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Hecke modules for arithmetic groups via bivariant K-theory

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Let Γ be a lattice in a locally compact group G. In another work, we used KK-theory to equip with Hecke operators the K-groups of any Γ-C^*-algebra on which the commensurator of Γ acts. When Γ is arithmetic, this gives Hecke operators on the K-theory of certain C^*-algebras that are naturally associated with Γ. In this paper, we first study the topological K-theory of the arithmetic manifold associated to Γ. We prove that the Chern character commutes with Hecke operators. Afterwards, we show that the Shimura product of double cosets naturally corresponds to the Kasparov product and thus that the KK-groups associated to an arithmetic group Γ become true Hecke modules. We conclude by discussing Hecke equivariant maps in KK-theory in great generality and apply this to the Borel–Serre compactification as well as various noncommutative compactifications associated with Γ. Along the way we discuss the relation between the K-theory and the integral cohomology of low-dimensional manifolds as Hecke modules.

1. Introduction

Let Γ be a lattice in a locally compact group G with commensurator $C_G(\Gamma)$. Let $S \subset C_G(\Gamma)$ be a group containing Γ. In [Mesland and Şengün 2016], for $g \in S$ and B a S-C^*-algebra (that is, a C^*-algebra on which S acts via automorphisms), we constructed elements $[T_g] \in KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$. We introduced *analytic Hecke operators* on any module over $KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$ as the endomorphisms arising from the classes $[T_g]$. In the present paper we prove several structural results about

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these Hecke operators, showing that they generalize the well-known cohomological Hecke operators in a way that is compatible with the Chern character and the double-coset Hecke ring of Shimura.

The double-coset Hecke ring of Shimura is well-known to number theorists. In the widely studied case where Γ is an arithmetic group, the Hecke ring acts linearly on various spaces of automorphic forms associated to Γ, providing a rich supply of symmetries [Shimura 1971, Chapter 3]. Those automorphic forms that are simultaneous eigenvectors of these symmetries are conjectured, and proven in many cases, to have deep connections to arithmetic [Clozel 1990; Taylor 1995]. The Hecke ring also acts on the cohomology of the arithmetic manifold M associated to Γ and there is a Hecke equivariant isomorphism between spaces of automorphic forms associated to Γ and cohomology of M twisted with suitable local systems [Franke 1998; Shimura 1971]. The passage to cohomology leads to many fundamental results and new insights on the arithmetic of automorphic forms. The results of this paper, together with those of [Mesland and Şengün 2016], offer an analytic habitat for the Hecke ring by providing ring homomorphisms from the Hecke ring to suitable KK-groups. The passage to KK-theory extends the scope of the action of the Hecke ring beyond cohomology and allows for the possibility of using tools from operator K-theory in the study of automorphic forms.

Let us describe the results of the paper more precisely. In Section 2, we consider the situation where S acts on a locally compact Hausdorff space X. Assume that Γ acts freely and properly on X and put $M = \Gamma\backslash X$. It is well-known that the C^*-algebras $C_0(X) \rtimes_r \Gamma$ and $C_0(M)$ are Morita equivalent, so

$$KK_0(C_0(X) \rtimes_r \Gamma, C_0(X) \rtimes_r \Gamma) \simeq KK_0(C_0(M), C_0(M)),$$

and thus for any $g \in S$ we obtain a class $[T_g] \in KK_0(C_0(M), C_0(M))$. The element g gives rise to a cover M_g of M and a pair of covering maps, forming the *Hecke correspondence* $M \xleftarrow{s} M_g \xrightarrow{t} M$. In [Mesland and Şengün 2016] it was shown that the class $[T_g]$ corresponds to the class of this Hecke correspondence, that is,

$$[T_g] = [M \leftarrow M_g \rightarrow M] \in KK_0(C_0(M), C_0(M)).$$

This class induces a Hecke operator $T_g : K^*(M) \to K^*(M)$ on topological K-theory. In this paper we show that the Chern character

$$\mathrm{Ch} : K^0(M) \oplus K^1(M) \to H^{\mathrm{ev}}(M, \mathbb{Q}) \oplus H^{\mathrm{odd}}(M, \mathbb{Q})$$

is *Hecke equivariant*. Here we equip $H^*(M, \mathbb{Q})$ with Hecke operators in the usual way using the Hecke correspondence $M \xleftarrow{s} M_g \xrightarrow{t} M$; see, for example, [Lee 2009].

In Section 3, we specialize to noncompact arithmetic hyperbolic 3-manifolds M. Let $\overline{M}$ be the Borel–Serre compactification of M. Consider the diagram

$$\begin{array}{ccccc} K^0(M) & \times & K_0(M) & \longrightarrow & \mathbb{Z} \\ \big\downarrow & & \big\uparrow & & \\ H^2(\overline{M}, \partial\overline{M}, \mathbb{Z}) & \times & H_2(\overline{M}, \partial\overline{M}, \mathbb{Z}) & \longrightarrow & \mathbb{Z} \end{array} \tag{1.1}$$

Here horizontal arrows are given by the standard pairings with respect to which the Hecke operators are adjoint. The vertical arrows are Hecke equivariant isomorphisms; we establish the one on the left via the results of Section 2 and the one on the right was proven in [Mesland and Şengün 2016]. Using the relative index theorem, we show that the diagram commutes. Using very different techniques, we proved a similar result in [Mesland and Şengün 2016] where the K-groups of M were replaced with those of the reduced group C^*-algebra $C_r^*(\Gamma)$ of Γ.

In Section 4 we prove the main result of the paper. The double-coset Hecke ring $\mathbb{Z}[\Gamma, S]$ is the free abelian group on the double cosets $\Gamma g\Gamma$, with $g \in S$, equipped with the Shimura product [Shimura 1971]. We show that the map $\Gamma g^{-1}\Gamma \mapsto [T_g]$ extends to a ring homomorphism

$$\mathbb{Z}[\Gamma, S] \to KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$$

for any S-C^*-algebra B. As mentioned in the second paragraph of this introduction, this homomorphism provides the Hecke ring $\mathbb{Z}[\Gamma, S]$ with a new habitat. The universality property of KK-theory [Higson 1987] implies that for any additive functor F on separable C^*-algebras that is homotopy invariant, split-exact and stable, the abelian groups $F(B \rtimes_r \Gamma)$ are modules over $\mathbb{Z}[\Gamma, S]$. For example, let Γ be an arithmetic group in a semisimple real Lie group G. By taking F to be local cyclic cohomology and $B = C_0(X)$ where X is the symmetric space of G, we recover the action of the Hecke ring on the cohomology of the arithmetic manifold X/Γ. In [Mesland and Şengün 2016], we took F to be K-homology and worked with three different S-C^*-algebras B that were naturally associated to Γ.

In Section 5, we show that a Γ-exact and S-equivariant extension

$$0 \to B \to E \to A \to 0$$

of C^*-algebras induces Hecke equivariant long exact sequences relating the KK-groups of the crossed products $B \rtimes_r \Gamma$, $E \rtimes_r \Gamma$ and $A \rtimes_r \Gamma$. In particular, suppose that X is a free and proper Γ-space on which S acts by homeomorphisms, and $\overline{X}$ a partial S-compactification of X with boundary $\partial X := \overline{X} \setminus X$. Then the extension

$$0 \to C_0(X) \to C_0(\overline{X}) \to C_0(\partial X) \to 0$$

induces a Hecke equivariant exact sequence

$$\begin{array}{ccccc} K_1(C_0(X) \rtimes_r \Gamma) & \longrightarrow & K_1(C_0(\overline{X}) \rtimes_r \Gamma) & \longrightarrow & K_1(C_0(\partial X) \rtimes_r \Gamma) \\ \uparrow & & & & \downarrow \\ K_0(C_0(\partial X) \rtimes_r \Gamma) & \longleftarrow & K_0(C_0(\overline{X}) \rtimes_r \Gamma) & \longleftarrow & K_0(C_0(X) \rtimes_r \Gamma) \end{array}$$

of $\mathbb{Z}[\Gamma, S]$-modules. The results of Sections 4 and 5 hold for the full crossed product algebras as well.

Let $\boldsymbol{G}$ be a reductive algebraic group and $\Gamma \subset \boldsymbol{G}(\mathbb{Q})$ an arithmetic group. Then the Borel–Serre partial compactification $\overline{X}$ of the associated global symmetric space X is a proper $\boldsymbol{G}(\mathbb{Q})$-compactification. The associated Morita equivalences provide a Hecke equivariant isomorphism of above six-term exact sequence with the topological K-theory exact sequence of the Borel–Serre compactification of the arithmetic manifold X/Γ and its boundary.

The generality of our methods also allows the consideration of various noncommutative compactifications. One family of examples are the Hecke equivariant Gysin exact sequences studied in [Mesland and Şengün 2016] coming from the geodesic compactification of hyperbolic n-space. Other examples of interest come from the Floyd boundary of Γ, such as the boundary of tree associated to $\mathrm{SL}(2, \mathbb{Z})$ and the Bruhat–Tits building of a p-adic group and its boundary. In most of these cases not all of the crossed products are Morita equivalent to a commutative C^*-algebra.

Set-up and notation. The following set-up will hold for the whole paper. Let G be a locally compact group and $\Gamma \subset G$ a torsion-free discrete subgroup. Recall that two subgroups H, K of G are called *commensurable* if $H \cap K$ is of finite index in both H and K. The commensurator $C_G(\Gamma)$ of Γ (in G) is the group of elements $g \in G$ for which Γ and $g\Gamma g^{-1}$ are commensurable. Moreover, S denotes a subgroup of $C_G(\Gamma)$ containing Γ.

2. Hecke equivariance of the Chern character

In this section, we assume that S acts on a locally compact Hausdorff space X and that the action of Γ on X is free and proper. Let M denote the Hausdorff space X/Γ. Given an element $g \in S$, we put $M_g := X/\Gamma_g$ and $M^g := X/\Gamma^g$, where $\Gamma^g := \Gamma \cap g^{-1}\Gamma g$ and $\Gamma_g := \Gamma \cap g\Gamma g^{-1} = g\Gamma^g g^{-1}$. Note that $s : M_g \to M$ and $s' : M^g \to M$ are finite sheeted covers (of the same degree) and the map $c : M_g \to M^g$ defined by $x\Gamma_g \mapsto g^{-1}x\Gamma^g$ is a homeomorphism. We obtain a second finite covering $t := s' \circ c : M_g \to M$.

We shall equip the topological K-theory of M with Hecke operators via two different constructions, one analytical, arising from a KK-class and the other topological, arising from a correspondence. We will see that these two constructions give rise to the same Hecke operator. Afterwards, we will show that the Chern character between the K-theory and the ordinary cohomology of M is Hecke equivariant.

2.1. ***Analytic Hecke operators.*** Let $g \in S$. As mentioned in the introduction, thanks to a Morita equivalence, the analytically constructed class

$$[T_g] \in KK_0(C_0(X) \rtimes \Gamma, C_0(X) \rtimes \Gamma)$$

gives rise to a class $[T_g^M] \in KK_0(C_0(M), C_0(M))$. This latter class has a simpler description, which we now recall.

The conditional expectation

$$\rho : C_0(M_g) \to C_0(M), \quad \rho(\psi)(m) = \sum_{x \in t^{-1}(m)} \psi(x),$$

and right module structure

$$\psi \cdot f(x) := \psi(x) f(t(x))$$

give $C_0(M_g)$ a right $C_0(M)$-module, which we denote by T_g^M. Because the map $s : M_g \to M$ is proper, there is a left action of $C_0(M)$ on T_g^M by compact operators

$$C_0(M) \to \mathbb{K}(T_g^M), \quad f \cdot \psi(x) = f(s(x))\psi(x).$$

Then $[T_g^M] \in KK_0(C_0(M), C_0(M))$ is the class of this bimodule.

We observe that $M \xleftarrow{s} M_g \xrightarrow{t} M$ defines a *correspondence* in the sense of [Connes and Skandalis 1984]. Associated to this correspondence, there exists a class $[s_*] \otimes [t!] \in KK_0(C_0(M), C_0(M))$, where $t!$ is the wrong way cycle arising from t. As t is simply a finite covering of manifolds, it follows from [Connes and Skandalis 1984, Proposition 2.9] that $t!$ acquires a simpler description and it is then not hard to see that $[s_*] \otimes [t!]$ equals $[T_g^M]$ above.

2.2. Definition. Let $M = X/\Gamma$ as above. For any separable C^*-algebra C, the *analytic Hecke operators*

$$T_g : KK_*(C_0(M), C) \to KK_*(C_0(M), C),$$
$$T_g : KK_*(C, C_0(M)) \to KK_*(C, C_0(M)),$$

are defined to be the Kasparov product with the class $[T_g^M] \in KK_0(C_0(M), C_0(M))$.

An important case is when one takes $C \simeq \mathbb{C}$. Then we obtain analytic Hecke operators on the topological K-theory of M:

$$T_g : K^*(M) \to K^*(M).$$

2.3. ***Topological Hecke operators.*** We now proceed to give an "elementary" description of our Hecke operators in the special case of topological K-theory. To do this, we follow the description of Hecke operators on ordinary cohomology from correspondences; see, for example, [Mesland and Şengün 2016]. To this end, we introduce the "transfer map" machinery from stable homotopy theory, which allows us to deal with generalized cohomology theories at no extra cost.

To a finite covering map $p:(Y, B)\to(X, A)$ of pairs of spaces (that is, a finite covering $p: Y\to X$ with subspaces $A\subset X$ and $B\subset Y$ such that $B=p^{-1}(A)$), there is a well-known construction [Adams 1978, Construction 4.1.1, Theorem 4.2.3; Kahn and Priddy 1972] that associates to the map p a map of suspension spectra $p^! : \Sigma^\infty(X/A)\to\Sigma^\infty(Y/B)$. Via precomposition with $p^!$, for any generalized cohomology theory h^* with spectrum E, we obtain a homomorphism called the *transfer map*

$$p^! : h^n(Y, B)=[\Sigma^\infty S^n\wedge\Sigma^\infty(Y/B), E]\to h^n(X, A)=[\Sigma^\infty S^n\wedge\Sigma^\infty(X/A), E].$$

This transfer map agrees with the usual one in the case of ordinary cohomology; see [Kahn and Priddy 1972, Proposition 2.1]. In the case of topological K-theory, the transfer map is induced by the direct image map of Atiyah [1961]; see [Kahn and Priddy 1972, Proposition 2.4]. Recall that if $f: Y\to X$ is a finite covering map and $E\to X$ is a vector bundle, then the direct image bundle $f^!E\to Y$ has fiber $(f^!E)_y$ at $y\in Y$ given by the direct sum $\bigoplus_{f(x)=y}E_x$.

2.4. Definition. Given any generalized cohomology theory h^* with spectrum E and $g\in S$, the *topological Hecke operator* T_g on $h^n(M)$ is defined as the composition

$$h^n(M)\xrightarrow{s^*}h^n(M_g)\xrightarrow{t^!}h^n(M).$$

In the case of topological K-theory, these topological Hecke operators agree with the analytic ones that we defined earlier.

2.5. Proposition. *Let $g\in S$. The analytic Hecke operator T_g on $K^*(M)$ agrees with the topological Hecke operator T_g on $K^*(M)$.*

Proof. Let us prove the statement for K^0 first. It suffices to show that, after we identify $K^0(M)\simeq K_0(C_0(M))$, the direct image map of Atiyah is induced by tensor product (from the right) with the $C_0(M)$-module T_g^M defined above in Section 2.1. To that end, we need to show that for any vector bundle $E\to M_g$, there is a unitary isomorphism between the $C_0(M)$-modules of sections

$$\alpha:\Gamma(E)\otimes_{C_0(M_g)}C_0(M_g)_{C_0(M)}\xrightarrow{\sim}\Gamma(t^!E).$$

This is achieved by choosing an open cover U_i of M_g for which the covering map t is homeomorphic. Let χ_i^2 be a partition of unity subordinate to the U_i. Define

$$\alpha(\psi \otimes f)(m) := \left(\sum_i \chi_i(x)\psi(x) f(x) \right)_{x \in t^{-1}(m)} \in t^! E.$$

It is straightforward to check that this induces the desired unitary isomorphism. Note that the above is also observed in [Ramras et al. 2013, Lemma 3.12].

To prove the claim for K^1, we descend to K^0 and exploit, as we did above, the fact that transfer is implemented by the direct image map. Consider the diagram below:

$$\begin{tikzcd} K^1(M_g) \arrow[r, "\simeq"] \arrow[d, "t^!"] & K^0(M_g \times \mathbb{R}) \arrow[d, "(t\times \mathrm{Id})^!"] \\ K^1(M) \arrow[r, "\simeq"] & K^0(M \times \mathbb{R}) \end{tikzcd} \tag{2.6}$$

The vertical arrows $t^!$ and $(t \times \mathrm{Id})^!$ are the transfer maps arising from the finite coverings $t : M_g \to M$ and $t \times \mathrm{Id} : M_g \times \mathbb{R} \to M \times \mathbb{R}$. The horizontal isomorphisms follow from long exact sequences in topological K-theory associated to suitable pairs of spaces. As the transfer map is natural and commutes with connecting morphisms [Adams 1978, p. 123–124], it follows that the diagram is commutative.

Note that $K^0(M \times \mathbb{R}) \simeq K_0(C_0(M) \otimes C_0(\mathbb{R}))$. Under the isomorphism

$$KK_0(C_0(M), C_0(M)) \xrightarrow{\simeq} KK_0(C_0(M) \otimes C_0(\mathbb{R}), C_0(M) \otimes C_0(\mathbb{R})),$$

our distinguished class $[T_g^M]$ gets sent to $[T_g^M \otimes C_0(\mathbb{R})]$. Now the same argument as in the first paragraph of this proof shows that the direct image map of Atiyah, for the finite covering $M_g \times \mathbb{R} \xrightarrow{t\times \mathrm{Id}} M \times \mathbb{R}$, is induced by tensor product with the $C_0(M) \otimes C_0(\mathbb{R})$-module $T_g^M \otimes C_0(\mathbb{R})$. ☐

2.7. Given a pair of compact Hausdorff spaces (X, A), we have the Chern character (see [Karoubi 1978, V.3.26])

$$\mathrm{Ch} : K^i(X, A) \to \mathrm{PH}^i(X, A, \mathbb{Q}), \quad i = 0, 1,$$

where PH^0 and PH^1 are the periodic cohomology groups given by the direct sums of the even and the odd degree ordinary cohomology groups, respectively. The Chern character commutes with suspension and thus is a stable cohomology operation (of degree 0).

Now let M be a noncompact arithmetic manifold. For $g \in C_G(M)$, let $\overline{M}$, $\overline{M_g}$ denote the Borel–Serre compactifications of M, M_g, respectively; see [Borel and Serre 1973; Mesland and Şengün 2016, Section 2.1.2]. It is well-known that the finite covering maps $s, t : M_g \to M$ extend to finite coverings of pairs of spaces $\bar{s}, \bar{t} : (\overline{M_g}, \partial\overline{M_g}) \to (\overline{M}, \partial\overline{M})$. From these, we obtain Hecke operators T_g on the relative groups $K^*(\overline{M}, \partial\overline{M})$ and $H^*(\overline{M}, \partial\overline{M}, \mathbb{Z})$. Notice that

$$K^*(\overline{M}, \partial\overline{M}) \simeq \widetilde{K}^*(M^+) = K^*(M) \simeq K_*(C_0(M)),$$

where M^+ is the one-point compactification of M. Furthermore, we have that $H^*(\overline{M}, \partial\overline{M}, \mathbb{Z}) \simeq H_c^*(M, \mathbb{Z})$, where H_c^* denotes compactly supported cohomology.

It follows that for a given arithmetic manifold M, by choosing $(X, A) = (M, \varnothing)$ if M is compact and $(X, A) = (\overline{M}, \partial\overline{M})$ if M is noncompact, we have the Chern character

$$\mathrm{Ch} : K^i(M) \to \mathrm{PH}_c^i(M, \mathbb{Q}), \quad i = 0, 1,$$

and both sides are Hecke modules. A most natural question is whether the Chern character commutes with the Hecke actions.

2.8. Proposition. *Let M be an arithmetic manifold and $g \in C_G(M)$. The Chern character*

$$\mathrm{Ch} : K^i(M) \to \mathrm{PH}_c^i(M, \mathbb{Q}), \quad i = 0, 1$$

commutes with the action of the Hecke operator T_g on both sides.

Proof. Consider a cohomology operation $\Psi : E^*(\cdot) \to F^*(\cdot)$ of degree 0 between two cohomology theories with spectra E, F. If Ψ is stable, there is in fact a map of spectra $\Psi : E \to F$ and the cohomology operation is simply the composition

$$E^n(X, A) = [\Sigma^\infty S^n \wedge \Sigma^\infty(A/X), E] \to F^n(X, A) = [\Sigma^\infty S^n \wedge \Sigma^\infty(X/A), F],$$
$$f \mapsto \Psi \circ f.$$

It immediately follows that the transfer operator associated to a finite cover of pairs of spaces $p : (Y, B) \to (X, A)$ commutes with Ψ, that is, the following diagram commutes:

$$\begin{tikzcd} E^n(Y, B) \arrow[r, "\Psi"] \arrow[d, "p^!"] & F^n(Y, B) \arrow[d, "p^!"] \\ E^n(X, A) \arrow[r, "\Psi"] & F^n(X, A) \end{tikzcd}$$

Now let us go back to our setting. Let us first assume that M is compact. Note that $H_c^*(M, \mathbb{Z}) = H^*(M, \mathbb{Z})$ in this case. As it is a stable cohomology operation, the Chern character commutes with the natural map s^* and also with the transfer map $t^!$, giving rise to the commutative diagram

$$\begin{tikzcd} K^*(M) \arrow[r, "s^*"] \arrow[d, "\mathrm{Ch}"'] & K^*(M_g) \arrow[r, "t^!"] \arrow[d, "\mathrm{Ch}"'] & K^*(M) \arrow[d, "\mathrm{Ch}"'] \\ \mathrm{PH}^*(M, \mathbb{Q}) \arrow[r, "s^*"] & \mathrm{PH}^*(M_g, \mathbb{Q}) \arrow[r, "t^!"] & \mathrm{PH}^*(M, \mathbb{Q}) \end{tikzcd}$$

showing that the Chern character map commutes with Hecke operators.

For the case where M is noncompact, the proof follows in the same way considering the diagram

$$\begin{tikzcd}
K^*(\overline{M}, \partial\overline{M}) \arrow[r, "\bar{s}^*"] \arrow[d, "\mathrm{Ch}"'] & K^*(\overline{M_g}, \partial\overline{M_g}) \arrow[r, "\bar{t}^!"] \arrow[d, "\mathrm{Ch}"'] & K^*(\overline{M}, \partial\overline{M}) \arrow[d, "\mathrm{Ch}"'] \\
\mathrm{PH}^*(\overline{M}, \partial\overline{M}, \mathbb{Q}) \arrow[r, "\bar{s}^*"] & \mathrm{PH}^*(\overline{M_g}, \partial\overline{M_g}, \mathbb{Q}) \arrow[r, "\bar{t}^!"] & \mathrm{PH}^*(\overline{M}, \partial\overline{M}, \mathbb{Q})
\end{tikzcd}$$

where $\bar{s}, \bar{t} : (\overline{M_g}, \partial\overline{M_g}) \to (\overline{M}, \partial\overline{M})$ are the extensions of $s, t : M_g \to M$ mentioned earlier. □

2.9. Remark. The transfer map used above is an example of what is known as a *wrong way map*. Connes and Skandalis [1984, Remark 2.10(a)] remark that given a K-oriented map $f : X \to Y$ between smooth manifolds, the wrong way maps $f^! : K(X) \to K(Y)$, induced by the Kasparov product with the class of the wrong way cycle $[f!] \in KK_*(C_0(X), C_0(Y))$, and $f^! : H_c(X, \mathbb{Q}) \to H_c(Y, \mathbb{Q})$ commute under the Chern character modulo an error term $\mathrm{Td}(f)$ defined via the Todd genus of certain bundles that naturally arise. In our case, this error term vanishes and we get that the transfer map commutes with the Chern character as we proved above.

2.10. Remark. Using the universal property of KK-theory, the Chern character can be obtained as the unique natural transformation

$$\mathrm{Ch} : KK_*(A, B) \to HL_*(A, B),$$

where HL_* denotes bivariant local cyclic homology; see [Meyer 2007; Puschnigg 1996]. For a locally compact space X, the local cyclic homology of $C_0(X)$ recovers the compactly supported sheaf cohomology of X [Puschnigg 1996, Theorem 11.7]. Thus ordinary cohomology admits an action of analytic Hecke operators via its structure as a module over KK-theory. It follows from the results of this section that the topological Hecke operators on ordinary cohomology arise from the analytic Hecke module structure.

3. Bianchi manifolds

In this section, we present a result about arithmetic noncompact hyperbolic 3-manifolds that complements the results obtained in our previous paper [Mesland and Şengün 2016, Section 5]. In that paper, for a Bianchi manifold M, we provided a Hecke equivariant isomorphism between $K_0(M)$ and $H_2(\overline{M}, \partial\overline{M}, \mathbb{Z})$, where $\overline{M}$ is the Borel–Serre compactification of M; see [Borel and Serre 1973]. We show below that $H^2(\overline{M}, \partial\overline{M}, \mathbb{Z})$ and $K^0(M)$ are isomorphic as Hecke modules and further argue that the cohomological pairing between H^2 and H_2 and the index pairing between K^0 and K_0 commute under these isomorphisms.

Let $\mathscr{O}$ be the ring of integers of an imaginary quadratic field and Γ be a torsion-free finite index subgroup of the Bianchi group $\mathrm{PSL}_2(\mathscr{O})$. Then Γ acts freely and properly on the hyperbolic 3-space $\boldsymbol{H}_3$. The associated hyperbolic 3-manifold $M = \boldsymbol{H}_3/\Gamma$ is known as a *Bianchi manifold*. It is well-known that any noncompact arithmetic hyperbolic 3-manifold is commensurable with a Bianchi manifold.

3.1. For compact connected spaces X, denote by $\widetilde{K}^0(X)$ the *reduced K-theory* of X, that is, the kernel of the map $K^0(X) \to \mathbb{Z}$ induced by $[E] \mapsto \dim_{\mathbb{C}}(E)$. Write $[n] \in K^0(X)$ for the class of the trivial bundle T^n of rank n over X. For a vector bundle E, the top exterior power $\bigwedge^{\dim E} E$ is called the *determinant line bundle* and denoted $\det E$. Let $\mathrm{Pic}(X)$ denote the *Picard group* of X, that is, the set of isomorphism classes of line bundles on X together with the tensor product operation.

Let M^+ denote the one-point compactification of the Bianchi manifold M. Since M^+ is a CW-complex of dimension 3, every complex vector bundle $E \to M^+$ splits as $E \simeq \det E \oplus T^{\dim_{\mathbb{C}}(E)-1}$; see [Weibel 2013, Corollary 4.4.1]. It follows from [Weibel 2013, Corollary 2.6.2] that the map

$$\dim \oplus \det : K^0(M^+) \to \mathbb{Z} \oplus \mathrm{Pic}(M^+), \qquad E \mapsto (\dim_{\mathbb{C}}(E), [\det E])$$

is an isomorphism. Noting $H^0(M^+, \mathbb{Z}) \simeq \mathbb{Z}$ and identifying $\mathrm{Pic}(M^+) \simeq H^2(M^+, \mathbb{Z})$ via the first Chern class c_1, we obtain the isomorphism

$$K^0(M^+) \to H^0(M^+, \mathbb{Z}) \oplus H^2(M^+, \mathbb{Z})$$

induced by $[E] \mapsto \dim_{\mathbb{C}}(E) + c_1(\det E)$. Note that this map agrees with the Chern character since $E \simeq T^{\dim_{\mathbb{C}}(E)-1} \oplus \det E$ as mentioned above. By Proposition 2.8, this isomorphism is Hecke equivariant.

Composing the Chern character with the projection map, we obtain a surjection $K^0(M^+) \to H^2(M^+, \mathbb{Z})$ whose kernel is $\widetilde{K}^0(M^+) = K^0(M)$. Noting that $H^2(M^+, \mathbb{Z})$ is isomorphic to the compactly supported cohomology $H^2_c(M, \mathbb{Z})$, which in turn is isomorphic to $H^2(\overline{M}, \partial\overline{M}, \mathbb{Z})$, we obtain an isomorphism

$$K^0(M) \xrightarrow{\sim} H^2(\overline{M}, \partial\overline{M}, \mathbb{Z}) \tag{3.2}$$

that is Hecke equivariant.

3.3. Given a line bundle $L \to \overline{M}$ and any connection ∇ on L, let

$$F_\nabla = \mathrm{Tr}\Big(\frac{-1}{2\pi i}\nabla^2\Big)$$

be the curvature 2-form of ∇. Then it is well-known that F_∇ is closed and its image in $H^2(M, \mathbb{R})$ is in fact integral and equals the first Chern class $c_1(L)$ of L.

3.4. Proposition. *Let $(N, \partial N) \subset (\overline{M}, \partial\overline{M})$ be an embedded surface, $L \to \overline{M}$ a line bundle that is trivial on $\partial\overline{M}$ and $\overline{N}$ the closed subspace of N obtained by removing an open neighborhood of ∂N over which L is trivial. View the interior $\mathring{N}$ of N as a* *spin^c surface with associated Dirac operator* $\not{D}_{\mathring{N}}$ (see [Mesland and Şengün 2016, Section 5]). *We have*

$$\langle [\not{D}_{\mathring{N}}], [L]-[1]\rangle = \int_{\overline{N}} F_\nabla$$

for any connection ∇ *on* L. *Here* $\langle\,\cdot\,,\,\cdot\,\rangle$ *is the index pairing.*

Proof. It follows from the relative index theorem of [Roe 1991, Theorem 4.6] that

$$\langle [\not{D}_{\mathring{N}}], [L]-[1]\rangle = \int_{\overline{N}} \widehat{A}(\mathring{N})\,\mathrm{Ch}(L|_{\mathring{N}}) - \int_{\overline{N}} \widehat{A}(\mathring{N}).$$

Here $L|_{\mathring{N}}$ is the restriction of L to the interior of N. Observe that

$$\mathrm{Ch}(L|_{\mathring{N}}) = 1 + c_1(L|_{\mathring{N}}) = 1 + [F_\nabla|_{\mathring{N}}],$$

where ∇ is any chosen connection on L and $F_\nabla|_{\mathring{N}}$ is the restriction of its curvature to $\mathring{N}$. The $\widehat{A}$-genus $\widehat{A}(\mathring{N})$ of $\mathring{N}$ equals 1 as it only has nonzero components in forms of degree 0 mod 4. The claim follows. □

The following is not necessary for the main result of this section, however we note it as it quickly follows from the above and [Ballmann and Brüning 2001, Lemma 2.22].

3.5. Corollary. *If* N *has finite volume*, *we have*

$$\langle [\not{D}_{\mathring{N}}], [L]-[1]\rangle = \int_{\mathring{N}} F_\nabla,$$

for any connection ∇ *on* L.

3.6. Proposition. *We have the equality*

$$\langle [\not{D}_{\mathring{N}}], [L]-[1]\rangle = \langle [(N, \partial N)], c_1(L)\rangle.$$

In particular, *the isomorphisms*

$$K^0(M) \xrightarrow{\simeq} H^2(\overline{M}, \partial\overline{M}, \mathbb{Z}), \qquad K_0(M) \xleftarrow{\simeq} H_2(\overline{M}, \partial\overline{M}, \mathbb{Z})$$

(see (3.2) and [Mesland and Şengün 2016, Proposition 5.6.]) *are compatible with the index pairing*

$$\langle\,\cdot\,,\,\cdot\,\rangle : K_0(M) \times K^0(M) \to \mathbb{Z}$$

and the integration pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle : H_2(\overline{M}, \partial\overline{M}, \mathbb{Z}) \times H^2(\overline{M}, \partial\overline{M}, \mathbb{Z}) \to \mathbb{Z}.$$

In other words, *diagram* (1.1) *of the introduction is commutative.*

Proof. It follows from our discussion in Section 3.1 that every element of $K^0(M)$ is of the form $[L]-[1]$, where 1 is the trivial line bundle and $L \to M$ is a line bundle that is trivial at infinity. Under the isomorphism (3.2), the image of $[L]-[1]$ is $c_1(L)$. Every class in $H_2(\overline{M}, \partial\overline{M}, \mathbb{Z})$ is represented by a properly embedded surface $(N, \partial N) \subset (\overline{M}, \partial\overline{M})$; see [Mesland and Şengün 2016, Section 5]. Then the pairing $\langle[(N, \partial N)], c_1(L)\rangle$ is given by the integral $\int_N F_\nabla$, where ∇ is any connection on L and F_∇ is the associated curvature 2-form as above. As L is trivial at infinity, we can choose closed $\overline{N} \subset \mathring{N}$ so that L is trivial outside $\overline{N}$ and it then follows that $\int_N F_\nabla = \int_{\overline{N}} F_\nabla$. Observe that the image of $[(N, \partial N)]$ in $K_0(M)$ under the isomorphism given in [Mesland and Şengün 2016, Proposition 5.6.] is $[\not{D}_{\mathring{N}}]$. Now by Proposition 3.4, we have the claim. □

4. The double-coset Hecke ring and *KK*-theory

We recall the construction of the Hecke operators via *KK*-theory as put forward in [Mesland and Şengün 2016]. We then show that the multiplication of double-cosets corresponds to the Kasparov product of the associated *KK*-classes.

4.1. ***Bimodules over the reduced crossed product.*** For a Γ-C^*-algebra B, the *reduced crossed product* $B \rtimes_r \Gamma$ is obtained as a completion of the convolution algebra $C_c(\Gamma, B)$; see, for example, [Kasparov 1995]. Let $g \in C_G(\Gamma)$ and $d := [\Gamma : \Gamma^g]$. The double coset $\Gamma g^{-1}\Gamma$ admits a decomposition as a disjoint union

$$\Gamma g^{-1}\Gamma = \bigsqcup_{i=1}^{d} g_i\Gamma, \quad g_i = \delta_i g^{-1}, \ \Gamma = \bigsqcup_{i=1}^{d} \delta_i \Gamma^g, \tag{4.2}$$

where the $\delta_i \in \Gamma$ form a complete set of coset representatives for Γ^g. We choose to work with g^{-1} in order for our formulae to be in line with those in [Mesland and Şengün 2016]. Consider the elements

$$t_i(\gamma) = t_i^g(\gamma) := g_{\gamma(i)}^{-1}\gamma g_i \in g\Gamma g^{-1},$$

where $i \mapsto \gamma(i)$ is induced by the permutation of the cosets in (4.2). From [Mesland and Şengün 2016, Lemma 2.3] we recall the relations

$$t_i(\gamma_1\gamma_2) = t_{\gamma_2(i)}(\gamma_1)t_i(\gamma_2), \qquad t_i(\gamma^{-1}) = t_{\gamma^{-1}(i)}(\gamma)^{-1},$$

which will be used in the sequel without further ado.

Let $S \subset C_G(\Gamma)$ be a subgroup containing Γ and B an S-C^*-algebra. The free right $B \rtimes_r \Gamma$-module $T_g^\Gamma \simeq (B \rtimes_r \Gamma)^d$ carries a left $B \rtimes_r \Gamma$-module structure given by

$$(t_g(f)\Psi)_i(\delta) = \sum_\gamma g_i^{-1} f(\gamma) t_i(\gamma^{-1})^{-1} \Psi_{\gamma^{-1}(i)}(t_i(\gamma^{-1})\delta). \tag{4.3}$$

Equivalently, we have the covariant representation

$$\begin{aligned} (t_g(b)\cdot\Psi)_i(\delta) &:= g_i^{-1}(b)\Psi_i(\delta), \\ (t_g(u_\gamma)\Psi)_i(\delta) &:= t_i(\gamma^{-1})^{-1}(\Psi_{\gamma^{-1}(i)}(t_i(\gamma^{-1})\delta)). \end{aligned} \tag{4.4}$$

Details of the construction, as well as the following definition, can be found in [Mesland and Şengün 2016, Section 2].

4.5. Definition. Let B be a separable S-C^*-algebra and C a separable C^*-algebra. The *Hecke operators*

$$T_g : KK_*(B \rtimes_r \Gamma, C) \to KK_*(B \rtimes_r \Gamma, C),$$
$$T_g : KK_*(C, B \rtimes_r \Gamma) \to KK_*(C, B \rtimes_r \Gamma)$$

are defined to be the Kasparov product with the class $[T_g^\Gamma] \in KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$.

We now give an equivalent description of the bimodules T_g^Γ. Consider the function space

$$C_c(\Gamma g^{-1}\Gamma, B) = \mathbb{C}[\Gamma g^{-1}\Gamma] \otimes_{\mathbb{C}}^{\mathrm{alg}} B.$$

The convolution product makes $C_c(\Gamma g^{-1}\Gamma, B)$ into a $C_c(\Gamma, B)$-bimodule:

$$f*\Psi(\xi) := \sum_{\gamma\in\Gamma} f(\gamma)\gamma(\Psi(\gamma^{-1}\xi)), \quad \Psi*f(\xi) := \sum_{\gamma\in\Gamma} \Psi(\xi\gamma)\xi f(\gamma^{-1}), \quad \xi \in \Gamma g^{-1}\Gamma.$$

Moreover, we define the inner product

$$\langle \Phi, \Psi\rangle(\delta) := \sum_{\xi\in\Gamma g^{-1}\Gamma} \xi^{-1}(\Phi(\xi)^*\Psi(\xi\delta)), \tag{4.6}$$

which makes $C_c(\Gamma g^{-1}\Gamma, B)$ into a pre-Hilbert-C^*-bimodule over $C_c(\Gamma, B)$.

4.7. Lemma. *For $g \in S \subset C_G(\Gamma)$ the map*

$$\alpha : C_c(\Gamma g^{-1}\Gamma, B) \to C_c(\Gamma, B)^d \subset T_g^\Gamma, \quad \alpha(\Psi)_i(\delta) := g_i^{-1}\Psi(g_i\delta),$$

induces a unitary isomorphism of $B \rtimes_r \Gamma$-bimodules.

Proof. The decomposition (4.2) shows that the map α has dense range. Moreover, α preserves the inner product

$$\begin{aligned}\langle \alpha(\Psi), \alpha(\Phi)\rangle(\delta) &= \sum_i \alpha(\Psi)_i^*\alpha(\Phi)_i(\delta) = \sum_i\sum_\gamma \alpha(\Psi)_i^*(\gamma)\gamma\alpha(\Phi)_i(\gamma^{-1}\delta)\\ &= \sum_i\sum_\gamma \gamma(\alpha(\Psi)_i(\gamma^{-1})^*\alpha(\Phi)_i(\gamma^{-1}\delta))\\ &= \sum_i\sum_\gamma \gamma g_i^{-1}(\Psi(g_i\gamma^{-1})^*\Phi(g_i\gamma^{-1}\delta))\\ &= \sum_{\xi\in\Gamma g^{-1}\Gamma} \xi^{-1}(\Phi(\xi)^*\Psi(\xi\delta)) = \langle\Psi, \Phi\rangle(\delta),\end{aligned}$$

from which it follows that α induces a unitary isomorphism on the C^*-module completions, which is in particular a right module map.

For the left module structure we compute

$$\begin{aligned}\alpha(f*\Psi)_i(\delta) &= g_i^{-1}\Big(\sum_{\gamma\in\Gamma} f(\gamma)\gamma\Psi(\gamma^{-1}g_i\delta)\Big)\\ &= \sum_{\gamma\in\Gamma} g_i^{-1}f(\gamma)g_i^{-1}\gamma\Psi(g_{\gamma^{-1}(i)}t_i(\gamma^{-1})\delta)\\ &= \sum_{\gamma\in\Gamma} g_i^{-1}f(\gamma)t_i(\gamma^{-1})^{-1}g_{\gamma^{-1}(i)}^{-1}\Psi(g_{\gamma^{-1}(i)}t_i(\gamma^{-1})\delta)\\ &= \sum_{\gamma\in\Gamma} g_i^{-1}f(\gamma)t_i(\gamma^{-1})^{-1}\alpha(\Psi)_{\gamma^{-1}(i)}(t_i(\gamma^{-1})\delta)\\ &= (t_g(f))(\alpha\Psi)_i(\delta), \end{aligned} \tag{4.8}$$

and we are done. □

Thus, the bimodules implementing the Hecke operators are completions of the B-valued functions on the associated double coset.

4.9. ***The double-coset Hecke ring.*** Let S be a subgroup of $C_G(\Gamma)$ that contains Γ. Following Shimura, we define the *Hecke ring* $\mathbb{Z}[\Gamma, S]$ as the free abelian group on the double cosets $\Gamma g\Gamma$ with $g \in S$, equipped with the product

$$[\Gamma g^{-1}\Gamma]\cdot[\Gamma h^{-1}\Gamma] := \sum_{k=1}^{K} m_k[\Gamma g_{i(k)}h_{j(k)}\Gamma], \tag{4.10}$$

where we have fixed finite sets I and J and coset representatives $\{g_i : i \in I\}$ and $\{h_j : j \in J\}$ for Γ^g and Γ^h in Γ, respectively. Moreover, m_k, $i(k)$ and $j(k)$ are such that $m_k := \#\{(i, j) : g_ih_j\Gamma = g_{i(k)}h_{j(k)}\Gamma\}$, and

$$\Gamma g^{-1}\Gamma h^{-1}\Gamma = \bigsqcup_{k=1}^{K} \Gamma g_{i(k)}h_{j(k)}\Gamma \tag{4.11}$$

is a disjoint union. For well-definedness and other details of the construction we refer to [Shimura 1971, Chapter 3]. We wish to show that, for an arbitrary S-C^*-algebra B, the map

$$T : \mathbb{Z}[\Gamma, S] \to KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma), \quad [\Gamma g^{-1}\Gamma] \mapsto T_g^{\Gamma} \tag{4.12}$$

is a ring homomorphism. To this end, we introduce the following notions. By a *bi-Γ-set* we mean a set V that carries both a left and a right Γ-action, and the actions commute in the sense that for all $\gamma, \delta \in \Gamma$ and $v \in V$ we have $\gamma(v\delta) = (\gamma v)\delta$.

The Γ-*product* of a pair (V, W) of bi-Γ-sets is the quotient of the Cartesian product $V \times W$ by the equivalence relation

$$(v, w) \sim (v', w') \Leftrightarrow \exists \gamma \in \Gamma \quad v' = v\gamma,\ w' = \gamma^{-1}w,$$

and is denoted by $V \times_\Gamma W$. The equivalence class of the pair (v, w) is denoted $[v, w]$. The Γ-product is a bi-Γ-set via the induced left and right Γ-actions

$$[v, w]\gamma := [v, w\gamma], \quad \gamma[v, w] := [\gamma v, w].$$

Let $\Gamma \subset S \subset C_G(\Gamma)$ be a subgroup and V a bi-Γ-set. We say that V is *anchored in S* if there is given a map $m : V \to S$ such that $m(\gamma v\delta) = \gamma m(v)\delta$ for all $v \in V$ and $\gamma, \delta \in \Gamma$. We refer to m as the *anchor*. Of course any double coset $\Gamma g\Gamma$ with $g \in S$ is anchored in S via the inclusion map.

4.13. Lemma. *Let V and W be bi-Γ-sets with anchor maps $m_V : V \to S$ and $m_W : W \to S$. Then their Γ-product $V \times_\Gamma W$ is anchored in S via the* product anchor $[v, w] \mapsto m_V(v)m_W(w)$.

The proof of this is straightforward. Note that if V and W are double Γ-cosets in S, anchored via their embeddings into S, then the product anchor of $V \times_\Gamma W$ need not be injective.

We wish to relate the anchored bi-Γ-sets $\Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma$ and $\bigsqcup_{k=1}^{K} \bigsqcup_{\ell=1}^{m_k} \Gamma z_k \Gamma$. By virtue of (4.11) we fix, once and for all, for each z_k and $1 \leqslant \ell \leqslant m_k$ a choice of *distinct* indices $i(k, \ell)$, $j(k, \ell)$ such that $z_k\Gamma = g_{i(k,\ell)}h_{j(k,\ell)}\Gamma$. We thus write $z_{(k,\ell)} = g_{i(k,\ell)}h_{j(k,\ell)}$. Consider the left action of Γ on the finite set $I \times J$ given by

$$\gamma(i, j) := (\gamma(i), t_i^g(\gamma)(j)). \tag{4.14}$$

4.15. Lemma. *With the above choices, the map*

$$\omega : \bigsqcup_{k=1}^{K} \bigsqcup_{\ell=1}^{m_k} \Gamma z_{(k,\ell)}\Gamma \to \Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma, \quad \gamma z_{(k,\ell)}\delta \mapsto [\gamma g_{i(k,\ell)}, h_{j(k,\ell)}\delta],$$

where $i = i(k, \ell)$ and $j = j(k, \ell)$, is a Γ-bi-equivariant bijection of S-anchored bi-Γ-sets.

Proof. By construction, ω is Γ-bi-equivariant and respects the anchors. We need only show that it is bijective. This is achieved as follows: For each k choose

$$\gamma_1^k = 1, \gamma_2^k, \ldots, \gamma_{d_k}^k \in \Gamma, \quad \text{with } \Gamma z_k \Gamma = \bigsqcup_{n=1}^{d_k} \gamma_n^k z_k \Gamma.$$

We thus have

$$\bigsqcup_{k=1}^{K} \bigsqcup_{\ell=1}^{m_k} \Gamma z_{(k,\ell)}\Gamma = \bigsqcup_{k=1}^{K} \bigsqcup_{\ell=1}^{m_k} \bigsqcup_{n=1}^{d_k} \gamma_n^k g_{i(k,\ell)}h_{j(k,\ell)}\Gamma. \tag{4.16}$$

The identities

$$[g_i\gamma, h_j\delta] = [g_i, \gamma h_j\delta] = [g_i, h_{\gamma(j)}t_j^h(\gamma)\delta]$$

show that every element in the Γ-product $\Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma$ has a representative of the form $[g_i, h_j\gamma]$ and such representatives are unique because g_i and h_j form a complete set of coset representatives. We so obtain a set bijection

$$\Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma \to \bigsqcup_{(i,j)\in I\times J} \{g_i\} \times h_j\Gamma, \quad [g_i\gamma, h_j\delta] \mapsto [g_i, h_{\gamma(j)} t_h^j(\gamma)\delta].$$

It follows that ω restricts to bijections

$$\omega : \gamma_n^k g_{i(k,\ell)} h_{j(k,\ell)}\Gamma \to \{g_{\gamma_n^k(i(k,\ell))}\} \times h_{t_i^g(\gamma_n^k)(j(k,\ell))}\Gamma.$$

Therefore it suffices to show that the map

$$N \times K \times L \to I \times J, \quad (n, k, \ell) \mapsto \gamma_n^k(i(k, \ell), j(k, \ell))$$

is bijective. By [Shimura 1971, Proposition 3.2] it holds that

$$\sum_{k=1}^{K} m_k d_k = |I||J| = |I \times J|,$$

and thus we need only show that this map is injective, and then use a counting argument to obtain surjectivity. To this end we prove that the equality

$$\gamma_n^k(i(k, \ell), j(k, \ell)) = \gamma_{n'}^{k'}(i(k', \ell'), j(k', \ell')) \tag{4.17}$$

implies that $(n, k, \ell) = (n', k', \ell')$. By (4.14), (4.17) implies that

$$\gamma_n^k g_{i(k,\ell)} h_{j(k,\ell)}\Gamma = \gamma_{n'}^{k'} g_{i(k',\ell')} h_{j(k',\ell')}\Gamma,$$

and thus

$$\Gamma g_{i(k,\ell)} h_{j(k,\ell)}\Gamma = \Gamma g_{i(k',\ell')} h_{j(k',\ell')}\Gamma.$$

This in turn implies that $k = k'$ and thus $\gamma_n^k z_k\Gamma = \gamma_{n'}^k z_k\Gamma$, so it follows that $n = n'$. Lastly, we are left with $\gamma_n^k(i(k, \ell)) = \gamma_n^k(i(k, \ell'))$, so $i(k, \ell) = i(k, \ell')$, which by construction implies $\ell = \ell'$. This shows that the map $(n, k, \ell) \mapsto \gamma_n^k(i(k, \ell), j(k, \ell))$ is injective. □

Now let V be a Γ-set with anchor $m : V \to S$ and X a S-(A, B)-bimodule. We always consider V as a discrete set. We equip $C_c(V, X)$ with a $C_c(\Gamma, B)$-valued inner product via

$$\langle \Phi, \Psi\rangle(\delta) := \sum_{v\in V} m(v)^{-1}\langle \Phi(v), \Psi(v\delta)\rangle$$

and left and right module structures via the Γ-action

$$f * \Psi(v) := \sum_{\gamma} f(\gamma)\gamma\Psi(\gamma^{-1}v), \qquad \Psi * f(v) := \sum_{\gamma} \Psi(v\gamma)m(v\gamma) f(\gamma^{-1}).$$

Thus the completion gives a C^*-$(A \rtimes_r \Gamma, B \rtimes_r \Gamma)$-bimodule. Note that if $u : X \to Y$ is an S-equivariant unitary bimodule isomorphism and $\omega : W \to V$ an isomorphism of S-anchored bi-Γ-sets, then

$$C_c(V, X) \to C_c(W, Y), \quad \Psi \mapsto u \circ \Psi \circ \omega$$

is a unitary bimodule isomorphism.

By Lemma 4.7, the bimodule T_g^Γ for $g \in S$ is isomorphic to the completion of $C_c(\Gamma g^{-1}\Gamma, B)$ with anchor $m : \Gamma g^{-1}\Gamma \to S$ the set inclusion, and is thus a special case of the above construction. The formalism of anchored bi-Γ-sets allows for an elegant description of tensor products of their associated modules.

4.18. Proposition. *Let $S \subset C_G(\Gamma)$ be a subgroup and A, B and C be S-C^*-algebras. Let V, W be S-anchored bi-Γ-sets, X an (A, B)-S-bimodule and Y a (B, C)-S-bimodule. Then the map*

$$\alpha : C_c(V, X) \otimes^{\mathrm{alg}}_{C_c(\Gamma, B)} C_c(W, Y) \to C_c(V \times_\Gamma W, X \otimes_B Y),$$

given by

$$\alpha(\Phi \otimes \Psi)[v, w] := \sum_\gamma \Phi(v\gamma) \otimes m(v)\gamma\Psi(\gamma^{-1}w),$$

is an inner product preserving map of $(C_c(\Gamma, A), C_c(\Gamma, C))$-bimodules with dense range. Consequently their respective C^*-module completions are unitarily isomorphic $(A \times_r \Gamma, C \rtimes_r \Gamma)$-bimodules.

Proof. The following calculation shows that α is unitary:

$$\begin{aligned}
&\langle \alpha(\Phi \otimes \Psi), \alpha(\Phi \otimes \Psi)\rangle(\delta) \\
&\quad = \sum_{[v,w]} m(w)^{-1} m(v)^{-1} \langle \alpha(\Phi \otimes \Psi)(v, w), \alpha(\Phi \otimes \Psi)(v, w\delta)\rangle \\
&\quad = \sum_{[v,w]} \sum_{\gamma,\varepsilon} m(w)^{-1} m(v)^{-1} \big\langle m(v)\gamma\Psi(\gamma^{-1}w), \langle \Phi(v\gamma), \Phi(v\varepsilon)\rangle m(v)\varepsilon\Psi(\varepsilon^{-1}w\delta)\big\rangle \\
&\quad = \sum_{[v,w]} \sum_{\gamma,\varepsilon} m(w)^{-1} \big\langle \gamma\Psi(\gamma^{-1}w), m(v)^{-1}\big(\langle \Phi(v\gamma), \Phi(v\varepsilon)\rangle\big)\varepsilon\Psi(\varepsilon^{-1}w\delta)\big\rangle \\
&\quad = \sum_{[v,w]} \sum_{\gamma,\varepsilon} m(\gamma^{-1}w)^{-1} \big\langle \Psi(\gamma^{-1}w), m(v\gamma)^{-1}\big(\langle \Phi(v\gamma), \Phi(v\varepsilon)\rangle\big)\gamma^{-1}\varepsilon\Psi(\varepsilon^{-1}w\delta)\big\rangle \\
&\quad = \sum_{[v,w]} \sum_{\gamma,\varepsilon} m(\gamma^{-1}w)^{-1} \big\langle \Psi(\gamma^{-1}w), m(v\gamma)^{-1}\big(\langle \Phi(v\gamma), \Phi(v\gamma\varepsilon)\rangle\big)\varepsilon\Psi(\varepsilon^{-1}\gamma^{-1}w\delta)\big\rangle.
\end{aligned}$$

By virtue of the equivalence relation on $V \times W$ we can replace the sum over equivalence classes $[v, w] \in V \times_\Gamma W$ and elements $\gamma \in \Gamma$ by a sum over $(v, w) \in V \times W$, and continue the calculation:

$$
\begin{aligned}
&= \sum_{v\in V}\sum_{w\in W}\sum_{\varepsilon} m(w)^{-1}\big\langle \Psi(w), m(v)^{-1}\big(\langle \Phi(v), \Phi(v\varepsilon)\rangle\big)\varepsilon\Psi(\varepsilon^{-1}w\delta)\big\rangle \\
&= \sum_{w}\sum_{\varepsilon} m(w)^{-1}\big\langle \Psi(w), \langle \Phi, \Phi\rangle(\varepsilon)\varepsilon\Psi(\varepsilon^{-1}w\delta)\big\rangle \\
&= \sum_{w} m(w)^{-1}\big\langle \Psi(w), \langle \Phi, \Phi\rangle * \Psi(w\delta)\big\rangle \\
&= \big\langle \Psi, \langle \Phi, \Phi\rangle\Psi\big\rangle(\delta).
\end{aligned}
$$

It is straightforward to establish that α is a bimodule map:

$$
\begin{aligned}
\alpha(f * \Phi \otimes \Psi)[v, w] &= \sum_{\gamma}(f * \Phi)(v\gamma) \otimes m(v)\gamma\Psi(\gamma^{-1}w) \\
&= \sum_{\gamma,\varepsilon} f(\varepsilon)\varepsilon\Phi(\varepsilon^{-1}v\gamma) \otimes m(v)\gamma\Psi(\gamma^{-1}w) \\
&= \sum_{\varepsilon} f(\varepsilon)\varepsilon\alpha(\Phi \otimes \Psi)[\varepsilon^{-1}v, w] = f * \alpha(\Phi \otimes \Psi)[v, w],
\end{aligned}
$$

$$
\begin{aligned}
\alpha(\Phi \otimes \Psi * f)[v, w] &= \sum_{\gamma} \Phi(v\gamma) \otimes m(v)\gamma(\Psi * f)(\gamma^{-1}w) \\
&= \sum_{\gamma,\varepsilon} \Phi(v\gamma) \otimes m(v)\gamma(\Psi(\gamma^{-1}w\varepsilon)m(\gamma^{-1}w\varepsilon)f(\varepsilon^{-1})) \\
&= \sum_{\gamma,\varepsilon} \Phi(v\gamma) \otimes m(v)\gamma\Psi(\gamma^{-1}w\varepsilon)m(w\varepsilon)f(\varepsilon^{-1}) \\
&= \sum_{\varepsilon} \alpha(\Phi \otimes \Psi)[v, w\varepsilon]m(w\varepsilon)f(\varepsilon^{-1}) \\
&= \alpha(\Phi \otimes \Psi) * f[v, w].
\end{aligned}
$$

Lastly, to see that α has dense range, denote by $\delta_v : V \to \mathbb{C}$ the indicator function at the element $v \in V$. The functions

$$
\chi^{[v,w]}_{x\otimes y}(v', w') := \delta_v(v')\delta_w(w')x \otimes y,
$$

with $v \in V$, $w \in W$, $x \in X$ and $y \in Y$, span a dense right $C_c(\Gamma, C)$-submodule. Now set

$$
\begin{aligned}
e^v_x(v') &:= \delta_v(v')x, \\
f^{(v,w)}_y(w') &:= \delta_w(w')m(v)^{-1}(y).
\end{aligned}
$$

Then it is easily verified that $\alpha(e^v_i \otimes f^{(v,w)}_y) = \chi^{[v,w]}_{x\otimes y}$, so α has dense range. This proves the proposition. $\square$

4.19. Theorem. *For any $g, h \in C_G(\Gamma)$ there is a unitary isomorphism of bimodules*

$$T_g^\Gamma \otimes_{B \rtimes_r \Gamma} T_h^\Gamma \xrightarrow{\sim} \bigoplus_{k=1}^{K} \big(T^\Gamma_{(g_{i(k)} h_{j(k)})^{-1}}\big)^{\oplus m_k}.$$

Consequently, for any S-C^*-algebra B, the map $T : [\Gamma g^{-1}\Gamma] \mapsto [T_g^\Gamma]$ extends to a ring homomorphism

$$T : \mathbb{Z}[\Gamma, S] \to KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma).$$

Proof. By Lemma 4.7, the modules T_g^Γ and T_h^Γ are unitarily isomorphic to those associated to the anchored bi-Γ-sets $\Gamma g^{-1}\Gamma$ and $\Gamma h^{-1}\Gamma$. By Proposition 4.18, their tensor product is given by

$$C_c(\Gamma g^{-1}\Gamma, B) \otimes^{\mathrm{alg}}_{C_c(\Gamma, B)} C_c(\Gamma h^{-1}\Gamma, B) \xrightarrow{\sim} C_c(\Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma, B \otimes_B B).$$

Since $B \otimes_B B \simeq B$ as S-modules and by Lemma 4.15, there is an isomorphism of anchored bi-Γ-sets

$$\Gamma g^{-1}\Gamma \times_\Gamma \Gamma h^{-1}\Gamma \simeq \bigsqcup_{k=1}^{K} \bigsqcup_{\ell=1}^{m_k} \Gamma z_{(k,\ell)} \Gamma.$$

Taking completions, we obtain the unitary bimodule isomorphism

$$T_g^\Gamma \otimes_{B \rtimes_r \Gamma} T_h^\Gamma \xrightarrow{\sim} \bigoplus_{k=1}^{K} \bigoplus_{\ell=1}^{m_k} \big(T^\Gamma_{z^{-1}_{(k,\ell)}}\big).$$

The definition of addition in KK-theory then yields

$$\begin{aligned} T[\Gamma g^{-1}\Gamma] \otimes T[\Gamma h^{-1}\Gamma] &\\ &= [T_g^\Gamma] \otimes [T_h^\Gamma] = \sum_{k=1}^{K} \sum_{\ell=1}^{m_k} [T^\Gamma_{z^{-1}_{(k,\ell)}}] = \sum_{k=1}^{K} \sum_{\ell=1}^{m_k} T[\Gamma z_{(k,\ell)} \Gamma] \\ &= \sum_{k=1}^{K} m_k T[\Gamma z_k \Gamma] = T\bigg(\sum_{k=1}^{K} m_k [\Gamma z_k \Gamma]\bigg) = T([\Gamma g^{-1}\Gamma] \cdot [\Gamma h^{-1}\Gamma]), \end{aligned}$$

showing that $[\Gamma g^{-1}\Gamma] \mapsto [T_g^\Gamma]$ is a ring homomorphism. □

We define $\mathscr{H}_B(\Gamma, S)$ to be the subring of $KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$ generated by T_g^Γ for $g \in C_G(\Gamma)$. The following corollary is now obvious.

4.20. Corollary. *If $\mathbb{Z}[\Gamma, S]$ is commutative, then $\mathscr{H}_B(\Gamma, S)$ is commutative.*

Similarly write $\mathscr{H}_M(S)$ for the subring of $KK_0(C_0(M), C_0(M))$ generated by the classes of the correspondences $M \xleftarrow{s} M_g \xrightarrow{t} M$ with $g \in S$.

4.21. Corollary. *Let X be an S-space on which Γ acts freely and properly with quotient $M := X/\Gamma$. The map $[\Gamma g^{-1}\Gamma] \mapsto [M \xleftarrow{t} M_g \xrightarrow{s} M]$ defines a ring homomorphism*

$$\mathbb{Z}[\Gamma, S] \to KK_0(C_0(M), C_0(M)).$$

In particular, the double-coset product $[\Gamma g^{-1}\Gamma] \cdot [\Gamma h^{-1}\Gamma]$ corresponds to the class of the composition of correspondences $[M \xleftarrow{s_g} M_g \, {}_{t_g}\!\times_{s_h} M_h \xrightarrow{t_h} M]$ and there is an isomorphism $\mathscr{H}_M(S) \simeq \mathscr{H}_{C_0(X)}(\Gamma, S)$.

Proof. By [Mesland and Şengün 2016, Proposition 3.8] the Morita equivalence isomorphism

$$KK_0(C_0(X) \rtimes \Gamma, C_0(X) \rtimes \Gamma) \to KK_0(C_0(M), C_0(M))$$

maps T_g^Γ to $T_g^M = [M \xleftarrow{s_g} M_g \xrightarrow{t_g} M]$. Thus the above map is the composition

$$\mathbb{Z}[\Gamma, S] \to KK_0(C_0(X) \rtimes \Gamma, C_0(X) \rtimes \Gamma) \to KK_0(C_0(M), C_0(M)),$$

whence a homomorphism. The last statement follows from [Connes and Skandalis 1984, Theorem 3.2]. Clearly $\mathscr{H}_M(S) \simeq \mathscr{H}_{C_0(X)}(\Gamma, S)$ under this isomorphism. □

4.22. Remark. Corollary 4.21 is the KK-theoretic analogue of the well-known fact that the double-coset Hecke ring can be interpreted in terms of (topological) correspondences, where the double-coset multiplication simply becomes composition of correspondences [Shimura 1971, Chapter 7].

5. Hecke equivariant exact sequences

As before, let S be a group such that $\Gamma \subset S \subset C_G(\Gamma)$. In this section we prove the following general result. For S-algebras A and B, and any element $[x]$ of $KK_i^S(A, B)$ we have that

$$[T_g^{A \rtimes_r \Gamma}] \otimes j_\Gamma([x]) = j_\Gamma([x]) \otimes [T_g^{B \rtimes_r \Gamma}] \in KK_i(A \rtimes_r \Gamma, B \rtimes_r \Gamma).$$

Here j_Γ denotes the Kasparov descent map [1988; 1995]

$$j_\Gamma : KK_*^S(A, B) \to KK_*^\Gamma(A, B) \to KK_*(A \rtimes_r \Gamma, B \rtimes_r \Gamma),$$

and we have written $T_g^{A \rtimes_r \Gamma}$ for T_g^Γ to emphasize the change of coefficient algebra. This result implies that for any S-equivariant semisplit extension

$$0 \to I \to A \to B \to 0$$

of C^*-algebras that is Γ-exact in the sense that

$$0 \to I \rtimes_r \Gamma \to A \rtimes_r \Gamma \to B \rtimes_r \Gamma \to 0$$

is exact, the long exact sequences in both variables of the KK-bifunctor are Hecke equivariant. In particular, we obtain Hecke equivariant exact sequences in K-theory and K-homology for various compactifications associated with locally symmetric spaces.

5.1. ***The descent theorem.*** Kasparov's descent construction associates to a Γ-equivariant C^*-B-module X a C^*-module $X \rtimes_r \Gamma$ over $B \rtimes_r \Gamma$ [Kasparov 1980; 1988; 1995]. To an S-equivariant C^*-module X and a double coset $\Gamma g^{-1}\Gamma$, with $g \in S$, we associate the $(C_c(\Gamma, A), C_c(\Gamma, B))$-bimodule

$$C_c(\Gamma g^{-1}\Gamma, X) = \mathbb{C}[\Gamma g^{-1}\Gamma] \otimes_{\mathbb{C}}^{\mathrm{alg}} X;$$

see Section 4.1. We denote the C^*-module completion so obtained by $T_g^{X \rtimes_r \Gamma}$. The following lemma is an application of Proposition 4.18.

5.2. Lemma. *Let A and B be S-C^*-algebras. Suppose that X is an S-equivariant right C^*-module over B and $\pi : A \to \mathrm{End}_B^*(X)$ an S-equivariant essential $*$-homomorphism. For every $g \in S$, there are inner product preserving bimodule homomorphisms*

$$\begin{aligned} &C_c(\Gamma g^{-1}\Gamma, A) \otimes_{C_c(\Gamma, A)}^{\mathrm{alg}} C_c(\Gamma, X) \\ &\qquad \xrightarrow{\sim} C_c(\Gamma g^{-1}\Gamma, X) \xleftarrow{\sim} C_c(\Gamma, X) \otimes_{C_c(\Gamma, B)}^{\mathrm{alg}} C_c(\Gamma g^{-1}\Gamma, B) \end{aligned} \tag{5.3}$$

of $(C_c(\Gamma, A), C_c(\Gamma, B))$-bimodules with dense range. Consequently the respective C^*-module completions are unitarily isomorphic $(A \rtimes_r \Gamma, B \rtimes_r \Gamma)$-bimodules.

From the identifications

$$\Gamma g^{-1}\Gamma \times_\Gamma \Gamma \simeq \Gamma g^{-1}\Gamma \simeq \Gamma \times_\Gamma \Gamma g^{-1}\Gamma$$

given by the multiplication maps and the S-equivariant isomorphisms

$$X \simeq A \otimes_A X \simeq X \otimes_B B,$$

coming from the bimodule structure we obtain the explicit from of the isomorphisms in (5.3):

$$\begin{aligned} \alpha : C_c(\Gamma g^{-1}\Gamma, A) \otimes_{C_c(\Gamma, A)}^{\mathrm{alg}} C_c(\Gamma, X) &\to C_c(\Gamma g^{-1}\Gamma, X), \\ \alpha(\Psi \otimes \Phi)(\xi) &:= \sum_{\gamma \in \Gamma} \Psi(\xi\gamma) \cdot \xi\gamma\Phi(\gamma^{-1}), \\ \beta : C_c(\Gamma, X) \otimes_{C_c(\Gamma, B)}^{\mathrm{alg}} C_c(\Gamma g^{-1}\Gamma, B) &\to C_c(\Gamma g^{-1}\Gamma, X), \\ \beta(\Phi \otimes \Psi)(\xi) &:= \sum_{\gamma \in \Gamma} \Phi(\gamma) \cdot \gamma\Psi(\gamma^{-1}\xi). \end{aligned}$$

As before, the elements g_i are such that $\Gamma g^{-1}\Gamma = \bigsqcup_{i=1}^d g_i\Gamma$. We construct from them the following operators.

5.4. Lemma. *The operator*

$$v_i : C_c(\Gamma, X) \to C_c(g_i\Gamma, X) \subset C_c(\Gamma g^{-1}\Gamma, X), \quad (v_i\Phi)(g_i\xi) := g_i\Phi(\xi),$$

extends to an adjointable isometry $X \rtimes_r \Gamma \to T_g^{X\rtimes\Gamma}$ *with adjoint given by*

$$(v_i)^*\Psi(\xi) := g_i^{-1}\Psi(g_i\xi).$$

Proof. The formula for the adjoint is easily verified. It follows that $(v_i)^*v_i = 1$ on $C_c(\Gamma, X)$, so v_i is isometric. The composition $v_iv_i^* = p_i$, the projection onto $C_c(g_i\Gamma, X)$, which is bounded as well. □

5.5. Theorem. *Let* (X, D) *be an* S*-equivariant left-essential unbounded Kasparov module of parity* j *and let* $g \in S$*. Then we have an equality*

$$j_\Gamma([(X, D)]) \otimes [T_g] = [T_g] \otimes j_\Gamma([(X, D)]) \in KK_j(A \rtimes_r \Gamma, B \rtimes_r \Gamma).$$

Proof. By Lemma 5.2 we have bimodule isomorphisms

$$(X \rtimes_r \Gamma) \otimes_{B\rtimes_r\Gamma} T_g^{B\rtimes_r\Gamma} \xrightarrow{\beta} T_g^{X\rtimes_r\Gamma} \xleftarrow{\alpha} T_g^{A\rtimes_r\Gamma} \otimes_{A\rtimes_r\Gamma} (X \rtimes_r \Gamma).$$

Define an operator $\widehat{D}$ on the dense submodule

$$C_c(\Gamma g^{-1}\Gamma, \operatorname{Dom} D) \subset T_g^{X\rtimes_r\Gamma}$$

via

$$(\widehat{D}\Upsilon)(\xi) := D(\Upsilon(\xi)).$$

Then $\widehat{D}\beta = \beta(D \otimes 1)$ and hence $\widehat{D}$ is essentially self-adjoint and regular, and has locally compact resolvent. We wish to show that $\widehat{D}$ represents the Kasparov product of $T_g^{A\rtimes_r\Gamma}$ and (X, D), under the isomorphism α. To this end we need to verify conditions 1–3 of [Kucerovsky 1997, Theorem 13]. Because the module $T_g^{A\rtimes_r\Gamma}$ carries the zero operator, only the connection condition 1 needs argument.

Let $\mathscr{A}$ denote the dense subalgebra of A such that $[D, a]$ is bounded for $a \in \mathscr{A}$. Then, for $\Psi \in C_c(\Gamma g^{-1}\Gamma, \mathscr{A})$, $\xi \in \Gamma$ and a fixed element g_i we have

$$\begin{aligned}
&\widehat{D}\alpha(\Psi \otimes \Phi)(g_i\xi) - \alpha(\Psi \otimes D\Phi)(g_i\xi) \\
&\qquad = \sum_{\gamma\in\Gamma} D\Psi(g_i\gamma) \cdot g_i\gamma\Phi(\gamma^{-1}\xi) - \Psi(g_i\gamma)g_i\gamma D\Phi(\gamma^{-1}\xi) \\
&\qquad = \sum_{\gamma\in\Gamma} \big([D, \Psi(g_i\gamma)] - \Psi(g_i\gamma)(D - g_i\gamma D\gamma^{-1}g_i^{-1})\big) g_i\gamma\Phi(\gamma^{-1}\xi) \\
&\qquad = g_i\Bigg(\sum_{\gamma\in\Gamma} g_i^{-1}\big([D, \Psi(g_i\gamma)] - \Psi(g_i\gamma)(D - g_i\gamma D\gamma^{-1}g_i^{-1})\big)\gamma\Phi(\gamma^{-1}\xi)\Bigg) \\
&\qquad = v_i(C_\Psi^i * \Phi)(g_i\xi).
\end{aligned}$$

Here C^i_Ψ denotes the map

$$C^i_\Psi : \Gamma \to \mathrm{End}^*_B(X), \quad \gamma \mapsto g_i^{-1}\big([D, \Psi(g_i\gamma)] - \Psi(g_i\gamma)(D - g_i\gamma D\gamma^{-1}g_i^{-1})\big),$$

which is of finite support since Ψ is. Such maps define adjointable operators on $C_c(\Gamma, X)$ via the convolution action. Writing $|\Psi\rangle : \Phi \to \Psi \otimes \Phi$ we have

$$\widehat{D}\alpha|\Psi\rangle - \alpha|\Psi\rangle D = \sum_{i=1}^{d} v^i \circ C^i_\Psi : X \rtimes_r \Gamma \to T_g^{X \rtimes_r \Gamma},$$

which defines a bounded adjointable operator. Thus $\widehat{D}$ satisfies Kucerovsky's connection condition as desired. □

5.6. Corollary. *For any $\alpha \in KK^S_j(A, B)$ and any separable C^*-algebra C, the induced maps*

$$\alpha_* : KK_i(C, A \rtimes_r \Gamma) \to KK_{i+j}(C, B \rtimes_r \Gamma),$$
$$\alpha^* : KK_i(B \rtimes_r \Gamma, C) \to KK_{i+j}(A \rtimes_r \Gamma, C)$$

are Hecke equivariant. In fact we can replace $KK(C, \cdot\,)$ and $KK(\,\cdot\,, C)$ by any co- or contravariant functor which is homotopy invariant, split exact and stable.

5.7. ***Extensions and Hecke equivariant exact sequences.*** The paper [Thomsen 2000] establishes, for any locally compact group G, an isomorphism

$$KK^G_1(A, B) \xrightarrow{\sim} \mathrm{Ext}^G(A \otimes \mathbb{K}_G, B \otimes \mathbb{K}_G),$$

where $\mathbb{K}_G \simeq \mathbb{K}(L^2(G \times \mathbb{N}))$. A G-equivariant semisplit extension

$$0 \to B \to E \to A \to 0$$

induces a G-equivariant semisplit extension

$$0 \to B \otimes \mathbb{K}_G \to E \otimes \mathbb{K}_G \to A \otimes \mathbb{K}_G \to 0,$$

and thus an element in $KK^G_1(A, B)$.

5.8. Theorem. *Let G be a locally compact group, $\Gamma \subset G$ a discrete subgroup, $C_G(\Gamma) \subset G$ its commensurator and S a group with $\Gamma \subset S \subset C_G(\Gamma)$. For any Γ-exact and S-equivariant semisplit extension*

$$0 \to B \to E \to A \to 0$$

of separable S-algebras and any separable C^*-algebra C, the exact sequences

$$\cdots \to KK_i(C, B \rtimes_r \Gamma) \to KK_i(C, E \rtimes_r \Gamma) \to KK_i(C, A \rtimes_r \Gamma) \to \cdots, \tag{5.9}$$
$$\cdots \to KK_i(A \rtimes_r \Gamma, C) \to KK_i(E \rtimes_r \Gamma, C) \to KK_i(B \rtimes_r \Gamma, C) \to \cdots \tag{5.10}$$

are $\mathbb{Z}[\Gamma, S]$-equivariant.

Proof. Exactness of Γ implies that we obtain a semisplit extension

$$0 \to B \rtimes_r \Gamma \to E \rtimes_r \Gamma \to A \rtimes_r \Gamma \to 0, \tag{5.11}$$

yielding the exact sequences (5.9) and (5.10). By Theorem 4.19 all groups in these exact sequences are Hecke modules. In sequence (5.9), the maps

$$KK_i(C, B \rtimes_r \Gamma) \to KK_i(C, E \rtimes_r \Gamma), \quad KK_i(C, E \rtimes_r \Gamma) \to KK_i(C, A \rtimes_r \Gamma)$$

are induced by elements in $KK_0(B \rtimes_r \Gamma, E \rtimes_r \Gamma)$ and $KK_0(A \rtimes_r \Gamma, E \rtimes_r \Gamma)$, respectively. These elements are in the image of the descent maps

$$KK_0^S(B, E) \to KK_0^\Gamma(B, E) \to KK_0(B \rtimes_r \Gamma, E \rtimes_r \Gamma),$$
$$KK_0^S(E, A) \to KK_0^\Gamma(E, A) \to KK_0(E \rtimes_r \Gamma, A \rtimes_r \Gamma),$$

and thus are Hecke equivariant by Theorem 5.5. Since the extension (5.11) is semisplit it defines a class $[\mathrm{Ext}] \in KK_1^S(A, B)$. The boundary maps in the exact sequence (5.9) are implemented by an element $\partial \in KK_1(A \rtimes_r \Gamma, B \rtimes_r \Gamma)$, and this element is the image of $[\mathrm{Ext}]$ under the composition

$$KK_1^S(A, B) \to KK_1^\Gamma(A, B) \to KK_1(A \rtimes_r \Gamma, B \rtimes_r \Gamma).$$

Thus by Theorem 5.5 the boundary maps in the sequence (5.9) are Hecke equivariant. The argument for sequence (5.10) is similar. □

Interesting examples of S-equivariant extensions come from partial compactifications of G-spaces. Let X be a locally compact space with a G-action. A partial S-compactification is an S-space $\overline{X}$ which contains X as an open dense subset. We write $\partial X := \overline{X} \setminus X$ and we obtain the S-equivariant exact sequence

$$0 \to C_0(X) \to C_0(\overline{X}) \to C_0(\partial X) \to 0.$$

5.12. Example. Let $G = \mathrm{Isom}(\boldsymbol{H})$, where $\boldsymbol{H}$ is the real hyperbolic n-space. The geodesic compactification $\overline{\boldsymbol{H}}$ of $\boldsymbol{H}$ is a G-compactification and thus, it is an S-compactification for any lattice $\Gamma \subset G$ and subgroup $\Gamma \subset S \subset C_G(\Gamma)$. The associated Hecke equivariant exact sequence in K-homology has been studied extensively in [Mesland and Şengün 2016]. For torsion-free Γ and $M := X/\Gamma$, there is a Morita equivalence $C_0(M) \sim C_0(X) \rtimes_r \Gamma$, and a KK-equivalence $C(\overline{\boldsymbol{H}}) \rtimes_r \Gamma \sim C_r^*(\Gamma)$. The exact sequence takes the form

$$\cdots \to K_*(C_0(M)) \to K_*(C_r^*(\Gamma)) \to K_*(C(\partial \boldsymbol{H}) \rtimes_r \Gamma) \to \cdots,$$

as in [Emerson and Meyer 2006; Emerson and Nica 2016].

5.13. Example. Let G be the group of real points of a reductive algebraic group $\boldsymbol{G}$ over $\mathbb{Q}$ and let X be its associated global symmetric space. The Borel–Serre partial compactification $\overline{X}$ of X is a $\boldsymbol{G}(\mathbb{Q})$-compactification but not a G-compactification;

see [Borel and Serre 1973]. However if $\Gamma \subset \boldsymbol{G}(\mathbb{Q})$ is an arithmetic subgroup, then $C_G(\Gamma) = \boldsymbol{G}(\mathbb{Q})$. So $\overline{X}$ is a $C_G(\Gamma)$-compactification. The action of Γ on $\overline{X}$ is cocompact and continues to be proper. Writing $M := X/\Gamma$ for torsion-free Γ, we obtain the Borel–Serre compactification $\overline{M} := \overline{X}/\Gamma$ of M and its boundary $\partial\overline{M} := \partial X/\Gamma$. There are Morita equivalences

$$C_0(X) \rtimes_r \Gamma \sim C_0(M), \quad C_0(\overline{X}) \rtimes_r \Gamma \sim C_0(\overline{M}), \quad C_0(\partial X) \rtimes_r \Gamma \sim C_0(\partial\overline{M}).$$

The exact sequence thus reduces to the topological K-theory sequence

$$\cdots \to K^*(M) \to K^*(\overline{M}) \to K^*(\partial\overline{M}) \to \cdots$$

of the pair $(\overline{M}, \partial\overline{M})$.

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