ANNALS OF K-THEORY

Orbital integrals and *K*-theory classes

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Let G be a semisimple Lie group with discrete series. We use maps $K_0(C_r^*G) \to \mathbb{C}$ defined by orbital integrals to recover group theoretic information about G, including information contained in K-theory classes not associated to the discrete series. An important tool is a fixed point formula for equivariant indices obtained by the authors in an earlier paper. Applications include a tool to distinguish classes in $K_0(C_r^*G)$, the (known) injectivity of Dirac induction, versions of Selberg's principle in K-theory and for matrix coefficients of the discrete series, a Tannaka-type duality, and a way to extract characters of representations from K-theory. Finally, we obtain a continuity property near the identity element of G of families of maps $K_0(C_r^*G) \to \mathbb{C}$, parametrised by semisimple elements of G, defined by stable orbital integrals. This implies a continuity property for L-packets of discrete series characters, which in turn can be used to deduce a (well-known) expression for formal degrees of discrete series representations from Harish-Chandra's character formula.

1. Introduction

Let G be a real semisimple Lie group. Its reduced C^*-algebra C_r^*G is the closure in $\mathcal{B}(L^2(G))$ of the algebra of convolution operators by functions in $L^1(G)$. It represents the tempered dual of G as a "noncommutative space" in the sense of noncommutative geometry, and encodes all tempered representations of G. Its K-theory $K_*(C_r^*G)$ is a natural invariant to consider. This K-theory is described explicitly in terms of equivariant indices of Dirac operators on G/K, for a maximal compact subgroup $K < G$, in the Connes–Kasparov conjecture. This was proved in various cases by Penington and Plymen [1983], Wassermann [1987], Lafforgue [2002b] and finally in general by Chabert, Echterhoff and Nest [Chabert et al. 2003].

Despite this explicit knowledge about the structure of $K_*(C_r^*G)$, it remains a challenge to extract explicit representation theoretic information from this K-theory group. There has been a good amount of success in this direction for classes

in $K_*(C_r^*G)$ corresponding to discrete series representations, for groups having such representations. For example, Lafforgue [2002a] used K-theory to recover Harish-Chandra's criterion $\text{rank}(G) = \text{rank}(K)$ for the existence of discrete series representations.

The von Neumann trace τ_e on C_r^*G, defined by $\tau_e(f) = f(e)$ for f in a dense subalgebra, induces a map on $K_0(C_r^*G)$. On classes corresponding to the discrete series, this gives the formal degrees of such representations. But this trace maps all other classes to zero (see Proposition 7.3 in [Connes and Moscovici 1982]). It has recently become clear that a natural generalisation of the von Neumann trace involving *orbital integrals* can be used to extract much more information from $K_0(C_r^*G)$. For a semisimple element $g \in G$, the orbital integral $\tau_g(f)$ of a function f on G is the integral of f over the conjugacy class of g. This integral converges for f in Harish-Chandra's Schwartz algebra, which has the same K-theory as C_r^*G. That leads to maps

$$\tau_g : K_0(C_r^*G) \to \mathbb{C}. \tag{1.1}$$

If D is an elliptic operator on a $\mathbb{Z}_2$-graded vector bundle over a manifold M, G-equivariant for a proper, cocompact action by G on M, then one has the equivariant index

$$\text{index}_G(D) \in K_0(C_r^*G).$$

In [Hochs and Wang 2018a], the authors proved a fixed point formula for the numbers

$$\tau_g(\text{index}_G(D)). \tag{1.2}$$

They showed that Harish-Chandra's character formula for the discrete series is a special case of this fixed point formula, much as Weyl's character formula is a special case of the Atiyah–Segal–Singer [Atiyah and Segal 1968] or Atiyah–Bott [Atiyah and Bott 1967] fixed point formulas, as proved in [Atiyah and Bott 1968]. Also, Shelstad's character identities for L-packets of representations follows from a K-theoretic argument involving τ_g, in the case of discrete series representations [Hochs and Wang 2018c].

Another approach to index theory involving orbital integrals is the work of Bismut on hypoelliptic Laplacians; see for example [Bismut 2011] or the survey [Ma 2017].

For discrete groups, orbital integrals (now sums over conjugacy classes) are also useful tools in K-theory. The main result in [Wang and Wang 2016] is a fixed point theorem for (1.2) in the discrete group case, which has consequences to orbifold geometry, positive scalar curvature metrics, and trace formulas. Gong [2015] and Samurkaş [2017] used such maps on the K-theory of maximal group C^*-algebras to deduce information about rigidity of manifolds. Lott [1999] used orbital integrals

for discrete groups to construct secondary invariants. Xie and Yu [2018] expressed Lott's delocalised η-invariant in terms of a K-theoretic ρ-invariant.

For semisimple Lie groups G, the results in [Hochs and Wang 2018a; 2018c; Lafforgue 2002b] mentioned above show that classes in $K_0(C_r^*G)$ corresponding to the discrete series contain a great deal of information about those representations. But it was long unclear what (representation theoretic) information can be recovered from other classes. That question was important motivation for this paper. As a concrete example, it was not known what information the generator of $K_0(C_r^* \mathrm{SL}(2, \mathbb{R}))$ corresponding to the limits of discrete series (or to the nonspherical principal series) contains.

In the present paper, we investigate further properties and applications of the maps (1.1) for semisimple Lie groups, many of them related to the fixed point formula for (1.2). This starts with an explicit expression for τ_g applied to K-theory generators defined via Dirac induction (Theorem 3.2). That result shows that τ_g is the zero map on K-theory if $\mathrm{rank}(G) \neq \mathrm{rank}(K)$, but it has interesting consequences if $\mathrm{rank}(G) = \mathrm{rank}(K)$. These include

- a way to use the maps τ_g to distinguish elements of $K_0(C_r^*G)$ (Corollary 4.1);

- an embedding of $K_0(C_r^*G)$ into the spaces of distributions on G^{reg} or G (Corollary 4.2);

- an induction formula from K-equivariant indices to G-equivariant indices (Corollary 4.8);

- versions of Selberg's vanishing principle for classes in $K_0(C_r^*G)$ (Corollary 4.9) and matrix coefficients of the discrete series (Corollary 4.10);

- a Tannaka-type duality result (Corollary 4.11);

- a result relating the value of τ_g on K-theory generators to characters of representations (Corollary 5.3).

Furthermore, Dirac induction is known to be injective (indeed, bijective), but we recover this injectivity independently as well.

In the last bullet point above, Corollary 5.3 explicitly states that τ_g maps a K-theory class to the value at g of the character of one of the irreducible direct summands of the representation it corresponds to naturally. The values at g of these characters are equal up to a sign, and they add up to zero if that representation is reducible. So the value at g of one of these characters is the most relevant information one could have expected to obtain by applying τ_g. This, to a large extent, answers the question if and what representation theoretic information is contained in classes in $K_0(C_r^*G)$ if $\mathrm{rank}(G) = \mathrm{rank}(K)$, even those not corresponding to the discrete series. In particular, the generator of $K_0(C_r^* \mathrm{SL}(2, \mathbb{R}))$ corresponding to the limits of discrete series determines the characters of these representations on K.

In work in preparation, Higson, Song and Tang compute the values of τ_g on generators of $K_0(C_r^*G)$ independently, without using index theory. This is part of their proof of the Connes–Kasparov conjecture, which states that Dirac induction is bijective.

For a fixed element $x \in K_0(C_r^*G)$, we will see that $\tau_g(x)$ does not depend continuously on g, for example at the identity element e. Theorem 6.2 states that a modified version of τ_g, related to L-packets of representations in the Langlands program, has better continuity properties at e. That implies continuity of certain finite sums of discrete series characters (Corollary 6.3). And that can be used to take the limit as $g \to e$ in Harish-Chandra's character formula for the discrete series to obtain expressions for formal degrees of discrete series representations.

We hope that the various applications of orbital integrals to K-theory of group C^*-algebras in this paper help to demonstrate the relevance of orbital integrals as a tool to study such K-theory groups. In future work, we hope to generalise the results and their applications in this paper to more general groups.

2. Preliminaries

Throughout this paper, let G be a connected semisimple Lie group with finite centre. Let $K < G$ be a maximal compact subgroup. For any Lie group, we denote its Lie algebra by the corresponding gothic letter. Fix a K-invariant inner product on $\mathfrak{g}$, and let $\mathfrak{p} \subset \mathfrak{g}$ be the orthogonal complement to $\mathfrak{k}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

2A. *Dirac induction.* The map $\mathrm{Ad} : K \to \mathrm{SO}(\mathfrak{p})$ lifts to $\widetilde{\mathrm{Ad}} : \widetilde{K} \to \mathrm{Spin}(\mathfrak{p})$, for a double cover $\widetilde{K}$ of K. Let $\Delta_{\mathfrak{p}}$ be the standard representation of $\mathrm{Spin}(\mathfrak{p})$, seen as a representation of $\widetilde{K}$ via $\widetilde{\mathrm{Ad}}$. Let $\hat{K}_{\mathrm{Spin}}$ be the set of irreducible representations V of $\widetilde{K}$ such that $\Delta_{\mathfrak{p}} \otimes V$ descends to a representation of K. Let $R_{\mathrm{Spin}}(K)$ be the free abelian group generated by $\hat{K}_{\mathrm{Spin}}$.

Let $V \in \hat{K}_{\mathrm{Spin}}$. Then we have the G-equivariant vector bundle

$$E_V := G \times_K (\Delta_{\mathfrak{p}} \otimes V) \to G/K.$$

Let $\{X_1, \ldots, X_{\dim(G/K)}\}$ be an orthonormal basis of $\mathfrak{p}$. Let $c_{\mathfrak{p}} : \mathfrak{p} \to \mathrm{End}(\Delta_{\mathfrak{p}})$ be the Clifford action. Let $L : \mathfrak{g} \to \mathrm{End}(C^\infty(G))$ be the infinitesimal left regular representation. Consider the Dirac operator

$$D_V := \sum_{j=1}^{\dim(G/K)} L(X_j) \otimes c_{\mathfrak{p}}(X_j) \otimes 1_V$$

on

$$\Gamma^\infty(E_V) = (C^\infty(G) \otimes \Delta_{\mathfrak{p}} \otimes V)^K.$$

If G/K has a G-invariant Spin-structure (which is the case precisely if $\Delta_{\mathfrak{p}}$ descends to K), then D_V is the Spin-Dirac operator on G/K coupled to the bundle

$G \times_K V \to G/K$; see Proposition 1.1 in [Parthasarathy 1972]. In any case, D_V is a G-equivariant elliptic differential operator, and has an index

$$\text{index}_G(D_V) \in K_*(C_r^*G).$$

Here C_r^*G is the reduced group C^*-algebra of G, and index_G is the analytic assembly map [Baum et al. 1994]. If $\dim(G/K)$ is even, then $\Delta_{\mathfrak{p}}$, and hence E_V, has a natural $\mathbb{Z}_2$-grading with respect to which D_V is odd. Then $\text{index}_G(D_V) \in K_0(C_r^*G)$. If $\dim(G/K)$ is odd, then there is no such grading, and $\text{index}_G(D_V) \in K_1(C_r^*G)$. So in general, we have

$$\text{index}_G(D_V) \in K_{\dim(G/K)}(C_r^*G).$$

Dirac induction is the map

$$\text{D-Ind}_K^G : R_{\text{Spin}}(K) \to K_{\dim(G/K)}(C_r^*G)$$

given by

$$\text{D-Ind}_K^G[V] = \text{index}_G(D_V),$$

with V as above. By the Connes–Kasparov conjecture, proved in [Chabert et al. 2003; Lafforgue 2002b; Wassermann 1987], this map is an isomorphism of abelian groups.

From now on, we suppose that G/K is even-dimensional, since the K-theory group $K_0(C_r^*G)$ we study is zero otherwise.

2B. *Orbital integrals and a fixed point formula.* Let $g \in G$ be a semisimple element. Let $Z_G(g) < G$ be its centraliser. Let $d(hZ_G(g))$ be the left invariant measure on $G/Z_G(g)$ determined by a Haar measure dg on G. The *orbital integral* with respect to g of a measurable function f on G is

$$\tau_g(f) := \int_{G/Z_G(g)} f(hgh^{-1}) \, d(hZ_G(g)),$$

if the integral converges. Harish-Chandra [1966, Theorem 6] proved that the integral converges for f in the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$. The subalgebra $\mathcal{C}(G) \subset C_r^*G$ is dense and closed under holomorphic functional calculus [Hochs and Wang 2018a, Theorem 2.3]. Hence we obtain a map

$$\tau_g : K_0(C_r^*G) = K_0(\mathcal{C}(G)) \to \mathbb{C}.$$

Note that τ_e is the usual von Neumann trace.

Let M be a Riemannian manifold with a proper, isometric, cocompact action by G. Let $E \to M$ be a G-equivariant, Hermitian, $\mathbb{Z}_2$-graded vector bundle. Let D be an odd, self-adjoint, G-equivariant, elliptic differential operator on E. Then we have

$$\text{index}_G(D) \in K_0(C_r^*G).$$

In [Hochs and Wang 2018a], the authors proved a fixed-point formula for the number $\tau_g(\mathrm{index}_G(D))$, for almost all $g \in G$. Consequences include Harish-Chandra's character formula for the discrete series [Harish-Chandra 1966, Theorem 16] (see [Hochs and Wang 2018a, Corollary 2.6]) and Shelstad's character identities in the case of discrete series representations [Shelstad 1979] (see [Hochs and Wang 2018c, Theorem 2.5]). In this paper, we explore further consequences.

To state the fixed point formula in [Hochs and Wang 2018a], let $\mathcal{N} \to M^g$ be the normal bundle to the fixed point set M^g of g in M. Let σ_D be the principal symbol of D. Let $c^g \in C_c(M^g)$ be nonnegative, and such that for all $m \in M^g$,

$$\int_{Z_G(g)} c^g(hm)\, dh = 1,$$

for a fixed Haar measure dh on $Z_G(g)$ compatible with dg and $d(hZ_G(g))$. If G/K is odd-dimensional, then $K_0(C_r^*G) = 0$, so $\tau_g(\mathrm{index}_G(D)) = 0$.

Theorem 2.1. *If G/K is even-dimensional, then for almost all semisimple $g \in G$, we have $\tau_g(\mathrm{index}_G(D)) = 0$ if g is not contained in any compact subgroup of G, and*

$$\tau_g(\mathrm{index}_G(D)) = \int_{TM^g} c^g \frac{\mathrm{ch}\big([\sigma_D|_{\mathrm{supp}(c^g)}](g)\big)\, \mathrm{Todd}(TM^g \otimes \mathbb{C})}{\mathrm{ch}\big([\bigwedge \mathcal{N} \otimes \mathbb{C}](g)\big)} \qquad (2.2)$$

if it is.

Here

$$\mathrm{ch} : K^0(\mathrm{supp}(c^g)) \to H^*(\mathrm{supp}(c^g)),$$

$$\mathrm{ch} : K^0(TM^g|_{\mathrm{supp}(c^g)}) \to H^*(TM^g|_{\mathrm{supp}(c^g)})$$

are Chern characters, and Todd denotes the Todd class.

Remark 2.3. Explicitly, Theorem 2.1 holds for the semisimple $g \in G$ with *finite Gaussian orbital integral* (FGOI) [Hochs and Wang 2018a, Definition 7]. That condition means that the integral

$$\int_{G/Z_G(g)} e^{-d(e, hgh^{-1})^2}\, d(hZ_G(g))$$

converges, where d is the G-invariant Riemannian distance on G. It was shown in [Hochs and Wang 2018a, Proposition 4.2] that almost every element of G has FGOI.

In this paper, whenever a result is stated for almost all g, what is meant is that it holds for semisimple elements with FGOI, and possibly also with dense powers in a maximal torus.

3. A fixed point formula on G/K

Let $T < K$ be a maximal torus. Let $\widetilde{T} < \widetilde{K}$ be its inverse image in $\widetilde{K}$. Fix a set R_c^+ of positive roots of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Let ρ_c be half the sum of the elements of R_c^+. Let $V \in \hat{K}_{\mathrm{Spin}}$. Let $\lambda \in i\mathfrak{t}^*$ be its highest weight with respect to R_c^+.

For any finite-dimensional (actual or virtual) representation W of K or $\widetilde{K}$, we denote its character by χ_W. For any function φ on $\widetilde{K}$ that descends to a function on K, we use the same notation φ for both the function on $\widetilde{K}$ and K. For example, we have $\chi_{\Delta_{\mathfrak{p}}} \chi_V \in C^\infty(K)$.

In the case where T is a Cartan subgroup of G, i.e., $\mathrm{rank}(G) = \mathrm{rank}(K)$, fix a set of positive noncompact roots R_n^+ of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ such that the character $\chi_{\Delta_{\mathfrak{p}}}$ of the graded representation $\Delta_{\mathfrak{p}}$ of $\widetilde{K}$ satisfies

$$\chi_{\Delta_{\mathfrak{p}}}|_{\widetilde{T}} = \prod_{\alpha \in R_n^+} (e^{\alpha/2} - e^{-\alpha/2}). \tag{3.1}$$

Such a choice of positive noncompact roots can always be made; see, for example, [Atiyah and Schmid 1977, pp. 17–18; Parthasarathy 1972, Remark 2.2; Atiyah and Singer 1968, (5.1)]. In the equal-rank case, we write $R^+ := R_c^+ \cup R_n^+$. We denote half the sums of the elements of R^+ and R_n^+ by ρ and ρ_n, respectively.

Let $W_K := N_K(T)/T$ be the Weyl group of (K, T).

Theorem 3.2. (a) *If* $\mathrm{rank}(G) = \mathrm{rank}(K)$, *then for almost all* $g \in T$,

$$\tau_g(\text{D-Ind}_K^G[V]) = (-1)^{\dim(G/K)/2} \frac{\chi_V}{\chi_{\Delta_{\mathfrak{p}}}}(g)$$

$$= (-1)^{\dim(G/K)/2} \frac{\sum_{w \in W_K} \varepsilon(w) e^{w(\lambda + \rho_c)}}{\prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2})}(g).$$

(*In particular, the right-hand sides are well-defined.*)

(b) *If* $\mathrm{rank}(G) \neq \mathrm{rank}(K)$, *then for almost all* $g \in T$,

$$\tau_g(\text{D-Ind}_K^G[V]) = 0.$$

Let $\mathfrak{a} \subset \mathfrak{p}$ be an abelian subspace such that $Z_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t} \oplus \mathfrak{a}$. Let $c \in C_c(\mathfrak{a})$ be a function whose integral over $\mathfrak{a}$ is 1. Let σ_{D_V} be the principal symbol of D_V.

Lemma 3.3. *For almost all* $g \in T$,

$$\tau_g(\text{D-Ind}_K^G[V]) = \int_{T\mathfrak{a}} c \frac{\text{ch}\big([\sigma_{D_V}|_{\text{supp}(c)}](g)\big)}{\text{ch}\big([\mathfrak{a} \times \bigwedge \mathfrak{p}/\mathfrak{a} \otimes \mathbb{C}](g)\big)}.$$

Proof. Let $g \in T$ be such that its powers are dense in T, and with FGOI (see Remark 2.3). By Proposition 4.2 in [Hochs and Wang 2018a], almost all elements of T have these two properties.

We have $G/K \cong \mathfrak{p}$ as K-spaces, and hence in particular as T-spaces. Therefore,

$$(G/K)^g = (G/K)^T = \mathfrak{p}^{\mathrm{Ad}(T)} = \mathfrak{a}.$$

Set $A := \exp(\mathfrak{a})$; this is the centraliser of g in $\exp(\mathfrak{p})$. We have $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}/\mathfrak{a}$ as representations of T. So the normal bundle in $G/K = \mathfrak{p}$ to $(G/K)^g = \mathfrak{a}$ is $\mathfrak{a} \times \mathfrak{p}/\mathfrak{a} \to \mathfrak{a}$. The Todd class of the trivial bundle $T(G/K)^g \otimes \mathbb{C} \to (G/K)^g$ is 1. Hence the claim follows from Theorem 2.1. $\qquad\square$

Let us compute $[\sigma_{D_V}|_{\mathrm{supp}(c)}]$. Let $\beta_{\mathfrak{a}} \in K^0(\mathfrak{a})$ be the Bott generator. (Note that $\mathfrak{a}$ is even-dimensional since G/K is.) Let $\pi : T\mathfrak{a} \to \mathfrak{a}$ be the tangent bundle projection, and $\pi|_{\mathrm{supp}(c)} : \mathrm{supp}(c) \times \mathfrak{a} \to \mathrm{supp}(c)$ its restriction. Note that

$$\Delta_{\mathfrak{p}} \cong \Delta_{\mathfrak{a}} \otimes \Delta_{\mathfrak{p}/\mathfrak{a}}$$

as graded representations of $\widetilde{T}$. These descend to T after tensoring with V.

Lemma 3.4. *Under the isomorphism*

$$K_0^T(\mathrm{supp}(c) \times \mathfrak{a}) \cong K_0(\mathrm{supp}(c) \times \mathfrak{a}) \otimes R(T),$$

we have

$$[\sigma_{D_V}|_{\mathrm{supp}(c)}] \mapsto \pi|_{\mathrm{supp}(c)}^* \beta_{\mathfrak{a}} \otimes [\Delta_{\mathfrak{p}/\mathfrak{a}} \otimes V].$$

Proof. Let $c_{\mathfrak{a}} : \mathfrak{a} \to \mathrm{End}(\Delta_{\mathfrak{a}})$ be the Clifford action. The class

$$\pi|_{\mathrm{supp}(c)}^* \beta_{\mathfrak{a}} \in K^0(\mathrm{supp}(c) \times \mathfrak{a})$$

is defined by[1] the vector bundle homomorphism

$$A : \mathrm{supp}(c) \times \Delta_{\mathfrak{a}}^+ \to \mathrm{supp}(c) \times \Delta_{\mathfrak{a}}^-$$

given by

$$A_Y = c_{\mathfrak{a}}(Y)$$

for all $Y \in \mathrm{supp}(c)$.

We have

$$(G \times_K (\Delta_{\mathfrak{p}}^\pm \otimes V))|_{\mathfrak{a}} \cong \mathfrak{a} \times \Delta_{\mathfrak{p}}^\pm \otimes V$$

as T-vector bundles. So

$$\pi|_{\mathrm{supp}(c)}^*\big((G \times_K (\Delta_{\mathfrak{p}}^\pm \otimes V))|_{\mathrm{supp}(c)}\big) = (\mathrm{supp}(c) \times \mathfrak{a}) \times \Delta_{\mathfrak{p}}^\pm \otimes V.$$

Let $X, Y \in \mathfrak{a}$, so that, using the above identification, we get

$$\sigma_{D_V}(X, Y) = c_{\mathfrak{p}}(Y) \otimes 1_V : \Delta_{\mathfrak{p}}^+ \otimes V \to \Delta_{\mathfrak{p}}^- \otimes V. \qquad (3.5)$$

Since $Y \in \mathfrak{a}$, the map (3.5) equals the odd endomorphism

$$c_{\mathfrak{a}}(Y) \otimes 1_{\Delta_{\mathfrak{p}/\mathfrak{a}} \otimes V} \in \mathrm{End}(\Delta_{\mathfrak{a}} \otimes \Delta_{\mathfrak{p}/\mathfrak{a}} \otimes V).$$

Together with the above form of the class $\pi|_{\mathrm{supp}(c)}^* \beta_{\mathfrak{a}}$, this implies the claim. $\qquad\square$

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[1] We absorb a possible sign in the definition of $\beta_{\mathfrak{a}}$; see [Connes and Moscovici 1982, Lemma 4.1].

Lemma 3.6. *Suppose that* $\mathrm{rank}(G) = \mathrm{rank}(K)$. *Then*

$$\bigwedge \mathfrak{p} \otimes \mathbb{C} = (-1)^{\dim(G/K)/2} \Delta_{\mathfrak{p}} \otimes \Delta_{\mathfrak{p}}$$

as graded representations of T.

Proof. The set of positive noncompact roots R_n^+ determines a complex structure on $\mathfrak{p}$ such that $\mathfrak{p}^{1,0}$ is the sum of the positive noncompact root systems. As graded representations of T, we have

$$\bigwedge \mathfrak{p} \otimes \mathbb{C} = \bigwedge \mathfrak{p}^{1,0} \otimes \bigwedge \mathfrak{p}^{0,1} = \bigwedge_{\mathbb{C}} \mathfrak{p} \otimes (\bigwedge_{\mathbb{C}} \mathfrak{p})^*.$$

The element $\rho_n \in i\mathfrak{t}^*$ is integral for $\widetilde{T}$, and $\Delta_{\mathfrak{p}} \otimes \mathbb{C}_{\rho_n}$ descends to a representation of T. We have

$$\bigwedge_{\mathbb{C}} \mathfrak{p} = (-1)^{\dim(G/K)/2} \Delta_{\mathfrak{p}} \otimes \mathbb{C}_{\rho_n}$$

as graded representations of T; see, for example, the proof of Lemma 5.5 in [Hochs and Wang 2018a]. Since $\Delta_{\mathfrak{p}}^* \cong (-1)^{\dim(G/K)/2} \Delta_{\mathfrak{p}}$, we conclude that

$$\bigwedge \mathfrak{p} \otimes \mathbb{C} = \Delta_{\mathfrak{p}} \otimes \Delta_{\mathfrak{p}}^* = (-1)^{\dim(G/K)/2} \Delta_{\mathfrak{p}} \otimes \Delta_{\mathfrak{p}}.$$

The nontrivial element of the kernel of the covering map $\widetilde{K} \to K$ acts on $\Delta_{\mathfrak{p}}$ as ± 1; therefore, $\Delta_{\mathfrak{p}} \otimes \Delta_{\mathfrak{p}}$ descends to a representation of T. $\square$

Lemma 3.7. *Let* c *be a nonnegative, compactly supported, continuous function on* $\mathbb{R}^{2n}$ *with integral* 1. *Let* $\beta \in K^0(\mathbb{R}^{2n})$ *be the Bott class, and consider*

$$\pi|_{\mathrm{supp}(c)} : \mathrm{supp}(c) \times \mathbb{R}^{2n} \to \mathrm{supp}(c),$$

where $\pi : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *is the natural projection. Then*

$$\int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} c\,\mathrm{ch}(\pi|_{\mathrm{supp}(c)}^* \beta) = 0. \tag{3.8}$$

Proof. By Proposition 6.11 in [Wang 2014], the integral (3.8) equals the L^2-index of the Spin-Dirac operator on $\mathbb{R}^{2n}$. That index is zero because the L^2-kernel of this Dirac operator is zero. Indeed, the Spin-Dirac operator on $\mathbb{R}^{2n}$ only has continuous spectrum; see, for example, Theorem 7.2.1 in [Ginoux 2009]. $\square$

Proof of Theorem 3.2. Lemma 3.4 implies that

$$\mathrm{ch}\big([\sigma_{D_V}|_{\mathrm{supp}(c)}](g)\big) = \mathrm{ch}(\pi|_{\mathrm{supp}(c)}^* \beta_{\mathfrak{a}})(\chi_{\Delta_{\mathfrak{p}/\mathfrak{a}}} \chi_V)(g).$$

Furthermore,

$$\mathrm{ch}\big([\mathfrak{a} \times \bigwedge \mathfrak{p}/\mathfrak{a} \otimes \mathbb{C}](g)\big) = \chi_{\bigwedge \mathfrak{p}/\mathfrak{a} \otimes \mathbb{C}}(g)$$

in the graded sense. So by Lemma 3.3,

$$\tau_g(\text{D-Ind}_K^G[V]) = \frac{\chi_{\Delta_{\mathfrak{p}/\mathfrak{a}}} \chi_V}{\chi_{\bigwedge \mathfrak{p}/\mathfrak{a} \otimes \mathbb{C}}}(g) \int_{T\mathfrak{a}} c\,\mathrm{ch}(\pi|_{\mathrm{supp}(c)}^* \beta_{\mathfrak{a}}).$$

If $\mathrm{rank}(G) \neq \mathrm{rank}(K)$, then $\mathfrak{a}$ is nonzero, and the claim follows from Lemma 3.7. If $\mathrm{rank}(G) = \mathrm{rank}(K)$, then Lemma 3.6 implies that

$$\tau_g(\text{D-Ind}_K^G[V]) = (-1)^{\dim(G/K)/2} \frac{\chi_V}{\chi_{\Delta_\mathfrak{p}}}(g);$$

in particular, the right-hand side is well-defined. The claim now follows from Weyl's character formula and (3.1). (Note that $(\widetilde{K}, \widetilde{T})$ and (K, T) have the same Weyl group W_K, since they have the same root system.) $\qquad\square$

Remark 3.9. If $g = e$, then $\tau_e(\text{D-Ind}_K^G[V])$ is the L^2-index of D_V by Proposition 4.4 in [Wang 2014]. That index is zero if the kernel of D_V is zero. Theorem 3.2 shows that, in the equal-rank case, the more general trace τ_g yields nonzero information even in cases where the kernel of D_V is zero (see also Section 5B).

4. Consequences

Suppose from now on that $\mathrm{rank}(G) = \mathrm{rank}(K)$.

4A. *Distinguishing K-theory classes.* As a consequence of Theorem 3.2, the traces τ_g "separate points" on $K_0(C_r^*G)$, or distinguish all elements of $K_0(C_r^*G)$, in the following sense.

Corollary 4.1. *Let $x \in K_0(C_r^*G)$. If $\tau_g(x) = 0$ for all g in a dense subset of T, then $x = 0$.*

Proof. Let $x \in K_0(C_r^*G)$. By surjectivity of Dirac induction, we can write

$$x = \sum_{V \in \hat{K}_{\mathrm{Spin}}} m_V \, \text{D-Ind}_K^G[V],$$

for $m_V \in \mathbb{Z}$, finitely many nonzero. By Theorem 3.2, we have for almost all $g \in T$,

$$\tau_g(x) = (-1)^{\dim(G/K)/2} \sum_{V \in \hat{K}_{\mathrm{Spin}}} m_V \frac{\chi_V}{\chi_{\Delta_\mathfrak{p}}}(g).$$

So if $\tau_g(x) = 0$ for all g in a dense subset of T, then by continuity and conjugation invariance of the characters χ_V, we find that

$$\sum_{V \in \hat{K}_{\mathrm{Spin}}} m_V \chi_V = 0.$$

So $m_V = 0$ for all V, i.e., $x = 0$. $\qquad\square$

4B. *K-theory and distributions.* Let $G^{\mathrm{reg}} \subset G$ be the subset of regular elements.

Corollary 4.2. *The map*

$$\tau : K_0(C_r^*G) \to \mathcal{D}'(G^{\mathrm{reg}})$$

defined by

$$\langle \tau(x), f \rangle = \int_{G^{\mathrm{reg}}} \tau_g(x) f(g) \, dg$$

for $x \in K_0(C_r^* G)$ and $f \in C_c^\infty(G^{\mathrm{reg}})$, is a well-defined, injective group homomorphism.

Proof. Let $x \in K_0(C_r^* G)$. By the surjectivity of Dirac induction, we can write $x = \text{D-Ind}_K^G[y]$, for some $y \in R_{\mathrm{Spin}}(K)$. Theorem 3.2 implies that the function $g \mapsto \tau_g(x)$ equals an analytic function almost everywhere on the set of elliptic elements of G. Theorem 2.1 implies that this function equals zero almost everywhere on the set of nonelliptic elements of G. So $g \mapsto \tau_g(x)$ equals an analytic function almost everywhere on G. Furthermore, that analytic function is bounded on compact subsets of G^{reg}. This implies that $\tau(x)$ is a well-defined distribution on G^{reg}.

If $\tau(x) = 0$, then $\tau_g(x) = 0$ for almost all $g \in G^{\mathrm{reg}}$, in particular for almost all elements of T. Hence Corollary 4.1 implies that $x = 0$. $\qquad\square$

Remark 4.3. As noted in the proof of Corollary 4.2, the first part of Theorem 2.1 implies that $\tau(x)$ is zero outside the set of regular elliptic elements of G.

Remark 4.4. We describe the map τ in Corollary 4.2 explicitly in terms of characters of representations in Section 5. There we see that $\tau(x)$ equals the character of a tempered representation of G almost everywhere on the set of regular elliptic elements, and zero almost everywhere outside the set of elliptic elements. Therefore, it extends to a distribution on all of G by Harish-Chandra's regularity theorem.

4C. *Injectivity of Dirac induction.* We have used the surjectivity of Dirac induction in the proof of Corollary 4.1 (which is justified because the Connes–Kasparov conjecture has been proved). Theorem 3.2 implies injectivity of Dirac induction.

Corollary 4.5. *Dirac induction is injective.*

Proof. Let $y \in R_{\mathrm{Spin}}(K)$, and suppose that $\text{D-Ind}_K^G(y) = 0$. Then $\tau_g(\text{D-Ind}_K^G(y)) = 0$ for all $g \in T$. Theorem 3.2 implies that for almost all $g \in T$,

$$\frac{\chi_y}{\chi_{\Delta_{\mathrm{p}}}}(g) = 0.$$

So $\chi_y = 0$, i.e., $y = 0$. $\qquad\square$

4D. *An induction formula.* Let M be an even-dimensional Riemannian manifold with a G-equivariant Spin^c-structure. Let $E \to M$ be a G-equivariant, Hermitian vector bundle. Let D_M^E be the Spin^c-Dirac operator on M twisted by E. By Abels' theorem [1974], there is a K-invariant submanifold $N \subset M$ such that $M \cong G \times_N N$ via the action map $G \times N \to M$. Furthermore, N has a K-equivariant Spin^c-structure on N compatible with the one on M; see Proposition 3.10 in [Hochs

and Mathai 2017]. The Spin^c-Dirac operator D_N^E on N, twisted by $E|_N$, has the property that

$$\text{D-Ind}_K^G(\text{index}_K(D_N^E)) = \text{index}_G(D_M^G) \in K_0(C_r^*G). \tag{4.6}$$

See Theorem 5.2 in [Hochs and Wang 2018a] and Proposition 4.7 in [Hochs 2009].

Theorem 3.2 and surjectivity of Dirac induction imply that the following diagram commutes for all g in the dense subset of T in Theorem 3.2:

$$
\begin{array}{c}
K_0(C_r^*G) \\
\text{D-Ind}_K^G \uparrow \qquad \searrow^{\tau_g} \\
R_{\text{Spin}}(K) \xrightarrow[(-1)^{\dim(G/K)/2}\,\text{ev}_g\,/\chi_{\Delta_\mathfrak{p}}(g)]{} \mathbb{C}
\end{array}
\tag{4.7}
$$

Here ev_g denotes evaluation of characters of representations at g; note that the bottom arrow is well-defined.

The equality (4.6) and commutativity of (4.7) imply the following formula for induction from slices.

Corollary 4.8. *We have, for almost all $g \in T$,*

$$\tau_g(\text{index}_G(D_M^E)) = (-1)^{\dim(G/K)/2}\,\text{index}_K(D_N^E)(g)/\chi_{\Delta_\mathfrak{p}}(g).$$

Note that the right-hand side can be computed via the Atiyah–Segal–Singer fixed point formula [Atiyah and Segal 1968].

Induction formulas like Corollary 4.8 we used in various settings to deduce results about G-equivariant indices from results about K-equivariant indices [Guo et al. 2018; Hochs 2009; Hochs and Mathai 2016; Hochs and Mathai 2017; Hochs and Wang 2018a]. The case $g = e$ is not covered by Corollary 4.8; that case is Corollary 53 in [Guo et al. 2018].

4E. *Selberg's principle.* The Selberg principle is a vanishing result for orbital integrals of certain convolution idempotents on G. See [Blanc and Brylinski 1992; Julg and Valette 1986; 1987] for approaches to this principle in the spirit of noncommutative geometry. Theorem 2.1 implies a version of this principle.

Corollary 4.9 (*K*-theoretic Selberg principle). *For almost all g not contained in compact subgroups of G, the map*

$$\tau_g : K_0(C_r^*G) \to \mathbb{C}$$

is zero.

Proof. Theorem 2.1 implies that for almost all g not contained in compact subgroups of G, and all $V \in R_{\text{Spin}}(K)$, we have

$$\tau_g(\text{D-Ind}_K^G[V]) = 0.$$

So surjectivity of Dirac induction implies the claim.   □

Corollary 4.9 has a purely representation theoretic consequence.

Corollary 4.10 (Selberg principle for matrix coefficients of the discrete series).
*Let π be a discrete series representation of G. Let v be a K-finite vector in the
representation space of π, and $m_{v,v}$ the corresponding matrix coefficient. For all
g not contained in compact subgroups of G, we have*

$$\tau_g(m_{v,v}) = 0.$$

Proof. Let d_π be the formal degree of π. By rescaling, we may assume that v
has norm 1. Then $d_\pi \overline{m}_{v,v}$ is an idempotent in $C_r^* G$. Let $[\pi] \in K_0(C_r^* G)$ be its
K-theory class. Since v is *K*-finite, the function $m_{v,v}$ lies in Harish-Chandra's
Schwartz algebra $\mathcal{C}(G)$. Therefore, for all semisimple $g \in G$,

$$\tau_g(m_{v,v}) = \frac{1}{d_\pi} \overline{\tau_g([\pi])}.$$

By Corollary 4.9, the number is zero for almost all g not contained in compact
subgroups. The claim therefore follows by continuity of $m_{v,v}$. $\qquad\square$

4F. *A Tannaka-type duality.* We now suppose that the representation $\Delta_\mathfrak{p}$ of $\widetilde{K}$
descends to K. This is true if we replace G by a double cover if necessary. Then
Dirac induction is defined on $R(K)$.

The *K*-theory group $K_0(C_r^* G)$ and its elements contain nontrivial information
about G and its representations; see, e.g., [Hochs and Wang 2018a; 2018c; Laf-
forgue 2002b]. But just the isomorphism class of $K_0(C_r^* G)$ as an abelian group con-
tains no information about G whatsoever: this group is always free, with countably
infinitely many generators. It turns out, however, that the combination of the iso-
morphism class of $K_0(C_r^* G)$, the topological space T and the maps $\tau_g \colon K_0(C_r^* G) \to \mathbb{C}$,
for g in a dense subset of T, together determine the Cartan motion group $K \ltimes \mathfrak{p}$ and
vice versa. The tempered representation theory of $K \ltimes \mathfrak{p}$ is closely related to that
of G; this is the Mackey analogy [Afgoustidis 2015; Higson 2008; 2011; Mackey
1975; Tan et al. 2017; Yu 2017]. Also, the analytic assembly map for G can be
defined in terms of a continuous deformation from $K \ltimes \mathfrak{p}$ to G; see pp. 23–24 of
[Baum et al. 1994] and [Higson 2008].

This is vaguely analogous to the fact that the irrational rotation algebras A_λ, for
irrational λ in $\left[0, \frac{1}{2}\right]$, have the same *K*-theory $\mathbb{Z} \oplus \mathbb{Z}$, but are determined up to
isomorphism by the pair $(K_0(A_\lambda), \tau)$, where τ is a natural trace. This is because
the image of τ is $\mathbb{Z} + \lambda \mathbb{Z}$.

Corollary 4.11. *The*

- *abelian group $K_0(C_r^* G)$ up to isomorphism,*

- *pointed topological space $(T, \{e\})$ up to homeomorphism, and*

- *family of group homomorphisms* $\tau_g : K_0(C_r^*G) \to \mathbb{C}$, *for g in a dense sub-set of T*,

together determine the Cartan motion group $K \ltimes \mathfrak{p}$, and vice versa.

Proof. Write

$$K_0(C_r^*G) = \bigoplus_{j \in \mathbb{Z}} \mathbb{Z}$$

and let e_j be a generator of the j-th copy of $\mathbb{Z}$. Let S be the intersection of the dense subset of T in the third point in the corollary and the set of $g \in T$ for which the formula in Theorem 3.2 holds. Then S is dense in T.

Consider the function $\chi_j : S \to \mathbb{R}$ given by $\chi_j(g) = \tau_g(e_j)$. By Theorem 3.2, there is a function $\psi \in C^\infty(T)$, not unique but independent of j, and there are uniquely determined integers d_j such that for all j,

$$\lim_{g \to e} \psi(g) \chi_j(g) = d_j,$$

where at least one of the integers d_j equals 1. (Indeed, take $\psi = \chi_{\Delta_\mathfrak{p}}|_S$ and d_j plus or minus the dimensions of the irreducible representations of K.) By replacing e_j by $-e_j$ where necessary, we can make sure that all integers d_j are positive.

Fix $j_0 \in \mathbb{Z}$ such that $d_{j_0} = 1$. Then, again by Theorem 3.2,

$$\left| \chi_{\Delta_\mathfrak{p}}|_S \right| = |\chi_{j_0}|^{-1}.$$

And $\overline{\chi_{\Delta_\mathfrak{p}}} = -\chi_{\Delta_\mathfrak{p}}$, so $\chi_{\Delta_\mathfrak{p}}$ is imaginary-valued. Hence

$$\chi_{\Delta_\mathfrak{p}}|_S = \pm i |\chi_{j_0}|^{-1}.$$

We cannot resolve the sign ambiguity with the data we have, but we do not need to.

The characters of irreducible representations V_j of K are continuous and conjugation invariant, so they are determined by

$$\chi_{V_j}|_S = (\chi_{\Delta_\mathfrak{p}}|_S) \chi_j = \pm i |\chi_{j_0}|^{-1} \chi_j,$$

with the sign chosen such that $\pm i |\chi_{j_0}|^{-1} \chi_j > 0$ near the identity element. This determines the representations V_j of K, and their tensor products and the underlying vector spaces. By Tannaka duality [Tannaka 1938], this determines K.

To recover $\mathfrak{p}$ as a K-representation, set $\psi := i |\chi_{j_0}|^{-1}$, extended continuously to T. Then

$$\psi = \pm \chi_{\Delta_\mathfrak{p}}|_T = \pm \prod_{\alpha \in R_n^+} (e^{\alpha/2} - e^{-\alpha/2}).$$

This implies that for all $X, Y \in \mathfrak{t}$,

$$\left. \frac{d}{dt} \right|_{t=0} \psi(X + tY) = \psi(\exp(X)) \sum_{\alpha \in R_n^+} \frac{\langle \alpha, Y \rangle}{2} \coth(\langle \alpha, X \rangle / 2).$$

The term on the right-hand side corresponding to α equals the same term with α replaced by $-\alpha$. But otherwise this expression determines the weights α up to signs. In this way, we recover the set R_n of $\mathfrak{t}$-weights of $\mathfrak{p} \otimes \mathbb{C}$ as a complex representation of T, and hence $\mathfrak{p}$ as a real representation of T, and therefore as a representation of K. This determines $K \ltimes \mathfrak{p}$.

Conversely, the Cartan motion group $K \ltimes \mathfrak{p}$ determines its maximal compact subgroup K and the quotient $\mathfrak{p} = (K \ltimes \mathfrak{p})/K$ as a representation of K. And K determines the pair $(T, \{e\})$ up to conjugacy. The K-theory group $K_0(C_r^* G)$ is isomorphic to $R(K)$ via Dirac induction. Furthermore, K and $\mathfrak{p}$ determine the characters χ_V, for $V \in \hat{K}$ and $\chi_{\Delta_{\mathfrak{p}}}$, and the dimension $\dim(G/K) = \dim(\mathfrak{p})$. Hence, by Theorem 3.2, this determines the maps $\tau_g : K_0(C_r^* G) \cong R(K) \to \mathbb{C}$, for g in a dense subset of T. $\qquad\qquad\square$

Remark 4.12. In Corollary 4.11, one only needs the neighbourhoods of the identity element, not all of its topology. And as stated in the corollary, one does not need the group structure of T.

Remark 4.13. If $G = K$ is compact, then the triple

$$(K_0(C_r^* G), (T, \{e\}), (\tau_g)_{g \in T})$$

determines the ring $R(G)$ of characters of G. That in turn determines the tensor products of representations of G, and forgetful maps to finite-dimensional complex vector spaces. So in this case, Corollary 4.11 reduces to Tannaka duality for compact groups [Tannaka 1938] (which was used in the proof of Corollary 4.11).

Remark 4.14. If the representation $\Delta_{\mathfrak{p}}$ of $\widetilde{K}$ does not descend to K, then we only recover the ring $R_{\mathrm{Spin}}(K)$ in the proof of Corollary 4.11 and cannot directly apply Tannaka duality.

5. Characters

Again, we suppose that the representation $\Delta_{\mathfrak{p}}$ of $\widetilde{K}$ descends to K. We may need to replace G by a double cover for this assumption to hold. This assumption is now not essential; see Remark 5.4.

5A. *Characters and τ_g.* The structure of the C^*-algebra $C_r^* G$ and its K-theory was described by Wassermann [1987] and Clare, Crisp and Higson [Clare et al. 2016]. We can use this to relate values of τ_g on K-theory classes to values of characters of representations.

Let $P = MAN < G$ be a cuspidal parabolic and σ in the set $\hat{M}_{\mathrm{ds}}$ of discrete series representations of M. Consider the bundle of Hilbert spaces $\mathcal{E}_{P,\sigma} \to \hat{A}$ whose fibre at $\nu \in \hat{A}$ is $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$. (This can be topologised by viewing it as a trivial bundle in the compact picture of induced representations.) Let $\mathrm{Ind}_P^G(\sigma)$

be the Hilbert $C_0(\hat{A})$-module of continuous sections of $\mathcal{E}_{P,\sigma}$ vanishing at infinity. The group

$$W_\sigma := \{w \in N_K(\mathfrak{a})/Z_K(\mathfrak{a}); w\sigma = \sigma\}$$

acts on $\mathcal{K}(\mathrm{Ind}_P^G(\sigma))$ via Knapp–Stein intertwiners; see Theorem 6.1 in [Clare et al. 2016]. Let $\mathcal{K}(\mathrm{Ind}_P^G(\sigma))^{W_\sigma}$ be the fixed point algebra of this action. Then

$$C_r^* G \cong \bigoplus_{P,\sigma} \mathcal{K}(\mathrm{Ind}_P^G(\sigma))^{W_\sigma},$$

where the sum runs over a set of cuspidal parabolics $P = MAN$ and $\sigma \in \hat{M}_{\mathrm{ds}}$. This is [Clare et al. 2016, Theorem 6.8]. See also Theorem 8 in [Wassermann 1987].

Now let P and σ be such that

$$K_0(\mathcal{K}(\mathrm{Ind}_P^G(\sigma))^{W_\sigma})$$

is nonzero, hence infinite cyclic. (This is equivalent to the condition that W_σ equals the R-group R_σ; see Lemma 10 in [Wassermann 1987].) Let $b(P,\sigma) \in K_0(C_r^* G)$ be the generator of this summand of $K_0(C_r^* G)$ in the image under Dirac induction of the $\mathbb{Z}_{\geq 1}$-span of $\hat{K}$ inside $R(K)$.

Let $\eta \in i\mathfrak{t}_M^*$ be the Harish-Chandra parameter of σ, and $\tilde{\eta} \in i\mathfrak{t}^*$ its extension by zero on the orthogonal complement of $\mathfrak{t}_M$ in $\mathfrak{t}$. For any positive root system $\widetilde{R}^+$ of $(\mathfrak{g}^\mathbb{C}, \mathfrak{t}^\mathbb{C})$ for which $\tilde{\eta}$ is dominant, let $\pi^G(\tilde{\eta}, \widetilde{R}^+)$ be the corresponding (limit of) discrete series representation of G. We need the following version of Schmid's character identities. This is Lemma 12 in [Wassermann 1987] in the equal rank case, but with information included about the infinitesimal characters of the limits of discrete series representations that occur.

Proposition 5.1. *There are $2^{\dim(A)}$ choices of positive roots $R_1^+, \ldots, R_{2^{\dim(A)}}^+ \subset R$, obtained from R^+ by the application of all combinations of $\dim(A)$ commuting reflections in simple noncompact roots, such that*

$$\mathrm{Ind}_P^G(\sigma \otimes 1_A \otimes 1_N) = \bigoplus_{j=1}^{2^{\dim(A)}} \pi^G(\tilde{\eta}, R_j^+).$$

Proof. This is a special case of Theorem 13.3 in [Knapp and Zuckerman 1982] for the maximal parabolic G in the equal-rank group G. □

As before, let ρ_c be half the sum of the compact positive roots. By Lemma 15(i) in [Wassermann 1987], the element $\tilde{\eta} - \rho_c$ is dominant for K. It is integral because $\Delta_\mathfrak{p}$ descends to K; this implies that ρ_n and hence $\tilde{\eta} - \rho + \rho_n$ is integral.

Proposition 5.2 (Wassermann). *Let $V_{\tilde{\eta}-\rho_c} \in \hat{K}$ have highest weight $\tilde{\eta} - \rho_c$. Then*

$$\mathrm{D\text{-}Ind}_K^G[V_{\tilde{\eta}-\rho_c}] = b(P,\sigma).$$

Proof. See the last page of [Wassermann 1987]. This uses Proposition 5.1. $\square$

Proposition 5.1 and Harish-Chandra's character formula for (limits of) discrete series representations imply that the character of the representation $\mathrm{Ind}_P^G(\sigma \otimes 1_A \otimes 1_N)$ naturally associated to the K-theory generator $b(P, \sigma)$ is zero on T, if this representation is reducible. (See Section 5B for an example.) Therefore, it is a useful property of the map τ_g that it maps $b(P, \sigma)$ to the possibly nonzero value of an irreducible summand of that representation.

Corollary 5.3. *For almost all $g \in T$, $\tau_g(b(P, \sigma))$ equals the value at g of the character of one of the irreducible summands of $\mathrm{Ind}_P^G(\sigma \otimes 1_A \otimes 1_N)$. The values at g of the characters of these summands at g are all equal up to a sign.*

Proof. Proposition 5.2 and Theorem 3.2 imply that

$$\tau_g(b(P, \sigma)) = \tau_g(\text{D-Ind}_K^G[V_{\tilde{\eta}-\rho_c}]) = (-1)^{\dim(G/K)/2} \frac{\sum_{w \in W_K} \varepsilon(w) e^{w\tilde{\eta}}}{\prod_{\alpha \in R^+}(e^{\alpha/2} - e^{-\alpha/2})}(g).$$

By Harish-Chandra's character formula (extended coherently to the limits of discrete series), the right-hand side is the value at g of the character of $\pi^G(\tilde{\eta}, R^+)$. That formula also shows that on T, the character of $\pi^G(\tilde{\eta}, R^+)$ equals the character of $\pi^G(\tilde{\eta}, R_j^+)$ modulo a sign, for $j = 1, \ldots, 2^{\dim(A)}$. Hence the claim follows from Proposition 5.1. $\square$

Remark 5.4. If the representation $\Delta_{\mathfrak{p}}$ does not descend to K, then the analogue of Corollary 5.3 relates $\tau_g(b(P, \sigma))$ to characters of the corresponding representations of a double cover of G.

5B. *Nonspherical principal series and limits of discrete series of* **SL$(2, \mathbb{R})$.** Consider the case where $G = \mathrm{SL}(2, \mathbb{R})$, $K = T = \mathrm{SO}(2)$, and $P = MAN < \mathrm{SL}(2, \mathbb{R})$ is the minimal parabolic of upper triangular matrices, where $M = \{\pm I\}$. Then $\hat{M}_{\mathrm{ds}} = \{\sigma_+, \sigma_-\}$, where σ_+ is the trivial representation of M in $\mathbb{C}$ and σ_- is the nontrivial one. Now we have Morita equivalences

$$\mathcal{K}(\mathrm{Ind}_P^G(\sigma_+))^{W_{\sigma_+}} \sim C_0([0, \infty)),$$

$$\mathcal{K}(\mathrm{Ind}_P^G(\sigma_-))^{W_{\sigma_-}} \sim C_0(\mathbb{R}) \rtimes \mathbb{Z}_2.$$

See Example 6.11 in [Clare et al. 2016]. So the pair (P, σ_+) does not contribute to $K_0(C_r^*(\mathrm{SL}(2, \mathbb{R})))$, whereas (P, σ_-) contributes a summand $\mathbb{Z}$, generated by

$$b(P, \sigma_-) = \text{D-Ind}_K^G[\mathbb{C}_0].$$

Let $\alpha \in i\mathfrak{t}^*$ be the root mapping $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to $2i$. Set $R^+ := \{\alpha\}$. Let

$$g = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \in T,$$

where $\varphi \in \mathbb{R} \setminus 2\pi\mathbb{Q}$. Theorem 3.2 now yields

$$\tau_g(b(P, \sigma_-)) = \frac{1}{2i \sin \varphi}.$$

This is the value at g of the character of the limit of discrete series representation $\pi^G(0, R^+)$, and minus the value at g of the character of the limit of discrete series representation $\pi^G(0, -R^+)$. The direct sum of these two representations is the nonspherical principal series representation $\mathrm{Ind}_P^G(\sigma_- \otimes 1_A \otimes 1_N)$. The character of that representation is zero at g.

Some authors, including the authors of this paper, have wondered if the K-theory generator $b(P, \sigma_-)$ can be detected by suitable maps out of $K_0(C_r^*(\mathrm{SL}(2, \mathbb{R})))$, and if representation theoretic information can be recovered from it. This example shows that the answer to both questions is yes.

6. Stable orbital integrals and continuity at the group identity

This section is independent of the rest of this paper. In particular, it does not depend on Theorem 3.2.

It follows from Theorem 3.2 that, for a fixed $x \in K_0(C_r^*G)$, the function

$$g \mapsto \tau_g(x)$$

on the set of semisimple elements g of G is not continuous if G is noncompact. In particular, it is not continuous at the identity element. Theorem 3.2 does imply that this function is continuous almost everywhere. Already in the compact case, it is a nontrivial question if the right-hand side of the fixed point formula (2.2) depends continuously on g, for example as $g \to e$ (as pointed out in Section 8.1 in [Berline et al. 2004]). It turns out that a version of τ_g involving *stable orbital integrals* has better continuity properties near the identity element. (This comes at the cost of mapping more elements to zero, however. See Section 5B, where the stable orbital integral of the class in $K_0(C_r^* \mathrm{SL}(2, \mathbb{R}))$ associated to the limits of discrete series is shown to be zero.)

6A. *Continuity at e.* Let $G_{\mathbb{C}}$ be a complex semisimple Lie group, and $G < G_{\mathbb{C}}$ a real form of $G_{\mathbb{C}}$. Let g be a semisimple element of G.

Definition 6.1. The *stable conjugacy class* of g in G is

$$(g)_s := \{hgh^{-1} \in G : h \in G_{\mathbb{C}}\},$$

the intersection of the conjugacy class $(g)_{G_{\mathbb{C}}}$ of g in $G_{\mathbb{C}}$ with G.

For every f in the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$, the *stable orbital integral* of f with respect to g is

$$\tau_g^s(f) := \sum_{g'} \tau_{g'}(f) = \sum_{g'} \int_{G/Z_G(g')} f(hg'h^{-1}) \, dh(Z_G(g')),$$

where the sum is over representatives g' of G-conjugacy classes in $(g)_s$, i.e., $(g)_s = \bigsqcup_{g'}(g')$.

Stable conjugacy classes are relevant to the notion of an L-packet of representations and Shelstad's character identities. See [Shelstad 1979].

The map $\tau_g^s : K_0(C_r^*G) = K_0(\mathcal{C}(G)) \to \mathbb{C}$ induced by τ_g^s has better continuity properties in g than τ_g. Let $S \subset G$ be the set of elements g for which Theorem 2.1 holds (see Remark 2.3). Then $G \setminus S$ has measure zero, so in particular S is dense.

Theorem 6.2. *For all $x \in K_0(C_r^*G)$,*

$$\lim_{g \to e; g \in S} \tau_g^s(x) = \tau_e(x).$$

(Note that $\tau_e = \tau_e^s$.)

Let $K < G$ be maximal compact. If $\mathrm{rank}(G) \neq \mathrm{rank}(K)$, then Theorem 6.2 follows from Theorem 3.2(b) and the fact that τ_e is identically zero on $K_0(C_r^*G)$. So assume from now on that $\mathrm{rank}(G) = \mathrm{rank}(K)$.

Theorem 6.2 implies a continuity property of characters of L-packets of discrete series representations.

As before, let $T < K$ be a maximal torus, and set $W_K := N_K(T)/T$. Let W_G be the Weyl group of the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Fix representatives $w \in W_G$ of all classes $[w] \in W_G/W_K$. For any discrete series representation with Harish-Chandra parameter λ, we denote its global character by Θ_λ.

Corollary 6.3. *Let π be a discrete series representation of G with Harish-Chandra parameter $\lambda \in i\mathfrak{t}^*$. Then*

$$\lim_{g \to e; g \in T^{\mathrm{reg}}} \sum_{[w] \in W_G/W_K} \Theta_{w\lambda}(g) = d_\pi,$$

where d_π is the formal degree of π.

This corollary will be proved after we prove Theorem 6.2. As a consequence, one can take the limit as $g \to e$ in Harish-Chandra's character formula to obtain an expression for d_π; see, e.g., page 25 of [Atiyah and Schmid 1977]. See also Proposition 50 in [Guo et al. 2018].

6B. *A K-theoretic character identity.* Let G_c be a compact inner form of G, which exists because $\mathrm{rank}(G) = \mathrm{rank}(K)$. Inner forms are defined for example in Chapter 2 of [Adams et al. 1992], but the only properties we need are that G_c is a real form of $G_{\mathbb{C}}$, and T identifies with a Cartan subgroup of G_c. So pairs (G, T) and (G_c, T) have the same root system. The positive root system R^+ determines a G-invariant complex structure on G_c/T. For any integral $\nu \in i\mathfrak{t}^*$, consider the holomorphic line bundles

$$L_\nu^G := G \times_T \mathbb{C}_\nu \to G/T,$$
$$L_\nu^{G_c} := G_c \times_T \mathbb{C}_\nu \to G_c/T.$$

Let $\bar{\partial}_{L_\nu^G}$ and $\bar{\partial}_{L_\nu^{G_c}}$ be the Dolbeault operators on G/T and G_c/T, respectively, coupled to these line bundles.

In [Hochs and Wang 2018c], the authors prove a K-theoretic analogue of Shelstad's character identities [Shelstad 1979], and deduce Shelstad's character identity in the case of the discrete series.

Theorem 6.4. *For all integral $\nu \in i\mathfrak{t}^*$ and all $g \in S$,*

$$\tau_g\big(\mathrm{index}_{G_c}\big(\bar{\partial}_{L_\nu^{G_c}} + \bar{\partial}^*_{L_\nu^{G_c}}\big)\big) = \sum_{[w] \in W_G/W_K} \tau_g\big(\mathrm{index}_G\big(\bar{\partial}_{L_{w^{-1}\nu}^G} + \bar{\partial}^*_{L_{w^{-1}\nu}^G}\big)\big).$$

Proof. This is (3.6) in [Hochs and Wang 2018c]. There, ν is regular but that property is not used in the proof of the above equality. $\square$

6C. *Dolbeault operators.* We will use some properties of the Dolbeault–Dirac operators in Theorem 6.4 to deduce Theorem 6.2.

First of all, every element of $K_0(C_r^*G)$ is the index of a Dolbeault–Dirac operator on G/T. Indeed, let $V \in \hat{K}_{\mathrm{Spin}}$, and let $\lambda \in i\mathfrak{t}^*$ be its highest weight with respect to the positive compact roots chosen earlier. Then $\lambda - \rho_n$ is a weight of $\Delta_\mathfrak{p} \otimes V$, so it is integral for T. Consider the holomorphic, G-equivariant line bundle

$$L_{\lambda-\rho_n}^G := G \times_T \mathbb{C}_{\lambda-\rho_n} \to G/T.$$

Let $\bar{\partial}_{L_{\lambda-\rho_n}^G}$ be the Dolbeault operator on G/T coupled to $L_{\lambda-\rho_n}^G$.

Proposition 6.5. *We have*

$$\mathrm{D\text{-}Ind}_K^G[V_\lambda] = (-1)^{\dim(G/K)} \mathrm{index}_G\big(\bar{\partial}_{L_{\lambda-\rho_n}^G} + \bar{\partial}^*_{L_{\lambda-\rho_n}^G}\big).$$

Proof. This is proved in Section 5 of [Hochs and Wang 2018b] in the case where $\lambda + \rho_c$ is regular for G, but that assumption is not necessary for the arguments. $\square$

Lemma 6.6. *We have, for all $w \in W_G$ and all $g \in S$,*

$$\tau_{wgw^{-1}}\big(\mathrm{index}_G\big(\bar{\partial}_{L_{\lambda-\rho}^G} + \bar{\partial}^*_{L_{\lambda-\rho}^G}\big)\big) = \tau_g\big(\mathrm{index}_G\big(\bar{\partial}_{L_{w^{-1}(\lambda-\rho)}^G} + \bar{\partial}^*_{L_{w^{-1}(\lambda-\rho)}^G}\big)\big).$$

Proof. In the case of Dolbeault operators twisted by holomorphic vector bundles, and finite fixed point sets, the fixed point formula in Theorem 2.1 simplifies considerably; see Corollary 6.3 in [Hochs and Wang 2018b]. For any $h \in T$ with dense powers, and any integral $\nu \in i\mathfrak{t}^*$, this yields

$$\tau_h\big(\mathrm{index}_G\big(\bar{\partial}_{L_\nu^G} + \bar{\partial}^*_{L_\nu^G}\big)\big) = \sum_{xT \in (G/T)^h} \frac{\mathrm{tr}(g|_{(L_\nu^G)_{xT}})}{\det_{\mathbb{C}}(1 - g^{-1}|_{T_{xT}G/T})}. \qquad (6.7)$$

Now, for $w \in W_G$, we have $(G/T)^{wgw^{-1}} = (G/T)^T = N_K(T)/T$, and for $x \in N_K(T)$,

$$(L_\nu^G)_{xT} = \mathbb{C}_{\mathrm{Ad}^*(x)\nu}, \qquad T_{xT}G/T = \bigoplus_{\alpha \in R^+} \mathbb{C}_{\mathrm{Ad}^*(x)\alpha}$$

as complex representations of T, where we use the complex structure on G/T defined by R^+. So

$$\tau_{wgw^{-1}}\left(\mathrm{index}_G\left(\bar{\partial}_{L_{\lambda-\rho}^G} + \bar{\partial}_{L_{\lambda-\rho}^G}^*\right)\right)$$

$$= \sum_{xT \in N_K(T)/T} \frac{\mathrm{tr}(wgw^{-1}|_{\mathbb{C}_{\mathrm{Ad}^*(x)(\lambda-\rho)}})}{\det_{\mathbb{C}}(1 - wg^{-1}w^{-1}|_{\bigoplus_{\alpha \in R^+} \mathbb{C}_{\mathrm{Ad}^*(x)\alpha}})}$$

$$= \sum_{xT \in N_K(T)/T} \frac{\mathrm{tr}(g|_{\mathbb{C}_{\mathrm{Ad}^*(w^{-1}x)(\lambda-\rho)}})}{\det_{\mathbb{C}}(1 - g^{-1}|_{\bigoplus_{\alpha \in R^+} \mathbb{C}_{\mathrm{Ad}^*(w^{-1}x)\alpha}})}$$

$$= \sum_{yT \in w^{-1}N_K(T)w/T} \frac{\mathrm{tr}(g|_{\mathbb{C}_{\mathrm{Ad}^*(yw^{-1})(\lambda-\rho)}})}{\det_{\mathbb{C}}(1 - g^{-1}|_{\bigoplus_{\alpha \in R^+} \mathbb{C}_{\mathrm{Ad}^*(yw^{-1})\alpha}})}. \quad (6.8)$$

(In the last step, we substituted $y = w^{-1}xw$.)

Finally, $w^{-1}N_K(T)w = N_K(T)$, and

$$\bigoplus_{\alpha \in R^+} \mathbb{C}_{\mathrm{Ad}^*(yw^{-1})\alpha} = T_{yT}G/T$$

as complex representations of T, with respect to the complex structure defined by the positive root system $w^{-1}R^+$ with respect to which $w^{-1}(\lambda - \rho)$ is dominant. So by (6.7), the expression (6.8) equals

$$\tau_g\left(\mathrm{index}_G\left(\bar{\partial}_{L_{w^{-1}(\lambda-\rho)}^G} + \bar{\partial}_{L_{w^{-1}(\lambda-\rho)}^G}^*\right)\right). \qquad \square$$

Lemma 6.9. *We have, for all integral $v \in i\mathfrak{t}^*$,*

$$\tau_e\left(\mathrm{index}_{G_c}\left(\bar{\partial}_{L_v^{G_c}} + \bar{\partial}_{L_v^{G_c}}^*\right)\right) = \tau_e\left(\mathrm{index}_G\left(\bar{\partial}_{L_v^G} + \bar{\partial}_{L_v^G}^*\right)\right).$$

Proof. By Connes and Moscovici's L^2-index formula [1982, Theorem 5.2], we have

$$\tau_e\left(\mathrm{index}_G\left(\bar{\partial}_{L_v^G} + \bar{\partial}_{L_v^G}^*\right)\right) = \varepsilon\left(\mathrm{ch}\left(\textstyle\bigwedge_{\mathbb{C}} \mathfrak{g}/\mathfrak{t} \otimes \mathbb{C}_v\right)\hat{A}(\mathfrak{g}, T)\right)[\mathfrak{g}/\mathfrak{t}],$$

$$\tau_e\left(\mathrm{index}_{G_c}\left(\bar{\partial}_{L_v^{G_c}} + \bar{\partial}_{L_v^{G_c}}^*\right)\right) = \varepsilon\left(\mathrm{ch}\left(\textstyle\bigwedge_{\mathbb{C}} \mathfrak{g}_c/\mathfrak{t} \otimes \mathbb{C}_v\right)\hat{A}(\mathfrak{g}_c, T)\right)[\mathfrak{g}_c/\mathfrak{t}],$$

for the same sign $\varepsilon \in \{\pm 1\}$. Here $\mathrm{ch} : R(T) \to H^*(\mathfrak{g}, T, \mathbb{R})$ is the relative Chern character, and the characteristic classes $\hat{A}$ in $H^*(\mathfrak{g}, T, \mathbb{R})$ are defined in Section 4 of [Connes and Moscovici 1982]. The right-hand side of the first line only depends on the representations $\bigwedge_{\mathbb{C}} \mathfrak{g}/\mathfrak{t} \otimes \mathbb{C}_v$ and $\mathfrak{g}/\mathfrak{t}$ of T, and similarly for the right-hand side of the second line. Since $\mathfrak{g}/\mathfrak{t}$ and $\mathfrak{g}_c/\mathfrak{t}$ are both equal to the sum of the positive root spaces as complex representations of T, we find that the two expressions are equal. $\qquad \square$

6D. *Proofs of Theorem 6.2 and Corollary 6.3.* To finish the proof of Theorem 6.2, we need a final lemma.

Lemma 6.10 (Arthur). *We have, for all $g \in T^{\mathrm{reg}}$,*

$$\tau_g^s = \sum_{[w] \in W_G/W_K} \tau_{wgw^{-1}}.$$

Proof. In Section 27 (p. 194) of [Arthur 2005], it is pointed out that two elements $g, g' \in T^{\mathrm{reg}}$ are conjugate if and only if $g = w_K g' w_K^{-1}$ for some $w_K \in W_K$, and stably conjugate if and only if $g = w_G g' w_G^{-1}$ for some $w_G \in W_G$. □

Proof of Theorem 6.2. By surjectivity of Dirac induction and Proposition 6.5, every $x \in K_0(C_r^* G)$ is represented by the equivariant index

$$x = \mathrm{index}_G\big(\bar{\partial}_{L_\nu^G} + \bar{\partial}_{L_\nu^G}^*\big)$$

for an integral element $\nu \in i\mathfrak{t}^*$.

Let $g \in S$. By Theorem 6.4 and Lemmas 6.6 and 6.10, we have

$$\tau_g^s(x) = \tau_g^s\big(\mathrm{index}_G\big(\bar{\partial}_{L_\nu^G} + \bar{\partial}_{L_\nu^G}^*\big)\big) = \tau_g\big(\mathrm{index}_{G_c}\big(\bar{\partial}_{L_\nu^{G_c}} + \bar{\partial}_{L_\nu^{G_c}}^*\big)\big).$$

Since G_c is compact, this expression is continuous in g. And by Lemma 6.9,

$$\tau_e\big(\mathrm{index}_{G_c}\big(\bar{\partial}_{L_\nu^{G_c}} + \bar{\partial}_{L_\nu^{G_c}}^*\big)\big) = \tau_e\big(\mathrm{index}_G\big(\bar{\partial}_{L_\nu^G} + \bar{\partial}_{L_\nu^G}^*\big)\big) = \tau_e(x). \qquad \square$$

Proof of Corollary 6.3. For $w \in W_G$, let $[\pi_{w\lambda}] \in K_0(C_r^* G)$ be the class defined by the discrete series representation with Harish-Chandra parameter $w\lambda$. By Propositions 5.1 and 5.2 in [Hochs and Wang 2018a], we have for all $g \in T^{\mathrm{reg}}$,

$$\sum_{[w] \in W_G/W_K} \Theta_{w\lambda}(g) = \sum_{[w] \in W_G/W_K} \tau_g([\pi_{w\lambda}])$$

$$= (-1)^{\dim(G/K)/2} \sum_{[w] \in W_G/W_K} \tau_g\big(\mathrm{index}_G\big(\bar{\partial}_{L_{w(\lambda-\rho)}^G} + \bar{\partial}_{L_{w(\lambda-\rho)}^G}^*\big)\big).$$

Lemmas 6.6 and 6.10 imply that the right-hand side equals

$$(-1)^{\dim(G/K)/2} \tau_g^s\big(\mathrm{index}_G\big(\bar{\partial}_{L_{\lambda-\rho}^G} + \bar{\partial}_{L_{\lambda-\rho}^G}^*\big)\big).$$

As $g \to e$ through the set S in Theorem 6.2, that result implies that the limit of the above expression is

$$(-1)^{\dim(G/K)/2} \tau_e\big(\mathrm{index}_G\big(\bar{\partial}_{L_{\lambda-\rho}^G} + \bar{\partial}_{L_{\lambda-\rho}^G}^*\big)\big) = \tau_e([\pi_\lambda]) = d_\pi.$$

The claim now follows from continuity of characters on the regular set. □

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