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Almost complex structures on connected sums of complex projective spaces

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We show that the m-fold connected sum $m\#\mathbb{CP}^{2n}$ admits an almost complex structure if and only if m is odd.

1. Introduction

A *complex structure* on a real vector bundle F over a connected CW complex X is a complex vector bundle E over X such that its underlying real vector bundle $E_{\mathbb{R}}$ is isomorphic to F. A *stable complex structure* on F is a complex structure on $F \oplus \varepsilon^d$, where ε^d is the d-dimensional trivial real vector bundle over X. For X a manifold we say that X has an *almost complex structure* if its tangent bundle admits a complex structure, and a *stable almost complex structure* if its tangent bundle admits a stable complex structure. Motivated by the question in [Miller 2015] we consider in this paper the m-fold connected sum of complex projective spaces $m\#\mathbb{CP}^{2n}$.

As shown by Hirzebruch [1987, Kommentare, p. 777], a necessary condition for the existence of an almost complex structure on a $4n$-dimensional compact manifold M is the congruence $\chi(M) \equiv (-1)^n \sigma(M) \bmod 4$, where $\chi(M)$ is the Euler characteristic and $\sigma(M)$ is the signature of M. Thus, for even m, the connected sums above cannot carry an almost complex structure. We show that for odd m they do admit almost complex structures, thus showing the following:

Theorem 1.1. *The m-fold connected sum $m\#\mathbb{CP}^{2n}$ admits an almost complex structure if and only if m is odd.*

In odd complex dimensions, the connected sums $m\#\mathbb{CP}^{2n+1}$ are Kähler: $\mathbb{CP}^{2n+1}$ admits an orientation reversing diffeomorphism, and therefore $m\#\mathbb{CP}^{2n+1}$ is diffeomorphic to $\mathbb{CP}^{2n+1}\#(m-1)\overline{\mathbb{CP}^{2n+1}}$, which is a blow–up of $\mathbb{CP}^{2n+1}$ in $m-1$ points. Furthermore Theorem 1.1 is known for $n = 1$ and $n = 2$; see [Audin 1991] and [Müller and Geiges 2000], respectively. In both cases the authors use general

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results on the existence of almost complex structures on manifolds of dimension 4 and 8, respectively.

In [Sutherland 1965, Theorem 1.1] or [Thomas 1967, Theorem 1.7] the authors showed the following.

Theorem 1.2. *Let M be a closed smooth $2d$-dimensional manifold. Then TM admits an almost complex structure if and only if it admits a stable almost complex structure E such that $c_d(E) = e(M)$, where c_d is the d-th Chern class of E and $e(M)$ is the Euler class of M.*

In Section 2 we describe the full set of stable almost complex structures in the reduced K-theory of $m\#\mathbb{CP}^{2n}$. In Section 3 we give, for odd m, an explicit example of a stable almost complex structure to which Theorem 1.2 applies, thus completing the proof of Theorem 1.1.

2. Stable almost complex structures on $m\#\mathbb{CP}^{2n}$

For a CW complex X let $K(X)$ and $KO(X)$ denote the complex and real K-groups, respectively. Moreover we denote by $\widetilde{K}(X)$ and $\widetilde{KO}(X)$ the reduced groups. Let $r : K(X) \to KO(X)$ denote the real reduction map, which can be restricted to a map $\widetilde{K}(X) \to \widetilde{KO}(X)$. We denote the restricted map again with r. A real vector bundle F over X has a stable almost complex structure if there is a an element $y \in \widetilde{K}(X)$ such that $r(y) = F - \dim F$. Since r is a group homomorphism, the set of all stable complex vector bundles such that the underlying real vector bundle is stably isomorphic to F is given by

$$y + \ker r \subset \widetilde{K}(X),$$

where y is such that $r(y) = F - \dim F$. Let $c : KO(X) \to K(X)$ denote the complexification map and $t : K(X) \to K(X)$ the map which is induced by complex conjugation of complex vector bundles. The maps t and c are ring homomorphisms, but r preserves only the group structure. The identities

$$c \circ r = 1 + t : K(X) \to K(X), \qquad r \circ c = 2 : KO(X) \to KO(X),$$

involving the maps r, c and t are well known. We write $\bar{y} = t(y)$ for an element $y \in K(X)$.

For two oriented manifolds M and N of the same dimension d, we denote by $M\#N$ the connected sum of M with N, which inherits an orientation from M and N. First, let us characterize the stable tangent bundle of $M\#N$.

Lemma 2.1. *Let $p_M : M\#N \to M$ and $p_N : M\#N \to N$ be collapsing maps to each factor of $M\#N$. Then we have*

$$p_M^*(M) \oplus p_N^*(N) \cong T(M\#N) \oplus \varepsilon^d.$$

Proof. Let $D_M \subset M$ and $D_N \subset N$ be embedded closed disks and W_M and W_N collar neighborhoods of $\partial(M \setminus \mathring{D}_M)$ and $\partial(N \setminus \mathring{D}_N)$, respectively, where $\mathring{D}$ denotes the interior of D. Thus $W_M \cong S^{d-1} \times [-2, 0]$ and $W_N \cong S^{d-1} \times [0, 2]$. The manifold $M\#N$ is obtained by identifying $S^{d-1} \times 0 \subset W_M$ with $S^{d-1} \times 0 \subset W_N$ by the identity map. Set $W := W_M \cup W_N \subset M\#N$ and note that $V_1 := p_M^*(M) \oplus p_N^*(N)$ as well as $V_2 := T(M\#N) \oplus \varepsilon^n$ are trivial over W. Moreover let $U_M \subset M\#N$ be the open set diffeomorphic to $(M \setminus W_M) \cup (S^{d-1} \times [-2, -1[)$, and analogously for $U_N \subset M\#N$.

Now, since $V_1|_{U_M} \cong p_M^*(TM) \oplus \varepsilon^d$ and $p_M^*(TM)|_{U_M} = T(M\#N)|_{U_M}$, we have an isomorphism given by $\Phi_M : V_2|_{U_M} \to V_1|_{U_M}$, $(\xi, w) \mapsto ((p_M)_*(\xi), w)$. For $\Phi_N : V_2|_{U_N} \to V_1|_{U_N}$, we set $\Phi_N(\eta, w) = (w, -(p_N)_*(\eta))$. Moreover, both vector bundles V_1 and V_2 are trivial over W and it is possible to choose trivializations of V_1 and V_2 over W such that Φ_M is given by $(v, w) \mapsto (v, w)$ over W_M and such that Φ_N is represented by $(v, w) \mapsto (w, -v)$ over W_N. Over $S^{d-1} \times [-1, 1]$ we can interpolate these isomorphisms by

$$\begin{pmatrix} v \\ w \end{pmatrix} \mapsto \begin{pmatrix} \cos\left(\frac{\pi}{4}(t+1)\right) & \sin\left(\frac{\pi}{4}(t+1)\right) \\ -\sin\left(\frac{\pi}{4}(t+1)\right) & \cos\left(\frac{\pi}{4}(t+1)\right) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

for $t \in [-1, 1]$. Using this interpolation we can glue Φ_M and Φ_N to a global isomorphism $V_2 \to V_1$. □

Hence, $T(M\#N) - d = TM + TN - 2d$ in $\widetilde{KO}(M\#N)$, where TM and TN denote the elements in $\widetilde{KO}(M\#N)$ induced by $p_M^*(TM)$ and $p_N^*(TN)$, respectively. This shows that if M and N admit stable almost complex structures so does $M\#N$; see [Kahn 1969]. For $M = N = \mathbb{CP}^{2n}$ we consider the natural orientation induced by the complex structure of $\mathbb{CP}^{2n}$.

We proceed with recalling some basic facts on complex projective spaces. Let H be the tautological line bundle over $\mathbb{CP}^d$ and let $x \in H^2(\mathbb{CP}^d; \mathbb{Z})$ be the generator, such that the total Chern class $c(H)$ is given by $1 + x$. The cohomology ring of $\mathbb{CP}^d$ is isomorphic to $\mathbb{Z}[x]/\langle x^{d+1}\rangle$. The K and KO theory of $\mathbb{CP}^d$ are completely understood. Let $\eta := H - 1 \in \widetilde{K}(\mathbb{CP}^d)$ and $\eta_R := r(\eta) \in \widetilde{KO}(\mathbb{CP}^d)$.

Theorem 2.2 (cf. [Sanderson 1964, Theorem 3.9; Fujii 1966, Lemma 3.5; Milnor and Stasheff 1974, p. 170; Thomas 1974, Proposition 4.3]).

(a) $K(\mathbb{CP}^d) = \mathbb{Z}[\eta]/\langle \eta^{d+1}\rangle$. *Letting n be the largest integer $\le d/2$, the following sets of elements are an integral basis of $K(\mathbb{CP}^d)$*:

 (i) $1, \eta, \eta(\eta+\bar\eta), \dots, \eta(\eta+\bar\eta)^{n-1}, (\eta+\bar\eta), \dots, (\eta+\bar\eta)^n$, *and also, in case d is odd*, $\eta^{2n+1} = \eta(\eta+\bar\eta)^n$;

 (ii) $1, \eta, \eta(\eta+\bar\eta), \dots, \eta(\eta+\bar\eta)^{n-1}, (\eta-\bar\eta)(\eta+\bar\eta), \dots, (\eta-\bar\eta)(\eta+\bar\eta)^{n-1}$, *and also, in case d is odd*, η^{2n+1}.

(b) (i) *If* $d = 2n$ *then* $KO(\mathbb{CP}^d) = \mathbb{Z}[\eta_R]/\langle \eta_R^{n+1} \rangle$.
(ii) *If* $d = 4n+1$ *then* $KO(\mathbb{CP}^d) = \mathbb{Z}[\eta_R]/\langle \eta_R^{2n+1}, 2\eta_R^{2n+2} \rangle$.
(iii) *If* $d = 4n+3$ *then* $KO(\mathbb{CP}^d) = \mathbb{Z}[\eta_R]/\langle \eta_R^{2n+2} \rangle$.

(c) *The complex stable tangent bundle is given by* $(2n+1)\overline{\eta} \in \widetilde{K}(\mathbb{CP}^{2n})$ *and the real stable tangent bundle is given by* $r((2n+1)\overline{\eta}) \in \widetilde{KO}(\mathbb{CP}^{2n})$.

(d) *The kernel of the real reduction map* $r : \widetilde{K}(\mathbb{CP}^d) \to \widetilde{KO}(\mathbb{CP}^d)$ *is freely generated by the elements*
(i) $\eta - \overline{\eta}, (\eta - \overline{\eta})(\eta + \overline{\eta}), \dots, (\eta - \overline{\eta})(\eta + \overline{\eta})^{(d/2)-1}$, *if* d *is even,*
(ii) $\eta - \overline{\eta}, (\eta - \overline{\eta})(\eta + \overline{\eta}), \dots, (\eta - \overline{\eta})(\eta + \overline{\eta})^{2n-1}, 2\eta^d$, *if* $d = 4n+1$,
(iii) $\eta - \overline{\eta}, (\eta - \overline{\eta})(\eta + \overline{\eta}), \dots, (\eta - \overline{\eta})(\eta + \overline{\eta})^{2n}, \eta^d$, *if* $d = 4n+3$.

Next we would like to describe the integer cohomology ring of $m\#\mathbb{CP}^{2n}$. For that we introduce the following notation. Let Λ denote either $\mathbb{Z}$ or $\mathbb{Q}$. We define an ideal $R_d(X_1, \dots, X_m)$ in $\Lambda[X_1, \dots, X_m]$, where $X_1, \dots, X_m$ are indeterminants, as the ideal generated by the following elements

$$X_i \cdot X_j, \quad i \neq j, \qquad X_i^d - X_j^d, \quad i \neq j, \qquad X_j^{d+1}, \quad j = 1, \dots, m.$$

Hence, we have

$$H^*(m\#\mathbb{CP}^d; \Lambda) \cong \Lambda[x_1, \dots, x_m]/R_d(x_1, \dots, x_m), \tag{2.3}$$

where $x_j = p_j^*(x) \in H^2(m\#\mathbb{CP}^d; \Lambda)$, for $x \in H^2(\mathbb{CP}^d; \Lambda)$ defined as above and $p_j : m\#\mathbb{CP}^d \to \mathbb{CP}^d$ the projection onto the j-th factor. Note that p_j induces an monomorphism on cohomology.

The stable tangent bundle of $m\#\mathbb{CP}^{2n}$ in $\widetilde{KO}(m\#\mathbb{CP}^{2n})$ is represented by

$$(2n+1)\sum_{j=1}^{m} r(\overline{\eta}_j),$$

where $\eta_j := p_j^*(\eta) \in \widetilde{K}(\mathbb{CP}^{2n})$ and $r : \widetilde{K}(m\#\mathbb{CP}^{2n}) \to \widetilde{KO}(m\#\mathbb{CP}^{2n})$ is the real reduction map. Hence the set of stable almost complex structures on $m\#\mathbb{CP}^{2n}$ is given by

$$(2n+1)\sum_{j=1}^{m} \overline{\eta}_j + \ker r. \tag{2.4}$$

For $k \in \mathbb{N}$ and $j = 1, \dots, m$, set $w_j^k = p_j^*(H)^k - p_j^*(H)^{-k}$, $e_j^{n-1} = \eta_j(\eta_j + \overline{\eta}_j)^{n-1}$ and $\omega = \eta_1^{2n}$.

Proposition 2.5. *The kernel of* $r : \widetilde{K}(m\#\mathbb{CP}^{2n}) \to \widetilde{KO}(m\#\mathbb{CP}^{2n})$ *is freely generated by*

$$\{w_j^k : k = 1, \dots, n-1,\ j = 1, \dots, m\} \cup \{e_1^{n-1} - e_j^{n-1} : j = 2, \dots, m\} \cup \{2e_1^{n-1} - \omega\}$$

for n even, and

$$\{w_j^k : k = 1, \dots, n,\ j = 1, \dots, m\}$$

for n odd.

Proof. Consider the cofiber sequence

$$\bigvee_{j=1}^{m} \mathbb{CP}^{2n-1} \xrightarrow{\ i\ } m\#\mathbb{CP}^{2n} \xrightarrow{\ \pi\ } S^{4n}. \tag{2.6}$$

Note that the line bundle $i^* p_j^*(H)$ is the tautological line bundle over the j-th summand of $\bigvee_{j=1}^m \mathbb{CP}^{2n-1}$ and the trivial bundle on the other summands, since the first Chern classes are the same. For the reduced groups we have

$$\widetilde{K}\Big(\bigvee_{j=1}^{m}\mathbb{CP}^{2n-1}\Big) \cong \bigoplus_{j=1}^{m} \widetilde{K}(\mathbb{CP}^{2n-1})$$

and $i^* p_j^*(\eta)$ generates the j-th summand of the above sum according to Theorem 2.2. The long exact sequence in K-theory of the cofibration (2.6) is given by

$$\begin{aligned} \cdots \to \widetilde{K}^{-1}\Big(\bigvee_{j=1}^{m}\mathbb{CP}^{2n-1}\Big) \to \widetilde{K}(S^{4n}) \to \widetilde{K}(m\#\mathbb{CP}^{2n}) \\ \to \widetilde{K}\Big(\bigvee_{j=1}^{m}\mathbb{CP}^{2n-1}\Big) \to \widetilde{K}^{1}(S^{4n}) \to \cdots. \end{aligned} \tag{2.7}$$

From Theorem 2 in [Fujii 1967], we have that $\widetilde{K}^{-1}(\mathbb{CP}^{2n-1}) = 0$, and hence $\widetilde{K}^{-1}\big(\bigvee_{j=1}^m \mathbb{CP}^{2n-1}\big) = 0$. Then from Bott periodicity we deduce the equality $\widetilde{K}^1(S^{4n}) = \widetilde{K}^{-1}(S^{4n}) = 0$. So we obtain a short exact sequence

$$0 \longrightarrow \widetilde{K}(S^{4n}) \xrightarrow{\ \pi^*\ } \widetilde{K}(m\#\mathbb{CP}^{2n}) \xrightarrow{\ i^*\ } \widetilde{K}\Big(\bigvee_{j=1}^{m}\mathbb{CP}^{2n-1}\Big) \longrightarrow 0$$

which splits, since the groups involved are finitely generated, torsion free abelian groups. Let $\omega_{\mathbb{C}}$ be the generator of $\widetilde{K}(S^{4n})$. Then the set

$$\{\pi^*(\omega_{\mathbb{C}})\} \cup \{\eta_j^k : j = 1, \dots, m,\ k = 1, \dots, 2n-1\}$$

is an integral basis of $\widetilde{K}(m\#\mathbb{CP}^{2n})$. We claim that $\eta_j^{2n} = \pi^*(\omega_{\mathbb{C}})$ for all j. Indeed, the elements η_j^{2n} lie in the kernel of i^*, and hence there are $k_j \in \mathbb{Z}$ such that $\eta_j^{2n} = k_j \cdot \pi^*(\omega_{\mathbb{C}})$. Let $\widetilde{\mathrm{ch}} : \widetilde{K}(X) \to \widetilde{H}(X;\mathbb{Q})$ denote the Chern character for a finite CW complex X, then $\widetilde{\mathrm{ch}}$ is a monomorphism for $X = m\#\mathbb{CP}^d$ (since $\widetilde{H}^*(m\#\mathbb{CP}^d;\mathbb{Z})$ has no torsion [Atiyah and Hirzebruch 1961, Section 2.5, Corollary]) and an isomorphism for $X = S^d$ onto $\widetilde{H}^*(S^d;\mathbb{Z})$ embedded in $\widetilde{H}^*(S^d;\mathbb{Q})$. Using the notation of (2.3) we have

$$\widetilde{\mathrm{ch}}(\eta_j^{2n}) = (e^{x_j} - 1)^{2n} = x_j^{2n}$$

and using the naturality of $\widetilde{\mathrm{ch}}$

$$\widetilde{\mathrm{ch}}(\pi^*(\omega_{\mathbb{C}})) = \pi^*(\widetilde{\mathrm{ch}}(\omega_{\mathbb{C}})) = \pm x_j^{2n},$$

since π^* is an isomorphism on cohomology in dimension $2n$. We can choose $\omega_{\mathbb{C}}$ such that $\widetilde{\mathrm{ch}}(\pi^*(\omega_{\mathbb{C}})) = x_j^{2n}$. This shows $k_j = 1$ for all j and $\widetilde{K}(m\#\mathbb{CP}^{2n})$ is freely generated by

$$\{\eta_j^k : j = 1, \dots, m,\ k = 1, \dots, 2n-1\} \cup \{\eta_1^{2n} = \cdots = \eta_m^{2n}\}.$$

Hence $K(m\#\mathbb{CP}^{2n}) = \mathbb{Z}[\eta_1, \dots, \eta_m]/R_{2n}(\eta_1, \dots, \eta_m)$. Since $p_j^*(H) \otimes p_j^*(\overline{H})$ is the trivial bundle we compute the identity

$$\overline{\eta}_j = \frac{-\eta_j}{1+\eta_j} = -\eta_j + \eta_j^2 - \cdots + \eta_j^{2n}.$$

The ring $\mathbb{Z}[\eta_1, \dots, \eta_m]/R_{2n}(\eta_1, \dots, \eta_m)$ is isomorphic to

$$\left(\bigoplus_{j=1}^m \mathbb{Z}[\eta_j]/\langle \eta_j^{2n+1}\rangle\right) \Big/ \left\langle \eta_j^{2n} - \eta_i^{2n} : j \neq i\right\rangle$$

and from Theorem 2.2 the set Γ_j which contains the elements

$$\eta_j, \eta_j(\eta_j + \overline{\eta}_j), \dots, \eta_j(\eta_j + \overline{\eta}_j)^{n-1},$$
$$\eta_j - \overline{\eta}_j, (\eta_j - \overline{\eta}_j)(\eta_j + \overline{\eta}_j), \dots, (\eta_j - \overline{\eta}_j)(\eta_j + \overline{\eta}_j)^{n-1}$$

together with $\{1\}$ is an integral basis of $\mathbb{Z}[\eta_j]/\langle \eta_j^{2n+1}\rangle$. Thus the set

$$\Gamma_1 \cup \cdots \cup \Gamma_m \subset \widetilde{K}(m\#\mathbb{CP}^{2n})$$

generates the group $\widetilde{K}(m\#\mathbb{CP}^{2n})$. Observe that

$$(\eta_j + \overline{\eta}_j)^k = 2\eta_j(\eta_j + \overline{\eta}_j)^{k-1} - (\eta_j - \overline{\eta}_j)(\eta_j + \overline{\eta}_j)^{k-1}. \tag{2.8}$$

Thus

$$\eta_j^{2n} = (\eta_j + \overline{\eta}_j)^n = 2\eta_j(\eta_j + \overline{\eta}_j)^{n-1} - (\eta_j - \overline{\eta}_j)(\eta_j + \overline{\eta}_j)^{n-1}. \tag{2.9}$$

We set $\omega := \eta_j^{2n}$ for any $j = 1, \dots, m$ and

$$e_j^k := \eta_j(\eta_j + \overline{\eta}_j)^k, \qquad j = 1, \dots, m, \quad k = 0, \dots, n-1,$$
$$f_j^k := (\eta_j - \overline{\eta}_j)(\eta_j + \overline{\eta}_j)^k, \quad j = 1, \dots, m, \quad k = 0, \dots, n-1,$$

and by virtue of relation (2.9) the set

$$B := \{\omega\} \cup \{e_j^k : j = 1, \dots, m,\ k = 0, \dots, n-1\} \cup \{f_j^k : j = 1, \dots, m,\ k = 0, \dots, n-2\}$$

is an integral basis of $\widetilde{K}(m\#\mathbb{CP}^{2n})$.

We proceed with the computation of $KO(m\#\mathbb{CP}^{2n})$. We have a long exact sequence for $\widetilde{KO}$-theory like in (2.7). From Theorem 2 in [Fujii 1967] we deduce $\widetilde{KO}^{-1}(\mathbb{CP}^{2n}) = 0$ and therefore $\widetilde{KO}^{-1}\big(\bigvee_{j=1}^{m} \mathbb{CP}^{2n}\big) = 0$. Moreover,

$$\widetilde{KO}^{1}(S^{4n}) = \widetilde{KO}^{-7}(S^{4n}) = \widetilde{KO}(S^{4n+7}) = 0$$

by Bott periodicity. Hence we obtain a short exact sequence

$$0 \to \widetilde{KO}(S^{4n}) \to \widetilde{KO}(m\#\mathbb{CP}^{2n}) \to \widetilde{KO}\Bigg(\bigvee_{j=1}^{m} \mathbb{CP}^{2n-1}\Bigg) \to 0. \tag{2.10}$$

Now we have to distinguish between the cases where n is even or odd. We first assume that $n = 2l$. In that case the ring $KO(\mathbb{CP}^{2n-1})$ is isomorphic to $\mathbb{Z}[\eta_R]/\langle \eta_R^n \rangle$; see Theorem 2.2(b). Hence all groups in (2.10) are torsion free. Therefore the kernel of $r : \widetilde{K}(m\#\mathbb{CP}^{2n}) \to \widetilde{KO}(m\#\mathbb{CP}^{2n})$ is the same as the kernel of

$$\varphi := c \circ r = 1 + t : \widetilde{K}(m\#\mathbb{CP}^{2n}) \to \widetilde{K}(m\#\mathbb{CP}^{2n})$$

since $r \circ c = 2$, and thus c is a monomorphism of the torsion free part of $\widetilde{KO}(m\#\mathbb{CP}^{2n})$.

Next we compute a basis of $\ker \varphi$. Using relation (2.8) we have $\varphi(\omega) = 2\omega$, $\varphi(e_j^k) = 2e_j^k - f_j^k$ and $\varphi(f_j^k) = 0$. Thus if

$$y = \lambda\omega + \sum_{j=1}^{m} \sum_{k=0}^{n-1} \lambda_j^k e_j^k,$$

then

$$\varphi(y) = 2\lambda\omega + \sum_{j=1}^{m} \sum_{k=0}^{n-1} \lambda_j^k (2e_j^k - f_j^k) = \Bigg(2\lambda + \sum_{j=1}^{m} \lambda_j^{n-1}\Bigg)\omega + \sum_{j=1}^{m} \sum_{k=0}^{n-2} \lambda_j^k (2e_j^k - f_j^k),$$

using the fact that $f_j^{n-1} = 2e_j^{n-1} - \omega$ by (2.9). As ω and $2e_j^k - f_j^k$, $j = 1, \dots, m$, $k = 0, \dots, n-2$, are linearly independent, we conclude that $\varphi(y) = 0$ if and only if $\lambda_j^k = 0$ for $j = 1, \dots, m$, $k = 1, \dots, n-2$ and

$$\sum_{j=1}^{m} \lambda_j^{n-1} + 2\lambda = 0.$$

This implies that the set

$$\{f_j^k : j = 1, \dots, m,\ k = 0, \dots, n-2\} \cup \{e_1^{n-1} - e_j^{n-1} : j = 2, \dots, m\} \cup \{2e_1^{n-1} - \omega\}$$

is an integral basis of $\ker \varphi$. Note that from (2.9) we have

$$2e_1^{n-1} - \omega = (\eta_1 - \overline{\eta}_1)(\eta_1 + \overline{\eta}_1)^{n-1}.$$

By an inductive argument we see that

$$(\eta_j - \overline{\eta}_j)(\eta_j + \overline{\eta}_j)^k = w_j^{k+1} + \text{linear combinations of } w_j^1, \dots, w_j^k \tag{2.11}$$

and
$$e_1^{n-1} - e_j^{n-1} = \eta_1^{2n-1} - \eta_j^{2n-1}.$$

Thus an integral basis of the kernel, in case n is even, is given by
$$\{w_j^k : j = 1, \ldots, m,\ k = 1, \ldots, n-1\} \cup \{w_1^n\} \cup \{\eta_1^{2n-1} - \eta_j^{2n-1} : j = 2, \ldots, m\}.$$

Now let us assume that $n = 2l + 1$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{K}(S^{4n}) & \xrightarrow{\pi^*} & \widetilde{K}(m\#\mathbb{CP}^{2n}) & \xrightarrow{i^*} & \widetilde{K}\big(\bigvee_{j=1}^m \mathbb{CP}^{2n-1}\big) & \longrightarrow & 0 \\
\downarrow & & \downarrow r_S & & \downarrow r_\# & & \downarrow r_\vee & & \downarrow \\
0 & \longrightarrow & \widetilde{KO}(S^{4n}) & \xrightarrow{\pi^*} & \widetilde{KO}(m\#\mathbb{CP}^{2n}) & \xrightarrow{i^*} & \widetilde{KO}\big(\bigvee_{j=1}^m \mathbb{CP}^{2n-1}\big) & \longrightarrow & 0
\end{array}$$

The map $r_S : \widetilde{K}(S^{8l+4}) \to \widetilde{KO}(S^{8l+4})$ is an isomorphism and therefore the map $i^*|_{\ker r_\#} : \ker r_\# \to \ker r_\vee$ is an isomorphism. Hence the rank of $\ker r_\#$ is mn. We see that the set
$$\{f_j^k : j = 1, \ldots, m,\ k = 0, \ldots, n-2\} \cup \{2e_j^{n-1} : j = 1, \ldots, m\} \cup \{\omega\}$$
is an integral basis of $(i^*)^{-1}(\ker r_\vee)$, which follows from $e_j^{n-1} = \eta_j^{2n-1} - (n-1)\omega$ and the structure of the kernel of $r_\vee$; see Theorem 2.2(d)(ii). The elements f_j^k for $j = 1, \ldots, m$ and $k = 0, \ldots, n-2$ lie in the kernel of $r_\#$. Let
$$y = \lambda\omega + \sum_{j=1}^m \lambda_j^{n-1} 2e_j^{n-1}$$
for $\lambda, \lambda_j^{n-1} \in \mathbb{Z}$ and suppose $r_\#(y) = 0$. From $\varphi(\omega) = 2\omega$ and $\varphi(e_j^{n-1}) = (\eta_j + \bar{\eta}_j)^n = \eta_j^{2n} = \omega$ it follows that
$$\lambda + \sum_{j=1}^m \lambda_j^{n-1} = 0.$$

Hence $\ker r_\#$ is freely generated by the elements f_j^k and $2e_j^{n-1} - \omega$. Observe from (2.9) that $2e_j^{n-1} - \omega = (\eta - \bar{\eta})(\eta + \bar{\eta})^{n-1}$. Thus in the case that n is odd we deduce like in (2.11) that the kernel of $r_\#$ is freely generated by w_j^k for $j = 1, \ldots, m$ and $k = 1, \ldots, n$. □

Hence by (2.4), stable almost complex structures of $m\#\mathbb{CP}^{2n}$ for n even are given by elements of the form
$$y = (2n+1)\sum_{i=1}^m \bar{\eta}_j + \sum_{j=1}^m \sum_{k=1}^{n-1} a_j^k w_j^k + a_1^n w_1^n + \sum_{j=2}^m b_j(\eta_1^{2n-1} - \eta_j^{2n-1}), \quad (2.12)$$
and for n odd,
$$y = (2n+1)\sum_{i=1}^m \bar{\eta}_j + \sum_{j=1}^m \sum_{k=1}^{n} a_j^k w_j^k \quad (2.13)$$

for $a_j^k, b_j \in \mathbb{Z}$. For Theorem 1.2 we have to compute the $2n$-th Chern class $c_{2n}(E)$ of a vector bundle E representing an element of the form (2.12) and (2.13). Let $\eta_1^{2n-1} - \eta_j^{2n-1}$ denote also a vector bundle over $m\#\mathbb{CP}^{2n}$ which represents the element $\eta_1^{2n-1} - \eta_j^{2n-1}$ in $\widetilde{K}(m\#\mathbb{CP}^{2n})$. The total Chern class of $\eta_1^{2n-1} - \eta_j^{2n-1}$ can be computed through the Chern character: we have

$$\widetilde{\mathrm{ch}}(\eta_1^{2n-1} - \eta_j^{2n-1}) = \widetilde{\mathrm{ch}}(\eta_1)^{2n-1} - \widetilde{\mathrm{ch}}(\eta_j)^{2n-1} = x_1^{2n-1} - x_j^{2n-1}.$$

The elements of degree k in the Chern character are given by $\nu_k(c_1, \dots, c_k)/k!$, where ν_k are the Newton polynomials. The coefficient in front of c_k in $\nu_k(c_1, \dots, c_k)$ is k (see [Mimura and Toda 1991, p. 195]) and the other terms are products of Chern classes of lower degree; hence the only nonvanishing Chern class is given by

$$c_{2n-1}(\eta_1^{2n-1} - \eta_j^{2n-1}) = (2n-2)!\,(x_1^{2n-1} - x_j^{2n-1}).$$

Thus the total Chern class of a vector bundle E representing an element of the form (2.12) is given by

$$c(E) = (1-(x_1+\cdots+x_m))^{2n+1} \cdot \Big(\frac{1+nx_1}{1-nx_1}\Big)^{a_1^n} \prod_{j=2}^{m} (1+(2n-2)!(x_1^{2n-1} - x_j^{2n-1}))^{b_j} \prod_{j=1}^{m}\prod_{k=1}^{n-1} \Big(\frac{1+kx_j}{1-kx_j}\Big)^{a_j^k},$$

and for (2.13),

$$c(E) = (1-(x_1+\cdots+x_m))^{2n+1} \prod_{j=1}^{m}\prod_{k=1}^{n} \Big(\frac{1+kx_j}{1-kx_j}\Big)^{a_j^k},$$

where the coefficient in front of $x_1^{2n} = \cdots = x_m^{2n}$ is equal to $c_{2n}(E)$.

Remark 2.14. Note that for $m = 1$ (and complex projective spaces of arbitrary dimension) this total Chern class was already computed by Thomas [1974, p. 130].

3. Almost complex structures on $m\#\mathbb{CP}^{2n}$

We now describe an explicit stable almost complex structure on $m\#\mathbb{CP}^{2n}$, where $m = 2u+1$, for which the assumptions of Theorem 1.2 are satisfied, thereby producing an almost complex structure on $m\#\mathbb{CP}^{2n}$. We choose, in the notation of (2.12) and (2.13), $a_j^k = 2$ for $j = 1, \dots, u$ and $k = 1$, and all other coefficients 0. Then the top Chern class is as desired:

Proposition 3.1. *Let $m = 2u+1$ be an odd number. In the cohomology ring of $m\#\mathbb{CP}^{2n}$, the coefficient c_{2n} of $x_1^{2n} = \cdots = x_m^{2n}$ of the class*

$$c = (1-(x_1+\cdots+x_{2u+1}))^{2n+1} \prod_{r=1}^{u} \Big(\frac{1+x_r}{1-x_r}\Big)^2$$

is $c_{2n} = m(2n-1)+2 = \chi(m\#\mathbb{CP}^{2n})$.

Proof. As $x_i \cdot x_j = 0$ for $i \neq j$, we have

$$\begin{aligned}(1-(x_1+\cdots+x_{2u+1}))^{2n+1} &= \sum_{j_0=0}^{2n+1}(-1)^{j_0}\binom{2n+1}{j_0}(x_1^{j_0}+\cdots+x_{2u+1}^{j_0}) \\ &= \sum_{r=1}^{2u+1}\sum_{j_0=0}^{2n+1}(-1)^{j_0}\binom{2n+1}{j_0}x_r^{j_0}.\end{aligned}$$

Thus,

$$c = \prod_{r=1}^{u}(1-x_r)^{2n-1}(1+x_r)^2 \prod_{s=u+1}^{2u+1}(1-x_s)^{2n+1}.$$

The factors $(1-x_s)^{2n+1}$ contribute $2n+1$ to c_{2n}, while the factors $(1-x_r)^{2n-1}(1+x_r)^2$ contribute $2n-3$. Thus,

$$\begin{aligned}c_{2n} &= u(2n-3)+(u+1)(2n+1) \\ &= (2u+1)(2n-1)+2 \\ &= \chi((2u+1)\#\mathbb{CP}^{2n}).\end{aligned}$$ □

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