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G-theory of root stacks and equivariant K-theory

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Using the description of the category of quasicoherent sheaves on a root stack, we compute the G-theory of root stacks via localization methods. We apply our results to the study of equivariant K-theory of algebraic varieties under certain conditions.

A list of notations and conventions can be found on page 182.

1. Introduction

Let X be an algebraic variety equipped with an action of a finite group G. One would like to compute the equivariant K-theory $K_G(X)$. A first answer was given in the paper [Ellingsrud and Lønsted 1984] in the case when X is a smooth curve. Let us briefly describe it. We set Y to be the quotient X/G, $\phi : X \to Y$ the quotient map, and B the branch locus. Then B is a finite union of G orbits $B_1, \dots, B_n$. Choosing a point $P_i \in B_i$ for each i, denote the inertia group of P_i by H_i. Note that it is a cyclic group. Using some basic properties of equivariant sheaves and the Borel construction, it was proved that there is a decomposition of abelian groups

$$K_G(X) = K(Y) \oplus \bigoplus_{i=1}^{n} R'_k(H_i),$$

where $R'_k(H)$ is the subgroup of a representation ring without invariants, that is, $x \in R'_k(H)$ if $x \in R_k(H)$ and $\langle x, 1_H \rangle = 0$. From here we can guess a flavor of the result in the general case: there should be some kind of a decomposition of $K_G(X)$ onto $K(Y)$ and the terms coming from ramification.

To generalize this to higher dimensions, there are two routes one may take. One may enter the realm of algebraic stacks. For example, Vistoli and Vezzosi [Vistoli 1991; Vezzosi and Vistoli 2002] proved the decomposition formula for $K_G(X)$ of a scheme X using (implicitly) a top-down description of the stack $[X/G]$.

Another route would be to enter the realm of logarithmic geometry; see [Niziol 2008; Hagihara 2003]. These two papers study the K-theory of the Kummer étale site on a logarithmic scheme. Note that, using the correspondence between sheaves on an infinite root stack and sheaves on the Kummer étale site [Talpo and Vistoli

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2018, §6], one can deduce the structure results of [Hagihara 2003, §4] and [Nizioł 2008, Theorem 1.1] from our Theorem 3.32 and Corollary 3.34.

We first discuss the general philosophy of our approach encompassing both of these routes. In algebraic geometry one frequently needs to consider equivariant objects on a scheme X with respect to the action of G. These objects correspond to objects over the quotient stack $[X/G]$. However, it can happen that $[X/G] \cong [X'/G']$ for seemingly unrelated X and X'. In such situation, it is useful to have a canonical description of the quotient stack $[X/G]$, perhaps in terms of its coarse moduli space Y. This may not always be possible but sometimes it is. In this paper we describe a situation in which this occurs (see Theorem 4.10). When our hypotheses are satisfied, the quotient stack becomes a root stack over its coarse moduli space Y.

The root stack construction goes back to [Olsson 2007]. If a quotient stack is "a tool" to take quotients, similarly a root stack can be used to "extract roots" from line bundles on a scheme. It turns out that this construction is quite useful, for example, in Gromov–Witten theory of a Deligne–Mumford stack; see [Abramovich et al. 2008; Cadman 2007; Olsson 2007]. The moduli stack of stable maps from a curve to a stack does not have nice properties, and instead one needs to consider so-called twisted stable maps from a twisted curve. As was shown in [Abramovich et al. 2008], one can replace a twisted curve by a root stack.

Another application of root stacks is the parabolic orbifold correspondence. In a nutshell, this correspondence describes sheaves and vector bundles on a root stack in terms of sheaves and vector bundles on the base with extra data. Parabolic bundles on a Riemann surface were defined in [Mehta and Seshadri 1980], and were shown to be related to a unitary representation of a homotopy group. Borne [2007] proved the equivalence of parabolic bundles and locally free sheaves on a root stack. Finally, Borne and Vistoli [2012] generalized it to the equivalence of quasicoherent sheaves on a root stack and parabolic sheaves.

The results of [Borne and Vistoli 2012] are the foundation of this work. Using their description of coherent sheaves on a root stack, we compute the algebraic G-theory of a root stack. See Theorem 3.32 for the statement of our first main result. The tool necessary for its proof is localization sequences associated with a quotient category. This method can be thought of as an algebraic analog of Segal's localization theorem [1968, Proposition 4.1] for equivariant topological K-theory.

The second result of this work is Theorem 4.10. It says that under certain assumptions a quotient stack is a root stack over its coarse moduli space. The main tool used in the proof is a generalization of Abhyankar's lemma; see [SGA 1 1971, Exposé XIII, Appendice I].

Combining these results gives an immediate application to equivariant K-theory of schemes. This is how we obtain a generalization of the aforementioned decomposition of [Ellingsrud and Lønsted 1984]. We formulate it as Theorem 5.1. If a finite group G acts on a scheme X, then, under some assumptions, we have the decomposition of $K_G(X)$ into the direct sum of groups $K(X/G)$ and G-theory of ramification divisors and their intersections. Note that our assumptions are always satisfied for tame actions of groups on smooth projective curves.

Let us give an outline of the paper for the convenience of the reader. In a short preliminary Section 2 we recall some necessary categorical techniques. We start by studying the G-theory of a root stack in Section 3. First, the description of the category of quasicoherent sheaves on a root stack by [Borne and Vistoli 2012] in Section 3A is recalled. After that we exploit localization methods to decompose the G-theory of parabolic sheaves. Finally, in Section 3D we combine all intermediate results and formulate Theorem 3.32, giving the G-theory of a root stack over a noetherian scheme. We finish the section with the observation in Corollary 3.34 that under some assumptions, the algebraic G-theory of a root stack coincides with its Waldhausen K-theory in the sense of [Joshua 2005].

In Section 4 we address the issue of when a quotient stack is a root stack. First we show that under our assumptions (tameness of the action and ramification divisor is normal crossing), the inertia group is generated in codimension one (see Theorem 4.9). We use Abhyankar's theorem [Grothendieck and Murre 1971, Theorem 2.3.2] in the proof. Then under the same hypothesis, we show that a quotient stack is a root stack (see Theorem 4.10).

The paper ends with Section 5, where we study equivariant K-theory of a scheme by combining the results of the previous two sections. As an example we compute the equivariant K-theory of the affine line and the Burniat surface.

2. Localization via Serre subcategories

2A. *Serre subcategories.* Let $\boldsymbol{A}$ be an abelian category. Recall that a *Serre subcategory* S of $\boldsymbol{A}$ is a nonempty full subcategory that is closed under extensions, subobjects and quotients. When $\boldsymbol{A}$ is well-powered the quotient category $\boldsymbol{A}/S$ exists; see [Swan 1968, p. 44, Theorem 2.1].

We need the following result to identify quotient categories.

Theorem 2.1. *Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be an exact functor between abelian categories. Denote by S the full subcategory whose objects are x with $F(x) \cong 0$. Then S is a Serre subcategory and we have a factorization*

$$\begin{array}{ccc} \boldsymbol{A} & \longrightarrow & \boldsymbol{A}/S \\ {\scriptstyle F}\downarrow & \swarrow & \\ \boldsymbol{B} & & \end{array}$$

Proof. See [Swan 1968, p. 114] □

Definition 2.2. The category S is called the *kernel of the functor* F and is denoted by $\mathbf{ker}(F)$.

Theorem 2.3. *In the situation of the previous theorem suppose the following hold*:

(1) *for every object* $y \in \boldsymbol{B}$ *there is* $x \in \boldsymbol{A}$ *such that* $F(x)$ *is isomorphic to* y, *and*

(2) *for every morphism* $f : F(x) \to F(x')$ *there is* $x'' \in \boldsymbol{A}$ *with* $h : x'' \to x$ *and* $g : x'' \to x'$ *such that* $F(h)$ *is an isomorphism and the following diagram commutes*:

$$\begin{array}{ccc} F(x'') & & \\ {\scriptstyle F(h)}\downarrow & \searrow^{F(g)} & \\ F(x) & \xrightarrow{\;f\;} & F(x') \end{array}$$

Then there is an equivalence of categories $\boldsymbol{A}/S \cong \boldsymbol{B}$.

Proof. See [Swan 1968, p. 114, Theorem 5.11]. □

2B. ***Some functor categories.*** Consider n-tuples of integers $\vec{r} = (r_1, r_2, \dots, r_n)$ and $\vec{s} = (s_1, s_2, \dots, s_n)$. We denote by $[\vec{r}, \vec{s}]$ the poset of n-tuples $(x_1, \dots, x_n)$ with

$$x_i \in \mathbb{Z} \quad \text{and} \quad r_i \le x_i \le s_i.$$

We make use of the shorthand notation

$$rI = [0, r] \quad \text{and} \quad \vec{r}I^n = [0, \vec{r}].$$

These intervals are naturally posets with

$$(x_1, x_2, \dots, x_n) \le (y_1, y_2, \dots, y_n) \quad \text{if and only if} \quad x_i \le y_i \text{ for all } i.$$

This poset structure allows us to view them as categories in the usual way.

Fix an abelian category $\boldsymbol{A}$ and consider the functor category

$$\mathrm{Func}(\vec{r}I^n, \boldsymbol{A}).$$

This category is abelian with kernels and cokernels formed pointwise. We are interested in the K-theory of such categories. In this subsection we try to understand some of their quotient categories. Given an object $\mathcal{F}$ in this category and an object u of $\vec{r}I^n$, we denote by $\mathcal{F}_u \in \boldsymbol{A}$ the value of the functor $\mathcal{F}$ on this object, and if $u \le v$, the arrow from F_u to F_v is denoted by

$$F_{+(v-u)} : F_u \to F_v.$$

In particular, we take $e_i = (0, 0, \dots, 1, 0, \dots, 0)$ to be a standard basis vector, so that we have a morphism

$$F_{+e_i} : F_{(u_1, \dots, u_n)} \to F_{u_1, \dots, u_{i-1}, u_i+1, u_{i+1}, \dots, u_n}.$$

Lemma 2.4. *Giving an object F of* $\mathrm{Func}(\vec{r}I^n, \boldsymbol{A})$ *is the same as providing the following data*:

(D1) *objects* $F_{(u_1,u_2,\dots,u_n)} \in \boldsymbol{A}$,

(D2) *arrows*

$$F_{+e_i} : F_u \to F_{u+e_i}$$

such that all diagrams of the form

$$\begin{array}{ccc} F_u & \longrightarrow & F_{u+e_j} \\ \downarrow & & \downarrow \\ F_{u+e_i} & \longrightarrow & F_{u+e_i+e_j} \end{array}$$

commute.

Proof. The hypotheses ensure that if $u \le v$ in $\vec{r}I^n$ then there is a well-defined map $F_u \to F_v$ which produces our functor. ☐

Proposition 2.5. (i) *Let* $\mathrm{tr}_{n-1}(\vec{r}) = (r_1, r_2, \dots, r_{n-1})$. *There is an exact functor*

$$\pi : \mathrm{Func}(\vec{r}I^n, \boldsymbol{A}) \to \mathrm{Func}(\mathrm{tr}_{n-1}(\vec{r})I^{n-1}, \boldsymbol{A})$$

defined on objects by

$$\pi(G)_{(u_1,u_2,\dots,u_{n-1})} = (G)_{(u_1,\dots,u_{n-1},0)}.$$

(ii) *The functor π has a left adjoint*, *denoted π^*. We have $\pi \circ \pi^* \simeq 1$.*

(iii) *The functor π^* is fully faithful.*

Proof. (i) There is an inclusion functor $\mathrm{tr}_{n-1}(\vec{r})I^{n-1} \hookrightarrow \vec{r}I^n$ defined by

$$(x_1, x_2, \dots, x_{n-1}) \mapsto (x_1, x_2, \dots, x_{n-1}, 0).$$

The functor π is just the restriction along this inclusion. The exactness follows from the fact that in functor categories, limits and colimits are computed pointwise.

(ii) Given a functor $F \in \mathrm{Func}(\mathrm{tr}_{n-1}(\vec{r})I^{n-1}, \boldsymbol{A})$, we need to construct an object $\pi^*(F) \in \mathrm{Func}(\vec{r}I^n, \boldsymbol{A})$. We set

$$\pi^*(F)_{(u_1,u_2,\dots,u_n)} = F_{(u_1,u_2,\dots,u_{n-1})}.$$

To produce a functor, we need maps

$$\lambda^i_{(u_1,\dots,u_n)} : \pi^*(\mathcal{F})_{(u_1,\dots,u_i,\dots,u_n)} \to \pi^*(F)_{(u_1,\dots,u_i+1,\dots,u_n)}.$$

We define

$$\lambda^i_{(u_1,\dots,u_n)} = \begin{cases} F_{(u_1,\dots,u_i,\dots,u_{n-1})} \to F_{(u_1,\dots,u_i+1,\dots,u_{n-1})} & \text{if } i < n, \\ \text{identity} & \text{if } i = n. \end{cases}$$

One checks that the hypotheses of Lemma 2.4 are satisfied. Observe that $\pi \circ \pi^* = 1$. This produces a natural map

$$\operatorname{Hom}(\pi^*(F), G) \to \operatorname{Hom}(F, \pi(G)).$$

To see that this is a bijection, suppose that we are given a morphism $\beta : F \to \pi(G)$. There is a diagram, where the dashed arrow is defined to be the composition,

$$\begin{array}{ccc} \pi^*(F)_{(u_1,\dots,u_n)} & \dashrightarrow & G_{(u_1,\dots,u_n)} \\ \| & & \uparrow \\ F_{(u_1,\dots,u_{n-1})} & \xrightarrow[\beta]{} & G_{(u_1,\dots,u_{n-1},0)} \end{array}$$

This produces a natural morphism

$$\operatorname{Hom}(\pi^*(F), G) \leftarrow \operatorname{Hom}(F, \pi(G))$$

and we check that it is inverse to the previous map.

(iii) We have

$$\operatorname{Hom}(\pi^*(F), \pi^*(F')) = \operatorname{Hom}(F, \pi\pi^*(F')) = \operatorname{Hom}(F, F'). \qquad \square$$

Theorem 2.6. (1) *The functor*

$$\pi : \operatorname{Func}(\vec{r} I^n, \boldsymbol{A}) \to \operatorname{Func}(\operatorname{tr}_{n-1}(\vec{r}) I^{n-1}, \boldsymbol{A})$$

satisfies the hypothesis of Theorem 2.3.

(2) *Let* $\vec{s} = (r_1, r_2, \dots, r_{n-1}, r_n - 1)$*. If* $r_n > 0$ *then the kernel of this functor is equivalent to* $\operatorname{Func}(\vec{s} I^n, \boldsymbol{A})$.

(3) *If* $r_n = 0$ *then there is an equivalence of categories*

$$\operatorname{Func}(\vec{r} I^n, \boldsymbol{A}) \cong \operatorname{Func}(\operatorname{tr}_{n-1}(\vec{r}) I^{n-1}, \boldsymbol{A}).$$

Proof. (1) The functor π is exact so it remains to check the two conditions of the theorem. The first condition follows from the fact that $\pi \circ \pi^*$ is the identity. Now suppose that we have a morphism $\pi(F) \to \pi(F')$. By adjointness we obtain a diagram

$$\begin{array}{ccc} \pi^*\pi(F) & & \\ \downarrow & \searrow & \\ F & & F' \end{array}$$

Applying π to this picture shows that the second condition holds.

(2) The functor π was defined by the rule $\pi(G)_{(u_1,u_2,\dots,u_{n-1})} = (G)_{(u_1,\dots,u_{n-1},0)}$. So it is clear that if $\pi G \cong 0$ then $(G)_{(u_1,\dots,u_{n-1},0)} \cong 0$ and giving an object G of $\mathbf{ker}\, \pi$ is the same (up to isomorphism) as giving the objects $(G)_{(u_1,\dots,u_n)} \in \boldsymbol{A}$ for

all $u \in \vec{r}I^n$, $u_n \neq 0$. And according to Lemma 2.4 it is the same as providing an object of the category $\mathrm{Func}(\vec{s}I^n, \boldsymbol{A})$.

(3) If $r_n = 0$ then we have an equivalence of categories $\mathrm{tr}_{n-1}(\vec{r}) \cong \vec{r}$. □

3. Coherent sheaves on root stacks

3A. ***Preliminary results.*** Recall that if M is a commutative monoid then $\widehat{M} = \mathrm{Hom}(M, \mathbb{G}_m)$ is its dual.

In this subsection we recall the main constructions and theorems from [Borne and Vistoli 2012], to which we refer the reader for further details. Let's start by defining a root stack.

Let X be a scheme. Denote by $\mathfrak{Div}\, X$ the groupoid of line bundles over X with sections. It has the structure of a symmetric monoidal category with tensor product given by

$$(L, s) \otimes (L', s') = (L \otimes L', s \otimes s').$$

Choosing n objects $(L_1, s_1), \dots, (L_n, s_n)$ of $\mathfrak{Div}\, X$ allows us to define a symmetric monoidal functor (see [Borne and Vistoli 2012, Definition 2.1])

$$L : \mathbb{N}^n \to \mathfrak{Div}\, X, \quad (k_1, \dots, k_n) \mapsto (L_1, s_1)^{\otimes k_1} \otimes \cdots \otimes (L_n, s_n)^{\otimes k_n}.$$

Such functors arise from morphisms $X \to [\mathrm{Spec}\, \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$. Let us recall how.

Proposition 3.1. (i) *Let $\boldsymbol{A}$ be the groupoid whose objects are quasicoherent $\mathcal{O}_X$-algebras $\mathcal{A}$ with a $\mathbb{Z}^n = \widehat{\widehat{\mathbb{N}^n}}$-grading $\mathcal{A} = \bigoplus_{u \in \mathbb{Z}^n} \mathcal{A}_u$ such that each summand $\mathcal{A}_u$ is an invertible sheaf. The morphisms are graded algebra isomorphisms. Then there is an equivalence of categories between $\boldsymbol{A}^{\mathrm{op}}$ and the groupoid of $\widehat{\mathbb{N}^n}$-torsors $P \to X$.*

(ii) *Let $\boldsymbol{B}$ be the groupoid whose objects are pairs $(\mathcal{A}, \alpha)$, where $\mathcal{A}$ is a sheaf of algebras satisfying the conditions in (i) and*

$$\alpha : \mathcal{O}_X[\mathbb{N}^n] \to \mathcal{A}$$

is a morphism respecting the grading. The morphisms in the category $\boldsymbol{B}$ are graded algebra morphisms commuting with the structure maps. Then there is an equivalence of categories between $\boldsymbol{B}^{\mathrm{op}}$ and the groupoid of morphisms $X \to [\mathrm{Spec}\, \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$.

Proof. This proposition is a summary of the discussion in [Borne and Vistoli 2012, p. 1343–1344], in particular the proof of Proposition 3.25. The detailed proof can be found there. Here we just illustrate the main idea behind the proof.

(i) The torsor $\pi : P \to X$ is determined by the sheaf of algebras $\pi_*(\mathcal{O}_P)$, which has a $\widehat{\mathbb{N}^n}$-action, and hence a weight grading. As the torsor is locally trivial, the condition about the summands being invertible follows by considering the algebra associated with the trivial torsor.

(ii) This follows from the standard description of the groupoid of X-points of a quotient stack. Finally, in [Borne and Vistoli 2012], the **fppf** topology is needed but in the present work it is not. The setting in [loc. cit.] is more general and the monoids in question may have torsion, so that the torsor P is a torsor over μ_n. Such a torsor may not be trivial in the Zariski topology, unlike a $\mathbb{G}_m$-torsor. Hence a finer topology is needed. See the proof of [Borne and Vistoli 2012, Lemma 3.26]. □

Corollary 3.2. *There is an equivalence of categories between the groupoid of symmetric monoidal functors*

$$\mathbb{N}^n \to \mathfrak{Div}\, X$$

and the groupoid of X-points of $[\operatorname{Spec} \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$.

Proof. For details see [Borne and Vistoli 2012, Proposition 3.25]. In essence, the symmetric monoidal functor determined by $(L_1, s_1), \dots, (L_n, s_n)$ produces the graded sheaf of algebras

$$\mathcal{A} = \bigoplus_{\vec{u} \in \mathbb{Z}^n} L_1^{u_1} \otimes \cdots \otimes L_n^{u_n}.$$

The sections produce an algebra map

$$\mathcal{O}_X[\mathbb{N}^n] \to \mathcal{A}.$$ □

Definition 3.3. Let $\vec{r} = (r_1, r_2, \dots, r_n)$ be a collection of positive natural numbers. We denote by $r_i\mathbb{N}$ the monoid $\{vr_i \mid v \in \mathbb{N}\}$. We denote by $\vec{r}\mathbb{N}^n$ the monoid

$$\vec{r}\mathbb{N}^n = r_1\mathbb{N} \times r_2\mathbb{N} \times \cdots \times r_n\mathbb{N}.$$

We view our symmetric monoidal functor above as a functor

$$L : \vec{r}\mathbb{N}^n \to \mathfrak{Div}\, X, \quad (r_1\alpha_1, r_2\alpha_2, \dots, r_n\alpha_n) \mapsto (L_1, s_1)^{\otimes\alpha_1} \otimes \cdots \otimes (L_n, s_n)^{\otimes\alpha_n}.$$

Consider the natural inclusion of monoids $j_{\vec{r}} : \vec{r}\mathbb{N}^n \hookrightarrow \mathbb{N}^n$. The *category of $\vec{r}$-th roots of L*, denoted by $(L)_{\vec{r}}$, is defined as follows.

Its objects are pairs (M, α), where $M : \mathbb{N}^n \to \mathfrak{Div}\, X$ is a symmetric monoidal functor, and $\alpha : L \to M \circ j$ is an isomorphism of symmetric monoidal functors.

An arrow from (M, α) to (M', α') is an isomorphism $h : M \to M'$ of symmetric monoidal functors $\mathbb{N}^n \to \mathfrak{Div}\, X$, such that the diagram

$$\begin{array}{ccc} & L & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\alpha'} \\ M \circ j & \xrightarrow{\ h \circ j\ } & M' \circ j \end{array}$$

commutes.

This category is in fact a groupoid, as a morphism ϕ in $\mathfrak{Div}\, X$, whose tensor power $\phi^{\otimes k}$ is an isomorphism, must be an isomorphism to begin with.

Given a morphism of schemes $t : T \to X$ there is pullback functor

$$t^* : \mathfrak{Div}\, X \to \mathfrak{Div}\, T.$$

Hence we can form the category of roots $(t^* \circ L)_{\vec{r}}$. This construction pastes together to produce a pseudofunctor $\mathfrak{Div}_X$, where

$$\mathfrak{Div}_X \to \mathrm{Sch}/X$$

is the symmetric monoidal stack described in [Borne and Vistoli 2012, p. 1335].

Definition 3.4. In the above situation, the fibered category associated with this pseudofunctor is called *the stack of roots* associated with L and $\vec{r}$. It is denoted by $X_{L,\vec{r}}$.

We often denote the stack of roots by

$$X_{L,\vec{r}} = X_{(L_1,s_1,r_1),\dots,(L_n,s_n,r_n)}.$$

There are also two equivalent definitions of the stack $X_{L,\vec{r}}$, and the equivalence is proved in [Borne and Vistoli 2012, Proposition 4.13 and Remark 4.14]. Let's recall the description of this stack as a fibered product.

Proposition 3.5. *The stack $X_{L,\vec{r}}$ is isomorphic to the fibered product*

$$X \times_{\operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]} [\operatorname{Spec} \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}].$$

According to (a slightly modified version of) Corollary 3.2, a symmetric monoidal functor $L : \vec{r}\mathbb{N}^n \to \mathfrak{Div}\, X$ corresponds to a morphism

$$X \to [\operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]/\widehat{\vec{r}\mathbb{N}^n}],$$

which in turn corresponds to an $\widehat{\vec{r}\mathbb{N}^n}$-torsor $\pi : P \to X$ and an $\widehat{\vec{r}\mathbb{N}^n}$-equivariant morphism $P \to \operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]$. This gives the next proposition.

Proposition 3.6. *The stack $X_{L,\vec{r}}$ is isomorphic to the quotient stack*

$$[P \times_{\operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]} \operatorname{Spec} \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}],$$

where the action on the first factor is defined through the dual of the inclusion $j_{\vec{r}} : \vec{r}\mathbb{N}^n \hookrightarrow \mathbb{N}^n$.

Proof. See [Borne and Vistoli 2012, p. 1350]. □

We recall the definition of parabolic sheaf; see [Borne and Vistoli 2012, Definition 5.6].

Definition 3.7. Consider a scheme X, an inclusion $\vec{r}\mathbb{Z}^n \subseteq \mathbb{Z}^n$ and a symmetric monoidal functor $L : \vec{r}\mathbb{Z}^n \to \mathfrak{Div}\, X$, defined by

$$L_u = L(u) = L_1^{\alpha_1} \otimes \cdots \otimes L_n^{\alpha_n},$$

where $u = (r_1\alpha_1, \ldots, r_n\alpha_n)$ and each $\alpha_i \in \mathbb{Z}$. A *parabolic sheaf* (E, ρ) on (X, L) with denominators $\vec{r}$ consists of the following data:

(a) A functor $E : \mathbb{Z}^n \to \mathfrak{QCoh}\, X$, denoted by $v \mapsto E_v$ on objects and $b \mapsto E_b$ on arrows.

(b) For any $u \in \vec{r}\mathbb{Z}^n$ and $v \in \mathbb{Z}^n$, an isomorphism

$$\rho^E_{u,v} : E_{u+v} \simeq L_u \otimes_{\mathcal{O}_X} E_v$$

of $\mathcal{O}_X$-modules. This map is called the *pseudoperiod isomorphism.*

These data are required to satisfy the following conditions. Take $u, u' \in \vec{r}\mathbb{Z}^n$, $a = (r_1\alpha_1, \ldots, r_n\alpha_n) \in \vec{r}\mathbb{N}^n$, $b \in \mathbb{N}^n$, $v \in \mathbb{Z}^n$. Then the following diagrams commute:

(i)
$$\begin{array}{ccc} E_v & \xrightarrow{\quad E_a \quad} & E_{a+v} \\ \downarrow{\simeq} & & \downarrow{\rho^E_{a,v}} \\ \mathcal{O}_X \otimes E_v & \xrightarrow{\sigma^L_a \otimes \mathrm{id}_{E_v}} & L_a \otimes E_v \end{array}$$

where $\sigma_a = \sigma^{\alpha_1} \otimes \cdots \otimes \sigma^{\alpha_n} \in \mathrm{H}^0(X, L_a)$.

(ii)
$$\begin{array}{ccc} E_{u+v} & \xrightarrow{\quad \rho^E_{u,v} \quad} & L_u \otimes E_v \\ \downarrow{E_b} & & \downarrow{\mathrm{id} \otimes E_b} \\ E_{u+b+v} & \xrightarrow{\quad \rho^E_{u,b+v} \quad} & L_u \otimes E_{b+v} \end{array}$$

(iii)
$$\begin{array}{ccc} E_{u+u'+v} & \xrightarrow{\quad \rho^E_{u+u',v} \quad} & L_{u+u'} \otimes E_v \\ \downarrow{\rho^E_{u,u'+v}} & & \downarrow{\mu \otimes \mathrm{id}} \\ L(u) \otimes E_{u'+v} & \xrightarrow{\mathrm{id} \otimes \rho^E_{u',v}} & L_u \otimes L_{u'} \otimes E_v \end{array}$$

(iv) The map

$$E_v = E_{0+v} \xrightarrow{\rho^E_{0,v}} \mathcal{O}_X \otimes E_v$$

is the natural isomorphism.

Definition 3.8. A parabolic sheaf (E, ρ) is said to be *coherent* if for each $v \in \mathbb{Z}^n$ the sheaf E_v is a coherent sheaf on X.

Theorem 3.9 (Borne, Vistoli). *Let X be a scheme and L a monoidal functor defined as in the beginning of this section. Then there is a canonical tensor equivalence of abelian categories between the category $\mathfrak{QCoh}\, X_{L,\vec{r}}$ and the category of parabolic sheaves on X, associated with L.*

Proof. See [Borne and Vistoli 2012, Proposition 5.10, Theorem 6.1] for details. The proof relies on the description of the stack as a quotient as in Proposition 3.6. From this description, sheaves on the stack are equivariant sheaves on

$$P \times_{\operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]} \times \operatorname{Spec} \mathbb{Z}[\mathbb{N}^n].$$

As remarked in the proof of Proposition 3.1, the torsor P is obtained from a sheaf of algebras on X. The sheaf of algebras $\mathcal{A}$ is constructed from the functor L by taking a direct sum construction; it has a natural grading. It follows that the scheme

$$P \times_{\operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]} \operatorname{Spec} \mathbb{Z}[\mathbb{N}^n] = \operatorname{Spec}(\mathcal{A} \otimes_{\mathbb{Z}[\vec{r}\mathbb{N}^n]} \mathbb{Z}[\mathbb{N}^n]).$$

The algebra on the right has a natural $\mathbb{Z}[\mathbb{N}^n]$-grading; see the corollary below for a local description. It follows that the equivariant sheaves on the scheme in question are just graded modules over this algebra. The proof follows by reinterpreting the graded modules in terms of the symmetric monoidal functor L. $\square$

Actually we can add the finiteness condition to the previous theorem and get the following:

Corollary 3.10. *Let X be a locally noetherian scheme. There is a canonical tensor equivalence of abelian categories between the category $\mathfrak{Coh}\, X_{L,\vec{r}}$ and the category of coherent parabolic sheaves on X, associated with L.*

Proof. We make use of the identifications in the above proof. The question is local on X, so we may assume that X is in fact an affine scheme $\operatorname{Spec}(R)$. By further restrictions we can assume that all the line bundles L_i are in fact trivial, and we identify them with R. In this situation the symmetric monoidal functor corresponds to a graded homomorphism

$$\mathbb{Z}[X_1, X_2, \dots, X_n] \to R[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}]$$

sending X_i to $x_i t_i$ with $x_i \in R$. Further, the morphism

$$\operatorname{Spec}(\mathbb{Z}[\mathbb{N}^n]) \to \operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]$$

comes from an integral extension of algebras

$$\mathbb{Z}[X_1, X_2, \dots, X_n][Y_1, \dots, Y_n]/(Y_1^{r_1} - X_1, \dots, Y_n^{r_n} - X_n).$$

Then taking tensor products yields a $\mathbb{Z}^n$-graded algebra

$$A = R[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}][s_1, \dots, s_n]/(s_1^{r_1} - x_1 t_1, \dots, s_n^{r_n} - x_n t_n),$$

where s_i has degree $(0, \dots, 0, 1, 0 \dots, 0) = e_i$. Now consider a finitely generated graded A-module M. We can assume that the generators of M are in fact homogeneous and hence there is an epimorphism

$$\bigoplus_{i=1}^{p} A(n_i) \to M.$$

The graded pieces of the module on the left are free of rank p and hence the graded pieces of M are finitely generated. It follows that a finitely generated A-module gives rise to a parabolic sheaf with values in the category of finitely generated R-modules — in other words, coherent sheaves on X.

Conversely, suppose that we have a graded A-module M with each graded piece a finitely generated R-module. We can find finitely many elements of M, let's say $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ of degrees

$$\deg(\alpha_i) = (\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{in}) \in \mathbb{Z}^n$$

with $0 \le \lambda_{ij} \le r_j$, such that the associated morphism

$$\phi : \bigoplus_{i=1}^{p} A(\deg(\alpha_i)) \to M$$

is an epimorphism in degrees

$$(\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{Z}^n$$

whenever $0 \le \mu_i \le r_i$. It follows that ϕ is an epimorphism and multiplication by t_i induces an isomorphism $M_v \xrightarrow{\sim} M_{v+e_i}$. □

3B. ***An extension lemma.*** The goal of this subsection is to slightly simplify the formulation of parabolic sheaves in the present context using the pseudoperiodicity condition. This will be needed to study K-theory in the next section. We let

$$e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^n,$$

where the 1 is in the i-th spot.

Definition 3.11. Let X be a scheme and L a symmetric monoidal functor

$$L : \vec{r}\mathbb{Z}^n \to \mathfrak{Div}_X,$$

determined by n divisors (L_i, s_i). An *extendable pair* (F, ρ) on (X, L) consists of the following data:

(a) A functor $F_\bullet : \vec{r}I^n \to \mathfrak{QCoh}(X)$.

(b) For any $\alpha \in \vec{r}I^n$ such that $\alpha_i = r_i$, an isomorphism of $\mathcal{O}_X$-modules

$$\rho_{\alpha,\alpha-r_ie_i} : F_\alpha \xrightarrow{\sim} L_i \otimes F_{\alpha-r_ie_i}.$$

We frequently drop the subscripts from the notation involving ρ, when they are clear from the context.

This data is required to satisfy the following three conditions:

(EX1) For all $i \in \{1, \dots, n\}$ and $\alpha \in \vec{r}I^n$, the diagram

$$\begin{array}{ccc} F_\alpha & \xrightarrow{F_{+(r_i-\alpha_i)e_i}} & F_{\alpha+(r_i-\alpha_i)e_i} \\ \downarrow{\scriptstyle \sigma_i} & & \downarrow{\scriptstyle \rho} \\ L_i \otimes F_\alpha & \xleftarrow[L_i \otimes F_{+\alpha_ie_i}]{} & L_i \otimes F_{\alpha-\alpha_ie_i} \end{array}$$

commutes, where σ_i is multiplication by the section s_i.

(EX2) For all $i \neq j$ and α with $\alpha_i = r_i$, the diagram

$$\begin{array}{ccc} F_\alpha & \xrightarrow{\rho} & L_i \otimes F_{\alpha-r_ie_i} \\ {\scriptstyle F_{\vec{e}_j}}\downarrow & & \downarrow{\scriptstyle F_{\vec{e}_j}} \\ F_{\alpha+e_j} & \xrightarrow{\rho} & L_i \otimes F_{\alpha+e_j-r_ie_i} \end{array}$$

commutes.

(EX3) For all i and j and $\alpha \in \vec{r}I^n$ with $\alpha_i = r_i$ and $\alpha_j = r_j$, the diagram

$$\begin{array}{ccc} F_\alpha & \xrightarrow{\rho} & L_i \otimes F_{\alpha-r_ie_i} \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ L_j \otimes F_{\alpha-r_je_j} & \xrightarrow{\rho} & L_i \otimes L_j \otimes F_{\alpha-r_ie_i-r_je_j} \end{array}$$

commutes.

Definition 3.12. An extendable pair (F, ρ) is called *coherent* if for each $v \in \vec{r}I^n$, the sheaf F_v is a coherent sheaf on X.

Proposition 3.13. *Let X be a scheme and L a symmetric monoidal functor as in Definition 3.7. Let (E, ρ) be a parabolic sheaf on (X, L) with denominators $\vec{r}$. Then the restricted functor $E|_{\vec{r}I^n}$ produces an extendable pair on (X, L).*

Proof. Note that the restricted functor has all the required data for an extendable pair by restricting the collection $\rho_{\alpha,\beta}$. We need to check that the axioms of an extendable pair are satisfied.

(EX1) We have that the composition

$$E_{\alpha+(r_i-\alpha_i)e_i} \xrightarrow{\rho} E_{\alpha-\alpha_i e_i} \otimes L_i \to E_\alpha \otimes L_i \xrightarrow{\rho^{-1}} E_{\alpha+r_i e_i}$$

is just the morphism $E_{+\alpha_i e_i}$ using axiom (ii) of parabolic sheaves. Precomposing with the map

$$E_{+(r_i-\alpha_i)e_i} : E_\alpha \to E_{\alpha+(r_i-\alpha_i)e_i}$$

gives the morphism $E_{+r_i e_i}$. The result now follows from axiom (i).

(EX2) This follows directly from axiom (ii).

(EX3) This follows directly from axiom (iii). □

Proposition 3.14. *Let X be a scheme and L a symmetric monoidal functor as in Definition 3.7. Given an extendable pair (F, ρ) on (X, L) we can extend it to a parabolic sheaf $(\hat{F}, \rho)$ on X, L and the extension is unique up to a canonical isomorphism. A coherent extendable pair extends to a coherent parabolic sheaf.*

Proof. For $v \in \mathbb{Z}^n$ we need to define its extension $\hat{F}_v$. We can write $v_i = r_i u_i + q_i$ with $0 \le q_i < r_i$ and $u_i \in \mathbb{Z}$. As before we let $L_u = \bigotimes_{i=1}^n L^{\otimes u_i}$ and $q = (q_1, \dots, q_n)$. Set $\hat{F}_v = L_u \otimes F_q$.

We need to construct maps

$$\hat{F}_{+e_i} : \hat{F}_v \to \hat{F}_{v+e_i}.$$

If $q_i < r_i - 1$ then the map is obtained by tensoring the map $F_{q_i} \to F_{q_i+e_i}$ with L_u. If $q_i = r_i - 1$ then the map is defined by

$$\begin{array}{ccc} \hat{F}_v = L_u \otimes F_q & \xrightarrow{\hat{F}_{e_i}} & \hat{F}_{v+e_i} = L_u \otimes L_i \otimes F_{q'} \\ & {\scriptstyle 1\otimes F_{\bar{e}_i}} \searrow \quad \nearrow {\scriptstyle 1\otimes\rho} & \\ & L_u \otimes F_{q+e_i} & \end{array}$$

where $q'_j = q_j$ for all $j \neq i$ and $q'_i = 0$.

In order to show that the construction above indeed produces a functor, we need to show that all diagrams of Lemma 2.4 commute. If both $q_i < r_i - 1$ and $q_j < r_j - 1$, then this is straightforward. If $q_i = r_i - 1$ and $q_j < r_j - 1$, then this follows from (EX2). This leaves the case $q_i = r_i - 1$ and $q_j = r_j - 1$. We have a diagram

$$\begin{array}{ccccc}
L_u \otimes F_q & \longrightarrow & L_u \otimes F_{q+e_i} & \longrightarrow & L_u \otimes L_i \otimes F_{q-q_i e_i} \\
\downarrow & & \downarrow & & \downarrow \\
L_u \otimes F_{q+e_j} & \longrightarrow & L_u \otimes F_{q+e_i+e_j} & \longrightarrow & L_u \otimes L_i \otimes F_{q-q_i e_i+e_j} \\
\downarrow & & \downarrow & & \downarrow \\
L_u \otimes L_j \otimes F_{q-q_j e_j} & \longrightarrow & L_u \otimes L_j \otimes F_{q-q_j e_j+e_i} & \longrightarrow & L_u \otimes L_i \otimes L_j \otimes F_{q-q_i e_i-q_j e_j}
\end{array}$$

The top left square commutes using the fact that F is a functor. The top right and bottom left squares commute using axiom (EX2). The bottom right square commutes using axiom (EX3). So indeed $\hat{F}_\bullet$ is a functor.

Note that we have canonical isomorphisms $L_u \otimes L_v \cong L_{u+v}$ for $u, v \in \vec{r}\mathbb{Z}$. These isomorphisms induce our pseudoperiod isomorphisms.

Finally, we need to check the conditions (i)–(iv) of a parabolic sheaf.

(i): For $\vec{r}\alpha, \vec{r}\alpha' \in \vec{r}\mathbb{N}^n$ the diagram

$$\begin{array}{ccccc}
\hat{F}_v & \xrightarrow{\hat{F}_{+\vec{r}\alpha}} & \hat{F}_{v+\vec{r}\alpha} & \xrightarrow{\hat{F}_{+\vec{r}\alpha'}} & \hat{F}_{v+\vec{r}\alpha+\vec{r}\alpha'} \\
 & & \downarrow & & \downarrow \\
 & & L_\alpha \otimes \hat{F}_v & \longrightarrow & L_\alpha \otimes L_{\alpha'} \otimes \hat{F}_v
\end{array}$$

commutes. This follows by the definition of the functor $\hat{F}_\bullet$ and the symmetric monoidal structure of L.

This allows us to make the following reduction: in order to check axiom (i), it suffices to check that the diagram

$$\begin{array}{ccc}
\hat{F}_v & \longrightarrow & \hat{F}_{v+r_i e_i} \\
 & \searrow^{\sigma_i} & \downarrow \rho \\
 & & L_i \otimes \hat{F}_v
\end{array}$$

commutes. And this follows directly from (EX1).

(ii): Once again we reduce to showing that

$$\begin{array}{ccc}
\hat{F}_{v+r_i e_i} & \longrightarrow & L_i \otimes \hat{F}_v \\
\downarrow \hat{F}_{+b} & & \downarrow L_i \otimes \hat{F}_{+b} \\
\hat{F}_{v+b+r_i e_i} & \longrightarrow & L_i \otimes \hat{F}_{v+b}
\end{array}$$

commutes. If we write $v = \vec{r}u + q$ then this diagram becomes

$$\begin{array}{ccc} L_{u+e_i} \otimes F_q & \longrightarrow & L_i \otimes (L_u \otimes F_q) \\ \Big\downarrow L_{u+e_i} \otimes \hat{F}_{+b} & & \Big\downarrow L_i \otimes L_u \otimes \hat{F}_{+b} \\ L_{u+e_i} \otimes \hat{F}_{q+b} & \longrightarrow & L_i \otimes (L_u \otimes \hat{F}_{q+b}) \end{array}$$

We can use the symmetric monoidal structure of L to show that this diagram indeed commutes.

(iii): We reduce to showing the commutativity of the diagram

$$\begin{array}{ccc} \hat{F}_{v+r_ie_i+r_je_j} & \longrightarrow & L_i \otimes \hat{F}_{v+r_je_j} \\ \Big\downarrow & & \Big\downarrow \\ L_j \otimes \hat{F}_{v+r_ie_i} & \longrightarrow & L_i \otimes L_j \otimes \hat{F}_v \end{array}$$

which follows from the monoidal structure of L.

Condition (iv) is by definition.

Finally, let $E_\bullet$ be another extension of $F_\bullet$. Again we can again write $v_i = r_i u_i + q_i$ with $0 \le q_i < r_i$ and $u_i \in \mathbb{Z}$. By pseudoperiodicity, $E_v \simeq L(u) \otimes E_q$, and $F_q = E_q$ because $E_\bullet$ is an extension. So, $E_v \cong \hat{F}_v$ for any $v \in \mathbb{Z}^n$.

It is clear from the construction that the finite generation condition is preserved under extension. □

Corollary 3.15. *Let X be a scheme and L a symmetric monoidal functor as in Definition 3.7. The category of parabolic sheaves* (*resp. coherent parabolic sheaves*) *on* (X, L) *is equivalent to the category of extendable pairs* (*resp. coherent extendable pairs*) *on* (X, L).

Proof. There is a pair of functors between these categories. The truncation functor sends a parabolic sheaf (E, ρ) to an extendable pair by forgetting all E_v when $v \notin \vec{r}I^n$. And the extension functor from extendable pairs to parabolic sheaves was defined in the previous proposition on objects by $F_\bullet \mapsto \hat{F}_\bullet$. It is easy to see that these functors are mutually inverse and preserve the finite generation condition. □

Remark 3.16. Let X be a scheme and L a symmetric monoidal functor as in Definition 3.7. We denote the category of coherent extendable pairs on (X, L) by $\mathcal{EP}(X, L, \vec{r})$. When X is locally noetherian this category is abelian.

3C. ***The localization sequence.*** In this subsection we localize the category of finitely generated extendable pairs so that it will be glued from simpler parts.

For this section X is a locally noetherian scheme and L a symmetric monoidal functor as in Definition 3.7.

First let us consider the functor $\pi_*^{L,\vec{r}} : \mathcal{EP}(X, L, \vec{r}) \to \mathfrak{Coh}\, X$, given by $F_\bullet \mapsto F_0$ on objects. It is an exact functor because exact sequences in diagram categories are defined pointwise.

Lemma 3.17. *The functor $\pi_*^{L,\vec{r}}$ has a left adjoint, denoted $\pi^*_{L,\vec{r}}$, and there is a natural isomorphism $\pi_*^{L,\vec{r}} \circ \pi^*_{L,\vec{r}} \simeq 1$.*

Proof. In what follows, we omit the superscripts and subscripts L and $\vec{r}$ in the notation for the appropriate functors. For $0 \le i \le n$, consider functions $\epsilon_i : \vec{r}I \to \{0, 1\}$, defined by $\epsilon_i(u) = 1$ if $u_i = r_i$ and zero otherwise. We define the functor π^* on a sheaf $F \in \mathfrak{Coh}\, X$ by the rule

$$(\pi^*(F))_u = \left(\bigotimes_{i=1}^{n} L_i^{\epsilon_i(u)} \right) \otimes F.$$

This forms a functor via the maps

$$(\pi^*(F))_u \to (\pi^*(F))_{u+e_i} = \begin{cases} \text{identity} & \text{if } u_i \in [0, r_i - 2], \\ \sigma_i & \text{if } u_i = r_i - 1, \end{cases}$$

where σ_i is the multiplication by the section s_i.

Define ρ to be the identity map. It is easy to see that all axioms of extendable pairs are satisfied.

Now let's take a coherent sheaf F and an extendable pair $E_\bullet$ and consider a map

$$\mathrm{Hom}_{\mathfrak{Coh}\, X}(F, \pi_* E) \to \mathrm{Hom}_{\mathcal{EP}}(\pi^* F, E)$$

given by sending $\phi \in \mathrm{Hom}_{\mathfrak{Coh}\, X}(F, \pi_* E)$ to precomposition of the structure maps of the extendable pair E with ϕ. It's obviously an injection. Surjectivity follows from commutativity of the squares in $\mathrm{Hom}_{\mathcal{EP}}(\pi^* F, E)$ and because all structure maps in $\pi^* F$ are identity. □

Proposition 3.18. *Suppose that X is a locally noetherian scheme. The functor $\pi_*^{L,\vec{r}} : \mathcal{EP}(X, L, \vec{r}) \to \mathfrak{Coh}\, X$ satisfies the hypothesis of Theorem 2.3.*

Proof. The only thing which is not completely obvious is the second condition. Consider two extendable pairs $E_\bullet$ and $F_\bullet$. Suppose that we have a morphism $\pi_*(E_\bullet) \to \pi_*(F_\bullet)$. By adjointness we obtain a diagram

$$\begin{array}{ccc} \pi^*\pi_*(E_\bullet) & & \\ \downarrow & \searrow & \\ E_\bullet & & F_\bullet \end{array}$$

Applying π to this picture shows that the second condition holds. □

Using Theorem 2.3 we obtain the following:

Corollary 3.19. *Let X be a locally noetherian scheme. There is an equivalence of abelian categories*

$$\mathcal{EP}(X, L, \vec{r}) \big/ \mathbf{ker}(\pi_*^{L,\vec{r}}) \to \mathfrak{Coh}\, X.$$

In the rest of this subsection we would like to give a description of the category $\mathbf{ker}(\pi_*^{L,\vec{r}})$. Let us study the objects first. Let $F_\bullet$ be an extendable pair. Then $\pi_*(F_\bullet) = F_0$, and if $F_\bullet \in \mathbf{ker}(\pi_*^{L,\vec{r}})$ then $F_0 \cong 0$. The pseudoperiod isomorphism implies in turn that $F_u \cong 0$ if all $u_i \in \{0, r_i\}$.

Let us consider the sheaves F_u such that $u_j \in \{0, r_j\}$ for $j \neq i$ (we can imagine them as sheaves on the edges of the cubical diagram $F_\bullet \in \mathrm{Func}(\vec{r} I^n, \boldsymbol{A})$). Using the axiom (EX1) we get that the multiplication by section map $s_i : F_u \to L_i \otimes F_u$ must factor through $F_{u+(r_i-u_i)e_i}$, which is a zero sheaf if $F_\bullet \in \mathbf{ker}(\pi_*^{L,\vec{r}})$. This implies the following lemma:

Lemma 3.20. *If $F_\bullet \in \mathbf{ker}(\pi_*^{L,\vec{r}})$ and $u \in \vec{r} I^n$ is such that $u_j \in \{0, r_j\}$ for all $j \neq i$, then* $\mathrm{supp}(F_u)$ *is contained in the divisor of zeroes of the section $s_i \in H^0(L_i)$.*

If $s_i = 0$ for some i, we say that $\mathrm{div}(s_i) = X$.

We apply the localization method (Theorem 2.3), to this partial description of the kernel. Let's fix some notation. Let

$$S_n(k) = \{T \subset \{1, \dots, n\} \mid |T| = k\}.$$

We often abuse notation and write $S(k)$ for $S_n(k)$ when it is clear from the context what n is. We view each interval $[0, r_i]$ as a pointed set, pointed at 0. It follows that we have order preserving inclusions

$$\iota_T : \prod_{i \in T} [0, r_i] \to \prod_{i=1}^{n} [0, r_i] := \vec{r} I^n.$$

Ignoring the pointed structure produces order preserving ($\leq$) projection maps

$$\pi_T : \vec{r} I^n \to \prod_{i \in T} [0, r_i].$$

Definition 3.21. As we agreed above, $L : \vec{r}\mathbb{Z}^n \to \mathfrak{Div}\, X$ is the symmetric monoidal functor as in Definition 3.7.

If $1 \leq k \leq n$ and $T \in S(k)$, then we define a symmetric monoidal functor $L_T : \vec{r}\mathbb{Z}^k \to \mathfrak{Div}\, X$ as a composition

$$\vec{r}\mathbb{Z}^k \xrightarrow{\iota_T} \vec{r}\mathbb{Z}^n \xrightarrow{L} \mathfrak{Div}\, X.$$

We say that L_T is obtained from L by the *restriction along ι_T*.

Now for $T \in S(k)$, let's consider the functor

$$\iota_T^* : \mathcal{EP}(X, L, \vec{r}) \to \mathcal{EP}(X, L_T, \pi_T(\vec{r})),$$

which is the restriction of an extendable pair $F_\bullet$ along the inclusion ι_T. The pseudo-period isomorphism is just obtained by restriction.

Definition 3.22. For any $1 \le k \le n$ we define functors

$$\mathrm{Face}^k := \prod_{T \in S(k)} \iota_T^* : \mathcal{EP}(X, L, \vec{r}) \to \prod_{T \in S(k)} \mathcal{EP}(X, L_T, \pi_T(\vec{r})).$$

Definition 3.23. For $1 \le k \le n$, we write $\mathbf{ker}^k = \mathbf{ker}(\mathrm{Face}^k)$ and $\mathbf{ker}^0 = \mathbf{ker}(\pi_*)$.

Lemma 3.24. *For any $1 \le k \le n$, any $F_\bullet \in \mathbf{ker}^{k-1}$ and any $T \in S(k)$ we can consider $(\iota_T^*(F_\bullet))_\bullet$ as an element of*

$$\mathrm{Func}\Big(\prod_{i \in T}[1, r_i - 1], \mathfrak{Coh}\Big(\bigcap_{i \in T} \mathrm{div}(s_i)\Big)\Big).$$

In other words, the images of these functors are supported on the indicated subschemes. As in Lemma 3.20, we say that if $s_i = 0$, then $\mathrm{div}(s_i) = X$.

Proof. If $k = 1$ then the result is proved in Lemma 3.20 and the observation before it.

Let's take any $2 \le k \le n$ and an extendable pair $F_\bullet \in \mathbf{ker}^{k-1}$. If we consider an extendable pair $(\iota_T^*(F_\bullet))_\bullet \in \mathcal{EP}(X, L_T, \pi_T(\vec{r}))$ then for any $v \in \prod_{i \in T}[0, r_i]$, we have isomorphisms of sheaves: $(\iota_T^*(F_\bullet))_v \cong 0$, whenever $v_i = 0$ for some $i \in T$. Because of the pseudoperiodicity isomorphism we also have that $(\iota_T(F_\bullet))_v \cong 0$, whenever $v_i = r_i$ for some $i \in T$.

The last step is an application of the axiom (EX1) to the extendable pair $(\iota_T^*(F_\bullet))_\bullet$. Because $(\iota_T^*(F_\bullet))_v \cong 0$ if $v_i = r_i$ for some $i \in T$, that implies that for any

$$w \in \prod_{i \in T}[1, r_i - 1]$$

the multiplication of the sheaf $(\iota_T^*(F_\bullet))_w$ by the sections $s_i \in H^0(X, L_i)$ for all $i \in T$ must factor through zero. So the support of the sheaf $(\iota_T^*(F_\bullet))_w$ is contained in $\bigcap_{i \in T} \mathrm{div}(s_i)$. □

Lemma 3.25. *If we restrict the domain of the functor Face^k to the full subcategory $\mathbf{ker}^{k-1}$ for any $1 \le k \le n$, then we obtain functors*

$$\mathrm{Face}^k|_{\mathbf{ker}^{k-1}} : \mathbf{ker}^{k-1} \to \prod_{T \in S(k)} \mathrm{Func}\Big(\prod_{i \in T}[1, r_i - 1], \Big(\bigcap_{i \in T} \mathrm{div}(s_i)\Big)\Big).$$

There is an equivalence of categories between $\mathbf{ker}^k$ and $\mathbf{ker}\big(\mathrm{Face}^k|_{\mathbf{ker}^{k-1}}\big)$.

Proof. The first part follows directly from the lemma before. The proof of the second part is straightforward and follows from the fact that $\mathbf{ker}^k$ is a full subcategory of $\mathbf{ker}^{k-1}$. □

Remark 3.26. In order to apply the localization procedure to the category $\mathbf{ker}^{k-1}$ we need to show that the functor $\mathrm{Face}^k|_{\mathbf{ker}^{k-1}}$ has a left adjoint. The existence of a left adjoint follows from the special adjoint functor theorem. But for the purpose of splitting of the corresponding short exact sequence of K-groups (see Section 3D for details), we need the unit of the adjunction to be the natural isomorphism. This doesn't follow from the abstract nonsense, so we need an explicit construction of a left adjoint functor. It is given in the proof of the following theorem.

Theorem 3.27. *Let X be a locally noetherian scheme and consider a symmetric monoidal functor $L : \vec{r}\mathbb{Z}^n \to \mathfrak{Div}\, X$.*

(i) *For any $1 \le k \le n$ there is an exact functor*

$$\mathrm{Face}^k|_{\mathbf{ker}^{k-1}} : \mathbf{ker}^{k-1} \to \prod_{T \in S(k)} \mathrm{Func}\Big(\prod_{i \in T}[1, r_i - 1], \mathfrak{Coh}\Big(\bigcap_{i \in T} \mathrm{div}(s_i)\Big)\Big),$$

where $\mathbf{ker}^k$ is a kernel of the functor Face^k and $\mathbf{ker}^0 := \mathbf{ker}(\pi_*^{L,\vec{r}})$.

(ii) *The functors $\mathrm{Face}^k|_{\mathbf{ker}^{k-1}}$ have left adjoints D^k such that*

$$\mathrm{Face}^k|_{\mathbf{ker}^{k-1}} \circ D^k \simeq 1.$$

(iii) $\mathrm{Face}^k|_{\mathbf{ker}^{k-1}}$ *satisfies the condition of Theorem 2.3*

(iv) *The functor*

$$\mathrm{Face}^n|_{\mathbf{ker}^{n-1}} : \mathbf{ker}^{n-1} \to \mathrm{Func}\Big(\prod_{i=1}^{n}[1, r_i - 1], \mathfrak{Coh}\Big(\bigcap_{i=1}^{n} \mathrm{div}(s_i)\Big)\Big)$$

is an equivalence of categories.

Proof. (i) These functors are obtained by restricting domains. As kernels and cokernels are computed pointwise, this is exact.

(ii) Given a functor $G_\bullet^T \in \mathrm{Func}\big(\prod_{i \in T}[1, r_i - 1], \ \mathfrak{Coh}\big(\bigcap_{i \in T} \mathrm{div}(s_i)\big)\big)$ for each $T \in S(k)$, we denote the corresponding object by

$$(G_\bullet^T)_{T \in S(k)} \in \prod_{T \in S(k)} \mathrm{Func}\Big(\prod_{i \in T}[1, r_i - 1], \mathfrak{Coh}\Big(\bigcap_{i \in T} \mathrm{div}(s_i)\Big)\Big).$$

Further, we view $G_\bullet^T$ as a functor $\prod_{i \in T}[0, r_i] \to \mathfrak{Coh}\big(\bigcap_{i \in T} \mathrm{div}(s_i)\big)$ by taking $G_u^T = 0$ if for some $i \in T$ we have $u_i \in \{0, r_i\}$, where 0 is some fixed zero object in $\mathfrak{Coh}(X)$. Also, for $i \in \{1, \dots, k\}$ if $u_i \in \{0, r_i - 1\}$ we define the morphisms $G_{+e_i}^T : G_u^T \to G_{u+e_i}^T$ as the initial and terminal map correspondingly.

Let us remind the reader of the definition of ϵ from Lemma 3.17. For any $0 \le i \le n$ we have functions $\epsilon_i : \vec{r}I \to \{0, 1\}$ such that for any $u \in \vec{r}I^n$, we have $\epsilon_i(u) = 1$ if $u_i = r_i$ and $\epsilon_i(u) = 0$ otherwise.

We define the functor D^k on objects as follows:

$$(D^k((G^T_\bullet)_{T\in S(k)}))_u = \left(\bigotimes_{i=1}^{n} L_i^{\epsilon_i(u)}\right) \otimes \left(\bigoplus_{T\in S(k)} G^T_{\pi_T(u)}\right).$$

Let's denote $(D^k((G^T_\bullet)_{T\in S(k)}))_\bullet$ by $D^k_\bullet$ for the simplicity of notations. First of all we want to view it as a functor $\vec{r}I^n \to \mathfrak{Coh}(X)$. For that we have to define the morphisms

$$D^k_{+e_i} : D^k_u \to D^k_{u+e_i}.$$

If $0 \le u_i < r_i - 1$, then this map is induced by $\bigoplus_{T\in S(k),\, i\in T} G^T_{+1}$. If $u_i = r_i - 1$, then it is induced by the terminal maps $\bigoplus_{T\in S(k),\, i\in T} G^T_{+1}$ and also by multiplication by the section s_i.

The pseudoperiod isomorphisms ρ are defined by the symmetric monoidal structure of the functor L. The proof of the axioms (EX2) and (EX3) is automatic, and the proof of (EX1) follows from the commutativity of the diagram

$$\begin{array}{ccc} D_u & \xrightarrow{\;D_{+(r_i-u_i)e_i}\;} & D_{u+(r_i-u_i)e_i} \\ \big\downarrow{\scriptstyle \sigma_i} & & \big\downarrow{\scriptstyle \rho} \\ L_i \otimes D_u & \xleftarrow[\;L_i \otimes D_{+u_ie_i}\;]{} & L_i \otimes D_{u-u_ie_i}. \end{array}$$

This diagram commutes because of the definition of $D_{+(r_i-\alpha_i)\vec{e}_i}$ and because $\operatorname{supp}(G^T_u) \subseteq \bigcap_{i\in T} \operatorname{div}(s_i)$ for any $u \in \prod_{i\in T}[0, r_i]$.

So we have shown that $D^k_\bullet$ is an extendable pair. If $k = 1$ then it's clear that $D^1_\bullet$ is in $\mathbf{ker}^0$, because $D^1_0 \cong 0$.

If $2 \le k \le n$, we want to see that $D^k_\bullet$ is in $\mathbf{ker}^{k-1}$. For that we have to see that for any $W \in S(k-1)$ and any $v \in \prod_{i\in W}[0, r_i]$, the sheaf $(\iota^*_W(D^k_\bullet))_v$ is isomorphic to zero. But this is true because for any $T \in S(k)$ we have that $G^T_u = 0$ if $u_i \in \{0, r_i\}$ for some $i \in T$.

Clearly, $\operatorname{Face}^k|_{\mathbf{ker}^{k-1}} \circ D^k = 1$.

Next we would like to show that D^k is indeed a left adjoint. Suppose that we have a morphism

$$(G^T_\bullet)_{T\in S(k)} \to \operatorname{Face}^k(F_\bullet).$$

Such a morphism consists of an $\binom{n}{k}$-tuple of morphisms

$$\phi_T : G^T_\bullet \to \iota^*_T(F_\bullet).$$

We wish to describe the adjoint map

$$\tilde{\phi} : D^k_\bullet \to F_\bullet.$$

Using the universal property of coproduct, this morphism is determined by maps

$$\tilde{\phi}(u)_T : \bigotimes_{i=1}^{n} L_i^{\epsilon(u)} \otimes G^T_{\pi_T(u)} \to F_u.$$

If u is such that $\epsilon_i(u) = 0$ for all $1 \le i \le n$, then these maps are just the compositions of ϕ_T with the morphisms $F_{+\alpha}$. If there is l such that $u_l = r_l$, then $\tilde{\phi}(u)_T$ is induced by the composition of ϕ_T with ρ_F^{-1} and with $F_{+\alpha}$.

We want to check that the map $\tilde{\phi}$ is indeed a natural transformation of functors. It's enough to check that the diagram

$$\begin{array}{ccc} D_u & \xrightarrow{\tilde{\phi}(u)} & F_u \\ \big\downarrow {\scriptstyle D_{+e_i}} & & \big\downarrow {\scriptstyle F_{+e_i}} \\ D_{u+e_i} & \xrightarrow[\tilde{\phi}(u+e_i)]{} & F_{u+\vec{e}_i} \end{array}$$

commutes. If $\epsilon_k(u) = 0$ for all $1 \le k \le n$ and also $u_i < r_i - 1$, then it commutes directly from the construction of the maps $\tilde{\phi}(u)$. Otherwise the commutativity follows from (EX1), (EX2) and (EX3) for $F_\bullet$.

Finally, we have the map

$$\mathrm{Hom}((G^T_\bullet)_{T\in S(k)}, \mathrm{Face}^k(F_\bullet)) \to \mathrm{Hom}(D^k((G^T_\bullet)_{T\in S(k)}), F_\bullet).$$

It's easy to see that this map is bijective, because the right Hom is uniquely defined by the restriction to k-faces.

(iii) This follows from (ii).

(iv) Because for $S(n)$ there is only one element, the set $\{1, \ldots, n\}$ itself, we have that $\iota_{\{1,\ldots,n\}} = \mathrm{id}$ and $\pi_{\{1,\ldots,n\}} = \mathrm{id}$. So $\mathrm{Face}|^n_{\mathbf{ker}^{n-1}}$ and D^n are identity functors. □

3D. *G-theory and K-theory of a root stack.* In this subsection we finally describe the G-theory of a root stack $X_{L,\vec{r}}$.

Lemma 3.28. *If X is a locally noetherian scheme and L a symmetric monoidal functor as in Definition 3.7, there is an equivalence of categories*

$$\mathfrak{Coh}\, X_{L,\vec{r}} \simeq \mathcal{EP}(X, L, \vec{r}).$$

Proof. This follows by combining Corollaries 3.10 and 3.15. □

So we have

$$G(X_{L,\vec{r}}) \cong K(\mathcal{EP}(X, L, \vec{r})),$$

and we reduced the problem to describing the K-theory of the (abelian) category of coherent extendable pairs $\mathcal{EP}(X, L, \vec{r})$.

We are going to use several splittings of the category of coherent extendable pairs to simplify the latter K-theory. The first step is this:

Lemma 3.29. *If X is a locally noetherian scheme, then in the notation of Section 3C one has*

$$K_i(\mathcal{EP}(X, L, \vec{r})) \cong G_i(X) \oplus K_i(\mathbf{ker}(\pi_*^{L,\vec{r}})) \quad \textit{for any } i \in \mathbb{Z}_+.$$

Proof. Using Corollary 3.19 and the localization property of K-theory (see for example [Quillen 1973]) we have the long exact sequence of groups

$$\cdots \to K_i(\mathbf{ker}(\pi_*^{L,\vec{r}})) \to K_i(\mathcal{EP}(X, L, \vec{r})) \to G_i(X) \to \cdots.$$

But this sequence splits because of the property $\pi_*^{L,\vec{r}} \circ \pi_{L,\vec{r}}^* \simeq 1$ proved in Lemma 3.17. □

Lemma 3.30. *If $\mathbf{A}$ is an abelian category then*

$$K_i(\mathrm{Func}(\vec{r}I^n, \mathbf{A})) \cong K_i(\mathbf{A})^{\oplus \prod_{j=1}^n r_j}.$$

Proof. The proof follows from the iterated application of Theorem 2.6 and localization property of the K-theory. □

Now we want to proceed with $K_\bullet(\mathbf{ker}(\pi_*^{L,\vec{r}}))$, exploiting the same ideas as in the previous lemmas.

Lemma 3.31. *Let X be a locally noetherian scheme, L a symmetric monoidal functor as in Definition 3.7 and $s_k \in H^0(L_k)$ for $k = 0, \dots, n$. Then for any $i \in \mathbb{Z}_+$,*

$$K_i(\mathbf{ker}(\pi_*^{L,\vec{r}})) \cong \bigoplus_{k=1}^{n} \bigoplus_{T \in S(k)} G_i\biggl(\bigcap_{l \in T} \mathrm{div}(s_l)\biggr)^{\oplus \prod_{l \in T}(r_l - 1)},$$

where $S(k) = \{T \subset \{1, \dots, n\} \mid |T| = k\}$.

Proof. This follows from application of the localization property of K-theory, Theorem 3.27 and the previous technical lemma. □

Combining Lemmas 3.28, 3.29 and 3.31 yields the main result of the section:

Theorem 3.32. *Let X be a locally noetherian scheme. Let (L_i, s_i) be objects of $\mathfrak{Div}\, X$ for $i = 1, \dots, n$ and $\vec{r} \in \mathbb{N}^n$. Then G-theory of a root stack $X_{L,\vec{r}}$ is given by the formula*

$$G_i(X_{L,\vec{r}}) \cong G_i(X) \oplus \Biggl(\bigoplus_{k=1}^{n} \bigoplus_{T \in S(k)} G_i\biggl(\bigcap_{l \in T} \mathrm{div}(s_l)\biggr)^{\oplus \prod_{l \in T}(r_l - 1)}\Biggr)$$

for any $i \in \mathbb{Z}_+$, where $S(k) = \{T \subset \{1, \dots, n\} \mid |T| = k\}$.

To finish the section we want to give sufficient conditions for a root stack to be smooth.

Proposition 3.33. *Let X be a smooth scheme over a field k. Let $D = \sum_{i=1}^{n} D_i$ be a normal crossing divisor. Assume that $\vec{r}$ is an n-tuple of natural numbers, such that each r_i is coprime to the characteristic of k. Then a root stack $X_{D,\vec{r}}$ is smooth.*

Proof. By definition a stack is smooth if its presentation is a smooth scheme. The question is local, so we can assume that $X = \mathrm{Spec}(R)$ and a divisor D is a strict normal crossing divisor. If we localize further, we can assume that R is a regular local ring, $D_i = (f_i)$ and $\{f_i\}$ forms a part of a regular sequence of parameters.

By [Cadman 2007, Example 2.4.1], the presentation of a root stack $X_{D,\vec{r}}$ is an affine scheme $A = R[t_1, \dots, t_n]/(t_1^{r_1} - f_1, \dots, t_n^{r_n} - f_n)$. By [Grothendieck and Murre 1971, Lemma 1.8.6], this scheme is smooth. □

Corollary 3.34. *Under the hypotheses of Proposition 3.33, $G(X_{D,\vec{r}}) = K(X_{D,\vec{r}})$, where the latter means the Waldhausen K-theory of perfect complexes on the stack as defined in [Joshua 2005].*

Proof. Indeed, if a stack is regular, its Waldhausen K-theory is the same as G-theory. See [Joshua 2005]. □

4. Quotient stacks as root stacks

4A. ***Generation of inertia groups.*** Let X be a scheme with an action of a finite group G. We always assume that this action is *admissible*. Let us recall, following [SGA 1 1971, V.1, Definition 1.7], that an action is called admissible if there exists an affine morphism $\phi : X \to Y$ such that $\mathcal{O}_Y \cong \phi_*(\mathcal{O}_X)^G$. This implies that the quotient X/G exists and is isomorphic to Y.

If $x \in X$ is a point (not necessarily closed), the subgroup of G stabilizing x is called the *decomposition group* and we denote it by $D(x, G)$. The subgroup of the decomposition group acting trivially on the residue field of x is called the *inertia group* of x and we denote it by $I(x, G)$.

Note that there is an induced action of $D(x, G)$ on the closure of the point x and $I(x, G)$ acts trivially on this closure. Hence if $x \in \bar{y}$ then there is an inclusion $I(y, G) \hookrightarrow I(x, G)$. We say that *the inertia groups are generated in codimension one* if for each point $x \in X$ we have that

$$I(x, G) = \prod_{x \in \bar{y}} I(y, G),$$

where the product is over all points of codimension one containing x and the identification is via the inclusions above. For a group acting on a smooth curve, all inertia groups are generated in codimension one. We will see under certain assumptions that this is also true in higher dimensions (see Theorem 4.9).

4B. ***Main theorem.*** In this subsection we provide sufficient conditions for a quotient stack to be a root stack. To illustrate the procedure we start with an example.

Example 4.1. Let $\mathcal{O}$ be a discrete valuation ring with an action of μ_r such that $\gcd(r, \operatorname{char}(\mathcal{O})) = 1$. Then the fixed ring $\mathcal{O}^{\mu_r}$ is also a discrete valuation ring. We assume that $\mathcal{O}$ contains a field so that its completion $\hat{\mathcal{O}}$ is a power series ring in one variable over the residue field. Note that μ_r must preserve the maximal ideal of $\mathcal{O}$. If we further assume that the action is generically free and inertial, i.e., μ_r acts trivially on the residue field, then if s is a local parameter for $\mathcal{O}$ we can conclude that $t = s^r$ is a local parameter for $R = \mathcal{O}^{\mu_r}$.

We set $Y = \operatorname{Spec}(R)$ and consider the root stack

$$\mathfrak{Y} = Y_{R,t,r} \to Y.$$

The parameter s induces a μ_r-equivariant morphism

$$X \to \mathfrak{Y}$$

corresponding to the triple $(\mathcal{O}, s, m)$, where m is the canonical isomorphism $\mathcal{O}^r \to \mathcal{O}$. We show in Proposition 4.6 that this morphism is in fact étale. Using the two out of three property for étale maps we get that the natural morphism

$$X \times \mu_r \to X \times_{\mathfrak{Y}} X$$

is étale. To show that $[X/\mu_r] \cong \mathfrak{Y}$ it suffices to show that this morphism is radicial (universally injective) and surjective. In other words we need to show that it is a bijection on K-points for each field K.

Given a pair of K-points a and b of X that give a K-point of $X \times_Y X$, the fiber of

$$X \times_{\mathfrak{Y}} X \to X \times_Y X$$

over this point consists of the space of isomorphisms between $a^*(\mathcal{O}, s, m)$ and $b^*(\mathcal{O}, s, m)$ in $\mathfrak{Y}$. If the support of the K-points is the generic point of $\mathcal{O}$ this is just a singleton and if the support is the closed point then the space is a bitorsor over μ_r. At any rate the morphism above is seen to be an isomorphism. Hence in this case we have

$$[X/\mu_r] \cong \mathfrak{Y}.$$

Remark 4.2. A μ_r-bundle P on a scheme Z is equivalent to the data of an invertible sheaf $\mathcal{K}$ and an isomorphism $\phi : \mathcal{K}^r \to \mathcal{O}_Z$. To construct P explicitly consider the sheaf of algebras $\operatorname{Sym}^\bullet \mathcal{K}^{-1}$. There is a distinguished global section $T \in \mathcal{K}^{-r}$ given by $(\phi \otimes 1_{\mathcal{K}^{-r}}(1))$. Then

$$P = \operatorname{Spec}(\operatorname{Sym}^\bullet \mathcal{K}^{-1}/(T-1)).$$

Remark 4.3. Suppose that there is on Y an invertible sheaf $\mathcal{N}$ and an isomorphism $\mathcal{N}^r \to \mathcal{L}$. Then $Y_{\mathcal{L},s,r}$ is a global quotient stack; see [Cadman 2007, Lemma 2.3.1

and Example 2.4.1; Borne 2007, §3.4]. We need this below, so let's recall some of the details. The coherent sheaf

$$\mathcal{A} = \mathcal{O}_Y \oplus \mathcal{N}^{-1} \oplus \cdots \oplus \mathcal{N}^{-(r-1)}$$

can be given the structure of an $\mathcal{O}_Y$-algebra via the composition

$$\mathcal{N}^{-r} \xrightarrow{\sim} \mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_Z.$$

There is an action of μ_r on this sheaf via the action of μ_r on $\mathcal{N}^{-1}$ given by scalar multiplication. Then $Y_{\mathcal{L},s,r} = [\mathrm{Spec}(\mathcal{A})/\mu_r]$. We need the explicit morphism

$$Y_{\mathcal{L},s,r} \to [\mathrm{Spec}(\mathcal{A})/\mu_r]$$

below so let's describe it. Consider a morphism $a : X \to Y$. A morphism $X \to Y_{\mathcal{L},s,r}$, lifting a, is a triple $(\mathcal{M}, t, \phi)$. As per the previous remark the sheaf $\mathcal{M}^{-1} \otimes \mathcal{N}^{-1}$ gives a μ_r-torsor. The torsor comes from the algebra

$$\mathcal{B} = \mathrm{Sym}^{\bullet}\, \mathcal{M} \otimes a^*\mathcal{N}^{-1}/(T-1).$$

To produce an X-point of $[\mathrm{Spec}(\mathcal{A})/\mu_r]$ we need to describe a μ_r-equivariant map

$$a^*\mathcal{A} \to \mathcal{B}.$$

This map comes from the section t via

$$t \in \mathrm{Hom}(\mathcal{O}, \mathcal{M}) = \mathrm{Hom}(a^*\mathcal{N}, \mathcal{M} \otimes a^*\mathcal{N}^{-1}).$$

This construction generalizes in the obvious way to a finite list of invertible sheaves with section.

Assumption 4.4. We assume X and Y are regular, separated, noetherian schemes over a field k. Let G be a finite group with cardinality coprime to the characteristic of k. We assume that G acts admissibly and generically freely on X with quotient $\phi : X \to Y$. Note that by [Görtz and Wedhorn 2010, Theorem 14.126] our hypotheses imply that the quotient map $X \to Y$ is flat.

Consider the map $\phi : X \to Y$, which is faithfully flat and finite. Recall that the set of points of X where ϕ is ramified is called the branch locus. It has a natural closed subscheme structure defined by $\mathrm{supp}(\Omega_{X/Y})$. Because the conditions of the purity theorem [Altman and Kleiman 1970, Chapter VI, Theorem 6.8] are satisfied, in our situation this closed subscheme gives rise to an effective Cartier divisor, which is called the branch divisor. We can write this divisor as

$$R = \sum_{i=1}^{n} (r_i - 1)\Bigl(\sum_{g \in G} g^* D_i\Bigr),$$

where each D_i is a prime divisor. As G acts generically freely, passing to generic points of our regular variety produces a Galois extension with Galois group G. We can view the D_i as points of the scheme X. The multiplicities r_i are related to the inertia groups of D_i via

$$r_i = |I(D_i, G)|;$$

see [Neukirch 1999, Chapter I, §9].

We let E_i be the image of D_i under ϕ. It is called the ramification divisor. We form the root stack

$$\mathfrak{Y} = Y_{((E_1,r_1),\dots,(E_n,r_n))}.$$

Note that we have assumed that the characteristic of our ground field is coprime to G and hence to each r_i. It follows, via a local calculation along the ring extension $\mathcal{O}_{X,D_i}/\mathcal{O}_{Y,E_i}$, that we have $\phi^*(E_i) = r_i\left(\sum_{g\in G} g^* D_i\right)$. This allows us to lift ϕ to produce a diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle\phi}\downarrow & \searrow^{\psi} & \\ Y & \xleftarrow{\ \pi\ } & \mathfrak{Y} \end{array}$$

The morphism ψ is equivariant in the sense that precomposition with $g \in G$ produces a two-commuting diagram. This gives us a morphism

$$[X/G] \to \mathfrak{Y}$$

that we would like to show is an isomorphism under our Assumption 4.4 and the extra condition that the ramification divisor is normal crossing.

For the proof of Proposition 4.6 we need the following lemma.

Proposition 4.5 (Abhyankar's lemma). *Let $Y = \mathrm{Spec}(A)$ be a regular local scheme and $D = \sum_{1\le i\le r} \mathrm{div}(f_i)$ a divisor with normal crossings, so that the f_i form part of a regular system of parameters for Y. Set $\overline{Y} = \mathrm{Supp}(D)$ and let $U = Y \setminus \overline{Y}$. Consider $V \to U$, an étale cover that is tamely ramified over D. If y_i are the generic points of $\mathrm{supp}(\mathrm{div}(f_i))$ then $\mathcal{O}_{Y,y_i}$ is a discrete valuation ring. If we let K_i be its field of fractions then, as V ramifies tamely, we have that*

$$V|_{K_i} = \mathrm{Spec}\Bigl(\prod_{j\in J_i} L_{ji}\Bigr),$$

where the L_{ji} are finite separable extensions of K_i. We let n_{ji} be the order of the inertia group of the Galois extension generated by L_{ji} and let

$$n_i = \mathrm{lcm}_{j\in J_i}\, n_{ji},$$

and set

$$A' = A[T_1,\dots,T_r]/(T_1^{n_1} - f_1,\dots,T_r^{n_r} - f_r), \qquad Y' = \mathrm{Spec}(A').$$

Then the étale cover $V' = V \times_X X'$ of $U \times_X X'$ extends uniquely up to isomorphism to an étale cover of X'.

Proof. This is [SGA 1 1971, Expose XIII, Proposition 5.2]. The proof given shows how to construct the extension of V', which we need below. The extension can be constructed as the normalization of X' in the generic point of $V \times_X X'$. □

Proposition 4.6. *Under Assumption 4.4, suppose that $\phi : X \to Y$ is ramified along a simple normal crossings divisor E. The morphism $\psi : X \to \mathfrak{Y}$ constructed above is étale.*

Proof. Étale maps are local on the source so we can assume that $Y = \operatorname{Spec}(S)$, and all E_i are trivial line bundles so that $s_i \in S$. Further, by shrinking X we can assume that the morphism $X \to \mathfrak{Y}$ is defined be trivial bundles on X. Because the map ϕ is finite we can write $X = \operatorname{Spec}(T)$. Here T and S are local regular Noetherian k-algebras, T is a finite S-module, s_i is part of a regular system of parameters and there are elements $t_i \in T$, such that $t_i^{r_i} = s_i$.

We may check étaleness after a faithfully flat base extension of the base field and hence may assume that the ground field k contains r_i-th roots of unity for all $1 \leq i \leq n$.

Using Remark 4.3, we see that the stack $\mathfrak{Y}$ is isomorphic to the quotient stack

$$[\operatorname{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}],$$

where $S' = S[y_1, \ldots, y_n]/(y_1^{r_1} - s_1, \ldots, y_n^{r_n} - s_n)$.

We want to show that the map $\operatorname{Spec}(T) \to [\operatorname{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}]$ is étale. Denote by T' the ring $T[x_1, \ldots, x_n]/(x^{r_1} - 1, \ldots, x^{r_n} - 1)$. Using Remark 4.3 again we obtain a Cartesian diagram

$$\begin{array}{ccc} \operatorname{Spec}(T') & \longrightarrow & \operatorname{Spec}(S') \\ \downarrow & & \downarrow \\ \operatorname{Spec}(T) & \longrightarrow & \left[\operatorname{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}\right] \end{array}$$

Because $\operatorname{Spec}(S')$ is a presentation of a quotient stack it is enough to show that the map $S' \to T'$ given by $y_i \mapsto t_i x_i$ is étale.

The morphism $S_{s_1 \ldots s_n} \to T_{t_1 \ldots t_n}$ is flat and unramified by assumption, and hence it is étale. By Abhyankar's lemma (Proposition 4.5), this morphism extends after base change to an étale cover of S'. By the proof of Abhyankar's lemma it suffices to show that T' is normal and the map $S' \to T'$ is integral. Both of these facts are easily checked and the result follows. □

For a point $p \in Y$ we define

$$I(p, Y) = \prod_{p \in \operatorname{supp}(E_i)} \mu_{r_i}.$$

Proposition 4.7. *Let Assumption 4.4 hold. Let K be a field and consider the morphism of K-points*

$$\pi_K : X \times_{\mathfrak{Y}} X(K) \to X \times_Y X(K).$$

The fiber $\pi_K^{-1}(x_1, x_2)$ over a K-point (x_1, x_2) is a bitorsor under the inertia group $I(\phi(x_1), Y)$.

Proof. In what follows, we use the shorthand G^* when we mean $\sum_{g \in G} g^*$. Recall that the morphism ψ is defined by $(\mathcal{O}(G^* E_i), s_{G^* E_i}, \alpha_i)$, where α_i are isomorphisms, coming from the fact that

$$r_i G^* E_i = r_i \phi^*(D_i).$$

The fiber over (x_1, x_2) is exactly the set of isomorphism from $x_1^* \mathcal{O}(G^* E_i)$ to $x_2^* \mathcal{O}(G^* E_i)$ as i varies. As in Example 4.1, this depends on whether the section $x_1^* s_{G^* E_i}$ vanishes or not. The vanishing condition precisely depends on $\phi(x_1)$, and the result follows. □

The final ingredient we need to finish the proof is that under our assumptions the inertia group of X is generated in codimension one. For that let us recall the following:

Proposition 4.8 (Abhyankar's theorem; see [Grothendieck and Murre 1971, Theorem 2.3.2]). *Let Y be a locally noetherian normal scheme, D a divisor with normal crossing, $\hat{Y} = \operatorname{supp}(D)$ and $U = Y \setminus \hat{Y}$. Assume that $X \to Y$ is a finite morphism and G is a finite group operating on X such that $X \mid U$ is a G-torsor. Then the following are equivalent*:

(i) *X is tamely ramified relative to D.*

(ii) *For every $y \in Y$ there exists an étale neighborhood Y' of y in Y, and a scheme $S = \mathcal{O}_Y[(T_i)_{i \in I'}]/((T_i^{r_i'} - f_i'))_{i \in I'}$, where $D_{Y'} = \sum_{i \in I'} D_i'$ and $\operatorname{div}(f_i') = D_i'$, such that there is an isomorphism of couples*

$$(X', G) \simeq (G \times^H S, G),$$

where $X' = X \times_Y Y'$ and $H = \prod_{i \in I'} \mu_{r_i'}$. Let us recall that $G \times^H S$ is the quotient $(G \times S)/H$, where H acts "by the formula" $h \cdot (g, s) = (gh^{-1}, hs)$.

Let us apply this fact to describe the inertia group.

Theorem 4.9. *Under Assumption 4.4, suppose that $\phi : X \to Y$ is ramified along a simple normal crossings divisor. Then the inertia groups of (X, G) are generated in codimension one.*

Proof. Firstly observe that condition (i) of Abhyankar's theorem (Proposition 4.8) is satisfied. Inertia is a local notion and also, clearly, the inertia group of (S, H) is generated in codimension one.

There is an isomorphism of quotient stacks $[(G \times^H S)/G] \cong [S/H]$. So inertia groups of $G \times^H S$ under the action of G and of S under the action of H are isomorphic for the corresponding points. This finishes the proof. □

Finally, we are ready to prove the main theorem of this section.

Theorem 4.10. *If Assumption 4.4 is satisfied and if also the ramification divisor is a normal crossing divisor, then we have the isomorphism of stacks* $[X/G] \cong \mathfrak{Y}$.

Proof. To prove this all we need to show is that the map

$$\chi : X \times G \to X \times_{\mathfrak{Y}} X, \quad (x, g) \mapsto (x, gx)$$

is an isomorphism.

By Proposition 4.6, the map $\psi : X \to \mathfrak{Y}$ is étale, and so the map $X \times_{\mathfrak{Y}} X \to X$ is étale as a pullback. Clearly two maps $X \times G \to X$ given by $(x, g) \mapsto x$ and $(x, g) \mapsto gx$ are étale and so the map χ must be étale.

We are going to show that the map

$$\chi(K) : X(K) \times G \to X \times_{\mathfrak{Y}} X(K)$$

is bijective for any field extension of the ground field $k \subset K$. The points of the scheme on the left is a pair (x, g), where $g \in G$ and $x : \mathrm{Spec}(K) \to X$ a K-point.

Consider the morphism $\Psi : X \times G \to X \times_Y X$. This morphism is surjective as we have a geometric quotient; see [Mumford et al. 1994, Definition 0.4]. Consider a K-point $(x_1, x_2) \in X \times_Y X(K)$. Using the properties of geometric quotients we have that $x_2 = gx_1$ for some $g \in G$. Using this we see the fiber $\Psi^{-1}(x_1, x_2)$ is a torsor over the inertia group $I(\mathrm{supp}(x_1), G)$. By Theorem 4.9 our inertia groups are generated in codimension one, so we see that we have an identification

$$I(\mathrm{supp}(x_1), G) = \mu_{r_{i_1}} \times \cdots \times \mu_{r_{i_l}}$$

as in Proposition 4.7. It follows that the morphism χ is étale and universally injective (radical). This implies that it is an open immersion. As it is also surjective, it is an isomorphism and the result follows. □

5. An application of root stacks to the equivariant K-theory of schemes

As an application of the theorems proved in Sections 3 and 4, we can formulate a result about equivariant K-theory.

Theorem 5.1. *Let X be a regular, separated, noetherian scheme over the field k with a generically free admissible action of a finite group G, such that the order of G is coprime to the characteristic of k. Let $X/G = Y$ and assume that all the* *conditions of Assumption 4.4 are satisfied. Also assume that $X \to Y$ is ramified along a simple normal crossing divisor E. Then there is an isomorphism of groups*

$$K_G^\bullet(X) \cong K^\bullet(Y) \oplus \Bigg(\bigoplus_{i=1}^{n}\Bigg(\bigoplus_{T \in S(i)} G^\bullet\Big(\bigcap_{l \in T} E_l\Big)^{\oplus \prod_{l \in T}(r_l - 1)}\Bigg)\Bigg),$$

where r_l are orders of inertia groups (see Section 4 for notation), *and*

$$S(i) = \{T \subset \{1, \ldots, n\} \mid |T| = i\}.$$

Proof. By assumption X is a regular scheme and the group G is finite, so for any G-equivariant sheaf we can always construct an equivariant locally free resolution by averaging the usual locally free resolution. This simple argument shows that the equivariant K-theory of X is the same as the equivariant G-theory.

The category of G-equivariant sheaves on X is equivalent to the category of sheaves on the quotient stack $[X/G]$, so we can see that

$$K_G(X) \cong G([X/G]).$$

In Theorem 4.10 we proved under our assumptions that there is an isomorphism of stacks $[X/G] \cong \mathfrak{Y}$, so we have an isomorphism of their G-theories

$$G([X/G]) \cong G(\mathfrak{Y}).$$

Finally the application of Theorem 3.32 gives the desired formula. □

Let us give some examples.

Example 5.2. Let's consider $\mathbb{A}^1$ over a field k with an action of μ_3 (it acts by multiplication). Assume that $\mathrm{char}(k) \neq 3$. Then $\mathbb{A}^1/\mu_3 \cong \mathbb{A}^1$ and ramification divisor is $\mathrm{div}(0)$. The inertia group is μ_3. So by Theorem 5.1,

$$K_{\mu_3}^\bullet(\mathbb{A}^1) \cong K^\bullet(\mathbb{A}^1) \oplus K^\bullet(k) \oplus K^\bullet(k) \cong K^\bullet(k)^{\oplus 3}.$$

Example 5.3. This example was inspired by the paper [Alexeev and Orlov 2013]. The Burniat surface X with $K_X^2 = 6$ is a Galois $G := C_2 \times C_2$-cover of $\mathrm{Bl}_3\,\mathbb{P}^2$ (a del Pezzo surface of degree 6). Let's assume that the ground field k is algebraically closed and $\mathrm{char}(k) \neq 2$. The ramification divisor is given in [loc. cit., Figure 1]: it is denoted by A_l, B_l, C_l, where $0 \le l \le 4$. The inertia group of each component is C_2, and the inertia group of an intersection point of any two components is G. The intersection of three components is empty. Also, $A_l \cong B_l \cong C_l \cong \mathbb{P}^1$ for all $l = 0, \ldots, 3$.

Applying Theorem 5.1, one gets

$$K_G^\bullet(X) \cong K^\bullet(\mathrm{Bl}_3\,\mathbb{P}^2) \oplus \Bigg(\bigoplus_{i=1}^{2} Z_i^\bullet\Bigg), \qquad Z_1^\bullet = K^\bullet(\mathbb{P}^1)^{\oplus 12}, \qquad Z_2^\bullet = K^\bullet(k)^{\oplus 30}.$$

Notations and conventions

k	our base field
$\mathbf{ker}$	the kernel of a functor (Definition 2.2)
$\vec{r}$	an n-tuple $(r_1, \dots, r_n)$ of real numbers
$\vec{r}I^n$	the poset of integer points in $\prod_{i=1}^{n}[0, r_i]$
$\mathrm{Func}(\mathbf{A}, \mathbf{B})$	the functor category between two abelian categories
$\widehat{M}$	the dual $\mathrm{Hom}(M, \mathbb{G}_{\mathrm{m}})$ of the monoid M
$\mathfrak{Div}\, X$	the symmetric monoidal category of line bundles with section (Section 3A)
$X_{L,\vec{r}}$	a stack of roots over the scheme X (Definition 3.4)
$\mathfrak{Coh}\, X$	category of coherent sheaves on X
$\mathcal{EP}(X, L, \vec{r})$	category of coherent extendable pairs (Remark 3.16)
$S_n(k) = S(k)$	The set of subsets of $\{1, 2, \dots, n\}$ of cardinality k (We often drop the subscript n when it is clear from context.)
Face^k	The k-th face functor (Definition 3.22)
$\mathbf{ker}^k$	The kernel of the face functor (Definition 3.23)

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ANNALS OF K-THEORY

2019 vol. 4 no. 2

G-theory of root stacks and equivariant *K*-theory 151
AJNEET DHILLON and IVAN KOBYZEV

Orbital integrals and *K*-theory classes 185
PETER HOCHS and HANG WANG

On derived categories of arithmetic toric varieties 211
MATTHEW BALLARD, ALEXANDER DUNCAN and PATRICK MCFADDIN

K-theory of Hermitian Mackey functors, real traces, and assembly 243
EMANUELE DOTTO and CRICHTON OGLE

On the K-theory coniveau epimorphism for products of Severi–Brauer varieties 317
NIKITA KARPENKO and EOIN MACKALL