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G-theory of root stacks and equivariant K-theory

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Using the description of the category of quasicoherent sheaves on a root stack, we compute the G-theory of root stacks via localization methods. We apply our results to the study of equivariant K-theory of algebraic varieties under certain conditions.

A list of notations and conventions can be found on page 182.

1. Introduction

Let X be an algebraic variety equipped with an action of a finite group G. One would like to compute the equivariant K-theory $K_G(X)$. A first answer was given in the paper [Ellingsrud and Lønsted 1984] in the case when X is a smooth curve. Let us briefly describe it. We set Y to be the quotient X/G, $\phi : X \to Y$ the quotient map, and B the branch locus. Then B is a finite union of G orbits $B_1, \dots, B_n$. Choosing a point $P_i \in B_i$ for each i, denote the inertia group of P_i by H_i. Note that it is a cyclic group. Using some basic properties of equivariant sheaves and the Borel construction, it was proved that there is a decomposition of abelian groups

$$K_G(X) = K(Y) \oplus \bigoplus_{i=1}^{n} R'_k(H_i),$$

where $R'_k(H)$ is the subgroup of a representation ring without invariants, that is, $x \in R'_k(H)$ if $x \in R_k(H)$ and $\langle x, 1_H \rangle = 0$. From here we can guess a flavor of the result in the general case: there should be some kind of a decomposition of $K_G(X)$ onto $K(Y)$ and the terms coming from ramification.

To generalize this to higher dimensions, there are two routes one may take. One may enter the realm of algebraic stacks. For example, Vistoli and Vezzosi [Vistoli 1991; Vezzosi and Vistoli 2002] proved the decomposition formula for $K_G(X)$ of a scheme X using (implicitly) a top-down description of the stack $[X/G]$.

Another route would be to enter the realm of logarithmic geometry; see [Niziół 2008; Hagihara 2003]. These two papers study the K-theory of the Kummer étale site on a logarithmic scheme. Note that, using the correspondence between sheaves on an infinite root stack and sheaves on the Kummer étale site [Talpo and Vistoli

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2018, §6], one can deduce the structure results of [Hagihara 2003, §4] and [Nizioł 2008, Theorem 1.1] from our Theorem 3.32 and Corollary 3.34.

We first discuss the general philosophy of our approach encompassing both of these routes. In algebraic geometry one frequently needs to consider equivariant objects on a scheme X with respect to the action of G. These objects correspond to objects over the quotient stack $[X/G]$. However, it can happen that $[X/G] \cong [X'/G']$ for seemingly unrelated X and X'. In such situation, it is useful to have a canonical description of the quotient stack $[X/G]$, perhaps in terms of its coarse moduli space Y. This may not always be possible but sometimes it is. In this paper we describe a situation in which this occurs (see Theorem 4.10). When our hypotheses are satisfied, the quotient stack becomes a root stack over its coarse moduli space Y.

The root stack construction goes back to [Olsson 2007]. If a quotient stack is "a tool" to take quotients, similarly a root stack can be used to "extract roots" from line bundles on a scheme. It turns out that this construction is quite useful, for example, in Gromov–Witten theory of a Deligne–Mumford stack; see [Abramovich et al. 2008; Cadman 2007; Olsson 2007]. The moduli stack of stable maps from a curve to a stack does not have nice properties, and instead one needs to consider so-called twisted stable maps from a twisted curve. As was shown in [Abramovich et al. 2008], one can replace a twisted curve by a root stack.

Another application of root stacks is the parabolic orbifold correspondence. In a nutshell, this correspondence describes sheaves and vector bundles on a root stack in terms of sheaves and vector bundles on the base with extra data. Parabolic bundles on a Riemann surface were defined in [Mehta and Seshadri 1980], and were shown to be related to a unitary representation of a homotopy group. Borne [2007] proved the equivalence of parabolic bundles and locally free sheaves on a root stack. Finally, Borne and Vistoli [2012] generalized it to the equivalence of quasicoherent sheaves on a root stack and parabolic sheaves.

The results of [Borne and Vistoli 2012] are the foundation of this work. Using their description of coherent sheaves on a root stack, we compute the algebraic G-theory of a root stack. See Theorem 3.32 for the statement of our first main result. The tool necessary for its proof is localization sequences associated with a quotient category. This method can be thought of as an algebraic analog of Segal's localization theorem [1968, Proposition 4.1] for equivariant topological K-theory.

The second result of this work is Theorem 4.10. It says that under certain assumptions a quotient stack is a root stack over its coarse moduli space. The main tool used in the proof is a generalization of Abhyankar's lemma; see [SGA 1 1971, Exposé XIII, Appendice I].

Combining these results gives an immediate application to equivariant K-theory of schemes. This is how we obtain a generalization of the aforementioned decomposition of [Ellingsrud and Lønsted 1984]. We formulate it as Theorem 5.1. If a finite group G acts on a scheme X, then, under some assumptions, we have the decomposition of $K_G(X)$ into the direct sum of groups $K(X/G)$ and G-theory of ramification divisors and their intersections. Note that our assumptions are always satisfied for tame actions of groups on smooth projective curves.

Let us give an outline of the paper for the convenience of the reader. In a short preliminary Section 2 we recall some necessary categorical techniques. We start by studying the G-theory of a root stack in Section 3. First, the description of the category of quasicoherent sheaves on a root stack by [Borne and Vistoli 2012] in Section 3A is recalled. After that we exploit localization methods to decompose the G-theory of parabolic sheaves. Finally, in Section 3D we combine all intermediate results and formulate Theorem 3.32, giving the G-theory of a root stack over a noetherian scheme. We finish the section with the observation in Corollary 3.34 that under some assumptions, the algebraic G-theory of a root stack coincides with its Waldhausen K-theory in the sense of [Joshua 2005].

In Section 4 we address the issue of when a quotient stack is a root stack. First we show that under our assumptions (tameness of the action and ramification divisor is normal crossing), the inertia group is generated in codimension one (see Theorem 4.9). We use Abhyankar's theorem [Grothendieck and Murre 1971, Theorem 2.3.2] in the proof. Then under the same hypothesis, we show that a quotient stack is a root stack (see Theorem 4.10).

The paper ends with Section 5, where we study equivariant K-theory of a scheme by combining the results of the previous two sections. As an example we compute the equivariant K-theory of the affine line and the Burniat surface.

2. Localization via Serre subcategories

2A. ***Serre subcategories.*** Let $\boldsymbol{A}$ be an abelian category. Recall that a *Serre subcategory* S of $\boldsymbol{A}$ is a nonempty full subcategory that is closed under extensions, subobjects and quotients. When $\boldsymbol{A}$ is well-powered the quotient category $\boldsymbol{A}/S$ exists; see [Swan 1968, p. 44, Theorem 2.1].

We need the following result to identify quotient categories.

Theorem 2.1. *Let $F : \boldsymbol{A} \to \boldsymbol{B}$ be an exact functor between abelian categories. Denote by S the full subcategory whose objects are x with $F(x) \cong 0$. Then S is a Serre subcategory and we have a factorization*

$$\begin{tikzcd} \boldsymbol{A} \arrow[r] \arrow[d, "F"] & \boldsymbol{A}/S \arrow[dl] \\ \boldsymbol{B} & \end{tikzcd}$$

Proof. See [Swan 1968, p. 114] □

Definition 2.2. The category S is called the *kernel of the functor F* and is denoted by $\mathbf{ker}(F)$.

Theorem 2.3. *In the situation of the previous theorem suppose the following hold*:

(1) *for every object $y \in \boldsymbol{B}$ there is $x \in \boldsymbol{A}$ such that $F(x)$ is isomorphic to y, and*

(2) *for every morphism $f : F(x) \to F(x')$ there is $x'' \in \boldsymbol{A}$ with $h : x'' \to x$ and $g : x'' \to x'$ such that $F(h)$ is an isomorphism and the following diagram commutes*:

$$\begin{tikzcd} F(x'') \arrow[d, "F(h)"'] \arrow[dr, "F(g)"] & \\ F(x) \arrow[r, "f"] & F(x') \end{tikzcd}$$

Then there is an equivalence of categories $\boldsymbol{A}/S \cong \boldsymbol{B}$.

Proof. See [Swan 1968, p. 114, Theorem 5.11]. □

2B. ***Some functor categories.*** Consider n-tuples of integers $\vec{r} = (r_1, r_2, \dots, r_n)$ and $\vec{s} = (s_1, s_2, \dots, s_n)$. We denote by $[\vec{r}, \vec{s}]$ the poset of n-tuples $(x_1, \dots, x_n)$ with

$$x_i \in \mathbb{Z} \quad \text{and} \quad r_i \le x_i \le s_i.$$

We make use of the shorthand notation

$$rI = [0, r] \quad \text{and} \quad \vec{r}I^n = [0, \vec{r}].$$

These intervals are naturally posets with

$$(x_1, x_2, \dots, x_n) \le (y_1, y_2, \dots, y_n) \quad \text{if and only if} \quad x_i \le y_i \text{ for all } i.$$

This poset structure allows us to view them as categories in the usual way.

Fix an abelian category $\boldsymbol{A}$ and consider the functor category

$$\mathrm{Func}(\vec{r}I^n, \boldsymbol{A}).$$

This category is abelian with kernels and cokernels formed pointwise. We are interested in the K-theory of such categories. In this subsection we try to understand some of their quotient categories. Given an object $\mathcal{F}$ in this category and an object u of $\vec{r}I^n$, we denote by $\mathcal{F}_u \in \boldsymbol{A}$ the value of the functor $\mathcal{F}$ on this object, and if $u \le v$, the arrow from F_u to F_v is denoted by

$$F_{+(v-u)} : F_u \to F_v.$$

In particular, we take $e_i = (0, 0, \dots, 1, 0, \dots, 0)$ to be a standard basis vector, so that we have a morphism

$$F_{+e_i} : F_{(u_1,\dots,u_n)} \to F_{u_1,\dots,u_{i-1},u_i+1,u_{i+1},\dots,u_n}.$$

Lemma 2.4. *Giving an object F of* $\mathrm{Func}(\vec{r}I^n, \boldsymbol{A})$ *is the same as providing the following data*:

(D1) *objects* $F_{(u_1,u_2,\dots,u_n)} \in \boldsymbol{A}$,

(D2) *arrows*

$$F_{+e_i} : F_u \to F_{u+e_i}$$

such that all diagrams of the form

$$\begin{tikzcd} F_u \arrow[r] \arrow[d] & F_{u+e_j} \arrow[d] \\ F_{u+e_i} \arrow[r] & F_{u+e_i+e_j} \end{tikzcd}$$

commute.

Proof. The hypotheses ensure that if $u \leq v$ in $\vec{r}I^n$ then there is a well-defined map $F_u \to F_v$ which produces our functor. $\square$

Proposition 2.5. (i) *Let* $\mathrm{tr}_{n-1}(\vec{r}) = (r_1, r_2, \dots, r_{n-1})$. *There is an exact functor*

$$\pi : \mathrm{Func}(\vec{r}I^n, \boldsymbol{A}) \to \mathrm{Func}(\mathrm{tr}_{n-1}(\vec{r})I^{n-1}, \boldsymbol{A})$$

defined on objects by

$$\pi(G)_{(u_1,u_2,\dots,u_{n-1})} = (G)_{(u_1,\dots,u_{n-1},0)}.$$

(ii) *The functor π has a left adjoint*, *denoted π^*. We have $\pi \circ \pi^* \simeq 1$.*

(iii) *The functor π^* is fully faithful.*

Proof. (i) There is an inclusion functor $\mathrm{tr}_{n-1}(\vec{r})I^{n-1} \hookrightarrow \vec{r}I^n$ defined by

$$(x_1, x_2, \dots, x_{n-1}) \mapsto (x_1, x_2, \dots, x_{n-1}, 0).$$

The functor π is just the restriction along this inclusion. The exactness follows from the fact that in functor categories, limits and colimits are computed pointwise.

(ii) Given a functor $F \in \mathrm{Func}(\mathrm{tr}_{n-1}(\vec{r})I^{n-1}, \boldsymbol{A})$, we need to construct an object $\pi^*(F) \in \mathrm{Func}(\vec{r}I^n, \boldsymbol{A})$. We set

$$\pi^*(F)_{(u_1,u_2,\dots,u_n)} = F_{(u_1,u_2,\dots,u_{n-1})}.$$

To produce a functor, we need maps

$$\lambda^i_{(u_1,\dots,u_n)} : \pi^*(\mathcal{F})_{(u_1,\dots,u_i,\dots,u_n)} \to \pi^*(F)_{(u_1,\dots,u_i+1,\dots,u_n)}.$$

We define

$$\lambda^i_{(u_1,\dots,u_n)} = \begin{cases} F_{(u_1,\dots,u_i,\dots,u_{n-1})} \to F_{(u_1,\dots,u_i+1,\dots,u_{n-1})} & \text{if } i < n, \\ \text{identity} & \text{if } i = n. \end{cases}$$

One checks that the hypotheses of Lemma 2.4 are satisfied. Observe that $\pi \circ \pi^* = 1$. This produces a natural map

$$\mathrm{Hom}(\pi^*(F), G) \to \mathrm{Hom}(F, \pi(G)).$$

To see that this is a bijection, suppose that we are given a morphism $\beta : F \to \pi(G)$. There is a diagram, where the dashed arrow is defined to be the composition,

$$\begin{tikzcd} \pi^*(F)_{(u_1,\dots,u_n)} \arrow[r, dashed] \arrow[d, equal] & G_{(u_1,\dots,u_n)} \\ F_{(u_1,\dots,u_{n-1})} \arrow[r, "\beta"'] & G_{(u_1,\dots,u_{n-1},0)} \arrow[u] \end{tikzcd}$$

This produces a natural morphism

$$\mathrm{Hom}(\pi^*(F), G) \leftarrow \mathrm{Hom}(F, \pi(G))$$

and we check that it is inverse to the previous map.

(iii) We have

$$\mathrm{Hom}(\pi^*(F), \pi^*(F')) = \mathrm{Hom}(F, \pi\pi^*(F')) = \mathrm{Hom}(F, F'). \qquad \square$$

Theorem 2.6. (1) *The functor*

$$\pi : \mathrm{Func}(\vec{r}I^n, \boldsymbol{A}) \to \mathrm{Func}(\mathrm{tr}_{n-1}(\vec{r})I^{n-1}, \boldsymbol{A})$$

satisfies the hypothesis of Theorem 2.3.

(2) *Let* $\vec{s} = (r_1, r_2, \dots, r_{n-1}, r_n - 1)$. *If* $r_n > 0$ *then the kernel of this functor is equivalent to* $\mathrm{Func}(\vec{s}I^n, \boldsymbol{A})$.

(3) *If* $r_n = 0$ *then there is an equivalence of categories*

$$\mathrm{Func}(\vec{r}I^n, \boldsymbol{A}) \cong \mathrm{Func}(\mathrm{tr}_{n-1}(\vec{r})I^{n-1}, \boldsymbol{A}).$$

Proof. (1) The functor π is exact so it remains to check the two conditions of the theorem. The first condition follows from the fact that $\pi \circ \pi^*$ is the identity. Now suppose that we have a morphism $\pi(F) \to \pi(F')$. By adjointness we obtain a diagram

$$\begin{tikzcd} \pi^*\pi(F) \arrow[d] \arrow[dr] & \\ F & F' \end{tikzcd}$$

Applying π to this picture shows that the second condition holds.

(2) The functor π was defined by the rule $\pi(G)_{(u_1,u_2,\dots,u_{n-1})} = (G)_{(u_1,\dots,u_{n-1},0)}$. So it is clear that if $\pi G \cong 0$ then $(G)_{(u_1,\dots,u_{n-1},0)} \cong 0$ and giving an object G of $\mathbf{ker}\,\pi$ is the same (up to isomorphism) as giving the objects $(G)_{(u_1,\dots,u_n)} \in \boldsymbol{A}$ for

all $u \in \vec{r}I^n$, $u_n \neq 0$. And according to Lemma 2.4 it is the same as providing an object of the category $\mathrm{Func}(\vec{s}I^n, \boldsymbol{A})$.

(3) If $r_n = 0$ then we have an equivalence of categories $\mathrm{tr}_{n-1}(\vec{r}) \cong \vec{r}$. □

3. Coherent sheaves on root stacks

3A. *Preliminary results.* Recall that if M is a commutative monoid then $\widehat{M} = \mathrm{Hom}(M, \mathbb{G}_m)$ is its dual.

In this subsection we recall the main constructions and theorems from [Borne and Vistoli 2012], to which we refer the reader for further details. Let's start by defining a root stack.

Let X be a scheme. Denote by $\mathfrak{Div}\, X$ the groupoid of line bundles over X with sections. It has the structure of a symmetric monoidal category with tensor product given by

$$(L, s) \otimes (L', s') = (L \otimes L', s \otimes s').$$

Choosing n objects $(L_1, s_1), \ldots, (L_n, s_n)$ of $\mathfrak{Div}\, X$ allows us to define a symmetric monoidal functor (see [Borne and Vistoli 2012, Definition 2.1])

$$L : \mathbb{N}^n \to \mathfrak{Div}\, X, \quad (k_1, \ldots, k_n) \mapsto (L_1, s_1)^{\otimes k_1} \otimes \cdots \otimes (L_n, s_n)^{\otimes k_n}.$$

Such functors arise from morphisms $X \to [\mathrm{Spec}\, \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$. Let us recall how.

Proposition 3.1. (i) *Let $\boldsymbol{A}$ be the groupoid whose objects are quasicoherent $\mathcal{O}_X$-algebras $\mathcal{A}$ with a $\mathbb{Z}^n = \widehat{\widehat{\mathbb{N}^n}}$-grading $\mathcal{A} = \bigoplus_{u \in \mathbb{Z}^n} \mathcal{A}_u$ such that each summand $\mathcal{A}_u$ is an invertible sheaf. The morphisms are graded algebra isomorphisms. Then there is an equivalence of categories between $\boldsymbol{A}^{\mathrm{op}}$ and the groupoid of $\widehat{\mathbb{N}^n}$-torsors $P \to X$.*

(ii) *Let $\boldsymbol{B}$ be the groupoid whose objects are pairs $(\mathcal{A}, \alpha)$, where $\mathcal{A}$ is a sheaf of algebras satisfying the conditions in (i) and*

$$\alpha : \mathcal{O}_X[\mathbb{N}^n] \to \mathcal{A}$$

is a morphism respecting the grading. The morphisms in the category $\boldsymbol{B}$ are graded algebra morphisms commuting with the structure maps. Then there is an equivalence of categories between $\boldsymbol{B}^{\mathrm{op}}$ and the groupoid of morphisms $X \to [\mathrm{Spec}\, \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$.

Proof. This proposition is a summary of the discussion in [Borne and Vistoli 2012, p. 1343–1344], in particular the proof of Proposition 3.25. The detailed proof can be found there. Here we just illustrate the main idea behind the proof.

(i) The torsor $\pi : P \to X$ is determined by the sheaf of algebras $\pi_*(\mathcal{O}_P)$, which has a $\widehat{\mathbb{N}^n}$-action, and hence a weight grading. As the torsor is locally trivial, the condition about the summands being invertible follows by considering the algebra associated with the trivial torsor.

(ii) This follows from the standard description of the groupoid of X-points of a quotient stack. Finally, in [Borne and Vistoli 2012], the **fppf** topology is needed but in the present work it is not. The setting in [loc. cit.] is more general and the monoids in question may have torsion, so that the torsor P is a torsor over μ_n. Such a torsor may not be trivial in the Zariski topology, unlike a $\mathbb{G}_m$-torsor. Hence a finer topology is needed. See the proof of [Borne and Vistoli 2012, Lemma 3.26]. □

Corollary 3.2. *There is an equivalence of categories between the groupoid of symmetric monoidal functors*

$$\mathbb{N}^n \to \mathfrak{Div}\, X$$

and the groupoid of X-points of $[\operatorname{Spec} \mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}]$.

Proof. For details see [Borne and Vistoli 2012, Proposition 3.25]. In essence, the symmetric monoidal functor determined by $(L_1, s_1), \dots, (L_n, s_n)$ produces the graded sheaf of algebras

$$\mathcal{A} = \bigoplus_{\vec{u} \in \mathbb{Z}^n} L_1^{u_1} \otimes \cdots \otimes L_n^{u_n}.$$

The sections produce an algebra map

$$\mathcal{O}_X[\mathbb{N}^n] \to \mathcal{A}.$$

□

Definition 3.3. Let $\vec{r} = (r_1, r_2, \dots, r_n)$ be a collection of positive natural numbers. We denote by $r_i\mathbb{N}$ the monoid $\{vr_i \mid v \in \mathbb{N}\}$. We denote by $\vec{r}\mathbb{N}^n$ the monoid

$$\vec{r}\mathbb{N}^n = r_1\mathbb{N} \times r_2\mathbb{N} \times \cdots \times r_n\mathbb{N}.$$

We view our symmetric monoidal functor above as a functor

$$L : \vec{r}\mathbb{N}^n \to \mathfrak{Div}\, X, \quad (r_1\alpha_1, r_2\alpha_2, \dots, r_n\alpha_n) \mapsto (L_1, s_1)^{\otimes\alpha_1} \otimes \cdots \otimes (L_n, s_n)^{\otimes\alpha_n}.$$

Consider the natural inclusion of monoids $j_{\vec{r}} : \vec{r}\mathbb{N}^n \hookrightarrow \mathbb{N}^n$. The *category of $\vec{r}$-th roots of L*, denoted by $(L)_{\vec{r}}$, is defined as follows.

Its objects are pairs (M, α), where $M : \mathbb{N}^n \to \mathfrak{Div}\, X$ is a symmetric monoidal functor, and $\alpha : L \to M \circ j$ is an isomorphism of symmetric monoidal functors.

An arrow from (M, α) to (M', α') is an isomorphism $h : M \to M'$ of symmetric monoidal functors $\mathbb{N}^n \to \mathfrak{Div}\, X$, such that the diagram

$$\begin{tikzcd} & L \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\ M \circ j \arrow[rr, "h \circ j"] & & M' \circ j \end{tikzcd}$$

commutes.

This category is in fact a groupoid, as a morphism ϕ in $\mathfrak{Div}\, X$, whose tensor power $\phi^{\otimes k}$ is an isomorphism, must be an isomorphism to begin with.

Given a morphism of schemes $t : T \to X$ there is pullback functor

$$t^* : \mathfrak{Div}\, X \to \mathfrak{Div}\, T.$$

Hence we can form the category of roots $(t^* \circ L)_{\vec{r}}$. This construction pastes together to produce a pseudofunctor $\mathfrak{Div}_X$, where

$$\mathfrak{Div}_X \to \mathrm{Sch}/X$$

is the symmetric monoidal stack described in [Borne and Vistoli 2012, p. 1335].

Definition 3.4. In the above situation, the fibered category associated with this pseudofunctor is called *the stack of roots* associated with L and $\vec{r}$. It is denoted by $X_{L,\vec{r}}$.

We often denote the stack of roots by

$$X_{L,\vec{r}} = X_{(L_1,s_1,r_1),\dots,(L_n,s_n,r_n)}.$$

There are also two equivalent definitions of the stack $X_{L,\vec{r}}$, and the equivalence is proved in [Borne and Vistoli 2012, Proposition 4.13 and Remark 4.14]. Let's recall the description of this stack as a fibered product.

Proposition 3.5. *The stack $X_{L,\vec{r}}$ is isomorphic to the fibered product*

$$X \times_{\mathrm{Spec}\,\mathbb{Z}[\vec{r}\mathbb{N}^n]} [\mathrm{Spec}\,\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}].$$

According to (a slightly modified version of) Corollary 3.2, a symmetric monoidal functor $L : \vec{r}\mathbb{N}^n \to \mathfrak{Div}\, X$ corresponds to a morphism

$$X \to [\mathrm{Spec}\,\mathbb{Z}[\vec{r}\mathbb{N}^n]/\widehat{\vec{r}\mathbb{N}^n}],$$

which in turn corresponds to an $\widehat{\vec{r}\mathbb{N}^n}$-torsor $\pi : P \to X$ and an $\widehat{\vec{r}\mathbb{N}^n}$-equivariant morphism $P \to \mathrm{Spec}\,\mathbb{Z}[\vec{r}\mathbb{N}^n]$. This gives the next proposition.

Proposition 3.6. *The stack $X_{L,\vec{r}}$ is isomorphic to the quotient stack*

$$[P \times_{\mathrm{Spec}\,\mathbb{Z}[\vec{r}\mathbb{N}^n]} \mathrm{Spec}\,\mathbb{Z}[\mathbb{N}^n]/\widehat{\mathbb{N}^n}],$$

where the action on the first factor is defined through the dual of the inclusion $j_{\vec{r}} : \vec{r}\mathbb{N}^n \hookrightarrow \mathbb{N}^n$.

Proof. See [Borne and Vistoli 2012, p. 1350]. □

We recall the definition of parabolic sheaf; see [Borne and Vistoli 2012, Definition 5.6].

Definition 3.7. Consider a scheme X, an inclusion $\vec{r}\mathbb{Z}^n \subseteq \mathbb{Z}^n$ and a symmetric monoidal functor $L : \vec{r}\mathbb{Z}^n \to \mathfrak{Div}\, X$, defined by

$$L_u = L(u) = L_1^{\alpha_1} \otimes \cdots \otimes L_n^{\alpha_n},$$

where $u = (r_1\alpha_1, \ldots, r_n\alpha_n)$ and each $\alpha_i \in \mathbb{Z}$. A *parabolic sheaf* (E, ρ) on (X, L) with denominators $\vec{r}$ consists of the following data:

(a) A functor $E : \mathbb{Z}^n \to \mathfrak{QCoh}\, X$, denoted by $v \mapsto E_v$ on objects and $b \mapsto E_b$ on arrows.

(b) For any $u \in \vec{r}\mathbb{Z}^n$ and $v \in \mathbb{Z}^n$, an isomorphism

$$\rho^E_{u,v} : E_{u+v} \simeq L_u \otimes_{\mathcal{O}_X} E_v$$

of $\mathcal{O}_X$-modules. This map is called the *pseudoperiod isomorphism*.

These data are required to satisfy the following conditions. Take $u, u' \in \vec{r}\mathbb{Z}^n$, $a = (r_1\alpha_1, \ldots, r_n\alpha_n) \in \vec{r}\mathbb{N}^n$, $b \in \mathbb{N}^n$, $v \in \mathbb{Z}^n$. Then the following diagrams commute:

(i)
$$\begin{tikzcd} E_v \arrow[r, "E_a"] \arrow[d, "\simeq"] & E_{a+v} \arrow[d, "\rho^E_{a,v}"] \\ \mathcal{O}_X \otimes E_v \arrow[r, "\sigma^L_a \otimes \mathrm{id}_{E_v}"] & L_a \otimes E_v \end{tikzcd}$$

where $\sigma_a = \sigma^{\alpha_1} \otimes \cdots \otimes \sigma^{\alpha_n} \in \mathrm{H}^0(X, L_a)$.

(ii)
$$\begin{tikzcd} E_{u+v} \arrow[r, "\rho^E_{u,v}"] \arrow[d, "E_b"] & L_u \otimes E_v \arrow[d, "\mathrm{id} \otimes E_b"] \\ E_{u+b+v} \arrow[r, "\rho^E_{u,b+v}"] & L_u \otimes E_{b+v} \end{tikzcd}$$

(iii)
$$\begin{tikzcd} E_{u+u'+v} \arrow[r, "\rho^E_{u+u',v}"] \arrow[d, "\rho^E_{u,u'+v}"] & L_{u+u'} \otimes E_v \arrow[d, "\mu \otimes \mathrm{id}"] \\ L(u) \otimes E_{u'+v} \arrow[r, "\mathrm{id} \otimes \rho^E_{u',v}"] & L_u \otimes L_{u'} \otimes E_v \end{tikzcd}$$

(iv) The map

$$E_v = E_{0+v} \xrightarrow{\rho^E_{0,v}} \mathcal{O}_X \otimes E_v$$

is the natural isomorphism.

Definition 3.8. A parabolic sheaf (E, ρ) is said to be *coherent* if for each $v \in \mathbb{Z}^n$ the sheaf E_v is a coherent sheaf on X.

Theorem 3.9 (Borne, Vistoli). *Let X be a scheme and L a monoidal functor defined as in the beginning of this section. Then there is a canonical tensor equivalence of abelian categories between the category $\mathfrak{QCoh}\, X_{L,\vec{r}}$ and the category of parabolic sheaves on X, associated with L.*

Proof. See [Borne and Vistoli 2012, Proposition 5.10, Theorem 6.1] for details. The proof relies on the description of the stack as a quotient as in Proposition 3.6. From this description, sheaves on the stack are equivariant sheaves on

$$P \times_{\operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]} \times \operatorname{Spec} \mathbb{Z}[\mathbb{N}^n].$$

As remarked in the proof of Proposition 3.1, the torsor P is obtained from a sheaf of algebras on X. The sheaf of algebras $\mathcal{A}$ is constructed from the functor L by taking a direct sum construction; it has a natural grading. It follows that the scheme

$$P \times_{\operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]} \operatorname{Spec} \mathbb{Z}[\mathbb{N}^n] = \operatorname{Spec}(\mathcal{A} \otimes_{\mathbb{Z}[\vec{r}\mathbb{N}^n]} \mathbb{Z}[\mathbb{N}^n]).$$

The algebra on the right has a natural $\mathbb{Z}[\mathbb{N}^n]$-grading; see the corollary below for a local description. It follows that the equivariant sheaves on the scheme in question are just graded modules over this algebra. The proof follows by reinterpreting the graded modules in terms of the symmetric monoidal functor L. $\square$

Actually we can add the finiteness condition to the previous theorem and get the following:

Corollary 3.10. *Let X be a locally noetherian scheme. There is a canonical tensor equivalence of abelian categories between the category $\mathfrak{Coh}\, X_{L,\vec{r}}$ and the category of coherent parabolic sheaves on X, associated with L.*

Proof. We make use of the identifications in the above proof. The question is local on X, so we may assume that X is in fact an affine scheme $\operatorname{Spec}(R)$. By further restrictions we can assume that all the line bundles L_i are in fact trivial, and we identify them with R. In this situation the symmetric monoidal functor corresponds to a graded homomorphism

$$\mathbb{Z}[X_1, X_2, \ldots, X_n] \to R[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$$

sending X_i to $x_i t_i$ with $x_i \in R$. Further, the morphism

$$\operatorname{Spec}(\mathbb{Z}[\mathbb{N}^n]) \to \operatorname{Spec} \mathbb{Z}[\vec{r}\mathbb{N}^n]$$

comes from an integral extension of algebras

$$\mathbb{Z}[X_1, X_2, \ldots, X_n][Y_1, \ldots, Y_n]/(Y_1^{r_1} - X_1, \ldots, Y_n^{r_n} - X_n).$$

Then taking tensor products yields a $\mathbb{Z}^n$-graded algebra

$$A = R[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_n^{\pm 1}][s_1, \dots, s_n]/(s_1^{r_1} - x_1 t_1, \dots, s_n^{r_n} - x_n t_n),$$

where s_i has degree $(0, \dots, 0, 1, 0 \dots, 0) = e_i$. Now consider a finitely generated graded A-module M. We can assume that the generators of M are in fact homogeneous and hence there is an epimorphism

$$\bigoplus_{i=1}^{p} A(n_i) \to M.$$

The graded pieces of the module on the left are free of rank p and hence the graded pieces of M are finitely generated. It follows that a finitely generated A-module gives rise to a parabolic sheaf with values in the category of finitely generated R-modules — in other words, coherent sheaves on X.

Conversely, suppose that we have a graded A-module M with each graded piece a finitely generated R-module. We can find finitely many elements of M, let's say $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ of degrees

$$\deg(\alpha_i) = (\lambda_{i1}, \lambda_{i2}, \dots, \lambda_{in}) \in \mathbb{Z}^n$$

with $0 \le \lambda_{ij} \le r_j$, such that the associated morphism

$$\phi : \bigoplus_{i=1}^{p} A(\deg(\alpha_i)) \to M$$

is an epimorphism in degrees

$$(\mu_1, \mu_2, \dots, \mu_n) \in \mathbb{Z}^n$$

whenever $0 \le \mu_i \le r_i$. It follows that ϕ is an epimorphism and multiplication by t_i induces an isomorphism $M_v \xrightarrow{\sim} M_{v+e_i}$. □

3B. ***An extension lemma.*** The goal of this subsection is to slightly simplify the formulation of parabolic sheaves in the present context using the pseudoperiodicity condition. This will be needed to study K-theory in the next section. We let

$$e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^n,$$

where the 1 is in the i-th spot.

Definition 3.11. Let X be a scheme and L a symmetric monoidal functor

$$L : \vec{r}\mathbb{Z}^n \to \mathfrak{Div}_X,$$

determined by n divisors (L_i, s_i). An *extendable pair* (F, ρ) on (X, L) consists of the following data:

(a) A functor $F_\bullet : \vec{r}I^n \to \mathfrak{QCoh}(X)$.

(b) For any $\alpha \in \vec{r}I^n$ such that $\alpha_i = r_i$, an isomorphism of $\mathcal{O}_X$-modules

$$\rho_{\alpha,\alpha-r_ie_i} : F_\alpha \xrightarrow{\sim} L_i \otimes F_{\alpha-r_ie_i}.$$

We frequently drop the subscripts from the notation involving ρ, when they are clear from the context.

This data is required to satisfy the following three conditions:

(EX1) For all $i \in \{1, \ldots, n\}$ and $\alpha \in \vec{r}I^n$, the diagram

$$\begin{tikzcd} F_\alpha \arrow[r, "F_{+(r_i-\alpha_i)e_i}"] \arrow[d, "\sigma_i"] & F_{\alpha+(r_i-\alpha_i)e_i} \arrow[d, "\rho"] \\ L_i \otimes F_\alpha & L_i \otimes F_{\alpha-\alpha_ie_i} \arrow[l, "L_i \otimes F_{+\alpha_ie_i}"] \end{tikzcd}$$

commutes, where σ_i is multiplication by the section s_i.

(EX2) For all $i \neq j$ and α with $\alpha_i = r_i$, the diagram

$$\begin{tikzcd} F_\alpha \arrow[r, "\rho"] \arrow[d, "F_{\vec{e}_j}"'] & L_i \otimes F_{\alpha-r_ie_i} \arrow[d, "F_{\vec{e}_j}"] \\ F_{\alpha+e_j} \arrow[r, "\rho"] & L_i \otimes F_{\alpha+e_j-r_ie_i} \end{tikzcd}$$

commutes.

(EX3) For all i and j and $\alpha \in \vec{r}I^n$ with $\alpha_i = r_i$ and $\alpha_j = r_j$, the diagram

$$\begin{tikzcd} F_\alpha \arrow[r, "\rho"] \arrow[d, "\rho"'] & L_i \otimes F_{\alpha-r_ie_i} \arrow[d, "\rho"] \\ L_j \otimes F_{\alpha-r_je_j} \arrow[r, "\rho"] & L_i \otimes L_j \otimes F_{\alpha-r_ie_i-r_je_j} \end{tikzcd}$$

commutes.

Definition 3.12. An extendable pair (F, ρ) is called *coherent* if for each $v \in \vec{r}I^n$, the sheaf F_v is a coherent sheaf on X.

Proposition 3.13. *Let X be a scheme and L a symmetric monoidal functor as in Definition 3.7. Let (E, ρ) be a parabolic sheaf on (X, L) with denominators $\vec{r}$. Then the restricted functor $E|_{\vec{r}I^n}$ produces an extendable pair on (X, L).*

Proof. Note that the restricted functor has all the required data for an extendable pair by restricting the collection $\rho_{\alpha,\beta}$. We need to check that the axioms of an extendable pair are satisfied.

(EX1) We have that the composition

$$E_{\alpha+(r_i-\alpha_i)e_i} \xrightarrow{\rho} E_{\alpha-\alpha_i e_i} \otimes L_i \to E_\alpha \otimes L_i \xrightarrow{\rho^{-1}} E_{\alpha+r_i e_i}$$

is just the morphism $E_{+\alpha_i e_i}$ using axiom (ii) of parabolic sheaves. Precomposing with the map

$$E_{+(r_i-\alpha_i)e_i} : E_\alpha \to E_{\alpha+(r_i-\alpha_i)e_i}$$

gives the morphism $E_{+r_i e_i}$. The result now follows from axiom (i).

(EX2) This follows directly from axiom (ii).

(EX3) This follows directly from axiom (iii). □

Proposition 3.14. *Let X be a scheme and L a symmetric monoidal functor as in Definition 3.7. Given an extendable pair (F, ρ) on (X, L) we can extend it to a parabolic sheaf $(\hat{F}, \rho)$ on X, L and the extension is unique up to a canonical isomorphism. A coherent extendable pair extends to a coherent parabolic sheaf.*

Proof. For $v \in \mathbb{Z}^n$ we need to define its extension $\hat{F}_v$. We can write $v_i = r_i u_i + q_i$ with $0 \le q_i < r_i$ and $u_i \in \mathbb{Z}$. As before we let $L_u = \bigotimes_{i=1}^n L^{\otimes u_i}$ and $q = (q_1, \ldots, q_n)$. Set $\hat{F}_v = L_u \otimes F_q$.

We need to construct maps

$$\hat{F}_{+e_i} : \hat{F}_v \to \hat{F}_{v+e_i}.$$

If $q_i < r_i - 1$ then the map is obtained by tensoring the map $F_{q_i} \to F_{q_i+e_i}$ with L_u. If $q_i = r_i - 1$ then the map is defined by

$$\begin{tikzcd}
\hat{F}_v = L_u \otimes F_q \arrow[rr, "\hat{F}_{e_i}"] \arrow[dr, "1 \otimes F_{\vec{e}_i}"'] & & \hat{F}_{v+e_i} = L_u \otimes L_i \otimes F_{q'} \\
& L_u \otimes F_{q+e_i} \arrow[ur, "1 \otimes \rho"'] &
\end{tikzcd}$$

where $q'_j = q_j$ for all $j \neq i$ and $q'_i = 0$.

In order to show that the construction above indeed produces a functor, we need to show that all diagrams of Lemma 2.4 commute. If both $q_i < r_i - 1$ and $q_j < r_j - 1$, then this is straightforward. If $q_i = r_i - 1$ and $q_j < r_j - 1$, then this follows from (EX2). This leaves the case $q_i = r_i - 1$ and $q_j = r_j - 1$. We have a diagram

$$\begin{tikzcd}
L_u \otimes F_q \arrow[r] \arrow[d] & L_u \otimes F_{q+e_i} \arrow[r] \arrow[d] & L_u \otimes L_i \otimes F_{q-q_ie_i} \arrow[d] \\
L_u \otimes F_{q+e_j} \arrow[r] \arrow[d] & L_u \otimes F_{q+e_i+e_j} \arrow[r] \arrow[d] & L_u \otimes L_i \otimes F_{q-q_ie_i+e_j} \arrow[d] \\
L_u \otimes L_j \otimes F_{q-q_je_j} \arrow[r] & L_u \otimes L_j \otimes F_{q-q_je_j+e_i} \arrow[r] & L_u \otimes L_i \otimes L_j \otimes F_{q-q_ie_i-q_je_j}
\end{tikzcd}$$

The top left square commutes using the fact that F is a functor. The top right and bottom left squares commute using axiom (EX2). The bottom right square commutes using axiom (EX3). So indeed $\hat{F}_\bullet$ is a functor.

Note that we have canonical isomorphisms $L_u \otimes L_v \cong L_{u+v}$ for $u, v \in \vec{r}\mathbb{Z}$. These isomorphisms induce our pseudoperiod isomorphisms.

Finally, we need to check the conditions (i)–(iv) of a parabolic sheaf.

(i): For $\vec{r}\alpha, \vec{r}\alpha' \in \vec{r}\mathbb{N}^n$ the diagram

$$\begin{tikzcd}
\hat{F}_v \arrow[r, "\hat{F}_{+\vec{r}\alpha}"] & \hat{F}_{v+\vec{r}\alpha} \arrow[r, "\hat{F}_{+\vec{r}\alpha'}"] \arrow[d] & \hat{F}_{v+\vec{r}\alpha+\vec{r}\alpha'} \arrow[d] \\
& L_\alpha \otimes \hat{F}_v \arrow[r] & L_\alpha \otimes L_{\alpha'} \otimes \hat{F}_v
\end{tikzcd}$$

commutes. This follows by the definition of the functor $\hat{F}_\bullet$ and the symmetric monoidal structure of L.

This allows us to make the following reduction: in order to check axiom (i), it suffices to check that the diagram

$$\begin{tikzcd}
\hat{F}_v \arrow[r] \arrow[dr, "\sigma_i"'] & \hat{F}_{v+r_ie_i} \arrow[d, "\rho"] \\
& L_i \otimes \hat{F}_v
\end{tikzcd}$$

commutes. And this follows directly from (EX1).

(ii): Once again we reduce to showing that

$$\begin{tikzcd}
\hat{F}_{v+r_ie_i} \arrow[r] \arrow[d, "\hat{F}_{+b}"] & L_i \otimes \hat{F}_v \arrow[d, "L_i \otimes \hat{F}_{+b}"] \\
\hat{F}_{v+b+r_ie_i} \arrow[r] & L_i \otimes \hat{F}_{v+b}
\end{tikzcd}$$

commutes. If we write $v = \vec{r}u + q$ then this diagram becomes

$$\begin{tikzcd} L_{u+e_i} \otimes F_q \arrow[r] \arrow[d, "L_{u+e_i} \otimes \hat{F}_{+b}"] & L_i \otimes (L_u \otimes F_q) \arrow[d, "L_i \otimes L_u \otimes \hat{F}_{+b}"] \\ L_{u+e_i} \otimes \hat{F}_{q+b} \arrow[r] & L_i \otimes (L_u \otimes \hat{F}_{q+b}) \end{tikzcd}$$

We can use the symmetric monoidal structure of L to show that this diagram indeed commutes.

(iii): We reduce to showing the commutativity of the diagram

$$\begin{tikzcd} \hat{F}_{v+r_ie_i+r_je_j} \arrow[r] \arrow[d] & L_i \otimes \hat{F}_{v+r_je_j} \arrow[d] \\ L_j \otimes \hat{F}_{v+r_ie_i} \arrow[r] & L_i \otimes L_j \otimes \hat{F}_v \end{tikzcd}$$

which follows from the monoidal structure of L.

Condition (iv) is by definition.

Finally, let $E_\bullet$ be another extension of $F_\bullet$. Again we can again write $v_i = r_iu_i + q_i$ with $0 \le q_i < r_i$ and $u_i \in \mathbb{Z}$. By pseudoperiodicity, $E_v \simeq L(u) \otimes E_q$, and $F_q = E_q$ because $E_\bullet$ is an extension. So, $E_v \cong \hat{F}_v$ for any $v \in \mathbb{Z}^n$.

It is clear from the construction that the finite generation condition is preserved under extension. $\square$

Corollary 3.15. *Let X be a scheme and L a symmetric monoidal functor as in Definition 3.7. The category of parabolic sheaves* (*resp. coherent parabolic sheaves*) *on* (X, L) *is equivalent to the category of extendable pairs* (*resp. coherent extendable pairs*) *on* (X, L).

Proof. There is a pair of functors between these categories. The truncation functor sends a parabolic sheaf (E, ρ) to an extendable pair by forgetting all E_v when $v \notin \vec{r}I^n$. And the extension functor from extendable pairs to parabolic sheaves was defined in the previous proposition on objects by $F_\bullet \mapsto \hat{F}_\bullet$. It is easy to see that these functors are mutually inverse and preserve the finite generation condition. $\square$

Remark 3.16. Let X be a scheme and L a symmetric monoidal functor as in Definition 3.7. We denote the category of coherent extendable pairs on (X, L) by $\mathcal{EP}(X, L, \vec{r})$. When X is locally noetherian this category is abelian.

3C. ***The localization sequence.*** In this subsection we localize the category of finitely generated extendable pairs so that it will be glued from simpler parts.

For this section X is a locally noetherian scheme and L a symmetric monoidal functor as in Definition 3.7.

First let us consider the functor $\pi_*^{L,\vec{r}} : \mathcal{EP}(X, L, \vec{r}) \to \mathfrak{Coh}\, X$, given by $F_\bullet \mapsto F_0$ on objects. It is an exact functor because exact sequences in diagram categories are defined pointwise.

Lemma 3.17. *The functor $\pi_*^{L,\vec{r}}$ has a left adjoint, denoted $\pi^*_{L,\vec{r}}$, and there is a natural isomorphism $\pi_*^{L,\vec{r}} \circ \pi^*_{L,\vec{r}} \simeq 1$.*

Proof. In what follows, we omit the superscripts and subscripts L and $\vec{r}$ in the notation for the appropriate functors. For $0 \le i \le n$, consider functions $\epsilon_i : \vec{r}I \to \{0, 1\}$, defined by $\epsilon_i(u) = 1$ if $u_i = r_i$ and zero otherwise. We define the functor π^* on a sheaf $F \in \mathfrak{Coh}\, X$ by the rule

$$(\pi^*(F))_u = \left(\bigotimes_{i=1}^{n} L_i^{\epsilon_i(u)} \right) \otimes F.$$

This forms a functor via the maps

$$(\pi^*(F))_u \to (\pi^*(F))_{u+e_i} = \begin{cases} \text{identity} & \text{if } u_i \in [0, r_i - 2], \\ \sigma_i & \text{if } u_i = r_i - 1, \end{cases}$$

where σ_i is the multiplication by the section s_i.

Define ρ to be the identity map. It is easy to see that all axioms of extendable pairs are satisfied.

Now let's take a coherent sheaf F and an extendable pair $E_\bullet$ and consider a map

$$\mathrm{Hom}_{\mathfrak{Coh}\, X}(F, \pi_* E) \to \mathrm{Hom}_{\mathcal{EP}}(\pi^* F, E)$$

given by sending $\phi \in \mathrm{Hom}_{\mathfrak{Coh}\, X}(F, \pi_* E)$ to precomposition of the structure maps of the extendable pair E with ϕ. It's obviously an injection. Surjectivity follows from commutativity of the squares in $\mathrm{Hom}_{\mathcal{EP}}(\pi^* F, E)$ and because all structure maps in $\pi^* F$ are identity. $\square$

Proposition 3.18. *Suppose that X is a locally noetherian scheme. The functor $\pi_*^{L,\vec{r}} : \mathcal{EP}(X, L, \vec{r}) \to \mathfrak{Coh}\, X$ satisfies the hypothesis of Theorem 2.3.*

Proof. The only thing which is not completely obvious is the second condition. Consider two extendable pairs $E_\bullet$ and $F_\bullet$. Suppose that we have a morphism $\pi_*(E_\bullet) \to \pi_*(F_\bullet)$. By adjointness we obtain a diagram

$$\begin{tikzcd} \pi^*\pi_*(E_\bullet) \arrow[d] \arrow[dr] & \\ E_\bullet & F_\bullet \end{tikzcd}$$

Applying π to this picture shows that the second condition holds. $\square$

Using Theorem 2.3 we obtain the following:

Corollary 3.19. *Let X be a locally noetherian scheme. There is an equivalence of abelian categories*

$$\mathcal{EP}(X, L, \vec{r}) \,\big/\, \mathbf{ker}(\pi_*^{L,\vec{r}}) \to \mathfrak{Coh}\, X.$$

In the rest of this subsection we would like to give a description of the category $\mathbf{ker}(\pi_*^{L,\vec{r}})$. Let us study the objects first. Let $F_\bullet$ be an extendable pair. Then $\pi_*(F_\bullet) = F_0$, and if $F_\bullet \in \mathbf{ker}(\pi_*^{L,\vec{r}})$ then $F_0 \cong 0$. The pseudoperiod isomorphism implies in turn that $F_u \cong 0$ if all $u_i \in \{0, r_i\}$.

Let us consider the sheaves F_u such that $u_j \in \{0, r_j\}$ for $j \neq i$ (we can imagine them as sheaves on the edges of the cubical diagram $F_\bullet \in \mathrm{Func}(\vec{r} I^n, \boldsymbol{A})$). Using the axiom (EX1) we get that the multiplication by section map $s_i : F_u \to L_i \otimes F_u$ must factor through $F_{u+(r_i-u_i)e_i}$, which is a zero sheaf if $F_\bullet \in \mathbf{ker}(\pi_*^{L,\vec{r}})$. This implies the following lemma:

Lemma 3.20. *If $F_\bullet \in \mathbf{ker}(\pi_*^{L,\vec{r}})$ and $u \in \vec{r} I^n$ is such that $u_j \in \{0, r_j\}$ for all $j \neq i$, then* $\mathrm{supp}(F_u)$ *is contained in the divisor of zeroes of the section $s_i \in H^0(L_i)$.*

If $s_i = 0$ for some i, we say that $\mathrm{div}(s_i) = X$.

We apply the localization method (Theorem 2.3), to this partial description of the kernel. Let's fix some notation. Let

$$S_n(k) = \{T \subset \{1, \dots, n\} \mid |T| = k\}.$$

We often abuse notation and write $S(k)$ for $S_n(k)$ when it is clear from the context what n is. We view each interval $[0, r_i]$ as a pointed set, pointed at 0. It follows that we have order preserving inclusions

$$\iota_T : \prod_{i \in T} [0, r_i] \to \prod_{i=1}^{n} [0, r_i] := \vec{r} I^n.$$

Ignoring the pointed structure produces order preserving ($\leq$) projection maps

$$\pi_T : \vec{r} I^n \to \prod_{i \in T} [0, r_i].$$

Definition 3.21. As we agreed above, $L : \vec{r}\mathbb{Z}^n \to \mathfrak{Div}\, X$ is the symmetric monoidal functor as in Definition 3.7.

If $1 \leq k \leq n$ and $T \in S(k)$, then we define a symmetric monoidal functor $L_T : \vec{r}\mathbb{Z}^k \to \mathfrak{Div}\, X$ as a composition

$$\vec{r}\mathbb{Z}^k \xrightarrow{\iota_T} \vec{r}\mathbb{Z}^n \xrightarrow{L} \mathfrak{Div}\, X.$$

We say that L_T is obtained from L by the *restriction along ι_T*.

Now for $T \in S(k)$, let's consider the functor

$$\iota_T^* : \mathcal{EP}(X, L, \vec{r}) \to \mathcal{EP}(X, L_T, \pi_T(\vec{r})),$$

which is the restriction of an extendable pair $F_\bullet$ along the inclusion ι_T. The pseudo-period isomorphism is just obtained by restriction.

Definition 3.22. For any $1 \leq k \leq n$ we define functors

$$\mathrm{Face}^k := \prod_{T \in S(k)} \iota_T^* : \mathcal{EP}(X, L, \vec{r}) \to \prod_{T \in S(k)} \mathcal{EP}(X, L_T, \pi_T(\vec{r})).$$

Definition 3.23. For $1 \leq k \leq n$, we write $\mathbf{ker}^k = \mathbf{ker}(\mathrm{Face}^k)$ and $\mathbf{ker}^0 = \mathbf{ker}(\pi_*)$.

Lemma 3.24. *For any $1 \leq k \leq n$, any $F_\bullet \in \mathbf{ker}^{k-1}$ and any $T \in S(k)$ we can consider $(\iota_T^*(F_\bullet))_\bullet$ as an element of*

$$\mathrm{Func}\Big(\prod_{i \in T} [1, r_i - 1], \mathfrak{Coh}\Big(\bigcap_{i \in T} \mathrm{div}(s_i)\Big)\Big).$$

In other words, the images of these functors are supported on the indicated sub-schemes. As in Lemma 3.20, we say that if $s_i = 0$, then $\mathrm{div}(s_i) = X$.

Proof. If $k = 1$ then the result is proved in Lemma 3.20 and the observation before it.

Let's take any $2 \leq k \leq n$ and an extendable pair $F_\bullet \in \mathbf{ker}^{k-1}$. If we consider an extendable pair $(\iota_T^*(F_\bullet))_\bullet \in \mathcal{EP}(X, L_T, \pi_T(\vec{r}))$ then for any $v \in \prod_{i \in T}[0, r_i]$, we have isomorphisms of sheaves: $(\iota_T^*(F_\bullet))_v \cong 0$, whenever $v_i = 0$ for some $i \in T$. Because of the pseudoperiodicity isomorphism we also have that $(\iota_T(F_\bullet))_v \cong 0$, whenever $v_i = r_i$ for some $i \in T$.

The last step is an application of the axiom (EX1) to the extendable pair $(\iota_T^*(F_\bullet))_\bullet$. Because $(\iota_T^*(F_\bullet))_v \cong 0$ if $v_i = r_i$ for some $i \in T$, that implies that for any

$$w \in \prod_{i \in T} [1, r_i - 1]$$

the multiplication of the sheaf $(\iota_T^*(F_\bullet))_w$ by the sections $s_i \in H^0(X, L_i)$ for all $i \in T$ must factor through zero. So the support of the sheaf $(\iota_T^*(F_\bullet))_w$ is contained in $\bigcap_{i \in T} \mathrm{div}(s_i)$. $\square$

Lemma 3.25. *If we restrict the domain of the functor* Face^k *to the full subcategory* $\mathbf{ker}^{k-1}$ *for any $1 \leq k \leq n$, then we obtain functors*

$$\mathrm{Face}^k|_{\mathbf{ker}^{k-1}} : \mathbf{ker}^{k-1} \to \prod_{T \in S(k)} \mathrm{Func}\Big(\prod_{i \in T} [1, r_i - 1], \Big(\bigcap_{i \in T} \mathrm{div}(s_i)\Big)\Big).$$

There is an equivalence of categories between $\mathbf{ker}^k$ *and* $\mathbf{ker}\big(\mathrm{Face}^k|_{\mathbf{ker}^{k-1}}\big)$.

Proof. The first part follows directly from the lemma before. The proof of the second part is straightforward and follows from the fact that $\mathbf{ker}^k$ is a full subcategory of $\mathbf{ker}^{k-1}$. □

Remark 3.26. In order to apply the localization procedure to the category $\mathbf{ker}^{k-1}$ we need to show that the functor $\mathrm{Face}^k|_{\mathbf{ker}^{k-1}}$ has a left adjoint. The existence of a left adjoint follows from the special adjoint functor theorem. But for the purpose of splitting of the corresponding short exact sequence of K-groups (see Section 3D for details), we need the unit of the adjunction to be the natural isomorphism. This doesn't follow from the abstract nonsense, so we need an explicit construction of a left adjoint functor. It is given in the proof of the following theorem.

Theorem 3.27. *Let X be a locally noetherian scheme and consider a symmetric monoidal functor $L : \vec{r}\mathbb{Z}^n \to \mathfrak{Div}\, X$.*

(i) *For any $1 \le k \le n$ there is an exact functor*

$$\mathrm{Face}^k|_{\mathbf{ker}^{k-1}} : \mathbf{ker}^{k-1} \to \prod_{T \in S(k)} \mathrm{Func}\Bigl(\prod_{i \in T} [1, r_i - 1], \mathfrak{Coh}\Bigl(\bigcap_{i \in T} \mathrm{div}(s_i)\Bigr)\Bigr),$$

where $\mathbf{ker}^k$ is a kernel of the functor Face^k and $\mathbf{ker}^0 := \mathbf{ker}(\pi_*^{L,\vec{r}})$.

(ii) *The functors $\mathrm{Face}^k|_{\mathbf{ker}^{k-1}}$ have left adjoints D^k such that*

$$\mathrm{Face}^k|_{\mathbf{ker}^{k-1}} \circ D^k \simeq 1.$$

(iii) $\mathrm{Face}^k|_{\mathbf{ker}^{k-1}}$ *satisfies the condition of Theorem 2.3*

(iv) *The functor*

$$\mathrm{Face}^n|_{\mathbf{ker}^{n-1}} : \mathbf{ker}^{n-1} \to \mathrm{Func}\Bigl(\prod_{i=1}^{n} [1, r_i - 1], \mathfrak{Coh}\Bigl(\bigcap_{i=1}^{n} \mathrm{div}(s_i)\Bigr)\Bigr)$$

is an equivalence of categories.

Proof. (i) These functors are obtained by restricting domains. As kernels and cokernels are computed pointwise, this is exact.

(ii) Given a functor $G_\bullet^T \in \mathrm{Func}\bigl(\prod_{i\in T}[1, r_i - 1],\ \mathfrak{Coh}\bigl(\bigcap_{i\in T} \mathrm{div}(s_i)\bigr)\bigr)$ for each $T \in S(k)$, we denote the corresponding object by

$$(G_\bullet^T)_{T \in S(k)} \in \prod_{T \in S(k)} \mathrm{Func}\Bigl(\prod_{i \in T} [1, r_i - 1], \mathfrak{Coh}\Bigl(\bigcap_{i \in T} \mathrm{div}(s_i)\Bigr)\Bigr).$$

Further, we view $G_\bullet^T$ as a functor $\prod_{i\in T}[0, r_i] \to \mathfrak{Coh}\bigl(\bigcap_{i\in T} \mathrm{div}(s_i)\bigr)$ by taking $G_u^T = 0$ if for some $i \in T$ we have $u_i \in \{0, r_i\}$, where 0 is some fixed zero object in $\mathfrak{Coh}(X)$. Also, for $i \in \{1, \dots, k\}$ if $u_i \in \{0, r_i - 1\}$ we define the morphisms $G_{+e_i}^T : G_u^T \to G_{u+e_i}^T$ as the initial and terminal map correspondingly.

Let us remind the reader of the definition of ϵ from Lemma 3.17. For any $0 \le i \le n$ we have functions $\epsilon_i : \vec{r}I \to \{0, 1\}$ such that for any $u \in \vec{r}I^n$, we have $\epsilon_i(u) = 1$ if $u_i = r_i$ and $\epsilon_i(u) = 0$ otherwise.

We define the functor D^k on objects as follows:

$$(D^k((G^T_\bullet)_{T\in S(k)}))_u = \left(\bigotimes_{i=1}^{n} L_i^{\epsilon_i(u)}\right) \otimes \left(\bigoplus_{T\in S(k)} G^T_{\pi_T(u)}\right).$$

Let's denote $(D^k((G^T_\bullet)_{T\in S(k)}))_\bullet$ by $D^k_\bullet$ for the simplicity of notations. First of all we want to view it as a functor $\vec{r}I^n \to \mathfrak{Coh}(X)$. For that we have to define the morphisms

$$D^k_{+e_i} : D^k_u \to D^k_{u+e_i}.$$

If $0 \le u_i < r_i - 1$, then this map is induced by $\bigoplus_{T\in S(k),\, i\in T} G^T_{+1}$. If $u_i = r_i - 1$, then it is induced by the terminal maps $\bigoplus_{T\in S(k),\, i\in T} G^T_{+1}$ and also by multiplication by the section s_i.

The pseudoperiod isomorphisms ρ are defined by the symmetric monoidal structure of the functor L. The proof of the axioms (EX2) and (EX3) is automatic, and the proof of (EX1) follows from the commutativity of the diagram

$$\begin{tikzcd}
D_u \arrow[r, "D_{+(r_i-u_i)e_i}"] \arrow[d, "\sigma_i"] & D_{u+(r_i-u_i)e_i} \arrow[d, "\rho"] \\
L_i \otimes D_u & L_i \otimes D_{u-u_ie_i}. \arrow[l, "L_i\otimes D_{+u_ie_i}"]
\end{tikzcd}$$

This diagram commutes because of the definition of $D_{+(r_i-\alpha_i)\vec{e}_i}$ and because $\operatorname{supp}(G^T_u) \subseteq \bigcap_{i\in T} \operatorname{div}(s_i)$ for any $u \in \prod_{i\in T}[0, r_i]$.

So we have shown that $D^k_\bullet$ is an extendable pair. If $k = 1$ then it's clear that $D^1_\bullet$ is in $\mathbf{ker}^0$, because $D^1_0 \cong 0$.

If $2 \le k \le n$, we want to see that $D^k_\bullet$ is in $\mathbf{ker}^{k-1}$. For that we have to see that for any $W \in S(k-1)$ and any $v \in \prod_{i\in W}[0, r_i]$, the sheaf $(\iota^*_W(D^k_\bullet))_v$ is isomorphic to zero. But this is true because for any $T \in S(k)$ we have that $G^T_u = 0$ if $u_i \in \{0, r_i\}$ for some $i \in T$.

Clearly, $\mathrm{Face}^k|_{\mathbf{ker}^{k-1}} \circ D^k = 1$.

Next we would like to show that D^k is indeed a left adjoint. Suppose that we have a morphism

$$(G^T_\bullet)_{T\in S(k)} \to \mathrm{Face}^k(F_\bullet).$$

Such a morphism consists of an $\binom{n}{k}$-tuple of morphisms

$$\phi_T : G^T_\bullet \to \iota^*_T(F_\bullet).$$

We wish to describe the adjoint map

$$\tilde{\phi} : D^k_\bullet \to F_\bullet.$$

Using the universal property of coproduct, this morphism is determined by maps

$$\tilde{\phi}(u)_T : \bigotimes_{i=1}^{n} L_i^{\epsilon(u)} \otimes G^T_{\pi_T(u)} \to F_u.$$

If u is such that $\epsilon_i(u) = 0$ for all $1 \le i \le n$, then these maps are just the compositions of ϕ_T with the morphisms $F_{+\alpha}$. If there is l such that $u_l = r_l$, then $\tilde{\phi}(u)_T$ is induced by the composition of ϕ_T with ρ_F^{-1} and with $F_{+\alpha}$.

We want to check that the map $\tilde{\phi}$ is indeed a natural transformation of functors. It's enough to check that the diagram

$$\begin{tikzcd} D_u \arrow[r, "\tilde{\phi}(u)"] \arrow[d, "D_{+e_i}"] & F_u \arrow[d, "F_{+e_i}"] \\ D_{u+e_i} \arrow[r, "\tilde{\phi}(u+e_i)"'] & F_{u+\vec{e}_i} \end{tikzcd}$$

commutes. If $\epsilon_k(u) = 0$ for all $1 \le k \le n$ and also $u_i < r_i - 1$, then it commutes directly from the construction of the maps $\tilde{\phi}(u)$. Otherwise the commutativity follows from (EX1), (EX2) and (EX3) for $F_\bullet$.

Finally, we have the map

$$\mathrm{Hom}((G^T_\bullet)_{T\in S(k)}, \mathrm{Face}^k(F_\bullet)) \to \mathrm{Hom}(D^k((G^T_\bullet)_{T\in S(k)}), F_\bullet).$$

It's easy to see that this map is bijective, because the right Hom is uniquely defined by the restriction to k-faces.

(iii) This follows from (ii).

(iv) Because for $S(n)$ there is only one element, the set $\{1, \ldots, n\}$ itself, we have that $\iota_{\{1,\ldots,n\}} = \mathrm{id}$ and $\pi_{\{1,\ldots,n\}} = \mathrm{id}$. So $\mathrm{Face}|^n_{\mathbf{ker}^{n-1}}$ and D^n are identity functors. $\square$

3D. *G-theory and K-theory of a root stack.* In this subsection we finally describe the G-theory of a root stack $X_{L,\vec{r}}$.

Lemma 3.28. *If X is a locally noetherian scheme and L a symmetric monoidal functor as in Definition 3.7, there is an equivalence of categories*

$$\mathfrak{Coh}\, X_{L,\vec{r}} \simeq \mathcal{EP}(X, L, \vec{r}).$$

Proof. This follows by combining Corollaries 3.10 and 3.15. $\square$

So we have
$$G(X_{L,\vec{r}}) \cong K(\mathcal{EP}(X, L, \vec{r})),$$
and we reduced the problem to describing the K-theory of the (abelian) category of coherent extendable pairs $\mathcal{EP}(X, L, \vec{r})$.

We are going to use several splittings of the category of coherent extendable pairs to simplify the latter K-theory. The first step is this:

Lemma 3.29. *If X is a locally noetherian scheme, then in the notation of Section 3C one has*
$$K_i(\mathcal{EP}(X, L, \vec{r})) \cong G_i(X) \oplus K_i(\mathbf{ker}(\pi_*^{L,\vec{r}})) \quad \textit{for any } i \in \mathbb{Z}_+.$$

Proof. Using Corollary 3.19 and the localization property of K-theory (see for example [Quillen 1973]) we have the long exact sequence of groups
$$\cdots \to K_i(\mathbf{ker}(\pi_*^{L,\vec{r}})) \to K_i(\mathcal{EP}(X, L, \vec{r})) \to G_i(X) \to \cdots.$$

But this sequence splits because of the property $\pi_*^{L,\vec{r}} \circ \pi_{L,\vec{r}}^* \simeq 1$ proved in Lemma 3.17. □

Lemma 3.30. *If $\mathbf{A}$ is an abelian category then*
$$K_i(\mathrm{Func}(\vec{r}I^n, \mathbf{A})) \cong K_i(\mathbf{A})^{\oplus \prod_{j=1}^n r_j}.$$

Proof. The proof follows from the iterated application of Theorem 2.6 and localization property of the K-theory. □

Now we want to proceed with $K_\bullet(\mathbf{ker}(\pi_*^{L,\vec{r}}))$, exploiting the same ideas as in the previous lemmas.

Lemma 3.31. *Let X be a locally noetherian scheme, L a symmetric monoidal functor as in Definition 3.7 and $s_k \in H^0(L_k)$ for $k = 0, \ldots, n$. Then for any $i \in \mathbb{Z}_+$,*
$$K_i(\mathbf{ker}(\pi_*^{L,\vec{r}})) \cong \bigoplus_{k=1}^{n} \bigoplus_{T \in S(k)} G_i\Big(\bigcap_{l \in T} \mathrm{div}(s_l)\Big)^{\oplus \prod_{l \in T}(r_l - 1)},$$
where $S(k) = \{T \subset \{1, \ldots, n\} \mid |T| = k\}$.

Proof. This follows from application of the localization property of K-theory, Theorem 3.27 and the previous technical lemma. □

Combining Lemmas 3.28, 3.29 and 3.31 yields the main result of the section:

Theorem 3.32. *Let X be a locally noetherian scheme. Let (L_i, s_i) be objects of $\mathfrak{Div}\, X$ for $i = 1, \ldots, n$ and $\vec{r} \in \mathbb{N}^n$. Then G-theory of a root stack $X_{L,\vec{r}}$ is given by the formula*
$$G_i(X_{L,\vec{r}}) \cong G_i(X) \oplus \Bigg(\bigoplus_{k=1}^{n} \bigoplus_{T \in S(k)} G_i\Big(\bigcap_{l \in T} \mathrm{div}(s_l)\Big)^{\oplus \prod_{l \in T}(r_l - 1)}\Bigg)$$
for any $i \in \mathbb{Z}_+$, where $S(k) = \{T \subset \{1, \ldots, n\} \mid |T| = k\}$.

To finish the section we want to give sufficient conditions for a root stack to be smooth.

Proposition 3.33. *Let X be a smooth scheme over a field k. Let $D = \sum_{i=1}^{n} D_i$ be a normal crossing divisor. Assume that $\vec{r}$ is an n-tuple of natural numbers, such that each r_i is coprime to the characteristic of k. Then a root stack $X_{D,\vec{r}}$ is smooth.*

Proof. By definition a stack is smooth if its presentation is a smooth scheme. The question is local, so we can assume that $X = \operatorname{Spec}(R)$ and a divisor D is a strict normal crossing divisor. If we localize further, we can assume that R is a regular local ring, $D_i = (f_i)$ and $\{f_i\}$ forms a part of a regular sequence of parameters.

By [Cadman 2007, Example 2.4.1], the presentation of a root stack $X_{D,\vec{r}}$ is an affine scheme $A = R[t_1, \dots, t_n]/(t_1^{r_1} - f_1, \dots, t_n^{r_n} - f_n)$. By [Grothendieck and Murre 1971, Lemma 1.8.6], this scheme is smooth. □

Corollary 3.34. *Under the hypotheses of Proposition 3.33, $G(X_{D,\vec{r}}) = K(X_{D,\vec{r}})$, where the latter means the Waldhausen K-theory of perfect complexes on the stack as defined in [Joshua 2005].*

Proof. Indeed, if a stack is regular, its Waldhausen K-theory is the same as G-theory. See [Joshua 2005]. □

4. Quotient stacks as root stacks

4A. ***Generation of inertia groups.*** Let X be a scheme with an action of a finite group G. We always assume that this action is *admissible*. Let us recall, following [SGA 1 1971, V.1, Definition 1.7], that an action is called admissible if there exists an affine morphism $\phi : X \to Y$ such that $\mathcal{O}_Y \cong \phi_*(\mathcal{O}_X)^G$. This implies that the quotient X/G exists and is isomorphic to Y.

If $x \in X$ is a point (not necessarily closed), the subgroup of G stabilizing x is called the *decomposition group* and we denote it by $D(x, G)$. The subgroup of the decomposition group acting trivially on the residue field of x is called the *inertia group* of x and we denote it by $I(x, G)$.

Note that there is an induced action of $D(x, G)$ on the closure of the point x and $I(x, G)$ acts trivially on this closure. Hence if $x \in \bar{y}$ then there is an inclusion $I(y, G) \hookrightarrow I(x, G)$. We say that *the inertia groups are generated in codimension one* if for each point $x \in X$ we have that

$$I(x, G) = \prod_{x \in \bar{y}} I(y, G),$$

where the product is over all points of codimension one containing x and the identification is via the inclusions above. For a group acting on a smooth curve, all inertia groups are generated in codimension one. We will see under certain assumptions that this is also true in higher dimensions (see Theorem 4.9).

4B. *Main theorem.* In this subsection we provide sufficient conditions for a quotient stack to be a root stack. To illustrate the procedure we start with an example.

Example 4.1. Let $\mathcal{O}$ be a discrete valuation ring with an action of μ_r such that $\gcd(r, \mathrm{char}(\mathcal{O})) = 1$. Then the fixed ring $\mathcal{O}^{\mu_r}$ is also a discrete valuation ring. We assume that $\mathcal{O}$ contains a field so that its completion $\hat{\mathcal{O}}$ is a power series ring in one variable over the residue field. Note that μ_r must preserve the maximal ideal of $\mathcal{O}$. If we further assume that the action is generically free and inertial, i.e., μ_r acts trivially on the residue field, then if s is a local parameter for $\mathcal{O}$ we can conclude that $t = s^r$ is a local parameter for $R = \mathcal{O}^{\mu_r}$.

We set $Y = \mathrm{Spec}(R)$ and consider the root stack

$$\mathfrak{Y} = Y_{R,t,r} \to Y.$$

The parameter s induces a μ_r-equivariant morphism

$$X \to \mathfrak{Y}$$

corresponding to the triple $(\mathcal{O}, s, m)$, where m is the canonical isomorphism $\mathcal{O}^r \to \mathcal{O}$. We show in Proposition 4.6 that this morphism is in fact étale. Using the two out of three property for étale maps we get that the natural morphism

$$X \times \mu_r \to X \times_{\mathfrak{Y}} X$$

is étale. To show that $[X/\mu_r] \cong \mathfrak{Y}$ it suffices to show that this morphism is radicial (universally injective) and surjective. In other words we need to show that it is a bijection on K-points for each field K.

Given a pair of K-points a and b of X that give a K-point of $X \times_Y X$, the fiber of

$$X \times_{\mathfrak{Y}} X \to X \times_Y X$$

over this point consists of the space of isomorphisms between $a^*(\mathcal{O}, s, m)$ and $b^*(\mathcal{O}, s, m)$ in $\mathfrak{Y}$. If the support of the K-points is the generic point of $\mathcal{O}$ this is just a singleton and if the support is the closed point then the space is a bitorsor over μ_r. At any rate the morphism above is seen to be an isomorphism. Hence in this case we have

$$[X/\mu_r] \cong \mathfrak{Y}.$$

Remark 4.2. A μ_r-bundle P on a scheme Z is equivalent to the data of an invertible sheaf $\mathcal{K}$ and an isomorphism $\phi : \mathcal{K}^r \to \mathcal{O}_Z$. To construct P explicitly consider the sheaf of algebras $\mathrm{Sym}^\bullet \mathcal{K}^{-1}$. There is a distinguished global section $T \in \mathcal{K}^{-r}$ given by $(\phi \otimes 1_{\mathcal{K}^{-r}}(1))$. Then

$$P = \mathrm{Spec}(\mathrm{Sym}^\bullet \mathcal{K}^{-1}/(T-1)).$$

Remark 4.3. Suppose that there is on Y an invertible sheaf $\mathcal{N}$ and an isomorphism $\mathcal{N}^r \to \mathcal{L}$. Then $Y_{\mathcal{L},s,r}$ is a global quotient stack; see [Cadman 2007, Lemma 2.3.1

and Example 2.4.1; Borne 2007, §3.4]. We need this below, so let's recall some of the details. The coherent sheaf

$$\mathcal{A} = \mathcal{O}_Y \oplus \mathcal{N}^{-1} \oplus \cdots \oplus \mathcal{N}^{-(r-1)}$$

can be given the structure of an $\mathcal{O}_Y$-algebra via the composition

$$\mathcal{N}^{-r} \xrightarrow{\sim} \mathcal{L}^{-1} \xrightarrow{s} \mathcal{O}_Z.$$

There is an action of μ_r on this sheaf via the action of μ_r on $\mathcal{N}^{-1}$ given by scalar multiplication. Then $Y_{\mathcal{L},s,r} = [\operatorname{Spec}(\mathcal{A})/\mu_r]$. We need the explicit morphism

$$Y_{\mathcal{L},s,r} \to [\operatorname{Spec}(\mathcal{A})/\mu_r]$$

below so let's describe it. Consider a morphism $a : X \to Y$. A morphism $X \to Y_{\mathcal{L},s,r}$, lifting a, is a triple $(\mathcal{M}, t, \phi)$. As per the previous remark the sheaf $\mathcal{M}^{-1} \otimes \mathcal{N}^{-1}$ gives a μ_r-torsor. The torsor comes from the algebra

$$\mathcal{B} = \operatorname{Sym}^\bullet \mathcal{M} \otimes a^*\mathcal{N}^{-1}/(T-1).$$

To produce an X-point of $[\operatorname{Spec}(\mathcal{A})/\mu_r]$ we need to describe a μ_r-equivariant map

$$a^*\mathcal{A} \to \mathcal{B}.$$

This map comes from the section t via

$$t \in \operatorname{Hom}(\mathcal{O}, \mathcal{M}) = \operatorname{Hom}(a^*\mathcal{N}, \mathcal{M} \otimes a^*\mathcal{N}^{-1}).$$

This construction generalizes in the obvious way to a finite list of invertible sheaves with section.

Assumption 4.4. We assume X and Y are regular, separated, noetherian schemes over a field k. Let G be a finite group with cardinality coprime to the characteristic of k. We assume that G acts admissibly and generically freely on X with quotient $\phi : X \to Y$. Note that by [Görtz and Wedhorn 2010, Theorem 14.126] our hypotheses imply that the quotient map $X \to Y$ is flat.

Consider the map $\phi : X \to Y$, which is faithfully flat and finite. Recall that the set of points of X where ϕ is ramified is called the branch locus. It has a natural closed subscheme structure defined by $\operatorname{supp}(\Omega_{X/Y})$. Because the conditions of the purity theorem [Altman and Kleiman 1970, Chapter VI, Theorem 6.8] are satisfied, in our situation this closed subscheme gives rise to an effective Cartier divisor, which is called the branch divisor. We can write this divisor as

$$R = \sum_{i=1}^{n} (r_i - 1)\left(\sum_{g \in G} g^* D_i\right),$$

where each D_i is a prime divisor. As G acts generically freely, passing to generic points of our regular variety produces a Galois extension with Galois group G. We can view the D_i as points of the scheme X. The multiplicities r_i are related to the inertia groups of D_i via

$$r_i = |I(D_i, G)|;$$

see [Neukirch 1999, Chapter I, §9].

We let E_i be the image of D_i under ϕ. It is called the ramification divisor. We form the root stack

$$\mathfrak{Y} = Y_{((E_1,r_1),\dots,(E_n,r_n))}.$$

Note that we have assumed that the characteristic of our ground field is coprime to G and hence to each r_i. It follows, via a local calculation along the ring extension $\mathcal{O}_{X,D_i}/\mathcal{O}_{Y,E_i}$, that we have $\phi^*(E_i) = r_i\big(\sum_{g\in G} g^*D_i\big)$. This allows us to lift ϕ to produce a diagram

$$\begin{tikzcd} X \arrow[d, "\phi"'] \arrow[dr, "\psi"] & \\ Y & \mathfrak{Y} \arrow[l, "\pi"'] \end{tikzcd}$$

The morphism ψ is equivariant in the sense that precomposition with $g \in G$ produces a two-commuting diagram. This gives us a morphism

$$[X/G] \to \mathfrak{Y}$$

that we would like to show is an isomorphism under our Assumption 4.4 and the extra condition that the ramification divisor is normal crossing.

For the proof of Proposition 4.6 we need the following lemma.

Proposition 4.5 (Abhyankar's lemma). *Let $Y = \operatorname{Spec}(A)$ be a regular local scheme and $D = \sum_{1\le i\le r} \operatorname{div}(f_i)$ a divisor with normal crossings, so that the f_i form part of a regular system of parameters for Y. Set $\overline{Y} = \operatorname{Supp}(D)$ and let $U = Y \setminus \overline{Y}$. Consider $V \to U$, an étale cover that is tamely ramified over D. If y_i are the generic points of $\operatorname{supp}(\operatorname{div}(f_i))$ then $\mathcal{O}_{Y,y_i}$ is a discrete valuation ring. If we let K_i be its field of fractions then, as V ramifies tamely, we have that*

$$V|_{K_i} = \operatorname{Spec}\Big(\prod_{j\in J_i} L_{ji}\Big),$$

where the L_{ji} are finite separable extensions of K_i. We let n_{ji} be the order of the inertia group of the Galois extension generated by L_{ji} and let

$$n_i = \operatorname{lcm}_{j\in J_i} n_{ji},$$

and set

$$A' = A[T_1,\dots,T_r]/(T_1^{n_1} - f_1,\dots,T_r^{n_r} - f_r), \qquad Y' = \operatorname{Spec}(A').$$

Then the étale cover $V' = V \times_X X'$ of $U \times_X X'$ extends uniquely up to isomorphism to an étale cover of X'.

Proof. This is [SGA 1 1971, Expose XIII, Proposition 5.2]. The proof given shows how to construct the extension of V', which we need below. The extension can be constructed as the normalization of X' in the generic point of $V \times_X X'$. $\square$

Proposition 4.6. *Under Assumption 4.4, suppose that $\phi : X \to Y$ is ramified along a simple normal crossings divisor E. The morphism $\psi : X \to \mathfrak{Y}$ constructed above is étale.*

Proof. Étale maps are local on the source so we can assume that $Y = \mathrm{Spec}(S)$, and all E_i are trivial line bundles so that $s_i \in S$. Further, by shrinking X we can assume that the morphism $X \to \mathfrak{Y}$ is defined be trivial bundles on X. Because the map ϕ is finite we can write $X = \mathrm{Spec}(T)$. Here T and S are local regular Noetherian k-algebras, T is a finite S-module, s_i is part of a regular system of parameters and there are elements $t_i \in T$, such that $t_i^{r_i} = s_i$.

We may check étaleness after a faithfully flat base extension of the base field and hence may assume that the ground field k contains r_i-th roots of unity for all $1 \le i \le n$.

Using Remark 4.3, we see that the stack $\mathfrak{Y}$ is isomorphic to the quotient stack

$$[\mathrm{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}],$$

where $S' = S[y_1, \ldots, y_n]/(y_1^{r_1} - s_1, \ldots, y_n^{r_n} - s_n)$.

We want to show that the map $\mathrm{Spec}(T) \to [\mathrm{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}]$ is étale. Denote by T' the ring $T[x_1, \ldots, x_n]/(x^{r_1} - 1, \ldots, x^{r_n} - 1)$. Using Remark 4.3 again we obtain a Cartesian diagram

$$\begin{tikzcd}
\mathrm{Spec}(T') \arrow[r] \arrow[d] & \mathrm{Spec}(S') \arrow[d] \\
\mathrm{Spec}(T) \arrow[r] & \left[\mathrm{Spec}(S')/\mu_{r_1} \times \cdots \times \mu_{r_n}\right]
\end{tikzcd}$$

Because $\mathrm{Spec}(S')$ is a presentation of a quotient stack it is enough to show that the map $S' \to T'$ given by $y_i \mapsto t_i x_i$ is étale.

The morphism $S_{s_1 \ldots s_n} \to T_{t_1 \ldots t_n}$ is flat and unramified by assumption, and hence it is étale. By Abhyankar's lemma (Proposition 4.5), this morphism extends after base change to an étale cover of S'. By the proof of Abhyankar's lemma it suffices to show that T' is normal and the map $S' \to T'$ is integral. Both of these facts are easily checked and the result follows. $\square$

For a point $p \in Y$ we define

$$I(p, Y) = \prod_{p \in \mathrm{supp}(E_i)} \mu_{r_i}.$$

Proposition 4.7. *Let Assumption 4.4 hold. Let K be a field and consider the morphism of K-points*

$$\pi_K : X \times_{\mathfrak{Y}} X(K) \to X \times_Y X(K).$$

The fiber $\pi_K^{-1}(x_1, x_2)$ over a K-point (x_1, x_2) is a bitorsor under the inertia group $I(\phi(x_1), Y)$.

Proof. In what follows, we use the shorthand G^* when we mean $\sum_{g\in G} g^*$. Recall that the morphism ψ is defined by $(\mathcal{O}(G^*E_i), s_{G^*E_i}, \alpha_i)$, where α_i are isomorphisms, coming from the fact that

$$r_i G^* E_i = r_i \phi^*(D_i).$$

The fiber over (x_1, x_2) is exactly the set of isomorphism from $x_1^*\mathcal{O}(G^*E_i)$ to $x_2^*\mathcal{O}(G^*E_i)$ as i varies. As in Example 4.1, this depends on whether the section $x_1^* s_{G^*E_i}$ vanishes or not. The vanishing condition precisely depends on $\phi(x_1)$, and the result follows. □

The final ingredient we need to finish the proof is that under our assumptions the inertia group of X is generated in codimension one. For that let us recall the following:

Proposition 4.8 (Abhyankar's theorem; see [Grothendieck and Murre 1971, Theorem 2.3.2]). *Let Y be a locally noetherian normal scheme, D a divisor with normal crossing, $\hat{Y} = \operatorname{supp}(D)$ and $U = Y \setminus \hat{Y}$. Assume that $X \to Y$ is a finite morphism and G is a finite group operating on X such that $X \mid U$ is a G-torsor. Then the following are equivalent*:

(i) *X is tamely ramified relative to D.*

(ii) *For every $y \in Y$ there exists an étale neighborhood Y' of y in Y, and a scheme $S = \mathcal{O}_Y[(T_i)_{i\in I'}]/((T_i^{r_i'} - f_i'))_{i\in I'}$, where $D_{Y'} = \sum_{i\in I'} D_i'$ and $\operatorname{div}(f_i') = D_i'$, such that there is an isomorphism of couples*

$$(X', G) \simeq (G \times^H S, G),$$

where $X' = X \times_Y Y'$ and $H = \prod_{i\in I'} \mu_{r_i'}$. Let us recall that $G \times^H S$ is the quotient $(G \times S)/H$, where H acts "by the formula" $h \cdot (g, s) = (gh^{-1}, hs)$.

Let us apply this fact to describe the inertia group.

Theorem 4.9. *Under Assumption 4.4, suppose that $\phi : X \to Y$ is ramified along a simple normal crossings divisor. Then the inertia groups of (X, G) are generated in codimension one.*

Proof. Firstly observe that condition (i) of Abhyankar's theorem (Proposition 4.8) is satisfied. Inertia is a local notion and also, clearly, the inertia group of (S, H) is generated in codimension one.

There is an isomorphism of quotient stacks $[(G \times^H S)/G] \cong [S/H]$. So inertia groups of $G \times^H S$ under the action of G and of S under the action of H are isomorphic for the corresponding points. This finishes the proof. □

Finally, we are ready to prove the main theorem of this section.

Theorem 4.10. *If Assumption 4.4 is satisfied and if also the ramification divisor is a normal crossing divisor, then we have the isomorphism of stacks* $[X/G] \cong \mathfrak{Y}$.

Proof. To prove this all we need to show is that the map

$$\chi : X \times G \to X \times_{\mathfrak{Y}} X, \quad (x, g) \mapsto (x, gx)$$

is an isomorphism.

By Proposition 4.6, the map $\psi : X \to \mathfrak{Y}$ is étale, and so the map $X \times_{\mathfrak{Y}} X \to X$ is étale as a pullback. Clearly two maps $X \times G \to X$ given by $(x, g) \mapsto x$ and $(x, g) \mapsto gx$ are étale and so the map χ must be étale.

We are going to show that the map

$$\chi(K) : X(K) \times G \to X \times_{\mathfrak{Y}} X(K)$$

is bijective for any field extension of the ground field $k \subset K$. The points of the scheme on the left is a pair (x, g), where $g \in G$ and $x : \mathrm{Spec}(K) \to X$ a K-point.

Consider the morphism $\Psi : X \times G \to X \times_Y X$. This morphism is surjective as we have a geometric quotient; see [Mumford et al. 1994, Definition 0.4]. Consider a K-point $(x_1, x_2) \in X \times_Y X(K)$. Using the properties of geometric quotients we have that $x_2 = gx_1$ for some $g \in G$. Using this we see the fiber $\Psi^{-1}(x_1, x_2)$ is a torsor over the inertia group $I(\mathrm{supp}(x_1), G)$. By Theorem 4.9 our inertia groups are generated in codimension one, so we see that we have an identification

$$I(\mathrm{supp}(x_1), G) = \mu_{r_{i_1}} \times \cdots \times \mu_{r_{i_l}}$$

as in Proposition 4.7. It follows that the morphism χ is étale and universally injective (radical). This implies that it is an open immersion. As it is also surjective, it is an isomorphism and the result follows. □

5. An application of root stacks to the equivariant K-theory of schemes

As an application of the theorems proved in Sections 3 and 4, we can formulate a result about equivariant K-theory.

Theorem 5.1. *Let X be a regular, separated, noetherian scheme over the field k with a generically free admissible action of a finite group G, such that the order of G is coprime to the characteristic of k. Let $X/G = Y$ and assume that all the* *conditions of Assumption 4.4 are satisfied. Also assume that $X \to Y$ is ramified along a simple normal crossing divisor E. Then there is an isomorphism of groups*

$$K_G^\bullet(X) \cong K^\bullet(Y) \oplus \left(\bigoplus_{i=1}^{n} \left(\bigoplus_{T \in S(i)} G^\bullet\left(\bigcap_{l \in T} E_l \right)^{\oplus \prod_{l \in T}(r_l - 1)} \right) \right),$$

where r_l are orders of inertia groups (see Section 4 for notation), *and*

$$S(i) = \{T \subset \{1, \ldots, n\} \mid |T| = i\}.$$

Proof. By assumption X is a regular scheme and the group G is finite, so for any G-equivariant sheaf we can always construct an equivariant locally free resolution by averaging the usual locally free resolution. This simple argument shows that the equivariant K-theory of X is the same as the equivariant G-theory.

The category of G-equivariant sheaves on X is equivalent to the category of sheaves on the quotient stack $[X/G]$, so we can see that

$$K_G(X) \cong G([X/G]).$$

In Theorem 4.10 we proved under our assumptions that there is an isomorphism of stacks $[X/G] \cong \mathfrak{Y}$, so we have an isomorphism of their G-theories

$$G([X/G]) \cong G(\mathfrak{Y}).$$

Finally the application of Theorem 3.32 gives the desired formula. □

Let us give some examples.

Example 5.2. Let's consider $\mathbb{A}^1$ over a field k with an action of μ_3 (it acts by multiplication). Assume that $\mathrm{char}(k) \neq 3$. Then $\mathbb{A}^1/\mu_3 \cong \mathbb{A}^1$ and ramification divisor is $\mathrm{div}(0)$. The inertia group is μ_3. So by Theorem 5.1,

$$K_{\mu_3}^\bullet(\mathbb{A}^1) \cong K^\bullet(\mathbb{A}^1) \oplus K^\bullet(k) \oplus K^\bullet(k) \cong K^\bullet(k)^{\oplus 3}.$$

Example 5.3. This example was inspired by the paper [Alexeev and Orlov 2013]. The Burniat surface X with $K_X^2 = 6$ is a Galois $G := C_2 \times C_2$-cover of $\mathrm{Bl}_3\,\mathbb{P}^2$ (a del Pezzo surface of degree 6). Let's assume that the ground field k is algebraically closed and $\mathrm{char}(k) \neq 2$. The ramification divisor is given in [loc. cit., Figure 1]: it is denoted by A_l, B_l, C_l, where $0 \le l \le 4$. The inertia group of each component is C_2, and the inertia group of an intersection point of any two components is G. The intersection of three components is empty. Also, $A_l \cong B_l \cong C_l \cong \mathbb{P}^1$ for all $l = 0, \ldots, 3$.

Applying Theorem 5.1, one gets

$$K_G^\bullet(X) \cong K^\bullet(\mathrm{Bl}_3\,\mathbb{P}^2) \oplus \left(\bigoplus_{i=1}^{2} Z_i^\bullet \right), \qquad Z_1^\bullet = K^\bullet(\mathbb{P}^1)^{\oplus 12}, \qquad Z_2^\bullet = K^\bullet(k)^{\oplus 30}.$$

Notations and conventions

k	our base field
$\mathbf{ker}$	the kernel of a functor (Definition 2.2)
$\vec{r}$	an n-tuple $(r_1, \dots, r_n)$ of real numbers
$\vec{r}I^n$	the poset of integer points in $\prod_{i=1}^n [0, r_i]$
$\mathrm{Func}(\mathbf{A}, \mathbf{B})$	the functor category between two abelian categories
$\widehat{M}$	the dual $\mathrm{Hom}(M, \mathbb{G}_{\mathrm{m}})$ of the monoid M
$\mathfrak{Div}\, X$	the symmetric monoidal category of line bundles with section (Section 3A)
$X_{L,\vec{r}}$	a stack of roots over the scheme X (Definition 3.4)
$\mathfrak{Coh}\, X$	category of coherent sheaves on X
$\mathcal{EP}(X, L, \vec{r})$	category of coherent extendable pairs (Remark 3.16)
$S_n(k) = S(k)$	The set of subsets of $\{1, 2, \dots, n\}$ of cardinality k (We often drop the subscript n when it is clear from context.)
Face^k	The k-th face functor (Definition 3.22)
$\mathbf{ker}^k$	The kernel of the face functor (Definition 3.23)

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Orbital integrals and K-theory classes

Peter Hochs and Hang Wang

Let G be a semisimple Lie group with discrete series. We use maps $K_0(C_r^*G) \to \mathbb{C}$ defined by orbital integrals to recover group theoretic information about G, including information contained in K-theory classes not associated to the discrete series. An important tool is a fixed point formula for equivariant indices obtained by the authors in an earlier paper. Applications include a tool to distinguish classes in $K_0(C_r^*G)$, the (known) injectivity of Dirac induction, versions of Selberg's principle in K-theory and for matrix coefficients of the discrete series, a Tannaka-type duality, and a way to extract characters of representations from K-theory. Finally, we obtain a continuity property near the identity element of G of families of maps $K_0(C_r^*G) \to \mathbb{C}$, parametrised by semisimple elements of G, defined by stable orbital integrals. This implies a continuity property for L-packets of discrete series characters, which in turn can be used to deduce a (well-known) expression for formal degrees of discrete series representations from Harish-Chandra's character formula.

1. Introduction

Let G be a real semisimple Lie group. Its reduced C^*-algebra C_r^*G is the closure in $\mathcal{B}(L^2(G))$ of the algebra of convolution operators by functions in $L^1(G)$. It represents the tempered dual of G as a "noncommutative space" in the sense of noncommutative geometry, and encodes all tempered representations of G. Its K-theory $K_*(C_r^*G)$ is a natural invariant to consider. This K-theory is described explicitly in terms of equivariant indices of Dirac operators on G/K, for a maximal compact subgroup $K < G$, in the Connes–Kasparov conjecture. This was proved in various cases by Penington and Plymen [1983], Wassermann [1987], Lafforgue [2002b] and finally in general by Chabert, Echterhoff and Nest [Chabert et al. 2003].

Despite this explicit knowledge about the structure of $K_*(C_r^*G)$, it remains a challenge to extract explicit representation theoretic information from this K-theory group. There has been a good amount of success in this direction for classes

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in $K_*(C^*_r G)$ corresponding to discrete series representations, for groups having such representations. For example, Lafforgue [2002a] used K-theory to recover Harish-Chandra's criterion $\mathrm{rank}(G) = \mathrm{rank}(K)$ for the existence of discrete series representations.

The von Neumann trace τ_e on $C^*_r G$, defined by $\tau_e(f) = f(e)$ for f in a dense subalgebra, induces a map on $K_0(C^*_r G)$. On classes corresponding to the discrete series, this gives the formal degrees of such representations. But this trace maps all other classes to zero (see Proposition 7.3 in [Connes and Moscovici 1982]). It has recently become clear that a natural generalisation of the von Neumann trace involving *orbital integrals* can be used to extract much more information from $K_0(C^*_r G)$. For a semisimple element $g \in G$, the orbital integral $\tau_g(f)$ of a function f on G is the integral of f over the conjugacy class of g. This integral converges for f in Harish-Chandra's Schwartz algebra, which has the same K-theory as $C^*_r G$. That leads to maps

$$\tau_g : K_0(C^*_r G) \to \mathbb{C}. \tag{1.1}$$

If D is an elliptic operator on a $\mathbb{Z}_2$-graded vector bundle over a manifold M, G-equivariant for a proper, cocompact action by G on M, then one has the equivariant index

$$\mathrm{index}_G(D) \in K_0(C^*_r G).$$

In [Hochs and Wang 2018a], the authors proved a fixed point formula for the numbers

$$\tau_g(\mathrm{index}_G(D)). \tag{1.2}$$

They showed that Harish-Chandra's character formula for the discrete series is a special case of this fixed point formula, much as Weyl's character formula is a special case of the Atiyah–Segal–Singer [Atiyah and Segal 1968] or Atiyah–Bott [Atiyah and Bott 1967] fixed point formulas, as proved in [Atiyah and Bott 1968]. Also, Shelstad's character identities for L-packets of representations follows from a K-theoretic argument involving τ_g, in the case of discrete series representations [Hochs and Wang 2018c].

Another approach to index theory involving orbital integrals is the work of Bismut on hypoelliptic Laplacians; see for example [Bismut 2011] or the survey [Ma 2017].

For discrete groups, orbital integrals (now sums over conjugacy classes) are also useful tools in K-theory. The main result in [Wang and Wang 2016] is a fixed point theorem for (1.2) in the discrete group case, which has consequences to orbifold geometry, positive scalar curvature metrics, and trace formulas. Gong [2015] and Samurkaş [2017] used such maps on the K-theory of maximal group C^*-algebras to deduce information about rigidity of manifolds. Lott [1999] used orbital integrals for discrete groups to construct secondary invariants. Xie and Yu [2018] expressed Lott's delocalised η-invariant in terms of a K-theoretic ρ-invariant.

For semisimple Lie groups G, the results in [Hochs and Wang 2018a; 2018c; Lafforgue 2002b] mentioned above show that classes in $K_0(C_r^*G)$ corresponding to the discrete series contain a great deal of information about those representations. But it was long unclear what (representation theoretic) information can be recovered from other classes. That question was important motivation for this paper. As a concrete example, it was not known what information the generator of $K_0(C_r^*\,\mathrm{SL}(2,\mathbb{R}))$ corresponding to the limits of discrete series (or to the nonspherical principal series) contains.

In the present paper, we investigate further properties and applications of the maps (1.1) for semisimple Lie groups, many of them related to the fixed point formula for (1.2). This starts with an explicit expression for τ_g applied to K-theory generators defined via Dirac induction (Theorem 3.2). That result shows that τ_g is the zero map on K-theory if $\mathrm{rank}(G) \neq \mathrm{rank}(K)$, but it has interesting consequences if $\mathrm{rank}(G) = \mathrm{rank}(K)$. These include

- a way to use the maps τ_g to distinguish elements of $K_0(C_r^*G)$ (Corollary 4.1);
- an embedding of $K_0(C_r^*G)$ into the spaces of distributions on G^{reg} or G (Corollary 4.2);
- an induction formula from K-equivariant indices to G-equivariant indices (Corollary 4.8);
- versions of Selberg's vanishing principle for classes in $K_0(C_r^*G)$ (Corollary 4.9) and matrix coefficients of the discrete series (Corollary 4.10);
- a Tannaka-type duality result (Corollary 4.11);
- a result relating the value of τ_g on K-theory generators to characters of representations (Corollary 5.3).

Furthermore, Dirac induction is known to be injective (indeed, bijective), but we recover this injectivity independently as well.

In the last bullet point above, Corollary 5.3 explicitly states that τ_g maps a K-theory class to the value at g of the character of one of the irreducible direct summands of the representation it corresponds to naturally. The values at g of these characters are equal up to a sign, and they add up to zero if that representation is reducible. So the value at g of one of these characters is the most relevant information one could have expected to obtain by applying τ_g. This, to a large extent, answers the question if and what representation theoretic information is contained in classes in $K_0(C_r^*G)$ if $\mathrm{rank}(G) = \mathrm{rank}(K)$, even those not corresponding to the discrete series. In particular, the generator of $K_0(C_r^*\,\mathrm{SL}(2,\mathbb{R}))$ corresponding to the limits of discrete series determines the characters of these representations on K.

In work in preparation, Higson, Song and Tang compute the values of τ_g on generators of $K_0(C_r^*G)$ independently, without using index theory. This is part of their proof of the Connes–Kasparov conjecture, which states that Dirac induction is bijective.

For a fixed element $x \in K_0(C_r^*G)$, we will see that $\tau_g(x)$ does not depend continuously on g, for example at the identity element e. Theorem 6.2 states that a modified version of τ_g, related to L-packets of representations in the Langlands program, has better continuity properties at e. That implies continuity of certain finite sums of discrete series characters (Corollary 6.3). And that can be used to take the limit as $g \to e$ in Harish-Chandra's character formula for the discrete series to obtain expressions for formal degrees of discrete series representations.

We hope that the various applications of orbital integrals to K-theory of group C^*-algebras in this paper help to demonstrate the relevance of orbital integrals as a tool to study such K-theory groups. In future work, we hope to generalise the results and their applications in this paper to more general groups.

2. Preliminaries

Throughout this paper, let G be a connected semisimple Lie group with finite centre. Let $K < G$ be a maximal compact subgroup. For any Lie group, we denote its Lie algebra by the corresponding gothic letter. Fix a K-invariant inner product on $\mathfrak{g}$, and let $\mathfrak{p} \subset \mathfrak{g}$ be the orthogonal complement to $\mathfrak{k}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

2A. ***Dirac induction.*** The map $\mathrm{Ad} : K \to \mathrm{SO}(\mathfrak{p})$ lifts to $\widetilde{\mathrm{Ad}} : \widetilde{K} \to \mathrm{Spin}(\mathfrak{p})$, for a double cover $\widetilde{K}$ of K. Let $\Delta_{\mathfrak{p}}$ be the standard representation of $\mathrm{Spin}(\mathfrak{p})$, seen as a representation of $\widetilde{K}$ via $\widetilde{\mathrm{Ad}}$. Let $\hat{K}_{\mathrm{Spin}}$ be the set of irreducible representations V of $\widetilde{K}$ such that $\Delta_{\mathfrak{p}} \otimes V$ descends to a representation of K. Let $R_{\mathrm{Spin}}(K)$ be the free abelian group generated by $\hat{K}_{\mathrm{Spin}}$.

Let $V \in \hat{K}_{\mathrm{Spin}}$. Then we have the G-equivariant vector bundle

$$E_V := G \times_K (\Delta_{\mathfrak{p}} \otimes V) \to G/K.$$

Let $\{X_1, \dots, X_{\dim(G/K)}\}$ be an orthonormal basis of $\mathfrak{p}$. Let $c_{\mathfrak{p}} : \mathfrak{p} \to \mathrm{End}(\Delta_{\mathfrak{p}})$ be the Clifford action. Let $L : \mathfrak{g} \to \mathrm{End}(C^\infty(G))$ be the infinitesimal left regular representation. Consider the Dirac operator

$$D_V := \sum_{j=1}^{\dim(G/K)} L(X_j) \otimes c_{\mathfrak{p}}(X_j) \otimes 1_V$$

on

$$\Gamma^\infty(E_V) = (C^\infty(G) \otimes \Delta_{\mathfrak{p}} \otimes V)^K.$$

If G/K has a G-invariant Spin-structure (which is the case precisely if $\Delta_{\mathfrak{p}}$ descends to K), then D_V is the Spin-Dirac operator on G/K coupled to the bundle

$G \times_K V \to G/K$; see Proposition 1.1 in [Parthasarathy 1972]. In any case, D_V is a G-equivariant elliptic differential operator, and has an index

$$\mathrm{index}_G(D_V) \in K_*(C^*_r G).$$

Here $C^*_r G$ is the reduced group C^*-algebra of G, and index_G is the analytic assembly map [Baum et al. 1994]. If $\dim(G/K)$ is even, then $\Delta_{\mathfrak{p}}$, and hence E_V, has a natural $\mathbb{Z}_2$-grading with respect to which D_V is odd. Then $\mathrm{index}_G(D_V) \in K_0(C^*_r G)$. If $\dim(G/K)$ is odd, then there is no such grading, and $\mathrm{index}_G(D_V) \in K_1(C^*_r G)$. So in general, we have

$$\mathrm{index}_G(D_V) \in K_{\dim(G/K)}(C^*_r G).$$

Dirac induction is the map

$$\text{D-Ind}_K^G : R_{\mathrm{Spin}}(K) \to K_{\dim(G/K)}(C^*_r G)$$

given by

$$\text{D-Ind}_K^G[V] = \mathrm{index}_G(D_V),$$

with V as above. By the Connes–Kasparov conjecture, proved in [Chabert et al. 2003; Lafforgue 2002b; Wassermann 1987], this map is an isomorphism of abelian groups.

From now on, we suppose that G/K is even-dimensional, since the K-theory group $K_0(C^*_r G)$ we study is zero otherwise.

2B. ***Orbital integrals and a fixed point formula.*** Let $g \in G$ be a semisimple element. Let $Z_G(g) < G$ be its centraliser. Let $d(hZ_G(g))$ be the left invariant measure on $G/Z_G(g)$ determined by a Haar measure dg on G. The *orbital integral* with respect to g of a measurable function f on G is

$$\tau_g(f) := \int_{G/Z_G(g)} f(hgh^{-1})\, d(hZ_G(g)),$$

if the integral converges. Harish-Chandra [1966, Theorem 6] proved that the integral converges for f in the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$. The subalgebra $\mathcal{C}(G) \subset C^*_r G$ is dense and closed under holomorphic functional calculus [Hochs and Wang 2018a, Theorem 2.3]. Hence we obtain a map

$$\tau_g : K_0(C^*_r G) = K_0(\mathcal{C}(G)) \to \mathbb{C}.$$

Note that τ_e is the usual von Neumann trace.

Let M be a Riemannian manifold with a proper, isometric, cocompact action by G. Let $E \to M$ be a G-equivariant, Hermitian, $\mathbb{Z}_2$-graded vector bundle. Let D be an odd, self-adjoint, G-equivariant, elliptic differential operator on E. Then we have

$$\mathrm{index}_G(D) \in K_0(C^*_r G).$$

In [Hochs and Wang 2018a], the authors proved a fixed-point formula for the number $\tau_g(\mathrm{index}_G(D))$, for almost all $g \in G$. Consequences include Harish-Chandra's character formula for the discrete series [Harish-Chandra 1966, Theorem 16] (see [Hochs and Wang 2018a, Corollary 2.6]) and Shelstad's character identities in the case of discrete series representations [Shelstad 1979] (see [Hochs and Wang 2018c, Theorem 2.5]). In this paper, we explore further consequences.

To state the fixed point formula in [Hochs and Wang 2018a], let $\mathcal{N} \to M^g$ be the normal bundle to the fixed point set M^g of g in M. Let σ_D be the principal symbol of D. Let $c^g \in C_c(M^g)$ be nonnegative, and such that for all $m \in M^g$,

$$\int_{Z_G(g)} c^g(hm)\, dh = 1,$$

for a fixed Haar measure dh on $Z_G(g)$ compatible with dg and $d(hZ_G(g))$. If G/K is odd-dimensional, then $K_0(C^*_r G) = 0$, so $\tau_g(\mathrm{index}_G(D)) = 0$.

Theorem 2.1. *If G/K is even-dimensional, then for almost all semisimple $g \in G$, we have $\tau_g(\mathrm{index}_G(D)) = 0$ if g is not contained in any compact subgroup of G, and*

$$\tau_g(\mathrm{index}_G(D)) = \int_{TM^g} c^g \frac{\mathrm{ch}\big([\sigma_D|_{\mathrm{supp}(c^g)}](g)\big)\,\mathrm{Todd}(TM^g \otimes \mathbb{C})}{\mathrm{ch}\big(\big[\bigwedge \mathcal{N} \otimes \mathbb{C}\big](g)\big)} \tag{2.2}$$

if it is.

Here

$$\mathrm{ch} : K^0(\mathrm{supp}(c^g)) \to H^*(\mathrm{supp}(c^g)),$$
$$\mathrm{ch} : K^0(TM^g|_{\mathrm{supp}(c^g)}) \to H^*(TM^g|_{\mathrm{supp}(c^g)})$$

are Chern characters, and Todd denotes the Todd class.

Remark 2.3. Explicitly, Theorem 2.1 holds for the semisimple $g \in G$ with *finite Gaussian orbital integral* (FGOI) [Hochs and Wang 2018a, Definition 7]. That condition means that the integral

$$\int_{G/Z_G(g)} e^{-d(e,hgh^{-1})^2}\, d(hZ_G(g))$$

converges, where d is the G-invariant Riemannian distance on G. It was shown in [Hochs and Wang 2018a, Proposition 4.2] that almost every element of G has FGOI.

In this paper, whenever a result is stated for almost all g, what is meant is that it holds for semisimple elements with FGOI, and possibly also with dense powers in a maximal torus.

3. A fixed point formula on G/K

Let $T < K$ be a maximal torus. Let $\widetilde{T} < \widetilde{K}$ be its inverse image in $\widetilde{K}$. Fix a set R_c^+ of positive roots of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Let ρ_c be half the sum of the elements of R_c^+. Let $V \in \hat{K}_{\mathrm{Spin}}$. Let $\lambda \in i\mathfrak{t}^*$ be its highest weight with respect to R_c^+.

For any finite-dimensional (actual or virtual) representation W of K or $\widetilde{K}$, we denote its character by χ_W. For any function φ on $\widetilde{K}$ that descends to a function on K, we use the same notation φ for both the function on $\widetilde{K}$ and K. For example, we have $\chi_{\Delta_{\mathfrak{p}}}\chi_V \in C^\infty(K)$.

In the case where T is a Cartan subgroup of G, i.e., $\operatorname{rank}(G) = \operatorname{rank}(K)$, fix a set of positive noncompact roots R_n^+ of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ such that the character $\chi_{\Delta_{\mathfrak{p}}}$ of the graded representation $\Delta_{\mathfrak{p}}$ of $\widetilde{K}$ satisfies

$$\chi_{\Delta_{\mathfrak{p}}}|_{\widetilde{T}} = \prod_{\alpha\in R_n^+} (e^{\alpha/2} - e^{-\alpha/2}). \tag{3.1}$$

Such a choice of positive noncompact roots can always be made; see, for example, [Atiyah and Schmid 1977, pp. 17–18; Parthasarathy 1972, Remark 2.2; Atiyah and Singer 1968, (5.1)]. In the equal-rank case, we write $R^+ := R_c^+ \cup R_n^+$. We denote half the sums of the elements of R^+ and R_n^+ by ρ and ρ_n, respectively.

Let $W_K := N_K(T)/T$ be the Weyl group of (K, T).

Theorem 3.2. (a) *If* $\operatorname{rank}(G) = \operatorname{rank}(K)$, *then for almost all* $g \in T$,

$$\begin{aligned}\tau_g(\text{D-Ind}_K^G[V]) &= (-1)^{\dim(G/K)/2}\frac{\chi_V}{\chi_{\Delta_{\mathfrak{p}}}}(g)\\ &= (-1)^{\dim(G/K)/2}\frac{\sum_{w\in W_K}\varepsilon(w)e^{w(\lambda+\rho_c)}}{\prod_{\alpha\in R^+}(e^{\alpha/2}-e^{-\alpha/2})}(g).\end{aligned}$$

(*In particular*, *the right-hand sides are well-defined.*)

(b) *If* $\operatorname{rank}(G) \neq \operatorname{rank}(K)$, *then for almost all* $g \in T$,

$$\tau_g(\text{D-Ind}_K^G[V]) = 0.$$

Let $\mathfrak{a} \subset \mathfrak{p}$ be an abelian subspace such that $Z_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t} \oplus \mathfrak{a}$. Let $c \in C_c(\mathfrak{a})$ be a function whose integral over $\mathfrak{a}$ is 1. Let σ_{D_V} be the principal symbol of D_V.

Lemma 3.3. *For almost all* $g \in T$,

$$\tau_g(\text{D-Ind}_K^G[V]) = \int_{T\mathfrak{a}} c\,\frac{\operatorname{ch}\big([\sigma_{D_V}|_{\operatorname{supp}(c)}](g)\big)}{\operatorname{ch}\big(\big[\mathfrak{a}\times\bigwedge\mathfrak{p}/\mathfrak{a}\otimes\mathbb{C}\big](g)\big)}.$$

Proof. Let $g \in T$ be such that its powers are dense in T, and with FGOI (see Remark 2.3). By Proposition 4.2 in [Hochs and Wang 2018a], almost all elements of T have these two properties.

We have $G/K \cong \mathfrak{p}$ as K-spaces, and hence in particular as T-spaces. Therefore,

$$(G/K)^g = (G/K)^T = \mathfrak{p}^{\mathrm{Ad}(T)} = \mathfrak{a}.$$

Set $A := \exp(\mathfrak{a})$; this is the centraliser of g in $\exp(\mathfrak{p})$. We have $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}/\mathfrak{a}$ as representations of T. So the normal bundle in $G/K = \mathfrak{p}$ to $(G/K)^g = \mathfrak{a}$ is $\mathfrak{a} \times \mathfrak{p}/\mathfrak{a} \to \mathfrak{a}$. The Todd class of the trivial bundle $T(G/K)^g \otimes \mathbb{C} \to (G/K)^g$ is 1. Hence the claim follows from Theorem 2.1. □

Let us compute $[\sigma_{D_V}|_{\mathrm{supp}(c)}]$. Let $\beta_{\mathfrak{a}} \in K^0(\mathfrak{a})$ be the Bott generator. (Note that $\mathfrak{a}$ is even-dimensional since G/K is.) Let $\pi : T\mathfrak{a} \to \mathfrak{a}$ be the tangent bundle projection, and $\pi|_{\mathrm{supp}(c)} : \mathrm{supp}(c) \times \mathfrak{a} \to \mathrm{supp}(c)$ its restriction. Note that

$$\Delta_{\mathfrak{p}} \cong \Delta_{\mathfrak{a}} \otimes \Delta_{\mathfrak{p}/\mathfrak{a}}$$

as graded representations of $\widetilde{T}$. These descend to T after tensoring with V.

Lemma 3.4. *Under the isomorphism*

$$K_0^T(\mathrm{supp}(c) \times \mathfrak{a}) \cong K_0(\mathrm{supp}(c) \times \mathfrak{a}) \otimes R(T),$$

we have

$$[\sigma_{D_V}|_{\mathrm{supp}(c)}] \mapsto \pi|_{\mathrm{supp}(c)}^* \beta_{\mathfrak{a}} \otimes [\Delta_{\mathfrak{p}/\mathfrak{a}} \otimes V].$$

Proof. Let $c_{\mathfrak{a}} : \mathfrak{a} \to \mathrm{End}(\Delta_{\mathfrak{a}})$ be the Clifford action. The class

$$\pi|_{\mathrm{supp}(c)}^* \beta_{\mathfrak{a}} \in K^0(\mathrm{supp}(c) \times \mathfrak{a})$$

is defined by[1] the vector bundle homomorphism

$$A : \mathrm{supp}(c) \times \Delta_{\mathfrak{a}}^+ \to \mathrm{supp}(c) \times \Delta_{\mathfrak{a}}^-$$

given by

$$A_Y = c_{\mathfrak{a}}(Y)$$

for all $Y \in \mathrm{supp}(c)$.

We have

$$(G \times_K (\Delta_{\mathfrak{p}}^{\pm} \otimes V))|_{\mathfrak{a}} \cong \mathfrak{a} \times \Delta_{\mathfrak{p}}^{\pm} \otimes V$$

as T-vector bundles. So

$$\pi|_{\mathrm{supp}(c)}^* \big((G \times_K (\Delta_{\mathfrak{p}}^{\pm} \otimes V))|_{\mathrm{supp}(c)}\big) = (\mathrm{supp}(c) \times \mathfrak{a}) \times \Delta_{\mathfrak{p}}^{\pm} \otimes V.$$

Let $X, Y \in \mathfrak{a}$, so that, using the above identification, we get

$$\sigma_{D_V}(X, Y) = c_{\mathfrak{p}}(Y) \otimes 1_V : \Delta_{\mathfrak{p}}^+ \otimes V \to \Delta_{\mathfrak{p}}^- \otimes V. \tag{3.5}$$

Since $Y \in \mathfrak{a}$, the map (3.5) equals the odd endomorphism

$$c_{\mathfrak{a}}(Y) \otimes 1_{\Delta_{\mathfrak{p}/\mathfrak{a}} \otimes V} \in \mathrm{End}(\Delta_{\mathfrak{a}} \otimes \Delta_{\mathfrak{p}/\mathfrak{a}} \otimes V).$$

Together with the above form of the class $\pi|_{\mathrm{supp}(c)}^* \beta_{\mathfrak{a}}$, this implies the claim. □

[1] We absorb a possible sign in the definition of $\beta_{\mathfrak{a}}$; see [Connes and Moscovici 1982, Lemma 4.1].

Lemma 3.6. *Suppose that* $\operatorname{rank}(G) = \operatorname{rank}(K)$. *Then*

$$\textstyle\bigwedge \mathfrak{p} \otimes \mathbb{C} = (-1)^{\dim(G/K)/2} \Delta_{\mathfrak{p}} \otimes \Delta_{\mathfrak{p}}$$

as graded representations of T.

Proof. The set of positive noncompact roots R_n^+ determines a complex structure on $\mathfrak{p}$ such that $\mathfrak{p}^{1,0}$ is the sum of the positive noncompact root systems. As graded representations of T, we have

$$\textstyle\bigwedge \mathfrak{p} \otimes \mathbb{C} = \bigwedge \mathfrak{p}^{1,0} \otimes \bigwedge \mathfrak{p}^{0,1} = \bigwedge_{\mathbb{C}} \mathfrak{p} \otimes (\bigwedge_{\mathbb{C}} \mathfrak{p})^*.$$

The element $\rho_n \in i\mathfrak{t}^*$ is integral for $\widetilde{T}$, and $\Delta_{\mathfrak{p}} \otimes \mathbb{C}_{\rho_n}$ descends to a representation of T. We have

$$\textstyle\bigwedge_{\mathbb{C}} \mathfrak{p} = (-1)^{\dim(G/K)/2} \Delta_{\mathfrak{p}} \otimes \mathbb{C}_{\rho_n}$$

as graded representations of T; see, for example, the proof of Lemma 5.5 in [Hochs and Wang 2018a]. Since $\Delta_{\mathfrak{p}}^* \cong (-1)^{\dim(G/K)/2} \Delta_{\mathfrak{p}}$, we conclude that

$$\textstyle\bigwedge \mathfrak{p} \otimes \mathbb{C} = \Delta_{\mathfrak{p}} \otimes \Delta_{\mathfrak{p}}^* = (-1)^{\dim(G/K)/2} \Delta_{\mathfrak{p}} \otimes \Delta_{\mathfrak{p}}.$$

The nontrivial element of the kernel of the covering map $\widetilde{K} \to K$ acts on $\Delta_{\mathfrak{p}}$ as ± 1; therefore, $\Delta_{\mathfrak{p}} \otimes \Delta_{\mathfrak{p}}$ descends to a representation of T. $\square$

Lemma 3.7. *Let c be a nonnegative, compactly supported, continuous function on* $\mathbb{R}^{2n}$ *with integral* 1. *Let $\beta \in K^0(\mathbb{R}^{2n})$ be the Bott class, and consider*

$$\pi|_{\operatorname{supp}(c)} : \operatorname{supp}(c) \times \mathbb{R}^{2n} \to \operatorname{supp}(c),$$

where $\pi : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the natural projection. Then

$$\int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} c \operatorname{ch}(\pi|_{\operatorname{supp}(c)}^* \beta) = 0. \tag{3.8}$$

Proof. By Proposition 6.11 in [Wang 2014], the integral (3.8) equals the L^2-index of the Spin-Dirac operator on $\mathbb{R}^{2n}$. That index is zero because the L^2-kernel of this Dirac operator is zero. Indeed, the Spin-Dirac operator on $\mathbb{R}^{2n}$ only has continuous spectrum; see, for example, Theorem 7.2.1 in [Ginoux 2009]. $\square$

Proof of Theorem 3.2. Lemma 3.4 implies that

$$\operatorname{ch}\big([\sigma_{D_V}|_{\operatorname{supp}(c)}](g)\big) = \operatorname{ch}(\pi|_{\operatorname{supp}(c)}^* \beta_{\mathfrak{a}})(\chi_{\Delta_{\mathfrak{p}/\mathfrak{a}}} \chi_V)(g).$$

Furthermore,

$$\textstyle\operatorname{ch}\big(\big[\mathfrak{a} \times \bigwedge \mathfrak{p}/\mathfrak{a} \otimes \mathbb{C}\big](g)\big) = \chi_{\bigwedge \mathfrak{p}/\mathfrak{a} \otimes \mathbb{C}}(g)$$

in the graded sense. So by Lemma 3.3,

$$\tau_g(\text{D-Ind}_K^G[V]) = \frac{\chi_{\Delta_{\mathfrak{p}/\mathfrak{a}}} \chi_V}{\chi_{\bigwedge \mathfrak{p}/\mathfrak{a} \otimes \mathbb{C}}}(g) \int_{T\mathfrak{a}} c \operatorname{ch}(\pi|_{\operatorname{supp}(c)}^* \beta_{\mathfrak{a}}).$$

If $\mathrm{rank}(G) \neq \mathrm{rank}(K)$, then $\mathfrak{a}$ is nonzero, and the claim follows from Lemma 3.7. If $\mathrm{rank}(G) = \mathrm{rank}(K)$, then Lemma 3.6 implies that

$$\tau_g(\text{D-Ind}_K^G[V]) = (-1)^{\dim(G/K)/2} \frac{\chi_V}{\chi_{\Delta_\mathfrak{p}}}(g);$$

in particular, the right-hand side is well-defined. The claim now follows from Weyl's character formula and (3.1). (Note that $(\widetilde{K}, \widetilde{T})$ and (K, T) have the same Weyl group W_K, since they have the same root system.) □

Remark 3.9. If $g = e$, then $\tau_e(\text{D-Ind}_K^G[V])$ is the L^2-index of D_V by Proposition 4.4 in [Wang 2014]. That index is zero if the kernel of D_V is zero. Theorem 3.2 shows that, in the equal-rank case, the more general trace τ_g yields nonzero information even in cases where the kernel of D_V is zero (see also Section 5B).

4. Consequences

Suppose from now on that $\mathrm{rank}(G) = \mathrm{rank}(K)$.

4A. *Distinguishing K-theory classes.* As a consequence of Theorem 3.2, the traces τ_g "separate points" on $K_0(C_r^*G)$, or distinguish all elements of $K_0(C_r^*G)$, in the following sense.

Corollary 4.1. *Let $x \in K_0(C_r^*G)$. If $\tau_g(x) = 0$ for all g in a dense subset of T, then $x = 0$.*

Proof. Let $x \in K_0(C_r^*G)$. By surjectivity of Dirac induction, we can write

$$x = \sum_{V \in \hat{K}_{\mathrm{Spin}}} m_V \, \text{D-Ind}_K^G[V],$$

for $m_V \in \mathbb{Z}$, finitely many nonzero. By Theorem 3.2, we have for almost all $g \in T$,

$$\tau_g(x) = (-1)^{\dim(G/K)/2} \sum_{V \in \hat{K}_{\mathrm{Spin}}} m_V \frac{\chi_V}{\chi_{\Delta_\mathfrak{p}}}(g).$$

So if $\tau_g(x) = 0$ for all g in a dense subset of T, then by continuity and conjugation invariance of the characters χ_V, we find that

$$\sum_{V \in \hat{K}_{\mathrm{Spin}}} m_V \chi_V = 0.$$

So $m_V = 0$ for all V, i.e., $x = 0$. □

4B. *K-theory and distributions.* Let $G^{\mathrm{reg}} \subset G$ be the subset of regular elements.

Corollary 4.2. *The map*

$$\tau : K_0(C_r^*G) \to \mathcal{D}'(G^{\mathrm{reg}})$$

defined by

$$\langle \tau(x), f \rangle = \int_{G^{\mathrm{reg}}} \tau_g(x) f(g)\, dg$$

for $x \in K_0(C^*_r G)$ and $f \in C^\infty_c(G^{\mathrm{reg}})$, is a well-defined, injective group homomorphism.

Proof. Let $x \in K_0(C^*_r G)$. By the surjectivity of Dirac induction, we can write $x = \text{D-Ind}_K^G[y]$, for some $y \in R_{\mathrm{Spin}}(K)$. Theorem 3.2 implies that the function $g \mapsto \tau_g(x)$ equals an analytic function almost everywhere on the set of elliptic elements of G. Theorem 2.1 implies that this function equals zero almost everywhere on the set of nonelliptic elements of G. So $g \mapsto \tau_g(x)$ equals an analytic function almost everywhere on G. Furthermore, that analytic function is bounded on compact subsets of G^{reg}. This implies that $\tau(x)$ is a well-defined distribution on G^{reg}.

If $\tau(x) = 0$, then $\tau_g(x) = 0$ for almost all $g \in G^{\mathrm{reg}}$, in particular for almost all elements of T. Hence Corollary 4.1 implies that $x = 0$. □

Remark 4.3. As noted in the proof of Corollary 4.2, the first part of Theorem 2.1 implies that $\tau(x)$ is zero outside the set of regular elliptic elements of G.

Remark 4.4. We describe the map τ in Corollary 4.2 explicitly in terms of characters of representations in Section 5. There we see that $\tau(x)$ equals the character of a tempered representation of G almost everywhere on the set of regular elliptic elements, and zero almost everywhere outside the set of elliptic elements. Therefore, it extends to a distribution on all of G by Harish-Chandra's regularity theorem.

4C. ***Injectivity of Dirac induction.*** We have used the surjectivity of Dirac induction in the proof of Corollary 4.1 (which is justified because the Connes–Kasparov conjecture has been proved). Theorem 3.2 implies injectivity of Dirac induction.

Corollary 4.5. *Dirac induction is injective.*

Proof. Let $y \in R_{\mathrm{Spin}}(K)$, and suppose that $\text{D-Ind}_K^G(y) = 0$. Then $\tau_g(\text{D-Ind}_K^G(y)) = 0$ for all $g \in T$. Theorem 3.2 implies that for almost all $g \in T$,

$$\frac{\chi_y}{\chi_{\Delta_{\mathfrak{p}}}}(g) = 0.$$

So $\chi_y = 0$, i.e., $y = 0$. □

4D. ***An induction formula.*** Let M be an even-dimensional Riemannian manifold with a G-equivariant Spin^c-structure. Let $E \to M$ be a G-equivariant, Hermitian vector bundle. Let D_M^E be the Spin^c-Dirac operator on M twisted by E. By Abels' theorem [1974], there is a K-invariant submanifold $N \subset M$ such that $M \cong G \times_N N$ via the action map $G \times N \to M$. Furthermore, N has a K-equivariant Spin^c-structure on N compatible with the one on M; see Proposition 3.10 in [Hochs and Mathai 2017]. The Spin^c-Dirac operator D_N^E on N, twisted by $E|_N$, has the property that

$$\text{D-Ind}_K^G(\text{index}_K(D_N^E)) = \text{index}_G(D_M^G) \in K_0(C_r^*G). \tag{4.6}$$

See Theorem 5.2 in [Hochs and Wang 2018a] and Proposition 4.7 in [Hochs 2009].

Theorem 3.2 and surjectivity of Dirac induction imply that the following diagram commutes for all g in the dense subset of T in Theorem 3.2:

$$\begin{tikzcd} K_0(C_r^*G) \arrow[drr, "\tau_g"] & & \\ R_{\text{Spin}}(K) \arrow[u, "\text{D-Ind}_K^G"] \arrow[rr, "(-1)^{\dim(G/K)/2}\,\text{ev}_g/\chi_{\Delta_\mathfrak{p}}(g)"'] & & \mathbb{C} \end{tikzcd} \tag{4.7}$$

Here ev_g denotes evaluation of characters of representations at g; note that the bottom arrow is well-defined.

The equality (4.6) and commutativity of (4.7) imply the following formula for induction from slices.

Corollary 4.8. *We have, for almost all* $g \in T$,

$$\tau_g(\text{index}_G(D_M^E)) = (-1)^{\dim(G/K)/2}\,\text{index}_K(D_N^E)(g)/\chi_{\Delta_\mathfrak{p}}(g).$$

Note that the right-hand side can be computed via the Atiyah–Segal–Singer fixed point formula [Atiyah and Segal 1968].

Induction formulas like Corollary 4.8 we used in various settings to deduce results about G-equivariant indices from results about K-equivariant indices [Guo et al. 2018; Hochs 2009; Hochs and Mathai 2016; Hochs and Mathai 2017; Hochs and Wang 2018a]. The case $g = e$ is not covered by Corollary 4.8; that case is Corollary 53 in [Guo et al. 2018].

4E. ***Selberg's principle.*** The Selberg principle is a vanishing result for orbital integrals of certain convolution idempotents on G. See [Blanc and Brylinski 1992; Julg and Valette 1986; 1987] for approaches to this principle in the spirit of noncommutative geometry. Theorem 2.1 implies a version of this principle.

Corollary 4.9 (K-theoretic Selberg principle). *For almost all* g *not contained in compact subgroups of* G, *the map*

$$\tau_g : K_0(C_r^*G) \to \mathbb{C}$$

is zero.

Proof. Theorem 2.1 implies that for almost all g not contained in compact subgroups of G, and all $V \in R_{\text{Spin}}(K)$, we have

$$\tau_g(\text{D-Ind}_K^G[V]) = 0.$$

So surjectivity of Dirac induction implies the claim. ∎

Corollary 4.9 has a purely representation theoretic consequence.

Corollary 4.10 (Selberg principle for matrix coefficients of the discrete series). *Let π be a discrete series representation of G. Let v be a K-finite vector in the representation space of π, and $m_{v,v}$ the corresponding matrix coefficient. For all g not contained in compact subgroups of G, we have*

$$\tau_g(m_{v,v}) = 0.$$

Proof. Let d_π be the formal degree of π. By rescaling, we may assume that v has norm 1. Then $d_\pi \bar{m}_{v,v}$ is an idempotent in C_r^*G. Let $[\pi] \in K_0(C_r^*G)$ be its K-theory class. Since v is K-finite, the function $m_{v,v}$ lies in Harish-Chandra's Schwartz algebra $\mathcal{C}(G)$. Therefore, for all semisimple $g \in G$,

$$\tau_g(m_{v,v}) = \frac{1}{d_\pi}\overline{\tau_g([\pi])}.$$

By Corollary 4.9, the number is zero for almost all g not contained in compact subgroups. The claim therefore follows by continuity of $m_{v,v}$. □

4F. ***A Tannaka-type duality.*** We now suppose that the representation $\Delta_{\mathfrak{p}}$ of $\widetilde{K}$ descends to K. This is true if we replace G by a double cover if necessary. Then Dirac induction is defined on $R(K)$.

The K-theory group $K_0(C_r^*G)$ and its elements contain nontrivial information about G and its representations; see, e.g., [Hochs and Wang 2018a; 2018c; Lafforgue 2002b]. But just the isomorphism class of $K_0(C_r^*G)$ as an abelian group contains no information about G whatsoever: this group is always free, with countably infinitely many generators. It turns out, however, that the combination of the isomorphism class of $K_0(C_r^*G)$, the topological space T and the maps $\tau_g: K_0(C_r^*G) \to \mathbb{C}$, for g in a dense subset of T, together determine the Cartan motion group $K \ltimes \mathfrak{p}$ and vice versa. The tempered representation theory of $K \ltimes \mathfrak{p}$ is closely related to that of G; this is the Mackey analogy [Afgoustidis 2015; Higson 2008; 2011; Mackey 1975; Tan et al. 2017; Yu 2017]. Also, the analytic assembly map for G can be defined in terms of a continuous deformation from $K \ltimes \mathfrak{p}$ to G; see pp. 23–24 of [Baum et al. 1994] and [Higson 2008].

This is vaguely analogous to the fact that the irrational rotation algebras A_λ, for irrational λ in $\left[0, \frac{1}{2}\right]$, have the same K-theory $\mathbb{Z} \oplus \mathbb{Z}$, but are determined up to isomorphism by the pair $(K_0(A_\lambda), \tau)$, where τ is a natural trace. This is because the image of τ is $\mathbb{Z} + \lambda\mathbb{Z}$.

Corollary 4.11. *The*

- *abelian group $K_0(C_r^*G)$ up to isomorphism,*
- *pointed topological space $(T, \{e\})$ up to homeomorphism, and*

- *family of group homomorphisms $\tau_g : K_0(C^*_r G) \to \mathbb{C}$, for g in a dense subset of T,*

together determine the Cartan motion group $K \ltimes \mathfrak{p}$, and vice versa.

Proof. Write

$$K_0(C^*_r G) = \bigoplus_{j \in \mathbb{Z}} \mathbb{Z}$$

and let e_j be a generator of the j-th copy of $\mathbb{Z}$. Let S be the intersection of the dense subset of T in the third point in the corollary and the set of $g \in T$ for which the formula in Theorem 3.2 holds. Then S is dense in T.

Consider the function $\chi_j : S \to \mathbb{R}$ given by $\chi_j(g) = \tau_g(e_j)$. By Theorem 3.2, there is a function $\psi \in C^\infty(T)$, not unique but independent of j, and there are uniquely determined integers d_j such that for all j,

$$\lim_{g \to e} \psi(g)\chi_j(g) = d_j,$$

where at least one of the integers d_j equals 1. (Indeed, take $\psi = \chi_{\Delta_\mathfrak{p}}|_S$ and d_j plus or minus the dimensions of the irreducible representations of K.) By replacing e_j by $-e_j$ where necessary, we can make sure that all integers d_j are positive.

Fix $j_0 \in \mathbb{Z}$ such that $d_{j_0} = 1$. Then, again by Theorem 3.2,

$$\left|\chi_{\Delta_\mathfrak{p}}|_S\right| = |\chi_{j_0}|^{-1}.$$

And $\overline{\chi_{\Delta_\mathfrak{p}}} = -\chi_{\Delta_\mathfrak{p}}$, so $\chi_{\Delta_\mathfrak{p}}$ is imaginary-valued. Hence

$$\chi_{\Delta_\mathfrak{p}}|_S = \pm i|\chi_{j_0}|^{-1}.$$

We cannot resolve the sign ambiguity with the data we have, but we do not need to.

The characters of irreducible representations V_j of K are continuous and conjugation invariant, so they are determined by

$$\chi_{V_j}|_S = (\chi_{\Delta_\mathfrak{p}}|_S)\chi_j = \pm i|\chi_{j_0}|^{-1}\chi_j,$$

with the sign chosen such that $\pm i|\chi_{j_0}|^{-1}\chi_j > 0$ near the identity element. This determines the representations V_j of K, and their tensor products and the underlying vector spaces. By Tannaka duality [Tannaka 1938], this determines K.

To recover $\mathfrak{p}$ as a K-representation, set $\psi := i|\chi_{j_0}|^{-1}$, extended continuously to T. Then

$$\psi = \pm\chi_{\Delta_\mathfrak{p}}|_T = \pm \prod_{\alpha \in R_n^+} (e^{\alpha/2} - e^{-\alpha/2}).$$

This implies that for all $X, Y \in \mathfrak{t}$,

$$\left.\frac{d}{dt}\right|_{t=0} \psi(X + tY) = \psi(\exp(X)) \sum_{\alpha \in R_n^+} \frac{\langle \alpha, Y\rangle}{2} \coth(\langle \alpha, X\rangle/2).$$

The term on the right-hand side corresponding to α equals the same term with α replaced by $-\alpha$. But otherwise this expression determines the weights α up to signs. In this way, we recover the set R_n of $\mathfrak{t}$-weights of $\mathfrak{p} \otimes \mathbb{C}$ as a complex representation of T, and hence $\mathfrak{p}$ as a real representation of T, and therefore as a representation of K. This determines $K \ltimes \mathfrak{p}$.

Conversely, the Cartan motion group $K \ltimes \mathfrak{p}$ determines its maximal compact subgroup K and the quotient $\mathfrak{p} = (K \ltimes \mathfrak{p})/K$ as a representation of K. And K determines the pair $(T, \{e\})$ up to conjugacy. The K-theory group $K_0(C_r^*G)$ is isomorphic to $R(K)$ via Dirac induction. Furthermore, K and $\mathfrak{p}$ determine the characters χ_V, for $V \in \hat{K}$ and $\chi_{\Delta_\mathfrak{p}}$, and the dimension $\dim(G/K) = \dim(\mathfrak{p})$. Hence, by Theorem 3.2, this determines the maps $\tau_g : K_0(C_r^*G) \cong R(K) \to \mathbb{C}$, for g in a dense subset of T. □

Remark 4.12. In Corollary 4.11, one only needs the neighbourhoods of the identity element, not all of its topology. And as stated in the corollary, one does not need the group structure of T.

Remark 4.13. If $G = K$ is compact, then the triple

$$(K_0(C_r^*G), (T, \{e\}), (\tau_g)_{g \in T})$$

determines the ring $R(G)$ of characters of G. That in turn determines the tensor products of representations of G, and forgetful maps to finite-dimensional complex vector spaces. So in this case, Corollary 4.11 reduces to Tannaka duality for compact groups [Tannaka 1938] (which was used in the proof of Corollary 4.11).

Remark 4.14. If the representation $\Delta_\mathfrak{p}$ of $\widetilde{K}$ does not descend to K, then we only recover the ring $R_{\mathrm{Spin}}(K)$ in the proof of Corollary 4.11 and cannot directly apply Tannaka duality.

5. Characters

Again, we suppose that the representation $\Delta_\mathfrak{p}$ of $\widetilde{K}$ descends to K. We may need to replace G by a double cover for this assumption to hold. This assumption is now not essential; see Remark 5.4.

5A. ***Characters and τ_g.*** The structure of the C^*-algebra C_r^*G and its K-theory was described by Wassermann [1987] and Clare, Crisp and Higson [Clare et al. 2016]. We can use this to relate values of τ_g on K-theory classes to values of characters of representations.

Let $P = MAN < G$ be a cuspidal parabolic and σ in the set $\hat{M}_{\mathrm{ds}}$ of discrete series representations of M. Consider the bundle of Hilbert spaces $\mathcal{E}_{P,\sigma} \to \hat{A}$ whose fibre at $\nu \in \hat{A}$ is $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1_N)$. (This can be topologised by viewing it as a trivial bundle in the compact picture of induced representations.) Let $\mathrm{Ind}_P^G(\sigma)$

be the Hilbert $C_0(\hat{A})$-module of continuous sections of $\mathcal{E}_{P,\sigma}$ vanishing at infinity. The group

$$W_\sigma := \{w \in N_K(\mathfrak{a})/Z_K(\mathfrak{a}); w\sigma = \sigma\}$$

acts on $\mathcal{K}(\mathrm{Ind}_P^G(\sigma))$ via Knapp–Stein intertwiners; see Theorem 6.1 in [Clare et al. 2016]. Let $\mathcal{K}(\mathrm{Ind}_P^G(\sigma))^{W_\sigma}$ be the fixed point algebra of this action. Then

$$C_r^*G \cong \bigoplus_{P,\sigma} \mathcal{K}(\mathrm{Ind}_P^G(\sigma))^{W_\sigma},$$

where the sum runs over a set of cuspidal parabolics $P = MAN$ and $\sigma \in \hat{M}_{\mathrm{ds}}$. This is [Clare et al. 2016, Theorem 6.8]. See also Theorem 8 in [Wassermann 1987].

Now let P and σ be such that

$$K_0(\mathcal{K}(\mathrm{Ind}_P^G(\sigma))^{W_\sigma})$$

is nonzero, hence infinite cyclic. (This is equivalent to the condition that W_σ equals the R-group R_σ; see Lemma 10 in [Wassermann 1987].) Let $b(P,\sigma) \in K_0(C_r^*G)$ be the generator of this summand of $K_0(C_r^*G)$ in the image under Dirac induction of the $\mathbb{Z}_{\geq 1}$-span of $\hat{K}$ inside $R(K)$.

Let $\eta \in i\mathfrak{t}_M^*$ be the Harish-Chandra parameter of σ, and $\tilde{\eta} \in i\mathfrak{t}^*$ its extension by zero on the orthogonal complement of $\mathfrak{t}_M$ in $\mathfrak{t}$. For any positive root system $\widetilde{R}^+$ of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ for which $\tilde{\eta}$ is dominant, let $\pi^G(\tilde{\eta}, \widetilde{R}^+)$ be the corresponding (limit of) discrete series representation of G. We need the following version of Schmid's character identities. This is Lemma 12 in [Wassermann 1987] in the equal rank case, but with information included about the infinitesimal characters of the limits of discrete series representations that occur.

Proposition 5.1. *There are $2^{\dim(A)}$ choices of positive roots $R_1^+, \dots, R_{2^{\dim(A)}}^+ \subset R$, obtained from R^+ by the application of all combinations of* $\dim(A)$ *commuting reflections in simple noncompact roots, such that*

$$\mathrm{Ind}_P^G(\sigma \otimes 1_A \otimes 1_N) = \bigoplus_{j=1}^{2^{\dim(A)}} \pi^G(\tilde{\eta}, R_j^+).$$

Proof. This is a special case of Theorem 13.3 in [Knapp and Zuckerman 1982] for the maximal parabolic G in the equal-rank group G. □

As before, let ρ_c be half the sum of the compact positive roots. By Lemma 15(i) in [Wassermann 1987], the element $\tilde{\eta} - \rho_c$ is dominant for K. It is integral because $\Delta_{\mathfrak{p}}$ descends to K; this implies that ρ_n and hence $\tilde{\eta} - \rho + \rho_n$ is integral.

Proposition 5.2 (Wassermann). *Let $V_{\tilde{\eta}-\rho_c} \in \hat{K}$ have highest weight $\tilde{\eta} - \rho_c$. Then*

$$\text{D-Ind}_K^G[V_{\tilde{\eta}-\rho_c}] = b(P,\sigma).$$

Proof. See the last page of [Wassermann 1987]. This uses Proposition 5.1. $\square$

Proposition 5.1 and Harish-Chandra's character formula for (limits of) discrete series representations imply that the character of the representation $\mathrm{Ind}_P^G(\sigma \otimes 1_A \otimes 1_N)$ naturally associated to the K-theory generator $b(P, \sigma)$ is zero on T, if this representation is reducible. (See Section 5B for an example.) Therefore, it is a useful property of the map τ_g that it maps $b(P, \sigma)$ to the possibly nonzero value of an irreducible summand of that representation.

Corollary 5.3. *For almost all $g \in T$, $\tau_g(b(P, \sigma))$ equals the value at g of the character of one of the irreducible summands of $\mathrm{Ind}_P^G(\sigma \otimes 1_A \otimes 1_N)$. The values at g of the characters of these summands at g are all equal up to a sign.*

Proof. Proposition 5.2 and Theorem 3.2 imply that

$$\tau_g(b(P, \sigma)) = \tau_g(\text{D-Ind}_K^G[V_{\tilde{\eta}-\rho_c}]) = (-1)^{\dim(G/K)/2} \frac{\sum_{w \in W_K} \varepsilon(w) e^{w\tilde{\eta}}}{\prod_{\alpha \in R^+}(e^{\alpha/2} - e^{-\alpha/2})}(g).$$

By Harish-Chandra's character formula (extended coherently to the limits of discrete series), the right-hand side is the value at g of the character of $\pi^G(\tilde{\eta}, R^+)$. That formula also shows that on T, the character of $\pi^G(\tilde{\eta}, R^+)$ equals the character of $\pi^G(\tilde{\eta}, R_j^+)$ modulo a sign, for $j = 1, \ldots, 2^{\dim(A)}$. Hence the claim follows from Proposition 5.1. $\square$

Remark 5.4. If the representation $\Delta_{\mathfrak{p}}$ does not descend to K, then the analogue of Corollary 5.3 relates $\tau_g(b(P, \sigma))$ to characters of the corresponding representations of a double cover of G.

5B. ***Nonspherical principal series and limits of discrete series of* SL(2, $\mathbb{R}$).** Consider the case where $G = \mathrm{SL}(2, \mathbb{R})$, $K = T = \mathrm{SO}(2)$, and $P = MAN < \mathrm{SL}(2, \mathbb{R})$ is the minimal parabolic of upper triangular matrices, where $M = \{\pm I\}$. Then $\hat{M}_{\mathrm{ds}} = \{\sigma_+, \sigma_-\}$, where σ_+ is the trivial representation of M in $\mathbb{C}$ and σ_- is the nontrivial one. Now we have Morita equivalences

$$\mathcal{K}(\mathrm{Ind}_P^G(\sigma_+))^{W_{\sigma_+}} \sim C_0([0, \infty)),$$
$$\mathcal{K}(\mathrm{Ind}_P^G(\sigma_-))^{W_{\sigma_-}} \sim C_0(\mathbb{R}) \rtimes \mathbb{Z}_2.$$

See Example 6.11 in [Clare et al. 2016]. So the pair (P, σ_+) does not contribute to $K_0(C_r^*(\mathrm{SL}(2, \mathbb{R})))$, whereas (P, σ_-) contributes a summand $\mathbb{Z}$, generated by

$$b(P, \sigma_-) = \text{D-Ind}_K^G[\mathbb{C}_0].$$

Let $\alpha \in i\mathfrak{t}^*$ be the root mapping $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ to $2i$. Set $R^+ := \{\alpha\}$. Let

$$g = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \in T,$$

where $\varphi \in \mathbb{R} \setminus 2\pi\mathbb{Q}$. Theorem 3.2 now yields

$$\tau_g(b(P, \sigma_-)) = \frac{1}{2i \sin\varphi}.$$

This is the value at g of the character of the limit of discrete series representation $\pi^G(0, R^+)$, and minus the value at g of the character of the limit of discrete series representation $\pi^G(0, -R^+)$. The direct sum of these two representations is the nonspherical principal series representation $\mathrm{Ind}_P^G(\sigma_- \otimes 1_A \otimes 1_N)$. The character of that representation is zero at g.

Some authors, including the authors of this paper, have wondered if the K-theory generator $b(P, \sigma_-)$ can be detected by suitable maps out of $K_0(C_r^*(\mathrm{SL}(2, \mathbb{R})))$, and if representation theoretic information can be recovered from it. This example shows that the answer to both questions is yes.

6. Stable orbital integrals and continuity at the group identity

This section is independent of the rest of this paper. In particular, it does not depend on Theorem 3.2.

It follows from Theorem 3.2 that, for a fixed $x \in K_0(C_r^*G)$, the function

$$g \mapsto \tau_g(x)$$

on the set of semisimple elements g of G is not continuous if G is noncompact. In particular, it is not continuous at the identity element. Theorem 3.2 does imply that this function is continuous almost everywhere. Already in the compact case, it is a nontrivial question if the right-hand side of the fixed point formula (2.2) depends continuously on g, for example as $g \to e$ (as pointed out in Section 8.1 in [Berline et al. 2004]). It turns out that a version of τ_g involving *stable orbital integrals* has better continuity properties near the identity element. (This comes at the cost of mapping more elements to zero, however. See Section 5B, where the stable orbital integral of the class in $K_0(C_r^*\,\mathrm{SL}(2, \mathbb{R}))$ associated to the limits of discrete series is shown to be zero.)

6A. *Continuity at e.* Let $G_{\mathbb{C}}$ be a complex semisimple Lie group, and $G < G_{\mathbb{C}}$ a real form of $G_{\mathbb{C}}$. Let g be a semisimple element of G.

Definition 6.1. The *stable conjugacy class* of g in G is

$$(g)_s := \{hgh^{-1} \in G : h \in G_{\mathbb{C}}\},$$

the intersection of the conjugacy class $(g)_{G_{\mathbb{C}}}$ of g in $G_{\mathbb{C}}$ with G.

For every f in the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$, the *stable orbital integral* of f with respect to g is

$$\tau_g^s(f) := \sum_{g'} \tau_{g'}(f) = \sum_{g'} \int_{G/Z_G(g')} f(hg'h^{-1})\, dh(Z_G(g')),$$

where the sum is over representatives g' of G-conjugacy classes in $(g)_s$, i.e., $(g)_s = \bigsqcup_{g'}(g')$.

Stable conjugacy classes are relevant to the notion of an L-packet of representations and Shelstad's character identities. See [Shelstad 1979].

The map $\tau_g^s : K_0(C_r^*G) = K_0(\mathcal{C}(G)) \to \mathbb{C}$ induced by τ_g^s has better continuity properties in g than τ_g. Let $S \subset G$ be the set of elements g for which Theorem 2.1 holds (see Remark 2.3). Then $G \setminus S$ has measure zero, so in particular S is dense.

Theorem 6.2. *For all $x \in K_0(C_r^*G)$,*

$$\lim_{g \to e;\, g \in S} \tau_g^s(x) = \tau_e(x).$$

(Note that $\tau_e = \tau_e^s$.)

Let $K < G$ be maximal compact. If $\operatorname{rank}(G) \neq \operatorname{rank}(K)$, then Theorem 6.2 follows from Theorem 3.2(b) and the fact that τ_e is identically zero on $K_0(C_r^*G)$. So assume from now on that $\operatorname{rank}(G) = \operatorname{rank}(K)$.

Theorem 6.2 implies a continuity property of characters of L-packets of discrete series representations.

As before, let $T < K$ be a maximal torus, and set $W_K := N_K(T)/T$. Let W_G be the Weyl group of the root system of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Fix representatives $w \in W_G$ of all classes $[w] \in W_G/W_K$. For any discrete series representation with Harish-Chandra parameter λ, we denote its global character by Θ_λ.

Corollary 6.3. *Let π be a discrete series representation of G with Harish-Chandra parameter $\lambda \in i\mathfrak{t}^*$. Then*

$$\lim_{g \to e;\, g \in T^{\mathrm{reg}}} \sum_{[w] \in W_G/W_K} \Theta_{w\lambda}(g) = d_\pi,$$

where d_π is the formal degree of π.

This corollary will be proved after we prove Theorem 6.2. As a consequence, one can take the limit as $g \to e$ in Harish-Chandra's character formula to obtain an expression for d_π; see, e.g., page 25 of [Atiyah and Schmid 1977]. See also Proposition 50 in [Guo et al. 2018].

6B. ***A K-theoretic character identity.*** Let G_c be a compact inner form of G, which exists because $\operatorname{rank}(G) = \operatorname{rank}(K)$. Inner forms are defined for example in Chapter 2 of [Adams et al. 1992], but the only properties we need are that G_c is a real form of $G_{\mathbb{C}}$, and T identifies with a Cartan subgroup of G_c. So pairs (G, T) and (G_c, T) have the same root system. The positive root system R^+ determines a G-invariant complex structure on G_c/T. For any integral $\nu \in i\mathfrak{t}^*$, consider the holomorphic line bundles

$$L_\nu^G := G \times_T \mathbb{C}_\nu \to G/T,$$
$$L_\nu^{G_c} := G_c \times_T \mathbb{C}_\nu \to G_c/T.$$

Let $\bar\partial_{L_\nu^G}$ and $\bar\partial_{L_\nu^{G_c}}$ be the Dolbeault operators on G/T and G_c/T, respectively, coupled to these line bundles.

In [Hochs and Wang 2018c], the authors prove a K-theoretic analogue of Shelstad's character identities [Shelstad 1979], and deduce Shelstad's character identity in the case of the discrete series.

Theorem 6.4. *For all integral $\nu \in i\mathfrak{t}^*$ and all $g \in S$,*

$$\tau_g\big(\mathrm{index}_{G_c}\big(\bar\partial_{L_\nu^{G_c}} + \bar\partial^*_{L_\nu^{G_c}}\big)\big) = \sum_{[w]\in W_G/W_K} \tau_g\big(\mathrm{index}_G\big(\bar\partial_{L^G_{w^{-1}\nu}} + \bar\partial^*_{L^G_{w^{-1}\nu}}\big)\big).$$

Proof. This is (3.6) in [Hochs and Wang 2018c]. There, ν is regular but that property is not used in the proof of the above equality. □

6C. *Dolbeault operators.* We will use some properties of the Dolbeault–Dirac operators in Theorem 6.4 to deduce Theorem 6.2.

First of all, every element of $K_0(C_r^*G)$ is the index of a Dolbeault–Dirac operator on G/T. Indeed, let $V \in \hat{K}_{\mathrm{Spin}}$, and let $\lambda \in i\mathfrak{t}^*$ be its highest weight with respect to the positive compact roots chosen earlier. Then $\lambda - \rho_n$ is a weight of $\Delta_{\mathfrak{p}} \otimes V$, so it is integral for T. Consider the holomorphic, G-equivariant line bundle

$$L^G_{\lambda-\rho_n} := G \times_T \mathbb{C}_{\lambda-\rho_n} \to G/T.$$

Let $\bar\partial_{L^G_{\lambda-\rho_n}}$ be the Dolbeault operator on G/T coupled to $L^G_{\lambda-\rho_n}$.

Proposition 6.5. *We have*

$$\text{D-Ind}_K^G[V_\lambda] = (-1)^{\dim(G/K)}\, \mathrm{index}_G\big(\bar\partial_{L^G_{\lambda-\rho_n}} + \bar\partial^*_{L^G_{\lambda-\rho_n}}\big).$$

Proof. This is proved in Section 5 of [Hochs and Wang 2018b] in the case where $\lambda + \rho_c$ is regular for G, but that assumption is not necessary for the arguments. □

Lemma 6.6. *We have, for all $w \in W_G$ and all $g \in S$,*

$$\tau_{wgw^{-1}}\big(\mathrm{index}_G\big(\bar\partial_{L^G_{\lambda-\rho}} + \bar\partial^*_{L^G_{\lambda-\rho}}\big)\big) = \tau_g\big(\mathrm{index}_G\big(\bar\partial_{L^G_{w^{-1}(\lambda-\rho)}} + \bar\partial^*_{L^G_{w^{-1}(\lambda-\rho)}}\big)\big).$$

Proof. In the case of Dolbeault operators twisted by holomorphic vector bundles, and finite fixed point sets, the fixed point formula in Theorem 2.1 simplifies considerably; see Corollary 6.3 in [Hochs and Wang 2018b]. For any $h \in T$ with dense powers, and any integral $\nu \in i\mathfrak{t}^*$, this yields

$$\tau_h\big(\mathrm{index}_G\big(\bar\partial_{L^G_\nu} + \bar\partial^*_{L^G_\nu}\big)\big) = \sum_{xT\in(G/T)^h} \frac{\mathrm{tr}(g|_{(L^G_\nu)_{xT}})}{\det_{\mathbb{C}}(1 - g^{-1}|_{T_{xT}G/T})}. \tag{6.7}$$

Now, for $w \in W_G$, we have $(G/T)^{wgw^{-1}} = (G/T)^T = N_K(T)/T$, and for $x \in N_K(T)$,

$$(L^G_\nu)_{xT} = \mathbb{C}_{\mathrm{Ad}^*(x)\nu}, \qquad T_{xT}G/T = \bigoplus_{\alpha\in R^+} \mathbb{C}_{\mathrm{Ad}^*(x)\alpha}$$

as complex representations of T, where we use the complex structure on G/T defined by R^+. So

$$\begin{aligned}
&\tau_{wgw^{-1}}\bigl(\mathrm{index}_G\bigl(\bar\partial_{L^G_{\lambda-\rho}}+\bar\partial^*_{L^G_{\lambda-\rho}}\bigr)\bigr)\\
&\qquad=\sum_{xT\in N_K(T)/T}\frac{\mathrm{tr}(wgw^{-1}|_{\mathbb{C}_{\mathrm{Ad}^*(x)(\lambda-\rho)}})}{\det_{\mathbb{C}}(1-wg^{-1}w^{-1}|_{\bigoplus_{\alpha\in R^+}\mathbb{C}_{\mathrm{Ad}^*(x)\alpha}})}\\
&\qquad=\sum_{xT\in N_K(T)/T}\frac{\mathrm{tr}(g|_{\mathbb{C}_{\mathrm{Ad}^*(w^{-1}x)(\lambda-\rho)}})}{\det_{\mathbb{C}}(1-g^{-1}|_{\bigoplus_{\alpha\in R^+}\mathbb{C}_{\mathrm{Ad}^*(w^{-1}x)\alpha}})}\\
&\qquad=\sum_{yT\in w^{-1}N_K(T)w/T}\frac{\mathrm{tr}(g|_{\mathbb{C}_{\mathrm{Ad}^*(yw^{-1})(\lambda-\rho)}})}{\det_{\mathbb{C}}(1-g^{-1}|_{\bigoplus_{\alpha\in R^+}\mathbb{C}_{\mathrm{Ad}^*(yw^{-1})\alpha}})}. \qquad (6.8)
\end{aligned}$$

(In the last step, we substituted $y=w^{-1}xw$.)

Finally, $w^{-1}N_K(T)w=N_K(T)$, and

$$\bigoplus_{\alpha\in R^+}\mathbb{C}_{\mathrm{Ad}^*(yw^{-1})\alpha}=T_{yT}G/T$$

as complex representations of T, with respect to the complex structure defined by the positive root system $w^{-1}R^+$ with respect to which $w^{-1}(\lambda-\rho)$ is dominant. So by (6.7), the expression (6.8) equals

$$\tau_g\bigl(\mathrm{index}_G\bigl(\bar\partial_{L^G_{w^{-1}(\lambda-\rho)}}+\bar\partial^*_{L^G_{w^{-1}(\lambda-\rho)}}\bigr)\bigr). \qquad \square$$

Lemma 6.9. *We have, for all integral $\nu\in i\mathfrak{t}^*$,*

$$\tau_e\bigl(\mathrm{index}_{G_c}\bigl(\bar\partial_{L^{G_c}_\nu}+\bar\partial^*_{L^{G_c}_\nu}\bigr)\bigr)=\tau_e\bigl(\mathrm{index}_G\bigl(\bar\partial_{L^G_\nu}+\bar\partial^*_{L^G_\nu}\bigr)\bigr).$$

Proof. By Connes and Moscovici's L^2-index formula [1982, Theorem 5.2], we have

$$\tau_e\bigl(\mathrm{index}_G\bigl(\bar\partial_{L^G_\nu}+\bar\partial^*_{L^G_\nu}\bigr)\bigr)=\varepsilon\bigl(\mathrm{ch}\bigl(\textstyle\bigwedge_{\mathbb{C}}\mathfrak{g}/\mathfrak{t}\otimes\mathbb{C}_\nu\bigr)\hat A(\mathfrak{g},T)\bigr)[\mathfrak{g}/\mathfrak{t}],$$
$$\tau_e\bigl(\mathrm{index}_{G_c}\bigl(\bar\partial_{L^{G_c}_\nu}+\bar\partial^*_{L^{G_c}_\nu}\bigr)\bigr)=\varepsilon\bigl(\mathrm{ch}\bigl(\textstyle\bigwedge_{\mathbb{C}}\mathfrak{g}_c/\mathfrak{t}\otimes\mathbb{C}_\nu\bigr)\hat A(\mathfrak{g}_c,T)\bigr)[\mathfrak{g}_c/\mathfrak{t}],$$

for the same sign $\varepsilon\in\{\pm1\}$. Here $\mathrm{ch}: R(T)\to H^*(\mathfrak{g},T,\mathbb{R})$ is the relative Chern character, and the characteristic classes $\hat A$ in $H^*(\mathfrak{g},T,\mathbb{R})$ are defined in Section 4 of [Connes and Moscovici 1982]. The right-hand side of the first line only depends on the representations $\bigwedge_{\mathbb{C}}\mathfrak{g}/\mathfrak{t}\otimes\mathbb{C}_\nu$ and $\mathfrak{g}/\mathfrak{t}$ of T, and similarly for the right-hand side of the second line. Since $\mathfrak{g}/\mathfrak{t}$ and $\mathfrak{g}_c/\mathfrak{t}$ are both equal to the sum of the positive root spaces as complex representations of T, we find that the two expressions are equal. $\square$

6D. *Proofs of Theorem 6.2 and Corollary 6.3.* To finish the proof of Theorem 6.2, we need a final lemma.

Lemma 6.10 (Arthur). *We have, for all $g \in T^{\mathrm{reg}}$,*

$$\tau_g^s = \sum_{[w]\in W_G/W_K} \tau_{wgw^{-1}}.$$

Proof. In Section 27 (p. 194) of [Arthur 2005], it is pointed out that two elements $g, g' \in T^{\mathrm{reg}}$ are conjugate if and only if $g = w_K g' w_K^{-1}$ for some $w_K \in W_K$, and stably conjugate if and only if $g = w_G g' w_G^{-1}$ for some $w_G \in W_G$. □

Proof of Theorem 6.2. By surjectivity of Dirac induction and Proposition 6.5, every $x \in K_0(C_r^*G)$ is represented by the equivariant index

$$x = \mathrm{index}_G\big(\bar\partial_{L_\nu^G} + \bar\partial_{L_\nu^G}^*\big)$$

for an integral element $\nu \in i\mathfrak{t}^*$.

Let $g \in S$. By Theorem 6.4 and Lemmas 6.6 and 6.10, we have

$$\tau_g^s(x) = \tau_g^s\big(\mathrm{index}_G\big(\bar\partial_{L_\nu^G} + \bar\partial_{L_\nu^G}^*\big)\big) = \tau_g\big(\mathrm{index}_{G_c}\big(\bar\partial_{L_\nu^{G_c}} + \bar\partial_{L_\nu^{G_c}}^*\big)\big).$$

Since G_c is compact, this expression is continuous in g. And by Lemma 6.9,

$$\tau_e\big(\mathrm{index}_{G_c}\big(\bar\partial_{L_\nu^{G_c}} + \bar\partial_{L_\nu^{G_c}}^*\big)\big) = \tau_e\big(\mathrm{index}_G\big(\bar\partial_{L_\nu^G} + \bar\partial_{L_\nu^G}^*\big)\big) = \tau_e(x). \qquad □$$

Proof of Corollary 6.3. For $w \in W_G$, let $[\pi_{w\lambda}] \in K_0(C_r^*G)$ be the class defined by the discrete series representation with Harish-Chandra parameter $w\lambda$. By Propositions 5.1 and 5.2 in [Hochs and Wang 2018a], we have for all $g \in T^{\mathrm{reg}}$,

$$\begin{aligned}\sum_{[w]\in W_G/W_K} \Theta_{w\lambda}(g) &= \sum_{[w]\in W_G/W_K} \tau_g([\pi_{w\lambda}])\\ &= (-1)^{\dim(G/K)/2} \sum_{[w]\in W_G/W_K} \tau_g\big(\mathrm{index}_G\big(\bar\partial_{L^G_{w(\lambda-\rho)}} + \bar\partial^*_{L^G_{w(\lambda-\rho)}}\big)\big).\end{aligned}$$

Lemmas 6.6 and 6.10 imply that the right-hand side equals

$$(-1)^{\dim(G/K)/2}\tau_g^s\big(\mathrm{index}_G\big(\bar\partial_{L^G_{\lambda-\rho}} + \bar\partial^*_{L^G_{\lambda-\rho}}\big)\big).$$

As $g \to e$ through the set S in Theorem 6.2, that result implies that the limit of the above expression is

$$(-1)^{\dim(G/K)/2}\tau_e\big(\mathrm{index}_G\big(\bar\partial_{L^G_{\lambda-\rho}} + \bar\partial^*_{L^G_{\lambda-\rho}}\big)\big) = \tau_e([\pi_\lambda]) = d_\pi.$$

The claim now follows from continuity of characters on the regular set. □

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On derived categories of arithmetic toric varieties

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We begin a systematic investigation of derived categories of smooth projective toric varieties defined over an arbitrary base field. We show that, in many cases, toric varieties admit full exceptional collections, making it possible to give concrete descriptions of their derived categories. Examples include all toric surfaces, all toric Fano 3-folds, some toric Fano 4-folds, the generalized del Pezzo varieties of Voskresenskiĭ and Klyachko, and toric varieties associated to Weyl fans of type A. Our main technical tool is a completely general Galois descent result for exceptional collections of objects on (possibly nontoric) varieties over nonclosed fields.

1. Introduction

Recently, several intriguing threads relating derived categories and arithmetic geometry have emerged and motivated general structure questions for k-linear triangulated categories for arbitrary fields k. Such exploration has yielded many nice applications as well as further enticing problems; see as a sampling [Antieau et al. 2017; Ananyevskiy et al. 2013; Ascher et al. 2017; Hassett and Tschinkel 2017; Honigs 2015; Lieblich et al. 2014]. Meanwhile, over $\mathbb{C}$, structural results for derived categories seem to have deep implications for the underlying birational geometry, e.g., [Addington and Thomas 2014; Auel et al. 2014; Bernardara and Bolognesi 2013; Bernardara et al. 2012; Kuznetsov 2010; Vial 2017]. Taking these together, derived categories become an important invariant for studying birational geometry over a general field [Auel and Bernardara 2018]. A further benefit of this noncommutative approach is direct utility for solving problems in algebraic K-theory, for example [Merkurjev and Panin 1997].

With such tantalizing ties, one would like a fertile testing ground for questions. In this paper, we begin a systematic study of one such area: derived categories of arithmetic toric varieties. Recall that if k is an arbitrary field with separable closure k^s, a *k-torus* is an algebraic group T over k such that extending scalars to k^s gives $T_{k^s} \simeq \mathbb{G}_m^n$. An *arithmetic toric variety* is a normal k-variety with a faithful

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action of a k-torus which has a dense open orbit. This area has the following nice features:

- rationality issues are deep in general but tractable in examples,
- robust tools already exist to investigate derived categories over the separable closure, and
- specific questions are often amenable to computational experimentation.

One of the best tools for understanding the structure of a derived category is an *exceptional collection* consisting of *exceptional objects*. As originally conceived in [Beĭlinson 1978], an exceptional object of a k-linear triangulated category (e.g., $\mathsf{D}^{\mathsf{b}}(X)$) is one whose endomorphism algebra is isomorphic to the base field k. When k is not algebraically closed, this definition is too restrictive and instead we use the existing notion: an object of $\mathsf{D}^{\mathsf{b}}(X)$ is *exceptional* if its endomorphism algebra is a division algebra (concentrated in homological degree zero). An exceptional collection is then given by a totally ordered set $\mathsf{E} = \{E_1, \dots, E_s\}$ of exceptional objects in $\mathsf{D}^{\mathsf{b}}(X)$ satisfying $\operatorname{Ext}^n(E_i, E_j) = 0$ for all integers n whenever $i > j$. An exceptional collection is *full* if it generates $\mathsf{D}^{\mathsf{b}}(X)$, i.e., the smallest thick subcategory of $\mathsf{D}^{\mathsf{b}}(X)$ containing E is all of $\mathsf{D}^{\mathsf{b}}(X)$. Details are discussed in Section 2 below.

We illustrate this more general notion for two arithmetic toric varieties. The real conic $X = \{x^2 + y^2 + z^2 = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$ has an exceptional collection $\{\mathcal{O}, \mathcal{F}\}$, where $\operatorname{End}(\mathcal{F})$ is isomorphic to the quaternion algebra $\mathbb{H}$. Over $\mathbb{C}$, we have $X_{\mathbb{C}} \simeq \mathbb{P}^1_{\mathbb{C}}$ and $\mathcal{F} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathcal{O}(1)^{\oplus 2}$. As another example, consider the Weil restriction Y of $\mathbb{P}^1_{\mathbb{C}}$ over $\mathbb{R}$ ("$\mathbb{P}^1(\mathbb{C})$ viewed as an $\mathbb{R}$-variety"). Here Y has an exceptional collection $\{\mathcal{O}, \mathcal{G}, \mathcal{H}\}$, where $\operatorname{End}(\mathcal{G}) \simeq \mathbb{C}$ and $\operatorname{End}(\mathcal{H}) \simeq \mathbb{R}$. Over $\mathbb{C}$, we have $Y \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with $\mathcal{G} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathcal{O}(1, 0) \oplus \mathcal{O}(0, 1)$ and $\mathcal{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathcal{O}(1, 1)$, where

$$\mathcal{O}(i, j) = \pi_1^* \mathcal{O}(i) \otimes \pi_2^* \mathcal{O}(j).$$

A central question for derived categories of arithmetic toric varieties is the following:

Question 1.1. Let X be a smooth projective toric variety over an arbitrary field. Does X admit a full exceptional collection? If so, does X possess a full exceptional collection of sheaves?

Over an algebraically closed field of characteristic zero, there is always a full exceptional collection of objects [Kawamata 2006; 2013] while the question of a full exceptional collection of sheaves is due to Orlov. Making allowances for different language, the question is also known to have a positive answer for Severi–Brauer varieties [Auel and Bernardara 2018; Bernardara 2009], minimal toric surfaces [Blunk et al. 2011], and smooth projective toric varieties with absolute Picard rank at most 2 [Yan 2014].

In this article, we provide further evidence for a positive answer to Question 1.1, treating cases with low dimension or a high degree of symmetry.

Theorem 1.2. *The following possess full exceptional collections of sheaves*:

- *smooth toric surfaces* (Proposition 4.7),
- *smooth toric Fano* 3*-folds* (Proposition 4.11),
- *all forms of* 43 *of the* 124 *smooth split toric Fano* 4*-folds* (Section 4C),
- *all forms of centrally symmetric toric Fano varieties* (Corollary 4.13), *and*
- *all forms in characteristic zero of toric varieties corresponding to Weyl fans of root systems of type A* (Proposition 4.21).

Our results leverage extant work in the algebraically closed case such as [Uehara 2014] for 3-folds and[Prabhu-Naik 2017] for 4-folds. We use Castravet and Tevelev's recently discovered exceptional collection for $X(A_n)$ [2017]. For the centrally symmetric toric Fano varieties (which are products of "generalized del Pezzo varieties" and projective lines [Voskresenskiĭ and Klyachko 1984]), we use an explicit exceptional collection (see also [Ballard et al. 2018]) closely related to the one found in [Castravet and Tevelev 2017]. Up to a twist by a line bundle, the authors had independently discovered the exact same collection! This suggests that symmetry imposes strong conditions on the possible exceptional collections, which paradoxically makes them easier to find.

To study arithmetic exceptional collections, we establish an effective Galois descent result for general exceptional collections. This applies to general varieties, not just toric ones.

Theorem 1.3 (Theorem 2.17, Lemma 2.20). *Let X be a k-scheme and L/k a G-Galois extension. Then X_L admits a full* (*resp. strong*) *G-stable exceptional collection of objects of $\mathsf{D}^{\mathsf{b}}(X_L)$* (*resp. sheaves*, *vector bundles*) *if and only if X admits a full* (*resp. strong*) *exceptional collection of objects of $\mathsf{D}^{\mathsf{b}}(X)$* (*resp. sheaves*, *vector bundles*).

We highlight one corollary of a positive answer to Question 1.1. Arithmetic toric varieties are also studied in [Merkurjev and Panin 1997], which focused on computing their algebraic K-groups via decompositions in a certain noncommutative motivic category of K_0-correspondences. They showed that for an arithmetic toric k-variety X with $G = \mathrm{Gal}(k^s/k)$, the group $K_0(X_{k^s})$ is a direct summand of a *permutation G-module* (there exists a $\mathbb{Z}$-basis permuted by G).

Question 1.4 [Merkurjev and Panin 1997]. Let X be an arithmetic toric variety over k and $G = \mathrm{Gal}(k^s/k)$. Is $K_0(X_{k^s})$ always a permutation G-module?

Question 1.1 can be viewed as a categorification of Question 1.4 as any such exceptional collection over k immediately gives a permutation basis.

In order to show that every toric variety has a full exceptional collection over $\mathbb{C}$, the main tool used in [Kawamata 2006; 2013] was the minimal model program (MMP) in birational geometry. The basic building blocks are toric stacks with Picard rank one, which always have full strong exceptional collections of line bundles. Indeed, runs of the MMP can be leveraged to effectively produce exceptional collections [Ballard et al. 2019].

Over a nonclosed field, one hopes to use the Galois-equivariant MMP, but the situation is more complicated. The most basic building blocks in this framework are those varieties X which have $\rho^G = \operatorname{rank}(\operatorname{Pic}(X)^G) = 1$. Based on the results above and the hope of using the MMP in the arithmetic situation, we ask the following question in the vein of [King 1997; Borisov and Hua 2009; Costa and Miró-Roig 2010]:

Question 1.5. Let X be a smooth toric k-variety and L/k a G-Galois splitting field. If $\operatorname{Pic}(X_L)^G$ is of rank 1, does X_L possess a full strong G-stable exceptional collection consisting of line bundles?

Organization. Section 2 treats Galois descent of exceptional collections consisting of objects on (possibly nontoric) varieties. In Section 3, we recall appropriate definitions of arithmetic toric varieties and establish additional descent results which are specific to toric varieties. In Section 4, we consider a range of examples. We begin by treating toric surfaces, followed by toric Fano 3-folds. For toric Fano 4-folds, we give partial results. We conclude by investigating the class of centrally symmetric toric Fano varieties, including the generalized del Pezzo varieties, and handling toric varieties associated to root systems of type A.

Notation. Throughout, k denotes an arbitrary field and k^s a separable closure. A *variety* is a geometrically integral separated scheme of finite type over k. All our schemes are quasicompact and quasiseparated. For a k-scheme X and field extension L/k, we write $X_L := X \times_{\operatorname{Spec} k} \operatorname{Spec} L$. If A is a k-algebra, we write $A_L = A \otimes_k L$. We use $\mathsf{D}^{\mathsf{b}}(X)$ to denote the bounded derived category $\mathsf{D}^{\mathsf{b}}(\operatorname{Coh}(X))$. For an $\mathcal{O}_X$-algebra A, we write $\mathsf{D}^{\mathsf{b}}(A)$ for the bounded derived category of complexes of A-modules which are coherent $\mathcal{O}_X$-modules.

2. Galois descent and exceptional collections

In this section we develop Galois descent for exceptional collections (in a generalized sense). We begin by recalling some definitions and conventions concerning structure theory of derived categories of schemes. We then give our main descent results for G-stable exceptional collections (Theorem 2.17). We complete the section by collecting a few useful consequences to be used in the sequel.

2A. ***Exceptional collections.*** We give some conventions for semiorthogonal decompositions of derived categories and in particular exceptional collections. Such collections have been widely studied over algebraically closed fields but have recently been treated in more generality [Ananyevskiy et al. 2013; Auel and Bernardara 2018; Auel et al. 2014; Bernardara 2009; Blunk et al. 2011; Elagin 2009; Xie 2017; Yan 2014]. We refer the reader to Remarks 2.15 and 2.19 for added details on some of these results.

For a triangulated category T, we use the standard notation

$$\mathrm{Ext}^n_{\mathsf{T}}(A, B) = \mathrm{Hom}_{\mathsf{T}}(A, B[n]).$$

For objects A, B of $\mathsf{D}^{\mathsf{b}}(X)$, we use $\mathrm{End}_X(A)$ and $\mathrm{Ext}^n_X(A, B)$ to denote $\mathrm{End}_{\mathsf{D}^{\mathsf{b}}(X)}(A)$ and $\mathrm{Ext}^n_{\mathsf{D}^{\mathsf{b}}(X)}(A, B)$, respectively.

Definition 2.1 (see [Bondal and Kapranov 1989]). Let T be a triangulated category. A full triangulated subcategory of T is *admissible* if its inclusion functor admits left and right adjoints. A *semiorthogonal decomposition* of T is a sequence of admissible subcategories $\mathsf{C}_1, \dots, \mathsf{C}_s$ such that

(1) $\mathrm{Hom}_{\mathsf{T}}(A_i, A_j) = 0$ for all $A_i \in \mathrm{Ob}(\mathsf{C}_i)$, $A_j \in \mathrm{Ob}(\mathsf{C}_j)$ whenever $i > j$;

(2) for each object T of T, there is a sequence of morphisms

$$0 = T_s \to \cdots \to T_0 = T$$

such that the cone of $T_i \to T_{i-1}$ is an object of C_i for all $i = 1, \dots, s$.

We use $\mathsf{T} = \langle \mathsf{C}_1, \dots, \mathsf{C}_s \rangle$ to denote such a decomposition.

Particularly nice examples of semiorthogonal decompositions are given by exceptional collections, the study of which goes back to [Beĭlinson 1978].

Definition 2.2. Let T be a k-linear triangulated category. An object E in T is *exceptional* if the following conditions hold:

(1) $\mathrm{End}_{\mathsf{T}}(E)$ is a division k-algebra.

(2) $\mathrm{Ext}^n_{\mathsf{T}}(E, E) = 0$ for $n \neq 0$.

A totally ordered set $\mathsf{E} = \{E_1, \dots, E_s\}$ of exceptional objects is an *exceptional collection* if $\mathrm{Ext}^n_{\mathsf{T}}(E_i, E_j) = 0$ for all integers n whenever $i > j$. An exceptional collection is *full* if it generates T, i.e., the smallest thick subcategory of T containing E is all of T. An exceptional collection is *strong* if $\mathrm{Ext}^n_{\mathsf{T}}(E_i, E_j) = 0$ whenever $n \neq 0$. An *exceptional block* is an exceptional collection $\mathsf{E} = \{E_1, \dots, E_s\}$ such that $\mathrm{Ext}^n_{\mathsf{T}}(E_i, E_j) = 0$ for every n whenever $i \neq j$. Given an exceptional collection $\mathsf{E} = \{E_1, \dots, E_s\}$, we denote by $\langle \mathsf{E} \rangle$ the category generated by the objects E_i.

Remark 2.3. Our notion of exceptional object generalizes the classical one, where Definition 2.2(1) is replaced by $\mathrm{End}_{\mathsf{T}}(E) = k$ [Bondal 1989, §2]. Over algebraically or separably closed fields, these definitions agree. Over nonclosed fields, the classical definition is too restrictive to allow for the use of interesting arithmetic invariants in the study of exceptional collections on twisted forms, e.g., Brauer classes.

Proposition 2.4 [Bondal 1989, Theorem 3.2]. *Let X be a k-scheme with exceptional collection $\{E_1, \dots, E_s\}$. If $\mathscr{E}_i$ is the category generated by E_i, there is a semiorthogonal decomposition $\mathsf{D}^{\mathsf{b}}(X) = \langle \mathscr{E}_1, \dots, \mathscr{E}_s, \mathsf{A}\rangle$, where A is the full subcategory with objects A such that $\mathrm{Hom}_X(A, E_i) = 0$ for all i.*

Remark 2.5. Bondal assumes smoothness and projectivity but the conclusion is independent of this. Note further that if X admits a full exceptional collection then it is automatically smooth and proper by [Orlov 2016, Propositions 3.30 and 3.31].

The existence of an exceptional collection on a scheme X provides a means of studying derived geometry of X in purely algebraic terms. Indeed, in such a situation, one may identify an "underlying" k-algebra which is derived equivalent to X. For exceptional blocks, one obtains a similar but slightly stronger fact.

Proposition 2.6 [Bondal 1989, Theorem 6.2]. *Let X be a smooth projective k-scheme and let $\{E_1, \dots, E_n\}$ be a full strong exceptional collection on $\mathsf{D}^{\mathsf{b}}(X)$. Let $\mathcal{E} = \bigoplus E_i$ and $A = \mathrm{End}(\mathcal{E})$. Then $\mathsf{R}\mathrm{Hom}_{\mathsf{D}^{\mathsf{b}}(X)}(\mathcal{E}, -) : \mathsf{D}^{\mathsf{b}}(X) \to \mathsf{D}^{\mathsf{b}}(A)$ is a k-linear equivalence.*

Proposition 2.7. *If $\mathsf{E} = \{E_1, \dots, E_s\}$ is an exceptional block with $\mathrm{End}(E_i) = D_i$, there is a k-algebra isomorphism $\mathrm{End}\big(\bigoplus E_i\big) \simeq D_1 \times \cdots \times D_s$, and hence a k-linear equivalence $\langle \mathsf{E}\rangle \simeq \mathsf{D}^{\mathsf{b}}(D_1 \times \cdots \times D_n)$.*

The object $\mathcal{E} = \bigoplus E_i$ of Proposition 2.6 is usually called a *tilting object*. If each E_i is a sheaf (resp. vector bundle), then E is called a *tilting sheaf* (resp. *tilting bundle*). Until recently, the theory of tilting objects has served as the main tool for extending the study of exceptional collections to nonclosed fields. The results above show that any exceptional collection gives rise to both a tilting object and a semiorthogonal decomposition, and thus the admission of such a collection is a particularly special property of a given triangulated category. Our aim in the following subsection is to extend descent results for semiorthogonal decompositions and tilting objects to (our more general notion of) exceptional collections. We give a formal definition of tilting object for completeness.

Definition 2.8. A *tilting object* for a k-scheme X is an object $\mathcal{E}$ of $\mathsf{D}^{\mathsf{b}}(X)$ which satisfies the following conditions:

(1) $\mathrm{Ext}^n_X(\mathcal{E}, \mathcal{E}) = 0$ for $n > 0$.

(2) $\mathcal{E}$ generates $\mathsf{D}^{\mathsf{b}}(X)$.

Remark 2.9 (K-theory and motivic decompositions). Exceptional collections have a particularly interesting manifestation in the realm of noncommutative motives. Indeed, an exceptional collection $\{E_1, \dots, E_s\}$ on a smooth projective variety X yields a decomposition $U(X) \simeq \bigoplus_i U(D_i)$ of its corresponding universal additive invariant [Tabuada 2015, §2.3], where $D_i = \mathrm{End}(E_i)$. This defines a motivic decomposition by viewing X as an object in the Merkurjev–Panin category of K-motives [Merkurjev and Panin 1997] or Kontsevich's category of noncommutative Chow motives [Tabuada 2014, Theorem 6.10] via its associated dg-category of perfect complexes.

One nice consequence is that this decomposition is detected by algebraic K-groups [Auel and Bernardara 2018, Proposition 1.10] in addition to a slew of other additive invariants in the sense of [Tabuada 2015, §2.2]. Such invariants include algebraic K-theory with coefficients, homotopy K-theory, étale K-theory, (topological) Hochschild homology, and (topological) cyclic homology.

2B. ***Galois descent.*** We develop Galois descent for exceptional collections consisting of objects in the derived category $\mathsf{D}^{\mathsf{b}}(X)$ of a (smooth projective) variety X. Throughout this section, pushforward and pullback functors are understood to be derived. For a k-scheme X and finite Galois extension L/k, any element $g \in \mathrm{Gal}(L/k)$ defines a morphism of k-schemes $g : X_L \to X_L$ which in turn defines the functor $g^* : \mathsf{D}^{\mathsf{b}}(X_L) \to \mathsf{D}^{\mathsf{b}}(X_L)$.

Definition 2.10. Let X be a scheme with an action of a group G. A *G-stable exceptional collection* on X is an exceptional collection $\mathsf{E} = \{E_1, \dots, E_s\}$ of objects in $\mathsf{D}^{\mathsf{b}}(X)$ such that for all $g \in G$ and $1 \le i \le s$ there exists $E \in \mathsf{E}$ such that $g^* E_i \simeq E$. We say a G-stable exceptional collection E is a *G-orbit* if, for every pair of objects $E, E' \in \mathsf{E}$, there exists a $g \in G$ such that $g^* E \simeq E'$.

Remark 2.11. A simple example of a G-stable exceptional collection is a *G-invariant* exceptional collection, i.e., an exceptional collection $\{E_1, \dots, E_s\}$ such that $g^* E_i \simeq E_i$ for all $1 \le i \le s$. It is often the case that toric varieties admit exceptional collections consisting of line bundles. If it is also the case that a group G acts trivially on $\mathrm{Pic}(X)$, such a collection is automatically G-invariant, and hence G-stable (see Lemma 2.21).

Lemma 2.12. *Any G-stable exceptional collection may be written as a collection of G-stable exceptional blocks* (*possibly after reordering*).

Proof. The decomposition of a G-stable exceptional collection into its G-orbits gives the desired exceptional blocks. Let E be a G-stable exceptional collection and for elements $E, E' \in \mathsf{E}$, we write $E \rightsquigarrow E'$ if $\mathrm{Ext}^n(E, E') \neq 0$ for some n.

Let $\mathsf{A} \subset \mathsf{E}$ be a G-orbit. To see that A is an exceptional block, suppose that $E \rightsquigarrow E'$ for $E, E' \in \mathsf{A}$. Since A is a G-orbit, $E' \simeq g^* E$ for some $g \in G$. Thus,

$E \rightsquigarrow g^*E$, and acting again by g, we have $g^*E \rightsquigarrow (g^2)^*E$. Since A is finite, we have $E \rightsquigarrow g^*E \rightsquigarrow \cdots \rightsquigarrow (g^s)^*E \rightsquigarrow E$ for some positive integer s. Thus, there is no ordering of the elements of A such that they form a subset of an exceptional collection — a contradiction.

If B is another G-orbit (distinct from A), we would like to see that these blocks can be ordered to form an exceptional collection. We claim that for any $E \in \mathsf{A}$ and $F \in \mathsf{B}$, one has $E \rightsquigarrow F$ only if A precedes B in the collection E (i.e., $\mathrm{Ext}^n(B, A) = 0$ for all n and all $A \in \mathsf{A}$, $B \in \mathsf{B}$). To see this, assume that $E \rightsquigarrow F$ and $F \rightsquigarrow E'$ for some $E' \in \mathsf{A}$. As A is a G-orbit, $E' \simeq g^*E$ for some $g \in G$. Hence, just as above, we have a sequence $E \rightsquigarrow F \rightsquigarrow g^*E \rightsquigarrow g^*F \rightsquigarrow \cdots \rightsquigarrow (g^s)^*F \rightsquigarrow E$. Thus, there is no ordering of the elements of A and B which forms an exceptional collection, contradicting the exceptionality of E. □

Lemma 2.13. *Let X be a Noetherian k-scheme, L/k a finite Galois extension with group G, and $\pi : X_L \to X$ the natural projection map. For any object M in $\mathsf{D}^{\mathsf{b}}(X_L)$ there is a natural isomorphism $\pi^*\pi_*(M) \simeq \bigoplus_{g\in G} g^*M$.*

Proof. As π is flat and affine, every coherent sheaf on X is acyclic for π^* and every coherent sheaf on X_L is acyclic for π_*. Hence, the derived functors coincide with the application of π^* or π_* componentwise to a complex. Thus, it suffices to establish a natural isomorphism at the level of coherent sheaves.

For any object M of $\mathrm{Coh}(X_L)$, we have $\pi_*M \simeq \pi_*g^*M$, and adjunction yields a natural transformation $\pi^*\pi_* \to g^*$. Summing over all $g \in G$ provides the transformation $\alpha : \pi^*\pi_* \to \bigoplus g^*$. We show this is an isomorphism.

It suffices to check that α is an isomorphism on any affine patch $\mathrm{Spec}\, R$ of X. Passing to modules, we abuse notation and let M be a finitely generated module over $R_L = R \otimes_k L$. Choose a presentation

$$R_L^{\oplus m} \to R_L^{\oplus n} \to M \to 0$$

of M and evaluate α on the sequence to get the commutative diagram

$$\begin{array}{ccccccc}
R^{\oplus m} \otimes_k (L \otimes_k L) & \longrightarrow & R^{\oplus n} \otimes_k (L \otimes_k L) & \longrightarrow & M \otimes_R R_L & \longrightarrow & 0 \\
{\scriptstyle \alpha_{R^{\oplus m}}}\downarrow & & {\scriptstyle \alpha_{R^{\oplus n}}}\downarrow & & {\scriptstyle \alpha_M}\downarrow & & \\
R^{\oplus m} \otimes_k \left(\bigoplus_g \Gamma_g(L)\right) & \longrightarrow & R^{\oplus m} \otimes_k \left(\bigoplus_g \Gamma_g(L)\right) & \longrightarrow & \bigoplus_g g^*M & \longrightarrow & 0
\end{array}$$

where $\Gamma_g(L)$ denotes the graph of g in $L \otimes_k L$. The left and middle maps are isomorphisms, so the right map must also be an isomorphism. □

Proposition 2.14 (descent for orbits). *Let X be a k-scheme, L/k a finite G-Galois extension, and $\pi : X_L \to X$ the natural projection map. If $\mathsf{E} = \{E_1, \ldots, E_s\}$ is a G-orbit forming an exceptional collection on X_L, and if E is any element of E,* *then there is an exceptional object F in $\mathsf{D}^{\mathsf{b}}(X)$ such that $\pi_* E \simeq F^{\oplus m}$ and $\pi^* F$ generates the category $\langle \mathsf{E} \rangle$.*

Proof. By Lemma 2.12, exceptional G-orbits are completely orthogonal (and by definition carry a transitive action of G), which is used throughout the proof. Fix an element $E \in \mathsf{E}$, so that $E = E_i$ for some i. Lemma 2.13 gives

$$\pi^* \pi_* E \simeq \bigoplus_{g \in G} g^* E.$$

We claim that $\operatorname{End}(\pi_* E)$ is a matrix algebra over a division algebra, and prove this by first showing that it is semisimple. Indeed, using $\operatorname{End}_X(M) \otimes_k L \simeq \operatorname{End}_{X_L}(\pi^* M)$ for any $M \in \mathsf{D}^{\mathsf{b}}(X)$ [Auel and Bernardara 2018, Remark 2.1], we have

$$\operatorname{End}_X(\pi_* E) \otimes_k L \simeq \operatorname{End}_{X_L}(\pi^* \pi_* E) \simeq \operatorname{End}_{X_L}\Bigl(\bigoplus_{g \in G} g^* E\Bigr).$$

Each $g^* E$ is exceptional, so that $\operatorname{End}_{X_L}(g^* E) =: D_g$ is a division algebra for each element $g \in G$. Let $H \leq G$ be the subgroup consisting of elements h satisfying $h^* E \simeq E$. For any system of coset representatives $g \in G/H$, we have $\operatorname{End}_X(\pi_* E)_L \simeq \prod_{g \in G/H} M_m(D_g)$, where $m = |H|$. This product of matrix algebras over division algebras is semisimple, i.e., the Jacobson radical $\operatorname{rad}(\operatorname{End}_X(\pi_* E)_L) = 0$. We then have $0 = \operatorname{rad}(\operatorname{End}_X(\pi_* E)_L) = \operatorname{rad}(\operatorname{End}_X(\pi_* E))_L$ by [Amitsur 1957, Theorem 1], and hence $\operatorname{rad}(\operatorname{End}_X(\pi_* E)) = 0$. Thus, $\operatorname{End}_X(\pi_* E)$ is semisimple and so must also be a product of matrix algebras over division algebras by the Artin–Wedderburn theorem.

Let Z be the center of $\operatorname{End}_X(\pi_* E)$ and Z_L the center of $\operatorname{End}_X(\pi_* E)_L$. Note that Z is an étale k-algebra, and to show that $\operatorname{End}(\pi_* E)$ is a matrix algebra, it suffices to show that Z has no zero divisors, and is thus a field. There is an embedding $Z \hookrightarrow Z_L = \prod_{g \in G/H} L_g$, where L_g is the center of the division algebra D_g. The transitive action of G on $\{E_1, \ldots, E_s\}$ implies that G acts transitively on a basis of Z_L, so that $Z = (Z_L)^G$ has no zero divisors.

We produce the object F using the identification $\operatorname{End}_X(\pi_* E) \simeq M_n(D)$, where D is a division algebra. Let $e_i = e_{ii}$ denote the usual idempotent matrices, so that $\{e_i\}$ is a complete set of primitive orthogonal idempotents. Notice that $F_i := \operatorname{Im}(e_i)$ is a simple $\operatorname{End}_X(\pi_* E)$-submodule of $\pi_* E$ for each i, and hence $F_i \simeq F_j$ for each i, j, and $\operatorname{End}_X(F_i) \simeq D$. Define $F = \operatorname{Im}(e_1) \subset \pi_* E$, included as a direct summand. We note that $\pi_* E \simeq \bigoplus F_i \simeq F^{\oplus n}$.

We now show that F is an exceptional object on X. As stated above, $\operatorname{End}_X(F)$ is a division algebra, so it suffices to show that $\operatorname{Ext}^n_X(F, F) = 0$ for $n \neq 0$. Using Lemma 2.13 and (π^*, π_*)-adjunction, we have

$$\operatorname{Ext}^n_X(\pi_* E, \pi_* E) = \bigoplus_{g \in G} \operatorname{Ext}^n_{X_L}(g^* E, E).$$

For $n \neq 0$, each summand of the right-hand side is 0, which follows from the mutual orthogonality of the exceptional block E (when $g^*E \not\simeq E$) and from exceptionality of E (when $g^*E \simeq E$). Since F is a direct summand of π_*E, it follows that $\mathrm{Ext}^n_X(F, F)$ is a summand of $\mathrm{Ext}^n_X(\pi_*E, \pi_*E) = 0$.

Lastly, we show that π^*F generates the category $\langle \mathsf{E} \rangle$. Since $F^{\oplus m} \simeq \pi_*E$, extending scalars to L gives $(\pi^*F)^{\oplus m} = \pi^*(F^{\oplus m}) \simeq \pi^*\pi_*E \simeq \bigoplus g^*E$. Thus,

$$\langle \pi^*F \rangle = \langle (\pi^*F)^{\oplus m} \rangle = \Big\langle \bigoplus g^*E \Big\rangle = \langle g^*E \rangle_{g \in G} = \langle \mathsf{E} \rangle. \qquad \square$$

Remark 2.15. Proposition 2.14 provides a very specific case of descent for triangulated categories, the main advantage of which is that it allows one to identify a specific exceptional object that base extends to the given orbit. Moreover, a G-orbit which forms an exceptional collection consisting of vector bundles or (resp. sheaves) descends to an exceptional collection consisting of vector bundles (resp. sheaves). Compare to the following descent result for semiorthogonal decompositions, which generalizes [Toën 2012, Corollary 2.15]. Although this result is useful for descending semiorthogonal decompositions, it does not identify exceptional objects.

Proposition 2.16 [Auel and Bernardara 2018, Proposition 2.12]. *Let T be a k-linear triangulated category such that T_{k^s} is k^s-equivalent to $\mathsf{D}^{\mathsf{b}}(k^s, (k^s)^n)$. Then there exists an étale algebra K of degree n over k, an Azumaya algebra A over K, and a k-linear equivalence $\mathsf{T} \simeq \mathsf{D}^{\mathsf{b}}(K/k, A)$.*

Let X, E, and F be as in Proposition 2.14, and note that taking $\mathsf{T} = \langle F \rangle$, we have $\mathsf{T}_{k^s} = \langle \pi^*F \rangle_{k^s} = \langle \mathsf{E} \rangle_{k^s}$. Since $\mathsf{E} = \{g^*E\}_{g \in G}$ is a full exceptional collection for $\langle \mathsf{E} \rangle$, the bundle $\mathcal{E} := \bigoplus (g^*E)_{k^s}$ is a tilting object for $\langle \mathsf{E} \rangle_{k^s}$. This defines an equivalence

$$\mathsf{T}_{k^s} \simeq \langle \mathsf{E} \rangle_{k^s} \simeq \mathsf{D}^{\mathsf{b}}(k^s, \mathrm{End}(\mathcal{E})) = \mathsf{D}^{\mathsf{b}}(k^s, (k^s)^n).$$

Proposition 2.16 yields an étale extension K/k, an Azumaya K-algebra A, and an equivalence $\mathsf{T} \simeq \mathsf{D}^{\mathsf{b}}(K/k, A)$. In this case, since $\mathsf{T} = \langle F \rangle$, we see that $A = \mathrm{End}_X(F)$ is an Azumaya algebra over its center Z (using the notation found in the proof of Proposition 2.14), which is simply a field extension of k.

Theorem 2.17 (descent for stable collections). *Let X be a k-scheme, L/k a finite G-Galois extension, and $\pi : X_L \to X$ the natural projection map. If X_L admits a full G-stable exceptional collection E of objects of $\mathsf{D}^{\mathsf{b}}(X_L)$, then X admits a full exceptional collection F of objects of $\mathsf{D}^{\mathsf{b}}(X)$. If E is strong, so is F. If the elements of E are vector bundles* (*resp. sheaves*), *the elements of F are vector bundles* (*resp. sheaves*).

Proof. By Lemma 2.12, we may write $\mathsf{E} = \{\mathsf{E}^1, \dots, \mathsf{E}^s\}$ as a collection of G-stable blocks, where each block is given by a G-orbit. Proposition 2.14 then associates to each block E^i an exceptional object F_i on X, and we show that $\mathsf{F} = \{F_1, \ldots, F_s\}$ is a full exceptional collection on X. We first show that $\operatorname{Ext}^n_X(F_i, F_j) = 0$ for all n whenever $i > j$. Let E^i and E^j be elements of the collections E^i and E^j, respectively. We then have

$$\operatorname{Ext}^n_X(\pi_* E^i, \pi_* E^j) \simeq \bigoplus_{g \in G} \operatorname{Ext}^n_{X_L}(g^* E^i, E^j). \tag{2.18}$$

Since E^i and E^j are elements of the exceptional collection E and $i < j$, each summand is 0 for all n, so that

$$\operatorname{Ext}^n_X(\pi_* E^i, \pi_* E^j) = 0 \quad \text{for all } n.$$

The objects F_i and F_j are direct summands of $\pi_* E^i$ and $\pi_* E^j$, respectively, and it follows that $\operatorname{Ext}^n_X(F_i, F_j) = 0$ for all n.

By Proposition 2.4, the exceptional collection $\{F_1, \ldots, F_s\}$ yields a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(X) = \langle \mathscr{F}_1, \ldots, \mathscr{F}_s, \mathsf{A} \rangle,$$

where $\mathscr{F}_i = \langle F_i \rangle$ and A is the full subcategory of objects A with $\operatorname{Hom}_{\mathsf{D}^{\mathsf{b}}(X)}(A, F_i) = 0$ for all i. In particular, the subcategories $\mathscr{F}_i$ are admissible. Extending scalars to L, we have $(\mathscr{F}_i)_L = \langle \mathsf{E}^i \rangle$, as both categories are generated by $\pi^* F$ by Proposition 2.14. The exceptional collection $\mathsf{E} = \{\mathsf{E}^1, \ldots, \mathsf{E}^s\}$ is full, and hence we have a semiorthogonal decomposition

$$\mathsf{D}^{\mathsf{b}}(X_L) = \langle (\mathscr{F}_1)_L, \ldots, (\mathscr{F}_s)_L \rangle.$$

Since our admissible subcategories $\mathscr{F}_i$ base extend to a semiorthogonal decomposition, [Auel et al. 2014, Lemma 2.9] gives a semiorthogonal decomposition $\mathsf{D}^{\mathsf{b}}(X) = \langle \mathscr{F}_1, \ldots, \mathscr{F}_s \rangle$. In particular, the collection $\{F_1, \ldots, F_s\}$ generates $\mathsf{D}^{\mathsf{b}}(X)$, so this collection is full.

If E is strong, the right side of (2.18) vanishes for $i \neq j$ (and any n). It follows exactly as above that $\operatorname{Ext}^n_X(F_i, F_j) = 0$ for all n when $i \neq j$, so that F is strong. □

Remark 2.19. Similar descent results for collections of sheaves are given by Elagin [2009] in the algebraically closed case (i.e., $k = \bar{k}$) using the framework of equivariant exceptional collections in equivariant derived categories. Indeed, for a variety X with an action of a finite group G and a G-invariant exceptional collection (see Remark 2.11) consisting of sheaves, this descent result is given in terms of α-twisted representations of G; see [Elagin 2009, Theorem 2.2]. For a G-stable exceptional collection consisting of sheaves, results are in terms of coinduced twisted representations of G; see [loc. cit., Theorem 2.3].

Lemma 2.20. *Let X be a k-scheme and L/k a finite G-Galois extension. If X admits an exceptional collection, then X_L admits a G-stable exceptional collection.*

Proof. Let $E_1, \dots, E_s$ be the given exceptional collection on X, and consider $\pi^* E_1, \dots, \pi^* E_s$ on X_L. To compute morphisms, we note that

$$\begin{aligned}\operatorname{Hom}_{X_L}(\pi^* E_i, \pi^* E_j) &= \operatorname{Hom}_X(E_i, \pi_* \pi^* E_j) \\ &= \operatorname{Hom}_X(E_i, E_j \otimes_k L) = \operatorname{Hom}_X(E_i, E_j) \otimes_k L.\end{aligned}$$

This vanishes if $j > i$. Let $A_i = \operatorname{Hom}_X(E_i, E_i)$. We can split $A_i \otimes_k L$ as a product of matrix algebras over division algebras $A_{i,j} = M_{N_{i,j}}(D_{i,j})$ and correspondingly decompose

$$\pi^* E_i = \bigoplus F_{i,j}^{N_{i,j}}$$

with

$$\operatorname{Hom}_{X_L}(F_{i,j}, F_{i,j}) = D_{i,j}.$$

Note that $F_{i,j}$ and $F_{i,j'}$ are orthogonal for $j \neq j'$. Thus, we have an exceptional collection. □

Lemma 2.21. *Let X be a k-scheme and L/k a finite extension with Galois group G. If G acts trivially on* $\operatorname{Pic}(X_L)$ *and X_L admits an exceptional collection of line bundles, then X admits an exceptional collection of vector bundles.*

Proof. The collection on X_L is automatically G-stable pointwise. Hence we can apply Theorem 2.17. □

Remark 2.22. Note that while we may start with a collection of line bundles, the descended collection may not consist only of line bundles. An example of this is the real conic discussed in the introduction.

Lemma 2.23. *Let X be a smooth k-variety and L/k a G-Galois extension. Let $Y_1, \dots, Y_s$ be a G-orbit of smooth transversal subvarieties of X_L. Let $Y_I = \bigcap_{i \in I} Y_i$ and let H_I be the normalizer of Y_I. If each Y_I admits a full H_I-stable exceptional collection, then $\widetilde{X}$ admits an exceptional collection, where $\widetilde{X}_L$ is the iterated blowup of X_L at the Y_i (in any order).*

Proof. This is an iterated application of Orlov's theorem; see [Castravet and Tevelev 2017, Lemma 7.2]. □

3. Arithmetic toric varieties

We introduce toric varieties over arbitrary fields. Such varieties, also known as *arithmetic toric varieties*, have been treated in [Duncan 2016; Elizondo et al. 2014; Merkurjev and Panin 1997; Voskresenskiĭ and Klyachko 1984].

Definition 3.1. A *torus* (over k) is an algebraic group T (over k) such that $T_{k^s} \simeq \mathbb{G}_m^n$. A torus is *split* if $T \simeq \mathbb{G}_m^n$. A field extension L/k satisfying $T_L \simeq \mathbb{G}_m^n$ is called a *splitting field* of the torus T. Any torus admits a finite Galois splitting field.

Definition 3.2. Given a torus T, a *toric T-variety* is a normal variety with a faithful T-action and a dense open T-orbit. A toric T-variety is *split* if T is a split torus. A *splitting field* of a toric T-variety is a splitting field of T. A variety is a *toric variety* if it is a toric T-variety for some torus T.

Definition 3.3. Let X be a toric T-variety whose dense open T-orbit contains a k-rational point. Then we say X is *neutral* [Duncan 2016] (or a *toric T-model* [Merkurjev and Panin 1997]). An orbit of a split torus always has a k-point, so a split toric variety is neutral, but the converse is not true in general.

Remark 3.4. In what follows, we use the term *toric variety* to mean toric T-variety for some fixed torus T, even though such a variety may have a toric structure for various tori. In fact, the choice of torus does not affect our analysis of toric varieties given below, and we refer interested readers to [Duncan 2016] for such considerations.

Recall that a *k-form* of a k-variety X is a k-variety X' such that $X_L \simeq X'_L$ for some field extension L/k. Any k-form of a toric variety is a toric variety [Duncan 2016].

3A. ***The split case.*** Let us begin by recalling some facts concerning toric varieties with $T \simeq \mathbb{G}_m^n$ (e.g., when $k = \mathbb{C}$ or $k = k^s$), which are studied in terms of combinatorial data, e.g., lattices, cones, fans. Good references for toric varieties over $\mathbb{C}$ include [Fulton 1993; Cox et al. 2011], and many results hold generally in the split case.

Let N be a finitely generated free abelian group of rank n and $M = \operatorname{Hom}(N, \mathbb{Z})$. A subsemigroup $\sigma \subset N_{\mathbb{R}}$ is a *cone* if $(\sigma^\vee)^\vee = \sigma$, where

$$\sigma^\vee = \{u \in M \mid u(v) \geq 0 \text{ for all } v \in \sigma\}.$$

A subsemigroup τ is a *face* of σ if it is of the form $\tau = \{v \in \sigma \mid u(v) = 0 \text{ for all } u \in S\}$ for some $S \subseteq \sigma^\vee$. A cone σ is *pointed* if 0 is a face of σ, and in this case $\sigma^\vee$ generates $M_{\mathbb{R}}$. Given a pointed cone σ, we associate the affine k-scheme $U_\sigma = \operatorname{Spec} k[\sigma^\vee]$, and for any face $\tau \subset \sigma$ the induced map $U_\tau \hookrightarrow U_\sigma$ is an open embedding.

A *fan* $\Sigma \subset N_{\mathbb{R}}$ is a finite collection of pointed cones such that

(1) any face of a cone in Σ is a cone in Σ and

(2) the intersection of any two cones in Σ is a face of each.

To any fan Σ we associate a k-variety X_Σ which is obtained by gluing the affine schemes U_σ along common subschemes U_τ corresponding to faces.

On the other hand, beginning with a split torus $T \simeq \mathbb{G}_m^n$ and toric T-variety X with fixed embedding $T \hookrightarrow X$, we recover M as the character lattice $\operatorname{Hom}(T, \mathbb{G}_m)$ of T and N as the cocharacter lattice $\operatorname{Hom}(\mathbb{G}_m, T)$. The association $\Sigma \mapsto X_\Sigma$ defines a bijective correspondence between fans $\Sigma \subset N_{\mathbb{R}}$ and toric T-varieties X

(we remind the reader that here we assume T is a split torus; in general, fans Σ admitting an action by $\mathrm{Gal}(k^s/k)$ are in bijection with neutral toric T-varieties).

Let $\Sigma(\ell)$ denote the collection of cones in Σ of dimension ℓ. Let $\mathrm{Div}_T(X)$ denote the free abelian group generated by the *rays* of Σ, i.e., elements of $\Sigma(1)$. By the orbit-cone correspondence [Cox et al. 2011, Theorem 3.2.6], $\mathrm{Div}_T(X)$ is isomorphic to the group of T-invariant Weil divisors of X. For X a (split) smooth projective toric variety, we have natural identifications

$$\mathrm{Pic}(X) = \mathrm{Pic}(X_{k^s}) = \mathrm{Cl}(X_{k^s}) = \mathrm{Cl}(X),$$

which yield an exact sequence

$$0 \to M \to \mathrm{Div}_T(X) \to \mathrm{Pic}(X) \to 0.$$

In particular, if X is of dimension n and m is the number of rays in Σ, the Picard rank of X is $\rho = m - n$.

Definition 3.5. A variety X is *Fano* (resp. *weak Fano*) if its anticanonical class $-K_X$ is ample (resp. nef and big). If X is a normal variety, a Cartier D divisor on X is *nef* ("numerically effective" or "numerically eventually free") if $D \cdot C \geq 0$ for every irreducible curve $C \subset X$. A divisor D is *very ample* if D is base point free and $\varphi_D : X \to \mathbb{P}(\Gamma(X, \mathcal{O}_X(D))^\vee)$ is an embedding. A divisor D is *ample* if ℓD is very ample for some $\ell \in \mathbb{Z}^+$. A line bundle $\mathcal{O}_X(D)$ is nef or (very) ample if the corresponding divisor D is nef or (very) ample. A Cartier divisor is *numerically trivial* if $D \cdot C = 0$ for every irreducible complete curve $C \subset X$. Let $N^1(X)$ be the quotient group of Cartier divisors by the subgroup of numerically trivial divisors. The *nef cone* $\mathrm{Nef}(X)$ is the cone in $N^1(X)$ generated by the nef divisors, and the *anti-nef cone* is the cone $-\mathrm{Nef}(X) \subset N^1(X)$. A line bundle $\mathcal{O}_X(D)$ is nef (ample) if D is nef (ample).

Proposition 3.6. *A Cartier divisor D on a split proper toric variety X is nef* (*resp. ample*) *if and only if $D \cdot C \geq 0$* (*resp. $D \cdot C > 0$*) *for all torus-invariant integral curves $C \subset X$.*

Proof. When k is algebraically closed, this is [Mustaţă 2002, Theorems 3.1 and 3.2]. One can see that the arguments remain valid in the split case more generally. □

3B. ***The not necessarily split case.*** Here we provide a "black box" for producing exceptional collections on arbitrary forms of toric varieties by identifying certain special exceptional collections on a *split* toric variety. This reduces an arithmetic question to a completely geometric question.

We begin by reviewing how to obtain arbitrary forms of toric varieties from the split case; see, for example, [Voskresenskiĭ 1982; Elizondo et al. 2014]. Let T be the split torus of a split smooth projective toric variety X with fan Σ in the space

$N \otimes \mathbb{R}$ associated to the lattice N. Let $\mathrm{Aut}(\Sigma)$ denote the subgroup of elements $g \in \mathrm{GL}(N)$ such that $g(\sigma) \in \Sigma$ for every cone $\sigma \in \Sigma$. There is a natural inclusion of $T \rtimes \mathrm{Aut}(\Sigma)$ into $\mathrm{Aut}(X)$ as the subgroup leaving the open orbit T-invariant.

Let k^s be the separable closure of k. The Galois cohomology set

$$H^1(k^s/k, \mathrm{Aut}(X)(k^s))$$

is in bijective correspondence with the k-forms of X. The natural map

$$H^1(k^s/k, T(k^s) \rtimes \mathrm{Aut}(\Sigma)) \to H^1(k^s/k, \mathrm{Aut}(X)(L))$$

in Galois cohomology is surjective; the failure of this map to be a bijection amounts to the fact that there may be several nonisomorphic toric variety structures on the same variety; see [Duncan 2016] for more details.

Suppose that $X' = {}^{\gamma}X$ is a twisted form of a split toric variety for a cocycle γ representing a class in $H^1(k^s/k, T(k^s) \rtimes \mathrm{Aut}(\Sigma))$. There is a "factorization" $X' = {}^{\alpha}({}^{\beta}X)$, where β represents a class in $H^1(k^s/k, \mathrm{Aut}(\Sigma))$ and α represents a class in $H^1(k^s/k, ({}^{\beta}T)(k^s))$. Informally, β changes the torus that acts on X, while α changes the torsor of the open orbit in X.

Suppose X is a toric T-variety. We say that an object $E \in \mathsf{D}^{\mathsf{b}}(X)$ is *T-equivariant* if E is in the image of the forgetful functor from $\mathsf{D}^{\mathsf{b}}(\mathrm{Coh}_T(X))$; see [Ballard et al. 2014, §2]. In particular, this implies that $t^*E \simeq E$ for all $t \in T(k)$.

Proposition 3.7. *Let X be a split toric T-variety over a field k and let Σ be the associated fan. Suppose that X admits an* $\mathrm{Aut}(\Sigma)$*-stable full exceptional collection* E *such that each object is T-equivariant. Then any k-form X' of X admits a full exceptional collection* E'*. Moreover*, E' *is strong* (*resp. consists of vector bundles*, *consists of sheaves*) *as soon as* E *is strong* (*resp. consists of vector bundles*, *consists of sheaves*).

Proof. By Lemma 2.20, there exists a G-stable exceptional collection F on X_L. From the proof of that lemma, the objects F of F are direct summands of π^*E for each object $E \in \mathsf{E}$, where each isomorphism class of a simple direct summand is represented by exactly one F. Since E is $\mathrm{Aut}(\Sigma)$-stable and each object is T-equivariant, we may conclude that F is $(T(L) \rtimes \mathrm{Aut}(\Sigma)) \rtimes G$-stable.

Let X' be a k-form of X; there exists a finite Galois extension L/k with Galois group G such that $X'_L \simeq X_L$. From Theorem 5.1 of [Duncan 2016], the natural map

$$H^1(L/k, T(L) \rtimes \mathrm{Aut}(\Sigma)) \to H^1(L/k, \mathrm{Aut}(X)(L))$$

in Galois cohomology is surjective. Thus, we may assume that $X' = {}^{c}X$ is the *twist* by a cocycle $c : G \to T(L) \rtimes \mathrm{Aut}(\Sigma)$. Recall that the cocycle condition is that $c(gh) = c(g){}^{g}c(h)$ for all $g, h \in G$, where ${}^{g}c(h)$ denotes the Galois action of g on $T(L) \rtimes \mathrm{Aut}(\Sigma)$.

Identifying $X_L = X'_L$, twisting gives $\sigma'(g) = c(g)\sigma(g)$, where σ is the action of G induced from X and σ' is induced from X'. The punchline is that the action σ' factors through the image of $(T(L) \rtimes \mathrm{Aut}(\Sigma)) \rtimes G$ described above. Thus the exceptional collection F is G-stable for the X' action as well. The proposition now follows by Theorem 2.17. □

Corollary 3.8. *Let X be a split toric T-variety over a field k and let Σ be the associated fan. If X admits an* $\mathrm{Aut}(\Sigma)$*-stable full* (*strong*) *exceptional collection of line bundles*, *then every k-form of X admits a full* (*strong*) *exceptional collection of vector bundles.*

Proof. Recall that every line bundle is isomorphic to a T-equivariant line bundle by standard results on toric varieties. The claim now follows by Proposition 3.7. □

Lemma 3.9. *Let X and Y be smooth projective toric varieties over k, and let $G = \mathrm{Gal}(k^s/k)$. Assume we have a K-positive toric flip $X \dashrightarrow Y$ such that over k^s the flipping loci F_i are disjoint and permuted by G. Let H_i be the normalizer of F_i. If X_L admits a full G-stable exceptional collection and Y_i admits a full H_i-stable exceptional collection*, *then Y admits a full exceptional collection.*

Proof. Passing to k^s, we are free to use [Ballard et al. 2019] giving semiorthogonal decompositions for the flip over each Y_i. Since the Y_i are disjoint, we can concatenate these collections to get a G-stable collection. □

3C. ***Products of toric varieties.*** Recall that, given groups G, H along with a homomorphism $\rho : H \hookrightarrow S_n$, the *wreath product* $G \wr H$ is the group $G^n \rtimes H$, where H acts on G^n by permuting the copies of G. We say a toric variety X is *indecomposable* if it cannot be written as a product $X_1 \times X_2$, where X_1 and X_2 are positive-dimensional toric varieties.

Lemma 3.10. *Suppose $Z = X_1^{n_1} \times \cdots \times X_r^{n_r}$ is a product of proper split toric varieties $X_1, \ldots, X_r$, where $X_i \not\simeq X_j$ for $i \neq j$ and each X_i is indecomposable. Then*

$$\mathrm{Aut}(\Sigma) \simeq (\mathrm{Aut}(\Sigma_1) \wr S_{n_1}) \times \cdots \times (\mathrm{Aut}(\Sigma_r) \wr S_{n_r}),$$

where Σ is the fan of Z and $\Sigma_1, \ldots, \Sigma_r$ are the fans of $X_1, \ldots, X_r$.

Proof. First, consider $Z = X_1 \times X_2$, where X_1, X_2 are proper split toric varieties. Let N (resp. N_1, N_2) be the cocharacter lattice and Σ (resp. Σ_1, Σ_2) be the fan of Z (resp. X_1, X_2). Here $N = N_1 \oplus N_2$ and Σ is the set of cones of the form $\sigma_1 \times \sigma_2$, where $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$. The faces of a cone $\sigma_1 \times \sigma_2$ are precisely the cones of the form $\sigma'_1 \times \sigma'_2$, where σ'_1 is a face of σ_1 and σ'_2 is a face of σ_2. The fan Σ_1 can be canonically identified with the subfan of Σ via the bijection $\sigma \mapsto \sigma \times \{0\}$.

Now, suppose also that $Z = Y \times W$ is a product of proper split toric varieties with Y indecomposable. Let Σ_Y be the fan of Y, which we can canonically identify with a subfan of Σ_Z. Every cone of Y is of the form $\sigma_1 \times \sigma_2$, where $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$. Since fans are closed under taking faces, $\sigma_1 \times \{0\}$ and $\{0\} \times \sigma_2$ are also cones in Σ_Y. Thus every cone in Σ_Y is a product of cones in the intersections $\Sigma_Y \cap \Sigma_1$ and $\Sigma_Y \cap \Sigma_2$.

In particular, since X is proper, we have that the space $N_Y \otimes \mathbb{R}$ is the direct sum of $(N_Y \otimes \mathbb{R}) \cap (N_1 \otimes \mathbb{R})$ and $(N_Y \otimes \mathbb{R}) \cap (N_2 \otimes \mathbb{R})$, and Σ_Y is a product of the fans $\Sigma_Y \cap \Sigma_1$ and $\Sigma_Y \cap \Sigma_2$. Since Y is indecomposable, one of these fans is indecomposable and Σ_Y must be a subfan of either Σ_1 or Σ_2.

Returning to the general case, we conclude that the decomposition of Σ as $\Sigma_1^{n_1} \times \cdots \times \Sigma_r^{n_r}$ is unique up to ordering. The description of the automorphism group is immediate. □

Lemma 3.11. *Let Z be a proper toric k-variety with splitting field L/k. Suppose $Z_L = \prod_{i=1}^n X_i$, where each X_i is an indecomposable split proper toric L-variety admitting a full* (*strong*) $\mathrm{Aut}(\Sigma_i)$*-stable exceptional collection of line bundles*, *where Σ_i is the fan of X_i. Then Z has a full* (*strong*) *exceptional collection of vector bundles.*

Proof. It is well known that the exterior product collection is an exceptional collection. For each isomorphism class among the X_i, fix a full (strong) $\mathrm{Aut}(\Sigma_{X_i})$-stable exceptional collection of line bundles. This ensures that the exterior product collection is stable under the action of $(\mathrm{Aut}(\Sigma_{X_1}) \wr S_{a_1}) \times \cdots \times (\mathrm{Aut}(\Sigma_{X_r}) \wr S_{a_r})$. Since this group is $\mathrm{Aut}(\Sigma)$ by Lemma 3.10, the exterior product collection descends by Corollary 3.8. □

4. Low dimension or high symmetry

We provide exceptional collections for smooth toric surfaces, Fano 3-folds, some Fano 4-folds, centrally symmetric toric varieties, and toric varieties corresponding to root systems of type A.

4A. ***Surfaces.*** Here we prove that every toric surface has a full exceptional collection. We begin by recalling the (classical) minimal model program for surfaces over nonclosed fields.

Suppose $f : X \to X'$ is a birational morphism of smooth projective surfaces over a field k. If k is separably closed, then by Proposition 5 of [Coombes 1988] the morphism factors into a sequence

$$X = X_0 \to X_1 \to \cdots \to X_r = X',$$

where each morphism $X_i \to X_{i+1}$ is the blowup of a point on X_{i+1}. Over a nonclosed field k, we can factor $f : X \to X'$ into a sequence where each morphism $X_i \to X_{i+1}$ is defined over k and is a blowup of a (necessarily finite) Galois orbit of k^s-points on X_{i+1}.

Blowing up a point produces an exceptional curve: a smooth rational curve with self-intersection -1. By Castelnuovo's contractibility criterion, such a curve can always be obtained as the result of a blow-up. If one finds a skew Galois orbit of such curves on X, then there exists a birational morphism $f : X \to X'$ contracting these curves. Repetition of this procedure eventually terminates.

Definition 4.1. A *minimal surface* X is a smooth projective surface over a field k such that every birational morphism $X \to X'$ to a smooth projective surface X' is an isomorphism.

Any smooth projective surface can be obtained by iteratively blowing up Galois orbits of separable points starting from a minimal model. A toric variety is geometrically rational. Minimal geometrically rational surfaces were classified by Manin [1966] and Iskovskikh [1979]. One checks that the toric surfaces in their collection are the following (see also a direct proof in [Xie 2017]):

Lemma 4.2. *A minimal smooth projective toric surface is a k^s/k-form of one of the following*:

(1) $\mathbb{P}^2$, $\mathrm{Aut}(\Sigma) = S_3$.

(2) $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathrm{Aut}(\Sigma) = D_8$.

(3) $\mathbb{F}_a = \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$, $a \geq 2$, $\mathrm{Aut}(\Sigma) = C_2$.

(4) $\mathsf{dP}_6 =$ *del Pezzo surface of degree* 6, $\mathrm{Aut}(\Sigma) = D_{12}$.

Proof. A minimal geometrically rational surface is either a del Pezzo surface or has a conic bundle structure [Manin 1966; Iskovskikh 1979]. Over the separable closure, a del Pezzo surface is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a blowup of $\mathbb{P}^2$ at up to 8 points in general position. Blowing up only one or two points never results in a minimal surface, and no more than three points can be simultaneously torus invariant and in general position. Thus every del Pezzo surface is a k^s/k-form of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or dP_6. Over the separable closure, a conic bundle structure has at most 2 singular fibers since their images must be torus invariant points on the base $\mathbb{P}^1$. A minimal conic bundle with at most two singular fibers over the separable closure must be either a del Pezzo surface or a minimal ruled surface. □

Here we exhibit full strong exceptional collections consisting of G-stable blocks for each minimal toric surface exhibited above (none of these collections are original). The fans associated to the split forms of these surfaces are given in Figure 1. In each case, we fix a torus T which gives X the structure of a toric T-surface. As remarked above, this gives a homomorphism $G \to \mathrm{Aut}(\Sigma)$ as well as an action of G on $\mathrm{Pic}(X_L)$, where L is a splitting field of T, $G = \mathrm{Gal}(L/k)$, and Σ is the fan corresponding to the split toric surface X_L. We produce G-stable exceptional collections in each case by exhibiting $\mathrm{Aut}(\Sigma)$-stable collections.

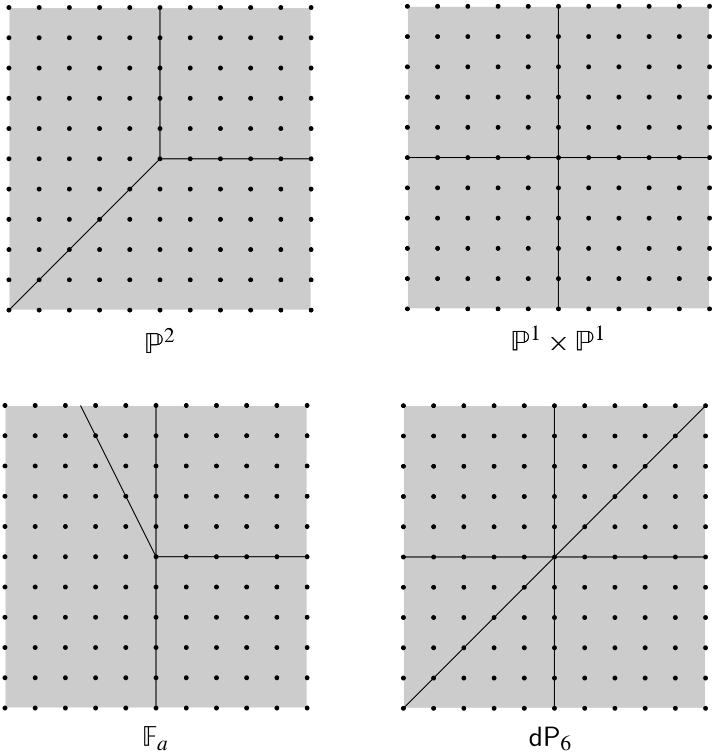

Figure 1. Fans for minimal toric surfaces.

Example 4.3. Let X be a toric T-surface whose split form is $\mathbb{P}^2$ with $\mathrm{Aut}(\Sigma) = S_3$. The S_3-action on $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}$ is clearly trivial, so that the exceptional collection $\{\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\}$ given in [Beĭlinson 1978] yields a full strong $\mathrm{Aut}(\Sigma)$-stable exceptional collection. By Corollary 3.8, X admits a full strong exceptional collection.

Example 4.4. Let X be a toric surface whose split form is $\mathbb{P}^1 \times \mathbb{P}^1$ with $\mathrm{Aut}(\Sigma) = D_8$, and consider the natural projections $p_1, p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Let

$$\mathcal{O}(p, q) = p_1^* \mathcal{O}(p) \otimes p_2^* \mathcal{O}(q).$$

By [Kvichansky and Nogin 1990], the collection $\{\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)\}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is exceptional since $\{\mathcal{O}, \mathcal{O}(1)\}$ is an exceptional collection for $\mathbb{P}^1$. The D_8-action preserves this collection, with orbits given by the blocks $\mathsf{E}^0 = \{\mathcal{O}\}$, $\mathsf{E}^1 = \{\mathcal{O}(1, 0), \mathcal{O}(0, 1)\}$, and $\mathsf{E}^2 = \{\mathcal{O}(1, 1)\}$. In particular, this collection is $\mathrm{Aut}(\Sigma)$-stable, and Corollary 3.8 yields an exceptional collection on X.

Example 4.5. Let X be a toric surface whose split form is the Hirzebruch surface $\mathbb{F}_a$; here $\mathrm{Aut}(\Sigma) = C_2$. Let e_1, e_2 be the standard basis for $\mathbb{Z}^2$. As in [Cox et al. 2011, Example 4.1.8], let $u_1 = -e_1 + ae_2$, $u_2 = e_2$, $u_3 = e_1$, and $u_4 = -e_2$ be the generators of $\Sigma(1)$ with corresponding toric divisors D_i. The Picard group of $\mathbb{F}_a$ is freely generated by $\{D_1, D_2\}$ and D_1 is linearly equivalent to D_3. The only nontrivial fan automorphism σ takes $e_1 \mapsto -e_1 + ae_2$ and $e_2 \mapsto e_2$. Thus σ leaves D_2, D_4 fixed and interchanges D_1 and D_3. We conclude that the action of C_2 on $\mathrm{Pic}(\mathbb{F}_a)$ is trivial, and thus, any exceptional collection is necessarily G-stable (see Lemma 2.21). An exceptional collection for $\mathbb{F}_a$ is given by $\{\mathcal{O}, \mathcal{O}(D_3), \mathcal{O}(D_4), \mathcal{O}(D_3 + D_4)\}$ [Kvichansky and Nogin 1990]. Corollary 3.8 then gives an exceptional collection on X.

Example 4.6. Let X be a toric surface whose split form is dP_6; here $\mathrm{Aut}(\Sigma) = D_{12}$. Viewing dP_6 as the blowup of $\mathbb{P}^2$ at 3 noncolinear points, let H be the pullback of the hyperplane divisor on $\mathbb{P}^2$ and E_i the exceptional divisors, $i = 1, 2, 3$. As shown in [King 1997, Proposition 6.2(ii)], the collection

$$\{\mathcal{O}, \mathcal{O}(H - E_1), \mathcal{O}(H - E_2), \mathcal{O}(H - E_3), \mathcal{O}(H), \mathcal{O}(2H - (E_1 + E_2 + E_3))\}$$

gives an exceptional collection for dP_6 which is $\mathrm{Aut}(\Sigma)$-stable.

Let us rephrase this in the notation of [Blunk et al. 2011]. There are two morphisms $\mathsf{dP}_6 \to \mathbb{P}^2$ realizing dP_2 as a blowup of $\mathbb{P}^2$, and we denote the collection of all six exceptional divisors by L_i and M_i, with $i = 1, 2, 3$. Let H and H' denote the pullbacks of the hyperplane divisors on $\mathbb{P}^2$ under the maps contracting M_i and L_i, respectively, where we identify H with the divisor given in King's collection above (and thus we also identify E_i with M_i). Then $H = L_1 + M_2 + M_3$, and using the relation $L_i + M_j = L_j + M_i$ it follows that

$$2H - (E_1 + E_2 + E_3) = L_1 + L_2 + M_3 = H'.$$

Furthermore, one checks that $H - E_1 = L_2 + M_3$, $H - E_2 = L_1 + M_3$, and $H - E_3 = L_1 + M_2$. As described in [Blunk et al. 2011, §2], the element σ in $S_3 \times C_2 = D_{12}$ which cyclically permutes the six lines L_i, M_i also satisfies $\sigma(H) = H'$ and $\sigma^2(H) = H$. We arrange the exceptional collection above into blocks

$$\begin{aligned}
\mathsf{E}^0 &= \{\mathcal{O}\},\\
\mathsf{E}^1 &= \{\mathcal{O}(H - E_1), \mathcal{O}(H - E_2), \mathcal{O}(H - E_3)\},\\
\mathsf{E}^2 &= \{\mathcal{O}(H), \mathcal{O}(2H - (E_1 + E_2 + E_3))\}.
\end{aligned}$$

In particular, the exceptional collection given above is $\mathrm{Aut}(\Sigma)$-stable, and so by Corollary 3.8 we have an exceptional collection on X.

Proposition 4.7. *Every toric surface admits a full exceptional collection of sheaves.*

Proof. There is a sequence of blowups $X = X_0 \to \cdots \to X_s = X'$, where X' is minimal and so must be one of the varieties given in Lemma 4.2. By Examples 4.3–4.6, X' admits a full strong exceptional collection of vector bundles, and thus X'_L admits a G-stable exceptional collection. By Lemma 2.23, X_L admits a G-stable exceptional collection. □

Remark 4.8. We would like to thank F. Xie for pointing out a mistake in the statement of a previous version of Proposition 4.7. Xie also discusses exceptional collections of toric surfaces in [Xie 2017], although her definition of exceptional object is not the same as ours. In the second arXiv version of that paper, Xie sketched in Remark 8.8 how one might construct an exceptional collection for toric surfaces. After the authors posted a preliminary version of this paper to the arXiv, Xie updated her preprint with Corollary 8.8, which proves the analog of the above proposition for collections of vector bundles but using her notion of exceptional collection.

4B. ***The toric Frobenius and toric Fano 3-folds.*** In Table 1 we present the classification of smooth toric Fano 3-folds given in [Batyrev 1999; Watanabe and Watanabe 1982], adopting Batyrev's enumeration. For each $X = X_\Sigma$, we record the following invariants:

- $\sigma(1) = |\Sigma(1)|$ is the number of rays of Σ [Bernardi and Tirabassi 2009].
- k_0 is the rank of the Grothendieck group $K_0(X)$, which coincides with the number of maximal cones in the fan Σ [Bernardi and Tirabassi 2009].
- $\mathrm{Aut}(\Sigma)$ is the automorphism group of the (lattice N which preserves the) fan Σ corresponding to X.
- ρ is the Picard rank of X [Watanabe and Watanabe 1982].
- ρ^G is the $\mathrm{Aut}(\Sigma)$-invariant Picard rank of X, i.e., the rank of $\mathrm{Pic}(X)^{\mathrm{Aut}(\Sigma)}$.
- $\mathfrak{fr} = |\mathsf{Frob}(X)|$ is the number of isomorphism classes of line bundles produced by the push forward of the structure sheaf under the Frobenius morphism [Bernardi and Tirabassi 2009; Uehara 2014].
- $\mathfrak{fr}^- = |\mathsf{Frob}(X) \cap -\mathrm{Nef}(X)|$ is the number of isomorphism classes of line bundles in $\mathsf{Frob}(X)$ which lie in the anti-nef cone of X [Uehara 2014].

Toric Frobenius. Let X be a split toric variety of dimension n with fixed torus embedding $T \hookrightarrow X$ and take $\ell \in \mathbb{Z}^+$. Define the ℓ-th Frobenius map on $T = \mathbb{G}_m^n$ to be $(x_1, \dots, x_n) \mapsto (x_1^\ell, \dots, x_n^\ell)$. The unique extension to X is denoted F_ℓ and called the *ℓ-th Frobenius morphism*. Alternatively, if $\Sigma \subset N$ is the fan associated to X, define a lattice $N' = \frac{1}{\ell}N$. The inclusion $N \subset N'$, which sends a cone in $N_\mathbb{R}$ to the cone with the same support in $N'_\mathbb{R}$, induces a finite surjective morphism which is precisely the ℓ-th Frobenius morphism $F_\ell : X \to X$.

The sheaf $(F_\ell)_*(\mathcal{O}_X)$ splits into line bundles and [Thomsen 2000] provides an algorithm for computing its direct summands. We let $\mathsf{Frob}(X)$ denote the union of all isomorphism classes of line bundles arising as direct summands of $(F_\ell)_*(\mathcal{O}_X)$ as ℓ varies over $\mathbb{Z}^+$. Note that $\mathsf{Frob}(X)$ is a finite set.

	Toric Fano 3-fold X	$\sigma(1)$	k_0	$\mathrm{Aut}(\Sigma)$	ρ	ρ^G	$\mathfrak{fr}$	$\mathfrak{fr}^-$
1.	$\mathbb{P}^3$	4	4	S_4	1	1	4	4
2.	$\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(2))$	5	6	S_3	2	2	7	6
3.	$\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(1))$	5	6	S_3	2	2	6	6
4.	$\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(1))$	5	6	$C_2\times C_2$	2	2	6	6
5.	$\mathbb{P}^2\times\mathbb{P}^1$	5	6	D_{12}	2	2	6	6
6.	$\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(1,1))$	6	8	D_8	3	2	8	8
7.	$\mathbb{P}_{\mathsf{dP}_8}(\mathcal{O}\oplus\mathcal{O}(l))$, $l^2=1$ on dP_8	6	8	D_8	3	3	8	8
8.	$\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$	6	8	$C_2\times S_4$	3	1	8	8
9.	$\mathsf{dP}_8\times\mathbb{P}^1$	6	8	$C_2\times C_2$	3	3	8	8
10.	$\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}\otimes\mathcal{O}(1,-1))$	6	8	D_8	3	2	8	8
11.	$\mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(1)))$	6	8	C_2	3	3	9	8
12.	$\mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}^2\times\mathbb{P}^1)$	6	8	C_2	3	3	8	8
13.	dP_7-bundle over $\mathbb{P}^1$	7	10	C_2	4	4	10	10
14.	dP_7-bundle over $\mathbb{P}^1$	7	10	$C_2\times C_2$	4	3	10	10
15.	$\mathsf{dP}_7\times\mathbb{P}^1$	7	10	$C_2\times C_2$	4	3	10	10
16.	dP_7-bundle over $\mathbb{P}^1$	7	10	C_2	4	4	10	10
17.	$\mathsf{dP}_6\times\mathbb{P}^1$	8	12	$C_2\times C_2\times S_3$	5	2	12	12
18.	dP_6-bundle over $\mathbb{P}^1$	8	12	$C_2\times C_2$	5	4	12	12

Table 1. Toric Fano 3-folds.

Conjecture 4.9 [Bondal 2006]. If X is a smooth proper toric variety then the collection $\mathsf{Frob}(X)$ generates $\mathsf{D}^{\mathsf{b}}(X)$.

For a toric variety X in which Bondal's Conjecture is true, we say that *the Frobenius generates the derived category of* X. In [loc. cit.], Bondal proves that if all summands of $\mathsf{Frob}(X)$ are nef, one actually gets a full strong exceptional collection, so that Conjecture 4.9 is true in this case. He also notes his arguments work for all but two (isomorphism classes of) toric Fano threefolds. To cover all toric Fano threefolds, Uehara noticed that discarding line bundles which do not lie in the set $-\mathrm{Nef}(X)$ yields a full strong exceptional collection [Uehara 2014].

Lemma 4.10. *Let X be a toric variety over k with splitting field L. Suppose E is a full* (*strong*) *exceptional collection for $\mathsf{D}^{\mathsf{b}}(X_L)$ where either $\mathsf{E}=\mathsf{Frob}(X_L)$ or $\mathsf{E}=\mathsf{Frob}(X_L)\cap-\mathrm{Nef}(X_L)$. Then there exists a full* (*strong*) *exceptional collection for $\mathsf{D}^{\mathsf{b}}(X)$.*

Proof. Both $\mathsf{Frob}(X_L)$ and $\mathrm{Nef}(X_L)$ are canonical constructions and thus are $\mathrm{Aut}(X_L)$-stable. In particular, E is $\mathrm{Aut}(\Sigma)$-stable and so Corollary 3.8 applies. □

Proposition 4.11. *Let X be a smooth projective toric Fano 3-fold over a field k. Then X admits a full strong exceptional collection consisting of vector bundles.*

Proof. Let X_L be the associated split toric Fano 3-fold. The main result of [Uehara 2014] guarantees that the set $\mathsf{E} = \mathsf{Frob}(X_L) \cap -\mathrm{Nef}(X_L)$ defines a full strong exceptional collection on X. Lemma 4.10 completes the proof. □

4C. ***Toric Fano 4-folds.*** There are 124 split smooth toric Fano 4-folds, which were first classified in [Batyrev 1999] (a missing case was added in [Sato 2000]). Full strong exceptional collections for all 124 of these 4-folds were exhibited in [Prabhu-Naik 2017]. However, it is not clear that these collections are $\mathrm{Aut}(\Sigma)$-stable, so they do not necessarily lead to full strong exceptional collections in the arithmetic case.

All collections obtained using Method 1 of [Prabhu-Naik 2017] produce $\mathrm{Aut}(\Sigma)$-stable collections (note that this is precisely the method used in [Uehara 2014] for toric Fano 3-folds, and we refer to this as the *Bondal–Uehara method*). Together with Lemmas 3.11 and 4.10, this gives stable exceptional collections for 43 of the 124 smooth toric Fano 4-folds. However, there are examples where the Bondal–Uehara method fails to produce an exceptional collection. In this case, all is not lost (see Section 4D).

More precisely, the varieties (61), (62), (63), (64), (77), (105), (107), (108), (110), (122), and (123) of [Prabhu-Naik 2017] are shown to have exceptional collections using the Bondal–Uehara method. Hence, they admit exceptional collections which are $\mathrm{Aut}(\Sigma)$-stable and thus provide exceptional collections for the arithmetic forms. Secondly, for the varieties (109), (114), and (115), the set $\mathsf{Frob}(X)$ is a full exceptional collection, which is G-stable by Lemma 4.10. Lastly, Lemma 3.11 guarantees the existence of exceptional collections on products. Hence, the following varieties also admit stable exceptional collections: (0), (4), (9), (17), (24), (25), (26), (27), (45), (52), (53), (54), (55), (56), (58), (67), (73), (88), (90), (92), (93), (97), (103), (111), (112), (113), (118), (119), (120).

4D. ***Centrally symmetric toric Fano varieties.*** Polytopes with the highest degree of symmetry are the *centrally symmetric* polytopes, i.e., $-P = P$. The smooth split toric varieties X whose anticanonical polytope is full-dimensional and centrally symmetric were classified in [Voskresenskiĭ and Klyachko 1984]. It was shown that any such variety (which we refer to as a *centrally symmetric toric Fano variety*) is isomorphic to a product of projective lines and *generalized del Pezzo varieties* V_n of dimension $n = 2m$. Note that $V_2 = \mathsf{dP}_6$ and V_4 is the missing (116) from the list in Section 4C (this is (118) in the enumeration found in [Batyrev 1999]). The goal of this section is to exhibit full stable exceptional collections on V_n, which in turn yields stable exceptional collections for any centrally symmetric toric Fano variety, in light of Lemma 3.11.

Castravet and Tevelev [2017, Theorem 6.6] found $\mathrm{Aut}(\Sigma)$-stable full strong exceptional collections for the varieties V_n. The present authors had independently discovered the same exceptional collection (up to a twist by a line bundle). Nevertheless, the perspective here may be of independent interest, so we sketch the argument. A more detailed analysis is given in [Ballard et al. 2018].

The variety V_n with $n = 2m$ has rays given by

$$\begin{aligned} e_1 &= (1, 0, \dots, 0), & \bar{e}_1 &= (-1, 0, \dots, 0), \\ e_2 &= (0, 1, \dots, 0), & \bar{e}_2 &= (0, -1, \dots, 0) \\ &\vdots & &\vdots \\ e_n &= (0, 0, \dots, 1), & \bar{e}_n &= (0, 0, \dots, -1), \\ e_{n+1} &= (-1, -1, \dots, -1), & \bar{e}_{n+1} &= (1, 1, \dots, 1), \end{aligned}$$

and maximal cones given as follows. From the rays $e_1, \dots, e_{n+1}$, omit a single e_i. From the remaining $n = 2m$ rays, choose $\frac{n}{2}$ of them and take their antipodes [Voskresenskiĭ and Klyachko 1984, proof of Theorem 5]. Note that $V_2 = \mathrm{dP}_6$ (whose fan is given in Figure 1). The number of maximal cones $c(n)$ of V_n is given by

$$c(n) = \frac{(n+1)!}{\left(\frac{n}{2}\right)!^2} = \frac{(2m+1)!}{m!^2}.$$

There's a natural action of $S_{n+1} \times C_2$, where S_{n+1} permutes $e_1, \dots, e_{n+1}$ and $\bar{e}_1, \dots, \bar{e}_{n+1}$ in the obvious way. The C_2-action is simply the antipodal map on the cocharacter lattice — we refer to it as "the involution". Clearly, the involution interchanges e_i and $\bar{e}_i$.

The variety V_n is of importance in birational geometry due to its appearance in the factorization of the standard Cremona transformation of $\mathbb{P}^n$. In fact, as is well known, V_n can be explicitly obtained from $\mathbb{P}^n$ as follows. First blow up the torus fixed points, then flip the (strict transforms) of the lines through these points, then flip the (strict transforms) of planes through these points, ..., up until, and not including, the half-dimensional linear subspaces. The resulting variety is V_n. For more, see [Casagrande 2003].

Since V_n and the blowup of $\mathbb{P}^n$ at its torus fixed points are isomorphic in codimension 1, they have isomorphic Picard groups. We use a basis

$$\{H, E_1, \dots, E_{n+1}\}$$

for $\mathrm{Pic}(V_n)$, which corresponds to the hyperplane section and the exceptional divisors of the blown up $\mathbb{P}^n$. We have

$$[e_i] = E_i, \qquad [\bar{e}_i] = \left(H - \sum_{j=1}^{n+1} E_j\right) + E_j,$$

where S_{n+1} permutes the E_i leaving H fixed, and the involution is represented by the following matrix:

$$\begin{pmatrix} n & 1 & 1 & \cdots & 1 \\ 1-n & 0 & -1 & \cdots & -1 \\ 1-n & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-n & -1 & -1 & \cdots & 0 \end{pmatrix}.$$

For each $c \in \mathbb{Z}$ and $J \subset \{1, \ldots, n+1\}$, define

$$F_{c,J} := c\biggl(\sum_{i=1}^{n+1} E_i - H\biggr) - \sum_{j \in J} E_j.$$

Note that the involution takes $F_{c,J}$ to $F_{|J|-c,J}$.

Proposition 4.12. *The set of $F_{c,J}$ with*

(1) $|J| - \frac{n}{4} \leq c \leq \frac{n}{4}$ *or*

(2) $\frac{n+2}{4} \leq c \leq |J| - \frac{n+2}{4}$

forms a full strong $(S_{n+1} \times C_2)$-stable exceptional collection on V_n under any ordering of the blocks such that $|J|$ is (nonstrictly) decreasing.

Proof sketch. This collection is the same as that of [Castravet and Tevelev 2017, Theorem 6.6] up to a twist by a line bundle. Thus, we only sketch an argument here, expanded in [Ballard et al. 2018]. One checks that the description of "forbidden cones" given in [Borisov and Hua 2009] shows that relevant cohomology groups vanish — this shows that it is a strong exceptional collection. To prove generation, one considers the series of flips required to reach $\mathbb{P}^n$ blown up at $n+1$ points. Using the description of the semiorthogonal decompositions in [Ballard et al. 2019], the line bundles can be shown to generate the necessary admissible subcategories of each intermediate birational model. □

Since any centrally symmetric toric Fano variety is a product of projective lines and the varieties V_n, Lemma 3.11 yields the following:

Corollary 4.13. *Any form of a centrally symmetric toric Fano variety admits a full strong exceptional collection consisting of vector bundles.*

4E. ***Toric varieties from the Weyl fans of type A.*** One method for identifying toric varieties with large symmetry groups is to start with root systems. Let R be a root system in a Euclidean space E. The $\mathbb{Z}$-lattice generated by R is denoted $M(R)$, while its dual in $E^\vee$ is denoted by $N(R)$. For every set S of simple roots in E, we have the dual cone corresponding to a closed Weyl chamber

$$\sigma_S := \{f \in E^\vee \mid \langle f, \alpha\rangle \geq 0, \ \forall \alpha \in S\}.$$

The cones σ_S are the maximal cones for a fan Σ_R in $E^\vee$. We denote the associated toric variety by $X(R)$. Recall that an automorphism of R is an element of $\mathrm{GL}(E)$ preserving R. Let $W(R)$ be the Weyl group and $\Gamma(R)$ the symmetry group of the Dynkin diagram of R. It is well known that

$$\mathrm{Aut}(R) \simeq W(R) \rtimes \Gamma(R).$$

Any automorphism of R induces an action on the fan $\Sigma(R)$, which yields a homomorphism $\phi : \mathrm{Aut}(R) \to \mathrm{Aut}(\Sigma(R))$.

Lemma 4.14. *The map* $\phi : \mathrm{Aut}(R) \to \mathrm{Aut}(\Sigma(R))$ *is an isomorphism.*

Proof. First note that the set R can be reconstructed from $\Sigma(R)$ by taking the union of the extremal rays generating the dual cones $\sigma_S^\vee$ for all σ_S. Thus any symmetry of the fan induces a symmetry of R. This gives the inverse map to ϕ. □

Here we focus on the case $R = A_n$. In [Losev and Manin 2000], the authors showed that $X(A_n)$ is a moduli space of rational curves with $(n+1)$ marked points and 2 poles. Another useful proof appeared in [Batyrev and Blume 2011].

Using this perspective, [Castravet and Tevelev 2017] exhibited an exceptional collection on $X(A_n)$ that is stable under the action of permuting the marked points and flipping the poles, i.e., an $(S_{n+1} \rtimes C_2)$-stable collection. Here we demonstrate that Castravet and Tevelev's exceptional collection satisfies the conditions of Proposition 3.7 and hence descends to an exceptional collection on any form of $X(A_n)$ (in characteristic 0).

To do this requires a bit of translating divisors and actions from the moduli-theoretic language to the toric language. We recall the moduli-theoretic language.

Definition 4.15. Let N be a set of order n. A *chain of polar* $\mathbb{P}^1$'s is a $(\{0, \infty\} \cup N)$-marked linear nodal chain of $\mathbb{P}^1$'s with 0 on the left tail and ∞ on the right tail. A chain of polar $\mathbb{P}^1$'s is *stable* if

(1) marked points do not coincide with nodes,

(2) only N-marked points are allowed to coincide,

(3) each component of the chain has at least three special points (nodes or marked points).

We write LM_N for the corresponding moduli space. We also use LM_n depending on the context. Note that the universal curve over LM_n is isomorphic to LM_{n+1}.

Theorem 4.16. *The toric variety* $X(A_{n-1})$ *is isomorphic to* LM_n. *Moreover, if we fix an embedding* $A_{n-1} \to A_n$, *the corresponding map* $X(A_n) \to X(A_{n-1})$ *is the universal curve. Moreover,* $X(A_n) \to X(A_{n-1})$ *is a toric morphism.*

Proof. This is [Losev and Manin 2000, Theorem 2.6.3]. See also [Batyrev and Blume 2011, Theorem 3.19]. The map is consequently toric by [Batyrev and Blume 2011, Proposition 1.4]. □

Under this isomorphism, the closures of the torus orbits on $X(A_n)$ have the following moduli-theoretic description. Fix a partition $N_1 \sqcup N_2 = N$ and let δ_{N_1} denote the divisor parametrizing polar chains of length exactly 2 having the first marked by N_1 and the last marked by N_2. For a partition with more parts

$$N_1 \sqcup N_2 \sqcup \cdots \sqcup N_t = N,$$

one has the locus $Z_{N_1,\dots,N_t}$ parametrizing polar chains of length exactly t, where the i-th $\mathbb{P}^1$ is marked by N_i. These loci are precisely the proper torus orbit closures on $X(A_n)$.

Note that each locus is a complete intersection

$$Z_{N_1,\dots,N_t} := \delta_{N_1} \cap \delta_{N_1 \cup N_2} \cap \cdots \cap \delta_{N_1 \cup \cdots \cup N_{t-1}}.$$

Moreover, we have an isomorphism

$$Z_{N_1,\dots,N_t} \simeq LM_{N_1} \times LM_{N_2} \times \cdots \times LM_{N_t},$$

where the left node of each $\mathbb{P}^1$ is marked with 0 and the right node is marked with ∞. Thus, we have toric morphisms

$$i_{N_1,\dots,N_t} : LM_{N_1} \times LM_{N_2} \times \cdots \times LM_{N_t} \to LM_N.$$

Also, for each subset $K \subset N$, we get a forgetful map $\pi_K : LM_N \to LM_K$, which is a toric morphism since it is a composition of maps from Theorem 4.16.

Recall there is a set of line bundles $\mathbb{G}_N$ on LM_N [Castravet and Tevelev 2017, Definition 1.5], and one generates a larger set H_N of sheaves via

$$\mathsf{H}_N := \left\{ (i_{N_1,\dots,N_t})_*(G_{l_1} \boxtimes \cdots \boxtimes G_{l_t}) \mid \forall N_1 \cup \cdots \cup N_t = N,\ G_{l_j} \in \mathbb{G}_{N_j} \right\},$$

where $i_{N_1,\dots,N_t} : Z_{N_1,\dots,N_t} \hookrightarrow LM_N$ is the inclusion.

Theorem 4.17. *There is an ordering on the set*

$$\mathsf{CT}_N := \mathsf{H}_N \cup \left(\bigcup_{K \subsetneq N} \{\pi_K^* E \mid E \in \mathsf{H}_K\} \right) \cup \{\mathcal{O}\}$$

making it into an $(S_N \rtimes C_2)$-stable exceptional collection under permutations of the two sets of markings.

Proof. This is [Castravet and Tevelev 2017, Proposition 1.5]. □

Proposition 4.18. *The action of $S_{n+1} \rtimes C_2$ given by permuting the two sets of marked points corresponds to the action of* $\mathrm{Aut}(A_n)$ *on $X(A_n)$.*

Proof. We use the standard presentation of the root system for A_n as $e_i - e_j$ for $1 \leq i < j \leq n+1$ and follow [Batyrev and Blume 2011, Construction 3.6]. The embedding $A_n \hookrightarrow A_{n+1}$ gives the universal curve $X(A_{n+1}) \to X(A_n)$. For $i \in \{1, \dots, n\}$, we take the $(n+1)$ projections $A_{n+1} \to A_n$, whose kernels are generated by $e_i - e_{n+1}$ for $1 \leq i \leq n+1$. These give sections $s_i : X(A_n) \to X(A_{n+1})$. Finally, for the polar sections, we have the dual vector v_{n+2}. The vectors v_{n+2} and $-v_{n+2}$ give toric invariant divisors which are isomorphic to $X(A_n)$ [Batyrev and Blume 2011, Proposition 1.9]. The isomorphisms give the other sections s_0 and s_∞.

The Weyl group is the permutation group of the e_i, and hence of the $e_i - e_{n+2}$. In particular, it permutes the s_i. The outer involution acts on the fan by negation and thus exchanges the cone corresponding to v_{n+2} with the cone corresponding to $-v_{n+2}$. □

Corollary 4.19. *The set* CT_N *is* $\mathrm{Aut}(\Sigma(A_n))$*-stable.*

Proof. This is an immediate corollary of Lemma 4.14 and Proposition 4.18. □

Proposition 4.20. *Each object in the collection* CT_N *is torus-equivariant.*

Proof. Line bundles are always isomorphic to torus-equivariant line bundles, so all objects in $\mathbb{G}_N$ are torus-equivariant. There is a canonical equivariant structure on tensor products and on pullbacks by equivariant morphisms (see [Ballard et al. 2014, §2]); thus each object $G_1 \boxtimes \cdots \boxtimes G_n$ is torus-equivariant for $G_{l_j} \in \mathbb{G}_{N_j}$. Let $i : Z \to X$ be shorthand for some map $i_{N_1,\dots,N_t}$. There is a splitting of tori $T = S \times S'$ where Z is an S-toric variety and S' acts trivially on $i(Z)$. Let $\psi : T \to S$ denote the projection. We have a composition of functors

$$\mathsf{D}^{\mathsf{b}}(\mathrm{Coh}_S\, Z) \to \mathsf{D}^{\mathsf{b}}(\mathrm{Coh}_T\, Z) \to \mathsf{D}^{\mathsf{b}}(\mathrm{Coh}_T\, X),$$

where the first map is the functor Res_ψ [Ballard et al. 2014, §2.9] and the second map is the T-equivariant pushforward [Ballard et al. 2014, §2.5]. This composition reduces to the ordinary pushforward $i_* : \mathsf{D}^{\mathsf{b}}(Z) \to \mathsf{D}^{\mathsf{b}}(X)$ when the equivariant structure is forgotten. We conclude that each object of H_K is torus-equivariant, and the result follows. □

We now prove the main result of this section.

Proposition 4.21. *Let k be a field of characteristic zero and X a form of $X(A_n)$ over k. Then X admits a full exceptional collection of sheaves.*

Proof. Combining Theorem 4.17, Corollary 4.19, and Proposition 4.20 allows us to appeal to Proposition 3.7 and conclude that CT_N descends to an exceptional collection of sheaves on X. □

Remark 4.22. To remove the characteristic zero assumption one needs to extend generation results of [Castravet and Tevelev 2017] to nonzero characteristic. This could conceivably be done by reversing the flow of reasoning in [Castravet and Tevelev 2017], using the fact that we know the collections for V_n in any characteristic. We do not pursue this.

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K-theory of Hermitian Mackey functors, real traces, and assembly

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We define a $\mathbb{Z}/2$-equivariant real algebraic K-theory spectrum $\mathrm{KR}(A)$, for every $\mathbb{Z}/2$-equivariant spectrum A equipped with a compatible multiplicative structure. This construction extends the real algebraic K-theory of Hesselholt and Madsen for discrete rings, and the Hermitian K-theory of Burghelea and Fiedorowicz for simplicial rings. It supports a trace map of $\mathbb{Z}/2$-spectra $\mathrm{tr}:\mathrm{KR}(A)\to\mathrm{THR}(A)$ to the real topological Hochschild homology spectrum, which extends the K-theoretic trace of Bökstedt, Hsiang and Madsen.

We show that the trace provides a splitting of the real K-theory of the spherical group-ring. We use the splitting induced on the geometric fixed points of KR, which we regard as an L-theory of $\mathbb{Z}/2$-equivariant ring spectra, to give a purely homotopy theoretic reformulation of the Novikov conjecture on the homotopy invariance of the higher signatures, in terms of the module structure of the rational L-theory of the "Burnside group-ring".

Introduction

In [Hesselholt and Madsen 2015], a $\mathbb{Z}/2$-equivariant spectrum $\mathrm{KR}(\mathscr{C})$ is constructed from an exact category with duality $\mathscr{C}$, whose underlying spectrum is the K-theory spectrum $\mathrm{K}(\mathscr{C})$ of [Quillen 1973] and whose fixed-points spectrum is the connective Hermitian K-theory spectrum, or Grothendieck–Witt spectrum, $\mathrm{GW}(\mathscr{C})$

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of [Schlichting 2010]. The associated bigraded cohomology theory is an algebraic analogue of Atiyah's KR-theory [1966], in the same way as algebraic K-theory is analogous to topological K-theory. Specified to the category of free modules over a discrete ring with anti-involution R, this construction provides a $\mathbb{Z}/2$-equivariant spectrum $\mathrm{KR}(R)$ whose fixed points are the connective Hermitian K-theory $\mathrm{GW}(R)$ of [Karoubi 1973] (when $\frac{1}{2} \in R$) and [Burghelea and Fiedorowicz 1985]. In the latter paper, the construction of $\mathrm{GW}(R)$ is extended from discrete rings to simplicial rings, and the homotopy type of $\mathrm{GW}(R)$ depends both on the homotopy types of R and of the fixed-points space $R^{\mathbb{Z}/2}$.

In this paper we propose a further extension of the real K-theory functor KR to the category of ring spectra with anti-involution. These are genuine $\mathbb{Z}/2$-equivariant spectra A with a suitably compatible multiplication (see Section 2.2). We model the homotopy theory of genuine $\mathbb{Z}/2$-equivariant spectra using orthogonal spectra with a strict $\mathbb{Z}/2$-action, with the weak equivalences defined from a complete $\mathbb{Z}/2$-universe, as developed in [Schwede 2013; Hill et al. 2016]. Thus, when referring to a genuine $\mathbb{Z}/2$-spectrum, we always mean an orthogonal spectrum with a strict involution. The output of our theory $\mathrm{KR}(A)$ is also a genuine $\mathbb{Z}/2$-equivariant spectrum, whose derived fixed-points spectrum and geometric fixed-points spectrum

$$\mathrm{GW}(A) := \mathrm{KR}(A)^{\mathbb{Z}/2} \quad \text{and} \quad \mathrm{L}^{g}(A) := \Phi^{\mathbb{Z}/2}\,\mathrm{KR}(A) := (\mathrm{KR}(A) \wedge S^{\infty\sigma})^{\mathbb{Z}/2}$$

behave, respectively, as a Hermitian K-theory and L-theory for ring spectra with anti-involution, by Propositions 2.6.3 and 2.6.7 (the model of [Schwede 2013] for the derived fixed-points is recalled in Section 2.2, and $S^{\infty\sigma}$ is the one-point compactification of the infinite countable sum of sign representations σ). They depend on the genuine equivariant homotopy type of A, that is, on the underlying spectrum of A and on the derived fixed-points spectrum $A^{\mathbb{Z}/2}$, and therefore differ from other constructions in the literature (e.g., from the Hermitian K-theory of [Spitzweck 2016] and from the quadratic or symmetric L-theory of [Lurie 2011], which are invariant for the morphisms which are equivalences of underlying nonequivariant spectra). The main application of this paper uses a trace map $\mathrm{tr} : \mathrm{KR} \to \mathrm{THR}$ to real topological Hochschild homology to reformulate the Novikov conjecture in terms of the monoidal structure of L^{g}.

There is an algebraic case of particular interest lying between discrete rings and ring spectra: the ring spectra with anti-involution whose underlying $\mathbb{Z}/2$-spectrum is the Eilenberg–Mac Lane spectrum HM of a $\mathbb{Z}/2$-Mackey functor M. In this case the multiplicative structure on HM specifies to a ring structure on the underlying abelian group $\pi_0 HM$ and a multiplicative action of $\pi_0 HM$ on $\pi_0(HM)^{\mathbb{Z}/2}$, suitably compatible with the restriction and the transfer maps. We call such an object a *Hermitian Mackey functor*. In the case where the restriction map is injective, these are form rings in the sense of [Bak 1981, §1.B]. A class of examples comes from

Tambara functors, where the underlying ring acts on the fixed-points datum via the multiplicative transfer. We start our paper by constructing the Hermitian K-theory of a Hermitian Mackey functor in Section 1, as the group completion of a certain symmetric monoidal category of Hermitian forms Herm_M over M

$$\mathrm{GW}(M) := \Omega B(Bi\,\mathrm{Herm}_M, \oplus).$$

The key idea for the definition of Herm_M is that the fixed-points datum of the Mackey functor specifies a refinement of the notion of "symmetry" used in the classical definition of Hermitian forms over a ring. In Section 2 we extend these ideas to ring spectra and we give the full construction of the functor KR.

The main feature of this real K-theory construction is that it comes equipped with a natural trace map to the real topological Hochschild homology spectrum $\mathrm{THR}(A)$ of [Hesselholt and Madsen 2015]; see also [Dotto 2012; Høgenhaven 2016; Dotto et al. 2017]. The following is in Section 3.2.

Theorem 1. *Let A be a connective ring spectrum with anti-involution. There is a natural transformation of $\mathbb{Z}/2$-spectra*

$$\mathrm{tr} : \mathrm{KR}(A) \to \mathrm{THR}(A)$$

which agrees on nonequivariant infinite loop spaces with the Bökstedt–Hsiang–Madsen trace map $\mathrm{K}(A) \to \mathrm{THH}(A)$ *from [Bökstedt et al. 1993].*

In the case of a discrete ring with anti-involution R the trace provides a map of spectra $\mathrm{GW}(R) \to \mathrm{THR}(R)^{\mathbb{Z}/2}$ which is a refinement of earlier constructions of the Chern character from Hermitian K-theory to dihedral homology appearing in [Cortiñas 1993]. In Section 2.7 we define, for any topological group π and ring spectrum with anti-involution A, an assembly map

$$\mathrm{KR}(A) \wedge B^{\sigma}\pi_+ \to \mathrm{KR}(A[\pi]),$$

where $A[\pi] := A \wedge \pi_+$ is the group-ring with the anti-involution induced by the inversion of π, and $B^{\sigma}\pi$ is a delooping of π with respect to the sign-representation. We define a map

$$Q : \mathrm{KR}(\mathbb{S}[\pi]) \xrightarrow{\mathrm{tr}} \mathrm{THR}(\mathbb{S}[\pi]) \simeq \mathbb{S} \wedge B^{\mathrm{di}}\pi_+ \to \mathbb{S} \wedge B^{\sigma}\pi_+,$$

where $B^{\mathrm{di}}\pi \to B^{\sigma}\pi$ is the projection from the dihedral nerve of π, and the equivalence is from [Høgenhaven 2016]. The following is proved in Section 3.3.

Theorem 2. *The map Q defines a natural retraction in the homotopy category of $\mathbb{Z}/2$-spectra for the restricted assembly map*

$$\mathbb{S} \wedge B^{\sigma}\pi_+ \xrightarrow{\eta\wedge\mathrm{id}} \mathrm{KR}(\mathbb{S}) \wedge B^{\sigma}\pi_+ \to \mathrm{KR}(\mathbb{S}[\pi]),$$

where $\eta : \mathbb{S} \to \mathrm{KR}(\mathbb{S})$ is the unit map. Thus the real K-theory of the spherical group-ring splits off a copy of the equivariant suspension spectrum $\mathbb{S} \wedge B^{\sigma}\pi_{+}$. If π is discrete, the Hermitian K-theory spectrum $\mathrm{GW}(\mathbb{S}[\pi]) := \mathrm{KR}(\mathbb{S}[\pi])^{\mathbb{Z}/2}$ splits off a copy of

$$(\mathbb{S} \wedge B^{\sigma}\pi_{+})^{\mathbb{Z}/2} \simeq \mathbb{S} \wedge ((B\pi \times \mathbb{RP}^{\infty}) \amalg \coprod_{\{[g] \mid g^2=1\}} BZ_{\pi}\langle g \rangle)_{+},$$

where the disjoint union runs through the conjugacy classes of the order two elements of π, and $Z_{\pi}\langle g \rangle$ is the centralizer of g in π.

Nonequivariantly this is the splitting of [Waldhausen et al. 2013], and the cofiber of the restricted assembly map is the Whitehead spectrum. It is unclear at the moment if the equivariant homotopy type of this cofiber directly relates to a geometric object. It was brought to our attention by Kristian Moi and Thomas Nikolaus that the fixed-points spectrum $\mathrm{GW}(\mathbb{S}[\pi])$ might relate to the spectrum $\mathrm{VLA}^{\bullet}$ from [Weiss and Williams 2014].

The geometric application that we propose uses the rationalization of $\mathrm{KR}(\mathbb{S}[\pi])$ to reformulate the Novikov conjecture. This conjecture, from [Novikov 1968], for a discrete group π is equivalent to the injectivity of the assembly map

$$\mathcal{A}_{\mathbb{Z}[\pi]} : \mathrm{L}^{\mathrm{q}}(\mathbb{Z}) \wedge B\pi_{+} \to \mathrm{L}^{\mathrm{q}}(\mathbb{Z}[\pi])$$

on rational homotopy groups, where L^{q} denotes the quadratic L-theory spectrum and $\mathbb{Z}[\pi]$ is the integral group-ring with the anti-involution induced by the inversion in π; see, e.g., [Kreck and Lück 2005]. Relying on Karoubi's periodicity theorem [Karoubi 1980], Burghelea and Fiedorowicz [1985] show that there is a rational decomposition

$$\mathrm{GW}_{*}(R) \otimes \mathbb{Q} \cong (\mathrm{L}^{\mathrm{q}}_{*}(R) \otimes \mathbb{Q}) \oplus (\mathrm{K}_{*}(R) \otimes \mathbb{Q})^{\mathbb{Z}/2}$$

for every discrete ring with anti-involution R, where $\mathrm{K}_{*}(R)$ are the algebraic K-theory groups. In Proposition 2.6.7 we reinterpret this result in terms of the splitting of the isotropy separation sequence of the rational KR spectrum, thus identifying rationally the connective L-theory spectrum with the geometric fixed-points spectrum $\Phi^{\mathbb{Z}/2}\,\mathrm{KR}(R)$; see [Schlichting 2017, Theorem 7.6] for a similar result. This justifies our definition of

$$\mathrm{L}^{\mathrm{g}}(A) := \Phi^{\mathbb{Z}/2}\,\mathrm{KR}(A),$$

the "genuine" L-theory spectrum of the ring spectrum with anti-involution A. The trace then induces a map on the rationalized geometric fixed-points spectra

$$\mathrm{tr} : \mathrm{L}^{\mathrm{q}}(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \simeq \mathrm{L}^{\mathrm{g}}(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \to \Phi^{\mathbb{Z}/2}(\mathrm{THR}(\mathbb{Z}[\pi])) \otimes \mathbb{Q},$$

which one could try to exploit to detect the injectivity of the L-theoretic assembly.

The rational geometric fixed points $\Phi^{\mathbb{Z}/2}\operatorname{THR}(R)\otimes\mathbb{Q}$ have however been computed to be contractible in [Dotto et al. 2017], as long as R is a *discrete* ring. This is in line with the results of [Cortiñas 1993], where the Chern character to dihedral homology factors through algebraic K-theory via the forgetful map, and therefore vanishes on the L-theory summand.

The rational geometric fixed points $\Phi^{\mathbb{Z}/2}\operatorname{THR}(A)\otimes\mathbb{Q}$ are generally nontrivial when the input A is not the Eilenberg–Mac Lane spectrum of a discrete ring. The starting point of our analysis is to replace the ring of integers with the Burnside Mackey functor, much in the same way one replaces the integers with the sphere spectrum in the proof of the K-theoretic Novikov conjecture of [Bökstedt et al. 1989]. We define a Hermitian Mackey functor

$$\mathbb{A}_{1/2}[\pi]:=\underline{\pi}_0^{\mathbb{Z}/2}(\mathbb{S}\wedge\pi_+)\big[\tfrac{1}{2}\big],$$

the "Burnside group-ring" of a discrete group π (see Definition 1.1.7) with 2 inverted. There is a restriction map $d:\mathbb{A}_{1/2}[\pi]\to\underline{\mathbb{Z}}_{1/2}[\pi]$ coming from the augmentation of the Burnside ring, where $\underline{\mathbb{Z}}_{1/2}[\pi]$ is the Mackey functor associated to the integral group-ring $\mathbb{Z}[\pi]$ with 2 inverted. The following is proved in Section 4.

Theorem 3. *Let π be a discrete group. There is a lift $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$ of the L-theoretic connective assembly map of the integral group-ring*

$$\begin{tikzcd}
 & \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q} \arrow[d, "d"] \\
\mathbb{Q}[\beta]\otimes H_*(B\pi;\mathbb{Q}) \arrow[ur, "\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}"] \arrow[r, "\mathcal{A}_{\mathbb{Z}[\pi]}"'] & \mathrm{L}^{\mathrm{q}}_{*\geq 0}(\mathbb{Z}[\pi])\otimes\mathbb{Q}\cong\mathrm{L}^{\mathrm{g}}_*(\underline{\mathbb{Z}}_{1/2}[\pi])\otimes\mathbb{Q}
\end{tikzcd}$$

For every polynomial $\underline{x}\in\mathbb{Q}[\beta]\otimes H_*(B\pi;\mathbb{Q})$ with nonzero constant term, we have $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})\neq 0$. It follows that the Novikov conjecture holds for π if and only if the image of $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$ intersects the kernel of d trivially.

We prove this theorem by detecting $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})$ using the trace map. We define a map

$$T:\mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q} \xrightarrow{\mathrm{tr}}\Phi^{\mathbb{Z}/2}\operatorname{THR}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q}\cong H_*((B^{\mathrm{di}}\pi)^{\mathbb{Z}/2};\mathbb{Q})\xrightarrow{p}H_*(B\pi;\mathbb{Q}),$$

where $B^{\mathrm{di}}\pi$ is the dihedral nerve of π, and $p:(B^{\mathrm{di}}\pi)^{\mathbb{Z}/2}\to B\pi$ is a certain projection map. The image of the constant term $T\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(1\otimes x_n)$ of a polynomial

$$\underline{x}=1\otimes x_n+\beta\otimes x_{n-4}+\cdots+\beta^k\otimes x_{n-4k}\in(\mathbb{Q}[\beta]\otimes H_*(B\pi;\mathbb{Q}))_n$$

of total degree n, with $x_n\neq 0$, is nonzero essentially by Theorem 2. A naturality argument then shows that $T\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$ vanishes on the positive powers of β, and therefore that $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})$ is not zero. By the periodicity of $\mathrm{L}^{\mathrm{q}}_*(\mathbb{Z}[\pi])$, if $\mathcal{A}_{\mathbb{Z}[\pi]}$ does not annihilate the polynomials with nonzero constant term, it must be injective (see Remark 4.1). Thus the Novikov conjecture holds precisely when d does not kill $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})$.

We further reduce the Novikov conjecture to an algebraic property of $\mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])$ as an $\mathrm{L}^{\mathrm{g}}_0(\mathbb{A}_{1/2})$-module. The rank map d above admits a natural splitting s_π which includes $\mathrm{L}^{\mathrm{q}}_{*\geq 0}(\mathbb{Z}[\pi])\otimes\mathbb{Q}$ as a summand of $\mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q}$. In particular, the unit of $\mathrm{L}^{\mathrm{q}}_{*\geq 0}(\mathbb{Z})\otimes\mathbb{Q}\cong\mathbb{Q}[\beta]$ defines an idempotent element a in the ring $\mathrm{L}^{\mathrm{g}}_0(\mathbb{A}_{1/2})\otimes\mathbb{Q}$.

Corollary 4. *There is an idempotent $a\in\mathrm{L}^{\mathrm{g}}_0(\mathbb{A}_{1/2})\otimes\mathbb{Q}$ such that every element $\underline{x}\in\mathbb{Q}[\beta]\otimes H_*(B\pi;\mathbb{Q})$ satisfies the identity*

$$s_\pi\mathcal{A}_{\mathbb{Z}[\pi]}(\underline{x})=a\cdot\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})\in\mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q},$$

where s_π is injective and $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})$ is nonzero when $\underline{x}$ has nonzero constant term. It follows that the Novikov conjecture holds for π if and only if the multiplication map

$$a\cdot(-):\mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])\to\mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])$$

is injective on the image of $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$.

The element a is sent to zero by the trace

$$\mathrm{tr}:\mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])\to\Phi^{\mathbb{Z}/2}\,\mathrm{THR}_*(\mathbb{A}_{1/2}[\pi]),$$

and therefore $a\cdot\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})$ cannot be detected by the trace map to THR. In future work we hope to be able to show that

$$\overline{\mathrm{tr}}(a\cdot\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x}))=\overline{\mathrm{tr}}(a)\cdot\mathcal{A}_{\mathrm{TCR}}(\overline{\mathrm{tr}}(\underline{x}))$$

is nonzero in $\Phi^{\mathbb{Z}/2}\,\mathrm{TCR}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q}$ by direct calculation, where $\overline{\mathrm{tr}}:\mathrm{KR}\to\mathrm{TCR}$ is a lift of the trace map to the real topological cyclic homology spectrum, and $\mathcal{A}_{\mathrm{TCR}}$ is the corresponding assembly map.

A brief outline of the paper follows. In Section 1 we define Hermitian Mackey functors, we construct some examples, and we define their Hermitian K-theory. In Section 2 we construct the real K-theory of a ring spectrum with anti-involution. We prove that its fixed points recover the connective Hermitian K-theory of simplicial rings of [Burghelea and Fiedorowicz 1985] and the Hermitian K-theory of Hermitian Mackey functors of Section 1, and we prove that its geometric fixed points are rationally equivalent to the connective L-theory of discrete rings. Under these equivalences we recover the L-theoretic and Hermitian assembly maps from the KR assembly. In Section 3 we recollect some of the basics on real topological Hochschild homology. We then construct the real trace map and the splitting of the real K-theory of the spherical group-ring. Finally in Section 4 we relate the trace map to the Novikov conjecture.

Notation and conventions. A space always means a compactly generated weak Hausdorff topological space. These form a category, which we denote by Top. We let Top^G be its category of G-objects, where G is a finite group, usually $G = \mathbb{Z}/2$. An equivalence of G-spaces is a continuous G-equivariant map which induces a weak equivalence on H-fixed-points, for every subgroup H of G.

By a spectrum, we always mean an orthogonal spectrum. A G-spectrum is a G-object in the category of orthogonal spectra, and an equivalence of G-spectra is a stable equivalence with respect to a complete G-universe, as in [Schwede 2013].

1. Hermitian Mackey functors and their *K*-theory

1.1. ***Hermitian Mackey functors.*** The standard input of Hermitian K-theory is a ring R with an anti-involution $w : R^{\mathrm{op}} \to R$, or in other words an abelian group R with a $\mathbb{Z}/2$-action and a ring structure

$$R \otimes_{\mathbb{Z}} R \to R$$

which is equivariant with respect to the $\mathbb{Z}/2$-action on the tensor product that swaps the two factors and acts on both variables. In equivariant homotopy theory abelian groups with $\mathbb{Z}/2$-actions are replaced by the more refined notion of $\mathbb{Z}/2$-Mackey functors. In what follows, we define a suitable multiplicative structure on a Mackey functor which extends the notion of a ring with anti-involution.

We recall that a $\mathbb{Z}/2$-Mackey functor L consists of two abelian groups $L(\mathbb{Z}/2)$ and $L(*)$, a $\mathbb{Z}/2$-action w on $L(\mathbb{Z}/2)$, and $\mathbb{Z}/2$-equivariant maps

$$R : L(*) \to L(\mathbb{Z}/2), \qquad T : L(\mathbb{Z}/2) \to L(*)$$

(with respect to the trivial action on $L(*)$), respectively called the restriction and the transfer, subject to the relation

$$RT(a) = a + w(a)$$

for every $a \in L(\mathbb{Z}/2)$.

Definition 1.1.1. A Hermitian Mackey functor is a $\mathbb{Z}/2$-Mackey functor L, together with a multiplication on $L(\mathbb{Z}/2)$ that makes it into a ring, and a multiplicative left action of $L(\mathbb{Z}/2)$ on the abelian group $L(*)$ which satisfy the following conditions:

(i) $w(aa') = w(a')w(a)$ for all $a, a' \in L(\mathbb{Z}/2)$, and $w(1) = 1$,

(ii) $R(a \cdot b) = aR(b)w(a)$ for all $a \in L(\mathbb{Z}/2)$ and $b \in L(*)$,

(iii) $a \cdot T(c) = T(acw(a))$ for all $a, c \in L(\mathbb{Z}/2)$,

(iv) $(a+a') \cdot b = a \cdot b + a' \cdot b + T(aR(b)w(a'))$ for all $a, a' \in L(\mathbb{Z}/2)$ and $b \in L(*)$, and $0 \cdot b = 0$.

Example 1.1.2. Let R be a ring with anti-involution $w : R^{\mathrm{op}} \to R$. The Mackey functor $\underline{R}$ associated to R has values $\underline{R}(\mathbb{Z}/2) = R$ and $\underline{R}(*) = R^{\mathbb{Z}/2}$, the abelian subgroup of fixed points. The restriction map is the inclusion of fixed points $R^{\mathbb{Z}/2} \to R$, and the transfer is $T(a) = a + w(a)$. The multiplication on R defines an action of R on $R^{\mathbb{Z}/2}$ by

$$a \cdot b = abw(a)$$

for $a \in R$ and $b \in R^{\mathbb{Z}/2}$. This gives $\underline{R}$ the structure of a Hermitian Mackey functor.

Example 1.1.3. Let $\mathbb{A}$ be the Burnside $\mathbb{Z}/2$-Mackey functor. The abelian group $\mathbb{A}(\mathbb{Z}/2)$ is the group completion of the monoid of isomorphism classes of finite sets, and it has the trivial involution. The abelian group $\mathbb{A}(*)$ is the group completion of the monoid of isomorphism classes of finite $\mathbb{Z}/2$-sets. The restriction forgets the $\mathbb{Z}/2$-action, and the transfer sends a set A to the free $\mathbb{Z}/2$-set $A \times \mathbb{Z}/2$. The underlying abelian group $\mathbb{A}(\mathbb{Z}/2)$ has a multiplication induced by the cartesian product, and it acts on $\mathbb{A}(*)$ by

$$A \cdot B = \Big(\prod_{\mathbb{Z}/2} A\Big) \times B.$$

Explicitly, $\mathbb{A}(\mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}$ as a ring, $\mathbb{A}(*)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ with generators the trivial $\mathbb{Z}/2$-set with one element and the free $\mathbb{Z}/2$-set $\mathbb{Z}/2$. The restriction is the identity on the first summand and multiplication by 2 on the second summand, and the transfer sends the generator of $\mathbb{Z}$ to the generator of the second $\mathbb{Z}$-summand. The underlying ring $\mathbb{Z}$ then acts on $\mathbb{Z} \oplus \mathbb{Z}$ by

$$a \cdot (b, c) = \big(ab, \tfrac{1}{2}ba(a-1) + a^2 c\big).$$

The Hermitian structure on the Burnside Mackey functor is a special case of the following construction. If the multiplication of a ring R is commutative, then an anti-involution on R is simply an action of $\mathbb{Z}/2$ by ring maps. The Mackey version of a commutative ring is a Tambara functor, and we show that there is indeed a forgetful functor from $\mathbb{Z}/2$-Tambara functors to Hermitian Mackey functors. We recall that a $\mathbb{Z}/2$-Tambara functor is a Mackey functor where both $L(\mathbb{Z}/2)$ and $L(*)$ are commutative rings, and with an additional equivariant multiplicative transfer

$$N : L(\mathbb{Z}/2) \to L(*),$$

called the norm, which satisfies the properties

(i) $T(a)b = T(aR(b))$ for all $a \in L(\mathbb{Z}/2)$ and $b \in L(*)$,

(ii) $RN(a) = aw(a)$ for all $a \in L(\mathbb{Z}/2)$,

(iii) $N(a+a') = N(a) + N(a') + T(aw(a'))$ for all $a, a' \in L(\mathbb{Z}/2)$, and $N(0) = 0$;

see [Tambara 1993; Strickland 2012].

Example 1.1.4. A Tambara functor L has the structure of a Hermitian Mackey functor by defining the $L(\mathbb{Z}/2)$-action on $L(*)$ as

$$a \cdot b = N(a)b,$$

where the right-hand product is the multiplication in $L(*)$, and then forgetting the multiplication of $L(*)$ and the norm.

Let us verify the axioms of a Hermitian Mackey functor. The first axiom is satisfied because the multiplication is commutative and equivariant. The second axiom is

$$R(a \cdot b) = R(N(a)b) = R(N(a))R(b) = aw(a)R(b) = aR(b)w(a)$$

and the third is

$$a \cdot T(c) = N(a)T(c) = T(R(N(a))c) = T(aw(a)c) = T(acw(a)).$$

The last axiom is clear from the third condition of a Tambara functor.

We conclude the section by extending to Hermitian Mackey functors two standard constructions of rings with anti-involution: the matrix ring and the group-ring.

If R is a ring with anti-involution and n is a positive integer, the ring $M_n(R)$ of $n \times n$-matrices has a natural anti-involution defined by conjugate transposition $w(A)_{ij} := w(A_{ji})$. A fixed point in $M_n(R)$ is a matrix whose diagonal entries belong to $R^{\mathbb{Z}/2}$, and where the entries $A_{i>j}$ are determined by the entries $A_{i<j}$, by $A_{i>j} = w(A_{j<i})$. Inspired by this example, we give the following definition.

Definition 1.1.5. Let L be a Hermitian Mackey functor. The Hermitian Mackey functor $M_n(L)$ of $n \times n$-matrices in L is defined by the abelian groups

$$\begin{aligned} M_n(L)(\mathbb{Z}/2) &= M_n(L(\mathbb{Z}/2)), \\ M_n(L)(*) &= \Big(\bigoplus_{1\leq i<j\leq n} L(\mathbb{Z}/2)\Big) \oplus \Big(\bigoplus_{1\leq i=j\leq n} L(*)\Big). \end{aligned}$$

The anti-involution on $M_n(L)(\mathbb{Z}/2)$ is the anti-involution of $L(\mathbb{Z}/2)$ applied entrywise followed by matrix transposition. The restriction of an element B of $M_n(L)(*)$ has entries

$$R(B)_{ij} = \begin{cases} B_{ij} & \text{if } i < j, \\ w(B_{ji}) & \text{if } i > j, \\ R(B_{ii}) & \text{if } i = j. \end{cases}$$

The transfer of an $n \times n$-matrix A with coefficients in $L(\mathbb{Z}/2)$ has components

$$T(A)_{ij} = \begin{cases} A_{ij} + w(A_{ji}) & \text{if } i < j, \\ T(A_{ii}) & \text{if } i = j. \end{cases}$$

The multiplication on $M_n(L)(\mathbb{Z}/2)$ is the standard matrix multiplication. The action of $M_n(L)(\mathbb{Z}/2)$ on $M_n(L)(*)$ is defined by

$$(A\cdot B)_{ij}=\begin{cases}(AR(B)w(A))_{ij} & \text{if } i<j,\\ T\Big(\sum_{1\le k<l\le n} A_{ik}B_{kl}w(A_{il})\Big)+\sum_{1\le k\le n} A_{ik}\cdot B_{kk} & \text{if } i=j,\end{cases}$$

that is, by the conjugation action on the off-diagonal entries, and through the Hermitian structure on the diagonal entries.

Lemma 1.1.6. *The object $M_n(L)$ defined above is a Hermitian Mackey functor, and if R is a ring with anti-involution then $\underline{M_n(R)}\cong M_n(\underline{R})$.*

Proof. It is clearly a well-defined Mackey functor, since

$$RT(A)_{ij}=\begin{cases}T(A)_{ij}=A_{ij}+w(A_{ji}) & \text{if } i<j,\\ w(T(A)_{ji})=w(A_{ji}+w(A_{ij}))=w(A_{ji})+A_{ij} & \text{if } i>j,\\ R(T(A)_{ii})=R(T(A_{ii}))=A_{ii}+w(A_{ii}) & \text{if } i=j\end{cases}$$

is equal to $(A+w(A))_{ij}$. Let us verify that the formula above indeed defines a monoid action. This is immediate for the components $i<j$. For the diagonal components let us first verify that the identity matrix I acts trivially. This is because

$$(I\cdot B)_{ii}=T\Big(\sum_{1\le k<l\le n} I_{ik}B_{kl}w(I_{il})\Big)+\sum_{1\le k\le n} I_{ik}\cdot B_{kk}=0+B_{kk}.$$

In order to show associativity we calculate the diagonal components of $(AC)\cdot B$ for matrices $A, C\in M_n(L(\mathbb{Z}/2))$ and $B\in M_n(L)(*)$. These are

$$\begin{aligned}((AC)\cdot B)_{ii}&=T\Big(\sum_{p<q}(AC)_{ip}B_{pq}w((AC)_{iq})\Big)+\sum_t (AC)_{it}\cdot B_{tt}\\ &=T\Big(\sum_{p<q}\sum_{k,l}A_{ik}C_{kp}B_{pq}w(A_{il}C_{lq})\Big)+\sum_t\Big(\sum_u A_{iu}C_{ut}\Big)\cdot B_{tt}.\end{aligned}$$

An easy induction argument on the fourth axiom of a Hermitian functor shows that

$$\Big(\sum_{1\le h\le n} a_h\Big)\cdot b=\sum_{1\le h\le n}(a_h\cdot b)+\sum_{1\le k<l\le n} T(a_kR(b)w(a_l)),$$

and the expression above becomes

$$\begin{aligned}((AC)\cdot B)_{ii}=T\Big(\sum_{p<q}\sum_{k,l}A_{ik}C_{kp}B_{pq}w(A_{il}C_{lq})\Big)+\sum_t\sum_u(A_{iu}C_{ut}\cdot B_{tt})\\ +\sum_t\sum_{k<l}T(A_{ik}C_{kt}R(B_{tt})w(A_{il}C_{lt})).\end{aligned}$$

On the other hand the diagonal components of $(A\cdot(C\cdot B))_{ii}$ are

$$
\begin{aligned}
T\Big(\sum_{k<l} A_{ik}(C\cdot B)_{kl}w(A_{il})\Big) &+ \sum_{u} A_{iu}\cdot(C\cdot B)_{uu}\\
&= T\Big(\sum_{k<l}\sum_{p,q} A_{ik}C_{kp}R(B)_{pq}w(A_{il}C_{lq})\Big)\\
&\qquad + \sum_{u} A_{iu}\cdot\Big(T\Big(\sum_{p<q} C_{up}B_{pq}w(C_{uq})\Big) + \sum_{t} C_{ut}\cdot B_{tt}\Big)\\
&= T\Big(\sum_{k<l}\sum_{p,q} A_{ik}C_{kp}R(B)_{pq}w(A_{il}C_{lq})\Big)\\
&\qquad + T\Big(\sum_{u}\sum_{p<q} A_{iu}C_{up}B_{pq}w(A_{iu}C_{uq})\Big) + \sum_{u}\sum_{t} A_{iu}\cdot C_{ut}\cdot B_{tt}\\
&= T\Big(\sum_{k<l}\sum_{p<q} A_{ik}C_{kp}(B_{pq}+w(B_{pq}))w(A_{il}C_{lq})\Big)\\
&\qquad + T\Big(\sum_{k<l}\sum_{t} A_{ik}C_{kt}R(B_{tt})w(A_{il}C_{lt})\Big)\\
&\qquad\qquad + T\Big(\sum_{u}\sum_{p<q} A_{iu}C_{up}B_{pq}w(A_{iu}C_{uq})\Big) + \sum_{u}\sum_{t} A_{iu}\cdot C_{ut}\cdot B_{tt}.
\end{aligned}
$$

We see that the second and the fourth term of this expression cancel with the third and second term, respectively, of the expression of $(AC)\cdot B$. Finally, by using that the transfer is equivariant we rewrite the sum of the first and third terms as

$$
\begin{aligned}
T\Big(\sum_{k<l}\sum_{p<q} &A_{ik}C_{kp}(B_{pq}+w(B_{pq}))w(A_{il}C_{lq})\Big) + T\Big(\sum_{u}\sum_{p<q} A_{iu}C_{up}B_{pq}w(A_{iu}C_{uq})\Big)\\
&= T\Big(\sum_{k<l}\sum_{p<q} A_{ik}C_{kp}B_{pq}w(A_{il}C_{lq})\Big) + T\Big(\sum_{k<l}\sum_{p<q} A_{il}C_{lq}B_{pq}w(A_{ik}C_{kp})\Big)\\
&\qquad + T\Big(\sum_{u}\sum_{p<q} A_{iu}C_{up}B_{pq}w(A_{iu}C_{uq})\Big)\\
&= T\Big(\sum_{k<l}\sum_{p<q} A_{ik}C_{kp}B_{pq}w(A_{il}C_{lq})\Big) + T\Big(\sum_{k>l}\sum_{p<q} A_{ik}C_{kq}B_{pq}w(A_{il}C_{lp})\Big)\\
&\qquad + T\Big(\sum_{u}\sum_{p<q} A_{iu}C_{up}B_{pq}w(A_{iu}C_{uq})\Big)\\
&= T\Big(\sum_{k,l}\sum_{p<q} A_{ik}C_{kp}B_{pq}w(A_{il}C_{lq})\Big).
\end{aligned}
$$

Let us now verify the other axioms of a Hermitian Mackey functor. The compatibility between the action and the restriction holds since

$$
\begin{aligned}
R(A\cdot B)_{ij} \\
&= \begin{cases} (A\cdot B)_{ij} & \text{if } i<j,\\ w((A\cdot B)_{ji}) = w((AR(B)w(A))_{ji}) & \text{if } i>j,\\ R((A\cdot B)_{ii}) = \sum_{1\le k<l\le n} RT(A_{ik}B_{kl}w(A_{il})) + \sum_{1\le k\le n} R(A_{ik}\cdot B_{kk}) & \text{if } i=j \end{cases}\\
&= \begin{cases} (AR(B)w(A))_{ij} & \text{if } i\neq j,\\ \displaystyle\sum_{1\le k<l\le n}(A_{ik}B_{kl}w(A_{il}) + A_{il}w(B_{kl})w(A_{ik})) \\ \qquad\qquad + \displaystyle\sum_{1\le k\le n} A_{ik}R(B_{kk})w(A_{ik}) & \text{if } i=j, \end{cases}
\end{aligned}
$$

which is equal to $(AR(B)w(A))_{ij}$. The compatibility between the action and the transfer is

$$
\begin{aligned}
(A\cdot T(C))_{ij} \\
&= \begin{cases} (AR(T(C))w(A))_{ij} = (A(C+w(C))w(A))_{ij} & \text{if } i<j,\\ T\Big(\sum_{1\le k<l\le n} A_{ik}T(C)_{kl}w(A_{il})\Big) + \sum_{1\le k\le n} A_{ik}\cdot T(C)_{kk} & \text{if } i=j \end{cases}\\
&= \begin{cases} (ACw(A)+w(ACw(A)))_{ij} = T(ACw(A))_{ij} & \text{if } i<j,\\ T\Big(\sum_{1\le k<l\le n} A_{ik}(C_{kl}+w(C_{lk}))w(A_{il})\Big) + \sum_{1\le k\le n} A_{ik}\cdot T(C_{kk}) & \text{if } i=j \end{cases}\\
&= \begin{cases} T(ACw(A))_{ij} & \text{if } i<j,\\ T\Big(\displaystyle\sum_{1\le k<l\le n} A_{ik}C_{kl}w(A_{il}) + A_{ik}w(C_{lk})w(A_{il})\Big) \\ \qquad\qquad + \displaystyle\sum_{1\le k\le n} T(A_{ik}C_{kk}w(A_{ik})) & \text{if } i=j, \end{cases}
\end{aligned}
$$

and this is by definition $T(ACw(A))_{ij}$. The distributivity of the action over the sum in $M_n(L)(*)$ is easy to verify for the components $i<j$. In the diagonal components we have that

$$
\begin{aligned}
((A+A')\cdot B)_{ii} &= T\Big(\sum_{1\le k<l\le n}(A+A')_{ik}B_{kl}w((A+A')_{il})\Big) + \sum_{1\le k\le n}(A+A')_{ik}\cdot B_{kk}\\
&= T\Big(\sum_{1\le k<l\le n} A_{ik}B_{kl}w(A_{il})\Big) + T\Big(\sum_{1\le k<l\le n} A'_{ik}B_{kl}w(A'_{il})\Big)\\
&\quad + T\Big(\sum_{1\le k<l\le n} A_{ik}B_{kl}w(A'_{il})\Big) + T\Big(\sum_{1\le k<l\le n} A'_{ik}B_{kl}w(A_{il})\Big)\\
&\quad + \sum_{1\le k\le n} A_{ik}\cdot B_{kk} + \sum_{1\le k\le n} A'_{ik}\cdot B_{kk} + \sum_{1\le k\le n} T(A_{ik}R(B_{kk})w(A'_{ik}))
\end{aligned}
$$

$$= (A \cdot B)_{ii} + (A' \cdot B)_{ii} + T\Big(\sum_{1 \le k < l \le n} A_{ik} B_{kl} w(A'_{il})\Big) \\ + T\Big(\sum_{1 \le k < l \le n} A'_{ik} B_{kl} w(A_{il})\Big) + \sum_{1 \le k \le n} T(A_{ik} R(B_{kk}) w(A'_{ik})).$$

By using that the transfer is equivariant and by reindexing the sum we rewrite the fourth summand as

$$T\Big(\sum_{1 \le k < l \le n} A'_{ik} B_{kl} w(A_{il})\Big) = T\Big(\sum_{1 \le k > l \le n} A'_{il} B_{lk} w(A_{ik})\Big) \\ = T\Big(\sum_{1 \le k > l \le n} A_{ik} w(B_{lk}) w(A'_{il})\Big).$$

Thus the expression above is equal to

$$(A \cdot B)_{ii} + (A' \cdot B)_{ii} + T\Big(\sum_{1 \le k, l \le n} A_{ik} R(B)_{kl} w(A'_{il})\Big) \\ = (A \cdot B)_{ii} + (A' \cdot B)_{ii} + T(AR(B)w(A'))_{ii}.$$

Finally, by inspection, we see that $\underline{M_n(R)} \cong M_n(\underline{R})$. □

Let π be a discrete group with an anti-involution $\tau : \pi^{\mathrm{op}} \to \pi$ (for example inversion). If R is a ring with anti-involution, the group-ring $R[\pi] = \bigoplus_\pi R$ inherits an anti-involution

$$w\Big(\sum_{g \in \pi} a_g g\Big) = \sum_{g \in \pi} w(a_{\tau g}) g.$$

A choice of section s of the quotient map $\pi \to \pi/(\mathbb{Z}/2)$ determines an isomorphism

$$(R[\pi])^{\mathbb{Z}/2} \cong R^{\mathbb{Z}/2}[\pi^{\mathbb{Z}/2}] \oplus R[\pi^{\mathrm{free}}/(\mathbb{Z}/2)],$$

where $\pi^{\mathrm{free}} = \pi - \pi^{\mathbb{Z}/2}$ is the subset of π on which $\mathbb{Z}/2$ acts freely. It is defined on the $R^{\mathbb{Z}/2}[\pi^{\mathbb{Z}/2}]$ summand by the inclusion, and on the second summand by sending cx to $cs(x) + w(c)\tau(s(x))$.

Definition 1.1.7 (group-Mackey functor). Let L be a Hermitian Mackey functor and π a discrete group with anti-involution $\tau : \pi^{\mathrm{op}} \to \pi$. The associated group-Mackey functor is the Hermitian Mackey functor $L[\pi]$ defined by the abelian groups

$$L[\pi](\mathbb{Z}/2) = L(\mathbb{Z}/2)[\pi], \qquad L[\pi](*) = L(*)[\pi^{\mathbb{Z}/2}] \oplus L(\mathbb{Z}/2)[\pi^{\mathrm{free}}/\mathbb{Z}/2].$$

The anti-involution on $L(\mathbb{Z}/2)[\pi]$ is the standard anti-involution on the group-ring. The restriction is induced by the restriction map $R : L(*) \to L(\mathbb{Z}/2)$ and by the inclusion of the fixed points of π on the first summand, and by the map

$$R(cx) = cs(x) + w(c)\tau(s(x))$$

on the second summand. The transfer is defined by

$$T(ag)=\begin{cases} T(a)g & \text{if } g\in\pi^{\mathbb{Z}/2},\\ a[g] & \text{if } g\in\pi^{\text{free}} \text{ and } g=s[g],\\ w(a)[g] & \text{if } g\in\pi^{\text{free}} \text{ and } g=\tau s[g].\end{cases}$$

The multiplication on $L[\pi](\mathbb{Z}/2)$ is that of the group-ring $L(\mathbb{Z}/2)[\pi]$. The action of a generator $ag\in L[\pi](\mathbb{Z}/2)$ on $L[\pi](*)$ is extended linearly from

$$ag\cdot bh=(a\cdot b)(gh\tau(g))$$

for $bh\in L(*)[\pi^{\mathbb{Z}/2}]$, and

$$ag\cdot cx=\begin{cases} acw(a)[gs(x)\tau(g)] & \text{if } gs(x)\tau(g)=s[gs(x)\tau(g)],\\ aw(c)w(a)[gs(x)\tau(g)] & \text{if } gs(x)\tau(g)=\tau s[gs(x)\tau(g)],\end{cases}$$

for $cx\in L(\mathbb{Z}/2)[\pi^{\text{free}}/(\mathbb{Z}/2)]$. It is then extended to the whole group-ring $L[\pi](\mathbb{Z}/2)$ by enforcing condition (iv) of Definition 1.1.1, namely by defining

$$\left(\sum_{g\in\pi} a_g g\right)\cdot\xi=\sum_{g\in\pi}(a_g g\cdot\xi)+\sum_{g<g'} T(a_g gR(\xi)w(a_{g'})\tau(g'))$$

for some choice of total order on the finite subset of π on which $a_g\neq 0$.

When $L=\mathbb{A}$ is the Burnside Mackey functor, we call $\mathbb{A}[\pi]$ the Burnside group-ring.

Remark 1.1.8. The definition of $L[\pi]$ depends on the choice of section up to isomorphism, and it is therefore not strictly functorial in π. However, it is independent of such choice for the Mackey functors that have trivial action w. This is the case, for example, for the Burnside Mackey functor $\mathbb{A}$. This construction is always functorial in L.

Lemma 1.1.9. *The functor $L[\pi]$ is a well-defined Hermitian Mackey functor, and if R is a ring with anti-involution $\underline{R[\pi]}\cong\underline{R}[\pi]$.*

Proof. We see that $L[\pi]$ is a Mackey functor, since

$$RT(ag)=\begin{cases} R(T(a)g)=(RT(a))g=(a+w(a))g & \text{if } g\in\pi^{\mathbb{Z}/2},\\ R(a[g])=ag+w(a)\tau(g) & \text{if } g\in\pi^{\text{free}} \text{ and } g=s[g],\\ R(w(a))[g]=w(a)\tau(g)+ag & \text{if } g\in\pi^{\text{free}} \text{ and } g=\tau s[g]\end{cases}$$

is equal to $ag+w(a)\tau(g)$. A calculation analogous to the one of Lemma 1.1.6 shows that the action of $L[\pi](\mathbb{Z}/2)$ on $L[\pi](*)$ is indeed associative (this also follows from Remark 2.7.1 if L is a Tambara functor, since it can be realized as the $\underline{\pi}_0$ of a commutative $\mathbb{Z}/2$-equivariant ring spectrum). The compatibility between the action and the restriction is

$$\begin{aligned} R(ag \cdot bh) &= R\big((a \cdot b)(gh\tau(g))\big) = R(a \cdot b)(gh\tau(g)) \\ &= aR(b)w(a)(gh\tau(g)) = (ag)R(bh)w(a)\tau(g) \end{aligned}$$

for the action on the first summand. On the second summand, this is

$$\begin{aligned} R(ag \cdot cx) &= \begin{cases} R(acw(a)[gs(x)\tau(g)]) & \text{if } gs(x)\tau(g) = s[gs(x)\tau(g)], \\ R(aw(c)w(a)[gs(x)\tau(g)]) & \text{if } gs(x)\tau(g) = \tau s[gs(x)\tau(g)] \end{cases} \\ &= \begin{cases} acw(a)gs(x)\tau(g) \\ \quad + w(acw(a))\tau(gs(x)\tau(g)) & \text{if } gs(x)\tau(g) = s[gs(x)\tau(g)], \\ aw(c)w(a)\tau(gs(x)\tau(g)) \\ \quad + w(aw(c)w(a))gs(x)\tau(g) & \text{if } gs(x)\tau(g) = \tau s[gs(x)\tau(g)], \end{cases} \end{aligned}$$

which is equal to $(ag)R(cx)(w(a)\tau(g))$. Let us verify the compatibility between the action and the transfer. We have that

$$\begin{aligned} ag \cdot T(bh) &= \begin{cases} (ag) \cdot (T(b)h) \\ \quad = T(abw(a))gh\tau(g) & \text{if } h \in \pi^{\mathbb{Z}/2}, \\ (ag) \cdot (b[h]) & \text{if } h \in \pi^{\text{free}}, h = s[h], \\ \quad = abw(a)[gh\tau(g)] & \quad gh\tau(g) = s[gh\tau(g)], \\ (ag) \cdot (b[h]) & \text{if } h \in \pi^{\text{free}}, h = s[h], \\ \quad = aw(b)w(a)[gh\tau(g)] & \quad gh\tau(g) = \tau s[gh\tau(g)], \\ (ag) \cdot (w(b)[h]) & \text{if } h \in \pi^{\text{free}}, h = \tau s[h], \\ \quad = aw(b)w(a)[g\tau(h)\tau(g)] & \quad g\tau(h)\tau(g) = s[g\tau(h)\tau(g)], \\ (ag) \cdot (w(b)[h]) & \text{if } h \in \pi^{\text{free}}, h = \tau s[h], \\ \quad = aw^2(b)w(a)[g\tau(h)\tau(g)] & \quad g\tau(h)\tau(g) = \tau s[g\tau(h)\tau(g)] \end{cases} \\ &= \begin{cases} T(abw(a))gh\tau(g) & \text{if } gh\tau(g) \in \pi^{\mathbb{Z}/2}, \\ abw(a)[gh\tau(g)] & \text{if } h \in \pi^{\text{free}} \text{ and } gh\tau(g) = s[gh\tau(g)], \\ aw(b)w(a)[gh\tau(g)] & \text{if } h \in \pi^{\text{free}} \text{ and } gh\tau(g) = \tau s[gh\tau(g)] \end{cases} \\ &= T(abw(a)gh\tau(g)). \end{aligned}$$

The last axiom is satisfied by construction. By inspection we see $\underline{R[\pi]} \cong \underline{R}[\pi]$. $\square$

1.2. ***The Hermitian K-theory of a Hermitian Mackey functor.*** Let L be a Hermitian Mackey functor. We use the Hermitian Mackey functors of matrices constructed in Definition 1.1.5 to define a symmetric monoidal category of Hermitian forms whose group completion is the Hermitian K-theory of L.

Definition 1.2.1. Let L be a Hermitian Mackey functor. An n-dimensional Hermitian form on L is an element of $M_n(L)(*)$ which restricts to an element of $\mathrm{GL}_n(L(\mathbb{Z}/2))$ under the restriction map

$$R : M_n(L)(*) \to M_n(L)(\mathbb{Z}/2) = M_n(L(\mathbb{Z}/2)).$$

We write $\mathrm{GL}_n(L)(*)$ for the set of n-dimensional Hermitian forms. A morphism $B \to B'$ of Hermitian forms is a matrix λ in $M_n(L(\mathbb{Z}/2))$ which satisfies

$$B = w(\lambda) \cdot B',$$

where the operation is the action of $M_n(L)(\mathbb{Z}/2)$ on $M_n(L)(*)$. The multiplication of matrices defines a category of Hermitian forms, which we denote by Herm_L.

Remark 1.2.2. Let R be a ring with anti-involution. An n-dimensional Hermitian form on the associated Hermitian Mackey functor $\underline{R}$ is an invertible matrix with entries in R which is fixed by the involution $w(A)_{ij} = w(A_{ji})$. This is the same as the datum of an antisymmetric nondegenerate bilinear pairing $R^{\oplus n} \otimes R^{\oplus n} \to R$, that is, a Hermitian form on $R^{\oplus n}$. Since the action of $M_n(\underline{R})(\mathbb{Z}/2)$ on $M_n(\underline{R})(*)$ is by conjugation, a morphism of Hermitian forms in the sense of Definition 1.2.1 corresponds to the classical notion of isometry.

The block-sum of matrices on objects and morphisms defines the structure of a permutative category on Herm_L. The symmetry isomorphism from $B \oplus B'$ to $B' \oplus B$, where B is n-dimensional and B' is m-dimensional, is given by the standard permutation matrix of $\mathrm{GL}_{n+m}(L(\mathbb{Z}/2))$ with blocks

$$\tau_{n,m} := \begin{pmatrix} O_{mn} & I_n \\ I_m & O_{nm} \end{pmatrix}.$$

Here O_{nm} is the null $n \times m$-matrix, where the diagonal zeros are those of $L(*)$ and the off-diagonal ones are in $L(\mathbb{Z}/2)$. The matrix I_n is the $n \times n$-identity matrix of $L(\mathbb{Z}/2)$. The classifying space $Bi\,\mathrm{Herm}_L$ of the category of invertible morphisms is therefore an E_∞-monoid.

Definition 1.2.3. Let L be a Hermitian Mackey functor. The Hermitian K-theory space of L is the group completion

$$\mathrm{GW}(L) := \Omega B(Bi\,\mathrm{Herm}_L, \oplus).$$

Segal's Γ-space construction for the symmetric monoidal category $(i\,\mathrm{Herm}_L, \oplus)$ provides a spectrum whose infinite loop space is equivalent to $\mathrm{GW}(L)$, that we also denote by $\mathrm{GW}(L)$.

Remark 1.2.4. If $\lambda : B \to B'$ is a morphism of Hermitian forms, the form B is determined by B' and the matrix λ. Thus a string of composable morphisms

$$B_0 \xrightarrow{\lambda_0} B_1 \xrightarrow{\lambda_1} \dots \xrightarrow{\lambda_n} B_n$$

is determined by the sequence of matrices $\lambda_1, \dots, \lambda_n$, and by the form B_n. This gives an isomorphism

$$Bi\,\mathrm{Herm}_L \cong \coprod_{n \geq 0} B(*, \mathrm{GL}_n(L(\mathbb{Z}/2)), \mathrm{GL}_n(L)(*)),$$

where $B\big(*, \mathrm{GL}_n(L(\mathbb{Z}/2)), \mathrm{GL}_n(L)(*)\big)$ is the Bar construction of the right action of the group $\mathrm{GL}_n(L(\mathbb{Z}/2))$ on the set of n-dimensional Hermitian forms $w(\lambda)\cdot B$, given by the Hermitian structure of the Mackey functor $M_n(L)$. The action indeed restricts to an action on $\mathrm{GL}_n(L)(*)$ because if λ is in $\mathrm{GL}_n(L(\mathbb{Z}/2))$ and the restriction of $B \in M_n(L)(*)$ is invertible, then

$$R(w(\lambda)\cdot B) = w(\lambda)R(B)\lambda$$

is also invertible. For rings with anti-involution this is [Burghelea and Fiedorowicz 1985, Remark 1.3].

Remark 1.2.5. Since the notion of Hermitian forms on Hermitian Mackey functors extends that of Hermitian forms on rings with anti-involution, it follows that our definition of Hermitian K-theory extends the Hermitian K-theory construction of [Burghelea and Fiedorowicz 1985] of the category of free modules over a discrete ring with anti-involution.

We now make our Hermitian K-theory construction functorial.

Definition 1.2.6. A morphism of Hermitian Mackey functors is a map of Mackey functors $f : L \to N$ such that $f_{\mathbb{Z}/2} : L(\mathbb{Z}/2) \to N(\mathbb{Z}/2)$ is a ring map, and such that $f_* : L(*) \to N(*)$ is $L(\mathbb{Z}/2)$-equivariant, where $N(*)$ is an $L(\mathbb{Z}/2)$-set via $f_{\mathbb{Z}/2}$.

Clearly a map of Hermitian Mackey functors $f : L \to N$ induces a symmetric monoidal functor $f_* : \mathrm{Herm}_L \to \mathrm{Herm}_N$, by applying $f_{\mathbb{Z}/2}$ and f_* to the matrices entrywise. Thus it induces a continuous map $f_* : \mathrm{GW}(L) \to \mathrm{GW}(N)$, and a map of spectra $f_* : \mathrm{GW}(L) \to \mathrm{GW}(N)$. We are mostly interested in the following example.

Example 1.2.7. Let $\mathbb{Z}$ be the ring of integers with the trivial anti-involution, and $\underline{\mathbb{Z}}$ the corresponding Hermitian Mackey functor. There is a morphism of Hermitian Mackey functors

$$d : \mathbb{A} \to \underline{\mathbb{Z}}$$

from the Burnside Mackey functor. The map $d_{\mathbb{Z}/2}$ is the identity of $\mathbb{Z}$, and the map

$$d_* : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$$

is the identity on the first summand and multiplication by 2 on the second. In terms of finite $\mathbb{Z}/2$-sets, it sends a set to its cardinality. This is in fact a morphism of Tambara functors for the standard multiplicative structures on $\mathbb{A}$ and $\underline{\mathbb{Z}}$, and since the Hermitian structures are defined via the multiplicative norms it follows that d is a map of Hermitian Mackey functors.

If moreover π is a discrete group with anti-involution, the map d induces a morphism on the associated group-Mackey functors $d : \mathbb{A}[\pi] \to \underline{\mathbb{Z}}[\pi]$. The underlying map $d_{\mathbb{Z}/2}$ is again the identity on $\mathbb{Z}[\pi]$, and the map

$$d_* : \mathbb{A}[\pi](*) = (\mathbb{Z}\oplus\mathbb{Z})[\pi] \oplus \mathbb{Z}[\pi^{\mathrm{free}}/(\mathbb{Z}/2)] \to (\mathbb{Z}[\pi])^{\mathbb{Z}/2} = \mathbb{Z}[\pi] \oplus \mathbb{Z}[\pi^{\mathrm{free}}/(\mathbb{Z}/2)]$$

is $d[\pi]$ on the first summand and the identity on the second summand. This map therefore induces a map on Hermitian K-theory spectra

$$d : \mathrm{GW}(\mathbb{A}[\pi]) \to \mathrm{GW}(\underline{\mathbb{Z}}[\pi]) = \mathrm{GW}(\mathbb{Z}[\pi]).$$

1.3. ***Multiplicative structures.*** We saw in Example 1.1.4 that $\mathbb{Z}/2$-Tambara functors provide a supply of Hermitian Mackey functors. In this section we show that the Hermitian K-theory spectrum of a Tambara functor is in fact a ring spectrum. We generalize this construction when the input is a commutative $\mathbb{Z}/2$-equivariant ring spectrum in Section 2.5, but the construction for Tambara functors gives us slightly more functoriality that will be useful in Section 4.

Let L be a Tambara functor. We define a pairing of categories

$$\otimes : \mathrm{Herm}_L \times \mathrm{Herm}_L \to \mathrm{Herm}_L$$

by means of an extension of the standard tensor product of matrices. On objects, we define the tensor product of two Hermitian forms B and B' on L of dimensions n and m, respectively, to be the nm-dimensional form $B \otimes B'$ with diagonal components

$$(B \otimes B')_{ii} = B_{kk} \cdot B'_{uu}, \quad \text{where } k = \left\lfloor \frac{i-1}{n} \right\rfloor + 1,\ u = i - n\left\lfloor \frac{i-1}{n} \right\rfloor,$$

where the multiplication denotes the multiplication in the commutative ring $L(*)$. The off-diagonal term $1 \le i < j \le nm$ of $B \otimes B'$ is defined by

$$(B \otimes B')_{ij} = R(B)_{kl} \cdot R(B')_{uv}, \quad \text{where } k = \left\lfloor \frac{i-1}{n} \right\rfloor + 1, \quad l = \left\lfloor \frac{j-1}{n} \right\rfloor + 1,$$
$$u = i - n\left\lfloor \frac{i-1}{n} \right\rfloor,\ v = j - n\left\lfloor \frac{j-1}{n} \right\rfloor,$$

and $R : M_n(L(\mathbb{Z}/2)) \to M_n(L)(*)$ is the restriction of the Mackey functor of matrices of Definition 1.1.5. This is the standard formula of the Kronecker product of matrices, where the diagonal elements are lifted to the fixed-points ring $L(*)$.

Example 1.3.1. In the case $m = n = 2$, the product above is given by the matrix

$$\begin{pmatrix} B_{11} & B_{12} \\ & B_{22} \end{pmatrix} \otimes \begin{pmatrix} B'_{11} & B'_{12} \\ & B'_{22} \end{pmatrix} = \begin{pmatrix} B_{11}B'_{11} & R(B_{11})B'_{12} & B_{12}R(B'_{11}) & B_{12}B'_{12} \\ & B_{11}B'_{22} & B_{12}w(B'_{12}) & B_{12}R(B'_{22}) \\ & & B_{22}B'_{11} & R(B_{22})B'_{12} \\ & & & B_{22}B'_{22} \end{pmatrix},$$

where the products appearing on the diagonal are taken in the ring $L(*)$, and the products of the off-diagonal terms are in the underlying ring $L(\mathbb{Z}/2)$.

Since the restriction map $R : L(*) \to L(\mathbb{Z}/2)$ is a ring map this operation lifts the standard Kronecker product of matrices, in the sense that

$$R(B \otimes B') = R(B) \otimes R(B')$$

as $nm \times nm$-matrices with coefficients in $L(\mathbb{Z}/2)$. We define the pairing $\otimes$ on morphisms by the standard Kronecker product of matrices with entries in $L(\mathbb{Z}/2)$.

Lemma 1.3.2. *The pairing* $\otimes \colon \mathrm{Herm}_L \times \mathrm{Herm}_L \to \mathrm{Herm}_L$ *is a well-defined functor.*

Proof. A tedious but straightforward verification shows that for every pair of matrices $A \in M_n(L(\mathbb{Z}/2))$ and $A' \in M_m(L(\mathbb{Z}/2))$, and forms $B \in M_n(L)(*)$ and $B' \in M_m(L)(*)$ we have that

$$(A \cdot B) \otimes (A' \cdot B') = (A \otimes A') \cdot (B \otimes B'),$$

where the dot is the action of the Hermitian structure of the Mackey functor $M_n(L)$. This uses the identities $T(a)b = T(aR(b))$ and $RN(a) = aw(a)$ of the Tambara structure.

Thus if $\lambda : B \to C$ and $\lambda' : B' \to C'$ are morphisms of Hermitian forms, we have that

$$w(\lambda \otimes \lambda') \cdot (C \otimes C') = (w(\lambda) \otimes w(\lambda')) \cdot (C \otimes C') = (w(\lambda) \cdot C) \otimes (w(\lambda') \cdot C') = B \otimes B',$$

which shows that $\lambda \otimes \lambda' : B \otimes B' \to C \otimes C'$ is a well-defined morphism. The composition of morphisms happens in the matrix rings $M_n(L(\mathbb{Z}/2))$, and therefore it is respected by $\otimes$. Similarly, $\otimes$ preserves the identity morphisms. $\square$

It is moreover immediate to verify that the standard compatibility conditions between $\otimes$ and the direct sum are satisfied for forms:

(i) $(B \oplus B') \otimes B'' = (B \otimes B'') \oplus (B' \otimes B'')$,

(ii) $B \otimes (B' \oplus B'') = \sigma (B \otimes B') \oplus (B \otimes B'')\sigma^{-1}$, where σ is a permutation matrix,

(iii) $0 \otimes B = 0$ and $B \otimes 0 = 0$,

(iv) $1 \otimes B = B \otimes 1 = B$, where 1 is the 1-form with entry the unit of the ring $L(*)$.

By property (ii) the permutation σ defines an isomorphism of forms

$$B \otimes (B' \oplus B'') \cong (B \otimes B') \oplus (B \otimes B''),$$

and one can easily verify that this isomorphism satisfies the higher coherences required to give the following.

Proposition 1.3.3. *The pairing* $\mathrm{Herm}_L \times \mathrm{Herm}_L \to \mathrm{Herm}_L$ *is a pairing of permutative categories, thus inducing a morphism of spectra*

$$\otimes : \mathrm{GW}(L) \wedge \mathrm{GW}(L) \to \mathrm{GW}(L)$$

which exhibits $\mathrm{GW}(L)$ *as a ring spectrum.* $\square$

The morphism $f : \mathrm{Herm}_L \to \mathrm{Herm}_N$ induced by a morphism of Tambara functors $f : L \to N$ clearly commutes with the monoidal structure $\otimes$, thus inducing a morphism of ring spectra $f : \mathrm{GW}(L) \to \mathrm{GW}(N)$.

Remark 1.3.4. Let L and N be Tambara functors, and suppose that $f : L \to N$ is a morphism of Hermitian Mackey functors such that $f_* : L(*) \to N(*)$ is multiplicative, but not necessarily unital. Then the induced functor $f : \mathrm{Herm}_L \to \mathrm{Herm}_N$ preserves the tensor product, but not its unit, and the map $f : \mathrm{GW}(L) \to \mathrm{GW}(N)$ is a morphism of nonunital ring spectra. The example we will be interested in is the morphism $\frac{T}{2} : \mathbb{Z}\left[\frac{1}{2}\right] \to \mathbb{A}\left[\frac{1}{2}\right]$, defined by the identity map on underlying rings, and by half the transfer $\left(0, \frac{1}{2}\right) : \mathbb{Z}\left[\frac{1}{2}\right] \to \mathbb{Z}\left[\frac{1}{2}\right] \oplus \mathbb{Z}\left[\frac{1}{2}\right]$ on fixed points.

2. Real K-theory

The aim of this section is to construct the free real K-theory $\mathbb{Z}/2$-spectrum of a ring spectrum with anti-involution and its assembly map, and to relate these objects to the classical constructions when the input ring spectrum is the Eilenberg–Mac Lane spectrum of a discrete ring.

2.1. ***Real semisimplicial spaces and the real and dihedral Bar constructions.*** In this section we investigate the Bar construction and the cyclic Bar construction associated to a monoid with an anti-involution. This is essentially a recollection of materials from [Loday 1987; Burghelea and Fiedorowicz 1985, §1; Hesselholt and Madsen 2015], but we need a context without degeneracies, which requires particular care.

We let Δ_+ be the subcategory of Δ of all objects and injective morphisms. The category Δ has an involution that fixes the objects and that sends a morphism $\alpha : [n] \to [k]$ to $\bar{\alpha}(i) = k - \alpha(n-i)$. This involution restricts to Δ_+. We recall from [Hesselholt and Madsen 2015] that a real simplicial space is a simplicial space X together with levelwise involutions $w : X_n \to X_n$ which satisfy $w \circ \alpha^* = \bar{\alpha}^* \circ w$ for every morphism $\alpha \in \Delta$. This can be conveniently reformulated as a $\mathbb{Z}/2$-diagram $X : \Delta^{\mathrm{op}} \to \mathrm{Top}$, in the sense of [Dotto and Moi 2016, Definition 1.1]. Similarly, we define a real semisimplicial space to be a $\mathbb{Z}/2$-diagram $X : \Delta_+^{\mathrm{op}} \to \mathrm{Top}$.

We are mostly concerned with the following two examples. By a nonunital topological monoid we mean a possibly nonunital monoid in the monoidal category of spaces with respect to the cartesian product. Let M be a nonunital topological monoid which is equipped with an anti-involution, that is, a continuous map of monoids $w : M^{\mathrm{op}} \to M$ that satisfies $w^2 = \mathrm{id}$.

Example 2.1.1. The real nerve of M is the semisimplicial space NM, the nerve of M, with n-simplices $N_n M = M^{\times n}$, and with the levelwise involution

$$(m_1, \ldots, m_n) \mapsto (w(m_n), \ldots, w(m_1)).$$

The resulting real semisimplicial space is denoted $N^\sigma M$; compare [Burghelea and Fiedorowicz 1985, Definition 1.12].

Example 2.1.2. The dihedral nerve of M is the cyclic nerve $N^{\mathrm{cy}}M$, with n-simplices $N^{\mathrm{cy}}_n M = M^{\times n+1}$, and with the involution defined degreewise by

$$(m_0, m_1, \dots, m_n) \mapsto (w(m_0), w(m_n), \dots, w(m_1)).$$

The resulting real semisimplicial space is denoted $N^{\mathrm{di}}M$. This involution combined with the semicyclic structure define a semidihedral object.

Segal's edgewise subdivision functor sd_e from [Segal 1973] turns a real semisimplicial space X into a semisimplicial $\mathbb{Z}/2$-space. It is defined by precomposing a real semisimplicial space $X : \Delta_+^{\mathrm{op}} \to \mathrm{Top}$ with the endofunctor of Δ_+ that sends $[n] = \{0, \dots, n\}$ to $[2n+1] = [n] \amalg [n]^{\mathrm{op}}$, and a morphism $\alpha : [n] \to [k]$ to $\alpha \amalg \bar{\alpha}$. Since $\mathrm{sd}_e(X^{\mathrm{op}}) = \mathrm{sd}_e X$, the levelwise involution on X defines a semisimplicial involution on $\mathrm{sd}_e X$. Thus the thick geometric realization $\|\mathrm{sd}_e X\|$ inherits a $\mathbb{Z}/2$-action.

Definition 2.1.3. The real Bar construction of a nonunital topological monoid with anti-involution M is the $\mathbb{Z}/2$-space $B^\sigma M$ defined as the geometric realization of the semisimplicial space

$$B^\sigma M := \|\mathrm{sd}_e N^\sigma M\|$$

with the involution induced by the semisimplicial involution of $\mathrm{sd}_e N^\sigma M$. Similarly, the dihedral Bar construction of M is the $\mathbb{Z}/2$-space $B^{\mathrm{di}}M$ defined as the geometric realization of the semisimplicial space

$$B^{\mathrm{di}}M := \|\mathrm{sd}_e N^{\mathrm{di}}M\|.$$

We note that in contrast to the usual cyclic Bar construction $|N^{\mathrm{cy}}M|$, which is defined using the thin geometric realization, $B^{\mathrm{di}}M$ does not have a circle action, nor does the unsubdivided $\|N^{\mathrm{di}}M\|$.

Example 2.1.4. Let π be a discrete group with the anti-involution defined by inversion $w = (-)^{-1} : \pi^{\mathrm{op}} \to \pi$. The $\mathbb{Z}/2$-space $B^\sigma \pi$ is a classifying space for principal π-bundles of $\mathbb{Z}/2$-spaces. A model for such a universal bundle is constructed in [May 1990] as the map

$$\mathrm{Map}(E\mathbb{Z}/2, E\pi) \to \mathrm{Map}(E\mathbb{Z}/2, B\pi),$$

where $E\pi$ denotes the free and contractible π-space. The base space is equivalent to the nerve of the functor category $\mathrm{Cat}(\mathcal{E}\mathbb{Z}/2, \pi)$ where $\mathcal{E}\mathbb{Z}/2$ is the translation category of the left $\mathbb{Z}/2$-set $\mathbb{Z}/2$ (whose nerve is the classical model for $E\mathbb{Z}/2$); see [Guillou et al. 2017]. It is easy to see that the nerve of $\mathrm{Cat}(\mathcal{E}\mathbb{Z}/2, \pi)$ and the edgewise subdivision of $N^\sigma \pi$ are equivariantly isomorphic.

Remark 2.1.5. In contrast with the simplicial case, the geometric realization of a semisimplicial space is in general not equivalent to the geometric realization of its subdivision. However, this is the case if the semisimplicial space X admits a (levelwise) weak equivalence $X \xrightarrow{\sim} Y$, where Y is a semisimplicial space which is the restriction of a proper simplicial space. This is because of the commutative diagram

$$\begin{tikzcd} \| \operatorname{sd}_e X\| \arrow[r, "\sim"] \arrow[d] & \| \operatorname{sd}_e Y\| \arrow[r, "\sim"] \arrow[d] & | \operatorname{sd}_e Y| \arrow[d, "\cong"] \\ \|X\| \arrow[r, "\sim"'] & \|Y\| \arrow[r, "\sim"'] & |Y| \end{tikzcd}$$

where $|-|$ denotes the thin geometric realization. For the nerve and the cyclic nerve this condition holds if the monoid M is weakly equivalent to a unital monoid. In the examples of interest in this paper we are always in this situation. Thus the underlying nonequivariant homotopy types of $B^\sigma M$ and $B^{\mathrm{di}} M$ are those of the Bar construction BM and the cyclic Bar construction $B^{\mathrm{cy}} M$, respectively.

If X is a real semisimplicial space, $\|X\|$ also inherits an involution, by the formula

$$[x \in X_n, (t_0, \ldots, t_n) \in \Delta^n] \mapsto [w(x) \in X_n, (t_n, \ldots, t_0) \in \Delta^n].$$

The map $\gamma : \| \operatorname{sd}_e X\| \to \|X\|$ that sends

$$[x, (t_0, \ldots, t_n)] \mapsto \left[x, \tfrac{1}{2}(t_0, \ldots, t_n, t_n, \ldots, t_0)\right]$$

is equivariant. If X admits a levelwise equivariant equivalence $X \xrightarrow{\sim} Y$, where Y is the restriction of a proper real simplicial space, then γ is an equivariant equivalence. This is again because of the above diagram, since in the presence of degeneracies the map γ descends to an equivariant homeomorphism $| \operatorname{sd}_e Y| \cong |Y|$. In general these two actions do not agree, and we choose to work with the subdivided version because it gives us control over the fixed points. We give a weaker condition that guarantees that these actions are equivalent for nerves of monoids in Lemma 2.1.11.

We now proceed by analyzing the fixed points of $B^\sigma M$ and $B^{\mathrm{di}} M$. The fixed points of $B^\sigma M$ are modeled not by a monoid, but by a category. Let us define a topological category $\operatorname{Sym} M$ (without identities) as follows. Its space of objects is the fixed-points space $M^{\mathbb{Z}/2}$, and the morphisms $m \to n$ consist of the subspace of elements $l \in M$ with $m = w(l)nl$. Composition is defined by $l \circ k = l \cdot k$. The following is analogous to [Burghelea and Fiedorowicz 1985, Proposition 1.13].

Proposition 2.1.6. *Let M be a nonunital topological monoid with anti-involution. The $\mathbb{Z}/2$-fixed points of $B^\sigma M$ are naturally homeomorphic to the classifying space of* $\operatorname{Sym} M$, *whose nerve is the Bar construction*

$$N \operatorname{Sym} M \cong N(M; M^{\mathbb{Z}/2})$$

of the right action of M on $M^{\mathbb{Z}/2}$ given by $n \cdot l = w(l)nl$.

Proof. The geometric realization of semisimplicial sets commutes with fixed points of finite groups. This can be easily proved by induction on the skeleton filtration, since fixed points commute with pushouts along closed inclusions and with filtered colimits along closed inclusions. Thus the fixed-points space $(B^\sigma M)^{\mathbb{Z}/2}$ is homeomorphic to the geometric realization of the semisimplicial space $(\mathrm{sd}_e N^\sigma M)^{\mathbb{Z}/2}$.

There is an equivariant isomorphism $\mathrm{sd}_e NM \cong N\,\mathrm{sd}_e M$ of semisimplicial $\mathbb{Z}/2$-spaces, where $\mathrm{sd}_e M$ is the edgewise subdivision of the category M (also known as the twisted arrow category). This is the topological category with $\mathbb{Z}/2$-action whose space of objects is M, and where the space of morphisms $m \to n$ is the subspace of $M \times M$ of pairs (l, k) such that $n = lmk$. Composition is defined by

$$(l, k) \circ (l', k') = (l'l, kk').$$

The involution on $\mathrm{sd}_e M$ sends an object m to $w(m)$, and a morphism (l, k) to $(w(k), w(l))$. Since the nerve functor commute with fixed points, the fixed points of $N\,\mathrm{sd}_e M$ are isomorphic to the nerve of the fixed-points category of $\mathrm{sd}_e M$. Its objects are the fixed objects $M^{\mathbb{Z}/2}$, and its morphisms the pairs (l, k) where $k = w(l)$. This is isomorphic to the category $\mathrm{Sym}\, M$. $\square$

A similar argument shows that the fixed points of the subdivided dihedral nerve of M are isomorphic to the two-sided Bar construction

$$(N^{\mathrm{di}}_{2n+1} M)^{\mathbb{Z}/2} \cong N_n(M^{\mathbb{Z}/2}; M; M^{\mathbb{Z}/2})$$

of the left action of M on $M^{\mathbb{Z}/2}$ defined by $m \cdot n := mnw(m)$ and the right action $n \cdot m := w(m)nm$. Thus the semisimplicial space $(\mathrm{sd}_e N^{\mathrm{di}} M)^{\mathbb{Z}/2}$ is isomorphic to the nerve of a category $\mathrm{Sym}^{\mathrm{cy}} M$. Its objects are the pairs (n_0, n_1) of fixed points of $M^{\mathbb{Z}/2}$. A morphism $m : (n_0, n_1) \to (n'_0, n'_1)$ is an element $m \in M$ such that $n'_0 = m \cdot n_0$ and $n_1 = n'_1 \cdot m$. We then obtain the following.

Proposition 2.1.7. *Let M be a nonunital topological monoid with anti-involution. The $\mathbb{Z}/2$-fixed points of $B^{\mathrm{di}} M$ are naturally homeomorphic to the classifying space of* $\mathrm{Sym}^{\mathrm{cy}} M$, *whose nerve is the two-sided Bar construction*

$$N\,\mathrm{Sym}^{\mathrm{cy}} M \cong N(M^{\mathbb{Z}/2}; M; M^{\mathbb{Z}/2}). \qquad \square$$

We are now able to determine the homotopy invariance property of B^σ and B^{di}. Here and throughout the paper, we call a $\mathbb{Z}/2$-equivariant map of $\mathbb{Z}/2$-spaces $f : X \to Y$ a weak equivalence if both f and its restriction on fixed points $f : X^{\mathbb{Z}/2} \to Y^{\mathbb{Z}/2}$ induce isomorphisms on all homotopy groups.

Lemma 2.1.8. *Let $f : M \to M'$ be a map of nonunital topological monoids with anti-involutions, and suppose that f is a weak equivalence of $\mathbb{Z}/2$-spaces. Then*

$$B^\sigma f : B^\sigma M \to B^\sigma M' \quad \textit{and} \quad B^{\mathrm{di}} f : B^{\mathrm{di}} M \to B^{\mathrm{di}} M'$$

are $\mathbb{Z}/2$-equivalences of spaces.

Proof. Nonequivariantly Bf is an equivalence, since realizations of semisimplicial spaces preserve levelwise equivalences. Since realizations commute with fixed points it remains to show that $(\mathrm{sd}_e NM)^{\mathbb{Z}/2} \to (\mathrm{sd}_e NM')^{\mathbb{Z}/2}$ is a levelwise equivalence. By Proposition 2.1.6 this is the map

$$f^{\times n} \times f^{\mathbb{Z}/2} : M^{\times n} \times M^{\mathbb{Z}/2} \to (M')^{\times n} \times (M')^{\mathbb{Z}/2},$$

which is an equivalence by assumption. A similar argument applies to $B^{\mathrm{di}} f$. $\square$

We further analyze the functors B^σ and B^{di}. The following property is crucial in the definition of the L-theoretic assembly map of Section 2.7.

Lemma 2.1.9. *Let π be a well-pointed topological group with the anti-involution $w = (-)^{-1}$ given by inversion. There is an equivariant map $\lambda : B\pi \to B^\sigma \pi$, where $B\pi$ has the trivial involution, which is nonequivariantly homotopic to the identity. On fixed points, the composite*

$$B\pi \xrightarrow{\lambda} (B^\sigma \pi)^{\mathbb{Z}/2} \xrightarrow{\iota} B\pi$$

with the fixed points inclusion $\iota : (B^\sigma \pi)^{\mathbb{Z}/2} \to B\pi$ is homotopic to the identity. This exhibits $B\pi$ as a retract of $(B^\sigma \pi)^{\mathbb{Z}/2}$. If moreover π is discrete, there is a further splitting

$$(B^\sigma \pi)^{\mathbb{Z}/2} \simeq \coprod_{\{[g] \,|\, g^2 = 1\}} B Z_\pi \langle g \rangle,$$

where the coproduct runs through the conjugacy classes of the elements of π of order 2, and $Z_\pi \langle g \rangle$ is the centralizer of g in π. Then λ corresponds to the inclusion of the summand $g = 1$.

Proof. By Remark 2.1.5 we may work with the thin realization of the nerve of π. We define a map $\lambda : N_p \pi \to (\mathrm{sd}_e N^\sigma \pi)_p = N^\sigma_{2p+1} \pi$ degreewise by

$$\lambda(g_1, \dots, g_p) = (g_1, \dots, g_p, 1, g_p^{-1}, \dots, g_1^{-1}).$$

This map is clearly simplicial and equivariant, and it induces an equivariant map $\lambda : |N\pi| \to |\,\mathrm{sd}_e N^\sigma \pi| \cong |N^\sigma \pi|$ on realizations. This map sends

$$\begin{aligned} &[(g_1, \dots, g_p); (t_0, \dots, t_p)] \\ &\qquad \mapsto \big[(g_1, \dots, g_p, 1, g_p^{-1}, \dots, g_1^{-1}); \tfrac{1}{2}(t_0, \dots, t_p, t_p, \dots, t_0)\big]. \end{aligned}$$

There is a homotopy $\frac{1}{2}(t, t) \simeq (t, 0)$ that keeps the sum of the two components constant. This induces a homotopy between λ and

$$\begin{aligned} &[(g_1, \dots, g_p, 1, g_p^{-1}, \dots, g_1^{-1}); (t_0, \dots, t_p, 0, \dots, 0)] \\ &\qquad = [(g_1, \dots, g_p); (t_0, \dots, t_p)] = \mathrm{id}\,. \end{aligned}$$

The same homotopy defines a homotopy between $\iota \circ \lambda$ and the identity.

Now let us assume that π is discrete. The fixed-points space $(B^\sigma\pi)^{\mathbb{Z}/2}$ is the classifying space of the category $\mathrm{Sym}\,\pi$ of Proposition 2.1.6. The objects of this category are the elements of π of order two. A morphism $g \to g'$ is an element h of π such that $g = h^{-1}g'h$. Each component of $\mathrm{Sym}\,\pi$ is then represented by the conjugacy class of an element g of order two, and the automorphism group of g is precisely $Z_\pi\langle g\rangle$. □

In the same way as the cyclic nerve of a group-like monoid G is a model for the free loop-space, the dihedral nerve is a $\mathbb{Z}/2$-equivariant model for the free loop space $\Lambda^\sigma B^\sigma G$, where $\Lambda^\sigma = \mathrm{Map}(S^\sigma, -)$ is the free loop space of the sign-representation sphere S^σ. Establishing this equivalence becomes delicate when G does not have a strict unit. Classically the map $B^{\mathrm{cy}}G \to \Lambda BG$ is constructed from the S^1-action on $B^{\mathrm{cy}}G$ induced by the cyclic structure, but this S^1-action is not well-defined on the thick realization. Let M be a nonunital topological monoid with anti-involution, and let M_+ denote M with a formally adjoined unit which is fixed by the anti-involution. Let us consider the diagram

$$\begin{array}{ccc} B^{\mathrm{di}}M = \|\operatorname{sd}_e N^{\mathrm{di}}M\| \longrightarrow \|\operatorname{sd}_e N^{\mathrm{di}}(M_+)\| \xrightarrow{\sim} |\operatorname{sd}_e N^{\mathrm{di}}(M_+)| & \xrightarrow{\cong} & |N^{\mathrm{di}}(M_+)| \\ & & \downarrow \\ \Lambda^\sigma B^\sigma M = \Lambda^\sigma \|\operatorname{sd}_e N^\sigma M\| \longrightarrow \Lambda^\sigma\|N^\sigma M\| & \xleftarrow{\cong} & \Lambda^\sigma|N^\sigma(M_+)| \end{array}$$

The first map is induced by the inclusion $M \to M_+$. The second map is the canonical map to the thin geometric realization, which is an equivalence since the inclusion of the disjoint unit is a cofibration. The third map is the canonical homeomorphism γ from Remark 2.1.5. The vertical map is adjoint to the composite $S^\sigma \times |N^{\mathrm{di}}(M_+)| \to |N^{\mathrm{di}}(M_+)| \to |N^\sigma(M_+)|$ of the circle action induced by the cyclic structure and the canonical projection. The next map is the isomorphism between $N(M_+)$ and the free simplicial space $E(NM)$ on the semisimplicial space NM, followed by the isomorphism $|E(NM)| \cong \|NM\|$; see [Ebert and Randal-Williams 2017, Lemma 1.8]. The last map is again the map γ from Remark 2.1.5.

Definition 2.1.10. We say that a nonunital topological monoid with anti-involution M is quasiunital if

(i) there is a unital well-pointed topological monoid A and a map of nonunital monoids $f : M \xrightarrow{\sim} A$ which is an equivalence on underlying spaces,

(ii) there is a right A-space B and a map of M-spaces $\phi : M^{\mathbb{Z}/2} \xrightarrow{\sim} B$ which is an equivalence on underlying spaces, where M acts on B via f,

(iii) there is a point $e \in B$ such that $e \cdot f(m) = \phi(w(m)m)$ for every $m \in M$, and B is well-pointed at e.

We say that M is group-like if $\pi_0 M$ is a group.

When M is unital, one can of course set f and ϕ to be the identity map. The element e plays the role of the unit as an element in the fixed-points space $M^{\mathbb{Z}/2}$. The next two results play a role in the construction of the trace of Section 3.2.

Lemma 2.1.11. *Let M be a nonunital topological monoid with anti-involution which is quasiunital. Then the map*

$$\gamma : B^{\sigma} M = \| \operatorname{sd}_e N^{\sigma} M \| \xrightarrow{\sim} \| N^{\sigma} M \|$$

is a $\mathbb{Z}/2$-equivariant weak equivalence.

Proof. Nonequivariantly, γ is an equivalence by Remark 2.1.5, since $NM \xrightarrow{\sim} NA$ is an equivalence and NA is a proper simplicial space. In order to show that γ is an equivalence on fixed points, we factor it as

$$\| \operatorname{sd}_e N^{\sigma} M \| \cong |E(\operatorname{sd}_e N^{\sigma} M)| \to | \operatorname{sd}_e(N^{\sigma} M_+)| \overset{\gamma}{\cong} |N^{\sigma} M_+| \cong \|N^{\sigma} M\|,$$

where the arrow is induced by the inclusion $M \to M_+$. We claim that this map is an equivalence on fixed points. By Proposition 2.1.6 this is the geometric realization of the map of simplicial spaces

$$|E(\operatorname{sd} N^{\sigma} M)|^{\mathbb{Z}/2} \cong |N(\operatorname{Sym} M)_+| \to |N \operatorname{Sym}(M_+)| \cong | \operatorname{sd}_e(N^{\sigma} M_+)|^{\mathbb{Z}/2},$$

where $(\operatorname{Sym} M)_+$ is the category $\operatorname{Sym} M$ with freely adjoined identities. We observe that the category $\operatorname{Sym}(M_+)$ also has freely added identities. Thus, denoting by $\operatorname{Sym}_+ M$ the category $\operatorname{Sym}(M_+)$ with the identities removed, we see that $\operatorname{Sym}_+ M$ is a well-defined category and that $\operatorname{Sym}(M_+) = (\operatorname{Sym}_+ M)_+$. Moreover, $\operatorname{Sym} M$ is the full subcategory of $\operatorname{Sym}_+ M$ on the objects in

$$M^{\mathbb{Z}/2} \subset M^{\mathbb{Z}/2} \amalg 1 = Ob \operatorname{Sym}_+ M.$$

Therefore we need to show that the map

$$\|N(\operatorname{Sym} M)\| \cong |N(\operatorname{Sym} M)_+| \to |N \operatorname{Sym}(M_+)| \cong \|N(\operatorname{Sym}_+ M)\|$$

induced by the inclusion of nonunital topological categories $\iota : \operatorname{Sym} M \to \operatorname{Sym}_+ M$ is a weak equivalence. The nerve of $\operatorname{Sym}_+ M$ is the Bar construction of the right action of M on the fixed-points space with a disjoint basepoint $M_+^{\mathbb{Z}/2}$, where M acts on $M^{\mathbb{Z}/2}$ as usual by $n \cdot l = w(l)nl$, and on the added basepoint by $+ \cdot l = w(l)l$. The nerve of the map ι identifies under the isomorphism of Proposition 2.1.6 with the canonical map

$$N(M; M^{\mathbb{Z}/2}) \to N(M; M_+^{\mathbb{Z}/2})$$

induced by the M-equivariant inclusion $M^{\mathbb{Z}/2} \to M_+^{\mathbb{Z}/2}$. Since M is quasiunital, the realization of this map is weakly equivalent to the thick realization of the map $\iota : N(A; B) \to N(A; B_+)$ induced by the inclusion $B \to B_+$, where A acts on $+$ by $+ \cdot a = e \cdot a$. The last condition of Definition 2.1.10 guarantees that the map

$(f,\phi):N(M;M_+^{\mathbb{Z}/2})\to N(A;B_+)$ is compatible with the last face map. Thus it is sufficient to show that $\iota:N(A;B)\to N(A;B_+)$ is an equivalence. Since A is unital and well-pointed these semisimplicial spaces admit degeneracies, and the thick realization of this map is weakly equivalent to its thin realization. Therefore we can exhibit a simplicial retraction $r:N(A;B_+)\to N(A;B)$ for ι, and a simplicial homotopy between $\iota\circ r$ and the identity. The map r is induced by the map of A-spaces $B_+\to B$ which is the identity on B and that sends $+$ to e. The homotopy is induced by the morphism $(e,e):e\to +$ in the topological category $\mathrm{Sym}(B_+)$ whose nerve is $N(A;B_+)$. □

Lemma 2.1.12. *Let M be a nonunital topological monoid with anti-involution which is quasiunital and group-like. Then the map*

$$B^{\mathrm{di}}M=\|\,\mathrm{sd}_e\,N^{\mathrm{di}}M\|\xrightarrow{\sim}\Lambda^\sigma\|N^\sigma M\|$$

is a $\mathbb{Z}/2$-equivariant weak equivalence.

Proof. This map is the middle vertical map of a commutative diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\sim"] & \|\,\mathrm{sd}_e\,N^{\mathrm{di}}M\| \arrow[r, "p"] \arrow[d] & \|\,\mathrm{sd}_e\,N^{\sigma}M\| \arrow[d, "\gamma"', "\sim"] \\
\Omega^\sigma\|N^\sigma M\| \arrow[r] & \Lambda^\sigma\|N^\sigma M\| \arrow[r, "\mathrm{ev}_0"'] & \|N^\sigma M\|
\end{tikzcd}$$

where ev_0 is the evaluation map, which is a fibration, and p projects off the first coordinate in each simplicial level. The left vertical map is an equivalence by an equivariant version of the group-completion theorem of [Moi 2013]; see also [Dotto 2012, §6.2; Stiennon 2013, Theorem 4.0.5]. We claim that when M is group-like, the top row is a fiber sequence of $\mathbb{Z}/2$-spaces, and this will end the proof.

We start by observing that since M is quasiunital and group-like, the maps

$$(-)\cdot m,\, m\cdot(-):M\to M \quad\text{and}\quad w(m)(-)m:M^{\mathbb{Z}/2}\to M^{\mathbb{Z}/2}$$

are weak equivalences. Indeed if we let m^{-1} denote an element of M whose class in π_0M is an inverse for the class of m, we see that the composites of $m\cdot(-)$ with $m^{-1}\cdot(-):M\to M$ are homotopic to multiplication with an element $1\in M$ whose component is the unit of π_0M. Since M is quasiunital, there is a square

$$\begin{tikzcd}
M \arrow[r, "\sim"] \arrow[d, "1\cdot(-)"'] & A \arrow[d, "1\cdot(-)=\mathrm{id}"] \\
M \arrow[r, "\sim"'] & A
\end{tikzcd}$$

that commutes up to homotopy. Thus $1\cdot(-)$ is a weak equivalence, and so is $m\cdot(-)$. A similar argument shows that $(-)\cdot m$ is a weak equivalence. Similarly, the compositions of $w(m)(-)m$ with $w(m^{-1})(-)m^{-1} : M^{\mathbb{Z}/2} \to M^{\mathbb{Z}/2}$ are homotopic to $w(1)(-)1$, and this is an equivalence since it compares to the action of the unit of A under the equivalence $M^{\mathbb{Z}/2} \to B$.

In order to show that M is the homotopy fiber of $p : \| \mathrm{sd}_e N^{\mathrm{di}} M \| \to \| \mathrm{sd}_e N^{\sigma} M \|$ we use a criterion of Segal, as stated in [Ebert and Randal-Williams 2017, Theorem 2.12]. The diagrams of $\mathbb{Z}/2$-spaces

$$\begin{tikzcd} M \times M^{\times 2n+1} \arrow[r, "d_n"] \arrow[d, "p"] & M \times M^{\times 2n-1} \arrow[d, "p"] \\ M^{\times 2n+1} \arrow[r, "d_n"'] & M^{\times 2n-1} \end{tikzcd} \qquad \begin{tikzcd} M \times M^{\times 3} \arrow[r, "d_0"] \arrow[d, "p"] & M \times M \arrow[d, "p"] \\ M^{\times 3} \arrow[r, "d_0"'] & M \end{tikzcd}$$

are homotopy cartesian. The left-hand square is a strict pull-back and the map p is a fibration. For the right-hand square, we see that the strict pull-back P is isomorphic to $M \times M^{\times 3}$, but under this isomorphism the map from the top left corner

$$M \times M^{\times 3} \to M \times M^{\times 3} \cong P$$

sends (m_0, m_1, m_2, m_3) to $(m_3 m_0 m_1, m_1, m_2, m_3)$. Since left and right multiplications in M are weak equivalences, this map is a nonequivariant equivalence. On fixed points, this is isomorphic to the map

$$M^{\mathbb{Z}/2} \times M \times M^{\mathbb{Z}/2} \to M^{\mathbb{Z}/2} \times M \times M^{\mathbb{Z}/2}$$

that sends (m_0, m_1, m_2) to $(w(m_1) m_0 m_1, m_1, m_2)$, and this is an equivalence since by the argument above, $w(m_1)(-)m_1 : M^{\mathbb{Z}/2} \to M^{\mathbb{Z}/2}$ is an equivalence. It follows by [Ebert and Randal-Williams 2017, Lemma 2.11, Theorem 2.12] that the square

$$\begin{tikzcd} M \times M = (\mathrm{sd}_e N^{\mathrm{di}} M)_0 \arrow[r] \arrow[d, "p"'] & \| \mathrm{sd}_e N^{\mathrm{di}} M \| \arrow[d, "p"] \\ M = (\mathrm{sd}_e N^{\sigma} M)_0 \arrow[r] & \| \mathrm{sd}_e N^{\sigma} M \| \end{tikzcd}$$

is homotopy cartesian. Therefore, the homotopy fibers of the vertical maps are equivalent. □

The constructions N^{σ} and N^{di} extend to categories with duality. We use this generalization occasionally, mostly in Section 2.4.

Remark 2.1.13. We recall that a category with strict duality is a category (possibly without identities) $\mathscr{C}$ equipped with a functor $D : \mathscr{C}^{\mathrm{op}} \to \mathscr{C}$ such that $D^2 = \mathrm{id}$. If $\mathscr{C}$ has one object this is the same as a monoid with anti-involution. There is a levelwise involution on the nerve $N\mathscr{C}$ which is defined by

$$\big(c_0 \xrightarrow{f_1} c_1 \to \cdots \xrightarrow{f_n} c_n\big) \mapsto \big(Dc_n \xrightarrow{Df_n} Dc_{n-1} \to \cdots \xrightarrow{Df_1} c_0\big).$$

We define $B^\sigma\mathscr{C} := \| \operatorname{sd}_e N^\sigma\mathscr{C}\|$. There is a category $\operatorname{Sym}\mathscr{C}$ whose objects are the morphisms $f : c \to Dc$ such that $Df = f$, and the morphisms $f \to f'$ are the maps $\gamma : c \to c'$ such that $f = D(\gamma) f'\gamma$. The considerations of Proposition 2.1.6 extend to give an isomorphism

$$\operatorname{sd}_e N^\sigma\mathscr{C} \cong N \operatorname{Sym}\mathscr{C}.$$

Similarly, there is a construction of the dihedral nerve of a category with strict duality. An n-simplex of the cyclic nerve $N^{\mathrm{cy}}\mathscr{C}$ is a string of composable morphisms

$$c_n \xrightarrow{f_0} c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \to \cdots \to c_{n-1} \xrightarrow{f_n} c_n,$$

and the levelwise involution of the dihedral nerve sends this string to

$$Dc_0 \xrightarrow{Df_0} Dc_n \xrightarrow{Df_n} Dc_{n-1} \xrightarrow{Df_{n-1}} Dc_{n-2} \to \cdots \to Dc_1 \xrightarrow{Df_1} Dc_0.$$

We define $B^{\mathrm{di}}\mathscr{C} := \| \operatorname{sd}_e N^{\mathrm{di}}\mathscr{C}\|$.

2.2. ***Ring spectra with anti-involution and their Hermitian forms.*** Let A be an orthogonal ring spectrum, that is, a (strictly associative) monoid in the symmetric monoidal category of orthogonal spectra with the smash product. An anti-involution on A is a map of ring spectra $w : A^{\mathrm{op}} \to A$ such that $w \circ w = \mathrm{id}$. Here A^{op} is the opposite ring spectrum, with underlying spectrum A and multiplication

$$A \wedge A \xrightarrow{\tau} A \wedge A \xrightarrow{\mu} A,$$

where τ is the symmetry isomorphism, and μ is the multiplication of A. Since the map w is a strict involution, it gives the underlying orthogonal spectrum of A the structure of an orthogonal $\mathbb{Z}/2$-spectrum; see [Schwede 2013]. We recall that the (genuine, or derived) fixed-points spectrum $A^{\mathbb{Z}/2}$ is the spectrum defined by the sequence of spaces

$$(A^{\mathbb{Z}/2})_n := (\Omega^{n\sigma} A_{n\rho})^{\mathbb{Z}/2},$$

where σ and ρ are the sign and the regular representation of $\mathbb{Z}/2$, respectively, and for every d-dimensional real $\mathbb{Z}/2$-representation V the pointed $\mathbb{Z}/2$-space A_V is the space A_d where $g \in \mathbb{Z}/2$ acts by $(g, \sigma_V(g)) \in \mathbb{Z}/2 \times O(d)$, where $\sigma_V : \mathbb{Z}/2 \to O(d)$ is the group homomorphism defined by the representation V. This is an invariant for the weak equivalences on orthogonal $\mathbb{Z}/2$-spectra defined by a complete $\mathbb{Z}/2$-universe, and the constructions of our paper depend on its homotopy type.

We remark that in general $A^{\mathbb{Z}/2}$ is no longer a ring spectrum, except in the case when A is commutative (this is completely analogous to the case of discrete rings). In this section we explain how such an object generalizes the Hermitian Mackey functors of Section 1.1 (see Proposition 2.2.6), and we define a spectral version of the category of Hermitian forms over A (Definition 2.2.4).

Example 2.2.1. The examples of ring spectra with anti-involution we are concerned with are the following.

(i) Let R be a simplicial ring with anti-involution. The usual Eilenberg–Mac Lane spectrum HR, defined as the sequence of Dold–Thom constructions

$$(HR)_n = R(S^n) = \left\{ \sum_{i=1}^{k} r_i x_i \;\middle|\; r_i \in R,\ x_i \in S^n \right\} \Big/ {0x \sim r*}$$

with the involution induced by the functoriality in R, is a ring spectrum with anti-involution.

(ii) The equivariant sphere spectrum $\mathbb{S}$, defined by the sequence of trivial $\mathbb{Z}/2$-spaces S^n and its usual multiplication, is a ring spectrum with anti-involution. More generally any commutative $\mathbb{Z}/2$-equivariant ring spectrum defines a ring spectrum with anti-involution.

(iii) If G is a topological monoid with an anti-involution $\iota : G^{\mathrm{op}} \to G$ (e.g., a group with inversion), the suspension spectrum $\mathbb{S}[G] := \mathbb{S} \wedge G_+$ is a ring spectrum with anti-involution. The multiplication is the usual one of the spherical group-ring, induced by the monoid structure of G, and the involution is $\mathrm{id} \wedge \iota : (\mathbb{S}[G])^{\mathrm{op}} = \mathbb{S}[G^{\mathrm{op}}] \to \mathbb{S}[G]$.

We let I be Bökstedt's category of finite sets and injective maps. Its objects are the natural numbers (zero included), and a morphism $i \to j$ is an injective map $\{1, \dots, i\} \to \{1, \dots, j\}$. We recall that the spectrum A induces a diagram $\Omega^\bullet A : I \to \mathrm{Top}$ (see, e.g., [Schlichtkrull 2004, §2.3]) by sending an integer i to the i-fold loop space $\Omega^i A$. We denote its homotopy colimit by

$$\Omega_I^\infty A := \operatorname*{hocolim}_I \Omega^\bullet A.$$

On the one hand the multiplication of A endows $\Omega_I^\infty A$ with the structure of a topological monoid; see [Schlichtkrull 2004, §2.2, 2.3]. On the other hand, the category I has an involution which is trivial on objects and that sends a morphism $\alpha : i \to j$ to

$$\bar{\alpha}(s) = j + 1 - \alpha(i + 1 - s).$$

The diagram $\Omega^\bullet A$ has a $\mathbb{Z}/2$-structure in the sense of [Dotto and Moi 2016, Definition 1.1], defined by the maps

$$\Omega^i A \xrightarrow{\Omega^i w} \Omega^i A \xrightarrow{\Omega^i \chi_i} \Omega^i A \xrightarrow{(-)\circ\chi_i} \Omega^i A.$$

Here $\chi_i \in \Sigma_i$ is the permutation that reverses the order $\{1, \dots, i\}$, applied both to the sphere S^i and through the orthogonal spectrum structure of A. This induces a $\mathbb{Z}/2$-action on the space $\Omega^\infty_I A$. These two structures make $\Omega^\infty_I A$ into a topological monoid with anti-involution. In case A is nonunital, $\Omega^\infty_I A$ is a nonunital topological monoid with anti-involution. We refer to [Dotto et al. 2017, §1.2] for the details.

Remark 2.2.2. Throughout the paper, we make extensive use of the fact that, as a $\mathbb{Z}/2$-space, $\Omega^\infty_I A$ is equivalent to the genuine equivariant infinite loop space of A. There is a comparison map

$$\iota_* : \operatorname*{hocolim}_{n\in\mathbb{N}} \Omega^{n\rho+1} A_{n\rho+1} \to \Omega^\infty_I A,$$

where ρ is the regular representation of $\mathbb{Z}/2$ and $A_V = \mathrm{Iso}(\mathbb{R}^d, V)_+ \wedge_{O(d)} A_d$ is the value of A at a d-dimensional $\mathbb{Z}/2$-representation V. This map is induced by the inclusion $\iota : \mathbb{N} \to I$ that sends n to $2n+1$ and the unique morphism $n \le m$ to the equivariant injection

$$\iota(n \le m)(k) := k + m - n.$$

The failure of ι_* to be a nonequivariant equivalence is measured by the action of the monoid of self-injections of $\mathbb{N}$, and this action is homotopically trivial since A is an orthogonal spectrum; see [Sagave and Schlichtkrull 2013, §2.5]. A similar comparison exists equivariantly, and the comparison map is an equivariant equivalence since A is an orthogonal $\mathbb{Z}/2$-spectrum. The details can be found in [Dotto et al. 2017, §1.2].

Since $\Omega^\infty_I A$ is a topological monoid with anti-involution, there is an action

$$\Omega^\infty_I A \times (\Omega^\infty_I A)^{\mathbb{Z}/2} \to (\Omega^\infty_I A)^{\mathbb{Z}/2}$$

defined by $a \cdot b := abw(a)$, where w denotes the anti-involution of $\Omega^\infty_I A$. We use this action to define a category of Hermitian forms over A in a way analogous to the category of Hermitian forms over a Hermitian Mackey functor of Section 1.2.

Definition 2.2.3. We let $M^\vee_n(A)$ be the (nonunital) ring spectrum

$$M^\vee_n(A) = \bigvee_{n\times n} A,$$

where the multiplication is defined by the maps $M^\vee_n(A_i) \wedge M^\vee_n(A_j) \to M^\vee_n(A_{i+j})$ that send $((k,l),a) \wedge ((k',l'),a')$, where $(k,l), (k',l') \in n \times n$ indicate the wedge component, to $((k,l'), a\cdot a')$ if $l = k'$, and to the basepoint otherwise.

The anti-involution $w : A^{\mathrm{op}} \to A$ induces an anti-involution on $M^\vee_n(A)$, defined as the composite

$$M^\vee_n(A)^{\mathrm{op}} = \Big(\bigvee_{n\times n} A\Big)^{\mathrm{op}} \xrightarrow{\tau} \bigvee_{n\times n} A^{\mathrm{op}} \xrightarrow{\vee w} \bigvee_{n\times n} A = M^\vee_n(A),$$

where τ is the automorphism of $n \times n$ which swaps the product factors. We now let $\widehat{M}_n^{\vee}(A)$ be the nonunital topological monoid with anti-involution

$$\widehat{M}_n^{\vee}(A) := \Omega_I^{\infty} M_n^{\vee}(A).$$

We let $\widehat{\mathrm{GL}}_n^{\vee}(A)$ be the subspace of invertible components, defined as the pull-back of nonunital topological monoids with anti-involution

$$\begin{tikzcd} \widehat{\mathrm{GL}}_n^{\vee}(A) \arrow[r] \arrow[d] & \widehat{M}_n^{\vee}(A) \arrow[d] \\ \mathrm{GL}_n(\pi_0 A) \arrow[r] & M_n(\pi_0 A) \end{tikzcd}$$

Here $\pi_0 A$ is the ring of components of A with the induced anti-involution, $M_n(\pi_0 A)$ is its ring of $(n \times n)$-matrices, and $\mathrm{GL}_n(\pi_0 A)$ is the subgroup of invertible matrices. The involutions on $M_n(\pi_0 A)$ and $\mathrm{GL}_n(\pi_0 A)$ are by entrywise involution and transposition. The right vertical map is the composite

$$\Omega_I^{\infty} M_n^{\vee}(A) \xrightarrow{\sim} \Omega_I^{\infty}\Bigl(\prod_{n \times n} A\Bigr) \twoheadrightarrow \pi_0 \Omega_I^{\infty}\Bigl(\prod_{n \times n} A\Bigr) \cong M_n(\pi_0 A),$$

which is both equivariant and multiplicative.

Definition 2.2.4. An n-dimensional Hermitian form on A is an element of the fixed-points space $\widehat{\mathrm{GL}}_n^{\vee}(A)^{\mathbb{Z}/2}$. These form a category $\mathrm{Sym}\,\widehat{\mathrm{GL}}_n^{\vee}(A)$ as in Proposition 2.1.6, and we define

$$\mathrm{Herm}_A := \coprod_{n \geq 0} \mathrm{Sym}\,\widehat{\mathrm{GL}}_n^{\vee}(A).$$

Remark 2.2.5. The anti-involution of A induces a functor $D : \mathscr{M}_A^{\mathrm{op}} \to \mathscr{M}_A$ on the category $\mathscr{M}_A$ of right A-module spectra. It is defined by the spectrum of module maps

$$D(P) = \mathrm{Hom}_A(P, A_w),$$

where A_w is the spectrum A equipped with the right A-module structure

$$A \wedge A \xrightarrow{\mathrm{id} \wedge w} A \wedge A^{\mathrm{op}} \xrightarrow{\tau} A^{\mathrm{op}} \wedge A \xrightarrow{\mu} A,$$

where τ is the symmetry isomorphism and μ is the multiplication of A. The ring spectrum of $(n \times n)$-matrices on A is usually defined as the endomorphism spectrum $\mathrm{End}\bigl(\bigvee_n A\bigr)$ of the sum of n copies of A. Since $\mathrm{Hom}_A(A, P)$ is canonically isomorphic to P, there is an isomorphism of ring spectra $\mathrm{End}\bigl(\bigvee_n A\bigr) \cong \prod_n \bigvee_n A$. The module $\bigvee_n A$ is homotopically self-dual, as the inclusion of wedges into products

$$\bigvee_n A \xrightarrow{\sim} \prod_n A \cong D\Bigl(\bigvee_n A\Bigr)$$

is a natural equivalence. The functor D then defines a homotopy coherent involution on $\mathrm{End}\big(\bigvee_n A\big)$, and one could define Hermitian forms as the homotopy fixed points of this action; this is essentially the approach of [Spitzweck 2016].

The point of our construction is to refine this homotopy coherent action to a genuine equivariant homotopy type that incorporates the fixed-points spectrum of A which, morally speaking, determines the notion of "symmetry" for the associated Hermitian forms. The inclusion $M_n^\vee(A) \to \mathrm{End}\big(\bigvee_n A\big)$ is a weak equivalence, it is coherently equivariant, and $M_n^\vee(A)$ has a strict involution which defines such a genuine homotopy type. The price we pay is that $M_n^\vee(A)$ is nonunital and it has only partially defined block sums (see Section 2.3), but it gives rise to small and manageable models for the real K-theory spectrum. A different such model for the matrix ring has been provided in [Kro 2005].

Since $\Omega_I^\infty A$ is a monoid with anti-involution, there is an action

$$\Omega_I^\infty A \times (\Omega_I^\infty A)^{\mathbb{Z}/2} \to (\Omega_I^\infty A)^{\mathbb{Z}/2}$$

defined by $a \cdot x := axw(a)$. On components this induces an action of the multiplicative monoid $\pi_0 A \cong \pi_0 \Omega_I^\infty A$ on the group of components $\pi_0 A^{\mathbb{Z}/2} \cong \pi_0(\Omega_I^\infty A)^{\mathbb{Z}/2}$ of the fixed-points spectrum. We recall that for every $\mathbb{Z}/2$-spectrum A, the abelian groups $\pi_0 A$ and $\pi_0 A^{\mathbb{Z}/2}$ form a $\mathbb{Z}/2$-Mackey functor $\underline{\pi}_0 A$.

Proposition 2.2.6. *Let A be a ring spectrum with anti-involution. The action of $\Omega_I^\infty A$ on $(\Omega_I^\infty A)^{\mathbb{Z}/2}$ defines the structure of a Hermitian Mackey functor on the $\mathbb{Z}/2$-Mackey functor $\underline{\pi}_0 A$. Moreover, there is an isomorphism of Hermitian Mackey functors*

$$\underline{\pi}_0 M_n^\vee A \cong M_n(\underline{\pi}_0 A),$$

where $M_n(\underline{\pi}_0 A)$ is the Mackey functor of matrices of Definition 1.1.5.

Proof. The multiplication of $\pi_0 A$ clearly anticommutes with the involution, since it does so for $\Omega_I^\infty A$. Similarly, the relation between the restriction and the action is satisfied, because the restriction $R : \pi_0 A^{\mathbb{Z}/2} \to \pi_0 A$ corresponds to the fixed-points inclusion of $\Omega_I^\infty A$. In order to verify the other conditions we describe the action of $\pi_0 A$.

Let us represent an element of $\pi_0 A$ by a map $f : S^n \to A_n$, and an element of $\pi_0 A^{\mathbb{Z}/2}$ by an equivariant map $x : S^{m\rho} \to A_{m\rho}$. We recall that there is a canonical isomorphism $S^{n\rho} \cong S^n \wedge S^n$, where the action on $S^n \wedge S^n$ swaps the two smash factors. The action is then the homotopy class of the map

$$f \cdot x : S^{(n+m)\rho} \cong S^n \wedge S^{m\rho} \wedge S^n \xrightarrow{f \wedge x \wedge (w \circ f)} A_n \wedge A_{m\rho} \wedge A_n \xrightarrow{\mu} A_{n+m\rho+n} \cong A_{(n+m)\rho},$$

where the involution on $S^n \wedge S^{m\rho} \wedge S^n$ swaps the two S^n smash factors and acts on $S^{m\rho}$, and on $A_n \wedge A_{m\rho} \wedge A_n$ it acts componentwise and swaps the two A_n factors.

The last map is the multiplication of A, and it is equivariant since the diagram

$$\begin{tikzcd}
A_n \wedge A_{m\rho} \wedge A_n \arrow[r, "\tau_3"] \arrow[d, "\mu"'] & A_n \wedge A_{m\rho} \wedge A_n \arrow[r, "w\wedge w\wedge w"] & A_n \wedge A_{m\rho} \wedge A_n \arrow[d, "\mu"] \\
A_{n+m\rho+n} \arrow[r, "w"'] & A_{n+m\rho+n} \arrow[r, "\chi_{n,n}"'] & A_{n+m\rho+n}
\end{tikzcd}$$

commutes by the definition of a ring spectrum with anti-involution, where $\tau_3 \in \Sigma_3$ reverses the order and $\chi_{n,n} \in \Sigma_{n+2m+n}$ is the permutation that swaps the first and last blocks of size n. The bottom map is by definition the action of $A_{(n+m)\rho}$.

The transfer of the class of a map $g : S^{2m} \to A_{2m}$ is defined as the class of the map

$$T(g) : S^{m\rho} \xrightarrow{p} S^{2m} \vee S^{2m} \xrightarrow{g\vee(w\circ g)} A_{m\rho} \vee A_{m\rho} \xrightarrow{\nabla} A_{m\rho},$$

where the first map collapses the fixed points $S^m \subset S^{m\rho}$, the last map is the fold, and the involutions act on, and permute, the wedge summands. The relations

$$f \cdot T(g) = T(fgw(f)) \quad \text{and} \quad (f + f') \cdot x = f \cdot x + f' \cdot x + T(fR(x)w(f'))$$

for f, f', g in $\pi_0 A$ and x in $\pi_0 A^{\mathbb{Z}/2}$ are now an immediate consequence of the naturality of the fold map, and of the distributivity of the smash product of pointed spaces over the wedge sum.

Let us now consider the Mackey functor $\underline{\pi}_0 M_n^{\vee}(A)$. Even though $M_n^{\vee}(A)$ is not unital, the map $M_n^{\vee}(A) \to \prod_n \bigvee_n A$ is an equivalence, and consequently $\pi_0 M_n^{\vee}(A) \cong M_n(\pi_0 A)$ is a unital ring. Moreover, the inclusion of indexed wedges into indexed products gives an equivalence

$$\begin{aligned}
&(\Omega_I^\infty M_n^{\vee}(A))^{\mathbb{Z}/2} \\
&\quad \xrightarrow{\sim} \Big(\Omega_I^\infty\Big(\prod_{n\times n} A\Big)\Big)^{\mathbb{Z}/2} \cong \Big(\prod_{n\times n} \Omega_I^\infty A\Big)^{\mathbb{Z}/2} \cong \Big(\prod_n (\Omega_I^\infty A)^{\mathbb{Z}/2}\Big) \times \prod_{1\le i<j\le n} (\Omega_I^\infty A).
\end{aligned}$$

Under this equivalence the action of $\Omega_I^\infty M_n^{\vee}(A)$ on $(\Omega_I^\infty M_n^{\vee}(A))^{\mathbb{Z}/2}$ corresponds to the action of $\Omega_I^\infty M_n^{\vee}(A)$ on $\big(\Omega_I^\infty \prod_{n\times n} A\big)^{\mathbb{Z}/2}$ given by $abw(a)$, where the multiplications are the infinite loop spaces of the left and right actions

$$M_n^{\vee}(A) \wedge \Big(\prod_{n\times n} A\Big) \to \prod_{n\times n} A, \qquad \Big(\prod_{n\times n} A\Big) \wedge M_n^{\vee}(A) \to \prod_{n\times n} A$$

defined by the standard matrix multiplication rules. This agrees with the action of Definition 1.1.5. □

2.3. ***The real algebraic K-theory $\mathbb{Z}/2$-space of a ring spectrum with anti-involution.*** The goal of this section is to define a $\mathbb{Z}/2$-action on the K-theory space of a ring spectrum A with anti-involution $w : A^{\mathrm{op}} \to A$. We define this action by adapting the group-completion construction of the free K-theory space

$$K(A) = \Omega B \coprod_n B\widehat{\mathrm{GL}}_n(A),$$

where $\coprod_n B\widehat{\mathrm{GL}}_n(A)$ is group-completed with respect to block-sum, to the model for the equivariant matrix ring constructed in the previous section.

We recall from Definition 2.1.3 that the classifying space of a nonunital monoid with anti-involution M inherits a natural $\mathbb{Z}/2$-action, and that we denote the corresponding $\mathbb{Z}/2$-space by $B^\sigma M$. Thus the anti-involution on $\widehat{\mathrm{GL}}_n^\vee(A)$ gives rise to a $\mathbb{Z}/2$-space $B^\sigma\widehat{\mathrm{GL}}_n^\vee(A)$. The space $\coprod_n B^\sigma\widehat{\mathrm{GL}}_n^\vee(A)$ does not have a strict monoid structure, since the standard block-sum of matrices does not restrict to the matrix rings $M_n^\vee(A)$. We can however define a Bar construction using a technique similar to Segal's group completion of partial monoids. The block-sum operation on the ring spectra $M_n^\vee(A)$ is a collection of maps

$$\oplus : M_n^\vee(A) \vee M_k^\vee(A) \to M_{n+k}^\vee(A)$$

induced by the inclusions $n \to n+k$ and $k \to n+k$. We observe that this map commutes with the anti-involutions. There is a simplicial $\mathbb{Z}/2$-space with p-simplices

$$\coprod_{n_1,\dots,n_p} B^\sigma\Omega_I^\infty(M_{n_1}^\vee(A) \vee \dots \vee M_{n_p}^\vee(A)).$$

The face maps are induced by the block-sum maps, and the degeneracies are the summand inclusions. This results in a well-defined simplicial object since $B^\sigma\Omega_I^\infty$ is functorial with respect to maps of ring spectra with anti-involution. This simplicial structure restricts to the $\mathbb{Z}/2$-spaces

$$\coprod_{n_1,\dots,n_p} B^\sigma\widehat{\mathrm{GL}}_{n_1,\dots,n_p}^\vee(A),$$

where $\widehat{\mathrm{GL}}_{n_1,\dots,n_p}^\vee(A)$ is defined as the pull-back of nonunital monoids with anti-involution

$$\begin{tikzcd}
\widehat{\mathrm{GL}}_{n_1,\dots,n_p}^\vee(A) \arrow[r] \arrow[d] & \Omega_I^\infty(M_{n_1}^\vee(A) \vee \dots \vee M_{n_p}^\vee(A)) \arrow[d] \\
(\mathrm{GL}_{n_1}(\pi_0 A) \times \dots \times \mathrm{GL}_{n_p}(\pi_0 A)) \arrow[r, hook] & (M_{n_1}(\pi_0 A) \times \dots \times M_{n_p}(\pi_0 A))
\end{tikzcd}$$

Definition 2.3.1. The free real K-theory space of a ring spectrum with anti-involution A is the $\mathbb{Z}/2$-space defined as the loop space of the thick geometric realization

$$\mathrm{KR}(A) := \Omega\left\| \coprod_{n_1,\dots,n_\bullet} B^\sigma\widehat{\mathrm{GL}}_{n_1,\dots,n_\bullet}^\vee(A)\right\|.$$

We observe that all of our constructions are homotopy invariant, and therefore the functor KR sends maps of ring spectra with anti-involution which are stable equivalences of underlying $\mathbb{Z}/2$-spectra to equivalences of $\mathbb{Z}/2$-spaces. We conclude this section by verifying that the underlying space of $\mathrm{KR}(A)$ has the right homotopy type.

Proposition 2.3.2. *Let A be a ring spectrum with anti-involution, and let us denote by $A|_1$ the underlying ring spectrum of A and by $\mathrm{KR}(A)|_1$ the underlying space of $\mathrm{KR}(A)$. There is a weak equivalence*

$$\mathrm{KR}(A)|_1 \xrightarrow{\sim} K(A|_1).$$

Proof. For convenience, we drop the restriction notation. The inclusion of wedges into products defines an equivalence of ring spectra

$$M_n^{\vee}(A) \to M_n(A) := \prod_n \bigvee_n A,$$

and therefore an equivalence of monoids $\widehat{M}_n^{\vee}(A) \to \widehat{M}_n(A)$ on infinite loop spaces. This induces an equivalence of spaces

$$\coprod_n B\widehat{M}_n^{\vee}(A) \to \coprod_n B\widehat{M}_n(A)$$

after taking the thick realization. The block-sum maps of $M_n^{\vee}(A)$ and $M_n(A)$ are compatible, in the sense that the diagram

$$\begin{tikzcd} M_n^{\vee}(A) \vee M_k^{\vee}(A) \arrow[r, "\oplus"] \arrow[d, "\sim"'] & M_{n+k}^{\vee}(A) \arrow[d, "\sim"] \\ M_n(A) \times M_k(A) \arrow[r, "\oplus"'] & M_{n+k}(A) \end{tikzcd}$$

commutes. It follows that the levelwise equivalences on the Bar constructions

$$\coprod_{n_1,\dots,n_p} B\Omega_I^{\infty}(M_{n_1}^{\vee}(A) \vee \cdots \vee M_{n_p}^{\vee}(A)) \xrightarrow{\sim} \coprod_{n_1,\dots,n_p} (B\Omega_I^{\infty} M_{n_1}(A)) \times \cdots \times (B\Omega_I^{\infty} M_{n_p}(A))$$

commute with the face maps. After restricting to invertible components and taking thick geometric realizations this gives the equivalence $\mathrm{KR}(A)|_1 \simeq K(A|_1)$. $\square$

In the construction of the trace map we need to compare the dihedral nerve $B^{\mathrm{di}}M$ and the free loop space $\Lambda^{\sigma} B^{\sigma} M$ for $M = \widehat{\mathrm{GL}}_{n_1,\dots,n_p}^{\vee}(A)$. In order to apply Lemma 2.1.12 we need the following.

Proposition 2.3.3. *Suppose that A is levelwise well-pointed, and that the unit map $S^0 \to A_0$ is an h-cofibration. Then the monoid with anti-involution $\widehat{\mathrm{GL}}_{n_1,\dots,n_p}^{\vee}(A)$ is quasiunital and group-like* (see Definition 2.1.10).

Proof. By definition $\widehat{\mathrm{GL}}^{\vee}_{n_1,\dots,n_p}(A)$ is the monoid of invertible components of

$$\widehat{M}^{\vee}_{n_1,\dots,n_p}(A) := \Omega^{\infty}_I(M^{\vee}_{n_1}(A) \vee \cdots \vee M^{\vee}_{n_p}(A)),$$

and it is therefore group-like. We show that $\widehat{M}^{\vee}_n(A)$ is quasiunital, which implies that $\widehat{\mathrm{GL}}^{\vee}_n(A)$ is quasiunital by restricting to the invertible components. The proof for general p is similar. The inclusion of wedges into products induces an equivalence of nonunital ring spectra

$$f : M^{\vee}_n(A) \xrightarrow{\sim} M_n(A) := \prod_n \bigvee_n A,$$

where the multiplication on $M_n A$ is defined by representing an element in a given spectral degree by a matrix with at most one nonzero entry in each column, and applying the standard matrix multiplication. This induces an equivalence of nonunital topological monoids

$$\widehat{M}^{\vee}_n(A) = \Omega^{\infty}_I M^{\vee}_n(A) \xrightarrow{\sim} \Omega^{\infty}_I M_n(A),$$

where $\Omega^{\infty}_I M_n(A)$ is unital and well-pointed.

The spectrum $\prod_{n\times n} A$ has a $\mathbb{Z}/2$-action, defined by applying the anti-involution entrywise and by composing with the involution of $n \times n$ that switches the factors. The inclusion of indexed wedges into indexed products induces an equivalence of equivariant spectra

$$\phi : M^{\vee}_n(A) = \bigvee_{n\times n} A \xrightarrow{\sim} \prod_{n\times n} A,$$

and therefore an equivalence on fixed points $(\Omega^{\infty}_I M^{\vee}_n(A))^{\mathbb{Z}/2} \xrightarrow{\sim} \left(\Omega^{\infty}_I \prod_{n\times n} A\right)^{\mathbb{Z}/2}$.

The monoid $\prod_n \bigvee_n A$ acts on the right on the spectrum $\prod_{n\times n} A$ by right matrix multiplication. Precisely, the action is determined by the maps

$$\Big(\prod_{n\times n} A_l\Big) \wedge \Big(\prod_n \bigvee_n A_k\Big) \to \prod_{n\times n} A_{l+k}$$

that send an element $\{b_{ij} \in A_l\} \wedge (1 \le h_1, \dots, h_n \le n, a_1, \dots, a_n \in A_k)$ to the matrix with (i, j)-entry $\mu(b_{ih_j}, a_j) \in A_{l+k}$, where μ is the multiplication of A. There is a second right-action which is defined by left matrix multiplication via the conjugate transposed, namely by sending $\{b_{ij}\} \wedge (1 \le i_1, \dots, i_n \le n, a_1, \dots, a_n)$ to the matrix with (i, j)-entry $\mu(w(a_i), b_{h_{ij}})$. These induce two commuting right actions of $\Omega^{\infty}_I M_n(A)$ on $\Omega^{\infty}_I\left(\prod_{n\times n} A\right)$, which we denote respectively by $xm := x \cdot_1 m$ and $w(m)x := x \cdot_2 m$. A straightforward argument shows that these actions satisfy $w(xm) = w(m)w(x)$, where w denotes the involution on $\Omega^{\infty}_I\left(\prod_{n\times n} A\right)$. We regard $\Omega^{\infty}_I\left(\prod_{n\times n} A\right)$ as a right $\Omega^{\infty}_I M_n(A)$-space via the action

$$x \cdot m := w(m)xm.$$

We observe that this action restricts to the fixed points space $\big(\Omega^\infty_I \prod_{n\times n} A\big)^{\mathbb{Z}/2}$, and that the equivalence

$$(\Omega^\infty_I M^\vee_n(A))^{\mathbb{Z}/2} \to \Big(\Omega^\infty_I \prod_{n\times n} A\Big)^{\mathbb{Z}/2}$$

is a map of $\Omega^\infty_I M^\vee_n(A)$-modules.

Finally, the unit of $\Omega^\infty_I M_n(A)$ is mapped to a fixed point under the nonequivariant map $\Omega^\infty_I M_n(A) \to \Omega^\infty_I \prod_{n\times n} A$ that includes wedges into products. We denote its image by e. Since f and ϕ are inclusions of wedges into products, the relation

$$e\cdot f(m) := w(f(m))ef(m) = \phi(w(m)m)$$

is satisfied for every $m \in M$. □

2.4. ***Connective equivariant deloopings of real algebraic K-theory.*** We show that the real K-theory space of a ring spectrum with anti-involution defined in Section 2.3 is the equivariant infinite loop space of a (special) $\mathbb{Z}/2$-equivariant Γ-space. Our construction of these deloopings is an adaptation of Segal's construction [Segal 1974; Shimada and Shimakawa 1979] for spectrally enriched symmetric monoidal categories, to a set-up where the symmetric monoidal structure is partially defined.

We start with an explicit definition of the $\mathbb{Z}/2$-Γ-space in question, and we relate it to Segal's construction in the proof of Proposition 2.4.2. Recall from [Shimakawa 1991] that a G-Γ-space, where G is a finite group, is a functor $X : \Gamma^{\mathrm{op}} \to \mathrm{Top}^G_*$ from the category Γ^{op} which is a skeleton for the category of pointed finite sets and pointed maps, to the category of pointed G-spaces. This induces a symmetric G-spectrum whose n-th space is the value at the n-sphere of the left Kan-extension of X to the category of finite pointed simplicial sets.

For every natural number n and sequence of nonnegative integers $\underline{a} = (a_1, \dots, a_n)$ we consider the collections of permutations $\alpha = \big\{\alpha_{S,T} \in \mathcal{S}_{\sum_{i\in S\amalg T} a_i}\big\}$, where the indices S, T run through the pairs of disjoint subsets $S \amalg T \subset \{1, \dots, n\}$, and $\mathcal{S}_k$ denotes the symmetric group. We require that these permutations satisfy the standard conditions of Segal's construction; see, e.g., [Dundas et al. 2013, Definition 2.3.1.1]. We denote by $\langle\underline{a}\rangle$ the set of such collections α for the n-tuple $\underline{a}$.

Given a ring spectrum with anti-involution A we let $\mathrm{KR}(A) : \Gamma^{\mathrm{op}} \to \mathrm{Top}^{\mathbb{Z}/2}_*$ be the functor that sends the pointed set $n_+ = \{+, 1, \dots, n\}$ to

$$\mathrm{KR}(A)_n := \coprod_{\underline{a}=(a_1,\dots,a_n)} B^\sigma\big(\widetilde{\langle\underline{a}\rangle} \times \widehat{\mathrm{GL}}^\vee_{\underline{a}}(A)\big),$$

where $\widehat{\mathrm{GL}}^\vee_{\underline{a}}(A) := \widehat{\mathrm{GL}}^\vee_{a_1,\dots,a_n}(A)$ is defined in Section 2.3, and $\widetilde{\langle\underline{a}\rangle}$ is the category with objects set $\langle\underline{a}\rangle$, and with a unique morphism between any pair of objects. The category $\widetilde{\langle\underline{a}\rangle}$ has a duality that is the identity on objects, and that sends the unique morphism $\alpha \to \beta$ to the unique morphism $\beta \to \alpha$. Thus $\widetilde{\langle \underline{a} \rangle} \times \widehat{\mathrm{GL}}^{\vee}_{a_1,\dots,a_n}(A)$ is a nonunital topological category with duality, and B^{σ} is the functor of Remark 2.1.13.

Remark 2.4.1. Since every object in $\widetilde{\langle \underline{a} \rangle}$ is both initial and final, the projection map

$$\widetilde{\langle \underline{a} \rangle} \times \widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A) \to \widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A)$$

is an equivalence of topological categories. Moreover, by the uniqueness of the morphisms of $\widetilde{\langle \underline{a} \rangle}$ we see that $\mathrm{Sym}\,\widetilde{\langle \underline{a} \rangle} = \widetilde{\langle \underline{a} \rangle}$. Thus the projection map

$$\mathrm{Sym}\big(\widetilde{\langle \underline{a} \rangle} \times \widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A)\big) \cong \mathrm{Sym}\,\widetilde{\langle \underline{a} \rangle} \times \mathrm{Sym}\,\widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A) \to \mathrm{Sym}\,\widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A)$$

is also an equivalence of categories, and $B^{\sigma}\big(\widetilde{\langle \underline{a} \rangle} \times \widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A)\big)$ and $B^{\sigma}\widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A)$ are equivariantly equivalent.

The extra $\widetilde{\langle \underline{a} \rangle}$-coordinate is used for the definition of $\mathrm{KR}(A)$ on morphisms. Given a pointed map $f : n_+ \to k_+$ and $\alpha \in \langle \underline{a} \rangle$ we let $f_*\underline{a} \in \mathbb{N}^{\times k}$ and $f_*\alpha \in \langle f_*\underline{a} \rangle$ respectively denote

$$(f_*\underline{a})_i := \sum_{j \in f^{-1}(i)} a_j \quad \text{and} \quad (f_*\alpha)_{S,T} := \alpha_{f^{-1}S, f^{-1}T},$$

for every $1 \le i \le k$ and $S \amalg T \subset \{1, \dots, k\}$. We define $f_* : \mathrm{KR}_n(A) \to \mathrm{KR}_k(A)$ by mapping the $\underline{a}$-summand to the $f_*\underline{a}$-summand by a map which is B^{σ} of the functor

$$\widetilde{\langle \underline{a} \rangle} \times \widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A) \to \widetilde{\langle f_*\underline{a} \rangle} \times \widehat{\mathrm{GL}}^{\vee}_{f_*\underline{a}}(A)$$

defined as follows. The first component is just the composite of the projection map with the map that takes α to $f_*\alpha$:

$$\widetilde{\langle \underline{a} \rangle} \times \widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A) \to \widetilde{\langle \underline{a} \rangle} \xrightarrow{f_*} \widetilde{\langle f_*\underline{a} \rangle}.$$

The second component is defined as follows. A pair of permutations $\alpha, \beta \in \langle \underline{a} \rangle$ gives rise to a morphism of monoids with anti-involution

$$(\alpha, \beta)_* : \widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A) \to \widehat{\mathrm{GL}}^{\vee}_{f_*\underline{a}}(A).$$

It is induced by the map of ring spectra with anti-involution obtained by wedging over $i \in \{1, \dots, k\}$ the maps

$$(\alpha, \beta)_j : \bigvee_{j \in f^{-1}(i)} M^{\vee}_{a_j}(A) \to M^{\vee}_{(f_*\underline{a})_i}(A)$$

defined by sending $x \in M^{\vee}_{a_j}(A)$ to $(\alpha, \beta)_j(x) := \beta_{(f^{-1}i)\backslash j, j}(0 \oplus x)\alpha^{-1}_{(f^{-1}i)\backslash j, j}$, where $0 \oplus x$ is the value at x of the block-sum map

$$\oplus : M^{\vee}_{(f_*\underline{a})_i - a_j}(A) \vee M^{\vee}_{a_j}(A) \to M^{\vee}_{(f_*\underline{a})_i}(A)$$

and α and β are considered as permutation matrices. Explicitly, an element of $M^{\vee}_{a_j}(A)$ in spectrum level l consists of a pair $(c,d) \in a_j \times a_j$ and a point $x \in A_l$. This is sent to

$$(\alpha,\beta)_j(c,d,x) = (\beta_{(f^{-1}i)\backslash j,j}(\iota c), \alpha_{(f^{-1}i)\backslash j,j}(\iota d), x),$$

where $\iota : a_j \to (f_*\underline{a})_j$ is the inclusion. The second component of the map $\mathrm{KR}(f)$ is then induced by the functor

$$\widetilde{\langle \underline{a}\rangle} \times \widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A) \to \widehat{\mathrm{GL}}^{\vee}_{f_*\underline{a}}(A)$$

that sends a morphism (α,β,x) in $\widetilde{\langle \underline{a}\rangle} \times \widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A)$ to $(\alpha,\beta)_*x$. Given a $\mathbb{Z}/2$-Γ-space X, we let X_{S^1} be the first space of the associated $\mathbb{Z}/2$-spectrum, defined as the geometric realization $X_{S^1} = |n \mapsto X_{S^1_n}|$, where S^1_n is the set of n-simplices of the simplicial circle $S^1_{\bullet} = \Delta^1/\partial$.

Proposition 2.4.2. *Let A be a ring spectrum with anti-involution. The functor* $\mathrm{KR}(A)$ *is a special* $\mathbb{Z}/2$-Γ*-space in the sense of [Shimakawa 1989]. The* $\mathbb{Z}/2$*-space* $\mathrm{KR}(A)_{S^1}$ *is equivalent to the real K-theory* $\mathbb{Z}/2$*-space of Definition 2.3.1. The underlying* Γ*-space of* $\mathrm{KR}(A)$ *is equivalent the K-theory* Γ*-space of A.*

Proof. Let $\mathcal{F}_A$ be the spectrally enriched category whose objects are the nonnegative integers and where the endomorphisms of k consist of the matrix ring $\prod_k \bigvee_k A$. We recall that the Segal construction on $\mathcal{F}_A$ is the Γ-category enriched in symmetric spectra defined by sending n_+ to the category $\mathcal{F}_A[n]$. Its objects are the pairs $\langle \underline{a}, \alpha\rangle$ where $\underline{a} = (a_1,\dots,a_n)$ is a collection of nonnegative integers and α is a collection of isomorphisms $\alpha = \{\alpha_{S,T} : \sum_{s\in S} a_s + \sum_{t\in T} a_t \to \sum_{i\in S\amalg T} a_i\}$ in the underlying category of $\mathcal{F}_A$ satisfying the conditions of [Dundas et al. 2013, Definition 2.3.1.1]. The spectrum of morphisms $\langle \underline{a},\alpha\rangle \to \langle \underline{b},\beta\rangle$ is nontrivial only if $\underline{a} = \underline{b}$, and it is defined by the collection of elements $\{f_S \in M_{\sum_{s\in S} a_s}(A)\}_{S\subset \underline{n}}$ which satisfy $\beta_{S,T}(f_S \oplus f_T) = f_{S\amalg T}\alpha_{S,T}$.

There is an equivalence of spectral categories $\mathcal{F}_A^{\times n} \to \mathcal{F}_A[n]$ that sends the tuple $\underline{a} = (a_1,\dots,a_n)$ to $\langle \underline{a},\alpha\rangle$, where $\alpha_{S,T}$ is the permutation matrix of the permutation of $S \amalg T$ that sends the order on $S \amalg T$ induced by the disjoint union of the orders of S and T to the order of $S \amalg T$ as a subset of n (the point is that $\mathcal{F}_A[n]$ is functorial in n with respect to all maps of pointed sets, whereas $\mathcal{F}_A^{\times n}$ is only functorial for order-preserving maps).

Now let $\mathcal{F}_A^{\vee}$ be the equivalent subcategory (without identities) of $\mathcal{F}_A$ with all the objects, but where the endomorphisms of a are the ring spectra $M^{\vee}_a(A) = \bigvee_{n\times n} A$. The space $\mathrm{KR}(A)_n$ is roughly the invertible components of the classifying space of the image of $(\mathcal{F}_A^{\vee})^{\vee n}$ inside $\mathcal{F}_A[n]$. More precisely, there is a commutative square of spectrally enriched categories

$$\begin{tikzcd}
(\mathcal{F}_A^{\vee})^{\vee n} \arrow[r] \arrow[d, "\sim"] & \mathcal{F}_A^{\vee}[n] \arrow[d] \\
\mathcal{F}_A^{\times n} \arrow[r, "\sim"'] & \mathcal{F}_A[n]
\end{tikzcd}$$

where $\mathcal{F}_A^{\vee}[n]$ is defined as the subcategory of $\mathcal{F}_A[n]$ on the objects $\langle \underline{a}, \alpha\rangle$, where $\alpha_{S,T}$ is a permutation representation, and where the morphisms $\{f_S\}_{S\subset \underline{n}}$ are such that there is a $j \in \underline{n}$ such that $f_S = 0$ if $j \notin S$. The top horizontal arrow is simply the restriction of the bottom horizontal one.

The spectral category $\mathcal{F}_A^{\vee}[n]$ has a strict duality, which is the identity on objects and the anti-involution on the matrix ring $M_a^{\vee}(A)$ on morphisms. We observe that a morphism $\{f_S\}_{S\subset \underline{n}}$ in $\mathcal{F}_A^{\vee}[n]$ is determined by the value f_j, since for every $S \subset n$ containing j we have that

$$f_S = f_{(S\backslash j)\amalg j} = \beta_{S\backslash j,j}(0_{S\backslash j} \oplus f_j)\alpha^{-1}_{S\backslash j,j}.$$

Moreover, this equation determines the relation $\beta_{S,T}(f_S \oplus f_T) = f_{S\amalg T}\alpha_{S,T}$, and it follows that the value at n_+ of the corresponding $\mathbb{Z}/2$-Γ-space is

$$\coprod_{\underline{a}} B^{\sigma}\big(\widetilde{\langle \underline{a}\rangle} \times \Omega_I^{\infty}(M_{a_1}^{\vee}(A) \vee \cdots \vee M_{a_n}^{\vee}(A))\big).$$

Its invertible components are then by definition $\mathrm{KR}_n(A)$ and the functoriality in Γ^{op} induced by the ambient category $\mathcal{F}_A[n]$ is the one described above. In particular $\mathrm{KR}_n(A)$ is functorial in n. The fact that $f_S := \beta_{S\backslash j,j}(0_{S\backslash j} \oplus f_j)\alpha^{-1}_{S\backslash j,j}$ determines a well-defined morphism $\langle \underline{a}, \alpha\rangle \to \langle \underline{a}, \beta\rangle$ follows from the following calculation:

$$\begin{aligned}
&\beta_{S,T}(f_S \oplus 0_T)\alpha^{-1}_{S,T}\\
&\quad= \beta_{S,T}((\beta_{S\backslash j,j}(0_{S\backslash j} \oplus f_j)\alpha^{-1}_{S\backslash j,j}) \oplus 0_T)\alpha^{-1}_{S,T}\\
&\quad= \beta_{S,T}(\beta_{S\backslash j,j} \amalg \mathrm{id}_T)(0_{S\backslash j} \oplus f_j \oplus 0_T)(\alpha^{-1}_{S\backslash j,j} \amalg \mathrm{id}_T)\alpha^{-1}_{S,T}\\
&\quad= \beta_{j,S\amalg T\backslash j}(\mathrm{id}_j \amalg \beta_{S\backslash j,T})(\tau_{S\backslash j,j} \amalg \mathrm{id}_T)(0_{S\backslash j} \oplus f_j \oplus 0_T)(\alpha^{-1}_{S\backslash j,j} \amalg \mathrm{id}_T)\alpha^{-1}_{S,T}\\
&\quad= \beta_{j,S\amalg T\backslash j}(\mathrm{id}_j \amalg \beta_{S\backslash j,T})(\tau_{S\backslash j,j} \amalg \mathrm{id}_T)(0_{S\backslash j} \oplus f_j \oplus 0_T)\\
&\qquad\qquad\qquad\qquad \circ (\tau_{j,S\backslash j} \amalg \mathrm{id}_T)(\mathrm{id}_j \amalg \alpha^{-1}_{S\backslash j,T})\alpha^{-1}_{j,S\amalg T\backslash j}\\
&\quad= \beta_{j,S\amalg T\backslash j}(\mathrm{id}_j \amalg \beta_{S\backslash j,T})(f_j \oplus 0_{S\amalg T\backslash j})(\mathrm{id}_j \amalg \alpha^{-1}_{S\backslash j,T})\alpha^{-1}_{j,S\amalg T\backslash j}\\
&\quad= \beta_{j,S\amalg T\backslash j}(f_j \oplus 0_{S\amalg T\backslash j})\alpha^{-1}_{j,S\amalg T\backslash j}\\
&\quad= \beta_{S\amalg T\backslash j,j}\tau_{j,S\amalg T\backslash j}(f_j \oplus 0_{S\amalg T\backslash j})\tau_{j,S\amalg T\backslash j}\alpha^{-1}_{S\amalg T\backslash j,j}\\
&\quad= \beta_{S\amalg T\backslash j,j}(0_{S\amalg T\backslash j} \oplus f_j)\alpha^{-1}_{S\amalg T\backslash j,j}\\
&\quad= f_{S\amalg T},
\end{aligned}$$

where $\tau_{S,T}:\sum_{s\in S}a_s+\sum_{t\in T}a_t\to\sum_{t\in T}a_t+\sum_{s\in S}a_s$ is the symmetry isomorphism of the symmetric monoidal structure. From this description of the morphisms of $\mathcal{F}^{\vee}_A[n]$ one can easily see that the top horizontal map of the square above, and hence all of its maps, are equivalences of categories. Thus the Γ-space underlying $\mathrm{KR}(A)$ is equivalent to the K-theory of A.

We show that KR is a special $\mathbb{Z}/2$-Γ-space. For every group homomorphism $\sigma:\mathbb{Z}/2\to\Sigma_n$ we need to show that the map

$$\mathcal{F}^{\vee}_A[n]\to(\mathcal{F}^{\vee}_A[1])^{\times n},$$

whose j-component is induced by the map $n_+\to 1_+$ that sends j to 1 and the rest to the basepoint, is a $\mathbb{Z}/2$-equivariant equivalence. Here the involution is induced by $\sigma:\mathbb{Z}/2\to\Sigma_n$ through the functoriality in n. The square above provides an equivalence of spectrally enriched categories $\mathcal{F}^{\vee}_A\to\mathcal{F}^{\vee}_A[1]$. This functor is in fact an isomorphism on mapping spectra, and it is therefore an equivariant equivalence. We show that the top horizontal arrow of

$$\begin{tikzcd}
(\mathcal{F}^{\vee}_A)^{\vee n} \arrow[r] \arrow[d, "\sim"'] & \mathcal{F}^{\vee}_A[n] \arrow[d] \\
(\mathcal{F}^{\vee}_A)^{\times n} \arrow[r, "\sim"'] & (\mathcal{F}^{\vee}_A[1])^{\times n}
\end{tikzcd}$$

defined as the restriction of $\mathcal{F}^{\times n}_A\to\mathcal{F}_A[n]$ is an equivariant equivalence, which finishes the proof. It has an equivariant inverse $\mathcal{F}^{\vee}_A[n]\to(\mathcal{F}^{\vee}_A)^{\vee n}$ that sends an object $\langle\underline{a},\alpha\rangle$ to $\underline{a}$ and a morphism $\{f_S\}$ to $f_{\{j\}}$. □

2.5. ***Pairings in real algebraic K-theory.*** Let A and B be ring spectra with anti-involution. Their smash product $A\wedge B$ is a ring spectrum with anti-involution, where the multiplication is defined componentwise and the anti-involution is diagonal. The aim of this section is to define a pairing in the homotopy category of $\mathbb{Z}/2$-spectra

$$\mathrm{KR}(A)\wedge\mathrm{KR}(B)\to\mathrm{KR}(A\wedge B),$$

which extends Loday's pairing of K-theory spectra of discrete rings [Loday 1976, Chapitre II]. The point-set construction of this pairing does not quite lift to the category of $\mathbb{Z}/2$-spectra because of the failure of thick realizations to commute with products strictly, but this is the only obstruction.

The standard formula for the Kronecker product of matrices restricts to an isomorphism

$$\otimes:M^{\vee}_n(X)\wedge M^{\vee}_k(Y)\cong\bigvee_{n\times n\times k\times k}(X\wedge Y)\to\bigvee_{nk\times nk}(X\wedge Y)=M^{\vee}_{nk}(X\wedge Y),$$

where X and Y are either pointed spaces or spectra, which is determined by the isomorphism $n\times n\times k\times k\cong nk\times nk$ sending (i,j,m,l) to $((i-1)n+m,(j-1)n+l)$.

This isomorphism is moreover $\mathbb{Z}/2$-equivariant with respect to the transposition of matrices.

Given $\underline{a}=(a_1,\dots,a_n)$, we define

$$\widehat{M}^\vee_{\underline{a}}(A):=\Omega^\infty_I(M^\vee_{a_1}(A)\vee\cdots\vee M^\vee_{a_n}(A)),$$

which is a topological monoid with anti-involution. Given $\underline{a}=(a_1,\dots,a_n)$ and $\underline{b}=(b_1,\dots,b_k)$, we let $\underline{a}\cdot\underline{b}$ be the nk-sequence of nonnegative integers

$$\underline{a}\cdot\underline{b}:=(a_1b_1,a_1b_2,\dots,a_1b_k,a_2b_1,\dots,a_2b_k,\dots,a_nb_1,\dots,a_nb_k).$$

Using the Kronecker pairing above we define $\mathbb{Z}/2$-equivariant maps

$$\begin{tikzcd}
B^\sigma\big(\widetilde{\langle\underline{a}\rangle}\times\widehat{M}^\vee_{\underline{a}}(A)\big)\times B^\sigma\big(\widetilde{\langle\underline{b}\rangle}\times\widehat{M}^\vee_{\underline{b}}(B)\big) \arrow[dddd, dashed] & B^\sigma\big(\widetilde{\langle\underline{a}\rangle}\times\widetilde{\langle\underline{b}\rangle}\times\widehat{M}^\vee_{\underline{a}}(A)\times\widehat{M}^\vee_{\underline{b}}(B)\big) \arrow[l, "\sim"'] \arrow[d] \\
 & B^\sigma\big(\widetilde{\langle\underline{a}\rangle}\times\widetilde{\langle\underline{b}\rangle}\times\operatorname*{hocolim}_{I\times I}\Omega^{i+j}(M^\vee_{\underline{a}}(A_i)\wedge M^\vee_{\underline{b}}(B_j))\big) \arrow[d, "\cong"'] \\
 & B^\sigma\Big(\widetilde{\langle\underline{a}\rangle\times\langle\underline{b}\rangle}\times\operatorname*{hocolim}_{I\times I}\Omega^{i+j}\Big(\bigvee_{(h,l)\in n\times k}(M^\vee_{a_h}(A_i)\wedge M^\vee_{b_l}(B_j))\Big)\Big) \arrow[d, "\cong"', "\otimes"] \\
 & B^\sigma\Big(\widetilde{\langle\underline{a}\rangle\times\langle\underline{b}\rangle}\times\operatorname*{hocolim}_{I\times I}\Omega^{i+j}\Big(\bigvee_{(h,l)\in n\times k}(M^\vee_{a_hb_l}(A_i\wedge B_j))\Big)\Big) \arrow[d, "\otimes\times\iota"] \\
B^\sigma\big(\widetilde{\langle\underline{a}\cdot\underline{b}\rangle}\times\widehat{M}^\vee_{\underline{a}\cdot\underline{b}}(A\wedge B)\big) & B^\sigma\Big(\widetilde{\langle\underline{a}\cdot\underline{b}\rangle}\times\operatorname*{hocolim}_{I\times I}\Omega^{i+j}\Big(\bigvee_{(h,l)\in n\times k}(M^\vee_{a_hb_l}((A\wedge B)_{i+j}))\Big)\Big) \arrow[l, "+_*"']
\end{tikzcd}$$

where $\otimes\times\iota$ is induced by the product of the map $\otimes:\langle\underline{a}\rangle\times\langle\underline{b}\rangle\to\langle\underline{a}\cdot\underline{b}\rangle$ defined by taking the Kronecker product of permutations (i.e., the Kronecker product of the associated permutation matrices), and of the canonical map $\iota: A_i\wedge B_j\to(A\wedge B)_{i+j}$. By restricting to the invertible components of $\widehat{M}^\vee_{\underline{a}}(A)$ and $\widehat{M}^\vee_{\underline{b}}(A)$ this gives $\mathbb{Z}/2$-equivariant maps

$$\mathrm{KR}(A)_n\wedge\mathrm{KR}(B)_k\xleftarrow{\sim}Z_{n,k}\to\mathrm{KR}(A\wedge B)_{nk},$$

where $Z_{n,k}=\coprod_{\underline{a},\underline{b}}B^\sigma\big(\widetilde{\langle\underline{a}\rangle}\times\widetilde{\langle\underline{b}\rangle}\times\widehat{\mathrm{GL}}^\vee_{\underline{a}}(A)\times\widehat{\mathrm{GL}}^\vee_{\underline{b}}(B)\big)$. This zig-zag is natural in both variables $n,k\in\Gamma^{\mathrm{op}}$, and it therefore induces a pairing in the homotopy category of $\mathbb{Z}/2$-spectra

$$\otimes:\mathrm{KR}(A)\wedge\mathrm{KR}(B)\to\mathrm{KR}(A\wedge B).$$

By taking fixed points and geometric fixed points, we respectively obtain pairings

$$\otimes : \mathrm{KR}(A)^{\mathbb{Z}/2} \wedge \mathrm{KR}(B)^{\mathbb{Z}/2} \to (\mathrm{KR}(A) \wedge \mathrm{KR}(B))^{\mathbb{Z}/2} \to \mathrm{KR}(A \wedge B)^{\mathbb{Z}/2},$$
$$\otimes : \Phi^{\mathbb{Z}/2}\,\mathrm{KR}(A) \wedge \Phi^{\mathbb{Z}/2}\,\mathrm{KR}(B) \to \Phi^{\mathbb{Z}/2}\,\mathrm{KR}(A \wedge B)$$

in the homotopy category of spectra. Let A be a ring spectrum with anti-involution and π a well-pointed topological group. The corresponding group-algebra is the ring spectrum

$$A[\pi] := A \wedge \pi_+$$

with the anti-involution defined diagonally from the anti-involution of A and the inversion map of π. This is an A-algebra via the map

$$A \wedge A[\pi] = A \wedge A \wedge \pi_+ \xrightarrow{\mu \wedge \mathrm{id}} A \wedge \pi_+ = A[\pi],$$

where μ denotes the multiplication of A. If moreover A is a commutative ring spectrum with anti-involution, that is, a commutative $\mathbb{Z}/2$-equivariant orthogonal ring spectrum, then the map $\mu \wedge \mathrm{id}$ is a morphism of ring spectra with anti-involution. Thus one can compose the pairings above for $B = A[\pi]$ with the induced map $\mathrm{KR}(A \wedge A[\pi]) \to \mathrm{KR}(A[\pi])$. The associativity of the Kronecker product of matrices then gives the following.

Proposition 2.5.1. *Let A be a commutative $\mathbb{Z}/2$-equivariant orthogonal ring spectrum and π a topological group. The graded abelian groups $\pi_*(\mathrm{KR}(A)^{\mathbb{Z}/2})$ and $\pi_*\Phi^{\mathbb{Z}/2}\,\mathrm{KR}(A)$ are graded rings. The graded abelian group $\pi_*(\mathrm{KR}(A[\pi])^{\mathbb{Z}/2})$ is a graded $\pi_*(\mathrm{KR}(A)^{\mathbb{Z}/2})$-module, and $\pi_*\Phi^{\mathbb{Z}/2}\,\mathrm{KR}(A[\pi])$ is a graded $\pi_*\Phi^{\mathbb{Z}/2}\,\mathrm{KR}(A)$-module.* □

2.6. *The Hermitian K-theory and genuine L-theory of a ring spectrum with anti-involution.* In this section we relate the fixed-points and geometric fixed-points spectra of KR of Eilenberg–Mac Lane spectra with the classical constructions of Hermitian K-theory and L-theory.

Definition 2.6.1. The free Hermitian K-theory space of a ring spectrum with anti-involution A is the fixed-points space

$$\mathrm{GW}(A) := \mathrm{KR}(A)^{\mathbb{Z}/2} \cong \Omega B\Big(\coprod_n \big(B^\sigma \widehat{\mathrm{GL}}{}^\vee_n(A)\big)^{\mathbb{Z}/2}\Big).$$

The free Hermitian K-theory spectrum of A is the spectrum $\mathrm{GW}(A)$ associated to the fixed-points Γ-space

$$\mathrm{GW}(A)_n := \mathrm{KR}(A)_n^{\mathbb{Z}/2} \cong \coprod_{\underline{a}=(a_1,\dots,a_n)} \big(B^\sigma\big(\widetilde{\langle \underline{a}\rangle} \times \widehat{\mathrm{GL}}{}^\vee_{\underline{a}}(A)\big)\big)^{\mathbb{Z}/2}.$$

Remark 2.6.2. The spectrum associated to the Γ-space $\mathrm{GW}(A)$ is the naïve fixed-points spectrum of the $\mathbb{Z}/2$-spectrum associated to the $\mathbb{Z}/2$-Γ-space $\mathrm{KR}(A)$. Since

$\mathrm{KR}(A)$ is special as an equivariant Γ-space (see Proposition 2.4.2), the canonical map of spectra $\mathrm{GW}(A) \to \mathrm{KR}(A)^{\mathbb{Z}/2}$ is a stable equivalence, where $\mathrm{KR}(A)^{\mathbb{Z}/2}$ is the genuine fixed-points spectrum of $\mathrm{KR}(A)$.

We recall the terminology from [Schwede 2013] (although our notation deviates slightly). If X is an orthogonal G-spectrum for a finite group G, the naïve fixed points of X is a spectrum with underlying sequence of spaces X_n^G. The genuine fixed points are obtained by deriving the naïve fixed points with respect to the model structure on orthogonal G-spectra associated to a complete G-universe. Concretely, these can be defined as a spectrum whose n-th space is $(\Omega^{n\bar{\rho}} X_{n\rho})^G$, where ρ is the regular representation of G with reduced regular representation $\bar{\rho}$, and for every d-dimensional real G-representation V the pointed G-space X_V is the space X_d, where $g \in G$ acts by $(g, \sigma_V(g)) \in G \times O(d)$ for $\sigma_V : G \to O(d)$ the group homomorphism defined by the representation V.

We now analyze the Γ-space $\mathrm{GW}(A)$ and interpret it as the Segal construction of a symmetric monoidal category of Hermitian forms on A. We recall from Proposition 2.1.6 that the fixed-points space of $B^\sigma \widehat{\mathrm{GL}}{}^\vee_{\underline{a}}(A)$ is the classifying space of a topological category $\mathrm{Sym}\, \widehat{\mathrm{GL}}{}^\vee_{\underline{a}}(A)$. Its space of objects is the space of invertible components of the fixed-points space

$$\widehat{M}^\vee_{\underline{a}}(A)^{\mathbb{Z}/2} := \big(\Omega^\infty_I (M^\vee_{a_1}(A) \vee \cdots \vee M^\vee_{a_n}(A))\big)^{\mathbb{Z}/2},$$

which is equivalent to the infinite loop space of the fixed-points spectrum

$$M^\vee_{a_1}(A)^{\mathbb{Z}/2} \times \cdots \times M^\vee_{a_1}(A)^{\mathbb{Z}/2}$$

(see Remark 2.2.2). A morphism $l : m \to n$ of $\mathrm{Sym}\, \widehat{\mathrm{GL}}{}^\vee_{\underline{a}}(A)$ is a homotopy invertible element of $\widehat{M}^\vee_{\underline{a}}(A)$ which satisfies $m = w(l)nl$, where w denotes the involution on $\widehat{M}^\vee_{\underline{a}}(A)$. Thus we think of $\mathrm{GW}(A)$ as the Segal construction of a symmetric monoidal category

$$\mathrm{Herm}_A = \coprod_n \mathrm{Sym}\, \widehat{\mathrm{GL}}{}^\vee_n(A)$$

of spectral Hermitian forms on A.

Proposition 2.6.3. *Suppose R is a simplicial ring with anti-involution $w : R^{\mathrm{op}} \to R$. There is a weak equivalence between* $\mathrm{GW}(HR)$ *and the connective cover of the Hermitian K-theory spectrum* $\mathrm{GW}(R) := {}_1\widetilde{L}(R)$ *of [Burghelea and Fiedorowicz 1985]. In particular if R is discrete this is equivalent to the connective Hermitian K-theory of free R-modules of [Karoubi 1973], when $2 \in R$ is invertible.*

If R is commutative, this equivalence induces a ring isomorphism on homotopy groups, where $\mathrm{GW}(HR)$ *is equipped with the multiplication of Proposition 2.5.1.*

Proof. The inclusion of wedges into products defines a map of ring spectra with anti-involution

$$M_n^\vee(HR) = \bigvee_{n\times n} HR \to \prod_{n\times n} HR \cong HM_n(R),$$

where $M_n(R) = \bigoplus_{n\times n} R$ is the ring of $n \times n$-matrices with entries in R. On the underlying $\mathbb{Z}/2$-spectra, this is an inclusion of indexed wedges into indexed products and it is therefore a stable equivalence. On the level of Γ-categories this shows that the composite

$$\mathcal{F}_{HR}^\vee[n] \to \mathcal{F}_{HR}[n] \to H\mathcal{F}_R[n]$$

is an equivalence, where $\mathcal{F}_R[n]$ is Segal's construction of the symmetric monoidal category of free R-modules $(\mathcal{F}_R, \oplus)$, with the duality induced by conjugate transposition of matrices (we observe that the middle term $\mathcal{F}_{HR}[n]$ does not have a duality). At the level of Γ-spaces this induces an equivalence

$$(B^\sigma\Omega^\infty H\mathcal{F}_{HR}^\vee[n])^{\mathbb{Z}/2} \xrightarrow{\sim} (B^\sigma\Omega^\infty H\mathcal{F}_R[n])^{\mathbb{Z}/2} \xleftarrow{\sim} (B^\sigma\mathcal{F}_R[n])^{\mathbb{Z}/2} \cong B\,\mathrm{Sym}(\mathcal{F}_R[n]),$$

which restricted to invertible components gives an equivalence

$$\mathrm{GW}_n(R) \simeq B\,\mathrm{Sym}(i\mathcal{F}_R[n]).$$

Moreover, there is a functor of Γ-categories $(\mathrm{Sym}\, i\mathcal{F}_R)[n] \to \mathrm{Sym}(i\mathcal{F}_R[n])$, and since both categories are equivalent to $\mathrm{Sym}\, i\mathcal{F}_R^{\times n}$ it is an equivalence. Finally, $\mathrm{Sym}\, i\mathcal{F}_R$ is the category of Hermitian forms over the simplicial ring R of [Burghelea and Fiedorowicz 1985].

When R is commutative, both ring structures on $\mathrm{GW}(R)$ and $\mathrm{GW}(HR)$ are defined from the Kronecker product of matrices, and by inspection the equivalence above is multiplicative. □

The previous proposition extends to the Hermitian K-theory of Mackey functors defined in Section 1.2, as we now show. We say that a ring spectrum with anti-involution A is Eilenberg–Mac Lane if $\pi_n A = 0 = \pi_n(A^{\mathbb{Z}/2})$ for all $n \neq 0$. In this case we write $A = HL$, where L denotes the Hermitian Mackey functor $L = \underline{\pi}_0 A$ of Proposition 2.2.6.

Proposition 2.6.4. *Let HL be an Eilenberg–Mac Lane ring spectrum with anti-involution. There is a stable equivalence of Γ-spaces*

$$\mathrm{GW}(HL) \xrightarrow{\sim} \mathrm{GW}(L)$$

induced by the projection maps $\Omega_I^\infty HL \to L(\mathbb{Z}/2)$ and $(\Omega_I^\infty HL)^{\mathbb{Z}/2} \to L(*)$ onto π_0. If moreover HL is commutative, this equivalence is multiplicative with respect to the ring structures of Proposition 1.3.3 and Proposition 2.5.1.

Proof. We recall that since $\Omega_I^\infty HL$ is a topological monoid with anti-involution, there is an action

$$\Omega_I^\infty HL \times (\Omega_I^\infty HL)^{\mathbb{Z}/2} \to (\Omega_I^\infty HL)^{\mathbb{Z}/2}$$

defined by sending (m,n) to $mnw(m)$, where w is the involution on $\Omega_I^\infty HL$. The Hermitian structure on $\underline{\pi}_0 HL$ is defined by taking π_0 of this map. We also recall from Proposition 2.2.6 that $M_n^\vee(HL)$ is a model for the Eilenberg–Mac Lane spectrum of the Hermitian Mackey functor of matrices $M_n(L)$ of Definition 1.1.5. Thus the projections onto π_0 define an equivalence of topological categories

$$\coprod_n \operatorname{Sym} \widehat{\mathrm{GL}}_n^\vee(HL) \xrightarrow{\sim} i\,\mathrm{Herm}_L$$

onto the category of Hermitian forms on M and isomorphisms. At the level of Γ-spaces this gives an equivalence

$$\mathrm{GW}(HL)_n \cong \coprod_{\underline{a}} B\big(\langle \underline{a}\rangle \times \operatorname{Sym} \widehat{\mathrm{GL}}_{\underline{a}}^\vee(HL)\big) \xrightarrow{\sim} i\,\mathrm{Herm}_M[n]$$

onto the Segal Γ-category associated to $(i\,\mathrm{Herm}_L, \oplus)$, by the same argument of Proposition 2.6.3. □

Remark 2.6.5. We do not know if every Hermitian Mackey functor can be realized as the Mackey functor of components of a nonunital ring spectrum with anti-involution (the nonunitality condition comes from the fact that an equivariant unit map $\mathbb{S} \to A$ induces a map $\mathbb{A} = \pi_0(\mathbb{S}^{\mathbb{Z}/2}) \to \pi_0(A^{\mathbb{Z}/2})$, and therefore a preferred element $1 \in \pi_0(A^{\mathbb{Z}/2})$). If there was an Eilenberg–Mac Lane spectrum functor H from Hermitian Mackey functors to nonunital ring spectra with anti-involution, Proposition 2.6.4 would in fact show that $\mathrm{GW}(L) \simeq \mathrm{KR}(HL)^{\mathbb{Z}/2}$ for every Hermitian Mackey functor L, allowing us to refine $\mathrm{GW}(L)$ to the fixed points of a genuine equivariant spectrum $\mathrm{KR}(HL)$.

For discrete rings with anti-involution R the Eilenberg–Mac Lane spectrum HR of Example 2.2.1 serves this purpose, and the Hermitian Mackey functor of components $\underline{\pi}_0 HR$ is the Hermitian Mackey functor defined by R as in Example 1.1.2. Thus Propositions 2.6.3 and 2.6.4 tell us that $\mathrm{GW}(HR) \simeq \mathrm{GW}(R)$, where $\mathrm{GW}(R)$ is by definition the group-completion of the category of Hermitian forms of free R-modules, which is denoted by ${}_1\widetilde{L}(R)$ in [Burghelea and Fiedorowicz 1985].

Definition 2.6.6. The free genuine L-theory spectrum of a ring spectrum with anti-involution A is the geometric fixed-points spectrum

$$\mathrm{L}^{\mathrm{g}}(A) := \Phi^{\mathbb{Z}/2}\,\mathrm{KR}(A).$$

When R is a discrete ring with anti-involution, we define $\mathrm{L}^{\mathrm{g}}(R) := \mathrm{L}^{\mathrm{g}}(HR)$.

Here the term "genuine" and the superscript of L^{g} refer to the fact that $\mathrm{L}^{\mathrm{g}}(A)$ depends on the genuine equivariant homotopy type of the input ring spectrum A.

Proposition 2.6.7. *Let R be a discrete ring with anti-involution. There is a natural isomorphism*

$$\mathrm{L}^{\mathrm{g}}_*(R)\left[\tfrac{1}{2}\right] \cong \mathrm{L}^{\mathrm{q}}_{*\geq 0}(R)\left[\tfrac{1}{2}\right]$$

after inverting 2, *where $\mathrm{L}^{\mathrm{q}}_{*\geq 0}$ are the quadratic L-groups. This isomorphism agrees with the restriction to the appropriate summands of the splittings*

$$\left(\mathrm{K}_*(R)\left[\tfrac{1}{2}\right]\right)^{\mathbb{Z}/2} \oplus \mathrm{L}^{\mathrm{g}}_*(R)\left[\tfrac{1}{2}\right] \cong \mathrm{GW}_*(R)\left[\tfrac{1}{2}\right] \cong \left(\mathrm{K}_*(R)\left[\tfrac{1}{2}\right]\right)^{\mathbb{Z}/2} \oplus \mathrm{L}^{\mathrm{q}}_{*\geq 0}(R)\left[\tfrac{1}{2}\right]$$

of the isotropy separation sequence away from 2 *and of the splitting of [Burghelea and Fiedorowicz 1985, Proposition* 6.2*], respectively.*

Proof. The isotropy separation sequence for the $\mathbb{Z}/2$-spectrum $\mathrm{KR}(R)$ (see, e.g., [Hesselholt and Madsen 1997, Proposition 2.1]) is a fiber sequence of spectra

$$\mathrm{KR}(R)_{h\mathbb{Z}/2} \xrightarrow{\text{tran}} \mathrm{KR}(R)^{\mathbb{Z}/2} \xrightarrow{\phi} \mathrm{L}^{\mathrm{g}}(R),$$

where the left map induces the transfer map

$$\pi_* \,\mathrm{KR}(R) \to \pi_* \,\mathrm{KR}(R)_{h\mathbb{Z}/2} \to \pi_* \,\mathrm{KR}(R)^{\mathbb{Z}/2}$$

of the Mackey structure of $\underline{\pi}_* \,\mathrm{KR}(R)$ on homotopy groups. Let us first identify the homotopy groups of the cofiber $\mathrm{L}^{\mathrm{g}}(R)$ of the transfer with the cokernel of the hyperbolic map, after inverting 2.

The composite $N : \mathrm{KR}(R)_{h\mathbb{Z}/2} \xrightarrow{\text{tran}} \mathrm{KR}(R)^{\mathbb{Z}/2} \to \mathrm{KR}(R)^{h\mathbb{Z}/2}$ of the transfer and the canonical map to the homotopy fixed points is the norm map, which is an equivalence if 2 is inverted in $\mathrm{KR}(R)$; this follows from the Tate spectral sequence of [Greenlees and May 1995, Theorem 10.5]. Thus the composite

$$\left(\mathrm{KR}(R)\left[\tfrac{1}{2}\right]\right)^{\mathbb{Z}/2} \to \left(\mathrm{KR}(R)\left[\tfrac{1}{2}\right]\right)^{h\mathbb{Z}/2} \simeq \left(\mathrm{KR}(R)\left[\tfrac{1}{2}\right]\right)_{h\mathbb{Z}/2}$$

of the canonical map and the inverse of the norm defines a natural splitting of the transfer on homotopy groups, giving naturally split short-exact sequences

$$0 \to \pi_n(\mathrm{KR}(R)_{h\mathbb{Z}/2})\left[\tfrac{1}{2}\right] \xrightarrow{\text{tran}} \pi_n(\mathrm{KR}(R)^{\mathbb{Z}/2})\left[\tfrac{1}{2}\right] \xrightarrow{\phi} \mathrm{L}^{\mathrm{g}}_n(R)\left[\tfrac{1}{2}\right] \to 0$$

for every $n \geq 0$. After inverting 2, the E_2-page of the Bousfield–Kan spectral sequence for the homotopy orbits spectrum $\left(\mathrm{KR}(R)\left[\tfrac{1}{2}\right]\right)_{h\mathbb{Z}/2}$ is concentrated on the 0-th horizontal line, and the homotopy groups

$$\pi_*\left(\left(\mathrm{KR}(R)\left[\tfrac{1}{2}\right]\right)_{h\mathbb{Z}/2}\right)$$

are then isomorphic to the coinvariants of the involution on $\mathrm{K}_*(R)\left[\frac{1}{2}\right]$. We then obtain short exact sequences

$$0 \to \left((\mathrm{K}_n(R))\left[\tfrac{1}{2}\right]\right)_{\mathbb{Z}/2} \xrightarrow{\text{tran}} \pi_n(\mathrm{KR}(R)^{\mathbb{Z}/2})\left[\tfrac{1}{2}\right] \xrightarrow{\phi} \mathrm{L}^{\mathrm{g}}_n(R)\left[\tfrac{1}{2}\right] \to 0,$$

where the transfer is induced on coinvariants by the transfer $\mathrm{K}_n(R) \to \pi_n(\mathrm{KR}(R)^{\mathbb{Z}/2})$ of the Mackey functor $\underline{\pi}_n\,\mathrm{KR}(R)$. The double coset formula tells us that

$$\text{res}\,\text{tran} = \text{id} + w,$$

where $\text{res} : \pi_n(\mathrm{KR}(R)^{\mathbb{Z}/2}) \to \mathrm{K}_n(R)$ is the restriction induced by the canonical map from the fixed points to the underline spectrum, and w is the involution induced by the $\mathbb{Z}/2$-action of $\mathrm{KR}(R)$. Therefore the left-hand transfer of the short exact sequence above is a section for the map

$$q : \pi_n(\mathrm{KR}(R)^{\mathbb{Z}/2})\left[\tfrac{1}{2}\right] \xrightarrow{\text{res}/2} (\mathrm{K}_n(R))\left[\tfrac{1}{2}\right] \twoheadrightarrow \left((\mathrm{K}_n(R))\left[\tfrac{1}{2}\right]\right)_{\mathbb{Z}/2}.$$

In particular, the cokernel $\mathrm{L}^{\mathrm{g}}_n(R)\left[\frac{1}{2}\right]$ of the transfer is naturally isomorphic to the kernel of q. Under the identification $\mathrm{KR}(R)^{\mathbb{Z}/2} \simeq \mathrm{GW}(R)$ of Proposition 2.6.3 the restriction map $\text{res} : \pi_n(\mathrm{KR}(R)^{\mathbb{Z}/2}) \to \mathrm{K}_n(R)$ corresponds to the map induced by the functor F that sends a Hermitian form to its underlying free R-module (this is immediate from the definitions, and the fact that under the isomorphism of Proposition 2.1.6 the inclusion $(B^\sigma M)^{\mathbb{Z}/2} \hookrightarrow BM$ corresponds to the realization of the map $N(M; M^{\mathbb{Z}/2}) \to N(M; *)$ that collapses $M^{\mathbb{Z}/2}$). The hyperbolic functor induces a map

$$H : \mathrm{K}_n(R) \to \pi_n\,\mathrm{GW}(R),$$

which also satisfies $FH = \text{id} + w$, and it follows that H also defines a section of q, and thus that the cokernel of H is also isomorphic to the kernel of q. Combining these isomorphisms we obtain a natural isomorphism

$$\mathrm{L}^{\mathrm{g}}_n(R)\left[\tfrac{1}{2}\right] \cong \text{coker}(\text{tran}) \cong \ker(q) \cong \text{coker}(H) =: W_n(R)\left[\tfrac{1}{2}\right],$$

where the cokernel of H is by definition the Witt-group $W_n(R)\left[\frac{1}{2}\right]$. This agrees with the L-group $\mathrm{L}^{\mathrm{q}}_n(R)$ after inverting 2; see, e.g., [Loday 1976, Théorème 3.2.6], or combine [Ranicki 2001, §9] and [Schlichting 2010, Lemma 4.10], at least when $n > 0$. For $n = 0$ one needs to make sure that the 0-th Witt group defined using finitely generated projective modules agrees with the 0-th Witt group defined using finitely generated free modules, away from 2. By performing algebraic surgery with the respective decorations, these are respectively isomorphic to the 0-th quadratic L-groups of bounded chain complexes of finitely generated projective and free modules; see [Ranicki 1980, §9, p. 180]. These agree after inverting 2 since the relative term provided by the Rothenberg sequence is 2-torsion.

Burghelea and Fiedorowicz [1985, Proposition 6.2] show that there is a natural isomorphism

$$\mathrm{GW}_*(R)\big[\tfrac{1}{2}\big] = \big(\mathrm{GW}_*(R)\big[\tfrac{1}{2}\big]\big)^s \oplus \big(\mathrm{GW}_*(R)\big[\tfrac{1}{2}\big]\big)^a \cong \big(\mathrm{K}_*(R)\big[\tfrac{1}{2}\big]\big)^{\mathbb{Z}/2} \oplus W_{*\geq 0}(R)\big[\tfrac{1}{2}\big],$$

where the superscripts s and a on GW_* denote, respectively, the 1- and (-1)-eigenspaces of the $\mathbb{Z}/2$-action σ on GW defined by taking the opposite sign of the entries of the matrix of a Hermitian form. Here $\big(\mathrm{K}_*(R)\big[\tfrac{1}{2}\big]\big)^{\mathbb{Z}/2}$ is the fixed points of the involution on homotopy groups induced by the involution on KR. This is stated only for $* \geq 1$, but the case $* = 0$ follows again by the argument above. The first splitting is the canonical decomposition into symmetric and antisymmetric part that sends x to $((x+\sigma x) \oplus (x - \sigma x))/2$. The second splitting is the direct sum of two isomorphisms, the first of which is induced by the forgetful functor $F : \big(\mathrm{GW}_*(R)\big[\tfrac{1}{2}\big]\big)^s \to \big(\mathrm{K}_*(R)\big[\tfrac{1}{2}\big]\big)^{\mathbb{Z}/2}$. Its inverse is the restriction on invariants of half the hyperbolic map

$$H/2 : \big(\mathrm{K}_*(R)\big[\tfrac{1}{2}\big]\big)^{\mathbb{Z}/2} \to \big(\mathrm{GW}_*(R)\big[\tfrac{1}{2}\big]\big)^s.$$

In other words, this is induced by the split short exact sequence

$$0 \to \big((\mathrm{K}_n(R))\big[\tfrac{1}{2}\big]\big)^{\mathbb{Z}/2} \xrightarrow{H/2} \pi_n \,\mathrm{GW}(R)\big[\tfrac{1}{2}\big] \xrightarrow{\pi} W_n(R)\big[\tfrac{1}{2}\big] \to 0.$$

Under the isomorphism $N : \big((\mathrm{K}_n(R))\big[\tfrac{1}{2}\big]\big)_{\mathbb{Z}/2} \to \big((\mathrm{K}_n(R))\big[\tfrac{1}{2}\big]\big)^{\mathbb{Z}/2}$ that sends the orbit of x to $x + w(x)$, half the hyperbolic corresponds to the hyperbolic map H, that is, the splitting of [Burghelea and Fiedorowicz 1985, Proposition 6.2] is induced by the split short exact sequence

$$0 \to \big((\mathrm{K}_n(R))\big[\tfrac{1}{2}\big]\big)_{\mathbb{Z}/2} \xrightarrow{H} \pi_n \,\mathrm{GW}(R)\big[\tfrac{1}{2}\big] \xrightarrow{\pi} W_n(R)\big[\tfrac{1}{2}\big] \to 0.$$

Now the fact that H and tran have a common retraction q provides the desired comparison. □

2.7. ***The assembly map of real algebraic K-theory.*** We define an assembly map for the real K-theory functor in the same spirit as Loday's definition [1976], using the multiplicative pairing of Section 2.5. We then relate this to the classical assemblies of Hermitian K-theory and L-theory in the case of Eilenberg–Mac Lane spectra.

Let A be a ring spectrum with anti-involution and π a well-pointed topological group. The corresponding group-algebra is the ring spectrum

$$A[\pi] := A \wedge \pi_+$$

with the anti-involution defined diagonally from the anti-involution of A and the inversion map of π.

Remark 2.7.1. Suppose that R is a discrete ring with anti-involution w and that π is discrete. Then the inclusion of indexed wedges into indexed products defines an equivalence of ring spectra with anti-involution

$$(HR)[\pi] = HR \wedge \pi_+ = \bigvee_{\pi} HR \xrightarrow{\sim} \bigoplus_{\pi} HR \cong H(R[\pi]),$$

where the anti-involution on the group-ring $R[\pi]$ sends $r \cdot g$ to $w(r) \cdot g^{-1}$, for all $r \in R$ and $g \in \pi$. More generally, the fixed-points spectrum of $A[\pi]$ decomposes as

$$(A[\pi])^{\mathbb{Z}/2} \xrightarrow{\sim} \Big(\bigoplus_{\pi} A\Big)^{\mathbb{Z}/2} \cong \Big(\bigoplus_{\pi^{\mathbb{Z}/2}} A^{\mathbb{Z}/2}\Big) \times \Big(\bigoplus_{\pi^{\mathrm{free}}/\mathbb{Z}/2} A\Big),$$

and an argument analogous to Proposition 2.2.6 shows that the Mackey functor of components $\underline{\pi}_0(A[\pi])$ is isomorphic to the group-Mackey functor $(\underline{\pi}_0 A)[\pi]$ of Definition 1.1.7.

There is a morphism of $\mathbb{Z}/2$-Γ-spaces $\gamma : \mathbb{S} \wedge B^{\sigma}\pi_+ \to \mathrm{KR}(\mathbb{S}[\pi])$, which is adjoint to the map of $\mathbb{Z}/2$-spaces

$$B^{\sigma}\pi \hookrightarrow B^{\sigma}\widehat{\mathrm{GL}}_1^{\vee}(\mathbb{S}[\pi]) \hookrightarrow \coprod_{n} B^{\sigma}\widehat{\mathrm{GL}}_n^{\vee}(\mathbb{S}[\pi]) = \mathrm{KR}(\mathbb{S}[\pi])_1,$$

where the first map is induced by the canonical map $\pi \to \operatorname{hocolim}_I \Omega^i(S^i \wedge \pi_+)$ which includes at the object $i = 0$.

Definition 2.7.2. Let A be a ring spectrum with anti-involution, and π a topological group. The assembly map of the real K-theory of $A[\pi]$ is the map in the homotopy category of $\mathbb{Z}/2$-spectra

$$\mathrm{KR}(A) \wedge B^{\sigma}\pi_+ \xrightarrow{\mathrm{id} \wedge \gamma} \mathrm{KR}(A) \wedge \mathrm{KR}(\mathbb{S}[\pi]) \xrightarrow{\otimes} \mathrm{KR}(A \wedge \mathbb{S}[\pi]) \cong \mathrm{KR}(A[\pi]),$$

where $\otimes$ is the pairing of Section 2.5. If A is commutative, this is a map of $\mathrm{KR}(A)$-modules for the module structures of Proposition 2.5.1.

We now explain how to extract from this map an assembly map for Hermitian K-theory and L-theory by taking fixed-points spectra. We recall that there are natural transformations

$$X^{\mathbb{Z}/2} \wedge K^{\mathbb{Z}/2} \to (X \wedge K)^{\mathbb{Z}/2} \quad \text{and} \quad (\Phi^{\mathbb{Z}/2}X) \wedge K^{\mathbb{Z}/2} \xrightarrow{\sim} \Phi^{\mathbb{Z}/2}(X \wedge K)$$

for any $\mathbb{Z}/2$-spectrum X and pointed $\mathbb{Z}/2$-space K, where the first map is generally not an equivalence. Thus by applying fixed points and geometric fixed points to the KR-assembly we obtain maps

$$\mathrm{GW}(A) \wedge (B^{\sigma}\pi)_+^{\mathbb{Z}/2} \to (\mathrm{KR}(A) \wedge B^{\sigma}\pi_+)^{\mathbb{Z}/2} \to \mathrm{GW}(A[\pi]),$$
$$\mathrm{L}^{\mathrm{g}}(A) \wedge (B^{\sigma}\pi)_+^{\mathbb{Z}/2} \xrightarrow{\sim} \Phi^{\mathbb{Z}/2}(\mathrm{KR}(A) \wedge B^{\sigma}\pi_+) \to \mathrm{L}^{\mathrm{g}}(A[\pi]).$$

We recall from Lemma 2.1.9 that there is an equivariant map $B\pi \to B^\sigma\pi$, which induces a summand inclusion $B\pi \to (B^\sigma\pi)^{\mathbb{Z}/2}$ on fixed points. By precomposing with this map we obtain the following.

Definition 2.7.3. The assembly map in Hermitian K-theory and genuine L-theory of $A[\pi]$ are the maps

$$\mathrm{GW}(A) \wedge B\pi_+ \to \mathrm{GW}(A) \wedge (B^\sigma\pi)_+^{\mathbb{Z}/2} \to \mathrm{GW}(A[\pi]),$$
$$\mathrm{L}^{\mathrm{g}}(A) \wedge B\pi_+ \to \mathrm{L}^{\mathrm{g}}(A) \wedge (B^\sigma\pi)_+^{\mathbb{Z}/2} \to \mathrm{L}^{\mathrm{g}}(A[\pi]),$$

respectively. If A is commutative, these are maps of $\mathrm{GW}(A)$-modules and $\mathrm{L}^{\mathrm{g}}(A)$-modules, respectively.

Proposition 2.7.4. *Let R be a discrete ring with anti-involution. Then the assembly map for the Hermitian K-theory of $(HR)[\pi]$ described above agrees with the connective assembly for the Hermitian K-theory of $R[\pi]$ of [Burghelea and Fiedorowicz 1985, §7], under the equivalence of Proposition 2.6.3 and the equivalence $(HR)[\pi] \to H(R[\pi])$ of Remark 2.7.1. It follows that the rationalized assembly map of $\mathrm{L}^{\mathrm{g}}(R[\pi])$ agrees with the rationalized assembly of $\mathrm{L}^{\mathrm{q}}(R[\pi])$ of [Ranicki 1992, Appendix B].*

Proof. By inspection, one sees that the pairing of Section 2.5 agrees with the pairing of [Burghelea and Fiedorowicz 1985, §6] under the equivalences of Proposition 2.6.3 and Remark 2.7.1, as do the maps $\mathbb{S}\wedge B\pi_+ \to \mathrm{GW}(\mathbb{S}[\pi])$ and $\mathbb{S}\wedge B\pi_+ \to \mathrm{GW}(\mathbb{Z}[\pi])$. More precisely, the diagram

$$\begin{tikzcd}
\mathrm{GW}(HR) \wedge B\pi_+ \arrow[r, "\mathrm{id}\wedge\gamma"] \arrow[d, "\sim"'] & \mathrm{GW}(HR) \wedge \mathrm{GW}(\mathbb{S}[\pi]) \arrow[r, "\otimes"] \arrow[d] & \mathrm{GW}(HR \wedge \mathbb{S}[\pi]) \arrow[r, "\cong"] \arrow[d] & \mathrm{GW}((HR)[\pi]) \arrow[d, "\sim"'] \\
\mathrm{GW}(R) \wedge B\pi_+ \arrow[r, "\mathrm{id}\wedge\gamma_{\mathbb{Z}}"] & \mathrm{GW}(R) \wedge \mathrm{GW}(\mathbb{Z}[\pi]) \arrow[r, "\otimes_{\mathbb{Z}}"] & \mathrm{GW}(R \otimes_{\mathbb{Z}} \mathbb{Z}[\pi]) \arrow[r, "\cong"] & \mathrm{GW}(R[\pi])
\end{tikzcd}$$

commutes in the homotopy category of spectra, where $\gamma_{\mathbb{Z}}$ and $\otimes_{\mathbb{Z}}$ are the maps whose composite is the assembly of [Burghelea and Fiedorowicz 1985, §7]. The top row is precisely the restriction of the assembly of Definition 2.7.2 to the Γ-space of fixed points, which compares to the assembly of Definition 2.7.3 by the diagram

$$\begin{tikzcd}
\mathrm{GW}(HR) \wedge B\pi_+ \arrow[r, equal] \arrow[d, "\sim"'] & \mathrm{GW}(HR) \wedge B\pi_+ \arrow[r] \arrow[d] & \mathrm{GW}(HR[\pi]) \arrow[d, "\sim"] \\
\mathrm{KR}(HR)^{\mathbb{Z}/2} \wedge B\pi_+ \arrow[r] \arrow[d] & (\mathrm{KR}(HR) \wedge B\pi_+)^{\mathbb{Z}/2} \arrow[r] \arrow[d] & \mathrm{KR}(HR[\pi])^{\mathbb{Z}/2} \\
\mathrm{KR}(HR)^{\mathbb{Z}/2} \wedge (B^\sigma\pi)_+^{\mathbb{Z}/2} \arrow[r] & (\mathrm{KR}(HR) \wedge B^\sigma\pi_+)^{\mathbb{Z}/2} \arrow[ur] &
\end{tikzcd}$$

The map from the first to the second row is the transformation from naïve to genuine fixed-points spectra, and therefore the upper rectangle commutes. The bottom left square commutes by naturality of the transformation $X^{\mathbb{Z}/2} \wedge K^{\mathbb{Z}/2} \to (X \wedge K)^{\mathbb{Z}/2}$. This shows that the assemblies in Hermitian K-theory agree.

By the naturality of the isotropy separation sequence, the assembly map of $\Phi^{\mathbb{Z}/2}\,\mathrm{KR}(A)$ is the cofiber of the assembly maps for $\mathrm{K}(A)_{h\mathbb{Z}/2}$ and $\mathrm{GW}(A)$ under the transfer map. The assembly map for the Witt groups is by definition the cokernel of the assembly maps of K and GW by the hyperbolic map, or equivalently the kernel of these assemblies by the forgetful functor. It follows from Proposition 2.6.7 that these assemblies agree after inverting 2. It is widely believed by the experts that the assembly maps in Witt theory and quadratic L-theory agree away from 2, but we were unfortunately unable to track down a reference; see, e.g., [Burghelea and Fiedorowicz 1985, 8.2, diagram (5) and footnote 8]. We prove that the assemblies agree rationally, at least for the ring of integers. Our argument is far from optimal, but is sufficient for our applications.

By periodicity, the Witt groups and the quadratic L-groups are rationally modules over the Laurent polynomial algebra $\mathbb{Q}[\beta, \beta^{-1}]$, where β is of degree 4; see [Karoubi 1973, 4.10] and [Ranicki 1992, Appendix B], respectively. Let us choose isomorphisms of $\mathbb{Q}[\beta]$-modules $\phi : \mathrm{L}^{\mathrm{q}}_{*\geq 0}(\mathbb{Z}) \otimes \mathbb{Q} \cong \mathbb{Q}[\beta] \cong W_{*\geq 0}(\mathbb{Z}) \otimes \mathbb{Q}$. Then any choice of isomorphisms

$$\phi_i^{\pi} : \mathrm{L}^{\mathrm{q}}_i(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \cong W_i(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \quad \text{for } i = 0, 1, 2, 3,$$

determines an isomorphism of $\mathbb{Q}[\beta]$-modules

$$\phi^{\pi} : \mathrm{L}^{\mathrm{q}}_{*\geq 0}(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \xrightarrow{\cong} W_{*\geq 0}(\mathbb{Z}[\pi]) \otimes \mathbb{Q},$$

which is given in degree $i = 4k + l$, for $k > 0$ and $l = 0, 1, 2, 3$, by $\phi_i^{\pi} = \beta^k \phi_l^{\pi} \beta^{-k}$. Since this is an isomorphism of $\mathbb{Q}[\beta]$-modules the right square of the diagram

$$\begin{tikzcd}
\mathrm{L}^{\mathrm{q}}_{*\geq 0}(\mathbb{Z}) \otimes H_*(B\pi; \mathbb{Q}) \arrow[r, "\cong\otimes\gamma_L"] \arrow[d, "\phi\otimes\mathrm{id}"] & \mathbb{Q}[\beta] \otimes \mathrm{L}^{\mathrm{q}}_{*\geq 0}(\mathbb{Z}[\pi]) \arrow[r] \arrow[d, "\mathrm{id}\otimes\phi^{\pi}"] & \mathrm{L}^{\mathrm{q}}_{*\geq 0}(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \arrow[d, "\phi_{\pi}"] \\
W_{*\geq 0}(\mathbb{Z}) \otimes H_*(B\pi; \mathbb{Q}) \arrow[r, "\cong\otimes\gamma_W"] & \mathbb{Q}[\beta] \otimes W_{*\geq 0}(\mathbb{Z}[\pi]) \arrow[r] & W_{*\geq 0}(\mathbb{Z}[\pi]) \otimes \mathbb{Q}
\end{tikzcd}$$

defining the assemblies commutes. It is therefore sufficient to show $\phi^{\pi}\gamma_L = \gamma_W$. The map γ_W is the composite $\mathbb{S} \wedge B\pi_+ \to \mathrm{GW}(\mathbb{Z}[\pi]) \to W(\mathbb{Z}[\pi])$, and the map $\gamma_L : \mathbb{S} \wedge B\pi_+ \to L(\mathbb{Z}[\pi])$ is the "preassembly" of [Ranicki 1992, Appendix B]. Since π is discrete, these maps are determined by the corresponding group homomorphisms $\delta_L : \pi \to L_1(\mathbb{Z}[\pi])$ and $\delta_W : \pi \to W_1(\mathbb{Z}[\pi])$. By the comparison of [Ranicki 2001, 9.11], one can see that the isomorphism ϕ_1^{π} can be chosen so that $\phi_1^{\pi}\delta_L = \delta_W$. Since the homotopy category of rational spectra is equivalent to graded $\mathbb{Q}$-vector spaces, the isomorphism ϕ_{π} can be realized as a zig-zag of maps of rational spectra. Since $\phi_1^\pi \delta_L = \delta_W$, the diagram

$$\begin{tikzcd} B\pi_+ \arrow[r] \arrow[dr] & \Omega^\infty(\mathrm{L}^{\mathrm{q}}(\mathbb{Z}[\pi]) \otimes \mathbb{Q}) \arrow[d, "\Omega^\infty \phi^\pi"] \\ & \Omega^\infty(W(\mathbb{Z}[\pi]) \otimes \mathbb{Q}) \end{tikzcd}$$

commutes in the homotopy category of spaces. It follows that $(\phi^\pi \otimes \mathbb{Q})\gamma_L = \gamma_W$ commutes in the homotopy category of spectra, which concludes the proof. $\square$

The unit of the ring spectrum with anti-involution A induces a map of $\mathbb{Z}/2$-spectra

$$\eta : \mathbb{S} \to \mathrm{KR}(A).$$

It is adjoint to the map of $\mathbb{Z}/2$-spaces $S^0 \to \coprod_{k\geq 0} B^\sigma \widehat{\mathrm{GL}}_k^\vee(A)$ that sends the basepoint of S^0 to the unique point in the component $k = 0$, and the nonbasepoint of S^0 to the point in the component $k = 1$ determined by the 0-simplex of $(\mathrm{sd}_e\, N^\sigma \widehat{\mathrm{GL}}_1^\vee(A))_0 = \widehat{\mathrm{GL}}_1^\vee(A)$ defined by the unit map $S^0 \to A_0$ of A.

Definition 2.7.5. The restricted assembly map of the real K-theory of $A[\pi]$ is the map of $\mathbb{Z}/2$-spectra

$$\mathbb{S} \wedge B^\sigma \pi_+ \xrightarrow{\eta \wedge \mathrm{id}} \mathrm{KR}(A) \wedge B^\sigma \pi_+ \to \mathrm{KR}(A[\pi]).$$

The geometric fixed points of the map $\eta : \mathbb{S} \to \mathrm{KR}(A)$ provide a map

$$\mathbb{S} \cong \Phi^{\mathbb{Z}/2}\mathbb{S} \to \Phi^{\mathbb{Z}/2}\,\mathrm{KR}(A) = \mathrm{L}^{\mathrm{g}}(A),$$

which immediately leads to the corresponding restricted assembly in genuine L-theory. In Hermitian K-theory, however, the tom Dieck splitting provides a map

$$\mathbb{S} \vee \mathbb{S} \to \mathbb{S} \vee \mathbb{S}_{h\mathbb{Z}/2} \xrightarrow{\sim} \mathbb{S}^{\mathbb{Z}/2} \to \mathrm{GW}(A)$$

from two summands of the sphere spectrum, where the first map is the wedge of the identity and the canonical map to the homotopy orbits.

Definition 2.7.6. The restricted assembly maps of the Hermitian K-theory and genuine L-theory of $A[\pi]$ are respectively the maps of spectra

$$(\mathbb{S} \vee \mathbb{S}) \wedge B\pi_+ \to \mathrm{GW}(A) \wedge B\pi_+ \to \mathrm{GW}(A[\pi]),$$
$$\mathbb{S} \wedge B\pi_+ \to \mathrm{L}^{\mathrm{g}}(A) \wedge B\pi_+ \to \mathrm{L}^{\mathrm{g}}(A[\pi]).$$

The restricted assembly map of the Hermitian K-theory of $\mathbb{Z}$ is usually defined on homotopy groups by the composite

$$\begin{aligned}\mathcal{A}^0_{\mathbb{Z}[\pi]} : \mathrm{GW}_0(\mathbb{Z}) \otimes \pi_*(\mathbb{S} \wedge B\pi_+) &\hookrightarrow \mathrm{GW}_*(\mathbb{Z}) \otimes \pi_*(\mathbb{S} \wedge B\pi_+) \\ &\to \pi_*(\mathrm{GW}(\mathbb{Z}) \wedge B\pi_+) \xrightarrow{\mathcal{A}_{\mathbb{Z}[\pi]}} \mathrm{GW}_*(\mathbb{Z}[\pi]),\end{aligned}$$

where the first map includes $\mathrm{GW}_0(\mathbb{Z})$ into $\mathrm{GW}_*(\mathbb{Z})$ as the degree zero summand. Both $\pi_0(\mathbb{S}\vee\mathbb{S})$ and $\mathrm{GW}_0(\mathbb{Z})$ are isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$, and the unit map $\eta:\mathbb{S}\to\mathrm{KR}(\mathbb{Z})$ provides such an isomorphism. The following compares the resulting assemblies.

Proposition 2.7.7. *The map $\mathbb{S}\vee\mathbb{S}\to\mathrm{GW}(\mathbb{Z})$ sends the two generators in π_0 to the hyperbolic form $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and to the unit form $\langle 1\rangle$, respectively. It follows that the restricted assembly*

$$(\mathbb{S}\vee\mathbb{S})\wedge B\pi_+\to\mathrm{GW}(\mathbb{Z}[\pi])$$

agrees on homotopy groups with the restricted assembly $\mathcal{A}^0_{\mathbb{Z}[\pi]}$ upon identifying $\pi_0(\mathbb{S}\vee\mathbb{S})$ and $\mathrm{GW}_0(\mathbb{Z})$ by the isomorphism

$$\pi_0(\mathbb{S}\vee\mathbb{S})\xrightarrow{\eta}\mathrm{GW}_0(\mathbb{Z})\xrightarrow{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)}\mathrm{GW}_0(\mathbb{Z}).$$

Proof. The isotropy separation sequences for the $\mathbb{Z}/2$-spectra $\mathbb{S}$ and $\mathrm{KR}(\mathbb{Z})$ give a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \pi_0\mathbb{S}_{h\mathbb{Z}/2} \arrow[r] \arrow[d, "\cong"] & \pi_0\mathbb{S}^{\mathbb{Z}/2} \arrow[r] \arrow[d] & \pi_0\Phi^{\mathbb{Z}/2}\mathbb{S}\cong\pi_0\mathbb{S} \arrow[r] \arrow[d, "\cong"] \arrow[l, bend right] & 0 \\
\cdots \arrow[r] & \pi_0\,\mathrm{KR}(\mathbb{Z})_{h\mathbb{Z}/2} \arrow[r] & \pi_0\,\mathrm{GW}(\mathbb{Z}) \arrow[r] & \pi_0\Phi^{\mathbb{Z}/2}\,\mathrm{KR}(\mathbb{Z})\cong W_0(\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The outer vertical maps are the units of the ring structures on $\mathrm{K}(\mathbb{Z})$ and $W(\mathbb{Z})$, respectively. The splitting of the upper sequence is the tom Dieck splitting. It follows that the bottom sequence splits as well, and that the middle map is an isomorphism. The diagram

$$\begin{tikzcd}
\pi_0\mathbb{S} \arrow[r, "\cong"] \arrow[d] & \pi_0\mathbb{S}_{h\mathbb{Z}/2} \arrow[r] \arrow[d, "\cong"] & \pi_0\mathbb{S}^{\mathbb{Z}/2} \arrow[d, "\cong"] \\
\pi_0\,\mathrm{K}(\mathbb{Z}) \arrow[r, "\cong"'] & \pi_0\,\mathrm{KR}(\mathbb{Z})_{h\mathbb{Z}/2} \arrow[r] & \pi_0\,\mathrm{GW}(\mathbb{Z})
\end{tikzcd}$$

commutes, and the composite of the two lower maps takes the isomorphism class of a free $\mathbb{Z}$-module to its hyperbolic form. Moreover, the composite

$$\pi_0\mathbb{S}\to\pi_0\mathbb{S}^{\mathbb{Z}/2}\to\pi_0\,\mathrm{GW}(\mathbb{Z})$$

is the unit of the ring structure of $\pi_0\,\mathrm{GW}(\mathbb{Z})$, and it takes the generator to $\langle 1\rangle$. The identification of the restricted assemblies now follows from Proposition 2.7.4. $\square$

3. The real trace map

3.1. ***Real topological Hochschild homology.*** We recollect some results on real topological Hochschild homology from [Hesselholt and Madsen 2015; Dotto 2012; Høgenhaven 2016; Dotto et al. 2017]. The real topological Hochschild homology of a ring spectrum with anti-involution A is a genuine $\mathbb{Z}/2$-spectrum $\mathrm{THR}(A)$. It is determined by a strict $\mathbb{Z}/2$-action on the Bökstedt model for topological Hochschild homology $\mathrm{THH}(A)$ of the underlying ring spectrum. We start by recalling its construction from [Hesselholt and Madsen 2015].

Let A be a ring spectrum with anti-involution, possibly nonunital, and I the category of finite sets and injective maps. For any nonnegative integer k there is a functor $\Omega^{\bullet} A : I^{\times k+1} \to \mathrm{Sp}$ to the category of orthogonal spectra that sends $\underline{i} = (i_0, i_1, \dots, i_k)$ to the spectrum

$$\Omega^{i_0+i_1+\cdots+i_k}(\mathbb{S} \wedge A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k}).$$

Its homotopy colimit constitutes the k-simplices of a semisimplicial orthogonal spectrum

$$\mathrm{THH}_k(A) := \operatorname*{hocolim}_{\underline{i}\in I^{\times k+1}} \Omega^{i_0+i_1+\cdots+i_k}(\mathbb{S} \wedge A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k});$$

see, e.g., [Dundas et al. 2013, Definition 4.2.2.1]. The involution on I described in Section 2.2 induces an involution on $I^{\times k+1}$, sending $(i_0, i_1, \dots, i_k)$ to $(i_0, i_k, \dots, i_1)$ (it is the k-simplices of the dihedral Bar construction on I with respect to the disjoint union). The diagram $\Omega^{\bullet} A$ admits a $\mathbb{Z}/2$-structure in the sense of [Dotto and Moi 2016, Definition 1.1], defined by conjugating a loop with the maps

$$S^{i_0+i_1+\cdots+i_k} \xrightarrow{\chi_{i_0}\wedge\chi_{i_1}\wedge\cdots\wedge\chi_k} S^{i_0+i_1+\cdots+i_k} \xrightarrow{\mathrm{id}_{i_0}\wedge\chi_k} S^{i_0+i_k+\cdots+i_1}$$

and

$$\begin{tikzcd}
A_{i_0}\wedge A_{i_1}\wedge\cdots\wedge A_{i_k} \arrow[r, "\chi_{i_0}\wedge\chi_{i_1}\wedge\cdots\wedge\chi_k"] & A_{i_0}\wedge A_{i_1}\wedge\cdots\wedge A_{i_k} \arrow[r, "\mathrm{id}_{i_0}\wedge\chi_k"] & A_{i_0}\wedge A_{i_k}\wedge\cdots\wedge A_{i_1} \arrow[d, "w\wedge\cdots\wedge w"] \\
 & & A_{i_0}\wedge A_{i_k}\wedge\cdots\wedge A_{i_1}
\end{tikzcd}$$

where $\chi_j \in \Sigma_j$ is the permutation that reverses the order of $\{1, \dots, j\}$. Thus the homotopy colimit $\mathrm{THH}_k(A)$ inherits a $\mathbb{Z}/2$-action which induces a semisimplicial map $\mathrm{THH}_{\bullet}(A)^{\mathrm{op}} \to \mathrm{THH}_{\bullet}(A)$. This therefore forms a real semisimplicial spectrum $\mathrm{THH}_{\bullet}(A)$.

Definition 3.1.1 [Hesselholt and Madsen 2015]. The real topological Hochschild homology of A is the $\mathbb{Z}/2$-spectrum $\mathrm{THR}(A)$ defined as the thick geometric realization of the semisimplicial $\mathbb{Z}/2$-spectrum $\mathrm{sd}_e \mathrm{THH}_{\bullet}(A)$, where sd_e is Segal's edgewise subdivision.

Remark 3.1.2. When A is unital the real semisimplicial spectrum $\mathrm{THH}_{\bullet}(A)$ is in fact simplicial. The map $\mathrm{THR}(A) \to |\mathrm{THH}_{\bullet}(A)|$ to the thin geometric realization is a stable equivalence of $\mathbb{Z}/2$-spectra provided that A is levelwise well-pointed and that the unit $S^0 \to A_0$ is an h-cofibration; see [Dotto et al. 2017, §2.3]. In order to consider the circle action induced by the cyclic structure, one should work under these extra assumptions with the thin realization.

We recall that the $\mathbb{Z}/2$-equivariant orthogonal spectrum A can be evaluated at any d-dimensional $\mathbb{Z}/2$-representation V by setting $A_V := \mathrm{Iso}(\mathbb{R}^d, V)_+ \wedge_{O(d)} A_d$.

Lemma 3.1.3. *Suppose that for every $n \geq 0$ the space A_n is $(n-1)$-connected, and the fixed-points space $A_{n\rho}^{\mathbb{Z}/2}$ is $(n-1)$-connected, where ρ is the regular representation of $\mathbb{Z}/2$. Then the $\mathbb{Z}/2$-spectrum $\mathrm{THR}(A)$ is an equivariant Ω-spectrum. In particular, the map*

$$\Big\| \mathrm{sd}_e \operatorname*{hocolim}_{\underline{i} \in I^{\times k+1}} \Omega^{i_0+i_1+\cdots+i_k}(A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k}) \Big\| \to \Omega^{\infty\mathbb{Z}/2}\,\mathrm{THR}(A)$$

is an equivalence, where $\Omega^{\infty\mathbb{Z}/2}$ denotes the genuine equivariant infinite loop space functor.

Example 3.1.4. (i) Any suspension spectrum satisfies the hypotheses of the above lemma. Indeed, $(\mathbb{S}\wedge X)_n = S^n \wedge X$ is $(n+\mathrm{conn}\, X)$-connected nonequivariantly, and $\mathrm{conn}\, X \geq -1$. Similarly, the fixed points $(\mathbb{S}\wedge X)_{n\rho}^{\mathbb{Z}/2} = S^n \wedge X^{\mathbb{Z}/2}$ are $(n + \mathrm{conn}\, X^{\mathbb{Z}/2})$-connected.

(ii) Eilenberg–Mac Lane spectra of abelian groups with $\mathbb{Z}/2$-action satisfy this condition as well; see, e.g., [Dotto 2016, Proposition A.1.1].

Proof of Lemma 3.1.3. We need to show that for every $\mathbb{Z}/2$-representation V, the adjoint structure map

$$\begin{aligned}\Big\| \operatorname*{hocolim}_{\underline{i} \in I^{\times 2k+2}} \Omega^{\underline{i}}(S^V \wedge A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_{2k+1}}) \Big\| \\ \xrightarrow{\sigma} \Omega^{\rho} \Big\| \operatorname*{hocolim}_{\underline{i} \in I^{\times 2k+2}} \Omega^{\underline{i}}(S^\rho \wedge S^V \wedge A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_{2k+1}}) \Big\|\end{aligned}$$

is an equivalence. It is shown in [Dotto et al. 2017, §2.3] using a semistability argument that there is an equivalence

$$\begin{aligned}\operatorname*{hocolim}_{n \in \mathbb{N}} \Omega^{n\rho\otimes(k\rho+2)}(S^\rho \wedge S^V \wedge A_{n\rho} \wedge A_{n\rho}^{\wedge 2k+1}) \\ \xrightarrow{\sim} \operatorname*{hocolim}_{\underline{i} \in I^{\times 2k+2}} \Omega^{\underline{i}}(S^\rho \wedge S^V \wedge A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_{2k+1}}),\end{aligned}$$

where the involution on $2k+1$ reverses the order. The source of this map is equivariantly connected for every k, by our connectivity assumption. Indeed, it is nonequivariantly

$$\begin{aligned}(2+\dim V) + (2k+2)(\mathrm{conn}\, A_{n\rho}) + 2k+1 - \dim(n\rho \otimes (k\rho+2)) \\ = (2+\dim V) + (2k+2)(2n-1) + 2k+1 - 2n(2k+2) \\ = (1+\dim V) + (2k+2)(2n) - 2n(2k+2) = (1+\dim V)\end{aligned}$$

connected, and its connectivity on fixed points is the minimum between $1+\dim V$ and

$$\begin{aligned}\dim(\rho+V)^{\mathbb{Z}/2}+\mathrm{conn}(A_{n\rho}^{\mathbb{Z}/2}\wedge A_{n\rho}^{\wedge k}\wedge A_{n\rho}^{\mathbb{Z}/2})-\dim(2n\rho+nk\rho\otimes\rho)^{\mathbb{Z}/2}\\ =1+\dim V^{\mathbb{Z}/2}+(n-1)+k(2n-1)+(n-1)+k+1-(2n+2nk)\\ =\dim V^{\mathbb{Z}/2}\geq 0.\end{aligned}$$

By [Hesselholt and Madsen 1997, Lemma 2.4] we can therefore commute realization and loops, and the map σ above is equivalent to the realization of the semisimplicial map

$$\begin{aligned}\operatorname*{hocolim}_{\underline{i}\in I^{\times 2k+2}}\Omega^{\underline{i}}(S^V\wedge A_{i_0}\wedge A_{i_1}\wedge\cdots\wedge A_{i_{2k+1}})\\ \xrightarrow{\sigma_k}\Omega^\rho\operatorname*{hocolim}_{\underline{i}\in I^{\times 2k+2}}\Omega^{\underline{i}}(S^\rho\wedge S^V\wedge A_{i_0}\wedge A_{i_1}\wedge\cdots\wedge A_{i_{2k+1}}).\end{aligned}$$

We show that σ_k is an equivariant equivalence for all $k\geq 0$. Again by [Dotto et al. 2017, §2.3] this map is equivalent to the map

$$\begin{aligned}\operatorname*{hocolim}_{n\in\mathbb{N}}\Omega^{n\rho\otimes(k\rho+2)}(S^V\wedge A_{n\rho}\wedge A_{n_\rho}^{\wedge 2k+1})\\ \to\operatorname*{hocolim}_{n\in\mathbb{N}}\Omega^{n\rho\otimes(k\rho+2)}\Omega^\rho(S^\rho\wedge S^V\wedge A_{n\rho}\wedge A_{n_\rho}^{\wedge 2k+1}),\end{aligned}$$

the homotopy colimit of $\Omega^{n\rho\otimes(k\rho+2)}$ applied to the unit $\eta_n: X_n\to\Omega^\rho(S^\rho\wedge X_n)$ of the loop-suspension adjunction, where $X_n:=S^V\wedge A_{n\rho}\wedge A_{n_\rho}^{\wedge 2k+1}$. By the equivariant Freudenthal suspension theorem, η_n is nonequivariantly roughly

$$2\,\mathrm{conn}\,X_n=2(\dim V+(2k+2)(2n-1)+2k+1)(\dim V+4n(k+1)-1)$$

connected, and its connectivity on fixed points is roughly the minimum of

$$\begin{aligned}\mathrm{conn}\,X_n=\dim V+4n(k+1)-1,\\ 2\,\mathrm{conn}\,X_n^{\mathbb{Z}/2}=2(\dim V^{\mathbb{Z}/2}+2(n-1)+k(2n-1)+k+1).\end{aligned}$$

Thus on fixed points, η_n is roughly $(\dim V+4n(k+1))$-connected. It follows that $\Omega^{n\rho\otimes(k\rho+2)}\eta_n$ is nonequivariantly approximately

$$\begin{aligned}c_n:=2(\dim V+4n(k+1))-\dim(n\rho\otimes(k\rho+2))\\ =2\dim V+8n(k+1)-2n(2k+2)=2\dim V+4n(k+1)\end{aligned}$$

connected. Its connectivity on fixed points is the minimum of $2\dim V+4n(k+1)$ and

$$\begin{aligned}(\dim V+4n(k+1))-\dim(n\rho\otimes(k\rho+2))^{\mathbb{Z}/2}=(\dim V+4n(k+1))-(2n+2nk)\\ =\dim V+2n(k+1),\end{aligned}$$

which is $d_n := \dim V + 2n(k+1)$. Since both c_n and d_n diverge with n for every $k \leq 0$, the maps $\Omega^{n\rho\otimes(k\rho+2)}\eta_n$ induce an equivalence on homotopy colimits. □

Remark 3.1.5. Under the connectivity assumptions of Lemma 3.1.3, the $\mathbb{Z}/2$-spectrum $\mathrm{THR}(A)$ arises as the $\mathbb{Z}/2$-spectrum of a $\mathbb{Z}/2$-Γ-space whose value at the pointed set $n_+ = \{+, 1, \dots, n\}$ is the $\mathbb{Z}/2$-space

$$\mathrm{THR}(A)_n := \Big\| \mathrm{sd}_e \operatorname*{hocolim}_{\underline{i}\in I^{\times k+1}} \Omega^{i_0+i_1+\cdots+i_k}(A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \wedge n_+) \Big\|.$$

Indeed, the value of the associated spectrum at a sphere S^n is the geometric realization

$$\begin{aligned} \mathrm{THR}(A)_{S^n} &:= \big| [p] \mapsto \mathrm{THR}(A)_{S^n_p} \big| \\ &\cong \Big\| \mathrm{sd}_e \operatorname*{hocolim}_{\underline{i}\in I^{\times k+1}} |\Omega^{i_0+i_1+\cdots+i_k}(A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \wedge S^n_p)| \Big\|, \end{aligned}$$

and under our connectivity assumptions the canonical map

$$|\Omega^{i_0+i_1+\cdots+i_k}(A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \wedge S^n_p)| \xrightarrow{\sim} \Omega^{i_0+i_1+\cdots+i_k}(A_{i_0} \wedge A_{i_1} \wedge \cdots \wedge A_{i_k} \wedge |S^n_p|)$$

is an equivariant equivalence with respect to the action of the stabilizer group of $(i_0, \dots, i_k) \in I^{\times k+1}$; see [Hesselholt and Madsen 1997, Lemma 2.4]. It follows from [Dotto and Moi 2016, Corollary 2.22] that the map on homotopy colimits is an equivariant equivalence.

Lemma 3.1.6. *The real topological Hochschild homology functor* THR *commutes with rationalizations on ring spectra with anti-involution which are levelwise well-pointed and whose unit $S^0 \to A_0$ is an h-cofibration.*

Proof. Under these assumptions the spectrum $\mathrm{THR}(A)$ is naturally equivalent to the dihedral Bar construction of a flat replacement $A^\flat$ of A with respect to the smash product. This result is a generalization of [Shipley 2000, Theorem 4.2.8; Patchkoria and Sagave 2016], and a proof can be found in [Dotto et al. 2017, §2.4].

Let $\mathbb{S}_\mathbb{Q}$ be a flat model for the rational $\mathbb{Z}/2$-equivariant sphere spectrum. We notice that if K_+ is any finite pointed $\mathbb{Z}/2$-set, the map

$$\mathbb{S}_\mathbb{Q} \cong \mathbb{S}_\mathbb{Q} \wedge \bigwedge_K \mathbb{S} \xrightarrow{\sim} \bigwedge_{K_+} \mathbb{S}_\mathbb{Q}$$

given by the K-fold smash product of the unit maps of $\mathbb{S}_\mathbb{Q}$ smashed with $\mathbb{S}_\mathbb{Q}$ is an equivalence. Nonequivariantly this is clear since $\mathbb{S}_\mathbb{Q} \simeq H\mathbb{Q}$ is idempotent. On geometric fixed points this is the map

$$\begin{aligned} &\Phi^{\mathbb{Z}/2}\mathbb{S}_\mathbb{Q} \wedge \bigwedge_{[k]\in K/(\mathbb{Z}/2)} \Phi^{(\mathbb{Z}/2)_k}\mathbb{S} \\ &\qquad \cong \Phi^{\mathbb{Z}/2}\mathbb{S}_\mathbb{Q} \wedge \bigwedge_{[k]\in K/(\mathbb{Z}/2)} \mathbb{S} \to \Phi^{\mathbb{Z}/2}\mathbb{S}_\mathbb{Q} \wedge \bigwedge_{K/(\mathbb{Z}/2)} \Phi^{(\mathbb{Z}/2)_k}\mathbb{S}_\mathbb{Q}, \end{aligned}$$

which is the smash of the identity with the $K/(\mathbb{Z}/2)$-fold smash of the unit maps of $\Phi^{(\mathbb{Z}/2)_k}\mathbb{S}_{\mathbb{Q}}$, where $(\mathbb{Z}/2)_k$ is the stabilizer group of $k \in K$.

Since the geometric fixed points $\Phi^{\mathbb{Z}/2}\mathbb{S}_{\mathbb{Q}}$ are also equivalent to $H\mathbb{Q}$ they are idempotent, and the map is an equivalence. Thus we have constructed natural equivalences

$$\begin{aligned}\mathrm{THR}(A) \wedge \mathbb{S}_{\mathbb{Q}} &\simeq \left|(A^{\flat})^{\wedge \bullet+1} \wedge \mathbb{S}_{\mathbb{Q}}\right| \\ &\xrightarrow{\sim} \left|(A^{\flat})^{\wedge \bullet+1} \wedge \mathbb{S}_{\mathbb{Q}}^{\wedge \bullet+1}\right| \\ &\cong \left|((A^{\flat}) \wedge \mathbb{S}_{\mathbb{Q}})^{\wedge \bullet+1}\right| \simeq \mathrm{THR}(A \wedge \mathbb{S}_{\mathbb{Q}}),\end{aligned}$$

completing the proof. □

The real topological Hochschild homology spectrum also supports an assembly map. Given a ring spectrum with anti-involution A and a well-pointed topological group π, it is a map

$$\mathrm{THR}(A) \wedge B^{\mathrm{di}}\pi_+ \to \mathrm{THR}(A[\pi]),$$

which is defined as the geometric realization of the map

$$\begin{aligned}&\Big(\operatorname*{hocolim}_{\underline{i}\in I^{\times k+1}} \Omega^{\underline{i}}(A_{i_0} \wedge \cdots \wedge A_{i_k} \wedge n_+)\Big) \wedge \pi_+^{\times k+1} \\ &\qquad\qquad \to \operatorname*{hocolim}_{\underline{i}\in I^{\times k+1}} \Omega^{\underline{i}}(A_{i_0} \wedge \pi_+ \wedge \cdots \wedge A_{i_k} \wedge \pi_+ \wedge n_+)\end{aligned}$$

that commutes the smash product with the homotopy colimit and the loops. It is shown in [Dotto et al. 2017, §5.2] that this map is in fact an equivalence. When $A = \mathbb{S}$ is the sphere, there is a unit map $\mathbb{S} \to \mathrm{THR}(\mathbb{S})$ which is defined by the map into the homotopy colimit from the object $\underline{i} = 0$.

Proposition 3.1.7 [Høgenhaven 2016]. *The composite*

$$\mathbb{S} \wedge B^{\mathrm{di}}\pi_+ \to \mathrm{THR}(\mathbb{S}) \wedge B^{\mathrm{di}}\pi_+ \to \mathrm{THR}(\mathbb{S}[\pi])$$

is an equivalence.

3.2. ***The definition of the real trace map.*** We adapt the construction of the trace of [Bökstedt et al. 1993] to define a natural map of $\mathbb{Z}/2$-Γ-spaces

$$\mathrm{tr}: \mathrm{KR}(A) \to \mathrm{THR}(A)$$

for every ring spectrum with anti-involution A which is levelwise well-pointed, whose unit $S^0 \to A_0$ is an h-cofibration, and which satisfies the connectivity hypothesis of Lemma 3.1.3. Under these assumptions the $\mathbb{Z}/2$-Γ-space $\mathrm{THR}(A)$ of Remark 3.1.5 models the real topological Hochschild spectrum of A, and we can use Proposition 2.3.3. At an object $n_+ \in \Gamma^{\mathrm{op}}$ the trace is defined as the following composite:

$$\begin{tikzcd}
\mathrm{KR}(A)_n = \coprod_{\underline{a}=(a_1,\dots,a_n)} B^\sigma\big(\widetilde{\langle\underline{a}\rangle} \times \widehat{\mathrm{GL}}^\vee_{\underline{a}}(A)\big) \arrow[r, "c"] & \coprod_{\underline{a}} \Lambda^\sigma \big\| N^\sigma\big(\widetilde{\langle\underline{a}\rangle} \times \widehat{\mathrm{GL}}^\vee_{\underline{a}}(A)\big)\big\| \\
\coprod_{\underline{a}} B^{\mathrm{di}}_\wedge\big(\widetilde{\langle\underline{a}\rangle} \times \Omega^\infty_I(M^\vee_{a_1}(A)\vee\cdots\vee M^\vee_{a_n}(A))\big) \arrow[d] & \coprod_{\underline{a}} B^{\mathrm{di}}\big(\widetilde{\langle\underline{a}\rangle} \times \widehat{\mathrm{GL}}^\vee_{\underline{a}}(A)\big) \arrow[l] \arrow[u, "\sim"'] \\
\coprod_{\underline{a}}\big(B^{\mathrm{di}}\widetilde{\langle\underline{a}\rangle}\big) \times \mathrm{THR}(M^\vee_{a_1}(A)\vee\cdots\vee M^\vee_{a_n}(A)) \arrow[r] & \mathrm{THR}(A)_n
\end{tikzcd}$$

All the maps except for the last one leave the $\langle\underline{a}\rangle$-coordinate untouched. The first map c is the composition of the equivalence

$$B^\sigma\big(\widetilde{\langle\underline{a}\rangle} \times \widehat{\mathrm{GL}}^\vee_{\underline{a}}(A)\big) \to \big\| N^\sigma\big(\widetilde{\langle\underline{a}\rangle} \times \widehat{\mathrm{GL}}^\vee_{\underline{a}}(A)\big)\big\|$$

of Lemma 2.1.11 and the inclusion of constant loops. Recall that $\Lambda^\sigma = \mathrm{Map}(S^\sigma, -)$ is the free loop space with respect to the sign representation. The second map is the map of Lemma 2.1.12, and it is an equivalence because $\widehat{\mathrm{GL}}^\vee_{\underline{a}}(A)$ is quasiunital and group-like, by Proposition 2.3.3. The third map includes the invertible components and projects the products onto the smash products, where $B^{\mathrm{di}}_\wedge$ denotes the dihedral Bar construction with respect to the smash product of spaces. The fourth map commutes the smash products and the loops. The fifth map projects off the $\widetilde{\langle\underline{a}\rangle}$-component, and on the THR factor it is induced by the maps of spaces

$$(M^\vee_{a_1}(A_{i_0})\vee\cdots\vee M^\vee_{a_n}(A_{i_0}))\wedge\cdots\wedge(M^\vee_{a_1}(A_{i_k})\vee\cdots\vee M^\vee_{a_n}(A_{i_k})) \to A_{i_0}\wedge\cdots\wedge A_{i_k}\wedge n_+$$

defined as follows. An element of $M^\vee_{a_1}(A_i)\vee\cdots\vee M^\vee_{a_n}(A_i)$ is an integer $1\le j\le n$, a pair $(c,d)\in a_j\times a_j$, and an element $x\in A_i$. The map above is then defined by the formula

$$(j_0,c_0,d_0,x_0)\wedge\cdots\wedge(j_k,c_k,d_k,x_k) \\ \mapsto \begin{cases} x_0\wedge\cdots\wedge x_k\wedge j_0 & \text{if } j_0=\cdots=j_k, \\ & \quad d_0=c_1, d_1=c_2,\dots,d_{k-1}=c_k, \\ & \quad \text{and } d_k=c_0, \\ * & \text{otherwise.} \end{cases}$$

This map remembers the entries of a string of matrices when they are composable, and it sends the remaining ones to the basepoint. The underlying map is analogous to the trace map of [Dundas and McCarthy 1996, §1.6.17], and it is a weak homotopy inverse for the map induced by the inclusion $A\to M^\vee_a(A)$ into the 1×1-component. Although we won't use this here, it is also an equivariant equivalence; see [Dotto et al. 2017, §4.3].

Let us denote by $\mathrm{tr}^{\mathrm{cy}}$ the composite

$$\mathrm{tr}^{\mathrm{cy}}_n : \mathrm{KR}^{\mathrm{cy}}(A)_n := \coprod_{\underline{a}=(a_1,\dots,a_n)} B^{\mathrm{di}}\big(\widetilde{\langle \underline{a}\rangle} \times \widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A)\big) \to \mathrm{THR}(A)_n.$$

All the spaces above extend to $\mathbb{Z}/2$-Γ-spaces by a construction similar to the definition of the Γ-structure on KR. It is immediate to verify that the map c and the upper-pointing equivalence are maps of $\mathbb{Z}/2$-Γ-spaces. We verify that $\mathrm{tr}^{\mathrm{cy}}$ is compatible with the Γ-structure.

Proposition 3.2.1. *The map* $\mathrm{tr}^{\mathrm{cy}} : \mathrm{KR}^{\mathrm{cy}}(A) \to \mathrm{THR}(A)$ *is a well-defined map of* $\mathbb{Z}/2$-Γ*-spaces.*

Proof. Let $f : n_+ \to k_+$ be a pointed map. We need to verify that for every collection of nonnegative integers $\underline{a} = (a_1, \dots, a_n)$, the square

$$\begin{array}{ccc} B^{\mathrm{di}}\big(\widetilde{\langle \underline{a}\rangle} \times \widehat{\mathrm{GL}}^{\vee}_{\underline{a}}(A)\big) & \xrightarrow{\mathrm{tr}^{\mathrm{cy}}} & \big\|\mathrm{sd}_e \operatorname*{hocolim}_{\underline{i}\in I^{\times p+1}} \Omega^{i_0+i_1+\dots+i_p}(A_{i_0}\wedge A_{i_1}\wedge\dots\wedge A_{i_p}\wedge n_+)\big\| \\ \Big\downarrow f_* & & \Big\downarrow f_* \\ B^{\mathrm{di}}\big(\widetilde{\langle f_*\underline{a}\rangle} \times \widehat{\mathrm{GL}}^{\vee}_{f_*\underline{a}}(A)\big) & \xrightarrow{\mathrm{tr}^{\mathrm{cy}}} & \big\|\mathrm{sd}_e \operatorname*{hocolim}_{\underline{i}\in I^{\times p+1}} \Omega^{i_0+i_1+\dots+i_p}(A_{i_0}\wedge A_{i_1}\wedge\dots\wedge A_{i_p}\wedge k_+)\big\| \end{array}$$

commutes. We prove that this diagram commutes in simplicial degree $p = 1$; the argument for higher p is similar. A 1-simplex of the upper left corner consists of two pairs of families of permutations (β, α) and (α, β), where $\alpha, \beta \in \langle \underline{a}\rangle$, and a pair of elements $x, y \in \Omega^{\infty}_I(M^{\vee}_{a_1}(A) \vee \dots \vee M^{\vee}_{a_n}(A))$ belonging to an invertible component. For a fixed pair (α, β), we need to show that the square

$$\begin{array}{ccc} (M^{\vee}_{a_1}(A_{i_0}) \vee\dots\vee M^{\vee}_{a_n}(A_{i_0})) \wedge (M^{\vee}_{a_1}(A_{i_1}) \vee\dots\vee M^{\vee}_{a_n}(A_{i_1})) & \longrightarrow & A_{i_0}\wedge A_{i_1}\wedge n_+ \\ \Big\downarrow {\scriptstyle (\vee_{j=1}^k(\beta,\alpha)_j)\wedge(\vee_{j=1}^k(\alpha,\beta)_j)} & & \Big\downarrow {\scriptstyle \mathrm{id}\wedge\mathrm{id}\wedge f} \\ (M^{\vee}_{a_1}(A_{i_0}) \vee\dots\vee M^{\vee}_{a_n}(A_{i_0})) \wedge (M^{\vee}_{a_1}(A_{i_1}) \vee\dots\vee M^{\vee}_{a_n}(A_{i_1})) & \longrightarrow & A_{i_0}\wedge A_{i_1}\wedge k_+ \end{array}$$

commutes, where $(\alpha, \beta)_j$ are the maps defined in Section 2.4 and the horizontal maps are defined at the beginning of the section. The upper composite takes $(j_0, (c_0, d_0), x_0) \wedge (j_1, (c_1, d_1), x_1)$, where $1 \le j_l \le n$, $(c_l, d_l) \in a_{j_l} \times a_{j_l}$ and $x \in A_{i_l}$ for $l = 0, 1$, to

$$x_0 \wedge x_1 \wedge f(j_0) \quad \text{if } j_0 = j_1 \text{ and } d_0 = c_1, d_1 = c_0,$$

and to the basepoint otherwise. The lower composite takes it to

$$x_0\wedge x_1\wedge f(j_0) \quad \text{if } f(j_0) = f(j_1) \text{ and } \beta_{f^{-1}f(j_1)\backslash j_0, j_0}(\iota_0 d_0) = \beta_{f^{-1}f(j_1)\backslash j_1, j_1}(\iota_1 c_1),$$
$$\alpha_{f^{-1}f(j_1)\backslash j_1, j_1}(\iota_1 d_1) = \alpha_{f^{-1}f(j_0)\backslash j_0, j_0}(\iota_0 c_0),$$

where $\iota_0 : a_{j_0} \to \coprod_{i\in f^{-1}f(j_0)} a_i$ is the inclusion, and similarly for ι_1. We need to show that these conditions are equivalent. Clearly the first condition implies the second one.

Suppose that the second condition holds, and set $i := f(j_0) = f(j_1)$. By construction, the family of permutations α satisfies the condition

$$\alpha_{(f^{-1}i)\backslash j_1, j_1} \circ (\alpha_{j_0,(f^{-1}i)\backslash\{j_0,j_1\}} \amalg \mathrm{id}_{a_{j_1}}) = \alpha_{(f^{-1}i)\backslash j_0, j_0} \circ (\mathrm{id}_{a_{j_0}} \amalg \alpha_{(f^{-1}i)\backslash\{j_0,j_1\}, j_1}).$$

By evaluating this expression at $\iota_0 c_0$ we obtain that

$$\alpha_{(f^{-1}i)\backslash j_1, j_1} \circ (\alpha_{j_0,(f^{-1}i)\backslash\{j_0,j_1\}} \amalg \mathrm{id}_{a_{j_1}})(\iota_0 c_0) = \alpha_{(f^{-1}i)\backslash j_0, j_0}(\iota_0 c_0) = \alpha_{(f^{-1}i)\backslash j_1, j_1}(\iota_1 d_1).$$

Since $\alpha_{(f^{-1}i)\backslash j_1, j_1}$ is invertible we must have that

$$(\alpha_{j_0,(f^{-1}i)\backslash\{j_0,j_1\}} \amalg \mathrm{id}_{a_{j_1}})(\iota_0 c_0) = \iota_1 d_1,$$

but since the left-hand map is the identity on a_{j_1} and ι_1 includes in a_{j_1} we must have that $j_0 = j_1$ and $c_0 = d_1$. A similar argument shows that $d_0 = c_1$. □

Let us finally verify that on underlying nonequivariant infinite loop spaces our trace agrees with the trace of [Bökstedt et al. 1993]. Since the Γ-spaces underlying KR and THR and the K-theory and THH spectra of [Bökstedt et al. 1993] are special, it is sufficient to compare the maps in spectrum level 1. Under the canonical equivalence $\mathrm{KR}(A)_{S^1} \simeq B\big(\coprod_n B\widehat{\mathrm{GL}}_n(A)\big)$ our trace is induced by the composite

$$\begin{array}{ccccccc} B\widehat{\mathrm{GL}}_n(A) & \longrightarrow & \Lambda B\widehat{\mathrm{GL}}_n(A) & \xleftarrow{\sim} & B^{\mathrm{cyc}}\widehat{\mathrm{GL}}_n(A) & \longrightarrow & B^{\mathrm{cyc}}\widehat{M}_n(A) \\ & & & & & & \downarrow \\ & & \mathrm{THH}(A) & \xleftarrow{\sim} & \mathrm{THH}(M_n^{\vee}(A)) & \xrightarrow{\sim} & \mathrm{THH}(M_n(A)) \end{array}$$

of the inclusion of constant loops, the canonical equivalence between the free loop space and the cyclic nerve, the inclusion of $\widehat{\mathrm{GL}}_n(A)$ in $\widehat{M}_n(A)$, the canonical map that commutes the loops and the Bar construction, and the last two maps which exhibit Morita invariance. This uses the naturality of the construction with respect to the inclusion $M_n^{\vee}(A) \to M_n(A)$. This composite is the same as the corresponding map of [Bökstedt et al. 1993].

3.3. ***The trace splits the restricted assembly map.*** Let A be a ring spectrum with anti-involution and π a well-pointed topological group. We recall that the restricted assembly map of KR is a map of $\mathbb{Z}/2$-spectra

$$\mathcal{A}^0 : \mathbb{S} \wedge B^{\sigma}\pi_+ \xrightarrow{\eta\wedge\mathrm{id}} \mathrm{KR}(A) \wedge B^{\sigma}\pi_+ \to \mathrm{KR}(A[\pi])$$

(see Definition 2.7.5). We let $p : B^{\mathrm{di}}\pi \to B^{\sigma}\pi$ denote the projection.

Theorem 3.3.1. *The restricted assembly map for the sphere spectrum*

$$\mathcal{A}^0 : \mathbb{S} \wedge B^\sigma \pi_+ \to \mathrm{KR}(\mathbb{S}[\pi])$$

is a split monomorphism in the homotopy category of $\mathbb{Z}/2$-spectra. A natural retraction is provided my the map

$$\mathrm{KR}(\mathbb{S}[\pi]) \xrightarrow{\mathrm{tr}} \mathrm{THR}(\mathbb{S}[\pi]) \xleftarrow{\sim} \mathbb{S} \wedge B^{\mathrm{di}}\pi_+ \xrightarrow{p} \mathbb{S} \wedge B^\sigma \pi_+.$$

Proof. We complete the following commutative diagram of $\mathbb{Z}/2$-Γ-spaces by defining the dashed arrows

$$\begin{tikzcd}
n_+ \wedge B^\sigma \pi_+ \arrow[r, "\mathcal{A}^0"] \arrow[d, "c"'] & \mathrm{KR}_n(\mathbb{S}[\pi]) \arrow[r, "c"] & \coprod_{\underline{a}} \Lambda^\sigma \big\| N^\sigma\big(\widetilde{\langle \underline{a}\rangle} \times \widehat{\mathrm{GL}}{}^\vee_{\underline{a}}(\mathbb{S}[\pi])\big)\big\| \\
n_+ \wedge (\Lambda^\sigma \| N^\sigma \pi \|)_+ \arrow[urr, dashed] & & \coprod_{\underline{a}} B^{\mathrm{di}}\big(\widetilde{\langle \underline{a}\rangle} \times \widehat{\mathrm{GL}}{}^\vee_{\underline{a}}(\mathbb{S}[\pi])\big) \arrow[u, "l", "\sim"'] \arrow[d, "\mathrm{tr}^{\mathrm{cy}}"] \\
n_+ \wedge B^{\mathrm{di}}\pi_+ \arrow[u, "\sim", "l"'] \arrow[urr, dashed] \arrow[rr, "\sim"'] & & \mathrm{THR}_n(\mathbb{S}[\pi])
\end{tikzcd}$$

Here the bottom map is the equivalence of Proposition 3.1.7, the map l is the equivalence of Lemma 2.1.12, and the vertical map c is induced by the composite

$$B^\sigma \pi \to \Lambda^\sigma B^\sigma \pi \xrightarrow{\sim} \Lambda^\sigma \| N^\sigma \pi \|$$

of the inclusion of constant loops and the canonical equivalence of Lemma 2.1.11. This shows that in the homotopy category the map

$$\mathrm{KR}(\mathbb{S}[\pi]) \xrightarrow{\mathrm{tr}} \mathrm{THR}(\mathbb{S}[\pi]) \simeq \mathbb{S} \wedge B^{\mathrm{di}}\pi_+$$

equals the composition of c and the inverse of l. Since $p = \mathrm{ev} \circ l$, where the evaluation map $\mathrm{ev} : \Lambda^\sigma \| N^\sigma \pi \| \to \| N^\sigma \pi \|$ splits c, we have that p splits $l^{-1} \circ c$, and this will conclude the proof.

The lower dashed map is defined as the adjoint of the map of $\mathbb{Z}/2$-spaces

$$B^{\mathrm{di}}\pi_+ \to \coprod_{k \geq 0} B^{\mathrm{di}} \widehat{\mathrm{GL}}{}^\vee_k(\mathbb{S}[\pi])$$

that sends $+$ to zero, and that includes $B^{\mathrm{di}}\pi$ in the $k = 1$ summand by the dihedral nerve of the map of monoids with anti-involution $\pi \to \widehat{\mathrm{GL}}{}^\vee_1(\mathbb{S}[\pi])$. The upper dashed map is adjoint to the map of $\mathbb{Z}/2$-spaces

$$(\Lambda^\sigma \| N^\sigma \pi \|)_+ \to \coprod_{k \geq 0} \Lambda^\sigma \| N^\sigma \widehat{\mathrm{GL}}{}^\vee_k(\mathbb{S}[\pi]) \|,$$

induced by the same map $\pi \to \widehat{\mathrm{GL}}{}^\vee_1(\mathbb{S}[\pi])$. The bottom right triangle commutes since the trace map leaves the $k = 1$ summand essentially untouched. The middle rectangle commutes by naturality of the map l. The upper left triangle of the diagram commutes by construction, since both $\mathcal{A}^0$ and the upper dashed map are induced by $\pi \to \widehat{\mathrm{GL}}_1^{\vee}(\mathbb{S}[\pi])$. $\square$

Corollary 3.3.2. *Let π be a topological group which is cofibrant as a $\mathbb{Z}/2$-space. The fixed-points spectrum* $\mathrm{GW}(\mathbb{S}[\pi])$ *splits off a copy of*

$$(\mathbb{S} \wedge B^{\sigma}\pi_+)^{\mathbb{Z}/2} \simeq \mathbb{S} \wedge ((B\pi \times \mathbb{RP}^{\infty}) \amalg (B^{\sigma}\pi)^{\mathbb{Z}/2})_+.$$

If π is discrete, the second term decomposes further as

$$(B^{\sigma}\pi)^{\mathbb{Z}/2} \cong \coprod_{\{[g] \mid g^2=1\}} BZ_{\pi}\langle g\rangle$$

by Lemma 2.1.9.

Proof. The splitting follows immediately from Theorem 3.3.1. By the Segal–tom Dieck splitting there is a natural decomposition

$$(\mathbb{S} \wedge B^{\sigma}\pi_+)^{\mathbb{Z}/2} \simeq \mathbb{S} \wedge ((B^{\sigma}\pi)_{h\mathbb{Z}/2} \amalg (B^{\sigma}\pi)^{\mathbb{Z}/2})_+.$$

We recall by Lemma 2.1.9 that there is an equivariant map $B\pi \to B^{\sigma}\pi$ which is a nonequivariant equivalence, where $B\pi$ has the trivial involution. This gives an equivalence

$$(B^{\sigma}\pi)_{h\mathbb{Z}/2} \xleftarrow{\sim} (B\pi)_{h\mathbb{Z}/2} = B\pi \times \mathbb{RP}^{\infty}. \qquad \square$$

Corollary 3.3.3. *The restricted assembly maps of the Hermitian K-theory and genuine L-theory of the spherical group-ring of Definition 2.7.6,*

$$\mathbb{S} \wedge (B\pi \amalg B\pi)_+ \to \mathrm{GW}(\mathbb{S}[\pi]),$$
$$\mathbb{S} \wedge B\pi_+ \to \mathrm{L}^{g}(\mathbb{S}[\pi]),$$

are naturally split monomorphisms in the homotopy category of spectra.

Proof. We start by proving the claim for L-theory. By Theorem 3.3.1 the second map in the composite

$$\mathbb{S} \wedge B\pi_+ \xrightarrow{\lambda} \mathbb{S} \wedge (B^{\sigma}\pi)_+^{\mathbb{Z}/2} \xrightarrow{\sim} \Phi^{\mathbb{Z}/2}(\mathbb{S} \wedge B^{\sigma}\pi_+) \longrightarrow \mathrm{L}^{g}(\mathbb{S}[\pi])$$

splits. Thus it is sufficient to show that the first map splits, and a retraction is provided by the inclusion of fixed points $\iota : (B^{\sigma}\pi^{\mathbb{Z}/2}) \to B\pi$, by Lemma 2.1.9. Similarly, the restricted assembly of the Hermitian K-theory is the composite

$$\mathbb{S}\wedge(B\pi \amalg B\pi)_+ \to \mathbb{S}\wedge((B^{\sigma}\pi)^{\mathbb{Z}/2} \amalg (B^{\sigma}\pi)_{h\mathbb{Z}/2})_+ \xrightarrow{\sim} (\mathbb{S}\wedge B^{\sigma}\pi_+)^{\mathbb{Z}/2} \to \mathrm{GW}(\mathbb{S}[\pi]),$$

and it is sufficient to show that the first map is a split monomorphism. The first summand is again split by the inclusion of fixed points. The second summand is split by the projection map

$$(B^\sigma\pi)_{h\mathbb{Z}/2} \xleftarrow{\sim} (B\pi)_{h\mathbb{Z}/2} \cong B\pi \times \mathbb{RP}^\infty \to B\pi. \qquad \square$$

We remark that the same argument of the proof of Corollary 3.3.3 shows that the restricted assembly map of any ring spectrum with anti-involution which is rationally equivalent to the sphere spectrum splits rationally. We will be particularly interested in the case where $A = H\mathbb{A}\left[\frac{1}{2}\right]$ is the Eilenberg–Mac Lane spectrum of the Burnside Mackey functor $\mathbb{A}\left[\frac{1}{2}\right]$ with 2 inverted. In particular, for L-theory we obtain the following.

Corollary 3.3.4. *Let A be a ring spectrum with anti-involution which satisfies the hypotheses of Lemma 3.1.6. Suppose that the unit map $\mathbb{S} \to A$ is a rational equivalence of underlying $\mathbb{Z}/2$-spectra. Then the rationalized restricted assembly map in L-theory*

$$\mathcal{A}^0 : H\mathbb{Q} \wedge B\pi_+ \simeq (\mathbb{S} \wedge B\pi_+) \otimes \mathbb{Q} \to \mathrm{L}^{\mathrm{g}}(A[\pi]) \otimes \mathbb{Q}$$

is naturally split by the map

$$T : \mathrm{L}^{\mathrm{g}}(A[\pi]) \xrightarrow{\mathrm{tr}} \Phi^{\mathbb{Z}/2}\,\mathrm{THR}(A[\pi]) \xleftarrow{\sim_{\mathbb{Q}}} \mathbb{S} \wedge (B^{\mathrm{di}}\pi)_+^{\mathbb{Z}/2} \xrightarrow{p} \mathbb{S} \wedge (B^\sigma\pi)_+^{\mathbb{Z}/2} \xrightarrow{2.1.9} \mathbb{S} \wedge B\pi_+. \qquad \square$$

4. Application to the Novikov conjecture

Let π be a discrete group and $\mathrm{L}^{\mathrm{q}}(\mathbb{Z}[\pi])$ the quadratic L-theory spectrum of the corresponding integral group-ring. The assembly map of quadratic L-theory is a map of spectra

$$\mathcal{A}_{\mathbb{Z}[\pi]} : \mathrm{L}^{\mathrm{q}}(\mathbb{Z}) \wedge B\pi_+ \to \mathrm{L}^{\mathrm{q}}(\mathbb{Z}[\pi]).$$

The Novikov conjecture for the discrete group π is equivalent to the injectivity on rational homotopy groups of the map $\mathcal{A}_{\mathbb{Z}[\pi]}$. Rationally, $\mathrm{L}^{\mathrm{q}}(\mathbb{Z})$ is a Laurent polynomial algebra on one generator β of degree 4. Thus, on rational homotopy groups the assembly map above takes the form

$$\mathcal{A}_{\mathbb{Z}[\pi]} : \mathbb{Q}[\beta, \beta^{-1}] \otimes H_*(B\pi; \mathbb{Q}) \to \mathrm{L}^{\mathrm{q}}_*(\mathbb{Z}[\pi]) \otimes \mathbb{Q}.$$

Remark 4.1. Since the assembly map is a map of $\mathbb{Q}[\beta, \beta^{-1}]$-modules, it is sufficient to show that $\mathcal{A}_{\mathbb{Z}[\pi]}$ does not annihilate the polynomials with nonzero constant term

$$\underline{x} = 1 \otimes x_n + \beta \otimes x_{n-4} + \cdots + \beta^k \otimes x_{n-4k} \in (\mathbb{Q}[\beta] \otimes H_*(B\pi; \mathbb{Q}))_n,$$

where $k \geq 0$ and $x_n \neq 0$ for every $n \geq 0$. Indeed, any degree j element $\underline{y}$ of $\mathbb{Q}[\beta, \beta^{-1}] \otimes H_*(B\pi; \mathbb{Q})$ can be written as $\underline{y} = \beta^{-l}\underline{x}$, where $\underline{x}$ is of the form above and l is the lowest power with nonzero coefficient of $\underline{y}$. Then $\mathcal{A}_{\mathbb{Z}[\pi]}(\underline{y}) = \beta^{-l}\mathcal{A}_{\mathbb{Z}[\pi]}(\underline{x})$

is nonzero if and only if $\mathcal{A}_{\mathbb{Z}[\pi]}(\underline{x})$ is nonzero. In particular, we can restrict our attention to the connective cover of the assembly map.

We let $\mathbb{A}_{1/2}$ be the Burnside Tambara functor with 2 inverted, and $H\mathbb{A}_{1/2}$ a cofibrant strictly commutative orthogonal $\mathbb{Z}/2$-equivariant ring spectrum model for its Eilenberg–Mac Lane spectrum (for the existence, see, e.g., [Ullman 2013, Theorem 5.1]). We let $d : \mathbb{A}_{1/2}[\pi] \to \underline{\mathbb{Z}}_{1/2}[\pi]$ be the rank map, as defined in Example 1.2.7.

Theorem 4.2. *For any discrete group π, there is a lift $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$ of the assembly map for the integral group-ring*

$$
\begin{tikzcd}
 & & \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q} \arrow[d, "d"] \\
\mathbb{Q}[\beta]\otimes H_*(B\pi;\mathbb{Q}) \arrow[urr, "\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}"] \arrow[r, "\mathcal{A}_{\mathbb{Z}[\pi]}"'] & \mathrm{L}^{\mathrm{q}}_{*\geq 0}(\mathbb{Z}[\pi])\otimes\mathbb{Q} \cong & \mathrm{L}^{\mathrm{g}}_*(\mathbb{Z}_{1/2}[\pi])\otimes\mathbb{Q}
\end{tikzcd}
$$

which does not annihilate the polynomials with nonzero constant term. Thus the Novikov conjecture holds for π if and only if the image of $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$ intersects the kernel of d trivially.

The way the lift $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$ is constructed allows us to further reduce the Novikov conjecture to a statement about the algebraic structure of $\mathrm{L}^{\mathrm{g}}{}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q}$ that does not involve the map d. After inverting 2 the morphism of Hermitian Mackey functors $d : \mathbb{A}_{1/2} \to \underline{\mathbb{Z}}_{1/2}$ splits (although not as a Tambara functor). This induces a map

$$s_\pi : \mathrm{L}^{\mathrm{g}}(\mathbb{Z}_{1/2}[\pi])\otimes\mathbb{Q} \to \mathrm{L}^{\mathrm{g}}(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q},$$

which is a section for d (see Lemma 4.4). For the trivial group $\pi = 1$ this provides an additive inclusion $s = s_1 : \mathbb{Q}[\beta] \to \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q}$. In particular, $s(1)$ defines an element in the ring $\mathrm{L}^{\mathrm{g}}_0(\mathbb{A}_{1/2})$ which acts on $\mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])$ by Proposition 2.5.1.

Corollary 4.3. *Every element $\underline{x} \in \mathbb{Q}[\beta]\otimes H_*(B\pi;\mathbb{Q})$ satisfies the identity*

$$s_\pi\mathcal{A}_{\mathbb{Z}[\pi]}(\underline{x}) = s(1)\cdot\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x}) \in \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q},$$

where s_π is injective and $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})$ is nonzero when $\underline{x}$ has nonzero constant term. It follows that the Novikov conjecture holds for π if and only if multiplication by $s(1)$

$$s(1) : \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q} \to \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi])\otimes\mathbb{Q}$$

is injective on the image of $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$.

Proof of Theorem 4.2. We first note that the isomorphism

$$\mathrm{L}^{\mathrm{g}}_*(\mathbb{Z}_{1/2}[\pi])\otimes\mathbb{Q} \cong \mathrm{L}^{\mathrm{q}}_{*\geq 0}(\mathbb{Z}[\pi])\otimes\mathbb{Q}$$

is a consequence of Proposition 2.6.7 and the fact that the map $\mathbb{Z}[\pi] \to \mathbb{Z}_{1/2}[\pi]$ induces an isomorphism $\mathrm{L}^{\mathrm{q}}_*(\mathbb{Z}[\pi])\left[\frac{1}{2}\right] \cong \mathrm{L}^{\mathrm{q}}_*(\mathbb{Z}_{1/2}[\pi])\left[\frac{1}{2}\right]$ on the quadratic L-groups. The latter is an immediate consequence of [Ranicki 1981, Proposition 3.6.4(ii)] (see also [Loday 1976, Théorème 3.2.6]). It follows from Proposition 2.7.4 that the assemblies agree under this isomorphism.

The additive section $s : \mathbb{Q}[\beta] \to \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}) \otimes \mathbb{Q}$ of Lemma 4.4 prescribes a lift $s(\beta)$ of the polynomial generator β. Since $\mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2})$ is a ring, $s(\beta)$ defines a multiplicative section $u : \mathbb{Q}[\beta] \to \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}) \otimes \mathbb{Q}$. We define $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$ to be the composite

$$\bar{\mathcal{A}}_{\mathbb{Z}[\pi]} : \mathbb{Q}[\beta] \otimes H_*(B\pi; \mathbb{Q}) \xrightarrow{u \otimes \mathrm{id}} \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}) \otimes H_*(B\pi; \mathbb{Q}) \xrightarrow{\mathcal{A}_{\mathbb{A}[\pi]}} \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi]) \otimes \mathbb{Q},$$

where $\mathcal{A}_{\mathbb{A}[\pi]}$ is the assembly of Definition 2.7.3. Since $d\mathcal{A}_{\mathbb{A}[\pi]} = \mathcal{A}_{\mathbb{Z}[\pi]}(d \otimes \mathrm{id})$ and u is a section for d, it follows that $d\bar{\mathcal{A}}_{\mathbb{Z}[\pi]} = \mathcal{A}_{\mathbb{Z}[\pi]}$.

Now let us show that $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$ does not annihilate the polynomials with nonzero constant term, that is, that for every

$$\underline{x} = 1 \otimes x_n + \beta \otimes x_{n-4} + \cdots + \beta^k \otimes x_{n-4k} \in (\mathbb{Q}[\beta] \otimes H_*(B\pi; \mathbb{Q}))_n$$

with $k \geq 0$ and $x_n \neq 0$, we have that $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})$ is nonzero. We let

$$T : \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi]) \otimes \mathbb{Q} \to H_*(B\pi; \mathbb{Q})$$

be the map of Corollary 3.3.4, which defines a retraction for the restricted assembly map $\mathcal{A}^0_{\mathbb{A}[\pi]}$. In order to show that $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})$ is nonzero it is sufficient to show that $T\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})$ is nonzero in $H_*(B\pi; \mathbb{Q})$. We write

$$T\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x}) = T\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(1 \otimes x_n) + T\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\beta \otimes x_{n-4} + \cdots + \beta^k \otimes x_{n-4k}).$$

We remark that since the section u is multiplicative, the restriction of $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$ to the summand $(\mathbb{Q} \cdot 1) \otimes H_*(B\pi; \mathbb{Q})$ agrees with the restricted assembly map $\mathcal{A}^0_{\mathbb{A}[\pi]}$. Since T splits $\mathcal{A}^0_{\mathbb{A}[\pi]}$ by Corollary 3.3.4, it follows that

$$T\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(1 \otimes x_n) = T\mathcal{A}^0_{\mathbb{A}[\pi]}(x_n) = x_n \neq 0.$$

It is therefore sufficient to show that $T\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\beta \otimes x_{n-4} + \cdots + \beta^k \otimes x_{n-4k})$ is zero. We prove this by invoking the naturality of T and $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$ with respect to group homomorphisms. By the Kan–Thurston theorem there is a group $\pi_{\langle n-1\rangle}$ and a map $B\pi_{\langle n-1\rangle} \to (B\pi)^{(n-1)}$ to the $(n-1)$-skeleton of $B\pi$ which is a homology isomorphism. Taking the first homotopy group of the composite $B\pi_{\langle n-1\rangle} \to (B\pi)^{(n-1)} \hookrightarrow B\pi$ gives a group homomorphism $\lambda : \pi_{\langle n-1\rangle} \to \pi$. In order to emphasize the naturality of our transformations in the group π we add a superscript to our notations. By naturality, there is a commutative diagram

$$\begin{tikzcd}
(\mathbb{Q}[\beta]\otimes H_*(B\pi_{\langle n-1\rangle};\mathbb{Q}))_n \arrow[r, "\bar{\mathcal{A}}_{\mathbb{Z}[\pi_{\langle n-1\rangle}]}"] \arrow[d, "\mathrm{id}\otimes\lambda"'] & \mathrm{L}^{\mathrm{g}}_n(\mathbb{Z}[\pi_{\langle n-1\rangle}])\otimes\mathbb{Q} \arrow[r, "T^{\pi_{\langle n-1\rangle}}"] \arrow[d, "\lambda"'] & H_n(B\pi_{\langle n-1\rangle};\mathbb{Q})=0 \arrow[d, "\lambda"'] \\
(\mathbb{Q}[\beta]\otimes H_*(B\pi;\mathbb{Q}))_n \arrow[r, "\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}"] & \mathrm{L}^{\mathrm{g}}_n(\mathbb{Z}[\pi])\otimes\mathbb{Q} \arrow[r, "T^{\pi}"] & H_n(B\pi;\mathbb{Q})
\end{tikzcd}$$

Since $\lambda : H_j((B\pi)^{(n-1)};\mathbb{Q})\cong H_j(B\pi_{\langle n-1\rangle};\mathbb{Q})\to H_j(B\pi;\mathbb{Q})$ is surjective for $j<n$, we can write

$$\beta\otimes x_{n-4}+\cdots+\beta^k\otimes x_{n-4k}=\beta\otimes\lambda(y_{n-4})+\cdots+\beta^k\otimes\lambda(y_{n-4k})$$

for some coefficients $y_j\in H_j(B\pi_{\langle n-1\rangle};\mathbb{Q})$. By the commutativity of the diagram above we see that

$$\begin{aligned}T^{\pi}\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}&(\beta\otimes x_{n-4}+\cdots+\beta^k\otimes x_{n-4k})\\ &=\lambda\,\underbrace{T^{\pi_{\langle n-1\rangle}}\bar{\mathcal{A}}_{\mathbb{Z}[\pi_{\langle n-1\rangle}]}(\beta\otimes y_{n-4}+\cdots+\beta^k\otimes y_{n-4k})}_{0}\end{aligned}$$

must vanish.

Now suppose that the kernel of d intersects the image of $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$ trivially, and let $\underline{x}\in\mathbb{Q}[\beta]\otimes H_*(B\pi;\mathbb{Q})$ be a polynomial with nonzero constant term. By the previous argument $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})$ is nonzero, and therefore it cannot belong to the kernel of d. So, we have that $d\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})=\mathcal{A}_{\mathbb{Z}[\pi]}(\underline{x})$ is nonzero. By Remark 4.1 the Novikov conjecture holds for π. Conversely, if $\mathcal{A}_{\mathbb{Z}[\pi]}$ is injective d must be injective on the image of $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$. □

Lemma 4.4. *The rank map*

$$d:\mathrm{L}^{\mathrm{g}}(\mathbb{A}_{1/2}[\pi])\left[\tfrac{1}{2}\right]\to\mathrm{L}^{\mathrm{g}}(\mathbb{Z}_{1/2}[\pi])\left[\tfrac{1}{2}\right]$$

admits a section s_π in the homotopy category, which is natural in π with respect to group homomorphisms. When $\pi=1$ is the trivial group, this section $s:\mathrm{L}^{\mathrm{g}}(\mathbb{Z}_{1/2})\left[\tfrac{1}{2}\right]\to\mathrm{L}^{\mathrm{g}}(\mathbb{A}_{1/2})\left[\tfrac{1}{2}\right]$ is multiplicative, but not unital, on rational homotopy groups.

Proof. The map of Hermitian Mackey functors $d:\mathbb{A}_{1/2}\to\underline{\mathbb{Z}}_{1/2}$ splits. A section $\frac{T}{2}:\underline{\mathbb{Z}}_{1/2}\to\mathbb{A}_{1/2}$ is defined by the identity on the underlying ring, and by the map

$$\left(0,\tfrac{1}{2}\right):\mathbb{Z}_{1/2}\to\mathbb{Z}_{1/2}\oplus\mathbb{Z}_{1/2}$$

on fixed points. We remark that $\frac{T}{2}$ is not a map of Tambara functors, since $\left(0,\frac{1}{2}\right)$ is not unital with respect to the ring structure of the Burnside ring $\mathbb{Z}_{1/2}\oplus\mathbb{Z}_{1/2}$. However, it is a morphism of Hermitian Mackey functors, and it extends to a morphism of Hermitian Mackey functors $\underline{\mathbb{Z}}_{1/2}[\pi]\to\mathbb{A}_{1/2}[\pi]$. This induces a section for d in

Hermitian K-theory

$$\bar{s}_\pi : \mathrm{GW}(\mathbb{Z}_{1/2}[\pi]) \to \mathrm{GW}(\mathbb{A}_{1/2}[\pi]).$$

Since $\frac{T}{2}$ is not a map of Tambara functors it is not clear if it can be realized as a map of $\mathbb{Z}/2$-equivariant commutative ring spectra $H\mathbb{Z}_{1/2} \to H\mathbb{A}_{1/2}$, and thus $\bar{s}_\pi$ does not a priori come from a map of real K-theory spectra. It is therefore not immediately clear if $\bar{s}_\pi$ induces a map on geometric fixed-points spectra.

The isotropy separation sequences for $\mathrm{KR}(\mathbb{A}_{1/2}[\pi])$ and $\mathrm{KR}(\mathbb{Z}_{1/2}[\pi])$ are compared by a commutative diagram

$$\begin{tikzcd}
\mathrm{K}(\mathbb{Z}_{1/2}[\pi])_{h\mathbb{Z}/2} \arrow[r, "\mathrm{tran}"] \arrow[d, equal] & \mathrm{GW}(\mathbb{Z}_{1/2}[\pi]) \arrow[r, "\phi"] \arrow[d, "\bar{s}_\pi"] & \mathrm{L}^{\mathrm{g}}(\mathbb{Z}_{1/2}[\pi]) \arrow[d, dotted, "s_\pi"] \\
\mathrm{K}(\mathbb{A}_{1/2}[\pi])_{h\mathbb{Z}/2} \arrow[r, "\mathrm{tran}"] & \mathrm{GW}(\mathbb{A}_{1/2}[\pi]) \arrow[r, "\phi"] & \mathrm{L}^{\mathrm{g}}(\mathbb{A}_{1/2}[\pi])
\end{tikzcd}$$

where the rows are cofiber sequences. After inverting 2 the transfer map of a $\mathbb{Z}/2$-spectrum X is naturally split by the map

$$\left(X\left[\tfrac{1}{2}\right]\right)^{\mathbb{Z}/2} \xrightarrow{r} \left(X\left[\tfrac{1}{2}\right]\right)^{h\mathbb{Z}/2} \simeq \left(X\left[\tfrac{1}{2}\right]\right)_{h\mathbb{Z}/2},$$

and therefore the cofiber of the transfer is equivalent to the homotopy fiber of the restriction map r. For the spectra $X = \mathrm{KR}$ the map r is induced by the forgetful functor that sends a Hermitian form to its underlying module. Since the functor induced by $\frac{T}{2}$ on Hermitian forms commutes with the forgetful functor, it induces a map s_π on the homotopy fibers of the maps r.

When π is trivial, the maps ϕ are maps of rings, and $\bar{s}$ is multiplicative by Remark 1.3.4. On rational homotopy groups (or in fact away from 2) the isotropy separation sequence gives short exact sequences

$$\begin{tikzcd}
\pi_n \mathrm{K}(\mathbb{Z}_{1/2})_{h\mathbb{Z}/2} \otimes \mathbb{Q} \arrow[r, tail, "\mathrm{tran}"] \arrow[d, equal] & \mathrm{GW}_n(\mathbb{Z}_{1/2}) \otimes \mathbb{Q} \arrow[r, two heads, "\phi_n"] \arrow[d, "\bar{s}_n"] & \mathrm{L}^{\mathrm{g}}_n(\mathbb{Z}_{1/2}) \otimes \mathbb{Q} \arrow[d, "s_n"] \\
\pi_n \mathrm{K}(\mathbb{A}_{1/2})_{h\mathbb{Z}/2} \otimes \mathbb{Q} \arrow[r, tail, "\mathrm{tran}"] & \mathrm{GW}_n(\mathbb{A}_{1/2}) \otimes \mathbb{Q} \arrow[r, two heads, "\phi_n"] & \mathrm{L}^{\mathrm{g}}_n(\mathbb{A}_{1/2}) \otimes \mathbb{Q}
\end{tikzcd}$$

for every $n \geq 0$. Then s_n is given by $s_n(x) = \phi_n \bar{s}_n(y)$ for some $y \in \mathrm{GW}_n(\mathbb{Z}_{1/2}) \otimes \mathbb{Q}$ such that $\phi_n(y) = x$, and it does not depend on such choice. Since ϕ is multiplicative, if $\phi_n(y) = x$ and $\phi_m(y') = x'$ we have that $\phi_{n+m}(yy') = xx'$. Thus

$$s_{n+m}(xx') = \phi_{n+m}\bar{s}_{n+m}(yy') = (\phi_n \bar{s}_n(y))(\phi_m \bar{s}_m(y')) = s_n(x)s_m(x'),$$

proving that s is multiplicative on rational homotopy groups. □

Proof of Corollary 4.3. Let s_π denote the natural section of Lemma 4.4. This fits into a diagram

$$\begin{tikzcd}
\mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}) \otimes H_*(B\pi;\mathbb{Q}) \arrow[rr, "\mathcal{A}_{\mathbb{A}[\pi]}"] & & \mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2}[\pi]) \otimes \mathbb{Q} \\
\mathbb{Q}[\beta] \otimes H_*(B\pi;\mathbb{Q}) \arrow[u, shift right, "u\otimes\mathrm{id}"'] \arrow[u, shift left, "s\otimes\mathrm{id}"] \arrow[r, "\mathcal{A}_{\mathbb{Z}[\pi]}"'] & \mathrm{L}^{\mathrm{q}}_{*\geq 0}(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \cong & \mathrm{L}^{\mathrm{g}}_*(\mathbb{Z}_{1/2}[\pi]) \otimes \mathbb{Q} \arrow[u, "s_\pi"']
\end{tikzcd}$$

where the right rectangle commutes. We observe that s does not commute with the unit maps of $\mathrm{KR}_*(\mathbb{A}_{1/2})$ and $\mathrm{KR}_*(\mathbb{Z}_{1/2})$, and therefore that s_π does not commute with the restricted assembly maps. The map s, however, is multiplicative by Lemma 4.4, and since u sends by definition β to $s(\beta)$ we have that $s = s(1)\cdot u$, and that $s(1)$ is idempotent. Since $\mathcal{A}_{\mathbb{A}[\pi]}$ is a map of $\mathrm{L}^{\mathrm{g}}_*(\mathbb{A}_{1/2})$-modules, we have that for every nonzero $\underline{x} \in \mathbb{Q}[\beta] \otimes H_*(B\pi;\mathbb{Q})$,

$$\begin{aligned} s_\pi \mathcal{A}_{\mathbb{Z}[\pi]}(\underline{x}) &= \mathcal{A}_{\mathbb{A}[\pi]}(s\otimes \mathrm{id})(\underline{x}) = \mathcal{A}_{\mathbb{A}[\pi]}((s(1)\cdot u)\otimes \mathrm{id})(\underline{x}) \\ &= s(1)\cdot (\mathcal{A}_{\mathbb{A}[\pi]}(u\otimes\mathrm{id})(\underline{x})) = s(1)\cdot \bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x}). \end{aligned}$$

If the Novikov conjecture holds, the left-hand term must be nonzero since s_π is injective. Thus $s(1)$ must act injectively on the image of $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$. Conversely, if $s(1)$ acts injectively on the image of $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}$, the right-hand term must be nonzero when $\underline{x}$ is a polynomial with nonzero constant term, since $\bar{\mathcal{A}}_{\mathbb{Z}[\pi]}(\underline{x})$ is nonzero by Theorem 4.2. It follows that $\mathcal{A}_{\mathbb{Z}[\pi]}(\underline{x})$ is nonzero, and this implies the Novikov conjecture by Remark 4.1. □

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On the K-theory coniveau epimorphism for products of Severi–Brauer varieties

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With appendices by Eoin Mackall

For X a product of Severi–Brauer varieties, we conjecture that if the Chow ring of X is generated by Chern classes, then the canonical epimorphism from the Chow ring of X to the graded ring associated to the coniveau filtration of the Grothendieck ring of X is an isomorphism. We show this conjecture is equivalent to the condition that if G is a split semisimple algebraic group of type AC, B is a Borel subgroup of G and E is a standard generic G-torsor, then the canonical epimorphism from the Chow ring of E/B to the graded ring associated with the coniveau filtration of the Grothendieck ring of E/B is an isomorphism. In certain cases we verify this conjecture.

Notation and Conventions. We fix a field k throughout. All of our objects are defined over k unless stated otherwise. Sometimes we use k as an index when no confusion will occur.

For any field F, we fix an algebraic closure $\bar{F}$.

A variety X is a separated scheme of finite type over a field.

Let $X = X_1 \times \cdots \times X_r$ be a product of varieties with projections $\pi_i : X \to X_i$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be sheaves of modules on $X_1, \ldots, X_r$. We use $\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_r$ for the external product $\pi_1^*\mathcal{F}_1 \otimes \cdots \otimes \pi_r^*\mathcal{F}_r$.

For a ring R with a $\mathbb{Z}$-indexed descending filtration $F_\nu^\bullet$ (e.g., $\nu = \gamma$ or τ as in Section 2), we write $\mathrm{gr}_\nu^i R$ for the corresponding quotient F_ν^i/F_ν^{i+1}. We write $\mathrm{gr}_\nu R = \bigoplus_{i\in\mathbb{Z}} \mathrm{gr}_\nu^i R$ for the associated graded ring.

A semisimple algebraic group G is of type AC if its Dynkin diagram is a union of diagrams of type A and type C. Similarly a semisimple group G is of type AA if its Dynkin diagram is a union of diagrams of type A.

For elements i, j of an index set $\mathcal{I}$, we write δ_{ij} for the function which is 0 when $i \neq j$ and 1 if $i = j$.

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Given two r-tuples of integers, say I, J, we write $I < J$ if the i-th component of I is less than the i-th component of J for any $1 \leq i \leq r$.

1. Introduction

For any smooth variety X, the coniveau spectral sequence for algebraic K-theory induces a canonical epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ from the Chow ring of X to the associated graded ring of the coniveau filtration on the Grothendieck ring of X (for notation related to Grothendieck rings see Section 2). The kernel of this epimorphism is torsion, as can be seen using the Grothendieck–Riemann–Roch without denominators. In general this can't be refined: there are examples of smooth varieties where the kernel of the K-theory coniveau epimorphism is nontrivial. With this in mind, a particularly difficult problem has been finding families of varieties where this epimorphism is, or fails to be, an isomorphism. In this direction we propose the following:

Conjecture 1.1. Let X be a product of Severi–Brauer varieties. If the Chow ring $\mathrm{CH}(X)$ of X is generated by Chern classes, then the canonical epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism.

Since the ring $\mathrm{gr}_\tau \mathrm{G}(X)$ is computable for such X (see Section 2 for recollections on the Grothendieck rings of Severi–Brauer varieties and their products), a positive answer to Conjecture 1.1 could then be interpreted as a method for computing the Chow ring of such varieties. This is carried out, for instance, in [Karpenko 2017b, Theorem 3.1], where Karpenko shows a special case of Conjecture 1.1 and, using this, is able to compute the Chow ring of certain generic Severi–Brauer varieties.

In Section 3, we give some evidence that a positive answer to Conjecture 1.1 is a likely one. The main result of this section, Theorem 3.3, shows that Conjecture 1.1 is equivalent to a particular case of an older conjecture of Karpenko:[1]

Conjecture 1.2. Let G be a split semisimple algebraic group, E a standard generic G-torsor, and P a special parabolic subgroup of G. Then the canonical epimorphism $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ is an isomorphism.

The proof uses an analysis of the products of Severi–Brauer varieties one obtains from a standard generic G-torsor for algebraic groups of type AA along with various specialization maps.

In Appendix A, we introduce the notion of the level of a central simple algebra. We show how the level gives a useful description of the Grothendieck ring of a

[1] In its original formulation [Karpenko 2017a, Conjecture 1.1], Conjecture 1.2 only asserts there is an isomorphism in the case P is a Borel subgroup. However, to prove Conjecture 1.2 for all special parabolic subgroups of G it suffices to check that the result holds for a particular choice of special parabolic subgroup P. These two forms of Conjecture 1.2 are then equivalent since a Borel subgroup is special.

Severi–Brauer variety and use this description in the main result of this section, Theorem A.15, where we prove Conjecture 1.1 for a single Severi–Brauer variety associated to a central simple algebra of level 1. This generalizes the previously known results obtained in [Karpenko 2017b, Theorem 3.1].

2. Grothendieck rings of Severi–Brauer varieties

By $\mathrm{K}(X)$, we mean the Grothendieck ring of locally free sheaves (equivalently vector bundles) on a variety X; by $\mathrm{G}(X)$ we mean the Grothendieck group of coherent sheaves on X. The i-th term of the γ-filtration on $\mathrm{K}(X)$ is denoted $F^i_\gamma(X)$; the i-th term of the coniveau filtration on $\mathrm{G}(X)$ is denoted $F^i_\tau(X)$.

There's a canonical map $\varphi_X : \mathrm{K}(X) \to \mathrm{G}(X)$ taking the class $[\mathcal{L}] \in \mathrm{K}(X)$ of a locally free sheaf $\mathcal{L}$ to the class $[\mathcal{L}] \in \mathrm{G}(X)$. When X is smooth, φ_X is an isomorphism giving $\mathrm{G}(X)$ the structure of a ring. The coniveau filtration is compatible with the ring structure on $\mathrm{G}(X)$, and $\varphi_X(F^i_\gamma(X)) \subset F^i_\tau(X)$. Moreover, if the Chow ring $\mathrm{CH}(X)$ is generated by Chern classes, then $\varphi_X(F^i_\gamma(X)) = F^i_\tau(X)$; see [Karpenko 1998, proof of Theorem 3.7].

We will often be working with the rings $\mathrm{K}(X)$ for X a Severi–Brauer variety and for X a product of Severi–Brauer varieties.

In the case X is a Severi–Brauer variety, $\mathrm{K}(X)$ has been determined by Quillen. To state this result, recall that X is the variety of right ideals of dimension $\deg(A)$ in the central simple algebra A associated with X. The tautological vector bundle ζ_X on X is a right A-module.

For any central simple algebra B, let us define $\mathrm{K}(B)$ as the Grothendieck group of the category of finitely generated left B-modules. The group $\mathrm{K}(B)$ is infinite cyclic with a canonical generator given by the class of a (unique up to isomorphism) simple B-module.

Theorem 2.1 [Quillen 1973, §8, Theorem 4.1]. *Let X be the Severi–Brauer variety of a central simple algebra A. The group homomorphism*

$$\bigoplus_{i=0}^{\deg(A)-1} \mathrm{K}(A^{\otimes i}) \to \mathrm{K}(X),$$

mapping the class of a left $A^{\otimes i}$-module M to the class of $\zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M$, is an isomorphism.

Note that if F is a field over k, the pullback $\mathrm{K}(X) \to \mathrm{K}(X_F)$ respects the decomposition of Theorem 2.1, is injective, and the image

$$\mathrm{K}(A^{\otimes i}) \subset \mathrm{K}(A_F^{\otimes i}) = \mathbb{Z}$$

is generated by $\mathrm{ind}(A^{\otimes i})/\mathrm{ind}(A_F^{\otimes i})$. For $i \geq 0$, let us write $\zeta_X(i)$ for the tensor product (over $A^{\otimes i}$) of $\zeta_X^{\otimes i}$ by a simple $A^{\otimes i}$-module. This is a vector bundle of rank $\mathrm{ind}(A^{\otimes i})$, and $\zeta_X^{\otimes i}$ decomposes into a direct sum of $\deg(A^{\otimes i})/\mathrm{ind}(A^{\otimes i})$ copies of $\zeta_X(i)$.

A similar description is afforded to the rings $\mathrm{K}(X)$ for products $X = X_1 \times \cdots \times X_r$ of Severi–Brauer varieties:

Theorem 2.2 [Peyre 1995, Corollary 3.2]. *Let $X = X_1 \times \cdots \times X_r$ be a product of Severi–Brauer varieties $X_1, \ldots, X_r$ corresponding to central simple algebras $A_1, \ldots, A_r$, respectively. Then the group homomorphism*

$$\bigoplus_{I<(\deg(A_1),\ldots,\deg(A_r))} \mathrm{K}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) \to \mathrm{K}(X),$$

as $I = (i_1, \ldots, i_r)$ ranges over r-tuples of nonnegative integers, is an isomorphism. Here the class of a left $A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$-module M is sent to the class $\zeta_{X_1}^{\otimes i_1} \boxtimes \cdots \boxtimes \zeta_{X_r}^{\otimes i_r} \otimes_{A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}} M$.

Similarly, if F is a field over k, the pullback $\mathrm{K}(X) \to \mathrm{K}(X_F)$ respects this decomposition, is injective, and the image

$$\mathrm{K}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) \subset \mathrm{K}((A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r})_F) = \mathbb{Z}$$

is generated by $\mathrm{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r})/\mathrm{ind}((A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r})_F)$.

Given two products $X = X_1 \times \cdots \times X_r$ and $Y = Y_1 \times \cdots \times Y_r$ of Severi–Brauer varieties, over possibly different fields F_1 and F_2 with $\dim(X_i) = \dim(Y_i)$ for every $1 \le i \le r$, let us identify $\mathrm{K}(X_{\overline{F_1}})$ with $\mathrm{K}(Y_{\overline{F_2}})$ via the isomorphism of Theorem 2.2. Let us also identify $\mathrm{K}(X)$ and $\mathrm{K}(Y)$ with their images in $\mathrm{K}(X_{\overline{F_1}}) = \mathrm{K}(Y_{\overline{F_2}})$. Note that we have $\mathrm{K}(X) = \mathrm{K}(Y)$ if and only if

$$\mathrm{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) = \mathrm{ind}(B_1^{\otimes i_1} \otimes \cdots \otimes B_r^{\otimes i_r})$$

for all integers $i_1, \ldots, i_r$, where $A_1, \ldots, A_r$ are the algebras associated to $X_1, \ldots, X_r$ and $B_1, \ldots, B_r$ are the algebras associated to $Y_1, \ldots, Y_r$.

The following statement shows that (unlike the coniveau filtration) the γ-filtration on $\mathrm{K}(X)$ is completely determined by $\mathrm{K}(X)$.

Theorem 2.3 [Izhboldin and Karpenko 1999, Theorem 1.1 and Corollary 1.2]. *If $\mathrm{K}(X) = \mathrm{K}(Y)$, then $F_\gamma^i(X) = F_\gamma^i(Y)$ for all $i \ge 0$.*

3. Equivalence of the two conjectures

Let G be an affine algebraic group, let U be a nonempty open G-invariant subset of a G-representation V. If the fppf quotient U/G is representable by a scheme, and if U is a G-torsor over U/G, then U has the property that for any G-torsor H over an infinite field $F \supset k$, there is an F-point x of U/G such that H is isomorphic to the fiber of the morphism $U \to U/G$ over x; see [Serre 2003, §5]. The generic fiber E of the quotient map $U \to U/G$ is called a *standard generic* G-torsor.

Example 3.1. If $G = \mathrm{SL}_n$, then G acts on $V = \mathrm{End}(k^n)$ with $\mathrm{GL}_n \subset V$ an open, G-invariant subset. The generic fiber $E = \mathrm{SL}_{n,k(\mathbb{G}_m)}$ of the quotient $\mathrm{GL}_n \to \mathrm{GL}_n/G = \mathbb{G}_m$ is a standard generic G-torsor.

A standard generic G-torsor E exists for any affine algebraic group G: one can take E to be the generic fiber of the quotient morphism $\mathrm{GL}_n \to \mathrm{GL}_n/G$ for any embedding $G \hookrightarrow \mathrm{GL}_n$.

Now assume G is a split semisimple algebraic group, with P a special parabolic subgroup of G, and E a standard generic G-torsor. Recall an algebraic group H over a field k is *special* if every H-torsor over any field extension of k is trivial. The quotient E/P is a *generic flag variety*, which is moreover *generically split*, meaning that E becomes trivial after scalar extension to the function field $k(E/P)$; see [Karpenko 2018, Lemma 7.1].

Example 3.2. Let $G = \mathrm{SL}_n/\mu_m$, where m is a divisor of n. Then G acts on $X = \mathbb{P}^{n-1}$ and, if P is the stabilizer of a rational point in X, the quotient G/P is isomorphic to X. The parabolic P is special: its conjugacy class is given by the subset of the Dynkin diagram of G corresponding to removing the first vertex; see [Karpenko 2018, §8].

If E is a standard generic G-torsor given as the generic fiber of a quotient map $U \to U/G$, then our identification of $G/P \cong X$ above shows that the generic flag variety E/P is a Severi–Brauer variety over the function field $k(U/G)$. The central simple $k(U/G)$-algebra associated to E/P is called a *generic central simple algebra of degree n and exponent m*. The index of such an algebra is equal to r, where $n = rs$ is a factorization of n with r having the same prime factors as m and with s prime to m.

In [Karpenko 2017b], Conjecture 1.1 is proved for the Severi–Brauer variety of a generic central simple algebra of degree n and exponent m and, as a corollary obtained by analysis similar to Example 3.2 above, Conjecture 1.2 is proved for split semisimple almost-simple algebraic groups of type A and C. In this section we prove an equivalence between Conjecture 1.1 and Conjecture 1.2 for algebraic groups of type AC similar to that obtained in [Karpenko 2017b] for a single Severi–Brauer variety and for a split semisimple almost-simple group of type A or C.

Theorem 3.3. *The following statements are equivalent*:

(1) *Conjecture 1.1 holds for all X.*

(2) *Conjecture 1.2 holds for all G of type AC and P given by removing the first vertex from each of the connected components of the Dynkin diagram of G.*

(3) *Conjecture 1.2 holds for all G of type AC and arbitrary P.*

(4) *Conjecture 1.2 holds for all G of type AA and arbitrary P.*

The proof is given below Lemma 3.6, after some preparation. It proceeds by showing (1) implies (2) implies (3) implies (4) implies (1). The most difficult part of the proof is in showing (4) implies (1). To do this, one realizes a product of Severi–Brauer varieties $X = X_1 \times \cdots \times X_r$ as a specialization of a generic flag variety E/P for a certain choice of split semisimple algebraic group G of type AA, standard generic G-torsor E, and special parabolic P. With mild hypotheses, one can show that this proves the following claim:

Lemma 3.4. *Let G be a split semisimple algebraic group of type AA, E a standard generic G-torsor, and P a special parabolic subgroup of G. Let X be a product of Severi–Brauer varieties such that X is a specialization of E/P. Assume the following conditions hold*:

(1) $\mathrm{CH}(X)$ *is generated by Chern classes.*

(2) *The canonical surjection* $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ *is an isomorphism.*

(3) *The specialization* $\mathrm{K}(E/P) \to \mathrm{K}(X)$ *is an isomorphism.*

Then the canonical surjection $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ *is an isomorphism.*

Proof. Since X is a specialization of E/P, there is a commutative diagram

$$\begin{tikzcd} \mathrm{CH}(E/P) \arrow[r, two heads] \arrow[d] & \mathrm{gr}_\tau \mathrm{G}(E/P) \arrow[d] \\ \mathrm{CH}(X) \arrow[r, two heads] & \mathrm{gr}_\tau \mathrm{G}(X) \end{tikzcd} \tag{D}$$

where the downward-pointing vertical arrows are specializations and the horizontal arrows are the canonical surjections.

In the diagram (D) above, the map $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ is an isomorphism by assumption and $\mathrm{CH}(X)$ is generated by Chern classes by assumption. Note that $\mathrm{CH}(E/P)$ is also generated by Chern classes, by [Karpenko 2018, Corollary 7.2 and Theorem 7.3]. Since the specialization $\mathrm{K}(E/P) \to \mathrm{K}(X)$ is an isomorphism it follows the specialization $\mathrm{CH}(E/P) \to \mathrm{CH}(X)$ is surjective.

The specialization $\mathrm{gr}_\tau \mathrm{G}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism: it fits into the commutative square

$$\begin{tikzcd} \mathrm{gr}_\gamma \mathrm{K}(E/P) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{gr}_\tau \mathrm{G}(E/P) \arrow[d] \\ \mathrm{gr}_\gamma \mathrm{K}(X) \arrow[r, "\sim"] & \mathrm{gr}_\tau \mathrm{G}(X) \end{tikzcd}$$

with the vertical arrows being specializations and the horizontal arrows being the canonical maps. The horizontal arrows are isomorphisms since the Chow rings $\mathrm{CH}(E/P)$ and $\mathrm{CH}(X)$ are generated by Chern classes [Karpenko 1998, proof of Theorem 3.7]; the left-vertical arrow is an isomorphism since by Theorem 2.3 the isomorphism $\mathrm{K}(E/P) \to \mathrm{K}(X)$ induces a bijection $F^i_\gamma(E/P) \cong F^i_\gamma(X)$ for all i.

Hence the specialization $\mathrm{CH}(E/P) \to \mathrm{CH}(X)$ is also an injection and therefore an isomorphism. It follows that the canonical surjection $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism as well, completing the proof. □

The problem is to find the correct G, P, and E that satisfy the conditions of Lemma 3.4. The naïve method, taking $E/P = E_1/P_1 \times \cdots \times E_r/P_r$ to be a product of generic flag varieties with each E_i/P_i having X_i as a specialization fails in at least one regard: the algebras associated to such an E/P are usually too unrelated. That is to say, the specialization in (3) of Lemma 3.4 is typically not a surjection.

The following result of Nguyen, giving a description to the central simple algebras obtained from a G-torsor for split semisimple algebraic groups G of type AA, provides at least one resolution to this problem.

Theorem 3.5 [Nguyen 2015, Theorem A.1]. *Let $\Gamma = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ be a product of r general linear groups for some integers $n_1, \dots, n_r$. Let C be a central subgroup of Γ, and write $G = \Gamma/C$. Let $\pi : G \to \Gamma/Z(\Gamma)$ be the natural projection. Then, for every field extension F of k, π_* identifies $H^1(F, G)$ with the set of isomorphism classes of r-tuples $(A_1, \dots, A_r)$ of central simple F-algebras such that the degree of each A_i is $\deg(A_i) = n_i$, and $A_1^{\otimes m_1} \otimes \cdots \otimes A_r^{\otimes m_r}$ is split over F for every r-tuple of*

$$\mathscr{X}^*(Z(\Gamma)/C) = \{(m_1, \dots, m_r) \in \mathbb{Z}^r \mid \tau_1^{m_1} \cdots \tau_r^{m_r} = 1 \textit{ for all } (\tau_1, \dots, \tau_r) \in C\}.$$

To apply the theorem above to get the same description for the algebras associated to a G-torsor for a split semisimple algebraic group G of type AA, one notes that such a G is isomorphic to a quotient of a product $G_{\mathrm{sc}} = \mathrm{SL}_{n_1} \times \cdots \times \mathrm{SL}_{n_r}$ by a central subgroup C of G_{sc}. One can then use the quotient $G' = G^{\mathrm{red}}/C$ of the reductive group $G^{\mathrm{red}} = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ and the canonical inclusion $\iota : G \to G'$, taking into account that the induced map on cohomology $\iota_* : H^1(F, G) \to H^1(F, G')$ is a surjection (with trivial kernel).

It turns out, with the description given in Theorem 3.5, one has sufficient control to ensure the conditions of Lemma 3.4 hold (up to introducing some additional factors, which won't matter in the end).

Lemma 3.6. *Let $X_1, \dots, X_r$ be a finite number of Severi–Brauer varieties corresponding to central simple k-algebras $A_1, \dots, A_r$ and let $X = X_1 \times \cdots \times X_r$ be their product. Let $n_i = \deg(A_i)$ for all $1 \le i \le r$. For every r-tuple of nonnegative integers $I = (i_1, \dots, i_r)$, write D_I for the underlying division algebra of the product*

$$A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$$

and write $Y_I = \mathrm{SB}(D_I)$ for the associated Severi–Brauer variety. Let

$$Z = X \times \prod_{I<(n_1,\dots,n_r)} Y_I.$$

In this setting, there exists a split semisimple algebraic group G of type AA and a special parabolic P of G so that for any standard generic G-torsor E, the variety Z is a specialization of E/P and the specialization map $\mathrm{K}(E/P) \to \mathrm{K}(Z)$ *is an isomorphism.*

Proof. For every such r-tuple $I = (i_1, \dots, i_r)$ we set $m_I := \operatorname{ind}(D_I)$ to be the index of D_I. The group

$$G_{\mathrm{sc}} = \prod_{j=1}^{r} \mathrm{SL}_{n_j} \times \prod_{I<(n_1,\dots,n_r)} \mathrm{SL}_{m_I}$$

is split, semisimple, and simply connected of type AA. We consider the quotient $G := G_{\mathrm{sc}}/S$, where S is the subgroup of the center of G_{sc} consisting of those elements

$$(x_1, \dots, x_r, x_{(0,\dots,0)}, \dots, x_{(n_1-1,\dots,n_r-1)})$$

satisfying the relation $x_{(i_1,\dots,i_r)} = x_1^{i_1} \cdots x_r^{i_r}$ (when identified with elements of $\mathbb{G}_m$). Let E be a standard generic G-torsor. We let

$$\sigma : G \to G_{\mathrm{ad}}, \quad \pi_i : G_{\mathrm{ad}} \to \mathrm{PGL}_{n_i}, \quad \pi_I : G_{\mathrm{ad}} \to \mathrm{PGL}_{m_I}$$

be the canonical isogeny, projection to the i-th factor for $i \le r$, and projection to the factor corresponding to the r-tuple I, respectively.

Let G^{red} be the reductive group

$$G^{\mathrm{red}} = \prod_{j=1}^{r} \mathrm{GL}_{n_j} \times \prod_{I<(n_1,\dots,n_r)} \mathrm{GL}_{m_I}$$

and set $G' = G^{\mathrm{red}}/S$. Let T be the kernel of the quotient $G^{\mathrm{red}} \to G_{\mathrm{ad}}$. We fix the isomorphism of the character group $\mathscr{X}^*(T) = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$ that identifies the character with weights $(i_1, \dots, i_n)$ with the element $(i_1, \dots, i_n)$. The subgroup S above is defined so that the inclusion $\mathscr{X}^*(T/S) \to \mathscr{X}^*(T)$ identifies $\mathscr{X}^*(T/S)$ with the sublattice generated by those elements

$$(i_1, \dots, i_r, -\delta_{I(0,\dots,0)}, \dots, -\delta_{I(n_1-1,\dots,n_r-1)}),$$

where $I = (i_1, \dots, i_r) < (n_1, \dots, n_r)$ is an r-tuple. For any field extension F of k, $\sigma_* : H^1(F, G) \to H^1(F, G_{\mathrm{ad}})$ factors through the map $H^1(F, G) \to H^1(F, G')$, induced by the inclusion of G into G'. This puts us in position to apply the description in Theorem 3.5 of the algebras $B_i := (\pi_i \circ \sigma)_*(E)$, $C_I := (\pi_I \circ \sigma)_*(E)$. In particular, our choice of S implies $B_1^{\otimes i_1} \otimes \cdots \otimes B_r^{\otimes i_r}$ is Brauer equivalent with $C_{(i_1,\dots,i_r)}$.

Again by Theorem 3.5, each of the algebras A_i are specializations of the algebras B_i and, additionally, for every r-tuple $I = (i_1, \dots, i_r)$ we have an equality

$$m_I = \operatorname{ind}(A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}) = \operatorname{ind}(B_1^{\otimes i_1} \otimes \cdots \otimes B_r^{\otimes i_r}),$$

since the underlying division algebra D_I of $A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$ is a specialization of C_I. The first claim then results from the fact that the variety

$$\prod_{i=1}^{r} \mathrm{SB}(B_i) \times \prod_{I<(n_1,\ldots,n_r)} \mathrm{SB}(C_I)$$

is isomorphic with E/P, which has Z as a specialization. The second claim results from the description of the rings $\mathrm{K}(E/P)$ and $\mathrm{K}(Z)$ given in Theorem 2.2. □

And now for the proof.

Proof of Theorem 3.3. We show (1) implies (2). To start, let G be a group of type AC and E be a standard generic G-torsor over a field extension F of our base k. Let G_{ad} be the adjoint group of G; it is isomorphic to a product

$$G_{\mathrm{ad}} = \prod_{i=1}^{n} G_i$$

with each G_i a simple adjoint group of type A or type C. We write $\sigma : G \to G_{\mathrm{ad}}$ for the canonical isogeny from G to its adjoint and $\pi_i : G_{\mathrm{ad}} \to G_i$ for the projection to the i-th factor of G_{ad}.

From the n maps $\pi_i \circ \sigma$ with varying i, we obtain n central simple F-algebras given by the images of E under the pushforwards on Galois cohomology

$$(\pi_i \circ \sigma)_*(E) \in \mathrm{im}(H^1(F, G) \to H^1(F, G_i)).$$

Let X be the product of the Severi–Brauer varieties associated to the n algebras $(\pi_i \circ \sigma)_*(E)$. Then X is isomorphic to E/P, where P is a parabolic subgroup of G whose conjugacy class is given by the subset of the set of vertices of the Dynkin diagram of G obtained by excluding the first vertex of each of its connected components. That the parabolic P obtained in this way is special is a consequence of Lemma 3.8 below since, by [Karpenko 2018, §8], the group $\sigma(P)$ is special. The claim now follows from [Karpenko 2018, Corollary 7.2 and Theorem 7.3], which shows $\mathrm{CH}(X)$ is generated by Chern classes, allowing us to apply (1) to $X \cong E/P$.

Next note that (2) implies (3) is a consequence of [Karpenko 2017b, Lemma 4.2], and that (3) implies (4) is obvious.

We finish by showing (4) implies (1). Let $X_1, \ldots, X_r$ be Severi–Brauer varieties over a field k corresponding to central simple algebras $A_1, \ldots, A_r$, respectively, and let $X = X_1 \times \cdots \times X_r$ be their product. Let $n_i = \deg(A_i)$ be the degree of the algebra A_i. For every r-tuple of nonnegative integers $I = (i_1, \ldots, i_r)$ we write D_I for the underlying division algebra of the tensor product $A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$. We write $Y_I := \mathrm{SB}(D_I)$ for the associated Severi–Brauer variety and $Z = X \times \prod_{I<(n_1,\ldots,n_r)} Y_I$ for the product of these varieties.

Let G be an algebraic group of type AA and P its special parabolic subgroup, obtained from Z as in Lemma 3.6. Let E be a standard generic G-torsor. By Lemma 3.7 below, to show the epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism, it's sufficient to show $\mathrm{CH}(Z) \to \mathrm{gr}_\tau \mathrm{G}(Z)$ is an isomorphism since the projection $Z \to X$ factors

$$Z \to X \times \prod_{I<(n_1,\dots,n_{r-1},n_r-1)} Y_I \to \cdots \to X \times Y_{(0,\dots,0)} \to X$$

with each arrow a projective bundle. Finally, the arrow $\mathrm{CH}(Z) \to \mathrm{gr}_\tau \mathrm{G}(Z)$ is an isomorphism by Lemma 3.4: $\mathrm{CH}(Z)$ is generated by Chern classes by repeated applications of the projective bundle formula and the assumption that $\mathrm{CH}(X)$ is generated by Chern classes, the map $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ is an isomorphism by assumption, and the specialization $\mathrm{K}(E/P) \to \mathrm{K}(Z)$ is an isomorphism. □

Lemma 3.7. *Assume Z is a projective bundle over a variety X. Then the canonical epimorphism $\mathrm{CH}(Z) \to \mathrm{gr}_\tau \mathrm{G}(Z)$ is an isomorphism if and only if the canonical epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism.*

Proof. The pullback along the projection $Z \to X$ gives a commuting diagram

$$\begin{array}{ccc} \mathrm{CH}(Z) & \twoheadrightarrow & \mathrm{gr}_\tau \mathrm{G}(Z) \\ \uparrow & & \uparrow \\ \mathrm{CH}(X) & \twoheadrightarrow & \mathrm{gr}_\tau \mathrm{G}(X) \end{array}$$

with both vertical arrows injections. It follows if the top-horizontal arrow is an isomorphism, then the bottom-horizontal arrow is an isomorphism.

The converse follows from the projective bundle formula: the groups $\mathrm{CH}(Z)$ and $\mathrm{gr}_\tau \mathrm{G}(Z)$ are direct sums of copies of the groups $\mathrm{CH}(X)$ and $\mathrm{gr}_\tau \mathrm{G}(X)$, respectively, and the coniveau epimorphism respects this direct sum decomposition. □

Lemma 3.8. *Let G be a split semisimple algebraic group over a field F, and let $\sigma : G \to G_{\mathrm{ad}}$ be the canonical isogeny with kernel C, the center of G. If P is a parabolic subgroup of G such that the image $\sigma(P)$ is special, then P is special.*

Proof. Let L be a Levi subgroup of P. By [Karpenko 2018, §3], P is special if and only if L is special. Since G is a split reductive group, P is also a split reductive group so that, by [Karpenko 2018, Theorem 2.1], L is special if and only if the semisimple commutator $L' \subset L$ is special. Similarly, $\sigma(P)$ is special if and only if $\sigma(L)'$ is special. Thus the proof of the lemma can be reduced to the statement that if L' is a split semisimple algebraic group and $L' \to \sigma(L)'$ is an isogeny with $\sigma(L)'$ split, semisimple, and special, then L' is special. The result then follows from the fact a split semisimple algebraic group is special if and only if it is a product of special linear or symplectic groups and all such groups are simply connected. □

We conclude this section with some remarks on, and special cases of, Conjectures 1.1 and 1.2.

Remark 3.9. One can construct a large class of products X of Severi–Brauer varieties which satisfy the condition that $\mathrm{CH}(X)$ is generated by Chern classes. To do so, let $G = \mathrm{PGL}_{n_1} \times \cdots \times \mathrm{PGL}_{n_r}$ for some $n_1, \ldots, n_r \geq 2$; let $A_1, \ldots, A_r$ be the central simple algebras associated to a standard generic G-torsor; let X be the product of the associated Severi–Brauer varieties. By [Karpenko 2018, Theorem 7.3], $\mathrm{CH}(X)$ has the desired property.

One can extend this class by base change: it's possible to lower the index of any tensor product $A = A_1^{\otimes i_1} \otimes \cdots \otimes A_r^{\otimes i_r}$ by extending the base to the function field of any generalized Severi–Brauer variety of A. The new variety X obtained from these algebras also has the property that $\mathrm{CH}(X)$ is generated by Chern classes [Karpenko 1998, Theorem 3.7]. This procedure can be repeated indefinitely.

In fact, to prove Conjecture 1.1 for all products of Severi–Brauer varieties, it suffices to prove Conjecture 1.1 for the varieties obtained by the above procedure (one can even restrict to the class whose construction involves the function field of *usual* Severi–Brauer varieties only); to go from the above case to the general case, one can use the specialization argument as in Theorem 3.3.

Example 3.10 ($\mathrm{A}_1 \times \mathrm{A}_1$ and $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$). In small rank cases, one can check Conjecture 1.2 for G of type AA by hand.

For G as in Conjecture 1.2 of type $\mathrm{A}_1 \times \mathrm{A}_1$, observe that for any projective homogeneous variety X of dimension less than or equal to 2, the epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is an isomorphism [Chernousov and Merkurjev 2006, Proposition 4.4].

For G as in Conjecture 1.2 of type $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$, one can proceed by cases. If G is a product of groups of smaller rank, then [Karpenko 2017a, Proposition 4.1] proves the claim. Otherwise, G is a quotient of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ by the diagonal of the center $\mu_2 \times \mu_2 \times \mu_2$ or by the subgroup generated by the partial 2-diagonals. In the first case, the corresponding generic flag variety is a product $C \times C \times C$ of a fixed conic C and the claim follows. In the second case, the corresponding generic flag variety is a product $X = C_1 \times C_2 \times C_3$, where each C_i is the conic of a quaternion algebra Q_i; here the sum of the classes $[Q_1] + [Q_2] + [Q_3]$ is trivial in the Brauer group. Since X is a projective bundle over any two of the factors, this proves the result by Lemma 3.7.

Example 3.11. Conjecture 1.2 holds for $G = \mathrm{SL}_n/\mu_m$ by [Karpenko 2017b, Theorem 1.1] and for products of such groups by [Karpenko 2017a, Proposition 4.1]. From this, one can show that Conjecture 1.1 holds for products $X = X_1 \times \cdots \times X_r$ satisfying the following conditions:

(1) For each $1 \leq i \leq r$ there is a prime p_i so that the algebra A_i associated to the variety X_i has index $p_i^{n_i}$ and exponent $p_i^{m_i}$ for some integers $n_i \geq m_i \geq 1$.

(2) The algebras A_i satisfy
$$\operatorname{ind}\Big(A_i^{\otimes p_i^{m_i-1}}\Big) = \operatorname{ind}(A_i)/p_i^{m_i-1}.$$

(3) The algebras A_i are disjoint in the sense that there are equalities
$$\operatorname{ind}\big(A_1^{\otimes i_r} \otimes \cdots \otimes A_r^{i_r}\big) = \operatorname{ind}\big(A_1^{\otimes i_1}\big) \cdots \operatorname{ind}\big(A_r^{\otimes i_r}\big)$$
for all integers $i_1, \ldots, i_r$.

To see this, one may assume that all A_i are division algebras and use Lemma 3.4. Property (2) allows one to realize such an X as a specialization of E/P, where E is a standard generic $G = \prod_{1 \leq i \leq r} \mathrm{SL}_{p_i^{n_i}}/\mu_{p_i^{m_i}}$-torsor and $P \subset G$ is a special parabolic subgroup whose conjugacy class can be obtained by removing the first vertex from each of the connected components of the Dynkin diagram of G. The canonical map $\mathrm{CH}(E/P) \to \mathrm{gr}_\tau \mathrm{G}(E/P)$ for this E/P is an isomorphism, as explained above. Now property (3), [Karpenko 2017a, Lemma 4.3], and Theorem 2.3 show the specialization $\mathrm{K}(E/P) \to \mathrm{K}(X)$ is an isomorphism.

Appendix A: Algebras with level 1

In this appendix we introduce the level of a central simple k-algebra. The level is a nonnegative integer that measures, roughly speaking, how far away the algebra is from having its index equal to its exponent. It's related to, and depends on, the reduced behavior of the primary components of the algebra as defined in [Karpenko 1998]. The same concept was considered in [Baek 2015], there as the length of a reduced sequence obtained from the reduced behavior of a p-primary algebra for a prime p; the length of this reduced sequence as defined by Baek is equal to the level of the p-primary algebra as defined here.

It turns out the level of a central simple algebra A can be used to obtain detailed information on λ-ring generators for the Grothendieck ring of the Severi–Brauer variety X of A; see Lemma A.6. A particular consequence of this is that the subring of $\mathrm{CH}(X)$ which is generated by Chern classes has an explicit and small set of generators that can be helpful for computational purposes. Using this more refined information based on the level, we're able to generalize the results of [Karpenko 2017b] to prove the main result, Theorem A.15, that the K-theory coniveau epimorphism is an isomorphism for Severi–Brauer varieties whose Chow ring is generated by Chern classes and whose associated central simple algebra has level 1.

Throughout this appendix we work with a fixed prime p and we continue to work over the fixed but arbitrary field k. We write $v_p(-)$ for the p-adic valuation. We've relegated some computations needed in this section to Appendix B.

Recall, the *reduced behavior* of an algebra A with index $\operatorname{ind}(A) = p^n$ and exponent $\exp(A) = p^m$, $0 < m \le n$, is defined to be the following sequence of p-adic orders of increasing p-primary tensor powers of A:

$$\begin{aligned} r\mathcal{B}eh(A) &= \big(v_p(\operatorname{ind}(A^{\otimes p^i}))\big)_{i=0}^{m} \\ &= \big(v_p(\operatorname{ind}(A)), v_p(\operatorname{ind}(A^{\otimes p})), \dots, v_p(\operatorname{ind}(A^{\otimes p^m}))\big). \end{aligned}$$

The reduced behavior of A is strictly decreasing; it starts with $v_p(\operatorname{ind}(A)) = n$ and ends with $v_p(\operatorname{ind}(A^{\otimes p^m})) = 0$.

Definition A.1. A is said to have *level* l, abbreviated $\operatorname{lev}(A) = l$, if there exist exactly l distinct integers $i_1, \dots, i_l \ge 1$ with $v_p(\operatorname{ind}(A^{\otimes p^{i_k}})) < v_p(\operatorname{ind}(A^{\otimes p^{i_k - 1}})) - 1$ for every $1 \le k \le l$. If no such integers exist, A is said to have level 0. An arbitrary central simple algebra B, not necessarily p-primary, is said to have *level* l if l is the maximum

$$l = \max_{q \text{ prime}} \{\operatorname{lev}(B_q)\}$$

of the levels of the q-primary components B_q of B.

Example A.2. A central simple algebra A has level 0, i.e., $\operatorname{lev}(A) = 0$, if and only if the index and exponent of A coincide: $\operatorname{ind}(A) = \exp(A)$.

Example A.3. If A is a generic algebra of degree p^n and exponent p^m with $m < n$, in the sense of Example 3.2, then the level of A is 1, i.e., $\operatorname{lev}(A) = 1$. The reduced behavior for this algebra is

$$\begin{aligned} r\mathcal{B}eh(A) &= \big(v_p(\operatorname{ind}(A)), v_p(\operatorname{ind}(A^{\otimes p})), \dots, v_p(\operatorname{ind}(A^{\otimes p^m}))\big) \\ &= (n, n-1, \dots, n-m+1, 0). \end{aligned}$$

To see this, note that with a large enough field extension F of k one may find a central division F-algebra B with index p^n, exponent p^m, and reduced behavior $r\mathcal{B}eh(B) = (n, n-1, \dots, n-m+1, 0)$ [Karpenko 1998, Lemma 3.10]. Since B is a specialization of A it follows that

$$p^{n-i} \ge \operatorname{ind}(A^{\otimes p^i}) \ge \operatorname{ind}(B^{\otimes p^i}) = p^{n-i}$$

for $i = 0, \dots, m-1$, so that equalities hold throughout.

We make the following definition for notational convenience.

Definition A.4. The *Chern subring* of a smooth variety X, denoted $\operatorname{CS}(X)$, is the subring of $\operatorname{CH}(X)$ which is generated by all Chern classes of elements of $\operatorname{K}(X)$.

Proposition A.5. *Let X be the Severi–Brauer variety of a central simple algebra A with* $\operatorname{ind}(A) = p^n$ *and* $\operatorname{lev}(A) = r$. *Then* $\operatorname{CS}(X)$ *is generated*, *as a ring*, *by the*

Chern classes of $r+1$ sheaves on X. Namely, the sheaves whose Chern classes generate $\mathrm{CS}(X)$ *are*

$$\zeta_X(1), \zeta_X(p^{i_1}), \ldots, \zeta_X(p^{i_r}),$$

where $1 \le i_1 < \cdots < i_r$ are the r distinct integers with

$$v_p(\mathrm{ind}(A^{\otimes p^{i_k}})) < v_p(\mathrm{ind}(A^{\otimes p^{i_k - 1}})) - 1.$$

Proof. It suffices to show that $\mathrm{K}(X)$ is generated by the classes of

$$\zeta_X(1), \zeta_X(p^{i_1}), \ldots, \zeta_X(p^{i_r})$$

as a λ-ring; this is because Chern classes of λ-operations of an element of $\mathrm{K}(X)$ are certain universal polynomials in the Chern classes of this element. This is done in the next lemma. □

Lemma A.6. *Let X be the Severi–Brauer variety of a central simple algebra A with $\mathrm{ind}(A) = p^n$ and $\mathrm{lev}(A) = r$. Then $\mathrm{K}(X)$ is generated, as a λ-ring, by $r+1$ elements. Namely, the sheaves whose classes generate $\mathrm{K}(X)$ are*

$$\zeta_X(1), \zeta_X(p^{i_1}), \ldots, \zeta_X(p^{i_r}),$$

where $1 \le i_1 < \cdots < i_r$ are the r distinct integers with

$$v_p(\mathrm{ind}(A^{\otimes p^{i_k}})) < v_p(\mathrm{ind}(A^{\otimes p^{i_k - 1}})) - 1.$$

Proof. Since the pullback $\pi^* : \mathrm{K}(X) \to \mathrm{K}(X_L)$ to a splitting field L of A is injective, we can work, instead of $\mathrm{K}(X)$ itself, with its image in $\mathrm{K}(X_L)$. We'll write ξ to denote the class of $\mathcal{O}(-1)$ in $\mathrm{K}(X_L)$. By the comments under Theorem 2.1 we have $\pi^*(\zeta_X(i)) = \mathrm{ind}(A^{\otimes i})\xi^i$. It follows that the elements $\mathrm{ind}(A^{\otimes i})\xi^i$ with $i \ge 0$ generate $\mathrm{K}(X)$ as an abelian group.

The λ-operations of any multiple of ξ^i are easy to compute:

$$\lambda^j(d\xi^i) = \binom{d}{j}\xi^{ij} \quad \text{for any } i, j, d \ge 0.$$

Let us first show that the elements $\mathrm{ind}(A^{\otimes p^j})\xi^{p^j}$ ($j \ge 0$) generate $\mathrm{K}(X)$ as a λ-ring. Since the λ-subring generated by these elements contains powers of $\mathrm{ind}(A)\xi = p^n\xi$, we only need to check that, for every $i \ge 1$, this subring contains an integer multiple of ξ^i whose coefficient has p-adic valuation equal to $v_p(\mathrm{ind}(A^{\otimes i}))$. For this, given any $i \ge 1$, we write $i = p^j s$ with $j \ge 0$ and s prime to p. We set $p^v := \mathrm{ind}(A^{\otimes i}) = \mathrm{ind}(A^{\otimes p^j})$. Write further $s = s_0 p^v + s_1$ with $0 \le s_1 < p^v$ and $s_0 \ge 0$. Then we have $\lambda^{p^v}(p^v \xi^{p^j}) = \xi^{p^j p^v}$ and $\lambda^{s_1}(p^v \xi^{p^j})$ is a multiple of $\xi^{p^j s_1}$ with p-adic valuation of the (binomial) coefficient of this multiple equal to v; see [Karpenko 1998, Lemma 3.5]. The claim we are checking follows.

It remains to show if $v_p(\mathrm{ind}(A^{\otimes p^j})) \geq v_p(\mathrm{ind}(A^{\otimes p^{j-1}})) - 1$ for some $j \geq 1$, then the generator $\mathrm{ind}(A^{\otimes p^j})\xi^{p^j}$ can be omitted. Let us set $p^v := \mathrm{ind}(A^{\otimes p^{j-1}})$. If $v = 0$, then we get ξ^{p^j} as a p-th power of $\xi^{p^{j-1}} = \mathrm{ind}(A^{\otimes p^{j-1}})\xi^{p^{j-1}}$. For $v > 0$, we consider the λ-operation $\lambda^p(p^v\xi^{p^{j-1}})$ which is a multiple of ξ^{p^j} with p-adic valuation of its coefficient equal to $v - 1 \leq v_p(\mathrm{ind}(A^{\otimes p^j}))$. □

To systematically study the relations between the Chern classes of the sheaves appearing in Proposition A.5, we introduce the following notation.

Definition A.7. Let A be a central simple algebra and X the Severi–Brauer variety of A. We write $\mathrm{CT}(i_1, \dots, i_r; X)$ for the graded subring of $\mathrm{CS}(X) \subset \mathrm{CH}(X)$ generated by the Chern classes of the sheaves $\zeta_X(i_1), \dots, \zeta_X(i_r)$.

Proposition A.8. *Let X be the Severi–Brauer variety of a central simple algebra A. Then, for any $i > 0$, $\mathrm{CT}(i; X) \otimes \mathbb{Z}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module. Furthermore, for $0 \leq j < \deg(A)$ the group $\mathrm{CT}^j(i; X) \otimes \mathbb{Z}_{(p)}$ is additively generated by*

$$\tau_i(j) := c_{p^v}(\zeta_X(i))^{s_0} c_{s_1}(\zeta_X(i)),$$

where p^v is the largest power of p dividing $\mathrm{ind}(A^{\otimes i})$ and $j = p^v s_0 + s_1$ with $0 \leq s_1 < p^v$.

Proof. By first extending to a prime-to-p extension (which is an injection when $\mathrm{CH}(X) \otimes \mathbb{Z}_{(p)}$ has $\mathbb{Z}_{(p)}$-coefficients) that splits the prime-to-p components of A, we can assume A is p-primary. We continue by reducing to the case $i = 1$.

Lemma A.9. *Let X be the Severi–Brauer variety of a central simple algebra A, and let Y be the Severi–Brauer variety of $A^{\otimes i}$. Then there is a functorial surjection*

$$\mathrm{CT}(1; Y) \twoheadrightarrow \mathrm{CT}(i; X).$$

Proof. Let

$$X \to X^{\times i} \to Y$$

be the composition of the diagonal embedding and the twisted Segre embedding. The corresponding maps on Grothendieck groups can be determined by moving to a splitting field L of X. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{K}(Y_L) \arrow[r] & \mathrm{K}(X_L^{\times i}) \arrow[r] & \mathrm{K}(X_L) \\
\mathrm{K}(Y) \arrow[r] \arrow[u, hook] & \mathrm{K}(X^{\times i}) \arrow[r] \arrow[u, hook] & \mathrm{K}(X) \arrow[u, hook]
\end{tikzcd}$$

defined so that under the top-horizontal maps we have

$$\mathcal{O}_{Y_L}(-1) \mapsto \mathcal{O}_{X_L}(-1) \boxtimes \cdots \boxtimes \mathcal{O}_{X_L}(-1) \mapsto \mathcal{O}_{X_L}(-i).$$

Thus, the class of $\zeta_Y(1)$ on Y is mapped to the class of $\zeta_X(i)$ on X.

So, under the composition of the diagonal $X \to X^{\times i}$ and the twisted Segre embedding $X^{\times i} \to Y$, there is a surjection $\mathrm{CT}(1; Y) \twoheadrightarrow \mathrm{CT}(i; X)$ induced by the pullback $\mathrm{CH}(Y) \to \mathrm{CH}(X)$. $\square$

Next we reduce to the case our algebra is division. Let D be the underlying division algebra of A, and Y the Severi–Brauer variety of D. Fix an embedding $Y \to X$ so that, over a splitting field of both, the inclusion is as a linear subvariety. The pullback

$$\mathrm{CH}(X) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}(Y) \otimes \mathbb{Z}_{(p)}$$

is an isomorphism in degrees where both groups are nonzero. If the claim is true for $\mathrm{CH}(Y) \otimes \mathbb{Z}_{(p)}$ then, since the pullback is functorial for Chern classes, we find $\mathrm{CT}^j(1; X) \otimes \mathbb{Z}_{(p)}$ is a free $\mathbb{Z}_{(p)}$-module of rank 1 in degrees $0 \leq j < \deg(D)$. That this holds is due to [Karpenko 2017b, Proposition 3.3], where it's shown that $\mathrm{CT}(1; X)$ is free if A is division. This serves as the base case for an induction proof.

In an arbitrary degree j with $\deg(D) \leq j < \deg(A)$, we assume the claim is true for all degrees $0 \leq k < j$. It suffices to show the map

$$\mathrm{CT}^{j-p^v}(1; X) \otimes \mathbb{Z}_{(p)} \to \mathrm{CT}^j(1; X) \otimes \mathbb{Z}_{(p)}$$

defined by multiplication by $\tau_1(p^v) = c_{p^v}(\zeta_X(1))$ is surjective and, by Nakayama's lemma, we can do this modulo p. Any element of $\mathrm{CT}^j(1; X)$ is a sum of monomials of the form $\tau_1(j - p^v) c_{i_1}^{n_1} \cdots c_{i_r}^{n_r}$ with $c_i = c_i(\zeta_X(1))$. We claim any such monomial which is not $\tau_1(j) = \tau_1(j - p^v)\tau_1(p^v)$ is congruent to 0 modulo p.

Indeed, if such a monomial was divisible by c_{i_1}, c_{i_2} then without loss of generality we can assume $v_p(i_2) \leq v_p(i_1) < v$. By [Karpenko 2017b, Proposition 3.5] there is a field F finite over the base so that $v_p \operatorname{ind}(A_F) = v_p(i_1)$, and $c_{i_1} = \pi_*(x)$ for an element x of $\mathrm{CH}(X_F) \otimes \mathbb{Z}_{(p)}$ and where $\pi : X_F \to X$ is the projection. Using the projection formula we find

$$c_{i_1} c_{i_2} = \pi_*(x) c_{i_2} = \pi_*(x \pi^*(c_{i_2})).$$

By Lemma A.10 below, it follows $\pi^*(c_{i_2})$ is divisible by p, which proves the claim.

To see the generators are as claimed for $i = 1$, one can compute the degrees of the images of the Chern classes of $\zeta_X(1)$ over an algebraic closure; for the other i, one can use Lemma A.9. $\square$

Lemma A.10. *Let X be the Severi–Brauer variety of a central simple algebra A with $\operatorname{ind}(A) = p^v$. Let F be a field with $p^{v-s} = \operatorname{ind}(A_F) < \operatorname{ind}(A) = p^v$ and let $\pi : X_F \to X$ be the projection. Then*

$$\pi^*(c_j(\zeta_X(1))) = 0 \pmod{p}$$

for all j not divisible by p^v.

Proof. We have $\pi^*(\zeta_X(1)) = \zeta_{X_F}(1)^{\oplus p^s}$ with $p^s = \mathrm{ind}(A)/\mathrm{ind}(A_F)$. By functoriality we have
$$\pi^*(c_j(\zeta_X(1))) = c_j(\zeta_{X_F}(1)^{\oplus p^s}).$$

We're going to compute the total Chern polynomial of $\zeta_{X_F}(1)^{\oplus p^s}$ modulo p. If F splits A then $c_t(\zeta_{X_F}(1)^{\oplus p^s}) = (1-h)^{p^s} = 1 \pm h^{p^s} \pmod p$, where h is the class of a hyperplane in $\mathrm{CH}(X_F)$. Otherwise $v \neq s$ and we have
$$c_t(\zeta_{X_F}(1)^{\oplus p^s}) = c_t(\zeta_{X_F}(1))^{p^s} = (1 + c_1 t + \cdots + c_{p^{v-s}} t^{p^{v-s}})^{p^s}$$
with $c_i = c_i(\zeta_{X_F}(1))$. Using the multinomial formula, the latter expression can be rewritten
$$(1 + c_1 t + \cdots + c_{p^{v-s}} t^{p^{v-s}})^{p^s} \\ = 1 + \sum_{j=1}^{p^v} \Bigg(\sum_{\substack{|I| = p^s \\ i_1 + 2i_2 + \cdots + p^{v-s} i_{p^{v-s}} = j}} \binom{p^s}{i_0, i_1, \ldots, i_{p^{v-s}}} c_1^{i_1} \cdots c_{p^{v-s}}^{i_{p^{v-s}}} \Bigg) t^j.$$

Here the notation means
$$\binom{n}{a_0, \ldots, a_i} = \frac{n!}{a_0! \cdots a_i!}$$
and $I = (i_0, \ldots, i_{p^{v-s}})$ is a tuple of nonnegative integers with $|I| = i_0 + \cdots + i_{p^{v-s}}$.

By Lemma B.3, p divides all of the coefficients
$$\binom{p^s}{i_0, \ldots, i_{p^{v-s}}}$$
except when p^s divides one of $i_0, \ldots, i_{p^{v-s}}$. We are left to show $c_{i_k}^{p^s} = 0$ modulo p for any $k = 0, \ldots, p^{v-s} - 1$. Using [Karpenko 2017b, Proposition 3.5], we can find a finite field extension E/F lowering the index of A_F and such that $c_{i_k} = \rho_*(x)$ for some x in $\mathrm{CH}(X_E) \otimes \mathbb{Z}_{(p)}$ and for $\rho : X_E \to X_F$ the projection. The projection formula then gives
$$c_{i_k}^{p^s} = \rho_*(x(\rho^*\rho_*(x))^{p^s-1}) = 0 \pmod p$$
since $\rho^*\rho_* = [E : F]$. □

Corollary A.11. *Let A be a central simple algebra and X its associated Severi–Brauer variety. The classes $\tau_i(j)$ of $\mathrm{CH}(X) \otimes \mathbb{Z}_{(p)}$ satisfy the following relations*:

(1) *For all $i \geq 1$, we have $\tau_i(0) = 1$.*

(2) *For any $j \geq 0$, we have $\tau_i(p^v)\tau_i(j) = \tau_i(p^v j)$, where $v = v_p(\mathrm{ind}(A^{\otimes i}))$.*

(3) *For any integers $a_1, \ldots, a_{p^v} \geq 0$, there is a relation*
$$\tau_i(1)^{a_1} \cdots \tau_i(p^v)^{a_{p^v}} = \alpha \tau_i(a_1 + 2a_2 + \cdots + p^v a_{p^v})$$

for some α in $\mathbb{Z}_{(p)}$ with

$$v_p(\alpha) = \begin{cases} 0 & \text{if } v = 0, \\ \sum_{k=1}^{p^v}(v - v_p(k))a_k & \text{if } v > 0 \text{ and } j = 0 \pmod{p^v}, \\ v_p(r) - v + \sum_{k=1}^{p^v}(v - v_p(k))a_k & \text{if } v > 0 \text{ and } j \neq 0 \pmod{p^v}, \end{cases}$$

where we write $j = a_1 + 2a_2 + \cdots + p^v a_{p^v}$ and $0 \le r < p^v$ is the remainder in the division of j by p^v.

Proof. We remark that the definition of the classes $\tau_i(j)$ makes sense for any integer $j \ge 0$, but when $j > \deg(A)$, these classes are 0. For simplifications below, we don't put any upper bound on the value j may have.

The relation (1) is obvious from the definition. The relation (2) is also clear from the definition. So we're left proving the complicated relation (3). To do this, we pullback, to a splitting field L, the left and right side of the equation in (3) and compare p-adic valuations of their coefficients on the element h^j, where h is the class of a hyperplane over L. Some immediate observations for the following: we can assume j isn't larger than the dimension of X and we can assume $v > 0$; otherwise the claim is trivial.

The pullback of $\tau_i(1)^{a_1} \cdots \tau_i(p^v)^{a_{p^v}}$ can be written βh^j, where

$$v_p(\beta) = \sum_{k=1}^{p^v}(v - v_p(k) + v_p(i)k)a_k.$$

Similarly, the pullback of $\tau_i(a_1 + \cdots + p^v a_{p^v})$ can be written γh^j with

$$v_p(\gamma) = \begin{cases} v_p(i)p^v s_0 & \text{if } j = 0 \pmod{p^v}, \\ v_p(i)p^v s_0 + v - v_p(s_1) + v_p(i)s_1 & \text{if } j \neq 0 \pmod{p^v}, \end{cases}$$

where $j = s_0 p^v + s_1$ and $0 \le s_1 < p^v$. Since $v_p(\gamma) \ge v_p(\beta)$ by Proposition A.8, the result follows by subtracting. □

Lemma A.12. *Let A be a central simple algebra with* $\operatorname{ind}(A) = p^n$ *and* $r\mathcal{B}eh(A) = (n_0, \ldots, n_m)$. *Let X be the Severi–Brauer variety of A. Then, for any pair of integers i, j with $0 \le i \le j \le m$, the total Chern polynomial*

$$c_t(\zeta_X(p^j))^{p^{n_i - n_j - (j-i)}} = 1 + \sum_{k=1}^{p^{n_i-(j-i)}} \beta_k \tau_{p^j}(k) t^k$$

is a polynomial with coefficients in $\mathrm{CT}(p^i; X) \otimes \mathbb{Z}_{(p)}$.

Moreover, the p-adic valuation of the coefficient β_k equals

$$v_p(\beta_k) = \begin{cases} n_i - n_j - (j-i) - v_p(k/p^{n_j}) & \text{if } k = 0 \pmod{p^{n_j}}, \\ n_i - n_j - (j-i) & \text{if } k \neq 0 \pmod{p^{n_j}}. \end{cases}$$

Proof. We identify $\mathrm{K}(X)$ with its image in $\mathrm{K}(X_L)$ for a splitting field L of X. We write ξ for the class of $\mathcal{O}(-1)$ in $\mathrm{K}(X_L)$. Then the class of $\zeta_X(p^i)$ is identified with $p^{n_i}\xi^{p^i}$ and the class of $\zeta_X(p^j)$ is identified with $p^{n_j}\xi^{p^j}$. We have

$$\lambda^{p^{j-i}}(p^{n_i}\xi^{p^i}) = \binom{p^{n_i}}{p^{j-i}}\xi^{p^j}.$$

It follows that

$$\begin{aligned} c_t(p^{n_i-(j-i)}\xi^{p^j}) &= c_t(p^{n_i-(j-i)-n_j}(p^{n_j}\xi^{p^j})) \\ &= c_t(\zeta_X(p^j))^{p^{n_i-n_j-(j-i)}} \\ &= (1+\tau_{p^j}(1)t+\cdots+\tau_{p^j}(p^{n_j})t^{p^{n_j}})^{p^{n_i-n_j-(j-i)}} \end{aligned}$$

is a polynomial with coefficients contained in $\mathrm{CT}(p^i;X)\otimes\mathbb{Z}_{(p)}$. This proves the first claim.

To prove the second claim, we write

$$(1+\tau_{p^j}(1)t+\cdots+\tau_{p^j}(p^{n_j})t^{p^{n_j}})^{p^{n_i-n_j-(j-i)}} = 1+\sum_{k=1}^{p^{n_i-(j-i)}}\beta_k\tau_{p^j}(k)t^k$$

using Proposition A.8. Explicitly there are equalities

$$\beta_k\tau_{p^j}(k) = \sum_I \binom{p^{n_i-(j-i)-n_j}}{I}\tau_{p^j}^I,$$

where the sum runs over tuples $I=(a_0,\dots,a_{p^{n_j}})$ such that $a_0+\cdots+a_{p^{n_j}} = p^{n_i-(j-i)-n_j}$ and $a_1+2a_2+\cdots+p^{n_j}a_{p^{n_j}}=k$; here we're using the notation

$$\binom{p^{n_i-(j-i)-n_j}}{I} = \binom{p^{n_i-(j-i)-n_j}}{a_0,\dots,a_{p^{n_j}}} = \frac{p^{n_i-(j-i)-n_j}!}{a_0!\cdots a_{p^{n_j}}!}$$

and

$$\tau_{p^j}^I = \tau_{p^j}(0)^{a_0}\tau_{p^j}(1)^{a_1}\cdots\tau_{p^j}(p^{n_j})^{a_{p^{n_j}}}$$

for a tuple $I=(a_0,\dots,a_{p^{n_j}})$. Thus

$$v_p(\beta_k) = v_p\Bigg(\sum_I\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I\Bigg) \geq \min\Bigg\{v_p\Bigg(\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I\Bigg)\Bigg\},$$

where α_I is the coefficient in $\tau_{p^j}^I=\alpha_I\tau_{p^j}(k)$ from Corollary A.11. In fact, the above inequality is an equality if there is a unique minimum over the given tuples I. The p-adic valuation of any coefficient

$$\binom{p^{n_i-(j-i)-n_j}}{I}\alpha_I$$

can be found using Corollary A.11 and Lemma B.2; the p-adic valuation of any such coefficient can also be bounded below using Corollary A.11 and Lemma B.3. With this bound, one can show there is a unique minimum among the p-adic valuation of these coefficients: set $s = n_i - (j-i)$ and $r = n_j$ in Lemma B.4. Finally, using Lemma B.2 to compute the valuation explicitly and using Lemma B.5, setting $s = n_i - (j-i)$ and $r = n_j$, shows the p-adic valuation of β_k is as claimed. □

The lemma above provides numbers β_k such that $\beta_k \mathrm{CT}^k(p^j; X) \subset \mathrm{CT}^k(1; X)$. Using a technique developed in [Karpenko 2017b], we can reduce the size of the β_j further. We assume A is a division algebra in the following as this is the only case we need.

Corollary A.13. *Let A be a division algebra with $\mathrm{ind}(A) = p^n$ and $r\mathcal{B}eh(A) = (n_0, \dots, n_m)$. Let X be the Severi–Brauer variety of A. Pick an integer $0 \leq j \leq m$, and let $0 \leq i \leq p^n - 1$ be a second integer.*

There exists a number α_i in $\mathbb{Z}_{(p)}$ so that $\alpha_i \tau_{p^j}(i)$ is contained in $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$. Moreover, the p-adic valuation of the α_i we find equals

$$v_p(\alpha_i) = \begin{cases} n - j - n_j & \text{if } 1 \leq i \leq p^{n_j}, \\ n - j - n_j - \lfloor \log_p(i/p^{n_j}) \rfloor & \text{if } p^{n_j} < i \leq p^{n-j}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let L be a maximal subfield of A, of degree p^n over the base, and let N be the image of the pushforward $\pi_* : \mathrm{CH}(X_L) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}(X) \otimes \mathbb{Z}_{(p)}$ along the projection $\pi : X_L \to X$. By [Karpenko 2017b, Proposition 3.5], the image N is contained in $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$. Recall also that the pullback π^* followed by the pushforward π^* is multiplication by p^n, the degree of L over the base. The proof of the corollary mimics that of [Karpenko 2017b, Proposition 3.12]; the idea of the proof is to use the explicit bounds of Lemma A.12 and the projection formula to get the result for any i. Note that the claim is trivial for $j = 0$ (or we can just set $\alpha_i = 1$ in this case) so, throughout the proof, it's safe to assume $j > 0$.

We first show, for each $i \leq p^{n-j}$ and using β_i for the coefficient such that $\beta_i \mathrm{CT}^i(p^j; X) \subset \mathrm{CT}^i(1; X)$ found in Lemma A.12, that $p^{v_p(\beta_i)} \tau_{p^j}(i)$ is in the image of the map π_*. Write $i = s_0 p^{n_j} + s_1$ with $0 \leq s_1 < p^{n_j}$. The image of $\tau_{p^j}(i)$ in $\mathrm{CH}(X_L) \otimes \mathbb{Z}_{(p)}$ is equal, up to prime-to-p parts, to

$$\pi^*(\tau_{p^j}(i)) = \begin{cases} p^{ij} h^i & \text{if } s_1 = 0, \\ p^{ij + n_j - v_p(s_1)} h^i & \text{if } s_1 > 0. \end{cases}$$

By Lemma A.12, the multiple $\beta_i \tau_{p^j}(i)$ has image, up to prime-to-p parts,

$$\pi^*(\beta_i \tau_{p^j}(i)) = p^{n + (i-1)j - v_p(i)} h^i$$

regardless of s_1. Thus,

$$\begin{aligned} p^{v_p(\beta_i)}\tau_{p^j}(i) &= \frac{1}{p^n}\pi_*\pi^*(p^{v_p(\beta_i)}\tau_{p^j}(i)) \\ &= \pi_*\Big(\frac{1}{p^n}\big(\pi^*(p^{v_p(\beta_i)}\tau_{p^j}(i))\big)\Big) = \pi_*(p^{(i-1)j-v_p(i)}h^i). \end{aligned}$$

Since $(i-1)j - v_p(i) \geq 0$, we find $p^{v_p(\beta_i)}\tau_{p^j}(i)$ is in N as claimed.

Now let i be an integer with $1 \leq i \leq p^n - 1$ and set $\ell = \lfloor \log_p(i/p^{n_j}) \rfloor$. To get the bounds on the p-adic valuation in the corollary statement, we work in cases. We first assume $\ell \geq n - j - n_j$, or equivalently, $i \geq p^{n-j}$. By the above and Lemma A.12, we can find an element x of $\mathrm{CH}(X_L)$ with

$$\pi_*(x) = \tau_{p^j}(p^{n-j}).$$

Set $k = i - p^{n-j}$. Then, using (2) and (3) of Corollary A.11,

$$\begin{aligned} \tau_{p^j}(i) = \tau_{p^j}(p^{n_j})^{n-j-n_j}\tau_{p^j}(k) &= \tau_{p^j}(p^{n-j})\tau_{p^j}(k) \\ &= \pi_*(x)\tau_{p^j}(k) = \pi_*(x\pi^*(\tau_{p^j}(k))). \end{aligned}$$

It follows from [Karpenko 2017b, Proposition 3.5] that $\tau_{p^j}(i)$ is contained in $N \subset \mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$ for all $i \geq p^{n-j}$.

For the other i, we act similarly. If $p^{n_j} < i \leq p^{n-j}$, then set $k = i - p^{n_j+\ell}$. Then there is a (different) element x with $\pi_*(x) = p^r\tau_{p^j}(p^{\ell+n_j})$, where $r = v_p(\beta_{p^{\ell+n_j}})$. Then

$$\begin{aligned} p^r\tau_{p^j}(i) = p^r\tau_{p^j}(p^{n_j})^{\ell}\tau_{p^j}(k) &= p^r\tau_{p^j}(p^{\ell+n_j})\tau_{p^j}(k) \\ &= \pi_*(x)\tau_{p^j}(k) = \pi_*(x\pi^*(\tau_{p^j}(k))) \end{aligned}$$

and the claim follows as before.

For the remaining i, when $i \leq p^{n_j}$, the claim actually follows immediately from Lemma A.12. □

We can do better still if we multiply the classes $\tau_1(i)$ and $\tau_{p^j}(k)$ for some integers $i, k \geq 0$.

Corollary A.14. *Let A be a division algebra with $\mathrm{ind}(A) = p^n$ and $r\mathcal{B}eh(A) = (n_0, \ldots, n_m)$. Let X be the Severi–Brauer variety of A. Pick an integer $0 \leq j \leq m$, and let $1 \leq i, k \leq p^n - 1$ be two integers with $i + k \leq p^n - 1$.*

There exists a number $\beta_{i,k}$ in $\mathbb{Z}_{(p)}$ such that $\beta_{i,k}\tau_1(i)\tau_{p^j}(k)$ is contained in $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$. Moreover, the p-adic valuation of the $\beta_{i,k}$ we find equals

$$v_p(\beta_{i,k}) = \begin{cases} \max\{v_p(i) - j - n_j, 0\} & \text{if } 1 \leq k \leq p^{n_j}, \\ \max\{v_p(i) - j - n_j - \lfloor \log_p(k/p^{n_j}) \rfloor, 0\} & \text{if } p^{n_j} < k \leq p^{n-j}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is the same as Corollary A.13 except that we use the equality

$$\pi^*(\beta_k \tau_1(i)\tau_{p^j}(k)) = p^{n+(k-1)j-v_p(k)+n-v_p(i)} h^{i+k},$$

up to prime-to-p parts, to find $p^{v_p(\beta_{i,k})}\tau_1(i)\tau_{p^j}(k)$ is contained in N. □

As an application, the above corollary can be used to settle the particular case of Conjecture 1.1 when X is the Severi–Brauer variety of an algebra A with level 1:

Theorem A.15. *Let A be a central simple k-algebra of level* 1 *and let X be the Severi–Brauer variety of A. Assume* $\mathrm{CH}(X)$ *is generated by Chern classes. Then the* K*-theory coniveau epimorphism* $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ *is an isomorphism.*

Proof. It's sufficient to show the claim when A is a division algebra of index p^n. In this case the kernel of the epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau \mathrm{G}(X)$ is p-primary-torsion so we can work with $\mathbb{Z}_{(p)}$ coefficients throughout the proof. Let L be a splitting field for A. Since $\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}$ is p-torsion free, the composition

$$\mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)} \to \mathrm{CH}(X) \otimes \mathbb{Z}_{(p)} \to \mathrm{gr}_\tau \mathrm{G}(X) \otimes \mathbb{Z}_{(p)}$$

is injective; we denote by C the image of this composition. We have an inequality

$$[\mathrm{CH}(X) \otimes \mathbb{Z}_{(p)} : \mathrm{CT}(1; X) \otimes \mathbb{Z}_{(p)}] \geq [\mathrm{gr}_\tau \mathrm{G}(X) \otimes \mathbb{Z}_{(p)} : C]. \tag{in}$$

We're going to use the bounds from Corollary A.14 to get an upper bound on the left of (in). We'll also bound the right of (in), by computing

$$[\mathrm{gr}_\tau \mathrm{G}(X) \otimes \mathbb{Z}_{(p)} : C] = \frac{[\mathrm{gr}_\tau \mathrm{G}(X_L) : C]}{[\mathrm{K}(X_L) : \mathrm{K}(X)]}$$

precisely; the equality of the ratio of these indices can be found in [Karpenko 2017b, proof of Theorem 3.1]. The proof will be completed once we show these two bounds are equal.

To get an upper bound on the left of (in), we sum the maximums of the p-adic valuations occurring in Corollaries A.13 and A.14. Plainly said, we compute an upper bound on p-adic valuations of the orders of the elements $\tau_1(i)\tau_{p^r}(k)$, where r is the (unique since A has level 1) smallest positive integer with

$$v_p(\mathrm{ind}(A^{\otimes p^r})) < v_p(\mathrm{ind}(A^{\otimes p^{r-1}})) - 1,$$

in the group $\mathrm{CH}(X)/\mathrm{CT}(1; X)$. Note that by Proposition A.5 and Proposition A.8, the elements $\tau_1(i)\tau_{p^r}(k)$ are exactly the generators of this quotient group, so that by computing an upper bound on their orders and raising p to this upper bound, we also compute an upper bound on the index in the left of (in). Once we have this upper bound, we'll move on to give a lower bound for the right-hand side of (in). These two bounds turn out to be equal, showing that our upper bound on the orders were in fact their precise order.

Set $n_r = v_p(\operatorname{ind}(A^{\otimes p^r}))$ and $\ell = n - r - n_r$. When $i = 0$, we sum the contributions from Corollary A.13,

$$\sum_{a=1}^{p^{n_r}-1} n - r - n_r + \sum_{a=p^{n_r}}^{p^{n-r}-1} n - r - n_r - \lfloor \log_p(a/p^{n_r}) \rfloor$$
$$= (p^{n_r} - 1)\ell + \sum_{b=0}^{\ell-1} \varphi(p^{n_r+b+1})(\ell - b),$$

where φ is the Euler totient function (we use this function to combine those terms a that have the same value of $\lfloor \log_p(a/p^{n_r}) \rfloor$; there are exactly $\varphi(p^{n_r+b+1}) = p^{n_r+b+1} - p^{n_r+b}$ such terms with value b, i.e., $p^{n_r+b}, \ldots, p^{n_r+b+1} - 1$). When $i > 0$, we only need to account for the terms with $v_p(i) > n - \ell$ (note if $\ell = 1$ then $r + n_r = n - 1$ and there are no terms of this kind):

$$\sum_{b=1}^{p^{n_r}-1} v_p(i) - r - n_r + \sum_{b=p^{n_r}}^{p^{v_p(i)-r}-1} v_p(i) - r - n_r - \lfloor \log_p(b/p^{n_r}) \rfloor$$
$$= (p^{n_r} - 1)(v_p(i) - r - n_r) + \sum_{b=0}^{v_p(i)-r-n_r-1} \varphi(p^{n_r+b+1})(v_p(i) - r - n_r - b).$$

Of the integers i satisfying $1 \le i < p^n$, there are $\varphi(p^{\ell-1})$ integers i with $v_p(i) = n - \ell + 1$, there are $\varphi(p^{\ell-2})$ integers i with $v_p(i) = n - \ell + 2$, and so on to $\varphi(p)$ integers i with $v_p(i) = n - \ell + (\ell - 1)$. Summing over all such i with $v_p(i) > n - \ell$, we get

$$\sum_{a=1}^{\ell-1} \varphi(p^{\ell-a}) \left((p^{n_r} - 1)a + \sum_{b=0}^{a} \varphi(p^{n_r+b+1})(a - b) \right).$$

Combining both the $i = 0$ and $i > 0$ contributions gives a definitive upper bound of

$$S = \sum_{a=1}^{\ell} \varphi(p^{\ell-a}) \left((p^{n_r} - 1)a + \sum_{b=0}^{a} \varphi(p^{n_r+b+1})(a - b) \right).$$

To get a lower bound on the right of (in), we calculate $[\operatorname{gr}_\tau \mathrm{G}(X) \otimes \mathbb{Z}_{(p)} : C]$ precisely. Since this index equals

$$\frac{[\operatorname{gr}_\tau \mathrm{G}(X_L) : C]}{[\mathrm{K}(X_L) : \mathrm{K}(X)]},$$

it's sufficient to calculate the numerator and denominator of this fraction. The numerator depends only on the dimension of X and equals

$$\prod_{i=1}^{p^n} (p^{n-v_p(i)}) = \prod_{j=1}^{n-1} (p^{n-j})^{\varphi(p^{n-j})}.$$

The denominator depends on the reduced behavior of A and equals

$$\prod_{i=0}^{p^n-1} \operatorname{ind}(A^{\otimes i}) = \left(\prod_{j=0}^{r-1}(p^{n-j})^{\varphi(p^{n-j})}\right)\left(\prod_{j=r}^{n_r+r}(p^{n_r+r-j})^{\varphi(p^{n-j})}\right).$$

Dividing the two gives

$$P = \left(\prod_{i=r}^{n_r+r}(p^{\ell})^{\varphi(p^{n-i})}\right)\left(\prod_{i=n_r+r+1}^{n}(p^{n-i})^{\varphi(p^{n-i})}\right).$$

What remains to be shown is the equality $\log_p(P) = S$. A computation of the logarithm gives

$$\begin{aligned}
\log_p(P) &= \log_p\left(\prod_{i=r}^{n_r+r}(p^{\ell})^{\varphi(p^{n-i})}\prod_{i=n_r+r+1}^{n}(p^{n-i})^{\varphi(p^{n-i})}\right) \\
&= \sum_{i=r}^{n_r+r} \ell\varphi(p^{n-i}) + \sum_{i=n_r+r+1}^{n}(n-i)\varphi(p^{n-i}) \\
&= \ell(p^{n-r} - p^{\ell-1}) + \sum_{i=1}^{n-r-n_r-1} i\varphi(p^i) \\
&= \ell(p^{n-r} - p^{\ell-1}) + \frac{(\ell-1)p^{\ell} - \ell p^{\ell-1} + 1}{p-1} \\
&= \ell p^{n-r} - \frac{p^{\ell}-1}{p-1}.
\end{aligned}$$

And by simplifying the sum S we find

$$\begin{aligned}
S &= \sum_{a=1}^{\ell}\varphi(p^{\ell-a})\left((p^{n_r}-1)a + \sum_{b=0}^{a}\varphi(p^{n_r+b+1})(a-b)\right) \\
&= \sum_{a=1}^{\ell}\varphi(p^{\ell-a})(p^{n_r}-1)a + \sum_{a=1}^{\ell}\varphi(p^{\ell-a})\sum_{b=0}^{a}\varphi(p^{n_r+b+1})(a-b) \\
&= \frac{p^{n-r}-p^{n_r}}{p-1} - \frac{p^{\ell}-1}{p-1} + \sum_{a=1}^{\ell}\varphi(p^{\ell-a})\left(\frac{p^{n_r}(p^{a+1}-(a+1)p+a)}{p-1}\right) \\
&= \frac{p^{n-r}-p^{n_r}}{p-1} - \frac{p^{\ell}-1}{p-1} + \frac{\ell p^{n-r+1} - (\ell+1)p^{n-r} + p^{n_r}}{p-1} \\
&= \ell p^{n-r} - \frac{p^{\ell}-1}{p-1},
\end{aligned}$$

as desired. $\square$

Appendix B: p-adic valuations

Fix a prime p to be used throughout this appendix. For any integer $n \geq 0$, let $S_p(n)$ denote the sum of the base-p digits of n. In other words, if $n = a_0 + a_1 p + \cdots + a_r p^r$ with $0 \leq a_0, \ldots, a_r \leq p-1$ then $S_p(n) = a_0 + a_1 + \cdots + a_r$. This appendix proves some simple results on the function S_p and on p-adic valuations involving this function. The proof for the next lemma is elementary and we omit it.

Lemma B.1. *Let $n \geq 0$ be an integer.*

(1) $S_p(p^n) = 1$.

(2) $S_p(p^n a) = S_p(a)$ *for any integer* $a \geq 0$.

(3) $S_p(p^n - 1) = n(p-1)$.

(4) *If* $0 \leq k \leq n$ *then* $S_p(p^n - p^k) = (n-k)(p-1)$.

(5) *If* $0 \leq a \leq p^n$ *then* $S_p(p^n - a) + S_p(a) = (n - v_p(a))(p-1) + 1$.

(6) *If* $0 \leq a \leq p^n - 1$ *then* $S_p(p^n - 1 - a) + S_p(a) = n(p-1)$.

We use the notation

$$\binom{n}{a_0, \ldots, a_r} = \frac{n!}{a_0! \cdots a_r!}.$$

If $a_0 + \cdots + a_r = n$ then we have the following:

Lemma B.2. *Let $n = a_0 + \cdots + a_r$ with $n, a_0, \ldots, a_r \geq 0$. Then*

$$v_p\left(\binom{n}{a_0, \ldots, a_r}\right) = \frac{1}{p-1}\left(\left(\sum_{i=0}^{r} S_p(a_i)\right) - S_p(n)\right).$$

Proof. See, for example, [Merkurjev 2003, Lemma 11.2]. □

Lemma B.3. *Let $a_0, \ldots, a_r \geq 0$ and $n > 0$ be integers with $a_0 + \cdots + a_r = n$. Then*

$$v_p\left(\binom{n}{a_0, \ldots, a_r}\right) \geq v_p(n) - \min_{0 \leq i \leq r}\{v_p(a_i)\}.$$

Proof. See, for example, [Merkurjev 2003, Lemma 11.3]. □

Lemma B.4. *Let $0 \leq r \leq s$ be integers. Fix an integer $0 < j \leq p^s$. Let $a_0, \ldots, a_{p^r} \geq 0$ be integers with $a_0 + \cdots + a_{p^r} = p^{s-r}$ and $a_1 + 2a_2 + \cdots + p^r a_{p^r} = j$. Write $j = s_0 p^r + s_1$ with $0 \leq s_1 < p^r$. Then if $s_1 = 0$, there is an inequality*

$$s - r - \min_{0 \leq k \leq p^r}\{v_p(a_k)\} + \sum_{i=1}^{p^r}(r - v_p(i))a_i \geq s - r - v_p(s_0),$$

and if $s_1 > 0$, there is an inequality

$$s - r - \min_{0 \leq k \leq p^r}\{v_p(a_k)\} - (r - v_p(s_1)) + \sum_{i=1}^{p^r}(r - v_p(i))a_i \geq s - r.$$

If $s_1 = 0$, then equality holds if and only if $a_0 = p^{s-r} - s_0$ and $a_{p^r} = s_0$. If $s_1 > 0$, then equality holds if and only if $a_0 = p^{s-r} - s_0 - 1$, $a_{s_1} = 1$, and $a_{p^r} = s_0$.

Proof. We first assume $s_1 = 0$. If $\ell = \min\{v_p(a_k)\}$ is 0, then the inequality clearly holds since $r - v_p(i) \geq 0$ for all $1 \leq i \leq p^r$. If $\ell > 0$ and $r = 0$, then $j = a_1$ and $j = s_0$. So ℓ is either $v_p(a_0) = v_p(p^s - j)$ or $v_p(a_1) = v_p(j) = v_p(s_0)$. Since $j \leq p^s$, it follows $\ell = v_p(s_0)$ and the claim follows with equality in this case. If $\ell = \min\{v_p(a_k)\} > 0$, then since $r - v_p(i) \geq 0$ for all $1 \leq i \leq p^r$, the inequality also holds if $r \neq 0$ and if there is a nonzero a_i with $i \neq 0, p^r$ as $(r - v_p(i))a_i - \ell \geq 0$.

Thus, to prove that the inequality holds in general (for $s_1 = 0$), it suffices to assume $\ell > 0$, $r > 0$, and $a_i = 0$ unless $i = 0$ or $i = p^r$. Assuming this is the case, it follows from the assumption $p^r a_{p^r} = j$ that $a_{p^r} = s_0$ and from the assumption $a_0 + a_{p^r} = p^{s-r}$ that $a_0 = p^{s-r} - s_0$. Since $s_0 \leq p^{s-r}$, we also have $v_p(a_{p^r}) \leq s - r$ so that $v_p(a_0) = v_p(a_{p^r})$ unless $a_{p^r} = p^{s-r}$ (in which case $v_p(a_0) = \infty$ and the claim is clear). Thus $\ell = v_p(s_0)$, the inequality holds, and it is even an equality in this case.

To see that $a_0 = p^{s-r} - s_0$ and $a_{p^r} = s_0$ is the only case the inequality is an equality, one can work through the same cases. If $\ell = 0$ and there is equality, then $v_p(s_0) = 0$ and the large summation must equal 0. Hence $p^r a_{p^r} = j$ and the claim follows. If $\ell > 0$, then either $r = 0$ or $r > 0$. If $r = 0$, the claim follows from the first paragraph. If $r > 0$, then either all a_i with $i \neq 0, p^r$ vanish or there is at least one $0 < i < p^r$ with $a_i \neq 0$. We can assume the latter case where the inequality is a strict inequality since $(r - v_p(i))a_i - \ell \geq a_i - \ell > 0$.

To show the claim when $s_1 > 0$, we work through cases similar to before. Note now $r > 0$ holds always, as otherwise we'd have $s_1 = 0$. If $\ell = \min\{v_p(a_k)\} = 0$ then since $r - v_p(i) \geq 0$, we're left to show that the summation

$$\sum_{i=1}^{p^r} (r - v_p(i))a_i$$

is greater than or equal to $r - v_p(s_1) \leq r$. Since $s_1 > 0$, there is a smallest integer k with $0 \leq k \leq r - 1$, $a_{bp^k} \neq 0$, and b relatively prime to p. It follows that p^k divides s_1 and $-(r - v_p(s_1)) \geq -r + k$. Since $(r - v_p(bp^k))a_{bp^k} = (r-k)a_{bp^k} \geq (r-k)$, we find that the inequality holds by summing

$$(r - v_p(bp^k))a_{bp^k} - (r - v_p(s_1)) \geq (r-k) - (r-k) = 0.$$

Thus to prove the inequality holds in general, it suffices to assume $\ell > 0$. Under our assumptions $\ell > 0$, $r > 0$, and $j \neq p^r a_{p^r}$, we have that there exists at least one i with $i \neq 0, p^r$ such that $a_i \neq 0$. Let k be the smallest integer between $0 \leq k < r$ such that $a_{bp^k} \neq 0$ for some b relatively prime to p. It follows p^k divides s_1, and hence $-(r-v_p(s_1)) \geq -r+k$. Now

$$\begin{aligned}(r-v_p(bp^k))a_{bp^k}-r+v_p(s_1)-\ell &\geq (r-k)p^\ell - r + v_p(s_1) - \ell \\ &= (r-k)(p^\ell-1)-\ell+v_p(s_1) \\ &\geq (p^\ell-1-\ell)+v_p(s_1) \\ &\geq 0.\end{aligned}$$

We end by showing that equality holds, assuming $s_1>0$, only in the specified case (it's clear equality holds in this case). We first assume $\ell=0$. For equality to hold, we must have

$$\sum_{i=1}^{p^r}(r-v_p(i))a_i = r-v_p(s_1).$$

Again there is a minimal $0\leq k<r$ with $a_{bp^k}\neq 0$ for some b relatively prime to p. We also get that p^k divides s_1. It follows that

$$(r-v_p(bp^k))a_{bp^k} = (r-k)a_{bp^k} \geq (r-k) \geq r-v_p(s_1)$$

must be an equality. Hence $a_{bp^k}=1$ and we are in the specified case.

We next assume $\ell>0$ and show our inequality is strict. Let k with $0\leq k<r$ be minimal with $a_{bp^k}\neq 0$ for some b relatively prime to p. Then

$$\sum_{i=1}^{p^r}(r-v_p(i))a_i \geq (r-k)p^\ell.$$

Since $\ell+r-v_p(s_1)\leq \ell+r-k$ it suffices to check $(r-k)p^\ell>\ell+r-k$ holds for all $(r-k), \ell>0$ in order to show this is a strict inequality in this case. But this is true since dividing by $r-k$ yields $p^\ell>\ell/(r-k)+1$; making another estimate we can show $p^\ell>\ell+1$ for all ℓ and this is always true for $\ell>0$ and $p\geq 2$. $\square$

Lemma B.5. *Let $0\leq r\leq s$ be integers. Fix an integer $1\leq j\leq p^s$ and write $j=s_0p^r+s_1$ with $0\leq s_1<p^r$.*

If $s_1=0$, let $I=(a_0,\dots,a_{p^r})$ be the tuple with $a_0=p^{s-r}-s_0$, $a_{p^r}=s_0$ and $a_i=0$ for all other i. Then

$$v_p\left(\binom{p^{s-r}}{I}\right) = \frac{1}{p-1}(S_p(a_0)+S_p(a_{p^r})-S_p(p^{s-r})) = s-r-v_p(s_0).$$

If $s_1>0$, let $I=(a_0,\dots,a_{p^r})$ be the tuple with $a_0=p^{s-r}-s_0-1$, $a_{s_1}=1$, $a_{p^r}=s_0$ and $a_i=0$ for all other i. Then

$$v_p\left(\binom{p^{s-r}}{I}\right) = \frac{1}{p-1}(S_p(a_0)+S_p(a_{s_1})+S_p(a_{p^r})-S_p(p^{s-r})) = s-r.$$

Proof. The first equality follows from Lemma B.2 and Lemma B.1 (1) and (5). The second equality follows from Lemma B.2 and Lemma B.1 (1) and (6). $\square$

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