ANNALS OF
K-THEORY

vol. 4 no. 1 2019

Segal operations in the algebraic K-theory of topological spaces

Thomas Gunnarsson and Ross Staffeldt

msp

A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 4, No. 1, 2019
dx.doi.org/10.2140/akt.2019.4.1

Segal operations in the algebraic K-theory of topological spaces

Thomas Gunnarsson and Ross Staffeldt

We extend earlier work of Waldhausen which defines operations on the algebraic K-theory of the one-point space. For a connected simplicial abelian group X and symmetric groups Σ_n, we define operations $\theta^n : A(X) \to A(X \times B\Sigma_n)$ in the algebraic K-theory of spaces. We show that our operations can be given the structure of E_∞-maps. Let $\phi_n : A(X \times B\Sigma_n) \to A(X \times E\Sigma_n) \simeq A(X)$ be the Σ_n-transfer. We also develop an inductive procedure to compute the compositions $\phi_n \circ \theta^n$, and outline some applications.

1. Introduction

Let X be a connected simplicial abelian group, let Σ_n be the symmetric group on n letters, and let $B\Sigma_n$ be the classifying space. Our goal is to define a family of Segal operations

$$\theta^n : A(X) \longrightarrow A(X \times B\Sigma_n)$$

satisfying the properties listed in Theorems 1.1 and 1.3 below. We follow [Waldhausen 1982] in our naming convention, which can be explained as follows. Around 1972, Graeme Segal [1974b] defined a set of operations in stable homotopy theory $\theta^n : \pi_i^s(S^0) \to \pi_i^s((B\Sigma_n)_+)$, verified certain properties and used the information to give a proof of the Kahn–Priddy theorem. The key to the Kahn–Priddy proof is a certain relation satisfied by the composition of an operation followed by a transfer homomorphism.

Waldhausen [1982] adapted the construction in [Segal 1974b] to define operations $\theta^n : A(*) \to A(B\Sigma_n)$, and proved these new operations have properties precisely analogous to fundamental properties of Segal's original operations. Consequently, Waldhausen used the same notation and called the operations "Segal operations".

We obtain the following result.

We thank Friedhelm Waldhausen for suggesting this problem, and we thank the Institut für Mathematik of the University of Osnabrück for its hospitality and institute faculty members for their interest and encouragement.

MSC2010: primary 19D10; secondary 19D23.

Keywords: algebraic K-theory of topological spaces, Segal operations, operations.

Theorem 1.1. *Given a connected simplicial abelian group X, there are maps $\theta^n : A(X) \to A(X \times B\Sigma_n)$ which have the following properties.*

(1) *The map θ^1 is the identity.*

(2) *The combined map*

$$\theta = \prod_{n \geq 1} \theta^n : A(X) \to \{1\} \times \prod_{n \geq 1} A(X \times B\Sigma_n)$$

has the structure of an E_∞-map if the target is equipped with the E_∞-structure arising from certain pairings $A(X \times B\Sigma_m) \times A(X \times B\Sigma_n) \to A(X \times B\Sigma_{n+m})$ derived from the box-tensor operation of Definition 3.10.

The first property is a normalization condition, as satisfied by the constructions of Segal and Waldhausen. The second property implies that for every $j > 0$ the operations induce homomorphisms $\pi_j A(X) \to \{1\} \times \prod_{n \geq 1} \pi_j A(X \times B\Sigma_n)$ when the target is given a particular algebraic structure. A third algebraic property of Waldhausen's operations is recalled in Proposition 8.1. This third property is crucial in the applications made by Segal and Waldhausen. Our extended operations exhibit a more technical algebraic property stated in Theorem 1.3 and Theorem 8.12.

A large part of our work follows [Gunnarsson and Schwänzl 2002] in which many results are developed for quite general situations, satisfying certain technical conditions. Part of this paper verifies these conditions. In order to explain the necessity of this technical work, we repeat several definitions from [Gunnarsson and Schwänzl 2002] and quote many results.

In Section 2 the main results are Proposition 2.17 and Theorem 2.1. For the purposes of algebraic K-theory we verify exactness properties of certain constructions; to prepare for the E_∞-structure we verify coherence properties.

In Section 3 we recall the $G_\bullet$-construction for algebraic K-theory [Gunnarsson et al. 1992; Grayson 1989] and prepare the constructions underlying the definition of the operations in Definition 3.29.

In Section 4 we set up to apply general machinery, taking the first step toward a main result: For X a connected simplicial abelian group, there is an operation

$$\omega = \prod_{n \geq 1} \omega^n : A(X) \longrightarrow \{1\} \times \prod_{n \geq 1} A_{\Sigma_n,\{\text{all}\}}(X) \tag{1.2}$$

which is a map of E_∞-spaces with respect to specific algebraic structures described in Section 4. The target of ω is the algebraic K-theory of Σ_n-spaces retracting to X (with the trivial Σ_n-action) and relatively finite with respect to X. See Definition 3.5. In the first step the E_∞-structure is only visible if we restrict to spherical objects. The next section addresses this problem.

In Section 5 we study how the functors from Definition 3.29 interact with suspension operators. At the end of the section we complete the construction of the operation displayed in (1.2).

In Theorem 6.1, we split $A_{\Sigma_n,\{\text{all}\}}(X)$ as a product of the algebraic K-theory of other spaces, one of which is $A(X \times B\Sigma_n)$. This corresponds to the subcategory of Σ_n-spaces retracting to X (with the trivial Σ_n action), relatively finite with respect to X, and with Σ_n acting freely outside of X. We also obtain an expression for the composite functors "projecting to the free part"

$$\theta^n : A(X) \xrightarrow{\omega_n} A_{\Sigma_n,\{\text{all}\}}(X) \to A(X \times B\Sigma_n).$$

This expression is used in Section 8.

In Section 7 we establish equivalences among various models for equivariant K-theory and discuss the functors that induce transfer operations.

In Section 8 our main computational result evaluates the composition

$$A(X) \xrightarrow{\theta^n} A(X \times B\Sigma_n) \xrightarrow{\phi_n} A(X),$$

where ϕ_n is the transfer map.

Theorem 1.3 (Theorem 8.12). *Let X be a connected simplicial abelian group, thinking of the group operation as a multiplication, and let $\tau^n : X \to X$ be the homomorphism that raises elements to the n-th power. Then*

$$\phi_n \theta^n_* = (-1)^{n-1} \cdot (n-1)! \cdot \tau^n_* : \pi_j A(X) \to \pi_j A(X)$$

for $j > 0$.

We conclude this introduction with some comments on applications. First, we recall one formulation of the Kahn–Priddy theorem in stable homotopy theory. Let $Q(X) = \Omega^\infty S^\infty(X_+)$ denote unreduced stable homotopy theory and define reduced stable homotopy theory $\widetilde{Q}(X) = \text{fiber}(Q(X) \to Q(*))$, the homotopy fiber. For each n there is a transfer map $Q(B\Sigma_n) \to Q(E\Sigma_n) \simeq Q(*)$, and, by composition, there results a map $\widetilde{Q}(B\Sigma_n) \to Q(*)$. The formulation of the Kahn–Priddy theorem that we prefer is that the map

$$\pi_j(\widetilde{Q}B\Sigma_p)_{(p)} \to \pi_j(Q(*))_{(p)}$$

of homotopy groups localized at a prime p is surjective for $j > 0$.

Waldhausen's analogue of this result applies to the algebraic K-theory of the one-point space. For the formulation we let $A(X)$ denote the algebraic K-theory of the space X and let $\widetilde{A}(X) = \text{fiber}(A(X) \to A(*))$ be the algebraic K-theory of X reduced relative to a point. Manipulations formally similar to those above yield a map $\widetilde{A}(B\Sigma_n) \to A(*)$ and the analogue of the Kahn–Priddy theorem is that the induced map

$$\pi_j(\widetilde{A}(B\Sigma_p)_{(p)} \to \pi_j(A(*))_{(p)}$$

of homotopy groups localized at p is surjective for $j > 0$. In [Waldhausen 1987] these operations are further developed and used to prove that the third factor $\mu(X)$ in the splitting

$$A(X) \simeq Q(X_+) \times Wh^{\mathrm{Diff}}(X) \times \mu(X)$$

is contractible, yielding the final result $A(X) \simeq Q(X_+) \times Wh^{\mathrm{Diff}}(X)$. The significance of this fact is developed in [Waldhausen et al. 2013].

In our situation we fix as base space a connected simplicial abelian group X and define reduced algebraic K-theory relative to X as

$$\widetilde{A}(X \times B\Sigma_n \operatorname{rel} X) = \operatorname{fiber}(A(X \times B\Sigma_n) \to A(X)).$$

The inclusion of a point into $B\Sigma_n$ combined with the definition of the algebraic K-theory of $X \times B\Sigma_n$ reduced relative to X yields a splitting

$$\pi_j A(X \times B\Sigma_n) \cong \pi_j \widetilde{A}(X \times B\Sigma_n \operatorname{rel} X) \oplus \pi_j A(X) \tag{1.4}$$

for any $j \geq 0$. We have transfer maps $\phi_n : A(X \times B\Sigma_n) \to A(X \times E\Sigma_n) \simeq A(X)$ and a basic calculation in Lemma 7.8 that the composition

$$A(X) \to A(X \times B\Sigma_n) \xrightarrow{\phi_n} A(X)$$

is multiplication by $n! = |\Sigma_n|$, where the first map is induced by inclusion of a point into $B\Sigma_n$.

When we specialize n to a prime number p, we have the following observations. Make the following diagram of homotopy groups reduced mod p, where the splitting (1.4) appears as the middle column:

$$\begin{tikzcd}
 & \pi_j \widetilde{A}(X \times B\Sigma_p \operatorname{rel} X)/p\mathbb{Z} \arrow[d] \arrow[dr, "\phi_{p*}"] & \\
\pi_j A(X)/p\mathbb{Z} \arrow[r, "\theta_*^p"] & \pi_j A(X \times B\Sigma_p)/p\mathbb{Z} \arrow[r, "\phi_{p*}"] & \pi_j A(X)/p\mathbb{Z} \\
 & \pi_j A(X)/p\mathbb{Z} \arrow[u, "i_*"] \arrow[ur, "0"'] &
\end{tikzcd}$$

The diagonal arrow from the bottom row is multiplication by $p! = |\Sigma_p|$, which is 0 modulo p. Thus, in terms of the splitting of $\pi_j A(X \times B\Sigma_p)/p\mathbb{Z}$ given above, on the second component of the image of θ_*^p, the map ϕ_{p*} is zero. Applying Theorem 8.12, ϕ_{p*} applied to the first component $\pi_j \widetilde{A}(X \times B\Sigma_p \operatorname{rel} X)/p\mathbb{Z}$ of the splitting contains the image of $\phi_{p*}\theta_*^p = (-1)^{p-1} \cdot (p-1)! \cdot \tau_*^p$, where $\tau^p : X \to X$ raises elements to the p-th power. The numerical factors are invertible mod p so that

$$\phi_{p*}(\pi_j \widetilde{A}(X \times B\Sigma_p \operatorname{rel} X)/p\mathbb{Z}) \supset \operatorname{Image} \tau_*^p,$$

viewing τ_*^p as an endomorphism of $\pi_j A(X)/p\mathbb{Z}$.

From these calculations one extracts various additional observations. It may happen that the p-th power homomorphism τ^p is an isomorphism, as in the case when X is a connected simplicial abelian group, finite in each simplicial dimension and p is relatively prime to the order of X_n for each n. Then for $j > 0$,

$$\phi_{p*} : \pi_j \widetilde{A}(X \times B\Sigma_p \operatorname{rel} X)/p\mathbb{Z} \to \pi_j A(X)/p\mathbb{Z}$$

is surjective. The next input is the following theorem.

Theorem 1.5 [Betley 1986, Theorem I]. *If $\pi_1(X)$ is a finite group, and $\pi_i(X)$ is finitely generated for all $i \geq 2$, then $\pi_j(A(X))$ is finitely generated for all j.*

Then Nakayama's lemma applies as in [Waldhausen 1982] to lift the result on mod p homotopy to a result on p-localized homotopy. We obtain the following theorem of Kahn–Priddy type.

Theorem 1.6. *Let X be a connected simplicial abelian group, finite in each dimension, such that the order of X_n is prime to p. For $j > 0$ and p an odd prime, the transfer induces surjections*

$$\pi_j \widetilde{A}(X \times B\Sigma_p \operatorname{rel} X)_{(p)} \to \pi_j A(X)_{(p)}$$

on homotopy groups localized at p. □

In particular, take $X = BC_2 = \mathbb{R}P^\infty$ and p an odd prime. There are similar statements for all the lens spaces BC_q, q prime to p.

A very interesting case is $X = BC_\infty$, the classifying space of the infinite cyclic group C_∞. Of course $X \simeq S^1$, and there are splittings-up-to-homotopy of infinite loop spaces

$$A^{fd}(S^1) \simeq A^{fd}(*) \times BA^{fd}(*) \times N_- A^{fd}(*) \times N_+ A^{fd}(*)$$

and

$$A^{fd}(S^1 \times B\Sigma_n) \simeq A^{fd}(B\Sigma_n) \times BA^{fd}(B\Sigma_n) \times N_- A^{fd}(B\Sigma_n) \times N_+ A^{fd}(B\Sigma_n).$$

These are studied in [Klein and Williams 2008] and the first is examined in great detail in [Grunewald et al. 2008]. In future work we would like to understand the operations we have constructed in terms of these splittings. As a first step in this direction we have shown in Section 4 that the operations we construct are morphisms of infinite loop spaces. Should the θ operations be compatible with the splitting, one must then investigate whether or not the θ operations commute with the Frobenius and Verschiebung operations on the nil-terms defined in [Grunewald et al. 2008].

Our work also admits a generalization where X may be any connected space. This result is a total operation

$$\widetilde{\omega} : A(X) \to \{1\} \times \prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X^n),$$

about which we know little at this point. Our experiments have also lead to the observation that if G is a simplicial group, not necessarily abelian, whose realization is homotopy equivalent to a finite CW-complex then there is a product structure on $A(BG)$. This will be the subject of a later paper. Finally, reversing the progression from Segal's original idea to Waldhausen's generalization, we can develop operations $\theta^n : \pi_*^s(X_+) \to \pi_*^s(X \times B\Sigma_n)$, where X is again a connected simplicial abelian group.

2. The symmetric bimonoidal category of retractive spaces over a connected simplicial abelian group

The category $\mathcal{R}(X)$ is the category of retractive simplicial sets (Y, r, s) over the simplicial set X, where $r : Y \to X$ is a retraction, $s : X \to Y$ is a section for r and morphisms $(Y_1, r_1, s_1) \to (Y_2, r_2, s_2)$ respect all the data. A cofibration $(Y_1, r_1, s_1) \rightarrowtail (Y_2, r_2, s_2)$ in $\mathcal{R}_f(X)$ is a map such that $Y_1 \to Y_2$ is injective. A weak equivalence is a map $(Y_1, r_1, s_1) \to (Y_2, r_2, s_2)$ whose realization $|Y_1| \to |Y_2|$ is a homotopy equivalence of spaces. For algebraic K-theory we use the full subcategory $\mathcal{R}_f(X)$ of relatively finite retractive simplicial sets with cofibrations and weak equivalences. "Relatively finite" means that there are only finitely many nondegenerate simplices in $Y - X$. For background on the terminology, see [Waldhausen 1985, Section 1.1].

We aim to construct a total operation

$$\theta : A(X) \to \{1\} \times \prod_{n\geq 1} A(X \times B\Sigma_n)$$

for X a connected simplicial abelian group with multiplication $\mu : X \times X \to X$ and to prove the operation has an E_∞-structure. In order to achieve this, the elements from which the construction is developed must be of high quality. The necessary qualities are recorded in the first part of Theorem 2.1; the second part of the theorem records algebraic properties of the product operation $\wedge_\mu$. We discuss first the definition of the product operation, prove the second part of the theorem, and finish this section with the proof of the first part of the theorem.

Concerning the first part of the theorem, our constructions require a coherence result for diagrams involving sum and product operations, as provided by LaPlaza [1972, Proposition 10]. His coherence theorem takes as input the commutativity of 24 diagrams, reducible to a smaller, but still relatively large, subset [Laplaza

1972, pp. 40–41]. We will see that the coherence properties we need rest on the well-understood coherence properties of the one-point union and smash product of pointed sets. On the other hand, the second part of the theorem involves properties of the operations not reducible to dimensionwise considerations.

Theorem 2.1. *Let X be a connected simplicial abelian group.*

(1) *The triple $(\mathcal{R}(X), \vee_X, \wedge_\mu)$, where $\vee_X$ denotes the operation of union along the common subspace X and $\wedge_\mu$ denotes the pairing* (2.7), *is a symmetric bimonoidal category.*

(2) *The pairing $\wedge_\mu$ restricts to $\mathcal{R}_f(X)$, where it is biexact, meaning exact in each variable separately. Explicitly, the functors defined by $- \wedge_\mu Y$ and $Y \wedge_\mu -$ preserve cofibrations, pushouts along cofibrations, and weak equivalences.*

Our product operation $\wedge_\mu$ derives from an exterior smash product $\wedge_e$ of retractive simplicial sets, following the exterior smash product of retractive spaces as described in [May and Sigurdsson 2006]. Since we are working with simplicial sets, our version of the exterior smash product has a description in terms of operations on discrete sets, applied dimensionwise. See the discussion at the start of the proof of part one of Theorem 2.1.

Definition 2.2. Let (Y_i, r_i, s_i) be objects of $\mathcal{R}(X_i)$, for $i = 1, 2$. The exterior smash product of (Y_1, r_1, s_1) with (Y_2, r_2, s_2) is in $\mathcal{R}(X_1 \times X_2)$, and the underlying space $Y_1 \wedge_e Y_2$ completes the following square to a pushout:

$$\begin{tikzcd} Y_1 \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 \arrow[r, tail] \arrow[d, "r_1 \times r_2"'] & Y_1 \times Y_2 \arrow[d] \\ X_1 \times X_2 \arrow[r, tail, "s_1 \wedge_e s_2"] & Y_1 \wedge_e Y_2 \end{tikzcd} \tag{2.3}$$

The square displays the section $s_1 \wedge_e s_2$; the retraction $r_1 \wedge_e r_2$ arises from the universal property of the pushout.

Note that if both X_1 and X_2 are the one-point space, then this is the smash product in the category of pointed spaces. Extending this idea, if $x_1 : \{*\} \to X_1$ and $x_2 : \{*\} \to X_2$ are two maps of the one-point space into X_1 and X_2, and we take preimages $r_1^{-1}(x_1)$ and $r_2^{-1}(x_2)$, then these are pointed spaces, and there is an injective map $r_1^{-1}(x_1) \wedge r_2^{-1}(x_2) \to Y_1 \wedge_e Y_2$ over the point $(x_1, x_2) \in X_1 \times X_2$. This observation helps explain the "fiberwise smash product" terminology and indicates how the coherence issues for products may be resolved at the level of pointed sets. Examples 2.4 and 2.5 here play roles in the proof of part one of Theorem 2.1. Also, since we work with simplicial sets, underlying the symmetric monoidal structure $(\mathcal{R}(X), \vee_X, \wedge_\mu)$ is the symmetric monoidal structure on the category of sets.

Example 2.4. For any $Y_2 \in R(X_2)$, note that $X_1 \wedge_e Y_2 \cong X_1 \times X_2$, the "zero" object in $\mathcal{R}(X_1 \times X_2)$. Colloquially, the exterior smash product of a terminal object with any object yields a terminal object. Explicitly, a natural isomorphism

$$\lambda^*_{Y_2} : X_1 \wedge_e Y_2 \to X_1 \times X_2$$

arises from the following diagram by mapping the pushout of the top row to the pushout of the bottom row:

$$\begin{tikzcd}
X_1 \times X_2 \arrow[d, "="'] & X_1 \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 \arrow[l] \arrow[r, tail] \arrow[d, "\cong"'] & X_1 \times Y_2 \arrow[d, "="'] \\
X_1 \times X_2 & X_1 \times Y_2 \arrow[l] \arrow[r, tail] & X_1 \times Y_2
\end{tikzcd}$$

Example 2.5. The bifunctor $\mathcal{R}(*) \times \mathcal{R}(X) \to \mathcal{R}(X)$ given by $(Y_1, Y_2) \mapsto Y_1 \wedge_e Y_2$ defines an action of $\mathcal{R}(*)$ on $\mathcal{R}(X)$ after identifying $\{*\} \times X$ with X in the canonical way. This bifunctor also restricts to an action $\mathcal{R}_f(*) \times \mathcal{R}_f(X) \to \mathcal{R}_f(X)$.

This action has an identity element. Indeed, for $S^0 = \{*, *'\}$ in $\mathcal{R}_f(*)$, with r the constant map to the basepoint $*$, s the inclusion, and $Y \in \mathcal{R}(X)$, the function $S^0 \times Y \to Y$ defined by $(*, y) \mapsto sr(y)$ and $(*', y) \mapsto y$ induces an isomorphism $S^0 \wedge_e Y \xrightarrow{\cong} Y$ of retractive spaces over X. An inverse to this isomorphism is provided by $y \mapsto [(*', y)] \in S^0 \wedge_e Y$.

Definition 2.6. Let X be a space with a multiplication $\mu : X^2 \to X$. We operate on the category $\mathcal{R}(X)$, using the pairing

$$\wedge_\mu = \mu_* \circ \wedge_e : \mathcal{R}(X) \times \mathcal{R}(X) \xrightarrow{\wedge_e} \mathcal{R}(X \times X) \xrightarrow{\mu_*} \mathcal{R}(X), \tag{2.7}$$

where $\wedge_e$ is the external smash product pairing defined in (2.3) and μ_* is the functor induced by the multiplication $\mu : X^2 \to X$. Explicitly, $(Y_1, r_1, s_1) \wedge_\mu (Y_2, r_2, s_2)$ completes the following diagram to a pushout:

$$\begin{tikzcd}
Y_1 \times X \cup_{X \times X} X \times Y_2 \arrow[r, tail] \arrow[d, "\mu(r_1,\mathrm{id}) \cup \mu(\mathrm{id}, r_2)"'] & Y_1 \times Y_2 \arrow[d] \\
X \arrow[r, tail, "s"] & Y_1 \wedge_\mu Y_2
\end{tikzcd} \tag{2.8}$$

We use these notations to bring this section close to conformity with [Gunnarsson and Schwänzl 2002]. Perfect conformity is not possible, for we must use both the one-point union of pointed spaces $\vee$ and the union of two spaces along a common subspace X, denoted $\vee_X$. We also point out that the usual notation $\wedge$ has been used in [May and Sigurdsson 2006] for a product defined by restricting the external smash product of two spaces over X to the diagonal of $X \times X$.

The next lemma is used to develop properties of the smash products; the proof will be given after demonstrating applications in Propositions 2.13 and 2.17.

Lemma 2.9. *Let $\mathcal{C}$ be a category with cofibrations and let*

$$\begin{tikzcd} A_2 & B_2 \arrow[l] \arrow[r, tail] & C_2 \\ A_1 \arrow[u, tail] \arrow[d] & B_1 \arrow[l] \arrow[r, tail] \arrow[u, tail] \arrow[d] & C_1 \arrow[u, tail] \arrow[d] \\ A_0 & B_0 \arrow[l] \arrow[r, tail] & C_0 \end{tikzcd} \tag{2.10}$$

be a commutative diagram in which the canonical map from $B_2 \cup_{B_1} C_1$ to C_2 is a cofibration. Passing to pushouts by columns results in a diagram

$$A_0 \cup_{A_1} A_2 \leftarrow B_0 \cup_{B_1} B_2 \rightarrowtail C_0 \cup_{C_1} C_2 \tag{2.11}$$

in which the right-pointing arrow is a cofibration. The diagram

$$A_0 \cup_{B_0} C_0 \leftarrow A_1 \cup_{B_1} C_1 \rightarrowtail A_2 \cup_{B_2} C_2 \tag{2.12}$$

obtained by passing to pushouts by rows has a similar property.

Proposition 2.13. *The exterior smash product $\wedge_e$ is functorial for pairs of maps. That is, given $f_1 : X_1 \to X_1'$ and $f_2 : X_2 \to X_2'$, the diagram*

$$\begin{tikzcd} \mathcal{R}_f(X_1) \times \mathcal{R}_f(X_2) \arrow[r, "\wedge_e"] \arrow[d, "f_{1*} \times f_{2*}"'] & \mathcal{R}_f(X_1 \times X_2) \arrow[d, "(f_1 \times f_2)_*"'] \\ \mathcal{R}_f(X_1') \times \mathcal{R}_f(X_2') \arrow[r, "\wedge_e"] & \mathcal{R}_f(X_1' \times X_2') \end{tikzcd} \tag{2.14}$$

commutes up to natural isomorphism.

Proof. For the naturality property of the external smash product, consider the diagram

$$\begin{tikzcd} X_1 \times X_2 & Y_1 \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 \arrow[l] \arrow[r, tail] & Y_1 \times Y_2 \\ X_1 \times X_2 \arrow[u, tail] \arrow[d, "f_1 \times f_2"'] & X_1 \times X_2 \arrow[l] \arrow[r, tail] \arrow[u, tail] \arrow[d, "f_1 \times f_2"'] & X_1 \times X_2 \arrow[u, tail] \arrow[d, "f_1 \times f_2"'] \\ X_1' \times X_2' & X_1' \times X_2' \arrow[l] \arrow[r, tail] & X_1' \times X_2' \end{tikzcd} \tag{2.15}$$

which fulfills the hypotheses of Lemma 2.9. Computing the colimits of the columns in this diagram yields the diagram

$$X_1' \times X_2' \xleftarrow{r_1' \times r_2'} (f_{1*}Y_1) \times X_2' \cup_{X_1' \times X_2'} X_1' \times (f_{2*}Y_2) \rightarrowtail f_{1*}Y_1 \times f_{2*}Y_2,$$

whose pushout is by definition $f_{1*}Y_1 \wedge_e f_{2*}Y_2$.

On the other hand, computing the colimits of the rows in the diagram yields the diagram

$$X_1' \times X_2' \xleftarrow{f_1 \times f_2} X_1 \times X_2 \rightarrowtail Y_1 \wedge_e Y_2,$$

whose pushout is $(f_1 \times f_2)_*(Y_1 \wedge_e Y_2)$. Since both iterative procedures compute the colimit of diagram (2.15), they are canonically isomorphic:

$$f_{1*}Y_1 \wedge_e f_{2*}Y_2 \cong (f_1 \times f_2)_*(Y_1 \wedge_e Y_2). \qquad \square$$

As a consequence, we have the following result.

Proposition 2.16. *Let X be a monoid with unit. The action of $\mathcal{R}(*)$ on $\mathcal{R}(X)$ set up in Example 2.5 may be made internal to $\mathcal{R}(X)$. Diagrammatically,*

$$\begin{tikzcd} \mathcal{R}(*) \times \mathcal{R}(X) \arrow[r, "i_{e*} \times \mathrm{id}"] \arrow[dr, "\wedge_e"'] & \mathcal{R}(X) \times \mathcal{R}(X) \arrow[d, "\wedge_\mu"] \\ & \mathcal{R}(X) \end{tikzcd}$$

commutes up to natural isomorphism.

Proof. Let $i_e : \{*\} \to X$ be the inclusion of the one-point space to the identity element of the monoid X. The functor $i_{e*} : \mathcal{R}(*) \to \mathcal{R}(X)$ sends a pointed retractive space Y to $X \vee Y$, where the base point of Y is identified with the unit element of X. The new retraction collapses $Y \subset X \vee Y$ to the identity $\{e\}$ in X. We have the diagram

$$\begin{tikzcd} \mathcal{R}(*) \times \mathcal{R}(X) \arrow[r, "\wedge_e"] \arrow[d, "i_{e*} \times \mathrm{id}"'] & \mathcal{R}(\{*\} \times X) \arrow[d, "(i_e \times \mathrm{id})_*"'] \arrow[dr, "p_{2*}"] & \\ \mathcal{R}(X) \times \mathcal{R}(X) \arrow[r, "\wedge_e"] & \mathcal{R}(X \times X) \arrow[r, "\mu_*"] & \mathcal{R}(X) \end{tikzcd}$$

The left-hand square commutes by Proposition 2.13, and the right-hand triangle commutes because e is the monoid identity. The bottom row defines $\wedge_\mu$ and the trip across the top defines the action of $\mathcal{R}(*)$ on $\mathcal{R}(X)$. $\square$

For example, this result has the consequence that coherent associativity for $\wedge_\mu$ implies corresponding coherent associativity for the $\wedge_e$ action of $\mathcal{R}(*)$ on $\mathcal{R}(X)$.

Next, we record the biexactness property of the external smash product as defined in the statement of Theorem 2.1.

Proposition 2.17. *The external smash product functor*

$$\wedge_e : \mathcal{R}_f(X_1) \times \mathcal{R}_f(X_2) \to \mathcal{R}_f(X_1 \times X_2)$$

is biexact.

Remark 2.18. In the approach of [May and Sigurdsson 2006] the external smash product is shown to preserve all colimits by exhibiting a left adjoint functor. Their approach uses properties of convenient categories of topological spaces.

For our applications in algebraic K-theory it seems more reasonable to give arguments modeled on those of [Waldhausen 1985, Lemma 1.1.1], which serve to illuminate other constructions we make.

Proof of Proposition 2.17. For simplicial sets, cofibrations are precisely the injections. Given a pair of cofibrations

$$(W_1, r_1, s_1) \rightarrowtail (W_1', r_1', s_1') \quad \text{and} \quad (W_2, r_2, s_2) \rightarrowtail (W_2', r_2', s_2')$$

in $\mathcal{R}_f(X_1)$ and $\mathcal{R}_f(X_2)$, respectively, the maps of differences of simplicial sets $W_1 - X_1 \to W_1' - X_1$ and $W_2 - X_2 \to W_2' - X_2$ are injective maps of sets in each simplicial dimension. The product of these maps is also injective. Since $(W_1 \wedge_e W_2) - X_1 \times X_2 = (W_1 - X_1) \times (W_2 - X_2)$, it follows that $W_1 \wedge_e W_2 \rightarrowtail W_1' \wedge_e W_2'$ is also a cofibration. Finally, if $W_1 - X_1$ and $W_2 - X_2$ contain only finitely many nondegenerate simplices, then the same is true of their product. Thus, the pairing $\wedge_e$ restricts to a pairing of $\mathcal{R}_f(X_1) \times \mathcal{R}_f(X_2)$ to $\mathcal{R}_f(X_1 \times X_2)$.

To prove that the functor $Z \wedge_e (-) : R_f(X_2) \to R_f(X_1 \times X_2)$ preserves pushouts of cofibrations, start by considering the diagram

$$\begin{tikzcd}
X_1 \times X_2 & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 \arrow[l] \arrow[r, tail] & Z \times Y_2 \\
X_1 \times X_2 \arrow[u, equal] \arrow[d, equal] & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1 \arrow[l] \arrow[r, tail] \arrow[u, tail] \arrow[d] & Z \times Y_1 \arrow[u, tail] \arrow[d] \\
X_1 \times X_2 & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_0 \arrow[l] \arrow[r, tail] & Z \times Y_0
\end{tikzcd} \tag{2.19}$$

where the right-pointing arrows are induced from the retractions and the left-pointing arrows are induced by inclusions. We verify the cofibration hypothesis of Lemma 2.9 using the following diagram to analyze the upper right-hand corner of (2.19):

$$\begin{tikzcd}
& Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 \arrow[dl, tail] & (Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1) \cup_{X_1 \times Y_1} X_1 \times Y_2 \arrow[l, "\cong"'] & X_1 \times Y_2 \arrow[l] \\
Z \times Y_2 & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1 \arrow[u, tail] \arrow[d, tail] & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1 \arrow[l, tail, "="'] \arrow[u, tail] \arrow[d, tail] & X_1 \times Y_1 \arrow[l, tail] \arrow[u, tail] \arrow[d, tail] \\
& Z \times Y_1 \arrow[ul, tail] & Z \times Y_1 \arrow[l, tail, "="'] & Z \times Y_1 \arrow[l, tail, "="']
\end{tikzcd}$$

Pass to pushouts in the columns, apply the universal mapping properties of the pushouts, and use isomorphism (2.24) to simplify the pushout of the middle column to obtain the commuting diagram

$$\begin{tikzcd}
Z\times Y_1 \cup_{X_1\times Y_1} X_1\times Y_2 \arrow[d, "\cong"'] & & Z\times Y_1 \cup_{X_1\times Y_1} X_1\times Y_2 \arrow[ll, "\cong"'] \arrow[d, tail] \\
Z\times Y_1 \cup_{(Z\times X_2 \cup_{X_1\times X_2} X_1\times Y_1)} (Z\times X_2 \cup_{X_1\times X_2} X_1\times Y_2) \arrow[rr] & & Z\times Y_2
\end{tikzcd}$$

The space $Z\times Y_1 \cup_{X_1\times Y_1} X_1\times Y_2$ is a subspace of $Z\times Y_2$, so the downward arrow on the right is a cofibration. Since isomorphisms are cofibrations, it follows that the lower arrow is also a cofibration. Thus, we have verified the cofibration condition of Lemma 2.9 for (2.19).

We may now calculate the colimit of diagram (2.19) by two different iterative procedures. Computing the pushouts of the rows first and applying Lemma 2.9 gives a diagram

$$Z\wedge_e Y_2 \leftarrowtail Z\wedge_e Y_1 \to Z\wedge_e Y_0 \tag{2.20}$$

and calculating the pushouts of the columns first and applying Lemma 2.9 again gives a another diagram

$$X_1\times X_2 \leftarrow Z\times X_2 \cup_{X_1\times X_2} X_1\times (Y_0\cup_{Y_1} Y_2) \rightarrowtail Z\times (Y_0\cup_{Y_1} Y_2). \tag{2.21}$$

To see this formula for the middle object in (2.21), make the following considerations. We have the diagram

$$\begin{tikzcd}
Z_2\times X_2 & X_1\times X_2 \arrow[l] \arrow[r, tail] & X_1\times Y_2 \\
Z_2\times X_2 \arrow[u, tail, "="] \arrow[d, "="'] & X_1\times X_2 \arrow[l] \arrow[r, tail] \arrow[u, tail, "="] \arrow[d, "="'] & X_1\times Y_1 \arrow[u, tail] \arrow[d] \\
Z_2\times X_2 & X_1\times X_2 \arrow[l] \arrow[r, tail] & X_1\times Y_0
\end{tikzcd} \tag{2.22}$$

meeting the conditions of Lemma 2.9, whose colimit we also compute iteratively. Computing the pushouts of the rows first gives precisely the middle column in (2.19), whose pushout we are now evaluating. On the other hand, computing the pushouts along the columns first gives a diagram

$$Z_2\times X_2 \leftarrow X_1\times X_2 \rightarrowtail X_1\times (Y_2\cup_{Y_1} Y_0)$$

whose pushout is the middle term displayed in (2.21). As the iterated pushouts of (2.22) are isomorphic to the colimit of the entire diagram, the iterated pushouts are isomorphic. This justifies (2.21).

Completing the analysis of diagram (2.19), the pushouts of (2.20) and (2.21) are isomorphic, because they both represent the colimit of the original diagram (2.19). Interpreting this statement, we have the result that $Z\wedge_e -$ preserves pushouts of cofibrations.

Suppose $f_1 : Y_1 \to Y_1'$ and $f_2 : Y_2 \to Y_2'$ are weak equivalences in $\mathcal{R}_f(X_1)$ and $\mathcal{R}_f(X_2)$, respectively. That is, the geometric realizations $|f_1|$ and $|f_2|$ are homotopy equivalences. Then $|f_1| \times \mathrm{id}_{|X_2|}$ and $\mathrm{id}_{|X_1|} \times |f_2|$ are homotopy equivalences. By the ordinary gluing lemma for homotopy equivalences applied to the diagram

$$\begin{tikzcd}
{|Y_1| \times |X_2|} \arrow[d, "{|f_1| \times \mathrm{id}_{|X_2|}}"'] & {|X_1| \times |X_2|} \arrow[l, tail] \arrow[r, tail] \arrow[d, "{\mathrm{id}_{|X_1|} \times \mathrm{id}_{|X_2|}}"'] & {|X_1| \times |Y_2|} \arrow[d, "{\mathrm{id}_{|X_1|} \times |f_2|}"'] \\
{|Y_1'| \times |X_2|} & {|X_1| \times |X_2|} \arrow[l, tail] \arrow[r, tail] & {|X_1| \times |Y_2'|}
\end{tikzcd}$$

the central arrow in

$$\begin{tikzcd}
{|X_1| \times |X_2|} \arrow[d, "{\mathrm{id}_{|X_1|} \times \mathrm{id}_{|X_2|}}"'] & {|Y_1| \times |X_2| \cup_{|X_1| \times |X_2|} |X_1| \times |Y_2|} \arrow[l] \arrow[r, tail] \arrow[d, "\simeq"'] & {|Y_1| \times |Y_2|} \arrow[d, "{|f_1| \times |f_2|}"'] \\
{|X_1| \times |X_2|} & {|Y_1'| \times |X_2| \cup_{|X_1| \times |X_2|} |X_1| \times |Y_2'|} \arrow[l] \arrow[r, tail] & {|Y_1'| \times |Y_2'|}
\end{tikzcd}$$

is also a homotopy equivalence. Since the pushout of the last diagram is homeomorphic to $|Y_1 \wedge_e Y_2| \to |Y_1' \wedge_e Y_2'|$ ("colimits commute"), $Y_1 \wedge_e Y_2 \to Y_1' \wedge_e Y_2'$ is a weak equivalence. ☐

Remark 2.23. The external smash product also preserves many colimits. However, our applications principally involve the special colimits that are pushouts of cofibration squares.

Here is the postponed proof of Lemma 2.9.

Proof of Lemma 2.9. We make frequent use of the isomorphism

$$(A \cup_B C) \cup_C D \cong A \cup_B D. \tag{2.24}$$

The canonical arrow $B_2 \cup_{B_1} B_0 \to C_2 \cup_{C_1} C_0$ factors into the composition of canonical arrows induced by passing to pushouts of the columns in the map of diagrams

$$\begin{tikzcd}
B_2 \arrow[r, "="] & B_2 \arrow[r, tail] & C_2 \\
B_1 \arrow[u, tail] \arrow[r, "="] \arrow[d] & B_1 \arrow[u, tail] \arrow[r, tail] \arrow[d] & C_1 \arrow[u, tail] \arrow[d] \\
B_0 \arrow[r, tail] & C_0 \arrow[r, "="] & C_0
\end{tikzcd}$$

We show each arrow in the factorization is a cofibration. The first arrow in the factorization appears as the lower row in the completed pushout diagram

$$\begin{tikzcd}
B_0 \arrow[r, tail] \arrow[d] & C_0 \arrow[d] & \\
B_2 \cup_{B_1} B_0 \arrow[r, tail] & (B_2 \cup_{B_1} B_0) \cup_{B_0} C_0 \arrow[r, "\cong"] & B_2 \cup_{B_1} C_0
\end{tikzcd}$$

augmented by an isomorphism, so the first arrow is a cofibration, as claimed. From the hypothesis on the canonical map from $B_2 \cup_{B_1} C_1$ to C_2, the upper arrow in the next diagram is a cofibration, so the lower arrow in the completed pushout diagram is as well:

$$\begin{tikzcd}
 & B_2 \cup_{B_1} C_1 \arrow[r, tail] \arrow[d] & C_2 \arrow[d] & \\
B_2 \cup_{B_1} C_0 & (B_2 \cup_{B_1} C_1) \cup_{C_1} C_0 \arrow[l, "\cong"'] \arrow[r, tail] & C_2 \cup_{B_2 \cup_{B_1} C_1} ((B_2 \cup_{B_1} C_1) \cup_{C_1} C_0) \arrow[r, "\cong"] & C_2 \cup_{C_1} C_0
\end{tikzcd}$$

Augmenting the completed pushout diagram by the two isomorphisms, the second arrow $B_2 \cup_{B_1} C_0 \to C_2 \cup_{C_1} C_0$ in the factorization is also a cofibration. Then the composition $B_2 \cup_{B_1} B_0 \rightarrowtail B_2 \cup_{B_0} C_0 \rightarrowtail C_2 \cup_{C_1} C_0$ is a cofibration and this arrow is isomorphic to the arrow in diagram (2.11).

To obtain the result for the row-wise pushouts from the result for columnwise pushouts, observe that the properties of the arrows in the diagram are symmetric with respect to reflection in the diagonal $A_0 B_1 C_2$. Therefore, it suffices to reflect the diagram in this diagonal and apply the columnwise result. □

Proof of the second part of Theorem 2.1. Since the functor

$$\mu_* : \mathcal{R}_f(X \times X) \to \mathcal{R}_f(X)$$

is exact [Waldhausen 1985, Lemma 2.1.6], and we have seen that $\wedge_e$ is biexact in Proposition 2.17, the composite $\wedge_\mu = \mu_* \circ \wedge_e$ is biexact. □

Now we take up coherence properties.

Proof of the first part of Theorem 2.1. It is well-known that the disjoint union of sets and the one-point union $\vee$ of pointed sets are categorical sum operations, so that all coherence conditions for these operations are automatically met. For the category of sets containing a fixed set S the union $\vee_S$ of two sets along the common subset is also the categorical sum, so $\vee_S$ fulfills all coherence conditions. Concerning products, the cartesian product of sets and the smash product of pointed sets are operations also meeting coherence conditions. When these operations of sum and product are considered together, they are related by distributivity isomorphisms, and the combined systems exhibit the coherence properties discussed in [Laplaza 1972]. It is possible to develop the coherence properties we need for operations on retractive spaces from these basic elements by developing the operation $\vee_X$ dimensionwise and pointwise over X from one-point union and the operation $\wedge_e$ dimensionwise and pointwise over $X_1 \times X_2$ from the smash product of pointed sets. Compare the remark following Definition 2.2. We take a different approach here.

For the sum $\vee_X$, we need a slight extension of the union of sets along a common subset to cover the case of the disjoint union of two simplicial sets along a common simplicial subset. Let $\mathcal{T}$ be the category of triples $(T, r : T \to S, s : S \to T)$, where

S and T are sets and the functions satisfy $r \circ s = \mathrm{id}_S$. Occasionally, it is convenient to view S as a subset of T. A morphism

$$(f, f') : (T_1, r_1 : T_1 \to S_1, s_1 : S_1 \to T_1) \to (T_0, r_0 : T_0 \to S_0, s_0 : S_0 \to T_0)$$

is a pair of maps $f : T_1 \to T_0$ and $f' : S_1 \to S_0$ such that $s_0 f' = f s_1$ and $r_0 f = f' r_1$. An object (Y, r, s) of $\mathcal{R}(X)$ can be viewed as a functor $\Delta^{\mathrm{op}} \to \mathcal{T}$, and conversely. There is a functor $u : \mathcal{T} \to \mathbf{Set}$ that selects the subset S and morphisms $f' : S_1 \to S_0$. On the pullback category

$$\begin{tikzcd}
\mathcal{T} \times_{\mathbf{Set}} \mathcal{T} \arrow[r] \arrow[d] & \mathcal{T} \times \mathcal{T} \arrow[d, "u \times u"'] \\
\mathbf{Set} \arrow[r, "\Delta"] & \mathbf{Set} \times \mathbf{Set}
\end{tikzcd}$$

define the operation $(T_1, r_1 : T_1 \to S, s_1 : S \to T_1) \vee_S (T_2, r_2 : T_2 \to S, s_2 : S \to T_2)$, abbreviated $(T_1, r_1, s_1) \vee_S (T_2, r_2, s_2)$, or even $T_1 \vee_S T_2$. Set

$$T_1 \vee_S T_2 = T_1 \amalg T_2 \,/ \sim,$$

where $\sim$ is the equivalence relation generated by setting $s_1(x) \sim s_2(x)$ for $x \in S$. Set $i_j : T_j \to T_1 \vee_S T_2$ to be the inclusion $T_j \to T_1 \amalg T_2$ followed by the quotient map to $T_1 \vee_s T_2$. For the rest of the structure, set

$$r : T_1 \vee_S T_2 \to S$$

to be the unique function satisfying $r i_j = r_j$, for $j = 1, 2$, and let

$$s : S \to T_1 \vee_S T_2$$

satisfy $s(x) = i_1 s_1(x) = i_2 s_2(x)$ for $x \in S$. Define

$$(i_1, i_1' = \mathrm{id}) : (T_1, s_1, r_1) \to (T_1 \vee_S T_2, r, s)$$

to obtain a morphism in $\mathcal{T}$. The identities $r i_1 = i_1' r_1$ and $s i_1' = i_1 s_1$ are satisfied by definition and by the condition $r_1 s_1 = \mathrm{id}$. Define

$$(i_2, i_2') : (T_2, s_2, r_2) \to (T_1 \vee_S T_2, r, s).$$

similarly. If (T', r', s') is another object of $\mathcal{T}$, let $(f_i, f_i') : (T_i, r_i, s_i) \to (T', r', s')$ be a morphism in $\mathcal{T} \times_{\mathbf{Set}} \mathcal{T}$ for $i = 1, 2$. This just means that $f_1' = f_2' : S \to S'$. Then the categorical sum properties of the disjoint union on the category **Set** and the quotient construction deliver a unique morphism

$$(h, h') : (T_1 \vee_S T_2, r', s') \to (T', r', s')$$

such that $(h, h') \circ (i_1, i'_1) = (f_1, f'_1)$ and $(h, h') \circ (i_2, i'_2) = (f_2, f'_2)$. When the base set is fixed, we obtain a categorical sum; in general, when the base set varies, we obtain a (partially defined) categorical sum on $\mathcal{T}$.

We have observed that an object of the category $\mathcal{R}(X)$ is a simplicial object in the category $\mathcal{T}$, that is, a functor $\Delta^{\mathrm{op}} \to \mathcal{T}$. A pair of objects (Y_1, r_1, s_1) and (Y_2, r_2, s_2) in $\mathcal{R}(X)$ defines a functor $\Delta^{\mathrm{op}} \to \mathcal{T} \times_{\mathbf{Set}} \mathcal{T}$. We obtain the operation $(Y_1, r_1, s_1) \vee_X (Y_2, r_2, s_2)$ based on the dimensionwise operation $(Y_1)_p \vee_{X_p} (Y_2)_p$. This makes $\vee_X$ a categorical sum in $\mathcal{R}(X)$, with unit (zero element, thinking additively) the space X. The commutativity isomorphisms γ', associativity isomorphisms α', and left and right unit isomorphisms λ' and ρ' are straightforward consequences of the analogous properties of the disjoint union operation on sets. Essentially, all the basic properties required for coherence of the sum operation $\vee_X$ are automatically fulfilled. That $\vee_X$ is the categorical sum simplifies almost all coherence considerations involving diagrams involving both $\vee_X$ and $\wedge_\mu$.

To complete the input for LaPlaza's coherence result we need to identify in $\mathcal{R}(X)$ an additive identity, a multiplicative zero element, a multiplicative identity, commutativity and associativity isomorphisms for $\wedge_\mu$, and, finally, distributivity isomorphisms.

Clearly $(X, \mathrm{id}, \mathrm{id})$ is the identity for $\vee_X$. Example 2.4 implies that $(X, \mathrm{id}, \mathrm{id})$ is a zero object from the left and the right for $\wedge_\mu$, in the sense that there are natural isomorphisms

$$\lambda^*_Y : X \wedge_\mu Y \to X \quad \text{and} \quad \rho^*_Y : Y \wedge_\mu X \to X.$$

Example 2.5 combined with Proposition 2.16 delivers the fact that $i_{e*}(S^0) = X \vee S^0$, where the base point of S^0 is identified with the multiplicative identity of X and the retraction collapses S^0 to the identity of X, is a multiplicative identity in the sense that there are natural isomorphisms

$$\lambda_Y : (X \vee S^0) \wedge_\mu Y \to Y \quad \text{and} \quad \rho_Y : Y \wedge_\mu (X \vee S^0) \to Y.$$

For commutativity of the product $\wedge_\mu = \mu_* \circ \wedge_e$, we have the following considerations. Use commutativity for cartesian products and apply the definitions from (2.8) of the internal smash product to obtain the following diagram:

$$\begin{tikzcd} X \arrow[d, equal] & X \times X \arrow[l, "\mu"'] \arrow[d, "\gamma"'] & Y_1 \times X \cup_{X \times X} X \times Y_2 \arrow[l] \arrow[r, tail] \arrow[d, "\gamma"'] & Y_1 \times Y_2 \arrow[d, "\gamma"'] \\ X & X \times X \arrow[l, "\mu"'] & Y_2 \times X \cup_{X \times X} X \times Y_1 \arrow[l] \arrow[r, tail] & Y_2 \times Y_1 \end{tikzcd} \tag{2.25}$$

In the diagram the arrows labeled γ are the isomorphisms switching the factors in the cartesian products. Note that

$$r_1 \wedge_\mu r_2 = \mu_*(r_1 \wedge_e r_2) = r_1 \cdot r_2 = r_2 \cdot r_1 = \mu_*(r_2 \wedge_e r_1) = r_2 \wedge_\mu r_1,$$

since X is abelian. Passage to pushouts yields an isomorphism

$$\gamma_{Y_1,Y_2} : (Y_1 \wedge_\mu Y_2, r_1 \wedge_\mu r_2, s_1 \wedge_\mu s_2) \xrightarrow{\cong} (Y_2 \wedge_\mu Y_1, r_2 \wedge_\mu r_1, s_2 \wedge_\mu s_1).$$

It is easily seen that $\gamma_{Y_2,Y_1}\gamma_{Y_1,Y_2} = \mathrm{id}$ holds (often written "$\gamma^2 = \mathrm{id}$" and called the inverse law), and that the left and right unit laws are compatible. These facts are recorded in the following commuting diagrams:

$$\begin{tikzcd}
 & Y_2 \wedge_\mu Y_1 \arrow[dr, "\gamma_{Y_2,Y_1}"] & \\
Y_1 \wedge_\mu Y_2 \arrow[ur, "\gamma_{Y_1,Y_2}"] \arrow[rr, equal] & & Y_1 \wedge_\mu Y_2
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \wedge_\mu (X \vee S^0) \arrow[rr, "\gamma_{Y,X\vee S^0}"] \arrow[dr, "\rho_Y"'] & & (X \vee S^0) \wedge_\mu Y \arrow[dl, "\lambda_Y"] \\
 & Y &
\end{tikzcd}$$

Consider now associativity, for which we use the diagram

$$\begin{tikzcd}
 & \begin{array}{c}((Y_1 \times Y_2) \times X) \cup ((Y_1 \times X) \times Y_3) \\ \cup((X \times Y_2) \times Y_3)\end{array} \arrow[dl, "\begin{array}{c}\mu(\mu(r_1,r_2),\mathrm{id})\cup\mu(\mu(r_1,\mathrm{id}),r_3)\\ \cup\mu(\mu(\mathrm{id},r_2),r_3)\end{array}"'] \arrow[r, tail] & (Y_1 \times Y_2) \times Y_3 \\
X & & \\
 & \begin{array}{c}((Y_1 \times X) \times X) \\ \cup((X \times Y_2) \times X)\end{array} \arrow[uu, tail] \arrow[dl, "\begin{array}{c}\mu(\mu(r_1,\mathrm{id}),\mathrm{id})\\ \cup\mu(\mu(\mathrm{id},r_2),\mathrm{id})\end{array}"'] \arrow[r, tail, "="] \arrow[dd, "\begin{array}{c}(\mu(r_1,\mathrm{id}),\mathrm{id})\\ \cup(\mu(\mathrm{id},r_2),\mathrm{id})\end{array}"] & \begin{array}{c}((Y_1 \times X) \times X) \\ \cup((X \times Y_2) \times X)\end{array} \arrow[uu, tail] \arrow[dd, "\begin{array}{c}(\mu(r_1,\mathrm{id}),\mathrm{id})\\ \cup(\mu(\mathrm{id},r_2),\mathrm{id})\end{array}"] \\
X \arrow[uu, "="] \arrow[dd, "="] & & \\
 & X \times X \arrow[dl, "\mu"'] \arrow[r, tail, "="] & X \times X \\
X & &
\end{tikzcd} \tag{2.26}$$

The point is that the associativity for $\wedge_\mu$ rests on associativity for $\times$, $\cup$, and associativity of the multiplication μ on X. By passage to colimits we obtain associativity for $\wedge_\mu$. For the usual smash product, associativity for cartesian products passes to associativity for smash products; our argument is similarly structured.

The first step is to obtain an expression for $(Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3$ that involves only cartesian products and colimits. Diagram (2.26) fulfills the hypotheses of Lemma 2.9, so we may calculate the colimit iteratively in two ways. Taking the colimit along the columns produces the diagram

$$X \xleftarrow{\mu(r_{12},\mathrm{id})\cup\mu(\mathrm{id},r_3)} ((Y_1 \wedge_\mu Y_2) \times X) \cup (X \times Y_3) \rightarrowtail (Y_1 \wedge_\mu Y_2) \times Y_3$$

whose colimit is by definition $(Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3$. On the other hand, computing the colimit along the rows produces the diagram

$$\begin{tikzcd}
 & \begin{array}{c}((Y_1 \times Y_2) \times X) \cup ((Y_1 \times X) \times Y_3) \\ \cup ((X \times Y_2) \times Y_3)\end{array} \arrow[r, tail] \arrow[dl, "\begin{array}{c}\mu(\mu(r_1,r_2),\mathrm{id})\cup\mu(\mu(r_1,\mathrm{id}),r_3) \\ \cup\mu(\mu(\mathrm{id},r_2),r_3)\end{array}"'] & (Y_1 \times Y_2) \times Y_3 \\
X & &
\end{tikzcd}$$

a copy of the top row in (2.26). Therefore, the colimit, or pushout, of this diagram is another representation of $(Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3$, and we record the completed diagram

$$\begin{tikzcd}
\begin{array}{c}((Y_1 \times Y_2) \times X) \cup ((Y_1 \times X) \times Y_3) \\ \cup ((X \times Y_2) \times Y_3)\end{array} \arrow[r, tail] \arrow[d, "\mu(\mu(r_1,r_2),\mathrm{id})\cup\mu(\mu(r_1,\mathrm{id}),r_3)\cup\mu(\mu(\mathrm{id},r_2),r_3)"'] & (Y_1 \times Y_2) \times Y_3 \arrow[d] \\
X \arrow[r, tail] & (Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3
\end{tikzcd} \tag{2.27}$$

as a preferred alternative representation of $(Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3$. Starting from a diagram similar to (2.26), but with parentheses shifted to the right, there is another completed pushout diagram

$$\begin{tikzcd}
\begin{array}{c}(Y_1 \times (Y_2 \times X)) \cup (Y_1 \times (X \times Y_3)) \\ \cup (X \times (Y_2 \times Y_3))\end{array} \arrow[r, tail] \arrow[d, "\mu(r_1,\mu(r_2,\mathrm{id}))\cup\mu(r_1,\mu(\mathrm{id},r_3))\cup\mu(\mathrm{id},\mu(r_2,r_3))"'] & Y_1 \times (Y_2 \times Y_3) \arrow[d] \\
X \arrow[r, tail] & Y_1 \wedge_\mu (Y_2 \wedge_\mu Y_3)
\end{tikzcd} \tag{2.28}$$

representing $Y_1 \wedge_\mu (Y_2 \wedge_\mu Y_3)$. Consequently, the associativity isomorphisms

$$\alpha_{Y_1,Y_2,Y_3} : Y_1 \times (Y_2 \times Y_3) \to (Y_1 \times Y_2) \times Y_3,$$
$$\alpha_{Y_1,Y_2,X} : Y_1 \times (Y_2 \times X) \to (Y_1 \times Y_2) \times X,$$

and so on, induce an isomorphism of diagram (2.28) with diagram (2.27) and an associativity isomorphism

$$\alpha_{Y_1,Y_2,Y_3} : Y_1 \wedge_\mu (Y_2 \wedge_\mu Y_3) \to (Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3. \tag{2.29}$$

In Laplaza's framework [1972], left distributivity of the product over the sum operation is encoded by a monomorphism

$$\delta_{Y_0,Y_1,Y_2} : Y_0 \wedge_\mu (Y_1 \vee_X Y_2) \to (Y_0 \wedge_\mu Y_1) \vee_X (Y_0 \wedge_\mu Y_2).$$

The fact that $\vee_X$ is a categorical sum enables us to construct an isomorphism $\delta^{-1}_{Y_0,Y_1,Y_2} : (Y_0 \wedge_\mu Y_1) \vee_X (Y_0 \wedge_\mu Y_2) \to Y_0 \wedge_\mu (Y_1 \vee_X Y_2)$ quite easily as follows. Applying the functor $Y_0 \wedge_\mu -$ to the sum diagram $Y_1 \to Y_1 \vee_X Y_2 \leftarrow Y_2$ provides a diagram $Y_0 \wedge_\mu Y_1 \to Y_0 \wedge_\mu (Y_1 \vee_X Y_2) \leftarrow Y_0 \wedge_\mu Y_2$. Since $\vee_X$ is a categorical sum, there results a map $(Y_0 \wedge_\mu Y_1) \vee_X (Y_0 \wedge_\mu Y_2) \to Y_0 \wedge_\mu (Y_1 \vee_X Y_2)$. To check that this map is an isomorphism observe that in a simplicial dimension p the p-simplices outside of X in the domain are $(Y_0 - X)_p \times (Y_1 - X)_p \amalg (Y_0 - X)_p \times (Y_2 - X)_p$, the p-simplices outside of X in the target are $(Y_0 - X)_p \times ((Y_1 - X)_p \amalg (Y_2 - X)_p)$, and the induced map is a one-to-one correspondence. Thus, we obtain the isomorphism $\delta^{-1}_{Y_0,Y_1,Y_2} : (Y_0 \wedge_\mu Y_1) \vee_X (Y_0 \wedge_\mu Y_2) \to Y_0 \wedge_\mu (Y_1 \vee_X Y_2)$, whose inverse

$$\delta_{Y_0,Y_1,Y_2} : Y_0 \wedge_\mu (Y_1 \vee_X Y_2) \cong (Y_0 \wedge_\mu Y_1) \vee_X (Y_0 \wedge_\mu Y_2) \tag{2.30}$$

can be shown to meet LaPlaza's conditions. Similarly, we obtain an isomorphism

$$\delta^{\#}_{Y_0,Y_1,Y_2} : (Y_0 \vee_X Y_1) \wedge_\mu Y_2 \cong (Y_0 \wedge_\mu Y_2) \vee_X (Y_1 \wedge_\mu Y_2). \tag{2.31}$$

This concludes the catalog of basic inputs for LaPlaza's theorem.

Given the basic inputs, the next step is to establish the commutativity of certain diagrams, twenty-four in number. Because $\vee_X$ is a categorical sum and $\wedge_\mu$ is biexact, preserving sums, checking the commutativity of seventeen of the diagrams is routine. The other seven diagrams involve the multiplicative or additive neutral objects or the multiplicative zero object and are straightforward to verify. LaPlaza's main theorem applies and "all diagrams that should commute do, in fact, commute". These remarks complete the proof of part one of Theorem 2.1. □

3. Defining the operations

The ingredients for the operations take values in categories of retractive spaces on which groups are acting. We first establish language and notation following [Gunnarsson and Schwänzl 2002, Definitions 5.1–5.4] for the following definitions.

Definition 3.1. A set $\mathcal{F}$ of subgroups of Σ_n is called a family of subgroups if it contains at most one member from each conjugacy class of subgroups.

Definition 3.2. For a finite group G, a G-simplicial set Y has orbit types in a family $\mathcal{F}$ relative to another G-simplicial set W if Y may be obtained from W by direct limit and by formation of pushouts of diagrams of the form

$$Y' \leftarrow \partial\Delta^n \times (G/H) \rightarrowtail \Delta^n \times (G/H), \tag{3.3}$$

where Δ^n is the standard simplicial n-simplex, $\partial\Delta^n$ is the simplicial boundary, and $H \in \mathcal{F}$.

Definition 3.4. For a Σ_n-simplicial set W, let $\mathcal{R}(W, \Sigma_n, \mathcal{F})$ denote the category whose objects are the triples (Y, r, s), where Y is a Σ_n-simplicial set with orbit types in $\mathcal{F}$ relative to a Σ_n-section $s : W \to Y$. The map $r : Y \to W$ is a Σ_n-retraction of Y to W, that is, $r \circ s = \mathrm{id}_W$. Morphisms are Σ_n-equivariant maps commuting with the retractions and sections.

Definition 3.5. Let $\mathcal{R}_f(W, \Sigma_n, \mathcal{F})$ denote the full subcategory of $\mathcal{R}(W, \Sigma_n, \mathcal{F})$ whose objects are the triples (Y, r, s) such that Y is built from W by formation of finitely many pushouts of the form of (3.3). The category $\mathcal{R}_f(W, \Sigma_n, \mathcal{F})$ is also equipped with cofibrations and weak equivalences. A cofibration $(W_1, r_1, s_1) \rightarrowtail (W_2, r_2, s_2)$ is an injective Σ_n-map and a weak equivalence $(Y_1, r_1, s_1) \to (Y_2, r_2, s_2)$ is a morphism for which the geometric realization of the underlying map $Y_1 \to Y_2$ is a Σ_n-equivariant homotopy equivalence.

For X a connected simplicial abelian group on which Σ_n acts trivially, we need the categories $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ of retractive left Σ_n-spaces $\widetilde{Y}$ over X which are finite relative to X. In principle, we may also allow X to be a connected commutative simplicial monoid with unit. We write $\Omega|wS_\bullet\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})| = A_{\Sigma_n,\{\text{all}\}}(X)$. The category of retractive left Σ_n-spaces on which Σ_n acts with trivial isotropy outside of X is then $\mathcal{R}_f(X, \Sigma_n, \{e\})$. In other words, the Σ_n-action on simplices outside of X is free on those simplices. Later, we abbreviate $\mathcal{R}_f(X, \Sigma_n, \{e\}) = \mathcal{R}_f(X, \Sigma_n)$. In Lemma 7.3 we justify the notation $\Omega|wS_\bullet\mathcal{R}_f(X, \Sigma_n, \{e\})| = A(X \times B\Sigma_n)$.

There are two constructions underlying our approach to the Segal operations. First is a family of biexact functors

$$\boxtimes_{k,\ell} : \mathcal{R}_f(X, \Sigma_k, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_{k+\ell}, \{\text{all}\})$$

defined for $k, \ell \geq 0$, called box-tensor operations (Definition 3.10). Second is a family of functors

$$\diamond_{n,k} : \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})^{[k]} \to \mathcal{R}_f(X, \Sigma_{kn}, \{\text{all}\}),$$

called diamond operations (Definition 3.16). Here $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})^{[k]}$ is the category of filtered objects

$$Y_1 \rightarrowtail Y_2 \rightarrowtail \cdots \rightarrowtail Y_k$$

with Y_i in $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ and natural transformations of such sequences.

First we set up the box-tensor operation. For a connected simplicial abelian group X, let $n = k + \ell$ and define an induction functor

$$\operatorname{Ind}_{\Sigma_k \times \Sigma_\ell}^{\Sigma_n} : \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}). \tag{3.6}$$

Let $\boldsymbol{n}$ be a finite set of cardinality n (for example, the standard example), let $\boldsymbol{k} \cup \boldsymbol{l}$ be the disjoint union of finite sets of cardinality k and l, respectively, and let $\operatorname{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})$ be the set of isomorphisms from $\boldsymbol{n}$ to the disjoint union. Let $\operatorname{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ = \operatorname{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l}) \cup \{*\}$ be viewed as an object of $\mathcal{R}_f(*)$, with the obvious section and with the retraction the constant map to $\{*\}$. The group Σ_n acts from the left on $\operatorname{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+$ by fixing the basepoint and by the rule $\sigma \cdot f = f \circ \sigma^{-1}$ for $\sigma \in \Sigma_n$ and $f : \boldsymbol{n} \to \boldsymbol{k} \cup \boldsymbol{l}$. Normally $\Sigma_k \times \Sigma_\ell$ also acts from the left by post-composition, but we find it convenient to use the right action defined by

$f \cdot (\sigma_1, \sigma_2) = (\sigma_1^{-1}, \sigma_2^{-1}) \circ f$. For $(Y, r, s) \in \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\})$ we unwind the defining pushout square

$$\begin{array}{ccc} (\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \times X) \cup_{*\times X} (* \times Y) & \rightarrowtail & \mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \times Y \\ \downarrow & & \downarrow \\ * \times X & \rightarrowtail & \mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e Y \end{array} \tag{3.7}$$

to find that the exterior smash product $\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e Y$ amounts to $\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})$-copies of Y, pasted together along the common subspace X. The retraction

$$r' : \mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e Y \to X$$

given by $r'([f, y]) = r(y)$ is Σ_n-equivariant when Σ_n acts trivially on X. We may also apply the principle of Proposition 2.16 to re-express the exterior smash product as an internal smash product and write

$$\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e Y \cong (X \vee \mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+) \wedge_\mu Y.$$

Define $\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e^{\Sigma_k \times \Sigma_\ell} Y$ to be the quotient space of $\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e Y$ by the equivalence relation generated by $[f \cdot (\sigma_1, \sigma_2), y] \sim [f, (\sigma_1, \sigma_2) \cdot y]$. The left action of Σ_n passes to the quotient, and, since the action of Σ_n on X is trivial, the retraction r' defined above also passes to the quotient, as does the section. Thus, we obtain the necessary structure maps

$$X \rightarrowtail \mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e^{\Sigma_k \times \Sigma_\ell} Y \xrightarrow{r} X.$$

This completes the definition of the induction functor

$$\mathrm{Ind}_{\Sigma_k \times \Sigma_\ell}^{\Sigma_n} : \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}). \tag{3.8}$$

Next we need an elementary pairing functor

$$\mathcal{R}_f(X, \Sigma_k, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\}). \tag{3.9}$$

The pairing sends $((Y_1, r_1, s_1), (Y_2, r_2, s_2))$ to $(Y_1, r_1, s_1) \wedge_\mu (Y_2, r_2, s_2)$.

Definition 3.10. Define the box-tensor operations by composing the pairing functor (3.9) with the induction functor (3.8):

$$\begin{aligned} \boxtimes_{k,\ell} : \mathcal{R}_f(X, \Sigma_k, \{\text{all}\}) &\times \mathcal{R}_f(X, \Sigma_\ell, \{\text{all}\}) \\ &\xrightarrow{\wedge_\mu} \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\}) \xrightarrow{\mathrm{Ind}_{\Sigma_k \times \Sigma_\ell}^{\Sigma_n}} \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \end{aligned} \tag{3.11}$$

Proposition 3.12. *The box-tensor operations are associative up to natural isomorphism.*

Proof. The associativity of the box-tensor operations is a consequence of the symmetric monoidal structure on $\mathcal{R}_f(X)$ associated with $\wedge_\mu$, along with properties of the cartesian product of groups and disjoint union of sets. Abbreviating $\mathrm{id}_{\mathcal{R}_f(X,\Sigma_{k_1},\{\mathrm{all}\})}$ by id_1 and $\mathrm{id}_{\mathcal{R}_f(X,\Sigma_{k_3},\{\mathrm{all}\})}$ by id_3, the assertion in detail is that the diagram

$$\begin{tikzcd}[column sep=tiny]
& \mathcal{R}_f(X,\Sigma_{k_1},\{\mathrm{all}\})\times\mathcal{R}_f(X,\Sigma_{k_2},\{\mathrm{all}\})\times\mathcal{R}_f(X,\Sigma_{k_3},\{\mathrm{all}\}) \arrow[dl, "\boxtimes_{k_1,k_2}\times\mathrm{id}_3"'] \arrow[dr, "\mathrm{id}_1\times\boxtimes_{k_2,k_3}"] & \\
\mathcal{R}_f(X,\Sigma_{k_1+k_2},\{\mathrm{all}\})\times\mathcal{R}_f(X,\Sigma_{k_3},\{\mathrm{all}\}) \arrow[dr, "\boxtimes_{k_1+k_2,k_3}"'] & & \mathcal{R}_f(X,\Sigma_{k_1},\{\mathrm{all}\})\times\mathcal{R}_f(X,\Sigma_{k_2+k_3},\{\mathrm{all}\}) \arrow[dl, "\boxtimes_{k_1,k_2+k_3}"] \\
& \mathcal{R}_f(X,\Sigma_{k_1+k_2+k_3},\{\mathrm{all}\}) &
\end{tikzcd}$$

commutes up to canonical isomorphism. Given a triple (Y_1, Y_2, Y_3) in the category at the top of the diagram, the value of the left-hand sequence of arrows is

$$\mathrm{Iso}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3, \boldsymbol{k}_1+\boldsymbol{k}_2\cup\boldsymbol{k}_3)_+ \\ \wedge_e^{\Sigma_{k_1+k_2}\times\Sigma_{k_3}} \Big(\big(\mathrm{Iso}(\boldsymbol{k}_1+\boldsymbol{k}_2, \boldsymbol{k}_1\cup\boldsymbol{k}_2)_+ \wedge_e^{\Sigma_{k_1}\times\Sigma_{k_2}} Y_1\wedge_\mu Y_2\big)\wedge_\mu Y_3\Big),$$

and we claim this space is isomorphic to

$$\mathrm{Iso}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3, (\boldsymbol{k}_1\cup\boldsymbol{k}_2)\cup\boldsymbol{k}_3)_+ \wedge_e^{(\Sigma_1\times\Sigma_2)\times\Sigma_3} (Y_1\wedge_\mu Y_2)\wedge_\mu Y_3. \tag{3.13}$$

To clarify the notation, $\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3$ denotes the standard finite set of cardinality $k_1+k_2+k_3$, $\boldsymbol{k}_1+\boldsymbol{k}_2\cup\boldsymbol{k}_3$ denotes the disjoint union of finite sets of cardinality k_1+k_2 and k_3, and so on. Similarly, the value of the right-hand sequence of arrows is

$$\mathrm{Iso}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3, \boldsymbol{k}_1\cup\boldsymbol{k}_2+\boldsymbol{k}_3)_+ \\ \wedge_e^{\Sigma_{k_1}\times\Sigma_{k_2+k_3}} \Big(Y_1\wedge_\mu \big(\mathrm{Iso}(\boldsymbol{k}_2+\boldsymbol{k}_3, \boldsymbol{k}_2\cup\boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_2}\times\Sigma_{k_3}} Y_2\wedge_\mu Y_3\big)\Big),$$

which we claim is isomorphic to

$$\mathrm{Iso}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3, \boldsymbol{k}_1\cup(\boldsymbol{k}_2\cup\boldsymbol{k}_3))_+ \wedge_e^{\Sigma_1\times(\Sigma_2\times\Sigma_3)} Y_1\wedge_\mu (Y_2\wedge_\mu Y_3). \tag{3.14}$$

The spaces in (3.13) and (3.14) are isomorphic by the combination of the associativity isomorphisms for disjoint union, cartesian products of groups, and the smash product $\wedge_\mu$. Thus, we have proved that the box-tensor operations are naturally associative, granting the two isomorphisms.

To establish one of these isomorphisms requires several steps. We concentrate on the first case, since the second is completely parallel. First, since $\mathrm{Iso}(\boldsymbol{k}_3,\boldsymbol{k}_3)=\Sigma_{k_3}$, there is an isomorphism

$$\mathrm{Iso}(\boldsymbol{k}_3,\boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_3}} Y_3 \xrightarrow{\cong} Y_3 \tag{3.15}$$

in $\mathcal{R}_f(X, \Sigma_3)$ induced by the formula $[f_3, y] \mapsto f_3^{-1} y$. With the right action of Σ_{k_3} on $\mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)$ given by $f \cdot \sigma = \sigma^{-1} \circ f$, we have $[f_3 \cdot \sigma, y] \mapsto (\sigma^{-1} f_3)^{-1} y = f_3^{-1} \sigma y$; starting from $[f_3, \sigma y]$, we also arrive at $f_3^{-1} \sigma y$. Thus, a map

$$\mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_3}} Y_3 \to Y_3$$

exists. Surjectivity is clear. For injectivity, if $[f_3, y]$ and $[f_3', y']$ map to the same element of Y, we have $f_3^{-1} y = (f_3')^{-1} y'$. Putting $\sigma = f_3' f_3^{-1}$, we have $y' = \sigma y$ and $f_3' \cdot \sigma = \sigma^{-1} f_3' = f_3 (f_3')^{-1} f_3' = f_3$, so $[f_3, y] = [f_3' \cdot \sigma, y] \sim [f_3', \sigma y] = [f_3', y']$. To get equivariance, recall that the left action of Σ_{k_3} on $\mathrm{Iso}(k_3, k_3)$ is given by $\sigma \cdot f_3 = f \circ \sigma^{-1}$. Thus,

$$[\sigma * f, y] = [f \circ \sigma^{-1}, y] \mapsto (f \circ \sigma^{-1})^{-1} y = \sigma(f^{-1} y)$$

shows equivariance.

Consequently,

$$\begin{aligned}&\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, \boldsymbol{k}_1 + \boldsymbol{k}_2 \cup \boldsymbol{k}_3)_+ \\ &\qquad \wedge_e^{\Sigma_{k_1+k_2} \times \Sigma_{k_3}} \big(\big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2, \boldsymbol{k}_1 \cup \boldsymbol{k}_2)_+ \wedge_e^{\Sigma_{k_1} \times \Sigma_{k_2}} Y_1 \wedge_\mu Y_2\big) \wedge_\mu Y_3\big)\end{aligned}$$

is isomorphic to

$$\begin{aligned}&\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, \boldsymbol{k}_1 + \boldsymbol{k}_2 \cup \boldsymbol{k}_3)_+ \\ &\wedge_e^{\Sigma_{k_1+k_2} \times \Sigma_{k_3}} \big(\big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2, \boldsymbol{k}_1 \cup \boldsymbol{k}_2)_+ \wedge_e^{\Sigma_{k_1} \times \Sigma_{k_2}} Y_1 \wedge_\mu Y_2\big) \wedge_\mu \big(\mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_3}} Y_3\big)\big).\end{aligned}$$

Applying a commutativity isomorphism of the product $\wedge_e$, this is isomorphic to

$$\begin{aligned}&\big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, \boldsymbol{k}_1 + \boldsymbol{k}_2 \cup \boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_1+k_2} \times \Sigma_{k_3}} \big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2, \boldsymbol{k}_1 \cup \boldsymbol{k}_2)_+ \wedge_e \mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)_+\big)\big) \\ &\qquad \wedge_e^{(\Sigma_{k_1} \times \Sigma_{k_2}) \times \Sigma_{k_3}} (Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3.\end{aligned}$$

Now we claim there is an isomorphism

$$\begin{aligned}&\big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, \boldsymbol{k}_1 + \boldsymbol{k}_2 \cup \boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_1+k_2} \times \Sigma_{k_3}} \big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2, \boldsymbol{k}_1 \cup \boldsymbol{k}_2)_+ \wedge_e \mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)_+\big)\big) \\ &\qquad \cong \mathrm{Iso}\big(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, (\boldsymbol{k}_1 \cup \boldsymbol{k}_2) \cup \boldsymbol{k}_3\big)_+\end{aligned}$$

induced by the formula $[f_{123}, [f_{12}, f_3]] \mapsto (f_{12}, f_3) \circ f_{123}$. We check that balanced expressions in

$$\big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, \boldsymbol{k}_1 + \boldsymbol{k}_2 \cup \boldsymbol{k}_3)_+ \wedge_e \big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2, \boldsymbol{k}_1 \cup \boldsymbol{k}_2)_+ \wedge_e \mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)_+\big)\big)$$

map to the same element of the target:

$$\begin{aligned}&[f_{123} \cdot (g_{12}, g_3), [f_{12}, f_3]] = [(g_{12}^{-1}, g_3^{-1}) \circ f_{123}, [f_{12}, f_3]] \\ &\qquad \mapsto (f_{12}, f_3) \circ \big((g_{12}^{-1}, g_3^{-1}) \circ f_{123}\big).\end{aligned}$$

On the other hand,

$$[f_{123}, (g_{12}, g_3)\cdot[f_{12}, f_3]] = [f_{123}, [f_{12}\circ g_{12}^{-1}, f_3\circ g_3^{-1}]] \mapsto (f_{12}\circ g_{12}^{-1}, f_3\circ g_3^{-1})\circ f_{123}$$

and these expressions are the same, by associativity of composition. Now suppose $[f_{123}, [f_{12}, f_3]]$ and $[f'_{123}, [f'_{12}, f'_3]]$ map to the same isomorphism. The equation $(f_{12}, f_3)\circ f_{123} = (f'_{12}, f'_3)\circ f'_{123}$ is equivalent to $(f'_{12}, f'_3)^{-1}\circ(f_{12}, f_3) = f'_{123}\circ f_{123}^{-1}$. Putting $(\sigma_{12}, \sigma_3) = (f'_{12}, f'_3)^{-1}\circ(f_{12}, f_3) = f'_{123}\circ f_{123}^{-1}$, we have

$$f'_{123}\cdot(\sigma_{12}, \sigma_3) = (f'_{123}\circ f_{123}^{-1})^{-1}\circ f'_{123} = f_{123},$$

and

$$(\sigma_{12}, \sigma_3)\cdot(f_{12}, f_3) = (f_{12}, f_3)\circ\big((f'_{12}, f'_3)^{-1}\circ(f_{12}, f_3)\big)^{-1} = (f'_{12}, f'_3),$$

so that

$$\begin{aligned}[f_{123}, [f_{12}, f_3]] &= [f'_{123}\cdot(\sigma_{12}, \sigma_3), [f_{12}, f_3]]\\ &\sim [f'_{123}, (\sigma_{12}, \sigma_3)\cdot[f_{12}, f_3]] = [f'_{123}, [f'_{12}, f'_3]]. \qquad \square\end{aligned}$$

The diamond operation $\diamond_{k,1} = \diamond_k$ requires some preliminary definitions. First recall the category of filtered objects $F_k\mathcal{R}_f(X)$ from [Waldhausen 1985, Section 1.1]; this is a category with cofibrations and weak equivalences. Let

$$\underline{P} = (P_1 \rightarrowtail P_2 \rightarrowtail \cdots \rightarrowtail P_k)$$

be an object of $F_k\mathcal{R}_f(X)$. For functions $f, g : \boldsymbol{k}\to\boldsymbol{k}$ we say $f \le g$ if $f(i) \le g(i)$ for all $i \in \boldsymbol{k}$. Let $I(\boldsymbol{k}) = \{f : \boldsymbol{k}\to\boldsymbol{k} \mid \text{there is } \sigma\in\Sigma_k \text{ such that } f\le\sigma\}$.

The set $I(\boldsymbol{k})$ is partially ordered by $\le$, and the sequence $\underline{P}$ defines a functor $\underline{\underline{P}} : I(\boldsymbol{k}) \to \mathcal{R}_f(X)$ by the rule $\underline{\underline{P}}(f) = P_{f(1)}\wedge_\mu P_{f(2)}\wedge_\mu\cdots\wedge_\mu P_{f(k)}$ on objects. We apply the convention that parentheses in iterated products are collected to the left. In particular, $P_{f(1)}\wedge_\mu P_{f(2)}\wedge_\mu P_{f(3)} = (P_{f(1)}\wedge_\mu P_{f(2)})\wedge_\mu P_{f(3)}$, and, in general,

$$P_{f(1)}\wedge_\mu P_{f(2)}\wedge_\mu\cdots\wedge_\mu P_{f(k)} = (\cdots(P_{f(1)}\wedge_\mu P_{f(2)})\wedge_\mu\cdots\wedge_\mu P_{f(k)}).$$

For arrows we observe that $f\le g$ implies there are cofibrations $P_{f(i)}\rightarrowtail P_{g(i)}$ which induce a cofibration $\underline{\underline{P}}(f)\rightarrowtail\underline{\underline{P}}(g)$. This depends on the exactness of $\wedge_\mu$, proved in Theorem 2.1.

Definition 3.16. Define the functor $\diamond_k : F_k\mathcal{R}_f(X)\to\mathcal{R}_f(X, \Sigma_k, \{\text{all}\})$ on objects by making a choice of $\operatorname{colim}_{I(\boldsymbol{k})}\underline{\underline{P}}$ and setting

$$\diamond_k(\underline{P}) = \operatorname*{colim}_{I(\boldsymbol{k})}\underline{\underline{P}}.$$

The definition extends to arrows by the universal property of the colimit. The Σ_k action is induced by the permutation of factors.

Example 3.17. Applied to a constant cofibration sequence $\underline{Y} = (Y \overset{=}{\rightarrowtail} Y \overset{=}{\rightarrowtail} \cdots \overset{=}{\rightarrowtail} Y)$ of length k, we obtain simply

$$\diamond_k(\underline{Y}) = Y \wedge_\mu Y \wedge_\mu \cdots \wedge_\mu Y$$

with the group Σ_k permuting the factors. Thus, the purpose of $\diamond_k$ is to extend $\wedge_\mu$-powers to filtered objects.

Definition 3.18. The generalized diamond operation

$$\diamond_{n,k} : F_k \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_{nk}, \{\text{all}\})$$

is a composition

$$\diamond_{n,k} : F_k \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \xrightarrow{\diamond_k} \mathcal{R}_f(X, B_{n,k}, \{\text{all}\}) \xrightarrow{\text{Ind}_{B_{n,k}}^{\Sigma_{nk}}} \mathcal{R}_f(X, \Sigma_{nk}, \{\text{all}\}),$$

with a basic diamond operation $\diamond_k$ followed by an induction construction $\text{Ind}_{B_{n,k}}^{\Sigma_{nk}}$. The intermediate group $B_{n,k}$ is the group of block permutations of nk objects blocked into k groups of n objects. Thus, the group $B_{n,k}$ is a wreath product: $B_{n,k} = \Sigma_k \wr \Sigma_n$. Explicitly, there is a short exact sequence of groups

$$1 \to (\Sigma_n)^k \to B_{n,k} \to \Sigma_k \to 1.$$

Recall $G_\bullet$ briefly here, following [Gunnarsson et al. 1992]. For a simplicial set Z the corresponding simplicial path set PZ is defined by $PZ_n = Z_{n+1}$. The face operator $d_i : PZ_n \to PZ_{n-1}$ coincides with $d_{i+1} : Z_{n+1} \to Z_n$; the degeneracy operator $s_i : PZ_n \to PZ_{n+1}$ coincides with $s_{i+1} : Z_{n+1} \to Z_{n+2}$. The face operator $d_0 : Z_{n+1} \to Z_n$ induces a simplicial map $d_0 : PZ \to Z$. The simplicial set PZ is simplicially homotopy equivalent to the constant simplicial set Z_0 [Waldhausen 1985, Lemma 1.5.1, p. 328]. Viewing $Z_1 = PZ_0$ as another constant simplicial set provides a simplicial map $Z_1 \to PZ$.

Definition 3.19 [Gunnarsson et al. 1992, p. 257]. For a category $\mathcal{C}$ with cofibrations and weak equivalences the simplicial category $G_\bullet\mathcal{C}$ is defined by the cartesian square

$$\begin{tikzcd} wG_\bullet\mathcal{C} \arrow[r] \arrow[d] & PwS_\bullet\mathcal{C} \arrow[d, "d_0"'] \\ PwS_\bullet\mathcal{C} \arrow[r, "d_0"] & wS_\bullet\mathcal{C} \end{tikzcd} \tag{3.20}$$

Recalling a few more details from [Gunnarsson et al. 1992], $G_\bullet\mathcal{C}$ has cofibrations and weak equivalences. As $G_n\mathcal{C} = (PS_\bullet\mathcal{C})_n \times_{S_n\mathcal{C}} (PS_\bullet\mathcal{C})_n = S_{n+1}\mathcal{C} \times_{S_n\mathcal{C}} S_{n+1}\mathcal{C}$, the weak equivalences and cofibrations in $wG_\bullet\mathcal{C}$ are given by pullback. There is also a stabilization map $\eta : \mathcal{C} \to G_\bullet\mathcal{C}$, where $\mathcal{C}$ is viewed as a constant simplicial category with cofibrations and weak equivalences, defined as follows. We have the map $\mathcal{C} = (PwS_\bullet\mathcal{C})_0 \to PwS_\bullet\mathcal{C}$ and the constant map $\mathcal{C} \to PwS_\bullet\mathcal{C}$ carrying $\mathcal{C}$ to the terminal object. These two combine to give an inclusion $\eta : \mathcal{C} \to G_\bullet \mathcal{C}$. After passing to diagonals, the construction may be iterated so there results a stabilization sequence

$$\mathcal{C} \to G_\bullet \mathcal{C} \to G_\bullet(G_\bullet \mathcal{C}) \to \cdots \to G^n_\bullet \mathcal{C} := G(G^{n-1}_\bullet \mathcal{C}) \to \cdots \to \operatorname*{colim}_n G^n_\bullet \mathcal{C} := G^\infty_\bullet \mathcal{C}$$

of simplicial categories with cofibrations and weak equivalences. Returning to the square (3.20), after passage to nerves in the w-direction, diagonalization, and geometric realization, there results a natural map

$$|wG_\bullet \mathcal{C}| \to \Omega |wS_\bullet \mathcal{C}|.$$

This may not always be a homotopy equivalence, but it is a homotopy equivalence when $\mathcal{C}$ has a property called pseudo-additivity [Gunnarsson et al. 1992, Definition 2.3 and Theorem 2.6]. In our case, with $\mathcal{C} = \mathcal{R}_f(X)$ we follow [Gunnarsson et al. 1992] to achieve the pseudo-additivity property by using the cylinder functor defined in [Waldhausen 1985, Section 1.6]. The cylinder functor induces a cone functor $c : \mathcal{R}_f(X) \to \mathcal{R}_f(X)$ and a suspension functor $\Sigma : \mathcal{R}_f(X) \to \mathcal{R}_f(X)$ so that we may define a category of prespectra

$$\Sigma^\infty \mathcal{R}_f(X) = \operatorname{colim}\big(\mathcal{R}_f(X) \xrightarrow{\Sigma} \mathcal{R}_f(X) \xrightarrow{\Sigma} \mathcal{R}_f(X) \xrightarrow{\Sigma} \cdots\big).$$

Then $\Sigma^\infty \mathcal{R}_f(X)$ has the pseudo-additivity property [Gunnarsson et al. 1992, Remark 2.4 and Lemma 2.5, p. 258–259], so

$$|wG_\bullet \Sigma^\infty \mathcal{R}_f(X)| \to \Omega |wS_\bullet \Sigma^\infty \mathcal{R}_f(X)|$$

is a weak homotopy equivalence. Also, by [Waldhausen 1985, Proposition 1.6.2], $|wS_\bullet \mathcal{R}_f(X)| \to |wS_\bullet \Sigma^\infty \mathcal{R}_f(X)|$ is a weak homotopy equivalence.

Additionally, we need the fact that there are weak homotopy equivalences

$$|wG^\infty_\bullet \mathcal{C}| \to \Omega |wG^\infty_\bullet S_\bullet \mathcal{C}| \leftarrow \Omega |wS_\bullet \mathcal{C}| \tag{3.21}$$

for any category $\mathcal{C}$ with cofibrations and weak equivalences [Gunnarsson et al. 1992, Theorem 4.1, p. 268].

The $G_\bullet \mathcal{C}$ construction has an explicit description as a category of exact functors. For full details refer to [Gunnarsson and Schwänzl 2002; Grayson 1989]. First, extend the partially ordered set $A \in \Delta$ to the set $\gamma(A) = \{L, R\} \amalg A$ with the ordering in which L and R are not comparable, $L < a$ and $R < a$ for every $a \in A$, and, for $a, a' \in A$, $a < a'$ in $\gamma(A)$ if and only if $a < a'$ in A. Pictorially, for $A = [n]$, $\gamma(A)$ looks like

$$\begin{tikzcd}
L \arrow[dr] & & & & \\
& 0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & n \\
R \arrow[ur] & & & &
\end{tikzcd}$$

The category $\Gamma(A)$ is the category of arrows in $\gamma(A)$, omitting the identity arrows on L and R. Diagrammatically, $\Gamma(A)$ looks like

$$
\begin{tikzcd}[column sep=small]
 & 0/L \arrow[rr, tail] \arrow[dd] & & 1/L \arrow[rr, tail] \arrow[dd] & & \cdots \arrow[rr, tail] & & n/L \arrow[dd] \\
0/R \arrow[rr, tail] \arrow[dr] & & 1/R \arrow[rr, tail] \arrow[dr] & & \cdots \arrow[rr, tail] & & n/R \arrow[dr] & \\
 & 0/0 \arrow[rr, tail] & & 1/0 \arrow[rr, tail] \arrow[d] & & \cdots \arrow[rr, tail] & & n/0 \arrow[d] \\
 & & & 1/1 \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & n/1 \arrow[d] \\
 & & & & & & \cdots & \vdots \arrow[d] \\
 & & & & & & & n/n
\end{tikzcd}
$$

Here a/b stands for $b \to a$ (or $b < a$), and an arrow $a/b \to c/d$ stands for a square

$$
\begin{array}{ccc}
a & \longrightarrow & c \\
\uparrow & & \uparrow \\
b & \longrightarrow & d
\end{array}
$$

in $\gamma(A)$. The exact sequences in $\Gamma(A)$ are sequences $j/k \to i/k \to i/j$ where $k \to j \to i$ in $\gamma(A)$. Then, for $A \in \Delta$,

$$G_A\mathcal{C} = \mathrm{Exact}(\Gamma(A), \mathcal{C}).$$

Since $\Gamma(A)$ is functorial in A, preserving exact sequences $j/k \to i/k \to i/j$, we have another description of $G_\bullet\mathcal{C} : \Delta^{\mathrm{op}} \to \mathrm{Cat}$. In this interpretation the stabilization $\eta : \mathcal{C} \to G_\bullet\mathcal{C}$ sends an object C of $\mathcal{C}$ to the functor $\eta(C) : \Gamma(A) \to \mathcal{C}$ whose value at i/L is C for all $i \in A$ and whose value at any other object of $\Gamma(A)$ is the zero object of $\mathcal{C}$. Given an arrow $i/L \to i'/L$ in $\Gamma(A)$, $\eta(C)$ assigns to it the identity on C; other arrows are assigned by the universal property of the zero object.

Definition 3.22 (cf. [Gunnarsson and Schwänzl 2002, Section 2.1, p. 268; Grayson 1989, Section 4]). Let Z be a simplicial object in a category $\mathcal{D}$. Define a concatenation operation $\mathrm{con} : \Delta^k \to \Delta$. For a sequence $(A_1, \dots, A_k)$ of finite nonempty ordered sets, order their disjoint union $A_1 \amalg \cdots \amalg A_k$ so that the subset A_i inherits the original order and so that, if $i \le j$ and $a_i \in A_i$ and $a_j \in A_j$, then $a_i < a_j$. Then define the k-fold edgewise subdivision of a simplicial object Z to be the composite functor

$$\mathrm{sub}_k\, Z : \Delta^k \xrightarrow{\mathrm{con}} \Delta \xrightarrow{Z} \mathcal{D}.$$

For a simplicial set Z there is a natural homeomorphism $|\mathrm{sub}_k\, Z| \to |Z|$.

Several more constructions are necessary before we can define for every integer $k \geq 1$ operations

$$\omega^k : w\,\mathrm{sub}_k\, G_\bullet \mathcal{R}_f(X) \to wG^k_\bullet \mathcal{R}_f(X, \Sigma_k, \{\mathrm{all}\}).$$

The original framework has proved to be quite robust, so we refer to [Grayson 1989, Sections 5–7; Gunnarsson and Schwänzl 2002, Section 2] for complete details and summarize what we use.

Theorem 3.23 [Grayson 1989, Sections 5–7, pp. 253–257]. *For $A \in \Delta$, let $\Gamma^1(A)$ be the category $\Gamma(A)$ discussed before Definition 3.22.*

(1) *For each $A \in \Delta$ and for each integer $k \geq 1$ there is a category with exact sequences $\Gamma^k(A)$. The category $\Gamma^k(A)$ is natural in the variable A.*
(2) *For $A_1, \dots, A_k \in \Delta$, let $A_1 \dots A_k$ be the concatenation. There is a functor*

$$\Xi_k : \Gamma(A_1) \times \cdots \times \Gamma(A_k) \to \Gamma^k(A_1 \dots A_k)$$

which is multi-exact, i.e., exact in each variable separately, and is natural in each of the variables. □

Grayson [1989, pp. 255–256] enumerates compatibility conditions (E1)–(E5) abstracted from properties of higher exterior powers and tensor products when applied to filtered modules. Given that the box-tensor operations $\boxtimes$ and diamond operations $\diamond_{n,k}$ fulfill (E1)–(E5) the robustness of the framework enables us to make the following observation.

Definition 3.24. For $A \in \Delta$, the collection of operations $\diamond_{n,k}$ and $\boxtimes$ define functors

$$\Lambda^k_{\diamond,\boxtimes} : \mathrm{Exact}\big(\Gamma(A), \mathcal{R}_f(X, \Sigma_n, \{\mathrm{all}\})\big) \to \mathrm{Exact}\big(\Gamma^k(A), \mathcal{R}_f(X, \Sigma_{n,k}, \{\mathrm{all}\})\big).$$

These functors are natural in A.

Remark 3.25. Since we don't need the explicit formula for $\Lambda^k_{\diamond,\boxtimes}$ except in a few specific cases, we refer the reader to the discussion in [Grayson 1989, p. 256–257] for all the details. For guidance, we point out that the categories $\Gamma^k(A)$ mentioned in Theorem 3.23 are constructed precisely to deliver the definition of $\Lambda^k_{\diamond,\boxtimes}$ on an object. Properties (E1) through (E4) ensure that the formulas on arrows yield a well-defined functor. Property (E5) of Grayson's list ensures that the functors $\Lambda^k_{\diamond,\boxtimes}$ carry an exact functor M to another exact functor.

In our situation we need the following property of a category with cofibrations.

Definition 3.26 (cf. [Gunnarsson et al. 1992, Definition 4.3, p. 274]). A category $\mathcal{C}$ with cofibrations has the extension property if for all commutative diagrams of cofibration sequences

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, tail] \arrow[d, "i"'] & C \arrow[d, tail] \\ A' \arrow[r, tail] & B' \arrow[r, tail] & C' \end{tikzcd}$$

in $\mathcal{C}$, with vertical cofibrations as indicated, it follows that the middle arrow i is also a cofibration.

Lemma 3.27. *Let $\mathcal{C}$ be a category with cofibrations, $A_1, \dots, A_k \in \Delta$, and let $A_1 \dots A_k$ be the concatenation.*

(1) *If $\mathcal{C}$ has the extension property, then the natural inclusion*

$$G^k_{A_1 \dots A_k}\mathcal{C} \to \operatorname{Exact}(\Gamma(A_1) \times \cdots \times \Gamma(A_k), \mathcal{C})$$

is an isomorphism.

(2) *The categories $\mathcal{R}_f(X, \Sigma_n, \mathcal{F})$ with cofibrations have the extension property.*

Proof. The first statement is [Gunnarsson and Schwänzl 2002, Remark 4.4, p. 274]. For (2), because we are working inside $\mathcal{R}(X)$ with simplicial sets, cofibrations are the injective maps. Therefore, the extension property holds for $\mathcal{R}_f(X, \Sigma_n, \mathcal{F})$. □

Proposition 3.28 [Gunnarsson and Schwänzl 2002, Proposition 4.5]. *The box-tensor operations and the diamond operations fulfill properties* (E1)–(E5).

Proof. Properties (E1)–(E4) are consequences of the symmetric bimonoidal structure of Theorem 2.1. Properties (E3) and (E4) also depend on the associativity of $\boxtimes$ established in Proposition 3.12. Property (E5) depends on the extension property of Definition 3.26 and takes some additional work manipulating cocartesian diagrams, cofibration sequences, and colimits. The necessary steps are laid out in [Gunnarsson and Schwänzl 2002, Lemmas 4.6–4.10, Corollary 4.11]. Because all those manipulations rely just on the coherence of the symmetric bimonoidal category structure, all steps work in the present, more general, situation. □

The subdivision construction (concatenation), the functors $\Lambda^k_{\diamond,\boxtimes}$, and the functors Ξ_k come into play in the following definition.

Definition 3.29. For $k \geq 1$, the components ω^k for the total Segal operation are defined as follows:

$$\begin{tikzcd} \operatorname{Exact}\big(\Gamma(A_1 \dots A_k), \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})\big) \arrow[r, "\mathrm{sub}_k\, \Lambda^k"] \arrow[d, "\omega^k"'] & \operatorname{Exact}\big(\Gamma^k(A_1 \dots A_k), \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})\big) \arrow[dl, "\Xi_k"] \\ \operatorname{Exact}\big(\Gamma(A_1) \times \cdots \times \Gamma(A_k), \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})\big) & \end{tikzcd} \tag{3.30}$$

By Lemma 3.27(1) we may interpret $\mathrm{Exact}\big(\Gamma(A_1)\times\cdots\times\Gamma(A_k),\, \mathcal{R}_f(X,\Sigma_n,\{\text{all}\})\big)$ as $G^k_{A_1\ldots A_k}\mathcal{R}_f(X,\Sigma_n,\{\text{all}\})$. The result is a family of functors

$$\omega^k : w\,\mathrm{sub}_k\, G_\bullet\mathcal{R}_f(X) \to wG^k_\bullet\mathcal{R}_f(X,\Sigma_k,\{\text{all}\})$$

for $k \geq 1$.

Referring to the discussion preceding Definition 3.22, the stabilization map $\eta : \mathcal{R}_f(X) \to G_0\mathcal{R}_f(X)$ has been concisely written in [Gunnarsson and Schwänzl 2002] as

$$(Y, r, s) \mapsto \eta((Y, r, s)) = \frac{Y}{\overline{\overline{X}}}\,.$$

The extension to higher simplicial dimensions admits the description $(s_0)^k(\eta(Y))$, where $s_0^k : G_0\mathcal{R}_f(X) \to G_k\mathcal{R}_f(X)$ is the iterated degeneracy. This can be denoted

$$\frac{Y = Y = \cdots = Y}{\overline{\overline{X = X = \cdots = X}}} \tag{3.31}$$

where the top row indicates constant filtered object and the bottom row indicates the constant filtration of the zero object. Since $\mathrm{sub}_k\, G_\bullet\mathcal{C}$ in simplicial dimension 0 can be identified with $G_k\mathcal{C}$, diagram (3.31) also represents

$$\eta : \mathcal{R}_f(X) \to \mathrm{sub}_k\, G_0\mathcal{R}_f(X)$$

for each $k \geq 1$. The next example incorporates Example 3.17 and is fundamental.

Example 3.32. The formula for the composite

$$\tilde{\alpha}_1^k : \mathcal{R}_f(X) \xrightarrow{\eta} \mathrm{sub}_k\, G_\bullet\mathcal{R}_f(X) \xrightarrow{\omega^k} G^k_\bullet\mathcal{R}_f(X,\Sigma_k,\{\text{all}\}) \tag{3.33}$$

is the functor $\Gamma([0])^k \to \mathcal{R}_f(X,\Sigma_k,\{\text{all}\})$ given by

$$\begin{cases} Y \wedge_\mu Y \wedge_\mu \cdots \wedge_\mu Y & \text{in positions } 0/L,\ 0^{(2)}/L^{(2)},\ \ldots,\ 0^{(k)}/L^{(k)}, \\ X & \text{in all other positions.} \end{cases}$$

4. E_∞-structure and restriction to spherical objects

We have already seen that, in order to obtain the algebraic K-theory of spaces using the $G_\bullet$-model, one uses a category of prespectra $\Sigma^\infty\mathcal{R}_f(X)$ obtained from $\mathcal{R}_f(X)$ by passage to a limit using a suspension operation. We are now going to deal with natural transformations of semigroup valued functors

$$[-, \mathcal{R}_f(X)] \to \Big[-, \{1\} \times \prod_{n\geq 1} A_{\Sigma_n,\{\text{all}\}}(X)\Big],$$

where the target is an abelian-group-valued functor. First we restrict to categories of n-spherical objects $\mathcal{R}^n_f(X)$, whose definition is recalled below. Segal's group completion theorem [1974a, Proposition 4.1] provides a unique natural transformation of abelian-group-valued functors $[-, \Omega|hN_\Gamma\mathcal{R}^n_f(X)|] \to \big[-, \{1\}\times\prod_{n\geq 1} A_{\Sigma_n,\{\text{all}\}}(X)\big]$.

In the domain, $hN_\bullet \mathcal{R}^n_f(X)$ is the simplicial category arising from the categorical sum operation $\vee$, as described in [Waldhausen 1985, Section 1.8], and maps are weak homotopy equivalences. The following diagram displays this result as the diagonal arrow:

$$
\begin{tikzcd}
{[-,\Omega|hN_\Gamma \mathcal{R}^n_f(X)|]} \arrow[d, dashed] & {[-,|h\mathcal{R}^n_f(X)|]} \arrow[l] \arrow[r] \arrow[dr] & {[-,|h\mathcal{R}_f(X)|]} \arrow[d, "\omega"'] \\
{[-,A(X)]} \arrow[rr, dashed] & & \left[-,\{1\}\times\prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X)\right]
\end{tikzcd}
\tag{4.1}
$$

In this section we show that the diagonal arrow is induced by an E_∞-map

$$\Omega|hN_\Gamma \mathcal{R}^n_f(X)| \to \{1\}\times\prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X).$$

But we want a natural transformation of abelian-group-valued functors $[-, A(X)] \to \left[-, 1\times\prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X)\right]$ as displayed by the lower horizontal arrow in the diagram, and we want it to be induced by an E_∞-map $A(X)\to\{1\}\times\prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X)$. There is a natural chain of equivalences

$$\lim_{n\to\infty} hN_\bullet \mathcal{R}^n(X) \simeq hS_\bullet \mathcal{R}_f(X) \simeq hS_\bullet \Sigma^\infty \mathcal{R}_f(X),$$

where the colimit is taken over suspension relative to X [Waldhausen 1985, Theorems 1.7.1 and 1.8.1]. This implies we have to examine the behavior of our constructions as they relate to suspension, which we analyze in Section 5.

We recall from [Waldhausen 1985, Section 1.7, p. 360] a definition of spherical objects in the category $\mathcal{R}_f(X)$, where X is a connected space. On this category we have the homology theory $H_*(Y, r, s) = H_*(Y, s(X); r^*(\mathbb{Z}[\pi_1 X]))$ (homology with local coefficients), and we say (Y, r, s) is n-spherical if $H_q(Y, r, s) = 0$ for $q \neq n$ and $H_n(Y, r, s)$ is a stably free $\mathbb{Z}[\pi_1 X]$-module. For $n \geq 0$ denote by $\mathcal{R}^n_f(X)$ the full subcategory of $\mathcal{R}_f(X)$ whose objects are n-spherical. For example, in case X is a connected simplicial abelian group, $\mathcal{R}^n_f(X)$ contains spaces homotopy equivalent to retractive spaces (Y, r, s) obtained by completing to pushouts diagrams of the form

$$X \xleftarrow{\vee\phi_i} \bigvee_{i=1}^N \partial\Delta^n \rightarrowtail \Delta^n,$$

where the attaching maps ϕ_i are constant maps to the identity element of X.

Let $\mathbb{N}$ be the natural numbers $\{0, 1, \ldots\}$, and F the category of finite subsets of $\mathbb{N}$ and injections. Let $F_+ \subset F$ be the full subcategory of nonempty finite subsets. Let $\amalg$ denote the associative sum on F_+ given by

$$\{x_i \mid 1\leq i\leq m\}\amalg\{y_j \mid 1\leq j\leq n\} = \{x_i \mid 1\leq i\leq m\}\cup\{y_j + x_m - y_1 + 1 \mid 1\leq j\leq n\},$$

where we assume $x_1 < \cdots < x_m$ and $y_1 < \cdots < y_n$.

Lemma 4.2 [Gunnarsson and Schwänzl 2002, Lemma 10.2, p. 289]. *The category F_+ is contractible.*

Proof. The functor $t : F_+ \to F_+$ defined by $t(x) = \{0\} \amalg x$ receives natural transformations from the identity functor on F_+ and from the constant functor with value $\{0\}$. Geometric realization of the nerve converts the natural transformations to homotopies, so the identity map on the realization of the nerve of F_+ is homotopic to a constant map. $\square$

Under the assumption that the category $\mathcal{C}$ satisfies the extension property for cofibrations, which has been verified for $\mathcal{R}_f(X)$ and $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ in Lemma 3.27(2), one may identify the iterated $G_\bullet$-construction $G^n_\bullet\mathcal{C}$ with $\text{Exact}(\Gamma(-)^n, \mathcal{C})$ according to Lemma 3.27(1). Using the adjointness relation, or diagonals, we have

$$\begin{aligned} G^3_A(\mathcal{C}) := \text{Exact}(\Gamma(A)^3, \mathcal{C}) &= \text{Exact}(\Gamma(A) \times \Gamma(A), \text{Exact}(\Gamma(A), \mathcal{C})) \\ &= \cdots = G_A(G_A(G_A\mathcal{C})), \end{aligned}$$

for example. Now extend $n \mapsto \text{Exact}(\Gamma(-)^n, \mathcal{C}) = G^n_\bullet\mathcal{C}$ to $G^{(-)}_\bullet : F \to \text{Cat}^{\Delta^{\text{op}}}$ following the recipe in [Gunnarsson and Schwänzl 2002]. Thus, on objects $x \in \text{Ob}(F_+)$ and $A \in \Delta$, put $G^x_A\mathcal{C} := \text{Exact}(\Gamma(A)^x, \mathcal{C})$. To obtain the extension to F, identify $\Gamma(A)^\varnothing$ with the one-point category, so that $G^\varnothing_\bullet\mathcal{C} := \text{Exact}(\Gamma(A)^\varnothing, \mathcal{C}) = \mathcal{C}$.

For the behavior on morphisms we distinguish cases. An isomorphism $x \to x'$ in F induces a natural morphism $G^x_\bullet\mathcal{C} \to G^{x'}_\bullet\mathcal{C}$ by permuting coordinates. An injection $i : x \to y$ induces $G^x_\bullet\mathcal{C} \to G^y_\bullet\mathcal{C}$ using stabilization

$$\begin{tikzcd} \Gamma(A)^{i(x)} \arrow[r, "\cong"] \arrow[d, "\equiv"] & \Gamma(A)^{i(x)} \times \{L/0\}^{y\setminus i(x)} \arrow[r] \arrow[d, dashed] & \Gamma(A)^{i(x)} \times \Gamma(0)^{y\setminus i(x)} \arrow[dl, dashed, "X'"'] \\ \Gamma(A)^x \arrow[r, "X"'] & \mathcal{C} & \Gamma(A)^x \times \Gamma(A)^{y\setminus i(x)} \equiv \Gamma(A)^y \arrow[l, dashed, "i_*X"] \arrow[u] \end{tikzcd}$$

where we recall $\Gamma(0) = \{L/0, R/0\}$ is the two point discrete category, and we define X' to be zero outside $\Gamma(A)^{i(x)} \times \Gamma(0)^{y\setminus i(x)}$. This is the η-stabilization given by inclusion of $\mathcal{C}$ on the L-line in $G_0\mathcal{C}$, as described before Definition 3.22.

Let $F\int G_A\mathcal{C}$ be Thomason's homotopy colimit construction, which is the category consisting of objects $(x, X : \Gamma(A)^x \to \mathcal{C})$ and morphisms $(x, X) \to (y, Y)$ given by $i : x \to y$ in F and a natural transformation $i_*X \to Y$ in $G^y_A\mathcal{C}$ [Thomason 1979, Definition 1.2.2]. The unique morphisms $\varnothing \to x$ in F provide functors $\mathcal{C} \to \text{Exact}(\Gamma(A)^x, \mathcal{C})$, eventually functors $\mathcal{C} \to F\int G_A\mathcal{C}$ natural in A, and finally a functor $\mathcal{C} \to F\int G_\bullet\mathcal{C}$. With the next result, we have made a step toward the right-hand column of diagram (4.1).

Theorem 4.3 (cf. [Gunnarsson and Schwänzl 2002, Theorem 10.3]). *The construction $F\int wG_\bullet\mathcal{C}$ gives a model for K-theory.*

Comments on the proof. The proof given in [Gunnarsson and Schwänzl 2002] can be summarized in the chain of weak homotopy equivalences

$$\begin{aligned}\Omega|wS_\bullet\mathcal{C}| \xleftarrow{(1)} \Omega|wG^\infty_\bullet S_\bullet\mathcal{C}| \xrightarrow{(1)} |wG^\infty_\bullet\mathcal{C}| \xrightarrow{(2)} |wG_\bullet G^\infty_\bullet\mathcal{C}| \xrightarrow{(3)} \\ \xrightarrow{(3)} |F_+\textstyle\int wG_\bullet G^\infty_\bullet\mathcal{C}| \xleftarrow{(4)} \operatorname{colim}_{\tilde{t}}|F_+\textstyle\int wG_\bullet\mathcal{C}| \xleftarrow{(5)} |F_+\textstyle\int wG_\bullet\mathcal{C}| \xrightarrow{(6)} |F\textstyle\int wG_\bullet\mathcal{C}|.\end{aligned}$$

Concerning the links in the chain, the arrows labeled (1) are recorded in (3.21); the arrow (2) results from swallowing the extra $G_\bullet$ into the colimit defining $G^\infty_\bullet$. That (3) is an equivalence depends on the fact that $|F_+\int wG_\bullet G^\infty_\bullet\mathcal{C}| \to |F_+|$ can be shown to be a quasifibration with $|F_+|$ contractible. To account for (4), the functor $t : F_+ \to F_+$ induces a functor $\tilde{t} : F_+\int wG_\bullet\mathcal{C} \to F_+\int wG_\bullet\mathcal{C}$ for which

$$\operatorname{colim}_{\tilde{t}} F_+\textstyle\int wG_\bullet\mathcal{C} = \operatorname{colim}\big(F_+\int wG_\bullet\mathcal{C} \xrightarrow{\tilde{t}} F_+\int wG_\bullet\mathcal{C}F_+ \xrightarrow{\tilde{t}} F_+\int wG_\bullet\mathcal{C} \xrightarrow{\tilde{t}} \cdots\big)$$

is naturally identifiable to $F_+\int wG_\bullet G^\infty_\bullet\mathcal{C}$. The realizations of the functors $\tilde{t}$ are all cofibrations, so the inclusion (5) into the base of the telescope is a weak equivalence. Finally, cofinality of F_+ in F implies that the arrow (6) is a weak homotopy equivalence. □

As in [Gunnarsson and Schwänzl 2002], the E_∞-structure on the total Segal operation is described in terms of the diagram

$$\begin{tikzcd} \mathcal{R}^n_f(X) \arrow[d, "\alpha_1"] \arrow[dr, "\tilde{\alpha}_1"] & \\ \{1\}\times\prod_{n\geq1}\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \arrow[r, "\beta_1"] \arrow[d, "\alpha_2"] & \{1\}\times\prod_{n\geq1}G^n_\bullet\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \arrow[d, "\beta_3"] \\ \{1\}\times\prod_{n\geq1}F\textstyle\int G_\bullet\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \arrow[r, "\beta_2"] & \{1\}\times\prod_{n\geq1}F\textstyle\int G_\bullet G^n_\bullet\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \end{tikzcd} \tag{4.4}$$

The components of the map $\tilde{\alpha}_1$ are defined in Example 3.32. The other maps in diagram (4.4) are defined as follows.

Definition 4.5. For α_1 the n-th component $\alpha_1(Y)_n$ is $Y \wedge_\mu \overset{n\text{ terms}}{\cdots} \wedge_\mu Y$, where Σ_n acts by permuting factors using the coherence data.

The maps β_1 and β_2 come from stabilizations

$$j^n : \mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \to G^n_\bullet\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}).$$

The maps α_2 and β_3 are given by the identification

$$\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \cong \{\varnothing\}\textstyle\int G_\bullet\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \cong G^\varnothing_\bullet\,\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}).$$

The category $\mathcal{R}^n_f(X)$ has the pairing derived from the categorical sum $\vee_X$. This feature allows us to dispense with the subdivision construction. Each of the four categories in the lower part of the diagram also has a natural pairing derived from the box tensor pairings

$$\boxtimes_{k,\ell} : \mathcal{R}_f(X, \Sigma_k, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_{k+\ell}, \{\text{all}\}).$$

Underlying the coherence properties of these pairings is the fact, established in Theorem 2.1, that $\mathcal{R}_f(X)$ is a category with cofibrations and weak equivalences, with a categorical sum $\vee$ and a symmetric monoidal biexact product $\wedge_\mu$. We refer to [Gunnarsson and Schwänzl 2002, pp. 291–292] for explicit formulas for the pairings, which are given in the abstract context of a category $\mathcal{C}$ with cofibrations and weak equivalences and subcategories $\mathcal{C}_{\Sigma_n}$ of Σ_n-equivariant objects. Here we record only notations for use in the next theorem.

(1) There is a product denoted $\widetilde{\boxtimes}$ on $\{1\} \times \prod_{n\geq 1} \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ and a product also denoted $\widetilde{\boxtimes}$ on $\{1\} \times \prod_{n\geq 1} G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$.

(2) There is a product denoted $\widehat{\boxtimes}$ on $\{1\} \times \prod_{n\geq 1} F\!\int\! G_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ and a product also denoted $\widehat{\boxtimes}$ on $\{1\} \times \prod_{n\geq 1} F\!\int\! G_\bullet G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$.

Theorem 4.6 (cf. [Gunnarsson and Schwänzl 2002, Theorem 10.7, p. 292]).

(1) *In the left column of* (4.4), *the categories* $\{1\} \times \prod_{n\geq 1} \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ *and* $\{1\} \times \prod_{n\geq 1} F\!\int\! G_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$, *with their composition laws* $\widetilde{\boxtimes}$ *and* $\widehat{\boxtimes}$, *inherit symmetric monoidal structures from the coherence data on* $\mathcal{R}_f(X)$.

(2) *In the right column of* (4.4), *the categories* $\{1\} \times \prod_{n\geq 1} G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ *and* $\{1\} \times \prod_{n\geq 1} F\!\int\! G_\bullet G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$, *with their composition laws* $\widetilde{\boxtimes}$ *and* $\widehat{\boxtimes}$, *inherit monoidal structures from the coherence data on* $\mathcal{R}_f(X)$.

(3) *The maps* α_1 *and* α_2 *in* (4.4) *are maps of symmetric monoidal categories.*

(4) *The maps* β_1, β_2, *and* β_3 *are maps of monoidal categories.*

(5) *The map* β_2 *is a homotopy equivalence, and in the pseudo-additive case* β_3 *is also a homotopy equivalence.*

(6) *The diagram* (4.4) *is commutative in the category of monoidal categories.*

Theorem 4.7. *Let X be a connected simplicial abelian group. The functor*

$$Z \mapsto \Big[Z, \{1\} \times \prod_{n\geq 1} A_{\Sigma_n,\{\text{all}\}}(X)\Big]$$

takes values in the category of abelian groups.

Proof. By Theorem 4.3, we take

$$\{1\} \times \prod_{n\geq 1} A_{\Sigma_n,\{\text{all}\}}(X) = \{1\} \times \prod_{n\geq 1} |F\!\int\! G_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})|.$$

Since the category $\{1\} \times \prod_{n\geq 1} F\!\int\! G_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ has a symmetric monoidal structure by part (1) of Theorem 4.6, the functor $\big[-, \{1\} \times \prod_{n\geq 1} A_{\Sigma_n,\{\text{all}\}}(X)\big]$ takes values in the category of abelian monoids. Repeating the argument of [Waldhausen 1982, Lemma 2.3, p. 404] shows that values taken are actually in the category of abelian groups. □

Remarks on the proof of Theorem 4.6.. The entire proof of the analogous result in [Gunnarsson and Schwänzl 2002, pp. 293–295] is essentially a formal appeal to LaPlaza's coherence theorem [1972], so it carries over completely.

The reader who investigates further will find the symmetry of the pairing on $\{1\} \times \prod_{n\geq 1} F\!\int G_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ involves manipulating products of values of functors

$$Y \in G^m_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \quad \text{and} \quad Z \in G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}).$$

What is required is comparison of expressions

$$Y(i_1/j_1, \dots, i_m/j_m) \wedge_\mu Z(i'_1/j'_1, \dots, i'_n/j'_n)$$

and

$$Z(i'_1/j'_1, \dots, i'_n/j'_n) \wedge_\mu Y(i_1/j_1, \dots, i_m/j_m),$$

and one sees that not only are commutativity isomorphisms for $\wedge_\mu$ involved, but so are permutations of inputs, which are taken care of by means of the homotopy colimit.

Another interesting part of the proof is the claims about the maps α_1 and α_2, so it deserves a comment. The biexactness and coherence of $\wedge_\mu$ give canonical natural isomorphisms γ^k_n called Cartan multinomial formulas:

$$\gamma^k_n : (\wedge_\mu)_n\Bigl(\bigvee_{i=1}^{k} c_i\Bigr) \xrightarrow{\cong} \bigvee_{s_1+\cdots+s_k=n} \operatorname{Ind}^{\Sigma_n}_{\Sigma_{s_1}\times\cdots\times\Sigma_{s_k}} (\wedge_\mu)^k_{i=1}((\wedge_\mu)_{s_i} c_i).$$

These induce natural isomorphisms

$$\gamma^k : \alpha_1 \circ \vee^k_X \stackrel{\cong}{\Longrightarrow} (\widetilde{\boxtimes})^k \circ \alpha^k_1.$$

Then the coherence theorem implies that α_1 has a (lax) symmetric monoidal structure. The functor α_2 is the inclusion of a symmetric monoid subcategory, so the assertion for α_2 is immediate.

In contrast to the algebraic roles played by α_1 and α_2, the roles of β_1, β_2, and β_3 are to assure us that we are ending in the correct target. Since the proof that β_3 is a homotopy equivalence requires the pseudo-additivity condition, which is fulfilled by suspension, this part of the argument actually depends on the next section. □

5. Suspension

Let us first state the main theorem of this section.

Theorem 5.1. *Let X be a simplicial abelian group. The total Segal operation*

$$\omega : A(X) \to \{1\} \times \prod_{n \geq 1} A_{\Sigma_n,\mathrm{all}}(X)$$

carries an infinite loop map structure.

Section 4 has delivered an infinite loop map

$$\Omega\,|hN_\Gamma \mathcal{R}^n_f(X)| \to \{1\} \times \prod_{n \geq 1} A_{\Sigma_n,\mathrm{all}}(X)$$

whose domain is the K-theory of a category of n-spherical objects. To obtain Theorem 5.1, we have to examine the passage to the limit over suspension in view of Waldhausen's result

$$\lim_{n\to\infty} hN_\bullet \mathcal{R}^n_f(X) \simeq hS_\bullet \mathcal{R}_f(X).$$

The technically challenging part is the compatibility of the operations with suspension. Fortunately, the machinery set up in [Gunnarsson and Schwänzl 2002, Section 10] is sufficiently general that we need only extend some definitions and quote a sequence of results to prove our generalization.

First we need a description of the suspension operation that is amenable to coherence considerations. To this end, we go step-by-step through Waldhausen's cone and suspension constructions and identify the result with a construction involving the operation $\wedge_e$. The cone construction for (Y, r, s) in $\mathcal{R}_f(X)$ takes the ordinary mapping cylinder of the retraction $M(r)$ and collapses out the cylinder $\Delta^1 \times X$ so that end result is in $\mathcal{R}_f(X)$. To amplify the definition, consider the diagram

$$\begin{tikzcd}
Y \amalg X & \partial\Delta^1 \times Y \arrow[l, "\mathrm{id}\amalg r"'] \arrow[r, tail] & \Delta^1 \times Y \\
X \amalg X \arrow[u, tail] \arrow[d] & \partial\Delta^1 \times X \arrow[l] \arrow[r, tail] \arrow[u, tail] \arrow[d] & \Delta^1 \times X \arrow[u, tail] \arrow[d] \\
X & X \arrow[l] \arrow[r, tail] & X
\end{tikzcd} \tag{5.2}$$

which fulfills the hypotheses of Lemma 2.9. Taking the pushouts of the rows produces a diagram

$$X \leftarrow \Delta^1 \times X \rightarrowtail M(r),$$

where $M(r)$ is the usual mapping cylinder of r and the pushout of the top row. As described above, taking the pushout of this diagram produces cY, the underlying space of the cone construction. The retraction to X arises from a map of diagram (5.2) to a trivial diagram of identity maps on X; the section $X \to cY$ and a cofibration $i : Y \to cY$ arise from canonical maps of ingredients of the diagram to the colimit. Then the suspension ΣY is defined as the pushout of the diagram $X \xleftarrow{r} Y \rightarrowtail cY$.

Lemma 5.3. *For $Y \in \mathcal{R}_f(X)$ there is a commuting diagram*

$$\begin{tikzcd} \{0\} \times Y \arrow[r, tail, "i_0"] \arrow[d] & \Delta^1_1 \wedge_e Y \arrow[d, "\cong"'] \\ Y \arrow[r, tail, "i"] & cY \end{tikzcd} \tag{5.4}$$

where $\Delta^1_1 \in \mathcal{R}_f(*)$ is the standard simplicial one-simplex given the base point 1, *and i_0 is induced from the inclusion $\{0\} \to \Delta^1$. Moreover,*

$$\Sigma Y := cY/Y \cong S^1 \wedge_e Y,$$

where $S^1 = \Delta^1/\partial\Delta^1$ is the standard simplicial circle.

Proof. Pass to pushouts in the commutative diagram

$$\begin{tikzcd} X \arrow[d] & \Delta^1_1 \times X \cup_{\{1\}\times X} \{1\} \times Y \arrow[l, "p_2 \cup r p_2"'] \arrow[r, tail] \arrow[d, "\mathrm{id} \cup r"'] & \Delta^1 \times Y \arrow[d] \\ X & \Delta^1 \times X \arrow[l, "p_2"'] \arrow[r, tail] & M(r) \end{tikzcd} \tag{5.5}$$

to obtain a unique natural map $\eta_1 : \Delta^1_1 \wedge_e Y \to cY$ making the diagram

$$\begin{tikzcd} & \Delta^1_1 \wedge_e Y \arrow[dd, "\eta_1"] & \\ \Delta^1_1 \times Y \arrow[ur] \arrow[dr] & & X \arrow[ul, tail, "s'"'] \arrow[dl, tail] \\ & cY & \end{tikzcd} \tag{5.6}$$

commute. Restricting $\Delta^1_1 \times Y \to \Delta^1_1 \wedge_e Y$ to $\partial\Delta^1_1 \times Y$ yields a diagram

$$\begin{tikzcd} \partial\Delta^1_1 \times Y \arrow[r, tail] \arrow[d, "r'"'] & \Delta^1_1 \times Y \arrow[d] \\ Y \amalg X \arrow[r, tail, "i'"] & \Delta^1_1 \wedge_e Y \end{tikzcd}$$

where $r'(0, y) = y$, $r'(1, y) = r(y)$ and $i'(y) = i_0(y)$, $i'(x) = s'(x)$. There results a canonical arrow $M(r) \to \Delta^1_1 \wedge_e Y$ such that the following square commutes:

$$\begin{tikzcd} \Delta^1_1 \times X \arrow[r, tail] \arrow[d, "p_2"'] & M(r) \arrow[d] \\ X \arrow[r, tail, "s'"] & \Delta^1_1 \wedge_e Y \end{tikzcd}$$

In turn, there is a unique map $\bar{\eta}_1 : cY \to \Delta^1_1 \wedge_e Y$ such that

$$\begin{tikzcd} & cY \arrow[dd, "\bar{\eta}_1"] & \\ \Delta^1_1 \times Y \arrow[ur] \arrow[dr] & & X \arrow[ul, tail] \arrow[dl, tail, "s'"] \\ & \Delta^1_1 \wedge_e Y & \end{tikzcd} \tag{5.7}$$

commutes. Combining diagrams (5.6) and (5.7) shows that η_1 and $\bar{\eta}_1$ are mutually inverse isomorphisms, relative to the common subspace X and compatible with the retractions.

Restricting the left half of (5.6) to $\{0\} \times Y \subset \Delta^1_1 \times Y$ gives (5.4):

$$\begin{tikzcd} & \Delta^1_1 \wedge_e Y \arrow[d, "\eta_1"'] \\ \{0\} \times Y \equiv Y \arrow[ur, "i_0"] \arrow[r, "i"'] & cY \end{tikzcd} \tag{5.8}$$

Replace $S^0 = \{*, *'\}$ with basepoint $*$ in Example 2.5 by $\partial\Delta^1_1$ with basepoint 1, and obtain the diagram

$$\begin{tikzcd} X \arrow[d, "="'] & \partial\Delta^1_1 \wedge_e Y \arrow[l, "r"'] \arrow[r, tail] \arrow[d, "\cong"'] & \Delta^1_1 \wedge_e Y \arrow[d, "\cong"] \\ X & Y \arrow[l, "r"] \arrow[r, tail, "i"] \arrow[ur, tail, "i_0"] & cY \end{tikzcd} \tag{5.9}$$

Passage to pushouts shows that the quotient $(\Delta^1_1 \wedge_e Y)/(\partial\Delta^1_1 \wedge_e Y)$ is isomorphic to ΣY in $\mathcal{R}_f(X)$. According to Proposition 2.17, the functor

$$- \wedge_e Y : \mathcal{R}_f(*) \to \mathcal{R}_f(X \times \{*\}) \cong \mathcal{R}_f(X)$$

preserves colimits, so we deduce

$$(\Delta^1_1 \wedge_e Y)/(\partial\Delta^1_1 \wedge_e Y) \cong (\Delta^1_1/\partial\Delta^1_1) \wedge_e Y \equiv S^1 \wedge_e Y,$$

where we define $S^1 := \Delta^1_1/\partial\Delta^1_1$ in $\mathcal{R}_f(*)$. □

According to Proposition 2.16, the action of $\mathcal{R}_f(*)$ on $\mathcal{R}_f(X)$ may be made internal. Explicitly, there is a natural isomorphism $i_{e*}S^1 \wedge_\mu Y \cong S^1 \wedge_e Y$. In the following we abuse notation slightly and write simply $S^1 \wedge_\mu Y$, leaving i_{e*} understood, where $i_e : \{*\} \to X$ is the inclusion of the one-point space as the identity element of X. We do this to emphasize the dependence of the rest of this section on the coherence of the operation $\wedge_\mu$.

Proposition 5.10 (cf. [Gunnarsson and Schwänzl 2002, Proposition 6.1, p. 283]). *The following diagram commutes up to natural isomorphism*:

$$\begin{tikzcd} w\,\mathrm{sub}_k\, G_\bullet \mathcal{R}_f(X) \arrow[r, "\omega^k"] \arrow[d, "S^1\wedge_\mu"'] & wG^k_\bullet \mathcal{R}_f(X, \Sigma_k, \{\mathrm{all}\}) \arrow[d, "\diamond_k S^1 \wedge_\mu"] \\ w\,\mathrm{sub}_k\, G_\bullet \mathcal{R}_f(X) \arrow[r, "\omega^k"] & wG^k_\bullet \mathcal{R}_f(X, \Sigma_k, \{\mathrm{all}\}) \end{tikzcd}$$

Proof. Write F_1 for the composite functor $\omega^k \circ (S^1 \wedge_\mu -)$ and F_2 for the composite $\diamond_k S^1 \wedge_\mu \omega^k(-)$. Although $\omega^k(S^1) = \diamond_k S^1 = S^1 \wedge_\mu \overset{k\text{ terms}}{\cdots} \wedge_\mu S^1$, we use the $\diamond_k$-notation for orientation purposes. Following [Gunnarsson and Schwänzl

2002, p. 297; Grayson 1989, p. 257], given a functor $M : \Gamma(A_1 \dots A_k) \to \mathcal{R}_f(X)$ representing an object of $\mathrm{sub}_k\, G_\bullet \mathcal{R}_f(X)$, the value of $\omega^k(M)$ on a typical element of $\Gamma^k(A_1 \dots A_k)$ has the form

$$(\diamond_{n_1} M(-)) \boxtimes (\diamond_{n_2} M(-)) \boxtimes \cdots \boxtimes (\diamond_{n_k} M(-)) = Z_{n_1} \boxtimes \cdots \boxtimes Z_{n_k},$$

where $Z_{n_i} := \diamond_{n_i} M(-)$ is an object of $\mathcal{R}_f(X, \Sigma_{n_i}, \{\text{all}\})$. Extending the formulas in the argument of Proposition 3.28 for the associativity of $\boxtimes$, we write

$$Z_{n_1} \boxtimes \cdots \boxtimes Z_{n_k} = \mathrm{Ind}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}^{\Sigma_{n_1+\cdots+n_k}} (Z_{n_1} \wedge_\mu \cdots \wedge_\mu Z_{n_k})$$

and set $n = n_1 + \cdots + n_k$.

Then a typical value of $F_1(M)$ has the form

$$\begin{aligned}
&\mathrm{Ind}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}^{\Sigma_n} \big((S^1 \wedge_\mu \overset{n_1 \text{ terms}}{\cdots} \wedge_\mu S^1 \wedge_\mu Z_{n_1}) \wedge_\mu \cdots \wedge_\mu (S^1 \wedge_\mu \overset{n_k \text{ terms}}{\cdots} \wedge_\mu S^1 \wedge_\mu Z_{n_k})\big) \\
&\quad \cong \mathrm{Ind}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}^{\Sigma_{n_1+\cdots+n_k}} \big((S^1 \wedge_\mu \overset{n_1 \text{ terms}}{\cdots} \wedge_\mu S^1) \wedge_\mu \overset{k \text{ groups}}{\cdots} \wedge_\mu (S^1 \wedge_\mu \overset{n_k \text{ terms}}{\cdots} \wedge_\mu S^1) \\
&\qquad\qquad \wedge_\mu (Z_{n_1} \wedge_\mu \cdots \wedge_\mu Z_{n_k})\big),
\end{aligned}$$

applying commutativity and associativity isomorphisms. Now Proposition 5.12 applies to deliver an isomorphism of $\Sigma_{n_1+\cdots+n_k}$-spaces:

$$\begin{aligned}
&\mathrm{Ind}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}^{\Sigma_n} \big((S^1 \wedge_\mu \overset{n_1 \text{ terms}}{\cdots} \wedge_\mu S^1) \wedge_\mu \overset{k \text{ groups}}{\cdots} \wedge_\mu (S^1 \wedge_\mu \overset{n_k \text{ terms}}{\cdots} \wedge_\mu S^1) \\
&\qquad\qquad \wedge_\mu (Z_{n_1} \wedge_\mu \cdots \wedge_\mu Z_{n_k})\big) \\
&\quad \xrightarrow{\cong} (\diamond_k S^1) \wedge_\mu \overset{n \text{ terms}}{\cdots} \wedge_\mu (\diamond_k S^1) \wedge_\mu \big(\mathrm{Ind}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}^{\Sigma_n} (Z_{n_1} \wedge_\mu \cdots \wedge_\mu Z_{n_k})\big).
\end{aligned}$$

This final expression is the value of F_2 on the same typical element M, so we have a natural isomorphism of functors $\epsilon : F_1 \Rightarrow F_2$. □

Now we prove the general Lemma 5.11 and its specialization Proposition 5.12.

Lemma 5.11. *Let H be a subgroup of G, let $Y \in \mathcal{R}(X, G)$, and let $Z \in \mathcal{R}(X, H)$. By restricting the G-action on Y to H, we obtain $Y \wedge_\mu Z \in \mathcal{R}(X, H)$, where the action is diagonal. Then there is a natural isomorphism of left G-spaces*

$$G_+ \wedge_e^H (Y \wedge_\mu Z) \xrightarrow{\cong} Y \wedge_\mu (G_+ \wedge_e^H Z),$$

where the G-action on the right-hand space is diagonal.

Proof. First define a G-map $f : G_+ \wedge_e (Y \wedge_\mu Z) \to Y \wedge_\mu (G_+ \wedge_e^H Z)$ by the formula

$$f(g, (y, z)) = (gy, [g, z]).$$

Applying the equivalence relation defining $Y \wedge_\mu (G_+ \wedge_e^H Z)$,

$$f(g, (hy, hz)) = (g(hy), [g, hz]) = ((gh)y, [gh, z]) = f(gh, (y, z)).$$

Therefore, there is an induced G-map

$$f' : G_+ \wedge_e^H (Y \wedge_\mu Z) \to Y \wedge_\mu (G_+ \wedge_e^H Z).$$

To reverse this map, define $F : Y \wedge_\mu (G_+ \wedge_e Z) \to G_+ \wedge_e^H (Y \wedge_\mu Z)$ by the formula

$$F(y, [g, z]) = [g, (g^{-1}y, z)].$$

Now

$$\begin{aligned} F(y, [gh, z]) &= [gh, (h^{-1}g^{-1}y, z)] \\ &= [g, (hh^{-1}g^{-1}y, hz)] = [g, (g^{-1}y, hz)] = F(y, [g, hz]), \end{aligned}$$

so there is an induced G-map

$$F' : Y \wedge_\mu (G_+ \wedge_e^H Z) \to G_+ \wedge_e^H (Y \wedge_\mu Z).$$

Clearly the composites $f'F'$ and $F'f'$ are the respective identities. □

Proposition 5.12. *Let $n = n_1 + \cdots + n_k$. Let $Z \in \mathcal{R}(X, \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}, \{\text{all}\})$. There is a natural isomorphism of Σ_n-spaces*

$$\begin{aligned} &\operatorname{Iso}(\boldsymbol{n}, \boldsymbol{n}_1 \cup \cdots \cup \boldsymbol{n}_k)_+ \wedge_e^{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} \big((S \diamond \overset{n_1 \text{ terms}}{\cdots} \diamond S) \wedge_\mu \cdots \wedge_\mu (S \diamond \overset{n_k \text{ terms}}{\cdots} \diamond S) \wedge_\mu Z\big) \\ &\qquad \xrightarrow{\cong} (S \diamond \overset{n \text{ terms}}{\cdots} \diamond S) \wedge_\mu \big(\operatorname{Iso}(\boldsymbol{n}, \boldsymbol{n}_1 \cup \cdots \cup \boldsymbol{n}_k)_+ \wedge_e^{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} Z\big). \end{aligned}$$

Proof. Apply Lemma 5.11, and observe that the operation $\diamond$ is defined in terms of $\wedge_\mu$, which is coherently associative. Collect all parentheses in expressions

$$(S \diamond \overset{n_1 \text{ terms}}{\cdots} \diamond S) \wedge_\mu \cdots \wedge_\mu (S \diamond \overset{n_k \text{ terms}}{\cdots} \diamond S)$$

to the left. Note that we need only the map $f' : G_+ \wedge_e^H (Y \wedge_\mu Z) \to Y \wedge_\mu (G_+ \wedge_e^H Z)$ from the lemma, so the choice of an identification of $\operatorname{Iso}(\boldsymbol{n}, \boldsymbol{n}_1 \cup \cdots \cup \boldsymbol{n}_k)$ with Σ_n is required to make sense of f'. This amounts to identifying $\boldsymbol{n}_1 \cup \cdots \cup \boldsymbol{n}_k$ with

$$\{1, \ldots, n_1, n_1 + 1, \ldots, n_1 + n_2, \ldots, n_1 + \cdots + n_k\}.$$ □

We use the Thomason homotopy colimit construction on functors defined on the category F to pass to the limit with suspensions. To treat suspension by S^1 on $\operatorname{sub}_k wG_\bullet\mathcal{R}_f(X)$, define an op-lax functor $\Phi_1 : F \to \mathrm{Cat}^{\Delta^{\mathrm{op}}}$ by

$$\begin{aligned} \Phi_1(x) &= \operatorname{sub}_k wG_\bullet\mathcal{R}_f(X) && \text{for an object } x \in F, \\ \Phi_1(\sigma) &= \mathrm{id} && \text{for an isomorphism } \sigma : x \to x, \\ \Phi_1(i : y \to x) & && \text{is induced by suspension by } x \backslash i(y) \text{ factors } S^1. \end{aligned}$$

Interpreting the smash product with an empty number of factors as S^0, the definitions coincide on isomorphisms. For $x \xleftarrow{i} y \xleftarrow{j} z$ we need to produce the natural transformation $\Phi_1(ij) \Rightarrow \Phi_1(i) \circ \Phi_1(j)$. On (Y, r, s) the value of $\Phi_1(j)$ is

$((S^1)^{y\backslash j(z)} \wedge_e Y, r', s')$ and the value of $\Phi_1(i)$ applied to this is

$$\big((S^1)^{x\backslash i(y)} \wedge_e ((S^1)^{y\backslash j(z)} \wedge_e Y), r'', s''\big).$$

Since i is injective, the set $y\backslash j(z)$ is identified with $i(y\backslash j(z))$. Since $x\backslash ij(z) = x\backslash i(y) \cup i(y\backslash j(z))$, we use associativity isomorphisms of the $\wedge_e$-action to write $\Phi_1(i \circ j) \overset{\cong}{\Longrightarrow} \Phi_1(i) \circ \Phi_1(j)$. The coherence properties of the action imply commutativity of the necessary diagrams [Thomason 1979, Definition 3.1.1, p. 99].

In a similar way we treat $\diamond_k S^1 \wedge_\mu -$ on $wG^k_\bullet \mathcal{R}_f(X, \Sigma_n\{\text{all}\})$, defining an op-lax functor $\Phi_2 : F \to \mathrm{Cat}^{\Delta^{\mathrm{op}}}$:

$$\begin{aligned} \Phi_2(x) = wG^k_\bullet \mathcal{R}_f(X, \Sigma_n\{\text{all}\}) &\quad \text{for an object } x \in F,\\ \Phi_2(\sigma) = \mathrm{id} &\quad \text{for an isomorphism } \sigma : x \to x,\\ \Phi_2(i : y \to x) &\quad \text{is induced by suspension by } x\backslash i(y) \text{ factors } \diamond_k S^1. \end{aligned}$$

The natural transformation $\Phi_2(i \circ j) \overset{\cong}{\Longrightarrow} \Phi_2(i) \circ \Phi_2(j)$ is treated in the same manner.

The results are two categories

$$\operatorname*{hocolim}_{S^1 \wedge_\mu -} \mathrm{sub}_k\, wG_\bullet \mathcal{R}_f(X) := F\textstyle\int \Phi_1 \quad \text{and} \quad \mathrm{hocolim}_{\diamond_k S^1 \wedge_\mu -} wG^k_\bullet \mathcal{R}_f(X) := F\textstyle\int \Phi_2.$$

Remark 5.13. There are a number of constructions in [Thomason 1979] that may justifiably be termed homotopy colimits. This particular construction $F\int\Phi_i$ is essential, but we use the hocolim notation to provide context for the reader.

Now we explain how Proposition 5.10 promotes

$$\omega^k : \mathrm{sub}_k\, wG_\bullet \mathcal{R}_f(X) \to wG^k_\bullet \mathcal{R}_f(X)$$

to a left-op natural transformation (lont) $\epsilon : \Phi_1 \Rightarrow \Phi_2$. First, we need to associate to an object x of F a functor $\epsilon(x) : \Phi_1(x) \to \Phi_2(x)$. This is just ω^k. Then we need for each arrow $i : y \to x$ in F a natural transformation $\epsilon(i) : \epsilon(x) \circ \Phi_1(i) \Rightarrow \Phi_2(i) \circ \epsilon(y)$. For any morphism i such that $x\backslash i(y)$ has cardinality 1, we obtain $\epsilon(i)$ by inverting the isomorphism of functors provided by Proposition 5.10. For the general case, one just goes back to the proof and replaces the symbol 1 by $x\backslash i(y)$ everywhere it occurs. The coherence results of Section 2 guarantee that the necessary diagrams commute, so ϵ is a lont. By [Thomason 1979, Definition 3.1.4, p. 101], ϵ induces a functor

$$F\textstyle\int\epsilon : F\int\Phi_1 \to F\int\Phi_2.$$

We have now proved the following result.

Theorem 5.14. *The operations ω^k pass through the Thomason homotopy colimit construction to deliver operations*

$$F\textstyle\int\epsilon := \omega^k : \mathrm{hocolim}_{S^1 \wedge_\mu -} \mathrm{sub}_k\, wG_\bullet \mathcal{R}_f(X) \to \mathrm{hocolim}_{\diamond_k S^1 \wedge_\mu -} wG^k_\bullet \mathcal{R}_f(X). \qquad \square$$

Proof of Theorem 5.1. The main result of Section 4 is that

$$\Omega|hN_\Gamma \mathcal{R}^n_f(X)| \to \{1\} \times \prod_{n\geq 1} A_{\Sigma_n,\mathrm{all}}(X)$$

is an infinite loop map, and this section shows these maps are compatible with suspension. Likewise for the equivalence $\Omega|hN_\Gamma\mathcal{R}^n_f(X)| \to \Omega|wS_\bullet\mathcal{R}^n_f(X)|$. The maps obtained by passing to the limit over suspension remain infinite loop maps, and we know $\Omega\operatorname{colim}|wS_\bullet\mathcal{R}^n_f(X)| \simeq \Omega|wS_\bullet\mathcal{R}_f(X)| = A(X)$. □

6. Projecting to the free part

As stated in Theorem 5.1, the constructions of [Gunnarsson and Schwänzl 2002] as modified in Section 5 deliver a total operation

$$\omega = \prod \omega^n : A(X) \to \prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X),$$

where $A_{\Sigma_n,\{\mathrm{all}\}}(X) = \Omega|hS_\bullet\mathcal{R}_{hf}(X, \Sigma_n, \{\mathrm{all}\})|$. We examine the target of this map, and introduce the Weyl group notation $W_{\Sigma_n}H = N_{\Sigma_n}H/H$, where H is a subgroup of the permutation group Σ_n and $N_{\Sigma_n}H$ is the normalizer in Σ_n of H.

Theorem 6.1. *Let X be a space on which symmetric groups Σ_n act trivially. For each n there is a homotopy equivalence*

$$h_n : A_{\Sigma_n,\{\mathrm{all}\}}(X) \to \prod_{H\in\{\mathrm{all}\}} A(X \times B(W_{\Sigma_n}H))$$

of infinite loop spaces. Here $A(X \times B(W_{\Sigma_n}H)) = \Omega|hS_\bullet\mathcal{R}_f(X, W_{\Sigma_n}H, \{e\})|$ is the K-theory of the category of retractive $W_{\Sigma_n}H$-spaces relative to X with the action being free outside of X.

Proof. The argument is largely formal, based on some well-known facts. Let $\mathcal{F}$ be the set of conjugacy classes (H_i) of subgroups of Σ_n. This set is finite and partially ordered in the usual way: $(H_i) \preceq (H_j)$ if some conjugate of H_i is contained in H_j. The partial ordering may be extended to a linear ordering, or enumeration $\{(H_0), (H_1), \dots, (H_N)\}$, so that $(H_i) \prec (H_j)$ implies $i < j$. Observe that $(H_0) = \{e\}$, we may take (H_1) as the class of transpositions, and $(H_N) = \Sigma_n$.

For any Σ_n-space Z we may define

$$\mathcal{F}_{\succ(H)}Z = \operatorname*{colim}_{(K)\succ(H)} Z^{(K)},$$

essentially the union of the fixed point sets of the conjugates of all the subgroups properly containing a conjugate of H. The space $\mathcal{F}_{\succ(H)}Z$ is by definition a Σ_n-invariant subspace of Z. If $(H_i) \prec (H_{i+1})$ in the enumeration then we may compute $\mathcal{F}_{\succ(H_{i+1})}(\mathcal{F}_{\succ(H_i)}Z)$, essentially the fixed points of conjugates of H_{i+1} inside the fixed points of H_i. On the complement $\mathcal{F}_{\succ(H_i)}Z\backslash(\mathcal{F}_{\succ(H_{i+1})}(\mathcal{F}_{\succ(H_i)}Z))$ the group Σ_n acts and the Weyl group $W_{\Sigma_n}H_i = N_{\Sigma_n}H_i/H_i$ acts freely.

Inductively define exact functors

$$S_i,\, Q_j : \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}), \quad -1 \le i \le N,\ 0 \le j \le N$$

by letting S_{-1} be the identity functor, and putting $S_i(Y) = \mathcal{F}_{\succ(H_i)}(S_{i-1}(Y))$ for $i \ge 0$. Then the functors Q_j are defined by the natural cofibration sequences

$$S_j(Y) \rightarrowtail S_{j-1}(Y) \twoheadrightarrow Q_j(Y), \quad 0 \le j \le N.$$

For us, the important case is S_0: Since $H_0 = \{e\}$, $S_0(Y)$ is the union of the fixed point sets of all the nonidentity subgroups of G. Then the quotient $Q_0(Y)$ can be thought of as extracting the part of Y on which G acts freely.

Let $i_k : \mathcal{R}_f(X, \Sigma_n, \{H_k\}) \to \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ be the inclusion. Since $Q_k(Y)$ actually lies in $\mathcal{R}_f(X, \Sigma_n, \{H_j\})$, we may formally write $Q_k = i_k \circ \bar{Q}_k$, where $\bar{Q}_k : \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_n, \{H_k\})$ is a retraction. We want to make an inductive application of the additivity theorem for the $G_\bullet$ construction, but this requires that the input be pseudo-additive. Passing to prespectra $\Sigma^\infty\mathcal{R}_f(X)$, by [Gunnarsson et al. 1992] there results a splitting

$$\operatorname{hocolim} wG_\bullet\mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \to \prod_{H\in\{\text{all}\}} \operatorname{hocolim} wG_\bullet\mathcal{R}_f(X, \Sigma_n, \{H\})$$

induced by the functors $\bar{Q}_k$ for $0 \le k \le N$. Recalling that $W_{\Sigma_n}H = N_{\Sigma_n}H/H$ is the Weyl group of H, consider the exact functor

$$\mathcal{R}_f(X, \Sigma_n, \{H\}) \to \mathcal{R}_f(X, W_{\Sigma_n}H, \{e\}), \quad Y \mapsto Y^H.$$

The induction construction $Z \mapsto Z \times^{W_{\Sigma_n}H} \Sigma_n$ provides an exact functor going the other way and the composites in either order are equivalent to the identities. Putting these equivalences together and specializing the notation establishes a chain of homotopy equivalences

$$\begin{aligned}\operatorname{hocolim} wG_\bullet\mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) &\to \prod_{H\in\{\text{all}\}} \operatorname{hocolim} wG_\bullet\mathcal{R}_f(X, \Sigma_n, \{H\}) \\ &\to \prod_{H\in\{\text{all}\}} \operatorname{hocolim} wG_\bullet\mathcal{R}_f(X, W_{\Sigma_n}H, \{e\}). \quad \square\end{aligned}$$

This completes the proof of Theorem 1.1; to explain Theorem 1.3 is the object of the next two sections. We are focusing on the composition

$$\begin{aligned}\theta^n : A(X) \xrightarrow{\omega^n} A_{\Sigma_n,\{\text{all}\}}(X) &\xrightarrow{h_n} \prod_{H\in\{\text{all}\}} \Omega|hS_\bullet\mathcal{R}_f(X, N_{\Sigma_n}H/H, \{e\})| \\ &\xrightarrow{p_e} \Omega|hS_\bullet\mathcal{R}_h f(X, \Sigma_n, \{e\})|.\end{aligned}$$

In Section 7 we justify the interpretation $\Omega|hS_\bullet\mathcal{R}_h f(X, \Sigma_n, \{e\})| = A(X \times B\Sigma_n)$. Then we want to understand what happens when we follow this composition by the transfer $\phi_n : A(X \times B\Sigma_n) \to A(X \times E\Sigma_n) \simeq A(X)$. We start by introducing notation for the composition

$$\mathcal{R}_f(X) \to \mathrm{sub}_n\, G_\bullet\mathcal{R}_f(X) \xrightarrow{\omega^n} G^n_\bullet\mathcal{R}_f(X, \Sigma_n, \{\mathrm{all}\}) \xrightarrow{Q_0 = S_{-1}/S_0} G^n_\bullet\mathcal{R}_f(X, \Sigma_n, \{e\}).$$

On $(Y, r, s) \in \mathcal{R}_f(X)$, the composition of the first two maps in the chain is $\tilde{\alpha}_n(Y)$ in the notation of Example 3.32, so we want to evaluate the functor $Q_0 = S_{-1}/S_0 \circ \tilde{\alpha}_n$ on the object (Y, r, s). By the terminology used in the proof of Theorem 6.1, S_{-1} is the identity and S_0 is the union of subobjects that are fixed by some nonidentity subgroup of Σ_n. The interpretation and transfer issues are taken up in Section 7; to prepare for the analysis of $\phi_n \circ \theta_n$ in Section 8 we introduce some notation.

The definitions of the Segal operations in [Waldhausen 1982] use certain subfunctors P^n_j of the smash power functor P^n on pointed sets. We extend the considerations to define certain subfunctors of $\wedge_e$ and $\wedge_\mu$ powers. For $(Y, r, s) \in \mathcal{R}(X)$, the set $Y^{\wedge_e n}$ is a quotient of the cartesian product Y^n. In a fixed simplicial dimension, we view this as the set of functions $y : \boldsymbol{n} \to Y$. The pushout construction identifies any such function y with at least one value y_i in X with the composite function $r \circ y$. Thus, to represent points of $Y^{\wedge_e n}$ in a given dimension, we just need to look at functions all of whose values are in $Y - X$ and functions all of whose values are in X. For $0 \le j \le n$ we define $\widetilde{P}^n_j Y$ to be the subset of functions $y : i \mapsto y_i$ such that the cardinality of $y^{-1}(Y - X)$ is less than or equal to j, if the image of y is contained in $(Y - X)$. Said another way, $\widetilde{P}^n_j Y$ is the set of n-tuples where at most j distinct elements of $Y - X$ are involved. For example, $\widetilde{P}^n_0 Y = X^n$ and $\widetilde{P}^n_1 Y$ is the union of X^n with the diagonal of $(Y - X)^n$. Most important for us, the subset $\widetilde{P}^n_{n-1} Y$ consists of all n-tuples involving no more than $n - 1$ distinct elements of Y, so that if no member of $(y_1, \ldots, y_n)$ is in X, then there are at least two distinct indices i, j with $y_i = y_j$.

When X is a connected abelian group, then we can push out along the iterated multiplication $X^n \to X$, obtaining functors $P^n_j Y$ relative to X. In particular, $P^n_{n-1} Y$ is the subset of $P^n Y$ consisting of points fixed by some nontrivial subgroup of Σ_n, so not all members of an n-tuple can be distinct. Thus $P^n_{n-1} Y = S_0\tilde{\alpha}_n(Y)$. In terms of functions $y : \boldsymbol{n} \to Y$, $P^n_{n-1} Y$ is the set of functions where the cardinality of $y^{-1}(Y - X)$ is at most $n - 1$.

Definition 6.2. Define $\tilde{\theta}^n Y$ and $\theta^n Y$ by means of the pushout squares

$$\begin{tikzcd} \widetilde{P}^n_{n-1} Y \arrow[r, tail] \arrow[d, "\wedge^n_e r"'] & \widetilde{P}^n Y \arrow[d] \\ X^n \arrow[r, tail] & \tilde{\theta}^n Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} P^n_{n-1} Y \arrow[r, tail] \arrow[d, "r"'] & P^n Y \arrow[d] \\ X \arrow[r, tail] & \theta^n Y \end{tikzcd}$$

Letting $j^n : \mathcal{R}_f(X, \Sigma_n, \{e\}) \to G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{e\})$ be the iterated stabilization, we combine the preceding observations with the definitions to immediately obtain the following proposition.

Proposition 6.3. *As functors from $\mathcal{R}_f(X)$ to $G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{e\})$,*

$$Q_0 \circ \tilde{\alpha}_n = j^n \circ \theta^n. \qquad \square$$

7. Transfer constructions

Our immediate goal is to interpret $\Omega|hS_\bullet \mathcal{R}_f(X^n, \Sigma_n, \{e\})|$ and $\Omega|hS_\bullet \mathcal{R}_f(X, \Sigma_n, \{e\})|$ in terms of the algebraic K-theory of topological spaces. In this section, families of subgroups play no role, so we revert to the less ornate notation $\Omega|hS_\bullet \mathcal{R}_f(X, G)|$ for $\Omega|hS_\bullet \mathcal{R}_f(X, G, \{e\})|$, the algebraic K-theory of G-spaces retracting to X, finite relative to X, and with G acting freely outside X.

There are two steps to this goal and each step uses arguments based on [Waldhausen 1985]. We let G be a finite group and Z a G-space. Let EG be the canonical contractible free left G-space. We prefer the model $EG_n = G^{n+1}$ with the G-action given by multiplication on the left in each factor, face maps defined by projecting away from a coordinate, and degeneracies defined by repeating a coordinate. An isomorphism of the quotient space $* \times^G EG \cong BG$ is induced by $(g_0, \dots, g_{i-1}, g_i, \dots, g_n) \mapsto (g_0^{-1} g_1, \dots, g_{i-1}^{-1} g_i, \dots, g_{n-1}^{-1} g_n)$.

First, [Waldhausen 1985, Lemma 2.1.3, p. 381] applies to yield the following result.

Lemma 7.1. *There is an equivalence of categories*

$$\mathcal{R}(EG \times^G Z) \sim \mathcal{R}(EG \times Z, G). \qquad \square$$

For reference, pullback along the projection

$$EG \times Z \to EG \times^G Z$$

defines a functor $\mathcal{R}(EG \times^G Z) \to \mathcal{R}(EG \times Z, G)$; the orbit map defines a functor in the opposite direction. The composites in either order are isomorphic to the respective identity functors. Moreover, these functors preserve weak equivalences and homotopy finite objects.

Next, we want the following lemma, which permits us to replace the G-action on Z with a free G-action on a homotopy equivalent space.

Lemma 7.2. *The projection $EG \times Z \to Z$ induces a homotopy equivalence*

$$hS_\bullet \mathcal{R}_{hf}(EG \times Z, G) \longrightarrow hS_\bullet \mathcal{R}_{hf}(Z, G).$$

Proof. The argument here is similar to that given to prove [Waldhausen 1985, Proposition 2.1.4, p. 382]. In detail, let $(Y', r', s') \in \mathcal{R}_{hf}(EG \times Z, G)$. Completing the diagram

$$Y' \xleftarrow{s'} EG \times Z \xrightarrow{p_2} Z$$

to a pushout defines an exact functor $\mathcal{R}_{hf}(EG \times Z, G) \to \mathcal{R}_{hf}(Z, G)$. Certainly, homotopy finite objects are carried to homotopy finite objects, and, incidentally, finite objects are carried to finite objects. Also, weak equivalences are mapped to weak equivalences.

Taking the product with EG gives an exact functor $\mathcal{R}_{hf}(Z, G) \to \mathcal{R}_{hf}(EG \times Z, G)$. In this case, when G is nontrivial, finite objects are carried to homotopy finite objects, since EG is contractible.

For (Y, r, s) in $\mathcal{R}_{hf}(Z, G)$, taking the induced map of pushouts in the diagram

$$\begin{tikzcd}
EG \times Y \arrow[d, "p_2"'] & EG \times Z \arrow[l, tail] \arrow[r] \arrow[d, "p_2"'] & Z \arrow[d, "\mathrm{id}"'] \\
Y & Z \arrow[l, tail, "s"'] \arrow[r, "\mathrm{id}"] & Z
\end{tikzcd}$$

provides a natural transformation from the composite endofunctor on $\mathcal{R}_{hf}(Z, G)$ to the identity functor. This natural transformation is a weak equivalence. For (Y', r', s') in $\mathcal{R}_{hf}(EG \times Z, G)$, taking the induced map of pushouts in the diagram

$$\begin{tikzcd}
Y' \arrow[d, "p_1 r' \times \mathrm{id}"'] & EG \times Z \arrow[l, tail, "s'"'] \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta \times \mathrm{id}"'] & EG \times Z \arrow[d, "\mathrm{id}"'] \\
EG \times Y' & EG \times EG \times Z \arrow[l, tail, "\mathrm{id} \times s'"'] \arrow[r, "p_1 \times p_3"] & EG \times Z
\end{tikzcd}$$

provides a natural transformation from the identity functor on $\mathcal{R}_{hf}(EG \times Z, G)$ to the other composite endofunctor. Again, this is a weak equivalence. By [Waldhausen 1985, Proposition 1.3.1, p. 330], $hS_\bullet \mathcal{R}_{hf}(Z \times EG, G) \to hS_\bullet \mathcal{R}_{hf}(Z, G)$ is a homotopy equivalence. □

Substituting for G the symmetric group Σ_n, we combine Lemmas 7.1 and 7.2 to record useful alternative models for $A(B\Sigma_n \times X)$ and $A(D_n X)$. The first part covers a remark made following Definition 3.5.

Lemma 7.3. *Let X have the trivial Σ_n-action, so that $B\Sigma_n \times X$ is the quotient of $E\Sigma_n \times X$ by the action of Σ_n. There are homotopy equivalences*

$$hS_\bullet \mathcal{R}_{hf}(B\Sigma_n \times X) \simeq hS_\bullet \mathcal{R}_{hf}(E\Sigma_n \times X, \Sigma_n) \simeq hS_\bullet \mathcal{R}_{hf}(X, \Sigma_n). \tag{7.4}$$

Thus, the space $\Omega|hS_\bullet \mathcal{R}_f(X, \Sigma_n)|$ is homotopy equivalent to $A(B\Sigma_n \times X)$.

Similarly, let X^n have the permutation action of Σ_n, and let $D_n X = E\Sigma_n \times^{\Sigma_n} X^n$ be the quotient of $E\Sigma_n \times X^n$ by the diagonal action of Σ_n. There are homotopy equivalences

$$hS_\bullet \mathcal{R}_{hf}(D_n X) \simeq hS_\bullet \mathcal{R}_{hf}(E\Sigma_n \times X^n, \Sigma_n) \simeq hS_\bullet \mathcal{R}_{hf}(X^n, \Sigma_n).$$

Thus, the space $\Omega|hS_\bullet \mathcal{R}_{hf}(X^n, \Sigma_n)|$ is homotopy equivalent to $A(D_n X)$. □

We recall here basic facts about the transfer in the algebraic K-theory of spaces adapted to our context. We are actually interested in two cases of transfer operations. For the first case the transfer operations are associated with finite subgroups of the symmetric groups Σ_n. In the second case the operations are associated with (injective) homomorphisms of simplicial abelian groups $\widetilde{X} \to X$, where the fiber is homotopy finite.

In terms of the description $A(X) = \Omega|hS_\bullet\mathcal{R}_f(X)|$, we have the following direct transfer construction. A fiber bundle projection $p : E \to B$ with finite fiber induces by pullback a functor $\mathcal{R}_f(B) \to \mathcal{R}_f(E)$, or with homotopy finite fiber, $\mathcal{R}_f(B) \to \mathcal{R}_{hf}(E)$. We then obtain a transfer morphism $p^* : A(B) \to A(E)$. In terms of equivariant models for algebraic K-theory, there are other descriptions of the transfer, as given below. We need to relate the various descriptions.

Eventually we need the transfer operations $A(B\Sigma_n \times X) \to A(BH \times X)$, where H is a subgroup of Σ_n. Our working definition is $A(B\Sigma_n \times X) = \Omega|hS_\bullet\mathcal{R}_{hf}(X, \Sigma_n)|$ but, in view of the equivalences (7.4), we have to compare three definitions in each context.

To this end, let G be a discrete group, H a subgroup of finite index, and Z a trivial G-space. Observe that $EG \times Z$ is the total space of a principal G-bundle with base $EG \times^G Z$. To make this transparent, and for use in the study of diagram (7.5), we replace the notation $EG \times^G Z$ by $* \times^G (EG \times Z)$. To explain the connection, $* \times^G (EG \times Z)$ is the orbit space of $EG \times Z$ under the diagonal left G-action, thought of as the balanced product of $EG \times Z$ with the trivial right G-space $*$. We can turn the left action of G on EG into a right action by setting $e \cdot_r g = g^{-1} \cdot_l e$. Then left G-orbits in $EG \times Z$ are seen to correspond to equivalence classes in $EG \times Z$ under the equivalence relation generated by $(e \cdot_r g, z) \sim (e, gz)$. The associated quotient space is usually denoted $EG \times^G Z$.

We consider the diagram

$$\begin{tikzcd}
\mathcal{R}(Z, H) \arrow[r] & \mathcal{R}(EG \times Z, H) & \mathcal{R}(EG \times^H Z) \arrow[l] \\
\mathcal{R}(Z, G) \arrow[r] \arrow[u, "p_1^*"'] & \mathcal{R}(EG \times Z, G) \arrow[u, "p_2^*"'] & \mathcal{R}(EG \times^G Z) \arrow[l] \arrow[u, "p_3^*"']
\end{tikzcd} \tag{7.5}$$

where the vertical arrows represent transfer constructions. The forgetful functor $p_1^* : \mathcal{R}(Z, G) \to \mathcal{R}(Z, H)$ just restricts the action to the subgroup H. This provides the simplest path to $p_1^* : A(BG \times Z) \to A(BH \times Z)$, using the basic model $A(BG \times Z) = \Omega|hS_\bullet(\mathcal{R}_{hf}(Z, G)|$. In the middle, the functor p_2^* is also a forgetful functor. At the right, the functor p_3^* is given by a pullback construction, explained in detail below.

To reach the categories in the middle column from those in the left column we compute products with EG. Along the top, the fact that EG is a nonstandard contractible H-space is an insignificant detail. Comparing with p_1^* on the left, the transfer p_2^* in the middle column is also obtained by restricting the action of G to H. Thus, the left-hand square in diagram (7.5) obviously commutes.

Before we compare p_3^* with p_2^*, we discuss p_3^*, the rightmost column in diagram (7.5), in detail. In order to manipulate pullback squares efficiently we replace the notation $EG \times^G Z$ by $* \times^G (EG \times Z)$ as discussed before Lemma 7.1. Suppose H is a subgroup of the group G, and let EG be the standard model for a contractible G-space on which G acts freely from the right. The space EG plays a similar role relative to the subgroup H. In order to compare situations, we take the standard model $X = * \times^G (EG \times Z)$ and a modified model $\widetilde{X} = * \times^H (EG \times Z)$. In this situation we have the basic pullback square

$$\begin{array}{ccc} (* \times^H G) \times (EG \times Z) & \longrightarrow & (* \times^H EG \times Z) = \widetilde{X} \\ {\scriptstyle p_2}\big\downarrow & & \big\downarrow \\ EG \times Z & \longrightarrow & * \times^G (EG \times Z) = X \end{array} \tag{7.6}$$

This displays the comparison map $\widetilde{X} \to X$ of the chosen models as a fiber bundle, with fiber $* \times^H G$. One may identify $* \times^H (EG \times Z) \cong (* \times^H G) \times^G (EG \times Z)$, and then the right-hand vertical arrow is isomorphic to the map

$$(* \times^H G) \times^G (EG \times Z) \to * \times^G (EG \times Z)$$

induced by projecting the coset space $* \times^H G$ to a point. This replacement also displays the upper horizontal map as the quotient projection

$$(* \times^H G) \times (EG \times Z) \to (* \times^H G) \times^G (Z \times EG).$$

The direct construction $p^* : \mathcal{R}(X) \to \mathcal{R}(\widetilde{X})$ maps (Y, r, s) to $(\widetilde{Y}, \tilde{r}, \tilde{s})$, derived from the pullback square

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\tilde{r}} & \widetilde{X} = * \times^H (EG \times Z) \\ \big\downarrow & & {\scriptstyle p}\big\downarrow \\ Y & \xrightarrow{r} & X = * \times^G (EG \times Z) \end{array} \tag{7.7}$$

Augmenting the right-hand column of (7.7) to the square of (7.6) shows that $\widetilde{Y} \to Y$ is a fiber bundle with fiber $* \times^H G$.

Now we address commutativity of the right-hand square in diagram (7.5). To reach the categories in the middle column from the categories in the right column, we also compute pullbacks. Recalling Lemma 7.1, the equivalence of categories $\mathcal{R}(EG \times^G Z) \simeq \mathcal{R}(EG \times Z, G)$ [Waldhausen 1985, Lemma 2.1.3] describes the functor moving left to the middle column. This functor assigns to a retractive space

(Y, r, s) over $EG \times^G Z$ the retractive G-space (Y', r', s') over $EG \times Z$ defined as the pullback in the following diagram:

$$\begin{tikzcd} Y' \arrow[r, "r'"] \arrow[d] & EG \times Z \arrow[d] \\ Y \arrow[r, "r"] & X = * \times^G (EG \times Z) \end{tikzcd}$$

Then moving up to $\mathcal{R}(EG \times Z, H)$ amounts to restricting the G-action in this pullback to H.

On the other hand, to move from the lower right to the upper middle by going up and then to the left, compute first the pullback (7.7) and then compute

$$\begin{tikzcd} \widetilde{Y}' \arrow[r, "\tilde{r}'"] \arrow[d] & EG \times Z \arrow[d] \\ \widetilde{Y} \arrow[r, "\tilde{r}"] & \widetilde{X} = * \times^H (EG \times Z). \end{tikzcd}$$

The composition of the two functors may be displayed in the stacked diagram

$$\begin{tikzcd} \widetilde{Y}' \arrow[r, "\tilde{r}'"] \arrow[d] & EG \times Z \arrow[d] \\ \widetilde{Y} \arrow[r, "\tilde{r}"] \arrow[d] & \widetilde{X} = * \times^H (EG \times Z) \arrow[d, "p"'] \\ Y \arrow[r, "r"] & X = * \times^G (EG \times Z) \end{tikzcd}$$

The end result is that $(\widetilde{Y}', \tilde{r}', \tilde{s}')$ is simply the G-space (Y', r', s') with the action restricted to H. Therefore, the right-hand square commutes.

Lemma 7.8 (cf. [Waldhausen 1982, Lemma 1.3, p. 399]). *Let G be a finite group, EG a universal G-bundle, $BG = * \times^G EG$ a classifying space, and Z a space with a trivial G-action. Then the composition*

$$A(Z) \xrightarrow{\textit{inclusion}} A(BG \times Z) \xrightarrow{\textit{transfer}} A(EG \times Z) \simeq A(Z)$$

is given by multiplication by the order of G, in the sense of the additive H-space structure. □

8. A fundamental cofibration sequence

Waldhausen's main result is this proposition.

Proposition 8.1 (cf. [Waldhausen 1982, Proposition 2.7, p. 407]). *The composition of the operation $\theta^n : A(*) \to A(B\Sigma_n \times *)$ with the transfer map $\phi_n : A(B\Sigma_n \times *) \to A(*)$*

is the same, up to weak homotopy, as the polynomial map on $A(*)$ given by the polynomial

$$p(x) = x(x-1)\cdots(x-n+1). \qquad \square$$

The analogous result for the present situation with the one-point space replaced by a simplicial abelian group X is more complicated to formulate and to work with. To prepare for the analogue of Waldhausen's result, we develop the following constructions, taking up where we left off with Definition 6.2 and Proposition 6.3. We make use of the maps

$$\delta^{n,k}_{n-1} : X^{n-1} \to X^n, \quad (x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, x_k),$$

and the respective induced functors $\delta^{n,k}_{n-1*} : \mathcal{R}_f(X^{n-1}) \to \mathcal{R}_f(X^n)$. The pushout construction

$$\begin{tikzcd} X^{n-1} \arrow[r, tail, "s"] \arrow[d, "\delta^{n,k}_{n-1}"'] & Z \arrow[d, "i^{n,k}_{n-1}"] \\ X^n \arrow[r, tail] & \delta^{n,k}_{n-1*}Z \end{tikzcd}$$

defines an exact functor $\delta^{n,k}_{n-1*} : \mathcal{R}_f(X^{n-1}) \to \mathcal{R}_f(X^n)$. For a retractive space (Z, r, s) over X^{n-1} with retraction $r : Z \to X^{n-1}$ written in terms of components as $r = (r_1, \ldots, r_{n-1})$, the composition of the canonical map $i^{n,k}_{n-1}$ followed by the retraction $\delta^{n,k}_{n-1*}r$ is given by the formula

$$(\delta^{n,k}_{n-1*}r) \circ i^{n,k}_{n-1}(z) = \delta^{n,k}_{n-1} \circ r(z) = (r_1(z), \ldots, r_k(z), \ldots, r_{n-1}(z), r_k(z)).$$

Note that in the special case $Z = \widetilde{P}^{n-1}Y = (\wedge_e)^{n-1}Y$, we have, for each k such that $1 \le k \le n-1$,

$$\big(\delta^{n,k}_{n-1*}(\widetilde{P}^{n-1}r)\big) \circ i^{n,k}_{n-1}(y_1, \ldots, y_{n-1}) = (r(y_1), \ldots, r(y_k), \ldots, r(y_{n-1}), r(y_k)). \quad (8.2)$$

Next we assemble these functors by gluing along the common space X^n, obtaining

$$\widetilde{\Delta}^n_{n-1} : \mathcal{R}_f(X^{n-1}) \to \mathcal{R}_f(X^n)$$

given on objects by $\widetilde{\Delta}^n_{n-1}(Z) = \delta^{n,1}_{n-1*}Z \cup_{X^n} \cdots \cup_{X^n} \delta^{n,n-1}_{n-1*}Z$, which can be viewed as an iterated pushout or as the colimit of a diagram modeled on the cone on $n-1$ points. We also need to push this construction forward to $\mathcal{R}_f(X)$ by μ_*, the iterated multiplication, obtaining

$$\Delta^n_{n-1} = \mu_* \circ \widetilde{\Delta}^n_{n-1} : \mathcal{R}_f(X^{n-1}) \to \mathcal{R}_f(X)$$

given on objects by $\Delta^n_{n-1}(Z) = \mu_*(\delta^{n,1}_{n-1*}Z) \cup_X \ldots \cup_X \mu_*(\delta^{n,n-1}_{n-1*}Z)$. If we start with

$$Z = Y \wedge_e \overset{n-1 \text{ factors}}{\cdots} \wedge_e Y = \widetilde{P}^{n-1}Y,$$

then the formula for the retraction on the k-th summand $\mu_*(\delta^{n,k}_{n-1*}\widetilde{P}^{n-1}Y)$ is

$$\begin{aligned}\big(\mu_*\delta^{n,k}_{n-1*}(\widetilde{P}^{n-1}r)\big)\circ i^{n,k}_{n-1}&(y_1,\dots,y_{n-1})\\ &=\mu(r(y_1),\dots,r(y_k),\dots,r(y_{n-1}),r(y_k)),\end{aligned} \tag{8.3}$$

where μ is the iterated multiplication.

We can now succinctly state our general results. Let

$$\tilde{\phi}_k:\mathcal{R}_f(X^k,\Sigma_k,\{\text{all}\})\to\mathcal{R}_f(X^k)\quad\text{and}\quad \phi_k:\mathcal{R}_f(X,\Sigma_k,\{\text{all}\})\to\mathcal{R}_f(X)$$

be the functors that forget the group action.

Proposition 8.4 (cf. [Waldhausen 1982, Proposition 2.7, p. 407]). *There is a cofibration sequence of functors $\mathcal{R}_f(X)\to\mathcal{R}_f(X^n)$*

$$\widetilde{\Delta}^n_{n-1}\tilde{\phi}_{n-1}\tilde{\theta}^{n-1}Y\rightarrowtail\tilde{\phi}_{n-1}\tilde{\theta}^{n-1}Y\wedge_e\tilde{\theta}^1Y\twoheadrightarrow\tilde{\phi}_n\tilde{\theta}^nY. \tag{8.5}$$

In the case that X is a connected simplicial abelian group, we have the cofibration sequence

$$\Delta^n_{n-1}\tilde{\phi}_{n-1}\tilde{\theta}^{n-1}Y\rightarrowtail\phi_{n-1}\theta^{n-1}Y\wedge_\mu\theta^1Y\twoheadrightarrow\phi_n\theta^nY \tag{8.6}$$

of functors $\mathcal{R}_f(X)\to\mathcal{R}_f(X)$.

Remark 8.7. The second cofibration sequence is obtained by applying the exact functor induced by the iterated multiplication $\mu: X^n\to X$ to the first sequence. The result in the middle term of the second sequence is open to interpretation. The formulation chosen amounts to interpretation of the factorization $\mu=\mu\circ(\mu\times\mathrm{id})$ along with the facts that $\mu_*\circ\wedge_e=\wedge_\mu$ and $\tilde{\theta}^1Y=\theta^1Y=Y$.

Proof of Proposition 8.4. Following Section 7, we interpret the transfer maps

$$\phi_n:A(D_nX)\to A(X^n)\quad\text{and}\quad\phi_n:A(X\times B\Sigma_n)\to A(X)$$

as induced by the forgetful functors

$$\mathcal{R}_f(X^n,\Sigma_n,\{\text{all}\})\to\mathcal{R}_f(X^n,\{e\})\quad\text{and}\quad\mathcal{R}_f(X,\Sigma_n,\{\text{all}\})\to\mathcal{R}_f(X,\{e\}),$$

respectively. This means we have to make nonequivariant analyses of the functors $\tilde{\theta}^n$ and θ^n, respectively.

To obtain the surjections, we consider the diagram

$$\begin{tikzcd} X \arrow[d] & X^{n-1}\times X \arrow[l, "\mu"'] \arrow[d] & \widetilde{P}^{n-1}_{n-2}Y\wedge_e Y \arrow[l, "((\wedge_e)^{n-1}r)\wedge_e r"'] \arrow[d] \arrow[r, tail] & \widetilde{P}^{n-1}Y\wedge_e Y \arrow[d, "\cong"] \\ X & X^n \arrow[l, "\mu"'] & \widetilde{P}^n_{n-1}Y \arrow[l, "r^n"'] \arrow[r, tail] & \widetilde{P}^nY \end{tikzcd} \tag{8.8}$$

Clearly, $\widetilde{P}^{n-1}_{n-2} \wedge_e Y$ maps into $\widetilde{P}^{n}_{n-1}$, because, if there are two indices i, j with $1 \le i, j \le n-1$ and $i \neq j$ and with $y_i = y_j$, then this still holds for $((y_1, \dots, y_{n-1}), y)$ rebracketed as $(y_1, \dots, y_{n-1}, y)$. Taking the pushouts along the rows using the columns two, three, and four produces a surjection

$$\phi_{n-1}\tilde{\theta}^{n-1}Y \wedge_e \tilde{\theta}^1 Y \twoheadrightarrow \phi_n \tilde{\theta}^n Y$$

in $\mathcal{R}_f(X^n)$ and pushing out along the rows using columns one, three and four yields

$$\phi_{n-1}\theta^{n-1}Y \wedge_\mu \theta^1 Y = \mu_*(\phi_{n-1}\theta^{n-1}Y \wedge_e \theta^1 Y) \twoheadrightarrow \phi_n \theta^n Y,$$

the surjection in $\mathcal{R}_f(X)$. Now we have to identify the “kernels”.

Reviewing the remarks at the end of Section 6, $\widetilde{P}^{n-1}Y \wedge_e Y = \widetilde{P}^n Y$ is the space whose simplices outside of X^n are n-tuples of simplices from $Y - X$; $\widetilde{P}^{n-1}_{n-2}Y \wedge_e Y$ is the space whose simplices outside of X^n are n-tuples $((y_1, \dots, y_{n-1}), y)$ with the condition that there are at least two distinct indices $1 \le i, j \le n-1$ with $y_i = y_j$; and $\widetilde{P}^n_{n-1}Y$ is the space whose simplices outside of X^n are n-tuples $(y_1, \dots, y_{n-1}, y_n)$ with the condition that there are at least two distinct indices $1 \le i, j \le n$ with $y_i = y_j$. Then the simplices of $\widetilde{P}^n_{n-1}Y$ not in the image of $\widetilde{P}^{n-1}_{n-2}Y \wedge_e Y$ are those n-tuples where the first $n-1$ are distinct but $y_n = y_k$ for some $1 \le k \le n-1$.

Using this observation we extend the diagram (8.8) by means of the following constructions. For $1 \le k \le n-1$, consider the diagrams

$$\begin{tikzcd}
X^n \arrow[d] & X^{n-1} \arrow[l, "\delta^{n,k}_{n-1}"'] \arrow[r] \arrow[d, "\delta^{n,k}_{n-1}"'] & \widetilde{P}^{n-1}Y \arrow[d, "\delta^{n,k}_{n-1}"] \\
X^n & \widetilde{P}^{n-1}Y \times X \cup_{X^n} X^{n-1} \times Y \arrow[l] \arrow[r] & \widetilde{P}^{n-1}Y \times Y
\end{tikzcd}$$

where $\delta^{n,k}_{n-1} : X^{n-1} \to X^n$ is given by $\delta^{n,k}_{n-1}(x_1, \dots, x_{n-1}) = (x_1, \dots, x_{n-1}, x_k)$ and the other maps labeled $\delta^{n,k}_{n-1}$ are given by similar formulas. For each k, taking the pushout of the first row extends $\widetilde{P}^{n-1}Y$ over X^{n-1} to the space $\delta^{n,k}_{n-1*}\widetilde{P}^{n-1}Y$ over X^n; taking the pushout of the second row yields $\widetilde{P}^{n-1}Y \wedge_e Y$. Since the diagram commutes, we obtain a family of maps over X^n

$$\delta^{n,k}_{n-1} : \delta^{n,k}_{n-1*}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y \to \tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y \wedge_e Y$$

with $\delta^{n,k}_{n-1}(y_1, \dots, y_{n-1}) = (y_1, \dots, y_k, \dots, y_{n-1}, y_k)$.

Now we are ready to augment diagram (8.8), after which we can compute the desired cofibration sequence. Having established the notation

$$\widetilde{\Delta}^n_{n-1}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y = \delta^{n,1}_{n-1*}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y \cup_{X^n} \cdots \cup_{X^n} \delta^{n,n-1}_{n-1*}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y,$$

write $\Delta^n_{n-1} : \Delta^n_{n-1}\phi_{n-1}\widetilde{P}^{n-1}Y \to \phi_{n-1}\widetilde{P}^{n-1}Y \wedge_e Y$ for the union of the maps $\delta^{n,k}_{n-1}$ just defined. Add this map above the upper right corner of (8.8) and fill out the following diagram:

$$
\begin{tikzcd}
X \arrow[d] & X^n \arrow[l, "\mu"'] \arrow[d, "\cong"] & \Delta^n_{n-1}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}_{n-2}Y \arrow[l] \arrow[r, tail, "i'"] \arrow[d, tail, "\Delta^n_{n-1}"'] & \Delta^n_{n-1}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y \arrow[d, tail, "\Delta^n_{n-1}"'] \\
X \arrow[d] & X^{n-1}\times X \arrow[l, "\mu"'] \arrow[d] & \tilde{\phi}_{n-1}\widetilde{P}^{n-1}_{n-2}Y \wedge_e Y \arrow[l, "r^{n-1}\wedge_e r"'] \arrow[r, tail, "i"] \arrow[d] & \tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y \wedge_e Y \arrow[d, "\cong"] \\
X & X^n \arrow[l, "\mu"'] & \widetilde{P}^n_{n-1}Y \arrow[l, "r^n"'] \arrow[r, tail, "i''"] & \widetilde{P}^nY
\end{tikzcd}
\tag{8.9}
$$

To explain the entry at the top of the third column, we identify the conditions on

$$(z_1, \dots, z_n) \in (\tilde{\phi}_{n-1}\widetilde{P}^{n-1}_{n-2}Y) \wedge_e Y \quad \text{and} \quad (y_1, \dots, y_{n-1}) \in \Delta^n_{n-1}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y$$

such that $i(z_1, \dots, z_n) = \Delta^n_{n-1}(y_1, \dots, y_{n-1})$. We see that $z_j = y_j$ for $1 \le j \le n-1$ and that there is k between 1 and $n-1$ such that $z_n = y_k$. Moreover, since no more than $n-2$ of the first $n-1$ simplices z_j are distinct, no more than $n-2$ of the simplices y_j are distinct. Hence, we obtain the description of the term at the top of the third column. Additionally we obtain the fact that the induced map

$$\bigl(\phi_{n-1}\widetilde{P}^{n-1}_{n-2}Y \wedge_e Y\bigr) \cup_{(\Delta^n_{n-1}\phi_{n-1}\widetilde{P}^{n-1}_{n-2}Y)} \bigl(\Delta^n_{n-1}\phi_{n-1}\widetilde{P}^{n-1}Y\bigr) \rightarrowtail \phi_{n-1}\widetilde{P}^{n-1}Y \wedge_e Y$$

is a cofibration, so Lemma 2.9 applies to diagram (8.9).

One takes the row-wise pushout of the three columns on the right and obtains the cofibration sequence in $\mathcal{R}_f(X^n)$

$$\widetilde{\Delta}^n_{n-1}\tilde{\phi}_{n-1}\widetilde{\theta}^{n-1}Y \overset{\widetilde{\Delta}^n_{n-1}}{\rightarrowtail} \tilde{\phi}_{n-1}\widetilde{\theta}^{n-1}Y \wedge_e \theta^1 Y \twoheadrightarrow \tilde{\phi}_n\widetilde{\theta}^n Y,$$

which is (8.5) from the statement.

One also composes the arrows pointing to the left in each row and takes the row-wise pushout of the resulting diagram, which consists of columns one, three, and four of the diagram (8.9), obtaining

$$\Delta^n_{n-1}\tilde{\phi}_{n-1}\tilde{\theta}^{n-1}Y \overset{\mu_*\widetilde{\Delta}^n_{n-1}}{\rightarrowtail} \phi_{n-1}\theta^{n-1}Y \wedge_\mu \theta^1 Y \twoheadrightarrow \phi_n\theta^n Y,$$

which is the second cofibration sequence (8.6) in the statement. $\square$

We want to apply the cofibration sequence (8.6) to evaluate the composite $\phi_n\theta^n$ on a homotopy class in $\pi_j A(X)$, where the basepoint is taken in the zero component. Two features of algebraic K-theory make this possible. The first feature is essentially a consequence of the additivity theorem and says that cofibration sequences imply additive relations.

Lemma 8.10. *Let Z be a space. The two composite maps*

$$|hS_2\mathcal{R}_f(Z)| \underset{s\vee q}{\overset{t}{\rightrightarrows}} |h\mathcal{R}_f(Z)| \longrightarrow \Omega|hS_\bullet\mathcal{R}(Z)|$$

are homotopic, where the right-hand arrow is the canonical map

$$|h\mathcal{R}_f(Z)| \to \Omega|hS_\bullet\mathcal{R}_f(Z)|. \qquad \square$$

The second feature is the triviality of products in higher homotopy groups, explained as follows. Since X is a simplicial abelian group, the homotopy functor $Y \mapsto [Y, A(X)]$ has a ring structure induced from the biexact pairing

$$\mathcal{R}(X) \times \mathcal{R}(X) \xrightarrow{\wedge_e} \mathcal{R}(X \times X) \xrightarrow{\mu_*} \mathcal{R}(X).$$

Now suppose $Y = \Sigma Y'$ is a suspension. Under this ring structure the product of two elements $[f_1]$ and $[f_2]$ in $[Y, A(X)]$ is zero, because $[f_1]$ may be represented by a map taking the upper cone C_+Y' in $\Sigma Y'$ to the point in $A(X)$ represented by the zero element in $\mathcal{R}_f(X)$, while $[f_2]$ is represented by a map taking the lower cone C_-Y' in $\Sigma Y'$ to the zero element. In a similar manner, there are pairings

$$\mathcal{R}(X^{n-1}) \times \mathcal{R}(X) \xrightarrow{\wedge_e} \mathcal{R}(X^{n-1} \times X) = \mathcal{R}(X^n)$$

and these are also zero on higher homotopy groups. Combining these observations means we have a chance to compute by induction the action of $\phi_n\theta^n$ on higher homotopy groups, because at each stage of the induction the middle term of the relevant cofibration contributes nothing to the final answer.

To start the induction, we compute $(\phi_2\theta^2)_*[f]$ for $f : S^j \to A(X)$. Applying the additivity theorem to the cofibration sequence (8.6), we can write

$$(\phi_2\theta^2)_*[f] = (\theta^1_*[f] \wedge_\mu \theta^1_*[f]) - (\Delta^2_1\theta^1)_*[f].$$

For the first term on the right side of the equation, we have observed that this product is zero. So we first obtain

$$(\phi_2\theta^2)_*[f] = -(\Delta^2_1\theta^1)_*[f]. \tag{8.11}$$

We analyze this expression as follows. First, ϕ_1 and θ^1 are identity functors. For $n = 2$, there is one diagonal map $\delta^{2,1}_1 : Z \to \delta^{2,1}_1 Z$, so

$$\widetilde{\Delta}^2_1\phi_1\theta^1 Y = \delta^{2,1}_{1*}\phi_1\theta^1(Y) = \delta^{2,1}_{1*} Y.$$

Then $\Delta^2_1\phi_1\theta^1 = \mu_* \circ \widetilde{\Delta}^2_1\phi_1\theta^1 = \mu_* \circ \delta^{2,1}_{1*}$, and the point is to see what is happening with the retraction $r : Y \to X$. Applying formula (8.3), the composition

$$\mu \circ (\widetilde{\Delta}^2_1 r) \circ i^{2,1}_1(y) = \mu(r(y), r(y)) = (r(y))^2 = (\tau_2 \circ r)(y),$$

where $\tau^2 : X \to X$ is the squaring homomorphism. That is, the action of $\Delta_1^2 = \mu_* \widetilde{\Delta}_1^2$ on homotopy is the same as the action on homotopy induced by the squaring homomorphism τ^2. Consequently,

$$(\phi_2\theta^2)_*[f] = -\tau_*^2[f].$$

The general result is the next theorem.

Theorem 8.12. *Let $\tau^n : X \to X$ be the homomorphism that raises elements to the n-th power, thinking of the operation in X as multiplication. Then*

$$\phi_n\theta_*^n = (-1)^{n-1} \cdot (n-1)! \cdot \tau_*^n : \pi_j A(X) \to \pi_j A(X)$$

for $j > 0$.

Proof. First we observe that on higher homotopy groups,

$$(\phi_n\theta^n)_* = (-1)^{n-1} \cdot (\Delta_{n-1}^n \widetilde{\Delta}_{n-2}^{n-1} \cdots \widetilde{\Delta}_1^2)_*.$$

An application of the cofibration sequence (8.6) and the vanishing product principle gives $(\phi_n\theta^n)_* = (-1) \cdot (\Delta_{n-1}^n \tilde{\phi}_{n-1}\tilde{\theta}^{n-1})_*$. Then one continues, with applications of the cofibration sequence (8.5) and the vanishing pairing principle,

$$\begin{aligned}(\phi_n\theta^n)_* &= (-1)^2 \cdot \left(\Delta_{n-1}^n \widetilde{\Delta}_{n-2}^{n-1} \tilde{\phi}_{n-2}\tilde{\theta}^{n-2}\right)_* = \cdots \\ &= (-1)^{n-1} \cdot \left(\Delta_{n-1}^n \widetilde{\Delta}_{n-2}^{n-1} \cdots \widetilde{\Delta}_1^2\right)_* = (-1)^{n-1} \cdot \left(\mu_* \widetilde{\Delta}_{n-1}^n \widetilde{\Delta}_{n-2}^{n-1} \cdots \widetilde{\Delta}_1^2\right)_*,\end{aligned}$$

recalling that $\tilde{\phi}_1$ and $\tilde{\theta}^1$ are identity functors.

Since the functors $\widetilde{\Delta}_{p-1}^p$ are built by unions from functors $\tilde{\delta}_{p-1*}^{p,k}$, we have to analyze composites

$$\delta_{n-1*}^{n,k_{n-1}} \circ \delta_{n-2*}^{n-1,k_{n-2}} \circ \cdots \circ \delta_{1*}^{2,1} : \mathcal{R}_f(X) \to \mathcal{R}_f(X^n)$$

for all choices of indices $1 \le k_{n-1} \le n-1$, $1 \le k_{n-2} \le n-2$, $\ldots$, $1 \le k_2 \le 2$. On (Y, r, s) the value of the chain is $(Y \cup_X X^n, r^n, s)$, where the retraction $r^n : Y \to X^n$ is evaluated by repeated application of formula (8.2). When we apply μ_* to this object, the value on (Y, r, s) is seen to be $(Y, \tau^n \circ r, s)$. Finally, we identify the numerical coefficient $(n-1)!$ by counting the number of terms in the composites $\widetilde{\Delta}_{n-1}^n \widetilde{\Delta}_{n-2}^{n-1} \cdots \widetilde{\Delta}_1^2$ according to the description above. ☐

References

[Betley 1986] S. Betley, "On the homotopy groups of $A(X)$", *Proc. Amer. Math. Soc.* **98**:3 (1986), 495–498. MR Zbl

[Grayson 1989] D. R. Grayson, "Exterior power operations on higher K-theory", *K-Theory* **3**:3 (1989), 247–260. MR Zbl

[Grunewald et al. 2008] J. Grunewald, J. R. Klein, and T. Macko, "Operations on the A-theoretic nil-terms", *J. Topol.* **1**:2 (2008), 317–341. MR Zbl

[Gunnarsson and Schwänzl 2002] T. Gunnarsson and R. Schwänzl, "Operations in A-theory", *J. Pure Appl. Algebra* **174**:3 (2002), 263–301. MR Zbl

[Gunnarsson et al. 1992] T. Gunnarsson, R. Schwänzl, R. M. Vogt, and F. Waldhausen, "An undelooped version of algebraic K-theory", *J. Pure Appl. Algebra* **79**:3 (1992), 255–270. MR Zbl

[Klein and Williams 2008] J. R. Klein and E. B. Williams, "The "fundamental theorem" for the algebraic K-theory of spaces, III: The nil-term", *Proc. Amer. Math. Soc.* **136**:9 (2008), 3025–3033. MR Zbl

[Laplaza 1972] M. L. Laplaza, "Coherence for distributivity", pp. 29–65 in *Coherence in categories*, edited by S. MacLane, Lecture Notes in Math. **281**, Springer, 1972. MR Zbl

[May and Sigurdsson 2006] J. P. May and J. Sigurdsson, *Parametrized homotopy theory*, Mathematical Surveys and Monographs **132**, American Mathematical Society, Providence, RI, 2006. MR Zbl

[Segal 1974a] G. Segal, "Categories and cohomology theories", *Topology* **13** (1974), 293–312. MR Zbl

[Segal 1974b] G. Segal, "Operations in stable homotopy theory", pp. 105–110 in *New developments in topology* (Oxford, 1972), London Math Soc. Lecture Note Ser. **11**, Cambridge University Press, 1974. MR Zbl

[Thomason 1979] R. W. Thomason, "Homotopy colimits in the category of small categories", *Math. Proc. Cambridge Philos. Soc.* **85**:1 (1979), 91–109. MR Zbl

[Waldhausen 1982] F. Waldhausen, "Operations in the algebraic K-theory of spaces", pp. 390–409 in *Algebraic K-theory, Part II* (Oberwolfach, 1980), Lecture Notes in Math. **967**, Springer, 1982. MR Zbl

[Waldhausen 1985] F. Waldhausen, "Algebraic K-theory of spaces", pp. 318–419 in *Algebraic and geometric topology* (New Brunswick, NJ, 1983), edited by A. Ranicki et al., Lecture Notes in Math. **1126**, Springer, 1985. MR Zbl

[Waldhausen 1987] F. Waldhausen, "Algebraic K-theory of spaces, concordance, and stable homotopy theory", pp. 392–417 in *Algebraic topology and algebraic K-theory* (Princeton, NJ, 1983), edited by W. Browder, Ann. of Math. Stud. **113**, Princeton Universty Press, 1987. MR Zbl

[Waldhausen et al. 2013] F. Waldhausen, B. Jahren, and J. Rognes, *Spaces of PL manifolds and categories of simple maps*, Annals of Mathematics Studies **186**, Princeton University Press, 2013. MR Zbl

Received 11 Jul 2017. Revised 9 Jul 2018. Accepted 31 Oct 2018.

THOMAS GUNNARSSON: thomasewgunnarsson@gmail.com
Department of Mathematics, Luleå University of Technology, Luleå, Sweden

ROSS STAFFELDT: ross@nmsu.edu
Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM, United States

ANNALS OF K-THEORY

msp.org/akt

Annals of K-Theory is a journal of the K-Theory Foundation (ktheoryfoundation.org). The K-Theory Foundation acknowledges the precious support of Foundation Compositio Mathematica, whose help has been instrumental in the launch of the Annals of K-Theory.

See inside back cover or msp.org/akt for submission instructions.

The subscription price for 2019 is US $490/year for the electronic version, and $550/year (+$25, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Annals of K-Theory (ISSN 2379-1681 electronic, 2379-1683 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

AKT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANNALS OF K-THEORY

2019 vol. 4 no. 1