(t) : t \in [0, n+1]\} \subseteq \dot{C}$ are compact, so we may assume on passing to a subsequence of (x_n) that

(i) $\|[d_k(t), x_n]\| < 1/(n+1)$ for all $1 \le k \le n$ and all $t \in [0, n+1]$, and

(ii) $\|(1 - x_n)c_k(t)\| < 1/(n+1)$ for all $1 \le k \le n$ and all $t \in [0, n+1]$.

For $t \in [n, n+1)$, write $s = t - n$ and set $x(t) = (1 - s)x_n + s x_{n+1}$; note that the function $x : t \mapsto x(t)$ is uniformly continuous. Then from (i) and (ii) above we have

(i) $\|[d_k(t), x(t)]\| < 1/(n+1)$ for all $1 \le k \le n$ and all $t \in [n, n+1)$, and

(ii) $\|(1 - x(t))c_k(t)\| < 1/(n+1)$ for all $1 \le k \le n$ and all $t \in [n, n+1)$.

This implies that x has the right properties. □

We have obvious inclusions $\mathcal{D}_L(\pi) \subset \mathcal{D}_T(\pi)$ and $\mathcal{C}_L(\pi) \subset \mathcal{C}_T(\pi)$, which induce a $*$-homomorphism

$$\eta : \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \to \mathcal{D}_T(\pi)/\mathcal{C}_T(\pi).$$

Proposition 4.2. *For any separable C^*-algebras A and B and any representation $\pi : A \to L(H_B)$, the map η is a $*$-isomorphism.*

Proof. It is clear from the definitions that $\mathcal{C}_L(\pi) = \mathcal{D}_L(\pi) \cap \mathcal{C}_T(\pi)$ and hence η is injective. It remains to prove that η is surjective. It suffices to show that for any $f \in \mathcal{D}_T(\pi)$ there is $\tilde{f} \in \mathcal{D}_L(\pi)$ such that $\tilde{f} - f \in \mathcal{C}_T(\pi)$. Let $f \in \mathcal{D}_T(\pi)$ be given.

Let D be the C^*-subalgebra of $C_u(T, L(H_B))$ generated by $\pi(A)$ (embedded as constant functions) and f, and let x be as in Lemma 4.1. With this choice of x (that depends on f) we define $\tilde{f} = (1-x)f$. Note that $\tilde{f} = f - xf \in \mathcal{D}_T(\pi)$ since $f, x \in \mathcal{D}_T(\pi)$, and $\tilde{f} - f = -xf \in C_u(T, K(H_B))$ since $x \in C_u(T, K(H_B))$. In particular, it follows that $\tilde{f} - f \in \mathcal{C}_T(\pi)$.

It remains to verify that $\tilde{f} \in \mathcal{D}_L(\pi)$. This follows as for any $a \in A$,

$$[\tilde{f}, \pi(a)] = [(1-x)f, \pi(a)] = [\pi(a), x]f + (1-x)[f, \pi(a)]. \qquad \square$$

An adaptation of the arguments from [Qiao and Roe 2010] gives the following:

Proposition 4.3. *Let A, B be separable C^*-algebras and let $\pi : A \to L(H_B)$ be an absorbing representation. Then*

(a) $K_*(\mathcal{D}_L(\pi)) = 0$ *and hence the boundary map*

$$\partial : K_*(\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)) \to K_{*+1}(\mathcal{C}_L(\pi))$$

is an isomorphism;

(b) *the evaluation map at $t = 0$ induces an isomorphism*

$$e_* : K_*(\mathcal{D}_T(\pi)/\mathcal{C}_T(\pi)) \to K_*(\mathcal{D}(\pi)/\mathcal{C}(\pi)).$$

Proof. Fix an ample representation π of A. One verifies that if $f \in \mathcal{D}_L(\pi)$, then the formula

$$F(t) := (f(t), f(t+1), \dots, f(t+n), \dots)$$

defines an element $F \in \mathcal{D}_L(\pi^\infty)$. Indeed,

$$[F(t), \pi(a)] = \big([f(t), \pi(a)], [f(t+1), \pi(a)], \dots, [f(t+n), \pi(a)], \dots\big)$$

and each entry belongs to $C_0(T, K(H_B))$ and is bounded by $\|[f, \pi(a)]\|$. This shows that $[F, \pi(a)] \in C_u(T, K(H_B^\infty))$. Since $[f, \pi(a)] \in C_0(T, K(H_B))$, it follows immediately that in fact $[F, \pi(a)] \in C_0(T, K(H_B^\infty))$.

With these remarks, the proof of (a) goes just like that of [Qiao and Roe 2010, Proposition 3.5]. Indeed, define $*$-homomorphisms $\alpha_i : \mathcal{D}_L(\pi) \to \mathcal{D}_L(\pi^\infty)$ for $i = 1, 2, 3, 4$ by

$$\begin{aligned}
\alpha_1(f) &= (f(t), 0, 0, \dots),\\
\alpha_2(f) &= (0, f(t+1), f(t+2), \dots),\\
\alpha_3(f) &= (0, f(t), f(t+1), \dots),\\
\alpha_4(f) &= (f(t), f(t+1), f(t+2), \dots).
\end{aligned}$$

It is clear that $\alpha_1 + \alpha_2 = \alpha_4$. The isometry $v \in L(H_B^\infty)$ defined by $v(h_0, h_1, h_2, \dots) = (0, h_0, h_1, h_2, \dots)$ commutes with $\pi^\infty(A)$ and hence $v \in \mathcal{D}_L(\pi^\infty)$. Moreover, $\alpha_4(a) = v\alpha_3(a)v^*$ and hence $(\alpha_4)_* = (\alpha_3)_*$ by [Higson and Roe 2000, Lemma 4.6.2].

Using uniform continuity, one shows that α_3 is homotopic to α_2 via the homotopy $f(t) \mapsto (0, f(t+s), f(t+s+1), \dots)$, $0 \le s \le 1$. We deduce that

$$(\alpha_1)_* + (\alpha_2)_* = (\alpha_1 + \alpha_2)_* = (\alpha_4)_* = (\alpha_3)_* = (\alpha_2)_*$$

and hence $(\alpha_1)_* = 0$. This concludes the proof of (a), since $(\alpha_1)_*$ is an isomorphism by Corollary 3.5.

To prove (b), one follows the proof of [Qiao and Roe 2010, Proposition 3.6] to show that both $K_*(\mathcal{D}_T^0(\pi)) = 0$ and $K_*(\mathcal{C}_T^0(\pi)) = 0$. The desired conclusion then follows in view of the split exact sequence

$$0 \to \mathcal{D}_T^0(\pi)/\mathcal{C}_T^0(\pi) \to \mathcal{D}_T(\pi)/\mathcal{C}_T(\pi) \to \mathcal{D}(\pi)/\mathcal{C}(\pi) \to 0.$$

Any $f \in \mathcal{D}_T^0(\pi)$ can be extended by 0 to an element of $C_u(\mathbb{R}, L(H_B))$. With this convention, define four maps $\beta_i : \mathcal{D}_T^0(\pi) \to \mathcal{D}_T^0(\pi^\infty)$, $i = 1, 2, 3, 4$, by

$$\begin{aligned}
\beta_1(f) &= (f(t), 0, 0, \dots),\\
\beta_2(f) &= (0, f(t-1), f(t-2), \dots),\\
\beta_3(f) &= (0, f(t), f(t-1), \dots),\\
\beta_4(f) &= (f(t), f(t-1), f(t-2), \dots).
\end{aligned}$$

This definition requires that one verifies that if $f \in \mathcal{D}_T^0(\pi)$, then

$$F'(t) := (f(t), f(t-1), \dots, f(t-n), \dots)$$

defines an element of $\mathcal{D}_T^0(\pi^\infty)$. This is clearly the case, since if f is uniformly continuous, then so is F' and moreover, just as argued in [Qiao and Roe 2010], for each t in a fixed bounded interval only finitely many components of $F'(t)$ are nonzero, and hence $[F'(t), \pi^\infty(a)] \in K(H_B^\infty)$ if $[f(t), \pi(a)] \in K(H_B)$ for all $t \in T$. Note that $(\beta_4)_* = (\beta_3)_*$ since $\beta_4(a) = v\beta_3(a)v^*$, where $v \in \mathcal{D}_T(\pi^\infty)$ is the same isometry as in part (a). Using uniform continuity, one observes that β_3 is homotopic to β_2 via the homotopy $f(t) \mapsto (0, f(t-s), f(t-s-1), \dots)$, $0 \le s \le 1$. We deduce that

$$(\beta_1)_* + (\beta_2)_* = (\beta_1 + \beta_2)_* = (\beta_4)_* = (\beta_3)_* = (\beta_2)_*$$

and hence $(\beta_1)_* = 0$. This shows that $K_*(\mathcal{D}_T^0(\pi)) = 0$, since $(\beta_1)_*$ is an isomorphism by Corollary 3.5. The proof for the vanishing of $K_*(\mathcal{C}_T^0(\pi))$ is entirely similar. Indeed, with the same notation as above, one observes that if $f \in \mathcal{C}_T^0(\pi)$ then $F' \in \mathcal{C}_T^0(\pi^\infty)$. Moreover, the four maps $\beta_i : \mathcal{D}_T^0(\pi) \to \mathcal{D}_T^0(\pi^\infty)$ restrict to maps $\beta_i' : \mathcal{C}_T^0(\pi) \to \mathcal{C}_T^0(\pi^\infty)$ with β_3' homotopic to β_2', and $(\beta_1')_*$ is an isomorphism by Corollary 3.5. $\square$

Theorem 4.4. *Let A, B be separable C^*-algebras and let $\pi : A \to L(H_B)$ be an absorbing representation. There are canonical isomorphisms of groups*

$$\alpha : KK_i(A, B) \xrightarrow{\cong} K_i(\mathcal{C}_L(\pi)), \quad i = 0, 1.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} KK_i(A,B) & \xrightarrow{P} & K_{i+1}(\mathcal{D}(\pi)/\mathcal{C}(\pi)) & \xrightarrow{\iota_*} & K_{i+1}(\mathcal{D}_T(\pi)/\mathcal{C}_T(\pi)) \\ & & & & \downarrow{\scriptstyle \eta_*^{-1}} \\ & & K_i(\mathcal{C}_L(\pi)) & \xleftarrow{\partial} & K_{i+1}(\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)) \end{array}$$

where P is the Paschke duality isomorphism — see [Paschke 1981; Skandalis 1988, Remarque 2.8; Thomsen 2001, Theorem 3.2] — and ι is the canonical inclusion. The maps ∂ and $\iota_* = e_*^{-1}$ are isomorphisms by Proposition 4.3 and η_* is an isomorphism by Proposition 4.2. □

As a corollary we obtain the following duality theorem, mentioned in the introduction. Recall from the introduction that $\mathcal{C}_L(A)$ stands for $\mathcal{C}_L(\pi)$, where π is ample (and thus absorbing, by Theorem 2.2), and A is identified with $\pi(A)$.

Theorem 4.5. *For any separable C^*-algebra A there are canonical isomorphisms of groups $K^i(A) \cong K_i(\mathcal{C}_L(A))$ for $i = 0, 1$.* □

5. An inverse map

Let $\alpha : KK_i(A, B) \xrightarrow{\cong} K_i(\mathcal{C}_L(\pi))$ be the isomorphism of Theorem 4.4. Recall that $K(H_B) \cong B \otimes K(H)$. Consider the $*$-homomorphism

$$\boldsymbol{\Phi} : \mathcal{D}_L(\pi) \otimes_{\max} A \to \frac{C_u(T, L(H_B))}{C_0(T, K(H_B))}$$

defined by $\boldsymbol{\Phi}(f \otimes a) = f\pi(a)$, and its restriction to $\mathcal{C}_L(\pi) \otimes_{\max} A$

$$\boldsymbol{\varphi} : \mathcal{C}_L(\pi) \otimes_{\max} A \to \frac{C_u(T, K(H_B))}{C_0(T, K(H_B))}.$$

We want $\boldsymbol{\varphi}$ to define a class in E-theory that we can take products with, but have to be a little careful due to the nonseparability of the C^*-algebra $\mathcal{C}_L(\pi) \otimes_{\max} A$. Just as in the case of the KK-groups [Skandalis 1988], if C is any C^*-algebra and B is a nonseparable C^*-algebra one defines $E_{\text{sep}}(B, C) = \varprojlim_{B_1} E(B_1, C)$, with $B_1 \subset B$ and B_1 separable. Moreover, if D is separable, then $E(D, B) = \varinjlim_{B_1} E(D, B_1)$, with $B_1 \subset B$ and B_1 separable. With these adjustments, one has a well-defined product

$$E(D, B) \times E_{\text{sep}}(B, C) \to E(D, C).$$

Moreover, it is clear that $[\![\boldsymbol{\varphi}]\!]$ defines an element of the group $E_{\text{sep}}(\mathcal{C}_L(\pi) \otimes_{\max} A, B)$.

Recall the isomorphism $K_i(\mathcal{C}_L(\pi)) \cong E_i(\mathbb{C}, \mathcal{C}_L(\pi))$. We use the product

$$E_i(\mathbb{C}, \mathcal{C}_L(\pi)) \times E_{\text{sep}}(\mathcal{C}_L(\pi) \otimes_{\max} A, B) \to E_i(A, B)$$

to define a map $\beta : K_i(\mathcal{C}_L(\pi)) \to E_i(A, B)$ by $\beta(z) = [\![\boldsymbol{\varphi}]\!] \circ (z \otimes \mathrm{id}_A)$. The map β is an inverse of α in the following sense.

Theorem 5.1. *The composition $\beta \circ \alpha$ coincides with the natural map $KK_i(A, B) \to E_i(A, B)$ for $i = 0, 1$.*

Proof. We prove the odd case $i = 1$ and leave the even case for the reader. Recall that the E-theory group $E_1(A, B)$ of Connes and Higson [1990] is isomorphic to $[\![SA, K(H_B)]\!]$ by a desuspension result from [Dadarlat and Loring 1994].

For two continuous functions $f, g : T \to L(H_B)$ we write $f(s) \sim g(s)$ (or $f(t) \sim g(t)$) if $f - g \in C_0(T, K(H_B))$. Let $\{\varphi_s : \mathcal{C}_L(\pi) \otimes_{\max} A \to K(H_B)\}_{s \in T}$ be an asymptotic homomorphism representing $\boldsymbol{\varphi}$. More precisely, take φ to be a set-theoretic lifting of $\boldsymbol{\varphi}$. This means that $\varphi_s(f \otimes a) \sim f(s)\pi(a)$.

The composition $\beta \circ \alpha : KK_1(A, B) \to E_1(A, B)$ is computed as follows. Let $y \in KK_1(A, B)$ and let $z = Py \in K_0(\mathcal{D}(\pi)/\mathcal{C}(\pi))$ be its image under the Paschke duality isomorphism $P : KK_1(A, B) \to K_0(\mathcal{D}(\pi)/\mathcal{C}(\pi))$. Let z be represented by a self-adjoint element $e \in \mathcal{D}(\pi) \subset \mathcal{D}_T(\pi)$ whose image in $\mathcal{D}(\pi)/\mathcal{C}(\pi)$ is an idempotent $\dot{e}$. We identify $\mathcal{D}(\pi)$ with the C^*-subalgebra of constant functions in $\mathcal{D}_T(\pi)$. Choose an element $x \in C_u(T, K(H_B))$ as in Lemma 4.1 with respect to the (separable) C^*-subalgebra D of $C_u(T, L(H_B))$ generated by $\pi(A)$, e, and $K(H_B)$. Therefore, both $[x, \pi(a)]$ and $(1 - x)[e, \pi(a)]$ belong to $C_0(T, K(H_B))$ for all $a \in A$, and moreover $(1 - x)e \in \mathcal{D}_L(\pi)$ as

$$[(1 - x)e, \pi(a)] = [1 - x, \pi(a)]e + (1 - x)[e, \pi(a)] \in C_0(T, K(H_B))$$

for all $a \in A$. Let $e_L = (1 - x)e$ and let $\dot{e}_L$ be its image in $\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)$. Under the isomorphism $\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \cong \mathcal{D}_T(\pi)/\mathcal{C}_T(\pi)$ of Proposition 4.2 we see that $\dot{e}_L$ is just the image of $e \in \mathcal{D}_T(\pi)$ in the quotient, which is an idempotent since $\dot{e}$ is so. It is then clear that $\eta_*^{-1}\iota_*(z) = [\dot{e}_L]$.

We define a $*$-homomorphism $\ell : \mathbb{C} \to \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)$ by $\ell(1) = \dot{e}_L$ and set $S = C_0(0, 1)$. Then $(\beta \circ \alpha)(y)$ is represented by the composition of the asymptotic homomorphisms from the diagram

$$S \otimes \mathbb{C} \otimes A \xrightarrow{1 \otimes \ell \otimes 1} S \otimes \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \otimes A \xrightarrow{\delta_t \otimes 1} \mathcal{C}_L(\pi) \otimes A \xrightarrow{\varphi_s} K(H_B), \quad (5.2)$$

where here and throughout the rest of the proof the tensor products are maximal ones, and the map labeled δ_t is defined by taking the product with a canonical element δ of $E_{1,\text{sep}}(\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi), \mathcal{C}_L(\pi))$ associated to the extension

$$0 \to \mathcal{C}_L(\pi) \to \mathcal{D}_L(\pi) \to \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \to 0,$$

which we now discuss. Fixing a separable C^*-subalgebra $\dot{M}$ of $\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)$, the image of δ in $E_1(\dot{M}, \mathcal{C}_L(\pi))$ is defined as follows. Choose a separable C^*-subalgebra M of $\mathcal{D}_L(\pi)$ that surjects onto $\dot{M}$, and for each $\dot{m} \in \dot{M}$ choose a lift $m \in M$. Let $(v_t)_{t\in T}$ be a positive, contractive, and continuous approximate unit for $M \cap \mathcal{C}_L(\pi)$ which is quasicentral in M. Then for $g \in S = C_0(0, 1)$, δ is characterized by stipulating that $\delta_t(g \otimes \dot{m})$ satisfies

$$\delta_t(g \otimes \dot{m}) \sim g(v_t)m$$

(the choices of (v_t) and the various lifts do not matter up to homotopy). In our case, to compute the composition we need, let M be a separable C^*-subalgebra of $\mathcal{D}_L(\pi)$ containing e and x, and let (v_t) be an approximate unit for $M \cap C_L(\pi)$ that is quasicentral in M.

On the level of elements, we can now concretely describe the composition in (5.2) as follows. If $g \in S = C_0(0, 1)$ and $a \in A$, then under the asymptotic morphism $\{\mu_t : SA \to K(H_B)\}_t$ defined by diagram (5.2), elementary tensors $g \otimes a$ are mapped as follows:

$$g\otimes a \mapsto g\otimes \dot{e}_L \otimes a \xmapsto{\delta_t} g(v_t)(1-x)e\otimes a \xmapsto{\varphi_{s(t)}} g(v_t(s(t)))(1-x(s(t)))e\pi(a) \quad (5.3)$$

for any positive map $t \mapsto s(t)$ which increases to ∞ sufficiently fast. Since the map $t \mapsto x(t)$ is an approximate unit of $K(H_B)$, $(1 - x)y \in C_0(T, K(H_B))$ for all $y \in K(H_B)$. In particular it follows that $(1 - x(s(t)))e[e, \pi(a)] \sim 0$ since $[e, \pi(a)] \in K(H_B)$. Since $e\pi(a) = e\pi(a)e + e[e, \pi(a)]$, it follows from (5.3) that

$$\mu_t(g \otimes a) \sim g(v_t(s(t)))(1 - x(s(t)))e\pi(a)e. \quad (5.4)$$

On the other hand, the natural map $KK_1(A, B) \to E_1(A, B)$ maps y to $[\![\gamma_t]\!]$, where $\{\gamma_t : S \otimes A \to K(H_B)\}_t$ is described in [Connes and Higson 1990] as follows. Consider the extension

$$0 \to K(H_B) \to e\pi(A)e + K(H_B) \to A \to 0.$$

Let $(u_t)_{t\in T}$ be a contractive, positive, and continuous approximate unit of $K(H_B)$ which is quasicentral in $e\pi(A)e + K(H_B)$. Then

$$\gamma_t(g \otimes a) \sim g(u_t)e\pi(a)e.$$

Applying Lemma 4.1 (this time with D the C^*-subalgebra of $C_u(T, L(H_B))$ generated by e, $\pi(A)$, $K(H_B)$, and $t \mapsto x(s(t))$), we can choose $(u_t)_t$ such that $\lim_{t\to\infty}(1 - u_t)x(s(t)) = 0$. Since the C^*-algebra $C_0[0, 1)$ is generated by the function $f(\theta) = 1 - \theta$, it follows that $\lim_{t\to\infty} g(u_t)x(s(t)) = 0$ for all $g \in C_0[0, 1)$, and in particular for all $g \in C_0(0, 1)$.

Our goal now is to verify that $(\mu_t)_t$ is homotopic to $(\gamma_t)_t$. Due to the choice of $(u_t)_t$ and the comments above, we have that

$$\gamma_t(g \otimes a) \sim g(u_t)e\pi(a)e \sim g(u_t)(1 - x(s(t)))e\pi(a)e \tag{5.5}$$

for all $a \in A$ and $g \in C_0(0, 1)$. Finally, define $w_t^{(r)} = (1-r)v_t(s(t))+ru_t$, $0 \le r \le 1$. As

$$\bigl[g(w_t^{(r)}), (1 - x(s(t)))e\pi(a)e\bigr] \to 0 \quad \text{as } t \to \infty$$

for all $r \in [0, 1]$ and $a \in A$, the condition

$$H_t^{(r)}(g \otimes a) \sim g(w_t^{(r)})(1 - x(s(t)))e\pi(a)e$$

defines an asymptotic morphism $H_t : SA \to C[0, 1] \otimes K(H_B)$. This gives the desired homotopy joining $(\mu_t)_t$ with $(\gamma_t)_t$. □

As suggested by the referee, we finish this section by sketching another proof which is maybe a little less self-contained, but more conceptual. The proof below is analogous to the approach used for [Qiao and Roe 2010, Proposition 4.3]. The basic idea in their approach is to apply naturality of the connecting map in E-theory for the diagram of strictly commutative asymptotic morphisms

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}_L(\pi) \otimes_{\max} A \arrow[r] \arrow[d, "\varphi_t"] & \mathcal{D}_L(\pi) \otimes_{\max} A \arrow[r] \arrow[d, "\phi_t"] & (\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)) \otimes_{\max} A \arrow[r] \arrow[d, "\bar{\phi}_t"] & 0 \\
0 \arrow[r] & K(H_B) \arrow[r] & L(H_B) \arrow[r] & L(H_B)/K(H_B) \arrow[r] & 0
\end{tikzcd}$$

where ϕ_t and φ_t represent the asymptotic morphisms induced by the $*$-homomorphisms $\boldsymbol{\Phi}$ and $\boldsymbol{\varphi}$ from the beginning of this section. The family $\bar{\phi}_t$ is the quotient family induced by ϕ_t, and consists of $*$-homomorphisms. Naturality of the boundary map in E-theory in this case amounts to the equality

$$[\![\varphi_t]\!] \circ [\![\delta_t \otimes \mathrm{id}_A]\!] = [\![\gamma_t]\!] \circ [\![\bar{\phi}_t]\!], \tag{5.6}$$

where δ_t is the boundary map for the top sequence of the diagram before tensoring with A, and γ_t is the boundary map for the bottom sequence. See [Connes and Higson 1990, Lemme 10] for the definition of the boundary maps associated to extensions (here and elsewhere one should use limits to deal with the nonseparable algebras involved in the way discussed earlier in this section). The naturality property of the boundary map with respect to general asymptotic morphisms that was discussed in [Guentner 1999, Theorem 5.3] seems to be the closest statement in the literature to the equality in (5.6), but it is nonetheless not sufficiently general to justify the equality. However, one can combine the arguments from the second part of the proof of Theorem 5.1 with those from [Guentner 1999] to verify naturality in full generality and in particular to justify (5.6).

Now (5.6) allows us to conceptualize the proof of Theorem 5.1. Let $y \in KK_i(A, B)$ and let $z = Py \in K_{i+1}(\mathcal{D}(\pi)/\mathcal{C}(\pi))$ be its image under the Paschke duality isomorphism $P : KK_i(A, B) \to K_{i+1}(\mathcal{D}(\pi)/\mathcal{C}(\pi))$. Consider

$$\eta_*^{-1}\iota_*(z) \in K_{i+1}(\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)) \cong E_{i+1}(\mathbb{C}, \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)),$$

where the maps ι_* and η_* are isomorphisms as in the proof of Theorem 4.4. We may view $\eta_*^{-1}\iota_*(z) \otimes [\![\mathrm{id}_A]\!]$ as an element of $E_{i+1}(A, \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \otimes_{\max} A)$. From (5.6) we obtain that

$$[\![\varphi_t]\!] \circ [\![\delta_t \otimes \mathrm{id}_A]\!] \circ (\eta_*^{-1}\iota_*(z) \otimes [\![\mathrm{id}_A]\!]) = [\![\gamma_t]\!] \circ [\![\bar{\phi}_t]\!] \circ (\eta_*^{-1}\iota_*(z) \otimes [\![\mathrm{id}_A]\!]). \quad (5.7)$$

The left-hand side of (5.7) represents the element $(\beta \circ \alpha)(y)$ of $E_i(A, B)$ by the very definition of α and β.

In order to identify the right-hand side of (5.7), it is useful to note that each individual map $\bar{\phi}_t$ is a $*$-homomorphism given by $\kappa \circ (\mathrm{ev}_t \otimes \mathrm{id}_A)$, where

$$\mathrm{ev}_t : \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \to \mathcal{D}(\pi)/\mathcal{C}(\pi)$$

is the evaluation map at t and

$$\kappa : (\mathcal{D}(\pi)/\mathcal{C}(\pi)) \otimes_{\max} A \to L(H_B)/K(H_B), \quad [b] \otimes a \mapsto [b \cdot \pi(a)]$$

is the "multiplication" $*$-homomorphism. Thus the asymptotic morphism $\{\bar{\phi}_t\}$ is homotopic to the constant asymptotic morphism given by $\bar{\phi}_0$, which is equal to $\kappa \circ (\mathrm{ev}_0 \otimes \mathrm{id}_A)$. Hence the right-hand side of (5.7) is equal to

$$[\![\gamma_t]\!] \circ [\![\kappa]\!] \circ ((\mathrm{ev}_0)_*\eta_*^{-1}\iota_*(z) \otimes [\![\mathrm{id}_A]\!]).$$

It follows from the following commutative diagram of $*$-homomorphisms

$$\begin{tikzcd}
\mathcal{D}(\pi)/\mathcal{C}(\pi) \arrow[r, "\mathrm{id}"] \arrow[d, "\iota"] & \mathcal{D}(\pi)/\mathcal{C}(\pi) \\
\mathcal{D}_T(\pi)/\mathcal{C}_T(\pi) \arrow[ur, "\mathrm{ev}_0"] & \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \arrow[l, "\eta"] \arrow[u, "\mathrm{ev}_0"']
\end{tikzcd}$$

that $(\mathrm{ev}_0)_*\eta_*^{-1}\iota_*(z) = z$. This allows us to simplify the right-hand side of (5.7) further to

$$[\![\gamma_t]\!] \circ [\![\kappa]\!] \circ (z \otimes [\![\mathrm{id}_A]\!]),$$

where z is viewed as an element in $E_{i+1}(\mathbb{C}, \mathcal{D}(\pi)/\mathcal{C}(\pi))$. This can be seen to be equal to the image of y under the natural map $KK_i(A, B) \to E_i(A, B)$.

Indeed, focusing on the odd case, where we have $y \in KK_1(A, B)$ and $z = Py \in K_0(\mathcal{D}(\pi)/\mathcal{C}(\pi))$, we may choose $e \in \mathcal{D}(\pi)$, as in the first part of the proof of Theorem 5.1, such that $z = [\dot{e}] \in K_0(\mathcal{D}(\pi)/\mathcal{C}(\pi))$. Then the $*$-homomorphism $a \in A \mapsto [e \cdot \pi(-)] \in L(H_B)/K(H_B)$, which represents $[\![\kappa]\!] \circ (z \otimes [\![\mathrm{id}_A]\!])$, is the

Busby invariant of the extension corresponding to $e \in \mathcal{D}(\pi)$. Hence its composition with the asymptotic morphism $\{\gamma_t\} : L(H_B)/K(H_B) \to K(H_B)$ represents the image of y under the natural map $KK_1(A, B) \to E_1(A, B)$.

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Hecke modules for arithmetic groups via bivariant K-theory

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Let Γ be a lattice in a locally compact group G. In another work, we used KK-theory to equip with Hecke operators the K-groups of any Γ-C^*-algebra on which the commensurator of Γ acts. When Γ is arithmetic, this gives Hecke operators on the K-theory of certain C^*-algebras that are naturally associated with Γ. In this paper, we first study the topological K-theory of the arithmetic manifold associated to Γ. We prove that the Chern character commutes with Hecke operators. Afterwards, we show that the Shimura product of double cosets naturally corresponds to the Kasparov product and thus that the KK-groups associated to an arithmetic group Γ become true Hecke modules. We conclude by discussing Hecke equivariant maps in KK-theory in great generality and apply this to the Borel–Serre compactification as well as various noncommutative compactifications associated with Γ. Along the way we discuss the relation between the K-theory and the integral cohomology of low-dimensional manifolds as Hecke modules.

1. Introduction

Let Γ be a lattice in a locally compact group G with commensurator $C_G(\Gamma)$. Let $S \subset C_G(\Gamma)$ be a group containing Γ. In [Mesland and Şengün 2016], for $g \in S$ and B a S-C^*-algebra (that is, a C^*-algebra on which S acts via automorphisms), we constructed elements $[T_g] \in KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$. We introduced *analytic Hecke operators* on any module over $KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$ as the endomorphisms arising from the classes $[T_g]$. In the present paper we prove several structural results about

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these Hecke operators, showing that they generalize the well-known cohomological Hecke operators in a way that is compatible with the Chern character and the double-coset Hecke ring of Shimura.

The double-coset Hecke ring of Shimura is well-known to number theorists. In the widely studied case where Γ is an arithmetic group, the Hecke ring acts linearly on various spaces of automorphic forms associated to Γ, providing a rich supply of symmetries [Shimura 1971, Chapter 3]. Those automorphic forms that are simultaneous eigenvectors of these symmetries are conjectured, and proven in many cases, to have deep connections to arithmetic [Clozel 1990; Taylor 1995]. The Hecke ring also acts on the cohomology of the arithmetic manifold M associated to Γ and there is a Hecke equivariant isomorphism between spaces of automorphic forms associated to Γ and cohomology of M twisted with suitable local systems [Franke 1998; Shimura 1971]. The passage to cohomology leads to many fundamental results and new insights on the arithmetic of automorphic forms. The results of this paper, together with those of [Mesland and Şengün 2016], offer an analytic habitat for the Hecke ring by providing ring homomorphisms from the Hecke ring to suitable KK-groups. The passage to KK-theory extends the scope of the action of the Hecke ring beyond cohomology and allows for the possibility of using tools from operator K-theory in the study of automorphic forms.

Let us describe the results of the paper more precisely. In Section 2, we consider the situation where S acts on a locally compact Hausdorff space X. Assume that Γ acts freely and properly on X and put $M = \Gamma\backslash X$. It is well-known that the C^*-algebras $C_0(X) \rtimes_r \Gamma$ and $C_0(M)$ are Morita equivalent, so

$$KK_0(C_0(X) \rtimes_r \Gamma, C_0(X) \rtimes_r \Gamma) \simeq KK_0(C_0(M), C_0(M)),$$

and thus for any $g \in S$ we obtain a class $[T_g] \in KK_0(C_0(M), C_0(M))$. The element g gives rise to a cover M_g of M and a pair of covering maps, forming the *Hecke correspondence* $M \xleftarrow{s} M_g \xrightarrow{t} M$. In [Mesland and Şengün 2016] it was shown that the class $[T_g]$ corresponds to the class of this Hecke correspondence, that is,

$$[T_g] = [M \leftarrow M_g \to M] \in KK_0(C_0(M), C_0(M)).$$

This class induces a Hecke operator $T_g : K^*(M) \to K^*(M)$ on topological K-theory. In this paper we show that the Chern character

$$\mathrm{Ch} : K^0(M) \oplus K^1(M) \to H^{\mathrm{ev}}(M, \mathbb{Q}) \oplus H^{\mathrm{odd}}(M, \mathbb{Q})$$

is *Hecke equivariant*. Here we equip $H^*(M, \mathbb{Q})$ with Hecke operators in the usual way using the Hecke correspondence $M \xleftarrow{s} M_g \xrightarrow{t} M$; see, for example, [Lee 2009].

In Section 3, we specialize to noncompact arithmetic hyperbolic 3-manifolds M. Let $\overline{M}$ be the Borel–Serre compactification of M. Consider the diagram

$$\begin{array}{ccccc} K^0(M) & \times & K_0(M) & \longrightarrow & \mathbb{Z} \\ \big\downarrow & & \big\uparrow & & \\ H^2(\overline{M}, \partial\overline{M}, \mathbb{Z}) & \times & H_2(\overline{M}, \partial\overline{M}, \mathbb{Z}) & \longrightarrow & \mathbb{Z} \end{array} \tag{1.1}$$

Here horizontal arrows are given by the standard pairings with respect to which the Hecke operators are adjoint. The vertical arrows are Hecke equivariant isomorphisms; we establish the one on the left via the results of Section 2 and the one on the right was proven in [Mesland and Şengün 2016]. Using the relative index theorem, we show that the diagram commutes. Using very different techniques, we proved a similar result in [Mesland and Şengün 2016] where the K-groups of M were replaced with those of the reduced group C^*-algebra $C^*_r(\Gamma)$ of Γ.

In Section 4 we prove the main result of the paper. The double-coset Hecke ring $\mathbb{Z}[\Gamma, S]$ is the free abelian group on the double cosets $\Gamma g \Gamma$, with $g \in S$, equipped with the Shimura product [Shimura 1971]. We show that the map $\Gamma g^{-1}\Gamma \mapsto [T_g]$ extends to a ring homomorphism

$$\mathbb{Z}[\Gamma, S] \to KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$$

for any S-C^*-algebra B. As mentioned in the second paragraph of this introduction, this homomorphism provides the Hecke ring $\mathbb{Z}[\Gamma, S]$ with a new habitat. The universality property of KK-theory [Higson 1987] implies that for any additive functor F on separable C^*-algebras that is homotopy invariant, split-exact and stable, the abelian groups $F(B \rtimes_r \Gamma)$ are modules over $\mathbb{Z}[\Gamma, S]$. For example, let Γ be an arithmetic group in a semisimple real Lie group G. By taking F to be local cyclic cohomology and $B = C_0(X)$ where X is the symmetric space of G, we recover the action of the Hecke ring on the cohomology of the arithmetic manifold X/Γ. In [Mesland and Şengün 2016], we took F to be K-homology and worked with three different S-C^*-algebras B that were naturally associated to Γ.

In Section 5, we show that a Γ-exact and S-equivariant extension

$$0 \to B \to E \to A \to 0$$

of C^*-algebras induces Hecke equivariant long exact sequences relating the KK-groups of the crossed products $B \rtimes_r \Gamma$, $E \rtimes_r \Gamma$ and $A \rtimes_r \Gamma$. In particular, suppose that X is a free and proper Γ-space on which S acts by homeomorphisms, and $\overline{X}$ a partial S-compactification of X with boundary $\partial X := \overline{X} \setminus X$. Then the extension

$$0 \to C_0(X) \to C_0(\overline{X}) \to C_0(\partial X) \to 0$$

induces a Hecke equivariant exact sequence

$$
\begin{array}{ccccc}
K_1(C_0(X)\rtimes_r\Gamma) & \longrightarrow & K_1(C_0(\overline{X})\rtimes_r\Gamma) & \longrightarrow & K_1(C_0(\partial X)\rtimes_r\Gamma) \\
\uparrow & & & & \downarrow \\
K_0(C_0(\partial X)\rtimes_r\Gamma) & \longleftarrow & K_0(C_0(\overline{X})\rtimes_r\Gamma) & \longleftarrow & K_0(C_0(X)\rtimes_r\Gamma)
\end{array}
$$

of $\mathbb{Z}[\Gamma, S]$-modules. The results of Sections 4 and 5 hold for the full crossed product algebras as well.

Let $\boldsymbol{G}$ be a reductive algebraic group and $\Gamma \subset \boldsymbol{G}(\mathbb{Q})$ an arithmetic group. Then the Borel–Serre partial compactification $\overline{X}$ of the associated global symmetric space X is a proper $\boldsymbol{G}(\mathbb{Q})$-compactification. The associated Morita equivalences provide a Hecke equivariant isomorphism of above six-term exact sequence with the topological K-theory exact sequence of the Borel–Serre compactification of the arithmetic manifold X/Γ and its boundary.

The generality of our methods also allows the consideration of various noncommutative compactifications. One family of examples are the Hecke equivariant Gysin exact sequences studied in [Mesland and Şengün 2016] coming from the geodesic compactification of hyperbolic n-space. Other examples of interest come from the Floyd boundary of Γ, such as the boundary of tree associated to $\mathrm{SL}(2, \mathbb{Z})$ and the Bruhat–Tits building of a p-adic group and its boundary. In most of these cases not all of the crossed products are Morita equivalent to a commutative C^*-algebra.

Set-up and notation. The following set-up will hold for the whole paper. Let G be a locally compact group and $\Gamma \subset G$ a torsion-free discrete subgroup. Recall that two subgroups H, K of G are called *commensurable* if $H \cap K$ is of finite index in both H and K. The commensurator $C_G(\Gamma)$ of Γ (in G) is the group of elements $g \in G$ for which Γ and $g\Gamma g^{-1}$ are commensurable. Moreover, S denotes a subgroup of $C_G(\Gamma)$ containing Γ.

2. Hecke equivariance of the Chern character

In this section, we assume that S acts on a locally compact Hausdorff space X and that the action of Γ on X is free and proper. Let M denote the Hausdorff space X/Γ. Given an element $g \in S$, we put $M_g := X/\Gamma_g$ and $M^g := X/\Gamma^g$, where $\Gamma^g := \Gamma \cap g^{-1}\Gamma g$ and $\Gamma_g := \Gamma \cap g\Gamma g^{-1} = g\Gamma^g g^{-1}$. Note that $s : M_g \to M$ and $s' : M^g \to M$ are finite sheeted covers (of the same degree) and the map $c : M_g \to M^g$ defined by $x\Gamma_g \mapsto g^{-1}x\Gamma^g$ is a homeomorphism. We obtain a second finite covering $t := s' \circ c : M_g \to M$.

We shall equip the topological K-theory of M with Hecke operators via two different constructions, one analytical, arising from a KK-class and the other topological, arising from a correspondence. We will see that these two constructions give rise to the same Hecke operator. Afterwards, we will show that the Chern character between the K-theory and the ordinary cohomology of M is Hecke equivariant.

2.1. ***Analytic Hecke operators.*** Let $g \in S$. As mentioned in the introduction, thanks to a Morita equivalence, the analytically constructed class

$$[T_g] \in KK_0(C_0(X) \rtimes \Gamma, C_0(X) \rtimes \Gamma)$$

gives rise to a class $[T_g^M] \in KK_0(C_0(M), C_0(M))$. This latter class has a simpler description, which we now recall.

The conditional expectation

$$\rho : C_0(M_g) \to C_0(M), \quad \rho(\psi)(m) = \sum_{x \in t^{-1}(m)} \psi(x),$$

and right module structure

$$\psi \cdot f(x) := \psi(x) f(t(x))$$

give $C_0(M_g)$ a right $C_0(M)$-module, which we denote by T_g^M. Because the map $s : M_g \to M$ is proper, there is a left action of $C_0(M)$ on T_g^M by compact operators

$$C_0(M) \to \mathbb{K}(T_g^M), \quad f \cdot \psi(x) = f(s(x))\psi(x).$$

Then $[T_g^M] \in KK_0(C_0(M), C_0(M))$ is the class of this bimodule.

We observe that $M \xleftarrow{s} M_g \xrightarrow{t} M$ defines a *correspondence* in the sense of [Connes and Skandalis 1984]. Associated to this correspondence, there exists a class $[s_*] \otimes [t!] \in KK_0(C_0(M), C_0(M))$, where $t!$ is the wrong way cycle arising from t. As t is simply a finite covering of manifolds, it follows from [Connes and Skandalis 1984, Proposition 2.9] that $t!$ acquires a simpler description and it is then not hard to see that $[s_*] \otimes [t!]$ equals $[T_g^M]$ above.

2.2. Definition. Let $M = X/\Gamma$ as above. For any separable C^*-algebra C, the *analytic Hecke operators*

$$T_g : KK_*(C_0(M), C) \to KK_*(C_0(M), C),$$
$$T_g : KK_*(C, C_0(M)) \to KK_*(C, C_0(M)),$$

are defined to be the Kasparov product with the class $[T_g^M] \in KK_0(C_0(M), C_0(M))$.

An important case is when one takes $C \simeq \mathbb{C}$. Then we obtain analytic Hecke operators on the topological K-theory of M:

$$T_g : K^*(M) \to K^*(M).$$

2.3. ***Topological Hecke operators.*** We now proceed to give an "elementary" description of our Hecke operators in the special case of topological K-theory. To do this, we follow the description of Hecke operators on ordinary cohomology from correspondences; see, for example, [Mesland and Şengün 2016]. To this end, we introduce the "transfer map" machinery from stable homotopy theory, which allows us to deal with generalized cohomology theories at no extra cost.

To a finite covering map $p : (Y, B) \to (X, A)$ of pairs of spaces (that is, a finite covering $p : Y \to X$ with subspaces $A \subset X$ and $B \subset Y$ such that $B = p^{-1}(A)$), there is a well-known construction [Adams 1978, Construction 4.1.1, Theorem 4.2.3; Kahn and Priddy 1972] that associates to the map p a map of suspension spectra $p^! : \Sigma^\infty(X/A) \to \Sigma^\infty(Y/B)$. Via precomposition with $p^!$, for any generalized cohomology theory h^* with spectrum E, we obtain a homomorphism called the *transfer map*

$$p^! : h^n(Y, B) = [\Sigma^\infty S^n \wedge \Sigma^\infty(Y/B), E] \to h^n(X, A) = [\Sigma^\infty S^n \wedge \Sigma^\infty(X/A), E].$$

This transfer map agrees with the usual one in the case of ordinary cohomology; see [Kahn and Priddy 1972, Proposition 2.1]. In the case of topological K-theory, the transfer map is induced by the direct image map of Atiyah [1961]; see [Kahn and Priddy 1972, Proposition 2.4]. Recall that if $f : Y \to X$ is a finite covering map and $E \to X$ is a vector bundle, then the direct image bundle $f^! E \to Y$ has fiber $(f^! E)_y$ at $y \in Y$ given by the direct sum $\bigoplus_{f(x)=y} E_x$.

2.4. Definition. Given any generalized cohomology theory h^* with spectrum E and $g \in S$, the *topological Hecke operator* T_g on $h^n(M)$ is defined as the composition

$$h^n(M) \xrightarrow{s^*} h^n(M_g) \xrightarrow{t^!} h^n(M).$$

In the case of topological K-theory, these topological Hecke operators agree with the analytic ones that we defined earlier.

2.5. Proposition. *Let $g \in S$. The analytic Hecke operator T_g on $K^*(M)$ agrees with the topological Hecke operator T_g on $K^*(M)$.*

Proof. Let us prove the statement for K^0 first. It suffices to show that, after we identify $K^0(M) \simeq K_0(C_0(M))$, the direct image map of Atiyah is induced by tensor product (from the right) with the $C_0(M)$-module T_g^M defined above in Section 2.1. To that end, we need to show that for any vector bundle $E \to M_g$, there is a unitary isomorphism between the $C_0(M)$-modules of sections

$$\alpha : \Gamma(E) \otimes_{C_0(M_g)} C_0(M_g)_{C_0(M)} \xrightarrow{\sim} \Gamma(t^! E).$$

This is achieved by choosing an open cover U_i of M_g for which the covering map t is homeomorphic. Let χ_i^2 be a partition of unity subordinate to the U_i. Define

$$\alpha(\psi \otimes f)(m) := \left(\sum_i \chi_i(x)\psi(x) f(x)\right)_{x \in t^{-1}(m)} \in t^! E.$$

It is straightforward to check that this induces the desired unitary isomorphism. Note that the above is also observed in [Ramras et al. 2013, Lemma 3.12].

To prove the claim for K^1, we descend to K^0 and exploit, as we did above, the fact that transfer is implemented by the direct image map. Consider the diagram below:

$$\begin{tikzcd} K^1(M_g) \arrow[r, "\simeq"] \arrow[d, "t^!"] & K^0(M_g \times \mathbb{R}) \arrow[d, "(t\times \mathrm{Id})^!"] \\ K^1(M) \arrow[r, "\simeq"] & K^0(M \times \mathbb{R}) \end{tikzcd} \tag{2.6}$$

The vertical arrows $t^!$ and $(t \times \mathrm{Id})^!$ are the transfer maps arising from the finite coverings $t : M_g \to M$ and $t \times \mathrm{Id} : M_g \times \mathbb{R} \to M \times \mathbb{R}$. The horizontal isomorphisms follow from long exact sequences in topological K-theory associated to suitable pairs of spaces. As the transfer map is natural and commutes with connecting morphisms [Adams 1978, p. 123–124], it follows that the diagram is commutative.

Note that $K^0(M \times \mathbb{R}) \simeq K_0(C_0(M) \otimes C_0(\mathbb{R}))$. Under the isomorphism

$$KK_0(C_0(M), C_0(M)) \xrightarrow{\simeq} KK_0(C_0(M) \otimes C_0(\mathbb{R}), C_0(M) \otimes C_0(\mathbb{R})),$$

our distinguished class $[T_g^M]$ gets sent to $[T_g^M \otimes C_0(\mathbb{R})]$. Now the same argument as in the first paragraph of this proof shows that the direct image map of Atiyah, for the finite covering $M_g \times \mathbb{R} \xrightarrow{t\times\mathrm{Id}} M \times \mathbb{R}$, is induced by tensor product with the $C_0(M) \otimes C_0(\mathbb{R})$-module $T_g^M \otimes C_0(\mathbb{R})$. $\square$

2.7. Given a pair of compact Hausdorff spaces (X, A), we have the Chern character (see [Karoubi 1978, V.3.26])

$$\mathrm{Ch} : K^i(X, A) \to \mathrm{PH}^i(X, A, \mathbb{Q}), \quad i = 0, 1,$$

where PH^0 and PH^1 are the periodic cohomology groups given by the direct sums of the even and the odd degree ordinary cohomology groups, respectively. The Chern character commutes with suspension and thus is a stable cohomology operation (of degree 0).

Now let M be a noncompact arithmetic manifold. For $g \in C_G(M)$, let $\overline{M}$, $\overline{M_g}$ denote the Borel–Serre compactifications of M, M_g, respectively; see [Borel and Serre 1973; Mesland and Şengün 2016, Section 2.1.2]. It is well-known that the finite covering maps $s, t : M_g \to M$ extend to finite coverings of pairs of spaces $\bar{s}, \bar{t} : (\overline{M_g}, \partial\overline{M_g}) \to (\overline{M}, \partial\overline{M})$. From these, we obtain Hecke operators T_g on the relative groups $K^*(\overline{M}, \partial\overline{M})$ and $H^*(\overline{M}, \partial\overline{M}, \mathbb{Z})$. Notice that

$$K^*(\overline{M}, \partial\overline{M}) \simeq \widetilde{K}^*(M^+) = K^*(M) \simeq K_*(C_0(M)),$$

where M^+ is the one-point compactification of M. Furthermore, we have that $H^*(\overline{M}, \partial\overline{M}, \mathbb{Z}) \simeq H_c^*(M, \mathbb{Z})$, where H_c^* denotes compactly supported cohomology.

It follows that for a given arithmetic manifold M, by choosing $(X, A) = (M, \varnothing)$ if M is compact and $(X, A) = (\overline{M}, \partial\overline{M})$ if M is noncompact, we have the Chern character

$$\mathrm{Ch} : K^i(M) \to \mathrm{PH}_c^i(M, \mathbb{Q}), \quad i = 0, 1,$$

and both sides are Hecke modules. A most natural question is whether the Chern character commutes with the Hecke actions.

2.8. Proposition. *Let M be an arithmetic manifold and $g \in C_G(M)$. The Chern character*

$$\mathrm{Ch} : K^i(M) \to \mathrm{PH}_c^i(M, \mathbb{Q}), \quad i = 0, 1$$

commutes with the action of the Hecke operator T_g on both sides.

Proof. Consider a cohomology operation $\Psi : E^*(\,\cdot\,) \to F^*(\,\cdot\,)$ of degree 0 between two cohomology theories with spectra E, F. If Ψ is stable, there is in fact a map of spectra $\Psi : E \to F$ and the cohomology operation is simply the composition

$$E^n(X, A) = [\Sigma^\infty S^n \wedge \Sigma^\infty(A/X), E] \to F^n(X, A) = [\Sigma^\infty S^n \wedge \Sigma^\infty(X/A), F],$$
$$f \mapsto \Psi \circ f.$$

It immediately follows that the transfer operator associated to a finite cover of pairs of spaces $p : (Y, B) \to (X, A)$ commutes with Ψ, that is, the following diagram commutes:

$$\begin{tikzcd} E^n(Y, B) \arrow[r, "\Psi"] \arrow[d, "p^!"] & F^n(Y, B) \arrow[d, "p^!"] \\ E^n(X, A) \arrow[r, "\Psi"] & F^n(X, A) \end{tikzcd}$$

Now let us go back to our setting. Let us first assume that M is compact. Note that $H_c^*(M, \mathbb{Z}) = H^*(M, \mathbb{Z})$ in this case. As it is a stable cohomology operation, the Chern character commutes with the natural map s^* and also with the transfer map $t^!$, giving rise to the commutative diagram

$$\begin{tikzcd} K^*(M) \arrow[r, "s^*"] \arrow[d, "\mathrm{Ch}"'] & K^*(M_g) \arrow[r, "t^!"] \arrow[d, "\mathrm{Ch}"'] & K^*(M) \arrow[d, "\mathrm{Ch}"'] \\ \mathrm{PH}^*(M, \mathbb{Q}) \arrow[r, "s^*"] & \mathrm{PH}^*(M_g, \mathbb{Q}) \arrow[r, "t^!"] & \mathrm{PH}^*(M, \mathbb{Q}) \end{tikzcd}$$

showing that the Chern character map commutes with Hecke operators.

For the case where M is noncompact, the proof follows in the same way considering the diagram

$$\begin{tikzcd} K^*(\overline{M}, \partial\overline{M}) \arrow[r, "\bar{s}^*"] \arrow[d, "\mathrm{Ch}"'] & K^*(\overline{M_g}, \partial\overline{M_g}) \arrow[r, "\bar{t}^!"] \arrow[d, "\mathrm{Ch}"'] & K^*(\overline{M}, \partial\overline{M}) \arrow[d, "\mathrm{Ch}"'] \\ \mathrm{PH}^*(\overline{M}, \partial\overline{M}, \mathbb{Q}) \arrow[r, "\bar{s}^*"] & \mathrm{PH}^*(\overline{M_g}, \partial\overline{M_g}, \mathbb{Q}) \arrow[r, "\bar{t}^!"] & \mathrm{PH}^*(\overline{M}, \partial\overline{M}, \mathbb{Q}) \end{tikzcd}$$

where $\bar{s}, \bar{t} : (\overline{M_g}, \partial\overline{M_g}) \to (\overline{M}, \partial\overline{M})$ are the extensions of $s, t : M_g \to M$ mentioned earlier. □

2.9. Remark. The transfer map used above is an example of what is known as a *wrong way map*. Connes and Skandalis [1984, Remark 2.10(a)] remark that given a K-oriented map $f : X \to Y$ between smooth manifolds, the wrong way maps $f^! : K(X) \to K(Y)$, induced by the Kasparov product with the class of the wrong way cycle $[f!] \in KK_*(C_0(X), C_0(Y))$, and $f^! : H_c(X, \mathbb{Q}) \to H_c(Y, \mathbb{Q})$ commute under the Chern character modulo an error term $\mathrm{Td}(f)$ defined via the Todd genus of certain bundles that naturally arise. In our case, this error term vanishes and we get that the transfer map commutes with the Chern character as we proved above.

2.10. Remark. Using the universal property of KK-theory, the Chern character can be obtained as the unique natural transformation

$$\mathrm{Ch} : KK_*(A, B) \to HL_*(A, B),$$

where HL_* denotes bivariant local cyclic homology; see [Meyer 2007; Puschnigg 1996]. For a locally compact space X, the local cyclic homology of $C_0(X)$ recovers the compactly supported sheaf cohomology of X [Puschnigg 1996, Theorem 11.7]. Thus ordinary cohomology admits an action of analytic Hecke operators via its structure as a module over KK-theory. It follows from the results of this section that the topological Hecke operators on ordinary cohomology arise from the analytic Hecke module structure.

3. Bianchi manifolds

In this section, we present a result about arithmetic noncompact hyperbolic 3-manifolds that complements the results obtained in our previous paper [Mesland and Şengün 2016, Section 5]. In that paper, for a Bianchi manifold M, we provided a Hecke equivariant isomorphism between $K_0(M)$ and $H_2(\overline{M}, \partial\overline{M}, \mathbb{Z})$, where $\overline{M}$ is the Borel–Serre compactification of M; see [Borel and Serre 1973]. We show below that $H^2(\overline{M}, \partial\overline{M}, \mathbb{Z})$ and $K^0(M)$ are isomorphic as Hecke modules and further argue that the cohomological pairing between H^2 and H_2 and the index pairing between K^0 and K_0 commute under these isomorphisms.

Let $\mathcal{O}$ be the ring of integers of an imaginary quadratic field and Γ be a torsion-free finite index subgroup of the Bianchi group $\mathrm{PSL}_2(\mathcal{O})$. Then Γ acts freely and properly on the hyperbolic 3-space $\boldsymbol{H}_3$. The associated hyperbolic 3-manifold $M = \boldsymbol{H}_3/\Gamma$ is known as a *Bianchi manifold.* It is well-known that any noncompact arithmetic hyperbolic 3-manifold is commensurable with a Bianchi manifold.

3.1. For compact connected spaces X, denote by $\widetilde{K}^0(X)$ the *reduced K-theory* of X, that is, the kernel of the map $K^0(X) \to \mathbb{Z}$ induced by $[E] \mapsto \dim_{\mathbb{C}}(E)$. Write $[n] \in K^0(X)$ for the class of the trivial bundle T^n of rank n over X. For a vector bundle E, the top exterior power $\bigwedge^{\dim E} E$ is called the *determinant line bundle* and denoted $\det E$. Let $\mathrm{Pic}(X)$ denote the *Picard group* of X, that is, the set of isomorphism classes of line bundles on X together with the tensor product operation.

Let M^+ denote the one-point compactification of the Bianchi manifold M. Since M^+ is a CW-complex of dimension 3, every complex vector bundle $E \to M^+$ splits as $E \simeq \det E \oplus T^{\dim_{\mathbb{C}}(E)-1}$; see [Weibel 2013, Corollary 4.4.1]. It follows from [Weibel 2013, Corollary 2.6.2] that the map

$$\dim \oplus \det : K^0(M^+) \to \mathbb{Z} \oplus \mathrm{Pic}(M^+), \quad E \mapsto (\dim_{\mathbb{C}}(E), [\det E])$$

is an isomorphism. Noting $H^0(M^+, \mathbb{Z}) \simeq \mathbb{Z}$ and identifying $\mathrm{Pic}(M^+) \simeq H^2(M^+, \mathbb{Z})$ via the first Chern class c_1, we obtain the isomorphism

$$K^0(M^+) \to H^0(M^+, \mathbb{Z}) \oplus H^2(M^+, \mathbb{Z})$$

induced by $[E] \mapsto \dim_{\mathbb{C}}(E) + c_1(\det E)$. Note that this map agrees with the Chern character since $E \simeq T^{\dim_{\mathbb{C}}(E)-1} \oplus \det E$ as mentioned above. By Proposition 2.8, this isomorphism is Hecke equivariant.

Composing the Chern character with the projection map, we obtain a surjection $K^0(M^+) \to H^2(M^+, \mathbb{Z})$ whose kernel is $\widetilde{K}^0(M^+) = K^0(M)$. Noting that $H^2(M^+, \mathbb{Z})$ is isomorphic to the compactly supported cohomology $H^2_c(M, \mathbb{Z})$, which in turn is isomorphic to $H^2(\overline{M}, \partial\overline{M}, \mathbb{Z})$, we obtain an isomorphism

$$K^0(M) \xrightarrow{\sim} H^2(\overline{M}, \partial\overline{M}, \mathbb{Z}) \tag{3.2}$$

that is Hecke equivariant.

3.3. Given a line bundle $L \to \overline{M}$ and any connection ∇ on L, let

$$F_\nabla = \mathrm{Tr}\Big(\frac{-1}{2\pi i}\nabla^2\Big)$$

be the curvature 2-form of ∇. Then it is well-known that F_∇ is closed and its image in $H^2(M, \mathbb{R})$ is in fact integral and equals the first Chern class $c_1(L)$ of L.

3.4. Proposition. *Let $(N, \partial N) \subset (\overline{M}, \partial\overline{M})$ be an embedded surface, $L \to \overline{M}$ a line bundle that is trivial on $\partial\overline{M}$ and $\overline{N}$ the closed subspace of N obtained by removing an open neighborhood of ∂N over which L is trivial. View the interior $\mathring{N}$ of N as a* *spin^c surface with associated Dirac operator* $\not{D}_{\mathring{N}}$ (see [Mesland and Şengün 2016, Section 5]). *We have*

$$\langle[\not{D}_{\mathring{N}}], [L]-[1]\rangle = \int_{\overline{N}} F_\nabla$$

for any connection ∇ *on L. Here* $\langle\,\cdot\,,\cdot\,\rangle$ *is the index pairing.*

Proof. It follows from the relative index theorem of [Roe 1991, Theorem 4.6] that

$$\langle[\not{D}_{\mathring{N}}], [L]-[1]\rangle = \int_{\overline{N}} \widehat{A}(\mathring{N})\,\mathrm{Ch}(L|_{\mathring{N}}) - \int_{\overline{N}} \widehat{A}(\mathring{N}).$$

Here $L|_{\mathring{N}}$ is the restriction of L to the interior of N. Observe that

$$\mathrm{Ch}(L|_{\mathring{N}}) = 1 + c_1(L|_{\mathring{N}}) = 1 + [F_\nabla|_{\mathring{N}}],$$

where ∇ is any chosen connection on L and $F_\nabla|_{\mathring{N}}$ is the restriction of its curvature to $\mathring{N}$. The $\widehat{A}$-genus $\widehat{A}(\mathring{N})$ of $\mathring{N}$ equals 1 as it only has nonzero components in forms of degree 0 mod 4. The claim follows. □

The following is not necessary for the main result of this section, however we note it as it quickly follows from the above and [Ballmann and Brüning 2001, Lemma 2.22].

3.5. Corollary. *If N has finite volume*, *we have*

$$\langle[\not{D}_{\mathring{N}}], [L]-[1]\rangle = \int_{\mathring{N}} F_\nabla,$$

for any connection ∇ *on L.*

3.6. Proposition. *We have the equality*

$$\langle[\not{D}_{\mathring{N}}], [L]-[1]\rangle = \langle[(N,\partial N)], c_1(L)\rangle.$$

In particular, *the isomorphisms*

$$K^0(M) \xrightarrow{\simeq} H^2(\overline{M}, \partial\overline{M}, \mathbb{Z}), \qquad K_0(M) \xleftarrow{\simeq} H_2(\overline{M}, \partial\overline{M}, \mathbb{Z})$$

(see (3.2) and [Mesland and Şengün 2016, Proposition 5.6.]) *are compatible with the index pairing*

$$\langle\,\cdot\,,\cdot\,\rangle : K_0(M)\times K^0(M) \to \mathbb{Z}$$

and the integration pairing

$$\langle\,\cdot\,,\cdot\,\rangle : H_2(\overline{M}, \partial\overline{M}, \mathbb{Z}) \times H^2(\overline{M}, \partial\overline{M}, \mathbb{Z}) \to \mathbb{Z}.$$

In other words, *diagram* (1.1) *of the introduction is commutative.*

Proof. It follows from our discussion in Section 3.1 that every element of $K^0(M)$ is of the form $[L]-[1]$, where 1 is the trivial line bundle and $L\to M$ is a line bundle that is trivial at infinity. Under the isomorphism (3.2), the image of $[L]-[1]$ is $c_1(L)$. Every class in $H_2(\overline{M}, \partial\overline{M}, \mathbb{Z})$ is represented by a properly embedded surface $(N, \partial N) \subset (\overline{M}, \partial \overline{M})$; see [Mesland and Şengün 2016, Section 5]. Then the pairing $\langle [(N, \partial N)], c_1(L)\rangle$ is given by the integral $\int_N F_\nabla$, where ∇ is any connection on L and F_∇ is the associated curvature 2-form as above. As L is trivial at infinity, we can choose closed $\overline{N} \subset \mathring{N}$ so that L is trivial outside $\overline{N}$ and it then follows that $\int_N F_\nabla = \int_{\overline{N}} F_\nabla$. Observe that the image of $[(N, \partial N)]$ in $K_0(M)$ under the isomorphism given in [Mesland and Şengün 2016, Proposition 5.6.] is $[\not{D}_{\mathring{N}}]$. Now by Proposition 3.4, we have the claim. $\square$

4. The double-coset Hecke ring and *KK*-theory

We recall the construction of the Hecke operators via *KK*-theory as put forward in [Mesland and Şengün 2016]. We then show that the multiplication of double-cosets corresponds to the Kasparov product of the associated *KK*-classes.

4.1. ***Bimodules over the reduced crossed product.*** For a Γ-C^*-algebra B, the *reduced crossed product* $B \rtimes_r \Gamma$ is obtained as a completion of the convolution algebra $C_c(\Gamma, B)$; see, for example, [Kasparov 1995]. Let $g \in C_G(\Gamma)$ and $d := [\Gamma : \Gamma^g]$. The double coset $\Gamma g^{-1}\Gamma$ admits a decomposition as a disjoint union

$$\Gamma g^{-1}\Gamma = \bigsqcup_{i=1}^{d} g_i\Gamma, \quad g_i = \delta_i g^{-1}, \ \Gamma = \bigsqcup_{i=1}^{d} \delta_i \Gamma^g, \tag{4.2}$$

where the $\delta_i \in \Gamma$ form a complete set of coset representatives for Γ^g. We choose to work with g^{-1} in order for our formulae to be in line with those in [Mesland and Şengün 2016]. Consider the elements

$$t_i(\gamma) = t_i^g(\gamma) := g_{\gamma(i)}^{-1}\gamma g_i \in g\Gamma g^{-1},$$

where $i \mapsto \gamma(i)$ is induced by the permutation of the cosets in (4.2). From [Mesland and Şengün 2016, Lemma 2.3] we recall the relations

$$t_i(\gamma_1\gamma_2) = t_{\gamma_2(i)}(\gamma_1)t_i(\gamma_2), \qquad t_i(\gamma^{-1}) = t_{\gamma^{-1}(i)}(\gamma)^{-1},$$

which will be used in the sequel without further ado.

Let $S \subset C_G(\Gamma)$ be a subgroup containing Γ and B an S-C^*-algebra. The free right $B \rtimes_r \Gamma$-module $T_g^\Gamma \simeq (B \rtimes_r \Gamma)^d$ carries a left $B \rtimes_r \Gamma$-module structure given by

$$(t_g(f)\Psi)_i(\delta) = \sum_\gamma g_i^{-1} f(\gamma) t_i(\gamma^{-1})^{-1} \Psi_{\gamma^{-1}(i)}(t_i(\gamma^{-1})\delta). \tag{4.3}$$

Equivalently, we have the covariant representation

$$\begin{aligned}(t_g(b)\cdot\Psi)_i(\delta) &:= g_i^{-1}(b)\Psi_i(\delta),\\ (t_g(u_\gamma)\Psi)_i(\delta) &:= t_i(\gamma^{-1})^{-1}(\Psi_{\gamma^{-1}(i)}(t_i(\gamma^{-1})\delta)).\end{aligned} \tag{4.4}$$

Details of the construction, as well as the following definition, can be found in [Mesland and Şengün 2016, Section 2].

4.5. Definition. Let B be a separable S-C^*-algebra and C a separable C^*-algebra. The *Hecke operators*

$$T_g : KK_*(B \rtimes_r \Gamma, C) \to KK_*(B \rtimes_r \Gamma, C),$$
$$T_g : KK_*(C, B \rtimes_r \Gamma) \to KK_*(C, B \rtimes_r \Gamma)$$

are defined to be the Kasparov product with the class $[T_g^\Gamma] \in KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$.

We now give an equivalent description of the bimodules T_g^Γ. Consider the function space

$$C_c(\Gamma g^{-1}\Gamma, B) = \mathbb{C}[\Gamma g^{-1}\Gamma] \otimes_{\mathbb{C}}^{\mathrm{alg}} B.$$

The convolution product makes $C_c(\Gamma g^{-1}\Gamma, B)$ into a $C_c(\Gamma, B)$-bimodule:

$$f * \Psi(\xi) := \sum_{\gamma \in \Gamma} f(\gamma)\gamma(\Psi(\gamma^{-1}\xi)), \quad \Psi * f(\xi) := \sum_{\gamma \in \Gamma} \Psi(\xi\gamma)\xi f(\gamma^{-1}), \quad \xi \in \Gamma g^{-1}\Gamma.$$

Moreover, we define the inner product

$$\langle \Phi, \Psi \rangle(\delta) := \sum_{\xi \in \Gamma g^{-1}\Gamma} \xi^{-1}(\Phi(\xi)^*\Psi(\xi\delta)), \tag{4.6}$$

which makes $C_c(\Gamma g^{-1}\Gamma, B)$ into a pre-Hilbert-C^*-bimodule over $C_c(\Gamma, B)$.

4.7. Lemma. *For $g \in S \subset C_G(\Gamma)$ the map*

$$\alpha : C_c(\Gamma g^{-1}\Gamma, B) \to C_c(\Gamma, B)^d \subset T_g^\Gamma, \quad \alpha(\Psi)_i(\delta) := g_i^{-1}\Psi(g_i\delta),$$

induces a unitary isomorphism of $B \rtimes_r \Gamma$-bimodules.

Proof. The decomposition (4.2) shows that the map α has dense range. Moreover, α preserves the inner product

$$\begin{aligned}
\langle \alpha(\Psi), \alpha(\Phi) \rangle(\delta) &= \sum_i \alpha(\Psi)_i^* \alpha(\Phi)_i(\delta) = \sum_i \sum_\gamma \alpha(\Psi)_i^*(\gamma)\gamma\alpha(\Phi)_i(\gamma^{-1}\delta) \\
&= \sum_i \sum_\gamma \gamma(\alpha(\Psi)_i(\gamma^{-1})^*\alpha(\Phi)_i(\gamma^{-1}\delta)) \\
&= \sum_i \sum_\gamma \gamma g_i^{-1}(\Psi(g_i\gamma^{-1})^*\Phi(g_i\gamma^{-1}\delta)) \\
&= \sum_{\xi \in \Gamma g^{-1}\Gamma} \xi^{-1}(\Phi(\xi)^*\Psi(\xi\delta)) = \langle \Psi, \Phi \rangle(\delta),
\end{aligned}$$

from which it follows that α induces a unitary isomorphism on the C^*-module completions, which is in particular a right module map.

For the left module structure we compute

$$\begin{aligned}
\alpha(f * \Psi)_i(\delta) &= g_i^{-1}\Big(\sum_{\gamma\in\Gamma} f(\gamma)\gamma\Psi(\gamma^{-1}g_i\delta)\Big)\\
&= \sum_{\gamma\in\Gamma} g_i^{-1}f(\gamma)g_i^{-1}\gamma\Psi(g_{\gamma^{-1}(i)}t_i(\gamma^{-1})\delta)\\
&= \sum_{\gamma\in\Gamma} g_i^{-1}f(\gamma)t_i(\gamma^{-1})^{-1}g_{\gamma^{-1}(i)}^{-1}\Psi(g_{\gamma^{-1}(i)}t_i(\gamma^{-1})\delta)\\
&= \sum_{\gamma\in\Gamma} g_i^{-1}f(\gamma)t_i(\gamma^{-1})^{-1}\alpha(\Psi)_{\gamma^{-1}(i)}(t_i(\gamma^{-1})\delta)\\
&= (t_g(f))(\alpha\Psi)_i(\delta), \qquad (4.8)
\end{aligned}$$

and we are done. □

Thus, the bimodules implementing the Hecke operators are completions of the B-valued functions on the associated double coset.

4.9. ***The double-coset Hecke ring.*** Let S be a subgroup of $C_G(\Gamma)$ that contains Γ. Following Shimura, we define the *Hecke ring* $\mathbb{Z}[\Gamma, S]$ as the free abelian group on the double cosets $\Gamma g\Gamma$ with $g \in S$, equipped with the product

$$[\Gamma g^{-1}\Gamma]\cdot[\Gamma h^{-1}\Gamma] := \sum_{k=1}^{K} m_k[\Gamma g_{i(k)}h_{j(k)}\Gamma], \qquad (4.10)$$

where we have fixed finite sets I and J and coset representatives $\{g_i : i \in I\}$ and $\{h_j : j \in J\}$ for Γ^g and Γ^h in Γ, respectively. Moreover, m_k, $i(k)$ and $j(k)$ are such that $m_k := \#\{(i, j) : g_ih_j\Gamma = g_{i(k)}h_{j(k)}\Gamma\}$, and

$$\Gamma g^{-1}\Gamma h^{-1}\Gamma = \bigsqcup_{k=1}^{K} \Gamma g_{i(k)}h_{j(k)}\Gamma \qquad (4.11)$$

is a disjoint union. For well-definedness and other details of the construction we refer to [Shimura 1971, Chapter 3]. We wish to show that, for an arbitrary S-C^*-algebra B, the map

$$T : \mathbb{Z}[\Gamma, S] \to KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma), \quad [\Gamma g^{-1}\Gamma] \mapsto T_g^{\Gamma} \qquad (4.12)$$

is a ring homomorphism. To this end, we introduce the following notions. By a *bi-Γ-set* we mean a set V that carries both a left and a right Γ-action, and the actions commute in the sense that for all $\gamma, \delta \in \Gamma$ and $v \in V$ we have $\gamma(v\delta) = (\gamma v)\delta$.

The Γ-*product* of a pair (V, W) of bi-Γ-sets is the quotient of the Cartesian product $V \times W$ by the equivalence relation

$$(v, w) \sim (v', w') \Leftrightarrow \exists \gamma \in \Gamma \quad v' = v\gamma,\ w' = \gamma^{-1}w,$$

and is denoted by $V \times_\Gamma W$. The equivalence class of the pair (v, w) is denoted $[v, w]$. The Γ-product is a bi-Γ-set via the induced left and right Γ-actions

$$[v, w]\gamma := [v, w\gamma], \quad \gamma[v, w] := [\gamma v, w].$$

Let $\Gamma \subset S \subset C_G(\Gamma)$ be a subgroup and V a bi-Γ-set. We say that V is *anchored in S* if there is given a map $m : V \to S$ such that $m(\gamma v\delta) = \gamma m(v)\delta$ for all $v \in V$ and $\gamma, \delta \in \Gamma$. We refer to m as the *anchor*. Of course any double coset $\Gamma g\Gamma$ with $g \in S$ is anchored in S via the inclusion map.

4.13. Lemma. *Let V and W be bi-Γ-sets with anchor maps $m_V : V \to S$ and $m_W : W \to S$. Then their Γ-product $V \times_\Gamma W$ is anchored in S via the* product anchor $[v, w] \mapsto m_V(v)m_W(w)$.

The proof of this is straightforward. Note that if V and W are double Γ-cosets in S, anchored via their embeddings into S, then the product anchor of $V \times_\Gamma W$ need not be injective.

We wish to relate the anchored bi-Γ-sets $\Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma$ and $\bigsqcup_{k=1}^{K} \bigsqcup_{\ell=1}^{m_k} \Gamma z_k\Gamma$. By virtue of (4.11) we fix, once and for all, for each z_k and $1 \leqslant \ell \leqslant m_k$ a choice of *distinct* indices $i(k, \ell)$, $j(k, \ell)$ such that $z_k\Gamma = g_{i(k,\ell)}h_{j(k,\ell)}\Gamma$. We thus write $z_{(k,\ell)} = g_{i(k,\ell)}h_{j(k,\ell)}$. Consider the left action of Γ on the finite set $I \times J$ given by

$$\gamma(i, j) := (\gamma(i), t_i^g(\gamma)(j)). \tag{4.14}$$

4.15. Lemma. *With the above choices, the map*

$$\omega : \bigsqcup_{k=1}^{K} \bigsqcup_{\ell=1}^{m_k} \Gamma z_{(k,\ell)}\Gamma \to \Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma, \quad \gamma z_{(k,\ell)}\delta \mapsto [\gamma g_{i(k,\ell)}, h_{j(k,\ell)}\delta],$$

where $i = i(k, \ell)$ and $j = j(k, \ell)$, is a Γ-bi-equivariant bijection of S-anchored bi-Γ-sets.

Proof. By construction, ω is Γ-bi-equivariant and respects the anchors. We need only show that it is bijective. This is achieved as follows: For each k choose

$$\gamma_1^k = 1, \gamma_2^k, \ldots, \gamma_{d_k}^k \in \Gamma, \quad \text{with } \Gamma z_k\Gamma = \bigsqcup_{n=1}^{d_k} \gamma_n^k z_k\Gamma.$$

We thus have

$$\bigsqcup_{k=1}^{K} \bigsqcup_{\ell=1}^{m_k} \Gamma z_{(k,\ell)}\Gamma = \bigsqcup_{k=1}^{K} \bigsqcup_{\ell=1}^{m_k} \bigsqcup_{n=1}^{d_k} \gamma_n^k g_{i(k,\ell)}h_{j(k,\ell)}\Gamma. \tag{4.16}$$

The identities

$$[g_i\gamma, h_j\delta] = [g_i, \gamma h_j\delta] = [g_i, h_{\gamma(j)}t_j^h(\gamma)\delta]$$

show that every element in the Γ-product $\Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma$ has a representative of the form $[g_i, h_j\gamma]$ and such representatives are unique because g_i and h_j form a complete set of coset representatives. We so obtain a set bijection

$$\Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma \to \bigsqcup_{(i,j)\in I\times J} \{g_i\} \times h_j\Gamma, \quad [g_i\gamma, h_j\delta] \mapsto [g_i, h_{\gamma(j)} t_h^j(\gamma)\delta].$$

It follows that ω restricts to bijections

$$\omega : \gamma_n^k g_{i(k,\ell)} h_{j(k,\ell)}\Gamma \to \{g_{\gamma_n^k(i(k,\ell))}\} \times h_{t_i^g(\gamma_n^k)(j(k,\ell))}\Gamma.$$

Therefore it suffices to show that the map

$$N \times K \times L \to I \times J, \quad (n, k, \ell) \mapsto \gamma_n^k(i(k, \ell), j(k, \ell))$$

is bijective. By [Shimura 1971, Proposition 3.2] it holds that

$$\sum_{k=1}^{K} m_k d_k = |I||J| = |I \times J|,$$

and thus we need only show that this map is injective, and then use a counting argument to obtain surjectivity. To this end we prove that the equality

$$\gamma_n^k(i(k, \ell), j(k, \ell)) = \gamma_{n'}^{k'}(i(k', \ell'), j(k', \ell')) \tag{4.17}$$

implies that $(n, k, \ell) = (n', k', \ell')$. By (4.14), (4.17) implies that

$$\gamma_n^k g_{i(k,\ell)} h_{j(k,\ell)}\Gamma = \gamma_{n'}^{k'} g_{i(k',\ell')} h_{j(k',\ell')}\Gamma,$$

and thus

$$\Gamma g_{i(k,\ell)} h_{j(k,\ell)}\Gamma = \Gamma g_{i(k',\ell')} h_{j(k',\ell')}\Gamma.$$

This in turn implies that $k = k'$ and thus $\gamma_n^k z_k\Gamma = \gamma_{n'}^k z_k\Gamma$, so it follows that $n = n'$. Lastly, we are left with $\gamma_n^k(i(k, \ell)) = \gamma_n^k(i(k, \ell'))$, so $i(k, \ell) = i(k, \ell')$, which by construction implies $\ell = \ell'$. This shows that the map $(n, k, \ell) \mapsto \gamma_n^k(i(k, \ell), j(k, \ell))$ is injective. □

Now let V be a Γ-set with anchor $m : V \to S$ and X a S-(A, B)-bimodule. We always consider V as a discrete set. We equip $C_c(V, X)$ with a $C_c(\Gamma, B)$-valued inner product via

$$\langle \Phi, \Psi\rangle(\delta) := \sum_{v\in V} m(v)^{-1}\langle \Phi(v), \Psi(v\delta)\rangle$$

and left and right module structures via the Γ-action

$$f * \Psi(v) := \sum_{\gamma} f(\gamma)\gamma\Psi(\gamma^{-1}v), \qquad \Psi * f(v) := \sum_{\gamma} \Psi(v\gamma)m(v\gamma)f(\gamma^{-1}).$$

Thus the completion gives a C^*-$(A \rtimes_r \Gamma, B \rtimes_r \Gamma)$-bimodule. Note that if $u : X \to Y$ is an S-equivariant unitary bimodule isomorphism and $\omega : W \to V$ an isomorphism of S-anchored bi-Γ-sets, then

$$C_c(V, X) \to C_c(W, Y), \quad \Psi \mapsto u \circ \Psi \circ \omega$$

is a unitary bimodule isomorphism.

By Lemma 4.7, the bimodule T_g^Γ for $g \in S$ is isomorphic to the completion of $C_c(\Gamma g^{-1}\Gamma, B)$ with anchor $m : \Gamma g^{-1}\Gamma \to S$ the set inclusion, and is thus a special case of the above construction. The formalism of anchored bi-Γ-sets allows for an elegant description of tensor products of their associated modules.

4.18. Proposition. *Let $S \subset C_G(\Gamma)$ be a subgroup and A, B and C be S-C^*-algebras. Let V, W be S-anchored bi-Γ-sets, X an (A, B)-S-bimodule and Y a (B, C)-S-bimodule. Then the map*

$$\alpha : C_c(V, X) \otimes^{\mathrm{alg}}_{C_c(\Gamma, B)} C_c(W, Y) \to C_c(V \times_\Gamma W, X \otimes_B Y),$$

given by

$$\alpha(\Phi \otimes \Psi)[v, w] := \sum_\gamma \Phi(v\gamma) \otimes m(v)\gamma\Psi(\gamma^{-1}w),$$

is an inner product preserving map of $(C_c(\Gamma, A), C_c(\Gamma, C))$-bimodules with dense range. Consequently their respective C^*-module completions are unitarily isomorphic $(A \times_r \Gamma, C \rtimes_r \Gamma)$-bimodules.

Proof. The following calculation shows that α is unitary:

$$\begin{aligned}
&\langle \alpha(\Phi \otimes \Psi), \alpha(\Phi \otimes \Psi)\rangle(\delta) \\
&\quad= \sum_{[v,w]} m(w)^{-1} m(v)^{-1} \langle \alpha(\Phi \otimes \Psi)(v, w), \alpha(\Phi \otimes \Psi)(v, w\delta)\rangle \\
&\quad= \sum_{[v,w]} \sum_{\gamma,\varepsilon} m(w)^{-1} m(v)^{-1} \big\langle m(v)\gamma\Psi(\gamma^{-1}w), \langle \Phi(v\gamma), \Phi(v\varepsilon)\rangle m(v)\varepsilon\Psi(\varepsilon^{-1}w\delta)\big\rangle \\
&\quad= \sum_{[v,w]} \sum_{\gamma,\varepsilon} m(w)^{-1} \big\langle \gamma\Psi(\gamma^{-1}w), m(v)^{-1}\big(\langle \Phi(v\gamma), \Phi(v\varepsilon)\rangle\big)\varepsilon\Psi(\varepsilon^{-1}w\delta)\big\rangle \\
&\quad= \sum_{[v,w]} \sum_{\gamma,\varepsilon} m(\gamma^{-1}w)^{-1} \big\langle \Psi(\gamma^{-1}w), m(v\gamma)^{-1}\big(\langle \Phi(v\gamma), \Phi(v\varepsilon)\rangle\big)\gamma^{-1}\varepsilon\Psi(\varepsilon^{-1}w\delta)\big\rangle \\
&\quad= \sum_{[v,w]} \sum_{\gamma,\varepsilon} m(\gamma^{-1}w)^{-1} \big\langle \Psi(\gamma^{-1}w), m(v\gamma)^{-1}\big(\langle \Phi(v\gamma), \Phi(v\gamma\varepsilon)\rangle\big)\varepsilon\Psi(\varepsilon^{-1}\gamma^{-1}w\delta)\big\rangle.
\end{aligned}$$

By virtue of the equivalence relation on $V \times W$ we can replace the sum over equivalence classes $[v, w] \in V \times_\Gamma W$ and elements $\gamma \in \Gamma$ by a sum over $(v, w) \in V \times W$, and continue the calculation:

$$
\begin{aligned}
&= \sum_{v\in V}\sum_{w\in W}\sum_{\varepsilon} m(w)^{-1}\big\langle \Psi(w), m(v)^{-1}\big(\langle \Phi(v), \Phi(v\varepsilon)\rangle\big)\varepsilon\Psi(\varepsilon^{-1}w\delta)\big\rangle \\
&= \sum_{w}\sum_{\varepsilon} m(w)^{-1}\big\langle \Psi(w), \langle \Phi, \Phi\rangle(\varepsilon)\varepsilon\Psi(\varepsilon^{-1}w\delta)\big\rangle \\
&= \sum_{w} m(w)^{-1}\big\langle \Psi(w), \langle \Phi, \Phi\rangle * \Psi(w\delta)\big\rangle \\
&= \big\langle \Psi, \langle \Phi, \Phi\rangle\Psi\big\rangle(\delta).
\end{aligned}
$$

It is straightforward to establish that α is a bimodule map:

$$
\begin{aligned}
\alpha(f * \Phi \otimes \Psi)[v, w] &= \sum_{\gamma} (f * \Phi)(v\gamma) \otimes m(v)\gamma\Psi(\gamma^{-1}w) \\
&= \sum_{\gamma,\varepsilon} f(\varepsilon)\varepsilon\Phi(\varepsilon^{-1}v\gamma) \otimes m(v)\gamma\Psi(\gamma^{-1}w) \\
&= \sum_{\varepsilon} f(\varepsilon)\varepsilon\alpha(\Phi \otimes \Psi)[\varepsilon^{-1}v, w] = f * \alpha(\Phi \otimes \Psi)[v, w],
\end{aligned}
$$

$$
\begin{aligned}
\alpha(\Phi \otimes \Psi * f)[v, w] &= \sum_{\gamma} \Phi(v\gamma) \otimes m(v)\gamma(\Psi * f)(\gamma^{-1}w) \\
&= \sum_{\gamma,\varepsilon} \Phi(v\gamma) \otimes m(v)\gamma(\Psi(\gamma^{-1}w\varepsilon)m(\gamma^{-1}w\varepsilon)f(\varepsilon^{-1})) \\
&= \sum_{\gamma,\varepsilon} \Phi(v\gamma) \otimes m(v)\gamma\Psi(\gamma^{-1}w\varepsilon)m(w\varepsilon)f(\varepsilon^{-1}) \\
&= \sum_{\varepsilon} \alpha(\Phi \otimes \Psi)[v, w\varepsilon]m(w\varepsilon)f(\varepsilon^{-1}) \\
&= \alpha(\Phi \otimes \Psi) * f[v, w].
\end{aligned}
$$

Lastly, to see that α has dense range, denote by $\delta_v : V \to \mathbb{C}$ the indicator function at the element $v \in V$. The functions

$$\chi^{[v,w]}_{x\otimes y}(v', w') := \delta_v(v')\delta_w(w')x \otimes y,$$

with $v \in V$, $w \in W$, $x \in X$ and $y \in Y$, span a dense right $C_c(\Gamma, C)$-submodule. Now set

$$
\begin{aligned}
e^v_x(v') &:= \delta_v(v')x, \\
f^{(v,w)}_y(w') &:= \delta_w(w')m(v)^{-1}(y).
\end{aligned}
$$

Then it is easily verified that $\alpha(e^v_i \otimes f^{(v,w)}_y) = \chi^{[v,w]}_{x\otimes y}$, so α has dense range. This proves the proposition. □

4.19. Theorem. *For any $g, h \in C_G(\Gamma)$ there is a unitary isomorphism of bimodules*

$$T_g^\Gamma \otimes_{B \rtimes_r \Gamma} T_h^\Gamma \xrightarrow{\sim} \bigoplus_{k=1}^{K} \big(T^\Gamma_{(g_{i(k)} h_{j(k)})^{-1}}\big)^{\oplus m_k}.$$

Consequently, for any S-C^*-algebra B, the map $T : [\Gamma g^{-1}\Gamma] \mapsto [T_g^\Gamma]$ extends to a ring homomorphism

$$T : \mathbb{Z}[\Gamma, S] \to KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma).$$

Proof. By Lemma 4.7, the modules T_g^Γ and T_h^Γ are unitarily isomorphic to those associated to the anchored bi-Γ-sets $\Gamma g^{-1}\Gamma$ and $\Gamma h^{-1}\Gamma$. By Proposition 4.18, their tensor product is given by

$$C_c(\Gamma g^{-1}\Gamma, B) \otimes^{\mathrm{alg}}_{C_c(\Gamma, B)} C_c(\Gamma h^{-1}\Gamma, B) \xrightarrow{\sim} C_c(\Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma, B \otimes_B B).$$

Since $B \otimes_B B \simeq B$ as S-modules and by Lemma 4.15, there is an isomorphism of anchored bi-Γ-sets

$$\Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma \simeq \bigsqcup_{k=1}^{K} \bigsqcup_{\ell=1}^{m_k} \Gamma z_{(k,\ell)} \Gamma.$$

Taking completions, we obtain the unitary bimodule isomorphism

$$T_g^\Gamma \otimes_{B \rtimes_r \Gamma} T_h^\Gamma \xrightarrow{\sim} \bigoplus_{k=1}^{K} \bigoplus_{\ell=1}^{m_k} \big(T^\Gamma_{z^{-1}_{(k,\ell)}}\big).$$

The definition of addition in KK-theory then yields

$$\begin{aligned} T[\Gamma g^{-1}\Gamma] \otimes T[\Gamma h^{-1}\Gamma] \\ &= [T_g^\Gamma] \otimes [T_h^\Gamma] = \sum_{k=1}^{K} \sum_{\ell=1}^{m_k} [T^\Gamma_{z^{-1}_{(k,\ell)}}] = \sum_{k=1}^{K} \sum_{\ell=1}^{m_k} T[\Gamma z_{(k,\ell)} \Gamma] \\ &= \sum_{k=1}^{K} m_k T[\Gamma z_k \Gamma] = T\Big(\sum_{k=1}^{K} m_k [\Gamma z_k \Gamma]\Big) = T([\Gamma g^{-1}\Gamma] \cdot [\Gamma h^{-1}\Gamma]), \end{aligned}$$

showing that $[\Gamma g^{-1}\Gamma] \mapsto [T_g^\Gamma]$ is a ring homomorphism. □

We define $\mathscr{H}_B(\Gamma, S)$ to be the subring of $KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$ generated by T_g^Γ for $g \in C_G(\Gamma)$. The following corollary is now obvious.

4.20. Corollary. *If $\mathbb{Z}[\Gamma, S]$ is commutative, then $\mathscr{H}_B(\Gamma, S)$ is commutative.*

Similarly write $\mathscr{H}_M(S)$ for the subring of $KK_0(C_0(M), C_0(M))$ generated by the classes of the correspondences $M \xleftarrow{s} M_g \xrightarrow{t} M$ with $g \in S$.

4.21. Corollary. *Let X be an S-space on which Γ acts freely and properly with quotient $M := X/\Gamma$. The map $[\Gamma g^{-1}\Gamma] \mapsto [M \xleftarrow{t} M_g \xrightarrow{s} M]$ defines a ring homomorphism*

$$\mathbb{Z}[\Gamma, S] \to KK_0(C_0(M), C_0(M)).$$

In particular, the double-coset product $[\Gamma g^{-1}\Gamma] \cdot [\Gamma h^{-1}\Gamma]$ corresponds to the class of the composition of correspondences $[M \xleftarrow{s_g} M_g \,{}_{t_g}\!\times_{s_h} M_h \xrightarrow{t_h} M]$ and there is an isomorphism $\mathscr{H}_M(S) \simeq \mathscr{H}_{C_0(X)}(\Gamma, S)$.

Proof. By [Mesland and Şengün 2016, Proposition 3.8] the Morita equivalence isomorphism

$$KK_0(C_0(X) \rtimes \Gamma, C_0(X) \rtimes \Gamma) \to KK_0(C_0(M), C_0(M))$$

maps T_g^Γ to $T_g^M = [M \xleftarrow{s_g} M_g \xrightarrow{t_g} M]$. Thus the above map is the composition

$$\mathbb{Z}[\Gamma, S] \to KK_0(C_0(X) \rtimes \Gamma, C_0(X) \rtimes \Gamma) \to KK_0(C_0(M), C_0(M)),$$

whence a homomorphism. The last statement follows from [Connes and Skandalis 1984, Theorem 3.2]. Clearly $\mathscr{H}_M(S) \simeq \mathscr{H}_{C_0(X)}(\Gamma, S)$ under this isomorphism. ☐

4.22. Remark. Corollary 4.21 is the KK-theoretic analogue of the well-known fact that the double-coset Hecke ring can be interpreted in terms of (topological) correspondences, where the double-coset multiplication simply becomes composition of correspondences [Shimura 1971, Chapter 7].

5. Hecke equivariant exact sequences

As before, let S be a group such that $\Gamma \subset S \subset C_G(\Gamma)$. In this section we prove the following general result. For S-algebras A and B, and any element $[x]$ of $KK_i^S(A, B)$ we have that

$$[T_g^{A\rtimes_r\Gamma}] \otimes j_\Gamma([x]) = j_\Gamma([x]) \otimes [T_g^{B\rtimes_r\Gamma}] \in KK_i(A \rtimes_r \Gamma, B \rtimes_r \Gamma).$$

Here j_Γ denotes the Kasparov descent map [1988; 1995]

$$j_\Gamma : KK_*^S(A, B) \to KK_*^\Gamma(A, B) \to KK_*(A \rtimes_r \Gamma, B \rtimes_r \Gamma),$$

and we have written $T_g^{A\rtimes_r\Gamma}$ for T_g^Γ to emphasize the change of coefficient algebra. This result implies that for any S-equivariant semisplit extension

$$0 \to I \to A \to B \to 0$$

of C^*-algebras that is Γ-exact in the sense that

$$0 \to I \rtimes_r \Gamma \to A \rtimes_r \Gamma \to B \rtimes_r \Gamma \to 0$$

is exact, the long exact sequences in both variables of the KK-bifunctor are Hecke equivariant. In particular, we obtain Hecke equivariant exact sequences in K-theory and K-homology for various compactifications associated with locally symmetric spaces.

5.1. ***The descent theorem.*** Kasparov's descent construction associates to a Γ-equivariant C^*-B-module X a C^*-module $X \rtimes_r \Gamma$ over $B \rtimes_r \Gamma$ [Kasparov 1980; 1988; 1995]. To an S-equivariant C^*-module X and a double coset $\Gamma g^{-1}\Gamma$, with $g \in S$, we associate the $(C_c(\Gamma, A), C_c(\Gamma, B))$-bimodule

$$C_c(\Gamma g^{-1}\Gamma, X) = \mathbb{C}[\Gamma g^{-1}\Gamma] \otimes^{\mathrm{alg}}_{\mathbb{C}} X;$$

see Section 4.1. We denote the C^*-module completion so obtained by $T_g^{X \rtimes_r \Gamma}$. The following lemma is an application of Proposition 4.18.

5.2. Lemma. *Let A and B be S-C^*-algebras. Suppose that X is an S-equivariant right C^*-module over B and $\pi : A \to \mathrm{End}^*_B(X)$ an S-equivariant essential $*$-homomorphism. For every $g \in S$, there are inner product preserving bimodule homomorphisms*

$$\begin{aligned}
&C_c(\Gamma g^{-1}\Gamma, A) \otimes^{\mathrm{alg}}_{C_c(\Gamma, A)} C_c(\Gamma, X) \\
&\qquad \xrightarrow{\sim} C_c(\Gamma g^{-1}\Gamma, X) \xleftarrow{\sim} C_c(\Gamma, X) \otimes^{\mathrm{alg}}_{C_c(\Gamma, B)} C_c(\Gamma g^{-1}\Gamma, B)
\end{aligned} \tag{5.3}$$

of $(C_c(\Gamma, A), C_c(\Gamma, B))$-bimodules with dense range. Consequently the respective C^*-module completions are unitarily isomorphic $(A \rtimes_r \Gamma, B \rtimes_r \Gamma)$-bimodules.

From the identifications

$$\Gamma g^{-1}\Gamma \times_\Gamma \Gamma \simeq \Gamma g^{-1}\Gamma \simeq \Gamma \times_\Gamma \Gamma g^{-1}\Gamma$$

given by the multiplication maps and the S-equivariant isomorphisms

$$X \simeq A \otimes_A X \simeq X \otimes_B B,$$

coming from the bimodule structure we obtain the explicit from of the isomorphisms in (5.3):

$$\begin{aligned}
\alpha : C_c(\Gamma g^{-1}\Gamma, A) \otimes^{\mathrm{alg}}_{C_c(\Gamma, A)} C_c(\Gamma, X) &\to C_c(\Gamma g^{-1}\Gamma, X), \\
\alpha(\Psi \otimes \Phi)(\xi) &:= \sum_{\gamma \in \Gamma} \Psi(\xi\gamma) \cdot \xi\gamma\Phi(\gamma^{-1}), \\
\beta : C_c(\Gamma, X) \otimes^{\mathrm{alg}}_{C_c(\Gamma, B)} C_c(\Gamma g^{-1}\Gamma, B) &\to C_c(\Gamma g^{-1}\Gamma, X), \\
\beta(\Phi \otimes \Psi)(\xi) &:= \sum_{\gamma \in \Gamma} \Phi(\gamma) \cdot \gamma\Psi(\gamma^{-1}\xi).
\end{aligned}$$

As before, the elements g_i are such that $\Gamma g^{-1}\Gamma = \bigsqcup_{i=1}^d g_i\Gamma$. We construct from them the following operators.

5.4. Lemma. *The operator*

$$v_i : C_c(\Gamma, X) \to C_c(g_i\Gamma, X) \subset C_c(\Gamma g^{-1}\Gamma, X), \quad (v_i\Phi)(g_i\xi) := g_i\Phi(\xi),$$

extends to an adjointable isometry $X \rtimes_r \Gamma \to T_g^{X\rtimes\Gamma}$ *with adjoint given by*

$$(v_i)^*\Psi(\xi) := g_i^{-1}\Psi(g_i\xi).$$

Proof. The formula for the adjoint is easily verified. It follows that $(v_i)^*v_i = 1$ on $C_c(\Gamma, X)$, so v_i is isometric. The composition $v_iv_i^* = p_i$, the projection onto $C_c(g_i\Gamma, X)$, which is bounded as well. □

5.5. Theorem. *Let* (X, D) *be an S-equivariant left-essential unbounded Kasparov module of parity* j *and let* $g \in S$. *Then we have an equality*

$$j_\Gamma([(X, D)]) \otimes [T_g] = [T_g] \otimes j_\Gamma([(X, D)]) \in KK_j(A \rtimes_r \Gamma, B \rtimes_r \Gamma).$$

Proof. By Lemma 5.2 we have bimodule isomorphisms

$$(X \rtimes_r \Gamma) \otimes_{B\rtimes_r\Gamma} T_g^{B\rtimes_r\Gamma} \xrightarrow{\beta} T_g^{X\rtimes_r\Gamma} \xleftarrow{\alpha} T_g^{A\rtimes_r\Gamma} \otimes_{A\rtimes_r\Gamma} (X \rtimes_r \Gamma).$$

Define an operator $\widehat{D}$ on the dense submodule

$$C_c(\Gamma g^{-1}\Gamma, \operatorname{Dom} D) \subset T_g^{X\rtimes_r\Gamma}$$

via

$$(\widehat{D}\Upsilon)(\xi) := D(\Upsilon(\xi)).$$

Then $\widehat{D}\beta = \beta(D \otimes 1)$ and hence $\widehat{D}$ is essentially self-adjoint and regular, and has locally compact resolvent. We wish to show that $\widehat{D}$ represents the Kasparov product of $T_g^{A\rtimes_r\Gamma}$ and (X, D), under the isomorphism α. To this end we need to verify conditions 1–3 of [Kucerovsky 1997, Theorem 13]. Because the module $T_g^{A\rtimes_r\Gamma}$ carries the zero operator, only the connection condition 1 needs argument.

Let $\mathscr{A}$ denote the dense subalgebra of A such that $[D, a]$ is bounded for $a \in \mathscr{A}$. Then, for $\Psi \in C_c(\Gamma g^{-1}\Gamma, \mathscr{A})$, $\xi \in \Gamma$ and a fixed element g_i we have

$$\begin{aligned}
&\widehat{D}\alpha(\Psi \otimes \Phi)(g_i\xi) - \alpha(\Psi \otimes D\Phi)(g_i\xi)\\
&\qquad= \sum_{\gamma\in\Gamma} D\Psi(g_i\gamma)\cdot g_i\gamma\Phi(\gamma^{-1}\xi) - \Psi(g_i\gamma)g_i\gamma D\Phi(\gamma^{-1}\xi)\\
&\qquad= \sum_{\gamma\in\Gamma}\big([D, \Psi(g_i\gamma)] - \Psi(g_i\gamma)(D - g_i\gamma D\gamma^{-1}g_i^{-1})\big)g_i\gamma\Phi(\gamma^{-1}\xi)\\
&\qquad= g_i\bigg(\sum_{\gamma\in\Gamma} g_i^{-1}\big([D, \Psi(g_i\gamma)] - \Psi(g_i\gamma)(D - g_i\gamma D\gamma^{-1}g_i^{-1})\big)\gamma\Phi(\gamma^{-1}\xi)\bigg)\\
&\qquad= v_i(C^i_\Psi * \Phi)(g_i\xi).
\end{aligned}$$

Here C^i_Ψ denotes the map

$$C^i_\Psi : \Gamma \to \mathrm{End}^*_B(X), \quad \gamma \mapsto g_i^{-1}\big([D, \Psi(g_i\gamma)] - \Psi(g_i\gamma)(D - g_i\gamma D\gamma^{-1}g_i^{-1})\big),$$

which is of finite support since Ψ is. Such maps define adjointable operators on $C_c(\Gamma, X)$ via the convolution action. Writing $|\Psi\rangle : \Phi \to \Psi \otimes \Phi$ we have

$$\widehat{D}\alpha|\Psi\rangle - \alpha|\Psi\rangle D = \sum_{i=1}^{d} v^i \circ C^i_\Psi : X \rtimes_r \Gamma \to T_g^{X\rtimes_r\Gamma},$$

which defines a bounded adjointable operator. Thus $\widehat{D}$ satisfies Kucerovsky's connection condition as desired. □

5.6. Corollary. *For any $\alpha \in KK^S_j(A, B)$ and any separable C^*-algebra C, the induced maps*

$$\alpha_* : KK_i(C, A \rtimes_r \Gamma) \to KK_{i+j}(C, B \rtimes_r \Gamma),$$
$$\alpha^* : KK_i(B \rtimes_r \Gamma, C) \to KK_{i+j}(A \rtimes_r \Gamma, C)$$

are Hecke equivariant. In fact we can replace $KK(C, \cdot\,)$ and $KK(\,\cdot\,, C)$ by any co- or contravariant functor which is homotopy invariant, split exact and stable.

5.7. *Extensions and Hecke equivariant exact sequences.* The paper [Thomsen 2000] establishes, for any locally compact group G, an isomorphism

$$KK^G_1(A, B) \xrightarrow{\sim} \mathrm{Ext}^G(A \otimes \mathbb{K}_G, B \otimes \mathbb{K}_G),$$

where $\mathbb{K}_G \simeq \mathbb{K}(L^2(G \times \mathbb{N}))$. A G-equivariant semisplit extension

$$0 \to B \to E \to A \to 0$$

induces a G-equivariant semisplit extension

$$0 \to B \otimes \mathbb{K}_G \to E \otimes \mathbb{K}_G \to A \otimes \mathbb{K}_G \to 0,$$

and thus an element in $KK^G_1(A, B)$.

5.8. Theorem. *Let G be a locally compact group, $\Gamma \subset G$ a discrete subgroup, $C_G(\Gamma) \subset G$ its commensurator and S a group with $\Gamma \subset S \subset C_G(\Gamma)$. For any Γ-exact and S-equivariant semisplit extension*

$$0 \to B \to E \to A \to 0$$

of separable S-algebras and any separable C^*-algebra C, the exact sequences

$$\cdots \to KK_i(C, B \rtimes_r \Gamma) \to KK_i(C, E \rtimes_r \Gamma) \to KK_i(C, A \rtimes_r \Gamma) \to \cdots, \tag{5.9}$$
$$\cdots \to KK_i(A \rtimes_r \Gamma, C) \to KK_i(E \rtimes_r \Gamma, C) \to KK_i(B \rtimes_r \Gamma, C) \to \cdots \tag{5.10}$$

are $\mathbb{Z}[\Gamma, S]$-equivariant.

Proof. Exactness of Γ implies that we obtain a semisplit extension

$$0 \to B \rtimes_r \Gamma \to E \rtimes_r \Gamma \to A \rtimes_r \Gamma \to 0, \tag{5.11}$$

yielding the exact sequences (5.9) and (5.10). By Theorem 4.19 all groups in these exact sequences are Hecke modules. In sequence (5.9), the maps

$$KK_i(C, B \rtimes_r \Gamma) \to KK_i(C, E \rtimes_r \Gamma), \quad KK_i(C, E \rtimes_r \Gamma) \to KK_i(C, A \rtimes_r \Gamma)$$

are induced by elements in $KK_0(B \rtimes_r \Gamma, E \rtimes_r \Gamma)$ and $KK_0(A \rtimes_r \Gamma, E \rtimes_r \Gamma)$, respectively. These elements are in the image of the descent maps

$$KK_0^S(B, E) \to KK_0^\Gamma(B, E) \to KK_0(B \rtimes_r \Gamma, E \rtimes_r \Gamma),$$
$$KK_0^S(E, A) \to KK_0^\Gamma(E, A) \to KK_0(E \rtimes_r \Gamma, A \rtimes_r \Gamma),$$

and thus are Hecke equivariant by Theorem 5.5. Since the extension (5.11) is semisplit it defines a class $[\mathrm{Ext}] \in KK_1^S(A, B)$. The boundary maps in the exact sequence (5.9) are implemented by an element $\partial \in KK_1(A \rtimes_r \Gamma, B \rtimes_r \Gamma)$, and this element is the image of $[\mathrm{Ext}]$ under the composition

$$KK_1^S(A, B) \to KK_1^\Gamma(A, B) \to KK_1(A \rtimes_r \Gamma, B \rtimes_r \Gamma).$$

Thus by Theorem 5.5 the boundary maps in the sequence (5.9) are Hecke equivariant. The argument for sequence (5.10) is similar. □

Interesting examples of S-equivariant extensions come from partial compactifications of G-spaces. Let X be a locally compact space with a G-action. A partial S-compactification is an S-space $\overline{X}$ which contains X as an open dense subset. We write $\partial X := \overline{X} \setminus X$ and we obtain the S-equivariant exact sequence

$$0 \to C_0(X) \to C_0(\overline{X}) \to C_0(\partial X) \to 0.$$

5.12. Example. Let $G = \mathrm{Isom}(\boldsymbol{H})$, where $\boldsymbol{H}$ is the real hyperbolic n-space. The geodesic compactification $\overline{\boldsymbol{H}}$ of $\boldsymbol{H}$ is a G-compactification and thus, it is an S-compactification for any lattice $\Gamma \subset G$ and subgroup $\Gamma \subset S \subset C_G(\Gamma)$. The associated Hecke equivariant exact sequence in K-homology has been studied extensively in [Mesland and Şengün 2016]. For torsion-free Γ and $M := X/\Gamma$, there is a Morita equivalence $C_0(M) \sim C_0(X) \rtimes_r \Gamma$, and a KK-equivalence $C(\overline{\boldsymbol{H}}) \rtimes_r \Gamma \sim C_r^*(\Gamma)$. The exact sequence takes the form

$$\cdots \to K_*(C_0(M)) \to K_*(C_r^*(\Gamma)) \to K_*(C(\partial \boldsymbol{H}) \rtimes_r \Gamma) \to \cdots,$$

as in [Emerson and Meyer 2006; Emerson and Nica 2016].

5.13. Example. Let G be the group of real points of a reductive algebraic group $\boldsymbol{G}$ over $\mathbb{Q}$ and let X be its associated global symmetric space. The Borel–Serre partial compactification $\overline{X}$ of X is a $\boldsymbol{G}(\mathbb{Q})$-compactification but not a G-compactification;

see [Borel and Serre 1973]. However if $\Gamma \subset \boldsymbol{G}(\mathbb{Q})$ is an arithmetic subgroup, then $C_G(\Gamma) = \boldsymbol{G}(\mathbb{Q})$. So $\overline{X}$ is a $C_G(\Gamma)$-compactification. The action of Γ on $\overline{X}$ is cocompact and continues to be proper. Writing $M := X/\Gamma$ for torsion-free Γ, we obtain the Borel–Serre compactification $\overline{M} := \overline{X}/\Gamma$ of M and its boundary $\partial\overline{M} := \partial X/\Gamma$. There are Morita equivalences

$$C_0(X) \rtimes_r \Gamma \sim C_0(M), \quad C_0(\overline{X}) \rtimes_r \Gamma \sim C_0(\overline{M}), \quad C_0(\partial X) \rtimes_r \Gamma \sim C_0(\partial\overline{M}).$$

The exact sequence thus reduces to the topological K-theory sequence

$$\cdots \to K^*(M) \to K^*(\overline{M}) \to K^*(\partial\overline{M}) \to \cdots$$

of the pair $(\overline{M}, \partial\overline{M})$.

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The slice spectral sequence for singular schemes and applications

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We examine the slice spectral sequence for the cohomology of singular schemes with respect to various motivic T-spectra, especially the motivic cobordism spectrum. When the base field k admits resolution of singularities and X is a scheme of finite type over k, we show that Voevodsky's slice filtration leads to a spectral sequence for MGL_X whose terms are the motivic cohomology groups of X defined using the cdh-hypercohomology. As a consequence, we establish an isomorphism between certain geometric parts of the motivic cobordism and motivic cohomology of X.

A similar spectral sequence for the connective K-theory leads to a cycle class map from the motivic cohomology to the homotopy invariant K-theory of X. We show that this cycle class map is injective for a large class of projective schemes. We also deduce applications to the torsion in the motivic cohomology of singular schemes.

1. Introduction

The motivic homotopy theory of schemes was put on a firm foundation by Voevodsky and his coauthors beginning with the work of Morel and Voevodsky [1999] and

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its stable counterpart [Voevodsky 1998]. It was observed by Voevodsky [2002b] that the motivic T-spectra in the stable homotopy category $\mathcal{SH}_X$ over a noetherian scheme X of finite Krull dimension can be understood via their slice filtration. This slice filtration leads to spectral sequences, which then become a very powerful tool in computing various cohomology theories for smooth schemes over X.

The main problem in the study of the slice filtration for a given motivic T-spectrum is twofold: the identification of its slices and the analysis of the convergence properties for the corresponding slice spectral sequence. When k is a field which admits resolution of singularities, the slices for many of these motivic T-spectra in $\mathcal{SH}_k$ are now known. In particular, we can compute these generalized cohomology groups of smooth schemes over k using the slice spectral sequence.

In this paper, we study a descent property of the motivic T-spectra in $\mathcal{SH}_X$ when X is a possibly singular scheme of finite type over k. This descent property tells us that the cohomology groups of a scheme $Y \in \mathbf{Sm}_X$, associated to an absolute motivic T-spectra in $\mathcal{SH}_X$ [Déglise 2014, §1.2], can be computed using only $\mathcal{SH}_k$.

Even though our methods apply to any of these absolute T-spectra, we restrict our study to the motivic cobordism spectrum MGL_X. We show using the above descent property of motivic spectra that MGL_X can be computed using the motivic cohomology groups of X. Recall from [Friedlander and Voevodsky 2000, Definitions 4.3 and 9.2] that the motivic cohomology groups of X are defined to be the cdh-hypercohomology groups $H^p(X, \mathbb{Z}(q)) = \mathbb{H}^{p-2q}_{\mathrm{cdh}}(X, C_* z_{\mathrm{equi}}(\mathbb{A}^q_k, 0)_{\mathrm{cdh}})$. Using these motivic cohomology groups, we show the following:

Theorem 1.1. *Let k be a field which admits resolution of singularities and let X be a separated scheme of finite type over k. Then for any integer $n \in \mathbb{Z}$, there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(n-q)) \otimes_{\mathbb{Z}} \mathbb{L}^q \Rightarrow \mathrm{MGL}^{p+q,n}(X), \tag{1.2}$$

and the differentials of this spectral sequence are given by $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$. Furthermore, this spectral sequence degenerates with rational coefficients.

If k is a perfect field of positive characteristic p, we obtain a similar spectral sequence after inverting p, except that we can not guarantee strong convergence unless X is smooth over k (see Remark 4.25).

As a consequence of Theorem 1.1 and its positive characteristic version, we get the following relation between the motivic cobordism and cohomology of singular schemes.

Theorem 1.3. *Let k be a field which admits resolution of singularities (resp. a perfect field of positive characteristic p). Then for any separated (resp. smooth) scheme X of finite type over k and dimension d and every $i \geq 0$, the edge map in* *the spectral sequence* (1.2)

$$\nu_X : \mathrm{MGL}^{2d+i,d+i}(X) \to H^{2d+i}(X, \mathbb{Z}(d+i))$$
$$\big(\textit{resp.}\quad \nu_X : \mathrm{MGL}^{2d+i,d+i}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]\big)$$

is an isomorphism.

We apply our descent result to obtain a similar spectral sequence for the connective *KH*-theory, KGL^0 (see Section 5). We use this spectral sequence and the canonical map $CKH(-) \to KH(-)$ from the connective *KH*-theory to obtain the following cycle class map from the motivic cohomology of a singular scheme to its homotopy invariant *K*-theory.

Theorem 1.4. *Let k be a field of exponential characteristic p and let X be a separated scheme of dimension d which is of finite type over k. Then the map* $\mathrm{KGL}^0_X \to s_0\, \mathrm{KGL}_X \cong H\mathbb{Z}$ *induces*, *for every integer $i \geq 0$*, *an isomorphism*

$$CKH^{2d+i,d+i}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \xrightarrow{\cong} H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big].$$

In particular, *there is a natural cycle class map*

$$\mathrm{cyc}_i : H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to KH_i(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big].$$

We use this cycle class map and the Chern class maps from the homotopy invariant *K*-theory to the Deligne cohomology of schemes over $\mathbb{C}$ to construct intermediate Jacobians and Abel–Jacobi maps for the motivic cohomology of singular schemes over $\mathbb{C}$. More precisely, we prove the following. This generalizes intermediate Jacobians and Abel–Jacobi maps of Griffiths and the torsion theorem of Roitman for smooth schemes.

Theorem 1.5. *Let X be a projective scheme over $\mathbb{C}$ of dimension d. Assume that either $d \leq 2$ or X is regular in codimension one. Then there is a semiabelian variety $J^d(X)$ and an Abel–Jacobi map* $\mathrm{AJ}^d_X : H^{2d}(X, \mathbb{Z}(d))_{\deg 0} \to J^d(X)$ *which is surjective and whose restriction to the torsion subgroups is an isomorphism.*

In a related work, Kohrita [2017, Theorem 6.5] has constructed an Abel–Jacobi map for the Lichtenbaum motivic cohomology $H^{2d}_L(X, \mathbb{Z}(d))$ of singular schemes over $\mathbb{C}$ using a different technique. He has also proven a version of the Roitman torsion theorem for the Lichtenbaum motivic cohomology. The natural map $H^{2d}(X, \mathbb{Z}(d)) \to H^{2d}_L(X, \mathbb{Z}(d))$ is not an isomorphism in general if $d \geq 3$. Note also that the Roitman torsion theorem for $H^{2d}(X, \mathbb{Z}(d))$ is a priori a finer statement than that for the analogous Lichtenbaum cohomology.

Using Theorem 1.5, we prove the following property of the cycle class map of Theorem 1.4, which is our final result. The analogous result for smooth projective schemes was proven by Marc Levine [1987, Theorem 3.2]. More generally, Levine shows that a relative Chow group of 0-cycles on a normal projective scheme over $\mathbb{C}$ injects inside $K_0(X)$.

Theorem 1.6. *Let X be a projective scheme of dimension d over $\mathbb{C}$. Assume that either $d \leq 2$ or X is regular in codimension one. Then the cycle class map* $\mathrm{cyc}_0 : H^{2d}(X, \mathbb{Z}(d)) \to KH_0(X)$ *is injective.*

We end this section with the comment that our motivation behind this work was to exploit powerful tools of the motivic homotopy theory to study several questions about the motivic cohomology and K-theory of singular schemes which were previously known only for smooth schemes. We hope that the methods and techniques of our proofs can be advanced further to answer many other cohomological questions about singular schemes. We refer to [Krishna and Pelaez 2018] for more results based on the techniques of this text.

2. A descent theorem for motivic spectra

In this section, we set up our notation, discuss various model structures used in our proofs and show the Quillen adjunction property of many functors among these model structures. The main objective of this section is to prove a cdh-descent property of the motivic T-spectra; see Theorem 2.14.

2.1. ***Notations and preliminary results.*** Let k be a perfect field of exponential characteristic p; in some instances we require that the field k admits resolution of singularities [Voevodsky 2010, Definition 4.1]. We write $\mathbf{Sch}_k$ for the category of separated schemes of finite type over k and $\mathbf{Sm}_k$ for the full subcategory of $\mathbf{Sch}_k$ consisting of smooth schemes over k. If $X \in \mathbf{Sch}_k$, let $\mathbf{Sm}_X$ denote the full subcategory of $\mathbf{Sch}_k$ consisting of smooth schemes over X. We write $(\mathbf{Sm}_k)_{\mathrm{Nis}}$ (resp. $(\mathbf{Sm}_X)_{\mathrm{Nis}}$, $(\mathbf{Sch}_k)_{\mathrm{cdh}}$, $(\mathbf{Sch}_k)_{\mathrm{Nis}}$) for $\mathbf{Sm}_k$ equipped with the Nisnevich topology (resp. $\mathbf{Sm}_X$ equipped with the Nisnevich topology, $\mathbf{Sch}_k$ equipped with the cdh-topology, $\mathbf{Sch}_k$ equipped with the Nisnevich topology). The product $X \times_{\mathrm{Spec}\, k} Y$ is denoted by $X \times Y$.

Let $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\mathrm{cdh}}$) be the category of pointed simplicial presheaves on $\mathbf{Sm}_k$ (resp. $\mathbf{Sm}_X$, $\mathbf{Sch}_k$) equipped with the motivic model structure described in [Isaksen 2005] considering the Nisnevich topology on $\mathbf{Sm}_k$ (resp. Nisnevich topology on $\mathbf{Sm}_X$, cdh-topology on $\mathbf{Sch}_k$) and the affine line $\mathbb{A}^1_k$ as an interval. A simplicial presheaf is often called a *motivic space*.

We define T in $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\mathrm{cdh}}$) as the pointed simplicial presheaf represented by $S^1_s \wedge S^1_t$, where S^1_t is $\mathbb{A}^1_k \setminus \{0\}$ (resp. $\mathbb{A}^1_X \setminus \{0\}$, $\mathbb{A}^1_k \setminus \{0\}$) pointed by 1, and S^1_s denotes the simplicial circle. Given an arbitrary integer $r \geq 1$, let S^r_s denote the iterated smash product $S^1_s \wedge \cdots \wedge S^1_s$ of S^1_s with r factors, and S^r_t the iterated smash product $S^1_t \wedge \cdots \wedge S^1_t$ of S^1_t with r factors; $S^0_s = S^0_t$ is by definition equal to the pointed simplicial presheaf represented by the base scheme $\mathrm{Spec}\, k$ (resp. X, $\mathrm{Spec}\, k$).

Since T is cofibrant in $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\mathrm{cdh}}$) we can apply freely the results in [Hovey 2001, §8]. Let $\mathrm{Spt}(\mathcal{M})$ (resp. $\mathrm{Spt}(\mathcal{M}_X)$, $\mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$) denote the category of symmetric T-spectra on $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\mathrm{cdh}}$) equipped with the motivic model structure defined in [Hovey 2001, Definition 8.7]. We write $\mathcal{SH}$ (resp. $\mathcal{SH}_X$, $\mathcal{SH}_{\mathrm{cdh}}$) for the homotopy category of $\mathrm{Spt}(\mathcal{M})$ (resp. $\mathrm{Spt}(\mathcal{M}_X)$, $\mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$), which is a tensor triangulated category. For any two integers $m, n \in \mathbb{Z}$, let $\Sigma^{m,n}$ denote the automorphism $\Sigma_s^{m-n} \circ \Sigma_t^n : \mathcal{SH} \to \mathcal{SH}$ (this also makes sense in $\mathcal{SH}_X$ and $\mathcal{SH}_{\mathrm{cdh}}$). We write Σ_T^n for $\Sigma^{2n,n}$, and $E \wedge F$ for the smash product of $E, F \in \mathcal{SH}$ (resp. $\mathcal{SH}_X$, $\mathcal{SH}_{\mathrm{cdh}}$).

Given a simplicial presheaf A, we write A_+ for the pointed simplicial presheaf obtained by adding a disjoint base point (isomorphic to the base scheme) to A. For any $B \in \mathcal{M}$, let $\Sigma_T^\infty(B)$ denote the object $(B, T \wedge B, \dots) \in \mathrm{Spt}(\mathcal{M})$. This functor makes sense for objects in $\mathcal{M}_{\mathrm{cdh}}$ and $\mathcal{M}_X$ as well.

If $F : \mathcal{A} \to \mathcal{B}$ is a functor with right adjoint $G : \mathcal{B} \to \mathcal{A}$, we say that $(F, G) : \mathcal{A} \to \mathcal{B}$ is an adjunction. We freely use the language of model and triangulated categories. We write Σ^1 for the suspension functor in a triangulated category, and Σ^n is the suspension (or desuspension in case $n < 0$) functor iterated n (or $-n$) times.

We use the following notation in all the categories under consideration: $*$ denotes the terminal object, and $\cong$ denotes that a map is an isomorphism or that a functor is an equivalence of categories.

2.2. ***Change of site.*** Let $X \in \mathbf{Sch}_k$ and let $v : X \to \operatorname{Spec} k$ denote the structure map. We write Pre_X and $\underline{\mathrm{Pre}}_k$ for the categories of pointed simplicial presheaves on $\mathbf{Sm}_X$ and $\mathbf{Sch}_k$, respectively. If $X = \operatorname{Spec} k$, where k is the base field, we write Pre_k instead of Pre_X. These categories are equipped with the objectwise flasque model structure [Isaksen 2005, §3]. To recall this model structure, we consider a finite set I of monomorphisms $\{V_i \to U\}_{i \in I}$ for any $U \in \mathbf{Sm}_X$. The categorical union $\bigcup_{i \in I} V_i$ is the coequalizer of the diagram

$$\coprod_{i,j \in I} V_i \times_U V_j \rightrightarrows \coprod_{i \in I} V_i$$

formed in Pre_X. We denote by i_I the induced monomorphism $\bigcup_{i \in I} V_i \to U$. Note that $\varnothing \to U$ arises in this way. The pushout product of maps of i_I and a map between simplicial sets exists in Pre_X. In particular, we may form the sets

$$I_{\mathrm{clo}}^{\mathrm{sch}}(\mathbf{Sm}_X) = \{i_I \square (\partial \Delta^n \subset \Delta^n)_+\}_{I, n \geq 0},$$
$$J_{\mathrm{clo}}^{\mathrm{sch}}(\mathbf{Sm}_X) = \{i_I \square (\Lambda_i^n \subset \Delta^n)_+\}_{I, n \geq 0, 0 \leq i \leq n},$$

where I is a finite set of monomorphisms $\{V_i \to U\}_{i \in I}$ with $U \in \mathbf{Sm}_X$, and $i_I : \bigcup_{i \in I} V_i \to U$ is the induced monomorphism defined above.

A map between simplicial presheaves is called a closed objectwise fibration if it has the right lifting property with respect to $J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X)$. A map $u : E \to F$ between simplicial presheaves is called a weak equivalence if $E(U) \to F(U)$ is a weak equivalence of simplicial sets for each $U \in \mathbf{Sm}_X$. A closed objectwise cofibration is a map having the left lifting property with respect to every trivial closed objectwise fibration. Note that this notion of weak equivalence, cofibrations and fibrations makes sense for simplicial presheaves in any category with finite products (e.g., $\mathbf{Sm}_k$, $\mathbf{Sch}_k$). It follows from [Isaksen 2005, Theorem 3.7] that the above notion of weak equivalence, cofibrations and fibrations forms a proper, simplicial and cellular model category structure on Pre_k, Pre_X and $\underline{\mathrm{Pre}}_k$. We call this the *objectwise flasque model structure*. Our reason for choosing this model structure is the following result.

Lemma 2.3 [Isaksen 2005, Lemma 6.2]. *If $V \to U$ is a monomorphism in $\mathbf{Sm}_k$ (resp. $\mathbf{Sm}_X$, $\mathbf{Sch}_k$), then U_+/V_+ is cofibrant in the flasque model structure on Pre_k (resp. Pre_X, $\underline{\mathrm{Pre}}_k$). In particular, $T^n \wedge U_+$ is cofibrant for any $n \geq 0$.*

It is clear that Pre_X and $\underline{\mathrm{Pre}}_k$ are cofibrantly generated model categories with generating cofibrations $I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X)$ and $I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k)$ and generating trivial cofibrations $J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X)$ and $J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k)$, respectively.

Let $\pi : (\mathbf{Sch}_k)_{\mathrm{cdh}} \to (\mathbf{Sm}_k)_{\mathrm{Nis}}$ be the continuous map of sites considered in [Voevodsky 2010, §4]. We write $(\pi^*, \pi_*) : \mathrm{Pre}_k \to \underline{\mathrm{Pre}}_k$ and $(v^*, v_*) : \mathrm{Pre}_k \to \mathrm{Pre}_X$ for the adjunctions induced by π and v, respectively.

We also consider the morphism of sites $\pi_X : (\mathbf{Sch}_k)_{\mathrm{cdh}} \to (\mathbf{Sm}_X)_{\mathrm{Nis}}$ and the corresponding adjunction $(\pi_X^*, \pi_{X*}) : \mathrm{Pre}_X \to \underline{\mathrm{Pre}}_k$. These adjunctions are related by the following lemma.

Lemma 2.4. *The following diagram commutes*:

$$\begin{tikzcd}
\mathrm{Pre}_k \arrow[r, "\pi^*"] \arrow[dr, "v^*"'] & \underline{\mathrm{Pre}}_k \arrow[d, "\pi_{X*}"] \\
& \mathrm{Pre}_X
\end{tikzcd}$$

Proof. We first notice that for every simplicial set K, $Y \in \mathbf{Sm}_k$ and $Z \in \mathbf{Sm}_X$, one has

$$\begin{aligned}
\pi^*(K \otimes Y_+) &= K \otimes Y_+ \in \underline{\mathrm{Pre}}_k, \\
v^*(K \otimes Y_+) &= K \otimes (Y \times X)_+ \in \mathrm{Pre}_X,
\end{aligned} \tag{2.5}$$

and

$$\pi_X^*(K \otimes Z_+) = K \otimes Z_+ \in \underline{\mathrm{Pre}}_k.$$

We observe that π^* and v^* commute with colimits since they are left adjoint, and that π_{X*} also commutes with colimits since it is a restriction functor. Hence, it suffices to show that for every simplicial set K and every $Y \in \mathbf{Sm}_k$, we have

$\pi_{X*}(\pi^*(K \otimes Y_+)) = v^*(K \otimes Y_+)$. Finally, a direct computation shows that

$$\pi_{X*}(K \otimes Y_+) = K \otimes (Y \times X)_+ \in \mathrm{Pre}_X$$

and we conclude by (2.5). □

Lemma 2.6. *The adjunctions* $(\pi^*, \pi_*) : \mathrm{Pre}_k \to \underline{\mathrm{Pre}}_k$, $(v^*, v_*) : \mathrm{Pre}_k \to \mathrm{Pre}_X$ *and* $(\pi_X^*, \pi_{X*}) : \mathrm{Pre}_X \to \underline{\mathrm{Pre}}_k$ *are all Quillen adjunctions. Moreover*, π_{X*} *and* π_* *preserve weak equivalences.*

Proof. We have seen above that all the three model categories (with the objectwise flasque model structure) are cofibrantly generated. Moreover, it follows from (2.5) that

$$\begin{aligned} \pi^*(I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_k)) &\subseteq I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k), & \pi^*(J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_k)) &\subseteq J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k), \\ v^*(I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_k)) &\subseteq I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X), & v^*(J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_k)) &\subseteq J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X), \\ \pi_X^*(I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X)) &\subseteq I^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k), & \pi_X^*(J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sm}_X)) &\subseteq J^{\mathrm{sch}}_{\mathrm{clo}}(\mathbf{Sch}_k). \end{aligned}$$

Hence, it follows from [Hovey 1999, Lemma 2.1.20] that (π^*, π_*), (v^*, v_*) and (π_X^*, π_{X*}) are Quillen adjunctions. The second part of the lemma is an immediate consequence of the fact that π_{X*} and π_* are restriction functors and the weak equivalences in the objectwise flasque model structure are defined schemewise. □

To show that the Quillen adjunction of Lemma 2.6 extends to the level of motivic model structures, we consider a distinguished square α [Voevodsky 2010, §2]

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd} \tag{2.7}$$

in $(\mathbf{Sm}_k)_{\mathrm{Nis}}$, $(\mathbf{Sm}_X)_{\mathrm{Nis}}$ or $(\mathbf{Sch}_k)_{\mathrm{cdh}}$, and write $P(\alpha)$ for the pushout of $Z \leftarrow Z' \to Y'$ in Pre_k, Pre_X or $\underline{\mathrm{Pre}}_k$, respectively.

The motivic model category $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\mathrm{cdh}}$, $\mathcal{M}_{\mathrm{ft}}$) is the left Bousfield localization of Pre_k (resp. Pre_X, $\underline{\mathrm{Pre}}_k$, $\underline{\mathrm{Pre}}_k$) with respect to the following two sets of maps:

- $P(\alpha) \to Y$ indexed by the distinguished squares in $(\mathbf{Sm}_k)_{\mathrm{Nis}}$ (resp. $(\mathbf{Sm}_X)_{\mathrm{Nis}}$, $(\mathbf{Sch}_k)_{\mathrm{cdh}}$, $(\mathbf{Sch}_k)_{\mathrm{Nis}}$),
- $p_Y : Y \times \mathbb{A}^1_k \to Y$ for $Y \in \mathbf{Sm}_k$ (resp. $Y \in \mathbf{Sm}_X$, $Y \in \mathbf{Sch}_k$, $Y \in \mathbf{Sch}_k$).

Notice that as we are working with the flasque model structures, by [Isaksen 2005, Theorems 4.8–4.9] it is possible to consider maps from the ordinary pushout $P(\alpha)$ instead of maps from the homotopy pushout of the diagram $Z \leftarrow Z' \to Y'$ in (2.7).

Remark 2.8. We also consider the Nisnevich (resp. cdh) local model structure, i.e., the left Bousfield localization of Pre_k (resp. $\underline{\mathrm{Pre}}_k$) with respect to the set of maps $P(\alpha) \to Y$ indexed by the distinguished squares in $(\mathbf{Sm}_k)_{\mathrm{Nis}}$ (resp. $(\mathbf{Sch}_k)_{\mathrm{cdh}}$).

We abuse notation and write $(\pi^*, \pi_*) : \mathcal{M} \to \mathcal{M}_{\mathrm{cdh}}$, $(v^*, v_*) : \mathcal{M} \to \mathcal{M}_X$ and $(\pi_X^*, \pi_{X*}) : \mathcal{M}_X \to \mathcal{M}_{\mathrm{cdh}}$ for the adjunctions induced by π, v and π_X, respectively.

Proposition 2.9. *The adjunctions* $(\pi^*, \pi_*) : \mathcal{M} \to \mathcal{M}_{\mathrm{cdh}}$, $(v^*, v_*) : \mathcal{M} \to \mathcal{M}_X$ *and* $(\pi_X^*, \pi_{X*}) : \mathcal{M}_X \to \mathcal{M}_{\mathrm{cdh}}$ *are Quillen adjunctions.*

Proof. We give the argument for (π^*, π_*), since the other cases are parallel. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Pre}_k \arrow[r, "\pi^*"] \arrow[d, "\mathrm{id}"'] & \underline{\mathrm{Pre}}_k \arrow[d, "\mathrm{id}"] \\
\mathcal{M} \arrow[r, dashed, "\pi^*"] & \mathcal{M}_{\mathrm{cdh}}
\end{tikzcd}$$

where the solid arrows are left Quillen functors by [Hirschhorn 2003, Lemma 3.3.4(1)] and Lemma 2.6. Thus, it follows from [Hirschhorn 2003, Definition 3.1.1(1)(b), Theorem 3.3.19] that it suffices to check that $\pi^*(P(\alpha) \to Y)$ and $\pi^*(Y \times \mathbb{A}_k^1 \to Y)$ are weak equivalences in $\mathcal{M}_{\mathrm{cdh}}$.

On the one hand, it is immediate that $\pi^*(Y \times \mathbb{A}_k^1 \to Y) = (Y \times \mathbb{A}_k^1 \to Y) \in \mathcal{M}_{\mathrm{cdh}}$, and is hence a weak equivalence in $\mathcal{M}_{\mathrm{cdh}}$. On the other hand, π^* commutes with pushouts since it is a left adjoint functor. It thus follows from (2.5) that

$$\pi^*(P(\alpha) \to Y) = (P(\alpha) \to Y) \in \mathcal{M}_{\mathrm{cdh}},$$

and is hence a weak equivalence in $\mathcal{M}_{\mathrm{cdh}}$. □

We write $\mathcal{H}$ (resp. $\mathcal{H}_X$, $\mathcal{H}_{\mathrm{cdh}}$) for the homotopy category of $\mathcal{M}$ (resp. $\mathcal{M}_X$, $\mathcal{M}_{\mathrm{cdh}}$) and $(\boldsymbol{L}\pi^*, \boldsymbol{R}\pi_*) : \mathcal{H} \to \mathcal{H}_{\mathrm{cdh}}$, $(\boldsymbol{L}v^*, \boldsymbol{R}v_*) : \mathcal{H} \to \mathcal{H}_X$, $(\boldsymbol{L}\pi_X^*, \boldsymbol{R}\pi_{X*}) : \mathcal{H}_X \to \mathcal{H}_{\mathrm{cdh}}$ for the derived adjunctions of the Quillen adjunctions in Proposition 2.9; see [Hirschhorn 2003, Theorem 3.3.20].

2.10. ***A* cdh*-descent for motivic spectra.*** It follows from (2.5) that the adjunctions between the categories of motivic spaces induce levelwise adjunctions

$$(\pi^*, \pi_*) : \mathrm{Spt}(\mathcal{M}) \to \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}}),$$
$$(v^*, v_*) : \mathrm{Spt}(\mathcal{M}) \to \mathrm{Spt}(\mathcal{M}_X),$$
$$(\pi_X^*, \pi_{X*}) : \mathrm{Spt}(\mathcal{M}_X) \to \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$$

between the corresponding categories of symmetric T-spectra such that the following diagram commutes (see Lemma 2.4):

$$\begin{tikzcd}
\mathrm{Spt}(\mathcal{M}) \arrow[r, "\pi^*"] \arrow[dr, "v^*"'] & \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}}) \arrow[d, "\pi_{X*}"] \\
& \mathrm{Spt}(\mathcal{M}_X)
\end{tikzcd} \tag{2.11}$$

We further conclude from Proposition 2.9 and [Hovey 2001, Theorem 9.3] the following:

Proposition 2.12. *The pairs*

(1) $(\pi^*, \pi_*) : \mathrm{Spt}(\mathcal{M}) \to \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$,

(2) $(v^*, v_*) : \mathrm{Spt}(\mathcal{M}) \to \mathrm{Spt}(\mathcal{M}_X)$ *and*

(3) $(\pi_X^*, \pi_{X*}) : \mathrm{Spt}(\mathcal{M}_X) \to \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$

are Quillen adjunctions between stable model categories.

We deduce from Proposition 2.12 that there are pairs of adjoint functors

$$(\boldsymbol{L}\pi^*, \boldsymbol{R}\pi_*) : \mathcal{SH} \to \mathcal{SH}_{\mathrm{cdh}},$$
$$(\boldsymbol{L}v^*, \boldsymbol{R}v_*) : \mathcal{SH} \to \mathcal{SH}_X,$$
$$(\boldsymbol{L}\pi_X^*, \boldsymbol{R}\pi_{X*}) : \mathcal{SH}_X \to \mathcal{SH}_{\mathrm{cdh}}$$

between the various stable homotopy categories of motivic T-spectra. We observe that for $a \geq b \geq 0$, the suspension functor $\Sigma^{a,b}$ in $\mathcal{SH}$ (resp. $\mathcal{SH}_X$, $\mathcal{SH}_{\mathrm{cdh}}$) is the derived functor of the left Quillen functor $E \mapsto S_s^{a-b} \wedge S_t^b \wedge E$ in $\mathrm{Spt}(\mathcal{M})$ (resp. $\mathrm{Spt}(\mathcal{M}_X)$, $\mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$). Since the functors π^*, v^*, π_X^* are simplicial and symmetric monoidal, we deduce that they commute with the suspension functors $\Sigma^{m,n}$, i.e., for every $m, n \in \mathbb{Z}$,

$$\boldsymbol{L}\pi^* \circ \Sigma^{m,n}(-) \cong \Sigma^{m,n} \circ \boldsymbol{L}\pi^*(-),$$
$$\boldsymbol{L}v^* \circ \Sigma^{m,n}(-) \cong \Sigma^{m,n} \circ \boldsymbol{L}v^*(-),$$
$$\boldsymbol{L}\pi_X^* \circ \Sigma^{m,n}(-) \cong \Sigma^{m,n} \circ \boldsymbol{L}\pi_X^*(-).$$

Recall that $\mathcal{M}_{\mathrm{ft}}$ is the motivic category for the Nisnevich topology in $\mathbf{Sch}_k$. We write $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ for the category of symmetric T-spectra on $\mathcal{M}_{\mathrm{ft}}$ equipped with the stable model structure considered in [Hovey 2001, Definition 8.7].

It is well known [Jardine 2003, p. 198] that $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ and $\mathrm{Spt}(\mathcal{M}_X)$ (for $X \in \mathbf{Sch}_k$) are simplicial model categories [Hirschhorn 2003, Definition 9.1.6]. For E, E' in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ or $\mathrm{Spt}(\mathcal{M}_X)$, we write $\mathrm{Map}(E, E')$ and $\mathrm{Map}_X(E, E')$ for the simplicial set of maps from E to E', i.e., the simplicial set with n-simplices of the form $\mathrm{Hom}_{\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})}(E \otimes \Delta^n, E')$ or $\mathrm{Hom}_{\mathrm{Spt}(\mathcal{M}_X)}(E \otimes \Delta^n, E')$, respectively.

For $f : X \to X'$, note that the Quillen adjunction $(f^*, f_*) : \mathrm{Spt}(\mathcal{M}_{X'}) \to \mathrm{Spt}(\mathcal{M}_X)$ [Ayoub 2007b, Théorème 4.5.14] is enriched on simplicial sets, i.e., we have $\mathrm{Map}_X(f^*E', E) \cong \mathrm{Map}_{X'}(E', f_*E)$ for $E \in \mathrm{Spt}(\mathcal{M}_X)$, $E' \in \mathrm{Spt}(\mathcal{M}_{X'})$.

The following result is a direct consequence of the proper base change theorem in motivic homotopy theory [Ayoub 2007a, Corollaire 1.7.18; Cisinski and Déglise 2012, Proposition 2.3.11(2); Cisinski 2013, Proposition 3.7].

Proposition 2.13. *$\boldsymbol{L}v^*$ is naturally equivalent to the composition $\boldsymbol{R}\pi_{X*} \circ \boldsymbol{L}\pi^*$.*

Proof. We observe that the following diagram of left Quillen functors commutes:

$$\begin{tikzcd} \mathrm{Spt}(\mathcal{M}) \arrow[r, "\pi^*"] \arrow[dr, "\pi^*_{\mathrm{ft}}"'] & \mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}}) \\ & \mathrm{Spt}(\mathcal{M}_{\mathrm{ft}}) \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

Let E be a motivic T-spectrum in $\mathrm{Spt}(\mathcal{M})$. Without any loss of generality, we can assume that E is cofibrant in $\mathrm{Spt}(\mathcal{M})$. Let $\nu : \pi^*_{\mathrm{ft}} E \to E'$ be a functorial fibrant replacement of $\pi^*_{\mathrm{ft}} E$ in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$.

The argument in [Jardine 2003, pp. 198–199] shows that the restriction functor π_{X*} maps weak equivalences in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ into weak equivalences in $\mathrm{Spt}(\mathcal{M}_X)$. Combining this with (2.11), we deduce that

$$\pi_{X*}(\nu) : \pi_{X*}(\pi^*_{\mathrm{ft}} E) = \pi_{X*}(\pi^* E) = v^* E \to \pi_{X*} E'$$

is a weak equivalence in $\mathrm{Spt}(\mathcal{M}_X)$. Since E is cofibrant in $\mathrm{Spt}(\mathcal{M})$, $\boldsymbol{L}v^* E \cong v^* E$. Hence, to conclude it suffices to show that E' is fibrant in $\mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$.

For the rest of the proof, for $Y \in \mathbf{Sch}_k$ we write $v_Y : Y \to \mathrm{Spec}\,(k)$ for the structure map. Notice that we have proved that $\boldsymbol{L}v^*_Y E \cong v^*_Y E \cong \pi_{Y*} E'$ in $\mathcal{SH}_Y$. Consider a distinguished abstract blow-up square in $\mathbf{Sch}_k$, i.e., a distinguished square in the lower cd-structure defined in [Voevodsky 2010, §2]:

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "f'"'] & Y' \arrow[d, "f"] \\ Z \arrow[r, "i"'] & Y \end{tikzcd}$$

Let $j = i \circ f'$. Then

$$\boldsymbol{R} f_* \boldsymbol{L} f^* (\boldsymbol{L} v^*_Y E) \cong \boldsymbol{R} f_* \boldsymbol{L} (v_Y \circ f)^* E \cong \boldsymbol{R} f_* \pi_{Y'*} E' \cong f_* \pi_{Y'*} E'$$

in $\mathcal{SH}_Y$. In particular, the last isomorphism above follows from the fact that $\pi_{Y'*} E'$ is fibrant in $\mathrm{Spt}(\mathcal{M}_{Y'})$, since E' is fibrant in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ and the restriction functor $\pi_{Y'} : \mathrm{Spt}(\mathcal{M}_{\mathrm{ft}}) \to \mathrm{Spt}(\mathcal{M}_{Y'})$ is a right Quillen functor (using the same argument as in Proposition 2.12). Similarly, we conclude that $\boldsymbol{R} i_* \boldsymbol{L} i^* (\boldsymbol{L} v^*_Y E) \cong i_* \pi_{Z*} E'$ and $\boldsymbol{R} j_* \boldsymbol{L} j^* (\boldsymbol{L} v^*_Y E) \cong j_* \pi_{Z'*} E'$ in $\mathcal{SH}_Y$.

Thus, by [Cisinski 2013, Proposition 3.7] we conclude that the commutative diagram

$$\begin{tikzcd} \pi_{Y*} E' \arrow[r] \arrow[d] & f_* \pi_{Y'*} E' \arrow[d] \\ i_* \pi_{Z*} E' \arrow[r] & j_* \pi_{Z'*} E' \end{tikzcd}$$

is a homotopy cofiber square in $\mathrm{Spt}(\mathcal{M}_Y)$ [Hirschhorn 2003, Definition 13.5.8], and thus also a homotopy fiber square since $\mathrm{Spt}(\mathcal{M}_Y)$ is a stable model category, i.e., its homotopy category is triangulated. Since $\Sigma_T^\infty Y_+$ is cofibrant in $\mathrm{Spt}(\mathcal{M}_Y)$ and $\pi_{Y*}E'$, $f_*\pi_{Y'*}E'$, $i_*\pi_{Z*}E'$ and $j_*\pi_{Z'*}E'$ are fibrant, combining [Hirschhorn 2003, Definition 9.1.6(M7)] and [Hirschhorn 2003, Corollary 9.7.5(1)] we conclude that the induced commutative diagram is a homotopy fiber square of simplicial sets:

$$\begin{array}{ccc}
\mathrm{Map}_Y(\Sigma_T^\infty Y_+, \pi_{Y*}E') & \longrightarrow & \mathrm{Map}_Y(\Sigma_T^\infty Y_+, f_*\pi_{Y'*}E') \\
\downarrow & & \downarrow \\
\mathrm{Map}_Y(\Sigma_T^\infty Y_+, i_*\pi_{Z*}E') & \longrightarrow & \mathrm{Map}_Y(\Sigma_T^\infty Y_+, j_*\pi_{Z'*}E')
\end{array}$$

Since the adjunction (f^*, f_*) is enriched in simplicial sets, we conclude that

$$\mathrm{Map}_Y(\Sigma_T^\infty Y_+, f_*\pi_{Y'*}E') \cong \mathrm{Map}_{Y'}(f^*\Sigma_T^\infty Y_+, \pi_{Y'*}E') \cong \mathrm{Map}_{Y'}(\Sigma_T^\infty Y'_+, \pi_{Y'*}E')$$

and by definition $\mathrm{Map}_{Y'}(\Sigma_T^\infty Y'_+, \pi_{Y'*}E') \cong \mathrm{Map}(\Sigma_T^\infty Y'_+, E')$. Similarly, we conclude that

$$\begin{aligned}
\mathrm{Map}_Y(\Sigma_T^\infty Y_+, \pi_{Y*}E') &\cong \mathrm{Map}(\Sigma_T^\infty Y_+, E'), \\
\mathrm{Map}_Y(\Sigma_T^\infty Y_+, i_*\pi_{Z*}E') &\cong \mathrm{Map}(\Sigma_T^\infty Z_+, E'), \\
\mathrm{Map}_Y(\Sigma_T^\infty Y_+, j_*\pi_{Z'*}E') &\cong \mathrm{Map}(\Sigma_T^\infty Z'_+, E').
\end{aligned}$$

Therefore, the following is a homotopy fiber square of simplicial sets:

$$\begin{array}{ccc}
\mathrm{Map}(\Sigma_T^\infty Y_+, E') & \longrightarrow & \mathrm{Map}(\Sigma_T^\infty Y'_+, E') \\
\downarrow & & \downarrow \\
\mathrm{Map}(\Sigma_T^\infty Z_+, E') & \longrightarrow & \mathrm{Map}(\Sigma_T^\infty Z'_+, E')
\end{array}$$

Since $\Sigma_T^\infty Z'_+ \to \Sigma_T^\infty Y'_+$ is a cofibration in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ and E' is fibrant in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$, we deduce that $\mathrm{Map}(\Sigma_T^\infty Y'_+, E') \to \mathrm{Map}(\Sigma_T^\infty Z'_+, E')$ is a fibration of simplicial sets; see [Hirschhorn 2003, Definition 9.1.6(M7)]. We observe that the functor $\mathrm{Map}(-, E')$ maps pushout squares in $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ into pullback squares of simplicial sets [Hirschhorn 2003, Proposition 9.1.8]; thus, by [Hirschhorn 2003, Corollary 13.3.8] we conclude that the map

$$\mathrm{Map}(\Sigma_T^\infty Y_+, E') \to \mathrm{Map}(\Sigma_T^\infty P(\alpha), E')$$

induced by $P(\alpha) \to Y$ is a weak equivalence of simplicial sets, where $P(\alpha)$ is the pushout of $Z \leftarrow Z' \to Y'$ in $\underline{\mathrm{Pre}}_k$. Finally, by [Hirschhorn 2003, Theorem 4.1.1(2)] we conclude that E' is fibrant in $\mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$, since by construction $\mathrm{Spt}(\mathcal{M}_{\mathrm{cdh}})$ is the left Bousfield localization of $\mathrm{Spt}(\mathcal{M}_{\mathrm{ft}})$ with respect to the maps of the form $\Sigma_T^\infty(P(\alpha) \to Y_+)$ indexed by the abstract blow-up squares in $\mathbf{Sch}_k$. □

The following result should be compared with [Cisinski 2013, Proposition 3.7].

Theorem 2.14. *Let $v : X \to \operatorname{Spec}(k)$ be in $\mathbf{Sch}_k$. Given a motivic T-spectrum $E \in \mathcal{SH}$, $Y \in \mathbf{Sm}_X$ and integers $m, n \in \mathbb{Z}$, there is a natural isomorphism*

$$\operatorname{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}v^* E) \cong \operatorname{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}\pi^* E).$$

Proof. By Proposition 2.13, $\boldsymbol{L}v^*(-) \cong (\boldsymbol{R}\pi_{X*} \circ \boldsymbol{L}\pi^*)(-)$ in $\mathcal{SH}_X$. Thus, by adjointness,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}v^* E) &\cong \operatorname{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \boldsymbol{L}v^*(\Sigma^{m,n} E)) \\ &\cong \operatorname{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\boldsymbol{L}\pi_X^* \Sigma_T^\infty Y_+, \boldsymbol{L}\pi^*(\Sigma^{m,n} E)) \\ &\cong \operatorname{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\boldsymbol{L}\pi_X^* \Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}\pi^* E).\end{aligned}$$

Finally, it follows from Lemma 2.3 that $\Sigma_T^\infty Y_+$ is cofibrant in the levelwise flasque model structure and hence in any of its localizations. In particular, it is cofibrant in the stable model structure of motivic T-spectra. We conclude that

$$\boldsymbol{L}\pi_X^* \Sigma_T^\infty Y_+ \cong \pi_X^* \Sigma_T^\infty Y_+ \cong \Sigma_T^\infty Y_+.$$

The corollary now follows. □

Remark 2.15. The above result could be called a cdh-descent theorem because it implies cdh-descent for many motivic spectra; see [Cisinski 2013, Proposition 3.7]. In particular, it implies cdh-descent for absolute motivic spectra (for example, KGL and MGL). Recall from [Déglise 2014, §1.2] that an absolute motivic spectrum E is a section of a 2-functor from $\mathbf{Sch}_k$ to triangulated categories such that for any $f : X' \to X$ in $\mathbf{Sch}_k$, the canonical map $f^* E_X \to E_{X'}$ is an isomorphism.

Lemma 2.16. *Let $f : Y \to X$ be a smooth morphism in $\mathbf{Sch}_k$. Let $v : X \to \operatorname{Spec}(k)$ be the structure map and $u = v \circ f$. Given any $E \in \mathcal{SH}$, the map*

$$\operatorname{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \boldsymbol{L}v^* E) \to \operatorname{Hom}_{\mathcal{SH}_Y}(\Sigma_T^\infty Y_+, \boldsymbol{L}u^* E)$$

is an isomorphism.

Proof. The functor $\boldsymbol{L}f^* : \mathcal{SH}_X \to \mathcal{SH}_Y$ admits a left adjoint $\boldsymbol{L}f_\sharp : \mathcal{SH}_Y \to \mathcal{SH}_X$ by [Ayoub 2007b, Proposition 4.5.19]; see also [Ayoub 2007a, Scholium 1.4.2]. Since $f : Y \to X$ is smooth, we have $\boldsymbol{L}f_\sharp(\Sigma_T^\infty Y_+) = \Sigma_T^\infty Y_+$ by [Morel and Voevodsky 1999, Proposition 3.1.23(1)] and we get

$$\begin{aligned}\operatorname{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \boldsymbol{L}v^* E) &\cong \operatorname{Hom}_{\mathcal{SH}_X}(\boldsymbol{L}f_\sharp(\Sigma_T^\infty Y_+), \boldsymbol{L}v^* E) \\ &\cong \operatorname{Hom}_{\mathcal{SH}_Y}(\Sigma_T^\infty Y_+, \boldsymbol{L}f^* \circ \boldsymbol{L}v^* E) \\ &\cong \operatorname{Hom}_{\mathcal{SH}_Y}(\Sigma_T^\infty Y_+, \boldsymbol{L}u^* E),\end{aligned}$$

and the lemma follows. □

A combination of Lemma 2.16 and Theorem 2.14 yields the following corollary:

Corollary 2.17. *Under the same hypotheses and notation of Theorem 2.14, assume in addition that $X \in \mathbf{Sm}_k$. Then there are natural isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{SH}}(\Sigma_T^\infty Y_+, \Sigma^{m,n} E) &\cong \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}v^* E) \\ &\cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma_T^\infty Y_+, \Sigma^{m,n} \boldsymbol{L}\pi^* E).\end{aligned}$$

3. Motivic cohomology of singular schemes

We continue to assume that k is a perfect field of exponential characteristic p. In this section, we show that the motivic cohomology of a scheme $X \in \mathbf{Sch}_k$, defined in terms of a cdh-hypercohomology (see Definition 3.1), is representable in the stable homotopy category $\mathcal{SH}_{\mathrm{cdh}}$.

Recall from [Mazza et al. 2006, Lecture 16] that given $T \in \mathbf{Sch}_k$ and an integer $r \geq 0$, the presheaf $z_{\mathrm{equi}}(T, r)$ on $\mathbf{Sm}_k$ is defined by letting $z_{\mathrm{equi}}(T, r)(U)$ be the free abelian group generated by the closed and irreducible subschemes $Z \subsetneq U \times T$ which are dominant and equidimensional of relative dimension r (any fiber is either empty or all its components have dimension r) over a component of U. It is known that $z_{\mathrm{equi}}(T, r)$ is a sheaf on the big étale site of $\mathbf{Sm}_k$.

Let $\underline{C}_* z_{\mathrm{equi}}(T, r)$ denote the chain complex of presheaves of abelian groups associated via the Dold–Kan correspondence to the simplicial presheaf on $\mathbf{Sm}_k$ given by $\underline{C}_n z_{\mathrm{equi}}(T, r)(U) = z_{\mathrm{equi}}(T, r)(U \times \Delta_k^n)$. The simplicial structure on $\underline{C}_* z_{\mathrm{equi}}(T, r)$ is induced by the cosimplicial scheme $\Delta_k^\bullet$. Recall the following definition of motivic cohomology of singular schemes from [Friedlander and Voevodsky 2000, Definition 9.2].

Definition 3.1. The motivic cohomology groups of $X \in \mathbf{Sch}_k$ are defined as the hypercohomology

$$H^m(X, \mathbb{Z}(n)) = \mathbb{H}_{\mathrm{cdh}}^{m-2n}(X, \underline{C}_* z_{\mathrm{equi}}(\mathbb{A}_k^n, 0)_{\mathrm{cdh}}) = A_{0,2n-m}(X, \mathbb{A}^n).$$

We also need to consider $\mathbb{Z}[1/p]$-coefficients. In this case, we write

$$H^m\big(X, \mathbb{Z}\big[\tfrac{1}{p}\big](n)\big) = \mathbb{H}_{\mathrm{cdh}}^{m-2n}\big(X, \underline{C}_* z_{\mathrm{equi}}(\mathbb{A}_k^n, 0)\big[\tfrac{1}{p}\big]\big).$$

For $n < 0$, we set $H^m(X, \mathbb{Z}(n)) = H^m(X, \mathbb{Z}[1/p](n)) = 0$.

3.2. ***The motivic cohomology spectrum.*** In order to represent the motivic cohomology of a singular scheme X in $\mathcal{SH}_X$, let us recall the Eilenberg–MacLane spectrum

$$H\mathbb{Z} = (K(0, 0), K(1, 2), \dots, K(n, 2n), \dots)$$

in $\mathrm{Spt}(\mathcal{M})$, where $K(n, 2n)$ is the presheaf of simplicial abelian groups on $\mathbf{Sm}_k$ associated to the presheaf of chain complexes $\underline{C}_* z_{\mathrm{equi}}(\mathbb{A}_k^n, 0)$ via the Dold–Kan correspondence. The external product of cycles induces product maps

$$K(m, 2m) \wedge K(n, 2n) \to K(m+n, 2(m+n)).$$

Notice $K(1,2) \cong \underline{C}_*(z_{\mathrm{equi}}(\mathbb{P}^1_k, 0)/z_{\mathrm{equi}}(\mathbb{P}^0_k, 0))$ [Mazza et al. 2006, Theorem 16.8], so composing the product maps with the canonical map

$$g : T \cong \mathbb{P}^1_k/\mathbb{P}^0_k \to \underline{C}_*(z_{\mathrm{equi}}(\mathbb{P}^1_k, 0)/z_{\mathrm{equi}}(\mathbb{P}^0_k, 0)) \cong K(1,2)$$

(where the first map assigns to any morphism $U \to \mathbb{P}^1_k$ its graph in $U \times \mathbb{P}^1_k$), we obtain the bonding maps. $H\mathbb{Z}$ is a symmetric spectrum whose symmetric structure is obtained by permuting the coordinates in $\mathbb{A}^n_k$. We shall not distinguish between a simplicial abelian group and the associated chain complex of abelian groups from now on in this text and will use them interchangeably.

3.3. *Motivic cohomology via $\mathcal{SH}_{\mathrm{cdh}}$.* Let $\mathbf{1} = \Sigma^\infty_T(S^0_s)$ be the sphere spectrum in $\mathcal{SH}$, and let $\mathbf{1}[1/p] \in \mathcal{SH}$ be the homotopy colimit [Neeman 2001, Definition 1.6.4] of the filtering diagram in $\mathcal{SH}$:

$$\mathbf{1} \xrightarrow{p} \mathbf{1} \xrightarrow{p} \mathbf{1} \xrightarrow{p} \cdots$$

where $\mathbf{1} \xrightarrow{r} \mathbf{1}$ is the composition of the sum map with the diagonal $\mathbf{1} \xrightarrow{\Delta} \bigoplus_{i=1}^r \mathbf{1} \xrightarrow{\Sigma} \mathbf{1}$. For $E \in \mathcal{SH}$, we define $E[1/p] \in \mathcal{SH}$ to be $E \wedge \mathbf{1}[1/p]$. This also makes sense in $\mathcal{SH}_X$ and $\mathcal{SH}_{\mathrm{cdh}}$.

The following is a reformulation of the main result in [Friedlander and Voevodsky 2000] when k admits resolution of singularities, and the main result in [Kelly 2012] when k has positive characteristic.

Theorem 3.4 [Cisinski and Déglise 2015]. *Let k be a perfect field of exponential characteristic p, and let $v : X \to \mathrm{Spec}\,(k)$ be a separated scheme of finite type. Then for any $m, n \in \mathbb{Z}$, there is a natural isomorphism*

$$\theta_X : H^m\big(X, \mathbb{Z}\big[\tfrac{1}{p}\big](n)\big) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{SH}_X}\big(\Sigma^\infty_T X_+, \Sigma^{m,n} \boldsymbol{L}v^* H\mathbb{Z}\big[\tfrac{1}{p}\big]\big). \tag{3.5}$$

Proof. Recall that $H^m(X, \mathbb{Z}[1/p](n)) = A_{0,2n-m}(X, \mathbb{A}^n)$ (Definition 3.1). We observe that $\underline{C}_* z_{\mathrm{equi}}(\mathbb{A}^n_k, 0)$ is the motive with compact supports $M^c(\mathbb{A}^n_k)$ of $\mathbb{A}^n_k$ [Voevodsky 2000, §4.1; Mazza et al. 2006, Definition 16.13]. Combining [Voevodsky 2000, Corollary 4.1.8] (or [Mazza et al. 2006, Theorem 16.7, Example 16.14]) with [Cisinski and Déglise 2015, 4.2, Proposition 4.3, Theorem 5.1 and Corollary 8.6], we conclude that

$$H^m\big(X, \mathbb{Z}\big[\tfrac{1}{p}\big](n)\big) \cong \mathrm{Hom}_{\mathcal{SH}_X}\big(\Sigma^{2n-m,0}(\Sigma^\infty_T X_+), \Sigma^{2n,n} \boldsymbol{L}v^* H\mathbb{Z}\big[\tfrac{1}{p}\big]\big),$$

which finishes the proof. □

As a combination of Theorem 2.14 and Theorem 3.4, we get a corollary:

Corollary 3.6. *Under the hypothesis and with the notation of Theorem 3.4, there are natural isomorphisms*

$$\begin{aligned} H^m\big(X, \mathbb{Z}[\tfrac{1}{p}](n)\big) &\cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma_T^\infty X_+,\, \Sigma^{m,n} \boldsymbol{L}\pi^* H\mathbb{Z}[\tfrac{1}{p}]\big) \\ &\cong \mathrm{Hom}_{\mathcal{SH}_X}\big(\Sigma_T^\infty X_+,\, \Sigma^{m,n} \boldsymbol{L}v^* H\mathbb{Z}[\tfrac{1}{p}]\big). \end{aligned}$$

4. Slice spectral sequence for singular schemes

Let k be a perfect field of exponential characteristic p. Given $X \in \mathbf{Sch}_k$, recall that Voevodsky's slice filtration of $\mathcal{SH}_X$ is given as follows. For an integer $q \in \mathbb{Z}$, let $\Sigma_T^q \mathcal{SH}_X^{\mathrm{eff}}$ denote the smallest full triangulated subcategory of $\mathcal{SH}_X$ which contains C_{eff}^q and is closed under arbitrary coproducts, where

$$C_{\mathrm{eff}}^q = \{\Sigma^{m,n}\Sigma_T^\infty Y_+ : m, n \in \mathbb{Z},\ n \geq q,\ Y \in \mathbf{Sm}_X\}. \tag{4.1}$$

In particular, $\mathcal{SH}_X^{\mathrm{eff}}$ is the smallest full triangulated subcategory of $\mathcal{SH}_X$ which is closed under infinite direct sums and contains all spectra of the type $\Sigma_T^\infty Y_+$ with $Y \in \mathbf{Sm}_X$. The slice filtration of $\mathcal{SH}_X$ [Voevodsky 2002b] is the sequence of full triangulated subcategories

$$\cdots \subseteq \Sigma_T^{q+1}\mathcal{SH}_X^{\mathrm{eff}} \subseteq \Sigma_T^{q}\mathcal{SH}_X^{\mathrm{eff}} \subseteq \Sigma_T^{q-1}\mathcal{SH}_X^{\mathrm{eff}} \subseteq \cdots.$$

It follows from [Neeman 1996; 2001] that the inclusion $i_q : \Sigma_T^q \mathcal{SH}_X^{\mathrm{eff}} \to \mathcal{SH}_X$ admits a right adjoint $r_q : \mathcal{SH}_X \to \Sigma_T^q \mathcal{SH}_X^{\mathrm{eff}}$ and the functors $f_q, s_{<q}, s_q : \mathcal{SH}_X \to \mathcal{SH}_X$ are triangulated, where $r_q \circ i_q$ is the identity, $f_q = i_q \circ r_q$ and $s_{<q}, s_q$ are characterized by the existence of the distinguished triangles

$$\begin{aligned} f_q E &\longrightarrow E \longrightarrow s_{<q} E, \\ f_{q+1} E &\longrightarrow f_q E \longrightarrow s_q E \end{aligned} \tag{4.2}$$

in $\mathcal{SH}_X$ for every $E \in \mathcal{SH}_X$.

Definition 4.3. Let $a, b, n \in \mathbb{Z}$ and $Y \in \mathbf{Sm}_X$. Let $F^n E^{a,b}(Y)$ be the image of the map induced by $f_n E \to E$ in (4.2):

$$\mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{a,b} f_n E) \to \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{a,b} E).$$

This determines a decreasing filtration $F^\bullet$ on $E^{a,b}(Y) = \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{a,b} E)$, and we write $\mathrm{gr}^n F^\bullet$ for the associated graded $F^n E^{a,b}(Y)/F^{n+1}E^{a,b}(Y)$.

The following result is well known; see [Voevodsky 2002b, §2].

Proposition 4.4. *The filtration $F^\bullet$ on $E^{a,b}(Y)$ is exhaustive* (in the sense of [Boardman 1999, Definition 2.1]).

Proof. Recall that $\mathcal{SH}_X$ is a compactly generated triangulated category in the sense of [Neeman 1996, Definition 1.7], with set of compact generators [Ayoub 2007b, Théorème 4.5.67] $\bigcup_{q\in\mathbb{Z}} C^q_{\text{eff}}$ (see (4.1)). Therefore a map $f : E_1 \to E_2$ in $\mathcal{SH}_X$ is an isomorphism if and only if for every $Y \in \mathbf{Sm}_X$ and every $m, n \in \mathbb{Z}$ the induced map of abelian groups $\operatorname{Hom}_{\mathcal{SH}_X}(\Sigma^{m,n}Y_+, E_1) \to \operatorname{Hom}_{\mathcal{SH}_X}(\Sigma^{m,n}Y_+, E_2)$ is an isomorphism. Thus, we conclude that $E \cong \operatorname{hocolim} f_q E$ in $\mathcal{SH}_X$.

Therefore, we deduce that for every $a, b \in \mathbb{Z}$ and every $Y \in \mathbf{Sm}_X$, there exist the isomorphisms

$$\begin{aligned}\operatorname*{colim}_{n\to-\infty} F^n E^{a,b}(Y) &\cong \operatorname*{colim}_{n\to-\infty} \operatorname{Hom}_{\mathcal{SH}_X}(\Sigma^\infty_T Y_+, \Sigma^{a,b} f_n E)\\ &\cong \operatorname{Hom}_{\mathcal{SH}_X}(\Sigma^\infty_T Y_+, \Sigma^{a,b} \operatorname{hocolim} f_q E) \cong E^{a,b}(Y)\end{aligned}$$

[Neeman 1996, Lemma 2.8; Isaksen 2005, Theorem 6.8], so the filtration $F^\bullet$ is exhaustive. □

4.5. *The slice spectral sequence.* Consider $Y \in \mathbf{Sm}_X$ a smooth X-scheme and $G \in \mathcal{SH}_X$. Since $\mathcal{SH}_X$ is a triangulated category, the collection of distinguished triangles $\{f_{q+1}G \to f_q G \to s_q G\}_{q\in\mathbb{Z}}$ determines a (slice) spectral sequence

$$E_1^{p,q} = \operatorname{Hom}_{\mathcal{SH}_X}(\Sigma^\infty_T Y_+, \Sigma^{p+q}_s s_p G)$$

with $G^{*,*}(Y)$ as its abutment and differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$.

In order to study the convergence of this spectral sequence, recall from [Voevodsky 2002b, p. 22] that $G \in \mathcal{SH}_X$ is called *bounded* with respect to the slice filtration if for every $m, n \in \mathbb{Z}$ and every $Y \in \mathbf{Sm}_X$, there exists $q \in \mathbb{Z}$ such that

$$\operatorname{Hom}_{\mathcal{SH}_X}(\Sigma^{m,n}\Sigma^\infty_T Y_+, f_{q+i}G) = 0 \tag{4.6}$$

for every $i > 0$. Clearly the slice spectral sequence is strongly convergent when G is bounded.

Proposition 4.7. *Let k be a field with resolution of singularities. Let $F \in \mathcal{SH}$ be bounded with respect to the slice filtration and let $G = \boldsymbol{L}v^*F \in \mathcal{SH}_X$ with $v : X \to \operatorname{Spec} k$. Then G is bounded with respect to the slice filtration.*

Proof. Since the base field k admits resolution of singularities, we deduce by [Pelaez 2013, Theorem 3.7] that $f_q G \cong \boldsymbol{L}v^* f_q F$ in $\mathcal{SH}_X$ for every $q \in \mathbb{Z}$. It follows from Theorem 2.14 that for every $m, n \in \mathbb{Z}$ and every $Y \in \mathbf{Sm}_X$, we have

$$\operatorname{Hom}_{\mathcal{SH}_X}(\Sigma^{m,n}\Sigma^\infty_T Y_+, f_{q+i}G) \cong \operatorname{Hom}_{\mathcal{SH}_{\text{cdh}}}(\Sigma^{m,n}\Sigma^\infty_T Y_+, \boldsymbol{L}\pi^*(f_{q+i}F))$$

for every $i > 0$. If $X \in \mathbf{Sm}_k$, then $Y \in \mathbf{Sm}_k$ and we have

$$\operatorname{Hom}_{\mathcal{SH}_{\text{cdh}}}(\Sigma^{m,n}\Sigma^\infty_T Y_+, \boldsymbol{L}\pi^*(f_{q+i}F)) \cong \operatorname{Hom}_{\mathcal{SH}}(\Sigma^{m,n}\Sigma^\infty_T Y_+, f_{q+i}F)$$

for every $i > 0$ by Corollary 2.17. Since F is bounded with respect to the slice filtration, we deduce from (4.6) that G is also bounded in $\mathcal{SH}_X$ in this case.

Finally, we proceed by induction on the dimension of Y, and assume that for every $m, n \in \mathbb{Z}$ and every $Y' \in \mathbf{Sch}_k$ with $\dim(Y') < \dim(Y)$, there exists $q \in \mathbb{Z}$ such that

$$\mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m,n}\Sigma_T^\infty Y'_+, \boldsymbol{L}\pi^*(f_{q+i}F)) = 0$$

for every $i > 0$. Since the base field k admits resolution of singularities, there exists a cdh-cover $\{X' \amalg Z \to Y\}$ of Y such that $X' \in \mathbf{Sm}_k$, $\dim(Z) < \dim(Y)$ and $\dim(W) < \dim(Y)$, where we set $W = X' \times_Y Z$.

Let q_1, q_2 and q_3 be the integers such that the vanishing condition (4.6) holds for (X', m, n), (Z, m, n) and $(W, m+1, n)$, respectively. Let q be the maximum of q_1, q_2 and q_3. Then by cdh-excision, for every $i > 0$, the following diagram is exact:

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m+1,n}\Sigma_T^\infty W, \boldsymbol{L}\pi^*(f_{q+i}F)) \\
&\qquad\qquad \to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m,n}\Sigma_T^\infty Y_+, \boldsymbol{L}\pi^*(f_{q+i}F)) \\
&\qquad\qquad \to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m,n}\Sigma_T^\infty X'_+, \boldsymbol{L}\pi^*(f_{q+i}F)) \\
&\qquad\qquad\qquad\qquad \oplus \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{m,n}\Sigma_T^\infty Z_+, \boldsymbol{L}\pi^*(f_{q+i}F)).
\end{aligned}$$

By choice of q, both ends in the diagram vanish. Hence the group in the middle also vanishes as we wanted. □

In order to get convergence results in positive characteristic, we need to restrict to spectra $E \in \mathcal{SH}$ which admit a structure of traces [Kelly 2012, Definitions 4.2.27 and 4.3.1].

Lemma 4.8. *With the notation of* (2.11), *let* $X \in \mathbf{Sch}_k$.

(1) *For every* $E \in \mathcal{SH}$, $\boldsymbol{L}\pi^*\big(E\big[\frac{1}{p}\big]\big) \cong (\boldsymbol{L}\pi^*E)\big[\frac{1}{p}\big]$ *and* $\boldsymbol{L}v^*\big(E\big[\frac{1}{p}\big]\big) \cong (\boldsymbol{L}v^*E)\big[\frac{1}{p}\big]$.

(2) *For every* $E \in \mathcal{SH}_{\mathrm{cdh}}$ *and every* $a, b \in \mathbb{Z}$,

$$\mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{a,b}\Sigma_T^\infty(X_+), E\big[\tfrac{1}{p}\big]\big) \cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma^{a,b}\Sigma_T^\infty(X_+), E) \otimes \mathbb{Z}\big[\tfrac{1}{p}\big].$$

Proof. (1): It follows from the definition of homotopy colimit [Neeman 2001, Definition 1.6.4] that $\boldsymbol{L}\pi^*$ and $\boldsymbol{L}v^*$ commute with homotopy colimits since they are left adjoint. This implies the result since $E[1/p]$ is given in terms of homotopy colimits.

(2): Since $\Sigma^{a,b}\Sigma_T^\infty(X_+)$ is compact in $\mathcal{SH}_{\mathrm{cdh}}$ [Ayoub 2007b, Théorème 4.5.67], the result follows from [Neeman 1996, Lemma 2.8]. □

Lemma 4.9. *Let* $X \in \mathbf{Sch}_k$ *and* $E \in \mathcal{SH}_X$. *Then for every* $r \in \mathbb{Z}$,

$$f_r\big(E\big[\tfrac{1}{p}\big]\big) \cong (f_rE)\big[\tfrac{1}{p}\big] \quad \textit{and} \quad s_r\big(E\big[\tfrac{1}{p}\big]\big) \cong (s_rE)\big[\tfrac{1}{p}\big].$$

Proof. Since the effective categories $\Sigma_T^q \mathcal{SH}_X^{\text{eff}}$ are closed under infinite direct sums, we conclude that the functors f_r, s_r commute with homotopy colimits. $\square$

Proposition 4.10. *Let $F \in \mathcal{SH}$ and $G = \boldsymbol{L}v^* F \in \mathcal{SH}_X$ with $v : X \to \operatorname{Spec} k$. Assume that for every $r \in \mathbb{Z}$, $s_r(F[1/p])$ has a weak structure of smooth traces* (in the sense of [Kelly 2012, Definition 4.2.27]), *and that $F[1/p]$ has a structure of traces* (in the sense of [Kelly 2012, Definition 4.3.1]). *If $F[1/p]$ is bounded with respect to the slice filtration, then $G[1/p]$ is bounded as well.*

Proof. Since the base field k is perfect and $F[1/p]$ is clearly $\mathbb{Z}[1/p]$-local, combining [Kelly 2012, Theorem 4.2.29] and Lemma 4.9, we conclude that $f_q G[1/p] \cong \boldsymbol{L}v^* f_q F[1/p]$ in $\mathcal{SH}_X$ for every $q \in \mathbb{Z}$.

It follows from Theorem 2.14 that for every $m, n \in \mathbb{Z}$ and every $Y \in \mathbf{Sm}_X$, we have

$$\operatorname{Hom}_{\mathcal{SH}_X}\big(\Sigma^{m,n}\Sigma_T^\infty(Y_+),\, f_{q+i} G\big[\tfrac{1}{p}\big]\big) \cong \operatorname{Hom}_{\mathcal{SH}_{\text{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Y_+),\, \boldsymbol{L}\pi^*\big(f_{q+i} F\big[\tfrac{1}{p}\big]\big)\big)$$

for every $i > 0$. If $X \in \mathbf{Sm}_k$, then $Y \in \mathbf{Sm}_k$ and we have

$$\operatorname{Hom}_{\mathcal{SH}_{\text{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Y_+),\, \boldsymbol{L}\pi^*\big(f_{q+i} F\big[\tfrac{1}{p}\big]\big)\big) \cong \operatorname{Hom}_{\mathcal{SH}}\big(\Sigma^{m,n}\Sigma_T^\infty(Y_+),\, f_{q+i} F\big[\tfrac{1}{p}\big]\big)$$

for every $i > 0$ by Corollary 2.17. Since $F[1/p]$ is bounded with respect to the slice filtration, we deduce from (4.6) that $G[1/p]$ is also bounded with respect to the slice filtration in $\mathcal{SH}_X$ in this case.

Finally, we proceed by induction on the dimension of Y, and assume that for every $m, n \in \mathbb{Z}$ and every $Z \in \mathbf{Sch}_k$ with $\dim_k(Z) < \dim_k(Y)$, there exists $q \in \mathbb{Z}$ such that

$$\operatorname{Hom}_{\mathcal{SH}_{\text{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Z_+),\, \boldsymbol{L}\pi^*\big(f_{q+i} F\big[\tfrac{1}{p}\big]\big)\big) = 0$$

for every $i > 0$.

Since k is perfect, by a theorem of Gabber [Illusie et al. 2014, Théorème 3(1)] and Temkin's strengthening [2017, Theorem 1.2.9] of Gabber's result, there exists $W \in \mathbf{Sm}_k$ and a surjective proper map $h : W \to Y$, which is generically étale of degree p^r, $r \geq 1$. In particular, h is generically flat, and thus by a theorem of Raynaud and Gruson [1971, Théorème 5.2.2], there exists a blow-up $g : Y' \to Y$ with center Z such that the following diagram commutes, where h' is finite flat surjective of degree p^r and $g' : W' \to W$ is the blow-up of W with center $h^{-1}(Z)$:

$$\begin{tikzcd} W' \arrow[r, "h'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ W \arrow[r, "h"'] & Y \end{tikzcd} \tag{4.11}$$

Thus we have a cdh-cover $\{Y' \amalg Z \to Y\}$ of Y such that $\dim_k(Z) < \dim_k(Y)$ and $\dim_k(E) < \dim_k(Y)$, where we set $E = Y' \times_Y Z$.

Let q_1 (resp. q_2, q_3) be the integers such that the vanishing condition (4.6) holds for (W, m, n) (resp. (Z, m, n), $(E, m+1, n)$). Let q be the maximum of q_1, q_2 and q_3. Then by cdh-excision, for every $i > 0$, the following diagram is exact:

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m+1,n}\Sigma_T^\infty(E_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)\\ &\qquad\to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Y_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)\\ &\qquad\to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Y'_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)\\ &\qquad\qquad\oplus \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Z_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big).\end{aligned}$$

By the choice of q, this reduces to the exact diagram

$$\begin{aligned}0 &\to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Y_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)\\ &\qquad\xrightarrow{g^*} \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Y'_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big).\end{aligned}$$

So it suffices to show that $g^* = 0$. In order to prove this, we observe that the diagram (4.11) commutes. Therefore, by the choice of q,

$$\mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(W_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big) = 0,$$

and we conclude that $h'^* \circ g^* = g'^* \circ h^* = 0$. Thus, it is enough to see that

$$\begin{aligned}h'^* &: \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(Y'_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)\\ &\qquad\to \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}\big(\Sigma^{m,n}\Sigma_T^\infty(W'_+), \boldsymbol{L}\pi^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)\big)\end{aligned}$$

is injective. Let $v' : Y' \to \operatorname{Spec} k$, and let

$$\epsilon : \boldsymbol{L}v'^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big) \to \boldsymbol{R}h'_*\boldsymbol{L}h'^*\boldsymbol{L}v'^*\big(f_{q+i}F\big[\tfrac{1}{p}\big]\big)$$

be the map given by the unit of the adjunction $(\boldsymbol{L}h'^*, \boldsymbol{R}h'_*)$. By the naturality of the isomorphism in Proposition 2.13 we deduce that h'^* gets identified with the map induced by ϵ:

$$\begin{aligned}\epsilon_* &: \mathrm{Hom}_{\mathcal{SH}_{Y'}}\big(\Sigma^{m,n}\Sigma_T^\infty Y'_+, \boldsymbol{L}v'^* f_{q+i}F\big[\tfrac{1}{p}\big]\big)\\ &\qquad\to \mathrm{Hom}_{\mathcal{SH}_{Y'}}\big(\Sigma^{m,n}\Sigma_T^\infty Y'_+, \boldsymbol{R}h'_*\boldsymbol{L}h'^*\boldsymbol{L}v'^* f_{q+i}F\big[\tfrac{1}{p}\big]\big).\end{aligned}$$

Since $F[1/p]$ has a structure of traces and $s_r(F[1/p])$ has a weak structure of smooth traces for every $r \in \mathbb{Z}$, it follows from [Kelly 2012, Proposition 4.3.7] that $f_{q+i}(F[1/p])$ has a structure of traces in the sense of [Kelly 2012, Definition 4.3.1]. Thus, we deduce from [Kelly 2012, Definition 4.3.1(Deg), p. 101] that ϵ_* is injective, since h' is finite flat surjective of degree p^r. This finishes the proof. □

If we only assume that the slices $s_r E$ have a structure of traces, then we get the weaker conditions of Proposition 4.15.

Corollary 4.12. *Let $F \in \mathcal{SH}$ and $G = \boldsymbol{L}v^*F \in \mathcal{SH}_X$, where $v : X \to \operatorname{Spec} k$ is the structure map. Assume that the following hold.*

(1) *For every $r \in \mathbb{Z}$, $s_r(F[1/p])$ has a structure of traces* (in the sense of [Kelly 2012, Definition 4.3.1]).

(2) *$F[1/p]$ is bounded with respect to the slice filtration.*

Then for every m, n in $\mathbb{Z}$, there exists $q \in \mathbb{Z}$ such that

$$\operatorname{Hom}_{\mathcal{SH}_X}\bigl(\Sigma^{m,n}\Sigma_T^\infty X_+,\, s_{q+i}G\bigl[\tfrac{1}{p}\bigr]\bigr) = 0$$

for every $i > 0$ (see (4.6)).

Proof. Since $s_r(F[1/p])$ has a structure of traces, we observe that in particular $s_r(F[1/p])$ has a weak structure of smooth traces [Kelly 2012, Definition 4.2.27]. Thus, combining Lemma 4.8, Lemma 4.9 and [Kelly 2012, Theorem 4.2.29] we conclude that for every $r \in \mathbb{Z}$,

$$s_r G\bigl[\tfrac{1}{p}\bigr] \cong \boldsymbol{L}v^* s_r F\bigl[\tfrac{1}{p}\bigr] \quad \text{and} \quad f_r G\bigl[\tfrac{1}{p}\bigr] \cong \boldsymbol{L}v^* f_r F\bigl[\tfrac{1}{p}\bigr].$$

If $X \in \mathbf{Sm}_k$, we have

$$\operatorname{Hom}_{\mathcal{SH}_X}\bigl(\Sigma^{m,n}\Sigma_T^\infty X_+,\, \boldsymbol{L}v^*\bigl(s_{q+i}F\bigl[\tfrac{1}{p}\bigr]\bigr)\bigr) \cong \operatorname{Hom}_{\mathcal{SH}}\bigl(\Sigma^{m,n}\Sigma_T^\infty X_+,\, s_{q+i}F\bigl[\tfrac{1}{p}\bigr]\bigr)$$

for every $i > 0$ by Corollary 2.17. Since $F[1/p]$ is bounded with respect to the slice filtration, there exist q_1 and $q_2 \in \mathbb{Z}$ such that the vanishing condition (4.6) holds for (X, m, n) and $(X, m-1, n)$, respectively. Let q be the maximum of q_1 and q_2. Then using the distinguished triangle $f_{q+i}F[1/p] \to s_{q+i}F[1/p] \to \Sigma_s^1 f_{q+i+1}F[1/p]$ in $\mathcal{SH}$ we conclude that $\operatorname{Hom}_{\mathcal{SH}}(\Sigma^{m,n}\Sigma_T^\infty(X_+), s_{q+i}F[1/p]) = 0$ for every $i > 0$, as we wanted.

When $X \in \mathbf{Sch}_k$, the argument in the proof of Proposition 4.10 works mutatis mutandis replacing $f_{q+i}F[1/p]$ with $s_{q+i}F[1/p]$, since for every $j \in \mathbb{Z}$, $s_jF[1/p]$ has a structure of traces. □

Corollary 4.13. *Assume the conditions (1) and (2) of Corollary 4.12 hold. Then for every $m, n \in \mathbb{Z}$, there exists $q \in \mathbb{Z}$ such that the map $f_{q+i+1}G[1/p] \to f_{q+i}G[1/p]$ induces an isomorphism*

$$\operatorname{Hom}_{\mathcal{SH}_X}\bigl(\Sigma^{m,n}\Sigma_T^\infty X_+,\, f_{q+i+1}G\bigl[\tfrac{1}{p}\bigr]\bigr) \cong \operatorname{Hom}_{\mathcal{SH}_X}\bigl(\Sigma^{m,n}\Sigma_T^\infty X_+,\, f_{q+i}G\bigl[\tfrac{1}{p}\bigr]\bigr)$$

for every $i > 0$.

Proof. Let $q_1, q_2 \in \mathbb{Z}$ be the integers corresponding to (m, n), $(m+1, n)$ in Corollary 4.12, respectively. Let q be the maximum of q_1 and q_2. Then the result follows by combining the vanishing in Corollary 4.12 with the distinguished triangle

$$\Sigma_s^{-1}s_{q+i}\bigl[\tfrac{1}{p}\bigr] \to f_{q+i+1}G\bigl[\tfrac{1}{p}\bigr] \to f_{q+i}G\bigl[\tfrac{1}{p}\bigr] \to s_{q+i}G\bigl[\tfrac{1}{p}\bigr]$$

in $\mathcal{SH}_X$. □

Remark 4.14. Combining Definition 4.3 and Corollary 4.13, we deduce that for every $a, b \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$ such that

$$F^n G\big[\tfrac{1}{p}\big]^{a,b}(X) = F^m G\big[\tfrac{1}{p}\big]^{a,b}(X)$$

for every $n \geq m$.

Proposition 4.15. *Assume the conditions (1) and (2) of Corollary 4.12 hold. Then for every $n \in \mathbb{Z}$, the slice spectral sequence*

$$E_1^{a,b}(X, n) = \mathrm{Hom}_{\mathcal{SH}_X}\big(\Sigma_T^\infty X_+,\, \Sigma^{a+b+n,n} s_a G\big[\tfrac{1}{p}\big]\big) \Rightarrow G\big[\tfrac{1}{p}\big]^{a+b+n,n}(X)$$

(see Section 4.5) *satisfies the following.*

(1) *For every $a, b \in \mathbb{Z}$, there exists $N > 0$ such that $E_r^{a,b} = E_\infty^{a,b}$ for $r \geq N$, where $E_\infty^{a,b}$ is the associated graded* $\mathrm{gr}^a F^\bullet$ *with respect to the descending filtration $F^\bullet$ on $G[1/p]^{a+b+n,n}(X)$* (see Definition 4.3).

(2) *For every $m, n \in \mathbb{Z}$, the descending filtration $F^\bullet$ on $G[1/p]^{m,n}(X)$ is exhaustive and complete* (see [Boardman 1999, Definition 2.1]).

Proof. (1): It suffices to show that for every $a, b \in \mathbb{Z}$ only finitely many of the differentials $d_r : E_r^{a,b} \to E_r^{a+r,b-r+1}$ are nonzero. But this follows from Corollary 4.12.

(2): By Proposition 4.4, the filtration $F^\bullet$ on $G[1/p]^{m,n}(X)$ is exhaustive. Finally, the completeness of $F^\bullet$ follows by combining Remark 4.14 with [Boardman 1999, Propositions 1.8 and 2.2(c)]. □

4.16. *The slice spectral sequence for* $\mathbf{MGL}(X)$. Our aim here is to apply the results of the previous sections to obtain a Hopkins–Morel type spectral sequence for $\mathrm{MGL}^{*,*}(X)$ when X is a singular scheme. For smooth schemes, the Hopkins–Morel spectral sequence has been studied in [Levine 2009; Hoyois 2015], and over Dedekind domains in [Spitzweck 2014].

Recall from [Voevodsky 1998, §6.3] that for any noetherian scheme S of finite Krull dimension, the scheme $\mathrm{Gr}_S(N, n)$ parametrizes n-dimensional linear subspaces of $\mathbb{A}_S^N$, and one writes $\mathrm{BGL}_{S,n} = \mathrm{colim}_N \mathrm{Gr}_S(N, n)$. There is a universal rank n bundle $U_{S,n} \to \mathrm{BGL}_{S,n}$, and one denotes the Thom space $\mathrm{Th}(U_{S,n})$ of this bundle by $\mathrm{MGL}_{S,n}$. Using the fact that the Thom space of a direct sum is the smash product of the corresponding Thom spaces and $T = \mathrm{Th}(\mathcal{O}_S)$, one gets a T-spectrum $\mathrm{MGL}_S = (\mathrm{MGL}_{S,0}, \mathrm{MGL}_{S,1}, \dots) \in \mathrm{Spt}(\mathcal{M}_S)$. There is a structure of symmetric spectrum on MGL_S, for which we refer to [Panin et al. 2008, §2.1].

We now let k be a field of characteristic zero and let $X \in \mathbf{Sch}_k$. We use MGL as a short hand for MGL_k throughout this text. It follows from the above definition of MGL_X (which shows that MGL_X is constructed from presheaves represented by smooth schemes) and Proposition 2.12 that the canonical map $\boldsymbol{L}v^*(\mathrm{MGL}) \to \mathrm{MGL}_X$ is an isomorphism.

Definition 4.17. We define $\mathrm{MGL}^{*,*}(X)$ to be the generalized cohomology groups

$$\begin{aligned}\mathrm{MGL}^{p,q}(X) &:= \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty X_+, \Sigma^{p,q}\,\mathrm{MGL}_X)\\ &\cong \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty X_+, \Sigma^{p,q}\,\boldsymbol{L}v^*\,\mathrm{MGL}).\end{aligned}$$

It follows from Theorem 2.14 that

$$\mathrm{MGL}^{p,q}(X) \cong \mathrm{Hom}_{\mathcal{SH}_{\mathrm{cdh}}}(\Sigma_T^\infty X_+, \Sigma^{p,q}\boldsymbol{L}\pi^*\,\mathrm{MGL}). \tag{4.18}$$

We now construct the spectral sequence for $\mathrm{MGL}^{*,*}(X)$ using the exact couple technique as follows. For $p, q, n \in \mathbb{Z}$, define

$$\begin{aligned}A^{p,q}(X, n) &:= [\Sigma_T^\infty X_+, \Sigma_s^{p+q-n}\Sigma_t^n(f_p\,\mathrm{MGL}_X)],\\ E^{p,q}(X, n) &:= [\Sigma_T^\infty X_+, \Sigma_s^{p+q-n}\Sigma_t^n s_p\,\mathrm{MGL}_X].\end{aligned}$$

Here, $[-,-]$ denotes the morphisms in $\mathcal{SH}_X$. It follows from (4.2) that there is an exact sequence

$$A^{p+1,q-1}(X, n) \xrightarrow{a_n^{p,q}} A^{p,q}(X, n) \xrightarrow{b_n^{p,q}} E^{p,q}(X, n) \xrightarrow{c_n^{p,q}} A^{p+1,q}(X, n). \tag{4.19}$$

Set $D_1(X, n) := \bigoplus_{p,q} A^{p,q}(X, n)$ and $E_1(X, n) := \bigoplus_{p,q} E^{p,q}(X, n)$. Write $a_n^1 := \bigoplus a_n^{p,q}$, $b_n^1 := \bigoplus b_n^{p,q}$ and $c_n^1 := \bigoplus c_n^{p,q}$. This gives an exact couple $\{D_n^1, E_n^1, a_n^1, b_n^1, c_n^1\}$ and the map $d_n^1 = b_n^1 \circ c_n^1 : E_n^1 \to E_n^1$ shows that (E_1, d_1) is a complex. Thus, by repeatedly taking the homology functors, we obtain a spectral sequence.

For the target of the spectral sequence, let $A^m(X, n) := \mathrm{colim}_{q\to\infty} A^{m-q,q}(X, n)$. Since X is a compact object of $\mathcal{SH}_X$ (see [Voevodsky 1998, Proposition 5.5; Ayoub 2007b, Théorème 4.5.67]), the colimit enters into $[-,-]$ so that

$$A^m(X, n) = [\Sigma_T^\infty X_+, \Sigma_s^{m-n}\Sigma_t^n\,\mathrm{MGL}_X] = \mathrm{MGL}_X^{m,n}(X).$$

The formalism of exact couples then yields a spectral sequence

$$E_1^{p,q}(X, n) = E_1^{p,q} \Rightarrow \mathrm{MGL}_X^{m,n}(X). \tag{4.20}$$

We now have

$$\begin{aligned}E_1^{p,q}(X, n) = {}& [\Sigma_T^\infty X_+, \Sigma_s^{p+q-n}\Sigma_t^n s_p\,\mathrm{MGL}_X]\\ \cong^1 {}& [\Sigma_T^\infty X_+, \Sigma_s^{p+q-n}\Sigma_t^n s_p\boldsymbol{L}v^*\,\mathrm{MGL}]\\ \cong^2 {}& [\Sigma_T^\infty X_+, \Sigma_s^{p+q-n}\Sigma_t^n\boldsymbol{L}v^*(s_p\,\mathrm{MGL})]\\ \cong^3 {}& [\Sigma_T^\infty X_+, \Sigma_s^{p+q-n}\Sigma_t^n\boldsymbol{L}v^*(\Sigma_T^p H(\mathbb{L}^{-p}))]\\ \cong {}& [\Sigma_T^\infty X_+, \Sigma_s^{p+q-n}\Sigma_t^n\Sigma_T^p\boldsymbol{L}v^*(H(\mathbb{L}^{-p}))].\end{aligned} \tag{4.21}$$

In this sequence of isomorphisms, $\cong^1$ is shown above, $\cong^2$ follows from [Pelaez 2013, Theorem 3.7] and $\cong^3$ follows from the isomorphism $s_p\,\mathrm{MGL} \xrightarrow{\cong} \Sigma_T^p H(\mathbb{L}^{-p})$, as shown, for example, in [Hoyois 2015, (8.6)], where $\mathbb{L} = \bigoplus_{i \le 0} \mathbb{L}^i \cong \bigoplus_{i \ge 0} MU_{2i}$ is the Lazard ring.

Since $\mathbb{L}$ is a torsion-free abelian group, it follows from Corollary 3.6 that the last term of (4.21) is the same as $H^{3p+q}(X, \mathbb{Z}(n+p)) \otimes_{\mathbb{Z}} \mathbb{L}^{-p}$.

The spectral sequence (4.20) is actually identical to an E_2-spectral sequence after reindexing. Indeed, letting

$$\widetilde{E}_2^{p',q'} = H^{p'-q'}(X, \mathbb{Z}(n-q')) \otimes_{\mathbb{Z}} \mathbb{L}^{q'}$$

and using (4.21), an elementary calculation shows that the invertible transformation $(3p+q, n+p) \mapsto (p'-q', n-q')$ yields

$$\begin{aligned} E_1^{p+1,q} &\cong [\Sigma_T^\infty X_+, \Sigma_s^{p+q+1-n} \Sigma_t^n s_{p+1} \mathrm{MGL}_X] \\ &\cong H^{(p'+2)-(q'-1)}(X, \mathbb{Z}(n-(q'-1))) \otimes_{\mathbb{Z}} \mathbb{L}^{q'-1} = \widetilde{E}_2^{p'+2,q'-1}. \end{aligned} \tag{4.22}$$

It is clear from (4.19) that the E_1-differential of the above spectral sequence is $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ and (4.22) shows that this differential is identified with the differential

$$d_2^{p',q'} = d_1^{p,q} : \widetilde{E}_2^{p',q'} \to \widetilde{E}_2^{p'+2,q'-1}.$$

Inductively, it follows that the chain complex $\{E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}\}$ is transformed to the chain complex $\{\widetilde{E}_{r+1}^{p',q'} \xrightarrow{d_r} \widetilde{E}_{r+1}^{p'+r+1,q'-r}\}$. Combining this with (4.18), we conclude the following.

Theorem 4.23. *Let k be a field which has characteristic zero and let $X \in$ $\mathbf{Sch}_k$. Then for any integer $n \in \mathbb{Z}$, there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(n-q)) \otimes_{\mathbb{Z}} \mathbb{L}^q \Rightarrow \mathrm{MGL}^{p+q,n}(X). \tag{4.24}$$

The differentials of this spectral sequence are given by $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, and for every p, $q \in \mathbb{Z}$, there exists $N > 0$ such that $E_r^{p,q} = E_\infty^{p,q}$ for $r \ge N$, where $E_\infty^{p,q}$ is the associated graded $\mathrm{gr}^{-q} F^\bullet$ *with respect to the descending filtration on* $\mathrm{MGL}^{p+q,n}(X)$ (see Definition 4.3). *Furthermore, this spectral sequence degenerates with rational coefficients.*

Proof. The construction of the spectral sequence is shown above. Since MGL is bounded by [Hoyois 2015, Theorem 8.12], it follows from Proposition 4.7 that the spectral sequence (4.24) is strongly convergent. Thus, we deduce the existence of $N > 0$ such that $E_r^{p,q} = E_\infty^{p,q}$ for $r \ge N$.

As for the degeneration with rational coefficients, we observe that the maps $f_p\, \mathrm{MGL} \to s_p\, \mathrm{MGL} \cong \Sigma_T^p H(\mathbb{L}^{-p})$ rationally split to yield an isomorphism of spectra $\mathrm{MGL}_{\mathbb{Q}} \xrightarrow{\cong} \oplus_{p \ge 0} \Sigma_T^p H(\mathbb{L}_{\mathbb{Q}}^{-p})$ in $\mathcal{SH}$ [Naumann et al. 2009, Theorem 10.5 and Corollary 10.6(i)]. The desired degeneration of the spectral sequence now follows immediately from its construction above. $\square$

Remark 4.25. If k is a perfect field of positive characteristic p, we observe that $s_r(\mathrm{MGL}[1/p]) \cong \Sigma^r_T H(\mathbb{L}^{-r})[1/p]$ for every $r \in \mathbb{Z}$ [Hoyois 2015, (8.6)], and so $s_r(\mathrm{MGL}[1/p])$ has a weak structure of smooth traces [Kelly 2012, Corollary 5.2.4]. Thus, we can apply [Kelly 2012, Theorem 4.2.29] to conclude $\boldsymbol{L}v^* s_r(\mathrm{MGL}[1/p]) \cong s_r(\boldsymbol{L}v^* \mathrm{MGL}[1/p])$. Except for this identification, the proof of Theorem 4.23 does not depend on the characteristic of k. We thus obtain a spectral sequence as in (4.24):

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(n-b)) \otimes_{\mathbb{Z}} \mathbb{L}^b\big[\tfrac{1}{p}\big] \Rightarrow \mathrm{MGL}^{a+b,n}(X)\big[\tfrac{1}{p}\big].$$

But we can only guarantee strong convergence when $X \in \mathbf{Sm}_k$ [Hoyois 2015, Theorem 8.12]. In general, for $X \in \mathbf{Sch}_k$, the spectral sequence satisfies the weaker convergence of Proposition 4.15(1)–(2). In this case, the strong convergence would follow if one knew that MGL has a structure of traces.

4.26. ***The slice spectral sequence for* KGL.** For any noetherian scheme X of finite Krull dimension, the motivic T-spectrum $\mathrm{KGL}_X \in \mathrm{Spt}(\mathcal{M}_X)$ was defined by Voevodsky [1998, §6.2]. It has the property that it represents algebraic K-theory of objects in $\mathbf{Sm}_X$ if X is regular. It was later shown by Cisinski [2013] that for X not necessarily regular, KGL_X represents Weibel's homotopy invariant K-theory $KH_*(Y)$ for $Y \in \mathbf{Sm}_X$. Like MGL_X, there is a structure of symmetric spectrum on KGL_X, for which we refer to [Jardine 2009, pp. 157 and 176].

Let k be a field of exponential characteristic p. The map $\boldsymbol{L}v^*(\mathrm{KGL}_k) \to \mathrm{KGL}_X$ is an isomorphism by [Cisinski 2013, Proposition 3.8]. It is also known that $s_r\, \mathrm{KGL}_k \cong \Sigma^r_T H\mathbb{Z}$ for $r \in \mathbb{Z}$; see [Levine 2008, Theorem 6.4.2] if k is perfect and [Röndigs and Østvær 2016, §1, p. 1158] in general. It follows from [Pelaez 2013, Theorem 3.7] (in positive characteristic we use [Kelly 2012, Theorem 4.2.29] instead) that $\boldsymbol{L}v^*(s_r\, \mathrm{KGL}[1/p]_k) \cong s_r(\boldsymbol{L}v^* \mathrm{KGL}[1/p]_k) \cong s_r\, \mathrm{KGL}[1/p]_X$. One also knows that $(\mathrm{KGL}_k)_{\mathbb{Q}} \cong \bigoplus_{p \in \mathbb{Z}} \Sigma^p_T H\mathbb{Q}$ in $\mathcal{SH}$ [Riou 2010, Definition 5.3.17 and Theorem 5.3.10]. We can thus use the Bott periodicity of KGL_X and repeat the construction of Section 4.16 mutatis mutandis (with $n = 0$) to conclude the following.

Theorem 4.27. *Let k be a field that admits resolution of singularities (resp. a field of exponential characteristic $p > 1$), and let $X \in \mathbf{Sch}_k$. Then there is a strongly convergent spectral sequence*

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(-b)) \Rightarrow KH_{-a-b}(X) \tag{4.28}$$

$$\Big(\textit{resp.}\quad E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(-b)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \Rightarrow KH_{-a-b}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]\Big). \tag{4.29}$$

The differentials of this spectral sequence are given by $d_r : E_r^{a,b} \to E_r^{a+r,b-r+1}$, and for every $a, b \in \mathbb{Z}$, there exists $N > 0$ such that $E_r^{a,b} = E_\infty^{a,b}$ for $r \geq N$, where $E_\infty^{a,b}$ is the associated graded $\mathrm{gr}^{-b} F^\bullet$ with respect to the descending filtration on

$KH_{-a-b}(X)$ (*resp.* $KH[1/p]_{-a-b}(X)$) (see Definition 4.3). *Furthermore, this spectral sequence degenerates with rational coefficients.*

Proof. If k admits resolution of singularities, we just need to show that the spectral sequence is convergent. For this, we observe that KGL_k is the spectrum associated to the Landweber exact $\mathbb{L}$-algebra $\mathbb{Z}[\beta, \beta^{-1}]$ that classifies the multiplicative formal group law [Spitzweck and Østvær 2009, Theorem 1.2]. Thus [Hoyois 2015, Theorem 8.12] implies that KGL_k is bounded with respect to the slice filtration (this argument also applies in positive characteristic). Hence, the convergence follows from Proposition 4.7.

In the case of positive characteristic, the existence of the spectral sequence follows by combining the argument of Section 4.16 with Lemmas 4.8 and 4.9. To establish the convergence, it suffices to check that $\mathrm{KGL}[1/p]_k$ satisfies the conditions in Proposition 4.10.

We have already seen that KGL_k is bounded with respect to the slice filtration. Thus, by Lemma 4.8(2) we conclude that $\mathrm{KGL}[1/p]_k$ is bounded with respect to the slice filtration as well. On the other hand, it follows from [Kelly 2012, Proposition 5.2.3] that $\mathrm{KGL}[1/p]_k$ has a structure of traces in the sense of [Kelly 2012, Definition 4.3.1]. Finally, since $s_r \, \mathrm{KGL}_k \cong \Sigma_T^r H\mathbb{Z}$ for $r \in \mathbb{Z}$, combining [Kelly 2012, Corollary 5.2.4] and Lemma 4.9, we deduce that $s_r(\mathrm{KGL}[1/p])_k$ has a weak structure of smooth traces in the sense of [Kelly 2012, Definition 4.2.27]. This finishes the proof. □

Remark 4.30. For $\operatorname{char} k = 0$, the spectral sequence of Theorem 4.27 is not new and was constructed by Haesemeyer [2004, Theorem 7.3] using a different approach. However, the expected degeneration (rationally) of this spectral sequence and its positive characteristic analogue are new.

As a combination of Theorem 4.27 and [Thomason and Trobaugh 1990, Theorems 9.5 and 9.6], we obtain the following spectral sequence for the algebraic K-theory $K^B(-)$ of singular schemes [Thomason and Trobaugh 1990].

Corollary 4.31. *Let k be a field of exponential characteristic $p > 1$. Let $\ell \neq p$ be a prime and $m \geq 0$ any integer. Given any $X \in \mathbf{Sch}_k$, there exist strongly convergent spectral sequences*

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(-b)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \Rightarrow K^B_{-a-b}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right], \tag{4.32}$$

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}/\ell^m(-b)) \Rightarrow K^B/\ell^m{}_{-a-b}(X). \tag{4.33}$$

5. Applications I: Comparing cobordism, *K*-theory and cohomology

In this section, we deduce some geometric applications of the slice spectral sequences for singular schemes. More applications will appear in the subsequent sections.

Consider the edge map $\mathrm{MGL} = f_0 \,\mathrm{MGL} \to s_0 \,\mathrm{MGL} \cong H\mathbb{Z}$ in the spectral sequence (4.24). This induces a natural map $\nu_X : \mathrm{MGL}^{i,j}(X) \to H^i(X, \mathbb{Z}(j))$ for every $X \in \mathbf{Sch}_k$ and $i, j \in \mathbb{Z}$.

The following result shows that there is no distinction between algebraic cycles and cobordism cycles at the level of 0-cycles.

Theorem 5.1. *Let k be a field which admits resolution of singularities (resp. a perfect field of positive characteristic p). Then for any $X \in \mathbf{Sch}_k$ of dimension d, we have $H^{2a-b}(X, \mathbb{Z}(a)) = 0$ (resp. $H^{2a-b}(X, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0$) whenever $a > d + b$. In particular, for every $X \in \mathbf{Sch}_k$ (resp. $X \in \mathbf{Sm}_k$), the map*

$$\nu_X \colon \mathrm{MGL}^{2d+i,d+i}(X) \to H^{2d+i}(X, \mathbb{Z}(d+i)) \tag{5.2}$$

$$\Big(\textit{resp.} \quad \nu_X \colon \mathrm{MGL}^{2d+i,d+i}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]\Big) \tag{5.3}$$

is an isomorphism for all $i \geq 0$.

Proof. Using the spectral sequence (4.24) (resp. Remark 4.25) and the fact that $\mathbb{L}^{>0} = 0$, the isomorphism of (5.2) (resp. (5.3)) follows immediately from the vanishing assertion for the motivic cohomology.

To prove the vanishing result, we note that for $X \in \mathbf{Sm}_k$, there is an isomorphism $H^{2a-b}(X, \mathbb{Z}(a)) \cong \mathrm{CH}^a(X, b)$ by [Voevodsky 2002a], and the latter group is clearly zero if $a > d + b$ by definition of Bloch's higher Chow groups.

If X is not smooth and k admits resolution of singularities, our assumption on k implies that there exists a cdh-cover $\{X' \amalg Z \to X\}$ of X such that $X' \in \mathbf{Sm}_k$, $\dim(Z) < \dim(X)$ and $\dim(W) < \dim(X)$, where we set $W = X' \times_X Z$. The cdh-descent for the motivic cohomology yields an exact sequence

$$H^{2a-b-1}(W, \mathbb{Z}(a)) \xrightarrow{\partial} H^{2a-b}(X, \mathbb{Z}(a)) \to H^{2a-b}(X', \mathbb{Z}(a)) \oplus H^{2a-b}(Z, \mathbb{Z}(a)).$$

The smooth case of our vanishing result shown above and an induction on the dimension together imply that the two end terms of this exact sequence vanish. Hence, the middle term vanishes too.

If X is not smooth and k is perfect of positive characteristic, we argue as in Proposition 4.10. Namely, by a theorem of Gabber [Illusie et al. 2014, Théorème 3(1)] and Temkin's strengthening [2017, Theorem 1.2.9] of Gabber's result, there exists $W \in \mathbf{Sm}_k$ and a surjective proper map $h : W \to X$, which is generically étale of degree p^r, $r \geq 1$. Then by a theorem of Raynaud and Gruson [1971, Theorem 5.2.2], there exists a blow-up $g : X' \to X$ with center Z such that the diagram

$$\begin{array}{ccc} W' & \xrightarrow{h'} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow[h]{} & X \end{array} \tag{5.4}$$

commutes, where h' is finite flat surjective of degree p^r and $g' : W' \to W$ is the blow-up of W with center $h^{-1}(Z)$.

Thus we have a cdh-cover $\{X' \amalg Z \to X\}$ of X, such that $\dim_k(Z) < \dim_k(X)$ and $\dim_k(E) < \dim_k(X)$, where we set $E = X' \times_X Z$. Then by cdh-excision, the following diagram is exact:

$$H^{2a-b-1}(E, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to H^{2a-b}(X, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]$$
$$\to H^{2a-b}(X', \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \oplus H^{2a-b}(Z, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big].$$

By induction on the dimension, this reduces to the exact sequence

$$0 \to H^{2a-b}(X, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \xrightarrow{g^*} H^{2a-b}(X', \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big].$$

So it suffices to show that $g^* = 0$. In order to prove this, we observe that (5.4) commutes. Therefore, since $W \in \mathbf{Sm}_k$, $H^{2a-b}(W, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0$. We conclude that $h'^* \circ g^* = g'^* \circ h^* = 0$. Thus, it is enough to see that

$$h'^* : H^{2a-b}(X', \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to H^{2a-b}(W', \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]$$

is injective. Let $v' : X' \to \operatorname{Spec} k$, and $\epsilon : \boldsymbol{L}v'^* H\mathbb{Z}[1/p] \to \boldsymbol{R}h'_* \boldsymbol{L}h'^* \boldsymbol{L}v'^* H\mathbb{Z}[1/p]$ be the map given by the unit of the adjunction $(\boldsymbol{L}h'^*, \boldsymbol{R}h'_*)$. By the naturality of the isomorphism in Proposition 2.13, we deduce that h'^* gets identified with the map induced by ϵ (see Corollary 3.6):

$$\epsilon_* : \operatorname{Hom}_{\mathcal{SH}_{X'}}\big(\Sigma^{m,n} \Sigma_T^\infty (X'_+), \boldsymbol{L}v'^* H\mathbb{Z}\big[\tfrac{1}{p}\big]\big)$$
$$\to \operatorname{Hom}_{\mathcal{SH}_{X'}}\big(\Sigma^{m,n} \Sigma_T^\infty (X'_+), \boldsymbol{R}h'_* \boldsymbol{L}h'^* \boldsymbol{L}v'^* H\mathbb{Z}\big[\tfrac{1}{p}\big]\big).$$

By [Kelly 2012, Corollary 5.2.4], $H\mathbb{Z}[1/p]$ has a structure of traces in the sense of [Kelly 2012, Definition 4.3.1]. Thus, we deduce from [Kelly 2012, Definition 4.3.1(Deg), p. 101] that ϵ_* is injective since h' is finite flat surjective of degree p^r. This finishes the proof. □

Remark 5.5. For $X \in \mathbf{Sm}_k$ and $i = 0$, the isomorphism of (5.2) was proved by Déglise [2013, Corollary 4.3.4].

When A is a field, the following result was proven by Morel [2012, Corollary 1.25] using methods of unstable motivic homotopy theory. Taking for granted the result for fields, Déglise [2013] proved Theorem 5.6 using homotopy modules. Spitzweck [2014, Corollary 7.3] proved Theorem 5.6 for localizations of a Dedekind domain.

Theorem 5.6. *Let k be a perfect field of exponential characteristic p. Then for any regular semilocal ring A which is essentially of finite type over k, and for any* *integer $n \geq 0$, the map*

$$\mathrm{MGL}^{n,n}(A) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \to H^n(A, \mathbb{Z}(n)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \tag{5.7}$$

is an isomorphism. In particular, there is a natural isomorphism

$$\mathrm{MGL}^{n,n}(A) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \cong K_n^M(A) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]$$

if k is also infinite.

Proof. Using the spectral sequence (4.24) and the fact that $\mathbb{L}^{>0} = 0$, it suffices to prove that $E_2^{n+i+j,-i}(A) = 0$ for every $j \geq 0$ and $i \geq 1$. In positive characteristic, we can use Remark 4.25 since A is regular. Notice that (4.24) and the spectral sequence in Remark 4.25 are strongly convergent for A by [Hoyois 2015, Lemmas 8.9 and 8.10].

On the one hand, we have isomorphisms

$$\begin{aligned} E_2^{n+i+j,-i}(A) &= H^{n+2i+j}(A, \mathbb{Z}(n+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \\ &\cong \mathrm{CH}^{n+i}(A, 2n+2i-n-2i-j) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \\ &= \mathrm{CH}^{n+i}(A, n-j) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]. \end{aligned}$$

On the other hand, letting F denote the fraction field of A, the Gersten resolution for the higher Chow groups (see [Bloch 1986, Theorem 10.1]) shows that the restriction map $\mathrm{CH}^{n+i}(A, n-j) \to \mathrm{CH}^{n+i}(F, n-j)$ is injective. But the term $\mathrm{CH}^{n+i}(F, n-j)$ is zero whenever $j \geq 0, i \geq 1$ for dimensional reasons. We conclude that $E_2^{n+i+j,-i}(A) = 0$. The last assertion of the theorem now follows from the isomorphism $\mathrm{CH}^n(A, n) \cong K_n^M(A)$ by [Kerz 2009, Theorem 1.1]. □

5.8. *Connective K-theory.* Let k be a field of exponential characteristic p and let $X \in \mathbf{Sch}_k$. Recall that the *connective K-theory* spectrum KGL_X^0 is defined to be the motivic T-spectrum $f_0\,\mathrm{KGL}_X$ in $\mathcal{SH}_X$ (see (4.2)). Strictly speaking, KGL_X^0 should be called effective K-theory. Nevertheless, we follow the terminology of [Dai and Levine 2014].

In particular, there is a canonical map $u_X : \mathrm{KGL}_X^0 \to \mathrm{KGL}_X$ which is universal for morphisms from objects of $\mathcal{SH}_X^{\mathrm{eff}}$ to KGL_X. For any $Y \in \mathbf{Sm}_X$, we let $CKH^{p,q}(Y) = \mathrm{Hom}_{\mathcal{SH}_X}(\Sigma_T^\infty Y_+, \Sigma^{p,q}\,\mathrm{KGL}_X^0)$. Using an analogue of Theorem 4.23 for KGL_X^0, one can prove the existence of the cycle class map for the higher Chow groups as follows.

Theorem 5.9. *Let k be a field of exponential characteristic p and let $X \in \mathbf{Sch}_k$ have dimension d. Then the map $\mathrm{KGL}_X^0[1/p] \to s_0\,\mathrm{KGL}_X[1/p] \cong H\mathbb{Z}[1/p]$ induces for every integer $i \geq 0$, an isomorphism*

$$CKH^{2d+i,d+i}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big] \xrightarrow{\cong} H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{p}\big]. \tag{5.10}$$

In particular, the canonical map $\mathrm{KGL}^0_X \to \mathrm{KGL}_X$ *induces a natural cycle class map*

$$\mathrm{cyc}_i : H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \to KH_i(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right]. \tag{5.11}$$

Proof. First we assume that k admits resolution of singularities. It follows from the definition that KGL^0_X is a connective T-spectrum, and $\boldsymbol{L}v^*(\mathrm{KGL}^0_k) \xrightarrow{\cong} \mathrm{KGL}^0_X$ by [Pelaez 2013, Theorem 3.7]. One also knows that $s_r\, \mathrm{KGL}^0_k \cong \Sigma^r_T H\mathbb{Z}$ for $r \geq 0$ [Levine 2008, Theorem 6.4.2] and is zero otherwise. The proof of Theorem 4.23 can now be repeated verbatim to conclude that for each $n \in \mathbb{Z}$, there is a strongly convergent spectral sequence

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(n-b)) \otimes_{\mathbb{Z}} \mathbb{Z}_{b \leq 0} \Rightarrow CKH^{a+b,n}(X), \tag{5.12}$$

where $\mathbb{Z}_{b \leq 0} = \mathbb{Z}$ if $b \leq 0$ and is zero otherwise. Furthermore, this spectral sequence degenerates with rational coefficients.

One now repeats the proof of Theorem 5.1 to conclude that the edge map $CKH^{2d+i,d+i}(X) \to H^{2d+i}(X, \mathbb{Z}(d+i))$ is an isomorphism for every $i \geq 0$. Finally, to get the desired cycle class map, we compose the inverse of this isomorphism with the canonical map $CKH^{2d+i,d+i}(X) \to KH_i(X)$.

If the characteristic of k is positive, then $s_r(\mathrm{KGL}^0_k) \cong \Sigma^r_T H\mathbb{Z}$ for every $r \geq 0$ and is zero otherwise [Levine 2008, Theorem 6.4.2]. So $s_r(\mathrm{KGL}^0_k[1/p])$ has a weak structure of traces [Kelly 2012, Corollary 5.2.4]. By Lemma 4.9, we deduce that $s_r(\mathrm{KGL}^0_k[1/p]) \cong \Sigma^r_T H\mathbb{Z}[1/p]$ for every $r \geq 0$ and is zero otherwise. Thus, we can apply [Kelly 2012, Theorem 4.2.29] to conclude $\boldsymbol{L}v^*(\mathrm{KGL}^0_k[1/p]) \cong \mathrm{KGL}^0_X[1/p]$. Then the argument of Theorem 4.27 applies, and we conclude that for each $n \in \mathbb{Z}$, there is a strongly convergent spectral sequence

$$E_2^{a,b} = H^{a-b}(X, \mathbb{Z}(n-b)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right]_{b \leq 0} \Rightarrow CKH^{a+b,n}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right]. \tag{5.13}$$

By Theorem 5.1, $H^{2a-b}(X, \mathbb{Z}(a)) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0$ whenever $a > d+b$. Thus, combining the spectral sequence (5.13) and the fact that $\mathbb{L}^{>0} = 0$, we deduce the isomorphism of (5.10) with $\mathbb{Z}[1/p]$-coefficients:

$$CKH^{2d+i,d+i}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \xrightarrow{\cong} H^{2d+i}(X, \mathbb{Z}(d+i)) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right]. \qquad \square$$

An argument identical to the proof of Theorem 5.6 shows that for any regular semilocal ring A which is essentially of finite type over an infinite field k and any integer $n \geq 0$, there is a natural isomorphism

$$CKH^{n,n}(A) \xrightarrow{\cong} K^M_n(A) \tag{5.14}$$

(notice that in positive characteristic, the spectral sequence is also strongly convergent integrally since A is regular).

Moreover, the canonical map $CKH^{n,n}(A) \to K_n(A)$ respects products [Pelaez 2011, Theorem 3.6.9], and hence coincides with the known map $K_n^M(A) \to K_n(A)$. This shows that the Milnor K-theory is represented by the connective K-theory, and one gets a lifting of the relation between the Milnor and Quillen K-theory of smooth semilocal schemes to the level of $\mathcal{SH}$. In particular, it is possible to recover Milnor K-theory and its map into Quillen K-theory from the T-spectrum KGL (which represents Quillen K-theory in $\mathcal{SH}$ for smooth k-schemes) by passing to its (-1)-effective cover $f_0\,\mathrm{KGL}_k \to \mathrm{KGL}_k$.

As another consequence of the slice spectral sequence, one gets the following comparison result between the connective and nonconnective versions of the homotopy K-theory. The homological analogue of this result was shown in [Dai and Levine 2014, Corollary 5.5].

Theorem 5.15. *Let k be a field of exponential characteristic p and let $X \in \mathbf{Sch}_k$ have dimension d. Then the canonical map*

$$CKH^{2n,n}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \to KH_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right]$$

is an isomorphism for every integer $n \leq 0$.

Proof. If k admits resolution of singularities, we observe that the slice spectral sequence is functorial for morphisms of motivic T-spectra. Since $H^{2q}(X, \mathbb{Z}(q)) = 0$ for $q < 0$, a comparison of the spectral sequences (4.28) and (5.12) shows that it is enough to prove that for every $r \geq 2$ and $q \leq 0$, either $q + r - 1 \leq 0$ or

$$H^{-q-r-(q+r-1)}(X, \mathbb{Z}(1-r-q)) = H^{1-2r-2q}(X, \mathbb{Z}(1-r-q)) = 0.$$

But this is true because $H^{1-2r-2q}(X, \mathbb{Z}(s)) = 0$ if $s < 0$.

In positive characteristic, we use the same argument as above for the spectral sequences (4.29) and (5.13). $\square$

Yet another consequence of the above spectral sequences is the following direct verification of Weibel's vanishing conjecture for negative KH-theory and negative CKH-theory of singular schemes. For KH-theory, there are other proofs of this conjecture by Haesemeyer [2004, Theorem 7.1] in characteristic zero and Kelly [2014, Theorem 3.5] and Kerz and Strunk [2017] in positive characteristic using different methods. We refer the reader to [Cisinski 2013; Cortiñas et al. 2008a; Geisser and Hesselholt 2010; Kerz et al. 2018; Krishna 2009; Weibel 2001] for more results associated to Weibel's conjecture. The vanishing result below for CKH-theory is new in any characteristic.

Theorem 5.16. *Let k be a field of exponential characteristic p and let $X \in \mathbf{Sch}_k$ have dimension d. Then $CKH^{m,n}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = KH_{2n-m}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0$ whenever $2n - m < -d$ and $KH_{-d}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong H^d_{\mathrm{cdh}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$.*

Proof. When k admits resolution of singularities, using the spectral sequences (4.28) and (5.10), it suffices to show $H^{p-q}(X, \mathbb{Z}(n-q)) = 0$ whenever $2n-p-q+d < 0$.

If $n-q < 0$, then we already know that this motivic cohomology group is zero. So we can assume $n-q \geq 0$. We set $a = n-q$ and $b = 2n-p-q$ so that $2a-b = 2n-2q-2n+p+q = p-q$. Since $2n-p-q+d < 0$ and $n-q \geq 0$ by our assumption, we get

$$b+d-a = 2n-p-q+d-n+q = n-p+d = (2n-p-q+d)-(n-q) < 0.$$

The theorem now follows because we have shown in the proof of Theorem 5.1 that $H^{p-q}(X, \mathbb{Z}(n-q)) = H^{2a-b}(X, \mathbb{Z}(a)) = 0$ as $a > b+d$. This argument also shows that $KH_{-d}(X) \cong H^d(X, \mathbb{Z}(0)) \cong H^d_{\mathrm{cdh}}(X, \mathbb{Z})$.

In positive characteristic, the same argument with the spectral sequences (4.29) and (5.13) gives that $CKH^{m,n}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = KH_{2n-m}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] = 0$ whenever $2n-m < -d$ and $KH_{-d}(X) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \cong H^d_{\mathrm{cdh}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$. □

Weibel's conjecture on the vanishing of certain negative K-theory was proven (after inverting the characteristic) by Kelly [2014]. Using our spectral sequence (which uses the methods of [Kelly 2012]), we can obtain the following result (which follows as well from [Kelly 2014] via the cdh-descent spectral sequence). The characteristic zero version of this computation was proven in [Cortiñas et al. 2008b, Theorem 0.2], and for arbitrary noetherian schemes, we refer the reader to [Kerz et al. 2018, Corollary D].

Corollary 5.17. *Let k be a field of exponential characteristic p and let $X \in \mathbf{Sch}_k$ have dimension d. Then*

$$K^B_{-d}(X) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right] \cong H^d_{\mathrm{cdh}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\tfrac{1}{p}\right].$$

6. The Chern classes on *KH*-theory

In order to obtain more applications of the slice spectral sequence for KH-theory and the cycle class map (see Theorem 5.9), we need to have a theory of Chern classes on the KH-theory of singular schemes.

Gillet [1981] showed that any cohomology theory satisfying the projective bundle formula and some other standard admissibility axioms admits a theory of Chern classes from algebraic K-theory of schemes over a field. These Chern classes are very powerful tools for understanding algebraic K-theory groups in terms of various cohomology theories such as motivic cohomology and Hodge theory. The Chern classes in Deligne cohomology are used to define various regulator maps on K-theory and they also give rise to the construction of intermediate Jacobians of smooth projective varieties over $\mathbb{C}$.

For a perfect field k of exponential characteristic $p \geq 1$, Kelly [2012, Corollary 5.5.10] showed that the motivic cohomology functor $X \mapsto \{H^i(X, \mathbb{Z}(j))[1/p]\}_{i,j \in \mathbb{Z}}$ satisfies the projective bundle formula in $\mathbf{Sch}_k$. This implies in particular by Gillet's theory that there are functorial Chern class maps

$$c_{i,j} : K_j(X) \to H^{2i-j}(X, \mathbb{Z}(i))\left[\tfrac{1}{p}\right]. \tag{6.1}$$

In this section, we show that in characteristic zero, Gillet's technique can be used to construct the above Chern classes on the homotopy invariant K-theory of singular schemes. Applications of these Chern classes to the understanding of the motivic cohomology and KH-theory of singular schemes will be given in the following two sections.

Let k be a field of characteristic zero and let $\mathbf{Sch}_{\mathrm{Zar}/k}$ denote the category of separated schemes of finite type over k equipped with the Zariski topology. Let $\mathbf{Sm}_{\mathrm{Zar}/k}$ denote the full subcategory of smooth schemes over k equipped with the Zariski topology. For any $X \in \mathbf{Sch}_k$, let X_{Zar} denote the small Zariski site of X. A presheaf of spectra on $\mathbf{Sch}_k$ or $\mathbf{Sm}_k$ means a presheaf of S^1-spectra.

Let $\mathrm{Pre}(\mathbf{Sch}_{\mathrm{Zar}/k})$ be the category of presheaves of simplicial sets on $\mathbf{Sch}_{\mathrm{Zar}/k}$ equipped with the injective Zariski local model structure, i.e., the weak equivalences are the maps that induce a weak equivalence of simplicial sets at every Zariski stalk and the cofibrations are given by monomorphisms. This model structure restricts to a similar model structure on the category $\mathrm{Pre}(X_{\mathrm{Zar}})$ of presheaves of simplicial sets on X_{Zar} for every $X \in \mathbf{Sch}_k$. We write $\mathcal{H}^{\mathrm{big}}_{\mathrm{Zar}}(k)$ and $\mathcal{H}^{\mathrm{sml}}_{\mathrm{Zar}}(X)$ for the homotopy categories of $\mathrm{Pre}(\mathbf{Sch}_{\mathrm{Zar}/k})$ and $\mathrm{Pre}(X_{\mathrm{Zar}})$, respectively.

6.2. ***Chern classes from KH-theory to motivic cohomology.*** For any $X \in \mathbf{Sch}_k$, let $\Omega BQP(X)$ denote the simplicial set obtained by taking the loop space of the nerve of the category $QP(X)$ obtained by applying Quillen's Q-construction to the exact category of locally free sheaves on X_{Zar}. Let $\mathcal{K}$ denote the presheaf of simplicial sets on $\mathbf{Sch}_{\mathrm{Zar}/k}$ given by $X \mapsto \Omega BQP(X)$. One knows that $\mathcal{K}$ is a presheaf of infinite loop spaces so that there is a presheaf of spectra $\widetilde{\mathcal{K}}$ on $\mathbf{Sch}_k$ such that $\mathcal{K} = (\widetilde{\mathcal{K}})_0$. Let $\widetilde{\mathcal{K}}^B$ denote the Thomason–Trobaugh presheaf of spectra on $\mathbf{Sch}_k$ such that $\widetilde{\mathcal{K}}^B(X) = K^B(X)$ for every $X \in \mathbf{Sch}_k$. There is a natural map of presheaves of spectra $\widetilde{\mathcal{K}} \to \widetilde{\mathcal{K}}^B$ which induces isomorphism between the nonnegative homotopy group presheaves.

Recall from [Jardine 1997, Theorem 2.34] that the category of presheaves of spectra on $\mathbf{Sch}_{\mathrm{Zar}/k}$ has a closed model structure, where the weak equivalences are given by the stalkwise stable equivalence of spectra, and a map $f : E \to F$ is a cofibration if f_0 is a monomorphism and $E_{n+1} \amalg_{S^1 \wedge E_n} S^1 \wedge F_n \to F_{n+1}$ is a monomorphism for each $n \geq 0$. Let $\mathcal{H}^s_{\mathrm{Zar}}(k)$ denote the associated homotopy category. There is a functor $\Sigma^\infty_s : \mathcal{H}^{\mathrm{big}}_{\mathrm{Zar}}(k) \to \mathcal{H}^s_{\mathrm{Zar}}(k)$ which has a right adjoint. We can consider the above model structure and the corresponding homotopy categories with respect to the Nisnevich and cdh-sites as well.

Let $\widetilde{\mathcal{K}}_{\mathrm{cdh}} \to \widetilde{\mathcal{K}}^{B}_{\mathrm{cdh}}$ denote the map between the functorial fibrant replacements in the above model structure on presheaves of spectra on $\mathbf{Sch}_k$ with respect to the cdh-topology. Let KH denote the presheaf of spectra on $\mathbf{Sch}_k$ such that $KH(X)$ is Weibel's homotopy invariant K-theory of X [Weibel 1989].

The following is a direct consequence of the main result of [Haesemeyer 2004].

Lemma 6.3. *Let k be a field of characteristic zero. For every $X \in \mathbf{Sch}_k$ and integer $p \in \mathbb{Z}$, there is a natural isomorphism $KH_p(X) \xrightarrow{\cong} \mathbb{H}^{-p}_{\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}})$.*

Proof. We have a natural isomorphism

$$\begin{aligned}\pi_p(\widetilde{\mathcal{K}}_{\mathrm{cdh}}(X)) &= \mathrm{Hom}_{\mathcal{H}^s_{\mathrm{cdh}}(k)}(\Sigma^{\infty}_s(S^p_s \wedge X), \widetilde{\mathcal{K}}) \\ &\cong \mathrm{Hom}_{\mathcal{H}_{\mathrm{cdh}}(k)}(S^p_s \wedge X, \mathcal{K}) \\ &\cong \mathbb{H}^{-p}_{\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}}). \end{aligned} \tag{6.4}$$

It is well known that the natural maps $K_p(X) \to \pi_p(\widetilde{\mathcal{K}}_{\mathrm{cdh}}(X)) \to \pi_p(\widetilde{\mathcal{K}}^{B}_{\mathrm{cdh}}(X))$ are isomorphisms for all $p \in \mathbb{Z}$ when X is smooth over k. In general, let $X \in \mathbf{Sch}_k$. We can find a Cartesian square

$$\begin{tikzcd} Z' \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd} \tag{6.5}$$

where $X' \in \mathbf{Sm}_k$ and f is a proper birational morphism which is an isomorphism outside the closed immersion $Z \hookrightarrow X$. Induction on dimension of X and cdh-descent for $\widetilde{\mathcal{K}}_{\mathrm{cdh}}$ as well as $\widetilde{\mathcal{K}}^{B}_{\mathrm{cdh}}$ now show that the map $\pi_p(\widetilde{\mathcal{K}}_{\mathrm{cdh}}(X)) \to \pi_p(\widetilde{\mathcal{K}}^{B}_{\mathrm{cdh}}(X))$ is an isomorphism for all $p \in \mathbb{Z}$. Composing the inverse of this isomorphism with the map in (6.4), we get a natural isomorphism $\pi_p(\widetilde{\mathcal{K}}^{B}_{\mathrm{cdh}}(X)) \xrightarrow{\cong} \mathbb{H}^{-p}_{\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}})$.

On the other hand, it follows from [Haesemeyer 2004, Theorem 6.4] that the natural map $KH(X) \to \widetilde{\mathcal{K}}^{B}_{\mathrm{cdh}}(X)$ is a homotopy equivalence. We conclude that there is a natural isomorphism $\nu_X : KH_p(X) \xrightarrow{\cong} \mathbb{H}^{-p}_{\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}})$ for every $X \in \mathbf{Sch}_k$ and $p \in \mathbb{Z}$. □

Let $\mathcal{BGL}$ be the simplicial presheaf on $\mathbf{Sch}_k$ with $\mathcal{BGL}(X) = \mathrm{colim}_n\, \mathrm{BGL}_n(\mathcal{O}(X))$. It is known (see [Gillet 1981, Proposition 2.15]) that there is a natural sectionwise weak equivalence $\mathcal{K}|_X \xrightarrow{\cong} \mathbb{Z} \times \mathbb{Z}_{\infty}\mathcal{BGL}|_X$ in $\mathrm{Pre}(\mathbf{Sch}_{\mathrm{Zar}/k})$ (see Section 6.2), where $\mathbb{Z}_{\infty}(-)$ is the $\mathbb{Z}$-completion functor of Bousfield–Kan.

To simplify the notation, for any integer $q \in \mathbb{Z}$, we write $\Gamma(q)$ for the presheaf on $\mathbf{Sch}_{\mathrm{Zar}/k}$ given by

$$\Gamma(q)(U) = \begin{cases} \underline{C}_* z_{\mathrm{equi}}(\mathbb{A}^q_k, 0)(U)[-2q] & \text{if } q \geq 0, \\ 0 & \text{if } q < 0. \end{cases}$$

(see Section 3). It is known that the restriction of $\Gamma(q)$ on $\mathbf{Sm}_{\mathrm{Zar}/k}$ is a sheaf (see, for instance, [Mazza et al. 2006, Definition 16.1]). We let $\Gamma(q)[2q] \to \mathcal{K}(\Gamma(q), 2q)$ denote a functorial fibrant replacement of $\Gamma(q)[2q]$ with respect to the injective Zariski local model structure.

It follows from [Asakura and Sato 2015, Section 3.1] that $\mathcal{K}(\Gamma(q), 2q)$ is a cohomology theory on $\mathbf{Sm}_{\mathrm{Zar}/k}$ which satisfies all of the conditions of [Gillet 1981, Definitions 1.1 and 1.2]. We conclude from Gillet's construction [1981, §2, p. 225] that for any $X \in \mathbf{Sm}_{\mathrm{Zar}/k}$, there is a morphism of simplicial presheaves $C_q : \mathcal{BGL}|_X \to \mathcal{K}(\Gamma(q), 2q)|_X$ in $\mathcal{H}^{\mathrm{sml}}_{\mathrm{Zar}}(X)$ which is natural in X. Composing with $\mathcal{K}|_X \xrightarrow{\cong} \mathbb{Z} \times \mathbb{Z}_\infty \mathcal{BGL}|_X$ and using the isomorphism $\mathbb{Z}_\infty \mathcal{K}(\Gamma(q), 2q) \cong \mathcal{K}(\Gamma(q), 2q)$, we obtain a map

$$C_q : \mathcal{K}|_X \xrightarrow{\cong} \mathbb{Z} \times \mathbb{Z}_\infty \mathcal{BGL}|_X \to \mathbb{Z} \times \mathcal{K}(\Gamma(q), 2q)|_X \to \mathcal{K}(\Gamma(q), 2q)|_X$$

in $\mathcal{H}^{\mathrm{sml}}_{\mathrm{Zar}}(X)$, where the last arrow is the projection.

Since $\mathcal{K}(\Gamma(q), 2q)$ is fibrant in $\mathrm{Pre}(\mathbf{Sch}_{\mathrm{Zar}/k})$, it follows from [Jardine 2015, Corollary 5.26] that the restriction $\mathcal{K}(\Gamma(q), 2q)|_X$ is fibrant in $\mathrm{Pre}(X_{\mathrm{Zar}})$. Since $\mathcal{K}|_X$ is cofibrant (in our local injective model structure), Gillet's construction [1981, p. 225] yields a map of simplicial presheaves $C_q : \mathcal{K}|_X \to \mathcal{K}(\Gamma(q), 2q)|_X$ in $\mathrm{Pre}(X_{\mathrm{Zar}})$. In particular, a map $\mathcal{K}(X) \to \mathcal{K}(\Gamma(q), 2q)(X)$. Furthermore, the naturality of the construction gives, for any morphism $f : Y \to X$ in $\mathbf{Sm}_k$, a diagram that commutes up to homotopy

$$\begin{array}{ccc} \mathcal{K}(X) & \xrightarrow{C_q} & \mathcal{K}(\Gamma(q), 2q)(X) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathcal{K}(Y) & \xrightarrow{C_q} & \mathcal{K}(\Gamma(q), 2q)(Y) \end{array} \tag{6.6}$$

(see, for instance, [Asakura and Sato 2015, (5.6.1)]). Equivalently, there is a morphism of simplicial presheaves $C_q : \mathcal{K} \to \mathcal{K}(\Gamma(q), 2q)$ in $\mathcal{H}^{\mathrm{big}}_{\mathrm{Zar}}(k)$ and hence a morphism in $(\mathbf{Sm}_k)_{\mathrm{Nis}}$ (see Section 2.1). Pulling back C_q via the morphism of sites $\pi : (\mathbf{Sch}_k)_{\mathrm{cdh}} \to (\mathbf{Sm}_k)_{\mathrm{Nis}}$ [Jardine 2015, p. 111], and considering the cohomologies of the associated cdh-sheaves, we obtain for any $X \in \mathbf{Sch}_k$, closed subscheme $Z \subseteq X$ and $p, q \geq 0$, the Chern class maps

$$\begin{aligned} c^Z_{X,p,q} : \mathbb{H}^{-p}_{Z,\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}}) &:= \mathbb{H}^{-p}_{Z,\mathrm{cdh}}(X, \boldsymbol{L}\pi^*(\mathcal{K})) \\ &\to \mathbb{H}^{-p}_{Z,\mathrm{cdh}}(X, \boldsymbol{L}\pi^*(\mathcal{K}(\Gamma(q), 2q))) \\ &= \mathbb{H}^{-p}_{Z,\mathrm{cdh}}(X, C_* z_{\mathrm{equi}}(\mathbb{A}^q_k, 0)_{\mathrm{cdh}}) := H^{2q-p}_Z(X, \mathbb{Z}(q)). \end{aligned} \tag{6.7}$$

It follows from Lemma 6.3 that $\mathbb{H}^{-p}_{Z,\mathrm{cdh}}(X, \mathcal{K}_{\mathrm{cdh}}) = KH^Z_p(X)$, where the $KH^Z(X)$ is the homotopy fiber of the map $KH(X) \to KH(X \setminus Z)$. Let (X, Z) denote the pair consisting of a scheme $X \in \mathbf{Sch}_k$ and a closed subscheme $Z \subseteq X$. A map of pairs $f : (Y, W) \to (X, Z)$ is a morphism $f : Y \to X$ such that $f^{-1}(Z) \subseteq W$. We have then shown the following.

Theorem 6.8. *Let k be a field of characteristic zero. Then for any pair (X, Z) in $\mathbf{Sch}_k$ and for any $p \geq 0$, $q \in \mathbb{Z}$, there are Chern class homomorphisms*

$$c^Z_{X,p,q} : KH^Z_p(X) \to H^{2q-p}_Z(X, \mathbb{Z}(q))$$

such that the composition of $c^X_{X,0,0}$ with $K_0(X) \to KH_0(X)$ is the rank map. For any map of pairs $f : (Y, W) \to (X, Z)$, there is a commutative diagram

$$\begin{array}{ccc} KH^Z_p(X) & \xrightarrow{c^Z_{X,p,q}} & H^{2q-p}_Z(X, \mathbb{Z}(q)) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ KH^W_p(Y) & \xrightarrow{c^W_{Y,p,q}} & H^{2q-p}_W(Y, \mathbb{Z}(q)) \end{array} \tag{6.9}$$

6.10. ***Chern classes from KH-theory to Deligne cohomology.*** Let $\mathcal{C}_{\mathrm{Zar}}$ denote the category of schemes which are separated and of finite type over $\mathbb{C}$ with the Zariski topology. We denote by $\mathcal{C}_{\mathrm{Nis}}$ the same category but with the Nisnevich topology. Let $\mathcal{C}_{\mathrm{an}}$ denote the category of complex analytic spaces with the analytic topology. There is a morphism of sites $\epsilon : \mathcal{C}_{\mathrm{an}} \to \mathcal{C}_{\mathrm{Zar}}$. For any $q \in \mathbb{Z}$, let $\Gamma(q)$ denote the complex of sheaves on $\mathcal{C}_{\mathrm{Zar}}$ defined as

$$\Gamma(q) = \begin{cases} \Gamma_{\mathcal{D}}(q) & \text{if } q \geq 0, \\ \boldsymbol{R}\epsilon_*((2\pi\sqrt{-1})\mathbb{Z}) & \text{if } q < 0, \end{cases} \tag{6.11}$$

where $\Gamma_{\mathcal{D}}(q)$ is the Deligne–Beilinson complex on $\mathcal{C}_{\mathrm{Zar}}$ in the sense of [Esnault and Viehweg 1988]. Then $\Gamma(q)$ is a cohomology theory on $\mathbf{Sm}_{\mathbb{C}}$ satisfying Gillet's conditions for a theory of Chern classes; see, for instance, [Asakura and Sato 2015, Section 3.4]. Applying the argument of Theorem 6.8 in verbatim, we obtain the Chern class homomorphisms

$$c^Z_{X,p,q} : KH^Z_p(X) \to \mathbb{H}^{2q-p}_{Z,\mathrm{cdh}}(X, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \tag{6.12}$$

for a pair of schemes (X, Z) in $\mathbf{Sch}_{\mathbb{C}}$ which is natural in (X, Z).

Let us now fix a scheme $X \in \mathbf{Sch}_{\mathbb{C}}$. Recall from [Deligne 1974, §6.2.5–6.2.8] that a smooth proper hypercovering of X is a smooth simplicial scheme $X_\bullet$ with a map of simplicial schemes $p_X : X_\bullet \to X$ such each map $X_i \to X$ is proper and p_X satisfies the universal cohomological descent in the sense of [Deligne 1974]. The resolution of singularities implies that such a hypercovering exists. The Deligne cohomology of X is defined in [Deligne 1974, §5.1.11] to be

$$H^p_{\mathcal{D}}(X, \mathbb{Z}(q)) := \mathbb{H}^p_{\mathrm{Zar}}(X, \boldsymbol{R}p_{X*}\Gamma_{\mathcal{D}}(q)) = \mathbb{H}^p_{\mathrm{Zar}}(X_\bullet, \Gamma_{\mathcal{D}}(q)). \tag{6.13}$$

Gillet's theory of Chern classes gives rise to the Chern class homomorphisms

$$c^{Q}_{X,p,q} : K_p(X) \to H^{2q-p}_{\mathcal{D}}(X, \mathbb{Z}(q)) \tag{6.14}$$

for any $X \in \mathbf{Sch}_{\mathbb{C}}$ which is contravariant functorial, where $K_i(X) = \pi_i(\Omega BQP(X))$ is the Quillen K-theory (see, for instance, [Barbieri-Viale et al. 1996, §2.4]). Our objective is to show that these Chern classes actually factor through the natural map $K_*(X) \to KH_*(X)$.

The construction of the Chern classes from KH-theory to the Deligne cohomology (see Theorem 6.20 below) will be achieved by the cdh-sheafification of Gillet's Chern classes at the level of presheaves of simplicial sets, followed by considering the induced maps on the hypercohomologies. Therefore, in order to factor the classical Chern classes $c^{Q}_{X,p,q}$ on Quillen K-theory through KH-theory, we only need to identify the target of the Chern class maps in (6.12) with the Deligne cohomology.

To do this, for any $X \in \mathbf{Sch}_{\mathbb{C}}$ we let $H^*_{\mathrm{an}}(X, \mathcal{F})$ denote the cohomology of the analytic space X_{an} with coefficients in the sheaf $\mathcal{F}$ on $\mathcal{C}_{\mathrm{an}}$. Let $\mathbb{Z} \to \mathrm{Sing}^*$ denote a fibrant replacement of the sheaf $\mathbb{Z}$ on $\mathcal{C}_{\mathrm{an}}$ so that $\boldsymbol{R}\epsilon_*(\mathbb{Z}) \xrightarrow{\cong} \epsilon_*(\mathrm{Sing}^*)$. Set $\mathbb{Z}(q) = (2\pi\sqrt{-1})^q \epsilon_*(\mathrm{Sing}^*) \cong \boldsymbol{R}\epsilon_*(\mathbb{Z})$.

Lemma 6.15. *For any $X \in \mathbf{Sm}_{\mathbb{C}}$, the map $H^p_{\mathrm{an}}(X, \mathbb{Z}) \to \mathbb{H}^p_{\mathrm{cdh}}(X, \mathbb{Z}(q)_{\mathrm{cdh}})$ is an isomorphism.*

Proof. Since $H^p_{\mathrm{an}}(X, \mathbb{Z}) \cong \mathbb{H}^p_{\mathrm{Zar}}(X, \mathbb{Z}(q))$, it is sufficient to show that the map $\mathbb{H}^p_{\mathrm{Zar}}(X, \mathbb{Z}(q)) \to \mathbb{H}^p_{\mathrm{cdh}}(X, \mathbb{Z}(q)_{\mathrm{cdh}})$ is an isomorphism.

Let $\mathcal{C}_{\mathrm{loc}}$ denote the category of schemes which are separated and of finite type over $\mathbb{C}$. We consider $\mathcal{C}_{\mathrm{loc}}$ as a Grothendieck site with coverings given by maps $Y' \to Y$ where the associated map of the analytic spaces is a local isomorphism of the corresponding topological spaces [SGA 4_3 1973, Exposé XI, p. 9]. Since a Nisnevich cover of schemes is a local isomorphism of the associated analytic spaces, there is a commutative diagram of morphisms of sites:

$$\begin{tikzcd} \mathcal{C}_{\mathrm{loc}} \arrow[r, "\delta"] \arrow[d, "\nu"'] & \mathcal{C}_{\mathrm{an}} \arrow[d, "\epsilon"] \\ \mathcal{C}_{\mathrm{Nis}} \arrow[r, "\tau"] & \mathcal{C}_{\mathrm{Zar}} \end{tikzcd} \tag{6.16}$$

Since every local isomorphism of analytic spaces is refined by open coverings, it is well known that the map $\mathbb{H}^p_{\mathrm{an}}(X, \mathcal{F}^*) \to H^p_{\mathrm{loc}}(X, \mathcal{F}^*)$ is an isomorphism for any complex of sheaves on $\mathcal{C}_{\mathrm{an}}$; see, for instance, [Milne 1980, Proposition 3.3, Theorem 3.12].

We set $(\mathbb{Z}(q))_{\mathrm{Nis}} = \tau^*(\mathbb{Z}(q)) = \nu_* \circ \delta^*(\mathrm{Sing}^*)$. We observe that for every $i \in \mathbb{Z}$, the cohomology sheaf $\mathcal{H}^i$ associated to the complex $\mathbb{Z}(q)$ is isomorphic to the Zariski (or Nisnevich) sheaf on $\mathbf{Sch}_{\mathbb{C}}$ associated to the presheaf $U \mapsto H^i_{\mathrm{an}}(U, \mathbb{Z})$. But this latter presheaf on $\mathbf{Sm}_{\mathbb{C}}$ is homotopy invariant with transfers. It follows from [Suslin and Voevodsky 2000, Corollary 1.1.1] that $\mathbb{H}^p_{\mathrm{Zar}}(X, \mathbb{Z}(q)) \to \mathbb{H}^p_{\mathrm{Nis}}(X, (\mathbb{Z}(q))_{\mathrm{Nis}})$ is an isomorphism. We are thus reduced to showing that for $X \in \mathbf{Sm}_{\mathbb{C}}$, the map $\mathbb{H}^p_{\mathrm{Nis}}(X, (\mathbb{Z}(q))_{\mathrm{Nis}}) \to \mathbb{H}^p_{\mathrm{cdh}}(X, (\mathbb{Z}(q))_{\mathrm{cdh}})$ is an isomorphism.

But this follows again from [Suslin and Voevodsky 2000, Corollary 1.1.1, 5.12.3, Theorem 5.13] because each $\mathcal{H}^i \cong \boldsymbol{R}^i \nu_*(\mathbb{Z})$ is a Nisnevich sheaf on $\mathbf{Sm}_{\mathbb{C}}$ associated to the homotopy invariant presheaf with transfers $U \mapsto H^i_{\mathrm{an}}(U, \mathbb{Z})$. The proof is therefore complete. □

For any $X \in \mathbf{Sch}_{\mathbb{C}}$, there are natural maps

$$\begin{aligned} H^p_{\mathcal{D}}(X, \mathbb{Z}(q)) \cong \mathbb{H}^p_{\mathrm{Zar}}(X_\bullet, \Gamma_{\mathcal{D}}(q)) \to \mathbb{H}^p_{\mathrm{Nis}}(X_\bullet, (\Gamma_{\mathcal{D}}(q))_{\mathrm{Nis}}) \\ \to \mathbb{H}^p_{\mathrm{cdh}}(X_\bullet, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}). \quad (6.17) \end{aligned}$$

Lemma 6.18. *For a projective scheme X over $\mathbb{C}$, the map*

$$H^p_{\mathcal{D}}(X, \mathbb{Z}(q)) \to \mathbb{H}^p_{\mathrm{cdh}}(X_\bullet, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}})$$

is an isomorphism.

Proof. Our assumption implies that each component X_p of the simplicial scheme $X_\bullet$ is smooth and projective. Given a complex of sheaves $\mathcal{F}^*_\bullet$ (in the Zariski or cdh-topology), there is a spectral sequence

$$E_1^{p,q} = \mathbb{H}^q_{\mathrm{Zar/cdh}}(X_p, (\mathcal{F}^*_\bullet)_{\mathrm{Zar/cdh}}) \Rightarrow \mathbb{H}^{p+q}_{\mathrm{Zar/cdh}}(X_\bullet, (\mathcal{F}^*_\bullet)_{\mathrm{Zar/cdh}});$$

see, for instance, [Asakura and Sato 2015, Appendix]. Using this spectral sequence and (6.17), it suffices to show that the map $H^p_{\mathrm{Zar}}(X, \Gamma_{\mathcal{D}}(q)) \to \mathbb{H}^p_{\mathrm{cdh}}(X, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}})$ is an isomorphism for any smooth projective scheme X over $\mathbb{C}$. For $q \leq 0$, this follows from Lemma 6.15. So we assume $q > 0$.

Since X is smooth and projective, the analytic Deligne complex $\mathbb{Z}(q)_{\mathcal{D}}$ is the complex of analytic sheaves $\mathbb{Z}(q) \to \mathcal{O}_{X_{\mathrm{an}}} \to \Omega^1_{X_{\mathrm{an}}} \to \cdots \to \Omega^{q-1}_{X_{\mathrm{an}}}$. In particular, there is a distinguished triangle

$$\boldsymbol{R}\epsilon_*(\Omega^{<q}_{X_{\mathrm{an}}}[-1]) \to \Gamma_{\mathcal{D}}(q) \to \mathbb{Z}(q) \to \boldsymbol{R}\epsilon_*(\Omega^{<q}_{X_{\mathrm{an}}})$$

in the derived category of sheaves on X_{Zar}.

As X is projective, it follows from GAGA that the natural map $\Omega^{<q}_{X/\mathbb{C}} \to \boldsymbol{R}\epsilon_*(\Omega^{<q}_{X_{\mathrm{an}}})$ is an isomorphism in the derived category of sheaves on X_{Zar}. In particular, we get a distinguished triangle in the derived category of sheaves on X_{Zar}:

$$\Omega^{<q}_{X/\mathbb{C}}[-1] \to \Gamma_{\mathcal{D}}(q) \to \mathbb{Z}(q) \to \Omega^{<q}_{X/\mathbb{C}}. \quad (6.19)$$

We thus have a commutative diagram of exact sequences

$$\begin{array}{ccccc}
\mathbb{H}^{p-1}_{\mathrm{Zar}}(X,\mathbb{Z}(q)) & \longrightarrow & \mathbb{H}^{p-1}_{\mathrm{Zar}}(X,\Omega^{<q}_{X/\mathbb{C}}) & \longrightarrow & H^{p}_{\mathrm{Zar}}(X,\Gamma_{\mathcal{D}}(q)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}^{p-1}_{\mathrm{cdh}}(X,(\mathbb{Z}(q))_{\mathrm{cdh}}) & \longrightarrow & \mathbb{H}^{p-1}_{\mathrm{cdh}}(X,(\Omega^{<q}_{X/\mathbb{C}})_{\mathrm{cdh}}) & \longrightarrow & H^{p}_{\mathrm{cdh}}(X,(\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}})
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \mathbb{H}^{p}_{\mathrm{Zar}}(X,\mathbb{Z}(q)) & \longrightarrow & \mathbb{H}^{p}_{\mathrm{Zar}}(X,\Omega^{<q}_{X/\mathbb{C}}) \\
& \downarrow & & \downarrow \\
\longrightarrow & \mathbb{H}^{p}_{\mathrm{cdh}}(X,(\mathbb{Z}(q))_{\mathrm{cdh}}) & \longrightarrow & \mathbb{H}^{p}_{\mathrm{cdh}}(X,(\Omega^{<q}_{X/\mathbb{C}})_{\mathrm{cdh}})
\end{array}$$

It follows from Lemma 6.15 that the first and the fourth vertical arrows from the left are isomorphisms. The second and the fifth vertical arrows are isomorphisms by [Cortiñas et al. 2008b, Corollary 2.5]. We conclude that the middle vertical arrow is also an isomorphism and this completes the proof. $\square$

As a combination of Lemma 6.3, (6.14) and Lemma 6.18, we obtain a theory of Chern classes from *KH*-theory to Deligne cohomology as follows.

Theorem 6.20. *For every projective scheme X over $\mathbb{C}$, there are Chern class homomorphisms*

$$c_{X,p,q} : KH_p(X) \to H^{2q-p}_{\mathcal{D}}(X,\mathbb{Z}(q))$$

such that for any morphism of projective $\mathbb{C}$-schemes $f : Y \to X$, one has

$$f^* \circ c_{X,p,q} = c_{Y,p,q} \circ f^*.$$

Proof. We only need to show that there is a natural isomorphism

$$\alpha_X : \mathbb{H}^{p}_{\mathrm{cdh}}(X,(\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \xrightarrow{\cong} H^{p}_{\mathcal{D}}(X,\mathbb{Z}(q)).$$

Given a morphism of projective $\mathbb{C}$-schemes $f : Y \to X$, there exists a commutative diagram

$$\begin{array}{ccc}
Y_\bullet & \xrightarrow{f_\bullet} & X_\bullet \\
{\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle p_X} \\
Y & \xrightarrow{f} & X
\end{array}$$

where the vertical arrows are the simplicial hypercovering maps. In particular, there is a commutative diagram

$$\begin{tikzcd}[column sep=small]
\mathbb{H}^p_{\mathrm{Zar}}(X, \Gamma_{\mathcal{D}}(q)) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathbb{H}^p_{\mathrm{Zar}}(Y, \Gamma_{\mathcal{D}}(q)) \arrow[dd] \arrow[dr] & \\
& \mathbb{H}^p_{\mathrm{cdh}}(X, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \arrow[rr] \arrow[dd] & & \mathbb{H}^p_{\mathrm{cdh}}(Y, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \arrow[dd] \\
H^p_{\mathcal{D}}(X, \mathbb{Z}(q)) \arrow[rr] \arrow[dr] & & H^p_{\mathcal{D}}(Y, \mathbb{Z}(q)) \arrow[dr] & \\
& \mathbb{H}^p_{\mathrm{cdh}}(X_\bullet, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \arrow[rr] & & \mathbb{H}^p_{\mathrm{cdh}}(Y_\bullet, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}})
\end{tikzcd}$$

Using Lemma 6.18, we get a map $\alpha_X : \mathbb{H}^p_{\mathrm{cdh}}(X, (\Gamma_{\mathcal{D}}(q))_{\mathrm{cdh}}) \to H^p_{\mathcal{D}}(X, \mathbb{Z}(q))$ such that $f^* \circ \alpha_X = \alpha_Y \circ f^*$ for any $f : Y \to X$ as above. Moreover, we have shown in the proof of Lemma 6.18 that this map is an isomorphism if $X \in \mathbf{Sm}_{\mathbb{C}}$. Since the source as well as the target of α_X satisfy cdh-descent by Lemma 6.18 (see [Suslin and Voevodsky 2000, Lemma 12.1]), we conclude as in the proof of Lemma 6.3 that α_X is an isomorphism for every projective $\mathbb{C}$-scheme X. □

7. Applications II: Intermediate Jacobian and Abel–Jacobi map for singular schemes

Recall that a very important object in the study of the geometric part of motivic cohomology of smooth projective varieties is an intermediate Jacobian. The intermediate Jacobians were defined by Griffiths and they receive the Abel–Jacobi maps from certain subgroups of the geometric part $H^{2*}(X, \mathbb{Z}(*))$ of the motivic cohomology groups.

A special case of these intermediate Jacobians is the Albanese variety of a smooth projective variety. The most celebrated result about the Albanese variety in the context of algebraic cycles is that the Abel–Jacobi map from the group of 0-cycles of degree zero to the Albanese variety is an isomorphism on the torsion subgroups. This theorem of Roitman tells us that the torsion part of the Chow group of 0-cycles on a smooth projective variety over $\mathbb{C}$ can be identified with the torsion subgroup of an abelian variety.

Roitman's torsion theorem has had enormous applications in the theory of algebraic cycles and algebraic K-theory. For example, it was predicted as part of the conjectures of Bloch and Beilinson that the Chow group of 0-cycles on smooth affine varieties of dimension at least two should be torsion-free. This is now a consequence of Roitman's torsion theorem. We hope to use the Roitman's torsion theorem of this paper to answer the analogous question about the motivic cohomology $H^{2d}(X, \mathbb{Z}(d))$ of a d-dimensional singular affine variety in a future project.

It was predicted as part of the relation between algebraic K-theory and motivic cohomology that the Chow group of 0-cycles should be (integrally) a subgroup of the Grothendieck group. This is also now a consequence of Roitman's theorem. We shall prove the analogue of this for singular schemes in the next section. Recall that the Riemann–Roch theorem says that this inclusion of the Chow group inside the Grothendieck group is always true rationally. For applications concerning the relation between Chow groups and étale cohomology, see [Bloch 1979].

In this section, we apply the theory of Chern classes from KH-theory to Deligne cohomology from Section 6 to construct the intermediate Jacobian and Abel–Jacobi map from the geometric part of the motivic cohomology of any singular projective variety over $\mathbb{C}$. In the next section, we shall use the Abel–Jacobi map to prove a Roitman torsion theorem for singular schemes. As another application of our Chern classes and the Roitman torsion theorem, we shall show that the cycle map from the geometric part of motivic cohomology to the KH groups, constructed in Theorem 5.9, is injective for a large class of schemes.

7.1. ***The Abel–Jacobi map.*** In the rest of this section, we consider all schemes over $\mathbb{C}$ and mostly deal with projective schemes. Let X be a projective scheme over $\mathbb{C}$ of dimension d. Let X_{sing} and X_{reg} denote the singular (with the reduced induced subscheme structure) and the smooth loci of X, respectively. Let r denote the number of d-dimensional irreducible components of X. We fix a resolution of singularities $f : \widetilde{X} \to X$ and let $E = f^{-1}(X_{\mathrm{sing}})$ throughout this section. The following is an immediate consequence of the cdh-descent for Deligne cohomology.

Lemma 7.2. *For any integer $q \geq d+1$, one has $H_{\mathcal{D}}^{q+d+i}(X, \mathbb{Z}(q)) = 0$ for $i \geq 1$.*

Proof. If X is smooth, this follows immediately from (6.19). In general, the cdh-descent for Deligne cohomology (see Lemma 6.18 or [Barbieri-Viale et al. 1996, Variant 3.2]) implies that there is an exact sequence

$$\begin{aligned} H_{\mathcal{D}}^{q+d+i-1}(E, \mathbb{Z}(q)) \to H_{\mathcal{D}}^{q+d+i}(X, \mathbb{Z}(q)) \\ \to H_{\mathcal{D}}^{q+d+i}(\widetilde{X}, \mathbb{Z}(q)) \oplus H_{\mathcal{D}}^{q+d+i}(X_{\mathrm{sing}}, \mathbb{Z}(q)). \end{aligned}$$

We conclude the proof by using this exact sequence and induction on $\dim(X)$. □

It follows from the definition of the Deligne cohomology that there is a natural map of complexes $\Gamma_{\mathcal{D}}(q)|_X \to \mathbb{Z}(q)|_X$ (see (6.19)) and in particular, there is a natural map $H_{\mathcal{D}}^p(X, \mathbb{Z}(q)) \xrightarrow{\kappa_X} H_{\mathrm{an}}^p(X, \mathbb{Z}(q))$. For any integer $0 \leq q \leq d$, the *intermediate Jacobian* $J^q(X)$ is defined so that we have an exact sequence

$$0 \to J^q(X) \to H_{\mathcal{D}}^{2q}(X, \mathbb{Z}(q)) \xrightarrow{\kappa_X} H_{\mathrm{an}}^{2q}(X, \mathbb{Z}(q)).$$

It follows from Theorem 6.20 that there is a commutative diagram

$$\begin{array}{ccccc} KH_0(X) & \xrightarrow{c_{X,d,0}} & H^{2d}_{\mathcal{D}}(X,\mathbb{Z}(d)) & \xrightarrow{\kappa_X} & H^{2d}_{\mathrm{an}}(X,\mathbb{Z}(d)) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ KH_0(\widetilde{X}) & \xrightarrow{c_{X,d,0}} & H^{2d}_{\mathcal{D}}(\widetilde{X},\mathbb{Z}(d)) & \xrightarrow{\kappa_{\widetilde{X}}} & H^{2d}_{\mathrm{an}}(\widetilde{X},\mathbb{Z}(d)) \end{array} \tag{7.3}$$

It follows from (6.19) that $\kappa_{\widetilde{X}}$ is surjective. The cdh-descent for the Deligne cohomology and Lemma 7.2 together imply that the middle vertical arrow in (7.3) is surjective. The cdh-excision property of singular cohomology (see [Deligne 1974, 8.3.10]) yields an exact sequence

$$\begin{aligned} &H^{2d-1}_{\mathrm{an}}(E,\mathbb{Z}(d)) \to H^{2d}_{\mathrm{an}}(X,\mathbb{Z}(d)) \\ &\qquad \to H^{2d}_{\mathrm{an}}(\widetilde{X},\mathbb{Z}(d)) \oplus H^{2d}_{\mathrm{an}}(X_{\mathrm{sing}},\mathbb{Z}(d)) \to H^{2d+1}_{\mathrm{an}}(E,\mathbb{Z}(d)). \end{aligned}$$

Since X_{sing} and E are projective schemes of dimension at most $d-1$, it follows that the right vertical arrow in (7.3) is an isomorphism. We conclude that there is a short exact sequence

$$0 \to J^d(X) \to H^{2d}_{\mathcal{D}}(X,\mathbb{Z}(d)) \xrightarrow{\kappa_X} H^{2d}_{\mathrm{an}}(X,\mathbb{Z}(d)) \to 0. \tag{7.4}$$

A similar Mayer–Vietoris property of the motivic cohomology yields an exact sequence

$$\begin{aligned} &H^{2d-1}(E,\mathbb{Z}(d)) \to H^{2d}(X,\mathbb{Z}(d)) \\ &\qquad \to H^{2d}(\widetilde{X},\mathbb{Z}(d)) \oplus H^{2d}(X_{\mathrm{sing}},\mathbb{Z}(d)) \to H^{2d+1}(E,\mathbb{Z}(d)). \end{aligned}$$

It follows from Theorem 5.1 that $H^{2d}(X_{\mathrm{sing}},\mathbb{Z}(d)) = H^{2d+1}(E,\mathbb{Z}(d)) = 0$. In particular, there exists a short exact sequence

$$\begin{aligned} 0 \to &\frac{H^{2d-1}(E,\mathbb{Z}(d))}{H^{2d-1}(\widetilde{X},\mathbb{Z}(d)) + H^{2d-1}(X_{\mathrm{sing}},\mathbb{Z}(d))} \\ &\qquad \to H^{2d}(X,\mathbb{Z}(d)) \to H^{2d}(\widetilde{X},\mathbb{Z}(d)) \to 0. \end{aligned} \tag{7.5}$$

Since the map $H^{2d}(\widetilde{X},\mathbb{Z}(d)) \cong \mathrm{CH}^d(\widetilde{X}) \to H^{2d}_{\mathrm{an}}(\widetilde{X},\mathbb{Z}(d))$ is the degree map, which is surjective, we deduce that the "degree" map $H^{2d}(X,\mathbb{Z}(d)) \to H^{2d}_{\mathrm{an}}(X,\mathbb{Z}(d))$ is also surjective. We let $A^d(X)$ denote the kernel of this degree map.

It follows from Theorem 6.20 that there is a Chern class map (take $p = 0$) $c_{X,q} : KH_0(X) \to H^{2q}_{\mathcal{D}}(X,\mathbb{Z}(q))$. Theorem 5.9 says that the spectral sequence (4.28) induces a cycle class map $\mathrm{cyc}_{X,0} : H^{2d}(X,\mathbb{Z}(d)) \to KH_0(X)$. Composing the two maps, we get a cycle class map from motivic to Deligne cohomology

$$\tilde{c}^d_X : H^{2d}(X,\mathbb{Z}(d)) \to H^{2q}_{\mathcal{D}}(X,\mathbb{Z}(q)) \tag{7.6}$$

and a commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^d(X) & \longrightarrow & H^{2d}(X, \mathbb{Z}(d)) & \longrightarrow & H^{2d}_{\mathrm{an}}(X, \mathbb{Z}(d)) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{AJ}^d_X} & & \Big\downarrow{\scriptstyle \tilde{c}^d_X} & & \Big\| & & \\ 0 & \longrightarrow & J^d(X) & \longrightarrow & H^{2q}_{\mathcal{D}}(X, \mathbb{Z}(q)) & \longrightarrow & H^{2d}_{\mathrm{an}}(X, \mathbb{Z}(d)) & \longrightarrow & 0 \end{array} \tag{7.7}$$

It is known that $J^d(X)$ is a semiabelian variety whose abelian variety quotient is the classical Albanese variety of $\widetilde{X}$; see [Biswas and Srinivas 1999, Theorem 1.1] or [Barbieri-Viale and Srinivas 2001]. The induced map $\mathrm{AJ}^d_X : A^d(X) \to J^d(X)$ is called the *Abel–Jacobi* map for the singular scheme X. We shall prove our main result about this Abel–Jacobi map in the next section. Here, we recall the following description of $J^d(X)$ in terms of 1-motives. Recall from [Barbieri-Viale and Kahn 2016, §12.12] that every projective scheme X of dimension d over $\mathbb{C}$ has a 1-motive $\mathrm{Alb}^+(X)$ associated to it. This is called the cohomological Albanese 1-motive of X. This is a generalization of the Albanese variety of smooth projective schemes.

Theorem 7.8 [Barbieri-Viale and Srinivas 2001, Corollary 3.3.2]. *For a projective scheme X of dimension d over $\mathbb{C}$, there is a canonical isomorphism*

$$J^d(X) \cong \mathrm{Alb}^+(X).$$

7.9. ***Levine–Weibel Chow group and motivic cohomology.*** In order to prove our main theorem of this section, we need to compare the motivic cohomology of singular schemes with another "motivic cohomology", called the (cohomological) Chow-group of 0-cycles, introduced by Levine and Weibel [1985]. We assume throughout our discussion that X is a reduced projective scheme of dimension d over $\mathbb{C}$. However, we remark that the following definition of the Chow group of 0-cycles makes sense over any ground field. Let $\mathcal{Z}_0(X)$ denote the free abelian group on the closed points of X_{reg}.

Definition 7.10. Let C be a pure dimension one reduced scheme in $\mathbf{Sch}_{\mathbb{C}}$. We say that a pair (C, Z) is *a good curve relative to X* if there exists a finite morphism $\nu : C \to X$ and a closed proper subscheme $Z \subsetneq C$ such that the following hold.

(1) No component of C is contained in Z.

(2) $\nu^{-1}(X_{\mathrm{sing}}) \cup C_{\mathrm{sing}} \subseteq Z$.

(3) ν is a local complete intersection morphism at every point $x \in C$ such that $\nu(x) \in X_{\mathrm{sing}}$.

Let (C, Z) be a good curve relative to X and let $\{\eta_1, \dots, \eta_r\}$ be the set of generic points of C. Let $\mathcal{O}_{C,Z}$ denote the semilocal ring of C at $S = Z \cup \{\eta_1, \dots, \eta_r\}$. Let $\mathbb{C}(C)$ denote the ring of total quotients of C and write $\mathcal{O}^\times_{C,Z}$ for the group of units in $\mathcal{O}_{C,Z}$. Notice that $\mathcal{O}_{C,Z}$ coincides with $k(C)$ if $|Z| = \varnothing$. As C is Cohen–Macaulay, $\mathcal{O}_{C,Z}^{\times}$ is the subgroup of $k(C)^{\times}$ consisting of those f which are regular and invertible in the local rings $\mathcal{O}_{C,x}$ for every $x \in Z$.

Given any $f \in \mathcal{O}_{C,Z}^{\times} \hookrightarrow \mathbb{C}(C)^{\times}$, we denote by $\operatorname{div}(f)$ the divisor of zeros and poles of f on C, which is defined as follows. If $C_1, \dots, C_r$ are the irreducible components of C, we set $\operatorname{div}(f)$ to be the 0-cycle $\sum_{i=1}^{r} \operatorname{div}(f_i)$, where $(f_1, \dots, f_r) = \theta_{(C,Z)}(f)$ and $\operatorname{div}(f_i)$ is the usual divisor of a rational function on an integral curve in the sense of [Fulton 1998]. Let $\mathcal{Z}_0(C, Z)$ denote the free abelian group on the closed points of $C \setminus Z$. As f is an invertible regular function on C along Z, $\operatorname{div}(f) \in \mathcal{Z}_0(C, Z)$.

By definition, given any good curve (C, Z) relative to X, we have a pushforward map $\mathcal{Z}_0(C, Z) \xrightarrow{\nu_*} \mathcal{Z}_0(X)$. We write $\mathcal{R}_0(C, Z, X)$ for the subgroup of $\mathcal{Z}_0(X)$ generated by the set $\{\nu_*(\operatorname{div}(f)) \mid f \in \mathcal{O}_{C,Z}^{\times}\}$. Let $\mathcal{R}_0^{\mathrm{BK}}(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the image of the map $\mathcal{Z}_0(C, Z, X) \to \mathcal{Z}_0(X)$, where $\mathcal{Z}_0(C, Z, X)$ runs through all good curves. We let $\mathrm{CH}_0^{\mathrm{BK}}(X) = \mathcal{Z}_0(X)/\mathcal{R}_0^{\mathrm{BK}}(X)$.

If we let $\mathcal{R}_0^{\mathrm{LW}}(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the divisors of rational functions on good curves as above, where we further assume that the map $\nu : C \to X$ is a closed immersion, then the resulting quotient group $\mathcal{Z}_0(X)/\mathcal{R}_0^{\mathrm{LW}}(X)$ is denoted by $\mathrm{CH}_0^{\mathrm{LW}}(X)$. There is a canonical surjection $\mathrm{CH}_0^{\mathrm{LW}}(X) \twoheadrightarrow \mathrm{CH}_0^{\mathrm{BK}}(X)$. However, we can say more about this map in the present context. This comparison will be an essential ingredient in the proof of Theorem 8.4.

Theorem 7.11. *For a projective scheme X over $\mathbb{C}$, the map $\mathrm{CH}_0^{\mathrm{LW}}(X) \twoheadrightarrow \mathrm{CH}_0^{\mathrm{BK}}(X)$ is an isomorphism.*

Proof. By [Binda and Krishna 2018, Lemma 3.13], there are cycle class maps $\mathrm{CH}_0^{\mathrm{LW}}(X) \twoheadrightarrow \mathrm{CH}_0^{\mathrm{BK}}(X) \to K_0(X)$, and one knows from [Levine 1987, Corollary 2.7] that the kernel of the composite map is $(d-1)!$-torsion. It follows that $\operatorname{Ker}(\mathrm{CH}_0^{\mathrm{LW}}(X) \to \mathrm{CH}_0^{\mathrm{BK}}(X))$ is torsion. In particular, it lies in $\mathrm{CH}_0^{\mathrm{LW}}(X)_{\deg 0}$.

On the other hand, it follows from [Binda and Krishna 2018, Proposition 9.7] that the Abel–Jacobi map $\mathrm{CH}_0^{\mathrm{LW}}(X)_{\deg 0} \to J^d(X)$ (see [Biswas and Srinivas 1999, Theorem 1.1]) factors through $\mathrm{CH}_0^{\mathrm{LW}}(X)_{\deg 0} \twoheadrightarrow \mathrm{CH}_0^{\mathrm{BK}}(X)_{\deg 0} \to J^d(X)$. Moreover, it follows from [Biswas and Srinivas 1999, Theorem 1.1] that the composite map is an isomorphism on the torsion subgroups. In particular,

$$\operatorname{Ker}(\mathrm{CH}_0^{\mathrm{LW}}(X)_{\deg 0} \twoheadrightarrow \mathrm{CH}_0^{\mathrm{BK}}(X)_{\deg 0})$$

is torsion-free. It must therefore be zero. □

In the rest of this text, we identify the above two Chow groups for projective schemes over $\mathbb{C}$ and write them as $\mathrm{CH}^d(X)$. There is a degree map

$$\deg_X : \mathrm{CH}^d(X) \to H^{2d}_{\mathrm{an}}(X, \mathbb{Z}(d)) \cong \mathbb{Z}^r.$$

Let $\mathrm{CH}^d(X)_{\deg 0}$ denote the kernel of this degree map. In order to obtain applications of the above Abel–Jacobi map, we connect $\mathrm{CH}^d(X)$ with the motivic cohomology as follows.

Lemma 7.12. *There is a canonical map $\gamma_X : \mathrm{CH}^d(X) \to H^{2d}(X, \mathbb{Z}(d))$ which restricts to a map $\gamma_X : \mathrm{CH}^d(X)_{\deg 0} \to A^d(X)$.*

Proof. We let U denote the smooth locus of X and let $x \in U$ be a closed point. The excision for the local cohomology with support in a closed subscheme tells us that the map

$$\mathbb{H}^0_{\{x\},\mathrm{cdh}}(X, C_*z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}}) \to \mathbb{H}^0_{\{x\},\mathrm{cdh}}(U, C_*z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}})$$

is an isomorphism. On the other hand, the purity theorem for the motivic cohomology of smooth schemes and the isomorphism between the motivic cohomology and higher Chow groups [Voevodsky 2002a] imply that the map

$$\mathbb{H}^0_{\{x\},\mathrm{cdh}}(U, C_*z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}}) \to \mathbb{H}^0_{\mathrm{cdh}}(U, C_*z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}})$$

is same as the map of the Chow groups $\mathbb{Z} \cong \mathrm{CH}_0(\{x\}) \to \mathrm{CH}_0(U)$. In particular, we obtain a map

$$\begin{aligned}\gamma_x : \mathbb{Z} &\to \mathbb{H}^0_{\{x\},\mathrm{cdh}}(X, C_*z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}}) \\ &\to \mathbb{H}^0_{\mathrm{cdh}}(X, C_*z_{\mathrm{equi}}(\mathbb{A}^d_{\mathbb{C}}, 0)_{\mathrm{cdh}}) = H^{2d}(X, \mathbb{Z}(d)).\end{aligned}$$

We let $\gamma_X([x])$ be the image of $1 \in \mathbb{Z}$ under this map. This yields a homomorphism $\gamma_X : \mathcal{Z}_0(X) \to H^{2d}(X, \mathbb{Z}(d))$. We now show that this map kills $\mathcal{R}_0(X)$.

We first assume that X is a reduced curve. In this case, an easy application of the spectral sequence of Theorem 4.27 and the vanishing result of Theorem 5.1 shows that there is a short exact sequence

$$0 \to H^2(X, \mathbb{Z}(1)) \to KH_0(X) \to H^0(X, \mathbb{Z}(0)) \to 0. \tag{7.13}$$

Using $H^0(X, \mathbb{Z}(0)) \xrightarrow{\cong} H^0_{\mathrm{an}}(X, \mathbb{Z})$ and the natural map $K_*(X) \to KH_*(X)$, we have a commutative diagram of the short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}(X) \arrow[r] & K_0(X) \arrow[r] \arrow[d] & H^0_{\mathrm{an}}(X, \mathbb{Z}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^2(X, \mathbb{Z}(1)) \arrow[r] & KH_0(X) \arrow[r] & H^0_{\mathrm{an}}(X, \mathbb{Z}) \arrow[r] & 0 \end{tikzcd} \tag{7.14}$$

It follows from [Binda and Krishna 2018, Lemma 3.11] that the map $\mathcal{Z}_0(X) \to K_0(X)$ given by $\mathrm{cyc}_X([x]) = [\mathcal{O}_{\{x\}}] \in K_0(X)$ defines an isomorphism $\mathrm{CH}^1(X) \xrightarrow{\cong} \mathrm{Pic}(X)$. Note that $x \in U$ and hence the class $[\mathcal{O}_{\{x\}}]$ in $K_0(X)$ makes sense. We conclude from this isomorphism and (7.14) that the composite map $\mathcal{Z}_0(X) \to K_0(X) \to KH_0(X)$ has image in $H^2(X, \mathbb{Z}(1))$ and it factors through $\mathrm{CH}^1(X)$.

We now assume $d \geq 2$ and $\nu : (C, Z) \to X$ be a good curve and let $f \in \mathcal{O}^{\times}_{C,Z}$. We need to show that $\gamma_X(\nu_*(\mathrm{div}(f))) = 0$. By [Binda and Krishna 2018, Lemma 3.4], we can assume that ν is an lci morphism. In particular, there is a functorial push-forward map $\nu_* : H^2(C, \mathbb{Z}(1)) \to H^{2d}(X, \mathbb{Z}(d))$ by Corollary 3.6 and [Navarro 2018, Definition 2.32, Theorem 2.33]. We thus have a commutative diagram

$$\begin{tikzcd} \mathcal{Z}_0(C, Z) \arrow[r, "\cong"'] \arrow[d, "\nu_*"'] \arrow[rr, bend left, "\gamma_C"] & \bigoplus_{x \notin Z} H^0(\{x\}, \mathbb{Z}(0)) \arrow[r] \arrow[d, "\nu_*"] & H^2(C, \mathbb{Z}(1)) \arrow[d, "\nu_*"] \\ \mathcal{Z}_0(X) \arrow[r, "\cong"] \arrow[rr, bend right, "\gamma_X"'] & \bigoplus_{x \notin X_{\mathrm{sing}}} H^0(\{x\}, \mathbb{Z}(0)) \arrow[r] & H^{2d}(X, \mathbb{Z}(d)) \end{tikzcd} \tag{7.15}$$

The two horizontal arrows on the right are the pushforward maps on the motivic cohomology since the inclusion $\{x\} \hookrightarrow X$ is an lci morphism for every $x \notin X_{\mathrm{sing}}$. We have shown that $\gamma_C(\mathrm{div}(f)) = 0$ and hence $\gamma_X(\nu_*(\mathrm{div}(f))) = \nu_*(\gamma_C(\mathrm{div}(f))) = 0$. Furthermore, the composite

$$\mathcal{Z}_0(X) \to H^{2d}(X, \mathbb{Z}(d)) \to H^{2d}(\widetilde{X}, \mathbb{Z}(d)) \to H^{2d}_{\mathrm{an}}(\widetilde{X}, \mathbb{Z}(d)) \cong \mathbb{Z}^r$$

is the degree map. This shows that $\gamma_X(\mathcal{Z}_0(X)_{\deg 0}) \subseteq A^d(X)$. □

8. Applications III: Roitman torsion and cycle class map

We now consider a projective scheme X of dimension d over $\mathbb{C}$. Using the map $\gamma_X : \mathrm{CH}^d(X) \to H^{2d}(X, \mathbb{Z}(d))$ and the Abel–Jacobi map AJ^d_X of (7.7), we now prove our main result on the Abel–Jacobi map and Roitman torsion for singular schemes. We shall use the following lemma in the proof.

Lemma 8.1. *Let X be a reduced projective scheme of dimension d over $\mathbb{C}$. There is a cycle class map $\mathrm{cyc}^{Q}_{X,0} : \mathrm{CH}^d(X) \to K_0(X)$ and a commutative diagram*

$$\begin{tikzcd} \mathrm{CH}^d(X) \arrow[r, "\mathrm{cyc}^{Q}_{X,0}"] \arrow[d, "\gamma_X"'] & K_0(X) \arrow[d] \\ H^{2d}(X, \mathbb{Z}(d)) \arrow[r, "\mathrm{cyc}_{X,0}"] & KH_0(X) \end{tikzcd} \tag{8.2}$$

Proof. Every closed point $x \in U$ defines the natural map

$$\mathbb{Z} = K_0(\{x\}) = K_0^{\{x\}}(X) \to K_0(X)$$

and hence a class $[\mathcal{O}_{\{x\}}] \in K_0(X)$. This defines a map $\mathrm{cyc}^{Q}_{X,0} : \mathcal{Z}_0(X) \to K_0(X)$ and it factors through $\mathrm{CH}^d(X)$ by [Levine and Weibel 1985, Proposition 2.1]. Since $\mathrm{CH}^d(X)$ is generated by the closed points in U, it suffices to show that for every closed point $x \in U$, the diagram

$$\begin{tikzcd} K_0^{\{x\}}(X) \arrow[r] \arrow[d, equal] & K_0(X) \arrow[d] \\ KH_0^{\{x\}}(X) \arrow[r] & KH_0(X) \end{tikzcd} \tag{8.3}$$

commutes. But this is clear from the functorial properties of the map of presheaves $K(-) \to KH(-)$ on $\mathbf{Sch}_{\mathbb{C}}$. $\square$

We can now prove:

Theorem 8.4. *Let X be a projective scheme over $\mathbb{C}$ of dimension d. Assume that either $d \leq 2$ or X is regular in codimension one. Then there is a semiabelian variety $J^d(X)$ and an Abel–Jacobi map $\mathrm{AJ}^d_X : A^d(X) \to J^d(X)$ which is surjective and whose restriction to the torsion subgroups $\mathrm{AJ}^d_X : A^d(X)_{\mathrm{tors}} \to J^d(X)_{\mathrm{tors}}$ is an isomorphism.*

Proof. We can assume that X is reduced. We first consider the case when X has dimension at most two but has arbitrary singularity. In this case, we only need to prove that AJ^d_X is surjective and its restriction to the torsion subgroups is an isomorphism.

The map AJ^d_X is induced by the Chern class map $c_{X,d,0} : KH_0(X) \to H^{2d}_{\mathcal{D}}(X, \mathbb{Z}(d))$ and the composite map $K_0(X) \to KH_0(X) \to H^{2d}_{\mathcal{D}}(X, \mathbb{Z}(d))$ is Gillet's Chern class map $C^{Q}_{X,d,0}$ of (6.14). Composing these maps with the cycle class maps and using Lemma 8.1, we get a commutative diagram

$$\begin{tikzcd} \mathrm{CH}^d(X)_{\deg 0} \arrow[r, "\gamma_X"] \arrow[dr, "\mathrm{AJ}^{d,Q}_X"'] & A^d(X) \arrow[d, "\mathrm{AJ}^d_X"] \\ & J^d(X) \end{tikzcd} \tag{8.5}$$

The map $\mathrm{AJ}^{d,Q}_X$ is surjective and is an isomorphism on the torsion subgroups by [Barbieri-Viale et al. 1996, Main Theorem]. It follows that AJ^d_X is also surjective. To prove that it is an isomorphism on the torsion subgroups, we apply Theorem 7.8 and [Barbieri-Viale and Kahn 2016, Corollary 13.7.5]. It follows from these results that there is indeed an isomorphism $\phi^d_X : J^d(X)_{\mathrm{tor}} \xrightarrow{\cong} A^d(X)_{\mathrm{tor}}$. Since $J^d(X)$ is a semiabelian variety, we know that for any given integer $n \geq 1$, the n-torsion subgroup ${}_nJ^d(X)$ is finite. It follows that ${}_nA^d(X)$ and ${}_nJ^d(X)$ are finite abelian groups of the same order. We conclude that the Abel–Jacobi map $\mathrm{AJ}^d_X : A^d(X) \to J^d(X)$ induces the map $\mathrm{AJ}^d_X : {}_nA^d(X) \to {}_nJ^d(X)$ between finite abelian groups which have same order. Therefore, this map is an isomorphism if and only if it is a surjection. But this is true by (8.5) because we have seen above that the composite map $\mathrm{AJ}^{d,Q}_X$

is an isomorphism between the n-torsion subgroups. Since $n \geq 1$ is arbitrary in this argument, we conclude the proof of the theorem.

We now consider the case when X has arbitrary dimension but is regular in codimension one. Let $f : \widetilde{X} \to X$ be a resolution of singularities of X. It is then known that $J^d(X) \cong J^d(\widetilde{X}) = \mathrm{Alb}(\widetilde{X})$; see [Mallick 2009, Remark 2, p. 505]. We have a commutative diagram

$$\begin{tikzcd} \mathrm{CH}^d(X)_{\deg 0} \arrow[r, "\gamma_X"] \arrow[dr, "\mathrm{AJ}^{\mathrm{LW}}_X"'] & A^d(X) \arrow[r, two heads, "f^*"] \arrow[d, dotted, "\mathrm{AJ}^d_X"] & A^d(\widetilde{X}) \arrow[d, "\mathrm{AJ}^d_{\widetilde{X}}"] \\ & J^d(X) \arrow[r, "\cong"] & J^d(\widetilde{X}) \end{tikzcd} \tag{8.6}$$

Since the lower horizontal arrow in this diagram is an isomorphism, it uniquely defines the Abel–Jacobi map AJ^d_X. The map $f^* \circ \gamma_X$ is known to be surjective by the moving lemma for 0-cycles on smooth schemes. In particular, f^* is surjective. The map $\mathrm{AJ}^d_{\widetilde{X}}$ is also known to be surjective. It follows that AJ^d_X is surjective.

To prove that this is an isomorphism on the torsion subgroups, we can argue exactly as in the first case of the theorem. This reduces us to showing that AJ^d_X is surjective on the n-torsion subgroups for every given integer $n \geq 1$. But this follows because $\mathrm{AJ}^{\mathrm{LW}}_X$ (and also $\mathrm{AJ}^d_{\widetilde{X}}$) is an isomorphism on the n-torsion subgroups by [Biswas and Srinivas 1999, Theorem 1.1], finishing the proof of the theorem. $\square$

Remark 8.7. For arbitrary $d \geq 1$, the map $\mathrm{AJ}^{d,Q}_X$ in (8.5) is known to be an isomorphism only up to multiplication by $(d-1)!$. This prevents us from extending Theorem 8.4 to higher dimensions if X has singularities in codimension one. We also warn the reader that unlike $\mathrm{AJ}^{d,Q}_X$ in (8.5), the map $\mathrm{AJ}^{\mathrm{LW}}_X$ in (8.6) is not defined via the Chern class map on $K_0(X)$. These maps coincide only up to multiplication by $(d-1)!$.

8.8. ***Injectivity of the cycle class map.*** Like the case of smooth schemes, the Roitman torsion theorem for singular schemes has many potential applications. Here, we use this to prove our next main result of this section. It was shown by Levine [1987, Theorem 3.2] that for a smooth projective scheme X of dimension d over $\mathbb{C}$, the cycle class map $H^{2d}(X, \mathbb{Z}(d)) \to K_0(X)$ (see (5.11)) is injective. We generalize this to singular schemes as follows.

Theorem 8.9. *Let X be a projective scheme of dimension d over $\mathbb{C}$. Assume that either $d \leq 2$ or X is regular in codimension one. Then the cycle class map* $\mathrm{cyc}_0 : H^{2d}(X, \mathbb{Z}(d)) \to KH_0(X)$ *is injective.*

Proof. We note that $\mathrm{cyc}_0 : H^{2d}(X, \mathbb{Z}(d)) \to KH_0(X)$ is induced by the spectral sequences (4.28) and (5.11), both of which degenerate with rational coefficients. In particular, $\mathrm{Ker}(\mathrm{cyc}_0)$ is a torsion group.

On the other hand, if $\dim(X) \le 2$, (7.7) and Theorem 8.4 tell us that the composite map $\tilde{c}^d_X : H^{2d}(X, \mathbb{Z}(d)) \xrightarrow{\mathrm{cyc}_0} KH_0(X) \xrightarrow{c_{X,0,d}} H^{2d}_{\mathcal{D}}(X, \mathbb{Z}(d))$ is an isomorphism on the torsion subgroups. We must therefore have $\mathrm{Ker}(\mathrm{cyc}_0) = 0$.

If X is regular in codimension one, we let $\widetilde{X} \to X$ be a resolution of singularities and consider the commutative diagram

$$\begin{tikzcd} H^{2d}(X, \mathbb{Z}(d)) \arrow[r, "\mathrm{cyc}_{X,0}"] \arrow[d, "f^*"'] & KH_0(X) \arrow[d, "f^*"] \\ H^{2d}(\widetilde{X}, \mathbb{Z}(d)) \arrow[r, "\mathrm{cyc}_{\widetilde{X},0}"] & K_0(\widetilde{X}) \end{tikzcd}$$

We have shown in the proof of Theorem 8.4 that the left vertical arrow is an isomorphism on the torsion subgroups. The bottom horizontal arrow is injective by [Levine 1987, Theorem 3.2]. It follows that $\mathrm{cyc}_{X,0}$ is injective on the torsion subgroup. We must therefore have $\mathrm{Ker}(\mathrm{cyc}_{X,0}) = 0$. This finishes the proof. □

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K-theory, local cohomology and tangent spaces to Hilbert schemes

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Using K-theory, we construct a map $\pi : T_Y\mathrm{Hilb}^p(X) \to H_y^p(\Omega^{p-1}_{X/\mathbb{Q}})$ from the tangent space to the Hilbert scheme at a point Y to the local cohomology group. We use this map π to answer (after slight modification) a question by Mark Green and Phillip Griffiths on constructing a map from the tangent space $T_Y\mathrm{Hilb}^p(X)$ to the Hilbert scheme at a point Y to the tangent space to the cycle group $TZ^p(X)$.

1. Introduction

Let X be a smooth projective variety over a field k of characteristic 0 and let $Y \subset X$ be a subvariety of codimension p. Considering Y as an element of $\mathrm{Hilb}^p(X)$, it is well known that the Zariski tangent space $T_Y\mathrm{Hilb}^p(X)$ can be identified with $H^0(Y, \mathcal{N}_{Y/X})$, where $\mathcal{N}_{Y/X}$ is the normal sheaf.

The element Y also defines an element of the cycle group $Z^p(X)$. We are interested in defining the tangent space $TZ^p(X)$ to the cycle group $Z^p(X)$. In [Green and Griffiths 2005], Mark Green and Phillip Griffiths define $TZ^p(X)$ for $p = 1$ (divisors) and $p = \dim(X)$ (0-cycles) and leave the general case as an open question. Much of their theory was extended by Benjamin Dribus, Jerome W. Hoffman and the author in [Dribus et al. 2018; Yang 2016a]. In [Yang 2016a], we define $TZ^p(X)$ for any integer p satisfying $1 \leqslant p \leqslant \dim(X)$, generalizing Green and Griffiths' definitions. We recall the following fact from [Yang 2016a] for our purpose, and refer to [Green and Griffiths 2005; Yang 2016a] for definition of $TZ^p(X)$.

Theorem 1.1 [Yang 2016a, Theorem 2.8]. *For X a smooth projective variety over a field k of characteristic* 0 *and for any integer $p \geqslant 1$, the tangent space $TZ^p(X)$ is identified with* $\mathrm{Ker}(\partial_1^{p,-p})$:

$$TZ^p(X) \cong \mathrm{Ker}(\partial_1^{p,-p}),$$

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where $\partial_1^{p,-p}$ *is the differential of the Cousin complex* [Hartshorne 1966] *of* $\Omega^{p-1}_{X/\mathbb{Q}}$ *in position* p:

$$0 \to \Omega^{p-1}_{k(X)/\mathbb{Q}} \to \cdots \to \bigoplus_{y \in X^{(p)}} H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}) \xrightarrow{\partial_1^{p,-p}} \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}}) \to \cdots .$$

We want to study the relation between $T_Y\mathrm{Hilb}^p(X)$ and $TZ^p(X)$. The following question is suggested in [Green and Griffiths 2005, pp. 18 and 87–89].

Question 1.2 [Green and Griffiths 2005]. For X a smooth projective variety over a field k of characteristic 0 and for any integer $p \geqslant 1$, is it possible to define a map from the tangent space $T_Y\mathrm{Hilb}^p(X)$ to the Hilbert scheme at a point Y to the tangent space to the cycle group $TZ^p(X)$?

For $p = \dim(X)$, this has been answered affirmatively in [Green and Griffiths 2005, Section 7.2].

Theorem 1.3 [Green and Griffiths 2005]. *For* $p = d := \dim(X)$, *there exists a map*

$$T_Y\mathrm{Hilb}^d(X) \to TZ^d(X)$$

from the tangent space to the Hilbert scheme at a point Y *to the tangent space to the cycle group.*

The main result of this short note is to construct a map

$$\pi : T_Y\mathrm{Hilb}^p(X) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$$

(see Definition 4.1), and use this map to study the above Question 1.2.

In Example 4.4, we show, for a general subvariety $Y \subset X$ of codimension p and $Y' \in T_Y\mathrm{Hilb}^p(X)$, that $\pi(Y')$ may not lie in $TZ^p(X)$ (the kernel of $\partial_1^{p,-p}$). However, we show in Theorem 4.6 that there exist $Z \subset X$ of codimension p and $Z' \in T_Z\mathrm{Hilb}^p(X)$ such that $\pi(Y') + \pi(Z') \in TZ^p(X)$.

As an application, we show how to find Milnor K-theoretic cycles in Theorem 4.7. In [Yang 2016b], we will apply these techniques to eliminate obstructions to deforming curves on a threefold.

Notations and conventions.

(1) K-theory used in this note is Thomason–Trobaugh nonconnective K-theory, if not stated otherwise.

(2) For any abelian group M, $M_{\mathbb{Q}}$ denotes the image of M in $M \otimes_{\mathbb{Z}} \mathbb{Q}$.

(3) $X[\varepsilon]$ denote the first-order trivial deformation of X, i.e.,

$$X[\varepsilon] = X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)),$$

where $k[\varepsilon]/(\varepsilon^2)$ is the ring of dual numbers.

2. K-theory and tangent spaces to Hilbert schemes

For X a smooth projective variety over a field k of characteristic 0 and $Y \subset X$ a subvariety of codimension p, let $i : Y \to X$ be the inclusion. Then $i_* O_Y$ is a coherent O_X-module and can be resolved by a bounded complex of vector bundles on X. Let $Y^{'}$ be a first-order deformation of Y, that is, $Y^{'} \subset X[\varepsilon]$ such that $Y^{'}$ is flat over $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ and $Y^{'} \otimes_{k[\varepsilon]/(\varepsilon^2)} k \cong Y$. Then $i_* O_{Y'}$ can be resolved by a bounded complex of vector bundles on $X[\varepsilon]$, where $i : Y^{'} \to X[\varepsilon]$.

Let $D^{\mathrm{perf}}(X[\varepsilon])$ denote the derived category of perfect complexes of $O_X[\varepsilon]$-modules, and let $\mathcal{L}_{(i)}(X[\varepsilon]) \subset D^{\mathrm{perf}}(X[\varepsilon])$ be defined as

$$\mathcal{L}_{(i)}(X[\varepsilon]) := \{E \in D^{\mathrm{perf}}(X[\varepsilon]) \mid \mathrm{codim}_{\mathrm{Krull}}(\mathrm{supph}(E)) \geq -i\},$$

where the closed subset $\mathrm{supph}(E) \subset X$ is the support of the total homology of the perfect complex E.

The resolution of $i_* O_{Y'}$, which is a perfect complex of $O_X[\varepsilon]$-modules supported on Y, defines an element of the Verdier quotient $\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon])$, denoted $[i_* O_{Y'}]$.

In general, the length of the perfect complex $[i_* O_{Y'}]$ may not be equal to p. Since $Y \subset X$ is of codimension p, we expect the perfect complex $[i_* O_{Y'}]$ to be of length p. To achieve this, instead of considering $[i_* O_{Y'}]$ as an element of the Verdier quotient $\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon])$, we consider its image in the idempotent completion $(\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon]))^{\#}$, denoted $[i_* O_{Y'}]^{\#}$, where the idempotent completion is in the sense of [Balmer and Schlichting 2001]. We have the following result:

Theorem 2.1 [Balmer 2007]. *For each $i \in \mathbb{Z}$, localization induces an equivalence*

$$(\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon]))^{\#} \simeq \coprod_{x[\varepsilon] \in X[\varepsilon]^{(-i)}} D^{\mathrm{perf}}_{x[\varepsilon]}(X[\varepsilon])$$

between the idempotent completion of $\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon])$ and the coproduct over $x[\varepsilon] \in X[\varepsilon]^{(-i)}$ of the derived category of perfect complexes of $O_{X[\varepsilon],x[\varepsilon]}$-modules with homology supported on the closed point $x[\varepsilon] \in \mathrm{Spec}(O_{X[\varepsilon],x[\varepsilon]})$. Consequently, one has

$$K_0\big((\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon]))^{\#}\big) \simeq \bigoplus_{x[\varepsilon] \in X[\varepsilon]^{(-i)}} K_0(D^{\mathrm{perf}}_{x[\varepsilon]}(X[\varepsilon])).$$

Let y be the generic point of Y and let $\mathcal{I}_Y$ be the ideal sheaf of Y. Then there exists the short exact sequence

$$0 \to \mathcal{I}_Y \to O_X \to i_* O_Y \to 0,$$

whose localization at y is the short exact sequence

$$0 \to (\mathcal{I}_Y)_y \to O_{X,y} \to (i_* O_Y)_y \to 0.$$

We have $O_{Y,y} = O_{X,y}/(\mathcal{I}_Y)_y$. Since $O_{Y,y}$ is a field, $(\mathcal{I}_Y)_y$ is the maximal ideal of the regular local ring (of dimension p) $O_{X,y}$. So the maximal ideal $(\mathcal{I}_Y)_y$ is generated by a regular sequence $f_1, \dots, f_p$ of length p.

Let $\mathcal{I}_{Y'}$ be the ideal sheaf of Y', so $\mathcal{I}_{Y'}/(\varepsilon)\mathcal{I}_{Y'} = \mathcal{I}_Y$ because of flatness. So we have $(\mathcal{I}_{Y'})_y/(\varepsilon)(\mathcal{I}_{Y'})_y = (\mathcal{I}_Y)_y$. Lift $f_1, \dots, f_p$ to $f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p$ in $(\mathcal{I}_{Y'})_y$, where $g_1, \dots, g_p \in O_{X,y}$. Then $f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p$ generates $(\mathcal{I}_{Y'})_y$ because of Nakayama's lemma:

$$(\mathcal{I}_{Y'})_y = (f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p).$$

Moreover, $f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p$ is a regular sequence, which can be checked directly.

We see that Y is generically defined by a regular sequence $f_1, \dots, f_p$ of length p, where $f_1, \dots, f_p \in O_{X,y}$. Moreover, Y' is generically given by lifting $f_1, \dots, f_p$ to $f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p$, where $g_1, \dots, g_p \in O_{X,y}$. Let $F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p)$ denote the Koszul complex associated to the regular sequence $f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p$, which is a resolution of $O_{X,y}[\varepsilon]/(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p)$:

$$0 \to F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \cdots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0 \to 0,$$

where each $F_i = \bigwedge^i (O_{X,y}[\varepsilon])^{\oplus p}$ and $A_i : \bigwedge^i (O_{X,y}[\varepsilon])^{\oplus p} \to \bigwedge^{i-1} (O_{X,y}[\varepsilon])^{\oplus p}$ are defined as usual.

Under the equivalence in Theorem 2.1, the localization at the generic point y sends $[i_* O_{Y'}]^\#$ to the Koszul complex $F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p)$:

$$[i_* O_{Y'}]^\# \to F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p).$$

Milnor K-groups with support are rationally defined in terms of eigenspaces of Adams operations in [Yang 2016c] as follows:

Definition 2.2 [Yang 2016c, Definition 3.2]. Let X be a finite equidimensional noetherian scheme and $x \in X^{(j)}$. For $m \in \mathbb{Z}$, the Milnor K-group with support $K_m^M(O_{X,x} \text{ on } x)$ is rationally defined to be

$$K_m^M(O_{X,x} \text{ on } x) := K_m^{(m+j)}(O_{X,x} \text{ on } x)_{\mathbb{Q}},$$

where $K_m^{(m+j)}$ is the eigenspace of $\psi^k = k^{m+j}$ and ψ^k are the Adams operations.

Theorem 2.3 [Gillet and Soulé 1987, Proposition 4.12]. *The Adams operations ψ^k defined on perfect complexes* (defined in [Gillet and Soulé 1987]) *satisfy*

$$\psi^k(F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p)) = k^p F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p).$$

Hence, $F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p)$ is of eigenweight p and can be considered as an element of $K_0^{(p)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}$:

$$F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p) \in K_0^{(p)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} = K_0^{M}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]).$$

Definition 2.4. We define a map $\mu : T_Y\mathrm{Hilb}^p(X) \to K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$ by

$$\mu : Y' \mapsto F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p).$$

3. Chern character

For any integer m, let $K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}}$ denote the weight i eigenspace of the relative K-group, that is, the kernel of the natural projection

$$K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} \xrightarrow{\varepsilon=0} K_m^{(i)}(O_{X,y} \text{ on } y)_{\mathbb{Q}}.$$

Recall that we have proved the following isomorphisms in [Dribus et al. 2018; Yang 2016c]:

Theorem 3.1 [Dribus et al. 2018, Corollary 9.5; Yang 2016c, Corollary 3.11]. *Let X be a smooth projective variety over a field k of characteristic 0 and let $y \in X^{(p)}$. The Chern character (from K-theory to negative cyclic homology) induces isomorphisms*

$$K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}} \cong H_y^p\big(\Omega_{X/\mathbb{Q}}^{\bullet,(i)}\big)$$

between relative K-groups and local cohomology groups, where

$$\begin{cases} \Omega_{X/\mathbb{Q}}^{\bullet,(i)} = \Omega_{X/\mathbb{Q}}^{2i-(m+p)-1} & \text{if } \frac{1}{2}(m+p) < i \le m+p, \\ \Omega_{X/\mathbb{Q}}^{\bullet,(i)} = 0 & \text{else.} \end{cases}$$

The main tool for proving these isomorphisms is the space-level versions of Goodwillie's and Cathelineau's isomorphisms, proved in [Cortiñas et al. 2009, Appendix B].

Let $K_m^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)$ denote the relative K-group, that is, the kernel of the natural projection

$$K_m^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \xrightarrow{\varepsilon=0} K_m^M(O_{X,y} \text{ on } y).$$

In other words, $K_m^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)$ is $K_m^{(m+p)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}}$. In particular, by taking $i = p$ and $m = 0$ in Theorem 3.1, we obtain the following formula:

Corollary 3.2. $$K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon) \xrightarrow{\cong} H_y^p(\Omega_{X/\mathbb{Q}}^{p-1}).$$

Definition 3.3. Let X be a smooth projective variety over a field k of characteristic 0 and let $y \in X^{(p)}$. There exists a natural surjective map

$$\mathrm{Ch} : K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to H_y^p(\Omega_{X/\mathbb{Q}}^{p-1}),$$

which is defined to be the composition of the natural projection

$$K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)$$

and the isomorphism

$$K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon) \xrightarrow{\cong} H_y^p(\Omega^{p-1}_{X/\mathbb{Q}}).$$

Now we recall a beautiful construction of Angéniol and Lejeune-Jalabert, which describes the map

$$\mathrm{Ch} : K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to H_y^p(\Omega^{p-1}_{X/\mathbb{Q}})$$

in Definition 3.3.

An element $M \in K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \subset K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}$ is represented by a strict perfect complex $L_\bullet$ supported at $y[\varepsilon]$:

$$0 \to F_n \xrightarrow{M_n} F_{n-1} \xrightarrow{M_{n-1}} \cdots \xrightarrow{M_2} F_1 \xrightarrow{M_1} F_0 \to 0,$$

where each $F_i = O_{X,y}[\varepsilon]^{r_i}$ and the M_i are matrices with entries in $O_{X,y}[\varepsilon]$.

Definition 3.4 [Angéniol and Lejeune-Jalabert 1989, p. 24]. The *local fundamental class* attached to this perfect complex is defined to be the collection

$$[L_\bullet]_{\mathrm{loc}} = \left\{ \frac{1}{p!} dM_i \circ dM_{i+1} \circ \cdots \circ dM_{i+p-1} \right\}, \quad i = 0, 1, \ldots,$$

where $d = d_{\mathbb{Q}}$ and each dM_i is the matrix of absolute differentials. In other words,

$$dM_i \in \mathrm{Hom}\big(F_i, F_{i-1} \otimes \Omega^1_{O_{X,y}[\varepsilon]/\mathbb{Q}}\big).$$

Theorem 3.5 [Angéniol and Lejeune-Jalabert 1989, Lemma 3.1.1, p. 24 and Definition 3.4, p. 29]. *The class* $[L_\bullet]_{\mathrm{loc}}$ *above is a cycle in* $\mathcal{H}om\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big)$, *and the image of* $[L_\bullet]_{\mathrm{loc}}$ *in* $H^p\big(\mathcal{H}om\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big)\big)$ *does not depend on the choice of the basis of* $L_\bullet$.

Since

$$H^p\big(\mathcal{H}om\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big)\big) = \mathcal{E}XT^p\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big),$$

the local fundamental class $[L_\bullet]_{\mathrm{loc}}$ defines an element in $\mathcal{E}XT^p\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big)$:

$$[L_\bullet]_{\mathrm{loc}} \in \mathcal{E}XT^p\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big).$$

Noting $L_\bullet$ is supported on y (same underlying space as $y[\varepsilon]$), there exists the following trace map (see [Angéniol and Lejeune-Jalabert 1989, p. 98–99] for details):

$$\mathrm{Tr} : \mathcal{E}XT^p\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big) \to H_y^p(\Omega^p_{X[\varepsilon]/\mathbb{Q}}).$$

Definition 3.6 [Angéniol and Lejeune-Jalabert 1989, Definition 2.3.2, p. 99]. The image of $[L_\bullet]_{\mathrm{loc}}$ under the above trace map, denoted $\mathcal{V}^p_{L_\bullet}$, is called the Newton class.

$K_0(O_{X,y}[\varepsilon]$ on $y[\varepsilon])$ is the Grothendieck group of the triangulated category $D^b(O_{X,y}[\varepsilon]$ on $y[\varepsilon])$, which is the derived category of perfect complexes of $O_{X,y}[\varepsilon]$-modules with homology supported on the closed point $y[\varepsilon] \in \operatorname{Spec}(O_{X,y}[\varepsilon])$. Recall that the Grothendieck group of a triangulated category is the monoid of isomorphism objects modulo the submonoid formed from distinguished triangles.

Theorem 3.7 [Angéniol and Lejeune-Jalabert 1989, Proposition 4.3.1, p. 113]. *The Newton class $\mathcal{V}^p_{L_\bullet}$ is well-defined on $K_0(O_{X,y}[\varepsilon]$ on $y[\varepsilon])$.*

The truncation map $\rfloor \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} : \Omega^p_{X[\varepsilon]/\mathbb{Q}} \to \Omega^{p-1}_{X/\mathbb{Q}}$ induces a map

$$\rfloor \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} : H^p_y(\Omega^p_{X[\varepsilon]/\mathbb{Q}}) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}).$$

Lemma 3.8. *The map*

$$\mathrm{Ch} : K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$$

from Definition 3.3 can be described as a composition

$$K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to \mathcal{E}XT^p\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big) \to H^p_y(\Omega^p_{X[\varepsilon]/\mathbb{Q}}) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}),$$
$$L_\bullet \mapsto [L_\bullet]_{\mathrm{loc}} \mapsto \mathcal{V}^p_{L_\bullet} \mapsto \mathcal{V}^p_{L_\bullet} \rfloor \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0}.$$

In particular, for the Koszul complex $F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p)$ in Definition 2.4, the Ch map can be described as follows. The diagram

$$\begin{cases} F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p) \longrightarrow O_{X,y}[\varepsilon]/(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p), \\ F_p(\cong O_{X,y}[\varepsilon]) \xrightarrow{[F_\bullet]_{\mathrm{loc}}} F_0 \otimes \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} (\cong \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}}), \end{cases}$$

where $[F_\bullet]_{\mathrm{loc}}$ is the local fundamental class attached to $F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p)$, gives an element in $\operatorname{Ext}^p_{O_{X,y}[\varepsilon]}(O_{X,y}[\varepsilon]/(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p), \Omega^p_{X[\varepsilon]/\mathbb{Q}})$. This, moreover, gives an element in $H^p_y(\Omega^p_{X[\varepsilon]/\mathbb{Q}})$, denoted $\mathcal{V}^p_{F_\bullet}$.

We use $F_\bullet(f_1, \dots, f_p)$ to denote the Koszul complex associated to the regular sequence $f_1, \dots, f_p$, which is a resolution of $O_{X,y}/(f_1, \dots, f_p)$. The truncation of $\mathcal{V}^p_{F_\bullet}$ in ε produces an element in $H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$, which can be represented by the diagram

$$\begin{cases} F_\bullet(f_1, \dots, f_p) \longrightarrow O_{X,y}/(f_1, \dots, f_p), \\ F_p(\cong O_{X,y}) \xrightarrow{[F_\bullet]_{\mathrm{loc}} \rfloor \frac{\partial}{\partial \varepsilon}|_{\varepsilon=0}} F_0 \otimes \Omega^{p-1}_{O_{X,y}/\mathbb{Q}} \big(\cong \Omega^{p-1}_{O_{X,y}/\mathbb{Q}}\big). \end{cases}$$

For simplicity, assuming $g_2 = \dots = g_p = 0$, we see that

$$[F_\bullet]_{\mathrm{loc}} \rfloor \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} = g_1 df_2 \wedge \dots \wedge df_p$$

and the truncation of $\mathcal{V}^p_{F_\bullet}$ in ε is represented by the diagram

$$\begin{cases} F_\bullet(f_1,\dots,f_p) & \longrightarrow & O_{X,y}/(f_1,\dots,f_p), \\ F_p(\cong O_{X,y}) & \xrightarrow{g_1 df_2\wedge\cdots\wedge df_p} & F_0\otimes\Omega^{p-1}_{O_{X,y}/\mathbb{Q}}(\cong \Omega^{p-1}_{O_{X,y}/\mathbb{Q}}). \end{cases}$$

Further concrete examples can be found in [Green and Griffiths 2005, Chapter 7, p. 90–91].

4. The map π

Definition 4.1. We define a map from $T_Y\mathrm{Hilb}^p(X)$ to $H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$ by composing Ch in Definition 3.3 with μ in Definition 2.4:

$$\pi : T_Y\mathrm{Hilb}^p(X) \xrightarrow{\mu} K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \xrightarrow{\mathrm{Ch}} H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}).$$

Recall that the Cousin complex of $\Omega^{p-1}_{X/\mathbb{Q}}$ is of the form

$$0\to\Omega^{p-1}_{k(X)/\mathbb{Q}}\to\cdots\to\bigoplus_{y\in X^{(p)}} H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}) \xrightarrow{\partial_1^{p,-p}} \bigoplus_{x\in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}})\to\cdots$$

and the tangent space $TZ^p(X)$ is identified with $\mathrm{Ker}(\partial_1^{p,-p})$ (see Theorem 1.1).

For $p=d:=\dim(X)$, $\partial_1^{d,-d}=0$ because of dimensional reasons. So

$$TZ^d(X)=\mathrm{Ker}(\partial_1^{d,-d})=\bigoplus_{y\in X^{(d)}} H^d_y(\Omega^{d-1}_{X/\mathbb{Q}}).$$

Corollary 4.2. *For $p=d:=\dim(X)$, the map π defines a map from $T_Y\mathrm{Hilb}^d(X)$ to $TZ^d(X)$ and it agrees with the map by Green and Griffiths in Theorem 1.3.*

We want to know, for general p, whether this map π defines a map from $T_Y\mathrm{Hilb}^p(X)$ to $TZ^p(X)$, as Green and Griffiths asked in Question 1.2.

Remark 4.3. In an email to the author, Christophe Soulé suggested considering the image of suitable Koszul complexes under the Ch map in Definition 3.3. This leads us to the following example, showing that π does not define a map from $T_Y\mathrm{Hilb}^p(X)$ to $TZ^p(X)$ in general. The Koszul complex technique is also used in Theorem 4.6.

The author sincerely thanks Christophe Soulé for very helpful suggestions.

Example 4.4. For a smooth projective threefold X over a field k of characteristic 0, let $Y\subset X$ be a curve with generic point y. We assume a point $x\in Y\subset X$ is defined by (f,g,h) and Y is generically defined by (f,g). Then $O_{X,y}=(O_{X,x})_{(f,g)}$.

We consider the infinitesimal deformation Y' of Y which is generically given by $(f+\varepsilon/h, g)$, where $1/h\in O_{X,y}=(O_{X,x})_{(f,g)}$. Note $1/h\notin O_{X,x}$. The Koszul complex of $(f+\varepsilon/h, g)$ is of the form

$$0 \to (O_{X,x})_{(f,g)}[\varepsilon] \xrightarrow{(g,-f-\varepsilon/h)^{\mathrm{T}}} (O_{X,x})^{\oplus 2}_{(f,g)}[\varepsilon] \xrightarrow{(f+\varepsilon/h,g)} (O_{X,x})_{(f,g)}[\varepsilon] \to 0,$$

where $(-,-)^{\mathrm{T}}$ denotes transpose.

The image $\pi(Y') \in H^2_y(\Omega^1_{X/\mathbb{Q}})$ is represented by the diagram

$$\begin{cases} (O_{X,x})_{(f,g)} \to (O_{X,x})^{\oplus 2}_{(f,g)} \to (O_{X,x})_{(f,g)} \to (O_{X,x})_{(f,g)}/(f,g) \to 0, \\ \qquad\qquad\qquad (O_{X,x})_{(f,g)} \xrightarrow{(1/h)dg} \Omega^1_{(O_{X,x})_{(f,g)}/\mathbb{Q}}. \end{cases}$$

Let $F_\bullet(f,g,h)$ be the Koszul complex of f, g, h:

$$0 \to O_{X,x} \to O^{\oplus 3}_{X,x} \to O^{\oplus 3}_{X,x} \to O_{X,x} \to 0.$$

Then $\partial_1^{2,-2}(\pi(Y'))$ in $H^3_x(\Omega^1_{X/\mathbb{Q}})$ is represented by the diagram

$$\begin{cases} F_\bullet(f,g,h) \longrightarrow O_{X,x}/(f,g,h), \\ \quad O_{X,x} \xrightarrow{1dg} \Omega^1_{O_{X,x}/\mathbb{Q}}, \end{cases}$$

which is not zero.

This example shows that, in general, the image of π may not lie in $TZ^p(X)$(the kernel of $\partial_1^{p,-p}$). However, we will show, in Theorem 4.6 below, that given $Y \subset X$ of codimension p and $Y' \in T_Y\mathrm{Hilb}^p(X)$, there exists $Z \subset X$ of codimension p and $Z' \in T_Z\mathrm{Hilb}^p(X)$ such that $\pi(Y') + \pi(Z')$ is a nontrivial element of $TZ^p(X)$.

To fix notation, let X be a smooth projective variety over a field k of characteristic 0 and let $Y \subset X$ be a subvariety of codimension p with generic point y. Let $W \subset Y$ be a subvariety of codimension 1 in Y with generic point w. One assumes W is generically defined by $f_1, f_2, \ldots, f_p, f_{p+1}$ and Y is generically defined by $f_1, f_2, \ldots, f_p$. So one has $O_{X,y} = (O_{X,w})_P$, where P is the ideal $(f_1, f_2, \ldots, f_p) \subset O_{X,w}$.

The element Y' is generically given by $(f_1+\varepsilon g_1, f_2+\varepsilon g_2, \ldots, f_p+\varepsilon g_p)$, where $g_1, \ldots, g_p \in O_{X,y}$. We assume $g_2 = \cdots = g_p = 0$. Since $O_{X,y} = (O_{X,w})_P$, we write $g_1 = a/b$, where $a, b \in O_{X,w}$ and $b \notin P$. In Theorem 4.6, we will consider the cases of whether or not b is in the maximal ideal $(f_1, f_2, \ldots, f_p, f_{p+1}) \subset O_{X,w}$.

Lemma 4.5. *If $b \notin (f_1, f_2, \ldots, f_p, f_{p+1})$, then $\partial_1^{p,-p}(\pi(Y')) = 0$.*

Proof. If $b \notin (f_1, f_2, \ldots, f_p, f_{p+1})$, then b is a unit in $O_{X,w}$, so $g_1 = a/b \in O_{X,w}$. Then $\pi(Y')$ is represented by the diagram

$$\begin{cases} F_\bullet(f_1, f_2, \ldots, f_p) \longrightarrow (O_{X,w})_P/(f_1, f_2, \ldots, f_p), \\ F_p(\cong (O_{X,w})_P) \xrightarrow{g_1 df_2\wedge\cdots\wedge df_p} F_0 \otimes \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}} (\cong \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}). \end{cases}$$

Here, $F_\bullet(f_1, f_2, \dots, f_p)$ is of the form

$$0 \to F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \cdots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,$$

where each $F_i = \bigwedge^i ((O_{X,w})_P)^{\oplus p}$. Since $f_{p+1} \notin P$, f_{p+1}^{-1} exists in $(O_{X,w})_P$, and we can write

$$g_1 df_2 \wedge \cdots \wedge df_p = \frac{g_1 f_{p+1}}{f_{p+1}} df_2 \wedge \cdots \wedge df_p.$$

Now $\partial_1^{p,-p}(\pi(Y'))$ is represented by the diagram

$$\begin{cases} F_\bullet(f_1, f_2, \dots, f_p, f_{p+1}) \longrightarrow O_{X,w}/(f_1, f_2, \dots, f_p, f_{p+1}), \\ F_{p+1}(\cong O_{X,w}) \xrightarrow{g_1 f_{p+1} df_2 \wedge \cdots \wedge df_p} F_0 \otimes \Omega^{p-1}_{O_{X,w}/\mathbb{Q}} (\cong \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}). \end{cases}$$

The complex $F_\bullet(f_1, f_2, \dots, f_p, f_{p+1})$ is of the form

$$0 \to \bigwedge^{p+1} (O_{X,w})^{\oplus p+1} \xrightarrow{A_{p+1}} \bigwedge^{p} (O_{X,w})^{\oplus p+1} \to \cdots.$$

Let $\{e_1, \dots, e_{p+1}\}$ be a basis of $(O_{X,w})^{\oplus p+1}$; the map A_{p+1} is

$$e_1 \wedge \cdots \wedge e_{p+1} \to \sum_{j=1}^{p+1} (-1)^j f_j e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots e_{p+1},$$

where $\hat{e}_j$ means to omit the j-th term.

Since f_{p+1} appears in A_{p+1},

$$g_1 f_{p+1} df_2 \wedge \cdots \wedge df_p \equiv 0 \in \operatorname{Ext}^{p+1}_{O_{X,w}}\big(O_{X,w}/(f_1, \dots, f_p, f_{p+1}), \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}\big)$$

and $\partial_1^{p,-p}(\pi(Y')) = 0$. $\square$

This lemma doesn't contradict Example 4.4, where $h \in (f, g, h) \subset O_{X,x}$.

Theorem 4.6. *For $Y' \in T_Y \mathrm{Hilb}^p(X)$ generically defined by $(f_1 + \varepsilon g_1, f_2, \dots, f_p)$, where $g_1 = a/b \in O_{X,y} = (O_{X,w})_P$, we have*:

Case 1: *If $b \notin (f_1, f_2, \dots, f_p, f_{p+1})$, then $\pi(Y') \in TZ^p(X)$, i.e., $\partial_1^{p,-p}(\pi(Y')) = 0$.*

Case 2: *If $b \in (f_1, f_2, \dots, f_p, f_{p+1})$, then there exist $Z \subset X$ of codimension p and $Z' \in T_Z \mathrm{Hilb}^p(X)$ with $\pi(Y') + \pi(Z') \in TZ^p(X)$, i.e., $\partial_1^{p,-p}(\pi(Y') + \pi(Z')) = 0$.*

Proof. Case 1 is Lemma 4.5. Now we consider the case $b \in (f_1, f_2, \dots, f_p, f_{p+1})$. Since $b \notin (f_1, f_2, \dots, f_p)$, we can write $b = \sum_{i=1}^p a_i f_i^{n_i} + a_{p+1} f_{p+1}^{n_{p+1}}$, where a_{p+1} is a unit in $O_{X,w}$ and each n_j is some integer. For simplicity, we assume each $n_j = 1$ and $a_{p+1} = 1$.

Since Y' is generically given by $(f_1+\varepsilon g_1, f_2, \dots, f_p)$, then $\pi(Y')$ is represented by the following diagram (where $g_1 = a/b$):

$$\begin{cases} F_\bullet(f_1, f_2, \dots, f_p) & \longrightarrow & (O_{X,w})_P/(f_1, f_2, \dots, f_p), \\ F_p(\cong (O_{X,w})_P) & \xrightarrow{(a/b)df_2\wedge\cdots\wedge df_p} & F_0 \otimes \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}(\cong \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}). \end{cases}$$

Here, $F_\bullet(f_1, f_2, \dots, f_p)$ is of the form

$$0 \to F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \cdots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,$$

where each $F_i = \bigwedge^i((O_{X,w})_P)^{\oplus p}$. Let $\{e_1, \dots, e_p\}$ be a basis of $(O_{X,w})^{\oplus p}$; the map A_p is

$$e_1 \wedge \cdots \wedge e_p \to \sum_{j=1}^{p}(-1)^j f_j e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots e_p,$$

where $\hat{e}_j$ means to omit the j-th term.

Noting

$$\frac{1}{b} - \frac{1}{f_{p+1}} = \frac{-\sum_{i=1}^{p} a_i f_i}{bf_{p+1}}$$

and each f_i $(i = 1, \dots, p)$ appears in A_p, the above diagram representing $\pi(Y')$ can be replaced by the following one:

$$\begin{cases} F_\bullet(f_1, f_2, \dots, f_p) & \longrightarrow & (O_{X,w})_P/(f_1, f_2, \dots, f_p), \\ F_p(\cong (O_{X,w})_P) & \xrightarrow{(a/f_{p+1})df_2\wedge\cdots\wedge df_p} & F_0 \otimes \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}(\cong \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}). \end{cases}$$

Then $\partial_1^{p,-p}(\pi(Y'))$ is represented by the diagram

$$\begin{cases} F_\bullet(f_1, f_2, \dots, f_p, f_{p+1}) & \longrightarrow & O_{X,w}/(f_1, f_2, \dots, f_p, f_{p+1}), \\ F_{p+1}(\cong O_{X,w}) & \xrightarrow{adf_2\wedge\cdots\wedge df_p} & F_0 \otimes \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}(\cong \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}). \end{cases}$$

Let P' denote the prime $(f_{p+1}, f_2, \dots, f_p) \subset O_{X,w}$. Then P' defines a generic point $z \in X^{(p)}$ and one has $O_{X,z} = (O_{X,w})_{P'}$. We define the subscheme

$$Z := \overline{\{z\}}.$$

Let Z' be a first-order infinitesimal deformation of Z, which is generically given by $(f_{p+1} + \varepsilon a/f_1, f_2, \dots, f_p)$. Then $\pi(Z')$ is represented by the diagram

$$\begin{cases} F_\bullet(f_{p+1}, f_2, \dots, f_p) & \longrightarrow & (O_{X,w})_{P'}/(f_{p+1}, f_2, \dots, f_p), \\ F_p(\cong (O_{X,w})_{P'}) & \xrightarrow{(a/f_1)df_2\wedge\cdots\wedge df_p} & F_0 \otimes \Omega^{p-1}_{(O_{X,w})_{P'}/\mathbb{Q}}(\cong \Omega^{p-1}_{(O_{X,w})_{P'}/\mathbb{Q}}), \end{cases}$$

and $\partial_1^{p,-p}(\pi(Z'))$ is represented by the diagram

$$\begin{cases} F_\bullet(f_{p+1}, f_2, \dots, f_p, f_1) & \longrightarrow & O_{X,w}/(f_{p+1}, f_2, \dots, f_p, f_1), \\ F_{p+1}(\cong O_{X,w}) & \xrightarrow{adf_2\wedge\cdots\wedge df_p} & F_0 \otimes \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}(\cong \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}). \end{cases}$$

Here, $F_\bullet(f_1, f_2, \dots, f_p, f_{p+1})$ and $F_\bullet(f_{p+1}, f_2, \dots, f_p, f_1)$ are Koszul resolutions of $O_{X,w}/(f_1, f_2, \dots, f_p, f_{p+1})$ and $O_{X,w}/(f_{p+1}, f_2, \dots, f_p, f_1)$, respectively.

These Koszul complexes $F_\bullet(f_1, f_2, \dots, f_p, f_{p+1})$ and $F_\bullet(f_{p+1}, f_2, \dots, f_p, f_1)$ are related by the commutative diagram

$$\begin{tikzcd} O_{X,w} \arrow[r, "D_{p+1}"] \arrow[d, "\det A_1"'] & \bigwedge^p O_{X,w}^{\oplus p+1} \arrow[r, "D_p"] \arrow[d, "\bigwedge^p A_1"'] & \cdots \arrow[r] \arrow[d] & O_{X,w}^{\oplus p+1} \arrow[r, "D_1"] \arrow[d, "A_1"'] & O_{X,w} \arrow[d, "="'] \\ O_{X,w} \arrow[r, "E_{p+1}"] & \bigwedge^p O_{X,w}^{\oplus p+1} \arrow[r, "E_p"] & \cdots \arrow[r] & O_{X,w}^{\oplus p+1} \arrow[r, "E_1"] & O_{X,w} \end{tikzcd}$$

(see [Griffiths and Harris 1978, p. 691]), where each D_i and E_i are defined as usual. In particular, $D_1 = (f_1, f_2, \dots, f_p, f_{p+1})$, $E_1 = (f_{p+1}, f_2, \dots, f_p, f_1)$, and A_1 is the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots 1 \\ 0 & 1 & 0 & \cdots 0 \\ 0 & 0 & 1 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots 0 \end{pmatrix}$$

Since $\det A_1 = -1$, one has

$$\partial_1^{p,-p}(\pi(Z')) = -\partial_1^{p,-p}(\pi(Y')) \in \operatorname{Ext}^{p+1}_{O_{X,w}}\big(O_{X,w}/(f_1, f_2, \dots, f_p, f_{p+1}), \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}\big),$$

and consequently, $\partial_1^{p,-p}(\pi(Z') + \pi(Y')) = 0 \in H^{p+1}_w\big(\Omega^{p-1}_{O_{X,w}/\mathbb{Q}}\big)$. In other words,

$$\pi(Z') + \pi(Y') \in TZ^p(X). \qquad \square$$

There exists the following commutative diagram, which is part of the commutative diagram of [Yang 2016c, Theorem 3.14] (taking $j = 1$):

$$\begin{tikzcd} \bigoplus_{x\in X^{(p)}} H^p_x\big(\Omega^{p-1}_{X/\mathbb{Q}}\big) \arrow[d, "\partial_1^{p,-p}"'] & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p)}} K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \arrow[l, "\mathrm{Ch}"'] \arrow[d, "d^{p,-p}_{1,X[\varepsilon]}"'] \\ \bigoplus_{x\in X^{(p+1)}} H^{p+1}_x\big(\Omega^{p-1}_{X/\mathbb{Q}}\big) & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p+1)}} K_{-1}^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \arrow[l, "\mathrm{Ch}"', "\cong"] \end{tikzcd}$$

For $Y' \in T_Y\mathrm{Hilb}^p(X)$, which is generically defined by $(f_1 + \varepsilon g_1, f_2, \dots, f_p)$ for $g_1 = a/b \in O_{X,y} = (O_{X,w})_P$, we use $F_\bullet(f_1 + \varepsilon g_1, f_2, \dots, f_p)$ to denote the Koszul complex associated to $f_1 + \varepsilon g_1, f_2, \dots, f_p$. Theorem 4.6 implies the following.

Case 1: If $b \notin (f_1, f_2, \dots, f_p, f_{p+1})$, $\partial_1^{p,-p}(\pi(Y')) = 0$, the commutative diagram

$$\begin{array}{ccc}
\pi(Y') & \xleftarrow{\text{Ch}} & F_\bullet(f_1+\varepsilon g_1, f_2, \dots, f_p) \\
{\scriptstyle \partial_1^{p,-p}}\big\downarrow & & \big\downarrow{\scriptstyle d^{p,-p}_{1,X[\varepsilon]}} \\
0 & \xleftarrow[\cong]{\text{Ch}} & d^{p,-p}_{1,X[\varepsilon]}(F_\bullet(f_1+\varepsilon g_1, f_2, \dots, f_p))
\end{array}$$

says $d^{p,-p}_{1,X[\varepsilon]}(F_\bullet(f_1+\varepsilon g_1, f_2, \dots, f_p)) = 0$.

Case 2: If $b \in (f_1, f_2, \dots, f_p, f_{p+1})$, we are reduced to considering $b = f_{p+1}$. Then there exist $Z \subset X$ which is generically defined by $(f_{p+1}, f_2, \dots, f_p)$ and $Z' \in T_Z\mathrm{Hilb}^p(X)$ which is generically defined by $(f_{p+1}+\varepsilon a/f_1, f_2, \dots, f_p)$ such that $\partial_1^{p,-p}(\pi(Y')+\pi(Z')) = 0$. We use $F_\bullet(f_{p+1}+\varepsilon a/f_1, f_2, \dots, f_p)$ to denote the Koszul complex associated to $f_{p+1}+\varepsilon a/f_1, f_2, \dots, f_p$.

The commutative diagram

$$\begin{array}{ccc}
\pi(Y')+\pi(Z') & \xleftarrow{\text{Ch}} & F_\bullet(f_1+\varepsilon a/f_{p+1}, f_2, \dots, f_p)+F_\bullet(f_{p+1}+\varepsilon a/f_1, f_2, \dots, f_p) \\
{\scriptstyle \partial_1^{p,-p}}\big\downarrow & & \big\downarrow{\scriptstyle d^{p,-p}_{1,X[\varepsilon]}} \\
0 & \xleftarrow[\cong]{\text{Ch}} & d^{p,-p}_{1,X[\varepsilon]}\big(F_\bullet(f_1+\varepsilon a/f_{p+1}, f_2, \dots, f_p) \\
 & & \qquad +F_\bullet(f_{p+1}+\varepsilon a/f_1, f_2, \dots, f_p)\big)
\end{array}$$

says $d^{p,-p}_{1,X[\varepsilon]}\big(F_\bullet(f_1+\varepsilon a/f_{p+1}, f_2, \dots, f_p)+F_\bullet(f_{p+1}+\varepsilon a/f_1, f_2, \dots, f_p)\big) = 0$.

Recall that in [Yang 2016c, Definition 3.4 and Corollary 3.15], the p-th Milnor K-theoretic cycle is defined as

$$Z_p^M(D^{\mathrm{Perf}}(X[\varepsilon])) := \mathrm{Ker}\big(d^{p,-p}_{1,X[\varepsilon]}\big).$$

The above can be summarized as follows:

Theorem 4.7. *For $Y' \in T_Y\mathrm{Hilb}^p(X)$ generically defined by $(f_1+\varepsilon g_1, f_2, \dots, f_p)$ for $g_1 = a/b \in O_{X,y} = (O_{X,w})_P$, we use $F_\bullet(f_1+\varepsilon g_1, f_2, \dots, f_p)$ to denote the Koszul complex associated to $f_1+\varepsilon g_1, f_2, \dots, f_p$.*

Case 1: *If $b \notin (f_1, f_2, \dots, f_p, f_{p+1})$, then*

$$F_\bullet(f_1+\varepsilon g_1, f_2, \dots, f_p) \in Z_p^M(D^{\mathrm{Perf}}(X[\varepsilon])).$$

Case 2: *If $b \in (f_1, f_2, \dots, f_p, f_{p+1})$, we are reduced to considering $b = f_{p+1}$. Then there exist $Z \subset X$ which is generically defined by $(f_{p+1}, f_2, \dots, f_p)$ and $Z' \in T_Z\mathrm{Hilb}^p(X)$ generically defined by $(f_{p+1}+\varepsilon a/f_1, f_2, \dots, f_p)$ such that*

$$F_\bullet(f_1+\varepsilon a/f_{p+1}, f_2, \dots, f_p)+F_\bullet(f_{p+1}+\varepsilon a/f_1, f_2, \dots, f_p) \in Z_p^M(D^{\mathrm{Perf}}(X[\varepsilon])).$$

The existence of Z and $Z' \in T_Z\mathrm{Hilb}^p(X)$ has applications in deformation of cycles; see [Yang 2016b] for a concrete example of eliminating obstructions to deforming curves on a threefold.

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Droites sur les hypersurfaces cubiques

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On montre que sur toute hypersurface cubique complexe de dimension au moins 2, le groupe de Chow des cycles de dimension 1 est engendré par les droites. Le cas lisse est un théorème connu. La démonstration ici donnée repose sur un résultat sur les surfaces géométriquement rationnelles sur un corps quelconque (1983), obtenu via la K-théorie algébrique.

Over any complex cubic hypersurface of dimension at least 2, the Chow group of 1-dimensional cycles is spanned by the lines lying on the hypersurface. The smooth case had already been given several other proofs.

1. Introduction

Soit X une variété sur un corps quelconque. On note $CH_i(X)$ le groupe de Chow des cycles de dimension i sur X modulo l'équivalence rationnelle.

Dans cette note, j'établis le théorème suivant qui était déjà connu dans le cas lisse :

Théorème 3.1. *Soit k un corps algébriquement clos de caractéristique zéro. Soit $X \subset \mathbf{P}^n_k$, avec $n \geq 3$ une hypersurface cubique. Le groupe de Chow $CH_1(X)$ est engendré par les droites contenues dans X.*

Commençons par rappeler les résultats établis dans le cas des hypersurfaces cubiques lisses. Pour $n = 3$, c'est un résultat classique. Pour $n = 6$, c'est établi par Paranjape [1994, §4]. Celui-ci utilise l'existence d'un $\mathbf{P}^2$ contenu dans $X \subset \mathbf{P}^6$ pour fibrer $X \subset \mathbf{P}^6$ en quadriques de dimension 2 au-dessus de $\mathbf{P}^3$. Paranjape écrit qu'une méthode analogue vaut pour tout $n \geq 6$. Pour tout $n \geq 4$, le théorème est établi par M. Shen [2014, théorème 1.1] par une méthode différente de celle de Paranjape. Pour $n \geq 5$, le théorème est aussi un cas particulier d'un résultat de Tian et Zong [2014, théorème 6.1] sur les intersections complètes de Fano dans $\mathbf{P}^m$ de multidegré $(d_1, \ldots, d_c)$ avec $d_1 + \cdots + d_c \leq m - 1$ (résultat obtenu par encore une autre méthode).

Comme le note déjà Paranjape [1994], pour $n \geq 6$, l'énoncé pour X lisse implique que le groupe de Chow $CH_1(X)$ est égal à $\mathbb{Z}$. En effet le schéma de Fano des

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droites de X est alors une variété de Fano (lisse, projective, faisceau anticanonique ample), et un théorème bien connu de Campana et de Kollár-Miyaoka-Mori dit que les variétés de Fano sont rationnellement connexes (par chaînes).

Il y a deux ingrédients dans la démonstration du théorème 3.1. Le premier ingrédient est un résultat sur les surfaces projectives lisses géométriquement rationnelles sur les corps de dimension cohomologique 1 (Théorème 2.1 ci-dessous), dont la démonstration utilise la K-théorie algébrique (théorème de Merkur'ev et Suslin). Le second ingrédient est classique : c'est la classification des types de surfaces cubiques singulières sur un corps algébriquement clos. La démonstration procède par sections hyperplanes et récurrence sur la dimension. Même pour une hypersurface cubique lisse donnée, elle impose de considérer toutes les hypersurfaces cubiques de dimension un de moins obtenues par section hyperplane, et celles-ci peuvent être singulières.

Récemment, pour $n \geq 4$, M. Shen [2016, théorème 4.1] établit un théorème qui généralise le cas lisse du théorème 3.1 sur un corps de base non nécessairement algébriquement clos, lorsque l'hypersurface cubique contient une droite définie sur ce corps. Le cas $n = 3$ est établi dans [Colliot-Thélène et Loughran 2017].

Mis à part les résultats de [Colliot-Thélène 1983], nous n'utilisons ici que les propriétés les plus simples des groupes de Chow des variétés, telles qu'on les trouve dans le chapitre 1 de [Fulton 1984], en particulier la suite de localisation et le comportement dans une fibration en droites affines (propositions 1.8 et 1.9 là-dedans).

Étant donnée une variété X projective sur un corps K, la R-équivalence sur l'ensemble $X(K)$ des points K-rationnels de X est la relation d'équivalence engendrée par la relation élémentaire suivante : deux K-points A et B sont élémentairement liés s'il existe un K-morphisme $f : \mathbf{P}^1_K \to X$ tel que A et B soient dans $f(\mathbf{P}^1(K)) \subset X(K)$. Si deux K-points A et B sont R-équivalents, alors $A - B = 0 \in CH_0(X)$.

2. Groupe de Chow des zéro-cycles d'une hypersurface cubique sur un corps de fonctions d'une variable

Le théorème suivant est une conséquence immédiate de [Colliot-Thélène 1983, proposition 4], puisque le groupe de cohomologie galoisienne $H^1(K, S)$ pour un K-tore S sur un corps K de dimension cohomologique 1 est nul.

Théorème 2.1 [Colliot-Thélène 1983, Theorem A (iv)]. *On suppose que K est un corps de caractéristique zéro et de dimension cohomologique égale à* 1. *Soit X une K-surface projective, lisse, géométriquement rationnelle. Le noyau de l'application degré* $\deg_K : CH_0(X) \to \mathbb{Z}$ *est nul. Si X possède un point rationnel, par exemple si K est un corps C_1, alors l'application degré* $\deg_K : CH_0(X) \to \mathbb{Z}$ *est un isomorphisme.* □

Ce théorème s'applique en particulier aux surfaces cubiques lisses. Étudions maintenant le cas des surfaces cubiques quelconques.

Proposition 2.2. *Soit K un corps de caractéristique zéro et de dimension cohomologique* 1. *Soit $X \subset \mathbf{P}^3_K$ une surface cubique. Supposons $X(K) \neq \varnothing$, ce qui est le cas si K est C_1, par exemple si K est un corps de fonctions d'une variable sur un corps algébriquement clos. Alors l'application degré* $\deg_K : CH_0(X) \to \mathbb{Z}$ *est un isomorphisme.*

Démonstration. Comme toute surface cubique lisse sur un corps algébriquement clos est rationnelle, le cas où X est lisse est un cas particulier du théorème 2.1.

Supposons X singulière. Si $X \subset \mathbf{P}^3_K$ est un cône, tout point fermé de X est rationnellement équivalent à un multiple d'un point K-rationnel du sommet du cône (cet argument vaut sur un corps quelconque).

Si X n'est pas un cône, mais n'est pas géométriquement intègre, alors c'est l'union d'un plan P et d'une quadrique Q géométriquement intègre, leur intersection est une conique C dans $\mathbf{P}^2_K$. Toute telle conique possède un point K-rationnel, puisque $cd(K) \leq 1$, et $\deg_K : CH_0(C) \to \mathbb{Z}$ est un isomorphisme. Fixons $m \in C(K)$. Tout point fermé du plan P est rationnellement équivalent à un multiple de m. Si la quadrique Q est un cône de sommet $q \in Q(K)$, tout point fermé de Q est rationnellement équivalent à un multiple de q, et m est rationnellement équivalent à q. Si la quadrique Q est lisse, alors elle est K-rationnelle car elle possède un K-point, et $\deg_K : CH_0(Q) \to \mathbb{Z}$ est un isomorphisme (en fait $Q(K)/R = \{*\}$). On conclut que $\deg_K : CH_0(X) \to \mathbb{Z}$ est un isomorphisme.

Supposons désormais que la surface cubique $X \subset \mathbf{P}^3_K$ n'est pas un cône et est géométriquement intègre. Elle est alors géométriquement rationnelle. Les diverses singularités possibles ont été analysées depuis longtemps (Schläffli, Cayley, B. Segre, Bruce–Wall [Bruce et Wall 1979], Demazure, Coray–Tsfasman [Coray et Tsfasman 1988]).

Si les points singuliers ne sont pas isolés, alors la surface cubique X contient une droite double $D \subset X$, qui est définie sur K. Tout K-point de X hors de D est situé sur une droite définie sur K rencontrant D, à savoir la droite résiduelle de l'intersection avec X du plan défini par D et le K-point. On a donc $X(K)/R = \{*\}$ et $\deg_K : CH_0(X) \to \mathbb{Z}$ est un isomorphisme.

Supposons désormais de plus que les points singuliers de X sont isolés.

Si X possède un point singulier K-rationnel, alors $X(K)/R = \{*\}$ [Madore 2008, lemme 1.3], sous la simple hypothèse que toute conique sur K possède un point rationnel. On a donc alors $X(L)/R = \{*\}$ pour toute extension finie de corps L/K. Ainsi $\deg_K : CH_0(X) \to \mathbb{Z}$ est un isomorphisme.

Supposons dorénavant de plus que l'on a $X_{sing}(K) = \varnothing$. Soit $f : Y \to X$ une résolution des singularités. Un argument simple (lemme de Nishimura) montre que l'application induite $Y(K) \to X(K)$ contient les K-points lisses de X dans son image. Donc $Y(K) \to X(K)$ est surjectif. Par hypothèse, on a $X(K) \neq \varnothing$. Soient P et Q deux K-points de X. Soient M, resp. N, dans $Y(K)$ d'image P, resp. Q, dans $X(K)$. La K-surface Y est projective, lisse, géométriquement rationnelle. Le théorème 2.1 assure $M - N = 0 \in CH_0(Y)$. Le morphisme propre f induit $f_* : CH_0(Y) \to CH_0(X)$. On a donc $P - Q = 0 \in CH_0(X)$. Si R est un point fermé de X, de corps résiduel $L = K(R)$, suivant que X_L possède un L-point singulier ou non, l'un des deux arguments ci-dessus garantit $R - M_L = 0 \in CH_0(X_L)$, et donc $\deg_K : CH_0(X) \to \mathbb{Z}$ est un isomorphisme. □

Théorème 2.3. *Soit K un corps de caractéristique zéro et de dimension cohomologique* 1. *Soient $n \geq 3$ et $X \subset \mathbf{P}^n_K$, $n \geq 3$ une hypersurface cubique. Si $X(K) \neq \varnothing$, par exemple si K est un corps C_1, alors l'application degré $\deg_K : CH_0(X) \to \mathbb{Z}$ est un isomorphisme.*

Démonstration. Soit O un point K-rationnel et P un point fermé de X, de corps résiduel $L = K(P)$. Sur $X_L \subset \mathbf{P}^n_L$, on dispose d'un point L-rationnel p défini par P et du L-point $q = O_L$. On choisit un espace linéaire $H \subset \mathbf{P}^n_L$ de dimension 3 qui contient p et q. Soit $Y := X_L \cap H$. Si $Y = H$, alors p et q sont R-équivalents sur X_L, donc $p - q = 0 \in CH_0(Y)$. Si $Y \subset H$ est une surface cubique, le théorème précédent assure aussi $p - q = 0 \in CH_0(Y)$ et donc $p - q = 0 \in CH_0(X_L)$. Ainsi $P - [L : K]O = 0 \in CH_0(X)$. □

Remarque 2.4. Pour tout corps K qui est C_1, et tout $n \geq 5$, un argument élémentaire [Madore 2008, proposition 1.4] montre que l'on a $X(K)/R = \{*\}$ pour toute hypersurface cubique (lisse ou non), d'où il résulte immédiatement que l'application $\deg_K : CH_0(X) \to \mathbb{Z}$ est un isomorphisme [Madore 2008, corollaire 1.6]. C'est une question ouverte si sur un tel corps K, et déjà sur un corps K de fonctions d'une variable sur le corps des complexes, on a $X(K)/R = \{*\}$ pour toute hypersurface cubique lisse $X \subset \mathbf{P}^n_K$ pour $n = 3, 4$.

3. Groupe de Chow des 1-cycles d'une hypersurface cubique sur un corps algébriquement clos

Théorème 3.1. *Soit k un corps algébriquement clos de caractéristique zéro. Soit $X \subset \mathbf{P}^n_k$, avec $n \geq 3$, une hypersurface cubique. Le groupe de Chow $CH_1(X)$ est engendré par les droites contenues dans X.*

Démonstration. On va établir cet énoncé par récurrence sur $n \geq 3$. On commence par établir le cas $n = 3$ par une discussion cas par cas.

Dans un plan $\mathbf{P}^2$ tout 1-cycle est rationnellement équivalent à un multiple d'une droite. Pour une quadrique $Q \subset \mathbf{P}^3$ non singulière, le groupe de Picard de Q est engendré par les deux classes de génératrices. Si $Y \subset \mathbf{P}^3$ de coordonnées (x, y, z, t)

est un cône défini par une équation $f(x, y, z) = 0$, et de sommet p de coordonnées $(0, 0, 0, 1)$, $CH_1(Y) = CH_1(Y \setminus p)$ est engendré par les génératrices du cône. Ceci établit le résultat dans le cas où la surface cubique n'est pas intègre, et aussi dans le cas où c'est un cône.

Supposons donc X intègre et non conique. Si les singularités de X ne sont pas isolées, alors X possède une droite double. On peut alors [Bruce et Wall 1979, §2, case E] trouver des coordonnées homogènes (x, y, z, t) de $\mathbf{P}^3$ telles que la surface soit donnée soit par l'équation

$$x^2z + y^2t = 0$$

soit par l'équation

$$x^2z + xyt + y^3 = 0.$$

Dans le premier cas, le complémentaire des deux droites $x = y = 0$ et $x = t = 0$, découpées par $x = 0$, est isomorphe au plan affine $\mathbf{A}^2$ de coordonnées (y, t). Dans le second cas, le complémentaire de la droite $x = y = 0$ découpée par $x = 0$ est isomorphe au plan affine $\mathbf{A}^2$ de coordonnées (y, t). Comme on a $CH_1(\mathbf{A}^2) = 0$, ceci établit que $CH_1(X)$ est engendré par des droites de X.

Sinon, X est normale, et si $f : X' \to X$ est sa désingularisation minimale, alors X' est une surface de del Pezzo généralisée de degré 3, et les “droites” de X' sont les transformées propres des vraies droites de X. Voir là-dessus [Coray et Tsfasman 1988, exemple 0.5]. La projection $CH_1(X') \to CH_1(X)$ est clairement surjective, et le groupe $CH_1(X') = \mathrm{Pic}(X')$ est engendré par les “droites” de X' (courbes D lisses de genre zéro avec $(D.D) = -1$ et les “racines irréductibles” (courbes lisses de genre zéro avec $(D.D) = -2$) qui sont des courbes contractées par f sur les points singuliers de X. Donc $CH_1(X)$ est engendré par les vraies droites de $X \subset \mathbf{P}^3$.

Soit $n \geq 4$. Supposons le cas $n - 1$ établi. Soit $X \subset \mathbf{P}^n$ une hypersurface cubique. On trouve dans X une droite D (il en existe sur toute surface cubique sur k algébriquement clos) et on choisit $Q \simeq \mathbf{P}^{n-2} \subset \mathbf{P}^n$ un espace linéaire de dimension $n - 2$ qui ne rencontre pas D et qui n'est pas contenu dans X. On considère le pinceau des espaces linéaires $\mathbf{P}^{n-1} \subset \mathbf{P}^n$ qui contiennent Q. On trouve ainsi une variété $Y \subset X \times \mathbf{P}^1$ munie d'un morphisme propre $Y \to X$ et d'une fibration $Y \to \mathbf{P}^1$ dont les fibres au-dessus de k-points $s \in \mathbf{P}^1(k)$ sont des hypersurfaces cubiques $Y_s \subset \mathbf{P}^{n-1}_k$ sections hyperplanes de $X \subset \mathbf{P}^n_k$ (l'hypothèse que X ne contient pas Q garantit qu'aucun Y_s n'est égal à $\mathbf{P}^{n-1}_k$) et dont la fibre générique est une hypersurface cubique $Y_\eta \subset \mathbf{P}^{n-1}_K$, avec $K = k(\mathbf{P}^1)$. La droite D définit une section de la fibration $Y \to \mathbf{P}^1$, soit une courbe $M \subset Y$, dont l'image se restreint en un K-point rationnel de Y_η. On dispose de la suite exacte

$$\bigoplus_{s \in \mathbf{P}^1(k)} CH_1(Y_s) \to CH_1(Y) \to CH_0(Y_\eta) \to 0.$$

D'après le théorème 2.3, la classe de M dans $CH_1(Y)$ s'envoie sur un générateur de $CH_0(Y_\eta) \simeq \mathbb{Z}$. L'application $CH_1(Y) \to CH_1(X)$ est surjective. En effet le morphisme $Y \to X$ induit un isomorphisme au-dessus du complémentaire du fermé propre $X \cap Q \subset Q$, et au-dessus de chaque point de $X \cap Q$, la fibre est une droite projective. L'image de M est la droite D de X, chaque groupe $CH_1(Y_s)$ est par hypothèse de récurrence engendré par des droites de Y_s, dont les images dans X sont des droites de X. $\square$

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