ANNALS OF K-THEORY

vol. 4 no. 2 2019

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A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 4, No. 2, 2019
dx.doi.org/10.2140/akt.2019.4.211

On derived categories of arithmetic toric varieties

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We begin a systematic investigation of derived categories of smooth projective toric varieties defined over an arbitrary base field. We show that, in many cases, toric varieties admit full exceptional collections, making it possible to give concrete descriptions of their derived categories. Examples include all toric surfaces, all toric Fano 3-folds, some toric Fano 4-folds, the generalized del Pezzo varieties of Voskresenskiĭ and Klyachko, and toric varieties associated to Weyl fans of type A. Our main technical tool is a completely general Galois descent result for exceptional collections of objects on (possibly nontoric) varieties over nonclosed fields.

1. Introduction

Recently, several intriguing threads relating derived categories and arithmetic geometry have emerged and motivated general structure questions for k-linear triangulated categories for arbitrary fields k. Such exploration has yielded many nice applications as well as further enticing problems; see as a sampling [Antieau et al. 2017; Ananyevskiy et al. 2013; Ascher et al. 2017; Hassett and Tschinkel 2017; Honigs 2015; Lieblich et al. 2014]. Meanwhile, over $\mathbb{C}$, structural results for derived categories seem to have deep implications for the underlying birational geometry, e.g., [Addington and Thomas 2014; Auel et al. 2014; Bernardara and Bolognesi 2013; Bernardara et al. 2012; Kuznetsov 2010; Vial 2017]. Taking these together, derived categories become an important invariant for studying birational geometry over a general field [Auel and Bernardara 2018]. A further benefit of this noncommutative approach is direct utility for solving problems in algebraic K-theory, for example [Merkurjev and Panin 1997].

With such tantalizing ties, one would like a fertile testing ground for questions. In this paper, we begin a systematic study of one such area: derived categories of arithmetic toric varieties. Recall that if k is an arbitrary field with separable closure k^s, a *k-torus* is an algebraic group T over k such that extending scalars to k^s gives $T_{k^s} \simeq \mathbb{G}_m^n$. An *arithmetic toric variety* is a normal k-variety with a faithful

MSC2010: primary 14F05, 14M25; secondary 14G27, 19E08.
Keywords: derived categories, exceptional collections, Galois descent, toric varieties.

action of a k-torus which has a dense open orbit. This area has the following nice features:

- rationality issues are deep in general but tractable in examples,
- robust tools already exist to investigate derived categories over the separable closure, and
- specific questions are often amenable to computational experimentation.

One of the best tools for understanding the structure of a derived category is an *exceptional collection* consisting of *exceptional objects*. As originally conceived in [Beĭlinson 1978], an exceptional object of a k-linear triangulated category (e.g., $\mathsf{D}^{\mathsf{b}}(X)$) is one whose endomorphism algebra is isomorphic to the base field k. When k is not algebraically closed, this definition is too restrictive and instead we use the existing notion: an object of $\mathsf{D}^{\mathsf{b}}(X)$ is *exceptional* if its endomorphism algebra is a division algebra (concentrated in homological degree zero). An exceptional collection is then given by a totally ordered set $\mathsf{E} = \{E_1, \dots, E_s\}$ of exceptional objects in $\mathsf{D}^{\mathsf{b}}(X)$ satisfying $\mathrm{Ext}^n(E_i, E_j) = 0$ for all integers n whenever $i > j$. An exceptional collection is *full* if it generates $\mathsf{D}^{\mathsf{b}}(X)$, i.e., the smallest thick subcategory of $\mathsf{D}^{\mathsf{b}}(X)$ containing E is all of $\mathsf{D}^{\mathsf{b}}(X)$. Details are discussed in Section 2 below.

We illustrate this more general notion for two arithmetic toric varieties. The real conic $X = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$ has an exceptional collection $\{\mathcal{O}, \mathcal{F}\}$, where $\mathrm{End}(\mathcal{F})$ is isomorphic to the quaternion algebra $\mathbb{H}$. Over $\mathbb{C}$, we have $X_{\mathbb{C}} \simeq \mathbb{P}^1_{\mathbb{C}}$ and $\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathcal{O}(1)^{\oplus 2}$. As another example, consider the Weil restriction Y of $\mathbb{P}^1_{\mathbb{C}}$ over $\mathbb{R}$ ("$\mathbb{P}^1(\mathbb{C})$ viewed as an $\mathbb{R}$-variety"). Here Y has an exceptional collection $\{\mathcal{O}, \mathcal{G}, \mathcal{H}\}$, where $\mathrm{End}(\mathcal{G}) \simeq \mathbb{C}$ and $\mathrm{End}(\mathcal{H}) \simeq \mathbb{R}$. Over $\mathbb{C}$, we have $Y \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with $\mathcal{G} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1)$ and $\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathcal{O}(1, 1)$, where

$$\mathcal{O}(i, j) = \pi_1^* \mathcal{O}(i) \otimes \pi_2^* \mathcal{O}(j).$$

A central question for derived categories of arithmetic toric varieties is the following:

Question 1.1. Let X be a smooth projective toric variety over an arbitrary field. Does X admit a full exceptional collection? If so, does X possess a full exceptional collection of sheaves?

Over an algebraically closed field of characteristic zero, there is always a full exceptional collection of objects [Kawamata 2006; 2013] while the question of a full exceptional collection of sheaves is due to Orlov. Making allowances for different language, the question is also known to have a positive answer for Severi–Brauer varieties [Auel and Bernardara 2018; Bernardara 2009], minimal toric surfaces [Blunk et al. 2011], and smooth projective toric varieties with absolute Picard rank at most 2 [Yan 2014].

In this article, we provide further evidence for a positive answer to Question 1.1, treating cases with low dimension or a high degree of symmetry.

Theorem 1.2. *The following possess full exceptional collections of sheaves*:

- *smooth toric surfaces* (Proposition 4.7),
- *smooth toric Fano* 3*-folds* (Proposition 4.11),
- *all forms of* 43 *of the* 124 *smooth split toric Fano* 4*-folds* (Section 4C),
- *all forms of centrally symmetric toric Fano varieties* (Corollary 4.13), *and*
- *all forms in characteristic zero of toric varieties corresponding to Weyl fans of root systems of type A* (Proposition 4.21).

Our results leverage extant work in the algebraically closed case such as [Uehara 2014] for 3-folds and[Prabhu-Naik 2017] for 4-folds. We use Castravet and Tevelev's recently discovered exceptional collection for $X(A_n)$ [2017]. For the centrally symmetric toric Fano varieties (which are products of "generalized del Pezzo varieties" and projective lines [Voskresenskiĭ and Klyachko 1984]), we use an explicit exceptional collection (see also [Ballard et al. 2018]) closely related to the one found in [Castravet and Tevelev 2017]. Up to a twist by a line bundle, the authors had independently discovered the exact same collection! This suggests that symmetry imposes strong conditions on the possible exceptional collections, which paradoxically makes them easier to find.

To study arithmetic exceptional collections, we establish an effective Galois descent result for general exceptional collections. This applies to general varieties, not just toric ones.

Theorem 1.3 (Theorem 2.17, Lemma 2.20). *Let X be a k-scheme and L/k a G-Galois extension. Then X_L admits a full* (*resp. strong*) *G-stable exceptional collection of objects of $\mathsf{D}^{\mathsf{b}}(X_L)$* (*resp. sheaves*, *vector bundles*) *if and only if X admits a full* (*resp. strong*) *exceptional collection of objects of $\mathsf{D}^{\mathsf{b}}(X)$* (*resp. sheaves*, *vector bundles*).

We highlight one corollary of a positive answer to Question 1.1. Arithmetic toric varieties are also studied in [Merkurjev and Panin 1997], which focused on computing their algebraic K-groups via decompositions in a certain noncommutative motivic category of K_0-correspondences. They showed that for an arithmetic toric k-variety X with $G = \mathrm{Gal}(k^s/k)$, the group $K_0(X_{k^s})$ is a direct summand of a *permutation G-module* (there exists a $\mathbb{Z}$-basis permuted by G).

Question 1.4 [Merkurjev and Panin 1997]. Let X be an arithmetic toric variety over k and $G = \mathrm{Gal}(k^s/k)$. Is $K_0(X_{k^s})$ always a permutation G-module?

Question 1.1 can be viewed as a categorification of Question 1.4 as any such exceptional collection over k immediately gives a permutation basis.

In order to show that every toric variety has a full exceptional collection over $\mathbb{C}$, the main tool used in [Kawamata 2006; 2013] was the minimal model program (MMP) in birational geometry. The basic building blocks are toric stacks with Picard rank one, which always have full strong exceptional collections of line bundles. Indeed, runs of the MMP can be leveraged to effectively produce exceptional collections [Ballard et al. 2019].

Over a nonclosed field, one hopes to use the Galois-equivariant MMP, but the situation is more complicated. The most basic building blocks in this framework are those varieties X which have $\rho^G = \mathrm{rank}(\mathrm{Pic}(X)^G) = 1$. Based on the results above and the hope of using the MMP in the arithmetic situation, we ask the following question in the vein of [King 1997; Borisov and Hua 2009; Costa and Miró-Roig 2010]:

Question 1.5. Let X be a smooth toric k-variety and L/k a G-Galois splitting field. If $\mathrm{Pic}(X_L)^G$ is of rank 1, does X_L possess a full strong G-stable exceptional collection consisting of line bundles?

Organization. Section 2 treats Galois descent of exceptional collections consisting of objects on (possibly nontoric) varieties. In Section 3, we recall appropriate definitions of arithmetic toric varieties and establish additional descent results which are specific to toric varieties. In Section 4, we consider a range of examples. We begin by treating toric surfaces, followed by toric Fano 3-folds. For toric Fano 4-folds, we give partial results. We conclude by investigating the class of centrally symmetric toric Fano varieties, including the generalized del Pezzo varieties, and handling toric varieties associated to root systems of type A.

Notation. Throughout, k denotes an arbitrary field and k^s a separable closure. A *variety* is a geometrically integral separated scheme of finite type over k. All our schemes are quasicompact and quasiseparated. For a k-scheme X and field extension L/k, we write $X_L := X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, L$. If A is a k-algebra, we write $A_L = A \otimes_k L$. We use $\mathsf{D}^{\mathsf{b}}(X)$ to denote the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathrm{Coh}(X))$. For an $\mathcal{O}_X$-algebra A, we write $\mathsf{D}^{\mathsf{b}}(A)$ for the bounded derived category of complexes of A-modules which are coherent $\mathcal{O}_X$-modules.

2. Galois descent and exceptional collections

In this section we develop Galois descent for exceptional collections (in a generalized sense). We begin by recalling some definitions and conventions concerning structure theory of derived categories of schemes. We then give our main descent results for G-stable exceptional collections (Theorem 2.17). We complete the section by collecting a few useful consequences to be used in the sequel.

2A. ***Exceptional collections.*** We give some conventions for semiorthogonal decompositions of derived categories and in particular exceptional collections. Such collections have been widely studied over algebraically closed fields but have recently been treated in more generality [Ananyevskiy et al. 2013; Auel and Bernardara 2018; Auel et al. 2014; Bernardara 2009; Blunk et al. 2011; Elagin 2009; Xie 2017; Yan 2014]. We refer the reader to Remarks 2.15 and 2.19 for added details on some of these results.

For a triangulated category T, we use the standard notation

$$\operatorname{Ext}^n_{\mathsf{T}}(A, B) = \operatorname{Hom}_{\mathsf{T}}(A, B[n]).$$

For objects A, B of $\mathsf{D}^{\mathsf{b}}(X)$, we use $\operatorname{End}_X(A)$ and $\operatorname{Ext}^n_X(A, B)$ to denote $\operatorname{End}_{\mathsf{D}^{\mathsf{b}}(X)}(A)$ and $\operatorname{Ext}^n_{\mathsf{D}^{\mathsf{b}}(X)}(A, B)$, respectively.

Definition 2.1 (see [Bondal and Kapranov 1989]). Let T be a triangulated category. A full triangulated subcategory of T is *admissible* if its inclusion functor admits left and right adjoints. A *semiorthogonal decomposition* of T is a sequence of admissible subcategories $\mathsf{C}_1, \dots, \mathsf{C}_s$ such that

(1) $\operatorname{Hom}_{\mathsf{T}}(A_i, A_j) = 0$ for all $A_i \in \operatorname{Ob}(\mathsf{C}_i)$, $A_j \in \operatorname{Ob}(\mathsf{C}_j)$ whenever $i > j$;

(2) for each object T of T, there is a sequence of morphisms

$$0 = T_s \to \cdots \to T_0 = T$$

such that the cone of $T_i \to T_{i-1}$ is an object of C_i for all $i = 1, \dots, s$.

We use $\mathsf{T} = \langle \mathsf{C}_1, \dots, \mathsf{C}_s \rangle$ to denote such a decomposition.

Particularly nice examples of semiorthogonal decompositions are given by exceptional collections, the study of which goes back to [Beĭlinson 1978].

Definition 2.2. Let T be a k-linear triangulated category. An object E in T is *exceptional* if the following conditions hold:

(1) $\operatorname{End}_{\mathsf{T}}(E)$ is a division k-algebra.

(2) $\operatorname{Ext}^n_{\mathsf{T}}(E, E) = 0$ for $n \neq 0$.

A totally ordered set $\mathsf{E} = \{E_1, \dots, E_s\}$ of exceptional objects is an *exceptional collection* if $\operatorname{Ext}^n_{\mathsf{T}}(E_i, E_j) = 0$ for all integers n whenever $i > j$. An exceptional collection is *full* if it generates T, i.e., the smallest thick subcategory of T containing E is all of T. An exceptional collection is *strong* if $\operatorname{Ext}^n_{\mathsf{T}}(E_i, E_j) = 0$ whenever $n \neq 0$. An *exceptional block* is an exceptional collection $\mathsf{E} = \{E_1, \dots, E_s\}$ such that $\operatorname{Ext}^n_{\mathsf{T}}(E_i, E_j) = 0$ for every n whenever $i \neq j$. Given an exceptional collection $\mathsf{E} = \{E_1, \dots, E_s\}$, we denote by $\langle \mathsf{E} \rangle$ the category generated by the objects E_i.

Remark 2.3. Our notion of exceptional object generalizes the classical one, where Definition 2.2(1) is replaced by $\mathrm{End}_{\mathsf{T}}(E) = k$ [Bondal 1989, §2]. Over algebraically or separably closed fields, these definitions agree. Over nonclosed fields, the classical definition is too restrictive to allow for the use of interesting arithmetic invariants in the study of exceptional collections on twisted forms, e.g., Brauer classes.

Proposition 2.4 [Bondal 1989, Theorem 3.2]. *Let X be a k-scheme with exceptional collection $\{E_1, \dots, E_s\}$. If $\mathscr{E}_i$ is the category generated by E_i, there is a semiorthogonal decomposition $\mathsf{D}^{\mathsf{b}}(X) = \langle \mathscr{E}_1, \dots, \mathscr{E}_s, \mathsf{A}\rangle$, where A is the full subcategory with objects A such that $\mathrm{Hom}_X(A, E_i) = 0$ for all i.*

Remark 2.5. Bondal assumes smoothness and projectivity but the conclusion is independent of this. Note further that if X admits a full exceptional collection then it is automatically smooth and proper by [Orlov 2016, Propositions 3.30 and 3.31].

The existence of an exceptional collection on a scheme X provides a means of studying derived geometry of X in purely algebraic terms. Indeed, in such a situation, one may identify an "underlying" k-algebra which is derived equivalent to X. For exceptional blocks, one obtains a similar but slightly stronger fact.

Proposition 2.6 [Bondal 1989, Theorem 6.2]. *Let X be a smooth projective k-scheme and let $\{E_1, \dots, E_n\}$ be a full strong exceptional collection on $\mathsf{D}^{\mathsf{b}}(X)$. Let $\mathcal{E} = \bigoplus E_i$ and $A = \mathrm{End}(\mathcal{E})$. Then $\mathsf{R}\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(X)}(\mathcal{E}, -) : \mathsf{D}^{\mathsf{b}}(X) \to \mathsf{D}^{\mathsf{b}}(A)$ is a k-linear equivalence.*

Proposition 2.7. *If $\mathsf{E} = \{E_1, \dots, E_s\}$ is an exceptional block with $\mathrm{End}(E_i) = D_i$, there is a k-algebra isomorphism $\mathrm{End}\big(\bigoplus E_i\big) \simeq D_1 \times \cdots \times D_s$, and hence a k-linear equivalence $\langle \mathsf{E}\rangle \simeq \mathsf{D}^{\mathsf{b}}(D_1 \times \cdots \times D_n)$.*

The object $\mathcal{E} = \bigoplus E_i$ of Proposition 2.6 is usually called a *tilting object*. If each E_i is a sheaf (resp. vector bundle), then E is called a *tilting sheaf* (resp. *tilting bundle*). Until recently, the theory of tilting objects has served as the main tool for extending the study of exceptional collections to nonclosed fields. The results above show that any exceptional collection gives rise to both a tilting object and a semiorthogonal decomposition, and thus the admission of such a collection is a particularly special property of a given triangulated category. Our aim in the following subsection is to extend descent results for semiorthogonal decompositions and tilting objects to (our more general notion of) exceptional collections. We give a formal definition of tilting object for completeness.

Definition 2.8. A *tilting object* for a k-scheme X is an object $\mathcal{E}$ of $\mathsf{D}^{\mathsf{b}}(X)$ which satisfies the following conditions:

(1) $\mathrm{Ext}^n_X(\mathcal{E}, \mathcal{E}) = 0$ for $n > 0$.

(2) $\mathcal{E}$ generates $\mathsf{D}^{\mathsf{b}}(X)$.

Remark 2.9 (K-theory and motivic decompositions). Exceptional collections have a particularly interesting manifestation in the realm of noncommutative motives. Indeed, an exceptional collection $\{E_1, \dots, E_s\}$ on a smooth projective variety X yields a decomposition $U(X) \simeq \bigoplus_i U(D_i)$ of its corresponding universal additive invariant [Tabuada 2015, §2.3], where $D_i = \mathrm{End}(E_i)$. This defines a motivic decomposition by viewing X as an object in the Merkurjev–Panin category of K-motives [Merkurjev and Panin 1997] or Kontsevich's category of noncommutative Chow motives [Tabuada 2014, Theorem 6.10] via its associated dg-category of perfect complexes.

One nice consequence is that this decomposition is detected by algebraic K-groups [Auel and Bernardara 2018, Proposition 1.10] in addition to a slew of other additive invariants in the sense of [Tabuada 2015, §2.2]. Such invariants include algebraic K-theory with coefficients, homotopy K-theory, étale K-theory, (topological) Hochschild homology, and (topological) cyclic homology.

2B. *Galois descent.* We develop Galois descent for exceptional collections consisting of objects in the derived category $\mathsf{D}^{\mathsf{b}}(X)$ of a (smooth projective) variety X. Throughout this section, pushforward and pullback functors are understood to be derived. For a k-scheme X and finite Galois extension L/k, any element $g \in \mathrm{Gal}(L/k)$ defines a morphism of k-schemes $g : X_L \to X_L$ which in turn defines the functor $g^* : \mathsf{D}^{\mathsf{b}}(X_L) \to \mathsf{D}^{\mathsf{b}}(X_L)$.

Definition 2.10. Let X be a scheme with an action of a group G. A *G-stable exceptional collection* on X is an exceptional collection $\mathsf{E} = \{E_1, \dots, E_s\}$ of objects in $\mathsf{D}^{\mathsf{b}}(X)$ such that for all $g \in G$ and $1 \le i \le s$ there exists $E \in \mathsf{E}$ such that $g^* E_i \simeq E$. We say a G-stable exceptional collection E is a *G-orbit* if, for every pair of objects $E, E' \in \mathsf{E}$, there exists a $g \in G$ such that $g^* E \simeq E'$.

Remark 2.11. A simple example of a G-stable exceptional collection is a *G-invariant* exceptional collection, i.e., an exceptional collection $\{E_1, \dots, E_s\}$ such that $g^* E_i \simeq E_i$ for all $1 \le i \le s$. It is often the case that toric varieties admit exceptional collections consisting of line bundles. If it is also the case that a group G acts trivially on $\mathrm{Pic}(X)$, such a collection is automatically G-invariant, and hence G-stable (see Lemma 2.21).

Lemma 2.12. *Any G-stable exceptional collection may be written as a collection of G-stable exceptional blocks* (*possibly after reordering*).

Proof. The decomposition of a G-stable exceptional collection into its G-orbits gives the desired exceptional blocks. Let E be a G-stable exceptional collection and for elements $E, E' \in \mathsf{E}$, we write $E \rightsquigarrow E'$ if $\mathrm{Ext}^n(E, E') \neq 0$ for some n.

Let $\mathsf{A} \subset \mathsf{E}$ be a G-orbit. To see that A is an exceptional block, suppose that $E \rightsquigarrow E'$ for $E, E' \in \mathsf{A}$. Since A is a G-orbit, $E' \simeq g^* E$ for some $g \in G$. Thus,

$E \rightsquigarrow g^*E$, and acting again by g, we have $g^*E \rightsquigarrow (g^2)^*E$. Since A is finite, we have $E \rightsquigarrow g^*E \rightsquigarrow \cdots \rightsquigarrow (g^s)^*E \rightsquigarrow E$ for some positive integer s. Thus, there is no ordering of the elements of A such that they form a subset of an exceptional collection — a contradiction.

If B is another G-orbit (distinct from A), we would like to see that these blocks can be ordered to form an exceptional collection. We claim that for any $E \in \mathsf{A}$ and $F \in \mathsf{B}$, one has $E \rightsquigarrow F$ only if A precedes B in the collection E (i.e., $\operatorname{Ext}^n(B, A) = 0$ for all n and all $A \in \mathsf{A}$, $B \in \mathsf{B}$). To see this, assume that $E \rightsquigarrow F$ and $F \rightsquigarrow E'$ for some $E' \in \mathsf{A}$. As A is a G-orbit, $E' \simeq g^*E$ for some $g \in G$. Hence, just as above, we have a sequence $E \rightsquigarrow F \rightsquigarrow g^*E \rightsquigarrow g^*F \rightsquigarrow \cdots \rightsquigarrow (g^s)^*F \rightsquigarrow E$. Thus, there is no ordering of the elements of A and B which forms an exceptional collection, contradicting the exceptionality of E. □

Lemma 2.13. *Let X be a Noetherian k-scheme, L/k a finite Galois extension with group G, and $\pi : X_L \to X$ the natural projection map. For any object M in $\mathsf{D}^{\mathsf{b}}(X_L)$ there is a natural isomorphism $\pi^*\pi_*(M) \simeq \bigoplus_{g \in G} g^*M$.*

Proof. As π is flat and affine, every coherent sheaf on X is acyclic for π^* and every coherent sheaf on X_L is acyclic for π_*. Hence, the derived functors coincide with the application of π^* or π_* componentwise to a complex. Thus, it suffices to establish a natural isomorphism at the level of coherent sheaves.

For any object M of $\operatorname{Coh}(X_L)$, we have $\pi_*M \simeq \pi_*g^*M$, and adjunction yields a natural transformation $\pi^*\pi_* \to g^*$. Summing over all $g \in G$ provides the transformation $\alpha : \pi^*\pi_* \to \bigoplus g^*$. We show this is an isomorphism.

It suffices to check that α is an isomorphism on any affine patch $\operatorname{Spec} R$ of X. Passing to modules, we abuse notation and let M be a finitely generated module over $R_L = R \otimes_k L$. Choose a presentation

$$R_L^{\oplus m} \to R_L^{\oplus n} \to M \to 0$$

of M and evaluate α on the sequence to get the commutative diagram

$$\begin{array}{ccccccc}
R^{\oplus m} \otimes_k (L \otimes_k L) & \longrightarrow & R^{\oplus n} \otimes_k (L \otimes_k L) & \longrightarrow & M \otimes_R R_L & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha_{R^{\oplus m}}} & & \downarrow{\scriptstyle \alpha_{R^{\oplus n}}} & & \downarrow{\scriptstyle \alpha_M} & & \\
R^{\oplus m} \otimes_k \left(\bigoplus_g \Gamma_g(L)\right) & \longrightarrow & R^{\oplus m} \otimes_k \left(\bigoplus_g \Gamma_g(L)\right) & \longrightarrow & \bigoplus_g g^*M & \longrightarrow & 0
\end{array}$$

where $\Gamma_g(L)$ denotes the graph of g in $L \otimes_k L$. The left and middle maps are isomorphisms, so the right map must also be an isomorphism. □

Proposition 2.14 (descent for orbits). *Let X be a k-scheme, L/k a finite G-Galois extension, and $\pi : X_L \to X$ the natural projection map. If $\mathsf{E} = \{E_1, \ldots, E_s\}$ is a G-orbit forming an exceptional collection on X_L, and if E is any element of E,* *then there is an exceptional object F in $\mathsf{D}^{\mathsf{b}}(X)$ such that $\pi_* E \simeq F^{\oplus m}$ and $\pi^* F$ generates the category $\langle \mathsf{E} \rangle$.*

Proof. By Lemma 2.12, exceptional G-orbits are completely orthogonal (and by definition carry a transitive action of G), which is used throughout the proof. Fix an element $E \in \mathsf{E}$, so that $E = E_i$ for some i. Lemma 2.13 gives

$$\pi^*\pi_* E \simeq \bigoplus_{g \in G} g^* E.$$

We claim that $\mathrm{End}(\pi_* E)$ is a matrix algebra over a division algebra, and prove this by first showing that it is semisimple. Indeed, using $\mathrm{End}_X(M) \otimes_k L \simeq \mathrm{End}_{X_L}(\pi^* M)$ for any $M \in \mathsf{D}^{\mathsf{b}}(X)$ [Auel and Bernardara 2018, Remark 2.1], we have

$$\mathrm{End}_X(\pi_* E) \otimes_k L \simeq \mathrm{End}_{X_L}(\pi^*\pi_* E) \simeq \mathrm{End}_{X_L}\Big(\bigoplus_{g \in G} g^* E\Big).$$

Each $g^* E$ is exceptional, so that $\mathrm{End}_{X_L}(g^* E) =: D_g$ is a division algebra for each element $g \in G$. Let $H \leq G$ be the subgroup consisting of elements h satisfying $h^* E \simeq E$. For any system of coset representatives $g \in G/H$, we have $\mathrm{End}_X(\pi_* E)_L \simeq \prod_{g \in G/H} M_m(D_g)$, where $m = |H|$. This product of matrix algebras over division algebras is semisimple, i.e., the Jacobson radical $\mathrm{rad}(\mathrm{End}_X(\pi_* E)_L) = 0$. We then have $0 = \mathrm{rad}(\mathrm{End}_X(\pi_* E)_L) = \mathrm{rad}(\mathrm{End}_X(\pi_* E))_L$ by [Amitsur 1957, Theorem 1], and hence $\mathrm{rad}(\mathrm{End}_X(\pi_* E)) = 0$. Thus, $\mathrm{End}_X(\pi_* E)$ is semisimple and so must also be a product of matrix algebras over division algebras by the Artin–Wedderburn theorem.

Let Z be the center of $\mathrm{End}_X(\pi_* E)$ and Z_L the center of $\mathrm{End}_X(\pi_* E)_L$. Note that Z is an étale k-algebra, and to show that $\mathrm{End}(\pi_* E)$ is a matrix algebra, it suffices to show that Z has no zero divisors, and is thus a field. There is an embedding $Z \hookrightarrow Z_L = \prod_{g \in G/H} L_g$, where L_g is the center of the division algebra D_g. The transitive action of G on $\{E_1, \ldots, E_s\}$ implies that G acts transitively on a basis of Z_L, so that $Z = (Z_L)^G$ has no zero divisors.

We produce the object F using the identification $\mathrm{End}_X(\pi_* E) \simeq M_n(D)$, where D is a division algebra. Let $e_i = e_{ii}$ denote the usual idempotent matrices, so that $\{e_i\}$ is a complete set of primitive orthogonal idempotents. Notice that $F_i := \mathrm{Im}(e_i)$ is a simple $\mathrm{End}_X(\pi_* E)$-submodule of $\pi_* E$ for each i, and hence $F_i \simeq F_j$ for each i, j, and $\mathrm{End}_X(F_i) \simeq D$. Define $F = \mathrm{Im}(e_1) \subset \pi_* E$, included as a direct summand. We note that $\pi_* E \simeq \bigoplus F_i \simeq F^{\oplus n}$.

We now show that F is an exceptional object on X. As stated above, $\mathrm{End}_X(F)$ is a division algebra, so it suffices to show that $\mathrm{Ext}^n_X(F, F) = 0$ for $n \neq 0$. Using Lemma 2.13 and (π^*, π_*)-adjunction, we have

$$\mathrm{Ext}^n_X(\pi_* E, \pi_* E) = \bigoplus_{g \in G} \mathrm{Ext}^n_{X_L}(g^* E, E).$$

For $n \neq 0$, each summand of the right-hand side is 0, which follows from the mutual orthogonality of the exceptional block E (when $g^*E \not\simeq E$) and from exceptionality of E (when $g^*E \simeq E$). Since F is a direct summand of π_*E, it follows that $\mathrm{Ext}^n_X(F, F)$ is a summand of $\mathrm{Ext}^n_X(\pi_*E, \pi_*E) = 0$.

Lastly, we show that π^*F generates the category $\langle \mathsf{E} \rangle$. Since $F^{\oplus m} \simeq \pi_*E$, extending scalars to L gives $(\pi^*F)^{\oplus m} = \pi^*(F^{\oplus m}) \simeq \pi^*\pi_*E \simeq \bigoplus g^*E$. Thus,

$$\langle \pi^*F \rangle = \langle (\pi^*F)^{\oplus m} \rangle = \Big\langle \bigoplus g^*E \Big\rangle = \langle g^*E \rangle_{g \in G} = \langle \mathsf{E} \rangle. \qquad \square$$

Remark 2.15. Proposition 2.14 provides a very specific case of descent for triangulated categories, the main advantage of which is that it allows one to identify a specific exceptional object that base extends to the given orbit. Moreover, a G-orbit which forms an exceptional collection consisting of vector bundles or (resp. sheaves) descends to an exceptional collection consisting of vector bundles (resp. sheaves). Compare to the following descent result for semiorthogonal decompositions, which generalizes [Toën 2012, Corollary 2.15]. Although this result is useful for descending semiorthogonal decompositions, it does not identify exceptional objects.

Proposition 2.16 [Auel and Bernardara 2018, Proposition 2.12]. *Let T be a k-linear triangulated category such that T_{k^s} is k^s-equivalent to $\mathsf{D}^{\mathsf{b}}(k^s, (k^s)^n)$. Then there exists an étale algebra K of degree n over k, an Azumaya algebra A over K, and a k-linear equivalence $\mathsf{T} \simeq \mathsf{D}^{\mathsf{b}}(K/k, A)$.*

Let X, E, and F be as in Proposition 2.14, and note that taking $\mathsf{T} = \langle F \rangle$, we have $\mathsf{T}_{k^s} = \langle \pi^*F \rangle_{k^s} = \langle \mathsf{E} \rangle_{k^s}$. Since $\mathsf{E} = \{g^*E\}_{g \in G}$ is a full exceptional collection for $\langle \mathsf{E} \rangle$, the bundle $\mathcal{E} := \bigoplus (g^*E)_{k^s}$ is a tilting object for $\langle \mathsf{E} \rangle_{k^s}$. This defines an equivalence

$$\mathsf{T}_{k^s} \simeq \langle \mathsf{E} \rangle_{k^s} \simeq \mathsf{D}^{\mathsf{b}}(k^s, \mathrm{End}(\mathcal{E})) = \mathsf{D}^{\mathsf{b}}(k^s, (k^s)^n).$$

Proposition 2.16 yields an étale extension K/k, an Azumaya K-algebra A, and an equivalence $\mathsf{T} \simeq \mathsf{D}^{\mathsf{b}}(K/k, A)$. In this case, since $\mathsf{T} = \langle F \rangle$, we see that $A = \mathrm{End}_X(F)$ is an Azumaya algebra over its center Z (using the notation found in the proof of Proposition 2.14), which is simply a field extension of k.

Theorem 2.17 (descent for stable collections). *Let X be a k-scheme, L/k a finite G-Galois extension, and $\pi : X_L \to X$ the natural projection map. If X_L admits a full G-stable exceptional collection E of objects of $\mathsf{D}^{\mathsf{b}}(X_L)$, then X admits a full exceptional collection F of objects of $\mathsf{D}^{\mathsf{b}}(X)$. If E is strong, so is F. If the elements of E are vector bundles (resp. sheaves), the elements of F are vector bundles (resp. sheaves).*

Proof. By Lemma 2.12, we may write $\mathsf{E} = \{\mathsf{E}^1, \dots, \mathsf{E}^s\}$ as a collection of G-stable blocks, where each block is given by a G-orbit. Proposition 2.14 then associates to each block E^i an exceptional object F_i on X, and we show that $\mathsf{F} = \{F_1, \ldots, F_s\}$ is a full exceptional collection on X. We first show that $\operatorname{Ext}^n_X(F_i, F_j) = 0$ for all n whenever $i > j$. Let E^i and E^j be elements of the collections E^i and E^j, respectively. We then have

$$\operatorname{Ext}^n_X(\pi_* E^i, \pi_* E^j) \simeq \bigoplus_{g \in G} \operatorname{Ext}^n_{X_L}(g^* E^i, E^j). \tag{2.18}$$

Since E^i and E^j are elements of the exceptional collection E and $i < j$, each summand is 0 for all n, so that

$$\operatorname{Ext}^n_X(\pi_* E^i, \pi_* E^j) = 0 \quad \text{for all } n.$$

The objects F_i and F_j are direct summands of $\pi_* E^i$ and $\pi_* E^j$, respectively, and it follows that $\operatorname{Ext}^n_X(F_i, F_j) = 0$ for all n.

By Proposition 2.4, the exceptional collection $\{F_1, \ldots, F_s\}$ yields a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(X) = \langle \mathscr{F}_1, \ldots, \mathscr{F}_s, \mathsf{A} \rangle,$$

where $\mathscr{F}_i = \langle F_i \rangle$ and A is the full subcategory of objects A with $\operatorname{Hom}_{\mathsf{D}^{\mathsf{b}}(X)}(A, F_i) = 0$ for all i. In particular, the subcategories $\mathscr{F}_i$ are admissible. Extending scalars to L, we have $(\mathscr{F}_i)_L = \langle \mathsf{E}^i \rangle$, as both categories are generated by $\pi^* F$ by Proposition 2.14. The exceptional collection $\mathsf{E} = \{\mathsf{E}^1, \ldots, \mathsf{E}^s\}$ is full, and hence we have a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(X_L) = \langle (\mathscr{F}_1)_L, \ldots, (\mathscr{F}_s)_L \rangle.$$

Since our admissible subcategories $\mathscr{F}_i$ base extend to a semiorthogonal decomposition, [Auel et al. 2014, Lemma 2.9] gives a semiorthogonal decomposition $\mathsf{D}^{\mathsf{b}}(X) = \langle \mathscr{F}_1, \ldots, \mathscr{F}_s \rangle$. In particular, the collection $\{F_1, \ldots, F_s\}$ generates $\mathsf{D}^{\mathsf{b}}(X)$, so this collection is full.

If E is strong, the right side of (2.18) vanishes for $i \neq j$ (and any n). It follows exactly as above that $\operatorname{Ext}^n_X(F_i, F_j) = 0$ for all n when $i \neq j$, so that F is strong. □

Remark 2.19. Similar descent results for collections of sheaves are given by Elagin [2009] in the algebraically closed case (i.e., $k = \bar{k}$) using the framework of equivariant exceptional collections in equivariant derived categories. Indeed, for a variety X with an action of a finite group G and a G-invariant exceptional collection (see Remark 2.11) consisting of sheaves, this descent result is given in terms of α-twisted representations of G; see [Elagin 2009, Theorem 2.2]. For a G-stable exceptional collection consisting of sheaves, results are in terms of coinduced twisted representations of G; see [loc. cit., Theorem 2.3].

Lemma 2.20. *Let X be a k-scheme and L/k a finite G-Galois extension. If X admits an exceptional collection, then X_L admits a G-stable exceptional collection.*

Proof. Let $E_1, \dots, E_s$ be the given exceptional collection on X, and consider $\pi^* E_1, \dots, \pi^* E_s$ on X_L. To compute morphisms, we note that

$$\begin{aligned}\operatorname{Hom}_{X_L}(\pi^* E_i, \pi^* E_j) &= \operatorname{Hom}_X(E_i, \pi_* \pi^* E_j) \\ &= \operatorname{Hom}_X(E_i, E_j \otimes_k L) = \operatorname{Hom}_X(E_i, E_j) \otimes_k L.\end{aligned}$$

This vanishes if $j > i$. Let $A_i = \operatorname{Hom}_X(E_i, E_i)$. We can split $A_i \otimes_k L$ as a product of matrix algebras over division algebras $A_{i,j} = M_{N_{i,j}}(D_{i,j})$ and correspondingly decompose

$$\pi^* E_i = \bigoplus F_{i,j}^{N_{i,j}}$$

with

$$\operatorname{Hom}_{X_L}(F_{i,j}, F_{i,j}) = D_{i,j}.$$

Note that $F_{i,j}$ and $F_{i,j'}$ are orthogonal for $j \neq j'$. Thus, we have an exceptional collection. □

Lemma 2.21. *Let X be a k-scheme and L/k a finite extension with Galois group G. If G acts trivially on* $\operatorname{Pic}(X_L)$ *and X_L admits an exceptional collection of line bundles, then X admits an exceptional collection of vector bundles.*

Proof. The collection on X_L is automatically G-stable pointwise. Hence we can apply Theorem 2.17. □

Remark 2.22. Note that while we may start with a collection of line bundles, the descended collection may not consist only of line bundles. An example of this is the real conic discussed in the introduction.

Lemma 2.23. *Let X be a smooth k-variety and L/k a G-Galois extension. Let $Y_1, \dots, Y_s$ be a G-orbit of smooth transversal subvarieties of X_L. Let $Y_I = \bigcap_{i \in I} Y_i$ and let H_I be the normalizer of Y_I. If each Y_I admits a full H_I-stable exceptional collection, then $\widetilde{X}$ admits an exceptional collection, where $\widetilde{X}_L$ is the iterated blowup of X_L at the Y_i (in any order).*

Proof. This is an iterated application of Orlov's theorem; see [Castravet and Tevelev 2017, Lemma 7.2]. □

3. Arithmetic toric varieties

We introduce toric varieties over arbitrary fields. Such varieties, also known as *arithmetic toric varieties*, have been treated in [Duncan 2016; Elizondo et al. 2014; Merkurjev and Panin 1997; Voskresenskiĭ and Klyachko 1984].

Definition 3.1. A *torus* (over k) is an algebraic group T (over k) such that $T_{k^s} \simeq \mathbb{G}_m^n$. A torus is *split* if $T \simeq \mathbb{G}_m^n$. A field extension L/k satisfying $T_L \simeq \mathbb{G}_m^n$ is called a *splitting field* of the torus T. Any torus admits a finite Galois splitting field.

Definition 3.2. Given a torus T, a *toric T-variety* is a normal variety with a faithful T-action and a dense open T-orbit. A toric T-variety is *split* if T is a split torus. A *splitting field* of a toric T-variety is a splitting field of T. A variety is a *toric variety* if it is a toric T-variety for some torus T.

Definition 3.3. Let X be a toric T-variety whose dense open T-orbit contains a k-rational point. Then we say X is *neutral* [Duncan 2016] (or a *toric T-model* [Merkurjev and Panin 1997]). An orbit of a split torus always has a k-point, so a split toric variety is neutral, but the converse is not true in general.

Remark 3.4. In what follows, we use the term *toric variety* to mean toric T-variety for some fixed torus T, even though such a variety may have a toric structure for various tori. In fact, the choice of torus does not affect our analysis of toric varieties given below, and we refer interested readers to [Duncan 2016] for such considerations.

Recall that a *k-form* of a k-variety X is a k-variety X' such that $X_L \simeq X'_L$ for some field extension L/k. Any k-form of a toric variety is a toric variety [Duncan 2016].

3A. ***The split case.*** Let us begin by recalling some facts concerning toric varieties with $T \simeq \mathbb{G}_m^n$ (e.g., when $k = \mathbb{C}$ or $k = k^s$), which are studied in terms of combinatorial data, e.g., lattices, cones, fans. Good references for toric varieties over $\mathbb{C}$ include [Fulton 1993; Cox et al. 2011], and many results hold generally in the split case.

Let N be a finitely generated free abelian group of rank n and $M = \operatorname{Hom}(N, \mathbb{Z})$. A subsemigroup $\sigma \subset N_{\mathbb{R}}$ is a *cone* if $(\sigma^\vee)^\vee = \sigma$, where

$$\sigma^\vee = \{u \in M \mid u(v) \geq 0 \text{ for all } v \in \sigma\}.$$

A subsemigroup τ is a *face* of σ if it is of the form $\tau = \{v \in \sigma \mid u(v) = 0 \text{ for all } u \in S\}$ for some $S \subseteq \sigma^\vee$. A cone σ is *pointed* if 0 is a face of σ, and in this case $\sigma^\vee$ generates $M_{\mathbb{R}}$. Given a pointed cone σ, we associate the affine k-scheme $U_\sigma = \operatorname{Spec} k[\sigma^\vee]$, and for any face $\tau \subset \sigma$ the induced map $U_\tau \hookrightarrow U_\sigma$ is an open embedding.

A *fan* $\Sigma \subset N_{\mathbb{R}}$ is a finite collection of pointed cones such that

(1) any face of a cone in Σ is a cone in Σ and

(2) the intersection of any two cones in Σ is a face of each.

To any fan Σ we associate a k-variety X_Σ which is obtained by gluing the affine schemes U_σ along common subschemes U_τ corresponding to faces.

On the other hand, beginning with a split torus $T \simeq \mathbb{G}_m^n$ and toric T-variety X with fixed embedding $T \hookrightarrow X$, we recover M as the character lattice $\operatorname{Hom}(T, \mathbb{G}_m)$ of T and N as the cocharacter lattice $\operatorname{Hom}(\mathbb{G}_m, T)$. The association $\Sigma \mapsto X_\Sigma$ defines a bijective correspondence between fans $\Sigma \subset N_{\mathbb{R}}$ and toric T-varieties X

(we remind the reader that here we assume T is a split torus; in general, fans Σ admitting an action by $\mathrm{Gal}(k^s/k)$ are in bijection with neutral toric T-varieties).

Let $\Sigma(\ell)$ denote the collection of cones in Σ of dimension ℓ. Let $\mathrm{Div}_T(X)$ denote the free abelian group generated by the *rays* of Σ, i.e., elements of $\Sigma(1)$. By the orbit-cone correspondence [Cox et al. 2011, Theorem 3.2.6], $\mathrm{Div}_T(X)$ is isomorphic to the group of T-invariant Weil divisors of X. For X a (split) smooth projective toric variety, we have natural identifications

$$\mathrm{Pic}(X) = \mathrm{Pic}(X_{k^s}) = \mathrm{Cl}(X_{k^s}) = \mathrm{Cl}(X),$$

which yield an exact sequence

$$0 \to M \to \mathrm{Div}_T(X) \to \mathrm{Pic}(X) \to 0.$$

In particular, if X is of dimension n and m is the number of rays in Σ, the Picard rank of X is $\rho = m - n$.

Definition 3.5. A variety X is *Fano* (resp. *weak Fano*) if its anticanonical class $-K_X$ is ample (resp. nef and big). If X is a normal variety, a Cartier D divisor on X is *nef* ("numerically effective" or "numerically eventually free") if $D \cdot C \geq 0$ for every irreducible curve $C \subset X$. A divisor D is *very ample* if D is base point free and $\varphi_D : X \to \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))^\vee)$ is an embedding. A divisor D is *ample* if ℓD is very ample for some $\ell \in \mathbb{Z}^+$. A line bundle $\mathcal{O}_X(D)$ is nef or (very) ample if the corresponding divisor D is nef or (very) ample. A Cartier divisor is *numerically trivial* if $D \cdot C = 0$ for every irreducible complete curve $C \subset X$. Let $N^1(X)$ be the quotient group of Cartier divisors by the subgroup of numerically trivial divisors. The *nef cone* $\mathrm{Nef}(X)$ is the cone in $N^1(X)$ generated by the nef divisors, and the *anti-nef cone* is the cone $-\mathrm{Nef}(X) \subset N^1(X)$. A line bundle $\mathcal{O}_X(D)$ is nef (ample) if D is nef (ample).

Proposition 3.6. *A Cartier divisor D on a split proper toric variety X is nef* (*resp. ample*) *if and only if $D \cdot C \geq 0$* (*resp. $D \cdot C > 0$*) *for all torus-invariant integral curves $C \subset X$.*

Proof. When k is algebraically closed, this is [Mustaţă 2002, Theorems 3.1 and 3.2]. One can see that the arguments remain valid in the split case more generally. □

3B. ***The not necessarily split case.*** Here we provide a "black box" for producing exceptional collections on arbitrary forms of toric varieties by identifying certain special exceptional collections on a *split* toric variety. This reduces an arithmetic question to a completely geometric question.

We begin by reviewing how to obtain arbitrary forms of toric varieties from the split case; see, for example, [Voskresenskiĭ 1982; Elizondo et al. 2014]. Let T be the split torus of a split smooth projective toric variety X with fan Σ in the space

$N \otimes \mathbb{R}$ associated to the lattice N. Let $\mathrm{Aut}(\Sigma)$ denote the subgroup of elements $g \in \mathrm{GL}(N)$ such that $g(\sigma) \in \Sigma$ for every cone $\sigma \in \Sigma$. There is a natural inclusion of $T \rtimes \mathrm{Aut}(\Sigma)$ into $\mathrm{Aut}(X)$ as the subgroup leaving the open orbit T-invariant.

Let k^s be the separable closure of k. The Galois cohomology set

$$H^1(k^s/k, \mathrm{Aut}(X)(k^s))$$

is in bijective correspondence with the k-forms of X. The natural map

$$H^1(k^s/k, T(k^s) \rtimes \mathrm{Aut}(\Sigma)) \to H^1(k^s/k, \mathrm{Aut}(X)(L))$$

in Galois cohomology is surjective; the failure of this map to be a bijection amounts to the fact that there may be several nonisomorphic toric variety structures on the same variety; see [Duncan 2016] for more details.

Suppose that $X' = {}^{\gamma}X$ is a twisted form of a split toric variety for a cocycle γ representing a class in $H^1(k^s/k, T(k^s) \rtimes \mathrm{Aut}(\Sigma))$. There is a "factorization" $X' = {}^{\alpha}({}^{\beta}X)$, where β represents a class in $H^1(k^s/k, \mathrm{Aut}(\Sigma))$ and α represents a class in $H^1(k^s/k, ({}^{\beta}T)(k^s))$. Informally, β changes the torus that acts on X, while α changes the torsor of the open orbit in X.

Suppose X is a toric T-variety. We say that an object $E \in \mathsf{D}^{\mathsf{b}}(X)$ is *T-equivariant* if E is in the image of the forgetful functor from $\mathsf{D}^{\mathsf{b}}(\mathrm{Coh}_T(X))$; see [Ballard et al. 2014, §2]. In particular, this implies that $t^*E \simeq E$ for all $t \in T(k)$.

Proposition 3.7. *Let X be a split toric T-variety over a field k and let Σ be the associated fan. Suppose that X admits an* $\mathrm{Aut}(\Sigma)$*-stable full exceptional collection* E *such that each object is T-equivariant. Then any k-form X' of X admits a full exceptional collection* E'. *Moreover*, E' *is strong* (*resp. consists of vector bundles*, *consists of sheaves*) *as soon as* E *is strong* (*resp. consists of vector bundles*, *consists of sheaves*).

Proof. By Lemma 2.20, there exists a G-stable exceptional collection F on X_L. From the proof of that lemma, the objects F of F are direct summands of π^*E for each object $E \in \mathsf{E}$, where each isomorphism class of a simple direct summand is represented by exactly one F. Since E is $\mathrm{Aut}(\Sigma)$-stable and each object is T-equivariant, we may conclude that F is $(T(L) \rtimes \mathrm{Aut}(\Sigma)) \rtimes G$-stable.

Let X' be a k-form of X; there exists a finite Galois extension L/k with Galois group G such that $X'_L \simeq X_L$. From Theorem 5.1 of [Duncan 2016], the natural map

$$H^1(L/k, T(L) \rtimes \mathrm{Aut}(\Sigma)) \to H^1(L/k, \mathrm{Aut}(X)(L))$$

in Galois cohomology is surjective. Thus, we may assume that $X' = {}^{c}X$ is the *twist* by a cocycle $c : G \to T(L) \rtimes \mathrm{Aut}(\Sigma)$. Recall that the cocycle condition is that $c(gh) = c(g){}^{g}c(h)$ for all $g, h \in G$, where ${}^{g}c(h)$ denotes the Galois action of g on $T(L) \rtimes \mathrm{Aut}(\Sigma)$.

Identifying $X_L = X'_L$, twisting gives $\sigma'(g) = c(g)\sigma(g)$, where σ is the action of G induced from X and σ' is induced from X'. The punchline is that the action σ' factors through the image of $(T(L) \rtimes \mathrm{Aut}(\Sigma)) \rtimes G$ described above. Thus the exceptional collection F is G-stable for the X' action as well. The proposition now follows by Theorem 2.17. □

Corollary 3.8. *Let X be a split toric T-variety over a field k and let Σ be the associated fan. If X admits an* $\mathrm{Aut}(\Sigma)$*-stable full* (*strong*) *exceptional collection of line bundles*, *then every k-form of X admits a full* (*strong*) *exceptional collection of vector bundles.*

Proof. Recall that every line bundle is isomorphic to a T-equivariant line bundle by standard results on toric varieties. The claim now follows by Proposition 3.7. □

Lemma 3.9. *Let X and Y be smooth projective toric varieties over k, and let $G = \mathrm{Gal}(k^s/k)$. Assume we have a K-positive toric flip $X \dashrightarrow Y$ such that over k^s the flipping loci F_i are disjoint and permuted by G. Let H_i be the normalizer of F_i. If X_L admits a full G-stable exceptional collection and Y_i admits a full H_i-stable exceptional collection*, *then Y admits a full exceptional collection.*

Proof. Passing to k^s, we are free to use [Ballard et al. 2019] giving semiorthogonal decompositions for the flip over each Y_i. Since the Y_i are disjoint, we can concatenate these collections to get a G-stable collection. □

3C. ***Products of toric varieties.*** Recall that, given groups G, H along with a homomorphism $\rho : H \hookrightarrow S_n$, the *wreath product* $G \wr H$ is the group $G^n \rtimes H$, where H acts on G^n by permuting the copies of G. We say a toric variety X is *indecomposable* if it cannot be written as a product $X_1 \times X_2$, where X_1 and X_2 are positive-dimensional toric varieties.

Lemma 3.10. *Suppose $Z = X_1^{n_1} \times \cdots \times X_r^{n_r}$ is a product of proper split toric varieties $X_1, \ldots, X_r$, where $X_i \not\simeq X_j$ for $i \neq j$ and each X_i is indecomposable. Then*

$$\mathrm{Aut}(\Sigma) \simeq (\mathrm{Aut}(\Sigma_1) \wr S_{n_1}) \times \cdots \times (\mathrm{Aut}(\Sigma_r) \wr S_{n_r}),$$

where Σ is the fan of Z and $\Sigma_1, \ldots, \Sigma_r$ are the fans of $X_1, \ldots, X_r$.

Proof. First, consider $Z = X_1 \times X_2$, where X_1, X_2 are proper split toric varieties. Let N (resp. N_1, N_2) be the cocharacter lattice and Σ (resp. Σ_1, Σ_2) be the fan of Z (resp. X_1, X_2). Here $N = N_1 \oplus N_2$ and Σ is the set of cones of the form $\sigma_1 \times \sigma_2$, where $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$. The faces of a cone $\sigma_1 \times \sigma_2$ are precisely the cones of the form $\sigma'_1 \times \sigma'_2$, where σ'_1 is a face of σ_1 and σ'_2 is a face of σ_2. The fan Σ_1 can be canonically identified with the subfan of Σ via the bijection $\sigma \mapsto \sigma \times \{0\}$.

Now, suppose also that $Z = Y \times W$ is a product of proper split toric varieties with Y indecomposable. Let Σ_Y be the fan of Y, which we can canonically identify with a subfan of Σ_Z. Every cone of Y is of the form $\sigma_1 \times \sigma_2$, where $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$. Since fans are closed under taking faces, $\sigma_1 \times \{0\}$ and $\{0\} \times \sigma_2$ are also cones in Σ_Y. Thus every cone in Σ_Y is a product of cones in the intersections $\Sigma_Y \cap \Sigma_1$ and $\Sigma_Y \cap \Sigma_2$.

In particular, since X is proper, we have that the space $N_Y \otimes \mathbb{R}$ is the direct sum of $(N_Y \otimes \mathbb{R}) \cap (N_1 \otimes \mathbb{R})$ and $(N_Y \otimes \mathbb{R}) \cap (N_2 \otimes \mathbb{R})$, and Σ_Y is a product of the fans $\Sigma_Y \cap \Sigma_1$ and $\Sigma_Y \cap \Sigma_2$. Since Y is indecomposable, one of these fans is indecomposable and Σ_Y must be a subfan of either Σ_1 or Σ_2.

Returning to the general case, we conclude that the decomposition of Σ as $\Sigma_1^{n_1} \times \cdots \times \Sigma_r^{n_r}$ is unique up to ordering. The description of the automorphism group is immediate. □

Lemma 3.11. *Let Z be a proper toric k-variety with splitting field L/k. Suppose $Z_L = \prod_{i=1}^n X_i$, where each X_i is an indecomposable split proper toric L-variety admitting a full* (*strong*) $\operatorname{Aut}(\Sigma_i)$-*stable exceptional collection of line bundles, where Σ_i is the fan of X_i. Then Z has a full* (*strong*) *exceptional collection of vector bundles.*

Proof. It is well known that the exterior product collection is an exceptional collection. For each isomorphism class among the X_i, fix a full (strong) $\operatorname{Aut}(\Sigma_{X_i})$-stable exceptional collection of line bundles. This ensures that the exterior product collection is stable under the action of $(\operatorname{Aut}(\Sigma_{X_1}) \wr S_{a_1}) \times \cdots \times (\operatorname{Aut}(\Sigma_{X_r}) \wr S_{a_r})$. Since this group is $\operatorname{Aut}(\Sigma)$ by Lemma 3.10, the exterior product collection descends by Corollary 3.8. □

4. Low dimension or high symmetry

We provide exceptional collections for smooth toric surfaces, Fano 3-folds, some Fano 4-folds, centrally symmetric toric varieties, and toric varieties corresponding to root systems of type A.

4A. ***Surfaces.*** Here we prove that every toric surface has a full exceptional collection. We begin by recalling the (classical) minimal model program for surfaces over nonclosed fields.

Suppose $f : X \to X'$ is a birational morphism of smooth projective surfaces over a field k. If k is separably closed, then by Proposition 5 of [Coombes 1988] the morphism factors into a sequence

$$X = X_0 \to X_1 \to \cdots \to X_r = X',$$

where each morphism $X_i \to X_{i+1}$ is the blowup of a point on X_{i+1}. Over a nonclosed field k, we can factor $f : X \to X'$ into a sequence where each morphism $X_i \to X_{i+1}$ is defined over k and is a blowup of a (necessarily finite) Galois orbit of k^s-points on X_{i+1}.

Blowing up a point produces an exceptional curve: a smooth rational curve with self-intersection -1. By Castelnuovo's contractibility criterion, such a curve can always be obtained as the result of a blow-up. If one finds a skew Galois orbit of such curves on X, then there exists a birational morphism $f : X \to X'$ contracting these curves. Repetition of this procedure eventually terminates.

Definition 4.1. A *minimal surface* X is a smooth projective surface over a field k such that every birational morphism $X \to X'$ to a smooth projective surface X' is an isomorphism.

Any smooth projective surface can be obtained by iteratively blowing up Galois orbits of separable points starting from a minimal model. A toric variety is geometrically rational. Minimal geometrically rational surfaces were classified by Manin [1966] and Iskovskikh [1979]. One checks that the toric surfaces in their collection are the following (see also a direct proof in [Xie 2017]):

Lemma 4.2. *A minimal smooth projective toric surface is a k^s/k-form of one of the following*:

(1) $\mathbb{P}^2$, $\mathrm{Aut}(\Sigma) = S_3$.
(2) $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{Aut}(\Sigma) = D_8$.
(3) $\mathbb{F}_a = \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$, $a \geq 2$, $\mathrm{Aut}(\Sigma) = C_2$.
(4) $\mathsf{dP}_6 =$ *del Pezzo surface of degree* 6, $\mathrm{Aut}(\Sigma) = D_{12}$.

Proof. A minimal geometrically rational surface is either a del Pezzo surface or has a conic bundle structure [Manin 1966; Iskovskikh 1979]. Over the separable closure, a del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blowup of $\mathbb{P}^2$ at up to 8 points in general position. Blowing up only one or two points never results in a minimal surface, and no more than three points can be simultaneously torus invariant and in general position. Thus every del Pezzo surface is a k^s/k-form of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or dP_6. Over the separable closure, a conic bundle structure has at most 2 singular fibers since their images must be torus invariant points on the base $\mathbb{P}^1$. A minimal conic bundle with at most two singular fibers over the separable closure must be either a del Pezzo surface or a minimal ruled surface. □

Here we exhibit full strong exceptional collections consisting of G-stable blocks for each minimal toric surface exhibited above (none of these collections are original). The fans associated to the split forms of these surfaces are given in Figure 1. In each case, we fix a torus T which gives X the structure of a toric T-surface. As remarked above, this gives a homomorphism $G \to \mathrm{Aut}(\Sigma)$ as well as an action of G on $\mathrm{Pic}(X_L)$, where L is a splitting field of T, $G = \mathrm{Gal}(L/k)$, and Σ is the fan corresponding to the split toric surface X_L. We produce G-stable exceptional collections in each case by exhibiting $\mathrm{Aut}(\Sigma)$-stable collections.

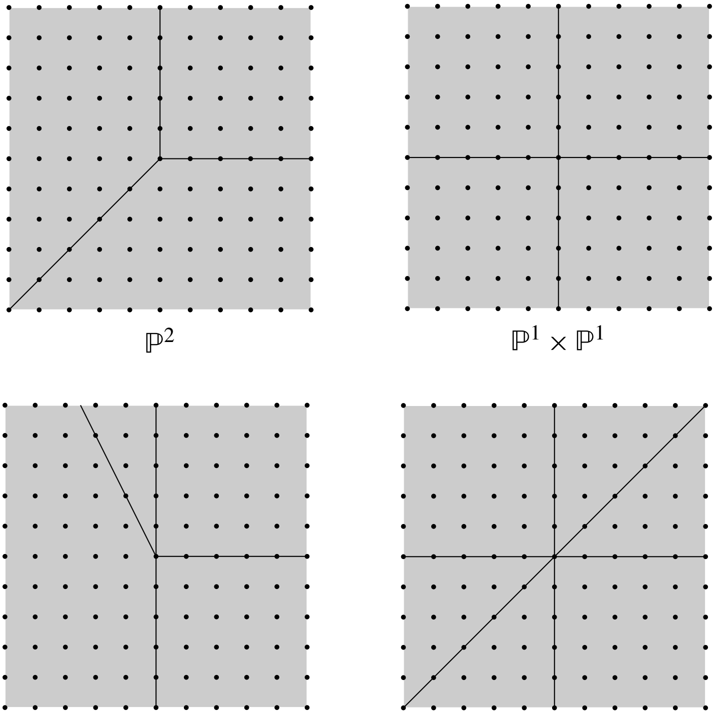

Figure 1. Fans for minimal toric surfaces.

Example 4.3. Let X be a toric T-surface whose split form is $\mathbb{P}^2$ with $\mathrm{Aut}(\Sigma) = S_3$. The S_3-action on $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}$ is clearly trivial, so that the exceptional collection $\{\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\}$ given in [Beĭlinson 1978] yields a full strong $\mathrm{Aut}(\Sigma)$-stable exceptional collection. By Corollary 3.8, X admits a full strong exceptional collection.

Example 4.4. Let X be a toric surface whose split form is $\mathbb{P}^1 \times \mathbb{P}^1$ with $\mathrm{Aut}(\Sigma) = D_8$, and consider the natural projections $p_1, p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Let

$$\mathcal{O}(p, q) = p_1^*\mathcal{O}(p) \otimes p_2^*\mathcal{O}(q).$$

By [Kvichansky and Nogin 1990], the collection $\{\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)\}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is exceptional since $\{\mathcal{O}, \mathcal{O}(1)\}$ is an exceptional collection for $\mathbb{P}^1$. The D_8-action preserves this collection, with orbits given by the blocks $\mathsf{E}^0 = \{\mathcal{O}\}$, $\mathsf{E}^1 = \{\mathcal{O}(1, 0), \mathcal{O}(0, 1)\}$, and $\mathsf{E}^2 = \{\mathcal{O}(1, 1)\}$. In particular, this collection is $\mathrm{Aut}(\Sigma)$-stable, and Corollary 3.8 yields an exceptional collection on X.

Example 4.5. Let X be a toric surface whose split form is the Hirzebruch surface $\mathbb{F}_a$; here $\mathrm{Aut}(\Sigma) = C_2$. Let e_1, e_2 be the standard basis for $\mathbb{Z}^2$. As in [Cox et al. 2011, Example 4.1.8], let $u_1 = -e_1 + ae_2$, $u_2 = e_2$, $u_3 = e_1$, and $u_4 = -e_2$ be the generators of $\Sigma(1)$ with corresponding toric divisors D_i. The Picard group of $\mathbb{F}_a$ is freely generated by $\{D_1, D_2\}$ and D_1 is linearly equivalent to D_3. The only nontrivial fan automorphism σ takes $e_1 \mapsto -e_1+ae_2$ and $e_2 \mapsto e_2$. Thus σ leaves D_2, D_4 fixed and interchanges D_1 and D_3. We conclude that the action of C_2 on $\mathrm{Pic}(\mathbb{F}_a)$ is trivial, and thus, any exceptional collection is necessarily G-stable (see Lemma 2.21). An exceptional collection for $\mathbb{F}_a$ is given by $\{\mathcal{O}, \mathcal{O}(D_3), \mathcal{O}(D_4), \mathcal{O}(D_3+D_4)\}$ [Kvichansky and Nogin 1990]. Corollary 3.8 then gives an exceptional collection on X.

Example 4.6. Let X be a toric surface whose split form is dP_6; here $\mathrm{Aut}(\Sigma) = D_{12}$. Viewing dP_6 as the blowup of $\mathbb{P}^2$ at 3 noncolinear points, let H be the pullback of the hyperplane divisor on $\mathbb{P}^2$ and E_i the exceptional divisors, $i = 1, 2, 3$. As shown in [King 1997, Proposition 6.2(ii)], the collection

$$\{\mathcal{O}, \mathcal{O}(H-E_1), \mathcal{O}(H-E_2), \mathcal{O}(H-E_3), \mathcal{O}(H), \mathcal{O}(2H-(E_1+E_2+E_3))\}$$

gives an exceptional collection for dP_6 which is $\mathrm{Aut}(\Sigma)$-stable.

Let us rephrase this in the notation of [Blunk et al. 2011]. There are two morphisms $\mathsf{dP}_6 \to \mathbb{P}^2$ realizing dP_2 as a blowup of $\mathbb{P}^2$, and we denote the collection of all six exceptional divisors by L_i and M_i, with $i = 1, 2, 3$. Let H and H' denote the pullbacks of the hyperplane divisors on $\mathbb{P}^2$ under the maps contracting M_i and L_i, respectively, where we identify H with the divisor given in King's collection above (and thus we also identify E_i with M_i). Then $H = L_1 + M_2 + M_3$, and using the relation $L_i + M_j = L_j + M_i$ it follows that

$$2H - (E_1+E_2+E_3) = L_1 + L_2 + M_3 = H'.$$

Furthermore, one checks that $H - E_1 = L_2 + M_3$, $H - E_2 = L_1 + M_3$, and $H - E_3 = L_1 + M_2$. As described in [Blunk et al. 2011, §2], the element σ in $S_3 \times C_2 = D_{12}$ which cyclically permutes the six lines L_i, M_i also satisfies $\sigma(H) = H'$ and $\sigma^2(H) = H$. We arrange the exceptional collection above into blocks

$$\begin{aligned} \mathsf{E}^0 &= \{\mathcal{O}\}, \\ \mathsf{E}^1 &= \{\mathcal{O}(H-E_1), \mathcal{O}(H-E_2), \mathcal{O}(H-E_3)\}, \\ \mathsf{E}^2 &= \{\mathcal{O}(H), \mathcal{O}(2H-(E_1+E_2+E_3))\}. \end{aligned}$$

In particular, the exceptional collection given above is $\mathrm{Aut}(\Sigma)$-stable, and so by Corollary 3.8 we have an exceptional collection on X.

Proposition 4.7. *Every toric surface admits a full exceptional collection of sheaves.*

Proof. There is a sequence of blowups $X = X_0 \to \cdots \to X_s = X'$, where X' is minimal and so must be one of the varieties given in Lemma 4.2. By Examples 4.3–4.6, X' admits a full strong exceptional collection of vector bundles, and thus X'_L admits a G-stable exceptional collection. By Lemma 2.23, X_L admits a G-stable exceptional collection. □

Remark 4.8. We would like to thank F. Xie for pointing out a mistake in the statement of a previous version of Proposition 4.7. Xie also discusses exceptional collections of toric surfaces in [Xie 2017], although her definition of exceptional object is not the same as ours. In the second arXiv version of that paper, Xie sketched in Remark 8.8 how one might construct an exceptional collection for toric surfaces. After the authors posted a preliminary version of this paper to the arXiv, Xie updated her preprint with Corollary 8.8, which proves the analog of the above proposition for collections of vector bundles but using her notion of exceptional collection.

4B. ***The toric Frobenius and toric Fano 3-folds.*** In Table 1 we present the classification of smooth toric Fano 3-folds given in [Batyrev 1999; Watanabe and Watanabe 1982], adopting Batyrev's enumeration. For each $X = X_\Sigma$, we record the following invariants:

- $\sigma(1) = |\Sigma(1)|$ is the number of rays of Σ [Bernardi and Tirabassi 2009].
- k_0 is the rank of the Grothendieck group $K_0(X)$, which coincides with the number of maximal cones in the fan Σ [Bernardi and Tirabassi 2009].
- $\mathrm{Aut}(\Sigma)$ is the automorphism group of the (lattice N which preserves the) fan Σ corresponding to X.
- ρ is the Picard rank of X [Watanabe and Watanabe 1982].
- ρ^G is the $\mathrm{Aut}(\Sigma)$-invariant Picard rank of X, i.e., the rank of $\mathrm{Pic}(X)^{\mathrm{Aut}(\Sigma)}$.
- $\mathfrak{fr} = |\mathsf{Frob}(X)|$ is the number of isomorphism classes of line bundles produced by the push forward of the structure sheaf under the Frobenius morphism [Bernardi and Tirabassi 2009; Uehara 2014].
- $\mathfrak{fr}^- = |\mathsf{Frob}(X) \cap -\mathrm{Nef}(X)|$ is the number of isomorphism classes of line bundles in $\mathsf{Frob}(X)$ which lie in the anti-nef cone of X [Uehara 2014].

Toric Frobenius. Let X be a split toric variety of dimension n with fixed torus embedding $T \hookrightarrow X$ and take $\ell \in \mathbb{Z}^+$. Define the ℓ-th Frobenius map on $T = \mathbb{G}_m^n$ to be $(x_1, \dots, x_n) \mapsto (x_1^\ell, \dots, x_n^\ell)$. The unique extension to X is denoted F_ℓ and called the *ℓ-th Frobenius morphism*. Alternatively, if $\Sigma \subset N$ is the fan associated to X, define a lattice $N' = \frac{1}{\ell} N$. The inclusion $N \subset N'$, which sends a cone in $N_\mathbb{R}$ to the cone with the same support in $N'_\mathbb{R}$, induces a finite surjective morphism which is precisely the ℓ-th Frobenius morphism $F_\ell : X \to X$.

The sheaf $(F_\ell)_*(\mathcal{O}_X)$ splits into line bundles and [Thomsen 2000] provides an algorithm for computing its direct summands. We let $\mathsf{Frob}(X)$ denote the union of all isomorphism classes of line bundles arising as direct summands of $(F_\ell)_*(\mathcal{O}_X)$ as ℓ varies over $\mathbb{Z}^+$. Note that $\mathsf{Frob}(X)$ is a finite set.

	Toric Fano 3-fold X	$\sigma(1)$	k_0	$\mathrm{Aut}(\Sigma)$	ρ	ρ^G	$\mathfrak{fr}$	$\mathfrak{fr}^-$
1.	$\mathbb{P}^3$	4	4	S_4	1	1	4	4
2.	$\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(2))$	5	6	S_3	2	2	7	6
3.	$\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(1))$	5	6	S_3	2	2	6	6
4.	$\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(1))$	5	6	$C_2\times C_2$	2	2	6	6
5.	$\mathbb{P}^2\times\mathbb{P}^1$	5	6	D_{12}	2	2	6	6
6.	$\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(1,1))$	6	8	D_8	3	2	8	8
7.	$\mathbb{P}_{\mathsf{dP}_8}(\mathcal{O}\oplus\mathcal{O}(l))$, $l^2=1$ on dP_8	6	8	D_8	3	3	8	8
8.	$\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$	6	8	$C_2\times S_4$	3	1	8	8
9.	$\mathsf{dP}_8\times\mathbb{P}^1$	6	8	$C_2\times C_2$	3	3	8	8
10.	$\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}\otimes\mathcal{O}(1,-1))$	6	8	D_8	3	2	8	8
11.	$\mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(1)))$	6	8	C_2	3	3	9	8
12.	$\mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}^2\times\mathbb{P}^1)$	6	8	C_2	3	3	8	8
13.	dP_7-bundle over $\mathbb{P}^1$	7	10	C_2	4	4	10	10
14.	dP_7-bundle over $\mathbb{P}^1$	7	10	$C_2\times C_2$	4	3	10	10
15.	$\mathsf{dP}_7\times\mathbb{P}^1$	7	10	$C_2\times C_2$	4	3	10	10
16.	dP_7-bundle over $\mathbb{P}^1$	7	10	C_2	4	4	10	10
17.	$\mathsf{dP}_6\times\mathbb{P}^1$	8	12	$C_2\times C_2\times S_3$	5	2	12	12
18.	dP_6-bundle over $\mathbb{P}^1$	8	12	$C_2\times C_2$	5	4	12	12

Table 1. Toric Fano 3-folds.

Conjecture 4.9 [Bondal 2006]. If X is a smooth proper toric variety then the collection $\mathsf{Frob}(X)$ generates $\mathsf{D}^{\mathsf{b}}(X)$.

For a toric variety X in which Bondal's Conjecture is true, we say that *the Frobenius generates the derived category of* X. In [loc. cit.], Bondal proves that if all summands of $\mathsf{Frob}(X)$ are nef, one actually gets a full strong exceptional collection, so that Conjecture 4.9 is true in this case. He also notes his arguments work for all but two (isomorphism classes of) toric Fano threefolds. To cover all toric Fano threefolds, Uehara noticed that discarding line bundles which do not lie in the set $-\mathrm{Nef}(X)$ yields a full strong exceptional collection [Uehara 2014].

Lemma 4.10. *Let X be a toric variety over k with splitting field L. Suppose E is a full (strong) exceptional collection for $\mathsf{D}^{\mathsf{b}}(X_L)$ where either $\mathsf{E}=\mathsf{Frob}(X_L)$ or $\mathsf{E}=\mathsf{Frob}(X_L)\cap-\mathrm{Nef}(X_L)$. Then there exists a full (strong) exceptional collection for $\mathsf{D}^{\mathsf{b}}(X)$.*

Proof. Both $\mathsf{Frob}(X_L)$ and $\mathrm{Nef}(X_L)$ are canonical constructions and thus are $\mathrm{Aut}(X_L)$-stable. In particular, E is $\mathrm{Aut}(\Sigma)$-stable and so Corollary 3.8 applies. □

Proposition 4.11. *Let X be a smooth projective toric Fano 3-fold over a field k. Then X admits a full strong exceptional collection consisting of vector bundles.*

Proof. Let X_L be the associated split toric Fano 3-fold. The main result of [Uehara 2014] guarantees that the set $\mathsf{E} = \mathsf{Frob}(X_L) \cap -\mathrm{Nef}(X_L)$ defines a full strong exceptional collection on X. Lemma 4.10 completes the proof. □

4C. ***Toric Fano 4-folds.*** There are 124 split smooth toric Fano 4-folds, which were first classified in [Batyrev 1999] (a missing case was added in [Sato 2000]). Full strong exceptional collections for all 124 of these 4-folds were exhibited in [Prabhu-Naik 2017]. However, it is not clear that these collections are $\mathrm{Aut}(\Sigma)$-stable, so they do not necessarily lead to full strong exceptional collections in the arithmetic case.

All collections obtained using Method 1 of [Prabhu-Naik 2017] produce $\mathrm{Aut}(\Sigma)$-stable collections (note that this is precisely the method used in [Uehara 2014] for toric Fano 3-folds, and we refer to this as the *Bondal–Uehara method*). Together with Lemmas 3.11 and 4.10, this gives stable exceptional collections for 43 of the 124 smooth toric Fano 4-folds. However, there are examples where the Bondal–Uehara method fails to produce an exceptional collection. In this case, all is not lost (see Section 4D).

More precisely, the varieties (61), (62), (63), (64), (77), (105), (107), (108), (110), (122), and (123) of [Prabhu-Naik 2017] are shown to have exceptional collections using the Bondal–Uehara method. Hence, they admit exceptional collections which are $\mathrm{Aut}(\Sigma)$-stable and thus provide exceptional collections for the arithmetic forms. Secondly, for the varieties (109), (114), and (115), the set $\mathsf{Frob}(X)$ is a full exceptional collection, which is G-stable by Lemma 4.10. Lastly, Lemma 3.11 guarantees the existence of exceptional collections on products. Hence, the following varieties also admit stable exceptional collections: (0), (4), (9), (17), (24), (25), (26), (27), (45), (52), (53), (54), (55), (56), (58), (67), (73), (88), (90), (92), (93), (97), (103), (111), (112), (113), (118), (119), (120).

4D. ***Centrally symmetric toric Fano varieties.*** Polytopes with the highest degree of symmetry are the *centrally symmetric* polytopes, i.e., $-P = P$. The smooth split toric varieties X whose anticanonical polytope is full-dimensional and centrally symmetric were classified in [Voskresenskiĭ and Klyachko 1984]. It was shown that any such variety (which we refer to as a *centrally symmetric toric Fano variety*) is isomorphic to a product of projective lines and *generalized del Pezzo varieties* V_n of dimension $n = 2m$. Note that $V_2 = \mathsf{dP}_6$ and V_4 is the missing (116) from the list in Section 4C (this is (118) in the enumeration found in [Batyrev 1999]). The goal of this section is to exhibit full stable exceptional collections on V_n, which in turn yields stable exceptional collections for any centrally symmetric toric Fano variety, in light of Lemma 3.11.

Castravet and Tevelev [2017, Theorem 6.6] found $\mathrm{Aut}(\Sigma)$-stable full strong exceptional collections for the varieties V_n. The present authors had independently discovered the same exceptional collection (up to a twist by a line bundle). Nevertheless, the perspective here may be of independent interest, so we sketch the argument. A more detailed analysis is given in [Ballard et al. 2018].

The variety V_n with $n = 2m$ has rays given by

$$\begin{aligned} e_1 &= (1, 0, \dots, 0), & \bar{e}_1 &= (-1, 0, \dots, 0), \\ e_2 &= (0, 1, \dots, 0), & \bar{e}_2 &= (0, -1, \dots, 0) \\ &\vdots & &\vdots \\ e_n &= (0, 0, \dots, 1), & \bar{e}_n &= (0, 0, \dots, -1), \\ e_{n+1} &= (-1, -1, \dots, -1), & \bar{e}_{n+1} &= (1, 1, \dots, 1), \end{aligned}$$

and maximal cones given as follows. From the rays $e_1, \dots, e_{n+1}$, omit a single e_i. From the remaining $n = 2m$ rays, choose $\frac{n}{2}$ of them and take their antipodes [Voskresenskiĭ and Klyachko 1984, proof of Theorem 5]. Note that $V_2 = \mathrm{dP}_6$ (whose fan is given in Figure 1). The number of maximal cones $c(n)$ of V_n is given by

$$c(n) = \frac{(n+1)!}{\left(\frac{n}{2}\right)!^2} = \frac{(2m+1)!}{m!^2}.$$

There's a natural action of $S_{n+1} \times C_2$, where S_{n+1} permutes $e_1, \dots, e_{n+1}$ and $\bar{e}_1, \dots, \bar{e}_{n+1}$ in the obvious way. The C_2-action is simply the antipodal map on the cocharacter lattice — we refer to it as "the involution". Clearly, the involution interchanges e_i and $\bar{e}_i$.

The variety V_n is of importance in birational geometry due to its appearance in the factorization of the standard Cremona transformation of $\mathbb{P}^n$. In fact, as is well known, V_n can be explicitly obtained from $\mathbb{P}^n$ as follows. First blow up the torus fixed points, then flip the (strict transforms) of the lines through these points, then flip the (strict transforms) of planes through these points, ..., up until, and not including, the half-dimensional linear subspaces. The resulting variety is V_n. For more, see [Casagrande 2003].

Since V_n and the blowup of $\mathbb{P}^n$ at its torus fixed points are isomorphic in codimension 1, they have isomorphic Picard groups. We use a basis

$$\{H, E_1, \dots, E_{n+1}\}$$

for $\mathrm{Pic}(V_n)$, which corresponds to the hyperplane section and the exceptional divisors of the blown up $\mathbb{P}^n$. We have

$$[e_i] = E_i, \qquad [\bar{e}_i] = \left(H - \sum_{j=1}^{n+1} E_j\right) + E_j,$$

where S_{n+1} permutes the E_i leaving H fixed, and the involution is represented by the following matrix:

$$\begin{pmatrix} n & 1 & 1 & \cdots & 1 \\ 1-n & 0 & -1 & \cdots & -1 \\ 1-n & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-n & -1 & -1 & \cdots & 0 \end{pmatrix}.$$

For each $c \in \mathbb{Z}$ and $J \subset \{1, \dots, n+1\}$, define

$$F_{c,J} := c\left(\sum_{i=1}^{n+1} E_i - H\right) - \sum_{j \in J} E_j.$$

Note that the involution takes $F_{c,J}$ to $F_{|J|-c,J}$.

Proposition 4.12. *The set of $F_{c,J}$ with*

(1) $|J| - \frac{n}{4} \leq c \leq \frac{n}{4}$ *or*

(2) $\frac{n+2}{4} \leq c \leq |J| - \frac{n+2}{4}$

forms a full strong $(S_{n+1} \times C_2)$-stable exceptional collection on V_n under any ordering of the blocks such that $|J|$ is (nonstrictly) decreasing.

Proof sketch. This collection is the same as that of [Castravet and Tevelev 2017, Theorem 6.6] up to a twist by a line bundle. Thus, we only sketch an argument here, expanded in [Ballard et al. 2018]. One checks that the description of "forbidden cones" given in [Borisov and Hua 2009] shows that relevant cohomology groups vanish — this shows that it is a strong exceptional collection. To prove generation, one considers the series of flips required to reach $\mathbb{P}^n$ blown up at $n+1$ points. Using the description of the semiorthogonal decompositions in [Ballard et al. 2019], the line bundles can be shown to generate the necessary admissible subcategories of each intermediate birational model. □

Since any centrally symmetric toric Fano variety is a product of projective lines and the varieties V_n, Lemma 3.11 yields the following:

Corollary 4.13. *Any form of a centrally symmetric toric Fano variety admits a full strong exceptional collection consisting of vector bundles.*

4E. ***Toric varieties from the Weyl fans of type A.*** One method for identifying toric varieties with large symmetry groups is to start with root systems. Let R be a root system in a Euclidean space E. The $\mathbb{Z}$-lattice generated by R is denoted $M(R)$, while its dual in $E^\vee$ is denoted by $N(R)$. For every set S of simple roots in E, we have the dual cone corresponding to a closed Weyl chamber

$$\sigma_S := \{f \in E^\vee \mid \langle f, \alpha \rangle \geq 0,\ \forall \alpha \in S\}.$$

The cones σ_S are the maximal cones for a fan Σ_R in $E^\vee$. We denote the associated toric variety by $X(R)$. Recall that an automorphism of R is an element of $\mathrm{GL}(E)$ preserving R. Let $W(R)$ be the Weyl group and $\Gamma(R)$ the symmetry group of the Dynkin diagram of R. It is well known that

$$\mathrm{Aut}(R) \simeq W(R) \rtimes \Gamma(R).$$

Any automorphism of R induces an action on the fan $\Sigma(R)$, which yields a homomorphism $\phi : \mathrm{Aut}(R) \to \mathrm{Aut}(\Sigma(R))$.

Lemma 4.14. *The map* $\phi : \mathrm{Aut}(R) \to \mathrm{Aut}(\Sigma(R))$ *is an isomorphism.*

Proof. First note that the set R can be reconstructed from $\Sigma(R)$ by taking the union of the extremal rays generating the dual cones $\sigma_S^\vee$ for all σ_S. Thus any symmetry of the fan induces a symmetry of R. This gives the inverse map to ϕ. □

Here we focus on the case $R = A_n$. In [Losev and Manin 2000], the authors showed that $X(A_n)$ is a moduli space of rational curves with $(n+1)$ marked points and 2 poles. Another useful proof appeared in [Batyrev and Blume 2011].

Using this perspective, [Castravet and Tevelev 2017] exhibited an exceptional collection on $X(A_n)$ that is stable under the action of permuting the marked points and flipping the poles, i.e., an $(S_{n+1} \rtimes C_2)$-stable collection. Here we demonstrate that Castravet and Tevelev's exceptional collection satisfies the conditions of Proposition 3.7 and hence descends to an exceptional collection on any form of $X(A_n)$ (in characteristic 0).

To do this requires a bit of translating divisors and actions from the moduli-theoretic language to the toric language. We recall the moduli-theoretic language.

Definition 4.15. Let N be a set of order n. A *chain of polar* $\mathbb{P}^1$'s is a $(\{0, \infty\} \cup N)$-marked linear nodal chain of $\mathbb{P}^1$'s with 0 on the left tail and ∞ on the right tail. A chain of polar $\mathbb{P}^1$'s is *stable* if

(1) marked points do not coincide with nodes,

(2) only N-marked points are allowed to coincide,

(3) each component of the chain has at least three special points (nodes or marked points).

We write LM_N for the corresponding moduli space. We also use LM_n depending on the context. Note that the universal curve over LM_n is isomorphic to LM_{n+1}.

Theorem 4.16. *The toric variety* $X(A_{n-1})$ *is isomorphic to* LM_n*. Moreover, if we fix an embedding* $A_{n-1} \to A_n$*, the corresponding map* $X(A_n) \to X(A_{n-1})$ *is the universal curve. Moreover,* $X(A_n) \to X(A_{n-1})$ *is a toric morphism.*

Proof. This is [Losev and Manin 2000, Theorem 2.6.3]. See also [Batyrev and Blume 2011, Theorem 3.19]. The map is consequently toric by [Batyrev and Blume 2011, Proposition 1.4]. □

Under this isomorphism, the closures of the torus orbits on $X(A_n)$ have the following moduli-theoretic description. Fix a partition $N_1 \sqcup N_2 = N$ and let δ_{N_1} denote the divisor parametrizing polar chains of length exactly 2 having the first marked by N_1 and the last marked by N_2. For a partition with more parts

$$N_1 \sqcup N_2 \sqcup \cdots \sqcup N_t = N,$$

one has the locus $Z_{N_1,\dots,N_t}$ parametrizing polar chains of length exactly t, where the i-th $\mathbb{P}^1$ is marked by N_i. These loci are precisely the proper torus orbit closures on $X(A_n)$.

Note that each locus is a complete intersection

$$Z_{N_1,\dots,N_t} := \delta_{N_1} \cap \delta_{N_1 \cup N_2} \cap \cdots \cap \delta_{N_1 \cup \cdots \cup N_{t-1}}.$$

Moreover, we have an isomorphism

$$Z_{N_1,\dots,N_t} \simeq LM_{N_1} \times LM_{N_2} \times \cdots \times LM_{N_t},$$

where the left node of each $\mathbb{P}^1$ is marked with 0 and the right node is marked with ∞. Thus, we have toric morphisms

$$i_{N_1,\dots,N_t} : LM_{N_1} \times LM_{N_2} \times \cdots \times LM_{N_t} \to LM_N.$$

Also, for each subset $K \subset N$, we get a forgetful map $\pi_K : LM_N \to LM_K$, which is a toric morphism since it is a composition of maps from Theorem 4.16.

Recall there is a set of line bundles $\mathbb{G}_N$ on LM_N [Castravet and Tevelev 2017, Definition 1.5], and one generates a larger set H_N of sheaves via

$$\mathsf{H}_N := \left\{ (i_{N_1,\dots,N_t})_*(G_{l_1} \boxtimes \cdots \boxtimes G_{l_t}) \mid \forall N_1 \cup \cdots \cup N_t = N,\ G_{l_j} \in \mathbb{G}_{N_j} \right\},$$

where $i_{N_1,\dots,N_t} : Z_{N_1,\dots,N_t} \hookrightarrow LM_N$ is the inclusion.

Theorem 4.17. *There is an ordering on the set*

$$\mathsf{CT}_N := \mathsf{H}_N \cup \left(\bigcup_{K \subsetneq N} \{ \pi_K^* E \mid E \in \mathsf{H}_K \} \right) \cup \{\mathcal{O}\}$$

making it into an $(S_N \rtimes C_2)$-stable exceptional collection under permutations of the two sets of markings.

Proof. This is [Castravet and Tevelev 2017, Proposition 1.5]. □

Proposition 4.18. *The action of $S_{n+1} \rtimes C_2$ given by permuting the two sets of marked points corresponds to the action of* $\operatorname{Aut}(A_n)$ *on $X(A_n)$.*

Proof. We use the standard presentation of the root system for A_n as $e_i - e_j$ for $1 \le i < j \le n+1$ and follow [Batyrev and Blume 2011, Construction 3.6]. The embedding $A_n \hookrightarrow A_{n+1}$ gives the universal curve $X(A_{n+1}) \to X(A_n)$. For $i \in \{1, \dots, n\}$, we take the $(n+1)$ projections $A_{n+1} \to A_n$, whose kernels are generated by $e_i - e_{n+1}$ for $1 \le i \le n+1$. These give sections $s_i : X(A_n) \to X(A_{n+1})$. Finally, for the polar sections, we have the dual vector v_{n+2}. The vectors v_{n+2} and $-v_{n+2}$ give toric invariant divisors which are isomorphic to $X(A_n)$ [Batyrev and Blume 2011, Proposition 1.9]. The isomorphisms give the other sections s_0 and s_∞.

The Weyl group is the permutation group of the e_i, and hence of the $e_i - e_{n+2}$. In particular, it permutes the s_i. The outer involution acts on the fan by negation and thus exchanges the cone corresponding to v_{n+2} with the cone corresponding to $-v_{n+2}$. □

Corollary 4.19. *The set* CT_N *is* $\mathrm{Aut}(\Sigma(A_n))$*-stable.*

Proof. This is an immediate corollary of Lemma 4.14 and Proposition 4.18. □

Proposition 4.20. *Each object in the collection* CT_N *is torus-equivariant.*

Proof. Line bundles are always isomorphic to torus-equivariant line bundles, so all objects in $\mathbb{G}_N$ are torus-equivariant. There is a canonical equivariant structure on tensor products and on pullbacks by equivariant morphisms (see [Ballard et al. 2014, §2]); thus each object $G_1 \boxtimes \cdots \boxtimes G_n$ is torus-equivariant for $G_{l_j} \in \mathbb{G}_{N_j}$. Let $i : Z \to X$ be shorthand for some map $i_{N_1,\dots,N_t}$. There is a splitting of tori $T = S \times S'$ where Z is an S-toric variety and S' acts trivially on $i(Z)$. Let $\psi : T \to S$ denote the projection. We have a composition of functors

$$\mathsf{D}^{\mathsf{b}}(\mathrm{Coh}_S\, Z) \to \mathsf{D}^{\mathsf{b}}(\mathrm{Coh}_T\, Z) \to \mathsf{D}^{\mathsf{b}}(\mathrm{Coh}_T\, X),$$

where the first map is the functor Res_ψ [Ballard et al. 2014, §2.9] and the second map is the T-equivariant pushforward [Ballard et al. 2014, §2.5]. This composition reduces to the ordinary pushforward $i_* : \mathsf{D}^{\mathsf{b}}(Z) \to \mathsf{D}^{\mathsf{b}}(X)$ when the equivariant structure is forgotten. We conclude that each object of H_K is torus-equivariant, and the result follows. □

We now prove the main result of this section.

Proposition 4.21. *Let k be a field of characteristic zero and X a form of $X(A_n)$ over k. Then X admits a full exceptional collection of sheaves.*

Proof. Combining Theorem 4.17, Corollary 4.19, and Proposition 4.20 allows us to appeal to Proposition 3.7 and conclude that CT_N descends to an exceptional collection of sheaves on X. □

Remark 4.22. To remove the characteristic zero assumption one needs to extend generation results of [Castravet and Tevelev 2017] to nonzero characteristic. This could conceivably be done by reversing the flow of reasoning in [Castravet and Tevelev 2017], using the fact that we know the collections for V_n in any characteristic. We do not pursue this.

Acknowledgements

Ballard was partially supported by NSF DMS-1501813. He would also like to thank the Institute for Advanced Study for providing a wonderful research environment. These ideas were developed during his membership. Ballard benefited from discussions with Alicia Lamarche. Duncan was partially supported by NSA grant H98230-16-1-0309. McFaddin would like to thank the Hausdorff Institute for their hospitality and lively research environment during the *K-theory and related fields* trimester program. A large portion of this manuscript was drafted during his time in Bonn. All authors also thank Fei Xie for pointing out that, due to an editing error in a previous version of this paper, Proposition 4.7 stated that all smooth toric surfaces have strong collections of vector bundles instead of a not-necessarily strong collection of sheaves as was proven. We would also like to thank the referees for useful comments.

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Received 13 Apr 2018. Revised 3 Jan 2019. Accepted 18 Jan 2019.

MATTHEW BALLARD: ballard@math.sc.edu
Department of Mathematics, University of South Carolina, Columbia, SC, United States

ALEXANDER DUNCAN: duncan@math.sc.edu
Department of Mathematics, University of South Carolina, Columbia, SC, United States

PATRICK MCFADDIN: pkmcfaddin@gmail.com
Department of Mathematics, University of South Carolina, Columbia, SC, United States

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