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K-theory, local cohomology and tangent spaces to Hilbert schemes

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Using K-theory, we construct a map $\pi : T_Y\mathrm{Hilb}^p(X) \to H_y^p(\Omega^{p-1}_{X/\mathbb{Q}})$ from the tangent space to the Hilbert scheme at a point Y to the local cohomology group. We use this map π to answer (after slight modification) a question by Mark Green and Phillip Griffiths on constructing a map from the tangent space $T_Y\mathrm{Hilb}^p(X)$ to the Hilbert scheme at a point Y to the tangent space to the cycle group $TZ^p(X)$.

1. Introduction

Let X be a smooth projective variety over a field k of characteristic 0 and let $Y \subset X$ be a subvariety of codimension p. Considering Y as an element of $\mathrm{Hilb}^p(X)$, it is well known that the Zariski tangent space $T_Y\mathrm{Hilb}^p(X)$ can be identified with $H^0(Y, \mathcal{N}_{Y/X})$, where $\mathcal{N}_{Y/X}$ is the normal sheaf.

The element Y also defines an element of the cycle group $Z^p(X)$. We are interested in defining the tangent space $TZ^p(X)$ to the cycle group $Z^p(X)$. In [Green and Griffiths 2005], Mark Green and Phillip Griffiths define $TZ^p(X)$ for $p = 1$ (divisors) and $p = \dim(X)$ (0-cycles) and leave the general case as an open question. Much of their theory was extended by Benjamin Dribus, Jerome W. Hoffman and the author in [Dribus et al. 2018; Yang 2016a]. In [Yang 2016a], we define $TZ^p(X)$ for any integer p satisfying $1 \leqslant p \leqslant \dim(X)$, generalizing Green and Griffiths' definitions. We recall the following fact from [Yang 2016a] for our purpose, and refer to [Green and Griffiths 2005; Yang 2016a] for definition of $TZ^p(X)$.

Theorem 1.1 [Yang 2016a, Theorem 2.8]. *For X a smooth projective variety over a field k of characteristic* 0 *and for any integer $p \geqslant 1$, the tangent space $TZ^p(X)$ is identified with* $\mathrm{Ker}(\partial_1^{p,-p})$:

$$TZ^p(X) \cong \mathrm{Ker}(\partial_1^{p,-p}),$$

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where $\partial_1^{p,-p}$ *is the differential of the Cousin complex* [Hartshorne 1966] *of* $\Omega^{p-1}_{X/\mathbb{Q}}$ *in position* p:

$$0 \to \Omega^{p-1}_{k(X)/\mathbb{Q}} \to \cdots \to \bigoplus_{y \in X^{(p)}} H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}) \xrightarrow{\partial_1^{p,-p}} \bigoplus_{x \in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}}) \to \cdots .$$

We want to study the relation between $T_Y\mathrm{Hilb}^p(X)$ and $TZ^p(X)$. The following question is suggested in [Green and Griffiths 2005, pp. 18 and 87–89].

Question 1.2 [Green and Griffiths 2005]. For X a smooth projective variety over a field k of characteristic 0 and for any integer $p \geqslant 1$, is it possible to define a map from the tangent space $T_Y\mathrm{Hilb}^p(X)$ to the Hilbert scheme at a point Y to the tangent space to the cycle group $TZ^p(X)$?

For $p = \dim(X)$, this has been answered affirmatively in [Green and Griffiths 2005, Section 7.2].

Theorem 1.3 [Green and Griffiths 2005]. *For* $p = d := \dim(X)$, *there exists a map*

$$T_Y\mathrm{Hilb}^d(X) \to TZ^d(X)$$

from the tangent space to the Hilbert scheme at a point Y *to the tangent space to the cycle group.*

The main result of this short note is to construct a map

$$\pi : T_Y\mathrm{Hilb}^p(X) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$$

(see Definition 4.1), and use this map to study the above Question 1.2.

In Example 4.4, we show, for a general subvariety $Y \subset X$ of codimension p and $Y' \in T_Y\mathrm{Hilb}^p(X)$, that $\pi(Y')$ may not lie in $TZ^p(X)$ (the kernel of $\partial_1^{p,-p}$). However, we show in Theorem 4.6 that there exist $Z \subset X$ of codimension p and $Z' \in T_Z\mathrm{Hilb}^p(X)$ such that $\pi(Y') + \pi(Z') \in TZ^p(X)$.

As an application, we show how to find Milnor K-theoretic cycles in Theorem 4.7. In [Yang 2016b], we will apply these techniques to eliminate obstructions to deforming curves on a threefold.

Notations and conventions.

(1) K-theory used in this note is Thomason–Trobaugh nonconnective K-theory, if not stated otherwise.

(2) For any abelian group M, $M_{\mathbb{Q}}$ denotes the image of M in $M \otimes_{\mathbb{Z}} \mathbb{Q}$.

(3) $X[\varepsilon]$ denote the first-order trivial deformation of X, i.e.,

$$X[\varepsilon] = X \times_k \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)),$$

where $k[\varepsilon]/(\varepsilon^2)$ is the ring of dual numbers.

2. K-theory and tangent spaces to Hilbert schemes

For X a smooth projective variety over a field k of characteristic 0 and $Y \subset X$ a subvariety of codimension p, let $i : Y \to X$ be the inclusion. Then i_*O_Y is a coherent O_X-module and can be resolved by a bounded complex of vector bundles on X. Let $Y^{'}$ be a first-order deformation of Y, that is, $Y^{'} \subset X[\varepsilon]$ such that $Y^{'}$ is flat over $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ and $Y^{'} \otimes_{k[\varepsilon]/(\varepsilon^2)} k \cong Y$. Then $i_*O_{Y'}$ can be resolved by a bounded complex of vector bundles on $X[\varepsilon]$, where $i : Y^{'} \to X[\varepsilon]$.

Let $D^{\mathrm{perf}}(X[\varepsilon])$ denote the derived category of perfect complexes of $O_X[\varepsilon]$-modules, and let $\mathcal{L}_{(i)}(X[\varepsilon]) \subset D^{\mathrm{perf}}(X[\varepsilon])$ be defined as

$$\mathcal{L}_{(i)}(X[\varepsilon]) := \{E \in D^{\mathrm{perf}}(X[\varepsilon]) \mid \mathrm{codim}_{\mathrm{Krull}}(\mathrm{supph}(E)) \geq -i\},$$

where the closed subset $\mathrm{supph}(E) \subset X$ is the support of the total homology of the perfect complex E.

The resolution of $i_*O_{Y'}$, which is a perfect complex of $O_X[\varepsilon]$-modules supported on Y, defines an element of the Verdier quotient $\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon])$, denoted $[i_*O_{Y'}]$.

In general, the length of the perfect complex $[i_*O_{Y'}]$ may not be equal to p. Since $Y \subset X$ is of codimension p, we expect the perfect complex $[i_*O_{Y'}]$ to be of length p. To achieve this, instead of considering $[i_*O_{Y'}]$ as an element of the Verdier quotient $\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon])$, we consider its image in the idempotent completion $(\mathcal{L}_{(-p)}(X[\varepsilon])/\mathcal{L}_{(-p-1)}(X[\varepsilon]))^\#$, denoted $[i_*O_{Y'}]^\#$, where the idempotent completion is in the sense of [Balmer and Schlichting 2001]. We have the following result:

Theorem 2.1 [Balmer 2007]. *For each $i \in \mathbb{Z}$, localization induces an equivalence*

$$(\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon]))^\# \simeq \coprod_{x[\varepsilon]\in X[\varepsilon]^{(-i)}} D^{\mathrm{perf}}_{x[\varepsilon]}(X[\varepsilon])$$

between the idempotent completion of $\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon])$ and the coproduct over $x[\varepsilon] \in X[\varepsilon]^{(-i)}$ of the derived category of perfect complexes of $O_{X[\varepsilon],x[\varepsilon]}$-modules with homology supported on the closed point $x[\varepsilon] \in \mathrm{Spec}(O_{X[\varepsilon],x[\varepsilon]})$. Consequently, one has

$$K_0\big((\mathcal{L}_{(i)}(X[\varepsilon])/\mathcal{L}_{(i-1)}(X[\varepsilon]))^\#\big) \simeq \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(-i)}} K_0(D^{\mathrm{perf}}_{x[\varepsilon]}(X[\varepsilon])).$$

Let y be the generic point of Y and let $\mathcal{I}_Y$ be the ideal sheaf of Y. Then there exists the short exact sequence

$$0 \to \mathcal{I}_Y \to O_X \to i_*O_Y \to 0,$$

whose localization at y is the short exact sequence

$$0 \to (\mathcal{I}_Y)_y \to O_{X,y} \to (i_* O_Y)_y \to 0.$$

We have $O_{Y,y} = O_{X,y}/(\mathcal{I}_Y)_y$. Since $O_{Y,y}$ is a field, $(\mathcal{I}_Y)_y$ is the maximal ideal of the regular local ring (of dimension p) $O_{X,y}$. So the maximal ideal $(\mathcal{I}_Y)_y$ is generated by a regular sequence $f_1, \dots, f_p$ of length p.

Let $\mathcal{I}_{Y'}$ be the ideal sheaf of Y', so $\mathcal{I}_{Y'}/(\varepsilon)\mathcal{I}_{Y'} = \mathcal{I}_Y$ because of flatness. So we have $(\mathcal{I}_{Y'})_y/(\varepsilon)(\mathcal{I}_{Y'})_y = (\mathcal{I}_Y)_y$. Lift $f_1, \dots, f_p$ to $f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p$ in $(\mathcal{I}_{Y'})_y$, where $g_1, \dots, g_p \in O_{X,y}$. Then $f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p$ generates $(\mathcal{I}_{Y'})_y$ because of Nakayama's lemma:

$$(\mathcal{I}_{Y'})_y = (f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p).$$

Moreover, $f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p$ is a regular sequence, which can be checked directly.

We see that Y is generically defined by a regular sequence $f_1, \dots, f_p$ of length p, where $f_1, \dots, f_p \in O_{X,y}$. Moreover, Y' is generically given by lifting $f_1, \dots, f_p$ to $f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p$, where $g_1, \dots, g_p \in O_{X,y}$. Let $F_\bullet(f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p)$ denote the Koszul complex associated to the regular sequence $f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p$, which is a resolution of $O_{X,y}[\varepsilon]/(f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p)$:

$$0 \to F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \cdots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0 \to 0,$$

where each $F_i = \bigwedge^i (O_{X,y}[\varepsilon])^{\oplus p}$ and $A_i : \bigwedge^i (O_{X,y}[\varepsilon])^{\oplus p} \to \bigwedge^{i-1} (O_{X,y}[\varepsilon])^{\oplus p}$ are defined as usual.

Under the equivalence in Theorem 2.1, the localization at the generic point y sends $[i_* O_{Y'}]^\#$ to the Koszul complex $F_\bullet(f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p)$:

$$[i_* O_{Y'}]^\# \to F_\bullet(f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p).$$

Milnor K-groups with support are rationally defined in terms of eigenspaces of Adams operations in [Yang 2016c] as follows:

Definition 2.2 [Yang 2016c, Definition 3.2]. Let X be a finite equidimensional noetherian scheme and $x \in X^{(j)}$. For $m \in \mathbb{Z}$, the Milnor K-group with support $K_m^M(O_{X,x} \text{ on } x)$ is rationally defined to be

$$K_m^M(O_{X,x} \text{ on } x) := K_m^{(m+j)}(O_{X,x} \text{ on } x)_{\mathbb{Q}},$$

where $K_m^{(m+j)}$ is the eigenspace of $\psi^k = k^{m+j}$ and ψ^k are the Adams operations.

Theorem 2.3 [Gillet and Soulé 1987, Proposition 4.12]. *The Adams operations ψ^k defined on perfect complexes* (defined in [Gillet and Soulé 1987]) *satisfy*

$$\psi^k(F_\bullet(f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p)) = k^p F_\bullet(f_1 + \varepsilon g_1, \dots, f_p + \varepsilon g_p).$$

Hence, $F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p)$ is of eigenweight p and can be considered as an element of $K_0^{(p)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}$:

$$F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p) \in K_0^{(p)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} = K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]).$$

Definition 2.4. We define a map $\mu : T_Y \mathrm{Hilb}^p(X) \to K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])$ by

$$\mu : Y' \mapsto F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p).$$

3. Chern character

For any integer m, let $K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}}$ denote the weight i eigenspace of the relative K-group, that is, the kernel of the natural projection

$$K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}} \xrightarrow{\varepsilon=0} K_m^{(i)}(O_{X,y} \text{ on } y)_{\mathbb{Q}}.$$

Recall that we have proved the following isomorphisms in [Dribus et al. 2018; Yang 2016c]:

Theorem 3.1 [Dribus et al. 2018, Corollary 9.5; Yang 2016c, Corollary 3.11]. *Let X be a smooth projective variety over a field k of characteristic 0 and let $y \in X^{(p)}$. The Chern character (from K-theory to negative cyclic homology) induces isomorphisms*

$$K_m^{(i)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}} \cong H_y^p\big(\Omega^{\bullet,(i)}_{X/\mathbb{Q}}\big)$$

between relative K-groups and local cohomology groups, where

$$\begin{cases} \Omega^{\bullet,(i)}_{X/\mathbb{Q}} = \Omega^{2i-(m+p)-1}_{X/\mathbb{Q}} & \text{if } \frac{1}{2}(m+p) < i \le m+p, \\ \Omega^{\bullet,(i)}_{X/\mathbb{Q}} = 0 & \text{else.} \end{cases}$$

The main tool for proving these isomorphisms is the space-level versions of Goodwillie's and Cathelineau's isomorphisms, proved in [Cortiñas et al. 2009, Appendix B].

Let $K_m^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)$ denote the relative K-group, that is, the kernel of the natural projection

$$K_m^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \xrightarrow{\varepsilon=0} K_m^M(O_{X,y} \text{ on } y).$$

In other words, $K_m^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)$ is $K_m^{(m+p)}(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)_{\mathbb{Q}}$. In particular, by taking $i = p$ and $m = 0$ in Theorem 3.1, we obtain the following formula:

Corollary 3.2. $$K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon) \xrightarrow{\cong} H_y^p(\Omega^{p-1}_{X/\mathbb{Q}}).$$

Definition 3.3. Let X be a smooth projective variety over a field k of characteristic 0 and let $y \in X^{(p)}$. There exists a natural surjective map

$$\mathrm{Ch} : K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to H_y^p(\Omega^{p-1}_{X/\mathbb{Q}}),$$

which is defined to be the composition of the natural projection

$$K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon)$$

and the isomorphism

$$K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon], \varepsilon) \xrightarrow{\cong} H_y^p(\Omega^{p-1}_{X/\mathbb{Q}}).$$

Now we recall a beautiful construction of Angéniol and Lejeune-Jalabert, which describes the map

$$\mathrm{Ch} : K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to H_y^p(\Omega^{p-1}_{X/\mathbb{Q}})$$

in Definition 3.3.

An element $M \in K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \subset K_0(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon])_{\mathbb{Q}}$ is represented by a strict perfect complex $L_\bullet$ supported at $y[\varepsilon]$:

$$0 \to F_n \xrightarrow{M_n} F_{n-1} \xrightarrow{M_{n-1}} \cdots \xrightarrow{M_2} F_1 \xrightarrow{M_1} F_0 \to 0,$$

where each $F_i = O_{X,y}[\varepsilon]^{r_i}$ and the M_i are matrices with entries in $O_{X,y}[\varepsilon]$.

Definition 3.4 [Angéniol and Lejeune-Jalabert 1989, p. 24]. The *local fundamental class* attached to this perfect complex is defined to be the collection

$$[L_\bullet]_{\mathrm{loc}} = \left\{ \frac{1}{p!} dM_i \circ dM_{i+1} \circ \cdots \circ dM_{i+p-1} \right\}, \quad i = 0, 1, \ldots,$$

where $d = d_{\mathbb{Q}}$ and each dM_i is the matrix of absolute differentials. In other words,

$$dM_i \in \mathrm{Hom}\big(F_i, F_{i-1} \otimes \Omega^1_{O_{X,y}[\varepsilon]/\mathbb{Q}}\big).$$

Theorem 3.5 [Angéniol and Lejeune-Jalabert 1989, Lemma 3.1.1, p. 24 and Definition 3.4, p. 29]. *The class $[L_\bullet]_{\mathrm{loc}}$ above is a cycle in $\mathcal{H}om\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big)$, and the image of $[L_\bullet]_{\mathrm{loc}}$ in $H^p\big(\mathcal{H}om\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big)\big)$ does not depend on the choice of the basis of $L_\bullet$.*

Since

$$H^p\big(\mathcal{H}om\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big)\big) = \mathcal{EXT}^p\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big),$$

the local fundamental class $[L_\bullet]_{\mathrm{loc}}$ defines an element in $\mathcal{EXT}^p\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big)$:

$$[L_\bullet]_{\mathrm{loc}} \in \mathcal{EXT}^p\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big).$$

Noting $L_\bullet$ is supported on y (same underlying space as $y[\varepsilon]$), there exists the following trace map (see [Angéniol and Lejeune-Jalabert 1989, p. 98–99] for details):

$$\mathrm{Tr} : \mathcal{EXT}^p\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big) \to H_y^p(\Omega^p_{X[\varepsilon]/\mathbb{Q}}).$$

Definition 3.6 [Angéniol and Lejeune-Jalabert 1989, Definition 2.3.2, p. 99]. The image of $[L_\bullet]_{\mathrm{loc}}$ under the above trace map, denoted $\mathcal{V}^p_{L_\bullet}$, is called the Newton class.

$K_0(O_{X,y}[\varepsilon]$ on $y[\varepsilon])$ is the Grothendieck group of the triangulated category $D^b(O_{X,y}[\varepsilon]$ on $y[\varepsilon])$, which is the derived category of perfect complexes of $O_{X,y}[\varepsilon]$-modules with homology supported on the closed point $y[\varepsilon] \in \mathrm{Spec}(O_{X,y}[\varepsilon])$. Recall that the Grothendieck group of a triangulated category is the monoid of isomorphism objects modulo the submonoid formed from distinguished triangles.

Theorem 3.7 [Angéniol and Lejeune-Jalabert 1989, Proposition 4.3.1, p. 113]. *The Newton class $\mathcal{V}^p_{L_\bullet}$ is well-defined on $K_0(O_{X,y}[\varepsilon]$ on $y[\varepsilon])$.*

The truncation map $\rfloor \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} : \Omega^p_{X[\varepsilon]/\mathbb{Q}} \to \Omega^{p-1}_{X/\mathbb{Q}}$ induces a map

$$\rfloor \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} : H^p_y(\Omega^p_{X[\varepsilon]/\mathbb{Q}}) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}).$$

Lemma 3.8. *The map*

$$\mathrm{Ch} : K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$$

from Definition 3.3 can be described as a composition

$$K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \to \mathcal{E}XT^p\big(L_\bullet, \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} \otimes L_\bullet\big) \to H^p_y(\Omega^p_{X[\varepsilon]/\mathbb{Q}}) \to H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}),$$
$$L_\bullet \mapsto [L_\bullet]_{\mathrm{loc}} \mapsto \mathcal{V}^p_{L_\bullet} \mapsto \mathcal{V}^p_{L_\bullet} \rfloor \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0}.$$

In particular, for the Koszul complex $F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p)$ in Definition 2.4, the Ch map can be described as follows. The diagram

$$\begin{cases} F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p) \longrightarrow O_{X,y}[\varepsilon]/(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p), \\ F_p(\cong O_{X,y}[\varepsilon]) \xrightarrow{[F_\bullet]_{\mathrm{loc}}} F_0 \otimes \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}} (\cong \Omega^p_{O_{X,y}[\varepsilon]/\mathbb{Q}}), \end{cases}$$

where $[F_\bullet]_{\mathrm{loc}}$ is the local fundamental class attached to $F_\bullet(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p)$, gives an element in $\mathrm{Ext}^p_{O_{X,y}[\varepsilon]}(O_{X,y}[\varepsilon]/(f_1+\varepsilon g_1, \dots, f_p+\varepsilon g_p), \Omega^p_{X[\varepsilon]/\mathbb{Q}})$. This, moreover, gives an element in $H^p_y(\Omega^p_{X[\varepsilon]/\mathbb{Q}})$, denoted $\mathcal{V}^p_{F_\bullet}$.

We use $F_\bullet(f_1, \dots, f_p)$ to denote the Koszul complex associated to the regular sequence $f_1, \dots, f_p$, which is a resolution of $O_{X,y}/(f_1, \dots, f_p)$. The truncation of $\mathcal{V}^p_{F_\bullet}$ in ε produces an element in $H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$, which can be represented by the diagram

$$\begin{cases} F_\bullet(f_1, \dots, f_p) \longrightarrow O_{X,y}/(f_1, \dots, f_p), \\ F_p(\cong O_{X,y}) \xrightarrow{[F_\bullet]_{\mathrm{loc}} \rfloor \frac{\partial}{\partial \varepsilon}\big|_{\varepsilon=0}} F_0 \otimes \Omega^{p-1}_{O_{X,y}/\mathbb{Q}} \big(\cong \Omega^{p-1}_{O_{X,y}/\mathbb{Q}}\big). \end{cases}$$

For simplicity, assuming $g_2 = \cdots = g_p = 0$, we see that

$$[F_\bullet]_{\mathrm{loc}} \rfloor \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} = g_1 df_2 \wedge \cdots \wedge df_p$$

and the truncation of $\mathcal{V}^p_{F_\bullet}$ in ε is represented by the diagram

$$\begin{cases} F_\bullet(f_1,\dots,f_p) \longrightarrow O_{X,y}/(f_1,\dots,f_p), \\ F_p(\cong O_{X,y}) \xrightarrow{g_1 df_2\wedge\cdots\wedge df_p} F_0\otimes\Omega^{p-1}_{O_{X,y}/\mathbb{Q}}(\cong \Omega^{p-1}_{O_{X,y}/\mathbb{Q}}).\end{cases}$$

Further concrete examples can be found in [Green and Griffiths 2005, Chapter 7, p. 90–91].

4. The map π

Definition 4.1. We define a map from $T_Y\mathrm{Hilb}^p(X)$ to $H^p_y(\Omega^{p-1}_{X/\mathbb{Q}})$ by composing Ch in Definition 3.3 with μ in Definition 2.4:

$$\pi : T_Y\mathrm{Hilb}^p(X) \xrightarrow{\mu} K_0^M(O_{X,y}[\varepsilon] \text{ on } y[\varepsilon]) \xrightarrow{\mathrm{Ch}} H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}).$$

Recall that the Cousin complex of $\Omega^{p-1}_{X/\mathbb{Q}}$ is of the form

$$0\to \Omega^{p-1}_{k(X)/\mathbb{Q}} \to \cdots \to \bigoplus_{y\in X^{(p)}} H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}) \xrightarrow{\partial_1^{p,-p}} \bigoplus_{x\in X^{(p+1)}} H^{p+1}_x(\Omega^{p-1}_{X/\mathbb{Q}}) \to \cdots$$

and the tangent space $TZ^p(X)$ is identified with $\mathrm{Ker}(\partial_1^{p,-p})$ (see Theorem 1.1).

For $p=d:=\dim(X)$, $\partial_1^{d,-d}=0$ because of dimensional reasons. So

$$TZ^d(X) = \mathrm{Ker}(\partial_1^{d,-d}) = \bigoplus_{y\in X^{(d)}} H^d_y(\Omega^{d-1}_{X/\mathbb{Q}}).$$

Corollary 4.2. *For $p=d:=\dim(X)$, the map π defines a map from $T_Y\mathrm{Hilb}^d(X)$ to $TZ^d(X)$ and it agrees with the map by Green and Griffiths in Theorem 1.3.*

We want to know, for general p, whether this map π defines a map from $T_Y\mathrm{Hilb}^p(X)$ to $TZ^p(X)$, as Green and Griffiths asked in Question 1.2.

Remark 4.3. In an email to the author, Christophe Soulé suggested considering the image of suitable Koszul complexes under the Ch map in Definition 3.3. This leads us to the following example, showing that π does not define a map from $T_Y\mathrm{Hilb}^p(X)$ to $TZ^p(X)$ in general. The Koszul complex technique is also used in Theorem 4.6.

The author sincerely thanks Christophe Soulé for very helpful suggestions.

Example 4.4. For a smooth projective threefold X over a field k of characteristic 0, let $Y\subset X$ be a curve with generic point y. We assume a point $x\in Y\subset X$ is defined by (f,g,h) and Y is generically defined by (f,g). Then $O_{X,y}=(O_{X,x})_{(f,g)}$.

We consider the infinitesimal deformation Y' of Y which is generically given by $(f+\varepsilon/h, g)$, where $1/h\in O_{X,y}=(O_{X,x})_{(f,g)}$. Note $1/h\notin O_{X,x}$. The Koszul complex of $(f+\varepsilon/h, g)$ is of the form

$$0 \to (O_{X,x})_{(f,g)}[\varepsilon] \xrightarrow{(g,-f-\varepsilon/h)^{\mathrm{T}}} (O_{X,x})^{\oplus 2}_{(f,g)}[\varepsilon] \xrightarrow{(f+\varepsilon/h,g)} (O_{X,x})_{(f,g)}[\varepsilon] \to 0,$$

where $(-,-)^{\mathrm{T}}$ denotes transpose.

The image $\pi(Y') \in H^2_y(\Omega^1_{X/\mathbb{Q}})$ is represented by the diagram

$$\begin{cases} (O_{X,x})_{(f,g)} \to (O_{X,x})^{\oplus 2}_{(f,g)} \to (O_{X,x})_{(f,g)} \to (O_{X,x})_{(f,g)}/(f,g) \to 0, \\ \qquad (O_{X,x})_{(f,g)} \xrightarrow{(1/h)dg} \Omega^1_{(O_{X,x})_{(f,g)}/\mathbb{Q}}. \end{cases}$$

Let $F_\bullet(f,g,h)$ be the Koszul complex of f, g, h:

$$0 \to O_{X,x} \to O^{\oplus 3}_{X,x} \to O^{\oplus 3}_{X,x} \to O_{X,x} \to 0.$$

Then $\partial_1^{2,-2}(\pi(Y'))$ in $H^3_x(\Omega^1_{X/\mathbb{Q}})$ is represented by the diagram

$$\begin{cases} F_\bullet(f,g,h) \longrightarrow O_{X,x}/(f,g,h), \\ \quad O_{X,x} \xrightarrow{1dg} \Omega^1_{O_{X,x}/\mathbb{Q}}, \end{cases}$$

which is not zero.

This example shows that, in general, the image of π may not lie in $TZ^p(X)$(the kernel of $\partial_1^{p,-p}$). However, we will show, in Theorem 4.6 below, that given $Y \subset X$ of codimension p and $Y' \in T_Y\mathrm{Hilb}^p(X)$, there exists $Z \subset X$ of codimension p and $Z' \in T_Z\mathrm{Hilb}^p(X)$ such that $\pi(Y') + \pi(Z')$ is a nontrivial element of $TZ^p(X)$.

To fix notation, let X be a smooth projective variety over a field k of characteristic 0 and let $Y \subset X$ be a subvariety of codimension p with generic point y. Let $W \subset Y$ be a subvariety of codimension 1 in Y with generic point w. One assumes W is generically defined by $f_1, f_2, \dots, f_p, f_{p+1}$ and Y is generically defined by $f_1, f_2, \dots, f_p$. So one has $O_{X,y} = (O_{X,w})_P$, where P is the ideal $(f_1, f_2, \dots, f_p) \subset O_{X,w}$.

The element Y' is generically given by $(f_1+\varepsilon g_1, f_2+\varepsilon g_2, \dots, f_p+\varepsilon g_p)$, where $g_1, \dots, g_p \in O_{X,y}$. We assume $g_2 = \cdots = g_p = 0$. Since $O_{X,y} = (O_{X,w})_P$, we write $g_1 = a/b$, where $a, b \in O_{X,w}$ and $b \notin P$. In Theorem 4.6, we will consider the cases of whether or not b is in the maximal ideal $(f_1, f_2, \dots, f_p, f_{p+1}) \subset O_{X,w}$.

Lemma 4.5. *If $b \notin (f_1, f_2, \dots, f_p, f_{p+1})$, then $\partial_1^{p,-p}(\pi(Y')) = 0$.*

Proof. If $b \notin (f_1, f_2, \dots, f_p, f_{p+1})$, then b is a unit in $O_{X,w}$, so $g_1 = a/b \in O_{X,w}$. Then $\pi(Y')$ is represented by the diagram

$$\begin{cases} F_\bullet(f_1, f_2, \dots, f_p) \longrightarrow (O_{X,w})_P/(f_1, f_2, \dots, f_p), \\ F_p(\cong (O_{X,w})_P) \xrightarrow{g_1 df_2\wedge\cdots\wedge df_p} F_0 \otimes \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}} (\cong \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}). \end{cases}$$

Here, $F_\bullet(f_1, f_2, \dots, f_p)$ is of the form

$$0 \to F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \cdots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,$$

where each $F_i = \bigwedge^i ((O_{X,w})_P)^{\oplus p}$. Since $f_{p+1} \notin P$, f_{p+1}^{-1} exists in $(O_{X,w})_P$, and we can write

$$g_1 df_2 \wedge \cdots \wedge df_p = \frac{g_1 f_{p+1}}{f_{p+1}} df_2 \wedge \cdots \wedge df_p.$$

Now $\partial_1^{p,-p}(\pi(Y'))$ is represented by the diagram

$$\begin{cases} F_\bullet(f_1, f_2, \dots, f_p, f_{p+1}) \longrightarrow O_{X,w}/(f_1, f_2, \dots, f_p, f_{p+1}), \\ F_{p+1}(\cong O_{X,w}) \xrightarrow{g_1 f_{p+1} df_2 \wedge \cdots \wedge df_p} F_0 \otimes \Omega^{p-1}_{O_{X,w}/\mathbb{Q}} (\cong \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}). \end{cases}$$

The complex $F_\bullet(f_1, f_2, \dots, f_p, f_{p+1})$ is of the form

$$0 \to \bigwedge^{p+1} (O_{X,w})^{\oplus p+1} \xrightarrow{A_{p+1}} \bigwedge^{p} (O_{X,w})^{\oplus p+1} \to \cdots.$$

Let $\{e_1, \dots, e_{p+1}\}$ be a basis of $(O_{X,w})^{\oplus p+1}$; the map A_{p+1} is

$$e_1 \wedge \cdots \wedge e_{p+1} \to \sum_{j=1}^{p+1} (-1)^j f_j e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots e_{p+1},$$

where $\hat{e}_j$ means to omit the j-th term.

Since f_{p+1} appears in A_{p+1},

$$g_1 f_{p+1} df_2 \wedge \cdots \wedge df_p \equiv 0 \in \mathrm{Ext}^{p+1}_{O_{X,w}} \big(O_{X,w}/(f_1, \dots, f_p, f_{p+1}), \Omega^{p-1}_{O_{X,w}/\mathbb{Q}} \big)$$

and $\partial_1^{p,-p}(\pi(Y')) = 0$. □

This lemma doesn't contradict Example 4.4, where $h \in (f, g, h) \subset O_{X,x}$.

Theorem 4.6. *For $Y' \in T_Y \mathrm{Hilb}^p(X)$ generically defined by $(f_1 + \varepsilon g_1, f_2, \dots, f_p)$, where $g_1 = a/b \in O_{X,y} = (O_{X,w})_P$, we have:*

Case 1: *If $b \notin (f_1, f_2, \dots, f_p, f_{p+1})$, then $\pi(Y') \in TZ^p(X)$, i.e., $\partial_1^{p,-p}(\pi(Y')) = 0$.*

Case 2: *If $b \in (f_1, f_2, \dots, f_p, f_{p+1})$, then there exist $Z \subset X$ of codimension p and $Z' \in T_Z \mathrm{Hilb}^p(X)$ with $\pi(Y') + \pi(Z') \in TZ^p(X)$, i.e., $\partial_1^{p,-p}(\pi(Y') + \pi(Z')) = 0$.*

Proof. Case 1 is Lemma 4.5. Now we consider the case $b \in (f_1, f_2, \dots, f_p, f_{p+1})$. Since $b \notin (f_1, f_2, \dots, f_p)$, we can write $b = \sum_{i=1}^p a_i f_i^{n_i} + a_{p+1} f_{p+1}^{n_{p+1}}$, where a_{p+1} is a unit in $O_{X,w}$ and each n_j is some integer. For simplicity, we assume each $n_j = 1$ and $a_{p+1} = 1$.

Since Y' is generically given by $(f_1+\varepsilon g_1, f_2, \dots, f_p)$, then $\pi(Y')$ is represented by the following diagram (where $g_1 = a/b$):

$$\begin{cases} F_\bullet(f_1, f_2, \dots, f_p) \xrightarrow{\qquad} (O_{X,w})_P/(f_1, f_2, \dots, f_p), \\ F_p(\cong (O_{X,w})_P) \xrightarrow{(a/b)df_2\wedge\cdots\wedge df_p} F_0 \otimes \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}} (\cong \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}). \end{cases}$$

Here, $F_\bullet(f_1, f_2, \dots, f_p)$ is of the form

$$0 \to F_p \xrightarrow{A_p} F_{p-1} \xrightarrow{A_{p-1}} \cdots \xrightarrow{A_2} F_1 \xrightarrow{A_1} F_0,$$

where each $F_i = \bigwedge^i ((O_{X,w})_P)^{\oplus p}$. Let $\{e_1, \dots, e_p\}$ be a basis of $(O_{X,w})^{\oplus p}$; the map A_p is

$$e_1 \wedge \cdots \wedge e_p \to \sum_{j=1}^{p} (-1)^j f_j e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots e_p,$$

where $\hat{e}_j$ means to omit the j-th term.

Noting

$$\frac{1}{b} - \frac{1}{f_{p+1}} = \frac{-\sum_{i=1}^{p} a_i f_i}{b f_{p+1}}$$

and each f_i $(i = 1, \dots, p)$ appears in A_p, the above diagram representing $\pi(Y')$ can be replaced by the following one:

$$\begin{cases} F_\bullet(f_1, f_2, \dots, f_p) \xrightarrow{\qquad} (O_{X,w})_P/(f_1, f_2, \dots, f_p), \\ F_p(\cong (O_{X,w})_P) \xrightarrow{(a/f_{p+1})df_2\wedge\cdots\wedge df_p} F_0 \otimes \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}} (\cong \Omega^{p-1}_{(O_{X,w})_P/\mathbb{Q}}). \end{cases}$$

Then $\partial_1^{p,-p}(\pi(Y'))$ is represented by the diagram

$$\begin{cases} F_\bullet(f_1, f_2, \dots, f_p, f_{p+1}) \xrightarrow{\qquad} O_{X,w}/(f_1, f_2, \dots, f_p, f_{p+1}), \\ F_{p+1}(\cong O_{X,w}) \xrightarrow{a df_2\wedge\cdots\wedge df_p} F_0 \otimes \Omega^{p-1}_{O_{X,w}/\mathbb{Q}} (\cong \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}). \end{cases}$$

Let P' denote the prime $(f_{p+1}, f_2, \dots, f_p) \subset O_{X,w}$. Then P' defines a generic point $z \in X^{(p)}$ and one has $O_{X,z} = (O_{X,w})_{P'}$. We define the subscheme

$$Z := \overline{\{z\}}.$$

Let Z' be a first-order infinitesimal deformation of Z, which is generically given by $(f_{p+1} + \varepsilon a/f_1, f_2, \dots, f_p)$. Then $\pi(Z')$ is represented by the diagram

$$\begin{cases} F_\bullet(f_{p+1}, f_2, \dots, f_p) \xrightarrow{\qquad} (O_{X,w})_{P'}/(f_{p+1}, f_2, \dots, f_p), \\ F_p(\cong (O_{X,w})_{P'}) \xrightarrow{(a/f_1)df_2\wedge\cdots\wedge df_p} F_0 \otimes \Omega^{p-1}_{(O_{X,w})_{P'}/\mathbb{Q}} (\cong \Omega^{p-1}_{(O_{X,w})_{P'}/\mathbb{Q}}), \end{cases}$$

and $\partial_1^{p,-p}(\pi(Z'))$ is represented by the diagram

$$\begin{cases} F_\bullet(f_{p+1}, f_2, \dots, f_p, f_1) & \longrightarrow & O_{X,w}/(f_{p+1}, f_2, \dots, f_p, f_1), \\ F_{p+1}(\cong O_{X,w}) & \xrightarrow{adf_2\wedge\cdots\wedge df_p} & F_0 \otimes \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}(\cong \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}). \end{cases}$$

Here, $F_\bullet(f_1, f_2, \dots, f_p, f_{p+1})$ and $F_\bullet(f_{p+1}, f_2, \dots, f_p, f_1)$ are Koszul resolutions of $O_{X,w}/(f_1, f_2, \dots, f_p, f_{p+1})$ and $O_{X,w}/(f_{p+1}, f_2, \dots, f_p, f_1)$, respectively.

These Koszul complexes $F_\bullet(f_1, f_2, \dots, f_p, f_{p+1})$ and $F_\bullet(f_{p+1}, f_2, \dots, f_p, f_1)$ are related by the commutative diagram

$$\begin{array}{ccccccccc} O_{X,w} & \xrightarrow{D_{p+1}} & \bigwedge^p O_{X,w}^{\oplus p+1} & \xrightarrow{D_p} & \cdots & \longrightarrow & O_{X,w}^{\oplus p+1} & \xrightarrow{D_1} & O_{X,w} \\ \downarrow{\scriptstyle \det A_1} & & \downarrow{\scriptstyle \bigwedge^p A_1} & & \downarrow & & \downarrow{\scriptstyle A_1} & & \downarrow{\scriptstyle =} \\ O_{X,w} & \xrightarrow{E_{p+1}} & \bigwedge^p O_{X,w}^{\oplus p+1} & \xrightarrow{E_p} & \cdots & \longrightarrow & O_{X,w}^{\oplus p+1} & \xrightarrow{E_1} & O_{X,w} \end{array}$$

(see [Griffiths and Harris 1978, p. 691]), where each D_i and E_i are defined as usual. In particular, $D_1 = (f_1, f_2, \dots, f_p, f_{p+1})$, $E_1 = (f_{p+1}, f_2, \dots, f_p, f_1)$, and A_1 is the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since $\det A_1 = -1$, one has

$$\partial_1^{p,-p}(\pi(Z')) = -\partial_1^{p,-p}(\pi(Y')) \in \mathrm{Ext}^{p+1}_{O_{X,w}}\big(O_{X,w}/(f_1, f_2, \dots, f_p, f_{p+1}), \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}\big),$$

and consequently, $\partial_1^{p,-p}(\pi(Z') + \pi(Y')) = 0 \in H_w^{p+1}\big(\Omega^{p-1}_{O_{X,w}/\mathbb{Q}}\big)$. In other words,

$$\pi(Z') + \pi(Y') \in TZ^p(X). \qquad \square$$

There exists the following commutative diagram, which is part of the commutative diagram of [Yang 2016c, Theorem 3.14] (taking $j = 1$):

$$\begin{array}{ccc} \bigoplus_{x\in X^{(p)}} H_x^p\big(\Omega^{p-1}_{X/\mathbb{Q}}\big) & \xleftarrow{\mathrm{Ch}} & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p)}} K_0^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \\ \downarrow{\scriptstyle \partial_1^{p,-p}} & & \downarrow{\scriptstyle d^{p,-p}_{1,X[\varepsilon]}} \\ \bigoplus_{x\in X^{(p+1)}} H_x^{p+1}\big(\Omega^{p-1}_{X/\mathbb{Q}}\big) & \underset{\cong}{\xleftarrow{\mathrm{Ch}}} & \bigoplus_{x[\varepsilon]\in X[\varepsilon]^{(p+1)}} K_{-1}^M(O_{X,x}[\varepsilon] \text{ on } x[\varepsilon]) \end{array}$$

For $Y' \in T_Y\mathrm{Hilb}^p(X)$, which is generically defined by $(f_1 + \varepsilon g_1, f_2, \dots, f_p)$ for $g_1 = a/b \in O_{X,y} = (O_{X,w})_P$, we use $F_\bullet(f_1 + \varepsilon g_1, f_2, \dots, f_p)$ to denote the Koszul complex associated to $f_1 + \varepsilon g_1, f_2, \dots, f_p$. Theorem 4.6 implies the following.

Case 1: If $b \notin (f_1, f_2, \ldots, f_p, f_{p+1})$, $\partial_1^{p,-p}(\pi(Y')) = 0$, the commutative diagram

$$\begin{array}{ccc} \pi(Y') & \xleftarrow{\text{Ch}} & F_\bullet(f_1 + \varepsilon g_1, f_2, \ldots, f_p) \\ \partial_1^{p,-p}\big\downarrow & & d_{1,X[\varepsilon]}^{p,-p}\big\downarrow \\ 0 & \xleftarrow[\cong]{\text{Ch}} & d_{1,X[\varepsilon]}^{p,-p}(F_\bullet(f_1 + \varepsilon g_1, f_2, \ldots, f_p)) \end{array}$$

says $d_{1,X[\varepsilon]}^{p,-p}(F_\bullet(f_1 + \varepsilon g_1, f_2, \ldots, f_p)) = 0$.

Case 2: If $b \in (f_1, f_2, \ldots, f_p, f_{p+1})$, we are reduced to considering $b = f_{p+1}$. Then there exist $Z \subset X$ which is generically defined by $(f_{p+1}, f_2, \ldots, f_p)$ and $Z' \in T_Z \operatorname{Hilb}^p(X)$ which is generically defined by $(f_{p+1} + \varepsilon a/f_1, f_2, \ldots, f_p)$ such that $\partial_1^{p,-p}(\pi(Y') + \pi(Z')) = 0$. We use $F_\bullet(f_{p+1} + \varepsilon a/f_1, f_2, \ldots, f_p)$ to denote the Koszul complex associated to $f_{p+1} + \varepsilon a/f_1, f_2, \ldots, f_p$.

The commutative diagram

$$\begin{array}{ccc} \pi(Y')+\pi(Z') & \xleftarrow{\text{Ch}} & F_\bullet(f_1+\varepsilon a/f_{p+1}, f_2, \ldots, f_p)+F_\bullet(f_{p+1}+\varepsilon a/f_1, f_2, \ldots, f_p) \\ \partial_1^{p,-p}\big\downarrow & & d_{1,X[\varepsilon]}^{p,-p}\big\downarrow \\ 0 & \xleftarrow[\cong]{\text{Ch}} & \begin{array}{l} d_{1,X[\varepsilon]}^{p,-p}\big(F_\bullet(f_1+\varepsilon a/f_{p+1}, f_2, \ldots, f_p) \\ \qquad +F_\bullet(f_{p+1}+\varepsilon a/f_1, f_2, \ldots, f_p)\big) \end{array} \end{array}$$

says $d_{1,X[\varepsilon]}^{p,-p}\big(F_\bullet(f_1 + \varepsilon a/f_{p+1}, f_2, \ldots, f_p) + F_\bullet(f_{p+1} + \varepsilon a/f_1, f_2, \ldots, f_p)\big) = 0$.

Recall that in [Yang 2016c, Definition 3.4 and Corollary 3.15], the p-th Milnor K-theoretic cycle is defined as

$$Z_p^M(D^{\text{Perf}}(X[\varepsilon])) := \operatorname{Ker}\big(d_{1,X[\varepsilon]}^{p,-p}\big).$$

The above can be summarized as follows:

Theorem 4.7. *For $Y' \in T_Y \operatorname{Hilb}^p(X)$ generically defined by $(f_1 + \varepsilon g_1, f_2, \ldots, f_p)$ for $g_1 = a/b \in O_{X,y} = (O_{X,w})_P$, we use $F_\bullet(f_1 + \varepsilon g_1, f_2, \ldots, f_p)$ to denote the Koszul complex associated to $f_1 + \varepsilon g_1, f_2, \ldots, f_p$.*

Case 1: *If $b \notin (f_1, f_2, \ldots, f_p, f_{p+1})$, then*

$$F_\bullet(f_1 + \varepsilon g_1, f_2, \ldots, f_p) \in Z_p^M(D^{\text{Perf}}(X[\varepsilon])).$$

Case 2: *If $b \in (f_1, f_2, \ldots, f_p, f_{p+1})$, we are reduced to considering $b = f_{p+1}$. Then there exist $Z \subset X$ which is generically defined by $(f_{p+1}, f_2, \ldots, f_p)$ and $Z' \in T_Z \operatorname{Hilb}^p(X)$ generically defined by $(f_{p+1} + \varepsilon a/f_1, f_2, \ldots, f_p)$ such that*

$$F_\bullet(f_1+\varepsilon a/f_{p+1}, f_2, \ldots, f_p)+F_\bullet(f_{p+1}+\varepsilon a/f_1, f_2, \ldots, f_p) \in Z_p^M(D^{\text{Perf}}(X[\varepsilon])).$$

The existence of Z and $Z' \in T_Z \operatorname{Hilb}^p(X)$ has applications in deformation of cycles; see [Yang 2016b] for a concrete example of eliminating obstructions to deforming curves on a threefold.

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