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Segal operations in the algebraic K-theory of topological spaces

Thomas Gunnarsson and Ross Staffeldt

We extend earlier work of Waldhausen which defines operations on the algebraic K-theory of the one-point space. For a connected simplicial abelian group X and symmetric groups Σ_n, we define operations $\theta^n : A(X) \to A(X \times B\Sigma_n)$ in the algebraic K-theory of spaces. We show that our operations can be given the structure of E_∞-maps. Let $\phi_n : A(X \times B\Sigma_n) \to A(X \times E\Sigma_n) \simeq A(X)$ be the Σ_n-transfer. We also develop an inductive procedure to compute the compositions $\phi_n \circ \theta^n$, and outline some applications.

1. Introduction

Let X be a connected simplicial abelian group, let Σ_n be the symmetric group on n letters, and let $B\Sigma_n$ be the classifying space. Our goal is to define a family of Segal operations

$$\theta^n : A(X) \longrightarrow A(X \times B\Sigma_n)$$

satisfying the properties listed in Theorems 1.1 and 1.3 below. We follow [Waldhausen 1982] in our naming convention, which can be explained as follows. Around 1972, Graeme Segal [1974b] defined a set of operations in stable homotopy theory $\theta^n : \pi_i^s(S^0) \to \pi_i^s((B\Sigma_n)_+)$, verified certain properties and used the information to give a proof of the Kahn–Priddy theorem. The key to the Kahn–Priddy proof is a certain relation satisfied by the composition of an operation followed by a transfer homomorphism.

Waldhausen [1982] adapted the construction in [Segal 1974b] to define operations $\theta^n : A(*) \to A(B\Sigma_n)$, and proved these new operations have properties precisely analogous to fundamental properties of Segal's original operations. Consequently, Waldhausen used the same notation and called the operations "Segal operations".

We obtain the following result.

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Theorem 1.1. *Given a connected simplicial abelian group X, there are maps $\theta^n : A(X) \to A(X \times B\Sigma_n)$ which have the following properties.*

(1) *The map θ^1 is the identity.*

(2) *The combined map*

$$\theta = \prod_{n \geq 1} \theta^n : A(X) \to \{1\} \times \prod_{n \geq 1} A(X \times B\Sigma_n)$$

has the structure of an E_∞-map if the target is equipped with the E_∞-structure arising from certain pairings $A(X \times B\Sigma_m) \times A(X \times B\Sigma_n) \to A(X \times B\Sigma_{n+m})$ derived from the box-tensor operation of Definition 3.10.

The first property is a normalization condition, as satisfied by the constructions of Segal and Waldhausen. The second property implies that for every $j > 0$ the operations induce homomorphisms $\pi_j A(X) \to \{1\} \times \prod_{n \geq 1} \pi_j A(X \times B\Sigma_n)$ when the target is given a particular algebraic structure. A third algebraic property of Waldhausen's operations is recalled in Proposition 8.1. This third property is crucial in the applications made by Segal and Waldhausen. Our extended operations exhibit a more technical algebraic property stated in Theorem 1.3 and Theorem 8.12.

A large part of our work follows [Gunnarsson and Schwänzl 2002] in which many results are developed for quite general situations, satisfying certain technical conditions. Part of this paper verifies these conditions. In order to explain the necessity of this technical work, we repeat several definitions from [Gunnarsson and Schwänzl 2002] and quote many results.

In Section 2 the main results are Proposition 2.17 and Theorem 2.1. For the purposes of algebraic K-theory we verify exactness properties of certain constructions; to prepare for the E_∞-structure we verify coherence properties.

In Section 3 we recall the $G_\bullet$-construction for algebraic K-theory [Gunnarsson et al. 1992; Grayson 1989] and prepare the constructions underlying the definition of the operations in Definition 3.29.

In Section 4 we set up to apply general machinery, taking the first step toward a main result: For X a connected simplicial abelian group, there is an operation

$$\omega = \prod_{n \geq 1} \omega^n : A(X) \longrightarrow \{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \{\text{all}\}}(X) \tag{1.2}$$

which is a map of E_∞-spaces with respect to specific algebraic structures described in Section 4. The target of ω is the algebraic K-theory of Σ_n-spaces retracting to X (with the trivial Σ_n-action) and relatively finite with respect to X. See Definition 3.5. In the first step the E_∞-structure is only visible if we restrict to spherical objects. The next section addresses this problem.

In Section 5 we study how the functors from Definition 3.29 interact with suspension operators. At the end of the section we complete the construction of the operation displayed in (1.2).

In Theorem 6.1, we split $A_{\Sigma_n,\{\text{all}\}}(X)$ as a product of the algebraic K-theory of other spaces, one of which is $A(X \times B\Sigma_n)$. This corresponds to the subcategory of Σ_n-spaces retracting to X (with the trivial Σ_n action), relatively finite with respect to X, and with Σ_n acting freely outside of X. We also obtain an expression for the composite functors "projecting to the free part"

$$\theta^n : A(X) \xrightarrow{\omega_n} A_{\Sigma_n,\{\text{all}\}}(X) \to A(X \times B\Sigma_n).$$

This expression is used in Section 8.

In Section 7 we establish equivalences among various models for equivariant K-theory and discuss the functors that induce transfer operations.

In Section 8 our main computational result evaluates the composition

$$A(X) \xrightarrow{\theta^n} A(X \times B\Sigma_n) \xrightarrow{\phi_n} A(X),$$

where ϕ_n is the transfer map.

Theorem 1.3 (Theorem 8.12). *Let X be a connected simplicial abelian group, thinking of the group operation as a multiplication, and let $\tau^n : X \to X$ be the homomorphism that raises elements to the n-th power. Then*

$$\phi_n \theta^n_* = (-1)^{n-1} \cdot (n-1)! \cdot \tau^n_* : \pi_j A(X) \to \pi_j A(X)$$

for $j > 0$.

We conclude this introduction with some comments on applications. First, we recall one formulation of the Kahn–Priddy theorem in stable homotopy theory. Let $Q(X) = \Omega^\infty S^\infty(X_+)$ denote unreduced stable homotopy theory and define reduced stable homotopy theory $\widetilde{Q}(X) = \text{fiber}(Q(X) \to Q(*))$, the homotopy fiber. For each n there is a transfer map $Q(B\Sigma_n) \to Q(E\Sigma_n) \simeq Q(*)$, and, by composition, there results a map $\widetilde{Q}(B\Sigma_n) \to Q(*)$. The formulation of the Kahn–Priddy theorem that we prefer is that the map

$$\pi_j(\widetilde{Q}B\Sigma_p)_{(p)} \to \pi_j(Q(*))_{(p)}$$

of homotopy groups localized at a prime p is surjective for $j > 0$.

Waldhausen's analogue of this result applies to the algebraic K-theory of the one-point space. For the formulation we let $A(X)$ denote the algebraic K-theory of the space X and let $\widetilde{A}(X) = \text{fiber}(A(X) \to A(*))$ be the algebraic K-theory of X reduced relative to a point. Manipulations formally similar to those above yield a map $\widetilde{A}(B\Sigma_n) \to A(*)$ and the analogue of the Kahn–Priddy theorem is that the induced map

$$\pi_j(\widetilde{A}(B\Sigma_p)_{(p)} \to \pi_j(A(*))_{(p)}$$

of homotopy groups localized at p is surjective for $j > 0$. In [Waldhausen 1987] these operations are further developed and used to prove that the third factor $\mu(X)$ in the splitting

$$A(X) \simeq Q(X_+) \times Wh^{\mathrm{Diff}}(X) \times \mu(X)$$

is contractible, yielding the final result $A(X) \simeq Q(X_+) \times Wh^{\mathrm{Diff}}(X)$. The significance of this fact is developed in [Waldhausen et al. 2013].

In our situation we fix as base space a connected simplicial abelian group X and define reduced algebraic K-theory relative to X as

$$\widetilde{A}(X \times B\Sigma_n \operatorname{rel} X) = \operatorname{fiber}(A(X \times B\Sigma_n) \to A(X)).$$

The inclusion of a point into $B\Sigma_n$ combined with the definition of the algebraic K-theory of $X \times B\Sigma_n$ reduced relative to X yields a splitting

$$\pi_j A(X \times B\Sigma_n) \cong \pi_j \widetilde{A}(X \times B\Sigma_n \operatorname{rel} X) \oplus \pi_j A(X) \tag{1.4}$$

for any $j \geq 0$. We have transfer maps $\phi_n : A(X \times B\Sigma_n) \to A(X \times E\Sigma_n) \simeq A(X)$ and a basic calculation in Lemma 7.8 that the composition

$$A(X) \to A(X \times B\Sigma_n) \xrightarrow{\phi_n} A(X)$$

is multiplication by $n! = |\Sigma_n|$, where the first map is induced by inclusion of a point into $B\Sigma_n$.

When we specialize n to a prime number p, we have the following observations. Make the following diagram of homotopy groups reduced mod p, where the splitting (1.4) appears as the middle column:

$$\begin{tikzcd}
& \pi_j \widetilde{A}(X \times B\Sigma_p \operatorname{rel} X)/p\mathbb{Z} \arrow[d] \arrow[dr, "\phi_{p*}"] & \\
\pi_j A(X)/p\mathbb{Z} \arrow[r, "\theta^p_*"] & \pi_j A(X \times B\Sigma_p)/p\mathbb{Z} \arrow[r, "\phi_{p*}"] & \pi_j A(X)/p\mathbb{Z} \\
& \pi_j A(X)/p\mathbb{Z} \arrow[u, "i_*"] \arrow[ur, "0"'] &
\end{tikzcd}$$

The diagonal arrow from the bottom row is multiplication by $p! = |\Sigma_p|$, which is 0 modulo p. Thus, in terms of the splitting of $\pi_j A(X \times B\Sigma_p)/p\mathbb{Z}$ given above, on the second component of the image of θ^p_*, the map ϕ_{p*} is zero. Applying Theorem 8.12, ϕ_{p*} applied to the first component $\pi_j \widetilde{A}(X \times B\Sigma_p \operatorname{rel} X)/p\mathbb{Z}$ of the splitting contains the image of $\phi_{p*}\theta^p_* = (-1)^{p-1} \cdot (p-1)! \cdot \tau^p_*$, where $\tau^p : X \to X$ raises elements to the p-th power. The numerical factors are invertible mod p so that

$$\phi_{p*}(\pi_j \widetilde{A}(X \times B\Sigma_p \operatorname{rel} X)/p\mathbb{Z}) \supset \operatorname{Image} \tau^p_*,$$

viewing τ^p_* as an endomorphism of $\pi_j A(X)/p\mathbb{Z}$.

From these calculations one extracts various additional observations. It may happen that the p-th power homomorphism τ^p is an isomorphism, as in the case when X is a connected simplicial abelian group, finite in each simplicial dimension and p is relatively prime to the order of X_n for each n. Then for $j > 0$,

$$\phi_{p*} : \pi_j \widetilde{A}(X \times B\Sigma_p \operatorname{rel} X)/p\mathbb{Z} \to \pi_j A(X)/p\mathbb{Z}$$

is surjective. The next input is the following theorem.

Theorem 1.5 [Betley 1986, Theorem I]. *If $\pi_1(X)$ is a finite group, and $\pi_i(X)$ is finitely generated for all $i \geq 2$, then $\pi_j(A(X))$ is finitely generated for all j.*

Then Nakayama's lemma applies as in [Waldhausen 1982] to lift the result on mod p homotopy to a result on p-localized homotopy. We obtain the following theorem of Kahn–Priddy type.

Theorem 1.6. *Let X be a connected simplicial abelian group, finite in each dimension, such that the order of X_n is prime to p. For $j > 0$ and p an odd prime, the transfer induces surjections*

$$\pi_j \widetilde{A}(X \times B\Sigma_p \operatorname{rel} X)_{(p)} \to \pi_j A(X)_{(p)}$$

on homotopy groups localized at p. □

In particular, take $X = BC_2 = \mathbb{R}P^\infty$ and p an odd prime. There are similar statements for all the lens spaces BC_q, q prime to p.

A very interesting case is $X = BC_\infty$, the classifying space of the infinite cyclic group C_∞. Of course $X \simeq S^1$, and there are splittings-up-to-homotopy of infinite loop spaces

$$A^{fd}(S^1) \simeq A^{fd}(*) \times BA^{fd}(*) \times N_-A^{fd}(*) \times N_+A^{fd}(*)$$

and

$$A^{fd}(S^1 \times B\Sigma_n) \simeq A^{fd}(B\Sigma_n) \times BA^{fd}(B\Sigma_n) \times N_-A^{fd}(B\Sigma_n) \times N_+A^{fd}(B\Sigma_n).$$

These are studied in [Klein and Williams 2008] and the first is examined in great detail in [Grunewald et al. 2008]. In future work we would like to understand the operations we have constructed in terms of these splittings. As a first step in this direction we have shown in Section 4 that the operations we construct are morphisms of infinite loop spaces. Should the θ operations be compatible with the splitting, one must then investigate whether or not the θ operations commute with the Frobenius and Verschiebung operations on the nil-terms defined in [Grunewald et al. 2008].

Our work also admits a generalization where X may be any connected space. This result is a total operation

$$\widetilde{\omega} : A(X) \to \{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \{\text{all}\}}(X^n),$$

about which we know little at this point. Our experiments have also lead to the observation that if G is a simplicial group, not necessarily abelian, whose realization is homotopy equivalent to a finite CW-complex then there is a product structure on $A(BG)$. This will be the subject of a later paper. Finally, reversing the progression from Segal's original idea to Waldhausen's generalization, we can develop operations $\theta^n : \pi_*^s(X_+) \to \pi_*^s(X \times B\Sigma_n)$, where X is again a connected simplicial abelian group.

2. The symmetric bimonoidal category of retractive spaces over a connected simplicial abelian group

The category $\mathcal{R}(X)$ is the category of retractive simplicial sets (Y, r, s) over the simplicial set X, where $r : Y \to X$ is a retraction, $s : X \to Y$ is a section for r and morphisms $(Y_1, r_1, s_1) \to (Y_2, r_2, s_2)$ respect all the data. A cofibration $(Y_1, r_1, s_1) \rightarrowtail (Y_2, r_2, s_2)$ in $\mathcal{R}_f(X)$ is a map such that $Y_1 \to Y_2$ is injective. A weak equivalence is a map $(Y_1, r_1, s_1) \to (Y_2, r_2, s_2)$ whose realization $|Y_1| \to |Y_2|$ is a homotopy equivalence of spaces. For algebraic K-theory we use the full subcategory $\mathcal{R}_f(X)$ of relatively finite retractive simplicial sets with cofibrations and weak equivalences. "Relatively finite" means that there are only finitely many nondegenerate simplices in $Y - X$. For background on the terminology, see [Waldhausen 1985, Section 1.1].

We aim to construct a total operation

$$\theta : A(X) \to \{1\} \times \prod_{n \geq 1} A(X \times B\Sigma_n)$$

for X a connected simplicial abelian group with multiplication $\mu : X \times X \to X$ and to prove the operation has an E_∞-structure. In order to achieve this, the elements from which the construction is developed must be of high quality. The necessary qualities are recorded in the first part of Theorem 2.1; the second part of the theorem records algebraic properties of the product operation $\wedge_\mu$. We discuss first the definition of the product operation, prove the second part of the theorem, and finish this section with the proof of the first part of the theorem.

Concerning the first part of the theorem, our constructions require a coherence result for diagrams involving sum and product operations, as provided by LaPlaza [1972, Proposition 10]. His coherence theorem takes as input the commutativity of 24 diagrams, reducible to a smaller, but still relatively large, subset [Laplaza

1972, pp. 40–41]. We will see that the coherence properties we need rest on the well-understood coherence properties of the one-point union and smash product of pointed sets. On the other hand, the second part of the theorem involves properties of the operations not reducible to dimensionwise considerations.

Theorem 2.1. *Let X be a connected simplicial abelian group.*

(1) *The triple $(\mathcal{R}(X), \vee_X, \wedge_\mu)$, where $\vee_X$ denotes the operation of union along the common subspace X and $\wedge_\mu$ denotes the pairing (2.7), is a symmetric bimonoidal category.*

(2) *The pairing $\wedge_\mu$ restricts to $\mathcal{R}_f(X)$, where it is biexact, meaning exact in each variable separately. Explicitly, the functors defined by $- \wedge_\mu Y$ and $Y \wedge_\mu -$ preserve cofibrations, pushouts along cofibrations, and weak equivalences.*

Our product operation $\wedge_\mu$ derives from an exterior smash product $\wedge_e$ of retractive simplicial sets, following the exterior smash product of retractive spaces as described in [May and Sigurdsson 2006]. Since we are working with simplicial sets, our version of the exterior smash product has a description in terms of operations on discrete sets, applied dimensionwise. See the discussion at the start of the proof of part one of Theorem 2.1.

Definition 2.2. Let (Y_i, r_i, s_i) be objects of $\mathcal{R}(X_i)$, for $i = 1, 2$. The exterior smash product of (Y_1, r_1, s_1) with (Y_2, r_2, s_2) is in $\mathcal{R}(X_1 \times X_2)$, and the underlying space $Y_1 \wedge_e Y_2$ completes the following square to a pushout:

$$\begin{tikzcd} Y_1 \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 \arrow[r, tail] \arrow[d, "r_1 \times r_2"'] & Y_1 \times Y_2 \arrow[d] \\ X_1 \times X_2 \arrow[r, tail, "s_1 \wedge_e s_2"] & Y_1 \wedge_e Y_2 \end{tikzcd} \tag{2.3}$$

The square displays the section $s_1 \wedge_e s_2$; the retraction $r_1 \wedge_e r_2$ arises from the universal property of the pushout.

Note that if both X_1 and X_2 are the one-point space, then this is the smash product in the category of pointed spaces. Extending this idea, if $x_1 : \{*\} \to X_1$ and $x_2 : \{*\} \to X_2$ are two maps of the one-point space into X_1 and X_2, and we take preimages $r_1^{-1}(x_1)$ and $r_2^{-1}(x_2)$, then these are pointed spaces, and there is an injective map $r_1^{-1}(x_1) \wedge r_2^{-1}(x_2) \to Y_1 \wedge_e Y_2$ over the point $(x_1, x_2) \in X_1 \times X_2$. This observation helps explain the "fiberwise smash product" terminology and indicates how the coherence issues for products may be resolved at the level of pointed sets. Examples 2.4 and 2.5 here play roles in the proof of part one of Theorem 2.1. Also, since we work with simplicial sets, underlying the symmetric monoidal structure $(\mathcal{R}(X), \vee_X, \wedge_\mu)$ is the symmetric monoidal structure on the category of sets.

Example 2.4. For any $Y_2 \in R(X_2)$, note that $X_1 \wedge_e Y_2 \cong X_1 \times X_2$, the "zero" object in $\mathcal{R}(X_1 \times X_2)$. Colloquially, the exterior smash product of a terminal object with any object yields a terminal object. Explicitly, a natural isomorphism

$$\lambda^*_{Y_2} : X_1 \wedge_e Y_2 \to X_1 \times X_2$$

arises from the following diagram by mapping the pushout of the top row to the pushout of the bottom row:

$$\begin{tikzcd}
X_1 \times X_2 \arrow[d, "="'] & X_1 \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 \arrow[l] \arrow[r, tail] \arrow[d, "\cong"'] & X_1 \times Y_2 \arrow[d, "="'] \\
X_1 \times X_2 & X_1 \times Y_2 \arrow[l] \arrow[r, tail] & X_1 \times Y_2
\end{tikzcd}$$

Example 2.5. The bifunctor $\mathcal{R}(*) \times \mathcal{R}(X) \to \mathcal{R}(X)$ given by $(Y_1, Y_2) \mapsto Y_1 \wedge_e Y_2$ defines an action of $\mathcal{R}(*)$ on $\mathcal{R}(X)$ after identifying $\{*\} \times X$ with X in the canonical way. This bifunctor also restricts to an action $\mathcal{R}_f(*) \times \mathcal{R}_f(X) \to \mathcal{R}_f(X)$.

This action has an identity element. Indeed, for $S^0 = \{*, *'\}$ in $\mathcal{R}_f(*)$, with r the constant map to the basepoint $*$, s the inclusion, and $Y \in \mathcal{R}(X)$, the function $S^0 \times Y \to Y$ defined by $(*, y) \mapsto sr(y)$ and $(*', y) \mapsto y$ induces an isomorphism $S^0 \wedge_e Y \xrightarrow{\cong} Y$ of retractive spaces over X. An inverse to this isomorphism is provided by $y \mapsto [(*', y)] \in S^0 \wedge_e Y$.

Definition 2.6. Let X be a space with a multiplication $\mu : X^2 \to X$. We operate on the category $\mathcal{R}(X)$, using the pairing

$$\wedge_\mu = \mu_* \circ \wedge_e : \mathcal{R}(X) \times \mathcal{R}(X) \xrightarrow{\wedge_e} \mathcal{R}(X \times X) \xrightarrow{\mu_*} \mathcal{R}(X), \tag{2.7}$$

where $\wedge_e$ is the external smash product pairing defined in (2.3) and μ_* is the functor induced by the multiplication $\mu : X^2 \to X$. Explicitly, $(Y_1, r_1, s_1) \wedge_\mu (Y_2, r_2, s_2)$ completes the following diagram to a pushout:

$$\begin{tikzcd}
Y_1 \times X \cup_{X \times X} X \times Y_2 \arrow[r, tail] \arrow[d, "\mu(r_1,\mathrm{id}) \cup \mu(\mathrm{id}, r_2)"'] & Y_1 \times Y_2 \arrow[d] \\
X \arrow[r, tail, "s"] & Y_1 \wedge_\mu Y_2
\end{tikzcd} \tag{2.8}$$

We use these notations to bring this section close to conformity with [Gunnarsson and Schwänzl 2002]. Perfect conformity is not possible, for we must use both the one-point union of pointed spaces $\vee$ and the union of two spaces along a common subspace X, denoted $\vee_X$. We also point out that the usual notation $\wedge$ has been used in [May and Sigurdsson 2006] for a product defined by restricting the external smash product of two spaces over X to the diagonal of $X \times X$.

The next lemma is used to develop properties of the smash products; the proof will be given after demonstrating applications in Propositions 2.13 and 2.17.

Lemma 2.9. *Let $\mathcal{C}$ be a category with cofibrations and let*

$$\begin{tikzcd} A_2 & B_2 \arrow[l] \arrow[r, tail] & C_2 \\ A_1 \arrow[u, tail] \arrow[d] & B_1 \arrow[l] \arrow[r, tail] \arrow[u, tail] \arrow[d] & C_1 \arrow[u, tail] \arrow[d] \\ A_0 & B_0 \arrow[l] \arrow[r, tail] & C_0 \end{tikzcd} \tag{2.10}$$

be a commutative diagram in which the canonical map from $B_2 \cup_{B_1} C_1$ to C_2 is a cofibration. Passing to pushouts by columns results in a diagram

$$A_0 \cup_{A_1} A_2 \leftarrow B_0 \cup_{B_1} B_2 \rightarrowtail C_0 \cup_{C_1} C_2 \tag{2.11}$$

in which the right-pointing arrow is a cofibration. The diagram

$$A_0 \cup_{B_0} C_0 \leftarrow A_1 \cup_{B_1} C_1 \rightarrowtail A_2 \cup_{B_2} C_2 \tag{2.12}$$

obtained by passing to pushouts by rows has a similar property.

Proposition 2.13. *The exterior smash product $\wedge_e$ is functorial for pairs of maps. That is, given $f_1 : X_1 \to X_1'$ and $f_2 : X_2 \to X_2'$, the diagram*

$$\begin{tikzcd} \mathcal{R}_f(X_1) \times \mathcal{R}_f(X_2) \arrow[r, "\wedge_e"] \arrow[d, "f_{1*} \times f_{2*}"'] & \mathcal{R}_f(X_1 \times X_2) \arrow[d, "(f_1 \times f_2)_*"'] \\ \mathcal{R}_f(X_1') \times \mathcal{R}_f(X_2') \arrow[r, "\wedge_e"] & \mathcal{R}_f(X_1' \times X_2') \end{tikzcd} \tag{2.14}$$

commutes up to natural isomorphism.

Proof. For the naturality property of the external smash product, consider the diagram

$$\begin{tikzcd} X_1 \times X_2 & Y_1 \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 \arrow[l] \arrow[r, tail] & Y_1 \times Y_2 \\ X_1 \times X_2 \arrow[u, tail] \arrow[d, "f_1 \times f_2"'] & X_1 \times X_2 \arrow[l] \arrow[r, tail] \arrow[u, tail] \arrow[d, "f_1 \times f_2"'] & X_1 \times X_2 \arrow[u, tail] \arrow[d, "f_1 \times f_2"'] \\ X_1' \times X_2' & X_1' \times X_2' \arrow[l] \arrow[r, tail] & X_1' \times X_2' \end{tikzcd} \tag{2.15}$$

which fulfills the hypotheses of Lemma 2.9. Computing the colimits of the columns in this diagram yields the diagram

$$X_1' \times X_2' \xleftarrow{r_1' \times r_2'} (f_{1*}Y_1) \times X_2' \cup_{X_1' \times X_2'} X_1' \times (f_{2*}Y_2) \rightarrowtail f_{1*}Y_1 \times f_{2*}Y_2,$$

whose pushout is by definition $f_{1*}Y_1 \wedge_e f_{2*}Y_2$.

On the other hand, computing the colimits of the rows in the diagram yields the diagram

$$X_1' \times X_2' \xleftarrow{f_1 \times f_2} X_1 \times X_2 \rightarrowtail Y_1 \wedge_e Y_2,$$

whose pushout is $(f_1 \times f_2)_*(Y_1 \wedge_e Y_2)$. Since both iterative procedures compute the colimit of diagram (2.15), they are canonically isomorphic:

$$f_{1*}Y_1 \wedge_e f_{2*}Y_2 \cong (f_1 \times f_2)_*(Y_1 \wedge_e Y_2). \qquad \square$$

As a consequence, we have the following result.

Proposition 2.16. *Let X be a monoid with unit. The action of $\mathcal{R}(*)$ on $\mathcal{R}(X)$ set up in Example 2.5 may be made internal to $\mathcal{R}(X)$. Diagrammatically,*

$$\begin{tikzcd} \mathcal{R}(*) \times \mathcal{R}(X) \arrow[r, "i_{e*} \times \mathrm{id}"] \arrow[dr, "\wedge_e"] & \mathcal{R}(X) \times \mathcal{R}(X) \arrow[d, "\wedge_\mu"] \\ & \mathcal{R}(X) \end{tikzcd}$$

commutes up to natural isomorphism.

Proof. Let $i_e : \{*\} \to X$ be the inclusion of the one-point space to the identity element of the monoid X. The functor $i_{e*} : \mathcal{R}(*) \to \mathcal{R}(X)$ sends a pointed retractive space Y to $X \vee Y$, where the base point of Y is identified with the unit element of X. The new retraction collapses $Y \subset X \vee Y$ to the identity $\{e\}$ in X. We have the diagram

$$\begin{tikzcd} \mathcal{R}(*) \times \mathcal{R}(X) \arrow[r, "\wedge_e"] \arrow[d, "i_{e*} \times \mathrm{id}"'] & \mathcal{R}(\{*\} \times X) \arrow[d, "(i_e \times \mathrm{id})_*"'] \arrow[dr, "p_{2*}"] & \\ \mathcal{R}(X) \times \mathcal{R}(X) \arrow[r, "\wedge_e"] & \mathcal{R}(X \times X) \arrow[r, "\mu_*"] & \mathcal{R}(X) \end{tikzcd}$$

The left-hand square commutes by Proposition 2.13, and the right-hand triangle commutes because e is the monoid identity. The bottom row defines $\wedge_\mu$ and the trip across the top defines the action of $\mathcal{R}(*)$ on $\mathcal{R}(X)$. $\square$

For example, this result has the consequence that coherent associativity for $\wedge_\mu$ implies corresponding coherent associativity for the $\wedge_e$ action of $\mathcal{R}(*)$ on $\mathcal{R}(X)$.

Next, we record the biexactness property of the external smash product as defined in the statement of Theorem 2.1.

Proposition 2.17. *The external smash product functor*

$$\wedge_e : \mathcal{R}_f(X_1) \times \mathcal{R}_f(X_2) \to \mathcal{R}_f(X_1 \times X_2)$$

is biexact.

Remark 2.18. In the approach of [May and Sigurdsson 2006] the external smash product is shown to preserve all colimits by exhibiting a left adjoint functor. Their approach uses properties of convenient categories of topological spaces.

For our applications in algebraic K-theory it seems more reasonable to give arguments modeled on those of [Waldhausen 1985, Lemma 1.1.1], which serve to illuminate other constructions we make.

Proof of Proposition 2.17. For simplicial sets, cofibrations are precisely the injections. Given a pair of cofibrations

$$(W_1, r_1, s_1) \rightarrowtail (W_1', r_1', s_1') \quad \text{and} \quad (W_2, r_2, s_2) \rightarrowtail (W_2', r_2', s_2')$$

in $\mathcal{R}_f(X_1)$ and $\mathcal{R}_f(X_2)$, respectively, the maps of differences of simplicial sets $W_1 - X_1 \to W_1' - X_1$ and $W_2 - X_2 \to W_2' - X_2$ are injective maps of sets in each simplicial dimension. The product of these maps is also injective. Since $(W_1 \wedge_e W_2) - X_1 \times X_2 = (W_1 - X_1) \times (W_2 - X_2)$, it follows that $W_1 \wedge_e W_2 \rightarrowtail W_1' \wedge_e W_2'$ is also a cofibration. Finally, if $W_1 - X_1$ and $W_2 - X_2$ contain only finitely many nondegenerate simplices, then the same is true of their product. Thus, the pairing $\wedge_e$ restricts to a pairing of $\mathcal{R}_f(X_1) \times \mathcal{R}_f(X_2)$ to $\mathcal{R}_f(X_1 \times X_2)$.

To prove that the functor $Z \wedge_e (-) : R_f(X_2) \to R_f(X_1 \times X_2)$ preserves pushouts of cofibrations, start by considering the diagram

$$\begin{tikzcd}
X_1 \times X_2 \arrow[d, equal] & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 \arrow[l] \arrow[r, tail] & Z \times Y_2 \\
X_1 \times X_2 \arrow[d, equal] & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1 \arrow[l] \arrow[r, tail] \arrow[u, tail] \arrow[d] & Z \times Y_1 \arrow[u, tail] \arrow[d] \\
X_1 \times X_2 & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_0 \arrow[l] \arrow[r, tail] & Z \times Y_0
\end{tikzcd} \tag{2.19}$$

where the right-pointing arrows are induced from the retractions and the left-pointing arrows are induced by inclusions. We verify the cofibration hypothesis of Lemma 2.9 using the following diagram to analyze the upper right-hand corner of (2.19):

$$\begin{tikzcd}
 & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_2 \arrow[dl, tail] & (Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1) \cup_{X_1 \times Y_1} X_1 \times Y_2 \arrow[l, tail, "\cong"'] & X_1 \times Y_2 \arrow[l, tail] \\
Z \times Y_2 & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1 \arrow[u, tail] \arrow[d, tail] & Z \times X_2 \cup_{X_1 \times X_2} X_1 \times Y_1 \arrow[l, tail, "="'] \arrow[u, tail] \arrow[d, tail] & X_1 \times Y_1 \arrow[l, tail] \arrow[u, tail] \arrow[d, tail] \\
 & Z \times Y_1 \arrow[ul, tail] & Z \times Y_1 \arrow[l, tail, "="'] & Z \times Y_1 \arrow[l, tail, "="']
\end{tikzcd}$$

Pass to pushouts in the columns, apply the universal mapping properties of the pushouts, and use isomorphism (2.24) to simplify the pushout of the middle column to obtain the commuting diagram

$$
\begin{tikzcd}
Z\times Y_1 \cup_{X_1\times Y_1} X_1\times Y_2 \arrow[d, "\cong"'] & & Z\times Y_1 \cup_{X_1\times Y_1} X_1\times Y_2 \arrow[ll, "\cong"'] \arrow[d, tail] \\
Z\times Y_1 \cup_{(Z\times X_2 \cup_{X_1\times X_2} X_1\times Y_1)} (Z\times X_2 \cup_{X_1\times X_2} X_1\times Y_2) \arrow[rr] & & Z\times Y_2
\end{tikzcd}
$$

The space $Z\times Y_1 \cup_{X_1\times Y_1} X_1\times Y_2$ is a subspace of $Z\times Y_2$, so the downward arrow on the right is a cofibration. Since isomorphisms are cofibrations, it follows that the lower arrow is also a cofibration. Thus, we have verified the cofibration condition of Lemma 2.9 for (2.19).

We may now calculate the colimit of diagram (2.19) by two different iterative procedures. Computing the pushouts of the rows first and applying Lemma 2.9 gives a diagram

$$Z \wedge_e Y_2 \leftarrowtail Z \wedge_e Y_1 \to Z \wedge_e Y_0 \tag{2.20}$$

and calculating the pushouts of the columns first and applying Lemma 2.9 again gives a another diagram

$$X_1 \times X_2 \leftarrow Z \times X_2 \cup_{X_1\times X_2} X_1 \times (Y_0 \cup_{Y_1} Y_2) \rightarrowtail Z \times (Y_0 \cup_{Y_1} Y_2). \tag{2.21}$$

To see this formula for the middle object in (2.21), make the following considerations. We have the diagram

$$
\begin{tikzcd}
Z_2\times X_2 & X_1\times X_2 \arrow[l] \arrow[r, tail] & X_1\times Y_2 \\
Z_2\times X_2 \arrow[u, tail, "="] \arrow[d, "="'] & X_1\times X_2 \arrow[l] \arrow[r, tail] \arrow[u, tail, "="] \arrow[d, "="'] & X_1\times Y_1 \arrow[u, tail] \arrow[d] \\
Z_2\times X_2 & X_1\times X_2 \arrow[l] \arrow[r, tail] & X_1\times Y_0
\end{tikzcd}
\tag{2.22}
$$

meeting the conditions of Lemma 2.9, whose colimit we also compute iteratively. Computing the pushouts of the rows first gives precisely the middle column in (2.19), whose pushout we are now evaluating. On the other hand, computing the pushouts along the columns first gives a diagram

$$Z_2 \times X_2 \leftarrow X_1 \times X_2 \rightarrowtail X_1 \times (Y_2 \cup_{Y_1} Y_0)$$

whose pushout is the middle term displayed in (2.21). As the iterated pushouts of (2.22) are isomorphic to the colimit of the entire diagram, the iterated pushouts are isomorphic. This justifies (2.21).

Completing the analysis of diagram (2.19), the pushouts of (2.20) and (2.21) are isomorphic, because they both represent the colimit of the original diagram (2.19). Interpreting this statement, we have the result that $Z \wedge_e -$ preserves pushouts of cofibrations.

Suppose $f_1 : Y_1 \to Y_1'$ and $f_2 : Y_2 \to Y_2'$ are weak equivalences in $\mathcal{R}_f(X_1)$ and $\mathcal{R}_f(X_2)$, respectively. That is, the geometric realizations $|f_1|$ and $|f_2|$ are homotopy equivalences. Then $|f_1| \times \mathrm{id}_{|X_2|}$ and $\mathrm{id}_{|X_1|} \times |f_2|$ are homotopy equivalences. By the ordinary gluing lemma for homotopy equivalences applied to the diagram

$$\begin{tikzcd} |Y_1| \times |X_2| \arrow[d, "|f_1|\times \mathrm{id}_{|X_2|}"'] & |X_1| \times |X_2| \arrow[l, tail] \arrow[r, tail] \arrow[d, "\mathrm{id}_{|X_1|}\times \mathrm{id}_{|X_2|}"'] & |X_1| \times |Y_2| \arrow[d, "\mathrm{id}_{|X_1|}\times |f_2|"'] \\ |Y_1'| \times |X_2| & |X_1| \times |X_2| \arrow[l, tail] \arrow[r, tail] & |X_1| \times |Y_2'| \end{tikzcd}$$

the central arrow in

$$\begin{tikzcd} |X_1| \times |X_2| \arrow[d, "\mathrm{id}_{|X_1|}\times \mathrm{id}_{|X_2|}"'] & |Y_1| \times |X_2| \cup_{|X_1|\times|X_2|} |X_1| \times |Y_2| \arrow[l] \arrow[r, tail] \arrow[d, "\simeq"'] & |Y_1| \times |Y_2| \arrow[d, "|f_1|\times|f_2|"'] \\ |X_1| \times |X_2| & |Y_1'| \times |X_2| \cup_{|X_1|\times|X_2|} |X_1| \times |Y_2'| \arrow[l] \arrow[r, tail] & |Y_1'| \times |Y_2'| \end{tikzcd}$$

is also a homotopy equivalence. Since the pushout of the last diagram is homeomorphic to $|Y_1 \wedge_e Y_2| \to |Y_1' \wedge_e Y_2'|$ ("colimits commute"), $Y_1 \wedge_e Y_2 \to Y_1' \wedge_e Y_2'$ is a weak equivalence. $\square$

Remark 2.23. The external smash product also preserves many colimits. However, our applications principally involve the special colimits that are pushouts of cofibration squares.

Here is the postponed proof of Lemma 2.9.

Proof of Lemma 2.9. We make frequent use of the isomorphism

$$(A \cup_B C) \cup_C D \cong A \cup_B D. \tag{2.24}$$

The canonical arrow $B_2 \cup_{B_1} B_0 \to C_2 \cup_{C_1} C_0$ factors into the composition of canonical arrows induced by passing to pushouts of the columns in the map of diagrams

$$\begin{tikzcd} B_2 \arrow[r, "="] & B_2 \arrow[r, tail] & C_2 \\ B_1 \arrow[u, tail] \arrow[r, "="] \arrow[d] & B_1 \arrow[u, tail] \arrow[r, tail] \arrow[d] & C_1 \arrow[u, tail] \arrow[d] \\ B_0 \arrow[r, tail] & C_0 \arrow[r, "="] & C_0 \end{tikzcd}$$

We show each arrow in the factorization is a cofibration. The first arrow in the factorization appears as the lower row in the completed pushout diagram

$$\begin{tikzcd} B_0 \arrow[r, tail] \arrow[d] & C_0 \arrow[d] & \\ B_2 \cup_{B_1} B_0 \arrow[r, tail] & (B_2 \cup_{B_1} B_0) \cup_{B_0} C_0 \arrow[r, "\cong"] & B_2 \cup_{B_1} C_0 \end{tikzcd}$$

augmented by an isomorphism, so the first arrow is a cofibration, as claimed. From the hypothesis on the canonical map from $B_2 \cup_{B_1} C_1$ to C_2, the upper arrow in the next diagram is a cofibration, so the lower arrow in the completed pushout diagram is as well:

$$\begin{tikzcd}
 & B_2 \cup_{B_1} C_1 \arrow[r, tail] \arrow[d] & C_2 \arrow[d] & \\
B_2 \cup_{B_1} C_0 & (B_2 \cup_{B_1} C_1) \cup_{C_1} C_0 \arrow[l, "\cong"'] \arrow[r, tail] & C_2 \cup_{B_2 \cup_{B_1} C_1} ((B_2 \cup_{B_1} C_1) \cup_{C_1} C_0) \arrow[r, "\cong"] & C_2 \cup_{C_1} C_0
\end{tikzcd}$$

Augmenting the completed pushout diagram by the two isomorphisms, the second arrow $B_2 \cup_{B_1} C_0 \to C_2 \cup_{C_1} C_0$ in the factorization is also a cofibration. Then the composition $B_2 \cup_{B_1} B_0 \rightarrowtail B_2 \cup_{B_0} C_0 \rightarrowtail C_2 \cup_{C_1} C_0$ is a cofibration and this arrow is isomorphic to the arrow in diagram (2.11).

To obtain the result for the row-wise pushouts from the result for columnwise pushouts, observe that the properties of the arrows in the diagram are symmetric with respect to reflection in the diagonal $A_0 B_1 C_2$. Therefore, it suffices to reflect the diagram in this diagonal and apply the columnwise result. □

Proof of the second part of Theorem 2.1. Since the functor

$$\mu_* : \mathcal{R}_f(X \times X) \to \mathcal{R}_f(X)$$

is exact [Waldhausen 1985, Lemma 2.1.6], and we have seen that $\wedge_e$ is biexact in Proposition 2.17, the composite $\wedge_\mu = \mu_* \circ \wedge_e$ is biexact. □

Now we take up coherence properties.

Proof of the first part of Theorem 2.1. It is well-known that the disjoint union of sets and the one-point union $\vee$ of pointed sets are categorical sum operations, so that all coherence conditions for these operations are automatically met. For the category of sets containing a fixed set S the union $\vee_S$ of two sets along the common subset is also the categorical sum, so $\vee_S$ fulfills all coherence conditions. Concerning products, the cartesian product of sets and the smash product of pointed sets are operations also meeting coherence conditions. When these operations of sum and product are considered together, they are related by distributivity isomorphisms, and the combined systems exhibit the coherence properties discussed in [Laplaza 1972]. It is possible to develop the coherence properties we need for operations on retractive spaces from these basic elements by developing the operation $\vee_X$ dimensionwise and pointwise over X from one-point union and the operation $\wedge_e$ dimensionwise and pointwise over $X_1 \times X_2$ from the smash product of pointed sets. Compare the remark following Definition 2.2. We take a different approach here.

For the sum $\vee_X$, we need a slight extension of the union of sets along a common subset to cover the case of the disjoint union of two simplicial sets along a common simplicial subset. Let $\mathcal{T}$ be the category of triples $(T, r : T \to S, s : S \to T)$, where

S and T are sets and the functions satisfy $r \circ s = \mathrm{id}_S$. Occasionally, it is convenient to view S as a subset of T. A morphism

$$(f, f') : (T_1, r_1 : T_1 \to S_1, s_1 : S_1 \to T_1) \to (T_0, r_0 : T_0 \to S_0, s_0 : S_0 \to T_0)$$

is a pair of maps $f : T_1 \to T_0$ and $f' : S_1 \to S_0$ such that $s_0 f' = f s_1$ and $r_0 f = f' r_1$. An object (Y, r, s) of $\mathcal{R}(X)$ can be viewed as a functor $\Delta^{\mathrm{op}} \to \mathcal{T}$, and conversely. There is a functor $u : \mathcal{T} \to \mathbf{Set}$ that selects the subset S and morphisms $f' : S_1 \to S_0$. On the pullback category

$$\begin{tikzcd}
\mathcal{T} \times_{\mathbf{Set}} \mathcal{T} \arrow[r] \arrow[d] & \mathcal{T} \times \mathcal{T} \arrow[d, "u \times u"'] \\
\mathbf{Set} \arrow[r, "\Delta"] & \mathbf{Set} \times \mathbf{Set}
\end{tikzcd}$$

define the operation $(T_1, r_1 : T_1 \to S, s_1 : S \to T_1) \vee_S (T_2, r_2 : T_2 \to S, s_2 : S \to T_2)$, abbreviated $(T_1, r_1, s_1) \vee_S (T_2, r_2, s_2)$, or even $T_1 \vee_S T_2$. Set

$$T_1 \vee_S T_2 = T_1 \amalg T_2 \,/ \sim,$$

where $\sim$ is the equivalence relation generated by setting $s_1(x) \sim s_2(x)$ for $x \in S$. Set $i_j : T_j \to T_1 \vee_S T_2$ to be the inclusion $T_j \to T_1 \amalg T_2$ followed by the quotient map to $T_1 \vee_s T_2$. For the rest of the structure, set

$$r : T_1 \vee_S T_2 \to S$$

to be the unique function satisfying $r i_j = r_j$, for $j = 1, 2$, and let

$$s : S \to T_1 \vee_S T_2$$

satisfy $s(x) = i_1 s_1(x) = i_2 s_2(x)$ for $x \in S$. Define

$$(i_1, i_1' = \mathrm{id}) : (T_1, s_1, r_1) \to (T_1 \vee_S T_2, r, s)$$

to obtain a morphism in $\mathcal{T}$. The identities $r i_1 = i_1' r_1$ and $s i_1' = i_1 s_1$ are satisfied by definition and by the condition $r_1 s_1 = \mathrm{id}$. Define

$$(i_2, i_2') : (T_2, s_2, r_2) \to (T_1 \vee_S T_2, r, s).$$

similarly. If (T', r', s') is another object of $\mathcal{T}$, let $(f_i, f_i') : (T_i, r_i, s_i) \to (T', r', s')$ be a morphism in $\mathcal{T} \times_{\mathbf{Set}} \mathcal{T}$ for $i = 1, 2$. This just means that $f_1' = f_2' : S \to S'$. Then the categorical sum properties of the disjoint union on the category **Set** and the quotient construction deliver a unique morphism

$$(h, h') : (T_1 \vee_S T_2, r', s') \to (T', r', s')$$

such that $(h,h')\circ(i_1,i_1')=(f_1,f_1')$ and $(h,h')\circ(i_2,i_2')=(f_2,f_2')$. When the base set is fixed, we obtain a categorical sum; in general, when the base set varies, we obtain a (partially defined) categorical sum on $\mathcal{T}$.

We have observed that an object of the category $\mathcal{R}(X)$ is a simplicial object in the category $\mathcal{T}$, that is, a functor $\Delta^{\mathrm{op}}\to\mathcal{T}$. A pair of objects (Y_1,r_1,s_1) and (Y_2,r_2,s_2) in $\mathcal{R}(X)$ defines a functor $\Delta^{\mathrm{op}}\to\mathcal{T}\times_{\mathbf{Set}}\mathcal{T}$. We obtain the operation $(Y_1,r_1,s_1)\vee_X(Y_2,r_2,s_2)$ based on the dimensionwise operation $(Y_1)_p\vee_{X_p}(Y_2)_p$. This makes $\vee_X$ a categorical sum in $\mathcal{R}(X)$, with unit (zero element, thinking additively) the space X. The commutativity isomorphisms γ', associativity isomorphisms α', and left and right unit isomorphisms λ' and ρ' are straightforward consequences of the analogous properties of the disjoint union operation on sets. Essentially, all the basic properties required for coherence of the sum operation $\vee_X$ are automatically fulfilled. That $\vee_X$ is the categorical sum simplifies almost all coherence considerations involving diagrams involving both $\vee_X$ and $\wedge_\mu$.

To complete the input for LaPlaza's coherence result we need to identify in $\mathcal{R}(X)$ an additive identity, a multiplicative zero element, a multiplicative identity, commutativity and associativity isomorphisms for $\wedge_\mu$, and, finally, distributivity isomorphisms.

Clearly $(X,\mathrm{id},\mathrm{id})$ is the identity for $\vee_X$. Example 2.4 implies that $(X,\mathrm{id},\mathrm{id})$ is a zero object from the left and the right for $\wedge_\mu$, in the sense that there are natural isomorphisms

$$\lambda_Y^*:X\wedge_\mu Y\to X\quad\text{and}\quad\rho_Y^*:Y\wedge_\mu X\to X.$$

Example 2.5 combined with Proposition 2.16 delivers the fact that $i_{e*}(S^0)=X\vee S^0$, where the base point of S^0 is identified with the multiplicative identity of X and the retraction collapses S^0 to the identity of X, is a multiplicative identity in the sense that there are natural isomorphisms

$$\lambda_Y:(X\vee S^0)\wedge_\mu Y\to Y\quad\text{and}\quad\rho_Y:Y\wedge_\mu(X\vee S^0)\to Y.$$

For commutativity of the product $\wedge_\mu=\mu_*\circ\wedge_e$, we have the following considerations. Use commutativity for cartesian products and apply the definitions from (2.8) of the internal smash product to obtain the following diagram:

$$\begin{tikzcd}
X \arrow[d, equal] & X\times X \arrow[l, "\mu"'] \arrow[d, "\gamma"'] & Y_1\times X\cup_{X\times X}X\times Y_2 \arrow[l] \arrow[r, tail] \arrow[d, "\gamma"'] & Y_1\times Y_2 \arrow[d, "\gamma"'] \\
X & X\times X \arrow[l, "\mu"'] & Y_2\times X\cup_{X\times X}X\times Y_1 \arrow[l] \arrow[r, tail] & Y_2\times Y_1
\end{tikzcd}\tag{2.25}$$

In the diagram the arrows labeled γ are the isomorphisms switching the factors in the cartesian products. Note that

$$r_1\wedge_\mu r_2=\mu_*(r_1\wedge_e r_2)=r_1\cdot r_2=r_2\cdot r_1=\mu_*(r_2\wedge_e r_1)=r_2\wedge_\mu r_1,$$

since X is abelian. Passage to pushouts yields an isomorphism

$$\gamma_{Y_1,Y_2} : (Y_1 \wedge_\mu Y_2, r_1 \wedge_\mu r_2, s_1 \wedge_\mu s_2) \xrightarrow{\cong} (Y_2 \wedge_\mu Y_1, r_2 \wedge_\mu r_1, s_2 \wedge_\mu s_1).$$

It is easily seen that $\gamma_{Y_2,Y_1}\gamma_{Y_1,Y_2} = \mathrm{id}$ holds (often written "$\gamma^2 = \mathrm{id}$" and called the inverse law), and that the left and right unit laws are compatible. These facts are recorded in the following commuting diagrams:

$$\begin{tikzcd}
 & Y_2 \wedge_\mu Y_1 \arrow[dr, "\gamma_{Y_2,Y_1}"] & \\
Y_1 \wedge_\mu Y_2 \arrow[ur, "\gamma_{Y_1,Y_2}"] \arrow[rr, equal] & & Y_1 \wedge_\mu Y_2
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \wedge_\mu (X \vee S^0) \arrow[rr, "\gamma_{Y,X\vee S^0}"] \arrow[dr, "\rho_Y"'] & & (X \vee S^0) \wedge_\mu Y \arrow[dl, "\lambda_Y"] \\
 & Y &
\end{tikzcd}$$

Consider now associativity, for which we use the diagram

$$\begin{tikzcd}
 & {\begin{array}{c}((Y_1 \times Y_2) \times X) \cup ((Y_1 \times X) \times Y_3) \\ \cup((X \times Y_2) \times Y_3)\end{array}} \arrow[r, tail] \arrow[dl, "{\begin{array}{c}\mu(\mu(r_1,r_2),\mathrm{id}) \cup \mu(\mu(r_1,\mathrm{id}),r_3) \\ \cup\mu(\mu(\mathrm{id},r_2),r_3)\end{array}}"'] & (Y_1 \times Y_2) \times Y_3 \\
X & & \\
 & {\begin{array}{c}((Y_1 \times X) \times X) \\ \cup((X \times Y_2) \times X)\end{array}} \arrow[uu, tail] \arrow[r, tail, "="] \arrow[dl, "{\begin{array}{c}\mu(\mu(r_1,\mathrm{id}),\mathrm{id}) \\ \cup\mu(\mu(\mathrm{id},r_2),\mathrm{id})\end{array}}"'] \arrow[dd, "{\begin{array}{c}(\mu(r_1,\mathrm{id}),\mathrm{id}) \\ \cup(\mu(\mathrm{id},r_2),\mathrm{id})\end{array}}"] & {\begin{array}{c}((Y_1 \times X) \times X) \\ \cup((X \times Y_2) \times X)\end{array}} \arrow[uu, tail] \arrow[dd, "{\begin{array}{c}(\mu(r_1,\mathrm{id}),\mathrm{id}) \\ \cup(\mu(\mathrm{id},r_2),\mathrm{id})\end{array}}"] \\
X \arrow[uu, "="] \arrow[dd, "="'] & & \\
 & X \times X \arrow[r, tail, "="] \arrow[dl, "\mu"'] & X \times X \\
X & &
\end{tikzcd} \tag{2.26}$$

The point is that the associativity for $\wedge_\mu$ rests on associativity for $\times$, $\cup$, and associativity of the multiplication μ on X. By passage to colimits we obtain associativity for $\wedge_\mu$. For the usual smash product, associativity for cartesian products passes to associativity for smash products; our argument is similarly structured.

The first step is to obtain an expression for $(Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3$ that involves only cartesian products and colimits. Diagram (2.26) fulfills the hypotheses of Lemma 2.9, so we may calculate the colimit iteratively in two ways. Taking the colimit along the columns produces the diagram

$$X \xleftarrow{\mu(r_{12},\mathrm{id})\cup\mu(\mathrm{id},r_3)} ((Y_1 \wedge_\mu Y_2) \times X) \cup (X \times Y_3) \rightarrowtail (Y_1 \wedge_\mu Y_2) \times Y_3$$

whose colimit is by definition $(Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3$. On the other hand, computing the colimit along the rows produces the diagram

$$\begin{tikzcd}
& \begin{array}{c}((Y_1 \times Y_2) \times X) \cup ((Y_1 \times X) \times Y_3) \\ \cup\, ((X \times Y_2) \times Y_3)\end{array} \arrow[r, tail] \arrow[dl, "\begin{array}{c}\mu(\mu(r_1,r_2),\mathrm{id})\cup\mu(\mu(r_1,\mathrm{id}),r_3)\\ \cup\mu(\mu(\mathrm{id},r_2),r_3)\end{array}"'] & (Y_1 \times Y_2) \times Y_3 \\
X & &
\end{tikzcd}$$

a copy of the top row in (2.26). Therefore, the colimit, or pushout, of this diagram is another representation of $(Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3$, and we record the completed diagram

$$\begin{tikzcd}
\begin{array}{c}((Y_1 \times Y_2) \times X) \cup ((Y_1 \times X) \times Y_3) \\ \cup\, ((X \times Y_2) \times Y_3)\end{array} \arrow[r, tail] \arrow[d, "\mu(\mu(r_1,r_2),\mathrm{id})\cup\mu(\mu(r_1,\mathrm{id}),r_3)\cup\mu(\mu(\mathrm{id},r_2),r_3)"'] & (Y_1 \times Y_2) \times Y_3 \arrow[d] \\
X \arrow[r, tail] & (Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3
\end{tikzcd} \tag{2.27}$$

as a preferred alternative representation of $(Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3$. Starting from a diagram similar to (2.26), but with parentheses shifted to the right, there is another completed pushout diagram

$$\begin{tikzcd}
\begin{array}{c}(Y_1 \times (Y_2 \times X)) \cup (Y_1 \times (X \times Y_3)) \\ \cup\, (X \times (Y_2 \times Y_3))\end{array} \arrow[r, tail] \arrow[d, "\mu(r_1,\mu(r_2,\mathrm{id}))\cup\mu(r_1,\mu(\mathrm{id},r_3))\cup\mu(\mathrm{id},\mu(r_2,r_3))"'] & Y_1 \times (Y_2 \times Y_3) \arrow[d] \\
X \arrow[r, tail] & Y_1 \wedge_\mu (Y_2 \wedge_\mu Y_3)
\end{tikzcd} \tag{2.28}$$

representing $Y_1 \wedge_\mu (Y_2 \wedge_\mu Y_3)$. Consequently, the associativity isomorphisms

$$\alpha_{Y_1,Y_2,Y_3} : Y_1 \times (Y_2 \times Y_3) \to (Y_1 \times Y_2) \times Y_3,$$
$$\alpha_{Y_1,Y_2,X} : Y_1 \times (Y_2 \times X) \to (Y_1 \times Y_2) \times X,$$

and so on, induce an isomorphism of diagram (2.28) with diagram (2.27) and an associativity isomorphism

$$\alpha_{Y_1,Y_2,Y_3} : Y_1 \wedge_\mu (Y_2 \wedge_\mu Y_3) \to (Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3. \tag{2.29}$$

In Laplaza's framework [1972], left distributivity of the product over the sum operation is encoded by a monomorphism

$$\delta_{Y_0,Y_1,Y_2} : Y_0 \wedge_\mu (Y_1 \vee_X Y_2) \to (Y_0 \wedge_\mu Y_1) \vee_X (Y_0 \wedge_\mu Y_2).$$

The fact that $\vee_X$ is a categorical sum enables us to construct an isomorphism $\delta^{-1}_{Y_0,Y_1,Y_2} : (Y_0 \wedge_\mu Y_1) \vee_X (Y_0 \wedge_\mu Y_2) \to Y_0 \wedge_\mu (Y_1 \vee_X Y_2)$ quite easily as follows. Applying the functor $Y_0 \wedge_\mu -$ to the sum diagram $Y_1 \to Y_1 \vee_X Y_2 \leftarrow Y_2$ provides a diagram $Y_0 \wedge_\mu Y_1 \to Y_0 \wedge_\mu (Y_1 \vee_X Y_2) \leftarrow Y_0 \wedge_\mu Y_2$. Since $\vee_X$ is a categorical sum, there results a map $(Y_0 \wedge_\mu Y_1) \vee_X (Y_0 \wedge_\mu Y_2) \to Y_0 \wedge_\mu (Y_1 \vee_X Y_2)$. To check that this map is an isomorphism observe that in a simplicial dimension p the p-simplices outside of X in the domain are $(Y_0 - X)_p \times (Y_1 - X)_p \amalg (Y_0 - X)_p \times (Y_2 - X)_p$, the p-simplices outside of X in the target are $(Y_0 - X)_p \times ((Y_1 - X)_p \amalg (Y_2 - X)_p)$, and the induced map is a one-to-one correspondence. Thus, we obtain the isomorphism $\delta^{-1}_{Y_0,Y_1,Y_2} : (Y_0 \wedge_\mu Y_1) \vee_X (Y_0 \wedge_\mu Y_2) \to Y_0 \wedge_\mu (Y_1 \vee_X Y_2)$, whose inverse

$$\delta_{Y_0,Y_1,Y_2} : Y_0 \wedge_\mu (Y_1 \vee_X Y_2) \cong (Y_0 \wedge_\mu Y_1) \vee_X (Y_0 \wedge_\mu Y_2) \tag{2.30}$$

can be shown to meet LaPlaza's conditions. Similarly, we obtain an isomorphism

$$\delta^{\#}_{Y_0,Y_1,Y_2} : (Y_0 \vee_X Y_1) \wedge_\mu Y_2 \cong (Y_0 \wedge_\mu Y_2) \vee_X (Y_1 \wedge_\mu Y_2). \tag{2.31}$$

This concludes the catalog of basic inputs for LaPlaza's theorem.

Given the basic inputs, the next step is to establish the commutativity of certain diagrams, twenty-four in number. Because $\vee_X$ is a categorical sum and $\wedge_\mu$ is biexact, preserving sums, checking the commutativity of seventeen of the diagrams is routine. The other seven diagrams involve the multiplicative or additive neutral objects or the multiplicative zero object and are straightforward to verify. LaPlaza's main theorem applies and "all diagrams that should commute do, in fact, commute". These remarks complete the proof of part one of Theorem 2.1. □

3. Defining the operations

The ingredients for the operations take values in categories of retractive spaces on which groups are acting. We first establish language and notation following [Gunnarsson and Schwänzl 2002, Definitions 5.1–5.4] for the following definitions.

Definition 3.1. A set $\mathcal{F}$ of subgroups of Σ_n is called a family of subgroups if it contains at most one member from each conjugacy class of subgroups.

Definition 3.2. For a finite group G, a G-simplicial set Y has orbit types in a family $\mathcal{F}$ relative to another G-simplicial set W if Y may be obtained from W by direct limit and by formation of pushouts of diagrams of the form

$$Y' \leftarrow \partial\Delta^n \times (G/H) \rightarrowtail \Delta^n \times (G/H), \tag{3.3}$$

where Δ^n is the standard simplicial n-simplex, $\partial\Delta^n$ is the simplicial boundary, and $H \in \mathcal{F}$.

Definition 3.4. For a Σ_n-simplicial set W, let $\mathcal{R}(W, \Sigma_n, \mathcal{F})$ denote the category whose objects are the triples (Y, r, s), where Y is a Σ_n-simplicial set with orbit types in $\mathcal{F}$ relative to a Σ_n-section $s : W \to Y$. The map $r : Y \to W$ is a Σ_n-retraction of Y to W, that is, $r \circ s = \mathrm{id}_W$. Morphisms are Σ_n-equivariant maps commuting with the retractions and sections.

Definition 3.5. Let $\mathcal{R}_f(W, \Sigma_n, \mathcal{F})$ denote the full subcategory of $\mathcal{R}(W, \Sigma_n, \mathcal{F})$ whose objects are the triples (Y, r, s) such that Y is built from W by formation of finitely many pushouts of the form of (3.3). The category $\mathcal{R}_f(W, \Sigma_n, \mathcal{F})$ is also equipped with cofibrations and weak equivalences. A cofibration $(W_1, r_1, s_1) \rightarrowtail (W_2, r_2, s_2)$ is an injective Σ_n-map and a weak equivalence $(Y_1, r_1, s_1) \to (Y_2, r_2, s_2)$ is a morphism for which the geometric realization of the underlying map $Y_1 \to Y_2$ is a Σ_n-equivariant homotopy equivalence.

For X a connected simplicial abelian group on which Σ_n acts trivially, we need the categories $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ of retractive left Σ_n-spaces $\widetilde{Y}$ over X which are finite relative to X. In principle, we may also allow X to be a connected commutative simplicial monoid with unit. We write $\Omega|wS_\bullet\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})| = A_{\Sigma_n,\{\text{all}\}}(X)$. The category of retractive left Σ_n-spaces on which Σ_n acts with trivial isotropy outside of X is then $\mathcal{R}_f(X, \Sigma_n, \{e\})$. In other words, the Σ_n-action on simplices outside of X is free on those simplices. Later, we abbreviate $\mathcal{R}_f(X, \Sigma_n, \{e\}) = \mathcal{R}_f(X, \Sigma_n)$. In Lemma 7.3 we justify the notation $\Omega|wS_\bullet\mathcal{R}_f(X, \Sigma_n, \{e\})| = A(X \times B\Sigma_n)$.

There are two constructions underlying our approach to the Segal operations. First is a family of biexact functors

$$\boxtimes_{k,\ell} : \mathcal{R}_f(X, \Sigma_k, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_{k+\ell}, \{\text{all}\})$$

defined for $k, \ell \geq 0$, called box-tensor operations (Definition 3.10). Second is a family of functors

$$\diamond_{n,k} : \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})^{[k]} \to \mathcal{R}_f(X, \Sigma_{kn}, \{\text{all}\}),$$

called diamond operations (Definition 3.16). Here $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})^{[k]}$ is the category of filtered objects

$$Y_1 \rightarrowtail Y_2 \rightarrowtail \cdots \rightarrowtail Y_k$$

with Y_i in $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ and natural transformations of such sequences.

First we set up the box-tensor operation. For a connected simplicial abelian group X, let $n = k + \ell$ and define an induction functor

$$\operatorname{Ind}_{\Sigma_k \times \Sigma_\ell}^{\Sigma_n} : \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}). \tag{3.6}$$

Let $\boldsymbol{n}$ be a finite set of cardinality n (for example, the standard example), let $\boldsymbol{k} \cup \boldsymbol{l}$ be the disjoint union of finite sets of cardinality k and l, respectively, and let $\operatorname{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})$ be the set of isomorphisms from $\boldsymbol{n}$ to the disjoint union. Let $\operatorname{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ = \operatorname{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l}) \cup \{*\}$ be viewed as an object of $\mathcal{R}_f(*)$, with the obvious section and with the retraction the constant map to $\{*\}$. The group Σ_n acts from the left on $\operatorname{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+$ by fixing the basepoint and by the rule $\sigma \cdot f = f \circ \sigma^{-1}$ for $\sigma \in \Sigma_n$ and $f : \boldsymbol{n} \to \boldsymbol{k} \cup \boldsymbol{l}$. Normally $\Sigma_k \times \Sigma_\ell$ also acts from the left by post-composition, but we find it convenient to use the right action defined by

$f \cdot (\sigma_1, \sigma_2) = (\sigma_1^{-1}, \sigma_2^{-1}) \circ f$. For $(Y, r, s) \in \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\})$ we unwind the defining pushout square

$$\begin{tikzcd} (\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \times X) \cup_{* \times X} (* \times Y) \arrow[r, tail] \arrow[d] & \mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \times Y \arrow[d] \\ * \times X \arrow[r, tail] & \mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e Y \end{tikzcd} \tag{3.7}$$

to find that the exterior smash product $\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e Y$ amounts to $\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})$-copies of Y, pasted together along the common subspace X. The retraction

$$r' : \mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e Y \to X$$

given by $r'([f, y]) = r(y)$ is Σ_n-equivariant when Σ_n acts trivially on X. We may also apply the principle of Proposition 2.16 to re-express the exterior smash product as an internal smash product and write

$$\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e Y \cong (X \vee \mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+) \wedge_\mu Y.$$

Define $\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e^{\Sigma_k \times \Sigma_\ell} Y$ to be the quotient space of $\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e Y$ by the equivalence relation generated by $[f \cdot (\sigma_1, \sigma_2), y] \sim [f, (\sigma_1, \sigma_2) \cdot y]$. The left action of Σ_n passes to the quotient, and, since the action of Σ_n on X is trivial, the retraction r' defined above also passes to the quotient, as does the section. Thus, we obtain the necessary structure maps

$$X \rightarrowtail \mathrm{Iso}(\boldsymbol{n}, \boldsymbol{k} \cup \boldsymbol{l})_+ \wedge_e^{\Sigma_k \times \Sigma_\ell} Y \xrightarrow{r} X.$$

This completes the definition of the induction functor

$$\mathrm{Ind}_{\Sigma_k \times \Sigma_\ell}^{\Sigma_n} : \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}). \tag{3.8}$$

Next we need an elementary pairing functor

$$\mathcal{R}_f(X, \Sigma_k, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\}). \tag{3.9}$$

The pairing sends $((Y_1, r_1, s_1), (Y_2, r_2, s_2))$ to $(Y_1, r_1, s_1) \wedge_\mu (Y_2, r_2, s_2)$.

Definition 3.10. Define the box-tensor operations by composing the pairing functor (3.9) with the induction functor (3.8):

$$\begin{aligned} \boxtimes_{k,\ell} : \mathcal{R}_f(X, \Sigma_k, \{\text{all}\}) &\times \mathcal{R}_f(X, \Sigma_\ell, \{\text{all}\}) \\ &\xrightarrow{\wedge_\mu} \mathcal{R}_f(X, \Sigma_k \times \Sigma_\ell, \{\text{all}\}) \xrightarrow{\mathrm{Ind}_{\Sigma_k \times \Sigma_\ell}^{\Sigma_n}} \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \end{aligned} \tag{3.11}$$

Proposition 3.12. *The box-tensor operations are associative up to natural isomorphism.*

Proof. The associativity of the box-tensor operations is a consequence of the symmetric monoidal structure on $\mathcal{R}_f(X)$ associated with $\wedge_\mu$, along with properties of the cartesian product of groups and disjoint union of sets. Abbreviating $\mathrm{id}_{\mathcal{R}_f(X,\Sigma_{k_1},\{\mathrm{all}\})}$ by id_1 and $\mathrm{id}_{\mathcal{R}_f(X,\Sigma_{k_3},\{\mathrm{all}\})}$ by id_3, the assertion in detail is that the diagram

$$\begin{tikzcd}[column sep=tiny] & \mathcal{R}_f(X,\Sigma_{k_1},\{\mathrm{all}\})\times\mathcal{R}_f(X,\Sigma_{k_2},\{\mathrm{all}\})\times\mathcal{R}_f(X,\Sigma_{k_3},\{\mathrm{all}\}) \arrow[dl, "\boxtimes_{k_1,k_2}\times\mathrm{id}_3"'] \arrow[dr, "\mathrm{id}_1\times\boxtimes_{k_2,k_3}"] & \\ \mathcal{R}_f(X,\Sigma_{k_1+k_2},\{\mathrm{all}\})\times\mathcal{R}_f(X,\Sigma_{k_3},\{\mathrm{all}\}) \arrow[dr, "\boxtimes_{k_1+k_2,k_3}"'] & & \mathcal{R}_f(X,\Sigma_{k_1},\{\mathrm{all}\})\times\mathcal{R}_f(X,\Sigma_{k_2+k_3},\{\mathrm{all}\}) \arrow[dl, "\boxtimes_{k_1,k_2+k_3}"] \\ & \mathcal{R}_f(X,\Sigma_{k_1+k_2+k_3},\{\mathrm{all}\}) & \end{tikzcd}$$

commutes up to canonical isomorphism. Given a triple (Y_1, Y_2, Y_3) in the category at the top of the diagram, the value of the left-hand sequence of arrows is

$$\mathrm{Iso}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3, \boldsymbol{k}_1+\boldsymbol{k}_2\cup\boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_1+k_2}\times\Sigma_{k_3}} \big(\big(\mathrm{Iso}(\boldsymbol{k}_1+\boldsymbol{k}_2, \boldsymbol{k}_1\cup\boldsymbol{k}_2)_+ \wedge_e^{\Sigma_{k_1}\times\Sigma_{k_2}} Y_1\wedge_\mu Y_2\big)\wedge_\mu Y_3\big),$$

and we claim this space is isomorphic to

$$\mathrm{Iso}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3, (\boldsymbol{k}_1\cup\boldsymbol{k}_2)\cup\boldsymbol{k}_3)_+ \wedge_e^{(\Sigma_1\times\Sigma_2)\times\Sigma_3} (Y_1\wedge_\mu Y_2)\wedge_\mu Y_3. \tag{3.13}$$

To clarify the notation, $\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3$ denotes the standard finite set of cardinality $k_1+k_2+k_3$, $\boldsymbol{k}_1+\boldsymbol{k}_2\cup\boldsymbol{k}_3$ denotes the disjoint union of finite sets of cardinality k_1+k_2 and k_3, and so on. Similarly, the value of the right-hand sequence of arrows is

$$\mathrm{Iso}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3, \boldsymbol{k}_1\cup\boldsymbol{k}_2+\boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_1}\times\Sigma_{k_2+k_3}} \big(Y_1\wedge_\mu\big(\mathrm{Iso}(\boldsymbol{k}_2+\boldsymbol{k}_3, \boldsymbol{k}_2\cup\boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_2}\times\Sigma_{k_3}} Y_2\wedge_\mu Y_3\big)\big),$$

which we claim is isomorphic to

$$\mathrm{Iso}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3, \boldsymbol{k}_1\cup(\boldsymbol{k}_2\cup\boldsymbol{k}_3))_+ \wedge_e^{\Sigma_1\times(\Sigma_2\times\Sigma_3)} Y_1\wedge_\mu(Y_2\wedge_\mu Y_3). \tag{3.14}$$

The spaces in (3.13) and (3.14) are isomorphic by the combination of the associativity isomorphisms for disjoint union, cartesian products of groups, and the smash product $\wedge_\mu$. Thus, we have proved that the box-tensor operations are naturally associative, granting the two isomorphisms.

To establish one of these isomorphisms requires several steps. We concentrate on the first case, since the second is completely parallel. First, since $\mathrm{Iso}(\boldsymbol{k}_3,\boldsymbol{k}_3)=\Sigma_{k_3}$, there is an isomorphism

$$\mathrm{Iso}(\boldsymbol{k}_3,\boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_3}} Y_3 \xrightarrow{\cong} Y_3 \tag{3.15}$$

in $\mathcal{R}_f(X, \Sigma_3)$ induced by the formula $[f_3, y] \mapsto f_3^{-1} y$. With the right action of Σ_{k_3} on $\mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)$ given by $f \cdot \sigma = \sigma^{-1} \circ f$, we have $[f_3 \cdot \sigma, y] \mapsto (\sigma^{-1} f_3)^{-1} y = f_3^{-1} \sigma y$; starting from $[f_3, \sigma y]$, we also arrive at $f_3^{-1} \sigma y$. Thus, a map

$$\mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_3}} Y_3 \to Y_3$$

exists. Surjectivity is clear. For injectivity, if $[f_3, y]$ and $[f_3', y']$ map to the same element of Y, we have $f_3^{-1} y = (f_3')^{-1} y'$. Putting $\sigma = f_3' f_3^{-1}$, we have $y' = \sigma y$ and $f_3' \cdot \sigma = \sigma^{-1} f_3' = f_3 (f_3')^{-1} f_3' = f_3$, so $[f_3, y] = [f_3' \cdot \sigma, y] \sim [f_3', \sigma y] = [f_3', y']$. To get equivariance, recall that the left action of Σ_{k_3} on $\mathrm{Iso}(k_3, k_3)$ is given by $\sigma \cdot f_3 = f \circ \sigma^{-1}$. Thus,

$$[\sigma * f, y] = [f \circ \sigma^{-1}, y] \mapsto (f \circ \sigma^{-1})^{-1} y = \sigma (f^{-1} y)$$

shows equivariance.

Consequently,

$$\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, \boldsymbol{k}_1 + \boldsymbol{k}_2 \cup \boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_1+k_2} \times \Sigma_{k_3}} \big((\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2, \boldsymbol{k}_1 \cup \boldsymbol{k}_2)_+ \wedge_e^{\Sigma_{k_1} \times \Sigma_{k_2}} Y_1 \wedge_\mu Y_2 \big) \wedge_\mu Y_3 \big)$$

is isomorphic to

$$\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, \boldsymbol{k}_1 + \boldsymbol{k}_2 \cup \boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_1+k_2} \times \Sigma_{k_3}} \big((\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2, \boldsymbol{k}_1 \cup \boldsymbol{k}_2)_+ \wedge_e^{\Sigma_{k_1} \times \Sigma_{k_2}} Y_1 \wedge_\mu Y_2 \big) \wedge_\mu \big(\mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_3}} Y_3 \big) \big).$$

Applying a commutativity isomorphism of the product $\wedge_e$, this is isomorphic to

$$\big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, \boldsymbol{k}_1 + \boldsymbol{k}_2 \cup \boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_1+k_2} \times \Sigma_{k_3}} \big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2, \boldsymbol{k}_1 \cup \boldsymbol{k}_2)_+ \wedge_e \mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)_+ \big) \big) \wedge_e^{(\Sigma_{k_1} \times \Sigma_{k_2}) \times \Sigma_{k_3}} (Y_1 \wedge_\mu Y_2) \wedge_\mu Y_3.$$

Now we claim there is an isomorphism

$$\big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, \boldsymbol{k}_1 + \boldsymbol{k}_2 \cup \boldsymbol{k}_3)_+ \wedge_e^{\Sigma_{k_1+k_2} \times \Sigma_{k_3}} \big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2, \boldsymbol{k}_1 \cup \boldsymbol{k}_2)_+ \wedge_e \mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)_+ \big) \big) \cong \mathrm{Iso}\big(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, (\boldsymbol{k}_1 \cup \boldsymbol{k}_2) \cup \boldsymbol{k}_3 \big)_+$$

induced by the formula $[f_{123}, [f_{12}, f_3]] \mapsto (f_{12}, f_3) \circ f_{123}$. We check that balanced expressions in

$$\big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3, \boldsymbol{k}_1 + \boldsymbol{k}_2 \cup \boldsymbol{k}_3)_+ \wedge_e \big(\mathrm{Iso}(\boldsymbol{k}_1 + \boldsymbol{k}_2, \boldsymbol{k}_1 \cup \boldsymbol{k}_2)_+ \wedge_e \mathrm{Iso}(\boldsymbol{k}_3, \boldsymbol{k}_3)_+ \big) \big)$$

map to the same element of the target:

$$[f_{123} \cdot (g_{12}, g_3), [f_{12}, f_3]] = [(g_{12}^{-1}, g_3^{-1}) \circ f_{123}, [f_{12}, f_3]] \mapsto (f_{12}, f_3) \circ \big((g_{12}^{-1}, g_3^{-1}) \circ f_{123} \big).$$

On the other hand,

$$[f_{123}, (g_{12}, g_3)\cdot[f_{12}, f_3]] = [f_{123}, [f_{12}\circ g_{12}^{-1}, f_3\circ g_3^{-1}]] \mapsto (f_{12}\circ g_{12}^{-1}, f_3\circ g_3^{-1})\circ f_{123}$$

and these expressions are the same, by associativity of composition. Now suppose $[f_{123}, [f_{12}, f_3]]$ and $[f'_{123}, [f'_{12}, f'_3]]$ map to the same isomorphism. The equation $(f_{12}, f_3)\circ f_{123} = (f'_{12}, f'_3)\circ f'_{123}$ is equivalent to $(f'_{12}, f'_3)^{-1}\circ(f_{12}, f_3) = f'_{123}\circ f_{123}^{-1}$. Putting $(\sigma_{12}, \sigma_3) = (f'_{12}, f'_3)^{-1}\circ(f_{12}, f_3) = f'_{123}\circ f_{123}^{-1}$, we have

$$f'_{123}\cdot(\sigma_{12}, \sigma_3) = (f'_{123}\circ f_{123}^{-1})^{-1}\circ f'_{123} = f_{123},$$

and

$$(\sigma_{12}, \sigma_3)\cdot(f_{12}, f_3) = (f_{12}, f_3)\circ\big((f'_{12}, f'_3)^{-1}\circ(f_{12}, f_3)\big)^{-1} = (f'_{12}, f'_3),$$

so that

$$\begin{aligned}[f_{123}, [f_{12}, f_3]] &= [f'_{123}\cdot(\sigma_{12}, \sigma_3), [f_{12}, f_3]]\\ &\sim [f'_{123}, (\sigma_{12}, \sigma_3)\cdot[f_{12}, f_3]] = [f'_{123}, [f'_{12}, f'_3]]. \quad\square\end{aligned}$$

The diamond operation $\diamond_{k,1} = \diamond_k$ requires some preliminary definitions. First recall the category of filtered objects $F_k\mathcal{R}_f(X)$ from [Waldhausen 1985, Section 1.1]; this is a category with cofibrations and weak equivalences. Let

$$\underline{P} = (P_1 \rightarrowtail P_2 \rightarrowtail \cdots \rightarrowtail P_k)$$

be an object of $F_k\mathcal{R}_f(X)$. For functions $f, g : \boldsymbol{k} \to \boldsymbol{k}$ we say $f \le g$ if $f(i) \le g(i)$ for all $i \in \boldsymbol{k}$. Let $I(\boldsymbol{k}) = \{f : \boldsymbol{k} \to \boldsymbol{k} \mid \text{there is } \sigma \in \Sigma_k \text{ such that } f \le \sigma\}$.

The set $I(\boldsymbol{k})$ is partially ordered by $\le$, and the sequence $\underline{P}$ defines a functor $\underline{\underline{P}} : I(\boldsymbol{k}) \to \mathcal{R}_f(X)$ by the rule $\underline{\underline{P}}(f) = P_{f(1)} \wedge_\mu P_{f(2)} \wedge_\mu \cdots \wedge_\mu P_{f(k)}$ on objects. We apply the convention that parentheses in iterated products are collected to the left. In particular, $P_{f(1)} \wedge_\mu P_{f(2)} \wedge_\mu P_{f(3)} = (P_{f(1)} \wedge_\mu P_{f(2)}) \wedge_\mu P_{f(3)}$, and, in general,

$$P_{f(1)} \wedge_\mu P_{f(2)} \wedge_\mu \cdots \wedge_\mu P_{f(k)} = (\cdots(P_{f(1)} \wedge_\mu P_{f(2)}) \wedge_\mu \cdots \wedge_\mu P_{f(k)}).$$

For arrows we observe that $f \le g$ implies there are cofibrations $P_{f(i)} \rightarrowtail P_{g(i)}$ which induce a cofibration $\underline{\underline{P}}(f) \rightarrowtail \underline{\underline{P}}(g)$. This depends on the exactness of $\wedge_\mu$, proved in Theorem 2.1.

Definition 3.16. Define the functor $\diamond_k : F_k\mathcal{R}_f(X) \to \mathcal{R}_f(X, \Sigma_k, \{\text{all}\})$ on objects by making a choice of $\operatorname{colim}_{I(\boldsymbol{k})} \underline{\underline{P}}$ and setting

$$\diamond_k(\underline{P}) = \operatorname*{colim}_{I(\boldsymbol{k})} \underline{\underline{P}}.$$

The definition extends to arrows by the universal property of the colimit. The Σ_k action is induced by the permutation of factors.

Example 3.17. Applied to a constant cofibration sequence $\underline{Y} = (Y \overset{=}{\rightarrowtail} Y \overset{=}{\rightarrowtail} \cdots \overset{=}{\rightarrowtail} Y)$ of length k, we obtain simply

$$\diamond_k(\underline{Y}) = Y \wedge_\mu Y \wedge_\mu \cdots \wedge_\mu Y$$

with the group Σ_k permuting the factors. Thus, the purpose of $\diamond_k$ is to extend $\wedge_\mu$-powers to filtered objects.

Definition 3.18. The generalized diamond operation

$$\diamond_{n,k} : F_k\mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_{nk}, \{\text{all}\})$$

is a composition

$$\diamond_{n,k} : F_k\mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \xrightarrow{\diamond_k} \mathcal{R}_f(X, B_{n,k}, \{\text{all}\}) \xrightarrow{\operatorname{Ind}_{B_{n,k}}^{\Sigma_{nk}}} \mathcal{R}_f(X, \Sigma_{nk}, \{\text{all}\}),$$

with a basic diamond operation $\diamond_k$ followed by an induction construction $\operatorname{Ind}_{B_{n,k}}^{\Sigma_{nk}}$. The intermediate group $B_{n,k}$ is the group of block permutations of nk objects blocked into k groups of n objects. Thus, the group $B_{n,k}$ is a wreath product: $B_{n,k} = \Sigma_k \wr \Sigma_n$. Explicitly, there is a short exact sequence of groups

$$1 \to (\Sigma_n)^k \to B_{n,k} \to \Sigma_k \to 1.$$

Recall $G_\bullet$ briefly here, following [Gunnarsson et al. 1992]. For a simplicial set Z the corresponding simplicial path set PZ is defined by $PZ_n = Z_{n+1}$. The face operator $d_i : PZ_n \to PZ_{n-1}$ coincides with $d_{i+1} : Z_{n+1} \to Z_n$; the degeneracy operator $s_i : PZ_n \to PZ_{n+1}$ coincides with $s_{i+1} : Z_{n+1} \to Z_{n+2}$. The face operator $d_0 : Z_{n+1} \to Z_n$ induces a simplicial map $d_0 : PZ \to Z$. The simplicial set PZ is simplicially homotopy equivalent to the constant simplicial set Z_0 [Waldhausen 1985, Lemma 1.5.1, p. 328]. Viewing $Z_1 = PZ_0$ as another constant simplicial set provides a simplicial map $Z_1 \to PZ$.

Definition 3.19 [Gunnarsson et al. 1992, p. 257]. For a category $\mathcal{C}$ with cofibrations and weak equivalences the simplicial category $G_\bullet\mathcal{C}$ is defined by the cartesian square

$$\begin{tikzcd} wG_\bullet\mathcal{C} \arrow[r] \arrow[d] & PwS_\bullet\mathcal{C} \arrow[d, "d_0"'] \\ PwS_\bullet\mathcal{C} \arrow[r, "d_0"] & wS_\bullet\mathcal{C} \end{tikzcd} \tag{3.20}$$

Recalling a few more details from [Gunnarsson et al. 1992], $G_\bullet\mathcal{C}$ has cofibrations and weak equivalences. As $G_n\mathcal{C} = (PS_\bullet\mathcal{C})_n \times_{S_n\mathcal{C}} (PS_\bullet\mathcal{C})_n = S_{n+1}\mathcal{C} \times_{S_n\mathcal{C}} S_{n+1}\mathcal{C}$, the weak equivalences and cofibrations in $wG_\bullet\mathcal{C}$ are given by pullback. There is also a stabilization map $\eta : \mathcal{C} \to G_\bullet\mathcal{C}$, where $\mathcal{C}$ is viewed as a constant simplicial category with cofibrations and weak equivalences, defined as follows. We have the map $\mathcal{C} = (PwS_\bullet\mathcal{C})_0 \to PwS_\bullet\mathcal{C}$ and the constant map $\mathcal{C} \to PwS_\bullet\mathcal{C}$ carrying $\mathcal{C}$ to the terminal object. These two combine to give an inclusion $\eta : \mathcal{C} \to G_\bullet \mathcal{C}$. After passing to diagonals, the construction may be iterated so there results a stabilization sequence

$$\mathcal{C} \to G_\bullet \mathcal{C} \to G_\bullet(G_\bullet \mathcal{C}) \to \cdots \to G^n_\bullet \mathcal{C} := G(G^{n-1}_\bullet \mathcal{C}) \to \cdots \to \operatorname*{colim}_n G^n_\bullet \mathcal{C} := G^\infty_\bullet \mathcal{C}$$

of simplicial categories with cofibrations and weak equivalences. Returning to the square (3.20), after passage to nerves in the w-direction, diagonalization, and geometric realization, there results a natural map

$$|wG_\bullet \mathcal{C}| \to \Omega |wS_\bullet \mathcal{C}|.$$

This may not always be a homotopy equivalence, but it is a homotopy equivalence when $\mathcal{C}$ has a property called pseudo-additivity [Gunnarsson et al. 1992, Definition 2.3 and Theorem 2.6]. In our case, with $\mathcal{C} = \mathcal{R}_f(X)$ we follow [Gunnarsson et al. 1992] to achieve the pseudo-additivity property by using the cylinder functor defined in [Waldhausen 1985, Section 1.6]. The cylinder functor induces a cone functor $c : \mathcal{R}_f(X) \to \mathcal{R}_f(X)$ and a suspension functor $\Sigma : \mathcal{R}_f(X) \to \mathcal{R}_f(X)$ so that we may define a category of prespectra

$$\Sigma^\infty \mathcal{R}_f(X) = \operatorname{colim}\big(\mathcal{R}_f(X) \xrightarrow{\Sigma} \mathcal{R}_f(X) \xrightarrow{\Sigma} \mathcal{R}_f(X) \xrightarrow{\Sigma} \cdots\big).$$

Then $\Sigma^\infty \mathcal{R}_f(X)$ has the pseudo-additivity property [Gunnarsson et al. 1992, Remark 2.4 and Lemma 2.5, p. 258–259], so

$$|wG_\bullet \Sigma^\infty \mathcal{R}_f(X)| \to \Omega |wS_\bullet \Sigma^\infty \mathcal{R}_f(X)|$$

is a weak homotopy equivalence. Also, by [Waldhausen 1985, Proposition 1.6.2], $|wS_\bullet \mathcal{R}_f(X)| \to |wS_\bullet \Sigma^\infty \mathcal{R}_f(X)|$ is a weak homotopy equivalence.

Additionally, we need the fact that there are weak homotopy equivalences

$$|wG^\infty_\bullet \mathcal{C}| \to \Omega |wG^\infty_\bullet S_\bullet \mathcal{C}| \leftarrow \Omega |wS_\bullet \mathcal{C}| \tag{3.21}$$

for any category $\mathcal{C}$ with cofibrations and weak equivalences [Gunnarsson et al. 1992, Theorem 4.1, p. 268].

The $G_\bullet \mathcal{C}$ construction has an explicit description as a category of exact functors. For full details refer to [Gunnarsson and Schwänzl 2002; Grayson 1989]. First, extend the partially ordered set $A \in \Delta$ to the set $\gamma(A) = \{L, R\} \amalg A$ with the ordering in which L and R are not comparable, $L < a$ and $R < a$ for every $a \in A$, and, for $a, a' \in A$, $a < a'$ in $\gamma(A)$ if and only if $a < a'$ in A. Pictorially, for $A = [n]$, $\gamma(A)$ looks like

$$\begin{tikzcd}
L \arrow[dr, tail] & & & & \\
 & 0 \arrow[r] & 1 \arrow[r] & \cdots \arrow[r] & n \\
R \arrow[ur, tail] & & & &
\end{tikzcd}$$

The category $\Gamma(A)$ is the category of arrows in $\gamma(A)$, omitting the identity arrows on L and R. Diagrammatically, $\Gamma(A)$ looks like

$$
\begin{tikzcd}
 & 0/L \arrow[rr, tail] \arrow[dd] & & 1/L \arrow[rr, tail] \arrow[dd] & & \cdots \arrow[rr, tail] & & n/L \arrow[dd] \\
0/R \arrow[rr, tail] \arrow[dr] & & 1/R \arrow[rr, tail] \arrow[dr] & & \cdots \arrow[rr, tail] & & n/R \arrow[dr] & \\
 & 0/0 \arrow[rr, tail] & & 1/0 \arrow[rr, tail] \arrow[d] & & \cdots \arrow[rr, tail] & & n/0 \arrow[d] \\
 & & & 1/1 \arrow[rr, tail] & & \cdots \arrow[rr, tail] & & n/1 \arrow[d] \\
 & & & & & & \cdots & \vdots \arrow[d] \\
 & & & & & & & n/n
\end{tikzcd}
$$

Here a/b stands for $b \to a$ (or $b < a$), and an arrow $a/b \to c/d$ stands for a square

$$
\begin{tikzcd}
a \arrow[r] & c \\
b \arrow[r] \arrow[u] & d \arrow[u]
\end{tikzcd}
$$

in $\gamma(A)$. The exact sequences in $\Gamma(A)$ are sequences $j/k \to i/k \to i/j$ where $k \to j \to i$ in $\gamma(A)$. Then, for $A \in \Delta$,

$$G_A\mathcal{C} = \mathrm{Exact}(\Gamma(A), \mathcal{C}).$$

Since $\Gamma(A)$ is functorial in A, preserving exact sequences $j/k \to i/k \to i/j$, we have another description of $G_\bullet\mathcal{C} : \Delta^{\mathrm{op}} \to \mathrm{Cat}$. In this interpretation the stabilization $\eta : \mathcal{C} \to G_\bullet\mathcal{C}$ sends an object C of $\mathcal{C}$ to the functor $\eta(C) : \Gamma(A) \to \mathcal{C}$ whose value at i/L is C for all $i \in A$ and whose value at any other object of $\Gamma(A)$ is the zero object of $\mathcal{C}$. Given an arrow $i/L \to i'/L$ in $\Gamma(A)$, $\eta(C)$ assigns to it the identity on C; other arrows are assigned by the universal property of the zero object.

Definition 3.22 (cf. [Gunnarsson and Schwänzl 2002, Section 2.1, p. 268; Grayson 1989, Section 4]). Let Z be a simplicial object in a category $\mathcal{D}$. Define a concatenation operation $\mathrm{con} : \Delta^k \to \Delta$. For a sequence $(A_1, \ldots, A_k)$ of finite nonempty ordered sets, order their disjoint union $A_1 \amalg \cdots \amalg A_k$ so that the subset A_i inherits the original order and so that, if $i \leq j$ and $a_i \in A_i$ and $a_j \in A_j$, then $a_i < a_j$. Then define the k-fold edgewise subdivision of a simplicial object Z to be the composite functor

$$\mathrm{sub}_k\, Z : \Delta^k \xrightarrow{\mathrm{con}} \Delta \xrightarrow{Z} \mathcal{D}.$$

For a simplicial set Z there is a natural homeomorphism $|\mathrm{sub}_k\, Z| \to |Z|$.

Several more constructions are necessary before we can define for every integer $k \geq 1$ operations

$$\omega^k : w\,\mathrm{sub}_k\, G_\bullet \mathcal{R}_f(X) \to wG^k_\bullet \mathcal{R}_f(X, \Sigma_k, \{\mathrm{all}\}).$$

The original framework has proved to be quite robust, so we refer to [Grayson 1989, Sections 5–7; Gunnarsson and Schwänzl 2002, Section 2] for complete details and summarize what we use.

Theorem 3.23 [Grayson 1989, Sections 5–7, pp. 253–257]. *For $A \in \Delta$, let $\Gamma^1(A)$ be the category $\Gamma(A)$ discussed before Definition 3.22.*

1. *For each $A \in \Delta$ and for each integer $k \geq 1$ there is a category with exact sequences $\Gamma^k(A)$. The category $\Gamma^k(A)$ is natural in the variable A.*
2. *For $A_1, \dots, A_k \in \Delta$, let $A_1 \dots A_k$ be the concatenation. There is a functor*

$$\Xi_k : \Gamma(A_1) \times \cdots \times \Gamma(A_k) \to \Gamma^k(A_1 \dots A_k)$$

which is multi-exact, i.e., exact in each variable separately, and is natural in each of the variables. □

Grayson [1989, pp. 255–256] enumerates compatibility conditions (E1)–(E5) abstracted from properties of higher exterior powers and tensor products when applied to filtered modules. Given that the box-tensor operations $\boxtimes$ and diamond operations $\diamond_{n,k}$ fulfill (E1)–(E5) the robustness of the framework enables us to make the following observation.

Definition 3.24. For $A \in \Delta$, the collection of operations $\diamond_{n,k}$ and $\boxtimes$ define functors

$$\Lambda^k_{\diamond,\boxtimes} : \mathrm{Exact}\big(\Gamma(A), \mathcal{R}_f(X, \Sigma_n, \{\mathrm{all}\})\big) \to \mathrm{Exact}\big(\Gamma^k(A), \mathcal{R}_f(X, \Sigma_{n,k}, \{\mathrm{all}\})\big).$$

These functors are natural in A.

Remark 3.25. Since we don't need the explicit formula for $\Lambda^k_{\diamond,\boxtimes}$ except in a few specific cases, we refer the reader to the discussion in [Grayson 1989, p. 256–257] for all the details. For guidance, we point out that the categories $\Gamma^k(A)$ mentioned in Theorem 3.23 are constructed precisely to deliver the definition of $\Lambda^k_{\diamond,\boxtimes}$ on an object. Properties (E1) through (E4) ensure that the formulas on arrows yield a well-defined functor. Property (E5) of Grayson's list ensures that the functors $\Lambda^k_{\diamond,\boxtimes}$ carry an exact functor M to another exact functor.

In our situation we need the following property of a category with cofibrations.

Definition 3.26 (cf. [Gunnarsson et al. 1992, Definition 4.3, p. 274]). A category $\mathcal{C}$ with cofibrations has the extension property if for all commutative diagrams of cofibration sequences

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, tail] \arrow[d, "i"'] & C \arrow[d, tail] \\ A' \arrow[r, tail] & B' \arrow[r, tail] & C' \end{tikzcd}$$

in $\mathcal{C}$, with vertical cofibrations as indicated, it follows that the middle arrow i is also a cofibration.

Lemma 3.27. *Let $\mathcal{C}$ be a category with cofibrations, $A_1, \ldots, A_k \in \Delta$, and let $A_1 \ldots A_k$ be the concatenation.*

(1) *If $\mathcal{C}$ has the extension property, then the natural inclusion*

$$G^k_{A_1 \ldots A_k}\mathcal{C} \to \mathrm{Exact}(\Gamma(A_1) \times \cdots \times \Gamma(A_k), \mathcal{C})$$

is an isomorphism.

(2) *The categories $\mathcal{R}_f(X, \Sigma_n, \mathcal{F})$ with cofibrations have the extension property.*

Proof. The first statement is [Gunnarsson and Schwänzl 2002, Remark 4.4, p. 274]. For (2), because we are working inside $\mathcal{R}(X)$ with simplicial sets, cofibrations are the injective maps. Therefore, the extension property holds for $\mathcal{R}_f(X, \Sigma_n, \mathcal{F})$. □

Proposition 3.28 [Gunnarsson and Schwänzl 2002, Proposition 4.5]. *The box-tensor operations and the diamond operations fulfill properties* (E1)–(E5).

Proof. Properties (E1)–(E4) are consequences of the symmetric bimonoidal structure of Theorem 2.1. Properties (E3) and (E4) also depend on the associativity of $\boxtimes$ established in Proposition 3.12. Property (E5) depends on the extension property of Definition 3.26 and takes some additional work manipulating cocartesian diagrams, cofibration sequences, and colimits. The necessary steps are laid out in [Gunnarsson and Schwänzl 2002, Lemmas 4.6–4.10, Corollary 4.11]. Because all those manipulations rely just on the coherence of the symmetric bimonoidal category structure, all steps work in the present, more general, situation. □

The subdivision construction (concatenation), the functors $\Lambda^k_{\diamond,\boxtimes}$, and the functors Ξ_k come into play in the following definition.

Definition 3.29. For $k \geq 1$, the components ω^k for the total Segal operation are defined as follows:

$$\begin{tikzcd} \mathrm{Exact}\big(\Gamma(A_1 \ldots A_k), \mathcal{R}_f(X, \Sigma_n, \{\mathrm{all}\})\big) \arrow[r, "\mathrm{sub}_k \Lambda^k"] \arrow[d, "\omega^k"'] & \mathrm{Exact}\big(\Gamma^k(A_1 \ldots A_k), \mathcal{R}_f(X, \Sigma_n, \{\mathrm{all}\})\big) \arrow[dl, "\Xi_k"] \\ \mathrm{Exact}\big(\Gamma(A_1) \times \cdots \times \Gamma(A_k), \mathcal{R}_f(X, \Sigma_n, \{\mathrm{all}\})\big) & \end{tikzcd} \tag{3.30}$$

By Lemma 3.27(1) we may interpret $\mathrm{Exact}\big(\Gamma(A_1)\times\cdots\times\Gamma(A_k), \mathcal{R}_f(X, \Sigma_n, \{\mathrm{all}\})\big)$ as $G^k_{A_1\ldots A_k}\mathcal{R}_f(X, \Sigma_n, \{\mathrm{all}\})$. The result is a family of functors

$$\omega^k : w\,\mathrm{sub}_k\, G_\bullet\mathcal{R}_f(X) \to wG^k_\bullet\mathcal{R}_f(X, \Sigma_k, \{\mathrm{all}\})$$

for $k \geq 1$.

Referring to the discussion preceding Definition 3.22, the stabilization map $\eta : \mathcal{R}_f(X) \to G_0\mathcal{R}_f(X)$ has been concisely written in [Gunnarsson and Schwänzl 2002] as

$$(Y, r, s) \mapsto \eta((Y, r, s)) = \frac{Y}{\overline{\overline{X}}}.$$

The extension to higher simplicial dimensions admits the description $(s_0)^k(\eta(Y))$, where $s_0^k : G_0\mathcal{R}_f(X) \to G_k\mathcal{R}_f(X)$ is the iterated degeneracy. This can be denoted

$$\frac{Y = Y = \cdots = Y}{\overline{\overline{X = X = \cdots = X}}} \tag{3.31}$$

where the top row indicates constant filtered object and the bottom row indicates the constant filtration of the zero object. Since $\mathrm{sub}_k\, G_\bullet\mathcal{C}$ in simplicial dimension 0 can be identified with $G_k\mathcal{C}$, diagram (3.31) also represents

$$\eta : \mathcal{R}_f(X) \to \mathrm{sub}_k\, G_0\mathcal{R}_f(X)$$

for each $k \geq 1$. The next example incorporates Example 3.17 and is fundamental.

Example 3.32. The formula for the composite

$$\tilde{\alpha}^k_1 : \mathcal{R}_f(X) \xrightarrow{\eta} \mathrm{sub}_k\, G_\bullet\mathcal{R}_f(X) \xrightarrow{\omega^k} G^k_\bullet\mathcal{R}_f(X, \Sigma_k, \{\mathrm{all}\}) \tag{3.33}$$

is the functor $\Gamma([0])^k \to \mathcal{R}_f(X, \Sigma_k, \{\mathrm{all}\})$ given by

$$\begin{cases} Y \wedge_\mu Y \wedge_\mu \cdots \wedge_\mu Y & \text{in positions } 0/L,\ 0^{(2)}/L^{(2)}, \ldots, 0^{(k)}/L^{(k)}, \\ X & \text{in all other positions.} \end{cases}$$

4. E_∞-structure and restriction to spherical objects

We have already seen that, in order to obtain the algebraic K-theory of spaces using the $G_\bullet$-model, one uses a category of prespectra $\Sigma^\infty\mathcal{R}_f(X)$ obtained from $\mathcal{R}_f(X)$ by passage to a limit using a suspension operation. We are now going to deal with natural transformations of semigroup valued functors

$$[-, \mathcal{R}_f(X)] \to \Big[-, \{1\} \times \prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X)\Big],$$

where the target is an abelian-group-valued functor. First we restrict to categories of n-spherical objects $\mathcal{R}^n_f(X)$, whose definition is recalled below. Segal's group completion theorem [1974a, Proposition 4.1] provides a unique natural transformation of abelian-group-valued functors $[-, \Omega|hN_\Gamma\mathcal{R}^n_f(X)|] \to \big[-, \{1\}\times\prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X)\big]$.

In the domain, $hN_\bullet\mathcal{R}^n_f(X)$ is the simplicial category arising from the categorical sum operation $\vee$, as described in [Waldhausen 1985, Section 1.8], and maps are weak homotopy equivalences. The following diagram displays this result as the diagonal arrow:

$$\begin{tikzcd}
{[-,\Omega|hN_\Gamma\mathcal{R}^n_f(X)|]} \arrow[d, dashed] & {[-,|h\mathcal{R}^n_f(X)|]} \arrow[l] \arrow[r] \arrow[dr] & {[-,|h\mathcal{R}_f(X)|]} \arrow[d, "\omega"'] \\
{[-,A(X)]} \arrow[rr, dashed] & & \Big[-,\{1\}\times\prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X)\Big]
\end{tikzcd} \tag{4.1}$$

In this section we show that the diagonal arrow is induced by an E_∞-map

$$\Omega|hN_\Gamma\mathcal{R}^n_f(X)| \to \{1\}\times\prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X).$$

But we want a natural transformation of abelian-group-valued functors $[-, A(X)]\to \big[-, 1\times\prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X)\big]$ as displayed by the lower horizontal arrow in the diagram, and we want it to be induced by an E_∞-map $A(X)\to\{1\}\times\prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X)$. There is a natural chain of equivalences

$$\lim_{n\to\infty} hN_\bullet\mathcal{R}^n(X)\simeq hS_\bullet\mathcal{R}_f(X)\simeq hS_\bullet\Sigma^\infty\mathcal{R}_f(X),$$

where the colimit is taken over suspension relative to X [Waldhausen 1985, Theorems 1.7.1 and 1.8.1]. This implies we have to examine the behavior of our constructions as they relate to suspension, which we analyze in Section 5.

We recall from [Waldhausen 1985, Section 1.7, p. 360] a definition of spherical objects in the category $\mathcal{R}_f(X)$, where X is a connected space. On this category we have the homology theory $H_*(Y,r,s)=H_*(Y,s(X);r^*(\mathbb{Z}[\pi_1 X]))$ (homology with local coefficients), and we say (Y,r,s) is n-spherical if $H_q(Y,r,s)=0$ for $q\neq n$ and $H_n(Y,r,s)$ is a stably free $\mathbb{Z}[\pi_1 X]$-module. For $n\geq 0$ denote by $\mathcal{R}^n_f(X)$ the full subcategory of $\mathcal{R}_f(X)$ whose objects are n-spherical. For example, in case X is a connected simplicial abelian group, $\mathcal{R}^n_f(X)$ contains spaces homotopy equivalent to retractive spaces (Y,r,s) obtained by completing to pushouts diagrams of the form

$$X\xleftarrow{\vee\phi_i}\bigvee_{i=1}^N\partial\Delta^n\rightarrowtail\Delta^n,$$

where the attaching maps ϕ_i are constant maps to the identity element of X.

Let $\mathbb{N}$ be the natural numbers $\{0,1,\ldots\}$, and F the category of finite subsets of $\mathbb{N}$ and injections. Let $F_+\subset F$ be the full subcategory of nonempty finite subsets. Let $\amalg$ denote the associative sum on F_+ given by

$$\{x_i\mid 1\leq i\leq m\}\amalg\{y_j\mid 1\leq j\leq n\}=\{x_i\mid 1\leq i\leq m\}\cup\{y_j+x_m-y_1+1\mid 1\leq j\leq n\},$$

where we assume $x_1<\cdots<x_m$ and $y_1<\cdots<y_n$.

Lemma 4.2 [Gunnarsson and Schwänzl 2002, Lemma 10.2, p. 289]. *The category F_+ is contractible.*

Proof. The functor $t : F_+ \to F_+$ defined by $t(x) = \{0\} \amalg x$ receives natural transformations from the identity functor on F_+ and from the constant functor with value $\{0\}$. Geometric realization of the nerve converts the natural transformations to homotopies, so the identity map on the realization of the nerve of F_+ is homotopic to a constant map. $\square$

Under the assumption that the category $\mathcal{C}$ satisfies the extension property for cofibrations, which has been verified for $\mathcal{R}_f(X)$ and $\mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ in Lemma 3.27(2), one may identify the iterated $G_\bullet$-construction $G^n_\bullet \mathcal{C}$ with $\operatorname{Exact}(\Gamma(-)^n, \mathcal{C})$ according to Lemma 3.27(1). Using the adjointness relation, or diagonals, we have

$$\begin{aligned} G^3_A(\mathcal{C}) := \operatorname{Exact}(\Gamma(A)^3, \mathcal{C}) &= \operatorname{Exact}(\Gamma(A) \times \Gamma(A), \operatorname{Exact}(\Gamma(A), \mathcal{C})) \\ &= \cdots = G_A(G_A(G_A\mathcal{C})), \end{aligned}$$

for example. Now extend $n \mapsto \operatorname{Exact}(\Gamma(-)^n, \mathcal{C}) = G^n_\bullet \mathcal{C}$ to $G^{(-)}_\bullet : F \to \mathrm{Cat}^{\Delta^{\mathrm{op}}}$ following the recipe in [Gunnarsson and Schwänzl 2002]. Thus, on objects $x \in \operatorname{Ob}(F_+)$ and $A \in \Delta$, put $G^x_A \mathcal{C} := \operatorname{Exact}(\Gamma(A)^x, \mathcal{C})$. To obtain the extension to F, identify $\Gamma(A)^\varnothing$ with the one-point category, so that $G^\varnothing_\bullet \mathcal{C} := \operatorname{Exact}(\Gamma(A)^\varnothing, \mathcal{C}) = \mathcal{C}$.

For the behavior on morphisms we distinguish cases. An isomorphism $x \to x'$ in F induces a natural morphism $G^x_\bullet \mathcal{C} \to G^{x'}_\bullet \mathcal{C}$ by permuting coordinates. An injection $i : x \to y$ induces $G^x_\bullet \mathcal{C} \to G^y_\bullet \mathcal{C}$ using stabilization

$$\begin{tikzcd} \Gamma(A)^{i(x)} \arrow[r, "\cong"] \arrow[d, "\equiv"] & \Gamma(A)^{i(x)} \times \{L/0\}^{y \backslash i(x)} \arrow[r] \arrow[d, dashed] & \Gamma(A)^{i(x)} \times \Gamma(0)^{y \backslash i(x)} \arrow[dl, dashed, "X'"'] \\ \Gamma(A)^x \arrow[r, "X"'] & \mathcal{C} & \Gamma(A)^x \times \Gamma(A)^{y \backslash i(x)} \equiv \Gamma(A)^y \arrow[l, dashed, "i_*X"] \arrow[u] \end{tikzcd}$$

where we recall $\Gamma(0) = \{L/0, R/0\}$ is the two point discrete category, and we define X' to be zero outside $\Gamma(A)^{i(x)} \times \Gamma(0)^{y \backslash i(x)}$. This is the η-stabilization given by inclusion of $\mathcal{C}$ on the L-line in $G_0\mathcal{C}$, as described before Definition 3.22.

Let $F \int G_A \mathcal{C}$ be Thomason's homotopy colimit construction, which is the category consisting of objects $(x, X : \Gamma(A)^x \to \mathcal{C})$ and morphisms $(x, X) \to (y, Y)$ given by $i : x \to y$ in F and a natural transformation $i_*X \to Y$ in $G^y_A \mathcal{C}$ [Thomason 1979, Definition 1.2.2]. The unique morphisms $\varnothing \to x$ in F provide functors $\mathcal{C} \to \operatorname{Exact}(\Gamma(A)^x, \mathcal{C})$, eventually functors $\mathcal{C} \to F \int G_A \mathcal{C}$ natural in A, and finally a functor $\mathcal{C} \to F \int G_\bullet \mathcal{C}$. With the next result, we have made a step toward the right-hand column of diagram (4.1).

Theorem 4.3 (cf. [Gunnarsson and Schwänzl 2002, Theorem 10.3]). *The construction $F \int wG_\bullet \mathcal{C}$ gives a model for K-theory.*

Comments on the proof. The proof given in [Gunnarsson and Schwänzl 2002] can be summarized in the chain of weak homotopy equivalences

$$\Omega|wS_\bullet\mathcal{C}| \xleftarrow{(1)} \Omega|wG_\bullet^\infty S_\bullet\mathcal{C}| \xrightarrow{(1)} |wG_\bullet^\infty\mathcal{C}| \xrightarrow{(2)} |wG_\bullet G_\bullet^\infty\mathcal{C}| \xrightarrow{(3)}$$
$$\xrightarrow{(3)} |F_+\textstyle\int wG_\bullet G_\bullet^\infty\mathcal{C}| \xleftarrow{(4)} \operatorname*{colim}_{\tilde{t}}|F_+\textstyle\int wG_\bullet\mathcal{C}| \xleftarrow{(5)} |F_+\textstyle\int wG_\bullet\mathcal{C}| \xrightarrow{(6)} |F\textstyle\int wG_\bullet\mathcal{C}|.$$

Concerning the links in the chain, the arrows labeled (1) are recorded in (3.21); the arrow (2) results from swallowing the extra $G_\bullet$ into the colimit defining $G_\bullet^\infty$. That (3) is an equivalence depends on the fact that $|F_+\int wG_\bullet G_\bullet^\infty\mathcal{C}| \to |F_+|$ can be shown to be a quasifibration with $|F_+|$ contractible. To account for (4), the functor $t : F_+ \to F_+$ induces a functor $\tilde{t} : F_+\int wG_\bullet\mathcal{C} \to F_+\int wG_\bullet\mathcal{C}$ for which

$$\operatorname*{colim}_{\tilde{t}} F_+\textstyle\int wG_\bullet\mathcal{C} = \operatorname{colim}\big(F_+\textstyle\int wG_\bullet\mathcal{C} \xrightarrow{\tilde{t}} F_+\textstyle\int wG_\bullet\mathcal{C}F_+ \xrightarrow{\tilde{t}} F_+\textstyle\int wG_\bullet\mathcal{C} \xrightarrow{\tilde{t}} \cdots\big)$$

is naturally identifiable to $F_+\int wG_\bullet G_\bullet^\infty\mathcal{C}$. The realizations of the functors $\tilde{t}$ are all cofibrations, so the inclusion (5) into the base of the telescope is a weak equivalence. Finally, cofinality of F_+ in F implies that the arrow (6) is a weak homotopy equivalence. □

As in [Gunnarsson and Schwänzl 2002], the E_∞-structure on the total Segal operation is described in terms of the diagram

$$\begin{tikzcd}
\mathcal{R}_f^n(X) \arrow[d, "\alpha_1"] \arrow[dr, "\tilde{\alpha}_1"] & \\
\{1\}\times\prod_{n\geq 1}\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \arrow[r, "\beta_1"] \arrow[d, "\alpha_2"] & \{1\}\times\prod_{n\geq 1}G_\bullet^n\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \arrow[d, "\beta_3"] \\
\{1\}\times\prod_{n\geq 1}F\textstyle\int G_\bullet\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \arrow[r, "\beta_2"] & \{1\}\times\prod_{n\geq 1}F\textstyle\int G_\bullet G_\bullet^n\mathcal{R}_f(X,\Sigma_n,\{\text{all}\})
\end{tikzcd} \tag{4.4}$$

The components of the map $\tilde{\alpha}_1$ are defined in Example 3.32. The other maps in diagram (4.4) are defined as follows.

Definition 4.5. For α_1 the n-th component $\alpha_1(Y)_n$ is $Y \wedge_\mu \overset{n\text{ terms}}{\cdots} \wedge_\mu Y$, where Σ_n acts by permuting factors using the coherence data.

The maps β_1 and β_2 come from stabilizations

$$j^n : \mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \to G_\bullet^n\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}).$$

The maps α_2 and β_3 are given by the identification

$$\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \cong \{\varnothing\}\textstyle\int G_\bullet\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \cong G_\bullet^\varnothing\,\mathcal{R}_f(X,\Sigma_n,\{\text{all}\}).$$

The category $\mathcal{R}_f^n(X)$ has the pairing derived from the categorical sum $\vee_X$. This feature allows us to dispense with the subdivision construction. Each of the four categories in the lower part of the diagram also has a natural pairing derived from the box tensor pairings

$$\boxtimes_{k,\ell} : \mathcal{R}_f(X, \Sigma_k, \{\text{all}\}) \times \mathcal{R}_f(X, \Sigma_\ell, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_{k+\ell}, \{\text{all}\}).$$

Underlying the coherence properties of these pairings is the fact, established in Theorem 2.1, that $\mathcal{R}_f(X)$ is a category with cofibrations and weak equivalences, with a categorical sum $\vee$ and a symmetric monoidal biexact product $\wedge_\mu$. We refer to [Gunnarsson and Schwänzl 2002, pp. 291–292] for explicit formulas for the pairings, which are given in the abstract context of a category $\mathcal{C}$ with cofibrations and weak equivalences and subcategories $\mathcal{C}_{\Sigma_n}$ of Σ_n-equivariant objects. Here we record only notations for use in the next theorem.

(1) There is a product denoted $\widetilde{\boxtimes}$ on $\{1\} \times \prod_{n\geq 1} \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ and a product also denoted $\widetilde{\boxtimes}$ on $\{1\} \times \prod_{n\geq 1} G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$.

(2) There is a product denoted $\widehat{\boxtimes}$ on $\{1\} \times \prod_{n\geq 1} F\!\int\! G_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ and a product also denoted $\widehat{\boxtimes}$ on $\{1\} \times \prod_{n\geq 1} F\!\int\! G_\bullet G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$.

Theorem 4.6 (cf. [Gunnarsson and Schwänzl 2002, Theorem 10.7, p. 292]).

(1) *In the left column of (4.4), the categories $\{1\} \times \prod_{n\geq 1} \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ and $\{1\} \times \prod_{n\geq 1} F\!\int\! G_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$, with their composition laws $\widetilde{\boxtimes}$ and $\widehat{\boxtimes}$, inherit symmetric monoidal structures from the coherence data on $\mathcal{R}_f(X)$.*

(2) *In the right column of (4.4), the categories $\{1\} \times \prod_{n\geq 1} G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ and $\{1\} \times \prod_{n\geq 1} F\!\int\! G_\bullet G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$, with their composition laws $\widetilde{\boxtimes}$ and $\widehat{\boxtimes}$, inherit monoidal structures from the coherence data on $\mathcal{R}_f(X)$.*

(3) *The maps α_1 and α_2 in (4.4) are maps of symmetric monoidal categories.*

(4) *The maps β_1, β_2, and β_3 are maps of monoidal categories.*

(5) *The map β_2 is a homotopy equivalence, and in the pseudo-additive case β_3 is also a homotopy equivalence.*

(6) *The diagram (4.4) is commutative in the category of monoidal categories.*

Theorem 4.7. *Let X be a connected simplicial abelian group. The functor*

$$Z \mapsto \Big[Z, \{1\} \times \prod_{n\geq 1} A_{\Sigma_n,\{\text{all}\}}(X)\Big]$$

takes values in the category of abelian groups.

Proof. By Theorem 4.3, we take

$$\{1\} \times \prod_{n\geq 1} A_{\Sigma_n,\{\text{all}\}}(X) = \{1\} \times \prod_{n\geq 1} |F\!\int\! G_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})|.$$

Since the category $\{1\} \times \prod_{n\geq 1} F\!\int\! G_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ has a symmetric monoidal structure by part (1) of Theorem 4.6, the functor $\big[-, \{1\} \times \prod_{n\geq 1} A_{\Sigma_n,\{\text{all}\}}(X)\big]$ takes values in the category of abelian monoids. Repeating the argument of [Waldhausen 1982, Lemma 2.3, p. 404] shows that values taken are actually in the category of abelian groups. □

Remarks on the proof of Theorem 4.6.. The entire proof of the analogous result in [Gunnarsson and Schwänzl 2002, pp. 293–295] is essentially a formal appeal to LaPlaza's coherence theorem [1972], so it carries over completely.

The reader who investigates further will find the symmetry of the pairing on $\{1\} \times \prod_{n\geq 1} F\int G_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ involves manipulating products of values of functors

$$Y \in G^m_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \quad \text{and} \quad Z \in G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}).$$

What is required is comparison of expressions

$$Y(i_1/j_1, \ldots, i_m/j_m) \wedge_\mu Z(i'_1/j'_1, \ldots, i'_n/j'_n)$$

and

$$Z(i'_1/j'_1, \ldots, i'_n/j'_n) \wedge_\mu Y(i_1/j_1, \ldots, i_m/j_m),$$

and one sees that not only are commutativity isomorphisms for $\wedge_\mu$ involved, but so are permutations of inputs, which are taken care of by means of the homotopy colimit.

Another interesting part of the proof is the claims about the maps α_1 and α_2, so it deserves a comment. The biexactness and coherence of $\wedge_\mu$ give canonical natural isomorphisms γ^k_n called Cartan multinomial formulas:

$$\gamma^k_n : (\wedge_\mu)_n\left(\bigvee_{i=1}^{k} c_i\right) \xrightarrow{\cong} \bigvee_{s_1+\cdots+s_k=n} \operatorname{Ind}^{\Sigma_n}_{\Sigma_{s_1}\times\cdots\times\Sigma_{s_k}} (\wedge_\mu)^k_{i=1}((\wedge_\mu)_{s_i} c_i).$$

These induce natural isomorphisms

$$\gamma^k : \alpha_1 \circ \vee^k_X \overset{\cong}{\Longrightarrow} (\widetilde{\boxtimes})^k \circ \alpha^k_1.$$

Then the coherence theorem implies that α_1 has a (lax) symmetric monoidal structure. The functor α_2 is the inclusion of a symmetric monoid subcategory, so the assertion for α_2 is immediate.

In contrast to the algebraic roles played by α_1 and α_2, the roles of β_1, β_2, and β_3 are to assure us that we are ending in the correct target. Since the proof that β_3 is a homotopy equivalence requires the pseudo-additivity condition, which is fulfilled by suspension, this part of the argument actually depends on the next section. □

5. Suspension

Let us first state the main theorem of this section.

Theorem 5.1. *Let X be a simplicial abelian group. The total Segal operation*

$$\omega : A(X) \to \{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \mathrm{all}}(X)$$

carries an infinite loop map structure.

Section 4 has delivered an infinite loop map

$$\Omega\, |hN_\Gamma \mathcal{R}^n_f(X)| \to \{1\} \times \prod_{n \geq 1} A_{\Sigma_n, \mathrm{all}}(X)$$

whose domain is the K-theory of a category of n-spherical objects. To obtain Theorem 5.1, we have to examine the passage to the limit over suspension in view of Waldhausen's result

$$\lim_{n \to \infty} hN_\bullet \mathcal{R}^n_f(X) \simeq hS_\bullet \mathcal{R}_f(X).$$

The technically challenging part is the compatibility of the operations with suspension. Fortunately, the machinery set up in [Gunnarsson and Schwänzl 2002, Section 10] is sufficiently general that we need only extend some definitions and quote a sequence of results to prove our generalization.

First we need a description of the suspension operation that is amenable to coherence considerations. To this end, we go step-by-step through Waldhausen's cone and suspension constructions and identify the result with a construction involving the operation $\wedge_e$. The cone construction for (Y, r, s) in $\mathcal{R}_f(X)$ takes the ordinary mapping cylinder of the retraction $M(r)$ and collapses out the cylinder $\Delta^1 \times X$ so that end result is in $\mathcal{R}_f(X)$. To amplify the definition, consider the diagram

$$\begin{tikzcd}
Y \amalg X & \partial\Delta^1 \times Y \arrow[l, "\mathrm{id} \amalg r"'] \arrow[r, tail] & \Delta^1 \times Y \\
X \amalg X \arrow[u, tail] \arrow[d] & \partial\Delta^1 \times X \arrow[l] \arrow[r, tail] \arrow[u, tail] \arrow[d] & \Delta^1 \times X \arrow[u, tail] \arrow[d] \\
X & X \arrow[l] \arrow[r, tail] & X
\end{tikzcd} \tag{5.2}$$

which fulfills the hypotheses of Lemma 2.9. Taking the pushouts of the rows produces a diagram

$$X \leftarrow \Delta^1 \times X \rightarrowtail M(r),$$

where $M(r)$ is the usual mapping cylinder of r and the pushout of the top row. As described above, taking the pushout of this diagram produces cY, the underlying space of the cone construction. The retraction to X arises from a map of diagram (5.2) to a trivial diagram of identity maps on X; the section $X \to cY$ and a cofibration $i : Y \to cY$ arise from canonical maps of ingredients of the diagram to the colimit. Then the suspension ΣY is defined as the pushout of the diagram $X \xleftarrow{r} Y \rightarrowtail cY$.

Lemma 5.3. *For $Y \in \mathcal{R}_f(X)$ there is a commuting diagram*

$$\begin{tikzcd} \{0\} \times Y \arrow[r, tail, "i_0"] \arrow[d] & \Delta^1_1 \wedge_e Y \arrow[d, "\cong"'] \\ Y \arrow[r, tail, "i"] & cY \end{tikzcd} \tag{5.4}$$

where $\Delta^1_1 \in \mathcal{R}_f(*)$ is the standard simplicial one-simplex given the base point 1, *and i_0 is induced from the inclusion $\{0\} \to \Delta^1$. Moreover,*

$$\Sigma Y := cY/Y \cong S^1 \wedge_e Y,$$

where $S^1 = \Delta^1/\partial\Delta^1$ is the standard simplicial circle.

Proof. Pass to pushouts in the commutative diagram

$$\begin{tikzcd} X \arrow[d] & \Delta^1_1 \times X \cup_{\{1\}\times X} \{1\} \times Y \arrow[l, "p_2 \cup r p_2"'] \arrow[r, tail] \arrow[d, "\mathrm{id} \cup r"'] & \Delta^1 \times Y \arrow[d] \\ X & \Delta^1 \times X \arrow[l, "p_2"'] \arrow[r, tail] & M(r) \end{tikzcd} \tag{5.5}$$

to obtain a unique natural map $\eta_1 : \Delta^1_1 \wedge_e Y \to cY$ making the diagram

$$\begin{tikzcd} & \Delta^1_1 \wedge_e Y \arrow[dd, "\eta_1"'] & \\ \Delta^1_1 \times Y \arrow[ur] \arrow[dr] & & X \arrow[ul, tail, "s'"'] \arrow[dl, tail] \\ & cY & \end{tikzcd} \tag{5.6}$$

commute. Restricting $\Delta^1_1 \times Y \to \Delta^1_1 \wedge_e Y$ to $\partial\Delta^1_1 \times Y$ yields a diagram

$$\begin{tikzcd} \partial\Delta^1_1 \times Y \arrow[r, tail] \arrow[d, "r'"'] & \Delta^1_1 \times Y \arrow[d] \\ Y \amalg X \arrow[r, tail, "i'"] & \Delta^1_1 \wedge_e Y \end{tikzcd}$$

where $r'(0, y) = y$, $r'(1, y) = r(y)$ and $i'(y) = i_0(y)$, $i'(x) = s'(x)$. There results a canonical arrow $M(r) \to \Delta^1_1 \wedge_e Y$ such that the following square commutes:

$$\begin{tikzcd} \Delta^1_1 \times X \arrow[r, tail] \arrow[d, "p_2"'] & M(r) \arrow[d] \\ X \arrow[r, tail, "s'"] & \Delta^1_1 \wedge_e Y \end{tikzcd}$$

In turn, there is a unique map $\bar{\eta}_1 : cY \to \Delta^1_1 \wedge_e Y$ such that

$$\begin{tikzcd} & cY \arrow[dd, "\bar{\eta}_1"'] & \\ \Delta^1_1 \times Y \arrow[ur] \arrow[dr] & & X \arrow[ul, tail] \arrow[dl, tail, "s'"] \\ & \Delta^1_1 \wedge_e Y & \end{tikzcd} \tag{5.7}$$

commutes. Combining diagrams (5.6) and (5.7) shows that η_1 and $\bar{\eta}_1$ are mutually inverse isomorphisms, relative to the common subspace X and compatible with the retractions.

Restricting the left half of (5.6) to $\{0\} \times Y \subset \Delta^1_1 \times Y$ gives (5.4):

$$\begin{tikzcd} & \Delta^1_1 \wedge_e Y \arrow[d, "\eta_1"'] \\ \{0\} \times Y \equiv Y \arrow[ur, "i_0"] \arrow[r, "i"'] & cY \end{tikzcd} \tag{5.8}$$

Replace $S^0 = \{*, *'\}$ with basepoint $*$ in Example 2.5 by $\partial\Delta^1_1$ with basepoint 1, and obtain the diagram

$$\begin{tikzcd} X \arrow[d, "="'] & \partial\Delta^1_1 \wedge_e Y \arrow[l, "r"'] \arrow[r, tail] \arrow[d, "\cong"'] & \Delta^1_1 \wedge_e Y \arrow[d, "\cong"] \\ X & Y \arrow[l, "r"] \arrow[ur, tail, "i_0"] \arrow[r, tail, "i"] & cY \end{tikzcd} \tag{5.9}$$

Passage to pushouts shows that the quotient $(\Delta^1_1 \wedge_e Y)/(\partial\Delta^1_1 \wedge_e Y)$ is isomorphic to ΣY in $\mathcal{R}_f(X)$. According to Proposition 2.17, the functor

$$- \wedge_e Y : \mathcal{R}_f(*) \to \mathcal{R}_f(X \times \{*\}) \cong \mathcal{R}_f(X)$$

preserves colimits, so we deduce

$$(\Delta^1_1 \wedge_e Y)/(\partial\Delta^1_1 \wedge_e Y) \cong (\Delta^1_1/\partial\Delta^1_1) \wedge_e Y \equiv S^1 \wedge_e Y,$$

where we define $S^1 := \Delta^1_1/\partial\Delta^1_1$ in $\mathcal{R}_f(*)$. □

According to Proposition 2.16, the action of $\mathcal{R}_f(*)$ on $\mathcal{R}_f(X)$ may be made internal. Explicitly, there is a natural isomorphism $i_{e*}S^1 \wedge_\mu Y \cong S^1 \wedge_e Y$. In the following we abuse notation slightly and write simply $S^1 \wedge_\mu Y$, leaving i_{e*} understood, where $i_e : \{*\} \to X$ is the inclusion of the one-point space as the identity element of X. We do this to emphasize the dependence of the rest of this section on the coherence of the operation $\wedge_\mu$.

Proposition 5.10 (cf. [Gunnarsson and Schwänzl 2002, Proposition 6.1, p. 283]). *The following diagram commutes up to natural isomorphism*:

$$\begin{tikzcd} w\,\mathrm{sub}_k\, G_\bullet \mathcal{R}_f(X) \arrow[r, "\omega^k"] \arrow[d, "S^1\wedge_\mu"'] & wG^k_\bullet \mathcal{R}_f(X, \Sigma_k, \{\mathrm{all}\}) \arrow[d, "\diamond_k S^1 \wedge_\mu"'] \\ w\,\mathrm{sub}_k\, G_\bullet \mathcal{R}_f(X) \arrow[r, "\omega^k"] & wG^k_\bullet \mathcal{R}_f(X, \Sigma_k, \{\mathrm{all}\}) \end{tikzcd}$$

Proof. Write F_1 for the composite functor $\omega^k \circ (S^1 \wedge_\mu -)$ and F_2 for the composite $\diamond_k S^1 \wedge_\mu \omega^k(-)$. Although $\omega^k(S^1) = \diamond_k S^1 = S^1 \wedge_\mu \overset{k \text{ terms}}{\cdots} \wedge_\mu S^1$, we use the $\diamond_k$-notation for orientation purposes. Following [Gunnarsson and Schwänzl

2002, p. 297; Grayson 1989, p. 257], given a functor $M : \Gamma(A_1 \dots A_k) \to \mathcal{R}_f(X)$ representing an object of $\mathrm{sub}_k\, G_\bullet \mathcal{R}_f(X)$, the value of $\omega^k(M)$ on a typical element of $\Gamma^k(A_1 \dots A_k)$ has the form

$$(\diamond_{n_1} M(-)) \boxtimes (\diamond_{n_2} M(-)) \boxtimes \cdots \boxtimes (\diamond_{n_k} M(-)) = Z_{n_1} \boxtimes \cdots \boxtimes Z_{n_k},$$

where $Z_{n_i} := \diamond_{n_i} M(-)$ is an object of $\mathcal{R}_f(X, \Sigma_{n_i}, \{\text{all}\})$. Extending the formulas in the argument of Proposition 3.28 for the associativity of $\boxtimes$, we write

$$Z_{n_1} \boxtimes \cdots \boxtimes Z_{n_k} = \mathrm{Ind}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}^{\Sigma_{n_1+\cdots+n_k}} (Z_{n_1} \wedge_\mu \cdots \wedge_\mu Z_{n_k})$$

and set $n = n_1 + \cdots + n_k$.

Then a typical value of $F_1(M)$ has the form

$$\begin{aligned}
&\mathrm{Ind}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}^{\Sigma_n} \big((S^1 \wedge_\mu \overset{n_1 \text{ terms}}{\cdots} \wedge_\mu S^1 \wedge_\mu Z_{n_1}) \wedge_\mu \cdots \wedge_\mu (S^1 \wedge_\mu \overset{n_k \text{ terms}}{\cdots} \wedge_\mu S^1 \wedge_\mu Z_{n_k}) \big) \\
&\quad \cong \mathrm{Ind}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}^{\Sigma_{n_1+\cdots+n_k}} \big((S^1 \wedge_\mu \overset{n_1 \text{ terms}}{\cdots} \wedge_\mu S^1) \wedge_\mu \overset{k \text{ groups}}{\cdots} \wedge_\mu (S^1 \wedge_\mu \overset{n_k \text{ terms}}{\cdots} \wedge_\mu S^1) \\
&\qquad \wedge_\mu (Z_{n_1} \wedge_\mu \cdots \wedge_\mu Z_{n_k}) \big),
\end{aligned}$$

applying commutativity and associativity isomorphisms. Now Proposition 5.12 applies to deliver an isomorphism of $\Sigma_{n_1+\cdots+n_k}$-spaces:

$$\begin{aligned}
&\mathrm{Ind}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}^{\Sigma_n} \big((S^1 \wedge_\mu \overset{n_1 \text{ terms}}{\cdots} \wedge_\mu S^1) \wedge_\mu \overset{k \text{ groups}}{\cdots} \wedge_\mu (S^1 \wedge_\mu \overset{n_k \text{ terms}}{\cdots} \wedge_\mu S^1) \\
&\qquad \wedge_\mu (Z_{n_1} \wedge_\mu \cdots \wedge_\mu Z_{n_k}) \big) \\
&\quad \xrightarrow{\cong} (\diamond_k S^1) \wedge_\mu \overset{n \text{ terms}}{\cdots} \wedge_\mu (\diamond_k S^1) \wedge_\mu \big(\mathrm{Ind}_{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}^{\Sigma_n} (Z_{n_1} \wedge_\mu \cdots \wedge_\mu Z_{n_k}) \big).
\end{aligned}$$

This final expression is the value of F_2 on the same typical element M, so we have a natural isomorphism of functors $\epsilon : F_1 \Rightarrow F_2$. □

Now we prove the general Lemma 5.11 and its specialization Proposition 5.12.

Lemma 5.11. *Let H be a subgroup of G, let $Y \in \mathcal{R}(X, G)$, and let $Z \in \mathcal{R}(X, H)$. By restricting the G-action on Y to H, we obtain $Y \wedge_\mu Z \in \mathcal{R}(X, H)$, where the action is diagonal. Then there is a natural isomorphism of left G-spaces*

$$G_+ \wedge_e^H (Y \wedge_\mu Z) \xrightarrow{\cong} Y \wedge_\mu (G_+ \wedge_e^H Z),$$

where the G-action on the right-hand space is diagonal.

Proof. First define a G-map $f : G_+ \wedge_e (Y \wedge_\mu Z) \to Y \wedge_\mu (G_+ \wedge_e^H Z)$ by the formula

$$f(g, (y, z)) = (gy, [g, z]).$$

Applying the equivalence relation defining $Y \wedge_\mu (G_+ \wedge_e^H Z)$,

$$f(g, (hy, hz)) = (g(hy), [g, hz]) = ((gh)y, [gh, z]) = f(gh, (y, z)).$$

Therefore, there is an induced G-map

$$f' : G_+ \wedge_e^H (Y \wedge_\mu Z) \to Y \wedge_\mu (G_+ \wedge_e^H Z).$$

To reverse this map, define $F : Y \wedge_\mu (G_+ \wedge_e Z) \to G_+ \wedge_e^H (Y \wedge_\mu Z)$ by the formula

$$F(y, [g, z]) = [g, (g^{-1}y, z)].$$

Now

$$\begin{aligned} F(y, [gh, z]) &= [gh, (h^{-1}g^{-1}y, z)] \\ &= [g, (hh^{-1}g^{-1}y, hz)] = [g, (g^{-1}y, hz)] = F(y, [g, hz]), \end{aligned}$$

so there is an induced G-map

$$F' : Y \wedge_\mu (G_+ \wedge_e^H Z) \to G_+ \wedge_e^H (Y \wedge_\mu Z).$$

Clearly the composites $f'F'$ and $F'f'$ are the respective identities. □

Proposition 5.12. *Let $n = n_1 + \cdots + n_k$. Let $Z \in \mathcal{R}(X, \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}, \{\text{all}\})$. There is a natural isomorphism of Σ_n-spaces*

$$\begin{aligned} &\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{n}_1 \cup \cdots \cup \boldsymbol{n}_k)_+ \wedge_e^{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} \big((S \diamond \overset{n_1 \text{ terms}}{\cdots} \diamond S) \wedge_\mu \cdots \wedge_\mu (S \diamond \overset{n_k \text{ terms}}{\cdots} \diamond S) \wedge_\mu Z\big) \\ &\qquad \xrightarrow{\cong} (S \diamond \overset{n \text{ terms}}{\cdots} \diamond S) \wedge_\mu \big(\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{n}_1 \cup \cdots \cup \boldsymbol{n}_k)_+ \wedge_e^{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}} Z\big). \end{aligned}$$

Proof. Apply Lemma 5.11, and observe that the operation $\diamond$ is defined in terms of $\wedge_\mu$, which is coherently associative. Collect all parentheses in expressions

$$(S \diamond \overset{n_1 \text{ terms}}{\cdots} \diamond S) \wedge_\mu \cdots \wedge_\mu (S \diamond \overset{n_k \text{ terms}}{\cdots} \diamond S)$$

to the left. Note that we need only the map $f' : G_+ \wedge_e^H (Y \wedge_\mu Z) \to Y \wedge_\mu (G_+ \wedge_e^H Z)$ from the lemma, so the choice of an identification of $\mathrm{Iso}(\boldsymbol{n}, \boldsymbol{n}_1 \cup \cdots \cup \boldsymbol{n}_k)$ with Σ_n is required to make sense of f'. This amounts to identifying $\boldsymbol{n}_1 \cup \cdots \cup \boldsymbol{n}_k$ with

$$\{1, \ldots, n_1, n_1 + 1, \ldots, n_1 + n_2, \ldots, n_1 + \cdots + n_k\}.$$ □

We use the Thomason homotopy colimit construction on functors defined on the category F to pass to the limit with suspensions. To treat suspension by S^1 on $\mathrm{sub}_k\, wG_\bullet \mathcal{R}_f(X)$, define an op-lax functor $\Phi_1 : F \to \mathrm{Cat}^{\Delta^{\mathrm{op}}}$ by

$$\begin{aligned} \Phi_1(x) = \mathrm{sub}_k\, wG_\bullet \mathcal{R}_f(X) \quad & \text{for an object } x \in F, \\ \Phi_1(\sigma) = \mathrm{id} \quad & \text{for an isomorphism } \sigma : x \to x, \\ \Phi_1(i : y \to x) \quad & \text{is induced by suspension by } x \backslash i(y) \text{ factors } S^1. \end{aligned}$$

Interpreting the smash product with an empty number of factors as S^0, the definitions coincide on isomorphisms. For $x \xleftarrow{i} y \xleftarrow{j} z$ we need to produce the natural transformation $\Phi_1(ij) \Rightarrow \Phi_1(i) \circ \Phi_1(j)$. On (Y, r, s) the value of $\Phi_1(j)$ is

$((S^1)^{y\backslash j(z)} \wedge_e Y, r', s')$ and the value of $\Phi_1(i)$ applied to this is

$$\big((S^1)^{x\backslash i(y)} \wedge_e ((S^1)^{y\backslash j(z)} \wedge_e Y), r'', s''\big).$$

Since i is injective, the set $y\backslash j(z)$ is identified with $i(y\backslash j(z))$. Since $x\backslash ij(z) = x\backslash i(y) \cup i(y\backslash j(z))$, we use associativity isomorphisms of the $\wedge_e$-action to write $\Phi_1(i \circ j) \overset{\cong}{\Longrightarrow} \Phi_1(i) \circ \Phi_1(j)$. The coherence properties of the action imply commutativity of the necessary diagrams [Thomason 1979, Definition 3.1.1, p. 99].

In a similar way we treat $\diamond_k S^1 \wedge_\mu -$ on $wG^k_\bullet \mathcal{R}_f(X, \Sigma_n\{\text{all}\})$, defining an op-lax functor $\Phi_2 : F \to \mathrm{Cat}^{\Delta^{\mathrm{op}}}$:

$$\begin{aligned} \Phi_2(x) = wG^k_\bullet \mathcal{R}_f(X, \Sigma_n\{\text{all}\}) \quad & \text{for an object } x \in F, \\ \Phi_2(\sigma) = \mathrm{id} \quad & \text{for an isomorphism } \sigma : x \to x, \\ \Phi_2(i : y \to x) \quad & \text{is induced by suspension by } x\backslash i(y) \text{ factors } \diamond_k S^1. \end{aligned}$$

The natural transformation $\Phi_2(i \circ j) \overset{\cong}{\Longrightarrow} \Phi_2(i) \circ \Phi_2(j)$ is treated in the same manner.

The results are two categories

$$\operatorname*{hocolim}_{S^1 \wedge_\mu -} \mathrm{sub}_k\, wG_\bullet \mathcal{R}_f(X) := F \textstyle\int \Phi_1 \quad \text{and} \quad \operatorname{hocolim}_{\diamond_k S^1 \wedge_\mu -} wG^k_\bullet \mathcal{R}_f(X) := F \textstyle\int \Phi_2.$$

Remark 5.13. There are a number of constructions in [Thomason 1979] that may justifiably be termed homotopy colimits. This particular construction $F \int \Phi_i$ is essential, but we use the hocolim notation to provide context for the reader.

Now we explain how Proposition 5.10 promotes

$$\omega^k : \mathrm{sub}_k\, wG_\bullet \mathcal{R}_f(X) \to wG^k_\bullet \mathcal{R}_f(X)$$

to a left-op natural transformation (lont) $\epsilon : \Phi_1 \Rightarrow \Phi_2$. First, we need to associate to an object x of F a functor $\epsilon(x) : \Phi_1(x) \to \Phi_2(x)$. This is just ω^k. Then we need for each arrow $i : y \to x$ in F a natural transformation $\epsilon(i) : \epsilon(x) \circ \Phi_1(i) \Rightarrow \Phi_2(i) \circ \epsilon(y)$. For any morphism i such that $x\backslash i(y)$ has cardinality 1, we obtain $\epsilon(i)$ by inverting the isomorphism of functors provided by Proposition 5.10. For the general case, one just goes back to the proof and replaces the symbol 1 by $x\backslash i(y)$ everywhere it occurs. The coherence results of Section 2 guarantee that the necessary diagrams commute, so ϵ is a lont. By [Thomason 1979, Definition 3.1.4, p. 101], ϵ induces a functor

$$F \textstyle\int \epsilon : F \int \Phi_1 \to F \int \Phi_2.$$

We have now proved the following result.

Theorem 5.14. *The operations ω^k pass through the Thomason homotopy colimit construction to deliver operations*

$$F \textstyle\int \epsilon := \omega^k : \operatorname{hocolim}_{S^1 \wedge_\mu -} \mathrm{sub}_k\, wG_\bullet \mathcal{R}_f(X) \to \operatorname{hocolim}_{\diamond_k S^1 \wedge_\mu -} wG^k_\bullet \mathcal{R}_f(X). \qquad \square$$

Proof of Theorem 5.1. The main result of Section 4 is that

$$\Omega|hN_\Gamma\mathcal{R}^n_f(X)| \to \{1\} \times \prod_{n\geq 1} A_{\Sigma_n,\mathrm{all}}(X)$$

is an infinite loop map, and this section shows these maps are compatible with suspension. Likewise for the equivalence $\Omega|hN_\Gamma\mathcal{R}^n_f(X)| \to \Omega|wS_\bullet\mathcal{R}^n_f(X)|$. The maps obtained by passing to the limit over suspension remain infinite loop maps, and we know $\Omega\,\mathrm{colim}|wS_\bullet\mathcal{R}^n_f(X)| \simeq \Omega|wS_\bullet\mathcal{R}_f(X)| = A(X)$. □

6. Projecting to the free part

As stated in Theorem 5.1, the constructions of [Gunnarsson and Schwänzl 2002] as modified in Section 5 deliver a total operation

$$\omega = \prod \omega^n : A(X) \to \prod_{n\geq 1} A_{\Sigma_n,\{\mathrm{all}\}}(X),$$

where $A_{\Sigma_n,\{\mathrm{all}\}}(X) = \Omega|hS_\bullet\mathcal{R}_{hf}(X, \Sigma_n, \{\mathrm{all}\})|$. We examine the target of this map, and introduce the Weyl group notation $W_{\Sigma_n}H = N_{\Sigma_n}H/H$, where H is a subgroup of the permutation group Σ_n and $N_{\Sigma_n}H$ is the normalizer in Σ_n of H.

Theorem 6.1. *Let X be a space on which symmetric groups Σ_n act trivially. For each n there is a homotopy equivalence*

$$h_n : A_{\Sigma_n,\{\mathrm{all}\}}(X) \to \prod_{H\in\{\mathrm{all}\}} A(X \times B(W_{\Sigma_n}H))$$

of infinite loop spaces. Here $A(X \times B(W_{\Sigma_n}H)) = \Omega|hS_\bullet\mathcal{R}_f(X, W_{\Sigma_n}H, \{e\})|$ is the K-theory of the category of retractive $W_{\Sigma_n}H$-spaces relative to X with the action being free outside of X.

Proof. The argument is largely formal, based on some well-known facts. Let $\mathcal{F}$ be the set of conjugacy classes (H_i) of subgroups of Σ_n. This set is finite and partially ordered in the usual way: $(H_i) \preceq (H_j)$ if some conjugate of H_i is contained in H_j. The partial ordering may be extended to a linear ordering, or enumeration $\{(H_0), (H_1), \ldots, (H_N)\}$, so that $(H_i) \prec (H_j)$ implies $i < j$. Observe that $(H_0) = \{e\}$, we may take (H_1) as the class of transpositions, and $(H_N) = \Sigma_n$.

For any Σ_n-space Z we may define

$$\mathcal{F}_{\succ(H)}Z = \underset{(K)\succ(H)}{\mathrm{colim}}\ Z^{(K)},$$

essentially the union of the fixed point sets of the conjugates of all the subgroups properly containing a conjugate of H. The space $\mathcal{F}_{\succ(H)}Z$ is by definition a Σ_n-invariant subspace of Z. If $(H_i) \prec (H_{i+1})$ in the enumeration then we may compute $\mathcal{F}_{\succ(H_{i+1})}(\mathcal{F}_{\succ(H_i)}Z)$, essentially the fixed points of conjugates of H_{i+1} inside the fixed points of H_i. On the complement $\mathcal{F}_{\succ(H_i)}Z\backslash(\mathcal{F}_{\succ(H_{i+1})}(\mathcal{F}_{\succ(H_i)}Z))$ the group Σ_n acts and the Weyl group $W_{\Sigma_n}H_i = N_{\Sigma_n}H_i/H_i$ acts freely.

Inductively define exact functors

$$S_i,\, Q_j : \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}), \quad -1 \le i \le N,\ 0 \le j \le N$$

by letting S_{-1} be the identity functor, and putting $S_i(Y) = \mathcal{F}_{\succ(H_i)}(S_{i-1}(Y))$ for $i \ge 0$. Then the functors Q_j are defined by the natural cofibration sequences

$$S_j(Y) \rightarrowtail S_{j-1}(Y) \twoheadrightarrow Q_j(Y), \quad 0 \le j \le N.$$

For us, the important case is S_0: Since $H_0 = \{e\}$, $S_0(Y)$ is the union of the fixed point sets of all the nonidentity subgroups of G. Then the quotient $Q_0(Y)$ can be thought of as extracting the part of Y on which G acts freely.

Let $i_k : \mathcal{R}_f(X, \Sigma_n, \{H_k\}) \to \mathcal{R}_f(X, \Sigma_n, \{\text{all}\})$ be the inclusion. Since $Q_k(Y)$ actually lies in $\mathcal{R}_f(X, \Sigma_n, \{H_j\})$, we may formally write $Q_k = i_k \circ \bar{Q}_k$, where $\bar{Q}_k : \mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \to \mathcal{R}_f(X, \Sigma_n, \{H_k\})$ is a retraction. We want to make an inductive application of the additivity theorem for the $G_\bullet$ construction, but this requires that the input be pseudo-additive. Passing to prespectra $\Sigma^\infty\mathcal{R}_f(X)$, by [Gunnarsson et al. 1992] there results a splitting

$$\operatorname{hocolim} wG_\bullet\mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) \to \prod_{H\in\{\text{all}\}} \operatorname{hocolim} wG_\bullet\mathcal{R}_f(X, \Sigma_n, \{H\})$$

induced by the functors $\bar{Q}_k$ for $0 \le k \le N$. Recalling that $W_{\Sigma_n}H = N_{\Sigma_n}H/H$ is the Weyl group of H, consider the exact functor

$$\mathcal{R}_f(X, \Sigma_n, \{H\}) \to \mathcal{R}_f(X, W_{\Sigma_n}H, \{e\}), \quad Y \mapsto Y^H.$$

The induction construction $Z \mapsto Z \times^{W_{\Sigma_n}H} \Sigma_n$ provides an exact functor going the other way and the composites in either order are equivalent to the identities. Putting these equivalences together and specializing the notation establishes a chain of homotopy equivalences

$$\begin{aligned}\operatorname{hocolim} wG_\bullet\mathcal{R}_f(X, \Sigma_n, \{\text{all}\}) &\to \prod_{H\in\{\text{all}\}} \operatorname{hocolim} wG_\bullet\mathcal{R}_f(X, \Sigma_n, \{H\})\\ &\to \prod_{H\in\{\text{all}\}} \operatorname{hocolim} wG_\bullet\mathcal{R}_f(X, W_{\Sigma_n}H, \{e\}). \qquad \square\end{aligned}$$

This completes the proof of Theorem 1.1; to explain Theorem 1.3 is the object of the next two sections. We are focusing on the composition

$$\begin{aligned}\theta^n : A(X) \xrightarrow{\omega^n} A_{\Sigma_n,\{\text{all}\}}(X) \xrightarrow{h_n} &\prod_{H\in\{\text{all}\}} \Omega|hS_\bullet\mathcal{R}_f(X, N_{\Sigma_n}H/H, \{e\})|\\ &\xrightarrow{p_e} \Omega|hS_\bullet\mathcal{R}_h f(X, \Sigma_n, \{e\})|.\end{aligned}$$

In Section 7 we justify the interpretation $\Omega|hS_\bullet\mathcal{R}_h f(X,\Sigma_n,\{e\})| = A(X\times B\Sigma_n)$. Then we want to understand what happens when we follow this composition by the transfer $\phi_n : A(X\times B\Sigma_n)\to A(X\times E\Sigma_n)\simeq A(X)$. We start by introducing notation for the composition

$$\mathcal{R}_f(X)\to \mathrm{sub}_n\, G_\bullet\mathcal{R}_f(X)\xrightarrow{\omega^n} G^n_\bullet\mathcal{R}_f(X,\Sigma_n,\{\mathrm{all}\})\xrightarrow{Q_0=S_{-1}/S_0} G^n_\bullet\mathcal{R}_f(X,\Sigma_n,\{e\}).$$

On $(Y,r,s)\in\mathcal{R}_f(X)$, the composition of the first two maps in the chain is $\tilde{\alpha}_n(Y)$ in the notation of Example 3.32, so we want to evaluate the functor $Q_0=S_{-1}/S_0\circ\tilde{\alpha}_n$ on the object (Y,r,s). By the terminology used in the proof of Theorem 6.1, S_{-1} is the identity and S_0 is the union of subobjects that are fixed by some nonidentity subgroup of Σ_n. The interpretation and transfer issues are taken up in Section 7; to prepare for the analysis of $\phi_n\circ\theta_n$ in Section 8 we introduce some notation.

The definitions of the Segal operations in [Waldhausen 1982] use certain subfunctors P^n_j of the smash power functor P^n on pointed sets. We extend the considerations to define certain subfunctors of $\wedge_e$ and $\wedge_\mu$ powers. For $(Y,r,s)\in\mathcal{R}(X)$, the set $Y^{\wedge_e n}$ is a quotient of the cartesian product Y^n. In a fixed simplicial dimension, we view this as the set of functions $y:\boldsymbol{n}\to Y$. The pushout construction identifies any such function y with at least one value y_i in X with the composite function $r\circ y$. Thus, to represent points of $Y^{\wedge_e n}$ in a given dimension, we just need to look at functions all of whose values are in $Y-X$ and functions all of whose values are in X. For $0\le j\le n$ we define $\widetilde{P}^n_j Y$ to be the subset of functions $y:i\mapsto y_i$ such that the cardinality of $y^{-1}(Y-X)$ is less than or equal to j, if the image of y is contained in $(Y-X)$. Said another way, $\widetilde{P}^n_j Y$ is the set of n-tuples where at most j distinct elements of $Y-X$ are involved. For example, $\widetilde{P}^n_0 Y = X^n$ and $\widetilde{P}^n_1 Y$ is the union of X^n with the diagonal of $(Y-X)^n$. Most important for us, the subset $\widetilde{P}^n_{n-1} Y$ consists of all n-tuples involving no more than $n-1$ distinct elements of Y, so that if no member of $(y_1,\ldots,y_n)$ is in X, then there are at least two distinct indices i, j with $y_i=y_j$.

When X is a connected abelian group, then we can push out along the iterated multiplication $X^n\to X$, obtaining functors $P^n_j Y$ relative to X. In particular, $P^n_{n-1}Y$ is the subset of P^nY consisting of points fixed by some nontrivial subgroup of Σ_n, so not all members of an n-tuple can be distinct. Thus $P^n_{n-1}Y = S_0\tilde{\alpha}_n(Y)$. In terms of functions $y:\boldsymbol{n}\to Y$, $P^n_{n-1}Y$ is the set of functions where the cardinality of $y^{-1}(Y-X)$ is at most $n-1$.

Definition 6.2. Define $\tilde{\theta}^n Y$ and $\theta^n Y$ by means of the pushout squares

$$\begin{tikzcd} \widetilde{P}^n_{n-1}Y \arrow[r, tail] \arrow[d, "\wedge^n_e r"'] & \widetilde{P}^n Y \arrow[d] \\ X^n \arrow[r, tail] & \tilde{\theta}^n Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} P^n_{n-1}Y \arrow[r, tail] \arrow[d, "r"'] & P^n Y \arrow[d] \\ X \arrow[r, tail] & \theta^n Y \end{tikzcd}$$

Letting $j^n : \mathcal{R}_f(X, \Sigma_n, \{e\}) \to G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{e\})$ be the iterated stabilization, we combine the preceding observations with the definitions to immediately obtain the following proposition.

Proposition 6.3. *As functors from $\mathcal{R}_f(X)$ to $G^n_\bullet \mathcal{R}_f(X, \Sigma_n, \{e\})$,*

$$Q_0 \circ \tilde{\alpha}_n = j^n \circ \theta^n. \qquad \square$$

7. Transfer constructions

Our immediate goal is to interpret $\Omega|hS_\bullet \mathcal{R}_f(X^n, \Sigma_n, \{e\})|$ and $\Omega|hS_\bullet \mathcal{R}_f(X, \Sigma_n, \{e\})|$ in terms of the algebraic K-theory of topological spaces. In this section, families of subgroups play no role, so we revert to the less ornate notation $\Omega|hS_\bullet \mathcal{R}_f(X, G)|$ for $\Omega|hS_\bullet \mathcal{R}_f(X, G, \{e\})|$, the algebraic K-theory of G-spaces retracting to X, finite relative to X, and with G acting freely outside X.

There are two steps to this goal and each step uses arguments based on [Waldhausen 1985]. We let G be a finite group and Z a G-space. Let EG be the canonical contractible free left G-space. We prefer the model $EG_n = G^{n+1}$ with the G-action given by multiplication on the left in each factor, face maps defined by projecting away from a coordinate, and degeneracies defined by repeating a coordinate. An isomorphism of the quotient space $* \times^G EG \cong BG$ is induced by $(g_0, \dots, g_{i-1}, g_i, \dots, g_n) \mapsto (g_0^{-1} g_1, \dots, g_{i-1}^{-1} g_i, \dots, g_{n-1}^{-1} g_n)$.

First, [Waldhausen 1985, Lemma 2.1.3, p. 381] applies to yield the following result.

Lemma 7.1. *There is an equivalence of categories*

$$\mathcal{R}(EG \times^G Z) \sim \mathcal{R}(EG \times Z, G). \qquad \square$$

For reference, pullback along the projection

$$EG \times Z \to EG \times^G Z$$

defines a functor $\mathcal{R}(EG \times^G Z) \to \mathcal{R}(EG \times Z, G)$; the orbit map defines a functor in the opposite direction. The composites in either order are isomorphic to the respective identity functors. Moreover, these functors preserve weak equivalences and homotopy finite objects.

Next, we want the following lemma, which permits us to replace the G-action on Z with a free G-action on a homotopy equivalent space.

Lemma 7.2. *The projection $EG \times Z \to Z$ induces a homotopy equivalence*

$$hS_\bullet \mathcal{R}_{hf}(EG \times Z, G) \longrightarrow hS_\bullet \mathcal{R}_{hf}(Z, G).$$

Proof. The argument here is similar to that given to prove [Waldhausen 1985, Proposition 2.1.4, p. 382]. In detail, let $(Y', r', s') \in \mathcal{R}_{hf}(EG \times Z, G)$. Completing the diagram

$$Y' \xleftarrow{s'} EG \times Z \xrightarrow{p_2} Z$$

to a pushout defines an exact functor $\mathcal{R}_{hf}(EG \times Z, G) \to \mathcal{R}_{hf}(Z, G)$. Certainly, homotopy finite objects are carried to homotopy finite objects, and, incidentally, finite objects are carried to finite objects. Also, weak equivalences are mapped to weak equivalences.

Taking the product with EG gives an exact functor $\mathcal{R}_{hf}(Z, G) \to \mathcal{R}_{hf}(EG \times Z, G)$. In this case, when G is nontrivial, finite objects are carried to homotopy finite objects, since EG is contractible.

For (Y, r, s) in $\mathcal{R}_{hf}(Z, G)$, taking the induced map of pushouts in the diagram

$$\begin{tikzcd}
EG \times Y \arrow[d, "p_2"'] & EG \times Z \arrow[l, tail] \arrow[r] \arrow[d, "p_2"'] & Z \arrow[d, "\mathrm{id}"'] \\
Y & Z \arrow[l, tail, "s"'] \arrow[r, "\mathrm{id}"] & Z
\end{tikzcd}$$

provides a natural transformation from the composite endofunctor on $\mathcal{R}_{hf}(Z, G)$ to the identity functor. This natural transformation is a weak equivalence. For (Y', r', s') in $\mathcal{R}_{hf}(EG \times Z, G)$, taking the induced map of pushouts in the diagram

$$\begin{tikzcd}
Y' \arrow[d, "p_1 r' \times \mathrm{id}"'] & EG \times Z \arrow[l, tail, "s'"'] \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta \times \mathrm{id}"'] & EG \times Z \arrow[d, "\mathrm{id}"'] \\
EG \times Y' & EG \times EG \times Z \arrow[l, tail, "\mathrm{id} \times s'"'] \arrow[r, "p_1 \times p_3"] & EG \times Z
\end{tikzcd}$$

provides a natural transformation from the identity functor on $\mathcal{R}_{hf}(EG \times Z, G)$ to the other composite endofunctor. Again, this is a weak equivalence. By [Waldhausen 1985, Proposition 1.3.1, p. 330], $hS_\bullet\mathcal{R}_{hf}(Z \times EG, G) \to hS_\bullet\mathcal{R}_{hf}(Z, G)$ is a homotopy equivalence. □

Substituting for G the symmetric group Σ_n, we combine Lemmas 7.1 and 7.2 to record useful alternative models for $A(B\Sigma_n \times X)$ and $A(D_n X)$. The first part covers a remark made following Definition 3.5.

Lemma 7.3. *Let X have the trivial Σ_n-action, so that $B\Sigma_n \times X$ is the quotient of $E\Sigma_n \times X$ by the action of Σ_n. There are homotopy equivalences*

$$hS_\bullet\mathcal{R}_{hf}(B\Sigma_n \times X) \simeq hS_\bullet\mathcal{R}_{hf}(E\Sigma_n \times X, \Sigma_n) \simeq hS_\bullet\mathcal{R}_{hf}(X, \Sigma_n). \tag{7.4}$$

Thus, the space $\Omega|hS_\bullet\mathcal{R}_f(X, \Sigma_n)|$ is homotopy equivalent to $A(B\Sigma_n \times X)$.

Similarly, let X^n have the permutation action of Σ_n, and let $D_n X = E\Sigma_n \times^{\Sigma_n} X^n$ be the quotient of $E\Sigma_n \times X^n$ by the diagonal action of Σ_n. There are homotopy equivalences

$$hS_\bullet\mathcal{R}_{hf}(D_n X) \simeq hS_\bullet\mathcal{R}_{hf}(E\Sigma_n \times X^n, \Sigma_n) \simeq hS_\bullet\mathcal{R}_{hf}(X^n, \Sigma_n).$$

Thus, the space $\Omega|hS_\bullet\mathcal{R}_{hf}(X^n, \Sigma_n)|$ is homotopy equivalent to $A(D_n X)$. □

We recall here basic facts about the transfer in the algebraic K-theory of spaces adapted to our context. We are actually interested in two cases of transfer operations. For the first case the transfer operations are associated with finite subgroups of the symmetric groups Σ_n. In the second case the operations are associated with (injective) homomorphisms of simplicial abelian groups $\widetilde{X} \to X$, where the fiber is homotopy finite.

In terms of the description $A(X) = \Omega|hS_\bullet\mathcal{R}_f(X)|$, we have the following direct transfer construction. A fiber bundle projection $p : E \to B$ with finite fiber induces by pullback a functor $\mathcal{R}_f(B) \to \mathcal{R}_f(E)$, or with homotopy finite fiber, $\mathcal{R}_f(B) \to \mathcal{R}_{hf}(E)$. We then obtain a transfer morphism $p^* : A(B) \to A(E)$. In terms of equivariant models for algebraic K-theory, there are other descriptions of the transfer, as given below. We need to relate the various descriptions.

Eventually we need the transfer operations $A(B\Sigma_n \times X) \to A(BH \times X)$, where H is a subgroup of Σ_n. Our working definition is $A(B\Sigma_n \times X) = \Omega|hS_\bullet\mathcal{R}_{hf}(X, \Sigma_n)|$ but, in view of the equivalences (7.4), we have to compare three definitions in each context.

To this end, let G be a discrete group, H a subgroup of finite index, and Z a trivial G-space. Observe that $EG \times Z$ is the total space of a principal G-bundle with base $EG \times^G Z$. To make this transparent, and for use in the study of diagram (7.5), we replace the notation $EG \times^G Z$ by $* \times^G (EG \times Z)$. To explain the connection, $* \times^G (EG \times Z)$ is the orbit space of $EG \times Z$ under the diagonal left G-action, thought of as the balanced product of $EG \times Z$ with the trivial right G-space $*$. We can turn the left action of G on EG into a right action by setting $e \cdot_r g = g^{-1} \cdot_l e$. Then left G-orbits in $EG \times Z$ are seen to correspond to equivalence classes in $EG \times Z$ under the equivalence relation generated by $(e \cdot_r g, z) \sim (e, gz)$. The associated quotient space is usually denoted $EG \times^G Z$.

We consider the diagram

$$\begin{array}{ccccc}
\mathcal{R}(Z, H) & \longrightarrow & \mathcal{R}(EG \times Z, H) & \longleftarrow & \mathcal{R}(EG \times^H Z) \\
\uparrow{\scriptstyle p_1^*} & & \uparrow{\scriptstyle p_2^*} & & \uparrow{\scriptstyle p_3^*} \\
\mathcal{R}(Z, G) & \longrightarrow & \mathcal{R}(EG \times Z, G) & \longleftarrow & \mathcal{R}(EG \times^G Z)
\end{array} \tag{7.5}$$

where the vertical arrows represent transfer constructions. The forgetful functor $p_1^* : \mathcal{R}(Z, G) \to \mathcal{R}(Z, H)$ just restricts the action to the subgroup H. This provides the simplest path to $p_1^* : A(BG \times Z) \to A(BH \times Z)$, using the basic model $A(BG \times Z) = \Omega|hS_\bullet(\mathcal{R}_{hf}(Z, G)|$. In the middle, the functor p_2^* is also a forgetful functor. At the right, the functor p_3^* is given by a pullback construction, explained in detail below.

To reach the categories in the middle column from those in the left column we compute products with EG. Along the top, the fact that EG is a nonstandard contractible H-space is an insignificant detail. Comparing with p_1^* on the left, the transfer p_2^* in the middle column is also obtained by restricting the action of G to H. Thus, the left-hand square in diagram (7.5) obviously commutes.

Before we compare p_3^* with p_2^*, we discuss p_3^*, the rightmost column in diagram (7.5), in detail. In order to manipulate pullback squares efficiently we replace the notation $EG \times^G Z$ by $* \times^G (EG \times Z)$ as discussed before Lemma 7.1. Suppose H is a subgroup of the group G, and let EG be the standard model for a contractible G-space on which G acts freely from the right. The space EG plays a similar role relative to the subgroup H. In order to compare situations, we take the standard model $X = * \times^G (EG \times Z)$ and a modified model $\widetilde{X} = * \times^H (EG \times Z)$. In this situation we have the basic pullback square

$$\begin{tikzcd}
(* \times^H G) \times (EG \times Z) \arrow[r] \arrow[d, "p_2"'] & (* \times^H EG \times Z) = \widetilde{X} \arrow[d] \\
EG \times Z \arrow[r] & * \times^G (EG \times Z) = X
\end{tikzcd} \tag{7.6}$$

This displays the comparison map $\widetilde{X} \to X$ of the chosen models as a fiber bundle, with fiber $* \times^H G$. One may identify $* \times^H (EG \times Z) \cong (* \times^H G) \times^G (EG \times Z)$, and then the right-hand vertical arrow is isomorphic to the map

$$(* \times^H G) \times^G (EG \times Z) \to * \times^G (EG \times Z)$$

induced by projecting the coset space $* \times^H G$ to a point. This replacement also displays the upper horizontal map as the quotient projection

$$(* \times^H G) \times (EG \times Z) \to (* \times^H G) \times^G (Z \times EG).$$

The direct construction $p^* : \mathcal{R}(X) \to \mathcal{R}(\widetilde{X})$ maps (Y, r, s) to $(\widetilde{Y}, \tilde{r}, \tilde{s})$, derived from the pullback square

$$\begin{tikzcd}
\widetilde{Y} \arrow[r, "\tilde{r}"] \arrow[d] & \widetilde{X} = * \times^H (EG \times Z) \arrow[d, "p"'] \\
Y \arrow[r, "r"] & X = * \times^G (EG \times Z)
\end{tikzcd} \tag{7.7}$$

Augmenting the right-hand column of (7.7) to the square of (7.6) shows that $\widetilde{Y} \to Y$ is a fiber bundle with fiber $* \times^H G$.

Now we address commutativity of the right-hand square in diagram (7.5). To reach the categories in the middle column from the categories in the right column, we also compute pullbacks. Recalling Lemma 7.1, the equivalence of categories $\mathcal{R}(EG \times^G Z) \simeq \mathcal{R}(EG \times Z, G)$ [Waldhausen 1985, Lemma 2.1.3] describes the functor moving left to the middle column. This functor assigns to a retractive space

(Y, r, s) over $EG \times^G Z$ the retractive G-space (Y', r', s') over $EG \times Z$ defined as the pullback in the following diagram:

$$\begin{tikzcd} Y' \arrow[r, "r'"] \arrow[d] & EG \times Z \arrow[d] \\ Y \arrow[r, "r"] & X = * \times^G (EG \times Z) \end{tikzcd}$$

Then moving up to $\mathcal{R}(EG \times Z, H)$ amounts to restricting the G-action in this pullback to H.

On the other hand, to move from the lower right to the upper middle by going up and then to the left, compute first the pullback (7.7) and then compute

$$\begin{tikzcd} \widetilde{Y}' \arrow[r, "\tilde{r}'"] \arrow[d] & EG \times Z \arrow[d] \\ \widetilde{Y} \arrow[r, "\tilde{r}"] & \widetilde{X} = * \times^H (EG \times Z). \end{tikzcd}$$

The composition of the two functors may be displayed in the stacked diagram

$$\begin{tikzcd} \widetilde{Y}' \arrow[r, "\tilde{r}'"] \arrow[d] & EG \times Z \arrow[d] \\ \widetilde{Y} \arrow[r, "\tilde{r}"] \arrow[d] & \widetilde{X} = * \times^H (EG \times Z) \arrow[d, "p"'] \\ Y \arrow[r, "r"] & X = * \times^G (EG \times Z) \end{tikzcd}$$

The end result is that $(\widetilde{Y}', \tilde{r}', \tilde{s}')$ is simply the G-space (Y', r', s') with the action restricted to H. Therefore, the right-hand square commutes.

Lemma 7.8 (cf. [Waldhausen 1982, Lemma 1.3, p. 399]). *Let G be a finite group, EG a universal G-bundle, $BG = * \times^G EG$ a classifying space, and Z a space with a trivial G-action. Then the composition*

$$A(Z) \xrightarrow{\textit{inclusion}} A(BG \times Z) \xrightarrow{\textit{transfer}} A(EG \times Z) \simeq A(Z)$$

is given by multiplication by the order of G, in the sense of the additive H-space structure. □

8. A fundamental cofibration sequence

Waldhausen's main result is this proposition.

Proposition 8.1 (cf. [Waldhausen 1982, Proposition 2.7, p. 407]). *The composition of the operation $\theta^n : A(*) \to A(B\Sigma_n \times *)$ with the transfer map $\phi_n : A(B\Sigma_n \times *) \to A(*)$*

is the same, up to weak homotopy, as the polynomial map on $A(*)$ given by the polynomial
$$p(x) = x(x-1)\cdots(x-n+1).\qquad\square$$

The analogous result for the present situation with the one-point space replaced by a simplicial abelian group X is more complicated to formulate and to work with. To prepare for the analogue of Waldhausen's result, we develop the following constructions, taking up where we left off with Definition 6.2 and Proposition 6.3. We make use of the maps

$$\delta^{n,k}_{n-1} : X^{n-1} \to X^n, \quad (x_1,\ldots,x_{n-1}) \mapsto (x_1,\ldots,x_{n-1},x_k),$$

and the respective induced functors $\delta^{n,k}_{n-1*} : \mathcal{R}_f(X^{n-1}) \to \mathcal{R}_f(X^n)$. The pushout construction

$$\begin{tikzcd} X^{n-1} \arrow[r, tail, "s"] \arrow[d, "\delta^{n,k}_{n-1}"'] & Z \arrow[d, "i^{n,k}_{n-1}"] \\ X^n \arrow[r, tail] & \delta^{n,k}_{n-1*}Z \end{tikzcd}$$

defines an exact functor $\delta^{n,k}_{n-1*} : \mathcal{R}_f(X^{n-1}) \to \mathcal{R}_f(X^n)$. For a retractive space (Z,r,s) over X^{n-1} with retraction $r : Z \to X^{n-1}$ written in terms of components as $r = (r_1,\ldots,r_{n-1})$, the composition of the canonical map $i^{n,k}_{n-1}$ followed by the retraction $\delta^{n,k}_{n-1*}r$ is given by the formula

$$(\delta^{n,k}_{n-1*}r)\circ i^{n,k}_{n-1}(z) = \delta^{n,k}_{n-1}\circ r(z) = (r_1(z),\ldots,r_k(z),\ldots,r_{n-1}(z),r_k(z)).$$

Note that in the special case $Z = \widetilde{P}^{n-1}Y = (\wedge_e)^{n-1}Y$, we have, for each k such that $1 \le k \le n-1$,

$$\big(\delta^{n,k}_{n-1*}(\widetilde{P}^{n-1}r)\big)\circ i^{n,k}_{n-1}(y_1,\ldots,y_{n-1}) = (r(y_1),\ldots,r(y_k),\ldots,r(y_{n-1}),r(y_k)).\quad (8.2)$$

Next we assemble these functors by gluing along the common space X^n, obtaining

$$\widetilde{\Delta}^n_{n-1} : \mathcal{R}_f(X^{n-1}) \to \mathcal{R}_f(X^n)$$

given on objects by $\widetilde{\Delta}^n_{n-1}(Z) = \delta^{n,1}_{n-1*}Z \cup_{X^n} \cdots \cup_{X^n} \delta^{n,n-1}_{n-1*}Z$, which can be viewed as an iterated pushout or as the colimit of a diagram modeled on the cone on $n-1$ points. We also need to push this construction forward to $\mathcal{R}_f(X)$ by μ_*, the iterated multiplication, obtaining

$$\Delta^n_{n-1} = \mu_* \circ \widetilde{\Delta}^n_{n-1} : \mathcal{R}_f(X^{n-1}) \to \mathcal{R}_f(X)$$

given on objects by $\Delta^n_{n-1}(Z) = \mu_*(\delta^{n,1}_{n-1*}Z) \cup_X \ldots \cup_X \mu_*(\delta^{n,n-1}_{n-1*}Z)$. If we start with

$$Z = Y \wedge_e \overset{n\,-\,1\text{ factors}}{\cdots} \wedge_e Y = \widetilde{P}^{n-1}Y,$$

then the formula for the retraction on the k-th summand $\mu_*(\delta^{n,k}_{n-1*}\widetilde{P}^{n-1}Y)$ is

$$\begin{aligned}\big(\mu_*\delta^{n,k}_{n-1*}(\widetilde{P}^{n-1}r)\big)\circ i^{n,k}_{n-1}&(y_1,\ldots,y_{n-1})\\ &= \mu(r(y_1),\ldots,r(y_k),\ldots,r(y_{n-1}),r(y_k)),\end{aligned} \tag{8.3}$$

where μ is the iterated multiplication.

We can now succinctly state our general results. Let

$$\tilde{\phi}_k : \mathcal{R}_f(X^k,\Sigma_k,\{\text{all}\}) \to \mathcal{R}_f(X^k) \quad \text{and} \quad \phi_k : \mathcal{R}_f(X,\Sigma_k,\{\text{all}\}) \to \mathcal{R}_f(X)$$

be the functors that forget the group action.

Proposition 8.4 (cf. [Waldhausen 1982, Proposition 2.7, p. 407]). *There is a cofibration sequence of functors $\mathcal{R}_f(X) \to \mathcal{R}_f(X^n)$*

$$\widetilde{\Delta}^n_{n-1}\tilde{\phi}_{n-1}\tilde{\theta}^{n-1}Y \rightarrowtail \tilde{\phi}_{n-1}\tilde{\theta}^{n-1}Y \wedge_e \tilde{\theta}^1 Y \twoheadrightarrow \tilde{\phi}_n\tilde{\theta}^n Y. \tag{8.5}$$

In the case that X is a connected simplicial abelian group, we have the cofibration sequence

$$\Delta^n_{n-1}\tilde{\phi}_{n-1}\tilde{\theta}^{n-1}Y \rightarrowtail \phi_{n-1}\theta^{n-1}Y \wedge_\mu \theta^1 Y \twoheadrightarrow \phi_n\theta^n Y \tag{8.6}$$

of functors $\mathcal{R}_f(X) \to \mathcal{R}_f(X)$.

Remark 8.7. The second cofibration sequence is obtained by applying the exact functor induced by the iterated multiplication $\mu : X^n \to X$ to the first sequence. The result in the middle term of the second sequence is open to interpretation. The formulation chosen amounts to interpretation of the factorization $\mu = \mu\circ(\mu\times \mathrm{id})$ along with the facts that $\mu_*\circ\wedge_e = \wedge_\mu$ and $\tilde{\theta}^1 Y = \theta^1 Y = Y$.

Proof of Proposition 8.4. Following Section 7, we interpret the transfer maps

$$\phi_n : A(D_n X) \to A(X^n) \quad \text{and} \quad \phi_n : A(X\times B\Sigma_n) \to A(X)$$

as induced by the forgetful functors

$$\mathcal{R}_f(X^n,\Sigma_n,\{\text{all}\}) \to \mathcal{R}_f(X^n,\{e\}) \quad \text{and} \quad \mathcal{R}_f(X,\Sigma_n,\{\text{all}\}) \to \mathcal{R}_f(X,\{e\}),$$

respectively. This means we have to make nonequivariant analyses of the functors $\tilde{\theta}^n$ and θ^n, respectively.

To obtain the surjections, we consider the diagram

$$\begin{tikzcd} X \arrow[d] & X^{n-1}\times X \arrow[l, "\mu"'] \arrow[d] & \widetilde{P}^{n-1}_{n-2}Y\wedge_e Y \arrow[l, "((\wedge_e)^{n-1}r)\wedge_e r"'] \arrow[d] \arrow[r, tail] & \widetilde{P}^{n-1}Y\wedge_e Y \arrow[d, "\cong"] \\ X & X^n \arrow[l, "\mu"'] & \widetilde{P}^n_{n-1}Y \arrow[l, "r^n"'] \arrow[r, tail] & \widetilde{P}^n Y \end{tikzcd} \tag{8.8}$$

Clearly, $\widetilde{P}^{n-1}_{n-2} \wedge_e Y$ maps into $\widetilde{P}^n_{n-1}$, because, if there are two indices i, j with $1 \le i, j \le n-1$ and $i \ne j$ and with $y_i = y_j$, then this still holds for $((y_1, \ldots, y_{n-1}), y)$ rebracketed as $(y_1, \ldots, y_{n-1}, y)$. Taking the pushouts along the rows using the columns two, three, and four produces a surjection

$$\phi_{n-1}\tilde{\theta}^{n-1}Y \wedge_e \tilde{\theta}^1 Y \twoheadrightarrow \phi_n\tilde{\theta}^n Y$$

in $\mathcal{R}_f(X^n)$ and pushing out along the rows using columns one, three and four yields

$$\phi_{n-1}\theta^{n-1}Y \wedge_\mu \theta^1 Y = \mu_*(\phi_{n-1}\theta^{n-1}Y \wedge_e \theta^1 Y) \twoheadrightarrow \phi_n\theta^n Y,$$

the surjection in $\mathcal{R}_f(X)$. Now we have to identify the "kernels".

Reviewing the remarks at the end of Section 6, $\widetilde{P}^{n-1}Y \wedge_e Y = \widetilde{P}^n Y$ is the space whose simplices outside of X^n are n-tuples of simplices from $Y - X$; $\widetilde{P}^{n-1}_{n-2}Y \wedge_e Y$ is the space whose simplices outside of X^n are n-tuples $((y_1, \ldots, y_{n-1}), y)$ with the condition that there are at least two distinct indices $1 \le i, j \le n-1$ with $y_i = y_j$; and $\widetilde{P}^n_{n-1}Y$ is the space whose simplices outside of X^n are n-tuples $(y_1, \ldots, y_{n-1}, y_n)$ with the condition that there are at least two distinct indices $1 \le i, j \le n$ with $y_i = y_j$. Then the simplices of $\widetilde{P}^n_{n-1}Y$ not in the image of $\widetilde{P}^{n-1}_{n-2}Y \wedge_e Y$ are those n-tuples where the first $n-1$ are distinct but $y_n = y_k$ for some $1 \le k \le n-1$.

Using this observation we extend the diagram (8.8) by means of the following constructions. For $1 \le k \le n-1$, consider the diagrams

$$\begin{tikzcd}
X^n \arrow[d] & X^{n-1} \arrow[l, "\delta^{n,k}_{n-1}"'] \arrow[r] \arrow[d, "\delta^{n,k}_{n-1}"'] & \widetilde{P}^{n-1}Y \arrow[d, "\delta^{n,k}_{n-1}"] \\
X^n & \widetilde{P}^{n-1}Y \times X \cup_{X^n} X^{n-1} \times Y \arrow[l] \arrow[r] & \widetilde{P}^{n-1}Y \times Y
\end{tikzcd}$$

where $\delta^{n,k}_{n-1} : X^{n-1} \to X^n$ is given by $\delta^{n,k}_{n-1}(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, x_k)$ and the other maps labeled $\delta^{n,k}_{n-1}$ are given by similar formulas. For each k, taking the pushout of the first row extends $\widetilde{P}^{n-1}Y$ over X^{n-1} to the space $\delta^{n,k}_{n-1*}\widetilde{P}^{n-1}Y$ over X^n; taking the pushout of the second row yields $\widetilde{P}^{n-1}Y \wedge_e Y$. Since the diagram commutes, we obtain a family of maps over X^n

$$\delta^{n,k}_{n-1} : \delta^{n,k}_{n-1*}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y \to \tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y \wedge_e Y$$

with $\delta^{n,k}_{n-1}(y_1, \ldots, y_{n-1}) = (y_1, \ldots, y_k, \ldots, y_{n-1}, y_k)$.

Now we are ready to augment diagram (8.8), after which we can compute the desired cofibration sequence. Having established the notation

$$\widetilde{\Delta}^n_{n-1}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y = \delta^{n,1}_{n-1*}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y \cup_{X^n} \cdots \cup_{X^n} \delta^{n,n-1}_{n-1*}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y,$$

write $\Delta^n_{n-1} : \Delta^n_{n-1}\phi_{n-1}\widetilde{P}^{n-1}Y \to \phi_{n-1}\widetilde{P}^{n-1}Y \wedge_e Y$ for the union of the maps $\delta^{n,k}_{n-1}$ just defined. Add this map above the upper right corner of (8.8) and fill out the following diagram:

$$
\begin{tikzcd}
X \arrow[d] & X^n \arrow[l, "\mu"'] \arrow[d, "\cong"'] & \Delta^n_{n-1}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}_{n-2}Y \arrow[l] \arrow[r, tail, "i'"] \arrow[d, tail, "\Delta^n_{n-1}"'] & \Delta^n_{n-1}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y \arrow[d, tail, "\Delta^n_{n-1}"'] \\
X \arrow[d] & X^{n-1}\times X \arrow[l, "\mu"'] \arrow[d] & \tilde{\phi}_{n-1}\widetilde{P}^{n-1}_{n-2}Y \wedge_e Y \arrow[l, "r^{n-1}\wedge_e r"'] \arrow[r, tail, "i"] \arrow[d] & \tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y \wedge_e Y \arrow[d, "\cong"] \\
X & X^n \arrow[l, "\mu"'] & \widetilde{P}^n_{n-1}Y \arrow[l, "r^n"'] \arrow[r, tail, "i''"] & \widetilde{P}^n Y
\end{tikzcd}
\tag{8.9}
$$

To explain the entry at the top of the third column, we identify the conditions on

$$(z_1, \ldots, z_n) \in (\tilde{\phi}_{n-1}\widetilde{P}^{n-1}_{n-2}Y) \wedge_e Y \quad \text{and} \quad (y_1, \ldots, y_{n-1}) \in \Delta^n_{n-1}\tilde{\phi}_{n-1}\widetilde{P}^{n-1}Y$$

such that $i(z_1, \ldots, z_n) = \Delta^n_{n-1}(y_1, \ldots, y_{n-1})$. We see that $z_j = y_j$ for $1 \le j \le n-1$ and that there is k between 1 and $n-1$ such that $z_n = y_k$. Moreover, since no more than $n-2$ of the first $n-1$ simplices z_j are distinct, no more than $n-2$ of the simplices y_j are distinct. Hence, we obtain the description of the term at the top of the third column. Additionally we obtain the fact that the induced map

$$\big(\phi_{n-1}\widetilde{P}^{n-1}_{n-2}Y \wedge_e Y\big) \cup_{(\Delta^n_{n-1}\phi_{n-1}\widetilde{P}^{n-1}_{n-2}Y)} \big(\Delta^n_{n-1}\phi_{n-1}\widetilde{P}^{n-1}Y\big) \rightarrowtail \phi_{n-1}\widetilde{P}^{n-1}Y \wedge_e Y$$

is a cofibration, so Lemma 2.9 applies to diagram (8.9).

One takes the row-wise pushout of the three columns on the right and obtains the cofibration sequence in $\mathcal{R}_f(X^n)$

$$\widetilde{\Delta}^n_{n-1}\tilde{\phi}_{n-1}\widetilde{\theta}^{n-1}Y \xrightarrow{\widetilde{\Delta}^n_{n-1}} \tilde{\phi}_{n-1}\widetilde{\theta}^{n-1}Y \wedge_e \theta^1 Y \twoheadrightarrow \tilde{\phi}_n\widetilde{\theta}^n Y,$$

which is (8.5) from the statement.

One also composes the arrows pointing to the left in each row and takes the row-wise pushout of the resulting diagram, which consists of columns one, three, and four of the diagram (8.9), obtaining

$$\Delta^n_{n-1}\tilde{\phi}_{n-1}\tilde{\theta}^{n-1}Y \xrightarrow{\mu_*\widetilde{\Delta}^n_{n-1}} \phi_{n-1}\theta^{n-1}Y \wedge_\mu \theta^1 Y \twoheadrightarrow \phi_n\theta^n Y,$$

which is the second cofibration sequence (8.6) in the statement. □

We want to apply the cofibration sequence (8.6) to evaluate the composite $\phi_n\theta^n$ on a homotopy class in $\pi_j A(X)$, where the basepoint is taken in the zero component. Two features of algebraic K-theory make this possible. The first feature is essentially a consequence of the additivity theorem and says that cofibration sequences imply additive relations.

Lemma 8.10. *Let Z be a space. The two composite maps*

$$|hS_2\mathcal{R}_f(Z)| \underset{s\vee q}{\overset{t}{\rightrightarrows}} |h\mathcal{R}_f(Z)| \longrightarrow \Omega|hS_\bullet\mathcal{R}(Z)|$$

are homotopic, where the right-hand arrow is the canonical map

$$|h\mathcal{R}_f(Z)| \to \Omega|hS_\bullet\mathcal{R}_f(Z)|. \qquad \square$$

The second feature is the triviality of products in higher homotopy groups, explained as follows. Since X is a simplicial abelian group, the homotopy functor $Y \mapsto [Y, A(X)]$ has a ring structure induced from the biexact pairing

$$\mathcal{R}(X) \times \mathcal{R}(X) \xrightarrow{\wedge_e} \mathcal{R}(X \times X) \xrightarrow{\mu_*} \mathcal{R}(X).$$

Now suppose $Y = \Sigma Y'$ is a suspension. Under this ring structure the product of two elements $[f_1]$ and $[f_2]$ in $[Y, A(X)]$ is zero, because $[f_1]$ may be represented by a map taking the upper cone C_+Y' in $\Sigma Y'$ to the point in $A(X)$ represented by the zero element in $\mathcal{R}_f(X)$, while $[f_2]$ is represented by a map taking the lower cone C_-Y' in $\Sigma Y'$ to the zero element. In a similar manner, there are pairings

$$\mathcal{R}(X^{n-1}) \times \mathcal{R}(X) \xrightarrow{\wedge_e} \mathcal{R}(X^{n-1} \times X) = \mathcal{R}(X^n)$$

and these are also zero on higher homotopy groups. Combining these observations means we have a chance to compute by induction the action of $\phi_n\theta^n$ on higher homotopy groups, because at each stage of the induction the middle term of the relevant cofibration contributes nothing to the final answer.

To start the induction, we compute $(\phi_2\theta^2)_*[f]$ for $f : S^j \to A(X)$. Applying the additivity theorem to the cofibration sequence (8.6), we can write

$$(\phi_2\theta^2)_*[f] = (\theta^1_*[f] \wedge_\mu \theta^1_*[f]) - (\Delta^2_1\theta^1)_*[f].$$

For the first term on the right side of the equation, we have observed that this product is zero. So we first obtain

$$(\phi_2\theta^2)_*[f] = -(\Delta^2_1\theta^1)_*[f]. \tag{8.11}$$

We analyze this expression as follows. First, ϕ_1 and θ^1 are identity functors. For $n = 2$, there is one diagonal map $\delta^{2,1}_1 : Z \to \delta^{2,1}_1 Z$, so

$$\widetilde{\Delta}^2_1\phi_1\theta^1 Y = \delta^{2,1}_{1*}\phi_1\theta^1(Y) = \delta^{2,1}_{1*}Y.$$

Then $\Delta^2_1\phi_1\theta^1 = \mu_* \circ \widetilde{\Delta}^2_1\phi_1\theta^1 = \mu_* \circ \delta^{2,1}_{1*}$, and the point is to see what is happening with the retraction $r : Y \to X$. Applying formula (8.3), the composition

$$\mu \circ (\widetilde{\Delta}^2_1 r) \circ i^{2,1}_1(y) = \mu(r(y), r(y)) = (r(y))^2 = (\tau_2 \circ r)(y),$$

where $\tau^2 : X \to X$ is the squaring homomorphism. That is, the action of $\Delta_1^2 = \mu_* \widetilde{\Delta}_1^2$ on homotopy is the same as the action on homotopy induced by the squaring homomorphism τ^2. Consequently,

$$(\phi_2\theta^2)_*[f] = -\tau_*^2[f].$$

The general result is the next theorem.

Theorem 8.12. *Let $\tau^n : X \to X$ be the homomorphism that raises elements to the n-th power, thinking of the operation in X as multiplication. Then*

$$\phi_n\theta_*^n = (-1)^{n-1} \cdot (n-1)! \cdot \tau_*^n : \pi_j A(X) \to \pi_j A(X)$$

for $j > 0$.

Proof. First we observe that on higher homotopy groups,

$$(\phi_n\theta^n)_* = (-1)^{n-1} \cdot (\Delta_{n-1}^n \widetilde{\Delta}_{n-2}^{n-1} \cdots \widetilde{\Delta}_1^2)_*.$$

An application of the cofibration sequence (8.6) and the vanishing product principle gives $(\phi_n\theta^n)_* = (-1) \cdot (\Delta_{n-1}^n \tilde{\phi}_{n-1} \tilde{\theta}^{n-1})_*$. Then one continues, with applications of the cofibration sequence (8.5) and the vanishing pairing principle,

$$\begin{aligned}(\phi_n\theta^n)_* &= (-1)^2 \cdot \big(\Delta_{n-1}^n \widetilde{\Delta}_{n-2}^{n-1} \tilde{\phi}_{n-2} \tilde{\theta}^{n-2}\big)_* = \cdots \\ &= (-1)^{n-1} \cdot \big(\Delta_{n-1}^n \widetilde{\Delta}_{n-2}^{n-1} \cdots \widetilde{\Delta}_1^2\big)_* = (-1)^{n-1} \cdot \big(\mu_* \widetilde{\Delta}_{n-1}^n \widetilde{\Delta}_{n-2}^{n-1} \cdots \widetilde{\Delta}_1^2\big)_*,\end{aligned}$$

recalling that $\tilde{\phi}_1$ and $\tilde{\theta}^1$ are identity functors.

Since the functors $\widetilde{\Delta}_{p-1}^p$ are built by unions from functors $\tilde{\delta}_{p-1*}^{p,k}$, we have to analyze composites

$$\delta_{n-1*}^{n,k_{n-1}} \circ \delta_{n-2*}^{n-1,k_{n-2}} \circ \cdots \circ \delta_{1*}^{2,1} : \mathcal{R}_f(X) \to \mathcal{R}_f(X^n)$$

for all choices of indices $1 \le k_{n-1} \le n-1$, $1 \le k_{n-2} \le n-2$, …, $1 \le k_2 \le 2$. On (Y, r, s) the value of the chain is $(Y \cup_X X^n, r^n, s)$, where the retraction $r^n : Y \to X^n$ is evaluated by repeated application of formula (8.2). When we apply μ_* to this object, the value on (Y, r, s) is seen to be $(Y, \tau^n \circ r, s)$. Finally, we identify the numerical coefficient $(n-1)!$ by counting the number of terms in the composites $\widetilde{\Delta}_{n-1}^n \widetilde{\Delta}_{n-2}^{n-1} \cdots \widetilde{\Delta}_1^2$ according to the description above. □

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On the Farrell–Jones conjecture for algebraic K-theory of spaces: the Farrell–Hsiang method

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We prove the Farrell–Jones conjecture for algebraic K-theory of spaces for virtually poly-$\mathbb{Z}$-groups. For this, we transfer the "Farrell–Hsiang method" from the linear case to categories of equivariant, controlled retractive spaces.

1. Introduction

The classification of high-dimensional manifolds and the understanding of their automorphism groups is a long-standing question in algebraic topology. The former turned out to be intimately related to the algebraic K- and L-theory of group rings [Wall 1999], while the latter has a deep connection to pseudo-isotopy theory and Waldhausen's algebraic K-theory of spaces [Waldhausen et al. 2013; Weiss and Williams 2014].

Since the 1970s, a lot of progress has been made to calculate the algebraic K- and L-theory of group rings. This culminated in what is now called the *Farrell–Jones conjecture*, first stated in [Farrell and Jones 1993]. For algebraic K-theory, it

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predicts that the algebraic K-theory of $R[G]$ can be computed from the algebraic K-theory of a certain set of subgroups by a homological recipe. The Farrell–Jones conjectures for algebraic K- and L-theory of group rings have been a highly active research area over the last 20 years. Though many cases of the conjectures are known by now, they remain open in general.

In the language of [Davis and Lück 1998], the conjecture takes the following form: Let $\mathbb{F}$ be a homotopy invariant functor from spaces to spectra. Let X be a connected CW-complex with fundamental group G. Then we obtain an induced functor $\mathbb{F}_X$ from the orbit category $\mathrm{Or}(G)$ to the category of spectra which sends G/H to $\mathbb{F}(\widetilde{X} \times_G G/H)$. By the methods of [Davis and Lück 1998], $\mathbb{F}_X$ gives rise to a G-homology theory $H_*^G(-;\mathbb{F}_X)$. The *Farrell–Jones conjecture* for $\mathbb{F}_X$ predicts that the *assembly map*

$$H_n^G(E_{\mathcal{VC}yc}G;\mathbb{F}_X) \to H_n^G(G/G;\mathbb{F}_X) \cong \pi_n\mathbb{F}(X), \tag{1.1}$$

which is induced by the projection map $E_{\mathcal{VC}yc}G \to G/G$ from the classifying space for virtually cyclic subgroups to a point, is an isomorphism for all $n \in \mathbb{Z}$. If we choose $\mathbb{F}$ to be nonconnective algebraic K-theory $\mathbb{K}^{-\infty}(\mathbb{Z}[\pi_1(-)])$ or Ranicki's ultimate lower L-theory $\mathbb{L}^{-\infty}(\mathbb{Z}[\pi_1(-)])$, we obtain the K- or L-theoretic Farrell–Jones conjecture for group rings, respectively.

In this article, we consider the case that $\mathbb{F}(-) = \mathbb{A}^{-\infty}(-)$, a nonconnective delooping of Waldhausen's algebraic K-theory of spaces. We obtain the following result:

1.2. Theorem. *Let X be a connected CW-complex with fundamental group G. If G is a virtually poly-$\mathbb{Z}$-group, then the assembly map*

$$H_n^G(E_{\mathcal{VC}yc}G;\mathbb{A}_X^{-\infty}) \to \pi_n\mathbb{A}^{-\infty}(X)$$

is an isomorphism for all $n \in \mathbb{Z}$.

In addition to the algebraic K- and L-theory of group rings, Farrell and Jones also stated conjecture (1.1) for pseudo-isotopy. They went on to prove the pseudo-isotopy version of the conjecture for spaces whose fundamental group is a cocompact lattice in an almost connected Lie group, assuming that it holds for spaces whose fundamental group is virtually poly-$\mathbb{Z}$. However, the announced proof of this special case was never published; see [Bartels et al. 2014a, Remark 7.1]. Since the isomorphism conjecture in A-theory is in fact equivalent to the (topological, PL and smooth) pseudo-isotopy version [Enkelmann et al. 2018, Theorem 3.2], the present article closes this gap in the published literature.

We also give a description of the A-theory of spaces with finite fundamental group, which is similar to Lemma 4.1 of [Bartels and Lück 2007]. Call a finite group D a *Dress group* if there are primes p and q and a normal series $P \trianglelefteq C \trianglelefteq D$ such that P is a p-group, C/P is cyclic and D/C is a q-group.

1.3. Theorem. *Let X be a connected CW-complex with finite fundamental group G. Let $\mathcal{D}$ denote the family of Dress subgroups of G. Then the assembly map*

$$H_n^G(E_{\mathcal{D}}G; \mathbb{A}_X^{-\infty}) \to \pi_n \mathbb{A}^{-\infty}(X)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Theorems 1.2 and 1.3 are not the first results of this type. For the algebraic K-theory and L-theory of group rings, the last decade has seen dramatic progress on the Farrell–Jones conjecture. To name some important results, the conjecture has been shown to hold for word-hyperbolic groups [Bartels et al. 2008b], CAT(0)-groups [Bartels and Lück 2012a; Wegner 2012], lattices in almost connected Lie groups [Bartels et al. 2014a; Kammeyer et al. 2016], subgroups of $\mathrm{GL}_n(\mathbb{Z})$ [Bartels et al. 2014b] and $\mathrm{GL}_n(\mathbb{Q})$ as well as $\mathrm{GL}_n(F(t))$ for any finite field F [Rüping 2016], solvable groups [Wegner 2015], and mapping class groups [Bartels and Bestvina 2019].

The proofs of these results make heavy use of a set of ideas known as "controlled algebra", which go back to work of Connell and Hollingsworth [1969] and Quinn [1979]. It was shown in [Bartels et al. 2004] that the methods of controlled algebra can be used to produce explicit models for the (equivariant) assembly map $H_n^G(E_{\mathcal{VC}yc}G; \mathbb{K}_R^{-\infty}) \to K_n(R[G])$. Precursors of this model appeared for example in [Pedersen and Weibel 1989] and [Anderson et al. 1994]. All recent proofs of the Farrell–Jones conjecture use this setup, and rely on at least one of two sufficient criteria to prove the conjecture: the notions of *transfer reducibility* and *being a Farrell–Hsiang group* (see [Bartels 2016]).

For the algebraic K-theory of spaces, known as A-theory, Vogell used the ideas of controlled algebra in the setting of retractive spaces to describe an A-theory assembly map [Vogell 1990; Carlsson et al. 1998]. These models were recast in [Weiss 2002] to repair some problems with the original approach.

In this article, we promote Weiss' categories of controlled retractive spaces to the equivariant setting (even though our notions of weak equivalence are closer to those of Carlsson–Pedersen–Vogell). We give a self-contained discussion of the categories of equivariant, controlled retractive spaces. We prove a number of theorems modeled after those of [Bartels et al. 2004], and produce a model for the equivariant A-theory assembly map. One particular feature of our treatment lies in the fact that we can reuse a considerable amount of results from [Bartels et al. 2004] and subsequent work.

One obtains a category (of controlled retractive spaces) whose K-theory vanishes if and only if the assembly map for G is an isomorphism. In the linear case, proofs of the Farrell–Jones conjecture proceed by using the notions of transfer reducibility or Farrell–Hsiang group to show that the K-theory of a similar "obstruction category" is trivial. We adapt the "Farrell–Hsiang method" to our setting:

1.4. Definition. Let G be a group and S be a finite, symmetric generating set of G. Let $\mathcal{F}$ be a family of subgroups of G.

Call (G, S) a *Dress–Farrell–Hsiang group with respect to* $\mathcal{F}$ if there exists $N \in \mathbb{N}$ such that for every $\varepsilon > 0$ there is an epimorphism $\pi : G \twoheadrightarrow F$ onto a finite group F such that the following holds: for every Dress group $D \leqslant F$, there are a $\overline{D} := \pi^{-1}(D)$-simplicial complex E_D of dimension at most N whose isotropy groups lie in $\mathcal{F}$, and a $\overline{D}$-equivariant map $\varphi_D : G \to E_D$ such that

$$d^{\ell^1}(\varphi_D(g), \varphi_D(g')) \leqslant \varepsilon$$

whenever $g^{-1}g' \in S$.

1.5. Theorem. *Let X be a connected CW-complex with fundamental group G. Let $\mathcal{F}$ be a family of subgroups of G. If G is a Dress–Farrell–Hsiang group with respect to $\mathcal{F}$, then the assembly map*

$$H_n^G(E_{\mathcal{F}}G; \mathbb{A}_X^{-\infty}) \to \pi_n \mathbb{A}^{-\infty}(X)$$

is an isomorphism for all $n \in \mathbb{Z}$.

Theorem 1.3 follows immediately from this result. Theorem 1.2 is deduced in Section 11 following the strategy of [Bartels et al. 2014a], using previous results from [Winges 2015] that all relevant instances of Farrell–Hsiang groups are actually Dress–Farrell–Hsiang.

Using the framework we develop here, the proof of the Farrell–Jones conjecture for transfer reducible groups can also be adapted to the A-theory setting; see [Enkelmann et al. 2018].

Structure of the article. Let us outline the structure of this article.

In the first half, we set up the technical background for our constructions. In Section 2, we define the notion of a coarse structure and explain how a coarse structure $\mathfrak{Z}$ gives rise to the notion of a *controlled space* relative to a base space W. In Section 3, we use these notions to construct the category of *controlled retractive spaces* $\mathcal{R}(W, \mathfrak{Z})$ relative to a base space W. Every subspace A of $\mathfrak{Z}$ gives rise to a class of weak equivalences h^A on $\mathcal{R}(W, \mathfrak{Z})$; if A is empty, this gives a notion of homotopy equivalence. We show that the category $\mathcal{R}(W, \mathfrak{Z})$, together with the weak equivalences h^A, is a Waldhausen category which has a cylinder functor and satisfies the saturation axiom and the cylinder axiom. As usual, we need some finiteness condition to make algebraic K-theory nontrivial, so we define subcategories of finite, homotopy finite and finitely dominated objects. In fact, we work G-equivariantly and obtain in particular a Waldhausen category of finite, G-equivariant controlled retractive spaces $\mathcal{R}_f^G(W, \mathfrak{Z})$.

In Section 4, we compare the different finiteness conditions and show that the resulting (connective) algebraic K-theory differs at most in degree 0. We show that we have a version of Waldhausen's fibration theorem which applies in our situation, even though the h^A-equivalences do not satisfy the extension axiom. We call this the modified fibration theorem and prove it as Proposition 4.14. (Such a statement was already used in [Weiss 2002].) We use this to construct homotopy fiber sequences which compare homotopy equivalences and h^A-equivalences and show an excision result, the "coarse Mayer–Vietoris theorem" in Theorem 4.23. These results still have a certain "defect" in degree 0, which is corrected in the next section. The section concludes with a criterion for the vanishing of the algebraic K-theory of the categories $\mathcal{R}^G_f(W, \mathfrak{Z})$.

In Section 5, we define a delooping of the algebraic K-theory space $K(\mathcal{R}^G_f(W, \mathfrak{Z}))$ to obtain the *nonconnective algebraic K-theory spectrum* $\mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z}))$. We establish nonconnective versions of the homotopy fiber sequences and "coarse Mayer–Vietoris theorem" from the previous section. In particular, this repairs the "defect" in degree 0 of the connective case.

The second half of the article discusses the Farrell–Jones conjecture for A-theory. Section 6 constructs a model for the assembly map. As in the linear case, there exists for any G-CW-complex X a coarse structure $\mathbb{J}(X)$ which, together with a certain class of weak equivalences h^∞, makes

$$\mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathbb{J}(-)), h^\infty)$$

into a G-homology theory. If W is a free G-CW-complex, we identify its coefficients with $\mathbb{A}^{-\infty}(H\backslash W)$. Here, $\mathbb{A}^{-\infty}(V)$ is a nonconnective delooping of Waldhausen's algebraic K-theory of spaces $A(V)$, which we define using the results of Section 5. Applying the G-homology theory to the map $E_{\mathcal{F}}G \to G/G$ gives the assembly map. We conclude with a criterion when this assembly map is a weak equivalence.

In Section 7, we recall the fibered isomorphism conjecture for A-theory. We define the notion of a Dress–Farrell–Hsiang group with respect to a family $\mathcal{F}$. Theorem 7.4 states that the fibered isomorphism conjecture is true for this class of groups. Imitating [Bartels and Lück 2012b], we show how the theorem follows once we know Corollary 9.6 and Theorem 10.1. Theorem 1.5 is a special case of Theorem 7.4.

In Section 8, we introduce the *A-theoretic Swan group* and show that it acts on the K-theory of the categories $\mathcal{R}^G_f(W, \mathbb{J}(X))$. We prove an analog of Swan's induction theorem as Theorem 8.7. This is used to construct a "transfer map" in Section 9. Section 10 contains a proof of the "squeezing theorem", Theorem 10.1.

Section 11 is devoted to applications. We prove Theorem 1.3 and proceed to show Theorem 1.2 following the strategy of [Bartels et al. 2014a]. We state the "fibered Farrell–Jones conjecture with wreath products in A-theory", establish the usual inheritance properties and generalize Theorem 7.4 to cover this case as well.

We conclude with the proof that virtually poly-$\mathbb{Z}$-groups satisfy the fibered Farrell–Jones conjecture with wreath products in A-theory.

2. Controlled equivariant CW-complexes

Throughout this article, G denotes a discrete group and W denotes a G-space.

2.1. Definition. Let Z be a G-space which is Hausdorff. A set of *morphism control conditions* $\mathfrak{C}$ is a collection of G-invariant subsets of $Z \times Z$ with the following properties:

(C1) Every $C \in \mathfrak{C}$ contains the diagonal $\Delta(Z) := \{(z, z) \mid z \in Z\}$.

(C2) Every $C \in \mathfrak{C}$ is symmetric.

(C3) For all $C, C' \in \mathfrak{C}$ there is some $C'' \in \mathfrak{C}$ such that $C \cup C' \subseteq C''$.

(C4) For all $C, C' \in \mathfrak{C}$ there is some $C'' \in \mathfrak{C}$ such that $C' \circ C \subseteq C''$, where the composition $C' \circ C$ is defined as

$$C' \circ C := \{(z'', z) \mid \exists z' : (z', z) \in C, (z'', z') \in C'\}.$$

A set of *object support conditions* $\mathfrak{S}$ is a collection of G-invariant subsets of Z with the following property:

(S1) For all $S, S' \in \mathfrak{S}$ there is some $S'' \in \mathfrak{S}$ such that $S \cup S' \subseteq S''$.

The triple $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ is called a *coarse structure*.

Note that conditions (C1) and (C4) imply condition (C3).

2.2. Example [Bartels et al. 2004, Sections 2.3.2 and 2.3.3; 2008b, Section 3.2].

(1) Let Z be a G-space. The *trivial object support condition* is $\mathfrak{S}_{\mathrm{triv}}(Z) = \{Z\}$. The *trivial morphism control condition* is given by $\mathfrak{C}_{\mathrm{triv}}(Z) := \{Z \times Z\}$. Together, these form the *trivial coarse structure* $\mathfrak{T}(Z)$.

(2) Let X be a G-space. The *G-compact support condition* is the object support condition defined to be

$$\mathfrak{C}_{G\text{-cpt}}(X) := \{K \subseteq X \mid K \text{ is } G\text{-compact}\}.$$

(3) Let M be a metric space with isometric G-action; metrics are allowed to map to the extended real line $\mathbb{R} \cup \{\infty\}$. The *bounded morphism control condition* is defined to be

$$\mathfrak{C}_{\mathrm{bdd}}(M) := \{B \in \mathcal{P}(M \times M) \mid \text{there is some } R > 0 \text{ such that } d(m_1, m_2) \leqslant R \text{ for all } (m_1, m_2) \in B\}.$$

Together with the trivial object support condition on M, we obtain the *bounded coarse structure*

$$\mathfrak{B}(M) := (M, \mathfrak{C}_{\mathrm{bdd}}(M), \mathfrak{S}_{\mathrm{triv}}(M)).$$

(4) Let X be a G-space. The *G-continuous control condition* $\mathfrak{C}_{G\text{-cc}}(X)$ is the morphism control condition given by the set of all subsets

$$C \subseteq (X \times [1, \infty[) \times (X \times [1, \infty[)$$

which satisfy the following:

(a) For every $x \in X$ and every G_x-invariant open neighborhood U of (x, ∞) in $X \times [1, \infty]$, there exists a G_x-invariant open neighborhood $V \subseteq U$ of (x, ∞) in $X \times [1, \infty]$ such that $((X \times [1, \infty[) \setminus U) \times V) \cap C = \varnothing$.

(b) Let $p_{[1,\infty[} : X \times [1, \infty[\to [1, \infty[$ be the projection map and equip $[1, \infty[$ with the Euclidean metric. Then the set $(p_{[1,\infty[} \times p_{[1,\infty[})(C)$ is a member of $\mathfrak{C}_{\mathrm{bdd}}([1, \infty[)$.

(c) C is symmetric, G-invariant and contains the diagonal.

(5) If $\mathfrak{C}_1$, $\mathfrak{C}_2$ are sets of morphism control conditions on the same space Z, then

$$\mathfrak{C}_1 \Cap \mathfrak{C}_2 := \{C_1 \cap C_2 \mid C_1 \in \mathfrak{C}_1, C_2 \in \mathfrak{C}_2\}$$

is again a set of control conditions. We refer to this construction as "pointwise intersection".

There are further constructions which allow us to produce new coarse structures out of these; see [Bartels et al. 2004, Section 2.3.1]. We will introduce these on the way as we need them; see, for example, Definitions 5.1, 5.2 and 6.1.

Recall that W is a space with a G-action. Let Y be a G-CW-complex relative to W. The *structural inclusion* of the relative G-CW-complex (Y, W) is usually denoted by $s : W \to Y$. If we speak about a "cell" of Y, this always means a (nonequivariant) open, relative cell. The closure of a cell e is denoted by $\bar{e}$, and ∂e is always the boundary of the cell e, i.e., the image of any attaching map for e. Let $\diamond_k Y$ denote the set of k-cells of Y. Set $\diamond Y := \bigcup_k \diamond_k Y$. If $e \in \diamond Y$ is a cell in Y, we define $\langle e \rangle \subseteq Y$ to be the smallest nonequivariant subcomplex of Y (relative W) which contains e. For a subgroup $H \leqslant G$, let $\langle e \rangle_H \subseteq Y$ denote the smallest H-CW-subcomplex of Y which contains e. Similarly, we define $\langle S \rangle$, $\langle S \rangle_H$ for any subset $S \subseteq Y$.

A nonequivariant version of the following definition was already considered in [Weiss 2002].

2.3. Definition. Let $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ be a coarse structure. Let Y be a G-CW-complex relative to W. A *control map* for Y is an equivariant function $\kappa : \diamond Y \to Z$.

A G-CW-complex Y relative W together with a control map κ is called a *labeled G-CW-complex relative W*.

Let (Y_1, κ_1) and (Y_2, κ_2) be labeled G-CW-complexes relative W. A *$\mathfrak{Z}$-controlled map* $f : Y_1 \to Y_2$ is an equivariant, cellular map (relative W) such that for all $k \in \mathbb{N}$ there is some $C \in \mathfrak{C}$ for which

$$(\kappa_2 \times \kappa_1)\big(\{(e_2, e_1) \mid e_1 \in \diamond_k Y_1, e_2 \in \diamond Y_2, \langle f(e_1)\rangle \cap e_2 \neq \varnothing\}\big) \subseteq C$$

holds.

A *$\mathfrak{Z}$-controlled G-CW-complex relative W* is a labeled G-CW-complex (Y, κ) such that the identity map on Y is a $\mathfrak{Z}$-controlled map and for all $k \in \mathbb{N}$ there is some $S \in \mathfrak{S}$ such that

$$\kappa(\diamond_k Y) \subseteq S.$$

We abbreviate the terminology to *controlled map* and *controlled G-CW-complex* if the coarse structure $\mathfrak{Z}$ is understood.

2.4. Remark. The $\mathfrak{Z}$-control condition for a labeled G-CW-complex (Y, κ) is a statement about attaching maps. Since $\mathfrak{C}$ is closed under composition and taking finite unions, the control condition in Definition 2.3 is equivalent to requiring that for each k, there is some $C_k \in \mathfrak{C}$ such that for every k-cell e and every cell e' intersecting the closed cell $\bar{e}$ nontrivially, we have $(\kappa(e'), \kappa(e)) \in C_k$.

Moreover, if C_k witnesses $\mathfrak{Z}$-controlledness for a given complex Y, we may assume that $C_k \subseteq C_{k+1}$, and the same holds for the support conditions. In particular, if Y is finite-dimensional, there are a single support condition S and a single control condition C witnessing that Y is $\mathfrak{Z}$-controlled.

Let (Y, κ) be a labeled G-CW-complex relative W. Define the *relative cylinder* $Y \leftthreetimes [0, 1]$ by the pushout

$$\begin{array}{ccc} W \times [0,1] & \longrightarrow & W \\ {\scriptstyle s \times \mathrm{id}_{[0,1]}}\big\downarrow & & \big\downarrow \\ Y \times [0,1] & \longrightarrow & Y \leftthreetimes [0,1] \end{array}$$

The projection map $p : Y \leftthreetimes [0, 1] \to Y$ induces a function $\diamond p : \diamond(Y \leftthreetimes [0, 1]) \to \diamond Y$, so $\kappa \circ \diamond p$ is a control map for $Y \leftthreetimes [0, 1]$. This turns $Y \leftthreetimes [0, 1]$ into a labeled G-CW complex relative to W. If (Y, κ) is $\mathfrak{Z}$-controlled, then $(Y \leftthreetimes [0, 1], \kappa \circ \diamond p)$ is also $\mathfrak{Z}$-controlled.

2.5. Definition. Let (Y_1, κ_1) and (Y_2, κ_2) be labeled G-CW-complexes relative W. A *$\mathfrak{Z}$-controlled homotopy* is a $\mathfrak{Z}$-controlled map $H : Y_1 \leftthreetimes [0, 1] \to Y_2$. Two maps $f_0, f_1 : Y_1 \to Y_2$ are *$\mathfrak{Z}$-controlled homotopic*, $f_0 \simeq_{\mathfrak{Z}} f_1$, if there is a $\mathfrak{Z}$-controlled homotopy whose restriction to $Y_1 \times \{0\}$ and $Y_1 \times \{1\}$ equals f_0 and f_1, respectively.

A $\mathfrak{Z}$-controlled map $f : Y_1 \to Y_2$ is a *$\mathfrak{Z}$-controlled homotopy equivalence* if there is a $\mathfrak{Z}$-controlled map $\bar{f} : Y_2 \to Y_1$ such that $\bar{f}f \simeq_{\mathfrak{Z}} \mathrm{id}_{Y_1}$ and $f\bar{f} \simeq_{\mathfrak{Z}} \mathrm{id}_{Y_2}$.

Suppose (Y, κ) is a labeled G-CW-complex relative to W, and that $B \subseteq Y$ is a G-invariant subcomplex of Y which contains W. Then B naturally becomes a labeled G-CW-complex relative to W by restricting κ to B. If we do not say otherwise, we always think about subcomplexes as labeled G-CW-complexes in this way.

2.6. Proposition ($\mathfrak{Z}$-controlled homotopy extension property [Weiss and Williams 1998, Section 1.A.6]). *Let (Y, κ) be a $\mathfrak{Z}$-controlled G-CW-complex relative W, and let $B \subseteq Y$ be a G-invariant subcomplex. Let Y_1 be a $\mathfrak{Z}$-controlled G-CW-complex relative W, and suppose that $h : Y \times \{0\} \cup B \leftthreetimes [0, 1] \to Y_1$ is a $\mathfrak{Z}$-controlled map.*

Then there is a $\mathfrak{Z}$-controlled map $Y \leftthreetimes [0, 1] \to Y_1$ extending h.

Proof. The proof follows the usual pattern. Subject to a choice of deformation retraction of $D^n \times [0, 1]$ to $D^n \times \{0\} \cup \partial D^n \times [0, 1]$, we can define a G-equivariant deformation retraction of $(\mathrm{sk}_n Y \cup B) \leftthreetimes [0, 1]$ onto $\mathrm{sk}_n Y \times \{0\} \cup B \leftthreetimes [0, 1]$ by composing with the characteristic maps of equivariant n-cells. The resulting deformation retraction can be chosen to be constant on all points which do not lie on an n-cell of Y which is not in B. It is $\mathfrak{Z}$-controlled because points on a given cell are moved at most into the image of the attaching sphere of the same cell (and attaching maps are controlled).

We obtain a deformation retraction of $Y \leftthreetimes [0, 1]$ onto $Y \times \{0\} \cup B \leftthreetimes [0, 1]$ by stacking the homotopies defined in the first step. This produces another $\mathfrak{Z}$-controlled homotopy since, for each n, all but finitely many of the stacked homotopies are constant on the n-skeleton. The endpoint of this homotopy is a retraction

$$r : Y \leftthreetimes [0, 1] \to Y \times \{0\} \cup B \leftthreetimes [0, 1],$$

so we may define an extension of h by $H := h \circ r$. □

3. Categories of controlled retractive spaces

The primary objective of the following discussion is to form a Waldhausen category of controlled G-CW-complexes relative W. This enables us to study the controlled A-theory of W in the sequel. Since the terminology can be considered standard by now, we freely use the notions of category with cofibrations [Waldhausen 1985, page 320], Waldhausen category [Waldhausen 1985, page 326] (where it is called "category with cofibrations and weak equivalences") and cylinder functor [Waldhausen 1985, page 348]. The saturation and extension axioms [Waldhausen 1985, page 327] as well as the cylinder axiom [Waldhausen 1985, page 349] play a role.

3.1. Definition. Let $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ be a coarse structure. The *category $\mathcal{R}^G(W, \mathfrak{Z})$ of $\mathfrak{Z}$-controlled retractive spaces over W* is the category whose objects are $\mathfrak{Z}$-controlled, free G-CW-complexes Y relative to W which come equipped with an equivariant retraction $r : Y \to W$ (i.e., $r \circ s = \mathrm{id}_W$, where s denotes the structural inclusion $W \to Y$). Morphisms in this category are $\mathfrak{Z}$-controlled maps over and under W.

We write (Y, s_Y, r_Y) or $Y \leftrightarrows W$ for objects of $\mathcal{R}^G(W, \mathfrak{Z})$, if we want to emphasize the section and retraction or the base space.

3.2. Remark. We would like to emphasize a few points about Definition 3.1 which may be easy to overlook.

By definition, the morphisms in $\mathcal{R}^G(W, \mathfrak{Z})$ are all *cellular* maps, and we never consider maps which are not cellular. This is important for inductive arguments, and also provides us with mapping cylinders.

Requiring the relative G-CW-complexes (Y, W) to have a retraction $Y \to W$ and morphisms to respect this retraction provides $\mathcal{R}^G(W, \mathfrak{Z})$ with a basepoint. However, the homotopy equivalences we define later are inherited from the category of relative G-CW-complexes. This means that some arguments lead us to consider maps which do not have to respect the retraction. We use the word *morphism* if a map respects the retractions, and speak about *maps* if the retractions do not need to be preserved.

3.3. Definition (finiteness conditions). Let $(Y \leftrightarrows W, \kappa)$ be a $\mathfrak{Z}$-controlled retractive space over W.

We call Y *finite* if it is finite-dimensional, the image of $Y \setminus W$ under the retraction meets the orbits of only finitely many path components of W, and for all $z \in Z$ there is some open neighborhood U of z such that $\kappa^{-1}(U)$ is finite.

An object Y is *homotopy finite* if there is a finite object F and a morphism $F \to Y$ which is a $\mathfrak{Z}$-controlled homotopy equivalence.

We call Y *finitely dominated* if there are a finite $\mathfrak{Z}$-controlled G-CW-complex D relative W, a $\mathfrak{Z}$-controlled morphism $p : D \to Y$ and a $\mathfrak{Z}$-controlled map $i : Y \to D$ such that $pi \simeq_{\mathfrak{Z}} \mathrm{id}_Y$ as $\mathfrak{Z}$-controlled maps.

Let us denote the full subcategories of finite, homotopy finite and finitely dominated $\mathfrak{Z}$-controlled retractive spaces by $\mathcal{R}^G_f(W, \mathfrak{Z})$, $\mathcal{R}^G_{hf}(W, \mathfrak{Z})$ and $\mathcal{R}^G_{fd}(W, \mathfrak{Z})$.

3.4. Remark. Note that the cells of finite objects can only be labeled with points whose isotropy group is finite. In fact, all control spaces we consider are free.

3A. $\boldsymbol{\mathcal{R}^G(W, \mathfrak{Z})}$ ***as a Waldhausen category.*** Observe that $\mathcal{R}^G(W, \mathfrak{Z})$ is canonically pointed by the zero object $* = (W \rightleftarrows W, \varnothing)$, and that $*$ is finite. Let $co\mathcal{R}^G(W, \mathfrak{Z})$ be the subcategory of all morphisms which are isomorphic to the inclusion of a G-invariant subcomplex. We call such morphisms *cofibrations* and denote them by "$\rightarrowtail$". Since isomorphisms are controlled, the controlled homotopy extension property (CHEP) holds with respect to cofibrations as a consequence of Proposition 2.6. As observed in [Weiss and Williams 1998, Section 1.A.6], the

CHEP is key to showing that $\mathcal{R}^G(W, \mathfrak{Z})$ is a Waldhausen category. In the remainder of this section, we elaborate on this remark, and also introduce a more general notion of weak equivalences in $\mathcal{R}^G(W, \mathfrak{Z})$, which is inspired by [Carlsson et al. 1998].

3.5. Lemma. *The subcategory $co\mathcal{R}^G(W, \mathfrak{Z})$ is a subcategory of cofibrations for $\mathcal{R}^G(W, \mathfrak{Z})$. If in a diagram $Y_2 \leftarrow Y_0 \rightarrowtail Y_1$ all three objects are finite, then so is the pushout $Y_1 \cup_{Y_0} Y_2$.*

Proof. The unique morphism $* \to Y$ (given by the structural inclusion) is clearly in $co\mathcal{R}^G(W, \mathfrak{Z})$, and the same holds true for any isomorphism. This shows Waldhausen's first two axioms. We are left to show that cofibrations admit cobase changes. Clearly isomorphisms do, so we can restrict ourselves to inclusions of G-invariant subcomplexes.

Let a diagram of the form $Y_2 \leftarrow Y_0 \rightarrowtail Y_1$ be given, where Y_0 is a subcomplex of Y_1. The pushout $Y := Y_1 \cup_{Y_0} Y_2$ exists in the category of G-CW-complexes relative W, and the resulting map $Y_2 \to Y$ is the inclusion of a subcomplex. By the universal property of the pushout, we obtain a structural retraction $Y \to W$.

We observe that $\diamond Y = \diamond Y_2 \sqcup (\diamond Y_1 \setminus \diamond Y_0)$. This allows us to define a control map $\kappa : \diamond Y \to Z$ by setting

$$\kappa(e) := \begin{cases} \kappa_2(e), & e \in \diamond Y_2, \\ \kappa_1(e), & e \in \diamond Y_1 \setminus Y_0. \end{cases}$$

Then (Y, κ) is an object in $\mathcal{R}^G(W, \mathfrak{Z})$ because the map $Y_0 \to Y_2$ is controlled. If Y_1 and Y_2 are finite, then so is Y.

By the universal property of Y, morphisms of retractive spaces out of Y are in bijection with compatible pairs of morphisms of retractive spaces out of Y_1 and Y_2. It is straightforward to check that this correspondence restricts to a bijection between controlled maps. Hence, (Y, κ) is also a pushout in $\mathcal{R}^G(W, \mathfrak{Z})$. □

Setting $co\mathcal{R}^G_f(W, \mathfrak{Z}) := \mathcal{R}^G_f(W, \mathfrak{Z}) \cap co\mathcal{R}^G(W, \mathfrak{Z})$, Lemma 3.5 shows that both $\mathcal{R}^G(W, \mathfrak{Z})$ and $\mathcal{R}^G_f(W, \mathfrak{Z})$ are categories with cofibrations.

The pushout of homotopy finite or finitely dominated objects is also homotopy finite or finitely dominated, respectively, and both $\mathcal{R}^G_{hf}(W, \mathfrak{Z})$ and $\mathcal{R}^G_{fd}(W, \mathfrak{Z})$ are therefore also categories with cofibrations. However, the proof requires us to know more about the Waldhausen category structure of $\mathcal{R}^G(W, \mathfrak{Z})$. It is given in Lemma 3.25.

3.6. Definition (cofinal subcomplexes). Let $A \subseteq Z$ be a G-invariant subspace. A $\mathfrak{Z}$-*thickening* A is a set of the form

$$A^C := \{z \in Z \mid (z, a) \in C, a \in A\}$$

for some $C \in \mathfrak{C}$.

Let additionally (Y, κ) be a labeled G-CW-complex relative W. A subcomplex $Y' \subseteq Y$ is called *cofinal away from* A if for every $k \in \mathbb{N}$ there is some $\mathfrak{Z}$-thickening A^C of A such that $\kappa^{-1}(Z \setminus A^C) \cap \diamond_k Y \subseteq \diamond_k Y'$.

In the following discussion, we tacitly assume the next lemma.

3.7. Lemma. *Let Y, Y_1, Y_2 be labeled G-CW-complexes.*

(1) *If $Y' \subseteq Y$ and $Y'' \subseteq Y$ are cofinal subcomplexes away from A, then so is $Y' \cap Y''$.*

(2) *If $Y'' \subseteq Y' \subseteq Y$ are inclusions of subcomplexes, $Y' \subseteq Y$ is cofinal away from A and $Y'' \subseteq Y'$ is cofinal away from A, then $Y'' \subseteq Y$ is cofinal away from A.*

(3) *Let $f : (Y_1, \kappa_1) \to (Y_2, \kappa_2)$ be a controlled map, and let $Y_2' \subseteq Y_2$ be cofinal away from A. Let f^*Y_2' be the largest subcomplex of Y_1 whose image under f is contained in Y_2'. Then $f^*Y_2' \subseteq Y_1$ is cofinal away from A.*

(4) *Suppose that Y is $\mathfrak{Z}$-controlled. Let $B \subseteq Y$ be a subcomplex and $B' \subseteq B$ a cofinal subcomplex. Then there is a cofinal subcomplex $Y' \subseteq Y$ with $Y' \cap B = B'$.*

Proof. (1) Let $k \in \mathbb{N}$. Choose morphism control conditions C' and C'' such that $\kappa^{-1}(Z \setminus A^{C'}) \cap \diamond_k Y \subseteq \diamond_k Y'$ and $\kappa^{-1}(Z \setminus A^{C''}) \cap \diamond_k Y \subseteq \diamond_k Y''$. Let $C \in \mathfrak{C}$ such that $C' \cup C'' \subseteq C$. Then $Z \setminus A^C \subseteq Z \setminus A^{C'}$ and $Z \setminus A^C \subseteq Z \setminus A^{C''}$, so

$$\kappa^{-1}(Z \setminus A^C) \cap \diamond_k Y \subseteq \diamond_k (Y' \cap Y'').$$

(2) Let $k \in \mathbb{N}$. This time, take morphism control conditions C' and C'' such that $\kappa^{-1}(Z \setminus A^{C'}) \cap \diamond_k Y \subseteq \diamond_k Y'$ and $\kappa^{-1}(Z \setminus A^{C''}) \cap \diamond_k Y' \subseteq \diamond_k Y''$. Let $C \in \mathfrak{C}$ such that $C' \cup C'' \subseteq C$. Then $Z \setminus A^C \subseteq Z \setminus A^{C'}$ and $Z \setminus A^C \subseteq Z \setminus A^{C''}$, so

$$\begin{aligned}\kappa^{-1}(Z \setminus A^C) \cap \diamond_k Y &\subseteq \kappa^{-1}(Z \setminus A^{C''}) \cap \kappa^{-1}(Z \setminus A^{C'}) \cap \diamond_k Y \\ &\subseteq \kappa^{-1}(Z \setminus A^{C''}) \cap \diamond_k Y' \\ &\subseteq \diamond_k Y''.\end{aligned}$$

(3) Let $k \in \mathbb{N}$. Choose $C \in \mathfrak{C}$ such that

$$(\kappa_2 \times \kappa_1)\big(\{(e_2, e_1) \mid e_1 \in \diamond_k Y_1, e_2 \in \diamond Y_2, \langle f(e_1)\rangle \cap e_2 \neq \varnothing\}\big) \subseteq C,$$

and let $C' \in \mathfrak{C}$ such that $\kappa_2^{-1}(Z \setminus A^{C'}) \cap \diamond_k Y_2 \subseteq \diamond_k Y_2'$. Choose $C'' \in \mathfrak{C}$ such that $C' \circ C \subseteq C''$. Let $e_1 \in \diamond_k Y_1$ such that $\kappa_1(e_1) \in Z \setminus A^{C''}$. Then $\kappa_2(e_2) \in Z \setminus A^{C'}$ for all $e_2 \in \langle f(e_1)\rangle$, so $e_2 \in Y_2'$ and hence $e_2 \in \diamond_k f^*Y_2'$.

(4) This is proven by induction over the skeleta. For $k = 0$, define the 0-skeleton of Y' to contain all 0-cells of B' and all 0-cells of Y which do not lie in B. Suppose that the k-skeleton $\mathrm{sk}_k Y'$ of Y' has been defined such that $\mathrm{sk}_k Y' \subseteq \mathrm{sk}_k Y$ is cofinal

and $\mathrm{sk}_k Y' \cap B = \mathrm{sk}_k B'$. There exists $C_k \in \mathfrak{C}$ such that $\kappa^{-1}(Z \setminus A^{C_k}) \cap \diamond_{\leqslant k} Y \subseteq \diamond_{\leqslant k} Y'$. Choose $C \in \mathfrak{C}$ such that

$$(\kappa \times \kappa)\big(\{(e', e) \mid e \in \diamond_{k+1} Y, e' \in \langle e \rangle\}\big) \subseteq C.$$

Choose $C'_{k+1} \in \mathfrak{C}$ such that $C_k \circ C \subseteq C'_{k+1}$. Define $\mathrm{sk}_{k+1} Y'$ by adding to $\mathrm{sk}_k Y'$ all $(k+1)$-cells of Y which lie in B' or which do not lie in B' and are labeled by points in $Z \setminus A^{C'_{k+1}}$. Then $\mathrm{sk}_{k+1} Y' \cap B = \mathrm{sk}_{k+1} B'$, and it is easy to check that $\mathrm{sk}_{k+1} Y' \subseteq \mathrm{sk}_{k+1} Y$ is cofinal.

The desired complex Y' is obtained by taking the union over all $\mathrm{sk}_k Y'$. □

3.8. Definition (partially defined maps). Let (Y_1, κ_1), (Y_2, κ_2) and (Y_3, κ_3) be labeled G-CW-complexes. A *partially defined $\mathfrak{Z}$-controlled map (away from A)* $Y_1 \to^A Y_2$ is a pair (Y'_1, f_1) where $Y'_1 \subseteq Y_1$ is cofinal away from A and $f_1 : Y'_1 \to Y_2$ is a controlled map.

For two partially defined controlled maps $(Y'_1, f_1) : Y_1 \to^A Y_2$ and $(Y'_2, f_2) : Y_2 \to^A Y_3$, their *composition* $(Y'_2, f_2) \circ^A (Y'_1, f_1)$ is the partially defined controlled map $(f_1^* Y'_2, f_2 \circ f_1|_{f_1^* Y'_2}) : Y_1 \to^A Y_3$.

Composition of partially defined maps is well-defined. It is also associative: Let $(Y'_1, f_1) : Y_1 \to^A Y_2$, $(Y'_2, f_2) : Y_2 \to^A Y_3$ and $(Y'_3, f_3) : Y_3 \to^A Y_4$ be partially defined maps, In order to show associativity, it is enough to check that the cofinal subcomplexes $(f_2 \circ f_1|_{f_1^* Y'_2})^* Y'_3$ and $f_1^*(f_2^* Y'_3)$ coincide. Observe that e is a cell in $(f_2 \circ f_1|_{f_1^* Y'_2})^* Y'_3$ if and only if $f_1(e) \subseteq Y'_2$ and $f_2(f_1(e)) \subseteq Y'_3$. If e is a cell in $f_1^*(f_2^* Y'_3)$, then $f_1(e) \subseteq f_2^* Y'_3 \subseteq Y'_2$ and hence $f_2(f_1(e)) \subseteq Y'_3$. So, we have $f_1^*(f_2^* Y'_3) \subseteq (f_2 \circ f_1|_{f_1^* Y'_2})^* Y'_3$. On the other hand, if $f_1(e) \subseteq Y'_2$ and $f_2(f_1(e)) \subseteq Y'_3$, then $f_2(\langle f_1(e) \rangle) \subseteq \langle f_2(f_1(e)) \rangle \subseteq Y'_3$. Therefore, $f_1(e) \subseteq f_2^* Y'_3$, so $(f_2 \circ f_1|_{f_1^* Y'_2})^* Y'_3 \subseteq f_1^*(f_2^* Y'_3)$.

3.9. Definition. Let (Y'_1, f_0), $(Y''_1, f_1) : Y_1 \to^A Y_2$ be partially defined controlled maps. Then (Y'_1, f_0) and (Y''_1, f_1) are *controlled homotopic away from A*, written $(Y'_1, f_0) \simeq^A (Y''_1, f_1)$, if there is a cofinal subcomplex $Y'''_1 \subseteq Y'_1 \cap Y''_1$ and a controlled homotopy $H : Y'''_1 \leftthreetimes [0, 1] \to Y_2$ from $f_0\big|_{Y'''_1}$ to $f_1\big|_{Y'''_1}$.

3.10. Lemma. *Let $H : Y_1 \leftthreetimes [0, 1] \to Y_2$ be a controlled homotopy, and suppose that $Y'_2 \subseteq Y_2$ is cofinal away from $A \subseteq Z$.*

Then there is a cofinal subcomplex $Y'_1 \subseteq Y_1$ away from A such that H restricts to a controlled homotopy $Y'_1 \leftthreetimes [0, 1] \to Y'_2$.

Proof. We construct Y'_1 by induction over the skeleta. Assume that we have constructed $\mathrm{sk}_{n-1} Y'_1 \subseteq Y_1$ such that $\mathrm{sk}_{n-1} Y'_1 \subseteq \mathrm{sk}_{n-1} Y_1$ is cofinal and such that $H(\mathrm{sk}_{n-1} Y'_1 \leftthreetimes [0, 1]) \subseteq Y'_2$. Define

$$I_n := \{e \in \diamond_n Y_1 \mid \partial e \subseteq \mathrm{sk}_{n-1} Y'_1, H(\langle e \rangle \leftthreetimes [0, 1]) \subseteq Y'_2\}.$$

Then $\mathrm{sk}_n Y_1' := \mathrm{sk}_{n-1} Y_1' \cup \bigcup_{e \in I_n} \langle e \rangle$ is a G-invariant subcomplex such that

$$H(\mathrm{sk}_n Y_1' \leftthreetimes [0,1]) \subseteq Y_2'.$$

So we only have to show that $\mathrm{sk}_n Y_1' \subseteq \mathrm{sk}_n Y_1$ is cofinal. There are control conditions C_1, C_1', C and C_2' with the following properties:

(1) For all $e, e' \in \diamond \mathrm{sk}_n Y_1$ with $e' \subseteq \langle e \rangle$ we have $(\kappa_1(e'), \kappa_1(e)) \in C_1$.

(2) $\kappa_1^{-1}(Z \setminus A^{C_1'}) \cap \diamond \mathrm{sk}_{n-1} Y_1 \subseteq \diamond \mathrm{sk}_{n-1} Y_1'$.

(3) For all $e \in \diamond_n Y_1$ and all $e' \in \diamond \langle H(\langle e \rangle \leftthreetimes [0,1]) \rangle$, we have $(\kappa_2(e'), \kappa_1(e)) \in C$.

(4) $\kappa_2^{-1}(Z \setminus A^{C_2'}) \cap \diamond \mathrm{sk}_{n+1} Y_2 \subseteq \diamond \mathrm{sk}_{n+1} Y_2'$.

Suppose $e \in \diamond_n Y_1$ such that $e \notin I_n$. If $\partial e \nsubseteq \mathrm{sk}_{n-1} Y_1'$, then $\kappa_1(e) \in A^{C_1' \circ C_1}$. If $H(\langle e \rangle \leftthreetimes [0,1]) \nsubseteq Y_2'$, then $\kappa_1(e) \in A^{C_2' \circ C}$. Hence, $\kappa_1(\diamond_n Y_1 \setminus I_n) \subseteq A^{C_1' \circ C_1 \cup C_2' \circ C}$. This proves that $\mathrm{sk}_n Y_1' \subseteq \mathrm{sk}_n Y_1$ is cofinal away from A.

Defining $Y_1' := \bigcup_n \mathrm{sk}_n Y_1'$ finishes the proof. □

3.11. Lemma. *Let $(Y_1^0, f_0), (Y_1^1, f_1) : Y_1 \to^A Y_2$ be partially defined controlled maps such that $(Y_1^0, f_0) \simeq^A (Y_1^1, f_1)$.*

(1) *For every partially defined controlled map $(Y_0', \alpha) : Y_0 \to^A Y_1$ we have*

$$(Y_1^0, f_0) \circ^A (Y_0', \alpha) \simeq^A (Y_1^1, f_1) \circ^A (Y_0', \alpha).$$

(2) *For every partially defined controlled map $(Y_2', \beta) : Y_2 \to^A Y_3$ we have*

$$(Y_2', \beta) \circ^A (Y_1^0, f_0) \simeq^A (Y_2', \beta) \circ^A (Y_1^1, f_1).$$

Proof. Consider the second claim. Since $(Y_1^0, f_0) \simeq^A (Y_1^1, f_1)$, there are a cofinal subcomplex $Y_1' \subseteq Y_1$ and a controlled homotopy $H : Y_1' \leftthreetimes [0,1] \to Y_2$ from $f_0|_{Y_1'}$ to $f_1|_{Y_1'}$. Consider $Y_2' \subseteq Y_2$. By Lemma 3.10, there is a cofinal subcomplex $Y_1'' \subseteq Y_1'$ such that H restricts to a homotopy $H' : Y_1'' \leftthreetimes [0,1] \to Y_2'$. Then $\beta \circ H'$ is the desired homotopy. The other claim is similar, but easier. □

3.12. Definition (homotopy equivalences away from A). A controlled map $f : Y_1 \to Y_2$ between controlled G-CW-complexes relative W is a *controlled homotopy equivalence away from A* if there is a partially defined controlled map $(Y_2', \bar{f}) : Y_2 \to^A Y_1$ such that $f \circ^A (Y_2', \bar{f}) \simeq^A \mathrm{id}_{Y_2}$ and $(Y_2', \bar{f}) \circ^A f \simeq^A \mathrm{id}_{Y_1}$.

Such maps are called *h^A-equivalences*, abbreviated to *h-equivalences* if $A = \varnothing$. We denote by $h^A \mathcal{R}^G(W, \mathfrak{Z})$ the collection of all morphisms in $\mathcal{R}^G(W, \mathfrak{Z})$ which are controlled homotopy equivalences away from A.

3.13. Remark. Note that maps in $h^A \mathcal{R}^G(W, \mathfrak{Z})$ are *morphisms*, hence required to respect the retractions, while in general partially defined maps and partially defined homotopy equivalences do not need to respect the retractions. This means that homotopy inverses of morphisms in $h^A \mathcal{R}^G(W, \mathfrak{Z})$ do not need to lie in $h^A \mathcal{R}^G(W, \mathfrak{Z})$.

See Section 2.1 of [Waldhausen 1985], where weak equivalences are defined in a similar way.

The following results are proven using *maps*. We obtain results about $h^A\mathcal{R}^G(W,\mathfrak{Z})$ because it is the intersection of the h^A-equivalences with the morphisms.

Note also that h-equivalences are h^A-equivalences for any choice of A.

The collection $h^A\mathcal{R}^G(W,\mathfrak{Z})$ is closed under composition of morphisms, and identity morphisms are controlled homotopy equivalences away from A. Hence $h^A\mathcal{R}_G(W,\mathfrak{Z})$ is a subcategory of $\mathcal{R}^G(W,\mathfrak{Z})$. Moreover, this subcategory satisfies the saturation axiom, i.e., whenever f_1 and f_2 are composable morphisms, and two out of f_1, f_2 and f_2f_1 are h^A-equivalences, so is the third.

We also need to discuss the cylinder functor on $\mathcal{R}^G_f(W,\mathfrak{Z})$ before we are ready to continue. Let $f : Y_1 \to Y_2$ be a controlled map of controlled G-CW-complexes relative W. Then we define $\mathsf{Cyl}(f)$ by the pushout

$$\begin{tikzcd}
Y_1 \times \{1\} = Y_1 \arrow[r, "f"] \arrow[d] & Y_2 \arrow[d] \\
Y_1 \rightthreetimes [0,1] \arrow[r] & \mathsf{Cyl}(f)
\end{tikzcd}$$

of G-CW-complexes relative W. We choose the canonical cofibration $Y_2 \rightarrowtail \mathsf{Cyl}(f)$ as the back inclusion of the cylinder, and let $Y_1 = Y_1 \times \{0\} \rightarrowtail Y_1 \rightthreetimes [0,1] \to \mathsf{Cyl}(f)$ be the front inclusion. The back projection $\mathsf{Cyl}(f) \to Y_2$ is induced by id_{Y_2} and f via the projection $Y_1 \rightthreetimes [0,1] \to Y_1$ and the universal property of the pushout. If f is a morphism in $\mathcal{R}^G(W,\mathfrak{Z})$, we can equip $\mathsf{Cyl}(f)$ with the induced structural retraction to obtain a retractive space $\mathsf{Cyl}_W(f)$. Then the above diagram becomes a pushout in $\mathcal{R}^G(W,\mathfrak{Z})$, and the front inclusion, back inclusion and back projection are morphisms in $\mathcal{R}^G(W,\mathfrak{Z})$.

If we use the construction of the pushout given in the proof of Lemma 3.5 and the fact that $Y_1 \rightarrowtail Y_1 \rightthreetimes [0,1]$ is the inclusion of a subcomplex, it is clear that this defines a functor from the category of arrows in $\mathcal{R}^G(W,\mathfrak{Z})$ to the category of diagrams of the shape

$$\begin{tikzcd}
Y_1 \arrow[r, tail] \arrow[dr, "f"'] & \mathsf{Cyl}_W(f) \arrow[d] & Y_2 \arrow[l, tail] \arrow[dl, "="] \\
& Y_2 &
\end{tikzcd} \tag{3.14}$$

in $\mathcal{R}^G(W,\mathfrak{Z})$. Observe also that the back projection is a controlled homotopy equivalence: the usual deformation retraction of $\mathsf{Cyl}(f)$ onto Y_2 is a controlled homotopy. We can choose $\mathsf{Cyl}_W(* \to A) = A$, which is needed for the following lemma.

3.15. Lemma. $\mathsf{Cyl}_W(-)$ *gives a* ***cylinder functor*** *on* $\mathcal{R}^G(W, \mathfrak{Z})$ *which satisfies the cylinder axiom with respect to h-equivalences.*

We are heading towards the following proposition.

3.16. Proposition (gluing lemma). *Assume we have the following commutative diagram in* $\mathcal{R}^G(W, \mathfrak{Z})$:

$$\begin{tikzcd} X_2 \arrow[d, "\sim^A"', "f_2"] & X_0 \arrow[l, "x_2"'] \arrow[r, tail, "x_1"] \arrow[d, "\sim^A"', "f_0"] & X_1 \arrow[d, "\sim^A"', "f_1"] \\ Y_2 & Y_0 \arrow[l, "y_2"'] \arrow[r, tail, "y_1"] & Y_1 \end{tikzcd} \tag{3.17}$$

Assume x_1, y_1 *are cofibrations and the* f_i *are* h^A*-equivalences. Then the induced map on the pushouts* $f : X_2 \cup_{X_0} X_1 \to Y_2 \cup_{Y_0} Y_1$ *is an* h^A*-equivalence.*

The uncontrolled version of Proposition 3.16 is well-known. Our proof follows the strategy pursued in [Kamps and Porter 1997, pages 33–59], which gives a detailed argument relying only on the homotopy extension property.

For the purpose of the proof, we introduce the following notation: if f and g are partially defined maps $X \to^A Y$ whose restrictions to some cofinal subcomplex of X are equal, we write $f =^A g$.

3.18. Remark. Note that Definition 3.12 could have been phrased in terms of equivalence classes with respect to the equivalence relation $=^A$, and that some of the more formal properties of the subcategory $h^A\mathcal{R}^G(W, \mathfrak{Z})$ can be easily derived by manipulating such equivalence classes.

However, "germs" of this kind are not adequate for proofs which require explicit constructions involving partially defined maps. The proof of the gluing lemma relies heavily on the CHEP, Proposition 2.6; see, e.g., Lemmas 3.19 and 3.20, which can only be applied to explicit choices of representatives. Similarly, the proof of Theorem 4.16 only makes sense with explicitly chosen partially defined maps.

For the proof of the gluing lemma, we need the following auxiliary results.

3.19. Lemma. *Let* $j_i : B \rightarrowtail Y_i$, *for* $i = 1, 2$, *be cofibrations. Let* $f : Y_1 \to Y_2$ *be an* h^A*-equivalence which satisfies* $fj_1 = j_2$. *Then there is a partially defined map* $(Y_2', g) : Y_2 \to^A Y_1$ *with* $(Y_2', g) \circ^A j_2 =^A j_1$ *and a homotopy* $H : (Y_2', g) \circ^A f \simeq^A \mathrm{id}_{Y_1}$ *away from* A *with* $H \circ^A (j_1 \times [0, 1]) =^A j_1 \times [0, 1]$.

Furthermore, $f \circ^A (Y_2', g)$ *is also homotopic to the inclusion via a homotopy under* B, *i.e.,* f *is an "*h^A*-equivalence under* B*".*

Proof. This is very similar to the standard proofs in the uncontrolled case, e.g., [May 1999, §6.5]. In our situation, one needs to take into account that maps and homotopies are only defined on cofinal subcomplexes. □

3.20. Lemma (left inverses for h^A-equivalences, relative case). *Assume we have the following diagram*:

$$\begin{array}{ccc} B_1 & \xrightarrow{b} & B_2 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ Y_1 & \xrightarrow{f} & Y_2 \end{array} \tag{3.21}$$

Assume that b is an h^A-equivalence with inverse $(B_2', b') : B_2 \to^A B_1$ and homotopy $H_{B_1} : (B_2', b') \circ^A b \simeq^A \mathrm{id}_{B_1}$. (We do not need to specify the other homotopy.)

If f is an h^A-equivalence, then there is a partially defined map $(Y_2', f') : Y_2 \to^A Y_1$ and a homotopy $H_{Y_1} : (Y_2', f') \circ^A f \simeq^A \mathrm{id}_{Y_1}$ such that $i_1 \circ^A (B_2', b') =^A (Y_2', f') \circ^A i_2$ and $H_{Y_1} \circ^A (i_1 \times [0, 1]) =^A i_1 \circ^A H_{B_1}$. (In short, f has a left h^A-inverse relative to B_i.)

Proof. Compare [Kamps and Porter 1997, I.7.3]. For the purpose of this proof, we omit the domains of partially defined maps from the notation. Let g be an h^A-inverse for f. The map $g \circ^A i_2$ is homotopic away from A to $g \circ^A i_2 \circ^A b \circ^A b'$ and hence to $i_1 \circ^A b'$. As i_2 is a cofibration, g is homotopic to a map $g' : Y_2 \to^A Y_1$ such that $g' \circ^A i_2 =^A i_1 \circ^A b'$. Now $i_1 \circ^A H_{B_1}$ is a homotopy away from A from $i_1 \circ^A b' \circ^A b =^A g' \circ^A f \circ^A i_1$ to i_1. As i_1 is a cofibration, homotopy extension gives a homotopy K, extending H_{B_1}, from $g' \circ^A f$ to a map l.

Then $l \circ^A i_1 =^A i_1$. Hence, Lemma 3.19 provides a left h^A-inverse l' of l under B_1. Define $f' := l' \circ^A g'$. Then, as a composition of h^A-equivalences, f' is itself an h^A-equivalence, and $f' \circ^A i_2 =^A i_1$.

We have homotopies $f' \circ^A f =^A l' \circ^A g' \circ^A f \simeq^A_K l' \circ^A l \simeq^A \mathrm{id}$. Restricting along i_1, this is the concatenation of the homotopy H_{B_1} and the constant homotopy. There is a cofinal subcomplex $B_1' \subseteq B_1$ such that we get a map $B_1' \leftthreetimes [0, 1] \leftthreetimes [0, 1] \to^A Y_1$ by projecting to the first two factors and then applying H_{B_1}. The homotopies above extend this to a map $Y_1' \leftthreetimes [0, 1] \leftthreetimes 0 \cup Y_1 \leftthreetimes \{0, 1\} \leftthreetimes [0, 1] \to^A Y_1$, defined on some cofinal subcomplex $Y_1' \subseteq Y_1$. We may assume that $B_1' \subseteq Y_1'$. The CHEP, Proposition 2.6, then gives the homotopy H_{Y_1}. □

3.22. Remark. We cannot make special assumptions about the cofinal subcomplex on which f' is defined. In particular, it could happen that $Y_2' \cap B_2 \not\subseteq B_2'$. We need to take care of this situation in the proof of Lemma 3.23 below.

3.23. Lemma (gluing lemma, special case). *Assume in (3.17) additionally that x_2 and y_2 are cofibrations. Then the conclusion of the proposition holds, i.e., the map f on the pushout is an h^A-equivalence.*

Proof. We can assume x_i, y_i are cellular inclusions, because they are so up to isomorphism.

Pick an h^A-inverse (Y_0', g_0) of f_0 and a homotopy $H_0\colon (Y_0', g_0)\circ^A f_0 \simeq^A \mathrm{id}_{X_0}$. By Lemma 3.20, for $i = 1, 2$ there are h^A-left inverses (Y_i', g_i) of f_i and homotopies $H_i : (Y_i', g_i) \circ^A f_i \simeq^A \mathrm{id}_{X_i}$ such that

$$(Y_i', g_i) \circ^A y_i =^A x_i \circ^A (Y_0', g_0) \quad \text{and} \quad H_i \circ^A (x_i \times [0,1]) =^A y_i \circ^A H_0.$$

Choose a cofinal subcomplex Y_0'' of Y_0 such that the diagram

$$\begin{tikzcd} Y_2' \arrow[d, "g_2", "\sim^A"'] & Y_0'' \arrow[l, tail, "y_2"'] \arrow[r, tail, "y_1"] \arrow[d, "g_0", "\sim^A"'] & Y_1' \arrow[d, "g_1", "\sim^A"'] \\ X_2 & X_0 \arrow[l, tail, "x_2"'] \arrow[r, tail, "x_1"] & X_1 \end{tikzcd}$$

commutes. However, $Y_1' \cup_{Y_0''} Y_2'$ does not need to be a subcomplex of $Y_1 \cup_{Y_0} Y_2$, as the Y_0'' provided by Lemma 3.20 could be too small. But by part (4) of Lemma 3.7 we can restrict further to cofinal subcomplexes Y_i'', $i = 1, 2$, such that $Y_i'' \cap Y_0 = Y_0''$. Then $Y'' := Y_1'' \cup_{Y_0''} Y_2''$ is canonically isomorphic to the cofinal subcomplex $Y_2'' \cup Y_1''$ of $Y_1 \cup_{Y_0} Y_2$. Thus we get a partially defined map $(Y'', g)\colon Y_1 \cup_{Y_0} Y_2 \to^A X_1 \cup_{X_0} X_2$.

By the same argument, we get a partially defined homotopy from $(Y'', g) \circ^A f$ to the inclusion.

Repeating the argument with g_i instead of f_i, we get a partially defined map $l\colon X_1 \cup_{X_0} X_2 \to^A Y_1 \cup_{Y_0} Y_2$ with $l \circ^A g \simeq^A \mathrm{id}_{Y_2 \cup_{Y_0} Y_1}$. It follows that $f \circ^A g \simeq^A \mathrm{id}_{Y_2 \cup_{Y_0} Y_1}$, and hence f is an h^A-equivalence. $\square$

3.24. Lemma. *Assume that (3.21) is a pushout square and b an h-equivalence. Then f is an h-equivalence.*

Proof. We can factor b into $B_1 \rightarrowtail \mathsf{Cyl}(b) \to B_2$, and by saturation both maps are h-equivalences. Taking the pushout along the first map, we obtain the diagram

$$\begin{tikzcd} B_1 \arrow[r, tail, "b'"] \arrow[d, tail, "i_1"'] & \mathsf{Cyl}(b) \arrow[r] \arrow[d, tail, "i'"'] & B_2 \arrow[d, tail, "i_2"'] \\ Y_1 \arrow[r, tail, "j"] \arrow[rr, bend right, "f"'] & M_{b,i_1} \arrow[r, "p"] & Y_2 \end{tikzcd}$$

Here M_{b,i_1} is the double mapping cylinder. One now shows that j is an h-equivalence using that b' is a cofibration and an h-equivalence, and that p is an h-equivalence because i_1 is a cofibration. The usual proofs of these facts apply almost verbatim. We refer to [Kamps and Porter 1997, Proposition I.7.4] for the details. $\square$

Proof of Proposition 3.16. See also [Kamps and Porter 1997, Theorem 7.1]. Using the mapping cylinder we can factor the diagram (3.17) as follows:

$$\begin{tikzcd}
X_2 \arrow[d, "\sim_A"', "f_2"] & X' \arrow[l, "\sim"', "x_3"] \arrow[d, "\sim_A"', "f'"] & X_0 \arrow[l, tail, "x_2'"] \arrow[r, tail, "x_1"] \arrow[ll, bend right, "x_2"'] \arrow[d, "\sim_A"', "f_0"] & X_1 \arrow[d, "\sim_A"', "f_1"] \\
Y_2 & Y' \arrow[l, "y_3"', "\sim"] & Y_0 \arrow[l, tail, "y_2'"'] \arrow[r, tail, "y_1"] \arrow[ll, bend left, "y_2"] & Y_1
\end{tikzcd}$$

The maps x_3 and y_3 are h-equivalences by Lemma 3.15, so f' is an h^A-equivalence by saturation. The right part of the diagram consisting of x_2', y_2', x_1, y_1 satisfies the assumptions of Lemma 3.23. Therefore, the induced map $f'' : X' \cup_{X_0} X_1 \to Y' \cup_{Y_0} Y_1$ is an h^A-equivalence. Abbreviate $X'' := X' \cup_{X_0} X_1$, $Y'' := Y' \cup_{Y_0} Y_1$. We get induced cofibrations $x_4 : X' \rightarrowtail X''$, $y_4 : Y' \rightarrowtail Y''$.

We obtain the cube

$$\begin{tikzcd}
& X' \arrow[rr, tail, "x_4"] \arrow[dl, "\sim"', "x_3"] \arrow[dd, "f'"] & & X'' \arrow[dl, "x_5"] \arrow[dd, "\sim_A"', "f''"] \\
X_2 \arrow[rr, tail] \arrow[dd, "f_2"] & & X_2 \cup_{X_0} X_1 \arrow[dd, "f"] & \\
& Y' \arrow[rr, tail] \arrow[dl, "\sim"', "y_3"] & & Y'' \arrow[dl, "y_5"] \\
Y_2 \arrow[rr, tail, "y_4"] & & Y_2 \cup_{Y_0} Y_1 &
\end{tikzcd}$$

where the top and bottom are pushout squares. By Lemma 3.24, the maps x_5, y_5 are h-equivalences. By saturation, f is an h^A-equivalence, which proves the proposition. □

3.25. Lemma. *Let $Y_2 \leftarrow Y_0 \rightarrowtail Y_1$ be a diagram of homotopy finite or finitely dominated objects. Then the pushout $Y_1 \cup_{Y_0} Y_2$ in $\mathcal{R}^G(W, \mathfrak{Z})$ is also homotopy finite or finitely dominated, respectively.*

Proof. For homotopy finite objects, this is a formal consequence of the gluing lemma, Proposition 3.16, for h-equivalences. For the second claim, we show that the following two statements are equivalent:

(1) $Y \in \mathcal{R}^G(W, \mathfrak{Z})$ is finitely dominated.

(2) $Y \in \mathcal{R}^G(W, \mathfrak{Z})$ is a retract of a homotopy finite object.

Suppose (Y, s_Y, r_Y) is finitely dominated, i.e., there are a finite object (D, s_D, r_D), a morphism $p : D \to Y$, a controlled map $i : Y \to D$ and a homotopy $h : pi \simeq \mathrm{id}_Y$. These data give a map $f : \mathsf{Cyl}(i) \to Y$ whose composition with the front inclusion is id_Y and whose composition with the back inclusion is p. Then $r := r_Y \circ f$ is a retraction which makes $\mathsf{Cyl}(i)$ into a retractive space over W, and both f and the front inclusion $Y \rightarrowtail \mathsf{Cyl}(i)$ are morphisms. By construction, Y is a retract of $\mathsf{Cyl}(i)$. The back inclusion $D \rightarrowtail \mathsf{Cyl}(i)$ is an h-equivalence. Hence $\mathsf{Cyl}(i)$ is a homotopy finite object.

Conversely, assume that there is a homotopy finite object F as well as morphisms $s : Y \to F$ and $q : F \to Y$ such that $qs = \mathrm{id}_Y$. Since F is homotopy finite, there is a finite object D and a morphism $e : D \to F$ which is an h-equivalence. Let $\bar{e} : F \to D$ be an inverse controlled map. Then $i := \bar{e}s : Y \to D$ is a controlled map to a finite object, and $p := qe : D \to Y$ is a morphism. Moreover, we have $pi = qe\bar{e}s \simeq qs = \mathrm{id}_Y$ by assumption, so Y is finitely dominated.

With the characterization of finitely dominated objects as retracts of homotopy finite objects at our disposal, it is a formal consequence of the first part of the lemma and the universal property of the pushout that pushouts of finitely dominated objects are finitely dominated.

Sections 7.3 and 7.4 of [Ullmann 2018] spell out the formal arguments we left out here. □

3.26. Corollary. *For any G-invariant subset $A \subseteq Z$, the categories $\mathcal{R}^G(W, \mathfrak{Z})$, $\mathcal{R}^G_f(W, \mathfrak{Z})$, $\mathcal{R}^G_{hf}(W, \mathfrak{Z})$ and $\mathcal{R}^G_{fd}(W, \mathfrak{Z})$ are Waldhausen categories with respect to $h^A\mathcal{R}^G(W, \mathfrak{Z})$. The saturation axiom holds for these categories.*

There is a cylinder functor on $\mathcal{R}^G(W, \mathfrak{Z})$ which restricts to a cylinder functor on the subcategories of finite, homotopy finite and finitely dominated objects; the h^A-equivalences satisfy the cylinder axiom.

Proof. We only need to summarize what we already know. Lemmas 3.5 and 3.25 state that the cofibrations indeed form a subcategory of cofibrations. The collection of h^A-equivalences defines a subcategory of weak equivalences by the gluing lemma, Proposition 3.16. Saturation and the cylinder functor have been discussed right before the statement of the gluing lemma. Since every h-equivalence is an h^A-equivalence, the cylinder axiom is obvious. □

3B. *Functoriality.* Let us turn to the question of in which sense the categories $\mathcal{R}^G(W, \mathfrak{Z})$ are functorial with respect to the space W and the coarse structure $\mathfrak{Z}$. Changing G is discussed in Section 6.

If $f : W_1 \to W_2$ is a G-equivariant (continuous) map, pushout along f and the structural inclusion of a given object defines an exact functor

$$\mathcal{R}^G(f, \mathfrak{Z}) : \mathcal{R}^G(W_1, \mathfrak{Z}) \to \mathcal{R}^G(W_2, \mathfrak{Z}).$$

For changing the coarse structure, we need to define a notion of morphism; compare [Bartels et al. 2004, Section 3.3].

3.27. Definition. Let $\mathfrak{Z}_1 = (Z_1, \mathfrak{C}_1, \mathfrak{S}_1)$, $\mathfrak{Z}_2 = (Z_2, \mathfrak{C}_2, \mathfrak{S}_2)$ be two coarse structures. A *morphism of coarse structures* $z : \mathfrak{Z}_1 \to \mathfrak{Z}_2$ is a G-equivariant map of sets $z : Z_1 \to Z_2$ satisfying the following properties:

(1) For every $S_1 \in \mathfrak{S}_1$, there is some $S_2 \in \mathfrak{S}_2$ such that $z(S_1) \subseteq S_2$.

(2) For every $S \in \mathfrak{S}_1$ and $C_1 \in \mathfrak{C}_1$, there is some $C_2 \in \mathfrak{C}_2$ such that
$$(z \times z)((S \times S) \cap C_1) \subseteq C_2.$$

(3) For every $S \in \mathfrak{S}_1$ and all subsets $A \subseteq S$ which are locally finite in Z_1, the set $z(A)$ is locally finite in Z_2 and for all $x \in z(A)$, the set $z^{-1}(x) \cap A$ is finite.

Note that z does not need to be continuous, but the topology of Z_1 and Z_2 is used in the third condition. Morphisms of coarse spaces induce morphisms of controlled categories:

3.28. Proposition. *The categories $\mathcal{R}^G(W, \mathfrak{Z})$, $\mathcal{R}^G_f(W, \mathfrak{Z})$, $\mathcal{R}^G_{hf}(W, \mathfrak{Z})$ and $\mathcal{R}^G_{fd}(W, \mathfrak{Z})$ are functorial in $\mathfrak{Z}$, i.e., they define functors from the category of coarse structures and their morphisms to the category of Waldhausen categories.*

The canonical inclusion functors yield natural transformations
$$\mathcal{R}^G_f(W, -) \to \mathcal{R}^G_{hf}(W, -) \to \mathcal{R}^G_{fd}(W, -) \to \mathcal{R}^G(W, -).$$

See also Remark 3.30 for some set-theoretical issues.

Proof. Let $z : \mathfrak{Z}_1 \to \mathfrak{Z}_2$ be a morphism of coarse structures. Define the induced functor
$$\mathcal{R}^G(W, z) : \mathcal{R}^G(W, \mathfrak{Z}_1) \to \mathcal{R}^G(W, \mathfrak{Z}_2)$$
by mapping an object (Y, κ) to $(Y, z \circ \kappa)$ and by the identity on morphisms. We only have to show that this is well-defined. Let $(Y, \kappa) \in \mathcal{R}^G(W, \mathfrak{Z}_1)$. For every $k \in \mathbb{N}$, there is some $S_1 \in \mathfrak{S}_1$ such that $\kappa(\diamond_k Y) \subseteq S_1$. Since z is a morphism of coarse structures, we can find some $S_2 \in \mathfrak{S}_2$ such that $z(\kappa(\diamond_k Y)) \subseteq z(S_1) \subseteq S_2$. The verification that controlled maps are sent to controlled maps is similar. Condition (3) of Definition 3.27 ensures that this construction preserves finiteness. Hence, homotopy finite and finitely dominated objects are also preserved. □

3.29. Example. One of the most frequent examples of a morphism of coarse structures is the following. Let $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ be a coarse structure, and suppose that $A \subseteq Z$ is a G-invariant subspace. Denote by $\mathfrak{Z} \cap A$ the coarse structure $(A, \mathfrak{C} \Cap \{A \times A\}, \mathfrak{S} \Cap \{A\})$, where $\Cap$ denotes pointwise intersection.

If A is closed in Z, the inclusion map of A into Z defines a morphism $\mathfrak{Z} \cap A \to \mathfrak{Z}$ of coarse structures. Here closedness is required to preserve local finiteness of subsets.

3.30. Remark (set-theoretical smallness requirements). In the following, we discuss the algebraic K-theory of the categories $\mathcal{R}^G_f(W, \mathfrak{Z})$ and $\mathcal{R}^G_{fd}(W, \mathfrak{Z})$. As always, one faces certain set-theoretic difficulties in making sense of the K-theory of these categories; see [Waldhausen 1985, Remark on page 379]. Possible solutions include the use of a change-of-universe functor to make the categories at hand small, or to choose small models to replace these categories. For example, we may redefine $\mathcal{R}^G(W, \mathfrak{Z})$ so that the underlying set of every retractive space is a subset of $W \times \lambda$, where λ is a sufficiently large cardinal.

The algebraic K-theory of $\mathcal{R}^G_f(W, \mathfrak{Z})$ does not depend, up to homotopy, on the set-theoretic model we choose, as long as λ is large enough compared to $\mathfrak{Z}$. Proposition 3.28 then only asserts functoriality on some small, but arbitrarily large subcategory of the category of all coarse structures. To avoid further complications, we ignore these matters from now on.

4. Comparison theorems and vanishing theorems

In addition to the notions used in the previous section, we now have the opportunity to use all three fundamental results of Waldhausen K-theory: the additivity theorem [Waldhausen 1985, Theorem 1.4.2], the fibration theorem [Waldhausen 1985, Theorem 1.6.4] and the approximation theorem [Waldhausen 1985, Theorem 1.6.7].

4A. ***Comparing finiteness conditions.*** We discuss to which extent the K-theory spaces arising from the various finiteness conditions differ. The answer is given in Proposition 4.8, but the proof requires two preparatory lemmas.

4.1. Lemma (mapping cylinder argument). *Let $f : Y \to Y'$ and $g : Y'' \to Y'$ be morphisms in $\mathcal{R}^G(W, \mathfrak{Z})$. Suppose that g is a retraction up to homotopy, i.e., that there exists a map $\bar{g} : Y' \to Y''$ such that $g\bar{g}$ is controlled homotopic to the identity map. Then there is an object Q in $\mathcal{R}^G(W, \mathfrak{Z})$ which fits into the commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, tail, "i_Y"'] & Y' \\ Q \arrow[ur, "q"] & Y'' \arrow[l, tail, "i_{Y''}"', "\sim"] \arrow[u, "g"'] \end{tikzcd} \tag{4.2}$$

in $\mathcal{R}^G(W, \mathfrak{Z})$, in which i_Y and $i_{Y''}$ are cofibrations. The underlying controlled G-CW-complex of Q can be chosen to be $\mathsf{Cyl}(\bar{g} f)$.

In particular, q is an h-equivalence if and only if g is one.

Proof. Denote the retractions of Y, Y', Y'' by r, r', r''. Note that $\bar{g}$ does not need to respect the retraction. Define $Q := \mathsf{Cyl}(\bar{g} f)$, and let i_Y and $i_{Y''}$ be the front and back inclusion. Since $g\bar{g}f \simeq f$, any choice of homotopy $g\bar{g} \simeq \mathrm{id}_{Y''}$ induces a map

$q : Q \to Y'$ which restricts to f and g on the front and back of the cylinder. We can turn q into a morphism of retractive spaces by defining a retraction on Q via $r_Q := r' \circ q$. Since q restricts to f and g on the two ends of the cylinder and both of these maps are morphisms of retractive spaces, i_Y and $i_{Y''}$ also respect the retractions. This proves the existence of the commutative diagram (4.2). □

The following lemma reflects the fact that something close to a Puppe sequence exists in any Waldhausen category $\mathcal{C}$ with a cylinder functor. Even though the extension axiom does not hold in $\mathcal{R}^G(W, \mathfrak{Z})$ [Waldhausen 1985, Section 1.2], it follows that the axiom does hold up to suspension. Recall that the suspension of an object $A \in \mathcal{C}$ is defined to be

$$\Sigma A := \mathsf{Cyl}(A \to *)/A,$$

and that this extends to an exact endofunctor on $\mathcal{C}$ [Waldhausen 1985, page 349].

4.3. Lemma. *Let $\mathcal{C}$ be a Waldhausen category which possesses a cylinder functor such that the cylinder axiom and the saturation axiom hold. Consider a morphism between cofiber sequences*

$$\begin{tikzcd} A \arrow[r, tail, "\alpha"] \arrow[d, "a"', "\sim"] & B \arrow[r, two heads, "\beta"] \arrow[d, "b"'] & C \arrow[d, "c"', "\sim"] \\ A' \arrow[r, tail, "\alpha'"] & B' \arrow[r, two heads, "\beta'"] & C' \end{tikzcd} \tag{4.4}$$

in which a and c are weak equivalences. Then Σb is a weak equivalence.

Proof. Repeated use of the cylinder functor gives rise to the commutative diagram in Figure 1.

The cylinder and saturation axioms imply that all vertical arrows in this diagram are weak equivalences. Moreover, we have the following commutative square in the category of arrows of $\mathcal{C}$:

$$\begin{tikzcd} (A \xrightarrow{\alpha} B) \arrow[r, "{(a, b)}"] \arrow[d, "{(\mathrm{id}, *)}"'] & (A' \xrightarrow{\alpha'} B') \arrow[d, "{(\mathrm{id}, *)}"] \\ (A \to *) \arrow[r, "{(a, *)}"'] & (A' \to *) \end{tikzcd}$$

Applying the cylinder functor to this square, and taking quotients with respect to the front and back inclusions of the cylinders, we obtain a commutative square

$$\begin{tikzcd} S = \mathsf{Cyl}(\alpha)/(A \vee B) \arrow[r, "s"] \arrow[d, "\sim"'] & \mathsf{Cyl}(\alpha')/(A' \vee B') = S' \arrow[d, "\sim"] \\ \Sigma A \arrow[r, "\Sigma a"'] & \Sigma A' \end{tikzcd} \tag{4.5}$$

$$
\begin{tikzcd}
 & A' \arrow[rr, tail, "\alpha'"] \arrow[dd, equal] & & B' \arrow[rr, two heads, "\beta'"] & & C' & & & & \\
A \arrow[ur, "a"] \arrow[rr, tail, "\alpha"] \arrow[dd, equal] & & B \arrow[ur, "b"] \arrow[rr, two heads, "\beta"] & & C \arrow[ur, "c"] & & & & & \\
 & A' \arrow[rr, tail] & & \mathsf{Cyl}(\alpha') \arrow[uu] \arrow[rr, two heads] & & C(\alpha') \arrow[uu] \arrow[dd, equal] & & & & \\
A \arrow[ur, "a"] \arrow[rr, tail] & & \mathsf{Cyl}(\alpha) \arrow[uu] \arrow[ur] \arrow[rr, two heads] & & C(\alpha) \arrow[uu] \arrow[ur] \arrow[dd, equal] & & & & & \\
 & & & B' \arrow[uu, tail] \arrow[rr, tail, "j'"] \arrow[dd, equal] & & C(\alpha') \arrow[rr, two heads] & & S' & & \\
 & & B \arrow[uu, tail] \arrow[ur, "b"] \arrow[rr, tail, "j"] \arrow[dd, equal] & & C(\alpha) \arrow[ur] \arrow[rr, two heads] & & S \arrow[ur, "s"] & & & \\
 & & & B' \arrow[rr, tail] & & \mathsf{Cyl}(j') \arrow[uu] \arrow[rr, two heads] & & C(j') \arrow[uu] \arrow[dd, equal] & & \\
 & & B \arrow[ur, "b"] \arrow[rr, tail] & & \mathsf{Cyl}(j) \arrow[uu] \arrow[ur] \arrow[rr, two heads] & & C(j) \arrow[uu] \arrow[ur] \arrow[dd, equal] & & & \\
 & & & & & C(\alpha') \arrow[uu, tail] \arrow[rr, tail] & & C(j') \arrow[rr, two heads] & & T' \\
 & & & & C(\alpha) \arrow[uu, tail] \arrow[ur] \arrow[rr, tail] & & C(j) \arrow[ur] \arrow[rr, two heads] & & T \arrow[ur, "t"] &
\end{tikzcd}
$$

Figure 1. The "Puppe sequence".

in which the vertical arrows are weak equivalences. Since we assumed a to be a weak equivalence, Σa is one by the gluing lemma. It follows that s, and therefore also the induced (nameless) morphism $C(j) \to C(j')$, is a weak equivalence. Note that the (also nameless) morphism $C(\alpha) \to C(\alpha')$ is also a weak equivalence because c is a weak equivalence. Hence, t is a weak equivalence by the gluing lemma. Just like s, the morphism t sits in a square like (4.5) together with the induced morphism $\Sigma b : \Sigma B \to \Sigma B'$. Hence, Σb is a weak equivalence. $\square$

We also need the following cofinality theorem [Vogell 1990, Theorem 1.6], which Vogell attributes to Thomason.

4.6. Theorem (Vogell cofinality). *Let $(\mathcal{C}, co\mathcal{C}, w\mathcal{C})$ be a Waldhausen category which has a cylinder functor such that the cylinder axiom holds. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory of $\mathcal{C}$ such that $(\mathcal{D}, co\mathcal{C} \cap \mathcal{D}, w\mathcal{C} \cap \mathcal{D})$ is also a Waldhausen category. Assume that*

(1) *$\mathcal{D} \subseteq \mathcal{C}$ is **weakly cofinal** in the sense that for all $C \in \mathcal{C}$ there exist $C' \in \mathcal{C}$ and $k \in \mathbb{N}$ such that $\Sigma^k C \vee C'$ is isomorphic to an object in $\mathcal{D}$;*

(2) *$\mathcal{D}$ is **saturated in** $\mathcal{C}$, i.e., any object weakly equivalent (via some zig-zag) to an object in $\mathcal{D}$ lies in $\mathcal{D}$.*

Then there is a homotopy fiber sequence

$$wS_\bullet\mathcal{D} \to wS_\bullet\mathcal{C} \to N_\bullet \operatorname{coker}(K_0\mathcal{D} \to K_0\mathcal{C}).$$

4.7. Remark. The conditions on $\mathcal{D}$ imply that $\mathcal{D} \subseteq \mathcal{C}$ is a subcategory with cofibrations and weak equivalences in the sense of [Waldhausen 1985, Section 1.1], and that $\mathcal{D}$ satisfies condition (iii) of [Vogell 1990, Theorem 1.6].

Let us also remark why Vogell's cofinality theorem holds true: As written, Vogell seems to prove only the cofinality theorem suggested by [Thomason and Trobaugh 1990, Exercise 1.10.2] since, in the last three lines of his proof, he chooses "C_0' such that $C_0 \vee C_0'$ is in $\mathcal{D}$". This proves Theorem 4.6 under the additional assumption that $k = 0$ in condition (1). In fact, the more general statement follows:

For $k \in \mathbb{N}$, set $\mathcal{C}_k := \{C \in \mathcal{C} \mid \exists C' \in \mathcal{C} : \Sigma^k C \vee C' \in \mathcal{D}\}$. Then $\mathcal{C} = \bigcup_{k\in\mathbb{N}} \mathcal{C}_k$, each $\mathcal{C}_k$ is a full subcategory of $\mathcal{C}$ and inherits a Waldhausen structure from $\mathcal{C}$ (one needs to check that each $\mathcal{C}_k$ is closed under pushouts), and $\mathcal{D}$ is cofinal in $\mathcal{C}_0$. Hence, we can apply the case $k = 0$ of Theorem 4.6 to conclude that there is a homotopy fiber sequence $wS_\bullet\mathcal{D} \to wS_\bullet\mathcal{C}_0 \to N_\bullet \operatorname{coker}(K_0\mathcal{D} \to K_0\mathcal{C}_0)$. Observe that the suspension functor induces a functor $\Sigma : \mathcal{C}_{k+1} \to \mathcal{C}_k$; since the suspension functor induces an equivalence on algebraic K-theory [Waldhausen 1985, Proposition 1.6.2], we conclude that the inclusion functor $\mathcal{C}_k \subseteq \mathcal{C}_{k+1}$ does so, too. Therefore, $wS_\bullet\mathcal{C} \simeq \operatorname{hocolim}_k wS_\bullet\mathcal{C}_k \simeq wS_\bullet\mathcal{C}_0$, and Theorem 4.6 follows.

4.8. Proposition. (1) *The natural inclusion of the finite into the homotopy finite objects induces a weak equivalence*

$$hS_\bullet\mathcal{R}^G_f(W, \mathfrak{Z}) \xrightarrow{\sim} hS_\bullet\mathcal{R}^G_{hf}(W, \mathfrak{Z}),$$

hence a weak equivalence on algebraic K-theory spaces.

(2) *The inclusion $\mathcal{R}^G_f(W, \mathfrak{Z}) \subseteq \mathcal{R}^G_{fd}(W, \mathfrak{Z})$ induces an isomorphism on K_i for $i \geqslant 1$ and an injection on K_0, where we take K-theory with respect to the h-equivalences.*

Proof. To prove the first part, we appeal to Waldhausen's approximation theorem. Both categories satisfy the saturation axiom and have a cylinder functor which satisfies the cylinder axiom. The first part of the approximation property is clear. We check the second part.

Let F be a finite object, let Y be homotopy finite, and let $f : F \to Y$ be a morphism. We have to construct a finite object F', and further, a morphism $F \to F'$ as well as an h-equivalence $F' \xrightarrow{\sim} Y$ such that their composition is f.

Since Y is homotopy finite, there is a finite object F_0 and an h-equivalence $e : F_0 \xrightarrow{\sim} Y$. The approximation property now follows from the mapping cylinder argument, Lemma 4.1.

We turn to the proof of the second part of the proposition, which uses Vogell's cofinality Theorem 4.6.

Since the inclusion of $\mathcal{R}^G_f(W,\mathfrak{Z})$ into $\mathcal{R}^G_{fd}(W,\mathfrak{Z})$ factors via $\mathcal{R}^G_{hf}(W,\mathfrak{Z})$, we need only consider the inclusion of the latter category. We show that Vogell's cofinality theorem applies in this situation.

We show first that $\mathcal{R}^G_{hf}(W,\mathfrak{Z})$ is saturated in $\mathcal{R}^G_{fd}(W,\mathfrak{Z})$. Let Y_1 be homotopy finite. There is a finite object (F, s_F, r_F) and an h-equivalence $a : F \to Y_1$. Let $b : Y_2 \to Y_1$ be an h-equivalence. There is an inverse map $\bar{b} : Y_1 \to Y_2$, but the composition $\bar{b}a$ does not have to respect the retractions. Define a retraction $r' := r_{Y_2}\bar{b}a$. Then $(F, s_F, r') \to (Y_2, s_{Y_2}, r_{Y_2})$ is an h-equivalence and (F, s_F, r') is a different object, but still finite. Hence Y_2 is homotopy finite. The case $b : Y_1 \to Y_2$ is obvious and the general case follows by induction.

So we are left with showing weak cofinality. Let $Y \in \mathcal{R}^G_{fd}(W,\mathfrak{Z})$ be arbitrary. Then we can find a finite object D as well as a morphism $d : D \to Y$ and a map $i : Y \to D$ such that $d \circ i$ is controlled homotopic to the identity map. Define an object $\widetilde{Y}$ which is the same controlled G-CW-complex as Y, but equipped with a new retraction which turns $i : \widetilde{Y} \to D$ into a morphism. Note that the composition $d \circ i : \widetilde{Y} \to Y$ is an h-equivalence.

Let $C\widetilde{Y}$ denote the cone $\mathsf{Cyl}(\mathrm{id}_{\widetilde{Y}})/\widetilde{Y}$, let $S\widetilde{Y}$ denote the object $C\widetilde{Y} \cup_{\widetilde{Y}} C\widetilde{Y}$, and define $S\mathsf{Cyl}(i)$ and $SC(i)$ analogously. Then we have a canonical h-equivalence $S\widetilde{Y} \to C\widetilde{Y} \cup_{\widetilde{Y}} * \cong \Sigma\widetilde{Y}$. Moreover, we have a morphism $S\mathsf{Cyl}(i) \to \Sigma\mathsf{Cyl}(i) \vee \Sigma\mathsf{Cyl}(i)$ given by the quotient map with respect to the canonical cofibration $\mathsf{Cyl}(i) \hookrightarrow S\mathsf{Cyl}(i)$. These objects fit into a commutative diagram

$$\begin{tikzcd}
S\widetilde{Y} \arrow[r, tail] \arrow[d] & S\mathsf{Cyl}(i) \arrow[r, two heads] \arrow[d] & SC(i) \arrow[d] \\
\Sigma Y \arrow[r, tail] & \Sigma Y \vee \Sigma C(i) \arrow[r, two heads] & \Sigma C(i)
\end{tikzcd}$$

in which the upper row comes from the cofiber sequence and the lower row is the split cofiber sequence. The vertical arrows are given as follows: The left vertical morphism is the composition $S\widetilde{Y} \xrightarrow{\sim} \Sigma\widetilde{Y} \xrightarrow{\Sigma di} \Sigma Y$, and the right vertical arrow is the canonical morphism $SC(i) \to * \cup_{C(i)} C(C(i)) \cong \Sigma C(i)$, i.e., the collapse of the other half of $SC(i)$. For the middle vertical morphism, we take the composition $S\mathsf{Cyl}(i) \to \Sigma\mathsf{Cyl}(i) \vee \Sigma\mathsf{Cyl}(i) \xrightarrow{\Sigma d' \vee \Sigma q} \Sigma Y \vee \Sigma C(i)$, where d' is the composition of the back projection $\mathsf{Cyl}(i) \to D$ with d, and q is the projection $\mathsf{Cyl}(i) \to C(i)$. Since both the left and the right vertical arrows are h-equivalences, it follows from Lemma 4.3 that the induced morphism $\Sigma S\mathsf{Cyl}(i) \to \Sigma^2 Y \vee \Sigma^2 C(i)$ is an h-equivalence. Recall that $\mathsf{Cyl}(i)$ is homotopy finite (since it is h-equivalent to D). Since $\Sigma S\mathsf{Cyl}(i)$ is h-equivalent to $\Sigma^2\mathsf{Cyl}(i)$, and suspension preserves finiteness, it follows that $\Sigma^2 Y \vee \Sigma^2 C(i)$ is homotopy finite. This proves weak cofinality, and we are done. $\square$

4.9. Definition. Let $D \in \mathcal{R}^G(W, \mathfrak{Z})$, and let $\alpha : D \to D$ be a controlled map. The *mapping telescope* $\mathrm{Tel}(\alpha)$ of α is the controlled G-CW-complex relative W

$$\mathsf{Cyl}(\alpha) \cup_D \mathsf{Cyl}(\alpha) \cup_D \cdots$$

obtained by taking countably many copies of the mapping cylinder of α and gluing the back and front end of each consecutive pair of cylinders.

Note that $\mathrm{Tel}(\alpha)$ does not need to be a retractive space. However, in certain cases it can be equipped with a retraction, and can then be used to replace dominated spaces by "nicer" ones.

4.10. Proposition. *Let $Y, D \in \mathcal{R}^G(W, \mathfrak{Z})$. Suppose we have maps $i : Y \to D$ and $d : D \to Y$ such that $d \circ i$ is (controlled) homotopic to* id_Y.

Then the canonical map $j : Y \xrightarrow{i} D \rightarrowtail \mathrm{Tel}(i \circ d)$ is a controlled homotopy equivalence. If d is a morphism, $\mathrm{Tel}(i \circ d)$ admits a retraction such that there exists an h-equivalence $\mathrm{Tel}(i \circ d) \xrightarrow{\sim} Y$ which is a homotopy inverse to j.

Proof. The proof of Proposition 1.4 in [Ferry and Ranicki 2001] works also in our setting. Note that we have an h-equivalence $Y \simeq Y \rtimes [0, \infty[$ because the control map disregards the cylinder coordinate.

The homotopy $d \circ i \simeq \mathrm{id}_Y$ induces a map $\bar{d} : \mathsf{Cyl}(i \circ d) \to Y$ which restricts to d on the front and back. Hence, countably many copies of $\bar{d}$ glue to a controlled map $T(d) : \mathrm{Tel}(i \circ d) \to Y$. Since $T(d) \circ j = d \circ i \simeq \mathrm{id}_Y$, the map $T(d)$ is homotopy inverse to j. If d is a morphism, we can define a retraction $r : \mathrm{Tel}(i \circ d) \to W$ by composing $T(d)$ with the retraction of Y. This makes $D \rightarrowtail \mathrm{Tel}(i \circ d)$ into a morphism, and $T(d)$ becomes an h-equivalence. □

4.11. Corollary. *Let $\mathcal{R}^G_{fd,\dim<\infty}(W, \mathfrak{Z}) \subseteq \mathcal{R}^G_{fd}(W, \mathfrak{Z})$ denote the full Waldhausen subcategory of finite-dimensional objects. Then the inclusion functor induces a weak equivalence*

$$hS_\bullet \mathcal{R}^G_{fd,\dim<\infty}(W, \mathfrak{Z}) \xrightarrow{\sim} hS_\bullet \mathcal{R}^G_{fd}(W, \mathfrak{Z}).$$

Proof. This comes from another application of the approximation theorem, using Proposition 4.10 and the mapping cylinder argument Lemma 4.1. □

4B. ***Comparing different notions of weak equivalences.*** Let $\mathfrak{Z}$ be a coarse structure, and let $A \subseteq Z$ be a G-invariant subspace. We would like to compare the K-theory spaces of $\mathcal{R}^G_f(W, \mathfrak{Z})$ with respect to the h- and h^A-equivalences. Unfortunately, the standard procedure to obtain homotopy fiber sequences relating these does not apply in our situation since the fibration theorem requires one subcategory of weak equivalences to satisfy the extension axiom.[1] We present a solution to this

[1] Added in proof: Meanwhile, Raptis [2018, Theorem 2.7] has observed that the extension axiom can be dropped in the assumptions of the fibration theorem.

problem which has also been employed by Weiss [2002, towards the end of the proof of Proposition 8.3]. For the sake of completeness, we record its validity for any suitable Waldhausen category.

Let $(\mathcal{C}, co\mathcal{C}, w\mathcal{C})$ be a small Waldhausen category which satisfies the saturation axiom and possesses a cylinder functor which satisfies the cylinder axiom with respect to $w\mathcal{C}$.

4.12. Definition. We call a morphism f in $\mathcal{C}$ an *equivalence after n-fold suspension* if $\Sigma^n f$ lies in $w\mathcal{C}$. We say that f is a *stable equivalence* if there is some $n \in \mathbb{N}$ such that f is an equivalence after n-fold suspension. Denote the class of equivalences after n-fold suspension by $w_{\Sigma,n}\mathcal{C}$, and the class of stable equivalences by $w_\Sigma\mathcal{C}$.

4.13. Lemma. *Let $n \geqslant 0$.*

(1) *The collections $w_{\Sigma,n}\mathcal{C}$ and $w_\Sigma\mathcal{C}$ are classes of weak equivalences which satisfy the saturation axiom. The cylinder functor satisfies the cylinder axiom with respect to both classes. Moreover, $w_\Sigma\mathcal{C}$ satisfies the extension axiom.*

(2) *The natural map $wS_\bullet\mathcal{C} \to w_\Sigma S_\bullet\mathcal{C}$ is a weak equivalence.*

Proof. Almost everything in (1) is straightforward; the only exception is the validity of the extension axiom for $w_\Sigma\mathcal{C}$.

Assume that we have a commutative diagram of exact sequences

$$\begin{tikzcd} A \arrow[r, rightarrowtail] \arrow[d, "a"'] & B \arrow[r, twoheadrightarrow] \arrow[d, "b"'] & C \arrow[d, "c"'] \\ A' \arrow[r, rightarrowtail] & B' \arrow[r, twoheadrightarrow] & C' \end{tikzcd}$$

in which a and c are weak equivalences after n-fold suspension. Suspend the diagram n times to obtain a diagram of the same shape in which the left and right arrows are weak equivalences. Then it follows from Lemma 4.3 that b is a weak equivalence after $(n+1)$-fold suspension.

For (2), we can apply the fibration theorem to the inclusion $w\mathcal{C} \subseteq w_\Sigma\mathcal{C}$ since we have just shown that $w_\Sigma\mathcal{C}$ satisfies the saturation and extension axioms, and that the cylinder axiom holds as well. So, it suffices to show that $wS_\bullet\mathcal{C}^{w_\Sigma}$ is contractible. Observe that $\mathcal{C}^{w_\Sigma}$ is the union of the ascending sequence $\mathcal{C}^w \subseteq \mathcal{C}^{w_{\Sigma,1}} \subseteq \mathcal{C}^{w_{\Sigma,2}} \subseteq \cdots$. Since K-theory commutes with directed colimits, it is enough to show that each $\mathcal{C}^{w_{\Sigma,n}}$ has contractible K-theory.

By the additivity theorem, the exact endofunctor $\Sigma^n : \mathcal{C}^{w_{\Sigma,n}} \to \mathcal{C}^{w_{\Sigma,n}}$ induces a self-homotopy equivalence in K-theory. Furthermore, it factors over $\mathcal{C}^w$. Since $wS_\bullet\mathcal{C}^w$ is contractible, the claim follows. $\square$

4.14. Proposition (modified fibration theorem). *Let $\mathcal{C}$ be a category with cofibrations. Let $v\mathcal{C} \subseteq w\mathcal{C}$ be two subcategories of weak equivalences. Suppose that $\mathcal{C}$ has* *a cylinder functor which satisfies the cylinder axiom with respect to $v\mathcal{C}$ (hence also with respect to $w\mathcal{C}$). Assume that $v\mathcal{C}$ and $w\mathcal{C}$ satisfy the saturation axiom.*

Then the canonical inclusion functors induce a homotopy pullback square

$$\begin{tikzcd} vS_\bullet\mathcal{C}^w \arrow[r] \arrow[d] & wS_\bullet\mathcal{C}^w \simeq * \arrow[d] \\ vS_\bullet\mathcal{C} \arrow[r] & wS_\bullet\mathcal{C} \end{tikzcd}$$

in which the corner on the top right is canonically contractible.

Proof. The square that we are claiming to be a homotopy pullback comes with a transformation to the square

$$\begin{tikzcd} vS_\bullet\mathcal{C}^{w_\Sigma} \arrow[r] \arrow[d] & w_\Sigma S_\bullet\mathcal{C}^{w_\Sigma} \simeq * \arrow[d] \\ vS_\bullet\mathcal{C} \arrow[r] & w_\Sigma S_\bullet\mathcal{C} \end{tikzcd} \tag{4.15}$$

This transformation is the identity on the lower left corner, and is induced by the canonical inclusion functors on the other three corners. As $w_\Sigma\mathcal{C}$ satisfies the extension axiom by Lemma 4.13, the square (4.15) is a homotopy pullback by the fibration theorem. The entries on the top right corners of both squares are contractible. The map between the lower right corners is a weak equivalence by Lemma 4.13. So all we have to check is that the canonical map $vS_\bullet\mathcal{C}^w \to vS_\bullet\mathcal{C}^{w_\Sigma}$ is a weak equivalence.

Just as in the proof of Lemma 4.13, we can write $\mathcal{C}^{w_\Sigma}$ as a directed union $\mathcal{C}^{w_\Sigma} = \bigcup_n \mathcal{C}^{w_{\Sigma,n}}$. In the direct limit system

$$\mathcal{C}^w \hookrightarrow \mathcal{C}^{w_{\Sigma,1}} \hookrightarrow \mathcal{C}^{w_{\Sigma,2}} \hookrightarrow \cdots,$$

each arrow induces an equivalence in K-theory (the suspension functor provides a homotopy inverse), and the claim follows from this. □

We can now begin to study the K-theory of categories of controlled retractive spaces. For technical reasons, we need to impose certain intermediate finiteness conditions on the objects as long as we are dealing with connective K-theory. This phenomenon is well-known in the linear setting; see [Cárdenas and Pedersen 1997].

Let $F \leqslant K_0(\mathcal{R}^G_{fd}(W,\mathfrak{Z}),h)$ be a subgroup. Denote by $\mathcal{R}^G_{fd,F}(W,\mathfrak{Z})$ the full subcategory of all those objects whose K_0-class lies in F. We think of these objects as being subject to an intermediate finiteness condition. Note that these objects can be equivalently characterized as those complexes whose K_0-class lies in the kernel of the projection homomorphism $K_0(\mathcal{R}^G_{fd}(W,\mathfrak{Z}),h) \to K_0(\mathcal{R}^G_{fd}(W,\mathfrak{Z}),h)/F$. It is now a consequence of Thomason's cofinality theorem [Thomason and Trobaugh 1990, Cofinality Theorem 1.10.1] that there is a homotopy fiber sequence

$$hS_\bullet\mathcal{R}^G_{fd,F}(W,\mathfrak{Z}) \to hS_\bullet\mathcal{R}^G_{fd}(W,\mathfrak{Z}) \to N_\bullet(K_0(\mathcal{R}^G_{fd}(W,\mathfrak{Z}),h)/F).$$

In particular, the change of finiteness condition only affects K_0; there, the induced map is a monomorphism which can be identified with the inclusion map $F \hookrightarrow K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h)$. A typical choice for F is $F := K_0(\mathcal{R}^G_f(W, \mathfrak{Z}), h)$, which we regard as a subgroup of $K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h)$ by virtue of Proposition 4.8.

Let $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ be a coarse structure and A be a G-invariant subset of Z. Let $\mathfrak{S}\langle A\rangle$ be the collection of all sets of the form $A^C \cap S$, where S is an element of $\mathfrak{S}$ and $C \in \mathfrak{C}$; see Definition 3.6. Define a new coarse structure $\mathfrak{Z}\langle A\rangle := (Z, \mathfrak{C}, \mathfrak{S}\langle A\rangle)$.

Recall from Definition 3.12 that A gives rise to a class of weak equivalences $h^A\mathcal{R}^G_f(W, \mathfrak{Z})$.

4.16. Theorem. *Let $\mathfrak{Z}$ be a coarse structure and let $A \subseteq Z$ be a G-invariant subset. Set $K := K_0(\mathcal{R}^G_f(W, \mathfrak{Z}), h) \leqslant K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h)$. Let $F \leqslant K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}\langle A\rangle), h)$ be the preimage of K under the natural homomorphism $K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}\langle A\rangle), h) \to K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h)$. Then*

$$hS_\bullet\mathcal{R}^G_{fd,F}(W, \mathfrak{Z}\langle A\rangle) \to hS_\bullet\mathcal{R}^G_{fd,K}(W, \mathfrak{Z}) \to h^A S_\bullet\mathcal{R}^G_{fd,K}(W, \mathfrak{Z}) \tag{4.17}$$

is a homotopy fiber sequence. Upon realization, there is a homotopy fiber sequence

$$|hS_\bullet\mathcal{R}^G_{fd,F}(W, \mathfrak{Z}\langle A\rangle)| \to |hS_\bullet\mathcal{R}^G_f(W, \mathfrak{Z})| \to |h^A S_\bullet\mathcal{R}^G_f(W, \mathfrak{Z})| \tag{4.18}$$

which is weakly equivalent to the former one.

Proof. The modified fibration theorem (Proposition 4.14) applies to our situation, so we have a homotopy fiber sequence

$$hS_\bullet\mathcal{R}^G_f(W, \mathfrak{Z})^{h^A} \to hS_\bullet\mathcal{R}^G_f(W, \mathfrak{Z}) \to h^A S_\bullet\mathcal{R}^G_f(W, \mathfrak{Z}).$$

Define $F' \leqslant K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z})^{h^A}, h)$ to be the preimage of $K_0(\mathcal{R}^G_f(W, \mathfrak{Z}), h)$ under the canonical homomorphism $K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z})^{h^A}, h) \to K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h)$. We first prove the following two assertions:

(1) The natural inclusion functor $\mathcal{R}^G_f(W, \mathfrak{Z})^{h^A} \to \mathcal{R}^G_{fd,F'}(W, \mathfrak{Z})^{h^A}$ induces an equivalence in K-theory.

(2) The natural inclusion functor $\mathcal{R}^G_{fd,F}(W, \mathfrak{Z}\langle A\rangle) \to \mathcal{R}^G_{fd,F'}(W, \mathfrak{Z})^{h^A}$ induces an equivalence in K-theory.

The first claim is proved in a fashion similar to that of Proposition 4.8. First of all, we may replace $\mathcal{R}^G_f(W, \mathfrak{Z})^{h^A}$ by the category of homotopy finite objects $\mathcal{R}^G_{hf}(W, \mathfrak{Z})^{h^A}$ since the inclusion of the former into the latter induces an equivalence in K-theory by the approximation theorem.

To conclude that the inclusion $\mathcal{R}^G_{hf}(W, \mathfrak{Z})^{h^A} \to \mathcal{R}^G_{fd,F'}(W, \mathfrak{Z})^{h^A}$ induces an equivalence as well, we rely on Vogell's cofinality Theorem 4.6 once again. As we already saw in the proof of Proposition 4.8, for every object $Y_1 \in \mathcal{R}^G_{fd,F'}(W, \mathfrak{Z})^{h^A}$ there are some finitely dominated object $Y_2 \in \mathcal{R}^G_{fd}(W, \mathfrak{Z})$ with $[Y_2] \in K_0(\mathcal{R}^G_f(W, \mathfrak{Z}), h)$, a finite object $Y \in \mathcal{R}^G_f(W, \mathfrak{Z})$ and an h-equivalence $f : Y \xrightarrow{\sim} \Sigma^2 Y_1 \vee Y_2$. However, there is no reason for Y_2 to be h^A-contractible, so we have to improve it.

Since Y_1 is h^A-contractible, there is some cofinal subcomplex $Y_1' \subseteq Y_1$ away from A such that the inclusion map $j_1 : Y_1' \hookrightarrow Y_1$ is nullhomotopic. By Lemma 3.7, there is a cofinal subcomplex $Y' \subseteq Y$ whose image under f is contained in $\Sigma^2 Y_1' \vee Y_2$; note that $Y' \subseteq Y$ is finite. Let $j : Y' \to Y$ be the inclusion map. Define $f' : Y' \to Y_2$ as the composition $Y' \xrightarrow{j} Y \xrightarrow{f} \Sigma^2 Y_1 \vee Y_2 \twoheadrightarrow Y_2$. Set $Y_3 := C(f')$ and observe that this is a finitely dominated object with $[Y_3] \in K_0(\mathcal{R}^G_f(W, \mathfrak{Z}), h)$. Let $i_2 : Y_2 \rightarrowtail \Sigma^2 Y_1 \vee Y_2$ be the canonical cofibration. Since j_1 is nullhomotopic, the map fj is homotopic to $i_2 f'$. Hence, there is an h-equivalence $C(fj) \xrightarrow{\sim_h} C(i_2 f') \simeq_h \Sigma^2 Y_1 \vee Y_3$. Observe that $C(fj)$ is homotopy finite, so $\Sigma^2 Y_1 \vee Y_3$ is homotopy finite, too.

Moreover, the natural map $C(fj) \to C(f)$ is an h^A-equivalence because j is one. As $C(f)$ is contractible, both $C(fj)$ and $\Sigma^2 Y_1 \vee Y_3$ are h^A-contractible. It follows that Y_3 is also h^A-contractible. Note $[Y_3] \in F'$. Therefore, Vogell's cofinality theorem applies. That is, all higher K-theory groups of $\mathcal{R}^G_{hf}(W, \mathfrak{Z})^{h^A}$ and $\mathcal{R}^G_{fd,F'}(W, \mathfrak{Z})^{h^A}$ coincide, and the induced homomorphism $K_0(\mathcal{R}^G_{hf}(W, \mathfrak{Z})^{h^A}, h) \to K_0(\mathcal{R}^G_{fd,F'}(W, \mathfrak{Z})^{h^A}, h)$ is injective.

On the level of K_0, we have a commutative diagram

$$\begin{tikzcd}[column sep=small]
K_0(\mathcal{R}^G_f(W, \mathfrak{Z})^{h^A}, h) \arrow[rr] \arrow[dr] & & K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z})^{h^A}, h) \\
& K_0(\mathcal{R}^G_{fd,F'}(W, \mathfrak{Z})^{h^A}, h) \arrow[ur] &
\end{tikzcd}$$

in which the left diagonal arrow is an injection. The right diagonal map is an injection by Thomason's cofinality theorem. Since the top horizontal and right diagonal homomorphism have the same image, it follows that the left diagonal map is an isomorphism. This shows claim (1).

Let us now turn to the second claim. We apply the approximation theorem. By Corollary 4.11, we may assume without loss of generality that all complexes are finite-dimensional. Only the second part of the approximation property needs checking. Let $Y_0 \in \mathcal{R}^G_{fd,F,\dim<\infty}(W, \mathfrak{Z}\langle A\rangle)$; then Y_0 is h^A-contractible. Let $f : Y_0 \to Y$ be a morphism in $\mathcal{R}^G_{fd,F',\dim<\infty}(W, \mathfrak{Z})^{h^A}$.

Let $r : Y \to W$ be the structural retraction and $s : W \to Y$ be the structural inclusion of Y. Since Y is h^A-contractible, there are a subcomplex $Y' \subseteq Y$ which is cofinal away from A and a homotopy h from the inclusion map $Y' \hookrightarrow Y$ to the composition $Y' \xrightarrow{r|_{Y'}} W \xrightarrow{s} Y$. By the CHEP, we find an extension of h to a controlled homotopy $H : Y \rtimes [0, 1] \to Y$ from id_Y to a controlled map $p' : Y \to Y$ which extends $s \circ r|_{Y'}$.

Let Y'' be the G-subcomplex of Y generated by the image of p'. Since p' is controlled and Y is finite-dimensional, Y'' is supported on some $\mathfrak{Z}$-thickening of A.

Note that we do not claim that Y'' is finitely dominated. Let $j : Y'' \hookrightarrow Y$ be the inclusion. Denote by p the map p', regarded as a map $Y \to Y''$. Then $Y \xrightarrow{p} Y'' \xrightarrow{j} Y$ is homotopic to the identity. Proposition 4.10 provides us with an h-equivalence $T(j) : \mathrm{Tel}(jp) \xrightarrow{\sim} Y$. In particular, $[\mathrm{Tel}(jp)] = [Y] \in K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h)$, and hence both lie in $K_0(\mathcal{R}^G_f(W, \mathfrak{Z}), h)$. Observe that $\mathrm{Tel}(jp)$ is also supported on a $\mathfrak{Z}$-thickening of A.

Pick now a finite domination $Y \xrightarrow{i} D \xrightarrow{d} Y$ of Y in $\mathcal{R}^G(W, \mathfrak{Z})$. Let D' be the smallest subcomplex of D which contains the image of $i \circ T(j)$. Since $i \circ T(j)$ is a controlled map and $\mathrm{Tel}(jp)$ is supported on a $\mathfrak{Z}$-thickening of A, the complex D' is also supported on some $\mathfrak{Z}$-thickening of A. The composition of the maps $\mathrm{Tel}(jp) \xrightarrow{i \circ T(j)} D'$ and $d' : D' \hookrightarrow D \xrightarrow{d} Y \to \mathrm{Tel}(jp)$ is homotopic to the identity. Redefining the retraction of D' to be d' composed with the retraction of $\mathrm{Tel}(jp)$, the latter map becomes a morphism. Hence, $\mathrm{Tel}(jp) \in \mathcal{R}^G_{fd}(W, \mathfrak{Z}\langle A\rangle)$.

Another application of the mapping cylinder argument (Lemma 4.1) to the diagram $Y_0 \xrightarrow{f} Y \xleftarrow{T(j)} \mathrm{Tel}(jp)$ yields the approximation property, and hence the second assertion.

There is a map of homotopy fiber sequences

$$\begin{array}{ccccc}
hS_\bullet\mathcal{R}^G_f(W, \mathfrak{Z})^{h^A} & \longrightarrow & hS_\bullet\mathcal{R}^G_f(W, \mathfrak{Z}) & \longrightarrow & h^A S_\bullet\mathcal{R}^G_f(W, \mathfrak{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
hS_\bullet\mathcal{R}^G_{fd,K}(W, \mathfrak{Z})^{h^A} & \longrightarrow & hS_\bullet\mathcal{R}^G_{fd,K}(W, \mathfrak{Z}) & \longrightarrow & h^A S_\bullet\mathcal{R}^G_{fd,K}(W, \mathfrak{Z})
\end{array}$$

The middle vertical map is a weak equivalence by Thomason cofinality. Observe that $\mathcal{R}^G_{fd,F'}(W, \mathfrak{Z})^{h^A} = \mathcal{R}^G_{fd,K}(W, \mathfrak{Z})^{h^A}$. Hence, assertion (1) implies that the left vertical map is a weak equivalence. Therefore, the right vertical map is a weak equivalence, too. Composing the weak equivalence

$$hS_\bullet\mathcal{R}^G_{fd,F}(W, \mathfrak{Z}\langle A\rangle) \xrightarrow{\sim} hS_\bullet\mathcal{R}^G_{fd,F'}(W, \mathfrak{Z})^{h^A}$$

from assertion (2) with the inclusion of the homotopy fiber yields sequence (4.17).

After taking realizations, we can invert the weak equivalence of assertion (1) to obtain sequence (4.18). □

4.19. Definition (cf. [Bartels et al. 2004, Definition 4.1]). A coarse structure $\mathfrak{Z}$ is called *G-proper with respect to A* if for every $C \in \mathfrak{C}$ and every $S \in \mathfrak{S}$ there are $S' \in \mathfrak{S}$, $C' \in \mathfrak{C}$ and a G-equivariant function $c : A^C \cap S \to A \cap S'$ such that

(1) $\{(c(z), z) \mid z \in A^C \cap S\} \subseteq C'$,

(2) for every set $B \subseteq A^C \cap S$ which is locally finite in Z, the image $c(B)$ is locally finite in Z and $c^{-1}(x) \cap B$ is finite for all $x \in c(B)$.

4.20. Proposition. *Let $A \subseteq Z$ be a closed, G-invariant subset. Let $\mathfrak{Z}$ be a coarse structure which is G-proper with respect to A. Recall the definition of the coarse* *structure $\mathfrak{Z}\cap A$ from Example 3.29. Let $F \leqslant K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}\cap A), h)$ be the preimage of $K_0(\mathcal{R}^G_f(W, \mathfrak{Z}), h)$ under the natural homomorphism $K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}\cap A), h) \to K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h)$, and define $F' \leqslant K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}\langle A\rangle), h)$ analogously.*

Then the exact inclusion functor

$$\mathcal{R}^G_{fd,F}(W, \mathfrak{Z}\cap A) \hookrightarrow \mathcal{R}^G_{fd,F'}(W, \mathfrak{Z}\langle A\rangle)$$

is an equivalence of Waldhausen categories.

Proof. It suffices to show that the inclusion functor is essentially surjective. So let $Y \in \mathcal{R}^G_{fd,F'}(W, \mathfrak{Z}\langle A\rangle)$. For each $k \in \mathbb{N}$, pick $S_k \in \mathfrak{S}$ such that $\kappa(\diamond_k Y) \subseteq A^{C_k} \cap S_k$. Since $\mathfrak{Z}$ is G-proper with respect to A, there is a G-equivariant function

$$c_k : A^{C_k} \cap S_k \to A \cap S'_k$$

as in Definition 4.19. The collection $\{c_k \circ \kappa|_{\diamond_k Y}\}_k$ defines a G-equivariant function $\kappa_A : \diamond Y \to A$ such that $\kappa(\diamond_k Y) \subseteq A \cap S'_k$. By construction, the identity map is a controlled isomorphism between (Y, κ) and (Y, κ_A), where the latter complex is now supported on A. The modification we make to the control map κ also preserves finiteness, and hence finite dominations. Note that $[(Y, \kappa)] = [(Y, \kappa_A)]$ is in $K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h)$, i.e., $[(Y, \kappa_A)] \in F$. This finishes the proof. □

4C. *The coarse Mayer–Vietoris theorem.* The main application of the homotopy fiber sequence established in the previous subsection is the excision result we prove next. Let $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ be a coarse structure.

4.21. Definition [Bartels et al. 2004, Proposition 4.3]. A pair (A, B) of G-invariant subspaces in Z is called *coarsely excisive* if for all $C \in \mathfrak{C}$ there is $C' \in \mathfrak{C}$ such that $A^C \cap B^C \subseteq (A \cap B)^{C'}$.

A *coarsely excisive triple* is a coarse structure $\mathfrak{Z}$ together with two closed, G-invariant subspaces $A_1, A_2 \subseteq Z$ such that $A_1 \cup A_2 = Z$ and the pair (A_1, A_2) is coarsely excisive.

We require a little more notation. For a closed, G-invariant subspace $A \subseteq Z$ we define $\mathfrak{Z}|_A := (Z, \mathfrak{C}, \mathfrak{S} \Cap A)$. Observe that $\mathcal{R}^G(W, \mathfrak{Z}|_A) \cong \mathcal{R}^G(W, \mathfrak{Z}\cap A)$.

4.22. Lemma. *Suppose that (A, B) is a coarsely excisive pair. Then*

$$h^{A\cap B} S_\bullet \mathcal{R}^G_f(W, \mathfrak{Z}|_A) = h^B S_\bullet \mathcal{R}^G_f(W, \mathfrak{Z}|_A).$$

Proof. Let (Y, κ) be an object in $\mathcal{R}^G_f(W, \mathfrak{Z}|_A)$. Note that the image of κ is contained in A. It suffices to show that a subcomplex $Y' \subseteq Y$ is cofinal away from $A \cap B$ if and only if it is cofinal away from B. Since $A \cap B \subseteq B$, it is obvious that every subcomplex which is cofinal away from $A \cap B$ is also cofinal away from B.

Now suppose that $Y' \subseteq Y$ is cofinal away from B. Let $k \in \mathbb{N}$. Then there is $C \in \mathfrak{C}$ such that $\kappa^{-1}(Z \setminus B^C) \cap \diamond_k Y \subseteq \diamond_k Y'$. By assumption, we can find $C' \in \mathfrak{C}$ such that $A^C \cap B^C \subseteq (A \cap B)^{C'}$. Then we have

$$\begin{aligned}\kappa^{-1}(Z \setminus (A \cap B)^{C'}) \cap \diamond_k Y &\subseteq \kappa^{-1}(Z \setminus (A^C \cap B^C)) \cap \diamond_k Y \\ &= \kappa^{-1}(Z \setminus A^C \cup Z \setminus B^C) \cap \diamond_k Y \\ &= (\kappa^{-1}(Z \setminus A^C) \cap \diamond_k Y) \cup (\kappa^{-1}(Z \setminus B^C) \cap \diamond_k Y) \\ &= \kappa^{-1}(Z \setminus B^C) \cap \diamond_k Y \\ &\subseteq \diamond_k Y'.\end{aligned}$$

This shows that $Y' \subseteq Y$ is cofinal away from $A \cap B$, and we are done. $\square$

4.23. Theorem (coarse Mayer–Vietoris, connective version). *Let $(\mathfrak{Z}, A_1, A_2)$ be a coarsely excisive triple, and assume that $\mathfrak{Z}$ is G-proper with respect to A_1, A_2 and $A_1 \cap A_2$. Let F be the preimage of $K := K_0(\mathcal{R}^G_f(W, \mathfrak{Z}), h)$ under the canonical homomorphism $K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z} \cap A_2), h) \to K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h)$, and let F' be the preimage of $K' := K_0(\mathcal{R}^G_f(W, \mathfrak{Z} \cap A_1), h)$ under the canonical homomorphism $K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z} \cap (A_1 \cap A_2)), h) \to K_0(\mathcal{R}^G_{fd}(W, \mathfrak{Z} \cap A_1), h)$.*

Then the natural inclusion maps induce a homotopy pullback square

$$\begin{tikzcd} hS_\bullet\mathcal{R}^G_{fd,F'}(W, \mathfrak{Z} \cap (A_1 \cap A_2)) \arrow[r] \arrow[d] & hS_\bullet\mathcal{R}^G_{fd,K'}(W, \mathfrak{Z} \cap A_1) \arrow[d] \\ hS_\bullet\mathcal{R}^G_{fd,F}(W, \mathfrak{Z} \cap A_2) \arrow[r] & hS_\bullet\mathcal{R}^G_{fd,K}(W, \mathfrak{Z}) \end{tikzcd}$$

Proof. Using Theorem 4.16 and Proposition 4.20, we have a map of homotopy fiber sequences

$$\begin{tikzcd} hS_\bullet\mathcal{R}^G_{fd,F'}(W, \mathfrak{Z} \cap (A_1 \cap A_2)) \arrow[r] \arrow[d] & hS_\bullet\mathcal{R}^G_{fd,K'}(W, \mathfrak{Z} \cap A_1) \arrow[r] \arrow[d] & h^{A_1 \cap A_2}S_\bullet\mathcal{R}^G_{fd,K'}(W, \mathfrak{Z} \cap A_1) \arrow[d] \\ hS_\bullet\mathcal{R}^G_{fd,F}(W, \mathfrak{Z} \cap A_2) \arrow[r] & hS_\bullet\mathcal{R}^G_{fd,K}(W, \mathfrak{Z}) \arrow[r] & h^{A_2}S_\bullet\mathcal{R}^G_{fd,K}(W, \mathfrak{Z}) \end{tikzcd}$$

Hence, it suffices to show that the right vertical map is a weak equivalence. We may replace $\mathcal{R}^G_{fd,K'}(W, \mathfrak{Z} \cap A_1)$ and $\mathcal{R}^G_{fd,K}(W, \mathfrak{Z})$ by $\mathcal{R}^G_f(W, \mathfrak{Z} \cap A_1)$ and $\mathcal{R}^G_f(W, \mathfrak{Z})$, respectively. Using Lemma 4.22, we can therefore identify the right vertical map as the natural inclusion map

$$\begin{aligned}h^{A_1 \cap A_2}S_\bullet\mathcal{R}^G_f(W, \mathfrak{Z} \cap A_1) \cong h^{A_1 \cap A_2}S_\bullet\mathcal{R}^G_f(W, \mathfrak{Z}|_{A_1}) = h^{A_2}S_\bullet\mathcal{R}^G_f(W, \mathfrak{Z}|_{A_1})& \\ \to h^{A_2}S_\bullet\mathcal{R}^G_f(W, \mathfrak{Z}).&\end{aligned}$$

Our claim is that the approximation theorem applies to show that this is an equivalence. We need only check the second part of the approximation property. So let

$f : Y_1 \to Y_2$ be a morphism in $\mathcal{R}^G_f(W, \mathfrak{Z})$ such that Y_1 is an object in $\mathcal{R}^G_f(W, \mathfrak{Z}|_{A_1})$. Define Y_2' as the smallest subcomplex of Y_2 which contains both the image of f and all cells which are labeled with points in A_1. Then Y_2' is supported on a $\mathfrak{Z}$-thickening of A_1. Since $\kappa^{-1}(Z \setminus A_2) \subseteq \kappa^{-1}(A_1) \subseteq \diamond Y_2'$, the subcomplex $Y_2' \subseteq Y_2$ is cofinal away from A_2. This implies that the inclusion map $Y_2' \hookrightarrow Y_2$ is an h^{A_2}-equivalence. As in the proof of Proposition 4.20, $\mathfrak{Z}$ being G-proper with respect to A_1 implies that Y_2' is isomorphic to an object Y_3 with support in A_1. Then f factors over Y_3, which shows the approximation property. □

4D. ***A vanishing result.*** To conclude this section, we also record a criterion which guarantees the vanishing of the K-theory of a category of controlled retractive spaces.

4.24. Proposition (Eilenberg swindles). *Let $\mathcal{C}$ be a small Waldhausen category. Let $\vee$ be a functorial coproduct on $\mathcal{C}$. Suppose that there is an exact endofunctor* sw *on $\mathcal{C}$ and a natural isomorphism* $\mathrm{id} \vee \mathrm{sw} \cong \mathrm{sw}$.

Then there is a contraction H_{sw} of $K(\mathcal{C})$ which is natural in the following sense: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be small Waldhausen categories, equipped with functorial coproducts $\vee_i$, $i = 1, 2$. Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be an exact functor which strictly preserves the coproduct, i.e., $F \circ \vee_1 = \vee_2 \circ (F \times F)$. Let sw_i be exact endofunctors on $\mathcal{C}_i$ together with natural isomorphisms $\eta_i : \mathrm{id}_{\mathcal{C}_i} \vee \mathrm{sw}_i \cong \mathrm{sw}_i$, $i = 1, 2$, such that $\mathrm{sw}_2 \circ F = F \circ \mathrm{sw}_1$ and $F \circ \eta_{1,A} = \eta_{2,FA}$. Then

$$H_{\mathrm{sw}_2} \circ (K(F) \times [0, 1]) = K(F) \circ H_{\mathrm{sw}_1}.$$

Proof. Recall that $K(\mathcal{C}) = \Omega|wS_\bullet\mathcal{C}|$, so concatenation of loops defines an H-space structure "+" on $K(\mathcal{C})$. Subject to the choice of an orientation-preserving homeomorphism $[0, 1] \cong [0, 2]$, the H-space product is naturally homotopy associative. Similarly, any choice of orientation-reversing homeomorphism $[0, 1] \cong [0, 1]$, say $t \mapsto 1-t$, induces a homotopy inverse inv such that $\mathrm{id} + \mathrm{inv}$ is nullhomotopic. This nullhomotopy depends on a choice of contraction of $[0, 1]$ to the point 0. Fixing, once and for all, suitable homeomorphisms $[0, 1] \cong [0, 2]$ and $[0, 1] \cong [0, 1]$ and a contraction of $[0, 1]$, all these homotopies become natural with respect to maps induced by exact functors.

The functorial coproduct $\vee : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ induces another H-space structure "$\vee$" on $K(\mathcal{C})$. Since $+$ and $\vee$ satisfy the interchange law, the Eckmann–Hilton argument shows that there is a natural homotopy $\vee \simeq +$. By abuse of notation, we use in the sequel the same name for functors and the maps they induce on K-theory. Let 0 denote the constant functor mapping everything to the zero object. Then we have

$$\begin{aligned}\mathrm{id} &\simeq \mathrm{id} + 0 \simeq \mathrm{id} + (\mathrm{sw} + (\mathrm{inv} \circ \mathrm{sw})) \simeq (\mathrm{id} + \mathrm{sw}) + (\mathrm{inv} \circ \mathrm{sw}) \\ &\simeq (\mathrm{id} \vee \mathrm{sw}) + (\mathrm{inv} \circ \mathrm{sw}) \simeq \mathrm{sw} + \mathrm{inv} \circ \mathrm{sw} \simeq 0.\end{aligned}$$

The fifth homotopy is induced by the natural isomorphism η. Hence, the concatenation of these homotopies defines a contraction of $K(\mathcal{C})$, and it is straightforward to check that this contraction is natural in the desired sense. □

4.25. Proposition. *Let $\mathfrak{Z}$ be a coarse structure and let $A \subseteq Z$ be a G-invariant subset. Suppose that there is a sequence of G-equivariant functions $(s_n : Z \to Z)_{n\in\mathbb{N}}$ which satisfies the following properties*:

(1) $s_0 = \mathrm{id}_Z$.

(2) *For every $C \in \mathfrak{C}$ and $S \in \mathfrak{S}$ there is some $C' \in \mathfrak{C}$ such that*
$$\bigcup_{n\geqslant 0}(s_n \times s_n)(C \cap (S \times S)) \subseteq C'.$$

(3) *For every $S \in \mathfrak{S}$ there is some $S' \in \mathfrak{S}$ such that $\bigcup_n s_n(S) \subseteq S'$.*

(4) *For every $S \in \mathfrak{S}$ and every $B \subseteq S$ which is locally finite in Z, each image $s_n(B)$ is locally finite in Z and $s_n^{-1}(x) \cap B$ is finite for all $x \in s_n(B)$. Furthermore, there are for every $z \in Z$ some n_0 and an open neighborhood U of z such that $s_n^{-1}(U) = \varnothing$ for all $n \geqslant n_0$.*

(5) *For every $C \in \mathfrak{C}$ there exists $C' \in \mathfrak{C}$ such that*
$$\bigcup_{n\geqslant 0} s_n(A^C) \subseteq A^{C'}.$$

(6) *For every $S \in \mathfrak{S}$ there is some $C \in \mathfrak{C}$ such that*
$$\bigcup_{n\geqslant 0}\{(s_{n+1}(x), s_n(x)) \mid x \in S\} \subseteq C.$$

Then there is an exact endofunctor on $(\mathcal{R}^G_f(W, \mathfrak{Z}), h^A)$ as in Proposition 4.24. This swindle is natural in the following sense: Let $\mathfrak{z} : \mathfrak{Z}_1 \to \mathfrak{Z}_2$ be a morphism of coarse structures. If $(s^i_n : Z_i \to Z_i)_n$, $i = 1, 2$, are as above such that $\mathfrak{z} \circ s^1_n = s^2_n \circ \mathfrak{z}$ for all n, then the induced exact functor $\mathcal{R}(\mathfrak{z})$ satisfies the assumptions of Proposition 4.24.

The same holds with $(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h^A)$ instead of $(\mathcal{R}^G_f(W, \mathfrak{Z}), h^A)$.

Proof. Define a functor $S : \mathcal{R}^G_f(W, \mathfrak{Z}) \to \mathcal{R}^G_f(W, \mathfrak{Z})$ as follows. Given a controlled retractive space (Y, κ) over W, consider the infinite coproduct $Y^\infty := \bigvee_{n\geqslant 0} Y$. So $\diamond Y^\infty = \coprod_{n\geqslant 0} \diamond Y$. Define a control map $\kappa^\infty : \diamond Y^\infty \to Z$ by $\kappa^\infty(e) := (s_n \circ \kappa)(e)$ if e is a cell in the n-th copy of Y. Then conditions (2) and (3) ensure that $(Y^\infty, \kappa^\infty)$ is a controlled retractive space over W. If $f : Y_1 \to Y_2$ is a controlled morphism, then $\bigvee_{n\geqslant 0} f : Y^\infty_1 \to Y^\infty_2$ is again a controlled morphism. Moreover, condition (4) guarantees that Y^∞ is finite if Y is finite.

Define $S(Y) := (Y^\infty, \kappa^\infty)$. We claim that this functor preserves h^A-equivalences. It suffices to check that for any subcomplex $Y' \subseteq Y$ which is cofinal away from A, the subcomplex $S(Y') \subseteq S(Y)$ is also cofinal away from A.

For the next paragraph, denote the n-th copy of Y by Y_n, and use Y'_n in the same way. Let $k \in \mathbb{N}$. Then there is some $C \in \mathfrak{C}$ such that $\kappa^{-1}(Z \setminus A^C) \cap \diamond_k Y \subseteq \diamond_k Y'$. Let $e \in (\kappa^\infty)^{-1}(Z \setminus s_n(A^C)) \cap \diamond_k Y_n$, and let e' be the corresponding k-cell in the original copy of Y. Since $s_n(\kappa(e')) = \kappa^\infty(e) \notin s_n(A^C)$, it follows that $e' \in \diamond_k Y'$. Consequently, $e \in \diamond_k Y'_n$, and we have shown that $(\kappa^\infty)^{-1}(Z \setminus s_n(A^C)) \cap \diamond_k Y_n \subseteq \diamond_k Y'_n$. Choosing $C' \in \mathfrak{C}$ as in (5), we have

$$\begin{aligned}(\kappa^\infty)^{-1}\big(Z \setminus A^{C'}\big) \cap \diamond_k Y^\infty &\subseteq (\kappa^\infty)^{-1}\Big(Z \setminus \bigcup_n s_n(A^C)\Big) \cap \diamond_k Y^\infty \\ &\subseteq \bigcup_{m \geqslant 0}\Big((\kappa^\infty)^{-1}\Big(Z \setminus \bigcup_n s_n(A^C)\Big) \cap \diamond_k Y_m\Big) \\ &\subseteq \bigcup_{m \geqslant 0}\big((\kappa^\infty)^{-1}(Z \setminus s_m(A^C)) \cap \diamond_k Y_m\big) \\ &\subseteq \bigcup_{m \geqslant 0} \diamond_k Y'_m = \diamond_k (Y')^\infty.\end{aligned}$$

So $S(Y') \subseteq S(Y)$ is also cofinal away from A, and it follows that S is an exact functor with respect to the h^A-equivalences.

The map $Y \vee Y^\infty \to Y^\infty$ which maps Y_n identically to Y_{n+1} and Y to Y_0 is a controlled isomorphism by condition (6). It induces a natural isomorphism $\mathrm{id} \vee S \cong S$.

Checking the naturality statement is straightforward. $\square$

4.26. Remark. Note that the conditions of Proposition 4.25 hold for $A = \varnothing$ whenever they are satisfied for some $A \subseteq Z$. Hence, the K-theory space $\Omega|hS_\bullet \mathcal{R}^G_f(W, \mathfrak{Z})|$ is also contractible.

Most of the time, the sequence of maps $(s_n)_n$ is induced by an *infinite shift map*, i.e., a G-equivariant function $s : Z \to Z$ with the following properties:

(1) For every $C \in \mathfrak{C}$ and $S \in \mathfrak{S}$ there is some $C' \in \mathfrak{C}$ such that

$$\bigcup_n (s \times s)^n (C \cap (S \times S)) \subseteq C'.$$

(2) For every $S \in \mathfrak{S}$ there is some $S' \in \mathfrak{S}$ such that $\bigcup_n s^n(S) \subseteq S'$.

(3) For every $S \in \mathfrak{S}$ and every $B \subseteq S$ which is locally finite in Z, the image $s(B)$ is locally finite in Z and $s^{-1}(x) \cap B$ is finite for all $x \in s(B)$. Furthermore, there are for every $z \in Z$ some n_0 and an open neighborhood U of z such that $(s^n)^{-1}(U) = \varnothing$ for all $n \geqslant n_0$.

(4) For every $C \in \mathfrak{C}$ there exists $C' \in \mathfrak{C}$ such that

$$\bigcup_{n \geqslant 0} s^n(A^C) \subseteq A^{C'}.$$

(5) For every $S \in \mathfrak{S}$ there is some $C \in \mathfrak{C}$ such that

$$\left\{(s(x), x) \mid x \in \bigcup_n s^n(S)\right\} \subseteq C.$$

In this case, the proposition applies with $s_n := s^n$, and the corresponding naturality statement applies whenever we have two infinite shift maps s_1 and s_2 as well as a morphism of coarse structures $\mathfrak{z}$ such that $\mathfrak{z} \circ s_1 = s_2 \circ \mathfrak{z}$.

5. Nonconnective A-theory spectra

We are now ready to put the results of the previous sections to use. Namely, we define (potentially) nonconnective deloopings of the K-theory spaces of controlled retractive spaces. The resulting spectra are insensitive to specific choices of finiteness conditions, and the main results of Section 4 simplify accordingly.

For linear K-theory, such deloopings have been defined previously by Pedersen and Weibel [1985]. Vogell [1990] adopted this approach to define a nonconnective delooping of $A(X)$.

5.1. Definition. Suppose that $Z = Z_1 \times Z_2$, and that $(Z_1, \mathfrak{C}, \mathfrak{S})$ is a coarse structure. Let $p : Z \to Z_1$ be the projection map. Then we define a coarse structure $(Z, p^*\mathfrak{C}, p^*\mathfrak{S})$ by setting

$$p^*\mathfrak{C} := \{(p \times p)^{-1}(C) \mid C \in \mathfrak{C}\} \quad \text{and} \quad p^*\mathfrak{S} := \{p^{-1}(S) \mid S \in \mathfrak{S}\}.$$

Let $\mathfrak{Z} = (Z, \mathfrak{C}, \mathfrak{S})$ be a coarse structure. Let $p_n : \mathbb{R}^n \times Z \to \mathbb{R}^n$ and $p_Z : \mathbb{R}^n \times Z \to Z$ denote the respective projection maps. Consider the bounded coarse structure $\mathfrak{B}(\mathbb{R}^n) = (\mathbb{R}^n, \mathfrak{C}_{\mathrm{bdd}}(\mathbb{R}^n), \mathfrak{S}_{\mathrm{triv}}(\mathbb{R}^n))$ from Example 2.2.

5.2. Definition. For $n \in \mathbb{N}$ define the coarse structure $\mathfrak{Z}(n) = (\mathbb{R}^n \times Z, \mathfrak{C}(n), \mathfrak{S}(n))$ by letting a set $C \subseteq (\mathbb{R}^n \times Z)^2$ be in $\mathfrak{C}(n)$ if and only if

(1) C is symmetric, G-invariant and contains the diagonal.

(2) There is a $C' \in p_n^*\mathfrak{C}_{\mathrm{bdd}}(\mathbb{R}^n)$ such that $C \subseteq C'$.

(3) For all $K \subseteq \mathbb{R}^n$ compact, there is a $C'' \in p_Z^*\mathfrak{C}$ such that

$$C \cap ((K \times Z) \times (K \times Z)) \subseteq C''.$$

Set $\mathfrak{S}(n) := p_Z^*\mathfrak{S}$.

Consider for all n also the restricted coarse structures

$$\mathfrak{Z}(n+1)^+ := \mathfrak{Z}(n+1) \cap (\mathbb{R}^n \times \mathbb{R}_{\geqslant 0} \times Z),$$
$$\mathfrak{Z}(n+1)^- := \mathfrak{Z}(n+1) \cap (\mathbb{R}^n \times \mathbb{R}_{\leqslant 0} \times Z).$$

Note that $\mathfrak{Z}(n+1) \cap (\mathbb{R}^n \times \{0\} \times Z) = \mathfrak{Z}(n)$.

Let $A \subseteq Z$ be a G-invariant subset. The obvious inclusion maps give rise to a commutative square

$$\begin{array}{ccc} h^A S_\bullet \mathcal{R}^G_f(W, \mathfrak{Z}(n)) & \longrightarrow & h^A S_\bullet \mathcal{R}^G_f(W, \mathfrak{Z}(n+1)^+) \\ \big\downarrow & & \big\downarrow \\ h^A S_\bullet \mathcal{R}^G_f(W, \mathfrak{Z}(n+1)^-) & \longrightarrow & h^A S_\bullet \mathcal{R}^G_f(W, \mathfrak{Z}(n+1)) \end{array} \tag{5.3}$$

Using the results of Section 4D, the top right and bottom left corners of this square are contractible since they admit infinite shift maps $(\vec{x}, x_{n+1}, z) \mapsto (\vec{x}, x_{n+1} \pm 1, z)$. This provides us with structure maps for a spectrum

$$\mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z}), h^A) := \big\{K(\mathcal{R}^G_f(W, \mathfrak{Z}(n)), h^A)\big\}_n.$$

These are the algebraic K-theory spectra we use for our main results. It follows from Propositions 3.28, 4.24 and 4.25 that the construction of this spectrum is natural in $\mathfrak{Z}$.

5.4. Remark. Definition 5.2 is more involved than one might expect. The coarse structure

$$\mathfrak{Z}[n] := (\mathbb{R}^n \times Z,\ p_n^*\mathfrak{C}_{\mathrm{bdd}}(\mathbb{R}^n) \Cap p_Z^*\mathfrak{C},\ p_Z^*\mathfrak{S})$$

might appear to be a more intuitive choice. There is a canonical inclusion functor $\mathcal{R}^G_f(W, \mathfrak{Z}[n]) \to \mathcal{R}^G_f(W, \mathfrak{Z}(n))$ which induces an isomorphism on homotopy groups in sufficiently high degrees, using $\mathfrak{Z}[0] = \mathfrak{Z}(0)$ and Proposition 5.5 below. We conjecture that this map is in fact a weak equivalence.

The difference between the coarse structures $\mathfrak{Z}(n)$ and $\mathfrak{Z}[n]$ is analogous to the linear situation (cf. [Pedersen and Weibel 1989]). Take, for example, categories $\mathcal{C}_X(R)$ of bounded morphisms over a metric space. Then $\mathfrak{Z}[n]$ corresponds to the category $\mathcal{C}_{\mathbb{R}^n \times X}(R)$, while $\mathfrak{Z}(n)$ corresponds to $\mathcal{C}_{\mathbb{R}^n}(\mathcal{C}_X(R))$. The inclusion functor $\mathcal{C}_{\mathbb{R}^n \times X}(R) \to \mathcal{C}_{\mathbb{R}^n}(\mathcal{C}_X(R))$ always induces an equivalence of algebraic K-theory spectra: apply nonconnective algebraic K-theory to the inclusion map and prove that both sides are equivalent to the spectrum $\Omega^n \mathbb{K}^{-\infty}(\mathcal{C}_X(R))$.

5.5. Proposition. (1) *The structure maps of the spectrum $\mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z}), h)$ induce isomorphisms on π_i for $i \geqslant 1$.*

(2) *The structure maps of the spectrum $\mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z}), h^A)$ induce isomorphisms on π_i for $i \geqslant 2$.*

Proof. By Theorem 4.23, there is a homotopy pullback square

$$\begin{array}{ccc} hS_\bullet \mathcal{R}^G_{fd,F'}(W, \mathfrak{Z}(n)) & \longrightarrow & hS_\bullet \mathcal{R}^G_{fd,K'}(W, \mathfrak{Z}(n+1)^+) \\ \big\downarrow & & \big\downarrow \\ hS_\bullet \mathcal{R}^G_{fd,F}(W, \mathfrak{Z}(n+1)^-) & \longrightarrow & hS_\bullet \mathcal{R}^G_{fd,K}(W, \mathfrak{Z}(n+1)) \end{array} \tag{5.6}$$

There is a transformation from square (5.3) to (5.6) induced by inclusion functors. By Thomason cofinality, this transformation is a weak equivalence on the top right and bottom right corners. Therefore, $hS_\bullet\mathcal{R}^G_{fd,K'}(W, \mathfrak{Z}(n+1)^+)$ is weakly contractible. In particular, its K_0 is trivial, so $\mathcal{R}^G_{fd,F'}(W, \mathfrak{Z}(n)) = \mathcal{R}^G_{fd}(W, \mathfrak{Z}(n))$.

We claim that $K_0(\mathcal{R}^G_f(W, \mathfrak{Z}(n+1)), h) = 0$. Since we can filter any object by its skeleta, and suspension corresponds to taking inverses in K_0, the class of any object in $\mathcal{R}^G_f(W, \mathfrak{Z}(n+1))$ equals its K-theoretic Euler characteristic, i.e., it equals an alternating sum of classes of 0-dimensional objects. The same argument as in the linear case [Pedersen and Weibel 1985, Corollary 1.3.1] now shows that the K_0-class of every 0-dimensional object is trivial.

From $K_0(\mathcal{R}^G_f(W, \mathfrak{Z}(n+1)), h) = 0$ it follows that

$$\mathcal{R}^G_{fd,F}(W, \mathfrak{Z}(n+1)^-) = \mathcal{R}^G_{fd}(W, \mathfrak{Z}(n+1)^-).$$

Since $\mathfrak{Z}(n+1)^-$ admits an infinite shift map, $hS_\bullet\mathcal{R}^G_{fd}(W, \mathfrak{Z}(n+1)^-)$ is weakly contractible by Section 4D. We already know that $hS_\bullet\mathcal{R}^G_f(W, \mathfrak{Z}(n+1)^-)$ is weakly contractible, so the transformation from (5.3) to (5.6) is also a weak equivalence on the bottom left corner.

As the square (5.6) is a homotopy pullback in which the bottom left and top right corners are weakly contractible, we get a weak equivalence

$$|hS_\bullet\mathcal{R}^G_{fd}(W, \mathfrak{Z}(n))| \xrightarrow{\sim} \Omega|hS_\bullet\mathcal{R}^G_{fd,K}(W, \mathfrak{Z}(n+1))|. \tag{5.7}$$

By Proposition 4.8, the map $|hS_\bullet\mathcal{R}^G_f(W, \mathfrak{Z}(n))| \to |hS_\bullet\mathcal{R}^G_{fd}(W, \mathfrak{Z}(n))|$ induces an isomorphism on π_i for $i \geqslant 2$. Hence, the structure map

$$K(\mathcal{R}^G_f(W, \mathfrak{Z}(n)), h) \to \Omega K(\mathcal{R}^G_f(W, \mathfrak{Z}(n+1)), h)$$

is an isomorphism on π_i for $i \geqslant 1$.

The structure map $K(\mathcal{R}^G_f(W, \mathfrak{Z}(n)), h^A) \to \Omega K(\mathcal{R}^G_f(W, \mathfrak{Z}(n+1)), h^A)$ sits in a map of homotopy fiber sequences arising from Theorem 4.16. The second assertion of the proposition follows from the first assertion and a five-lemma argument. □

5.8. Remark. We can also define a nonconnective spectrum $\mathbb{K}^{-\infty}(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h^A)$ using the finitely dominated objects. The natural maps

$$K(\mathcal{R}^G_f(W, \mathfrak{Z}(n)), h^A) \to K(\mathcal{R}^G_{fd}(W, \mathfrak{Z}(n)), h^A)$$

are isomorphisms on π_i for $i \geqslant 1$; hence, the induced map

$$\mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z}), h^A) \to \mathbb{K}^{-\infty}(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h^A)$$

is a stable equivalence of spectra by Proposition 5.5.

For convenience, we record the nonconnective versions of the main results of the previous section.

5.9. Theorem. *Let $\mathfrak{Z}$ be a coarse structure and let $A \subseteq Z$ be a closed, G-invariant subset such that $\mathfrak{Z}$ is G-proper with respect to A. Then the inclusion functors induce a homotopy fiber sequence*

$$\mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z} \cap A), h) \to \mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z}), h) \to \mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z}), h^A).$$

Proof. This is Theorem 4.16 together with Proposition 4.20 and Remark 5.8. □

5.10. Theorem (coarse Mayer–Vietoris theorem). *Let $(\mathfrak{Z}, A_1, A_2)$ be a coarsely excisive triple, and assume that $\mathfrak{Z}$ is G-proper with respect to A_1, A_2 and $A_1 \cap A_2$. Then the obvious inclusion maps give rise to a homotopy pullback square of spectra*

$$\begin{tikzcd}
\mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z} \cap (A_1 \cap A_2)), h) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z} \cap A_1), h) \arrow[d] \\
\mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z} \cap A_2), h) \arrow[r] & \mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z}), h)
\end{tikzcd}$$

Proof. This is Theorem 4.23 together with Remark 5.8. □

5.11. Theorem (Eilenberg swindle). *Let $\mathfrak{Z}$ be a coarse structure and let $A \subseteq Z$ be a G-invariant subset. Suppose that there is a sequence of G-equivariant functions $(s_n : Z \to Z)_n$ as in Proposition 4.25. Then $\mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathfrak{Z}), h^A)$ is weakly contractible.*

Proof. This follows from Section 4D. □

6. The Davis–Lück assembly map

We can now translate the model of the assembly map given in [Bartels et al. 2004] to A-theory. Assume from now on that G is a countable discrete group.

6.1. Definition. Let X be a G-CW-complex and M a metric space with free, isometric G-action.

Define the coarse structure $\mathbb{J}(M, X) = (M \times X \times [1, \infty[, \mathfrak{C}(M, X), \mathfrak{S}(M, X))$ as follows. Let p_M, $p_{M\times X}$, $p_{X\times[1,\infty[}$ and $p_{[1,\infty[}$ denote the projection maps from $M \times X \times [1, \infty[$ to the factor indicated by the index of p. Set

$$\mathfrak{C}(M, X) := p_M^* \mathfrak{B}(M) \Cap p^*_{X\times[1,\infty[} \mathfrak{C}_{G\text{-cc}}(X),$$
$$\mathfrak{S}(M, X) := p^*_{M\times X} \mathfrak{S}_{G\text{-cpt}}(M \times X).$$

The bounded coarse structure, G-compact support condition and G-continuous control condition have been defined in Example 2.2.

One particular instance of this definition is the case where $M = G$, equipped with a left invariant and proper metric, "proper" meaning that every ball of finite radius is finite. Such metrics exist [Dranishnikov and Smith 2006, Proposition 1.3];

if G is finitely generated, we can pick a word metric. Whenever d and d' are two left invariant, proper metrics on G, the identity map $\mathrm{id} : (G, d) \to (G, d')$ is a coarse equivalence by [Dranishnikov and Smith 2006, Proposition 1.1]. In particular, every R-ball with respect to d is contained in some R'-ball with respect to d', and vice versa. Hence, the bounded control condition on G is independent of the choice of left invariant, proper metric, and we can suppress the metric in our notation.

6.2. Definition. We abbreviate $\mathbb{J}(X) := \mathbb{J}(G, X)$.

When considering $\mathcal{R}^G_f(W, \mathbb{J}(M, X))$, we denote the class of weak equivalences $h^{M\times X\times\{1\}}$ by h^∞. Observe also that for a G-invariant subcomplex $A \subseteq X$, we have $\mathbb{J}(M, X)\cap(M\times A\times[1, \infty[) = \mathbb{J}(M, A)$. Finally, we note that $\mathbb{J}(M, X)$ is G-proper with respect to subspaces of the form $M \times A \times [1, \infty[$ for $A \subseteq X$ a G-invariant subcomplex.

6.3. Definition. Let us introduce the following shorthands:

$$\begin{aligned}
\mathbb{T}(G, W, X) &:= \mathbb{K}^{-\infty}\big(\mathcal{R}^G_f(W, \mathbb{J}(X) \cap (G \times X \times \{1\})), h\big),\\
\mathbb{F}(G, W, X) &:= \mathbb{K}^{-\infty}\big(\mathcal{R}^G_f(W, \mathbb{J}(X)), h\big),\\
\mathbb{D}(G, W, X) &:= \mathbb{K}^{-\infty}\big(\mathcal{R}^G_f(W, \mathbb{J}(X)), h^\infty\big).
\end{aligned}$$

As a consequence of Theorem 5.9, these spectra fit into a natural homotopy fiber sequence

$$\mathbb{T}(G, W, X) \to \mathbb{F}(G, W, X) \to \mathbb{D}(G, W, X). \tag{6.4}$$

6.5. Definition. An *(unreduced) G-homology theory* is a functor $\mathbb{H}$ from the category of G-CW-complexes to the category of spectra such that the following hold:

(1) Every G-equivariant homotopy equivalence $f : X_1 \xrightarrow{\sim} X_2$ induces a weak equivalence $\mathbb{H}(f) : \mathbb{H}(X_1) \to \mathbb{H}(X_2)$.

(2) Every homotopy pushout square of G-CW-complexes induces a homotopy pullback square of spectra upon application of $\mathbb{H}(-)$.

(3) If $X = \mathrm{colim}_i\, X_i$ is a directed colimit, the natural map $\mathrm{hocolim}_i\, \mathbb{H}(X_i) \to \mathbb{H}(X)$ is a weak equivalence.

6.6. Remark. Observe that any unreduced G-homology theory in the sense of Definition 6.5 automatically respects finite coproducts because

$$\begin{array}{ccc}
\varnothing & \longrightarrow & X_1\\
\big\downarrow & & \big\downarrow\\
X_2 & \longrightarrow & X_1 \amalg X_2
\end{array}$$

is a homotopy pushout square. From the direct limit axiom Definition 6.5(3), conclude that any unreduced G-homology theory commutes with arbitrary coproducts.

6.7. Theorem. (1) *The projection $X \to G/G$ induces a weak equivalence*

$$\mathbb{T}(G, W, X) \xrightarrow{\sim} \mathbb{T}(G, W, G/G)$$

for every G-CW-complex X.

(2) *The assignment $X \mapsto \mathbb{D}(G, W, X)$ is an unreduced G-equivariant homology theory.*

(3) *The connecting map $\Omega\mathbb{D}(G, W, G/G) \to \mathbb{T}(G, W, G/G)$ is a weak equivalence.*

Proof. For part (1), consider the functor

$$p : \mathcal{R}^G_f(W, \mathbb{J}(X)(n) \cap (G \times X \times \{1\})(n)) \to \mathcal{R}^G_f(W, \mathbb{J}(G/G) \cap (G \times G/G \times \{1\})(n))$$

induced by the projection map $X \to G/G$; it is well-defined because of the G-compact support condition on $G \times X$. Let $(Y \leftrightarrows W, \kappa)$ be any object from the category on the right-hand side. Then any choice of a point $x \in X$ induces a control map

$$\tilde{\kappa} : \diamond Y \to \mathbb{R}^n \times G \times X, \quad e \mapsto (\kappa_{\mathbb{R}^n}(e), \kappa_G(e), \kappa_G(e) \cdot x)$$

which turns Y into an object $\widetilde{Y}$ of the left-hand side. This construction provides an inverse to p, showing that p is an exact equivalence of Waldhausen categories.

For the second part of the theorem, observe first that $X \mapsto \mathcal{R}^G_f(W, \mathbb{J}(X))$ is indeed a functor on G-CW-complexes; this follows from Proposition 3.28 using Lemma 3.3 from [Bartels et al. 2004] and the G-compact support condition. Also, due to the G-compact support condition, in conjunction with the fact that algebraic K-theory commutes with directed colimits, we immediately obtain the direct limit axiom (3). Hence, it suffices to consider only cocompact G-CW-complexes. The remainder of the proof is formally the same as in [Bartels et al. 2004, §5]; note that in the proof of the property Definition 6.5(2), the special case of a coproduct is missing in [Bartels et al. 2004] and has to be treated separately. For more details, see also [Ullmann 2010, Section 7.2].

For the last part of the theorem, consider $\mathcal{R}^G_f(W, \mathbb{J}(G/G))$. The map

$$s : G \times G/G \times [1, \infty[, \quad (g, G, t) \mapsto (g, G, t + 1)$$

is an infinite shift map, so $\mathbb{F}(G, W, G/G)$ is weakly contractible by Theorem 5.11. The claim follows. □

Let $\mathrm{Or}(G)$ denote the *orbit category* of G, i.e., the category of left G-sets G/H and G-equivariant maps between them.

Let V be a topological space. Then $\mathcal{R}_f(V)$, the category of finite retractive spaces over V, is isomorphic to the category $\mathcal{R}^{\{1\}}_f(V, \mathfrak{T}(*))$, where $\mathfrak{T}(*)$ is the trivial coarse structure over a point from Example 2.2. The results of Section 5 provide us with a spectrum

$$\mathbb{A}^{-\infty}(V) := \mathbb{K}^{-\infty}(\mathcal{R}_f^{\{1\}}(V, \mathfrak{T}(*)), h)$$

which is a (potentially nonconnective) delooping of $A(V)$. We call this the *nonconnective algebraic K-theory spectrum of V*. Given any G-space W, we may therefore define an Or(G)-spectrum $\mathbb{A}_W^{-\infty}$ by setting

$$\mathbb{A}_W^{-\infty}(G/H) := \mathbb{A}^{-\infty}(W^{\mathrm{op}} \times_G G/H) \cong \mathbb{A}^{-\infty}(H\backslash W),$$

where W^{op} denotes the space W equipped with the right action of G induced by the original left action via $w \cdot g := g^{-1}w$.

6.8. Theorem. *Let W be a free G-CW-complex. Then there is a zig-zag of equivalences of* Or(G)*-spectra*

$$\Omega\mathbb{D}(G, W, -) \simeq \mathbb{A}_W^{-\infty}(-).$$

With the exception of Corollary 6.17 below, the proof of this theorem occupies the rest of this section. Both the strategy of proof and the method to use Theorem 6.8 to relate the connecting map $\Omega\mathbb{D}(G, W, X) \to \mathbb{T}(G, W, X)$ to the Davis–Lück assembly map go back to work of Hambleton and Pedersen [2004, Sections 7 and 8].

Consider $\mathbb{D}(G, W, G/H)$ for some $G/H \in \mathrm{Or}(G)$. Define a coarse structure

$$\mathbb{J}^{\mathrm{dis}}(G/H) = (G \times G/H \times [1, \infty[, \mathfrak{C}^{\mathrm{dis}}(G, G/H), \mathfrak{S}(G, G/H)),$$

where $\mathfrak{C}^{\mathrm{dis}}(G, G/H)$ is the collection of all $C \in \mathfrak{C}(G, G/H)$ such that $\gamma H = \gamma' H$ for all $((g, \gamma H, t), (g', \gamma' H, t')) \in C$.

6.9. Lemma. *For all n, the natural inclusion functor*

$$\mathcal{R}_f^G(W, \mathbb{J}^{\mathrm{dis}}(G/H)(n)) \hookrightarrow \mathcal{R}_f^G(W, \mathbb{J}(G/H)(n))$$

induces an equivalence in K-theory with respect to the h^∞-equivalences.

Proof. Let $f : Y_1 \to Y_2$ be an arbitrary morphism in $\mathcal{R}_f^G(W, \mathbb{J}(G/H)(n))$. Let C be a control condition witnessing that f is a controlled map. For each closed ball $B_R \subseteq \mathbb{R}^n$, $C \cap (B_R \times G \times G/H \times [1, \infty[)^2$ satisfies the continuous control condition. Therefore, there is some $t_0 > 1$ such that $((x, g, \gamma H, t), (x', g', \gamma' H, t')) \in C$ implies $\gamma H = \gamma' H$ whenever $x, x' \in B_R$ and $t, t' \geqslant t_0$. Since we require bounded control over $\mathbb{R}^n$, and since the G-continuous control condition includes bounded control over $[1, \infty[$, there exists some cofinal subcomplex $Y_1' \subseteq Y_1$ away from $\mathbb{R}^n \times G \times G/H \times \{1\}$ such that $f|_{Y_1'}$ satisfies a control condition in $\mathfrak{C}^{\mathrm{dis}}(G, G/H)$.

We want to prove the approximation property. For the first part, consider a morphism $f : Y_1 \to Y_2$ in $\mathcal{R}_f^G(W, \mathbb{J}^{\mathrm{dis}}(G/H)(n))$ which is an h^∞-equivalence in $\mathcal{R}^G_f(W, \mathbb{J}(G/H)(n))$. If g is an h^∞-inverse to f, we can restrict it to a suitable cofinal subcomplex such that its restriction satisfies a control condition in $\mathfrak{C}^{\mathrm{dis}}(G, G/H)$ by the previous paragraph. The same works for homotopies. This shows the first part of the approximation property.

For the second part, let $f : Y_1 \to Y_2$ be a morphism in $\mathcal{R}^G_f(W, \mathbb{J}(G/H))$, where Y_1 is an object in $\mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H))$. Again by the first paragraph, there is some cofinal subcomplex $Y_2' \subseteq Y_2$ which satisfies a control condition in $\mathfrak{C}^{\mathrm{dis}}(G, G/H)$. Then there exists a cofinal subcomplex $Y_1' \subseteq Y_1$ such that $f|_{Y_1'}$ maps into Y_2'. Let Y be the pushout of

$$Y_1 \leftarrowtail Y_1' \xrightarrow{f|_{Y_1'}} Y_2'.$$

Then Y is an object in $\mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)(n))$ and the canonical morphism $Y_2' \rightarrowtail Y$ is an h^∞-equivalence. Hence, the morphism $Y \to Y_2$ induced by the universal property of the pushout is also an h^∞-equivalence by saturation. This proves the second part of the approximation property. □

Defining $\mathbb{D}'(G, W, G/H) := \mathbb{K}^{-\infty}(\mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)), h^\infty)$, Lemma 6.9 states that the natural map $\mathbb{D}'(G, W, G/H) \to \mathbb{D}(G, W, G/H)$ is a weak equivalence. Observe that, by considering $\mathbb{J}^{\mathrm{dis}}(G/H)$, we have effectively eliminated the G-continuous control condition. It has been replaced by bounded control over $[1, \infty[$ together with discrete control over G/H.

6.10. Lemma. *For all $n \geqslant 1$, there is a zig-zag of exact functors*

$$\big(\mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)(n-1)), h^\infty\big) \to \cdots \leftarrow \big(\mathcal{R}^G_f(W, (\mathbb{J}^{\mathrm{dis}}(G/H) \cap (G \times G/H \times \{1\}))(n), h)\big)$$

which induces equivalences in K-theory and which is natural in G/H.

Proof. Recall the temporary notation $\mathfrak{Z}[n]$ we introduced in Remark 5.4. We only need to use

$$\mathfrak{Z}[1] := (\mathbb{R} \times Z,\, p_{\mathbb{R}}^*\mathfrak{C}_{\mathrm{bdd}}(\mathbb{R}) \Cap p_Z^*\mathfrak{C},\, p_Z^*\mathfrak{S}).$$

In analogy to the delooping construction we discussed in Section 5, we also use coarse structures $\mathfrak{Z}[1]^+$ and $\mathfrak{Z}[1]^-$.

For the purpose of this proof, define

$$\mathbb{J}^{\mathrm{dis}}(G/H)_1 := \mathbb{J}^{\mathrm{dis}}(G/H) \cap (G \times G/H \times \{1\}).$$

The underlying space of the coarse structure $\mathbb{J}^{\mathrm{dis}}(G/H)(n-1)$ is the product $\mathbb{R}^{n-1} \times G \times G/H \times [1, \infty[$. The obvious isometry $[1, \infty[\cong [0, \infty[$ induces a homeomorphism $\mathbb{R}^{n-1} \times G \times G/H \times [1, \infty[\cong \mathbb{R}^{n-1} \times [0, \infty[\times G \times G/H$. This homeomorphism gives rise to an isomorphism of Waldhausen categories

$$\mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)(n-1)) \cong \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1[1]^+(n-1)). \tag{6.11}$$

Under this isomorphism, the class of h^∞-equivalences corresponds to the homotopy equivalences h^0 away from $\mathbb{R}^{n-1} \times \{0\} \times G \times G/H \times \{1\}$. As in the proof of Theorem 4.23, we obtain a weak equivalence

$$\begin{aligned} &h^0 S_\bullet \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1[1]^+(n-1)) \\ &\qquad\qquad\qquad \xrightarrow{\sim} h^- S_\bullet \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1[1](n-1)), \quad (6.12) \end{aligned}$$

where in the second term h^- refers to the class of homotopy equivalences away from $\mathbb{R}^{n-1} \times \mathbb{R}_{\leqslant 0} \times G \times G/H \times \{1\}$.

There is a natural, exact inclusion functor

$$\mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1[1](n-1)) \hookrightarrow \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1(1)(n-1)).$$

Analogous to (5.7) in the proof of Proposition 5.5, there are weak equivalences

$$\begin{aligned} |hS_\bullet \mathcal{R}^G_{fd}(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1(n-1))| &\xrightarrow{\sim} \Omega |hS_\bullet \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1[1](n-1))|, \\ |hS_\bullet \mathcal{R}^G_{fd}(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1(n-1))| &\xrightarrow{\sim} \Omega |hS_\bullet \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1(1)(n-1))|. \end{aligned}$$

Since the inclusion maps

$$\begin{aligned} hS_\bullet \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1[1](n-1)) &\to h^- S_\bullet \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1[1](n-1)), \\ hS_\bullet \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1(1)(n-1)) &\to h^- S_\bullet \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1(1)(n-1)) \end{aligned}$$

are weak equivalences, too, we conclude that the map

$$h^- S_\bullet \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1[1](n-1)) \to h^- S_\bullet \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1(1)(n-1)) \quad (6.13)$$

is also a weak equivalence. There is another exact inclusion functor

$$\mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1(n)) \hookrightarrow \mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H)_1(1)(n-1)), \quad (6.14)$$

which induces an equivalence on K-theory with respect to the h^--equivalences for similar reasons. The desired zig-zag is then formed by the equivalences arising from (6.11), (6.12), (6.13) and (6.14). □

Since there is a weak equivalence from the shifted spectrum $\{\mathbb{D}'(G, W, G/H)_{n-1}\}_n$, where we set $\mathbb{D}'(G, W, G/H)_{-1} = *$, to $\Omega\mathbb{D}'(G, W, G/H)$, Lemma 6.10 provides us with a zig-zag of natural weak equivalences

$$\Omega\mathbb{D}'(G, W, G/H) \simeq \mathbb{K}^{-\infty}\big(\mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(G/H) \cap (G \times G/H \times \{1\})), h\big). \quad (6.15)$$

In order to prove Theorem 6.8, it is therefore sufficient to identify the latter $\mathrm{Or}(G)$-spectrum.

6.16. Lemma. *There is a zig-zag of weak equivalences of* $\mathrm{Or}(G)$*-spectra*

$$\mathbb{K}^{-\infty}\big(\mathcal{R}^G_f(W, \mathbb{J}^{\mathrm{dis}}(-) \cap (G \times - \times \{1\})), h\big) \simeq \mathbb{A}^{-\infty}_W(-).$$

Proof. Let $\hat{\mathcal{R}} \subseteq \mathcal{R}^G_f\big(W, (\mathbb{J}^{\mathrm{dis}}(G/H) \cap (G \times G/H \times \{1\}))\big)$ be the full Waldhausen subcategory of those objects (Y, κ) for which the set of cells $\kappa^{-1}(\{1_G\} \times \{H\} \times \{1\})$ intersects every G-orbit of cells.

We claim that the inclusion functor $\hat{\mathcal{R}} \hookrightarrow \mathcal{R}^G_f\big(W, (\mathbb{J}^{\mathrm{dis}}(G/H) \cap (G \times G/H \times \{1\}))\big)$ is an exact equivalence. What we need to show is that every object is isomorphic to some object in $\hat{\mathcal{R}}$. Let $(Y, \kappa) \in \mathcal{R}^G_f\big(W, (\mathbb{J}^{\mathrm{dis}}(G/H) \cap (G \times G/H \times \{1\}))\big)$. Due to the G-compact support condition, we can find a set of representatives R for the G-cells of Y such that $\kappa(R) \subseteq F_1 \times F_2 \times \{1\}$ for some finite sets $F_1 \subseteq G$, $F_2 \subseteq G/H$. Multiplying by appropriate group elements, we can assume without loss of generality that $\kappa(R) \subseteq F \times \{H\} \times \{1\}$ for some finite set $F \subseteq G$. Let $c : F \to \{1_G\}$ be the unique function. By requiring G-equivariance, c induces a G-equivariant function $\kappa_c : \diamond Y \to G \times G/H \times \{1\}$. Since there are only finitely many equivariant cells in Y, the labeled G-CW-complex (Y, κ_c) satisfies bounded control over G. By construction, (Y, κ_c) is an object of $\hat{\mathcal{R}}$. The identity map on Y defines an isomorphism $(Y, \kappa) \cong (Y, \kappa_c)$. This proves that the inclusion functor is an equivalence.

Next, we define an exact functor $Q : \hat{\mathcal{R}} \to \mathcal{R}_f(W^{\mathrm{op}} \times_G G/H, \mathfrak{T}(*))$. Let $(Y, \kappa) \in \hat{\mathcal{R}}$. Define $Y_H \subseteq Y$ to be the H-invariant subcomplex given by $\kappa^{-1}(H \times \{H\} \times \{1\})$. Then $H \backslash Y_H$ is naturally a retractive space over $H \backslash \operatorname{res}^G_H W \cong W^{\mathrm{op}} \times_G G/H$. Set $Q(Y) := H \backslash Y_H$.

We claim that this functor is also an equivalence of Waldhausen categories. The following argument is similar to [Waldhausen 1985, Lemma 2.1.3].

Let $(X, \kappa) \in \mathcal{R}_f(W^{\mathrm{op}} \times_G G/H, \mathfrak{T}(*))$. Let $\pi : W \times G/H \to W^{\mathrm{op}} \times_G G/H$ denote the G-equivariant map sending (w, gH) to $(g^{-1}w, H)$. By pulling back along π, we obtain a retractive space $\widetilde{X}$ relative $W \times G/H$. Define $\Phi(X)$ as the pushout

$$\begin{tikzcd} W \times G/H \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ W \arrow[r, "\Phi(s)"] & \Phi(X) \end{tikzcd}$$

The retraction of $\widetilde{X}$ induces a retraction $\Phi(r)$ on $\Phi(X)$. Note that there is a canonical bijection $\diamond\Phi(X) \xrightarrow{\sim} \diamond\widetilde{X}$. The projection map $\widetilde{X} \to W \times G/H \to G/H$ induces a G-equivariant function $\tilde{\kappa} : \diamond\widetilde{X} \to G/H$ with the property that, if e, e' are cells in $\widetilde{X}$ such that $e' \subseteq \langle e \rangle$, then $\tilde{\kappa}(e) = \tilde{\kappa}(e')$.

Choose a set of representatives S for the G-orbits of cells in $\Phi(X)$ such that $\tilde{\kappa}(e) = H$ for all $e \in S$. Define the G-equivariant function

$$\Phi(\kappa) : \diamond\Phi(X) \to G \times G/H \times \{1\}$$

by $\Phi(\kappa)(e) := (1_G, H, 1)$ for all $e \in S$ and extending G-equivariantly. This turns $\Phi(X)$ into an object $(\Phi(X), \Phi(\kappa)) \in \hat{\mathcal{R}}$. As W is a free G-CW complex, $\Phi(Q(Y))$

is canonically isomorphic to Y. Since $Q(\Phi(X))$ is canonically isomorphic to X, this shows that Q is essentially surjective and fully faithful. This finishes the proof of the lemma. □

Combining Lemmas 6.9 and 6.16 with the zig-zag (6.15), we obtain the zig-zag of weak equivalences of $\mathrm{Or}(G)$-spectra

$$\begin{aligned}\Omega\mathbb{D}(G, W, -) &\simeq \Omega\mathbb{D}(G, W, -)\\ &\simeq \mathbb{K}^{-\infty}\big(\mathcal{R}_f^G(W, \mathbb{J}^{\mathrm{dis}}(-) \cap (G \times - \times \{1\})), h\big)\\ &\simeq \mathbb{A}_W^{-\infty}(-),\end{aligned}$$

whose existence we claimed in Theorem 6.8.

As explained in [Davis and Lück 1998], any $\mathrm{Or}(G)$-spectrum $\mathbb{E}$ gives rise to a G-homology theory $\mathbb{H}^G(-;\mathbb{E})$. By considering the map induced by the projection $X \to G/G$, one obtains for every G-CW-complex X a *Davis–Lück assembly map*

$$\alpha_X : \mathbb{H}^G(X;\mathbb{E}) \to \mathbb{E}(G/G).$$

The upshot of our discussion is that we have constructed a model for the assembly map associated to the $\mathrm{Or}(G)$-spectrum $\mathbb{A}_W^{-\infty}$:

6.17. Corollary. *Let W be a free G-CW-complex. Then the following holds*:

(1) *The connecting map $\Omega\mathbb{D}(G, W, X) \to \mathbb{T}(G, W, X)$ is equivalent to the equivariant A-theory assembly map*

$$\alpha_{X,W} : \mathbb{H}^G(X;\mathbb{A}_W^{-\infty}) \to \mathbb{A}_W^{-\infty}(G/G) \simeq \mathbb{A}^{-\infty}(G\backslash W).$$

(2) *The assembly map $\alpha_{X,W}$ is a weak equivalence if and only if $\mathbb{F}(G, W, X)$ is weakly contractible.*

Proof. The first claim follows from Theorem 6.8 by the argument given in [Bartels et al. 2004, Section 6.2]. The second part of the corollary is then evident from the homotopy fiber sequence $\mathbb{T}(G, W, X) \to \mathbb{F}(G, W, X) \to \mathbb{D}(G, W, X)$. □

7. The isomorphism conjecture for Dress–Farrell–Hsiang groups

Recall (e.g., from [Lück and Reich 2005, Conjecture 113]) the statement of the isomorphism conjecture for A-theory:

7.1. Conjecture (A-theoretic fibered isomorphism conjecture). Let $\mathcal{F}$ be a family of groups and let G be a countable discrete group. Then for every free G-CW-complex W the assembly map

$$\alpha_{\mathcal{F},W} : \mathbb{H}^G(E_{\mathcal{F}}G;\mathbb{A}_W^{-\infty}) \to \mathbb{H}^G(G/G;\mathbb{A}_W^{-\infty}) \cong \mathbb{A}^{-\infty}(G\backslash W)$$

is a weak equivalence, where $E_{\mathcal{F}}G$ is the classifying space of G for the family $\mathcal{F}$.

Whenever Conjecture 7.1 holds for some group G, we say that G *satisfies the A-theoretic fibered isomorphism conjecture with respect to $\mathcal{F}$*. For the special case that $\mathcal{F} = \mathcal{VC}yc$ is the family of virtually cyclic groups, we also say that G *satisfies the A-theoretic fibered Farrell–Jones conjecture*.

Due to Corollary 6.17, the fibered isomorphism conjecture is equivalent to the weak contractibility of the spectra $\mathbb{F}(G, W, E_{\mathcal{F}}G)$ introduced in the previous section. Thus, the A-theoretic isomorphism conjecture becomes accessible via the methods employed in [Bartels et al. 2008b; Bartels and Lück 2012a; 2012b] for the algebraic K-theory and L-theory of group rings. Our goal is to establish an analog of the main result of [Bartels and Lück 2012b].

Let us recall the definition of Dress–Farrell–Hsiang groups.

7.2. Definition. Let D be a finite group. We call D a *Dress group* if there are primes p and q and subgroups $P \trianglelefteq C \trianglelefteq D$ such that P is a p-group, C/P is cyclic and D/C is a q-group.

Recall the definition of the *ℓ^1-metric* on a simplicial complex. If X is a simplicial complex and $\xi = \sum_x \xi_x \cdot x$, $\eta = \sum_x \eta_x \cdot x$ are points in X, this metric is given by

$$d^{\ell^1}(\xi, \eta) = \sum_x |\xi_x - \eta_x|.$$

All simplicial complexes we consider are equipped with this metric.

We call a generating set S of a group G *symmetric* if $s \in S$ implies $s^{-1} \in S$.

7.3. Definition. Let G be a group and S be a symmetric, finite generating set of G. Let $\mathcal{F}$ be a family of subgroups of G.

Call (G, S) a *Dress–Farrell–Hsiang group with respect to $\mathcal{F}$* if there exists $N \in \mathbb{N}$ such that for every $\varepsilon > 0$ there is an epimorphism $\pi : G \twoheadrightarrow F$ onto a finite group F such that the following holds: for every Dress group $D \leqslant F$, there are a $\overline{D} := \pi^{-1}(D)$-simplicial complex E_D of dimension at most N whose isotropy groups lie in $\mathcal{F}$, and a $\overline{D}$-equivariant map $\varphi_D : G \to E_D$ such that

$$d^{\ell^1}(\varphi_D(g), \varphi_D(g')) \leqslant \varepsilon$$

whenever $g^{-1}g' \in S$.

A slightly stricter version of this definition appeared previously in [Winges 2015, Definition 3.1]. The notion of Dress–Farrell–Hsiang groups generalizes that of Farrell–Hsiang groups from [Bartels and Lück 2012b, Definition 1.1; 2014a, Definition 2.14]. For examples, we refer to Section 11 and [Winges 2015].

7.4. Theorem. *Let G be a discrete group. Suppose that there are a symmetric, finite generating set $S \subseteq G$ and a family of subgroups $\mathcal{F}$ of G such that (G, S) is a Dress–Farrell–Hsiang group with respect to $\mathcal{F}$. Then G satisfies the fibered* *isomorphism conjecture in A-theory*, Conjecture 7.1, *with respect to* $\mathcal{F}$, *i.e., the assembly map*

$$\mathbb{H}^G(E_{\mathcal{F}}G; \mathbb{A}_W^{-\infty}) \to \mathbb{A}^{-\infty}(G\backslash W)$$

is a weak equivalence for every free G*-CW-complex* W.

Choosing W to be the universal cover of a given connected CW-complex whose fundamental group is G, Theorem 7.4 implies Theorem 1.5.

Before we can turn to the proof of Theorem 7.4, we need to extend the definition of the obstruction category $\mathcal{R}_f^G(W, \mathbb{J}(X))$. Recall the definition of the coarse structure $\mathbb{J}(M, X) = (M \times X \times [1, \infty[, \mathfrak{C}(M, X), \mathfrak{S}(M, X))$ from Definition 6.1. Let $(M_k)_{k\in\mathbb{N}}$ be a sequence of metric spaces with a free, isometric G-action, and suppose that X is a G-CW-complex. Equip G with a proper, left invariant metric. Then we define a coarse structure

$$\mathbb{J}((M_k)_k, X) = \left(\coprod_{k\in\mathbb{N}} M_k \times X \times [1, \infty[, \mathfrak{C}((M_k)_k, X), \mathfrak{S}((M_k)_k, X) \right)$$

as follows:

(1) A set C lies in $\mathfrak{C}((M_k)_k, X)$ if it is of the form $C = \coprod_k C_k$ with $C_k \in \mathfrak{C}(M_k, X)$, and it additionally satisfies the following *uniform metric control condition*: there is some $R > 0$ such that for all pairs $((m, x, t), (m', x', t')) \in C$ we have $d(m, m') < R$ (i.e., the bound does not depend on k).

(2) A set S lies in $\mathfrak{S}((M_k)_k, X)$ if it is of the form $S = \coprod_k S_k$ with $S_k \in \mathfrak{S}(M_k, X)$.

We consider the Waldhausen category $\mathcal{R}_f^G(W, \mathbb{J}((M_k)_k, X))$. Note that this is a subcategory of $\prod_{k\in\mathbb{N}} \mathcal{R}_f^G(W, \mathbb{J}(M_k, X))$ in a natural way, and that we therefore typically write objects and morphisms as sequences $(Y_k)_k$ and $(f_k)_k$. Moreover, the category

$$\prod^{\mathrm{fin}} \mathcal{R}_f^G(W, \mathbb{J}(M_k, X)) := \operatorname{colim}_l \prod_{k=1}^{l} \mathcal{R}_f^G(W, \mathbb{J}(M_k, X))$$

of eventually trivial sequences is a full subcategory of $\mathcal{R}_f^G(W, \mathbb{J}((M_k)_k, X))$ and inherits a Waldhausen structure. With this additional notation at our disposal, the proof of Theorem 7.4 proceeds as follows.

Suppose that (G, S) is a Dress–Farrell–Hsiang group with respect to $\mathcal{F}$. Pick N to be as in Definition 7.3, so that for every $k \geqslant 1$, there are a finite group F_k, an epimorphism $\pi_k : G \twoheadrightarrow F_k$ and a family of maps $(\varphi_D : G \to E_D)_{D\in\mathcal{D}_k}$ such that

(1) the space E_D is a $\overline{D} := \pi_k^{-1}(D)$-simplicial complex of dimension at most N whose stabilizers lie in $\mathcal{F}$,

(2) the map φ_D is $\overline{D}$-equivariant and $d^{\ell^1}(\varphi_D(g), \varphi_D(g')) \leqslant 1/k$ whenever $g^{-1}g' \in S$,

where $\mathcal{D}_k$ denotes the family of Dress subgroups of F_k. Then the proof is organized around a sequence of diagrams, indexed over $j \in \mathbb{N}$, of the form

$$\begin{tikzcd}
& & \prod^{\mathrm{fin}} \mathcal{R}^G_f(W, \mathbb{J}(E_k \times G, E_{\mathcal{F}}G)) \arrow[d] \\
\mathcal{R}^G_f(W, \mathbb{J}((T_k \times G)_k, E_{\mathcal{F}}G)) \arrow[rr, "((\varphi_k)_k)_*"] \arrow[dr, "P_j"] & & \mathcal{R}^G_f(W, \mathbb{J}((E_k \times G)_k, E_{\mathcal{F}}G)) \arrow[dl, "Q_j"'] \\
& \mathcal{R}^G_f(W, \mathbb{J}(G, E_{\mathcal{F}}G)) \arrow[ul, dashed, bend left, "\mathrm{tr}"] &
\end{tikzcd}$$

Define $T_k := \coprod_{D \in \mathcal{D}_k} G/\overline{D}$, equipped with the discrete metric which assigns distance ∞ to any two points which are not equal. Define $E_k := \coprod_{D \in \mathcal{D}_k} G \times_{\overline{D}} E_D$, equipped with the diagonal G-action. Consider the metric $k \cdot d^{\ell^1}$ on E_k. Equip G with the word metric given by S. The products $T_k \times G$ and $E_k \times G$ become metric spaces by summing up the metrics on the two factors.

Define P_j to be the projection functor which takes the inclusion into the full product category, projects onto the j-th component, and then applies the functor induced by the projection $T_j \times G \times E_{\mathcal{F}}G \times [1, \infty[\to G \times E_{\mathcal{F}}G \times [1, \infty[$. Define Q_j analogously, and let the unlabeled arrow be the canonical inclusion functor.

To show that the K-theory of the obstruction category $\mathcal{R}^G_f(W, \mathbb{J}(G, E_{\mathcal{F}}G))$ is trivial, the following input is required:

(1) There is a sequence of G-equivariant maps $\varphi_k : T_k \times G \to E_k \times G$ inducing the functor $((\varphi_k)_k)_*$ such that $Q_j \circ ((\varphi_k)_k)_* = P_j$ for all j. This is Lemma 7.5.

(2) For each n, there is a *transfer functor*

$$\mathrm{tr} : \mathcal{R}^G_f(W, \mathbb{J}(G, E_{\mathcal{F}}G)(n)) \to \mathcal{R}^G_f(W, \mathbb{J}((T_k \times G)_k, E_{\mathcal{F}}G)(n))$$

(the dashed arrow in the above diagram) such that $P_j \circ \mathrm{tr}$ induces the identity in K-theory for all j; in fact, there is even an appropriate map on nonconnective K-theory, but we do not need to know that. This is covered in Section 9; see Corollary 9.6 in particular.

(3) The canonical inclusion functor

$$\prod^{\mathrm{fin}} \mathcal{R}^G_f(W, \mathbb{J}(E_k \times G, E_{\mathcal{F}}G)) \to \mathcal{R}^G_f(W, \mathbb{J}((E_k \times G)_k, E_{\mathcal{F}}G))$$

induces a weak equivalence in nonconnective K-theory. This follows from Theorem 10.1.

Using the fact that $K_{-n}\big(\mathcal{R}^G_f(W, \mathbb{J}(X))\big) \cong K_1\big(\mathcal{R}^G_f(W, \mathbb{J}(X)(n+1))\big)$ for any G-CW-complex X (Proposition 5.5), a diagram chase shows that $K_n\big(\mathcal{R}^G_f(W, \mathbb{J}(X))\big) = 0$

for all $n \in \mathbb{Z}$ under these assumptions; see [Bartels and Lück 2012b, Section 4]. We remark that the only part of the proof which uses the presence of the classifying space $E_{\mathcal{F}}G$ is (3); the other two parts still work if we replace $E_{\mathcal{F}}G$ with an arbitrary G-CW-complex.

7.5. Lemma (cf. [Bartels and Lück 2012b, Section 7]). *Let X be a G-CW-complex. For each D, the $\overline{D}$-equivariant map φ_D gives rise to a G-equivariant map*

$$\widetilde{\varphi_D} : G/\overline{D} \times G \to G \times_{\overline{D}} E_D,$$
$$(\gamma \overline{D}, g) \mapsto (\gamma, \varphi_D(\gamma^{-1} g)).$$

Then the equivariant maps

$$\varphi_k : T_k \times G \times X \times [1, \infty[\to E_k \times G \times X \times [1, \infty[,$$
$$(\gamma \overline{D}, g, x, t) \mapsto (\widetilde{\varphi_D}(\gamma \overline{D}, g), g, x, t)$$

induce an exact functor

$$((\varphi_k)_k)_* : \mathcal{R}^G_f(W, \mathbb{J}((T_k \times G)_k, X)) \to \mathcal{R}^G_f(W, \mathbb{J}((E_k \times G)_k, X))$$

such that $Q_j \circ ((\varphi_k)_k)_* = P_j$ for all j.

Proof. If the given maps induce a well-defined functor, this functor has the required property. So we have to check that composing with the maps φ_k preserves the uniform metric control condition. Let k be arbitrary. Suppose that $d_{T_k \times G}((\gamma \overline{D}, g), (\gamma' \overline{D'}, g')) < R$. Then $d_G(g, g') < R$ and $d_{T_k}(\gamma \overline{D}, \gamma' \overline{D'}) < R$, which implies that $\gamma \overline{D} = \gamma' \overline{D'}$. Hence, $\overline{D} = \overline{D'}$ and there is some $\delta \in \overline{D}$ such that $\gamma' = \gamma \delta$. Moreover, we can find $m < R$ and $s_1, \dots, s_m \in S$ such that $g' = g s_1 \cdots s_m$. It follows that

$$\begin{aligned}
&d^{\ell^1}_{G \times_{\overline{D}} E_D}\big((\gamma, \varphi_D(\gamma^{-1} g)), (\gamma', \varphi_D(\gamma'^{-1} g'))\big) \\
&\quad = d^{\ell^1}_{G \times_{\overline{D}} E_D}\big((\gamma, \varphi_D(\gamma^{-1} g)), (\gamma \delta, \varphi_D(\delta^{-1} \gamma^{-1} g'))\big) \\
&\quad = d^{\ell^1}_{G \times_{\overline{D}} E_D}\big((\gamma, \varphi_D(\gamma^{-1} g)), (\gamma, \varphi_D(\gamma^{-1} g'))\big) \\
&\quad = d^{\ell^1}_{E_D}(\varphi_D(\gamma^{-1} g), \varphi_D(\gamma^{-1} g')) \\
&\quad \leqslant \frac{m}{k} < \frac{R}{k}
\end{aligned}$$

due to the S-equivariance of φ_D up to $1/k$. We conclude that

$$\begin{aligned}
&d_{E_k \times G}\big((\widetilde{\varphi_D}(\gamma \overline{D}, g), g), (\widetilde{\varphi_D}(\gamma' \overline{D'}, g'), g')\big) \\
&\quad = d_{E_k \times G}\big((\widetilde{\varphi_D}(\gamma \overline{D}, g), g), (\widetilde{\varphi_D}(\gamma \overline{D}, g'), g')\big) \\
&\quad < d_G(g, g') + k \cdot \frac{R}{k} < 2R,
\end{aligned}$$

so uniform metric control is preserved. □

8. The A-theoretic Swan group

In the linear setting, the transfer functors mentioned in the previous section are defined via the action of the *Swan group* of G on the K-theory of the obstruction category. This group arises as the Grothendieck group of the category of integral, finite-rank G-representations, and the action is induced by tensoring such a representation with geometric modules. See [Bartels and Lück 2012b, Section 5].

To establish the existence of a transfer functor, we need a nonlinear analog of this action. For this purpose, recall the notion of *biexact functor* from [Waldhausen 1985, page 342]: If $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are Waldhausen categories, a *biexact functor* is a functor

$$\wedge : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_3$$

with the following properties:

(E1) The functor is *exact in the first variable*, i.e., for all $A_2 \in \mathcal{C}_2$ the functor $- \wedge A_2 : \mathcal{C}_1 \to \mathcal{C}_3$ is exact.

(E2) The functor is *exact in the second variable*, i.e., for all $A_1 \in \mathcal{C}_1$ the functor $A_1 \wedge - : \mathcal{C}_2 \to \mathcal{C}_3$ is exact.

(TC) The functor satisfies the "more technical condition" that for every pair of cofibrations $A_1 \rightarrowtail A_1'$ in $\mathcal{C}_1$ and $A_2 \rightarrowtail A_2'$ in $\mathcal{C}_2$, the canonical morphism $(A_1 \wedge A_2') \cup_{A_1 \wedge A_2} (A_1' \wedge A_2) \to A_1' \wedge A_2'$ is a cofibration in $\mathcal{C}_3$.

As explained in [loc. cit.], such a functor induces pairings on homotopy groups

$$K_i(\mathcal{C}_1) \times K_j(\mathcal{C}_2) \to K_{i+j}(\mathcal{C}_3)$$

for all $i, j \in \mathbb{N}$.

Define $\mathcal{R}ep(G)$ to be the category of pointed (right) G-CW-complexes whose underlying CW-complex is finite; the morphisms of this category are those maps which are pointed, equivariant and cellular. This category can be equipped with a Waldhausen structure in which the cofibrations are the morphisms isomorphic to a cellular inclusion, and the weak equivalences are the morphisms which are homotopy equivalences *in the nonequivariant sense*. We denote the subcategory of these weak equivalences by $h\mathcal{R}ep(G)$.

8.1. Definition. Define the *A-theoretic Swan group* $\mathrm{Sw}^A(G)$ to be

$$\mathrm{Sw}^A(G) := K_0(\mathcal{R}ep(G), h).$$

Explicitly, $\mathrm{Sw}^A(G)$ is generated by h-equivalence classes of objects in $\mathcal{R}ep(G)$, subject to the condition that $[D_0] + [D_2] = [D_1]$ whenever there is a cofibration sequence $D_0 \rightarrowtail D_1 \twoheadrightarrow D_2$ in $\mathcal{R}ep(G)$. As $-[D] = [\Sigma D]$, every element $s \in \mathrm{Sw}^A(G)$ can be written as $s = [D]$ for some object $D \in \mathcal{R}ep(G)$.

We can extend the abelian group structure on $\mathrm{Sw}^A(G)$ to a ring structure using the smash product. The proof of the following proposition amounts to a number of well-known facts about the smash product of CW-complexes.

8.2. Proposition. *The functor*

$$\wedge : \mathcal{R}ep(G) \times \mathcal{R}ep(G) \to \mathcal{R}ep(G), \quad (D, D') \mapsto D \wedge D'$$

is biexact. The functors $- \wedge S^0$ and $S^0 \wedge -$ are naturally equivalent to the identity functor, and the square

$$\begin{tikzcd} \mathcal{R}ep(G) \times \mathcal{R}ep(G) \times \mathcal{R}ep(G) \arrow[r, "\mathrm{id} \times \wedge"] \arrow[d, "\wedge \times \mathrm{id}"'] & \mathcal{R}ep(G) \times \mathcal{R}ep(G) \arrow[d, "\wedge"] \\ \mathcal{R}ep(G) \times \mathcal{R}ep(G) \arrow[r, "\wedge"'] & \mathcal{R}ep(G) \end{tikzcd}$$

commutes up to natural isomorphism.

Thus, $\mathrm{Sw}^A(G)$ becomes a ring under the product $[D] \cdot [D'] := [D \wedge D']$. The main point about $\mathrm{Sw}^A(G)$ is that it admits an action on A-theory. Suppose that $(Y \leftrightarrows W, \kappa)$ is a labeled G-CW-complex and retractive space relative W; let $s : W \to Y$ be the structural inclusion and $r : Y \to W$ be the structural retraction. Then we can form the pushout

$$\begin{tikzcd} (* \times Y) \cup (D \times W) \arrow[r, "r \cup \mathrm{pr}_W"] \arrow[d, "\mathrm{inc} \cup (\mathrm{id} \times s)"'] & W \arrow[d, "D \wedge_W s"] \\ D \times Y \arrow[r] & D \wedge_W Y \end{tikzcd}$$

to obtain a G-CW-complex under W. We equip the product $D \times Y$ with the diagonal action $g \cdot (d, y) := (dg^{-1}, gy)$. By the universal property of the pushout, every map of retractive spaces $f : Y_1 \to Y_2$ induces a map $D \wedge_W f : D \wedge_W Y_1 \to D \wedge_W Y_2$. In particular, we can equip $D \wedge_W Y$ with a structural retraction

$$D \wedge_W r : D \wedge_W Y \to D \wedge_W W \cong W.$$

Regarding D as a G-CW-complex relative to the basepoint, we let $\diamond D$ denote the set of relative cells in D. Then $D \wedge_W Y$ becomes a labeled G-CW-complex via the control map

$$D \wedge_W \kappa : \diamond(D \wedge_W Y) \cong \diamond D \times \diamond Y \xrightarrow{\mathrm{pr}} \diamond Y \xrightarrow{\kappa} Z.$$

Our goal is to show that this pairing defines a biexact functor. To do this, we need the controlled version of a well-known statement about homotopy equivalences of free G-CW-complexes.

Let $\mathfrak{Z}$ be a coarse structure and consider the category of controlled retractive spaces $\mathcal{R}^G(W, \mathfrak{Z})$. Observe that we have a notion of control for nonequivariant maps between labeled G-CW-complexes.

8.3. Lemma. *Let Y_1 and Y_2 be objects in $\mathcal{R}^G(W, \mathfrak{Z})$. Suppose $f : Y_1 \to Y_2$ is a morphism in $\mathcal{R}^G(W, \mathfrak{Z})$ such that there are a nonequivariant controlled map $\bar{g} : Y_2 \to Y_1$ as well as nonequivariant controlled homotopies $\overline{H} : \mathrm{id}_{Y_1} \simeq_{\mathfrak{Z}} \bar{g}f$ and $\overline{K} : \mathrm{id}_{Y_2} \simeq_{\mathfrak{Z}} f\bar{g}$.*

Then f is an h-equivalence, i.e., there are a G-equivariant, controlled inverse g and G-equivariant, controlled homotopies $gf \simeq \mathrm{id}_{Y_1}$ and $fg \simeq \mathrm{id}_{Y_2}$.

Proof. The proof works as in the uncontrolled case; cf. [tom Dieck 1987, Proposition II.2.7]. □

8.4. Proposition. *The smash product $\wedge_W$ over W induces a biexact functor*

$$\wedge_W : \mathcal{R}ep(G) \times \mathcal{R}^G(W, \mathfrak{Z}) \to \mathcal{R}^G(W, \mathfrak{Z}),$$
$$(D, (Y \leftrightarrows W, \kappa)) \mapsto (D \wedge_W Y \leftrightarrows W, D \wedge_W \kappa)$$

which preserves the property of being finite.

The functor $S^0 \wedge_W -$ is naturally equivalent to the identity functor, and the diagram

$$\begin{array}{ccc} \mathcal{R}ep(G) \times \mathcal{R}ep(G) \times \mathcal{R}^G(W, \mathfrak{Z}) & \xrightarrow{\wedge \times \mathrm{id}} & \mathcal{R}ep(G) \times \mathcal{R}^G(W, \mathfrak{Z}) \\ {\scriptstyle \mathrm{id} \times \wedge_W}\downarrow & & \downarrow{\scriptstyle \wedge_W} \\ \mathcal{R}ep(G) \times \mathcal{R}^G(W, \mathfrak{Z}) & \xrightarrow[\wedge_W]{} & \mathcal{R}^G(W, \mathfrak{Z}) \end{array}$$

commutes up to natural isomorphism.

Proof. Observe that $D \wedge_W Y$ contains only free G-cells because Y is assumed to be free (relative W). Every cell (e_D, e_Y) of $D \wedge_W Y$ is labeled by the same point in Z as e_Y, so it is immediate that the support and control conditions are preserved by $\wedge_W$. Moreover, for any subset $A \subseteq Z$ we have $(D \wedge_W \kappa)^{-1}(A) = \diamond D \times \kappa^{-1}(A)$; since D is finite, $D \wedge_W Y$ lies in $\mathcal{R}^G_f(W, \mathfrak{Z})$ whenever Y is an object in $\mathcal{R}^G_f(W, \mathfrak{Z})$.

Let us turn to exactness in the first variable. Fix $Y \in \mathcal{R}^G(W, \mathfrak{Z})$. We have $* \wedge_W Y = W$. Let $i : D \rightarrowtail D'$ be a cofibration in $\mathcal{R}ep(G)$. Then

$$i \wedge_W Y : D \wedge_W Y \to D' \wedge_W Y$$

is also a cofibration, because the same holds for cellular inclusions.

Suppose that D is the pushout of $D_2 \leftarrow D_0 \rightarrowtail D_1$ in $\mathcal{R}ep(G)$. Then $D \times Y$ is also the pushout of $(D_2 \leftarrow D_0 \rightarrowtail D_1) \times Y$, and similarly for $(* \times Y) \cup (D \times W)$. Since pushouts commute with each other, we see that $D \wedge_W Y$ is also the pushout of $(D_2 \wedge_W Y) \leftarrow (D_0 \wedge_W Y) \rightarrowtail (D_1 \wedge_W Y)$.

The interesting part of the argument is to show that $- \wedge_W Y$ preserves h-equivalences. Suppose that $\delta : D \xrightarrow{\sim} D'$ is a weak equivalence, i.e., there is a nonequivariant map $\bar{\delta} : D' \to D$ such that $\delta\bar{\delta}$ and $\bar{\delta}\delta$ are (nonequivariantly) homotopic to the identity map. Taking smash products with id_Y and the constant homotopy on Y, we observe that $\delta \wedge_W Y$ is a morphism in $\mathcal{R}^G(W, \mathfrak{Z})$ which is an h-equivalence in $\mathcal{R}(W, \mathfrak{Z})$, i.e., upon forgetting all G-actions. By Lemma 8.3, $\delta \wedge_W Y$ is an h-equivalence in $\mathcal{R}^G(W, \mathfrak{Z})$.

Exactness in the second variable is similar, but easier. To show condition (TC), one has to show that for $D' \in \mathcal{R}ep(G)$, $Y' \in \mathcal{R}^G(W, \mathfrak{Z})$ and subcomplexes $D \subseteq D'$ and $Y \subseteq Y'$, the complex $(D \wedge_W Y') \cup (D' \wedge_W Y)$ is naturally a subcomplex of $D' \wedge_W Y'$, which is the case.

Finally, $S^0 \wedge_W Y \cong Y$, and associativity of the pairing follows again from the fact that pushouts commute with each other. $\square$

As explained at the beginning of this section, the biexact functor $- \wedge_W -$ from Proposition 8.4 turns $K_i(\mathcal{R}^G_f(W, \mathfrak{Z}), h)$ and $K_i(\mathcal{R}^G_{fd}(W, \mathfrak{Z}), h)$ into $\mathrm{Sw}^A(G)$-modules for all $i \in \mathbb{Z}$ (using that $K_{-i}(\mathcal{R}^G_f(W, \mathfrak{Z}), h) := K_1\big(\mathcal{R}^G_f(W, \mathfrak{Z}(i+1), h)\big)$ for $i \geqslant 0$).

8.5. Remark. Let us digress for a moment to outline the connection between the pairing induced by the biexact functor $\wedge_W$ and bivariant A-theory [Williams 2000, Section 4] (see also [Raptis and Steimle 2014, Section 3]). For the purpose of this remark, we relax the definition of retractive spaces to allow for spaces which are not CW-complexes.

Let $p : V_1 \to V_2$ be a fibration. Then the category $\mathcal{R}(p)$ consists of those retractive spaces (Y, r, s) over V_1 such that the composition $p \circ r$ is a fibration, and for every $v \in V_2$ the (homotopy) fiber $\mathcal{F}_v(p \circ r)$ of $p \circ r$ at v is finitely dominated in $\mathcal{R}(\mathcal{F}_v(p))$. Note that $\mathcal{R}(V \to *)$ is simply the category of (finitely dominated) retractive spaces over V.

For two composable fibrations $p : V_1 \to V_2$ and $q : V_2 \to V_3$, there is defined an exact functor
$$\mathcal{R}(q) \times \mathcal{R}(p) \to \mathcal{R}(q \circ p)$$
given on objects by first pulling back along p, then taking the external smash product and finally pulling back once more along the diagonal map $\Delta : V_1 \to V_1 \times V_1$.

Let W be a free G-CW-complex (or more generally, a principal G-bundle). Let $V := * \times_G W = G \backslash W$ denote the quotient. Taking quotients with respect to the G-action defines a functor $\mathcal{R}^G_{fd}(W) \to \mathcal{R}_{fd}(V) = \mathcal{R}(V \to *)$ which induces an equivalence in K-theory [Waldhausen 1985, Lemma 2.1.3].

Moreover, there exists an exact functor $F : \mathcal{R}ep(G) \to \mathcal{R}(V \xrightarrow{\mathrm{id}} V)$ sending D to $D \times_G W$; since D comes equipped with a base point, there is an induced section to the canonical retraction map $D \times_G W \to * \times_G W = V$.

Combining these functors, we obtain a diagram

$$\begin{array}{ccc} \mathcal{R}ep(G) \times \mathcal{R}^G(W) & \xrightarrow{\wedge_W} & \mathcal{R}^G(W) \\ \downarrow & & \downarrow \\ \mathcal{R}(\mathrm{id}_V) \times \mathcal{R}(* \times_G W \to *) & \longrightarrow & \mathcal{R}(V \to *) \end{array}$$

commutative up to natural isomorphism, which relates the action of $\mathrm{Sw}^A(G)$ on A-theory to the bivariant theory.

In fact, Malkiewich and Merling [2016, Proposition 3.7] have shown that the functor F induces an equivalence in K-theory for $W = EG$; they have the standing assumption that the group G is finite, but this specific part of the argument works for arbitrary discrete groups. Hence, the action of $\mathrm{Sw}^A(G)$ on $A(BG)$ coincides with that of the "upside-down-A-theory" of BG.

This ends the digression.

We need to consider functoriality of Sw^A in G to some extent. For any group homomorphism $\varphi : H \to G$, restriction defines an exact functor

$$\mathrm{res}_\varphi : \mathcal{R}ep(G) \to \mathcal{R}ep(H), \quad D \mapsto \mathrm{res}_\varphi D.$$

If H is a subgroup of G and $[G : H] < \infty$, we can also define an exact induction functor

$$\mathrm{ind}_H^G : \mathcal{R}ep(H) \to \mathcal{R}ep(G), \quad D \mapsto D \wedge_H (G_+).$$

Note that this does not preserve the unit object S^0; in fact, $\mathrm{ind}_H^G S^0 = (H\backslash G)_+$.

We also consider the case of A-theory. Abbreviate the category $\mathcal{R}^G(W, \mathfrak{B}(G)(n))$ by $\mathcal{R}^G(W, n)$. There we have for an arbitrary subgroup $H \leqslant G$ an induction functor

$$\mathrm{ind}_H^G : \mathcal{R}^H(\mathrm{res}_G^H W, n) \to \mathcal{R}^G(W, n), \quad (Y, \kappa) \mapsto (\mathrm{ind}_H^G Y, \mathrm{ind}_H^G \kappa),$$

where $\mathrm{ind}_H^G Y$ is defined as the pushout

$$\begin{array}{ccc} G \times_H (\mathrm{res}_G^H W) & \xrightarrow{(g, w) \mapsto g \cdot w} & W \\ {\scriptstyle G \times_H s}\downarrow & & \downarrow{\scriptstyle \mathrm{ind}_H^G s} \\ G \times_H Y & \longrightarrow & \mathrm{ind}_H^G Y \end{array}$$

and the control map $\mathrm{ind}_H^G \kappa$ is given by

$$\mathrm{ind}_H^G \kappa : \diamond(\mathrm{ind}_H^G Y) = G \times_H (\diamond Y) \to G \times_H (\mathbb{R}^n \times H) \cong \mathbb{R}^n \times G,$$
$$(g, e) \mapsto (\kappa(e), g).$$

Suppose $[G : H] < \infty$. Choose a set-theoretic section σ to the projection map $G \to H\backslash G$ which satisfies $\sigma(H) = 1$. Then σ induces an H-equivariant map

$p_\sigma : G \to H$, $g \mapsto g\sigma(Hg)^{-1}$ which defines a morphism of coarse structures $\mathfrak{B}(G)(n) \to \mathfrak{B}(H)(n)$; we retain bounded control because $H\backslash G$ is finite. Then define an exact functor

$$\operatorname{res}_G^H : \mathcal{R}^G(W, n) \to \mathcal{R}^H(\operatorname{res}_G^H W, n), \quad (Y, \kappa) \mapsto (\operatorname{res}_G^H Y, (\mathbb{R}^n \times p_\sigma) \circ \kappa).$$

Note that this functor also preserves finiteness. The functor res_G^H depends on the choice of σ, but a different choice of σ yields a naturally isomorphic functor. Hence, we suppress σ in what follows.

The restriction and induction functors are related in the expected way:

8.6. Lemma (Frobenius reciprocity). *Let G be a group and let $H \leqslant G$ be a subgroup of finite index. Then we have*

$$\operatorname{ind}_H^G(s) \cdot t = \operatorname{ind}_H^G(s \cdot \operatorname{res}_G^H t),$$
$$\operatorname{ind}_H^G(s) \cdot a = \operatorname{ind}_H^G(s \cdot \operatorname{res}_G^H a)$$

for all $s \in \operatorname{Sw}^A(H)$, $t \in \operatorname{Sw}^A(G)$ and $a \in K_i(\mathcal{R}_f^G(W, n))$. More precisely, there are natural equivalences

$$\wedge \circ (\operatorname{ind}_H^G \times \mathrm{id}) \xrightarrow{\sim} \operatorname{ind}_H^G \circ \wedge \circ (\mathrm{id} \times \operatorname{res}_G^H) : \mathcal{R}ep(H) \times \mathcal{R}ep(G) \to \mathcal{R}ep(G)$$

and

$$\wedge_W \circ (\operatorname{ind}_H^G \times \mathrm{id}) \xrightarrow{\sim} \operatorname{ind}_H^G \circ \wedge_W \circ (\mathrm{id} \times \operatorname{res}_G^H) : \mathcal{R}ep(H) \times \mathcal{R}^G(W, \mathfrak{Z}) \to \mathcal{R}^G(W, \mathfrak{Z}).$$

Proof. Let $D \in \mathcal{R}ep(H)$ and $D' \in \mathcal{R}ep(G)$. Then the first equivalence is implemented by the G-equivariant homeomorphism

$$(D \wedge_H G_+) \wedge D' \xrightarrow{\cong} (D \wedge \operatorname{res}_G^H D') \wedge_H G_+, \quad ((d, g), d') \mapsto ((d, d'g^{-1}), g).$$

For $D \in \mathcal{R}ep(H)$ and $Y \in \mathcal{R}^G(W, \mathfrak{Z})$, the G-equivariant homeomorphism

$$(D \wedge_H G_+) \wedge_W Y \xrightarrow{\cong} \operatorname{ind}_H^G\big(D \wedge_{\operatorname{res}_G^H W} \operatorname{res}_G^H Y\big), \quad ((d, g), y) \mapsto (g^{-1}, (d, gy))$$

yields the second equivalence. □

8.7. Theorem. *Let G be a finite group. The homomorphism*

$$\sum_H \operatorname{ind}_H^G : \bigoplus_{H \leqslant G \text{ Dress}} \operatorname{Sw}^A(H) \to \operatorname{Sw}^A(G)$$

is a surjection.

Proof. By Frobenius reciprocity, it suffices to show that $1_G = [S^0]$ lies in the image of the homomorphism. Since we can filter $D \in \mathcal{R}ep(G)$ by its skeleta and suspension of objects corresponds to taking inverses in K_0, the class of D equals its (equivariant) Euler characteristic. Hence, if $[S^0] = [D_+]$ for some finite G-CW-complex D which has no G-fixed-point, then $[S^0]$ is a sum of elements which are induced from proper subgroups.

If G is not a Dress group, then G acts on a finite, contractible CW-complex D without G-fixed points by a theorem of Oliver [1975]. Then $[D_+] = [S^0]$ in $\mathrm{Sw}^A(G)$. The claim follows by induction. □

8.8. Remark. For the sake of completeness, note that Oliver's theorem [1975] even says that a finite group acts without a global fixed-point on a finite, contractible CW-complex if and only if the group is not Dress.

One can do slightly better than the induction argument in the proof of Theorem 8.7. As shown in [Winges 2015, Corollary 2.10], Oliver's theorem implies the existence of a finite, contractible G-CW-complex, all of whose stabilizers are Dress groups.

8.9. Corollary. *Let G be a finite group, and let W be a G-CW-complex. Then the homomorphism*

$$\sum_H \mathrm{ind}_H^G : \bigoplus_{H \leqslant G \text{ Dress}} K_i(\mathcal{R}_f^H(\mathrm{res}_G^H W, n)) \to K_i(\mathcal{R}_f^G(W, n))$$

is surjective for all $i \in \mathbb{Z}$.

Proof. Immediate from Lemma 8.6 and Theorem 8.7. □

We are also able to describe the kernel of the surjection in Corollary 8.9 once we have proven Theorem 7.4 (see Theorem 11.1). As a second application, we obtain a variant of Swan's induction theorem [1960, Corollary 4.2]. Recall that the Swan group $\mathrm{Sw}(G)$ is the Grothendieck group of integral, finite-rank G-representations.

8.10. Corollary. *Let G be a finite group. Then the unit element $1_G = [\mathbb{Z}] \in \mathrm{Sw}(G)$ can be written as a sum of permutation modules*

$$1_G = \sum_{i=1}^{k} n_i \cdot \big[\mathbb{Z}[G/H_i]\big],$$

where each H_i is a Dress group and $n_i \in \mathbb{Z}$.

Proof. We define a linearization homomorphism

$$\mathrm{Sw}^A(G) \to \mathrm{Sw}(G), \quad [D] \mapsto \sum_{k=0}^{\infty} (-1)^k [\widetilde{C}_k(D)],$$

where $\widetilde{C}_*$ denotes the reduced cellular chain complex. This is a well-defined ring homomorphism. Then the claim follows from the proof of Theorem 8.7. □

Corollary 8.10 differs from Swan's theorem [1960, Corollary 4.2] in that we obtain a description of 1_G in terms of permutation modules instead of arbitrary representations, at the expense of considering a larger family of subgroups.

9. The transfer functor

We proceed to construct the transfer functors tr from Section 7. This uses the action of Sw^A induced by "$\wedge_W$" on A-theory from the previous section. The proof proceeds as in [Bartels and Lück 2012b, Section 6].

Let G be a countable discrete group and X a G-CW-complex. Let $\pi : G \to Q$ be a surjective group homomorphism, and let $H \leqslant Q$ be a subgroup of finite index. Then we define a biexact functor

$$T_{\pi,H} : \mathcal{R}ep(H) \times \mathcal{R}^G_f(W, \mathbb{J}(X)) \to \mathcal{R}^G_f(W, \mathbb{J}(X)),$$
$$(D, (Y, \kappa)) \mapsto \mathrm{res}_\pi (\mathrm{ind}_H^Q D) \wedge_W (Y, \kappa).$$

Recall the coarse structure $\mathbb{J}(X) = \mathbb{J}(G, X)$ from Definition 6.1. Let $\overline{H} := \pi^{-1}(H)$, and equip $G/\overline{H} \times G$ with the metric

$$d_{G,\overline{H}}((\gamma_1 \overline{H}, g_1), (\gamma_2 \overline{H}, g_2)) := \begin{cases} d_G(g_1, g_2), & \gamma_1 \overline{H} = \gamma_2 \overline{H}, \\ \infty, & \text{otherwise.} \end{cases}$$

Next, we define another functor

$$\hat{T}_{\pi,H} : \mathcal{R}ep(H) \times \mathcal{R}^G_f(W, \mathbb{J}(G, X)) \to \mathcal{R}^G_f(W, \mathbb{J}(G/\overline{H} \times G, X))$$

which lifts $T_{\pi,H}$ along the projection functor induced by $G/\overline{H} \times G \to G$. To do so, we equip $\mathrm{res}_\pi (\mathrm{ind}_H^Q D) \wedge_W Y$ with a different control map whose definition we give next.

The unique map $\diamond D \to H\backslash H$ induces a Q-equivariant function $\diamond(\mathrm{ind}_H^Q D) = (\diamond D) \times_H Q \to H\backslash Q$. Restricting the Q-actions along π, we obtain a G-equivariant function $c'_D : \diamond(\mathrm{res}_\pi \, \mathrm{ind}_H^Q D) \to \overline{H}\backslash G$. Regarding source and target as left G-sets by letting g act via g^{-1} on the right, we obtain a map of left G-sets. Moreover, we can identify $\overline{H}\backslash G$ with its left G-action with $G/\overline{H}$. Then we regard c'_D as a G-equivariant function of left G-sets

$$c_D : \diamond(\mathrm{res}_\pi \, \mathrm{ind}_H^Q D) \to G/\overline{H}.$$

For $D \in \mathcal{R}ep(H)$ and $(Y, \kappa) \in \mathcal{R}^G_f(W, \mathbb{J}(G, X))$, define

$$\hat{T}_{\pi,H}(Y) := \mathrm{res}_\pi (\mathrm{ind}_H^Q D) \wedge_W Y$$

and its control map

$$\hat{T}_{\pi,H}(\kappa) : \diamond \hat{T}_{\pi,H}(Y) \cong \diamond(\mathrm{res}_\pi \, \mathrm{ind}_H^Q D) \times \diamond Y \xrightarrow{c_D \times \kappa} G/\overline{H} \times G \times X \times [1, \infty[.$$

On morphisms, we set $\hat{T}_{\pi,H}(\delta, f) := \mathrm{res}_\pi (\mathrm{ind}_H^Q \delta) \wedge_W f$.

9.1. Remark. In this section and the next, we need to pay special attention to the behavior of morphisms over the metric space M appearing in the control space $M \times X \times [1, \infty[$ underlying $\mathbb{J}(M, X)$. We call a morphism f *R-controlled* if there exists a morphism control condition C such that f is C-controlled and $d_M(m, m') \leqslant R$ for all $((m, x, t), (m', x', t')) \in C$.

9.2. Lemma. *This defines a biexact functor*

$$\hat{T}_{\pi,H} : \mathcal{R}ep(H) \times \mathcal{R}^G_f(W, \mathbb{J}(G, X)) \to \mathcal{R}^G_f(W, \mathbb{J}(G/\overline{H} \times G, X))$$

with the following properties:

(1) *If f is a morphism in $\mathcal{R}^G_f(W, \mathbb{J}(G, X))$ which is R-controlled over G and δ is any morphism in $\mathcal{R}ep(H)$, then $\hat{T}_{\pi,H}(\delta, f)$ is R-controlled over $G/\overline{H} \times G$.*

(2) *Let $P : \mathcal{R}^G_f(W, \mathbb{J}(G/\overline{H} \times G, X)) \to \mathcal{R}^G_f(W, \mathbb{J}(G, X))$ denote the canonical projection functor. Then $P \circ \hat{T}_{\pi,H} = T_{\pi,H}$.*

Proof. Let $\delta : D \to D'$ be an arbitrary morphism in $\mathcal{R}ep(H)$. Then the induced morphism $\mathrm{ind}_H^G \delta = \delta \wedge_H Q_+$ has the property that any cell $(e, q) \in \diamond D \times_H Q$ is mapped to $D \wedge_H (Hq)_+ \subseteq D \wedge_H Q_+$. It follows that $\hat{T}_{\pi,H}(\delta, f)$ is 0-controlled over $G/\overline{H}$. Since the control map of $\hat{T}_{\pi,H}(D, Y)$ is defined as a product, $\hat{T}_{\pi,H}(\delta, f)$ is R-controlled if f is R-controlled. In particular, $\hat{T}_{\pi,H}$ is well-defined.

The equality $P \circ \hat{T}_{\pi,H} = T_{\pi,H}$ is obvious. □

9.3. Proposition. *Let G be a countable discrete group and let $\pi : G \to F$ be a surjective group homomorphism onto a finite group F. Suppose that G is equipped with a proper, left invariant metric. Let $\mathcal{D}$ denote the family of Dress subgroups of F. Define $M := \coprod_{H \in \mathcal{D}} G/\overline{H} \times G$; we equip $G/\overline{H} \times G$ with the metric in which different summands are infinitely far apart, and where each summand carries the metric $d_{G,\overline{H}}$.*

Then there is an exact functor $\mathrm{tr}_\pi : \mathcal{R}^G_f(W, \mathbb{J}(X)) \to \mathcal{R}^G_f(W, \mathbb{J}(M, X))$ *with the following properties*:

(1) *If f is a morphism which is R-controlled over G, then $\mathrm{tr}_\pi(f)$ is R-controlled over M.*

(2) *Let $P : \mathcal{R}^G_f(W, \mathbb{J}(M, X)) \to \mathcal{R}^G_f(W, \mathbb{J}(X))$ denote the functor induced by the projection map $M \to G$. Then $P \circ \mathrm{tr}_\pi$ induces the identity map on K-groups.*

Proof. Using Theorem 8.7, we can find a sequence $(D_H)_{H \in \mathcal{D}}$ with $D_H \in \mathcal{R}ep(H)$ such that

$$\sum_{H \in \mathcal{D}} [\mathrm{ind}_H^F D_H] = 1_F \in \mathrm{Sw}^A(F). \tag{9.4}$$

Define the transfer by

$$\mathrm{tr}_\pi(Y, \kappa) := \bigvee_{H \in \mathcal{D}} \hat{T}_{\pi,H}(D_H, (Y, \kappa)),$$

where we regard $\hat{T}_{\pi,H}(D_H, (Y, \kappa))$ as an object over $M \times X \times [1, \infty[$ via the natural inclusion $G/\overline{H} \times G \times X \times [1, \infty[\subseteq M \times X \times [1, \infty[$. Similarly, we set $\mathrm{tr}_\pi(f) := \bigvee_H \hat{T}_{\pi,H}(\mathrm{id}_{D_H}, f)$ for morphisms.

As a consequence of Lemma 9.2(1), this functor preserves R-controlled morphisms. Moreover, we have

$$P \circ \mathrm{tr}_\pi = P \circ \Big(\bigvee_{H \in \mathcal{D}} \hat{T}_{\pi,H}(D_H, -) \Big) \cong \bigvee_{H \in \mathcal{D}} P \circ \hat{T}_{\pi,H}(D_H, -) = \bigvee_{H \in \mathcal{D}} T_{\pi,H}(D_H, -).$$

Using the action of $\mathrm{Sw}^A(G)$ on $K_i(\mathcal{R}^G_f(W, \mathbb{J}(X))$ and the identity (9.4), we conclude that for $a \in K_i(\mathcal{R}^G_f(W, \mathbb{J}(X))$,

$$\begin{aligned} K_i(P \circ \mathrm{tr}_\pi)(a) &= \sum_{H \in \mathcal{D}} [\mathrm{res}_\pi(\mathrm{ind}_H^F D_H)] \cdot a = \Big(\sum_{H \in \mathcal{D}} [\mathrm{res}_\pi(\mathrm{ind}_H^F D_H)] \Big) \cdot a \\ &= \mathrm{res}_\pi \Big(\sum_{H \in \mathcal{D}} [\mathrm{ind}_H^F D_H] \Big) \cdot a = \mathrm{res}_\pi(1_F) \cdot a \\ &= 1_G \cdot a = a, \end{aligned}$$

so $P \circ \mathrm{tr}_\pi$ induces the identity map as claimed. □

9.5. Corollary. *Let G be a countable discrete group. For every $k \in \mathbb{N}$, let $\pi_k : G \twoheadrightarrow F_k$ be an epimorphism onto a finite group. Let $\mathcal{D}_k$ be the family of Dress subgroups of F_k, and define $T_k := \coprod_{H \in \mathcal{D}_k} G/\overline{H} \times G$. Recall the definition of the coarse structure $\mathbb{J}((T_k)_k, X)$ from Section 7.*

Then there is an exact functor

$$\mathrm{tr} : \mathcal{R}^G_f(W, \mathbb{J}(X)) \to \mathcal{R}^G_f(W, \mathbb{J}((T_k)_k, X))$$

such that each composition $P_k \circ \mathrm{tr}$ of tr with the functor

$$P_k : \mathcal{R}^G_f(W, \mathbb{J}((T_k)_k, X)) \to \mathcal{R}^G_f(W, \mathbb{J}(X))$$

from Section 7 induces the identity on K-groups.

Proof. Define $\mathrm{tr} := (\mathrm{tr}_{\pi_k})_{k \in \mathbb{N}}$ and use Proposition 9.3. □

9.6. Corollary. *Assume we are in the same situation as in Corollary 9.5. For every $n \in \mathbb{N}$, there is an exact functor*

$$\mathrm{tr} : \mathcal{R}^G_f(W, \mathbb{J}(X)(n)) \to \mathcal{R}^G_f(W, \mathbb{J}((T_k)_k, X)(n))$$

such that each composition $P_k \circ \mathrm{tr}$ of tr with the functor

$$P_k : \mathcal{R}^G_f(W, \mathbb{J}((T_k)_k, X)(n)) \to \mathcal{R}^G_f(W, \mathbb{J}(X)(n))$$

from Section 7 induces the identity on K-groups.

Proof. The definitions above generalize to $\mathcal{R}^G_f(W, \mathbb{J}(X)(n))$. The statements follow from the case $n = 0$ because the $\mathbb{R}^n$-coordinate remains untouched. □

Corollary 9.6 even provides us with a sequence of functors which induces a map on nonconnective algebraic K-theory spectra. This map splits the map induced by each functor P_k up to homotopy.

10. The "squeezing" theorem

The main result of this section is the following analog of [Bartels et al. 2008b, Theorem 7.2], which is the final ingredient for the proof of Theorem 7.4. We freely use the notation from Section 7. Recall also Remark 9.1.

10.1. Theorem (squeezing theorem). *Let G be a countable discrete group, and let $\mathcal{F}$ be a family of subgroups of G. Let $(E_k)_k$ be a sequence of G-simplicial complexes whose isotropy lies in $\mathcal{F}$. Suppose that there is some N such that the dimension of E_k is at most N for all k. Equip E_k with the metric $k \cdot d^{\ell^1}$. Then the inclusion functor induces a weak equivalence*

$$\mathbb{K}^{-\infty}\Big(\prod\nolimits^{\mathrm{fin}} \mathcal{R}_f^G(W, \mathbb{J}(E_k \times G, E_{\mathcal{F}}G))\Big) \xrightarrow{\sim} \mathbb{K}^{-\infty}\big(\mathcal{R}_f^G(W, \mathbb{J}((E_k \times G)_k, E_{\mathcal{F}}G))\big).$$

For the purposes of this section, abbreviate

$$\mathcal{B}_{\mathrm{fin}}((M_k)_k) := \prod\nolimits^{\mathrm{fin}} \mathcal{R}_f^G(W, \mathbb{J}(M_k, E_{\mathcal{F}}G)),$$
$$\mathcal{B}((M_k)_k) := \mathcal{R}_f^G(W, \mathbb{J}((M_k)_k, E_{\mathcal{F}}G))$$

for any sequence $(M_k)_k$ of metric spaces with free, isometric G-action (in our case $M_k = E_k \times G$). Observe that $\mathcal{B}_{\mathrm{fin}}((M_k)_k)$ can be described as the full subcategory of objects in $\mathcal{B}((M_k)_k)$ with support on a finite disjoint union.

Let $Y = (Y_k)_k$ be an object in $\mathcal{B}((M_k)_k)$. For $K \in \mathbb{N}$, define $(Y_k)_{k>K}$ to be the sequence $(X_k)_k$ with $X_k = *$ for $k \leqslant K$ and $X_k = Y_k$ for $k > K$. Define $h^{\mathrm{fin}}\mathcal{B}((M_k)_k)$ to be the category of those morphisms $f = (f_k)_k : (Y_k^1)_k \to (Y_k^2)_k$ for which there is some $K > 0$ such that the induced morphism $(f_k)_{k>K} : (Y_k^1)_{k>K} \to (Y_k^2)_{k>K}$ is an h-equivalence in $\mathcal{B}((M_k)_k)$. Note that this is a stronger condition than requiring f_k to be a controlled homotopy equivalence for all $k > K$. Using the modified fibration theorem (Proposition 4.14), there is a homotopy fiber sequence

$$hS_\bullet\mathcal{B}((M_k)_k)^{h^{\mathrm{fin}}} \to hS_\bullet\mathcal{B}((M_k)_k) \to h^{\mathrm{fin}}S_\bullet\mathcal{B}((M_k)_k).$$

It is straightforward to check that this homotopy fiber sequence can be delooped, and that the approximation theorem applies to the inclusion $\mathcal{B}_{\mathrm{fin}}((M_k)_k) \hookrightarrow \mathcal{B}((M_k)_k)^{h^{\mathrm{fin}}}$. We conclude that there is a homotopy fiber sequence

$$\mathbb{K}^{-\infty}\big(\mathcal{B}_{\mathrm{fin}}((M_k)_k), h\big) \to \mathbb{K}^{-\infty}\big(\mathcal{B}((M_k)_k), h\big) \to \mathbb{K}^{-\infty}\big(\mathcal{B}((M_k)_k), h^{\mathrm{fin}}\big). \quad (10.2)$$

Consequently, it suffices to show that $\mathbb{K}^{-\infty}\big(\mathcal{B}((E_k \times G)_k), h^{\mathrm{fin}}\big)$ is weakly contractible in order to prove Theorem 10.1. As in [Bartels et al. 2008b], the proof is by induction on N.

10.3. Remark. Before we start with the actual proof, let us provide some intuition why Theorem 10.1 holds true. We consider in Lemma 10.4 the case that each E_k is a disjoint union of equivariant simplices in which different simplices are infinitely far apart. In this case, the desired vanishing result is easily obtained since we can define an Eilenberg swindle by contracting each simplex to a point. This provides almost the start of the induction. However, in the situation considered in Theorem 10.1 different simplices in E_k are only distance $2k$ apart from each other. The basic observation is that this is ultimately the same as considering different simplices to have distance ∞: since the notion of h^{fin}-equivalence allows us to ignore finitely many E_k and each morphism has a uniform control bound, morphisms cannot propagate between different simplices for sufficiently large k. Similarly, objects are forced to decompose over the various simplices provided k is large enough. This observation is formalized in an application of the approximation theorem; see Corollary 10.5.

To perform the induction step, we need to show that $(\mathcal{B}((E_k)_k), h^{\mathrm{fin}})$ is sufficiently excisive after taking K-theory; see Lemma 10.7. The proof of Lemma 10.7 involves technicalities similar to the ones encountered in showing Corollary 10.5. Morally, excision is accomplished since bounded neighborhoods of the N-skeleton in the $(N+1)$-skeleton become arbitrarily small as k grows, due to the fact that we blow up the ℓ^1-metric as k becomes larger, and since we are allowed to ignore finitely many components. Consequently, cells which are labeled by points sufficiently "deep" in a simplex can only be attached to cells which are based on the same simplex; i.e., ignoring finitely many of the E_k and modulo neighborhoods of the N-skeleton, each object decomposes disjointly over the $(N+1)$-simplices.

We turn now to the actual proof of Theorem 10.1.

10.4. Lemma. *Suppose that $(\Delta_k)_k$ is a sequence of G-simplicial complexes of the form*

$$\Delta_k = \coprod_{i \in I_k} G/H_i \times \Delta^N$$

such that $H_i \in \mathcal{F}$ for all i. Equip Δ_k with the metric which assigns distance ∞ to points in different path components, and equals $k \cdot d^{\ell^1}$ for points on the same simplex. Then

$$\mathbb{K}^{-\infty}(\mathcal{B}_{\mathrm{fin}}((\Delta_k \times G)_k), h), \quad \mathbb{K}^{-\infty}(\mathcal{B}((\Delta_k \times G)_k), h), \quad \mathbb{K}^{-\infty}(\mathcal{B}((\Delta_k \times G)_k), h^{\mathrm{fin}})$$

are all weakly contractible.

Proof. It is shown in [Bartels et al. 2008b, proof of Proposition 7.4] that there is a sequence of maps on the underlying control spaces such that Theorem 5.11 applies. ☐

10.5. Corollary. *Theorem 10.1 holds for $N = 0$.*

Proof. Since each E_k is 0-dimensional, it is a disjoint union of transitive G-sets, $E_k = \coprod_{i \in I_k} G/H_i$ with $H_i \in \mathcal{F}$. Define Δ_k to be the simplicial complex E_k, equipped with the metric from Lemma 10.4. There is an exact functor

$$F : \mathcal{B}((\Delta_k \times G)_k) \to \mathcal{B}((E_k \times G)_k).$$

We claim that this functor induces a weak equivalence

$$\mathbb{K}^{-\infty}(F) : \mathbb{K}^{-\infty}\big(\mathcal{B}((\Delta_k \times G)_k), h^{\mathrm{fin}}\big) \xrightarrow{\sim} \mathbb{K}^{-\infty}\big(\mathcal{B}((E_k \times G)_k), h^{\mathrm{fin}}\big).$$

Obviously, F maps h^{fin}-equivalences to h^{fin}-equivalences. We claim that F satisfies the approximation property.

Let $f : (Y^1_k)_k \to (Y^2_k)_k$ be a morphism in $\mathcal{B}((\Delta_k)_k)$ such that $F(f)$ is an h^{fin}-equivalence. Since we require uniform metric control, there is some $K > 0$ such that $(f_k)_{k>K}$ is an h-equivalence which is 0-controlled over $(E_k)_k$. We can assume that $(f_k)_{k>K}$ has an inverse which is 0-controlled over $(E_k)_k$, and that we can find homotopies between the compositions which are 0-controlled over $(E_k)_k$ as well. Hence, f is also an h^{fin}-equivalence in $\mathcal{B}((\Delta_k \times G)_k)$.

For the second part of the approximation property, let $Y^1 = (Y^1_k)_k \in \mathcal{B}((\Delta_k \times G)_k)$, $Y^2 = (Y^2_k)_k \in \mathcal{B}((E_k \times G)_k)$, and let $f = (f_k)_k : F((Y^1_k)_k) \to (Y^2_k)_k$ be a morphism in $\mathcal{B}((E_k \times G)_k)$. Then there is some $K > 0$ such that $(Y^2_k)_{k>K}$ and $(f_k)_{k>K}$ are 0-controlled over $(E_k)_k$. Define $Y = (Y_k)_k$ via $Y_k := Y^1_k$ for $k \leqslant K$ and $Y_k := Y^2_k$ for $k > K$. Then f factors canonically as $Y^1 \to Y \to Y^2$, where the first morphism is 0-controlled over $(E_k)_k$ and the latter morphism is an h^{fin}-equivalence. Since Y is also 0-controlled over $(E_k)_k$, this proves the approximation property.

Hence, $\mathbb{K}^{-\infty}(F)$ is a weak equivalence by the approximation theorem. The claim follows from Lemma 10.4. □

Suppose now that Theorem 10.1 holds for N, and let $(E_k)_k$ be a sequence of G-simplicial complexes of dimension at most $N+1$. Consider for each k the pushout diagram

$$\begin{array}{ccc} \coprod_{i \in I^N_k} G/H_i \times \partial\Delta^{N+1} & \longrightarrow & \mathrm{sk}_N E_k \\ \downarrow & & \downarrow \\ \coprod_{i \in I^N_k} G/H_i \times \Delta^{N+1} & \longrightarrow & E_k \end{array} \tag{10.6}$$

describing the attachment of the $(N+1)$-simplices of E_k.

10.7. Lemma. *Let $N \geqslant 0$. The commutative square of nonconnective K-theory spectra*

$$\begin{array}{ccc} \mathbb{K}^{-\infty}\Big(\mathcal{B}\big(\big(\coprod_{i \in I^N_k} G/H_i \times \partial\Delta^{N+1} \times G\big)_k\big), h^{\mathrm{fin}}\Big) & \longrightarrow & \mathbb{K}^{-\infty}\big(\mathcal{B}((\mathrm{sk}_N E_k \times G)_k), h^{\mathrm{fin}}\big) \\ \downarrow & & \downarrow \\ \mathbb{K}^{-\infty}\Big(\mathcal{B}\big(\big(\coprod_{i \in I^N_k} G/H_i \times \Delta^{N+1} \times G\big)_k\big), h^{\mathrm{fin}}\Big) & \longrightarrow & \mathbb{K}^{-\infty}\big(\mathcal{B}((E_k \times G)_k), h^{\mathrm{fin}}\big) \end{array}$$

induced by diagram (10.6) is a homotopy pullback square of spectra.

Lemma 10.7 provides the induction step: The top left and top right corners of the square from Lemma 10.7 are weakly contractible by the induction hypothesis. The bottom left corner is weakly contractible by Lemma 10.4. Hence, the bottom right corner is also weakly contractible, and Theorem 10.1 follows.

In the rest of this section, we prove Lemma 10.7.

10.8. Lemma. *Let $(M_k)_k$ be a sequence of metric spaces with free, isometric G-action, and let $X_k \subseteq M_k$ be G-invariant, closed subspaces.*

Define $X := \coprod_k X_k \times G \times E_{\mathcal{F}}G \times [1, \infty[$. Let $h^X\mathcal{B}((M_k)_k)$ be the subcategory of controlled homotopy equivalences away from X. Let $h^{X,\mathrm{fin}}\mathcal{B}((M_k)_k)$ denote the subcategory of those morphisms $f : (Y_k^1)_k \to (Y_k^2)_k$ for which there is some $K \in \mathbb{N}$ such that the induced morphism $(f_k)_{k>K} : (Y_k^1)_{k>K} \to (Y_k^2)_{k>K}$ is an h^X-equivalence.

Then there is a homotopy fiber sequence

$$\mathbb{K}^{-\infty}\big(\mathcal{B}((X_k)_k), h^{\mathrm{fin}}\big) \to \mathbb{K}^{-\infty}\big(\mathcal{B}((M_k)_k), h^{\mathrm{fin}}\big) \to \mathbb{K}^{-\infty}\big(\mathcal{B}((M_k)_k), h^{X,\mathrm{fin}}\big).$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{K}^{-\infty}\big(\mathcal{B}_{\mathrm{fin}}((X_k)_k), h\big) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}\big(\mathcal{B}((X_k)_k), h\big) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}\big(\mathcal{B}((X_k)_k), h^{\mathrm{fin}}\big) \arrow[d] \\
\mathbb{K}^{-\infty}\big(\mathcal{B}_{\mathrm{fin}}((M_k)_k), h\big) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}\big(\mathcal{B}((M_k)_k), h\big) \arrow[r] \arrow[d] & \mathbb{K}^{-\infty}\big(\mathcal{B}((M_k)_k), h^{\mathrm{fin}}\big) \arrow[d] \\
\mathbb{K}^{-\infty}\big(\mathcal{B}_{\mathrm{fin}}((M_k)_k), h^X\big) \arrow[r] & \mathbb{K}^{-\infty}\big(\mathcal{B}((M_k)_k), h^X\big) \arrow[r] & \mathbb{K}^{-\infty}\big(\mathcal{B}((M_k)_k), h^{X,\mathrm{fin}}\big)
\end{tikzcd}$$

in which all maps are induced by the appropriate inclusion functors. The left and middle columns are homotopy fiber sequences by Theorem 5.9. The top and middle rows are instances of the homotopy fiber sequence (10.2). By a straightforward modification of the argument for (10.2), the bottom row is also a homotopy fiber sequence. Hence, the right column is a homotopy fiber sequence as claimed. □

Proof of Lemma 10.7. Let $\Delta_k := \coprod_{i \in I_k^N} G/H_i \times \Delta^{N+1}$, equipped with the metric from Lemma 10.4, and $\partial\Delta_k := \coprod_{i \in I_k^N} G/H_i \times \partial\Delta^{N+1}$. The inclusion of metric spaces $\partial\Delta_k \times G \subseteq \Delta_k \times G$ gives rise to a class of weak equivalences $h^{\partial,\mathrm{fin}}\mathcal{B}((\Delta_k \times G)_k)$ and to a corresponding homotopy fiber sequence as in Lemma 10.8. Similarly, $\mathrm{sk}_N E_k \times G \subseteq E_k \times G$ gives rise to a class of weak equivalences $h^{N,\mathrm{fin}}\mathcal{B}((E_k \times G)_k)$ and a corresponding homotopy fiber sequence.

Diagram (10.6) induces a map between these homotopy fiber sequences. To prove the lemma, it suffices to show that the induced map on the homotopy cofibers

$$\mathbb{K}^{-\infty}\big(\mathcal{B}((\Delta_k \times G)_k), h^{\partial,\mathrm{fin}}\big) \to \mathbb{K}^{-\infty}\big(\mathcal{B}((E_k \times G)_k), h^{N,\mathrm{fin}}\big)$$

is a weak equivalence. Note that this map is induced by an exact functor F, namely the one induced by the characteristic maps of the $(N+1)$-simplices. The claim is that the approximation theorem applies again, but as for Corollary 10.5, we have to prove both parts of the approximation property.

We start with a preliminary observation. Let (Y^1, κ^1) and (Y^2, κ^2) be objects in $\mathcal{B}((E_k \times G)_k)$, and let $f : Y^1 \to Y^2$ be a morphism. Suppose that f is R-controlled, and let e be a cell in Y^1_k such that the E_k-component x of $\kappa^1(e)$ is a point in an $(N+1)$-simplex σ. Let e' be a cell in Y^2_k with $e' \subseteq \langle f(e) \rangle$, and suppose that the E_k-component y of $\kappa^2(e')$ does not lie in σ. Then $kd^{\ell^1}(x, y) \leqslant R$. According to [Bartels et al. 2008b, Lemma 7.15], there is a point z on the boundary of σ such that $kd^{\ell^1}(x, z) \leqslant 2R$. Hence, if the distance of x to the boundary of σ is greater than $2R$, then for every cell e' in Y^2_k with $e' \subseteq \langle f(e) \rangle$, the E_k-component of $\kappa^2(e')$ also lies in σ.

Let us now turn to the first part of the approximation property. Let $f : Y^1 \to Y^2$ be a morphism in $\mathcal{B}((\Delta_k \times G)_k)$ such that $F(f)$ is an $h^{N,\mathrm{fin}}$-equivalence. Choose $R > 0$ such that Y^1, Y^2 and f are all R-controlled, and further, such that $F(f)$ has a (partially defined) homotopy inverse and homotopies which are also R-controlled. Let $Y^1_k(6R)$ be the subobject of Y^1_k spanned by those cells $e \in \diamond Y^1_k$ such that the Δ_k-component of $\kappa^1(e)$ has distance at least $6R$ to $\partial\Delta_k$. If e is any cell in Y^1_k, the E_k-component of $\kappa^1(e)$ has distance at least $5R$ to the boundary of the $(N+1)$-simplex in which it lies; to see this, combine the preliminary observation with the fact that Y^1 is R-controlled. Since $Y^1(6R) := (Y^1_k(6R))_k \subseteq Y^1$ is cofinal away from $\coprod_k \partial\Delta_k \times G \times E_{\mathcal{F}}G \times [1, \infty[$, the inclusion is an h^∂-equivalence. In particular, it is an $h^{\partial,\mathrm{fin}}$-equivalence. We can similarly define a subcomplex $Y^2(4R) \subseteq Y^2$, and this inclusion is also an $h^{\partial,\mathrm{fin}}$-equivalence.

Since f is R-controlled, there is an induced morphism $f' : Y^1(6R) \to Y^2(4R)$. The morphism $F(f')$ is still an $h^{N,\mathrm{fin}}$-equivalence; the inverse and homotopies arise by restricting the inverse and homotopies of $F(f)$ to appropriate cofinal subcomplexes. Hence, they are still R-controlled. It follows that they do not cross the boundaries of simplices, so they lift to $\mathcal{B}((\Delta_k \times G)_k)$. This shows that f' is an $h^{\partial,\mathrm{fin}}$-equivalence.

For the second part of the approximation property, let $Y^1 \in \mathcal{B}((\Delta_k \times G)_k)$, $Y^2 \in \mathcal{B}((E_k \times G)_k)$, and let $f : F(Y^1) \to Y^2$ be a morphism in $\mathcal{B}((E_k \times G)_k)$. The argument is similar to the first part. Again, choose $R > 0$ such that Y_1, Y_2 and f are all R-controlled. Then $Y^1(6R)$, defined as before, is a subcomplex of Y^1 which is cofinal away from $\coprod_k \partial\Delta_k \times G \times E_{\mathcal{F}}G \times [1, \infty[$; similarly, $Y^2(4R)$ is a subcomplex of Y^2 which is cofinal away from $\coprod_k \mathrm{sk}_N E_k \times G \times E_{\mathcal{F}}G \times [1, \infty[$. Moreover, $Y^2(4R)$ is supported on the interiors of the $(N+1)$-simplices, and if e is a cell in $Y^2(4R)$, the subcomplex $\langle e \rangle$ spanned by e is based on the same simplex as e by the preliminary observation. Since the characteristic maps of the

$(N+1)$-simplices are homeomorphisms on the interiors (and restrict to isometries on individual simplices), we can lift $Y^2(4R)$ to an object in $\mathcal{B}((\Delta_k \times G)_k)$. Now define Y to be the pushout of $Y^1 \leftarrowtail Y^1(6R) \xrightarrow{f|_{Y^1(6R)}} Y^2(4R)$ in $\mathcal{B}((\Delta_k \times G)_k)$. Since the inclusion $Y^1(6R) \rightarrowtail Y^1$ is an $h^{\partial,\mathrm{fin}}$-equivalence, the canonical inclusion $Y^2(4R) \rightarrowtail Y$ is also an $h^{\partial,\mathrm{fin}}$-equivalence. Since F is exact, $F(Y)$ is the pushout of $F(Y^1) \leftarrowtail F(Y^1(6R)) \to F(Y^2(4R))$. Let $g : F(Y) \to Y^2$ be the map induced by the universal property of the pushout. Since

$$\begin{tikzcd} Y^2(4R) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & Y^2 \\ & F(Y) \arrow[ur, "g"'] & \end{tikzcd}$$

commutes, g is an $h^{N,\mathrm{fin}}$-equivalence by the saturation axiom. We conclude that

$$\begin{tikzcd} F(Y^1) \arrow[rr, "f"] \arrow[dr] & & Y^2 \\ & F(Y) \arrow[ur, "g"'] & \end{tikzcd}$$

is the required factorization. Therefore, the approximation property holds, and we are done. □

11. Applications

To conclude, we turn to some applications of Theorem 7.4. As an immediate corollary, we obtain Theorem 1.3, which gives a description of the A-theory of spaces with finite fundamental group.

11.1. Theorem. *Let V be a connected CW-complex with finite fundamental group G. Let $\widetilde{V}$ be the universal cover of V. Denote by $\mathcal{D}$ the family of Dress subgroups of G. Then the Davis–Lück assembly map*

$$\mathbb{H}^G(E_{\mathcal{D}}G; \mathbb{A}^{-\infty}_{\widetilde{V}}) \to \mathbb{A}^{-\infty}(V)$$

is a weak equivalence.

Proof. The group G is Dress–Farrell–Hsiang with respect to $\mathcal{D}$: for every $\varepsilon > 0$, choose $\pi = \mathrm{id}_G$ and let f_D be the projection onto a point for all $D \in \mathcal{D}$. Now apply Theorem 7.4 with $W = \widetilde{V}$. □

Our ultimate goal is the proof of Theorem 1.2. Formally, everything we do is very close to the treatment in [Bartels et al. 2014a]. This involves a rather intricate induction process which relies on a number of inheritance properties of the isomorphism conjecture. These will be established along the way. The reader is encouraged to refer to [loc. cit.] for definitions.

11.2. Proposition (transitivity principle). *Let $\mathcal{F}_0 \subseteq \mathcal{F}_1$ be two families of subgroups of G. Suppose that G satisfies the fibered isomorphism conjecture in A-theory with respect to $\mathcal{F}_1$, and that every $H \in \mathcal{F}_1$ satisfies the fibered isomorphism conjecture in A-theory with respect to $\mathcal{F}_0|_H := \{H \cap K \mid K \in \mathcal{F}_0\}$.*

Then G satisfies the fibered isomorphism conjecture in A-theory with respect to $\mathcal{F}_0$.

Proof. The proof is analogous to the linear case. However, the published proofs (e.g., [Bartels and Lück 2006, Theorem 2.4; Bartels et al. 2008a, Theorem 3.3]) all rely on the formalism of equivariant homology theories. Since we want to avoid discussing to what extent the homology theories associated to A-theory spectra form equivariant homology theories, we give a proof using the language of $\mathrm{Or}(G)$-spectra.

Let $\mathbb{E}$ be an arbitrary $\mathrm{Or}(G)$-spectrum, and let E be a G-CW-complex. Observe that $G/H \times E$ is naturally G-homeomorphic to $\mathrm{ind}_H^G \mathrm{res}_G^H E = G \times_H \mathrm{res}_G^H E$. Induction defines a functor $\mathrm{ind}_H^G : \mathrm{Or}(H) \to \mathrm{Or}(G)$, so we obtain an $\mathrm{Or}(H)$-spectrum $\mathbb{E} \circ \mathrm{ind}_H^G$. The same arguments as in the proof of Proposition 157 of [Lück and Reich 2005] show that there is a natural isomorphism

$$\mathbb{H}^H(\mathrm{res}_G^H E; \mathbb{E} \circ \mathrm{ind}_H^G) \cong \mathbb{H}^G(\mathrm{ind}_H^G \mathrm{res}_G^H E; \mathbb{E}).$$

Now let $\mathbb{A}_W^{-\infty}$ be the $\mathrm{Or}(G)$-spectrum from Section 6 associated to a free G-CW-complex W. Since $W \times_G (\mathrm{ind}_H^G H/L) \cong \mathrm{res}_G^H W \times_H H/L$, we have

$$\mathbb{H}^G(G/H \times E; \mathbb{A}_W^{-\infty}) \cong \mathbb{H}^H\big(\mathrm{res}_G^H E; \mathbb{A}_{\mathrm{res}_G^H W}^{-\infty}\big).$$

In particular, the map $\mathbb{H}^G(G/H \times E_{\mathcal{F}_0}G; \mathbb{A}_W^{-\infty}) \to \mathbb{H}^G(G/H; \mathbb{A}_W^{-\infty})$ induced by the projection map is weakly equivalent to the map

$$\mathbb{H}^H\big(\mathrm{res}_G^H E_{\mathcal{F}_0}G; \mathbb{A}_{\mathrm{res}_G^H W}^{-\infty}\big) \to \mathbb{H}^H\big(H/H; \mathbb{A}_{\mathrm{res}_G^H W}^{-\infty}\big) = \mathbb{A}^{-\infty}(\mathrm{res}_G^H W/H).$$

Since $\mathrm{res}_G^H W$ is a free H-CW-complex and $\mathrm{res}_G^H E_{\mathcal{F}_0}G = E_{\mathcal{F}_0|_H}H$, this map is an equivalence for all $H \in \mathcal{F}_1$ by assumption. It follows that the maps

$$\mathbb{H}^G\bigg(\coprod_i E_{\mathcal{F}_0}G \times G/H_i \times D^n; \mathbb{A}_W^{-\infty}\bigg) \to \mathbb{H}^G\bigg(\coprod_i G/H_i \times D^n; \mathbb{A}_W^{-\infty}\bigg)$$

are weak equivalences whenever H_i lies in $\mathcal{F}_1$ for all i because $G/H_i \times D^n$ is G-homotopy equivalent to G/H_i and the homology theory under consideration commutes with coproducts. By an induction along the skeleta, it follows that the projection map $E_{\mathcal{F}_0}G \times X \to X$ induces an equivalence in $\mathbb{H}^G(-, \mathbb{A}_W^{-\infty})$ for every finite-dimensional G-CW-complex X whose isotropy groups lie in $\mathcal{F}_1$. Since homology commutes with filtered colimits, the same holds for all G-CW-complexes X whose isotropy groups lie in $\mathcal{F}_1$.

In particular, we can pick $X = E_{\mathcal{F}_1}G$. Then $E_{\mathcal{F}_0}G \times E_{\mathcal{F}_1}G$ is G-homotopy equivalent to $E_{\mathcal{F}_0}G$, so we conclude that the G-map $E_{\mathcal{F}_0}G \to E_{\mathcal{F}_1}G$ (which is unique up to G-homotopy) induces a weak equivalence. This implies the claim. □

11.3. Proposition. *Let $\varphi : K \to G$ be a group homomorphism. Suppose that G satisfies the fibered isomorphism conjecture in A-theory with respect to the family $\mathcal{F}$ of subgroups of G.*

Then K satisfies the fibered isomorphism conjecture in A-theory with respect to the family of subgroups

$$\varphi^*\mathcal{F} := \{\varphi^{-1}(H) \mid H \in \mathcal{F}\}.$$

Proof. Let $\widetilde{\mathbb{E}}$ be a functor from the category of K-sets to the category of spectra; let $\mathbb{E}$ denote its restriction to $\mathrm{Or}(K)$. It has been shown in the proof of [Bartels and Reich 2007, Proposition 4.2] that there is for every G-CW-complex X a weak equivalence

$$\mathbb{H}^K(\mathrm{res}_\varphi X; \mathbb{E}) \cong \mathrm{map}_G(-, X)_+ \wedge_{\mathrm{Or}(G)} \mathrm{map}_K(?, \mathrm{res}_\varphi -)_+ \wedge_{\mathrm{Or}(K)} \widetilde{\mathbb{E}}(?) \tag{11.4}$$

which is natural in X. Let $G/H \in \mathrm{Or}(G)$, and let $K\backslash G/H$ denote the orbit space of $\mathrm{res}_\varphi G/H$. Subject to a choice of (set-theoretic) section $\sigma : K\backslash G/H \to G$ of the obvious projection map, there is an isomorphism

$$\mathrm{res}_\varphi G/H \cong \coprod_{KgH \in K\backslash G/H} T_\sigma(KgH), \qquad \text{where } T_\sigma(KgH) := K/(K \cap \sigma(KgH)H\sigma(KgH)^{-1}).$$

This isomorphism gives rise to a commutative diagram

$$\begin{array}{ccc}
\left(\bigvee_{KgH\in K\backslash G/H} \mathrm{map}_K(?, T_\sigma(KgH))_+\right) \wedge_{\mathrm{Or}(K)} \mathbb{E}(?) & \xrightarrow{\cong} & \mathrm{map}_K(?, \mathrm{res}_\varphi G/H)_+ \wedge_{\mathrm{Or}(K)} \mathbb{E}(?) \\
\downarrow & & \downarrow \\
\bigvee_{KgH\in K\backslash G/H} \mathbb{E}(T_\sigma(KgH)) & \longrightarrow & \widetilde{\mathbb{E}}(\mathrm{res}_\varphi G/H)
\end{array}$$

in which the vertical maps are induced by evaluating $\widetilde{\mathbb{E}}$. The right vertical map is natural in G/H. The left vertical map is easily seen to be a weak equivalence. Whenever $\widetilde{\mathbb{E}}$ commutes with coproducts, the lower horizontal map is a weak equivalence. In this case, the right vertical map is also a weak equivalence, and we obtain a weak equivalence of $\mathrm{Or}(G)$-spectra

$$\mathrm{map}_K(?, \mathrm{res}_\varphi -)_+ \wedge_{\mathrm{Or}(K)} \mathbb{E}(?) \simeq \widetilde{\mathbb{E}} \circ \mathrm{res}_\varphi\,. \tag{11.5}$$

The weak equivalences (11.4) and (11.5) combine to a weak equivalence, natural in X,

$$\mathbb{H}^K(\mathrm{res}_\varphi X; \mathbb{E}) \simeq \mathbb{H}^G(X; \widetilde{\mathbb{E}} \circ \mathrm{res}_\varphi).$$

Let W be a free K-CW-complex. Since $\mathbb{A}_W^{-\infty}$ extends to a functor on all K-sets and commutes with coproducts (see Lemma 11.6 below), we obtain a natural equivalence

$$\mathbb{H}^K(\mathrm{res}_\varphi\, E_{\mathcal{F}}G;\, \mathbb{A}_W^{-\infty}) \simeq \mathbb{H}^G(E_{\mathcal{F}}G;\, \mathbb{A}_W^{-\infty} \circ \mathrm{res}_\varphi).$$

Since $\mathrm{ind}_\varphi\, W = W \times_K \mathrm{res}_\varphi\, G$ is a free G-CW-complex and

$$W \times_K \mathrm{res}_\varphi\, G/H \cong \mathrm{ind}_\varphi\, W \times_G G/H,$$

we have a natural weak equivalence of $\mathrm{Or}(G)$-spectra $\mathbb{A}_W^{-\infty} \circ \mathrm{res}_\varphi \simeq \mathbb{A}_{\mathrm{ind}_\varphi W}^{-\infty}$. Hence, there is a natural weak equivalence

$$\mathbb{H}^K(\mathrm{res}_\varphi\, E_{\mathcal{F}}G;\, \mathbb{A}_W^{-\infty}) \simeq \mathbb{H}^G(E_{\mathcal{F}}G;\, \mathbb{A}_{\mathrm{ind}_\varphi W}^{-\infty}).$$

Observe that $\mathrm{res}_\varphi\, E_{\mathcal{F}}G = E_{\varphi^*\mathcal{F}}K$. We conclude that the assembly map

$$\mathbb{H}^K(E_{\varphi^*\mathcal{F}}K;\, \mathbb{A}_W^{-\infty}) \to \mathbb{H}^K(K/K;\, \mathbb{A}_W^{-\infty})$$

is weakly equivalent to the assembly map

$$\mathbb{H}^G(E_{\mathcal{F}}G;\, \mathbb{A}_{\mathrm{ind}_\varphi W}^{-\infty}) \to \mathbb{H}^G(G/G;\, \mathbb{A}_{\mathrm{ind}_\varphi W}^{-\infty}).$$

The latter map is assumed to be a weak equivalence, so we are done. □

11.6. Lemma. *Let W be a CW-complex, and let $W = \coprod_{i \in I} W_i$ be a decomposition of W into subspaces. Then the natural map*

$$\bigvee_{i \in I} \mathbb{A}^{-\infty}(W_i) \to \mathbb{A}^{-\infty}(W)$$

is a weak equivalence.

Proof. Let Y be a CW-complex relative W together with a retraction $r : Y \to W$. Then the partition $W = \coprod_i W_i$ induces a partition of Y into subcomplexes $Y = \coprod_i Y_i$, where $Y_i := r^{-1}(W_i)$. Similarly, every morphism of retractive spaces $f : Y^1 \to Y^2$ over W decomposes into a coproduct $f = \coprod_i f_i$ since f is compatible with the retractions. Restricting to finite objects, this shows that there is an isomorphism

$$\mathrm{colim}_{J \subseteq I \text{ finite}} \prod_{i \in J} \mathcal{R}_f(W_i, \mathfrak{T}(*)(n)) \xrightarrow{\cong} \mathcal{R}_f(W, \mathfrak{T}(*)(n)).$$

Here we have used the fact that the image of the retraction of a finite object intersects only finitely many path components of W. It follows that the map of spectra $\bigvee_{i \in I} \mathbb{A}^{-\infty}(W_i) \to \mathbb{A}^{-\infty}(W)$ is a levelwise equivalence. □

11.7. Corollary. *Let G be a discrete group and $H \leqslant G$ a subgroup. If G satisfies the fibered isomorphism conjecture in A-theory with respect to the family $\mathcal{F}$, then H satisfies the fibered isomorphism conjecture in A-theory with respect to $\mathcal{F}|_H$.*

Proof. Apply Proposition 11.3 to the inclusion $H \hookrightarrow G$. □

11.8. Corollary. *Let $\pi : G \to Q$ be a surjective group homomorphism. Suppose that Q satisfies the fibered Farrell–Jones conjecture in A-theory, and that for every virtually cyclic subgroup $V \leqslant Q$, the preimage $\pi^{-1}(V)$ satisfies the fibered Farrell–Jones conjecture in A-theory. Then G satisfies the fibered Farrell–Jones conjecture in A-theory.*

Proof. Note that $\pi^*\mathcal{VC}yc = \{\pi^{-1}(V) \mid V \leqslant Q \text{ virtually cyclic}\}$. Thus, the claim is a combination of Proposition 11.3 and the transitivity principle, Proposition 11.2. □

11.9. Corollary. *Let $\pi : G \to Q$ be a surjective group homomorphism with finite kernel. If Q satisfies the fibered Farrell–Jones conjecture in A-theory, then so does G.*

Proof. If $V \leqslant Q$ is virtually cyclic, then $\pi^{-1}(V)$ is also virtually cyclic, and thus G satisfies the conjecture by Corollary 11.8. □

The next two statements and their proofs are analogous to [Bartels et al. 2014a, Sections 3.2 and 3.3]. We only sketch their proofs and refer to [loc. cit.] for details.

11.10. Lemma (cf. [Bartels et al. 2014a, Lemma 3.15]). *Let Γ be a crystallographic group of virtual cohomological dimension 2 which possesses a normal, infinite cyclic subgroup. Then Γ satisfies the fibered Farrell–Jones conjecture in A-theory.*

Proof. Do an induction on the order of the smallest finite group F such that there is a short exact sequence $1 \to \mathbb{Z}^2 \to \Gamma \to F \to 1$. Using Theorem 7.4, the claim follows from [Winges 2015, Lemma 5.2 and Proposition 5.3] in conjunction with the induction hypothesis and the transitivity principle, Proposition 11.2. □

11.11. Proposition (cf. [Bartels et al. 2014a, Section 3.3]). *Let Γ be a virtually finitely generated abelian group. Then Γ satisfies the fibered Farrell–Jones conjecture in A-theory.*

Proof. We do an induction on the virtual cohomological dimension of Γ. If $\mathrm{vcd}(\Gamma) \leqslant 1$, the group Γ is virtually cyclic and there is nothing to show. So assume $\mathrm{vcd}(\Gamma) \geqslant 2$. Then do a subinduction on the cardinality of the smallest finite group F such that Γ admits an epimorphism onto F whose kernel is isomorphic to $\mathbb{Z}^{\mathrm{vcd}(\Gamma)}$.

Since Γ admits a surjection with finite kernel onto a crystallographic group [Quinn 2012, Lemma 4.2.1], we may assume by Corollary 11.9 that Γ is crystallographic of the same virtual cohomological dimension. Now fix an epimorphism $p : \Gamma \twoheadrightarrow F$ onto a finite group F such that the kernel of p is isomorphic to $\mathbb{Z}^{\mathrm{vcd}(\Gamma)}$ and such that the cardinality of F is minimal among all finite groups which admit such an epimorphism. By induction and the transitivity principle, Proposition 11.2, it suffices to show that Γ satisfies the fibered isomorphism conjecture with respect to the family of all virtually finitely generated abelian subgroups A of Γ which satisfy either of the following:

- $\mathrm{vcd}(A) < \mathrm{vcd}(\Gamma)$ or
- $\mathrm{vcd}(A) = \mathrm{vcd}(\Gamma)$ and A admits an epimorphism $p' : A \to F'$ onto a finite group F' such that $|F'| < |F|$ and the kernel of p' is isomorphic to $\mathbb{Z}^{\mathrm{vcd}(\Gamma)}$.

Suppose that Γ possesses a normal, infinite cyclic subgroup $C \trianglelefteq \Gamma$. We want to apply Corollary 11.8. Since $\mathrm{vcd}(\Gamma/C) < \mathrm{vcd}(\Gamma)$, the quotient Γ/C satisfies the fibered Farrell–Jones conjecture. Let $\pi : \Gamma \to \Gamma/C$ be the projection. For every virtually cyclic subgroup $V \leqslant G/C$, the preimage $\pi^{-1}(V)$ has virtual cohomological dimension 2. Again, $\pi^{-1}(V)$ admits a surjection with finite kernel onto a crystallographic group K [Quinn 2012, Lemma 4.2.1], so we may assume that $\pi^{-1}(V)$ is crystallographic by Corollary 11.9. Since $\mathrm{vcd}(\pi^{-1}(V)) = 2$ and there exists a normal, infinite cyclic subgroup, it follows from Lemma 11.10 that $\pi^{-1}(V)$ satisfies the fibered Farrell–Jones conjecture. So Corollary 11.8 applies.

Suppose that there is no normal, infinite cyclic subgroup in Γ. Then Γ is a Dress–Farrell–Hsiang group with respect to a family containing only groups to which the induction hypothesis applies [Winges 2015, Proposition 5.4]. Theorem 7.4 and Proposition 11.2 imply that Γ satisfies the fibered Farrell–Jones conjecture. $\square$

Proposition 11.11 is the stepping stone to proving an even slightly stronger version of Theorem 1.2; see Theorem 11.19 below. Recall that the (unrestricted) wreath product $G_1 \wr G_2$ of a group G_1 with another group G_2 is the semidirect product $\left(\prod_{G_2} G_1\right) \rtimes G_2$, where G_2 acts on the left factor by left translations.

11.12. Definition. Let $\mathcal{F}$ be a family of groups and let G be a discrete group. We say that G *satisfies the fibered isomorphism conjecture with wreath products in A-theory* with respect to $\mathcal{F}$ if for every finite group F, the wreath product $G \wr F$ satisfies the fibered isomorphism conjecture in A-theory with respect to $\mathcal{F}$.

If $\mathcal{F}$ is the family of virtually cyclic groups, we say that G *satisfies the fibered Farrell–Jones conjecture with wreath products in A-theory*.

11.13. Corollary. *Every virtually finitely generated abelian group satisfies the fibered Farrell–Jones conjecture with wreath products in A-theory.*

Proof. This is an immediate consequence of Proposition 11.11 since the wreath product of a virtually finitely generated abelian group with a finite group is again virtually finitely generated abelian. $\square$

Let us record some additional inheritance properties of the fibered isomorphism conjecture with wreath products. The following results have been worked out in [Kühl 2008]; we collect them here for reference and the convenience of the reader.

11.14. Lemma. *Let G, G_1, G_2 be discrete groups, and let $\mathcal{F}$ be a family of groups.*

(1) *Let $H \leqslant G$ be a subgroup. If G satisfies the fibered isomorphism conjecture with wreath products with respect to $\mathcal{F}$, then so does H.*

(2) *Let $H \leqslant G$ be a subgroup of finite index. If H satisfies the fibered isomorphism conjecture with wreath products with respect to $\mathcal{F}$, so does G.*

(3) *If G_1 and G_2 satisfy the fibered Farrell–Jones conjecture with wreath products, so does $G_1 \times G_2$.*

(4) *Suppose G satisfies the fibered isomorphism conjecture with wreath products with respect to $\mathcal{F}$, and that every subgroup $H \leqslant G$ which lies in $\mathcal{F}$ satisfies the fibered Farrell–Jones conjecture with wreath products. If $\mathcal{F}$ is closed under taking quotients, then G satisfies the fibered Farrell–Jones conjecture with wreath products.*

(5) *Let $\pi : G \twoheadrightarrow Q$ be a surjective homomorphism. Suppose that Q satisfies the fibered Farrell–Jones conjecture with wreath products, and that for every virtually cyclic subgroup $V \leqslant Q$ the preimage $\pi^{-1}(V)$ satisfies the fibered Farrell–Jones conjecture with wreath products. Then G satisfies the fibered Farrell–Jones conjecture with wreath products.*

(6) *Let $\pi : G \twoheadrightarrow Q$ be a surjective homomorphism with finite kernel. If Q satisfies the fibered Farrell–Jones conjecture with wreath products, so does G.*

Proof. Claim (1) is a consequence of Corollary 11.7 since $H \wr F$ is a subgroup of $G \wr F$ for every group F.

For (2), assume first that H is normal in G. Set $F := G/H$. Choose a set-theoretic section $s : F \to G$ of the projection map $\pi : G \twoheadrightarrow F$. For $g \in G$ and $f \in F$ define

$$h(g, f) := s(f)^{-1} g s(\pi(g)^{-1} f).$$

Then $g \mapsto ((h(g, f))_f, \pi(g))$ defines a monomorphism $G \hookrightarrow H \wr F$. Thus, for every finite group F', the wreath product $G \wr F'$ is a subgroup of $(H \wr F) \wr F'$. Since $(H \wr F) \wr F'$ itself embeds into $H \wr (F \wr F')$ [Kühl 2008, Lemma 1.21], the claim follows from (1). If H is not normal, (1) allows us to replace H by $\bigcap_{g \in G} gHg^{-1}$.

For (3), observe that $(G_1 \times G_2) \wr F$ is a subgroup of $(G_1 \wr F) \times (G_2 \wr F) =: \Gamma$. By (1), it suffices to check that the latter group satisfies the fibered Farrell–Jones conjecture. Consider the projection map $p_1 : \Gamma \to G_1 \wr F$. We want to apply Corollary 11.8, so we need to check that $V \times (G_2 \wr F)$ satisfies the fibered Farrell–Jones conjecture for every virtually cyclic subgroup V of $G_1 \wr F$. This can be done by another application of Corollary 11.8. The target of the projection map $p_2 : V \times (G_2 \wr F) \to G_2 \wr F$ satisfies the fibered Farrell–Jones conjecture, so the only thing left to verify is that every product $V \times V'$ of virtually cyclic groups satisfies the fibered Farrell–Jones conjecture. Since the product of two virtually cyclic groups is virtually finitely generated abelian, this is true by Proposition 11.11.

Let us turn to (4). Consider a wreath product $G \wr F$, where F is finite. Our goal is to apply the transitivity principle, so we need to check that every subgroup

$H \leqslant G \wr F$ which lies in $\mathcal{F}$ satisfies the fibered Farrell–Jones conjecture. Let H be such a subgroup. Since $H' := H \cap \bigl(\prod_F G\bigr)$ is normal in H and has finite index, it suffices to show that H' satisfies the fibered Farrell–Jones conjecture with wreath products by (2). Observe that $H' \in \mathcal{F}$. Let H_f denote the image of H' under the projection map $\bigl(\prod_F G\bigr) \to G$ onto the factor indexed by $f \in F$. Then H' embeds into $\prod_{f\in F} H_f$. Since $\mathcal{F}$ is closed under quotients, H_f satisfies the fibered Farrell–Jones conjecture with wreath products, and so does the product $\prod_{f\in F} H_f$ by (3). Now (1) implies that H' satisfies the fibered Farrell–Jones conjecture with wreath products, and we are done.

For (5), observe that π induces a surjective homomorphism $\pi_F : G \wr F \twoheadrightarrow Q \wr F$ for every finite group F. The quotient $Q \wr F$ satisfies the fibered Farrell–Jones conjecture by assumption. We want to apply Corollary 11.8. So let $V \leqslant Q \wr F$ be virtually cyclic. In order to show that $\pi_F^{-1}(V)$ satisfies the fibered Farrell–Jones conjecture, it suffices to show that $\widetilde{V} := \pi_F^{-1}(V) \cap \bigl(\prod_F G\bigr)$ satisfies the fibered Farrell–Jones conjecture with wreath products. Denote by V_f the image of $V \cap \bigl(\prod_F Q\bigr)$ under the projection $\prod_F Q \to Q$ onto the factor indexed by $f \in F$. Then $\widetilde{V}$ embeds into $\prod_{f\in F} \pi^{-1}(V_f)$. Since V_f is a virtually cyclic subgroup of Q, the preimage $\pi^{-1}(V_f)$ satisfies the fibered Farrell–Jones conjecture with wreath products by assumption, and hence so does $\prod_{f\in F} \pi^{-1}(V_f)$ by (3). Then $\widetilde{V}$ satisfies the fibered Farrell–Jones conjecture with wreath products by (1).

The last part of the lemma follows from (5) because the preimage of each virtually cyclic subgroup of Q is again virtually cyclic (and these satisfy the fibered Farrell–Jones conjecture with wreath products by Corollary 11.13). □

In analogy to [Wegner 2015, Proposition 2.19], we are going to show next that the Dress–Farrell–Hsiang condition (Definition 7.3) is well-behaved with respect to wreath products with finite groups. Let G be a group, $\mathcal{F}$ a family of subgroups and Φ a finite group. Denote by $\mathcal{F}^{\wr\Phi}$ the family of subgroups of $G \wr \Phi$ consisting of those groups which contain a finite-index subgroup of the form $\prod_{\psi\in\Phi} H_\psi$, where each H_ψ lies in $\mathcal{F}$.

Recall the following construction of the product of simplicial complexes. Let $E_1, \dots, E_k$ be (abstract) ordered simplicial complexes. Then define $E_1 \otimes \cdots \otimes E_k$ to be the simplicial complex whose r-simplices are ascending chains $(e_1^0, \dots, e_k^0) < \cdots < (e_1^r, \dots, e_k^r)$ with respect to the lexicographic ordering such that $\{e_i^0, \dots, e_i^r\}$ is a simplex in E_i for all i. The map $|E_1 \otimes \cdots \otimes E_k| \to |E_1| \times \cdots \times |E_k|$ induced by the obvious projections $E_1 \otimes \cdots \otimes E_k \to E_i$ is a homeomorphism (with respect to the topologies induced by the ℓ^1-metric).

11.15. Proposition. *Let G be a discrete group and let $\mathcal{F}$ be a family of subgroups. Let S be a finite, symmetric generating set such that (G, S) is a Dress–Farrell–Hsiang group with respect to $\mathcal{F}$.*

Then there is for every finite group Φ a generating set $S^{\wr\Phi}$ of $G \wr \Phi$ such that $(G \wr \Phi, S^{\wr\Phi})$ is a Dress–Farrell–Hsiang group with respect to $\mathcal{F}^{\wr\Phi}$.

Proof. We start with a preliminary observation. Let Φ be a finite group, and let $\pi : G \twoheadrightarrow F$ be an epimorphism onto some finite group F. Then π induces a surjective homomorphism $\pi^{\wr\Phi} : G \wr \Phi \twoheadrightarrow F \wr \Phi$ given by $((g_\psi)_\psi, \varphi) \mapsto ((\pi(g_\psi))_\psi, \varphi)$.

Let $H \leqslant F \wr \Phi$ be a Dress group. Let Φ_H be the image of H under the canonical projection $F \wr \Phi \twoheadrightarrow \Phi$, and let H_ξ denote the image of $H \cap \left(\prod_\Phi F\right)$ under the map $p_\xi : \left(\prod_\Phi F\right) \to F$ given by projection onto the ξ-th component. Since the class of Dress groups is closed under quotients, each H_ξ is a Dress group. For each $\varphi \in \Phi_H$, pick a preimage $\kappa^\varphi = ((\kappa^\varphi_\psi)_\psi, \varphi) \in H$. Choose a section $s : \Phi_H\backslash\Phi \to \Phi$ of the obvious projection map such that $s(\Phi_H) = 1$. Now define

$$\kappa := \left(\left(\kappa_{s(\Phi_H\psi)}^{s(\Phi_H\psi)\psi^{-1}}\right)_\psi, 1\right) \in F \wr \Phi,$$

and let $\hat{H}$ denote the group

$$\left(\prod_{\Phi_H\psi\in\Phi_H\backslash\Phi}\left(\prod_{\psi'\in\Phi_H\psi} H_{s(\Phi_H\psi)}\right)\right) \rtimes \Phi_H,$$

where Φ_H acts on the left-hand side by permuting the index set of every factor $\prod_{\Phi_H\psi} H_{s(\Phi_H\psi)}$. Observe that $\hat{H}$ is naturally a subgroup of $F \wr \Phi$. We claim that κ subconjugates H into $\hat{H}$.

To see this, compute first, for an arbitrary element $((\alpha_\psi)_\psi, \varphi) \in H$,

$$\begin{aligned}
\kappa((\alpha_\psi)_\psi, \varphi)\kappa^{-1} &= \left(\left(\kappa_{s(\Phi_H\psi)}^{s(\Phi_H\psi)\psi^{-1}}\right)_\psi, 1\right)((\alpha_\psi)_\psi, \varphi)\left(\left(\left[\kappa_{s(\Phi_H\psi)}^{s(\Phi_H\psi)\psi^{-1}}\right]^{-1}\right)_\psi, 1\right) \\
&= \left(\left(\kappa_{s(\Phi_H\psi)}^{s(\Phi_H\psi)\psi^{-1}}\alpha_\psi\left[\kappa_{s(\Phi_H\varphi^{-1}\psi)}^{s(\Phi_H\varphi^{-1}\psi)\psi^{-1}\varphi}\right]^{-1}\right)_\psi, \varphi\right) \\
&= \left(\left(\kappa_{s(\Phi_H\psi)}^{s(\Phi_H\psi)\psi^{-1}}\alpha_\psi\left[\kappa_{s(\Phi_H\psi)}^{s(\Phi_H\psi)\psi^{-1}\varphi}\right]^{-1}\right)_\psi, \varphi\right).
\end{aligned}$$

In order to show that this element lies in $\hat{H}$, we need to check that for every $\xi \in \Phi$, $\kappa_{s(\Phi_H\xi)}^{s(\Phi_H\xi)\xi^{-1}}\alpha_\xi\left[\kappa_{s(\Phi_H\xi)}^{s(\Phi_H\xi)\xi^{-1}\varphi}\right]^{-1}$ lies in $H_{s(\Phi_H\xi)}$. Indeed,

$$\begin{aligned}
&\kappa^{s(\Phi_H\xi)\xi^{-1}}((\alpha_\psi)_\psi, \varphi)(\kappa^{s(\Phi_H\xi)\xi^{-1}\varphi})^{-1} \\
&= \left((\kappa_\psi^{s(\Phi_H\xi)\xi^{-1}}1)_\psi, s(\Phi_H\xi)\xi^{-1}\right)((\alpha_\psi)_\psi, \varphi)\left(\left(\left[\kappa^{s(\Phi_H\xi)\xi^{-1}\varphi}_{s(\Phi_H\xi)\xi^{-1}\varphi\psi}\right]^{-1}\right)_\psi, \varphi^{-1}\xi s(\Phi_H\xi)^{-1}\right) \\
&= \left(\left(\kappa_\psi^{s(\Phi_H\xi)\xi^{-1}}\alpha_{\xi s(\Phi_H\xi)^{-1}\psi}\left[\kappa_\psi^{s(\Phi_H\xi)\xi^{-1}\varphi}\right]^{-1}\right)_\psi, 1\right).
\end{aligned}$$

Since this is an element in $H \cap \left(\prod_\Phi F\right)$, we obtain

$$\begin{aligned}
&\kappa_{s(\Phi_H\xi)}^{s(\Phi_H\xi)\xi^{-1}}\alpha_\xi\left[\kappa_{s(\Phi_H\xi)}^{s(\Phi_H\xi)\xi^{-1}\varphi}\right]^{-1} \\
&\qquad = p_{s(\Phi_H\xi)}\left(\kappa^{s(\Phi_H\xi)\xi^{-1}}((\alpha_\psi)_\psi, \varphi)[\kappa^{s(\Phi_H\xi)\xi^{-1}\varphi}]^{-1}\right) \in H_{s(\Phi_H\xi)}.
\end{aligned}$$

Hence, $\kappa H\kappa^{-1} \subseteq \hat{H}$.

Since (G, S) is Dress–Farrell–Hsiang, there is some N as in Definition 7.3. Let $\varepsilon' > 0$. Let $\pi = \pi_{\varepsilon'} : G \twoheadrightarrow F$ be some epimorphism satisfying the conditions in Definition 7.3. Define $\pi^{\wr\Phi}$ as above. According to our preliminary observation, it suffices to consider subgroups of $F \wr \Phi$ which have the form

$$H = \Bigg(\prod_{\Phi_H \psi \in \Phi_H \backslash \Phi} \Bigg(\prod_{\psi' \in \Phi_H \psi} H_{\Phi_H \psi} \Bigg) \Bigg) \rtimes \Phi_H,$$

where Φ_H is some subgroup of Φ and each $H_{\Phi_H \psi}$ is a Dress subgroup of F. Define a generating set $S^{\wr\Phi}$ of $G \wr \Phi$ by

$$S^{\wr\Phi} := \{((g_\psi)_\psi, \varphi) \mid g_\psi \in S \text{ for all } \psi,\ \varphi \in \Phi\}.$$

For each $\Phi_H \psi \in \Phi_H \backslash \Phi$, choose a $\pi^{-1}(H_{\Phi_H \psi})$-equivariant map $f_{\Phi_H \psi} : G \to E_{\Phi_H \psi}$ to a $\pi^{-1}(H_{\Phi_H \psi})$-simplicial complex of dimension at most N whose stabilizers lie in $\mathcal{F}$, and such that $d(f_{\Phi_H \psi}(g), f_{\Phi_H \psi}(g')) \leqslant \varepsilon'$ whenever $g^{-1} g' \in S$. Define

$$f_H : G \wr \Phi \to \prod_{\Phi_H \psi \in \Phi_H \backslash \Phi} \ \prod_{\psi' \in \Phi_H \psi} E_{\Phi_H \psi} =: E_H,$$

$$((g_\psi)_\psi, \varphi) \mapsto \big((f_{\Phi_H \psi}(g_{\psi'}))_{\psi' \in \Phi_H \psi} \big)_{\Phi_H \psi \in \Phi_H \backslash \Phi}.$$

We regard E_H as a simplicial complex via the product construction described previously. Let H act on E_H by

$$((h_{\psi'})_{\psi' \in \Phi_H \psi})_{\Phi_H \psi}, \varphi) \cdot ((x_{\psi'})_{\psi' \in \Phi_H \psi})_{\Phi_H \psi} := ((h_{\psi'} x_{\varphi^{-1} \psi'})_{\psi' \in \Phi_H \psi})_{\Phi_H \psi}.$$

This induces a $(\pi^{\wr\Phi})^{-1}(H)$-action on E_H by restriction, and f_H is $(\pi^{\wr\Phi})^{-1}(H)$-equivariant with respect to this action. Observe that the dimension of E_H is bounded by $|\Phi| N$, and that this number only depends on Φ.

Consider a point $x := ((x_{\psi'})_{\psi' \in \Phi_H \psi})_{\Phi_H \psi}$ in E_H, and the stabilizer $(\pi^{\wr\Phi})^{-1}(H)_x$. The intersection $(\pi^{\wr\Phi})^{-1}(H)_x \cap \big(\prod_\Phi G\big)$ is a finite-index subgroup of $(\pi^{\wr\Phi})^{-1}(H)_x$, and is equal to $\prod_{\Phi_H \psi \in \Phi_H \backslash \Phi} \prod_{\psi' \in \Phi_H \psi} H_{x_{\psi'}}$. Since each $H_{x_{\psi'}}$ lies in $\mathcal{F}$, this shows that the stabilizer of x lies in $\mathcal{F}^{\wr\Phi}$.

What is left to show is that the map f_H has the desired contracting property. So let $g = ((g_\psi)_\psi, \varphi)$ and $g' = ((g'_\psi)_\psi, \varphi')$ be elements in $G \wr \Phi$ such that $g^{-1} g' \in S^{\wr\Phi}$; equivalently, $g_\psi^{-1} g'_\psi \in S$ for all $\psi \in \Phi$. For each $\Phi_H \psi \in \Phi_H \backslash \Phi$ and $\psi' \in \Phi_H \psi$, we have

$$d^{\ell^1}_{E_{\Phi_H \psi}}(f_{\Phi_H \psi}(g_{\psi'}), f_{\Phi_H \psi}(g'_{\psi'})) \leqslant \varepsilon'.$$

Let $\varepsilon > 0$. By Lemma 11.16 below, $d^{\ell^1}_{E_H}(f_H(g), f_H(g')) \leqslant \varepsilon$ as long as ε' was initially chosen to be small enough. ☐

11.16. Lemma. *Let $N, K \in \mathbb{N}$. For every $\varepsilon > 0$ there is some $\varepsilon' > 0$ such that for every sequence $E_1, \dots, E_K$ of (abstract) ordered simplicial complexes, each of which has dimension at most N, the following holds:*

Let $E := E_1 \otimes \cdots \otimes E_K$. For $x \in |E|$, let $(x_1, \ldots, x_K)$ denote the image of x under the canonical map $|E| \to |E_1| \times \cdots \times |E_K|$. Denote by d_i the ℓ^1-metric on $|E_i|$, and let d be the ℓ^1-metric on $|E|$.

Then for all $x, x' \in |E|$, we have $d(x, x') \leqslant \varepsilon$ whenever $d_i(x_i, x_i') \leqslant \varepsilon'$.

Proof. The argument is very similar to the one employed in the proof of [Bartels et al. 2014b, Lemma 5.5]. Since distances with respect to the ℓ^1-metric are independent of the ambient complex, we may assume that $E_i = \Delta^{2N+1}$. Consider the composition

$$\prod_{1 \leqslant i \leqslant K} |\Delta^{2N+1}| = |E_1| \times \cdots \times |E_K| \xrightarrow{\cong} |E| \subseteq |\Delta^{(2N+2)^K - 1}|$$

of the inverse of the canonical homeomorphism with the inclusion into the full simplex. Consider the domain of this map as a metric space by taking the metric $d_\Sigma := \sum_i d_i$ and equip the target with its natural ℓ^1-metric d_Δ. This map is uniformly continuous since the source is compact; hence, there is some $\varepsilon'' > 0$ such that $d_\Delta(x, x') \leqslant \varepsilon$ whenever $d_\Sigma((x_1, \ldots, x_K), (x_1', \ldots, x_K')) \leqslant \varepsilon''$. Thus, the claim holds for $\varepsilon' := \varepsilon''/K$. □

11.17. Corollary. *Let G be a discrete group and let $\mathcal{F}$ be a family of groups such that all groups in $\mathcal{F}$ satisfy the fibered Farrell–Jones conjecture with wreath products in A-theory. If there is a finite, symmetric generating set S of G such that (G, S) is a Dress–Farrell–Hsiang group with respect to $\mathcal{F}$, then G satisfies the fibered Farrell–Jones conjecture with wreath products in A-theory.*

Proof. Let Φ be a finite group. By Proposition 11.15, the wreath product $G \wr \Phi$ is a Dress–Farrell–Hsiang group with respect to $\mathcal{F}^{\wr\Phi}$, so $G \wr \Phi$ satisfies the fibered isomorphism conjecture with respect to $\mathcal{F}^{\wr\Phi}$. Since all groups in $\mathcal{F}$ satisfy the fibered Farrell–Jones conjecture with wreath products, parts (3) and (2) of Lemma 11.14 imply that all groups in $\mathcal{F}^{\wr\Phi}$ satisfy the fibered Farrell–Jones conjecture. Hence, $G \wr \Phi$ satisfies the fibered Farrell–Jones conjecture by the transitivity principle, Proposition 11.2. □

11.18. Theorem. *Let Γ be an irreducible special affine group. Then Γ satisfies the fibered Farrell–Jones conjecture with wreath products in A-theory.*

Proof. By [Winges 2015, Theorem 6.1], Γ is a Dress–Farrell–Hsiang group with respect to the family of virtually finitely generated abelian groups. Since we have already shown that all virtually finitely generated abelian groups satisfy the fibered Farrell–Jones conjecture with wreath products in Corollary 11.13, the theorem is an immediate consequence of Corollary 11.17. □

11.19. Theorem (cf. [Bartels et al. 2014a, Section 5]). *Let G be a virtually poly-$\mathbb{Z}$-group. Then G satisfies the fibered Farrell–Jones conjecture with wreath products in A-theory.*

Proof. Repeat the argument on page 377 of [Bartels et al. 2014a], which relies only on the inheritance properties of the conjecture. □

Theorem 1.2 from the introduction follows as a special case.

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Almost complex structures on connected sums of complex projective spaces

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We show that the m-fold connected sum $m\#\mathbb{CP}^{2n}$ admits an almost complex structure if and only if m is odd.

1. Introduction

A *complex structure* on a real vector bundle F over a connected CW complex X is a complex vector bundle E over X such that its underlying real vector bundle $E_{\mathbb{R}}$ is isomorphic to F. A *stable complex structure* on F is a complex structure on $F \oplus \varepsilon^d$, where ε^d is the d-dimensional trivial real vector bundle over X. For X a manifold we say that X has an *almost complex structure* if its tangent bundle admits a complex structure, and a *stable almost complex structure* if its tangent bundle admits a stable complex structure. Motivated by the question in [Miller 2015] we consider in this paper the m-fold connected sum of complex projective spaces $m\#\mathbb{CP}^{2n}$.

As shown by Hirzebruch [1987, Kommentare, p. 777], a necessary condition for the existence of an almost complex structure on a $4n$-dimensional compact manifold M is the congruence $\chi(M) \equiv (-1)^n\sigma(M) \bmod 4$, where $\chi(M)$ is the Euler characteristic and $\sigma(M)$ is the signature of M. Thus, for even m, the connected sums above cannot carry an almost complex structure. We show that for odd m they do admit almost complex structures, thus showing the following:

Theorem 1.1. *The m-fold connected sum $m\#\mathbb{CP}^{2n}$ admits an almost complex structure if and only if m is odd.*

In odd complex dimensions, the connected sums $m\#\mathbb{CP}^{2n+1}$ are Kähler: $\mathbb{CP}^{2n+1}$ admits an orientation reversing diffeomorphism, and therefore $m\#\mathbb{CP}^{2n+1}$ is diffeomorphic to $\mathbb{CP}^{2n+1}\#(m-1)\overline{\mathbb{CP}^{2n+1}}$, which is a blow–up of $\mathbb{CP}^{2n+1}$ in $m-1$ points. Furthermore Theorem 1.1 is known for $n=1$ and $n=2$; see [Audin 1991] and [Müller and Geiges 2000], respectively. In both cases the authors use general

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results on the existence of almost complex structures on manifolds of dimension 4 and 8, respectively.

In [Sutherland 1965, Theorem 1.1] or [Thomas 1967, Theorem 1.7] the authors showed the following.

Theorem 1.2. *Let M be a closed smooth $2d$-dimensional manifold. Then TM admits an almost complex structure if and only if it admits a stable almost complex structure E such that $c_d(E) = e(M)$, where c_d is the d-th Chern class of E and $e(M)$ is the Euler class of M.*

In Section 2 we describe the full set of stable almost complex structures in the reduced K-theory of $m\#\mathbb{CP}^{2n}$. In Section 3 we give, for odd m, an explicit example of a stable almost complex structure to which Theorem 1.2 applies, thus completing the proof of Theorem 1.1.

2. Stable almost complex structures on $m\#\mathbb{CP}^{2n}$

For a CW complex X let $K(X)$ and $KO(X)$ denote the complex and real K-groups, respectively. Moreover we denote by $\widetilde{K}(X)$ and $\widetilde{KO}(X)$ the reduced groups. Let $r : K(X) \to KO(X)$ denote the real reduction map, which can be restricted to a map $\widetilde{K}(X) \to \widetilde{KO}(X)$. We denote the restricted map again with r. A real vector bundle F over X has a stable almost complex structure if there is a an element $y \in \widetilde{K}(X)$ such that $r(y) = F - \dim F$. Since r is a group homomorphism, the set of all stable complex vector bundles such that the underlying real vector bundle is stably isomorphic to F is given by

$$y + \ker r \subset \widetilde{K}(X),$$

where y is such that $r(y) = F - \dim F$. Let $c : KO(X) \to K(X)$ denote the complexification map and $t : K(X) \to K(X)$ the map which is induced by complex conjugation of complex vector bundles. The maps t and c are ring homomorphisms, but r preserves only the group structure. The identities

$$c \circ r = 1 + t : K(X) \to K(X), \qquad r \circ c = 2 : KO(X) \to KO(X),$$

involving the maps r, c and t are well known. We write $\bar{y} = t(y)$ for an element $y \in K(X)$.

For two oriented manifolds M and N of the same dimension d, we denote by $M\#N$ the connected sum of M with N, which inherits an orientation from M and N. First, let us characterize the stable tangent bundle of $M\#N$.

Lemma 2.1. *Let $p_M : M\#N \to M$ and $p_N : M\#N \to N$ be collapsing maps to each factor of $M\#N$. Then we have*

$$p_M^*(M) \oplus p_N^*(N) \cong T(M\#N) \oplus \varepsilon^d.$$

Proof. Let $D_M \subset M$ and $D_N \subset N$ be embedded closed disks and W_M and W_N collar neighborhoods of $\partial(M \setminus \mathring{D}_M)$ and $\partial(N \setminus \mathring{D}_N)$, respectively, where $\mathring{D}$ denotes the interior of D. Thus $W_M \cong S^{d-1} \times [-2, 0]$ and $W_N \cong S^{d-1} \times [0, 2]$. The manifold $M\#N$ is obtained by identifying $S^{d-1} \times 0 \subset W_M$ with $S^{d-1} \times 0 \subset W_N$ by the identity map. Set $W := W_M \cup W_N \subset M\#N$ and note that $V_1 := p_M^*(M) \oplus p_N^*(N)$ as well as $V_2 := T(M\#N) \oplus \varepsilon^n$ are trivial over W. Moreover let $U_M \subset M\#N$ be the open set diffeomorphic to $(M \setminus W_M) \cup (S^{d-1} \times [-2, -1[)$, and analogously for $U_N \subset M\#N$.

Now, since $V_1|_{U_M} \cong p_M^*(TM) \oplus \varepsilon^d$ and $p_M^*(TM)|_{U_M} = T(M\#N)|_{U_M}$, we have an isomorphism given by $\Phi_M : V_2|_{U_M} \to V_1|_{U_M}$, $(\xi, w) \mapsto ((p_M)_*(\xi), w)$. For $\Phi_N : V_2|_{U_N} \to V_1|_{U_N}$, we set $\Phi_N(\eta, w) = (w, -(p_N)_*(\eta))$. Moreover, both vector bundles V_1 and V_2 are trivial over W and it is possible to choose trivializations of V_1 and V_2 over W such that Φ_M is given by $(v, w) \mapsto (v, w)$ over W_M and such that Φ_N is represented by $(v, w) \mapsto (w, -v)$ over W_N. Over $S^{d-1} \times [-1, 1]$ we can interpolate these isomorphisms by

$$\begin{pmatrix} v \\ w \end{pmatrix} \mapsto \begin{pmatrix} \cos\bigl(\frac{\pi}{4}(t+1)\bigr) & \sin\bigl(\frac{\pi}{4}(t+1)\bigr) \\ -\sin\bigl(\frac{\pi}{4}(t+1)\bigr) & \cos\bigl(\frac{\pi}{4}(t+1)\bigr) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

for $t \in [-1, 1]$. Using this interpolation we can glue Φ_M and Φ_N to a global isomorphism $V_2 \to V_1$. $\square$

Hence, $T(M\#N) - d = TM + TN - 2d$ in $\widetilde{KO}(M\#N)$, where TM and TN denote the elements in $\widetilde{KO}(M\#N)$ induced by $p_M^*(TM)$ and $p_N^*(TN)$, respectively. This shows that if M and N admit stable almost complex structures so does $M\#N$; see [Kahn 1969]. For $M = N = \mathbb{CP}^{2n}$ we consider the natural orientation induced by the complex structure of $\mathbb{CP}^{2n}$.

We proceed with recalling some basic facts on complex projective spaces. Let H be the tautological line bundle over $\mathbb{CP}^d$ and let $x \in H^2(\mathbb{CP}^d; \mathbb{Z})$ be the generator, such that the total Chern class $c(H)$ is given by $1 + x$. The cohomology ring of $\mathbb{CP}^d$ is isomorphic to $\mathbb{Z}[x]/\langle x^{d+1}\rangle$. The K and KO theory of $\mathbb{CP}^d$ are completely understood. Let $\eta := H - 1 \in \widetilde{K}(\mathbb{CP}^d)$ and $\eta_R := r(\eta) \in \widetilde{KO}(\mathbb{CP}^d)$.

Theorem 2.2 (cf. [Sanderson 1964, Theorem 3.9; Fujii 1966, Lemma 3.5; Milnor and Stasheff 1974, p. 170; Thomas 1974, Proposition 4.3]).

(a) $K(\mathbb{CP}^d) = \mathbb{Z}[\eta]/\langle \eta^{d+1}\rangle$. *Letting n be the largest integer $\leq d/2$, the following sets of elements are an integral basis of $K(\mathbb{CP}^d)$:*

   (i) $1, \eta, \eta(\eta+\bar\eta), \dots, \eta(\eta+\bar\eta)^{n-1}, (\eta+\bar\eta), \dots, (\eta+\bar\eta)^n$, *and also, in case d is odd,* $\eta^{2n+1} = \eta(\eta+\bar\eta)^n$;

   (ii) $1, \eta, \eta(\eta+\bar\eta), \dots, \eta(\eta+\bar\eta)^{n-1}, (\eta-\bar\eta)(\eta+\bar\eta), \dots, (\eta-\bar\eta)(\eta+\bar\eta)^{n-1}$, *and also, in case d is odd,* η^{2n+1}.

(b) (i) *If* $d = 2n$ *then* $KO(\mathbb{CP}^d) = \mathbb{Z}[\eta_R]/\langle \eta_R^{n+1} \rangle$.
(ii) *If* $d = 4n+1$ *then* $KO(\mathbb{CP}^d) = \mathbb{Z}[\eta_R]/\langle \eta_R^{2n+1}, 2\eta_R^{2n+2} \rangle$.
(iii) *If* $d = 4n+3$ *then* $KO(\mathbb{CP}^d) = \mathbb{Z}[\eta_R]/\langle \eta_R^{2n+2} \rangle$.

(c) *The complex stable tangent bundle is given by* $(2n+1)\bar{\eta} \in \widetilde{K}(\mathbb{CP}^{2n})$ *and the real stable tangent bundle is given by* $r((2n+1)\bar{\eta}) \in \widetilde{KO}(\mathbb{CP}^{2n})$.

(d) *The kernel of the real reduction map* $r : \widetilde{K}(\mathbb{CP}^d) \to \widetilde{KO}(\mathbb{CP}^d)$ *is freely generated by the elements*
(i) $\eta - \bar{\eta}, (\eta - \bar{\eta})(\eta + \bar{\eta}), \dots, (\eta - \bar{\eta})(\eta + \bar{\eta})^{(d/2)-1}$, *if* d *is even,*
(ii) $\eta - \bar{\eta}, (\eta - \bar{\eta})(\eta + \bar{\eta}), \dots, (\eta - \bar{\eta})(\eta + \bar{\eta})^{2n-1}, 2\eta^d$, *if* $d = 4n+1$,
(iii) $\eta - \bar{\eta}, (\eta - \bar{\eta})(\eta + \bar{\eta}), \dots, (\eta - \bar{\eta})(\eta + \bar{\eta})^{2n}, \eta^d$, *if* $d = 4n+3$.

Next we would like to describe the integer cohomology ring of $m\#\mathbb{CP}^{2n}$. For that we introduce the following notation. Let Λ denote either $\mathbb{Z}$ or $\mathbb{Q}$. We define an ideal $R_d(X_1, \dots, X_m)$ in $\Lambda[X_1, \dots, X_m]$, where $X_1, \dots, X_m$ are indeterminants, as the ideal generated by the following elements

$$X_i \cdot X_j, \quad i \neq j, \qquad X_i^d - X_j^d, \quad i \neq j, \qquad X_j^{d+1}, \quad j = 1, \dots, m.$$

Hence, we have

$$H^*(m\#\mathbb{CP}^d; \Lambda) \cong \Lambda[x_1, \dots, x_m]/R_d(x_1, \dots, x_m), \tag{2.3}$$

where $x_j = p_j^*(x) \in H^2(m\#\mathbb{CP}^d; \Lambda)$, for $x \in H^2(\mathbb{CP}^d; \Lambda)$ defined as above and $p_j : m\#\mathbb{CP}^d \to \mathbb{CP}^d$ the projection onto the j-th factor. Note that p_j induces an monomorphism on cohomology.

The stable tangent bundle of $m\#\mathbb{CP}^{2n}$ in $\widetilde{KO}(m\#\mathbb{CP}^{2n})$ is represented by

$$(2n+1)\sum_{j=1}^{m} r(\bar{\eta}_j),$$

where $\eta_j := p_j^*(\eta) \in \widetilde{K}(\mathbb{CP}^{2n})$ and $r : \widetilde{K}(m\#\mathbb{CP}^{2n}) \to \widetilde{KO}(m\#\mathbb{CP}^{2n})$ is the real reduction map. Hence the set of stable almost complex structures on $m\#\mathbb{CP}^{2n}$ is given by

$$(2n+1)\sum_{j=1}^{m} \bar{\eta}_j + \ker r. \tag{2.4}$$

For $k \in \mathbb{N}$ and $j = 1, \dots, m$, set $w_j^k = p_j^*(H)^k - p_j^*(H)^{-k}$, $e_j^{n-1} = \eta_j(\eta_j + \bar{\eta}_j)^{n-1}$ and $\omega = \eta_1^{2n}$.

Proposition 2.5. *The kernel of* $r : \widetilde{K}(m\#\mathbb{CP}^{2n}) \to \widetilde{KO}(m\#\mathbb{CP}^{2n})$ *is freely generated by*

$$\{w_j^k : k = 1, \dots, n-1,\ j = 1, \dots, m\} \cup \{e_1^{n-1} - e_j^{n-1} : j = 2, \dots, m\} \cup \{2e_1^{n-1} - \omega\}$$

for n even, and

$$\{w_j^k : k = 1, \dots, n,\ j = 1, \dots, m\}$$

for n odd.

Proof. Consider the cofiber sequence

$$\bigvee_{j=1}^{m} \mathbb{CP}^{2n-1} \xrightarrow{i} m\#\mathbb{CP}^{2n} \xrightarrow{\pi} S^{4n}. \tag{2.6}$$

Note that the line bundle $i^* p_j^*(H)$ is the tautological line bundle over the j-th summand of $\bigvee_{j=1}^m \mathbb{CP}^{2n-1}$ and the trivial bundle on the other summands, since the first Chern classes are the same. For the reduced groups we have

$$\widetilde{K}\Big(\bigvee_{j=1}^{m} \mathbb{CP}^{2n-1}\Big) \cong \bigoplus_{j=1}^{m} \widetilde{K}(\mathbb{CP}^{2n-1})$$

and $i^* p_j^*(\eta)$ generates the j-th summand of the above sum according to Theorem 2.2. The long exact sequence in K-theory of the cofibration (2.6) is given by

$$\begin{aligned}\cdots \to \widetilde{K}^{-1}\Big(\bigvee_{j=1}^{m} \mathbb{CP}^{2n-1}\Big) \to \widetilde{K}(S^{4n}) \to \widetilde{K}(m\#\mathbb{CP}^{2n})&\\ \to \widetilde{K}\Big(\bigvee_{j=1}^{m} \mathbb{CP}^{2n-1}\Big) \to \widetilde{K}^{1}(S^{4n}) \to \cdots .&\end{aligned} \tag{2.7}$$

From Theorem 2 in [Fujii 1967], we have that $\widetilde{K}^{-1}(\mathbb{CP}^{2n-1}) = 0$, and hence $\widetilde{K}^{-1}\big(\bigvee_{j=1}^m \mathbb{CP}^{2n-1}\big) = 0$. Then from Bott periodicity we deduce the equality $\widetilde{K}^{1}(S^{4n}) = \widetilde{K}^{-1}(S^{4n}) = 0$. So we obtain a short exact sequence

$$0 \longrightarrow \widetilde{K}(S^{4n}) \xrightarrow{\pi^*} \widetilde{K}(m\#\mathbb{CP}^{2n}) \xrightarrow{i^*} \widetilde{K}\Big(\bigvee_{j=1}^{m} \mathbb{CP}^{2n-1}\Big) \longrightarrow 0$$

which splits, since the groups involved are finitely generated, torsion free abelian groups. Let $\omega_{\mathbb{C}}$ be the generator of $\widetilde{K}(S^{4n})$. Then the set

$$\{\pi^*(\omega_{\mathbb{C}})\} \cup \{\eta_j^k : j = 1, \dots, m,\ k = 1, \dots, 2n-1\}$$

is an integral basis of $\widetilde{K}(m\#\mathbb{CP}^{2n})$. We claim that $\eta_j^{2n} = \pi^*(\omega_{\mathbb{C}})$ for all j. Indeed, the elements η_j^{2n} lie in the kernel of i^*, and hence there are $k_j \in \mathbb{Z}$ such that $\eta_j^{2n} = k_j \cdot \pi^*(\omega_{\mathbb{C}})$. Let $\widetilde{\mathrm{ch}} : \widetilde{K}(X) \to \widetilde{H}(X; \mathbb{Q})$ denote the Chern character for a finite CW complex X, then $\widetilde{\mathrm{ch}}$ is a monomorphism for $X = m\#\mathbb{CP}^d$ (since $\widetilde{H}^*(m\#\mathbb{CP}^d; \mathbb{Z})$ has no torsion [Atiyah and Hirzebruch 1961, Section 2.5, Corollary]) and an isomorphism for $X = S^d$ onto $\widetilde{H}^*(S^d; \mathbb{Z})$ embedded in $\widetilde{H}^*(S^d; \mathbb{Q})$. Using the notation of (2.3) we have

$$\widetilde{\mathrm{ch}}(\eta_j^{2n}) = (e^{x_j} - 1)^{2n} = x_j^{2n}$$

and using the naturality of $\widetilde{\mathrm{ch}}$

$$\widetilde{\mathrm{ch}}(\pi^*(\omega_{\mathbb{C}})) = \pi^*(\widetilde{\mathrm{ch}}(\omega_{\mathbb{C}})) = \pm x_j^{2n},$$

since π^* is an isomorphism on cohomology in dimension $2n$. We can choose $\omega_{\mathbb{C}}$ such that $\widetilde{\mathrm{ch}}(\pi^*(\omega_{\mathbb{C}})) = x_j^{2n}$. This shows $k_j = 1$ for all j and $\widetilde{K}(m\#\mathbb{CP}^{2n})$ is freely generated by

$$\{\eta_j^k : j = 1, \dots, m,\ k = 1, \dots, 2n-1\} \cup \{\eta_1^{2n} = \dots = \eta_m^{2n}\}.$$

Hence $K(m\#\mathbb{CP}^{2n}) = \mathbb{Z}[\eta_1, \dots, \eta_m]/R_{2n}(\eta_1, \dots, \eta_m)$. Since $p_j^*(H) \otimes p_j^*(\overline{H})$ is the trivial bundle we compute the identity

$$\overline{\eta}_j = \frac{-\eta_j}{1+\eta_j} = -\eta_j + \eta_j^2 - \dots + \eta_j^{2n}.$$

The ring $\mathbb{Z}[\eta_1, \dots, \eta_m]/R_{2n}(\eta_1, \dots, \eta_m)$ is isomorphic to

$$\left(\bigoplus_{j=1}^m \mathbb{Z}[\eta_j]/\langle \eta_j^{2n+1}\rangle\right) \Big/ \left\langle \eta_j^{2n} - \eta_i^{2n} : j \neq i \right\rangle$$

and from Theorem 2.2 the set Γ_j which contains the elements

$$\eta_j, \eta_j(\eta_j + \overline{\eta}_j), \dots, \eta_j(\eta_j + \overline{\eta}_j)^{n-1},$$
$$\eta_j - \overline{\eta}_j, (\eta_j - \overline{\eta}_j)(\eta_j + \overline{\eta}_j), \dots, (\eta_j - \overline{\eta}_j)(\eta_j + \overline{\eta}_j)^{n-1}$$

together with $\{1\}$ is an integral basis of $\mathbb{Z}[\eta_j]/\langle \eta_j^{2n+1}\rangle$. Thus the set

$$\Gamma_1 \cup \dots \cup \Gamma_m \subset \widetilde{K}(m\#\mathbb{CP}^{2n})$$

generates the group $\widetilde{K}(m\#\mathbb{CP}^{2n})$. Observe that

$$(\eta_j + \overline{\eta}_j)^k = 2\eta_j(\eta_j + \overline{\eta}_j)^{k-1} - (\eta_j - \overline{\eta}_j)(\eta_j + \overline{\eta}_j)^{k-1}. \tag{2.8}$$

Thus

$$\eta_j^{2n} = (\eta_j + \overline{\eta}_j)^n = 2\eta_j(\eta_j + \overline{\eta}_j)^{n-1} - (\eta_j - \overline{\eta}_j)(\eta_j + \overline{\eta}_j)^{n-1}. \tag{2.9}$$

We set $\omega := \eta_j^{2n}$ for any $j = 1, \dots, m$ and

$$e_j^k := \eta_j(\eta_j + \overline{\eta}_j)^k, \qquad j = 1, \dots, m, \quad k = 0, \dots, n-1,$$
$$f_j^k := (\eta_j - \overline{\eta}_j)(\eta_j + \overline{\eta}_j)^k, \quad j = 1, \dots, m, \quad k = 0, \dots, n-1,$$

and by virtue of relation (2.9) the set

$$B := \{\omega\} \cup \{e_j^k : j = 1, \dots, m,\ k = 0, \dots, n-1\} \cup \{f_j^k : j = 1, \dots, m,\ k = 0, \dots, n-2\}$$

is an integral basis of $\widetilde{K}(m\#\mathbb{CP}^{2n})$.

We proceed with the computation of $KO(m\#\mathbb{CP}^{2n})$. We have a long exact sequence for $\widetilde{KO}$-theory like in (2.7). From Theorem 2 in [Fujii 1967] we deduce $\widetilde{KO}^{-1}(\mathbb{CP}^{2n}) = 0$ and therefore $\widetilde{KO}^{-1}\big(\bigvee_{j=1}^{m} \mathbb{CP}^{2n}\big) = 0$. Moreover,

$$\widetilde{KO}^{1}(S^{4n}) = \widetilde{KO}^{-7}(S^{4n}) = \widetilde{KO}(S^{4n+7}) = 0$$

by Bott periodicity. Hence we obtain a short exact sequence

$$0 \to \widetilde{KO}(S^{4n}) \to \widetilde{KO}(m\#\mathbb{CP}^{2n}) \to \widetilde{KO}\bigg(\bigvee_{j=1}^{m} \mathbb{CP}^{2n-1}\bigg) \to 0. \tag{2.10}$$

Now we have to distinguish between the cases where n is even or odd. We first assume that $n = 2l$. In that case the ring $KO(\mathbb{CP}^{2n-1})$ is isomorphic to $\mathbb{Z}[\eta_R]/\langle \eta_R^n \rangle$; see Theorem 2.2(b). Hence all groups in (2.10) are torsion free. Therefore the kernel of $r : \widetilde{K}(m\#\mathbb{CP}^{2n}) \to \widetilde{KO}(m\#\mathbb{CP}^{2n})$ is the same as the kernel of

$$\varphi := c \circ r = 1 + t : \widetilde{K}(m\#\mathbb{CP}^{2n}) \to \widetilde{K}(m\#\mathbb{CP}^{2n})$$

since $r \circ c = 2$, and thus c is a monomorphism of the torsion free part of $\widetilde{KO}(m\#\mathbb{CP}^{2n})$.

Next we compute a basis of $\ker \varphi$. Using relation (2.8) we have $\varphi(\omega) = 2\omega$, $\varphi(e_j^k) = 2e_j^k - f_j^k$ and $\varphi(f_j^k) = 0$. Thus if

$$y = \lambda\omega + \sum_{j=1}^{m} \sum_{k=0}^{n-1} \lambda_j^k e_j^k,$$

then

$$\varphi(y) = 2\lambda\omega + \sum_{j=1}^{m} \sum_{k=0}^{n-1} \lambda_j^k (2e_j^k - f_j^k) = \bigg(2\lambda + \sum_{j=1}^{m} \lambda_j^{n-1}\bigg)\omega + \sum_{j=1}^{m} \sum_{k=0}^{n-2} \lambda_j^k (2e_j^k - f_j^k),$$

using the fact that $f_j^{n-1} = 2e_j^{n-1} - \omega$ by (2.9). As ω and $2e_j^k - f_j^k$, $j = 1, \dots, m$, $k = 0, \dots, n-2$, are linearly independent, we conclude that $\varphi(y) = 0$ if and only if $\lambda_j^k = 0$ for $j = 1, \dots, m$, $k = 1, \dots, n-2$ and

$$\sum_{j=1}^{m} \lambda_j^{n-1} + 2\lambda = 0.$$

This implies that the set

$$\{f_j^k : j = 1, \dots, m,\ k = 0, \dots, n-2\} \cup \{e_1^{n-1} - e_j^{n-1} : j = 2, \dots, m\} \cup \{2e_1^{n-1} - \omega\}$$

is an integral basis of $\ker \varphi$. Note that from (2.9) we have

$$2e_1^{n-1} - \omega = (\eta_1 - \bar{\eta}_1)(\eta_1 + \bar{\eta}_1)^{n-1}.$$

By an inductive argument we see that

$$(\eta_j - \bar{\eta}_j)(\eta_j + \bar{\eta}_j)^k = w_j^{k+1} + \text{linear combinations of } w_j^1, \dots, w_j^k \tag{2.11}$$

and
$$e_1^{n-1} - e_j^{n-1} = \eta_1^{2n-1} - \eta_j^{2n-1}.$$
Thus an integral basis of the kernel, in case n is even, is given by
$$\{w_j^k : j = 1, \dots, m,\ k = 1, \dots, n-1\} \cup \{w_1^n\} \cup \{\eta_1^{2n-1} - \eta_j^{2n-1} : j = 2, \dots, m\}.$$
Now let us assume that $n = 2l + 1$. Consider the commutative diagram
$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \widetilde{K}(S^{4n}) \arrow[r, "\pi^*"] \arrow[d, "r_S"] & \widetilde{K}(m\#\mathbb{CP}^{2n}) \arrow[r, "i^*"] \arrow[d, "r_\#"] & \widetilde{K}\big(\bigvee_{j=1}^m \mathbb{CP}^{2n-1}\big) \arrow[r] \arrow[d, "r_\vee"] & 0 \arrow[d] \\
0 \arrow[r] & \widetilde{KO}(S^{4n}) \arrow[r, "\pi^*"] & \widetilde{KO}(m\#\mathbb{CP}^{2n}) \arrow[r, "i^*"] & \widetilde{KO}\big(\bigvee_{j=1}^m \mathbb{CP}^{2n-1}\big) \arrow[r] & 0
\end{tikzcd}$$

The map $r_S : \widetilde{K}(S^{8l+4}) \to \widetilde{KO}(S^{8l+4})$ is an isomorphism and therefore the map $i^*|_{\ker r_\#} : \ker r_\# \to \ker r_\vee$ is an isomorphism. Hence the rank of $\ker r_\#$ is mn. We see that the set
$$\{f_j^k : j = 1, \dots, m,\ k = 0, \dots, n-2\} \cup \{2e_j^{n-1} : j = 1, \dots, m\} \cup \{\omega\}$$
is an integral basis of $(i^*)^{-1}(\ker r_\vee)$, which follows from $e_j^{n-1} = \eta_j^{2n-1} - (n-1)\omega$ and the structure of the kernel of $r_\vee$; see Theorem 2.2(d)(ii). The elements f_j^k for $j = 1, \dots, m$ and $k = 0, \dots, n-2$ lie in the kernel of $r_\#$. Let
$$y = \lambda\omega + \sum_{j=1}^m \lambda_j^{n-1} 2e_j^{n-1}$$
for $\lambda, \lambda_j^{n-1} \in \mathbb{Z}$ and suppose $r_\#(y) = 0$. From $\varphi(\omega) = 2\omega$ and $\varphi(e_j^{n-1}) = (\eta_j + \bar\eta_j)^n = \eta_j^{2n} = \omega$ it follows that
$$\lambda + \sum_{j=1}^m \lambda_j^{n-1} = 0.$$
Hence $\ker r_\#$ is freely generated by the elements f_j^k and $2e_j^{n-1} - \omega$. Observe from (2.9) that $2e_j^{n-1} - \omega = (\eta - \bar\eta)(\eta + \bar\eta)^{n-1}$. Thus in the case that n is odd we deduce like in (2.11) that the kernel of $r_\#$ is freely generated by w_j^k for $j = 1, \dots, m$ and $k = 1, \dots, n$. $\square$

Hence by (2.4), stable almost complex structures of $m\#\mathbb{CP}^{2n}$ for n even are given by elements of the form
$$y = (2n+1)\sum_{i=1}^m \bar\eta_j + \sum_{j=1}^m \sum_{k=1}^{n-1} a_j^k w_j^k + a_1^n w_1^n + \sum_{j=2}^m b_j(\eta_1^{2n-1} - \eta_j^{2n-1}), \tag{2.12}$$
and for n odd,
$$y = (2n+1)\sum_{i=1}^m \bar\eta_j + \sum_{j=1}^m \sum_{k=1}^{n} a_j^k w_j^k \tag{2.13}$$

for $a_j^k, b_j \in \mathbb{Z}$. For Theorem 1.2 we have to compute the $2n$-th Chern class $c_{2n}(E)$ of a vector bundle E representing an element of the form (2.12) and (2.13). Let $\eta_1^{2n-1} - \eta_j^{2n-1}$ denote also a vector bundle over $m\#\mathbb{CP}^{2n}$ which represents the element $\eta_1^{2n-1} - \eta_j^{2n-1}$ in $\widetilde{K}(m\#\mathbb{CP}^{2n})$. The total Chern class of $\eta_1^{2n-1} - \eta_j^{2n-1}$ can be computed through the Chern character: we have

$$\widetilde{\mathrm{ch}}(\eta_1^{2n-1} - \eta_j^{2n-1}) = \widetilde{\mathrm{ch}}(\eta_1)^{2n-1} - \widetilde{\mathrm{ch}}(\eta_j)^{2n-1} = x_1^{2n-1} - x_j^{2n-1}.$$

The elements of degree k in the Chern character are given by $\nu_k(c_1, \dots, c_k)/k!$, where ν_k are the Newton polynomials. The coefficient in front of c_k in $\nu_k(c_1, \dots, c_k)$ is k (see [Mimura and Toda 1991, p. 195]) and the other terms are products of Chern classes of lower degree; hence the only nonvanishing Chern class is given by

$$c_{2n-1}(\eta_1^{2n-1} - \eta_j^{2n-1}) = (2n-2)!\,(x_1^{2n-1} - x_j^{2n-1}).$$

Thus the total Chern class of a vector bundle E representing an element of the form (2.12) is given by

$$c(E) = (1 - (x_1 + \dots + x_m))^{2n+1} \cdot \Big(\frac{1+nx_1}{1-nx_1}\Big)^{a_1^n} \prod_{j=2}^{m} (1 + (2n-2)!(x_1^{2n-1} - x_j^{2n-1}))^{b_j} \prod_{j=1}^{m} \prod_{k=1}^{n-1} \Big(\frac{1+kx_j}{1-kx_j}\Big)^{a_j^k},$$

and for (2.13),

$$c(E) = (1 - (x_1 + \dots + x_m))^{2n+1} \prod_{j=1}^{m} \prod_{k=1}^{n} \Big(\frac{1+kx_j}{1-kx_j}\Big)^{a_j^k},$$

where the coefficient in front of $x_1^{2n} = \dots = x_m^{2n}$ is equal to $c_{2n}(E)$.

Remark 2.14. Note that for $m = 1$ (and complex projective spaces of arbitrary dimension) this total Chern class was already computed by Thomas [1974, p. 130].

3. Almost complex structures on $m\#\mathbb{CP}^{2n}$

We now describe an explicit stable almost complex structure on $m\#\mathbb{CP}^{2n}$, where $m = 2u+1$, for which the assumptions of Theorem 1.2 are satisfied, thereby producing an almost complex structure on $m\#\mathbb{CP}^{2n}$. We choose, in the notation of (2.12) and (2.13), $a_j^k = 2$ for $j = 1, \dots, u$ and $k = 1$, and all other coefficients 0. Then the top Chern class is as desired:

Proposition 3.1. *Let $m = 2u+1$ be an odd number. In the cohomology ring of $m\#\mathbb{CP}^{2n}$, the coefficient c_{2n} of $x_1^{2n} = \dots = x_m^{2n}$ of the class*

$$c = (1 - (x_1 + \dots + x_{2u+1}))^{2n+1} \prod_{r=1}^{u} \Big(\frac{1+x_r}{1-x_r}\Big)^2$$

is $c_{2n} = m(2n-1) + 2 = \chi(m\#\mathbb{CP}^{2n})$.

Proof. As $x_i \cdot x_j = 0$ for $i \neq j$, we have

$$\begin{aligned}(1-(x_1+\cdots+x_{2u+1}))^{2n+1} &= \sum_{j_0=0}^{2n+1}(-1)^{j_0}\binom{2n+1}{j_0}(x_1^{j_0}+\cdots+x_{2u+1}^{j_0})\\ &= \sum_{r=1}^{2u+1}\sum_{j_0=0}^{2n+1}(-1)^{j_0}\binom{2n+1}{j_0}x_r^{j_0}.\end{aligned}$$

Thus,

$$c = \prod_{r=1}^{u}(1-x_r)^{2n-1}(1+x_r)^2\prod_{s=u+1}^{2u+1}(1-x_s)^{2n+1}.$$

The factors $(1-x_s)^{2n+1}$ contribute $2n+1$ to c_{2n}, while the factors $(1-x_r)^{2n-1}(1+x_r)^2$ contribute $2n-3$. Thus,

$$\begin{aligned}c_{2n} &= u(2n-3)+(u+1)(2n+1)\\ &= (2u+1)(2n-1)+2\\ &= \chi((2u+1)\#\mathbb{CP}^{2n}).\end{aligned}$$

□

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