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Localization C^*-algebras and K-theoretic duality

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Based on the localization algebras of Yu, and their subsequent analysis by Qiao and Roe, we give a new picture of KK-theory in terms of time-parametrized families of (locally) compact operators that asymptotically commute with appropriate representations.

1. Introduction

Let A be a unital C^*-algebra, unitally represented on a Hilbert space H. Assume that there is a continuous family $(q_t)_{t\in[0,\infty)}$ of compact projections on H that asymptotically commutes with A, meaning that $[q_t, a] \to 0$ as $t \to \infty$ for all $a \in A$. Note that if p is a projection in A, then the family $t \mapsto pq_t$ of compact operators gets close to being a projection, and is thus close to a projection that is uniquely defined up to homotopy; in particular, there is a well-defined K-theory class $[pq_t] \in K_0(K(H)) = \mathbb{Z}$. It is moreover not difficult to see that this idea can be bootstrapped up to define a homomorphism

$$[q_t] : K_0(A) \to \mathbb{Z}, \quad [p] \mapsto [pq_t]. \tag{1.1}$$

This suggests using such parametrized families $(q_t)_{t\in[0,\infty)}$ to define elements of K-homology.

Indeed, something like this has been done when $A = C(X)$ is commutative. In this case, the condition that $[q_t, a] \to 0$ is equivalent to the condition that the "propagation" of q_t (in the sense of [Roe 1993, Definition 4.5]) tends to zero, up to an arbitrarily good approximation. Motivated by considerations like the above, and by the heat kernel approach to the Atiyah–Singer index theorem, Yu [1997] described K-homology for simplicial complexes in terms of families with asymptotically vanishing propagation using his localization algebras. Subsequently, Qiao and Roe [2010] gave a new approach to this result of Yu that works for all compact (in fact, all proper) metric spaces.

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In this paper, we present a new picture of Kasparov's KK groups [Kasparov 1980b] based on asymptotically commuting families. Thanks to the relationship between asymptotically vanishing propagation and asymptotic commutation, our picture can be thought of as an extension of the results of Yu and Qiao–Roe from commutative to general (separable) C^*-algebras, and from K-homology to KK-theory. We think this gives an attractive picture of KK-theory. We also suspect that the ease with which the pairing in (1.1) is defined — note that unlike in the case of Paschke duality, there is no dimension shift, and unlike in the case of E-theory, there is no suspension — should be useful for future applications. Having said this, we should note that the picture of the pairing in (1.1) is overly simplified, as in general to get the whole KK group one needs to consider formal differences of such families of projections (q_t) in an appropriate sense.

We now give precise statements of our main results. For a C^*-algebra B, we denote by $C_u(T, B)$ the C^*-algebra of bounded and uniformly continuous functions from $T = [0, \infty)$ to B. Inspired by [Yu 1997; Qiao and Roe 2010], we define the localization algebra $\mathcal{C}_L(\pi)$ associated to a representation π of a separable C^*-algebra A on a separable Hilbert space to be the C^*-subalgebra of $C_u(T, L(H))$ consisting of all the functions f such that for all $a \in A$,

$$[f, \pi(a)] \in C_0(T, K(H)) \quad \text{and} \quad \pi(a) f \in C_u(T, K(H)).$$

Let us recall that a representation π is ample if it is nondegenerate, faithful and $\pi(A) \cap K(H) = \{0\}$. One verifies that the isomorphism class of $\mathcal{C}_L(\pi)$ does not depend on the choice of an ample representation π. In this case, we write $\mathcal{C}_L(A)$ in place of $\mathcal{C}_L(\pi)$ and view A as a C^*-subalgebra of $L(H)$. Note that if A is unital, then

$$\mathcal{C}_L(A) = \{f \in C_u(T, K(H)) : [f, a] \in C_0(T, K(H)),\ \forall a \in A\}.$$

In this paper we establish canonical isomorphisms $K^i(A) \cong K_i(\mathcal{C}_L(A))$, $i = 0, 1$, between the K-homology of A and the K-theory of the localization algebra $\mathcal{C}_L(A)$. More generally, we use results of [Thomsen 2001] to show that for separable C^*-algebras A, B and any absorbing representation $\pi : A \to L(H_B)$ on the standard infinite dimensional countably generated right Hilbert B-module H_B, there are canonical isomorphisms of groups

$$KK_i(A, B) \xrightarrow{\cong} K_i(\mathcal{C}_L(\pi)), \quad i = 0, 1, \tag{1.2}$$

where the localization C^*-algebra $\mathcal{C}_L(\pi)$ consists of those functions $f \in C_u(T, L(H_B))$ such that for all $a \in A$,

$$[f, \pi(a)] \in C_0(T, K(H_B)) \quad \text{and} \quad \pi(a) f \in C_u(T, K(H_B)).$$

The isomorphism in (1.2) is defined and proved by combining Paschke duality with a generalization of the techniques used by Roe and Qiao in the commutative case.

The paper is structured as follows. In Section 2, we discuss absorbing representations and give a version of Voiculescu's theorem appropriate to localization algebras. In Section 3, we define the various dual algebras and localization algebras that we use, and show that they do not depend on the choice of absorbing representation. In Section 4, we prove the isomorphism in (1.2). Finally, in Section 5, we construct maps $K_i(\mathcal{C}_L(\pi)) \to E_i(A, B)$ and show that they "invert" the isomorphism in (1.2) in the sense that the composition $KK_i(A, B) \to K_i(\mathcal{C}_L(\pi)) \to E_i(A, B)$ is the canonical natural transformation from KK-theory to E-theory.

2. Absorbing representations

Let A and B be separable C^*-algebras. If E and F are countably generated right Hilbert B-modules, we denote by $L(E, F)$ the C^*-algebra of bounded B-linear adjointable operators from E to F. The corresponding C^*-algebra of "compact" operators is denoted by $K(E, F)$ [Kasparov 1980a]. Set $L(E) = L(E, E)$ and $K(E) = K(E, E)$. Recall that H_B is the standard infinite dimensional countably generated right Hilbert B-module.

We shall use the notion of (unitally) absorbing $*$-representations $\pi : A \to L(H_B)$; see [Thomsen 2001].

Definition 2.1. (i) Suppose that A is a unital separable C^*-algebra. A unital representation $\pi : A \to L(H_B)$ is called *unitally absorbing* for the pair (A, B) if for any other unital representation $\sigma : A \to L(E)$, there is an isometry $v \in C_b(\mathbb{N}, L(E, H_B))$ such that $v\sigma(a) - \pi(a)v \in C_0(\mathbb{N}, K(E, H_B))$ for all $a \in A$.

(ii) Suppose that A is a separable C^*-algebra. We denote by $\widetilde{A}$ the unitalization of A, with the convention that $\widetilde{A} = A$ if A is already unital. A representation $\pi : A \to L(H_B)$ is called *absorbing* for the pair (A, B) if its unitalization $\widetilde{\pi} : \widetilde{A} \to L(H_B)$ is unitally absorbing for the pair $(\widetilde{A}, B)$.

Note that in Definition 2.1, if we denote the components of v by v_n, we have $v_n\sigma(a) - \pi(a)v_n \in K(E, H_B)$ and $\lim_{n\to\infty} \|v_n\sigma(a) - \pi(a)v_n\| = 0$ for all $a \in A$.

Theorem 2.2 [Voiculescu 1976]. *Any ample representation of a separable C^*-algebra on a separable infinite dimensional Hilbert space is absorbing.*

Theorem 2.3 [Kasparov 1980a]. *Let A be a unital separable C^*-algebra and let B be a σ-unital C^*-algebra. If either A or B are nuclear, then any unital ample representation $\pi : A \to L(H) \subset L(H_B)$ is absorbing for the pair (A, B).*

Theorem 2.4 [Thomsen 2001]. *For any separable C^*-algebras A and B there exist absorbing representations $\pi : A \to L(H_B)$.*

Given two $*$-representations $\pi_i : A \to L(E_i)$ we write that $\pi_1 \underset{v}{\precsim} \pi_2$ if there is an isometry $v \in C_u(T, L(E_1, E_2))$ such that

$$v\pi_1(a) - \pi_2(a)v \in C_0(T, K(E_1, E_2)).$$

If in addition $v \in C_u(T, L(E_1, E_2))$ is a unitary with the same property, then we write $\pi_1 \underset{v}{\approx} \pi_2$.

Let $w^\infty : E_1^\infty \to E_1 \oplus E_1^\infty$ be the unitary defined by

$$w^\infty(h_0, h_1, h_2, \dots) = h_0 \oplus (h_1, h_2, \dots).$$

Lemma 2.5 [Dadarlat and Eilers 2002, Lemma 2.16]. *Let $\pi_i : A \to L(E_i)$ for $i = 1, 2$ be two representations and let $v \in L(E_1^\infty, E_2)$ be an isometry such that $v\pi_1^\infty(a) - \pi_2(a)v \in K(E_1^\infty, E_2)$ for all $a \in A$. Then*

$$u = (1_{E_1} \oplus v)w^\infty v^* + (1_{E_2} - vv^*) \in L(E_2, E_1 \oplus E_2)$$

is a unitary operator such that $\pi_1(a) \oplus \pi_2(a) - u\pi_2(a)u^* \in K(E_1 \oplus E_2)$ for all $a \in A$ and moreover,

$$\|\pi_1(a) \oplus \pi_2(a) - u\pi_2(a)u^*\| \le 6\|v\pi_1^\infty(a) - \pi_2(a)v\| + 4\|v\pi_1^\infty(a^*) - \pi_2(a^*)v\|.$$

Using this lemma, one gets the following strengthened variation of Voiculescu's theorem [1976]. This result appears in [Dadarlat and Eilers 2001] as Theorem 3.11, except that the uniform continuity of the isometry v and the unitary u were not addressed explicitly in the statement.

Theorem 2.6. *Let A, B be separable C^*-algebras and let $\pi_i : A \to L(E_i)$, $i = 1, 2$, be two representations where $E_i \cong H_B$. If π_2 is absorbing, then $\pi_1 \underset{v}{\precsim} \pi_2$ for some isometry $v \in C_u(T, L(E_1, E_2))$. If both π_1 and π_2 are absorbing, then $\pi_1 \underset{u}{\approx} \pi_2$ for some unitary $u \in C_u(T, L(E_1, E_2))$.*

Proof. As π_2 absorbs π_2^∞ there is an isometry $u = (u_n)_n \in C_b(\mathbb{N}, L(E_2^\infty, E_2))$ such that $u\pi_2^\infty(a) - \pi_2(a)u \in C_0(\mathbb{N}, K(E_2^\infty, E_2))$ for all $a \in A$. As π_2 absorbs π_1, there is a sequence of isometries $w_n \in L(E_1, E_2^\infty)$ with mutually orthogonal ranges such that $w_n\pi_1(a) - \pi_2^\infty(a)w_n \in K(E_1, E_2^\infty)$ and $\lim_{n\to\infty} \|w_n\pi_1(a) - \pi_2^\infty(a)w_n\| = 0$ for all $a \in A$. Then $v_n = u_n w_n \in L(E_1, E_2)$ is a sequence of isometries with orthogonal ranges such that the corresponding isometry $v \in C_b(\mathbb{N}, L(E_1, E_2))$ satisfies $v\pi_1(a) - \pi_2(a)v \in C_0(\mathbb{N}, K(E_1, E_2))$ for all $a \in A$. This follows from the identity

$$u_n w_n \pi_1(a) - \pi_2(a)u_n w_n = u_n(w_n\pi_1(a) - \pi_2^\infty(a)w_n) + (u_n\pi_2^\infty(a) - \pi_2(a)u_n)w_n.$$

Since $v_n^* v_m = 0$ for $n \ne m$, one observes that $\boldsymbol{v}(n + s) = (1 - s)^{1/2} v_n + s^{1/2} v_{n+1}$, $0 \le s \le 1$, extends v to a uniformly continuous isometry $\boldsymbol{v} \in C_u(T, L(E_1, E_2))$ that satisfies $\pi_1 \underset{\boldsymbol{v}}{\precsim} \pi_2$.

For the second part of the statement, we note that by the first part $\pi_1^\infty \underset{v}{\precsim} \pi_2$. Thus, $v\pi_1^\infty(a) - \pi_2(a)v \in C_0(T, K(E_1^\infty, E_2))$ for all $a \in A$ where $v = (v_t)_{t\in T}$ is a uniformly continuous isometry with $v_t \in L(E_1^\infty, E_2)$. It follows by Lemma 2.5 that

$$u_t = (1_{E_1} \oplus v_t)w^\infty v_t^* + (1_{E_2} - v_t v_t^*)$$

is a uniformly continuous unitary such that $\pi_1 \oplus \pi_2 \underset{u}{\approx} \pi_2$. By symmetry we have that $\pi_1 \oplus \pi_2 \underset{u}{\approx} \pi_1$ and hence $\pi_1 \underset{u}{\approx} \pi_2$. □

3. Dual algebras

Let A and B be separable C^*-algebras and let $\pi : A \to L(H_B)$ be a $*$-representation.

Definition 3.1. The *localization algebra* $\mathcal{C}_L(\pi)$ associated to π is the C^*-subalgebra of $C_u(T, L(H_B))$ consisting of all functions f such that $[f, \pi(a)] \in C_0(T, K(H_B))$ and $\pi(a)f \in C_u(T, K(H_B))$ for all $a \in A$.

While $\mathcal{C}_L(\pi)$ is the central object of the paper, we also need to consider a series of pairs of C^*-algebras and ideals which will play a supporting role:

$$\mathcal{D}(\pi) = \{b \in L(H_B) : [b, \pi(a)] \in K(H_B),\ \forall a \in A\},$$
$$\mathcal{C}(\pi) = \{b \in L(H_B) : \pi(a)b \in K(H_B),\ \forall a \in A\},$$

and their parametrized versions,

$$\mathcal{D}_T(\pi) = \{f \in C_u(T, L(H_B)) : [f, \pi(a)] \in C_u(T, K(H_B)),\ \forall a \in A\} \cong C_u(T, \mathcal{D}(\pi)),$$
$$\mathcal{C}_T(\pi) = \{f \in C_u(T, L(H_B)) : \pi(a)f \in C_u(T, K(H_B)),\ \forall a \in A\} \cong C_u(T, \mathcal{C}(\pi)).$$

The evaluation map at 0 leads to the pair

$$\mathcal{D}_T^0(\pi) = \{f \in \mathcal{D}_T(\pi) : f(0) = 0\},$$
$$\mathcal{C}_T^0(\pi) = \{f \in \mathcal{C}_T(\pi) : f(0) = 0\}.$$

Finally, we view the localization algebra $\mathcal{C}_L(\pi)$ as an ideal of

$$\mathcal{D}_L(\pi) = \{f \in C_u(T, L(H_B)) : [f, \pi(a)] \in C_0(T, K(H_B)),\ \forall a \in A\},$$
$$\mathcal{C}_L(\pi) = \{f \in \mathcal{D}_L(\pi) : \pi(a)f \in C_u(T, K(H_B)),\ \forall a \in A\}.$$

In order to simplify some of the statements, it is useful to introduce the following notation: $A_1(\pi) = \mathcal{D}_T(\pi)$, $A_2(\pi) = \mathcal{C}_T(\pi)$, $A_3(\pi) = \mathcal{D}_T^0(\pi)$, $A_4(\pi) = \mathcal{C}_T^0(\pi)$, $A_5(\pi) = \mathcal{D}_L(\pi)$ and $A_6(\pi) = \mathcal{C}_L(\pi)$. We are going to see that the isomorphism classes of these C^*-algebras are independent of π, provided that π is an absorbing representation. We follow the presentation from [Higson and Roe 2000, Section 5.2], where analogous properties of $\mathcal{D}(\pi)$ and $\mathcal{C}(\pi)$ are established, except that we need to employ a strengthened version of Voiculescu's theorem, contained in Theorem 2.6 above.

Let $\pi_1, \pi_2 : A \to L(H_B)$ be two representations.

Lemma 3.2. *If $\pi_1 \underset{v}{\precsim} \pi_2$, then the equation $\Phi_v(f) = vfv^*$ defines a $*$-homomorphism*

$$\Phi_v : \mathcal{D}_T(\pi_1) \to \mathcal{D}_T(\pi_2)$$

with the property that $\Phi_v(A_j(\pi_1)) \subset A_j(\pi_2)$ for all $1 \le j \le 6$.

Proof. This follows from the identities

$$\begin{aligned}[vfv^*, \pi_2(a)] &= v[f, \pi_1(a)]v^* + (v\pi_1(a) - \pi_2(a)v)fv^* \\ &\qquad\qquad - vf(v\pi_1(a^*) - \pi_2(a^*)v)^*, \\ \pi_2(a)vfv^* &= v\pi_1(a)fv^* - (v\pi_1(a) - \pi_2(a)v)fv^*. \end{aligned}$$ □

Corollary 3.3. *Let $\pi_1, \pi_2 : A \to L(H_B)$ be two absorbing representations. Then $A_j(\pi_1) \cong A_j(\pi_2)$ for all $1 \le j \le 6$.*

Proof. Theorem 2.6 yields a unitary $v \in C_u(T, L(H_B))$ such that $\pi_1 \underset{v}{\approx} \pi_2$. The corresponding maps $\Phi_v : A_j(\pi_1) \to A_j(\pi_2)$ are isomorphisms. □

Lemma 3.4. *Let $\pi_1, \pi_2 : A \to L(H_B)$ be two representations of A and suppose that v_1, v_2 are two isometries such that $\pi_1 \underset{v_i}{\precsim} \pi_2$, $i = 1, 2$. Then*

$$(\Phi_{v_1})_* = (\Phi_{v_2})_* : K_*(A_j(\pi_1)) \to K_*(A_j(\pi_2))$$

for all $1 \le j \le 6$.

Proof. The unitary

$$u = \begin{pmatrix} 1 - v_1 v_1^* & v_1 v_2^* \\ v_2 v_1^* & 1 - v_2 v_2^* \end{pmatrix} \in M_2(\mathcal{D}_L(\pi_2))$$

conjugates $\left(\begin{smallmatrix} \Phi_{v_1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ over $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \Phi_{v_1} \end{smallmatrix}\right)$. It follows that

$$(\Phi_{v_1})_* = (\Phi_{v_2})_* : K_*(\mathcal{D}_T(\pi_1)) \to K_*(\mathcal{D}_T(\pi_2)).$$

Similarly, one verifies that the equality $(\Phi_{v_1})_* = (\Phi_{v_2})_* : K_*(A_j(\pi_1)) \to K_*(A_j(\pi_2))$ holds for all $1 \le j \le 6$. □

Denote by π^∞ the direct sum $\pi^\infty = \bigoplus_{n=1}^\infty \pi : A \to L(H_B^\infty) = L\big(\bigoplus_{n=1}^\infty H_B\big)$.

Corollary 3.5. *If $\pi : A \to L(H_B)$ is an absorbing representation, then the inclusion $\mathcal{D}_T(\pi) \to \mathcal{D}_T(\pi^\infty)$, $f \mapsto (f, 0, 0, \dots)$ induces isomorphisms on K-theory: $K_*(A_j(\pi)) \to K_*(A_j(\pi^\infty))$, for all $1 \le j \le 6$.*

Proof. We have $\pi \underset{v}{\precsim} \pi^\infty$, where $v \in C_u(T, L(H_B, H_B^\infty))$ is the constant isometry defined by $v(t)(h) = (h, 0, 0, \dots)$ for any $t \in T$ and $h \in H_B$. The inclusion map from the statement coincides with Φ_v. On the other hand, $\pi \underset{u}{\approx} \pi^\infty$ since π is absorbing, and hence Φ_u is an isomorphism. We conclude the proof by noting that $(\Phi_v)_* = (\Phi_u)_*$ by Lemma 3.4. □

4. A duality isomorphism

Let A and B be separable C^*-algebras. We are going to show that when we fix an absorbing representation $\pi : A \to L(H_B)$ — the existence of such an absorbing representation is guaranteed by Theorem 2.4 — the K-theory of $\mathcal{C}_L(\pi)$ is canonically isomorphic to the KK-theory of the pair (A, B).

We start with a technical lemma that will be used several times later.

Lemma 4.1. *For any separable C^*-algebra $D \subset C_u(T, L(H_B))$, there is a positive contraction $x \in C_u(T, K(H_B))$ such that*

(a) $[x, d] \in C_0(T, K(H_B))$ *for all $d \in D$, and*

(b) $(1 - x)d \in C_0(T, K(H_B))$ *for all $d \in D \cap C_u(T, K(H_B))$.*

Proof. Our arguments will in fact show that the statement holds true in the more general situation where $L(H_B)$ is replaced by a C^*-algebra L and $K(H_B)$ is replaced by a two-sided closed ideal I of L. Let $\dot{D}$ denote the C^*-subalgebra of L generated by all images $d(t)$ as d ranges over D and t over T. This is separable, and contains $\dot{C} = \dot{D} \cap I$ as an ideal. Let $(x_n)_n$ be a positive contractive approximate unit for $\dot{C}$ which is quasicentral in $\dot{D}$. Choose countable dense subsets $(d_k)_{k=1}^{\infty}$ and $(c_k)_{k=1}^{\infty}$ of D and $D \cap C_u(T, I)$, respectively. As for each n, the subsets $\bigcup_{k=1}^{n} \{d_k(t) : t \in [0, n+1]\} \subseteq \dot{D}$ and $\bigcup_{k=1}^{n} \{c_k(t) : t \in [0, n+1]\} \subseteq \dot{C}$ are compact, so we may assume on passing to a subsequence of (x_n) that

(i) $\|[d_k(t), x_n]\| < 1/(n+1)$ for all $1 \le k \le n$ and all $t \in [0, n+1]$, and

(ii) $\|(1 - x_n)c_k(t)\| < 1/(n+1)$ for all $1 \le k \le n$ and all $t \in [0, n+1]$.

For $t \in [n, n+1)$, write $s = t - n$ and set $x(t) = (1-s)x_n + s x_{n+1}$; note that the function $x : t \mapsto x(t)$ is uniformly continuous. Then from (i) and (ii) above we have

(i) $\|[d_k(t), x(t)]\| < 1/(n+1)$ for all $1 \le k \le n$ and all $t \in [n, n+1)$, and

(ii) $\|(1 - x(t))c_k(t)\| < 1/(n+1)$ for all $1 \le k \le n$ and all $t \in [n, n+1)$.

This implies that x has the right properties. □

We have obvious inclusions $\mathcal{D}_L(\pi) \subset \mathcal{D}_T(\pi)$ and $\mathcal{C}_L(\pi) \subset \mathcal{C}_T(\pi)$, which induce a $*$-homomorphism

$$\eta : \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \to \mathcal{D}_T(\pi)/\mathcal{C}_T(\pi).$$

Proposition 4.2. *For any separable C^*-algebras A and B and any representation $\pi : A \to L(H_B)$, the map η is a $*$-isomorphism.*

Proof. It is clear from the definitions that $\mathcal{C}_L(\pi) = \mathcal{D}_L(\pi) \cap \mathcal{C}_T(\pi)$ and hence η is injective. It remains to prove that η is surjective. It suffices to show that for any $f \in \mathcal{D}_T(\pi)$ there is $\tilde{f} \in \mathcal{D}_L(\pi)$ such that $\tilde{f} - f \in \mathcal{C}_T(\pi)$. Let $f \in \mathcal{D}_T(\pi)$ be given.

Let D be the C^*-subalgebra of $C_u(T, L(H_B))$ generated by $\pi(A)$ (embedded as constant functions) and f, and let x be as in Lemma 4.1. With this choice of x (that depends on f) we define $\tilde{f} = (1-x)f$. Note that $\tilde{f} = f - xf \in \mathcal{D}_T(\pi)$ since $f, x \in \mathcal{D}_T(\pi)$, and $\tilde{f} - f = -xf \in C_u(T, K(H_B))$ since $x \in C_u(T, K(H_B))$. In particular, it follows that $\tilde{f} - f \in \mathcal{C}_T(\pi)$.

It remains to verify that $\tilde{f} \in \mathcal{D}_L(\pi)$. This follows as for any $a \in A$,

$$[\tilde{f}, \pi(a)] = [(1-x)f, \pi(a)] = [\pi(a), x]f + (1-x)[f, \pi(a)]. \qquad \square$$

An adaptation of the arguments from [Qiao and Roe 2010] gives the following:

Proposition 4.3. *Let A, B be separable C^*-algebras and let $\pi : A \to L(H_B)$ be an absorbing representation. Then*

(a) $K_*(\mathcal{D}_L(\pi)) = 0$ *and hence the boundary map*

$$\partial : K_*(\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)) \to K_{*+1}(\mathcal{C}_L(\pi))$$

is an isomorphism;

(b) *the evaluation map at $t = 0$ induces an isomorphism*

$$e_* : K_*(\mathcal{D}_T(\pi)/\mathcal{C}_T(\pi)) \to K_*(\mathcal{D}(\pi)/\mathcal{C}(\pi)).$$

Proof. Fix an ample representation π of A. One verifies that if $f \in \mathcal{D}_L(\pi)$, then the formula

$$F(t) := (f(t), f(t+1), \dots, f(t+n), \dots)$$

defines an element $F \in \mathcal{D}_L(\pi^\infty)$. Indeed,

$$[F(t), \pi(a)] = \big([f(t), \pi(a)], [f(t+1), \pi(a)], \dots, [f(t+n), \pi(a)], \dots\big)$$

and each entry belongs to $C_0(T, K(H_B))$ and is bounded by $\|[f, \pi(a)]\|$. This shows that $[F, \pi(a)] \in C_u(T, K(H_B^\infty))$. Since $[f, \pi(a)] \in C_0(T, K(H_B))$, it follows immediately that in fact $[F, \pi(a)] \in C_0(T, K(H_B^\infty))$.

With these remarks, the proof of (a) goes just like that of [Qiao and Roe 2010, Proposition 3.5]. Indeed, define $*$-homomorphisms $\alpha_i : \mathcal{D}_L(\pi) \to \mathcal{D}_L(\pi^\infty)$ for $i = 1, 2, 3, 4$ by

$$\begin{aligned}
\alpha_1(f) &= (f(t), 0, 0, \dots),\\
\alpha_2(f) &= (0, f(t+1), f(t+2), \dots),\\
\alpha_3(f) &= (0, f(t), f(t+1), \dots),\\
\alpha_4(f) &= (f(t), f(t+1), f(t+2), \dots).
\end{aligned}$$

It is clear that $\alpha_1 + \alpha_2 = \alpha_4$. The isometry $v \in L(H_B^\infty)$ defined by $v(h_0, h_1, h_2, \dots) = (0, h_0, h_1, h_2, \dots)$ commutes with $\pi^\infty(A)$ and hence $v \in \mathcal{D}_L(\pi^\infty)$. Moreover, $\alpha_4(a) = v\alpha_3(a)v^*$ and hence $(\alpha_4)_* = (\alpha_3)_*$ by [Higson and Roe 2000, Lemma 4.6.2].

Using uniform continuity, one shows that α_3 is homotopic to α_2 via the homotopy $f(t) \mapsto (0, f(t+s), f(t+s+1), \dots)$, $0 \le s \le 1$. We deduce that

$$(\alpha_1)_* + (\alpha_2)_* = (\alpha_1 + \alpha_2)_* = (\alpha_4)_* = (\alpha_3)_* = (\alpha_2)_*$$

and hence $(\alpha_1)_* = 0$. This concludes the proof of (a), since $(\alpha_1)_*$ is an isomorphism by Corollary 3.5.

To prove (b), one follows the proof of [Qiao and Roe 2010, Proposition 3.6] to show that both $K_*(\mathcal{D}_T^0(\pi)) = 0$ and $K_*(\mathcal{C}_T^0(\pi)) = 0$. The desired conclusion then follows in view of the split exact sequence

$$0 \to \mathcal{D}_T^0(\pi)/\mathcal{C}_T^0(\pi) \to \mathcal{D}_T(\pi)/\mathcal{C}_T(\pi) \to \mathcal{D}(\pi)/\mathcal{C}(\pi) \to 0.$$

Any $f \in \mathcal{D}_T^0(\pi)$ can be extended by 0 to an element of $C_u(\mathbb{R}, L(H_B))$. With this convention, define four maps $\beta_i : \mathcal{D}_T^0(\pi) \to \mathcal{D}_T^0(\pi^\infty)$, $i = 1, 2, 3, 4$, by

$$\begin{aligned}
\beta_1(f) &= (f(t), 0, 0, \dots), \\
\beta_2(f) &= (0, f(t-1), f(t-2), \dots), \\
\beta_3(f) &= (0, f(t), f(t-1), \dots), \\
\beta_4(f) &= (f(t), f(t-1), f(t-2), \dots).
\end{aligned}$$

This definition requires that one verifies that if $f \in \mathcal{D}_T^0(\pi)$, then

$$F'(t) := (f(t), f(t-1), \dots, f(t-n), \dots)$$

defines an element of $\mathcal{D}_T^0(\pi^\infty)$. This is clearly the case, since if f is uniformly continuous, then so is F' and moreover, just as argued in [Qiao and Roe 2010], for each t in a fixed bounded interval only finitely many components of $F'(t)$ are nonzero, and hence $[F'(t), \pi^\infty(a)] \in K(H_B^\infty)$ if $[f(t), \pi(a)] \in K(H_B)$ for all $t \in T$. Note that $(\beta_4)_* = (\beta_3)_*$ since $\beta_4(a) = v\beta_3(a)v^*$, where $v \in \mathcal{D}_T(\pi^\infty)$ is the same isometry as in part (a). Using uniform continuity, one observes that β_3 is homotopic to β_2 via the homotopy $f(t) \mapsto (0, f(t-s), f(t-s-1), \dots)$, $0 \le s \le 1$. We deduce that

$$(\beta_1)_* + (\beta_2)_* = (\beta_1 + \beta_2)_* = (\beta_4)_* = (\beta_3)_* = (\beta_2)_*$$

and hence $(\beta_1)_* = 0$. This shows that $K_*(\mathcal{D}_T^0(\pi)) = 0$, since $(\beta_1)_*$ is an isomorphism by Corollary 3.5. The proof for the vanishing of $K_*(\mathcal{C}_T^0(\pi))$ is entirely similar. Indeed, with the same notation as above, one observes that if $f \in \mathcal{C}_T^0(\pi)$ then $F' \in \mathcal{C}_T^0(\pi^\infty)$. Moreover, the four maps $\beta_i : \mathcal{D}_T^0(\pi) \to \mathcal{D}_T^0(\pi^\infty)$ restrict to maps $\beta_i' : \mathcal{C}_T^0(\pi) \to \mathcal{C}_T^0(\pi^\infty)$ with β_3' homotopic to β_2', and $(\beta_1')_*$ is an isomorphism by Corollary 3.5. $\square$

Theorem 4.4. *Let A, B be separable C^*-algebras and let $\pi : A \to L(H_B)$ be an absorbing representation. There are canonical isomorphisms of groups*

$$\alpha : KK_i(A, B) \xrightarrow{\cong} K_i(\mathcal{C}_L(\pi)), \quad i = 0, 1.$$

Proof. Consider the diagram

$$\begin{tikzcd} KK_i(A, B) \arrow[r, "P"] & K_{i+1}(\mathcal{D}(\pi)/\mathcal{C}(\pi)) \arrow[r, "\iota_*"] & K_{i+1}(\mathcal{D}_T(\pi)/\mathcal{C}_T(\pi)) \arrow[d, "\eta_*^{-1}"] \\ & K_i(\mathcal{C}_L(\pi)) & K_{i+1}(\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)) \arrow[l, "\partial"'] \end{tikzcd}$$

where P is the Paschke duality isomorphism — see [Paschke 1981; Skandalis 1988, Remarque 2.8; Thomsen 2001, Theorem 3.2] — and ι is the canonical inclusion. The maps ∂ and $\iota_* = e_*^{-1}$ are isomorphisms by Proposition 4.3 and η_* is an isomorphism by Proposition 4.2. □

As a corollary we obtain the following duality theorem, mentioned in the introduction. Recall from the introduction that $\mathcal{C}_L(A)$ stands for $\mathcal{C}_L(\pi)$, where π is ample (and thus absorbing, by Theorem 2.2), and A is identified with $\pi(A)$.

Theorem 4.5. *For any separable C^*-algebra A there are canonical isomorphisms of groups $K^i(A) \cong K_i(\mathcal{C}_L(A))$ for $i = 0, 1$.* □

5. An inverse map

Let $\alpha : KK_i(A, B) \xrightarrow{\cong} K_i(\mathcal{C}_L(\pi))$ be the isomorphism of Theorem 4.4. Recall that $K(H_B) \cong B \otimes K(H)$. Consider the $*$-homomorphism

$$\mathbf{\Phi} : \mathcal{D}_L(\pi) \otimes_{\max} A \to \frac{C_u(T, L(H_B))}{C_0(T, K(H_B))}$$

defined by $\mathbf{\Phi}(f \otimes a) = f\pi(a)$, and its restriction to $\mathcal{C}_L(\pi) \otimes_{\max} A$

$$\boldsymbol{\varphi} : \mathcal{C}_L(\pi) \otimes_{\max} A \to \frac{C_u(T, K(H_B))}{C_0(T, K(H_B))}.$$

We want $\boldsymbol{\varphi}$ to define a class in E-theory that we can take products with, but have to be a little careful due to the nonseparability of the C^*-algebra $\mathcal{C}_L(\pi) \otimes_{\max} A$. Just as in the case of the KK-groups [Skandalis 1988], if C is any C^*-algebra and B is a nonseparable C^*-algebra one defines $E_{\mathrm{sep}}(B, C) = \varprojlim_{B_1} E(B_1, C)$, with $B_1 \subset B$ and B_1 separable. Moreover, if D is separable, then $E(D, B) = \varinjlim_{B_1} E(D, B_1)$, with $B_1 \subset B$ and B_1 separable. With these adjustments, one has a well-defined product

$$E(D, B) \times E_{\mathrm{sep}}(B, C) \to E(D, C).$$

Moreover, it is clear that $[\![\boldsymbol{\varphi}]\!]$ defines an element of the group $E_{\mathrm{sep}}(\mathcal{C}_L(\pi) \otimes_{\max} A, B)$.

Recall the isomorphism $K_i(\mathcal{C}_L(\pi)) \cong E_i(\mathbb{C}, \mathcal{C}_L(\pi))$. We use the product

$$E_i(\mathbb{C}, \mathcal{C}_L(\pi)) \times E_{\text{sep}}(\mathcal{C}_L(\pi) \otimes_{\max} A, B) \to E_i(A, B)$$

to define a map $\beta : K_i(\mathcal{C}_L(\pi)) \to E_i(A, B)$ by $\beta(z) = [\![\boldsymbol{\varphi}]\!] \circ (z \otimes \mathrm{id}_A)$. The map β is an inverse of α in the following sense.

Theorem 5.1. *The composition $\beta \circ \alpha$ coincides with the natural map $KK_i(A, B) \to E_i(A, B)$ for $i = 0, 1$.*

Proof. We prove the odd case $i = 1$ and leave the even case for the reader. Recall that the E-theory group $E_1(A, B)$ of Connes and Higson [1990] is isomorphic to $[\![SA, K(H_B)]\!]$ by a desuspension result from [Dadarlat and Loring 1994].

For two continuous functions $f, g : T \to L(H_B)$ we write $f(s) \sim g(s)$ (or $f(t) \sim g(t)$) if $f - g \in C_0(T, K(H_B))$. Let $\{\varphi_s : \mathcal{C}_L(\pi) \otimes_{\max} A \to K(H_B))\}_{s \in T}$ be an asymptotic homomorphism representing $\boldsymbol{\varphi}$. More precisely, take φ to be a set-theoretic lifting of $\boldsymbol{\varphi}$. This means that $\varphi_s(f \otimes a) \sim f(s)\pi(a)$.

The composition $\beta \circ \alpha : KK_1(A, B) \to E_1(A, B)$ is computed as follows. Let $y \in KK_1(A, B)$ and let $z = Py \in K_0(\mathcal{D}(\pi)/\mathcal{C}(\pi))$ be its image under the Paschke duality isomorphism $P : KK_1(A, B) \to K_0(\mathcal{D}(\pi)/\mathcal{C}(\pi))$. Let z be represented by a self-adjoint element $e \in \mathcal{D}(\pi) \subset \mathcal{D}_T(\pi)$ whose image in $\mathcal{D}(\pi)/\mathcal{C}(\pi)$ is an idempotent $\dot{e}$. We identify $\mathcal{D}(\pi)$ with the C^*-subalgebra of constant functions in $\mathcal{D}_T(\pi)$. Choose an element $x \in C_u(T, K(H_B))$ as in Lemma 4.1 with respect to the (separable) C^*-subalgebra D of $C_u(T, L(H_B))$ generated by $\pi(A)$, e, and $K(H_B)$. Therefore, both $[x, \pi(a)]$ and $(1 - x)[e, \pi(a)]$ belong to $C_0(T, K(H_B))$ for all $a \in A$, and moreover $(1 - x)e \in \mathcal{D}_L(\pi)$ as

$$[(1 - x)e, \pi(a)] = [1 - x, \pi(a)]e + (1 - x)[e, \pi(a)] \in C_0(T, K(H_B))$$

for all $a \in A$. Let $e_L = (1 - x)e$ and let $\dot{e}_L$ be its image in $\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)$. Under the isomorphism $\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \cong \mathcal{D}_T(\pi)/\mathcal{C}_T(\pi)$ of Proposition 4.2 we see that $\dot{e}_L$ is just the image of $e \in \mathcal{D}_T(\pi)$ in the quotient, which is an idempotent since $\dot{e}$ is so. It is then clear that $\eta_*^{-1}\iota_*(z) = [\dot{e}_L]$.

We define a $*$-homomorphism $\ell : \mathbb{C} \to \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)$ by $\ell(1) = \dot{e}_L$ and set $S = C_0(0, 1)$. Then $(\beta \circ \alpha)(y)$ is represented by the composition of the asymptotic homomorphisms from the diagram

$$S \otimes \mathbb{C} \otimes A \xrightarrow{1 \otimes \ell \otimes 1} S \otimes \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \otimes A \xrightarrow{\delta_t \otimes 1} \mathcal{C}_L(\pi) \otimes A \xrightarrow{\varphi_s} K(H_B), \quad (5.2)$$

where here and throughout the rest of the proof the tensor products are maximal ones, and the map labeled δ_t is defined by taking the product with a canonical element δ of $E_{1,\text{sep}}(\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi), \mathcal{C}_L(\pi))$ associated to the extension

$$0 \to \mathcal{C}_L(\pi) \to \mathcal{D}_L(\pi) \to \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \to 0,$$

which we now discuss. Fixing a separable C^*-subalgebra $\dot{M}$ of $\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)$, the image of δ in $E_1(\dot{M}, \mathcal{C}_L(\pi))$ is defined as follows. Choose a separable C^*-subalgebra M of $\mathcal{D}_L(\pi)$ that surjects onto $\dot{M}$, and for each $\dot{m} \in \dot{M}$ choose a lift $m \in M$. Let $(v_t)_{t\in T}$ be a positive, contractive, and continuous approximate unit for $M \cap \mathcal{C}_L(\pi)$ which is quasicentral in M. Then for $g \in S = C_0(0, 1)$, δ is characterized by stipulating that $\delta_t(g \otimes \dot{m})$ satisfies

$$\delta_t(g \otimes \dot{m}) \sim g(v_t)m$$

(the choices of (v_t) and the various lifts do not matter up to homotopy). In our case, to compute the composition we need, let M be a separable C^*-subalgebra of $\mathcal{D}_L(\pi)$ containing e and x, and let (v_t) be an approximate unit for $M \cap C_L(\pi)$ that is quasicentral in M.

On the level of elements, we can now concretely describe the composition in (5.2) as follows. If $g \in S = C_0(0, 1)$ and $a \in A$, then under the asymptotic morphism $\{\mu_t : SA \to K(H_B)\}_t$ defined by diagram (5.2), elementary tensors $g\otimes a$ are mapped as follows:

$$g\otimes a \mapsto g\otimes \dot{e}_L \otimes a \xmapsto{\delta_t} g(v_t)(1-x)e\otimes a \xmapsto{\varphi_{s(t)}} g(v_t(s(t)))(1-x(s(t)))e\pi(a) \quad (5.3)$$

for any positive map $t \mapsto s(t)$ which increases to ∞ sufficiently fast. Since the map $t \mapsto x(t)$ is an approximate unit of $K(H_B)$, $(1 - x)y \in C_0(T, K(H_B))$ for all $y \in K(H_B)$. In particular it follows that $(1 - x(s(t)))e[e, \pi(a)] \sim 0$ since $[e, \pi(a)] \in K(H_B)$. Since $e\pi(a) = e\pi(a)e + e[e, \pi(a)]$, it follows from (5.3) that

$$\mu_t(g \otimes a) \sim g(v_t(s(t)))(1 - x(s(t)))e\pi(a)e. \quad (5.4)$$

On the other hand, the natural map $KK_1(A, B) \to E_1(A, B)$ maps y to $[\![\gamma_t]\!]$, where $\{\gamma_t : S \otimes A \to K(H_B)\}_t$ is described in [Connes and Higson 1990] as follows. Consider the extension

$$0 \to K(H_B) \to e\pi(A)e + K(H_B) \to A \to 0.$$

Let $(u_t)_{t\in T}$ be a contractive, positive, and continuous approximate unit of $K(H_B)$ which is quasicentral in $e\pi(A)e + K(H_B)$. Then

$$\gamma_t(g \otimes a) \sim g(u_t)e\pi(a)e.$$

Applying Lemma 4.1 (this time with D the C^*-subalgebra of $C_u(T, L(H_B))$ generated by e, $\pi(A)$, $K(H_B)$, and $t \mapsto x(s(t))$), we can choose $(u_t)_t$ such that $\lim_{t\to\infty}(1 - u_t)x(s(t)) = 0$. Since the C^*-algebra $C_0[0, 1)$ is generated by the function $f(\theta) = 1 - \theta$, it follows that $\lim_{t\to\infty} g(u_t)x(s(t)) = 0$ for all $g \in C_0[0, 1)$, and in particular for all $g \in C_0(0, 1)$.

Our goal now is to verify that $(\mu_t)_t$ is homotopic to $(\gamma_t)_t$. Due to the choice of $(u_t)_t$ and the comments above, we have that

$$\gamma_t(g \otimes a) \sim g(u_t)e\pi(a)e \sim g(u_t)(1 - x(s(t)))e\pi(a)e \tag{5.5}$$

for all $a \in A$ and $g \in C_0(0, 1)$. Finally, define $w_t^{(r)} = (1-r)v_t(s(t)) + ru_t$, $0 \le r \le 1$. As

$$\left[g(w_t^{(r)}),\, (1 - x(s(t)))e\pi(a)e\right] \to 0 \quad \text{as } t \to \infty$$

for all $r \in [0, 1]$ and $a \in A$, the condition

$$H_t^{(r)}(g \otimes a) \sim g(w_t^{(r)})(1 - x(s(t)))e\pi(a)e$$

defines an asymptotic morphism $H_t : SA \to C[0, 1] \otimes K(H_B)$. This gives the desired homotopy joining $(\mu_t)_t$ with $(\gamma_t)_t$. □

As suggested by the referee, we finish this section by sketching another proof which is maybe a little less self-contained, but more conceptual. The proof below is analogous to the approach used for [Qiao and Roe 2010, Proposition 4.3]. The basic idea in their approach is to apply naturality of the connecting map in E-theory for the diagram of strictly commutative asymptotic morphisms

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{C}_L(\pi) \otimes_{\max} A & \longrightarrow & \mathcal{D}_L(\pi) \otimes_{\max} A & \longrightarrow & (\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)) \otimes_{\max} A & \longrightarrow 0 \\
& \Big\downarrow{\scriptstyle \varphi_t} & & \Big\downarrow{\scriptstyle \phi_t} & & \Big\downarrow{\scriptstyle \bar{\phi}_t} & \\
0 \longrightarrow & K(H_B) & \longrightarrow & L(H_B) & \longrightarrow & L(H_B)/K(H_B) & \longrightarrow 0
\end{array}$$

where ϕ_t and φ_t represent the asymptotic morphisms induced by the $*$-homomorphisms $\mathbf{\Phi}$ and $\boldsymbol{\varphi}$ from the beginning of this section. The family $\bar{\phi}_t$ is the quotient family induced by ϕ_t, and consists of $*$-homomorphisms. Naturality of the boundary map in E-theory in this case amounts to the equality

$$[\![\varphi_t]\!] \circ [\![\delta_t \otimes \mathrm{id}_A]\!] = [\![\gamma_t]\!] \circ [\![\bar{\phi}_t]\!], \tag{5.6}$$

where δ_t is the boundary map for the top sequence of the diagram before tensoring with A, and γ_t is the boundary map for the bottom sequence. See [Connes and Higson 1990, Lemme 10] for the definition of the boundary maps associated to extensions (here and elsewhere one should use limits to deal with the nonseparable algebras involved in the way discussed earlier in this section). The naturality property of the boundary map with respect to general asymptotic morphisms that was discussed in [Guentner 1999, Theorem 5.3] seems to be the closest statement in the literature to the equality in (5.6), but it is nonetheless not sufficiently general to justify the equality. However, one can combine the arguments from the second part of the proof of Theorem 5.1 with those from [Guentner 1999] to verify naturality in full generality and in particular to justify (5.6).

Now (5.6) allows us to conceptualize the proof of Theorem 5.1. Let $y \in KK_i(A, B)$ and let $z = Py \in K_{i+1}(\mathcal{D}(\pi)/\mathcal{C}(\pi))$ be its image under the Paschke duality isomorphism $P : KK_i(A, B) \to K_{i+1}(\mathcal{D}(\pi)/\mathcal{C}(\pi))$. Consider

$$\eta_*^{-1}\iota_*(z) \in K_{i+1}(\mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)) \cong E_{i+1}(\mathbb{C}, \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi)),$$

where the maps ι_* and η_* are isomorphisms as in the proof of Theorem 4.4. We may view $\eta_*^{-1}\iota_*(z) \otimes [\![\mathrm{id}_A]\!]$ as an element of $E_{i+1}(A, \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \otimes_{\max} A)$. From (5.6) we obtain that

$$[\![\varphi_t]\!] \circ [\![\delta_t \otimes \mathrm{id}_A]\!] \circ (\eta_*^{-1}\iota_*(z) \otimes [\![\mathrm{id}_A]\!]) = [\![\gamma_t]\!] \circ [\![\bar{\phi}_t]\!] \circ (\eta_*^{-1}\iota_*(z) \otimes [\![\mathrm{id}_A]\!]). \quad (5.7)$$

The left-hand side of (5.7) represents the element $(\beta \circ \alpha)(y)$ of $E_i(A, B)$ by the very definition of α and β.

In order to identify the right-hand side of (5.7), it is useful to note that each individual map $\bar{\phi}_t$ is a $*$-homomorphism given by $\kappa \circ (\mathrm{ev}_t \otimes \mathrm{id}_A)$, where

$$\mathrm{ev}_t : \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \to \mathcal{D}(\pi)/\mathcal{C}(\pi)$$

is the evaluation map at t and

$$\kappa : (\mathcal{D}(\pi)/\mathcal{C}(\pi)) \otimes_{\max} A \to L(H_B)/K(H_B), \quad [b] \otimes a \mapsto [b \cdot \pi(a)]$$

is the "multiplication" $*$-homomorphism. Thus the asymptotic morphism $\{\bar{\phi}_t\}$ is homotopic to the constant asymptotic morphism given by $\bar{\phi}_0$, which is equal to $\kappa \circ (\mathrm{ev}_0 \otimes \mathrm{id}_A)$. Hence the right-hand side of (5.7) is equal to

$$[\![\gamma_t]\!] \circ [\![\kappa]\!] \circ ((\mathrm{ev}_0)_*\eta_*^{-1}\iota_*(z) \otimes [\![\mathrm{id}_A]\!]).$$

It follows from the following commutative diagram of $*$-homomorphisms

$$\begin{tikzcd}
\mathcal{D}(\pi)/\mathcal{C}(\pi) \arrow[r, "\mathrm{id}"] \arrow[d, "\iota"] & \mathcal{D}(\pi)/\mathcal{C}(\pi) \\
\mathcal{D}_T(\pi)/\mathcal{C}_T(\pi) \arrow[ur, "\mathrm{ev}_0"] & \mathcal{D}_L(\pi)/\mathcal{C}_L(\pi) \arrow[l, "\eta"] \arrow[u, "\mathrm{ev}_0"']
\end{tikzcd}$$

that $(\mathrm{ev}_0)_*\eta_*^{-1}\iota_*(z) = z$. This allows us to simplify the right-hand side of (5.7) further to

$$[\![\gamma_t]\!] \circ [\![\kappa]\!] \circ (z \otimes [\![\mathrm{id}_A]\!]),$$

where z is viewed as an element in $E_{i+1}(\mathbb{C}, \mathcal{D}(\pi)/\mathcal{C}(\pi))$. This can be seen to be equal to the image of y under the natural map $KK_i(A, B) \to E_i(A, B)$.

Indeed, focusing on the odd case, where we have $y \in KK_1(A, B)$ and $z = Py \in K_0(\mathcal{D}(\pi)/\mathcal{C}(\pi))$, we may choose $e \in \mathcal{D}(\pi)$, as in the first part of the proof of Theorem 5.1, such that $z = [\dot{e}] \in K_0(\mathcal{D}(\pi)/\mathcal{C}(\pi))$. Then the $*$-homomorphism $a \in A \mapsto [e \cdot \pi(-)] \in L(H_B)/K(H_B)$, which represents $[\![\kappa]\!] \circ (z \otimes [\![\mathrm{id}_A]\!])$, is the

Busby invariant of the extension corresponding to $e \in \mathcal{D}(\pi)$. Hence its composition with the asymptotic morphism $\{\gamma_t\} : L(H_B)/K(H_B) \to K(H_B)$ represents the image of y under the natural map $KK_1(A, B) \to E_1(A, B)$.

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