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Motivic analogues of MO and MSO

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We construct algebraic unoriented and oriented cobordism, named MGLO and MSLO, respectively. MGLO is defined and its homotopy groups are explicitly computed, giving an answer to a question of Jack Morava. MSLO is also defined and its coefficients are explicitly computed after completing at a prime p. Similarly to MSO, the homotopy type of MSLO depends on whether the prime p is even or odd. Finally, a computation of a localization of the homotopy groups of MGLR is given.

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1. Introduction

Motivic homotopy theory on smooth schemes over a field was introduced by Morel and Voevodsky [1999] with the purpose of proving the Bloch–Kato conjecture, which was accomplished by Voevodsky [2003a]. Motivic analogues of well known spectra of algebraic topology carry additional deep algebraic information. For example, motivic "ordinary" homology computes Bloch's higher Chow groups, motivic K-theory is algebraic K-theory, and motivic cobordism has a geometric interpretation as algebraic cobordism [Levine and Morel 2007].

In [Hu et al. 2011], Hu, Kriz, and Ormsby (following notes of Deligne [2009]) introduced equivariant stable motivic homotopy theory, and motivic real K-theory (an analogue of Atiyah's KR) to solve Thomason's homotopy limit problem on

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algebraic Hermitian K-theory. A follow-up paper [Berrick et al. 2015] generalized their result.

The authors of [Hu et al. 2011] introduced a motivic analogue of MR, which they denote by MGLR. The computation of the coefficients of MGLR remains a difficult problem.

In this paper we introduce a nonequivariant motivic spectrum MGLO which is related to MGLR and analogous to unoriented (topological) cobordism MO. We prove MGLO is a wedge of suspensions of ordinary motivic homology with coefficients in C_2. Although our result is similar to the analogous result for MO, the pattern of suspensions in MGLO is more subtle due to the Tate twist. Our result is stated in Theorem 3.12. This answers a question of Jack Morava.

An important subtlety arises in the construction of MGLO, and a new concept is developed in the process. The point is that in topology MO can be obtained from MR by a construction called geometric fixed points [Lewis et al. 1986, Chapter 2, Definition 9.7]. In more detail, let EC_2 be a free contractible C_2-equivariant CW complex. Then we have a cofiber sequence

$$EC_{2+} \to S^0 \to \widetilde{EC}_2.$$

For a C_2-equivariant topological spectrum E, we define the geometric fixed points of E as

$$\Phi^{C_2}(\mathsf{E}) = (\widetilde{EC}_2 \wedge \mathsf{E})^{C_2}.$$

In particular,

$$\mathsf{MO} = \Phi^{C_2}(\mathsf{MR}),$$

and this is the spectrum we compute with. A similar point is also relevant in [Hill et al. 2016]. There is also a motivic geometric fixed point functor $\Phi^{C_2}_{\text{ét}}$ (see Section 6). Applying this functor to MGLR gives

$$\mathsf{MGLO} = \Phi^{C_2}_{\text{ét}}(\mathsf{MGLR}).$$

Extending this construction, we also define a motivic analogue of oriented cobordism MSO, which we denote by MSLO. In Theorem 4.12 we compute the coefficients of MSLO completed at an odd prime, and in Theorems 4.25 and 4.23 we show that the 2-completion of MSLO splits as a wedge sum of copies of motivic homology.

We would like to point out that the spectrum MSLO is not the same as the spectrum MSL defined by Panin and Walter [2010]. The topological realization of MSL is MSU, the special unitary cobordism spectrum. The topological counterparts of MSLO and MSL (i.e., MSO and MSU, respectively) are discussed in [Pengelley 1982]. Furthermore, using almost the same construction used to form MGLR [Hu et al. 2011], one can form a spectrum MSLR, which we call special hermitian algebraic cobordism. The underlying nonequivariant spectrum of MSLR is MSL, and the underlying geometric fixed points spectrum of MSL is MSLO.

In Sections 5 and 6 we use our computation of MGLO to obtain some results on the coefficients of MGLR. In particular, we compute the coefficients of 2-completed MGLR localized at two elements θ and λ in Theorem 6.6. In Theorem 5.5 and Corollary 5.6 we show that MGLR is not motivically real-oriented, solving a question asked in [Hu et al. 2011].

Notation and conventions. Throughout the paper, k is a field of characteristic 0. The stable motivic homotopy category of Morel and Voevodsky, as constructed in [Morel and Voevodsky 1999], is denoted by $\mathrm{SH}(\mathsf{k})$. An important feature of motivic homotopy theory is that we have two circles. These we denote as S^1 and S^α, as opposed to the other common notation of $S^{1,0}$ and $S^{1,1}$, respectively. The topological circle S^1 is formed in the usual way as $\Delta^1/\partial\Delta^1$, which we point at 1. The geometric sphere S^α is $\mathbb{G}_m \simeq \mathrm{Spec}(\mathsf{k}[z, z^{-1}])$ pointed at 1.

For a finite group G, let $G\mathrm{Sm}/\mathsf{k}$ denote the category of smooth schemes of finite type over k with left G actions and equivariant maps. The construction of the stable G-equivariant motivic homotopy category $\mathrm{SH}_G(\mathsf{k})$ can be found in [Hu et al. 2011]. We write $[-,-]_G$ for maps in $\mathrm{SH}_G(\mathsf{k})$. An important feature of the C_2-equivariant motivic homotopy category $\mathrm{SH}_{C_2}(\mathsf{k})$ is that we have four circles. These are denoted S^1, S^α, S^σ, and $S^{\sigma\alpha}$. The topological sphere S^1 is the usual simplicial sphere and S^σ the simplicial sphere with action $z \to -z$. The geometric sphere S^α is the pointed scheme $(\mathbb{G}_m, 1)$ equipped with trivial action and $S^{\sigma\alpha}$ is the pointed scheme $(\mathbb{G}_m, 1)$ equipped with the inversion action $z \mapsto z^{-1}$. For this reason we often use the notation $\mathbb{G}_m^{1/z}$ instead of $S^{\sigma\alpha}$.

We adopt the convention that $*$ refers to an integer grading of homotopy or (co)homology groups while $\star$ refers to multidimensional grading. In more detail, $\star$ grading refers to either $\mathbb{Z}^2$ grading in the cases of $\mathrm{SH}(\mathsf{k})$ and the classical stable C_2-equivariant category, or to $\mathbb{Z}^4$ grading in the case of $\mathrm{SH}_{C_2}(\mathsf{k})$.

2. A motivic analogue of MO

In this section, we give a detailed account of how to construct a motivic analogue of the unoriented cobordism spectrum MO. In Section 3, we give a full computation of the coefficients of this spectrum, which we call MGLO, up to knowledge of the coefficients of motivic $H\mathbb{Z}/2$. In particular, one can compute the coefficients explicitly for the fields $\mathbb{R}$ and $\mathbb{C}$. Moreover, the topological realization of MGLO over the field $\mathbb{C}$ is MO.

***The construction of* MGLO.** The idea behind our definition of MGLO is that, just as the geometric fixed points of MO is MR, the geometric fixed points of MGLR should be MGLO. The definition presented in this paper is different from the definition given in [Hu et al. 2011]. Using simplicial EC_2, the authors of [Hu et al. 2011] define MGLO as

$$(\widetilde{EC}_2 \wedge \mathsf{MGLR})^{C_2}. \tag{2.1}$$

However, the functor

$$(\widetilde{EC}_2 \wedge (-))^{C_2}$$

in (2.1) fails to satisfy a crucial property for general motivic spectra. Topologically, given a G-equivariant spectrum E, the functor

$$\Phi^G(-) := (\widetilde{EG} \wedge (-))^G$$

applied to E produces a nonequivariant spectrum $\Phi^G(\mathsf{E})$, which is equivalent to forgetting E to the prespectrum level and then simultaneously taking G-fixed points of the spaces making up the prespectrum of E and the connecting maps to form a nonequivariant prespectrum. One can then promote this to a nonequivariant spectrum in the usual way. Similarly, in our definition, MGLO is defined by forgetting MGLR to the level of prespectra and then taking C_2-fixed points of the spaces and connecting maps to form a nonequivariant prespectrum. Promoting this to a spectrum defines MGLO.

We suspect this alternative definition of MGLO to be different than (2.1), the reason being that simplicial $\widetilde{EC}_2$ is a model for $S^{\infty\sigma}$. This only takes into account the σ grading. However, we need to also take into account the $\sigma\alpha$ grading. In other words, our $\widetilde{EC}_2$ should really be a model of $S^{\infty\sigma+\infty\sigma\alpha}$. It turns out that there is an alternative version of $\widetilde{EC}_2$, whose definition was originally given in [Morel and Voevodsky 1999, Chapter 4.2], and which we redefine in Section 6. We refer to this alternative as the *geometric* model. Our primary definition for MGLO is Definition 2.19. By Theorem 6.1 our primary definition of MGLO is equivalent to

$$(\mathsf{MGLR} \wedge S^{\infty\sigma+\infty\sigma\alpha})^{C_2}. \tag{2.2}$$

While we do not have a proof that (2.2) and (2.1) are different spectra, the nonequivalence of the geometric and simplicial classifying spaces for C_2 imply a general nonequivalence of (2.2) and (2.1) whenever MGLR is replaced by a general C_2-equivariant motivic spectrum E. For this reason, we do not assume an equivalence between (2.2) and (2.1) in this paper. For more detail, see Section 6.

Quadratic forms. The classical Milnor spectrum MO has as its prespectrum the Thom spaces, defined as the quotient $\mathrm{BO}_n \,/\, \mathrm{BO}_{n-1}$ induced by inclusion into the zero section. This is well defined because of the well known equivalence of the geometric realization of the two-sided bar construction

$$|B(*, O_n, S^{n-1})| \simeq |B(*, O_{n-1}, *)| \simeq \mathrm{BO}_{n-1}\,.$$

In other words, a key ingredient in the construction of MO is the orthogonal groups O_n along with their associated transitive action on an appropriate model of a sphere. It is well known that the classical orthogonal group O_n is a special case of a generalized class of orthogonal groups which are defined in terms of symmetric bilinear forms. In more detail, given a symmetric bilinear form $b : k^n \times k^n \to k$, we can define the transpose of a matrix $A \in \mathrm{GL}_n(\mathsf{k})$ to be the unique matrix A^{T_b} such that

$$b(Ax, y) = b(x, A^{T_b} y) \quad \forall x, y \in \mathsf{k}^n.$$

Using this, we can then define the group of orthogonal matrices by

$$O_n^b := \{A \in \mathrm{GL}_n(\mathsf{k}) \mid AA^{T_b} = I\}.$$

We often suppress b in our notation whenever the underlying symmetric bilinear form b is understood from context.

While MGLO is supposed to be a motivic version of MO, it is also supposed to be the geometric fixed points of the C_2-motivic spectrum MGLR, which in turn is a motivic version of the C_2-equivariant spectrum MR. The C_2 action on MR comes from an action on the group $\mathrm{GL}_n(\mathbb{C})$ given by complex conjugation,

$$A \leftrightarrow \bar{A}.$$

However, complex conjugation is trivial over fields which do not contain $\sqrt{-1}$. This motivates the discussion which follows.

Following [Hu et al. 2011, Section 6.1], we instead consider the hyperbolic quadratic form on k^{2n}:

$$q(x_1, \dots, x_{2n}) = x_1 x_2 + \cdots + x_{2n-1} x_{2n}.$$

The associated symmetric bilinear form is

$$b((x_1, \dots, x_{2n}), (y_1, \dots, y_{2n})) = \sum_{i=1}^{n} x_{2i} y_{2i-1} + x_{2i-1} y_{2i}.$$

The b-adjoint of a matrix $A = (a_{i,j})_{i,j=0}^{2n}$ is a $2n \times 2n$ matrix A^{T_b} such that

$$b(Ax, y) = b(x, A^{T_b} y). \tag{2.3}$$

Explicitly, putting $A^{T_b} = (b_{i,j=1}^{2n})$, one has

$$\begin{aligned} b_{2i,2j} &= a_{2j-1,2i-1}, & b_{2i,2j-1} &= a_{2j,2i-1}, \\ b_{2i-1,2j-1} &= a_{2j,2i}, & b_{2i-1,2j} &= a_{2j-1,2i}. \end{aligned}$$

Notice that there is a C_2 action on the quadric

$$\mathcal{Q}_n := \mathbb{V}(x, y \mid b(x, y) = 1)$$

given by

$$x \leftrightarrow y,$$

where $\mathbb{V}(x_i \mid E)$ (sometimes abbreviated to $\mathbb{V}(E)$) denotes the locus of the equations E in the variables x_i.

Taking C_2 fixed points of the quadric under this action, we have

$$\begin{aligned}(\mathcal{Q}_n)^{C_2} &= \mathbb{V}(x, y \mid b(x, y) = 1, x = y)\\ &= \mathbb{V}\bigg(\sum_{i=1}^{n} x_{2i} y_{2i-1} + x_{2i-1} y_{2i} - 1, x = y\bigg).\end{aligned} \tag{2.4}$$

The projection from (2.4) onto the x coordinate scaled by a factor of 2 gives an equivalence to $Q_{2n-1} := \mathbb{V}(x \in \mathsf{k}^{2n} \mid x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n} - 1)$. But the projection from (2.4) onto the x-axis gives the same thing as projecting $\mathcal{Q}_n$ onto the x-axis. So long as $x \neq 0$ there exists a y such that $b(x, y) = 1$. But this means that the image of the projection map is $\mathbb{A}^{2n} \smallsetminus 0$. It is a standard result that $\mathbb{A}^{2n} \smallsetminus 0$ has the homotopy type of $S^{2n-1,n} = S^{n-1+n\alpha}$. Using (2.3) we can define the even-dimensional orthogonal groups by

$$O_{2n} := \{A \in \mathrm{GL}_{2n}(k) \mid AA^{T_b} = I\}.$$

The group O_{2n} acts on the quadric Q_{2n-1} in a natural way. We can write Q_{2n-1} as

$$\mathbb{V}\big(\tfrac{1}{2}b(x, x) - 1\big).$$

The action on Q_{2n-1} is given elementwise by $A \cdot x = Ax$. Notice that

$$b(Ax, Ax) = b(x, A^{T_b}Ax) = b(x, x).$$

Therefore we have defined an O_{2n} action on Q_{2n-1}. We define O_{2n-1} to be

$$O_{2n-1} := \{A \in O_{2n} \mid A(1, 1, 0, \ldots, 0) = (1, 1, 0, \ldots, 0)\}.$$

Lemma 2.5. *For each positive integer n, the even-dimensional orthogonal group O_{2n} acts transitively on the motivic sphere Q_{2n-1}.*

Proof. The quadratic form

$$q(x) = \sum_{i=1}^{n} x_{2i-1}x_{2i}$$

is uniquely defined by a $2n \times 2n$ symmetric matrix A consisting of all zeros, except for n copies of the 2×2 matrix

$$\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix} \tag{2.6}$$

along its diagonal. The matrix A is in turn congruent to the matrix B consisting of all zeros except for n copies of the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.7}$$

along its diagonal. Therefore, the claim about transitivity is equivalent to proving transitivity with respect to the orthogonal group and sphere induced from the symmetric bilinear form induced by the matrix B. The symmetric bilinear form represented by B is given by,

$$\sum_{i=1}^{n} x_{2i-1}y_{2i-1} - x_{2i}y_{2i}.$$

Under this symmetric bilinear form b_B, orthogonal matrices consist of a set of vectors $\mathcal{B} = \{b_i\}_{i=1}^{2n}$ such that

$$b_B(b_i, b_j) = \begin{cases} \pm 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Under our equivalent symmetric bilinear form b_B, our sphere is given by

$$Q_{2n-1}^{B} := \mathbb{V}\left(x \in \mathsf{k}^{2n} \;\middle|\; -1 + \sum_{i=1}^{n}(x_{2i-1}^2 - x_{2i}^2)\right).$$

Now, to prove our claim about transitivity, let $\{e_i\}_{i=1}^{2n}$ denote the standard basis for k^{2n}. Proving transitivity is equivalent to proving that for any point $p \in Q_{2n-1}^{B}$ there exists a matrix M such that $Me_1 = p$. Under this assumption, the set of vectors $\mathcal{A} = \{p\} \cup \{p + e_i\}_{i=1}^{2n-1}$ are linearly independent. Using the Gram–Schmidt process with respect to the inner product induced by b_B, we can form an orthonormal set of vectors with respect to the basis $\mathcal{A} = \{p\} \cup \{p + e_i\}_{i=1}^{2n-1}$. The basis will become the rows of M, and our claim will be proven. However, we need to show that the points obtained from the Gram–Schmidt process still live inside of k^{2n}, rather than some potentially bigger field $\mathsf{k}' \supset \mathsf{k}$. To this end, note that

$$\mathrm{proj}_a(a + e_i) = a - \frac{b(a + e_i, a)}{b(a, a)}a = (1 - b(e_i, a))a.$$

Therefore,

$$b_B(\mathrm{proj}_a(a + e_i), \mathrm{proj}_a(a + e_i)) = (1 - b(e_i, a))^2 \cdot b_B(a, a) = (1 - b(e_i, a))^2.$$

This proves the claim. $\square$

Definition 2.8. The odd-dimensional orthogonal groups O_{2n-1} are defined to be the stabilizer of the point $(1, 1, 0, \ldots, 0)$.

Next we define the even-dimensional quadrics as

$$\begin{aligned} Q_{2n-2} &:= \mathbb{V}(x \in \mathsf{k}^{2n} \mid b(x, x^0), b(x, x) + 1) \\ &= \{x \in \mathsf{k}^{2n} \mid x_1x_2 + \cdots + x_{2n-1}x_{2n} + 1 = x_1 + x_2 = 0\}. \end{aligned}$$

We would like to make analogous statements to Lemma 2.5 for O_{2n-1} and Q_{2n-2}, but first we show that Q_{2n-2} is homotopy equivalent to a familiar space.

Lemma 2.9. *The motivic space Q_{2n-2} is homotopy equivalent to the motivic sphere $S^{n-1+(n-1)\alpha}$.*

Proof. We have that

$$Q_{2n-2} = \mathbb{V}(x \in \mathsf{k}^{2n} \mid x_1x_2 + \cdots + x_{2n-1}x_{2n} + 1, x_1 + x_2).$$

We note that this space is homotopy equivalent to

$$\mathbb{V}((y, x_3, x_4, \ldots, x_{2n}) \in \mathsf{k}^{2n-1} \mid -y^2 + x_3x_4 + \cdots + x_{2n-1}x_{2n} + 1).$$

But this is easily seen to be equivalent to

$$\operatorname{Spec}(\mathsf{k}[y, x_3, x_4, \ldots, x_{2n-1}, x_{2n}]/((1-y)(1+y) + x_3x_4 + \cdots + x_{2n-1}x_{2n})).$$

Now, by [Asok et al. 2017, Theorem 2], we notice that

$$S^{n-1+(n-1)\alpha} \simeq \operatorname{Spec}(\mathsf{k}[z, a_3, a_4, \ldots, a_{2n-1}, a_{2n}]/(a_3a_4 + \cdots + a_{2n-1}a_{2n} - z(1+z))).$$

Using the change of variables $z \mapsto -\frac{1}{2}(1+y)$, $a_i \mapsto \frac{1}{2}x_i$, we have that

$$\begin{aligned} &\operatorname{Spec}\big(\mathsf{k}[z, a_3, a_4, \ldots, a_{2n-1}, a_{2n}]/(a_3a_4 + \cdots + a_{2n-1}a_{2n} - z(1+z))\big) \\ &\qquad \simeq \operatorname{Spec}\big(\mathsf{k}\big[-\tfrac{1}{2}(1+y), \tfrac{1}{2}x_3, \tfrac{1}{2}x_4, \ldots, \tfrac{1}{2}x_{2n-1}, \tfrac{1}{2}x_{2n}\big] \\ &\qquad\qquad\qquad /\big(\tfrac{1}{4}(x_3x_4 + \cdots + x_{2n-1}x_{2n} + (1-y)(1+y))\big)\big) \\ &\qquad \simeq \operatorname{Spec}\big(\mathsf{k}[y, x_3, x_4, \ldots, x_{2n-1}, x_{2n}] \\ &\qquad\qquad\qquad /(x_3x_4 + \cdots + x_{2n-1}x_{2n} + (1-y)(1+y))\big). \end{aligned}$$

□

The O_{2n} action on Q_{2n-1} induces an O_{2n-1} action on Q_{2n-2}, which we prove presently. Recall that O_{2n-1} acts pointwise on the quadric Q_{2n-2} by $A \cdot x \mapsto Ax$. Notice that Q_{2n-2} is induced from the form $b_{2n}(x, y)$, and $x \in Q_{2n-2}$ implies that $\frac{1}{2}b_{2n}(x, x) = -1$. Since

$$b(Ax, Ax) = b(x, A^{T_b}Ax) = b(x, x),$$

it only remains to show that if $x_1 = -x_2$ and $y = (y_1, y_2, \ldots, y_{2n})$ is the image of x, then $y_1 = -y_2$. But notice that for $x \in Q_{2n-2}$ we have that $b(x, (1, 1, 0, \ldots, 0)) = 0$.

Let $A \in O_{2n-1}$ and let $y = (y_1, y_2, \dots, y_{2n})$ be the image of x. Then

$$\begin{aligned} y_1 + y_2 &= b(y, (1, 1, 0, \dots, 0)) = b(Ax, (1, 1, 0, \dots, 0)) \\ &= b(x, A^{T_b}(1, 1, 0, \dots, 0)) = b(x, (1, 1, 0, \dots, 0)) = x_1 + x_2 = 0. \end{aligned}$$

This proves that O_{2n-1} acts on the quadric Q_{2n-2}.

Lemma 2.10. *O_{2n-1} acts transitively on Q_{2n-2}, and the fixed point subgroup of $y^0 = (1, -1, 0, \dots, 0)$ can be naturally identified with O_{2n-2}.*

Proof. We prove the transitivity claim in a similar manner to Lemma 2.5. It is enough to show that for any $x \in Q_{2n-2}$ there is a matrix $A \in O_{2n-1}$ such that $Ax = y^0$.

Notice that technically our O_{2n-1} lives inside of O_{2n}. We choose orthonormal bases

$$\mathcal{B}_1 = \left\{ \frac{x^0}{\|x^0\|}, \frac{y^0}{\|y^0\|}, e_3, \dots, e_{2n} \right\},$$
$$\mathcal{B}_2 = \left\{ \frac{x^0}{\|x^0\|}, \frac{x}{\|x\|}, v_3, \dots, v_{2n} \right\}.$$

Notice there exists a change of basis matrix P from $\mathcal{B}_2$ to $\mathcal{B}_1$ which sends x^0 to x^0 and x to $y^0/\|y^0\|$.

This implies that for $x \in Q_{2n-2}$ we have that $Px = \lambda y_0$. We have that

$$-1 = \tfrac{1}{2} b(x, x) = \tfrac{1}{2} b(Px, Px) = \tfrac{1}{2} b(\lambda y^0, \lambda y^0) = \tfrac{1}{2}\lambda^2 b(y^0, y^0) = -\lambda^2 \Longrightarrow \lambda = \pm 1.$$

If $\lambda = 1$ then we are done. If $\lambda = -1$ then we have that $(-P)x = y^0$. This proves the transitivity claim.

Now notice that the subgroup of O_{2n-1} which fixes $y^0 = (1, -1, 0, \dots, 0) \in \mathsf{k}^{2n}$ is

$$\{A \in O_{2n} \mid Ax^0 = x^0 \text{ and } Ay^0 = y^0\} = \{A \in O_{2n-1} \mid Ae_1 = e_1 \text{ and } Ae_2 = e_2\}.$$

But this is just matrices $A \in O_{2n}$ of the form

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & x_{3,3} & \dots & x_{3,2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & x_{2n,3} & \dots & x_{2n,2n} \end{bmatrix}.$$

This shows that O_{2n-2} can be naturally identified with the subgroup of O_{2n-1}, which fixes the point y^0. $\square$

Cellularity. The following definition is due to [Dugger and Isaksen 2005, Definition 2.1]. Let $\mathcal{M}$ be a pointed model category, and let $\mathcal{A}$ be a set of objects in $\mathcal{M}$.

Definition 2.11. The class of $\mathcal{A}$-cellular objects is the smallest class of objects of $\mathcal{M}$ such that

(1) every object of $\mathcal{A}$ is $\mathcal{A}$-cellular;

(2) if X is weakly equivalent to an $\mathcal{A}$-cellular object, then X is cellular;

(3) if $\mathcal{D} : I \to \mathcal{M}$ is a diagram such that $\mathcal{D}$ is $\mathcal{A}$-cellular, then so is $\operatorname{hocolim} \mathcal{D}$.

Choosing $\mathcal{M}$ to be the stable motivic homotopy category, and choosing $\mathcal{A}$ to be

$$\{S^{m+n\alpha} \mid m, n \in \mathbb{Z}\},$$

we obtain the cellular stable motivic homotopy category.

Adapting the proof of [Dugger and Isaksen 2005, Proposition 4.1], we prove the following.

Proposition 2.12. *The variety O_n is stably cellular for every $n \geq 1$.*

Proof. We first suppose that $n = 2k$. Let $x = (1, 1, 0, \dots, 0)$. Now consider the fiber bundle $O_n \to \mathbb{P}^{n-1}$ given by

$$O_n \xrightarrow{m_x} \mathbb{A}^n \to \mathbb{A}^n / \mathbb{A}^n \setminus 0 \simeq \mathbb{P}^{n-1}.$$

Here m_x denotes the map $A \mapsto Ax$. Notice that m_x induces a transitive action of O_n on the motivic sphere Q_{n-1}. The fiber over the point $(1, 0, 0, \dots, 0)$ consists of all $A \in O_n$ such that $a_{11} \neq 1$, and $a_{j1} = 0$ for $j \geq 2$. Recall that

$$O_{n-1} \cong \{A \in O_n \mid A(1, 0, 0, \dots) = (1, 0, 0, \dots)\}.$$

But this is just $\{A \in m_x^{-1}((1, 0, 0, \dots)) \mid a_{11} = 1\}$. Since

$$\det(AA^T) = \det(A)\det(A^T) = \det(A)^2 = 1,$$

it follows that $a_{11} = \pm 1$, and so $m_x^{-1}((1, 0, 0, \dots)) = O_{n-1} \times \{\pm 1\}$. As a scheme, but not as a group, this is isomorphic to

$$\{\pm 1\} \times \mathbb{A}^{n-1} \times O_{n-1},$$

which is stably cellular by induction and [Dugger and Isaksen 2005, Lemma 3.4]. The usual cover of $\mathbb{P}^n$ by affines is a completely trivializing cover for the bundle, so [Dugger and Isaksen 2005, Lemma 3.8] applies. $\square$

Two-sided bar construction. Recall that we have the following equivalences,

$$Q_n \simeq \begin{cases} S^{k+k\alpha} & \text{if } n = 2k, \\ S^{k-1+k\alpha} & \text{if } n = 2k-1. \end{cases}$$

The groups O_n act on the quadrics Q_{n-1}, allowing us to form the two-sided bar construction, which we now discuss.

Let G be a finite group and X and Y motivic spaces. If $X \times G \to X$ is a right G action and $G \times Y \to Y$ is a left G action, then we form the two-sided bar construction $B(X, G, Y)$ as the left derived functor of the coequalizer of $X \times G \times Y \rightrightarrows X \times Y$. We denote the geometric realization of $B(X, G, Y)$ by $|B(X, G, Y)|$.

Definition 2.13. In the special case $X = Y = *$, we define $BG := |B(*, G, *)|$.

Lemma 2.14. *The geometric realization of $B(O_n, O_{n-1}, *)$ is homotopy equivalent to Q_{n-1}.*

Proof. It is well known for $H \hookrightarrow G$ an inclusion of groups that the left coset G/H is isomorphic to $|B(G, H, *)|$. Taking $G = O_n$ and $H = O_{n-1}$, this gives

$$O_n/O_{n-1} \cong |B(O_n, O_{n-1}, *)|.$$

Notice that by the above discussion, O_n acts on Q_{n-1}, and the stabilizer of a point is O_{n-1}. This induces an isomorphism between O_n/O_{n-1} and Q_{n-1}, proving that

$$Q_{n-1} \cong |B(O_n, O_{n-1}, *)|. \qquad \square$$

Lemma 2.15. *The geometric realization of the two-sided bar $B(G, G, *)$ is contractible. In particular, we have $|B(O_n, O_n, *)| \simeq *$.*

Proof. Notice that $* \cong G/G \cong |B(G, G, *)|$. $\square$

Proposition 2.16. *The geometric realization of the two-sided bar construction $B(*, O_n, Q_{n-1})$ is homotopy equivalent to* BO_{n-1}.

Proof. We have that

$$\begin{aligned} |B(*, O_n, Q_{n-1})| &\simeq \big|B\big(*, O_n, |B(O_n, O_{n-1}, *)|\big)\big| \\ &\simeq \big|B\big(|B(*, O_n, O_n)|, O_{n-1}, *\big)\big| \simeq |B(*, O_{n-1}, *)|. \qquad \square \end{aligned}$$

***The prespectrum for* MGLO.** We define a motivic prespectrum as follows.

Definition 2.17. A motivic prespectrum E is defined to be a collection of based spaces $E_1, E_2, \ldots$ equipped with connecting maps $S^{1+\alpha} \wedge E_n \xrightarrow{\sigma_n} E_{n+1}$. If the adjoint maps $E_n \xrightarrow{\tilde{\sigma}_n} [S^{1+\alpha}, E_{n+1}]$ are homotopy equivalences, then we say that E is a motivic spectrum.

The identifications from Proposition 2.16 imply that we have a canonical map

$$\mathrm{BO}_{n-1+} \xrightarrow{\pi} \mathrm{BO}_{n+} \tag{2.18}$$

which is defined by projection maps $(\mathrm{BO}_{n-1})_m \xrightarrow{\pi_m} (\mathrm{BO}_n)_m$ given by

$$\underbrace{O_n \times O_n \times \cdots \times O_n}_{m \text{ times}} \times Q_{n-1} \mapsto \underbrace{O_n \times O_n \times \cdots \times O_n}_{m \text{ times}}.$$

Therefore, we can think of (2.18) as a sphere bundle. This allows us to define Thom space-like objects as the homotopy cofiber of π. The Thom space of BO_n, which we denote as $\mathrm{Thom}(\mathrm{BO}_n)$, is defined to be the pushout of the diagram

$$\begin{tikzcd} \mathrm{BO}_{n-1+} \arrow[r] \arrow[d] & \mathrm{BO}_{n+} \arrow[d] \\ * \arrow[r] & \mathrm{Thom}(\mathrm{BO}_n) \end{tikzcd}$$

The spaces $\mathrm{Thom}(\mathrm{BO}_{2n})$ form the spaces for the prespectrum of MGLO.

Definition 2.19 (MGLO). At the level of prespectra, MGLO is defined by

$$(\mathsf{MGLO})_n := \mathrm{Thom}(\mathrm{BO}_{2n}).$$

Notice the natural inclusions $O_{n-1} \times O_{m-1} \subset O_n \times O_m$ induce maps

$$B(O_{n-1} \times O_{m-1}) \to B(O_n \times O_m).$$

We define $\mathrm{Thom}(B(O_{2n} \times O_{2m}))$ to be

$$B(O_{2n} \times O_{2m})/B(O_{2n-1} \times O_{2m-1}) \simeq \mathrm{Thom}(\mathrm{BO}_{2n}) \wedge \mathrm{Thom}(\mathrm{BO}_{2m}).$$

The structure maps

$$S^{1+\alpha} \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) \xrightarrow{\sigma_n} \mathrm{Thom}(\mathrm{BO}_{2n+2})$$

are defined by

$$\begin{aligned} S^{1+\alpha} \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) &\simeq \Sigma\mathbb{G}_m \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) \\ &\to |B(*, O_2, \mathbb{G}_m)|_+ \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) \to \mathrm{BO}_{2+} \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) \\ &\to \mathrm{Thom}(\mathrm{BO}_2) \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) \xrightarrow{\simeq} \mathrm{Thom}(\mathrm{BO}_2 \times \mathrm{BO}_{2n}) \to \mathrm{Thom}(\mathrm{BO}_{2n+2}). \end{aligned}$$

This defines the prespectrum MGLO and we can promote it to a spectrum in the usual way.

Notice that since the orthogonal groups are stably cellular by Proposition 2.12, it follows that the classifying spaces BO_n are also stably cellular. Since each of the Thom spaces $\mathrm{Thom}(\mathrm{BO}_n)$ is constructed as the homotopy cofiber of the inclusion $\mathrm{BO}_{n-1} \to \mathrm{BO}_n$, it follows that the spaces $\mathrm{Thom}(\mathrm{BO}_n)$ are also cellular. Since these are the spaces defining the prespectrum of MGLO, it follows that MGLO is cellular.

3. Computing the coefficients of MGLO

Combining Proposition 2.16 with a Mayer–Vietoris argument as in [Milnor and Stasheff 1974] gives us two Thom isomorphisms in motivic $H\mathbb{Z}/2$ (co)homology:

$$H^\star(\mathrm{BO}_{n+}) \cong H^{\star+\omega_n}(\mathrm{Thom}(\mathrm{BO}_n)),$$
$$H_\star(\mathrm{BO}_{n+}) \cong H_{\star+\omega_n}(\mathrm{Thom}(\mathrm{BO}_n)).$$

Here $\omega_{2k} := k + k\alpha$ and $\omega_{2k+1} := k + 1 + k\alpha$.

For each space BO_n, we get a unique Thom class $\mathrm{Thom}(\mathrm{BO}_n) \xrightarrow{w_n} \Sigma^{\omega_n} H\mathbb{Z}/2$. Composing w_n with the homotopy cofiber of the map $\mathrm{BO}_{n-1+} \to \mathrm{BO}_{n+}$, we get a class $w_n \in H^{\omega_n}(\mathrm{BO}_{n+})$. The following theorem has essentially been proved by A. Smirnov and A. Vishik [2014, Theorem 3.1.1] using different language from the present paper. The biggest difference between [Smirnov and Vishik 2014] and the theorem presented here is that the former only applies to fields of characteristic 0 for which $\sqrt{-1} \in \mathsf{k}$, whereas the present theorem holds for any field k of characteristic 0.

Theorem 3.1. *There is a unique set of classes $w_1, w_2, \ldots, w_n$ belonging to motivic $\mathbb{Z}/2$ cohomology for which*

$$H^\star(\mathrm{BO}_{n+}) \cong H^\star[w_1, \ldots, w_n].$$

Here $\deg(w_{2i}) = i + i\alpha$ *and* $\deg(w_{2i+1}) = i + 1 + i\alpha$.

Proof. Notice that the cofibration $\mathrm{BO}_{n-1+} \to \mathrm{BO}_{n+} \to \mathrm{Thom}(\mathrm{BO}_n)$ induces a long exact sequence in cohomology given by

$$\cdots \to H^\star(\mathrm{Thom}(\mathrm{BO}_n)) \to H^\star(\mathrm{BO}_{n+}) \to H^\star(\mathrm{BO}_{n-1+}) \to H^{\star+1}(\mathrm{Thom}(\mathrm{BO}_n)) \to \cdots .$$

Using the Thom isomorphism $H^\star(\mathrm{BO}_{n+}) \xrightarrow{\cong} H^{\star+\omega_n}(\mathrm{Thom}(\mathrm{BO}_n))$ we get the long exact sequence

$$\cdots \to H^\star(\mathrm{BO}_{n+}) \xrightarrow{f_n^*} H^{\star+\omega_n}(\mathrm{BO}_{n+}) \xrightarrow{g_n^*} H^{\star+\omega_n}(\mathrm{BO}_{n-1+}) \xrightarrow{h_n^*} H^{\star+1}(\mathrm{BO}_{n+}) \to \cdots .$$

Notice that f_n^* is multiplication by some nonzero class w_n. By induction,

$$H^\star(\mathrm{BO}_{n-1+}) = H^\star[w_1, \ldots, w_{n-1}].$$

Since BO_n is cellular, we have that $H^{p+q\alpha}(\mathrm{BO}_{n+}) = 0$ for $q < 0$. It is also clear that the map f_n^* is injective on $\mathbb{Z}/2 \cong H^0(\mathrm{BO}_{n+})$. We can start with the case $n = 0$ by identifying BO_0 with $|B(*, O_1, Q_0)|$, which is contractible. Therefore, we have that $h_n^*(w_i) = 0$ for $i = 0, \ldots, n-1$. It follows that each of the w_i can be uniquely lifted to $H^\star(\mathrm{BO}_{n+})$.

Moreover, since $h_n^*(w_i) = 0$ for $i = 0, \dots, n-1$, it follows that $h_n^* = 0$. Thus, the long exact sequence splits and we get the short exact sequence

$$0 \to H^\star(\mathrm{BO}_{n+}) \xrightarrow{f_n^*} H^{\star+\omega_n}(\mathrm{BO}_{n+}) \xrightarrow{g_n^*} H^{\star+\omega_n}(\mathrm{BO}_{n-1_+}) \to 0.$$

The key point is that f_n^* is multiplication by the cohomology class $w_n \in H^{\omega_n}(\mathrm{BO}_{n+})$. In other words, $f_n^* = \smile w_n$.

From this the claim follows. We have

$$\begin{aligned} H^\star(\mathrm{BO}_{n+}) &\cong H^\star[w_1, \dots, w_{n-1}] \oplus H^\star[w_1, \dots, w_{n-1}] \smile w_n \\ &\cong H^\star[w_1, \dots, w_n]. \end{aligned}$$

□

A quick word is in order. We have a Thom isomorphism in (co)homology. We have computed the cohomology of BO_n, but there is a motivic universal coefficient theorem, and so the (co)homologies are essentially the same and there is a duality between the (co)homology classes. Motivically, this is not always the case. However, $\mathrm{BO}_{n+} \wedge H\mathbb{Z}/2$ is a wedge sum of suspensions of $H\mathbb{Z}/2$ of dimensions $p + q\alpha$ with $p \geq q$ and so we can show that the (co)homology classes are dual to one another [Hoyois 2015, Section 5.1]. This gives us the following theorem.

Theorem 3.2. *There is a unique set of classes $w_1, w_2, \dots, w_n$ belonging to motivic $\mathbb{Z}/2$ homology for which*

$$H_\star(\mathrm{BO}_{n+}) \cong H_\star[w_1, \dots, w_n].$$

Here $\deg(w_{2i}) = i + i\alpha$ *and* $\deg(w_{2i+1}) = i + 1 + i\alpha$.

Using analogous arguments to those found in [Milnor and Stasheff 1974], we get the corollary below.

Corollary 3.3. *We have*

$$H_\star(\mathsf{MGLO}) \cong H_\star[w_1, w_2, \dots].$$

Here $\deg(w_{2i}) = i + i\alpha$ *and* $\deg(w_{2i+1}) = i + 1 + i\alpha$.

Since $\mathsf{MGLO} \wedge H\mathbb{Z}/2$ is a wedge sum of suspensions of $H\mathbb{Z}/2$ of dimensions $p + q\alpha$ with $p \geq q$, it follows that the (co)homology classes are dual to one another

Dual motivic Steenrod algebra. We review some results on the dual motivic Steenrod algebra. These results can be found in [Kylling 2017].

The dual motivic Steenrod algebra $\mathcal{A}^\vee_{\mathrm{Mot}}$ is defined to be $H\mathbb{Z}/2 \wedge H\mathbb{Z}/2$. As an $H_\star$-algebra, the coefficients of $\mathcal{A}^\vee_{\mathrm{Mot}}$ are given by

$$H_\star[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/(\tau_i^2 - \tau\xi_{i+1} - \rho\tau_{i+1} - \rho\tau_0\xi_{i+1}). \tag{3.4}$$

Here $|\xi_i| = (2^i - 1)(1 + \alpha)$, $|\tau_i| = (2^i - 1)(1 + \alpha) + 1$, τ is the generator of $H_{1-\alpha} \cong \mu_2(\mathsf{k})$, and ρ is the class of -1 in $H_{-\alpha} \cong \mathsf{k}^\times/(\mathsf{k}^\times)^2$. Let $E = (\epsilon_0, \dots, \epsilon_n)$,

$\epsilon_i \in \{0, 1\}$, and $R = (r_1, \dots, r_m)$. The dual motivic Steenrod algebra is a free $H_\star$-module with basis consisting of the monomials,

$$\tau(E)\xi(R) := \prod_E \tau_i^{\epsilon_i} \prod_R \xi_i^{r_i}.$$

By comparing the $H_\star$-module basis for the coefficients of $\mathsf{MGLO} \wedge H\mathbb{Z}/2$ and $\mathcal{A}^\vee_{\mathrm{Mot}}$, we see that $\mathsf{MGLO} \wedge H\mathbb{Z}/2$ is a wedge sum of suspensions of $\mathcal{A}^\vee_{\mathrm{Mot}}$. Consider the submodule $\mathcal{M}$ of $H_\star(\mathsf{MGLO})$ obtained by deleting all generators of degree $|\xi_i|$ and squaring all generators of degree $|\tau_i|$. Let $\mathfrak{M}$ be an $H_\star$-module basis for this submodule. Then

$$\mathsf{MGLO} \wedge H\mathbb{Z}/2 \simeq \bigvee_{m_i \in \mathfrak{M}} \Sigma^{|m_i|} \mathcal{A}^\vee_{\mathrm{Mot}}.$$

***The equivalence between* MGLO *and* $\mathsf{MGLO}/(2, \eta)$.** We begin with the definitions of the mod 2 Moore spectrum, and the motivic Hopf map.

Definition 3.5. The mod 2 Moore spectrum is defined to be the stable cofiber $\mathsf{M}(2)$ of the following map induced by multiplication by 2:

$$S^0 \xrightarrow{2} S^0 \to \mathsf{M}(2).$$

Notice that the map 2 is induced by the stable homotopy class represented by $2 \in \mathbb{Z} \subseteq \pi_0(S^0)$, where $2 = 1 + 1$ and 1 is the class representing the unit map.

It is a well known fact that $H\mathbb{Z} \wedge \mathsf{M}(2) \simeq H\mathbb{Z}/2$. Recall that classically $2 = 0$ in the coefficients of MO. The analogous statement is true for MGLO.

Proposition 3.6. *We have* $2 = 0$ *in the coefficients of* MGLO.

Proof. We have a map

$$\pi_{1+\alpha}(\mathrm{BO}_2) \to \pi_{1+\alpha}(\mathrm{Thom}(\mathrm{BO}_2)) \to \pi_0(\mathsf{MGLO}).$$

The unit is the image of $1 \in \pi_{1+\alpha}(B\mathbb{G}_m)$ via the map

$$h : \pi_{1+\alpha}(B\mathbb{G}_m) \to \pi_{1+\alpha}(\mathrm{BO}_2).$$

The map $z \mapsto z^{-1}$ sends $1 \mapsto -1 \in \pi_{1+\alpha}(B\mathbb{G}_m)$, but becomes identified with the identity under h. Thus, $1 = -1 \in \pi_{1+\alpha}(\mathrm{BO}_2)$. $\square$

Consider the Hopf map given by the projection $h : \mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$. Recall that $\mathbb{A}^2 \setminus 0 \simeq S^{1+2\alpha}$ and $\mathbb{P}^1 \simeq S^{1+\alpha}$. It follows that h induces a stable map $\eta : \Sigma^\alpha S^0 \to S^0$. We denote the cokernel of this map by S^0/η. For a general spectrum E, we denote the cokernel of the map $\eta \wedge \mathsf{E}$ by E/η.

Proposition 3.7. *We have* $\eta = 0$ *in the coefficients of* MGLO.

Proof. It is well known that $\eta = 0$ in the coefficients of MGL. We stably prove MGLO is an E_∞-ring spectrum in Corollary 6.2. Therefore, it is enough to produce a map of ring spectra from MGL to MGLO. We accomplish this by producing a surjective map $\mathrm{GL}_n \to O_{2n}$. This map is given by

$$A \mapsto A \oplus (A^{T_b})^{-1}.$$

This in turn induces a map $\mathsf{MGL} \to \mathsf{MGLO}$ as desired. □

Applying the motivic Hurewicz theorem. We use a modified version of the motivic Hurewicz theorem of [Bachmann 2018]. We recall what it means to be $(n-1)$-connected in the motivic sense.

Definition 3.8. A motivic spectrum E is *finite type* $(n-1)$-connected if the following hold:

(1) $\pi_{i+j\alpha}(\mathsf{E}) = 0$ for all $0 < i < n$.

(2) For each fixed $i \in \mathbb{Z}$, $\pi_{i+j\alpha}(\mathsf{E}) = 0$ for all but at most a finite number of $j \in \mathbb{Z}$.

Theorem 3.9. *Let* k *have characteristic* 0, *and suppose that* E *is a* finite type $(n-1)$*-connected cellular stable motivic spectrum for which* 2 *and* η *are* 0. *Then*

$$H_{n+*\alpha}(\mathsf{E}; \mathbb{Z}/2) \cong \pi_{n+*\alpha}(\mathsf{E}).$$

Proof. This follows from [Bachmann 2018, Theorem 3]. □

Consider the basis elements $v_i \in \mathfrak{M} \subset H_\star(\mathsf{MGLO})$. Then each of the v_i is dual to a cohomology class $c_i \in H^\star(\mathsf{MGLO})$, and so there exists a map

$$\mathsf{MGLO} \xrightarrow{f} \bigvee_{m_i \in \mathfrak{M}} \Sigma^{|m_i|} H\mathbb{Z}/2 \tag{3.10}$$

which induces an equivalence on homology.

Theorem 3.11. *The map* f *in* (3.10) *is a homotopy equivalence*, *and so* MGLO *splits as a wedge sum of* $H\mathbb{Z}/2$.

Proof. Taking the cofiber of the map f we obtain a cofibration

$$\mathsf{MGLO} \xrightarrow{f} \bigvee_{m_i \in \mathfrak{M}} \Sigma^{|m_i|} H\mathbb{Z}/2 \to \mathsf{F}.$$

The idea is that we know that F is cellular, and the coefficients of $\mathsf{F} \wedge H\mathbb{Z}/2$ are 0 by construction. Since 2 and η are 0 in $\bigvee_{m_i \in \mathfrak{M}} \Sigma^{|m_i|} H\mathbb{Z}/2$, it follows that 2^2 and η^2 are 0 in F. Then the motivic Hurewicz theorem combined with the Nakayama lemma implies that $\mathsf{F} = 0$, and so f is an equivalence. □

$\mathsf{MGLO}_\star$ *and a comparison with* MO_*. Combining everything, we have:

Theorem 3.12. *As an $H_\star$ algebra,*

$$\mathsf{MGLO}_\star \cong H_\star[u_{n+n\alpha}, u_{n+1+n\alpha}, u_{(2^i-1)(1+\alpha)+2} \mid n, i \in \mathbb{Z}^{\geq 0}, n \neq 2^i - 1].$$

Let $t^{\mathbb{C}}$ denote the complex topological realization functor. Then

$$t^{\mathbb{C}}(S^1) = S^1, \qquad t^{\mathbb{C}}(S^\alpha) = S^1, \qquad t^{\mathbb{C}}(H\mathbb{Z}/2_{\mathrm{Mot}}) = H\mathbb{Z}/2.$$

From this it follows that $t^{\mathbb{C}}(\mathsf{MGLO}) = \mathsf{MO}$. Over $\mathsf{k} = \mathbb{C}$, we have that

$$\begin{aligned}\mathsf{MGLO}_\star &= H\mathbb{Z}/2_{\mathrm{Mot}\star}[x_2, x_{2+2\alpha}, x_{3+2\alpha}, u_{4+2\alpha}, u_{4+4\alpha}, u_{5+4\alpha}, x_{5+5\alpha}, \dots] \\ &= \mathbb{Z}/2[\theta][u_2, u_{2+2\alpha}, x_{3+2\alpha}, u_{4+2\alpha}, u_{4+4\alpha}, u_{5+4\alpha}, u_{5+5\alpha}, \dots] \\ &= \mathbb{Z}/2[\theta, u_2, u_{2+2\alpha}, u_{3+2\alpha}, u_{4+2\alpha}, u_{4+4\alpha}, u_{5+4\alpha}, u_{5+5\alpha}, \dots].\end{aligned}$$

Recall that

$$\mathsf{MO}_* = \mathbb{Z}/2[a_2, a_4, a_5, a_6, a_8, a_9, a_{10}, \dots]. \tag{3.13}$$

So the generators of MO_* correspond to generators in $\mathsf{MGLO}_\star$ twisted by powers of θ.

4. A motivic analogue of MSO

Recall that the classical oriented cobordism spectrum MSO is closely related to MO. Similarly to MO, the spectrum MSO can be constructed from the Thom spaces of the classifying spaces of SO_n, which we denote by BSO_n. Recall that the group SO_n is defined as

$$\{A \in O_n \mid \det(A) = 1\}.$$

Although many results found in the this section can be generalized to more general fields, many of the proofs rely on the coefficients of the motivic $\mathbb{Z}/p$ cohomology of the mod p Eilenberg–Mac Lane spectrum being equal to $\mathbb{Z}/p[\tau]$, where τ denotes the Tate twist of degree $\alpha - 1$. Therefore, for the entirety of Section 4, the reader should always assume that k is a field of characteristic 0 containing all p-th roots of unity, and for which all its elements are p-th powers.

***Computing the coefficients of* MSLO.** Having constructed a motivic analogue of MO, it becomes apparent that it would be possible to construct a motivic analogue of MSO by mimicking the construction of MGLO. The simple observation is that we can again consider the quadratic form,

$$q(x_1, x_2, \dots, x_{2n}) = x_1x_2 + x_3x_4 + \dots + x_{2n-1}x_{2n}.$$

To this we can associate a unique orthogonal group O_{2n}. Since the determinant function is algebraic, we can define the $2n$-dimensional special orthogonal groups as

$$\mathrm{SO}_{2n} := \{A \in O_{2n} \mid \det A = 1\}.$$

Again, for $n \geq 1$ we get a transitive group action of SO_{2n} on

$$Q_{2n-1} := \mathbb{V}(x \in \mathsf{k}^{2n} \mid q(x) - 1) \simeq S^{n-1+n\alpha}.$$

Letting $x^0 = (1, 1, 0, \dots, 0)$, the stabilizer of x^0 with respect to the group action of SO_{2n} on Q_{2n-1} is defined to be SO_{2n-1}. One easily sees that this is exactly equal to

$$\{A \in O_{2n-1} \mid \det(A) = 1\}.$$

Defining, as before,

$$Q_{2n-2} := \mathbb{V}(x \in \mathsf{k}^{2n} \mid q(x) + 1, x_1 + x_2) \simeq S^{n-1+(n-1)\alpha},$$

we get a group action of SO_{2n-1} on Q_{2n-2}. This action is transitive, and defining $y^0 \in \mathsf{k}^{2n}$ to be $(1, -1, 0, \dots, 0)$, we can show that the stabilizer of y^0 is SO_{2n-2}.

In the lower-dimensional cases, we note that $\mathrm{SO}_2 \simeq \mathbb{G}_m$, and $\mathrm{SO}_1 \simeq *$. The later equivalence is obvious. For the former, we have to do a bit of work.

Proposition 4.1. $$\mathrm{SO}_2 \simeq \mathbb{G}_m.$$

Proof. We consider the symmetric bilinear form $b((x_1, x_2), (y_1, y_2))$ to see how A is related to A^T. Recall that A^T is defined to be the unique matrix $A \in \mathrm{GL}_2(\mathsf{k})$ for which $b(Ax, y) = b(x, A^T y)$. We write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad A^T = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \qquad x = (x_1, x_2), \qquad y = (y_1, y_2). \tag{4.2}$$

Recall that $b(x, y) = x_1 y_2 + x_2 y_1$. Therefore,

$$b(Ax, y) = a x_1 y_2 + b x_2 y_2 + c x_1 y_1 + d x_2 y_1$$

and

$$b(x, A^T y) = c' x_1 y_1 + d' x_1 y_2 + a' x_2 y_1 + b' x_2 y_2.$$

Comparing, we see that

$$A^T = \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

Now notice that we have the further relations $\det(A) = 1$ and $AA^T = I$. Explicitly multiplying the matrices, we see that

$$AA^T = \begin{bmatrix} ad + bc & 2ab \\ 2cd & ad + bc \end{bmatrix}.$$

Since $\det(A) = ad - bc = 1$, we have that $ad + bc = (ad - bc) + 2bc = 1 + 2bc$. Therefore, we get the relations $2bc = 2ab = 2cd = 0$. It follows, from these relations alone, that either $a = c = 0$, $b = c = 0$, or $b = d = 0$. But we also have the relation $ad - bc = 1$. Therefore, it must be the case that $b = c = 0$. Therefore,

$$\mathrm{SO}_2 = \{(a, b, c, d) \in \mathsf{k}^4 \mid b = c = 0, ad = 1\} \simeq \{(v, w) \in \mathsf{k}^2 \mid vw = 1\} \simeq \mathbb{G}_m. \qquad \square$$

Using a two-sided bar construction as before, we have

$$|B(\mathrm{SO}_n, \mathrm{SO}_{n-1}, *)| \simeq Q_{n-1}.$$

Moreover, we are able to show that

$$|B(*, \mathrm{SO}_n, Q_{n-1})| \simeq \mathrm{BSO}_{n-1}\,.$$

Definition 4.3 (MSLO). The n-th Thom space defining the prespectrum for MSLO is given by the homotopy cofiber of the map

$$\mathrm{BSO}_{n-1+} \to \mathrm{BSO}_{n+}.$$

Notice that in particular we have the following lemma.

Lemma 4.4. $$\mathbb{P}^\infty \simeq B\mathbb{G}_m \simeq \mathrm{BSO}_2 \simeq \mathrm{Thom}(\mathrm{BSO}_2).$$

Proof. Since $\mathrm{SO}_1 \simeq *$, we have $\mathrm{BSO}_1 \simeq *$. By definition of Thom(BSO_2), the statement follows. □

***Calculating the C_2 cohomology of* MSLO.** The goal of this section is to calculate the motivic C_2 cohomology of MSLO. To do this, we first note that O_n acts on the unit sphere $S^0 \simeq \{\pm 1\}$ by $A \cdot g \mapsto (\det(A))g$ for $A \in O_n$, $g \in \{\pm 1\}$. This action is easily seen to be transitive, and the stabilizer of $1 \in S^0$ is

$$\{A \in O_n \mid \det(A) = 1\} = \mathrm{SO}_n\,.$$

It follows that $|B(*, O_n, S^0)| \simeq \mathrm{BSO}_n$. As before, we get a Thom isomorphism

$$H^\star(\mathrm{BO}_{n+}) \cong H^{\star+1}(\mathrm{BO}_n\,/\,\mathrm{BSO}_n).$$

We can use this to get a Gysin sequence. We consider the long exact sequence

$$\cdots \to H^\star(\mathrm{BO}_n/\mathrm{BSO}_n) \to H^\star(\mathrm{BO}_{n+}) \to H^\star(\mathrm{BSO}_{n+}) \to H^{\star+1}(\mathrm{BO}_n/\mathrm{BSO}_n) \to \cdots.$$

Substituting in the Thom isomorphism gives us

$$\begin{aligned}\cdots \to H^{\star-1}(\mathrm{BO}_{n+}) \to H^\star(\mathrm{BO}_{n+}) \to H^\star(\mathrm{BSO}_{n+})& \\ \to H^\star(\mathrm{BO}_{n+}) \to H^{\star+1}(\mathrm{BO}_{n+}) &\to \cdots\,.\end{aligned}$$

Proposition 4.5. *There exists a surjective map*

$$H^\star(\mathrm{BO}_{n+}) \to H^\star(\mathrm{BSO}_{n+}),$$

with kernel generated by w_1 as an $H^\star$-module. Hence,

$$H^\star(\mathrm{BSO}_{n+}) \cong H^\star[w_2, w_3, \ldots, w_n]$$

with $|w_{2i}| = i + i\alpha$, and $|w_{2i+1}| = i + 1 + i\alpha$.

Proof. Let $x \in H^1(\mathrm{BO}_{n+})$ be the composition of the Thom class $u \in H^1(\mathrm{BO}_n/\mathrm{BSO}_n)$ with the homotopy cofiber of the map

$$\mathrm{BSO}_{n+} \to \mathrm{BO}_{n+}.$$

This gives a nonzero class $x \in H^1(\mathrm{BO}_{n+})$. Since there is only one nonzero class $H^\star(\mathrm{BO}_{n+})$ of degree 1, it is clear that x is the same class as $w_1 \in H^1(\mathrm{BO}_{n+})$ from Theorem 3.1.

Thus, we can write

$$\cdots \to H^\star(\mathrm{BO}_{n+}) \to H^\star(\mathrm{BSO}_{n+}) \to H^\star(\mathrm{BO}_{n+}) \xrightarrow{\smile w_1} H^{\star+1}(\mathrm{BO}_{n+}) \to \cdots .$$

Since $H^\star(\mathrm{BO}_{n+}) = H^\star[w_1, \dots, w_n]$, the map $\smile w_1$ is injective in all dimensions, and so the Gysin sequence breaks up into short exact sequences

$$0 \to H^{r+s\alpha-1}(\mathrm{BO}_{n+}) \xrightarrow{\smile w_1} H^{r+s\alpha}(\mathrm{BO}_{n+}) \to H^{r+s\alpha}(\mathrm{BSO}_{n+}) \to 0.$$

The conclusion follows. □

Calculating the $\mathbb{Z}/p$ cohomology of MSLO for p an odd prime.

Definition 4.6. The Euler class $x_n \in H^{\omega_n}(\mathrm{BSO}_{n+})$ is defined to be the composition of the Thom class $c \in H^{\omega_n}(\mathrm{Thom}(\mathrm{BSO}_n))$ with the homotopy cofiber f of

$$\mathrm{BSO}_{n-1+} \to \mathrm{BSO}_{n+} \xrightarrow{f} \mathrm{Thom}(\mathrm{BSO}_n).$$

Theorem 4.7. *$H^\star(\mathrm{BSO}_{n+}; \mathbb{Z}/p)$ is the polynomial ring $H\mathbb{Z}/p^\star[x_1^2, \dots, x_k^2]$ for $n = 2k+1$ and $H\mathbb{Z}/p^\star[x_1^2, \dots, x_{k-1}^2, x_k]$ for $n = 2k$.*

Proof. The sphere bundle $S(n-1) \to \mathrm{BSO}_{n-1} \to \mathrm{BSO}_n$ induces a Gysin sequence with $\mathbb{Z}/p$ coefficients

$$\cdots \to H^i(\mathrm{BSO}_{n+}) \xrightarrow{\smile x_n} H^{i+\omega_n}(\mathrm{BSO}_{n+}) \xrightarrow{g_n^*} H^{i+\omega_n}(\mathrm{BSO}_{n-1+}) \xrightarrow{h_n^*} H^{i+1}(\mathrm{BSO}_{n+}) \to \cdots .$$

Now, if $n = 2k$, then by induction we have that $H^\star(\mathrm{BSO}_{n-1+}) \cong H^\star[x_1^2, \dots, x_{k-1}^2]$. Recall that by [Voevodsky 1999], $H\mathbb{Z}/p_\star^{m+n\alpha}(\mathrm{BO}_{n+}) = 0$ for $n < 0$. Using the fact that $\smile x_n$ is an isomorphism on $H^0(\mathrm{BSO}_{n+}) \cong \mathbb{Z}/p$, we see that $h_n^* = 0$ and so g_n^* is surjective and the map breaks into short exact sequences. The proof then follows that of Theorem 3.1.

If $n = 2k+1$, then x_n is zero in $H^{\omega_n}(\mathrm{BSO}_{n+})$ since it has order 2. To see that x_n has order 2, we note that x_n is the element corresponding to $x_n \smile x_n$ under the Thom isomorphism. Therefore, $x_n \smile x_n = -x_n \smile x_n$ by the commutativity relation of the cup product. It follows that $\smile x_n = 0$, and so the Gysin sequence splits into short exact sequences

$$0 \to H^{i+\omega_n}(\mathrm{BSO}_{n+}) \xrightarrow{g_n^*} H^{i+\omega_n}(\mathrm{BSO}_{n-1+}) \xrightarrow{h_n^*} H^{i+1}(\mathrm{BSO}_{n+}) \to 0.$$

Therefore g_n^* injects $H^\star(\mathrm{BSO}_{n+})$ as a subring of

$$H^\star(\mathrm{BSO}_{n-1+}) \cong H^\star[w_1^2, \dots, w_{k-1}^2, w_k].$$

The subring $\mathrm{im}(g_n^*)$ contains $H^\star[x_1^2, \dots, x_k^2]$, and we can show it equals this ring by comparing ranks in each dimension. □

Calculating the coefficients of MSLO_p for p an odd prime. Recall that the computation of MSO at an odd prime is more or less the same as the computation of complex cobordism MU. Similarly, the computation of MSLO will be no harder than the computation of MGL.

We denote the Milnor primitives by $Q_i \in \mathcal{A}^\star$, $|Q_i| = p^i(1+\alpha) - \alpha$. Recall that if p is odd, then the mod p motivic cohomology of MSLO is generated by classes x_i of degree $2(1+\alpha)i$ as a free $H^\star$-module.

The following proof is based on the proof of a similar result due to S. Borghesi [2003, Proposition 6].

Theorem 4.8. *Let p be an odd prime. The mod p cohomology of* MSLO *takes the form*

$$H^\star(\mathsf{MSLO}) = (\mathcal{A}^\star/(Q_0, Q_1, \dots))[m_i \mid i \neq p^n - 1]$$

as an $\mathcal{A}^\star$-module, where $|m_i| = 2i(1+\alpha)$.

Proof. For c a cohomology class of degree $p + q\alpha$, we define $\|c\| := p - q$. We call the number $\|c\|$ the invariance of the cohomology class c. Now note that the motivic Steenrod algebra $\mathcal{A}^\star$ acts on the cohomology of MSLO. Let Q_i denote the Milnor primitives in degree $2^i(1+\alpha) - \alpha$. Notice that $\|Q_i\| = 1$. Recall that as an $H^\star$ module, the cohomology of MSLO has a basis in monomials whose invariance is equal to 0. Call this basis $\mathfrak{M}$. Therefore, $\|Q_i c\| = 1$ implies that $Q_i c = 0$. The reason is because for any $x \in H^\star$, $\|x\| \leq 0$. Putting this together, we have that if $m \in \mathfrak{M}$, and y is a basis element of $\mathcal{A}^\star$ as an $H^\star$ module, then the action of y on m sends m to a sum of elements in $\mathfrak{M}$ with coefficients in $\mathbb{Z}/2$. Now, since $Q_i c = 0$ for all $c \in \mathfrak{M}$, it follows that the action of $\mathcal{A}^\star$ on $H^\star(\mathsf{MSLO})$ factors through $\mathcal{A}^\star/(Q_0, Q_1, \dots)$. By discussion of the $\mathcal{A}^\star$ action on the cohomology of MSLO, it now follows that the action produces an $H^\star$ linear map in which there is no interplay between the $H^\star$ coefficients. Therefore, any dependencies must be topologically induced. But topologically, there are no dependencies, and so the theorem is proved. □

Corollary 4.9. *Let p be an odd prime. The mod p cohomology of* MSLO *takes the form*

$$H^\star(\mathsf{MSLO}) = H^\star(\mathsf{BPGL})[m_i \mid i \neq p^n - 1]$$

as an $\mathcal{A}^\star$-module, where $|m_i| = 2i(1+\alpha)$.

For the remainder of this subsection, we work over the field $\mathsf{k} = \mathbb{C}$. By [Stahn 2016], we know that over $\mathbb{C}$, the motivic $\mathbb{Z}/p$ cohomology of a point is equal to $\mathbb{Z}/p[\tau]$, where $|\tau| = \alpha - 1$. Dually, the motivic $\mathbb{Z}/p$ homology of a point is equal to $\mathbb{Z}/p[\theta]$, where $|\theta| = 1 - \alpha$. Furthermore, we have that $\mathcal{A}_\star \cong \mathcal{A}_\star^{\mathrm{top}} \otimes_{\mathbb{Z}/p} \mathbb{Z}/p[\theta]$.

Definition 4.10. Let $\mathcal{E}(n)$, $0 \leq n < \infty$, denote the quotient Hopf algebroid

$$\mathcal{E}(n) := \mathcal{A}_\star//(\xi_1, \xi_2, \dots, \tau_{n+1}, \tau_{n+2}, \dots) = H_\star[\tau_0, \dots, \tau_n]/(\tau_i^2 \mid 0 \leq i \leq n).$$

If $n = \infty$, let

$$\mathcal{E}(\infty) := \mathcal{A}_\star//(\xi_1, \xi_2, \dots) = H_\star[\tau_0, \tau_1, \dots]/(\tau_i^2 \mid 0 \leq i).$$

There is a way of switching between $\mathcal{A}^\star$ structures on cohomology and $\mathcal{A}_\star$ structures on homology. In our case we have the following.

Proposition 4.11. *As an $\mathcal{A}_\star$-comodule algebra, $H_\star\mathsf{BPGL} = \mathcal{A}_\star \,\square_{\mathcal{E}(\infty)} H_\star$.*

Using a change of rings isomorphism, we have

$$\mathrm{Ext}_{\mathcal{A}_\star}(H_\star, H_\star(\mathsf{BPGL})) \cong \mathrm{Ext}_{\mathcal{A}_\star}(H_\star, \mathcal{A}_\star \,\square_{\mathcal{E}(\infty)} H_\star) \cong \mathrm{Ext}_{\mathcal{E}(\infty)}(H_\star, H_\star).$$

If we let $\mathcal{E}(\infty)^{\mathrm{top}}$ and $H_\star^{\mathrm{top}}$ denote the topological analogues of $\mathcal{E}(\infty)$ and $H_\star$, respectively, then it follows that over $\mathsf{k} = \mathbb{C}$,

$$\mathrm{Ext}_{\mathcal{E}(\infty)}(H_\star, H_\star) \cong \mathrm{Ext}_{\mathcal{E}(\infty)^{\mathrm{top}}}(H_\star^{\mathrm{top}}, H_\star^{\mathrm{top}}) \otimes_{\mathbb{Z}/p} \mathbb{Z}/p[\theta].$$

From here the proof proceeds classically, and so we have the following theorem.

Theorem 4.12. *After completing at an odd prime p, the coefficients of MSLO are given by*

$$\pi_\star(\mathsf{MSLO}^{\wedge}_p) \cong \mathbb{Z}_{(p)}[\theta, x_1, x_2, x_3, \dots],$$

where $|x_i| = 2i(1+\alpha)$.

$H\mathbb{Z}/2_\star$-algebra structure of $H_\star(H\mathbb{Z}; \mathbb{Z}/2)$. By [Voevodsky 2003b], the map

$$\psi_\star : \mathcal{A}_\star \to \mathcal{A}_\star \otimes_{H_\star} \mathcal{A}_\star$$

is given by

$$\psi_\star(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i, \qquad \psi_\star(\tau_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i + \tau_k \otimes 1.$$

As in [Milnor 1958], we define the conjugates of ξ_i and τ_i inductively as

$$\sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes c(\xi_i) = 0 \quad \text{and} \quad \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes c(\tau_i) + \tau_k \otimes 1 = 0,$$

respectively.

This gives us

$$c(\xi_k) = -\xi_k - c(\xi_1)\xi_{k-1}^2 - \cdots - c(\xi_{k-1})\xi_1^{2^{k-1}},$$
$$c(\tau_k) = -\tau_k - c(\tau_0)\xi_k - c(\tau_1)\xi_{k-1}^2 - \cdots - c(\tau_{k-1})\xi_1^{2^{k-1}},$$

respectively.

As in topology, motivically we have a cofibration

$$H\mathbb{Z} \xrightarrow{2} H\mathbb{Z} \xrightarrow{\text{mod } 2} H\mathbb{Z}/2$$

induced from the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \to 0.$$

Taking motivic $H\mathbb{Z}/2$ homology of $H\mathbb{Z}$, we get a long exact sequence

$$\cdots \to H^\star(H\mathbb{Z}) \xrightarrow{2} H^\star(H\mathbb{Z}) \xrightarrow{\text{mod } 2} H^\star(H\mathbb{Z}/2) \xrightarrow{\partial} \cdots.$$

This gives us an exact couple and so induces a Bockstein spectral sequence. In particular, we get the diagram

$$\begin{tikzcd}
H_\star(H\mathbb{Z}) \arrow[rr, "\beta"] & & H_\star(H\mathbb{Z}) \arrow[dl, "\text{mod } 2"] \arrow[rr, "\beta"] & & H_\star(H\mathbb{Z}) \arrow[dl, "\text{mod } 2"] \\
& H_\star(H\mathbb{Z}/2) \arrow[ul, "\partial"] \arrow[rr, "d"] & & H_\star(H\mathbb{Z}/2) \arrow[ul, "\partial"] &
\end{tikzcd}$$

Notice that $2 = 0$ in $H_\star(H\mathbb{Z})$, and so we have that

$$H_\star(H\mathbb{Z}) \xrightarrow{\text{mod } 2} H_\star(H\mathbb{Z}/2)$$

is injective. Thus we have a short exact sequence

$$0 \to H_\star(H\mathbb{Z}) \xrightarrow{\text{mod } 2} H_\star(H\mathbb{Z}/2) \xrightarrow{d} H_\star(H\mathbb{Z}/2) \to 0.$$

Here d is the dual of the Steenrod operation Sq^1. Notice that $H_\star(H\mathbb{Z}) = \ker(d)$.

Lemma 4.13. *The motivic cohomology of $H_\star(H\mathbb{Z})$ over $\mathsf{k} = \mathbb{C}$ is isomorphic to*

$$\mathbb{Z}/2[\theta, \tau_1, \tau_2, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \theta\xi_{i+1}).$$

Proof. First, one observes that $d(\tau_0) = 1$ and $d(\tau_i) = \xi_i$ for $i \in \mathbb{Z}^{>0}$. Next, one observes that since d commutes with the Tate twist θ, and since $\tau_i^2 = \theta\xi_{i+1}$, we have

$$0 = 2\tau_i d(\tau_i) = d(\tau_i^2) = \theta d(\xi_{i+1}).$$

Therefore $d(\xi_{i+1}) = 0$. Now, as a $\mathbb{Z}/2[\theta]$-algebra, the classes $\{\xi_i\}_{i=1}^\infty$ and the classes $\{c(\xi_i)\}_{i=1}^\infty$ both generate the same algebra. Looking now at the inductive formula for the conjugate of τ_i, and acknowledging that $2 = 0$ in the coefficients, we have

$$c(\tau_k) = \tau_k + c(\tau_0)\xi_k + c(\tau_1)\xi_{k-1}^2 + \cdots + c(\tau_{k-1})\xi_1^{2^{k-1}}.$$

First we notice that $c(\tau_0) = \tau_0$, and so $d(c(\tau_0)) = 1$. We claim that $d(c(\tau_i)) = 0$ for $i \in \mathbb{Z}^{>0}$. For τ_1, we have that $c(\tau_1) = \tau_1 + \tau_0\xi_1$. Taking the differential of each side, we have that

$$d(c(\tau_1)) = d(\tau_1) + \tau_0 d(\xi_1) + \xi_1 d(\tau_0) = d(\tau_1) + \xi_1 = \xi_1 + \xi_1 = 0.$$

Now, by induction we can assume $d(c(\tau_{n-1})) = 0$. Therefore,

$$\begin{aligned} d(c(\tau_n)) &= d(\tau_n) + d(c(\tau_0)\xi_n) + d(c(\tau_1)\xi_{n-1}^2) + \cdots + d(c(\tau_{n-1})\xi_1^{2^{n-1}}) \\ &= d(\tau_n) + d(c(\tau_0)\xi_n) = d(\tau_n) + \xi_n = \xi_n + \xi_n = 0. \end{aligned}$$

Thus,

$$\ker(d) = \mathbb{Z}/2[\theta, c(\tau_1), c(\tau_2), \dots, c(\xi_1), c(\xi_2), \dots].$$

One can show that $c(\tau_i)^2 = \theta c(\xi_{i+1})$. This proves the claim. □

The* $\mathbf{Sq^1}$ *cohomology. Notice that the motivic Steenrod operation Sq^1 has the property that $\mathrm{Sq}^1 \circ \mathrm{Sq}^1 = 0$. Therefore, we can think of Sq^1 as a differential of $H^\star(\mathsf{MSLO})$. We use the notation $H^\star(M; \mathrm{Sq}^1)$ to denote the cohomology of the $\mathcal{A}^\star$ module M with respect to the differential Sq^1.

Following [Voevodsky 2003b], let $I = (\epsilon_0, s_1, \epsilon_1, s_2, \dots, s_k, \epsilon_k)$ be a sequence where $\epsilon_i \in \{0, 1\}$ and s_i are nonnegative integers. Denote by P^I the product

$$P^I = \beta^{\epsilon_0} P^{s_1} \cdots P^{s_k} \beta^{\epsilon_k}.$$

A sequence I is called admissible if $s_i \geq 2s_{i+1} + \epsilon_i$. Monomials P^I corresponding to admissible sequences are called admissible monomials. Here $\beta = \mathrm{Sq}^1$.

Lemma 4.14. *Admissible monomials generate $\mathcal{A}^\star$ as a left $H^\star$-module.*

Proof. See [Voevodsky 2003b]. □

Lemma 4.15. *Suppose that $I = (0, s_1, \dots, s_k, 0)$ and $J = (0, t_1, \dots, t_r, 0)$ with $s_1, \dots, s_k, t_1, \dots, t_r \in \mathbb{Z}^{>0}$. Then $\beta P^I \neq P^J \beta$. Also, $\beta P^s \neq P^t \beta$ for $s, t \in \mathbb{Z}^{>0}$.*

Proof. This follows immediately from Lemma 4.14. □

Lemma 4.16. $H^\star(\mathcal{A}^\star; \mathrm{Sq}^1) = 0$ *and* $H^\star(\mathcal{A}^\star/\mathcal{A}^\star \mathrm{Sq}^1; \mathrm{Sq}^1) = H^\star$.

Proof. To prove the first statement, note that $\mathrm{im}(\mathrm{Sq}^1) = \ker(\mathrm{Sq}^1) = \mathrm{Sq}^1 \mathcal{A}^\star$. For the second statement, we notice that $\mathrm{im}(\mathrm{Sq}^1) = \mathrm{Sq}^1 \mathcal{A}^\star/\mathcal{A}^\star \mathrm{Sq}^1$. Since $\mathrm{Sq}^1 \mathcal{A}^\star/\mathcal{A}^\star \mathrm{Sq}^1$ is clearly in both the kernel and image of Sq^1, and using Lemma 4.15, we know that if $I = (0, s_1, \dots, s_k, 0)$ with $s_1, \dots, s_k \in \mathbb{Z}^{>0}$ or $I = (s)$ with $s \in \mathbb{Z}^{>0}$, then $\mathrm{Sq}^1 P^I \notin \mathcal{A}^\star \mathrm{Sq}^1$. We have shown what happens to admissible monomials. We only have to look at what happens to elements of $H^\star$. Clearly these elements get sent to zero since they commute with the Sq^1 operation. Since elements of $H^\star$ are clearly not in the image of Sq^1, it follows that $H^\star(\mathcal{A}^\star/\mathcal{A}^\star \mathrm{Sq}^1; \mathrm{Sq}^1) = H^\star$. □

We need the following proposition from [Smirnov and Vishik 2014].

Proposition 4.17. *Recall that $H^\star(\mathrm{BO}_{n+}) \cong H^\star[w_1, \dots, w_n]$ as an $H^\star$-module. If -1 is a square in* k, *then*

$$\mathrm{Sq}^k(w_m) = \sum_{j=0}^{k} \binom{m-k}{j} w_{k-j} w_{m+j}.$$

The Cartan formula over $\mathsf{k} = \mathbb{C}$ gives the following.

Proposition 4.18. *Let τ be the Tate twist of degree $\alpha - 1$ in $H^\star$, and suppose $H^\star(\mathrm{BO}_{n+}) \cong H^\star[w_1, \dots, w_n]$. We define*

$$\epsilon_{i,j} = \begin{cases} 1 & \text{if } k \text{ is even and } i, j \text{ are odd,} \\ 0 & \text{otherwise.} \end{cases}$$

If -1 is a square in k, *then*

$$\mathrm{Sq}^k(w_r w_s) = \sum_{i+j=k} \tau^{\epsilon_{i,j}} \,\mathrm{Sq}^i(w_r)\,\mathrm{Sq}^j(w_s).$$

Proof. This follows from the formulas given in [Voevodsky 2003b], along with relations between the geometric and simplicial classifying spaces of O_n found in [Smirnov and Vishik 2014]. □

Lemma 4.19. $\mathrm{Sq}^1 t_n = 0$, *where $t_n \in H^\star(\mathrm{Thom}(\mathrm{BSO}_n))$ is the Thom class.*

Proof. Let $H^\star(\mathrm{BO}_{n+}) = H^\star[w_1, \dots, w_n]$. Recall that by Proposition 4.5, we can identify $H^\star(\mathrm{BSO}_{n+})$ with $H^\star[w_2, w_3, \dots, w_n] \subset H^\star(\mathrm{BO}_{n+})$. Recall also that there is a Thom isomorphism

$$H^\star(\mathrm{BSO}_{n+}) \smile w_n \cong H^\star(\mathrm{Thom}(\mathrm{BSO}_{n+})). \tag{4.20}$$

Therefore, $\mathrm{Sq}^1(t_n)$ can be identified with $\mathrm{Sq}^1(w_n)$ under (4.20) and so we can work out the Steenrod operation on $H^\star(\mathrm{Thom}(\mathrm{BSO}_n))$ by comparison with $H^\star(\mathrm{BO}_{n+})$. In particular, $\mathrm{Sq}^1(w_n) = w_n w_1$. Since $w_1 = 0$ in $H^\star(\mathrm{BSO}_{n+})$, the claim follows. □

Since $H^\star(\mathsf{MSLO})$ is an $\mathcal{A}^\star$ module, we can compute its Sq^1 cohomology.

Proposition 4.21. $H^\star(H^\star(\mathsf{MSLO}); \mathrm{Sq}^1) = H^\star[u_2^2, u_4^2, u_6^2, \dots]$.

Proof. By Lemma 4.19, Sq^1 commutes with the Thom isomorphism. Therefore, it is enough to show that $H^\star(H^\star(\mathsf{BSO}); \mathrm{Sq}^1) = H^\star[w_2^2, w_4^2, w_6^2, \dots]$. We note that $\mathrm{Sq}^1(w_{2n}) = w_{2n+1}$. From this it follows that $H^\star[u_3, u_5, u_7, \dots] \subset \mathrm{im}(\mathrm{Sq}^1)$. This implies that the only elements which can be in the kernel but not in the image of Sq^1 are $H^\star[w_2^2, w_4^2, w_6^2, \dots] \subset H^\star(\mathsf{BSO})$. Noting that $\mathrm{Sq}^1(w_{2n}^2) = 0$ for all n, the claim follows. □

A motivic version of Wall's theorem.

Lemma 4.22. *The morphism of $\mathcal{A}^\star$-modules*

$$\mathcal{A}^\star \to H^\star(\mathsf{MSLO})$$

given by $a \mapsto a \cdot 1$, where 1 denotes the Thom class $t_0 \in H^{0,0}(\mathsf{MSLO})$, has kernel $J = \mathcal{A}^\star \operatorname{Sq}^1$.

Proof. To simplify notation, we write $\mathcal{A}^\star/\beta := \mathcal{A}^\star/\mathcal{A}^\star \operatorname{Sq}^1$.

First, it is clear that $\operatorname{Sq}^i(w_j) = 0$ if $i > j$ by Proposition 4.17. If $i \leq j$, then $\operatorname{Sq}^1(w_j)$ is a sum of monomials $w_k w_l$ with $k, l < 2j$. The monomials $\operatorname{Sq}^{i_n} \cdots \operatorname{Sq}^{i_1}$ with $i_n \geq 2i_{n-1}$ and $i_1 > 1$ form an $H^\star$-module basis for $\mathcal{A}^\star/\beta$. Therefore, it is enough to show that the polynomials $\operatorname{Sq}^{i_n} \cdots \operatorname{Sq}^{i_1}(t)$ are linearly independent in $H^\star(\mathsf{MSLO})$. Let $I = (i_k, \dots, i_1)$ with $i_s \geq 2i_{s-1}$ and $i_1 > 1$. We order the monomials $w^I = w^{i_k} w^{i_{k-1}} \cdots w^{i_1}$ lexicographically. For example, $w_8 w_4$ is of higher order than $w_4 w_2$ and $w_8 w_2$, but lower order than $w_8 w_4 w_2$ and $w_{10} w_2$. By induction, we assume that $\operatorname{Sq}^{i_{n-1}} \cdots \operatorname{Sq}^{i_1}(t) = w_{i_{n-1}} \cdots w_{i_1} t + \text{lower order terms}$.

Now suppose $w_{j_{n-1}} \cdots w_{j_1} t \in H^\star(\mathsf{MSLO})$ is such that $j_{n-1} \geq j_{n-1} \geq \cdots \geq j_1$. If $i \geq 2 j_{n-1}$, then we show $\operatorname{Sq}^i(w_{j_{n-1}} \cdots w_{j_1} t) = w_i w_{j_{n-1}} \cdots w_{j_1} t + \text{lower order terms}$. Using the Cartan formula, we have

$$\begin{aligned}\operatorname{Sq}^i(w_{j_{n-1}} \cdots w_{j_1} t) &= \operatorname{Sq}^i(t) \cdot w_{j_{n-1}} \cdots w_{j_1} + \text{lower order terms} \\ &= w_i w_{j_{n-1}} \cdots w_{j_1} t + \text{lower order terms}.\end{aligned}$$

This proves the lemma. □

Theorem 4.23. *Over $\mathsf{k} = \mathbb{C}$, $H^\star(\mathsf{MSLO})$ is a wedge sum of suspensions of $\mathcal{A}^\star$ and $\mathcal{A}^\star/\mathcal{A}^\star \operatorname{Sq}^1$.*

Proof. Our approach is to define a map from a wedge sum of suspensions of $\mathcal{A}^\star/\beta$ to M which induce an isomorphism in Sq^1 cohomology.

Choose classes $\{x_\alpha\}_{\alpha \in I} \in M$ whose images in $H^\star(M; \operatorname{Sq}^1)$ form a basis of $H^\star(M; \operatorname{Sq}^1)$ as a $\mathbb{Z}/2[\theta]$-module. By Proposition 4.21, we can choose the classes $u_2^2, u_4^2, \dots \in H^\star(\mathsf{MSLO}) \cong H^\star[u_2, u_3, u_4, \dots]$. The x_α are killed by Sq^1 and so we can define a map

$$\phi_1 : \bigoplus_{\alpha \in I} \mathcal{A}^\star/\beta[-\deg(x_\alpha)] \to M.$$

Next, we define

$$\overline{\mathcal{A}^\star} := \{\text{admissible monomials } x \in \mathcal{A}^\star \mid |x| > 0\}.$$

Using this definition, we define

$$\overline{M} := M/\overline{\mathcal{A}^\star} M.$$

Notice that $\bigoplus_{\alpha \in I} \mathcal{A}^\star/\beta[-\deg(x_\alpha)] \cong \mathcal{A}^\star/\beta \otimes_{H^\star} C$ for $C = \mathbb{Z}/2[\theta][u_2^2, u_4^2, \dots]$. We consider the projection map

$$M \xrightarrow{\pi} \overline{M}.$$

We then choose a $\mathbb{Z}/2[\theta]$-submodule $Z \subset M$ such that $\pi_{|Z}$ is injective, and

$$\overline{M} \cong \pi(\phi_1(\mathcal{A}^\star/\beta \otimes_{H^\star} C)) \oplus \pi(Z).$$

Now set

$$N = \mathcal{A}^\star/\beta \otimes_{H^\star} C \oplus \mathcal{A}^\star \otimes_{H^\star} Z.$$

The natural map

$$\phi_2 : \mathcal{A}^\star \otimes_{H^\star} Z \to M$$

gives a map

$$\Phi := \phi_1 \oplus \phi_2 : N \to M.$$

Writing $N = \mathcal{A}^\star/\beta \otimes_{H^\star} C \oplus \mathcal{A}^\star \otimes_{H^\star} Z$, we let N_i denote the $\mathcal{A}^\star$-submodule of N given by $N_i = \mathcal{A}^\star/\beta \otimes_{H^\star} C_i \oplus \mathcal{A}^\star \otimes_{H^\star} Z_i$. Here C_i and Z_i denote all elements in C and Z, respectively, of total degree i. We say the class x with degree $n + m\alpha$ has total degree $n + m$. We define M_i to be the image of N_i under the map Φ. We then define $N^{(n)}$ and $M^{(n)}$ to be $\bigoplus_{i \le n} N_i$ and $\bigoplus_{i \le n} \Phi(N_i)$, respectively.

We show by induction that the map $\Phi : N^{(n)} \to M^{(n)}$ is an isomorphism. Starting with $n = 0$, $N^{(0)} = \mathcal{A}^\star/\beta$ and $M^{(0)} = \mathcal{A}^\star \cdot t$, where t is the Thom class. By Lemma 4.22 this map is an isomorphism.

Suppose we have proved $\Phi : N^{(n-1)} \to M^{(n-1)}$ is an isomorphism and let $\lambda : N/N^{(n-1)} \to M/M^{(n-1)}$ be the map induced by Φ. We show $\lambda_{|(N^{(n)}/N^{(n-1)})}$ is injective. Let P be the $\mathbb{Z}/2[\theta]$-module generated by elements of the form c, z, $\mathrm{Sq}^1(z)$ for $c \in C_n$, $z \in Z_n$. We can regard P as a $\mathbb{Z}/2[\theta]$-submodule of the $\mathbb{Z}/2[\theta]$-module $N/N^{(n-1)}$.

We first prove that $\lambda_{|P}$ is injective. Notice that since $H^\star(\mathcal{A}^\star; \mathrm{Sq}^1) = 0$, the map

$$\Phi^* : H^\star(N; \mathrm{Sq}^1) \to H^\star(M; \mathrm{Sq}^1)$$

is still an isomorphism. Since

$$\Phi : N^{(n-1)} \to M^{(n-1)}$$

is an isomorphism by induction, it follows that

$$\lambda^* : H^\star(N/N^{(n-1)}; \mathrm{Sq}^1) \to H^\star(M/M^{(n-1)}; \mathrm{Sq}^1)$$

is also an isomorphism.

Suppose $v \in P$ and $\lambda(v) = 0$. Notice that the total dimension of v is n or $n+1$. We consider the two cases separately. If the total dimension of v is n, then $v = c + z$ for $c \in C_n$, $z \in Z_n$. Now $\lambda(v) = 0$ implies $\Phi(c+z) \in M^{(n-1)}$ for $\Phi : N^{(n)} \to M^{(n)}$. However, by choice of Z, $\lambda(z) \in M_n$, and so $z = 0$. Then $v = c$, and so $\lambda(c) = 0$.

Since λ^* is an isomorphism, it follows that $\mathrm{Sq}^1(c) = 0$, and $c = \mathrm{Sq}^1(c')$ for some $c' \in N/N^{(n-1)}$ with total degree n. But every element in $N/N^{(n-1)}$ has total degree $\geq n$ (or = 0), and so $c' = 0$, which implies $c = 0$.

Now, suppose that the total dimension of v is $n+1$. Then $v = \mathrm{Sq}^1(z)$ for some $z \in Z_n$. Suppose $\lambda(v) = 0$. By definition of v, it follows that $\lambda(v) = \lambda(\mathrm{Sq}^1(z)) = 0$. Since Sq^1 commutes with λ, it follows that $\mathrm{Sq}^1(\lambda(z)) = 0$. Now, notice that in Sq^1 homology, $\mathrm{Sq}^1(\lambda(z)) = 0$. But this means $\lambda(z) = \lambda(c) + \mathrm{Sq}^1(z')$ for some $c \in C_n$, and $z' \in (M/M^{(n-1)})$ of total degree $n-1$. Thus $z' = 0$, and we reduce to the previous case.

Now, returning to the induction step, we have that the multiplication map

$$\mu : \mathsf{MSLO} \wedge \mathsf{MSLO} \to \mathsf{MSLO}$$

induces a coproduct map

$$\mu^* : H^\star(\mathsf{MSLO}) \to H^\star(\mathsf{MSLO}) \otimes_{H^\star} H^\star(\mathsf{MSLO}).$$

We define a projection map

$$p : M \to M/M^{(n-1)}.$$

Let $u \in N^{(n)}$, and $\Phi : N^{(n)} \to M^{(n)}$. Then

$$\mu^*\Phi(u) = 1 \otimes_{H^\star} \Phi(u) \quad \text{modulo } M \otimes_{H^\star} M^{(n-1)}.$$

Therefore, for any $v \in P$ we have

$$(1 \otimes_{H^\star} p)\mu^*\Phi(v) = 1 \otimes_{H^\star} \lambda(v).$$

Now choose a $\mathbb{Z}/2[\theta]$-basis $c_1, c_2, \ldots, c_r$ for C_n, and $z_1, z_2, \ldots, z_s$ for Z_n. Then we can give P a $\mathbb{Z}/2[\theta]$-basis

$$\{v_i\} = \{c_1, \ldots, c_r, z_1, z_2, \ldots, z_s, \mathrm{Sq}^1(z_1), \mathrm{Sq}^1(z_2), \ldots, \mathrm{Sq}^1(z_s)\}.$$

Any $v \in N^{(n)}/N^{(n-1)} = N_n$ then has a unique expression in the form $v = \sum_i a_i v_i$ for $a_i \in \mathcal{A}^\star \backslash \mathcal{A}^\star \mathrm{Sq}^1 \cup \{0\}$. Now, we let m denote the maximum total dimension of all of the a_i. Next, let $\{a_{m_1}, a_{m_2}, \ldots, a_{m_v}\}$ denote all of the a_i of total dimension m.

Notice that if $\lambda(v) = 0$, then $\Phi(v) \in M^{(n-1)}$, and hence

$$0 = (1 \otimes_{H^\star} p)\mu^*\Phi(v) = \sum a_{m_j} \cdot 1 \otimes_{H^\star} \lambda(v_{m_j}) + \sum b_k \cdot 1 \otimes_{H^\star} m_k$$

for some $m_k \in M$, $b_k \in \mathcal{A}^\star$ with total dim $b_k < m$.

The fact $\sum a_{m_j} \cdot 1 \otimes_{H^\star} \lambda(v_{m_j}) = \sum b_k \cdot 1 \otimes_{H^\star} m_k$ implies $\sum a_{m_j} \cdot 1 \otimes_{H^\star} \lambda(v_{m_j}) = 0$. However, we showed that $\lambda_{|P}$ is injective, and so the $\lambda(v_{i_j})$ are linearly independent. This then implies $a_{m_j} \cdot 1 = 0$ for all j. But then $a_{m_j} \in \mathcal{A}^\star \mathrm{Sq}^1$, which is a contradiction, and so $\lambda(v) = 0$ implies $v = 0$. $\square$

Corollary 4.24. *Over the field* $\mathsf{k} = \mathbb{C}$,

$$H_\star(\mathsf{MSLO}) \cong H_\star(H\mathbb{Z}/2) \otimes_{\mathbb{Z}/2[\theta]} C \oplus \mathcal{A}_\star \otimes_{\mathbb{Z}/2[\theta]} Z.$$

Here C *is the algebra* $\mathbb{Z}/2[\theta, x_4, x_8, \dots]$, *where the* x_{4i} *are generators of degree* $2(1+\alpha)i$. Z *is a* $\mathbb{Z}/2[\theta]$ *polynomial algebra.*

The homotopy type of MSLO**.**

Theorem 4.25. *For* $\mathsf{k} = \mathbb{C}$, 2*-completed* MSLO *splits as a wedge sum of suspensions of motivic homology with coefficients in* $\mathbb{Z}/2$ *and* $\mathbb{Z}_2$.

Proof. Once we know that the motivic $\mathbb{Z}/2$ homology of MSLO is a wedge sum of suspensions of $\mathcal{A}^\star$ and $\mathcal{A}^\star/\mathcal{A}^\star \operatorname{Sq}^1$, we can again construct a map

$$\mathsf{MSLO} \to \bigvee_{i\in I} H\mathbb{Z}/2[r_i] \vee \bigvee_{j\in J} H\mathbb{Z}[s_j]$$

which is an equivalence on motivic $\mathbb{Z}/2$ homology. Then, by applying the Nakayama lemma and the motivic Hurewicz theorem [Bachmann 2018], one can show that the map is a homotopy equivalence. $\square$

The dimension of the $H\mathbb{Z}/2$ ***suspensions.*** We already showed in Corollary 4.24 that the $H\mathbb{Z}$ suspensions of MSLO must live in degrees generated by monomials x_{4i} of degrees $2i(1+\alpha)$. It remains to describe the degrees of the $H\mathbb{Z}/2$ suspensions. To answer this question we use well known combinatorial counting techniques, as this question very much resembles the coin change problem well known to combinatorists [Harris et al. 2008, Section 2.6.3] and computer scientists [Abelson et al. 1996, Section 1.2.2] alike.

Definition 4.26. Let M be a bigraded module with basis $\mathfrak{B}$. Let $\mathfrak{B}_{n,m}$ denote all elements of $\mathfrak{B}$ with bidegree (n, m). The basis $\mathfrak{B}$ is said to be a *special* basis if the following conditions hold:

(1) $\mathfrak{B}_{n,m} = \{\,\}$ if $n < 0$.

(2) $\mathfrak{B}_{n,m} = \{\,\}$ if $m < 0$.

(3) The size of the set $\mathfrak{B}_{n,m}$ is finite for all $(n, m) \in \mathbb{Z}\times\mathbb{Z}$.

(4) $\mathfrak{B} = \bigcup_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} \mathfrak{B}_{n,m}$.

Clearly $H_\star(\mathsf{MSLO})$, $H\mathbb{Z}/2_\star(H\mathbb{Z}/2)$, and $H\mathbb{Z}/2_\star(H\mathbb{Z})$ each have a special basis under their induced $n + m\alpha$ grading.

We can associate a unique polynomial $f_{\mathfrak{B}}(x, y) = \sum c_{n,m} x^n y^m \in \mathbb{Z}[[x, y]]$ to any special basis $\mathfrak{B}$. Here $c_{n,m}$ represents the number of elements in $\mathfrak{B}$ of bidegree (n, m). Notice that we can order the words $x^n y^m$ by the length of the word followed by the alphabetical order of the word. For example, $x^2 y = xxy$ comes before $xy^2 = xyy$, and $y^2 = yy$ comes before $x^4 = xxxx$. Let $\mathfrak{B}_{H_\star \mathsf{MSLO}}$ be an $H_\star$ basis for $H_\star(\mathsf{MSLO})$, $\mathfrak{B}_{H_\star H\mathbb{Z}}$ an $H_\star$ basis for $H_\star(H\mathbb{Z})$, and $\mathfrak{B}_{H_\star H\mathbb{Z}/2}$ an $H_\star$ basis for $H_\star(H\mathbb{Z}/2)$.

Proposition 4.27. *Let $f_{H_\star \mathsf{MSLO}}(x, y)$, $f_{H_\star H\mathbb{Z}}(x, y)$, and $f_{H_\star H\mathbb{Z}/2}(x, y)$ be the associated formal polynomials for the special bases $\mathfrak{B}_{H_\star \mathsf{MSLO}}$, $\mathfrak{B}_{H_\star H\mathbb{Z}}$, and $\mathfrak{B}_{H_\star H\mathbb{Z}/2}$, respectively. The number of $H\mathbb{Z}/2$ suspensions of $H_\star(\mathsf{MSLO})$ in dimension $n + m\alpha$ is given by the coefficient $c_{n,m}$ in*

$$g(x, y) = \sum c_{n,m} x^n y^m = \frac{f_{H_\star \mathsf{MSLO}}(x, y) - f_{H_\star H\mathbb{Z}}(x, y) \prod_{i=0}^{\infty}(1 - (xy)^{2i})^{-1}}{f_{H_\star H\mathbb{Z}/2}(x, y)}.$$

Proof. The function $a(x, y) = f_{H_\star \mathsf{MSLO}}(x, y)$ represents the number of basis elements of $H_\star \mathsf{MSLO}$ in each degree. The function

$$b(x, y) = f_{H_\star H\mathbb{Z}}(x, y) \prod_{i=0}^{\infty}(1 - (xy)^{2i})^{-1}$$

represents all elements in $a(x, y)$ generated by an $H\mathbb{Z}$ suspension. Therefore, $a(x, y) - b(x, y)$ represents all basis elements of $H_\star \mathsf{MSLO}$ generated by $H\mathbb{Z}/2$ suspensions. Thus, dividing by $f_{H_\star H\mathbb{Z}/2}(x, y)$ gives the number of $H\mathbb{Z}/2$ suspensions in each degree after applying a Taylor expansion around the point (0, 0). □

5. MGLR, an analogue of MR

There is a C_2-equivariant spectrum belonging to classical topology, which was constructed by Landweber. The coefficients of this spectrum were computed by Hu and Kriz [2001]. The coefficients are bigraded. While the bigrading given in [Hu and Kriz 2001] is $\mathsf{MR}_{*+*'\alpha}$, we use σ grading instead of α. The reason for this is that Hu and Kriz used the α to signify the relationship between motivic homotopy theory and classical C_2-equivariant homotopy theory. The topological realization functor over $\mathbb{R}$ sends motivic α grading to the C_2 grading. However, in the present case, we want to stress the relationship between C_2 motivic homotopy theory and C_2 classical homotopy theory using the topological realization over $\mathbb{C}$.

In this section we discuss a C_2-equivariant motivic spectrum MGLR which was constructed by Hu, Kriz, and Ormsby [Hu et al. 2011]. There is a complex topological realization functor $t^{\mathbb{C}}_{C_2}$ for C_2-equivariant motivic spectra, and $t^{\mathbb{C}}_{C_2}(\mathsf{MGLR}) = \mathsf{MR}$.

One should think of MGLR as a motivic analogue of MR. Roughly speaking, the spectrum MR can be thought of as complex cobordism MU endowed with a C_2 action. At its heart, MU is built from the classifying spaces BU_n, where U_n denotes the n-dimensional unitary group. We get an involution on this group given by $A \leftrightarrow \bar{A}^T$. The groups U_n equipped with this involution action determine the construction of MR. If one wanted to mimic this construction motivically, one would immediately be faced with a problem: complex conjugation is not algebraic. A priori this means that the groups U_n are not definable; however, it turns out that over the complex numbers, $U_n \cong \mathrm{GL}_n(\mathbb{C})$. In fact, the motivic analogue of MU is the well known algebraic cobordism MGL.

In analogy with MR, MGLR should be thought of as algebraic cobordism MGL endowed with a C_2 action. Consider the symmetric bilinear form

$$b((x_1, \ldots, x_{2n}), (y_1, \ldots, y_{2n})) = \sum_{i=1}^{n} x_{2i} y_{2i-1} + x_{2i-1} y_{2i}.$$

For any $A \in \mathrm{GL}_{2n}(\mathsf{k})$, there is a unique matrix A^{T_b} for which $b(Ax, y) = b(x, A^{T_b} y)$ for all $x, y \in \mathsf{k}^{2n}$. The C_2 action of MGLR is induced from the involution action $A \leftrightarrow (A^{T_b})^{-1}$.

The λ twist. In [Hu and Kriz 2001], the authors show that MR completed at 2 splits as a wedge sum of suspensions of a spectrum BPR whose suspensions are in degrees $m_i(1+\sigma)$ for $m_i \neq 2^{i+1}-1$, $\Phi^{C_2}(\mathsf{BPR}) = H\mathbb{Z}/2$, and nonequivariantly $\mathsf{BPR} = \mathsf{BP}$. This splitting comes from applying the Quillen idempotent to the formal group law on $\mathsf{MR}_{*(1+\sigma)}$. From this, it follows that $\mathsf{MR}_\star$ is freely generated by generators x_n of degree $n(1+\sigma)$ for $n \neq 2^{i+1}-1$ as a $\mathsf{BPR}_\star$ algebra. One could ask whether MGLR splits as a wedge sum of suspensions of BPGLR, with $\Phi^{C_2}(\mathsf{BPGLR}) = H\mathbb{Z}/2$ and $\mathsf{BPGLR} = \mathsf{BPGL}$ nonequivariantly, in such a way that $\mathsf{MGLR}_\star$ is free as a $\mathsf{BPGLR}_\star$ algebra. Unfortunately, there does not appear to be any way to construct such a splitting. However, there exists an element $\lambda \in \pi_{1-\sigma+\sigma\alpha-\alpha}(\mathsf{MGLR})$. If we invert this element, then we get a formal group law and we can use the Quillen idempotent construction to get a splitting. First, let us elaborate on this mysterious element λ.

In the topological setting there is the notion of real-oriented spectra and it turns out that MR is universal among real-oriented spectra. There is also a notion of real orientation found in [Hu et al. 2011]. Following that paper's notation, we define $\widetilde{X}$ to be the functorial fibrant replacement of $\overline{X}$, the reduced suspension of X.

Definition 5.1. A C_2-equivariant ring spectrum E is real-oriented if the following two conditions are satisfied. Here MGLR(1) denotes the first term of the prespectrum defining MGLR.

(1) The unit in $E^\star(S^{1+\sigma\alpha+\sigma+\alpha})$ restricts to the unit ϕ_E of $E^\star(\mathsf{MGLR}(1))$.

(2) The map

$$S^{2+2\sigma\alpha} \simeq \widetilde{\mathbb{G}_m^{1/z}} \wedge \widetilde{\mathbb{G}_m^{1/z}} \to \widetilde{\mathbb{G}_m^{1/z}} \times \widetilde{\mathbb{G}_m^{1/z}} \to B(\mathbb{G}_m^{1/z} \times \mathbb{G}_m^{1/z}) \to \mathrm{BGL}_2 \to \mathsf{MGLR}(1),$$

with representative $\omega \in \pi_{2+2\sigma\alpha}$, composes with ϕ_E to give a unit λ_E.

Whenever this is satisfied we get many results analogous to those found in [Hu and Kriz 2001].

Theorem 5.2. *If the C_2-equivariant ring spectrum E is real-oriented, then we have* $E^\star(B\mathbb{G}_m^{1/z}) = E^\star[u]$, *where* $\deg(u) = -(1+\sigma\alpha)$.

Unfortunately, it is not clear whether or not MGLR satisfies Definition 5.1. Clearly MGLR satisfies condition (1) of Definition 5.1. However, it is not clear that λ_{MGLR} is invertible. Using the methods of [Elmendorf et al. 1997] we can "invert" λ_{MGLR} to construct a spectrum $\lambda^{-1}\mathsf{MGLR}$ satisfying both conditions of Definition 5.1. The formal group law of Theorem 5.2 then gives a canonical map

$$\mathsf{L} \to \lambda^{-1}\mathsf{MGLR}_{*(1+\sigma\alpha)}.$$

Here L denotes the Lazard ring.

Notice that the topological realization functor over $\mathbb{C}$, which we denote by $t^{\mathbb{C}}$, is a symmetric monoidal functor, and so if it is applied to the spectrum MGLR, we get a ring homomorphism

$$\mathsf{MGLR}_\star \to \mathsf{MR}_\star.$$

One can show that λ_{MGLR} is sent to the unit 1 under this ring homomorphism, and so we get a ring homomorphism

$$\lambda^{-1}\mathsf{MGLR}_\star \to \mathsf{MR}_\star. \tag{5.3}$$

Since the homomorphism $t^{\mathbb{C}}$ sends $1+\sigma\alpha$ grading to $1+\sigma$ grading, and since $\lambda^{-1}\mathsf{MGLR}_{*(1+\sigma\alpha)} \subset \lambda^{-1}\mathsf{MGLR}_\star$ and $\mathsf{MR}_{*(1+\sigma)} \subset \mathsf{MR}_\star$ are commutative rings, we have the following result.

Lemma 5.4. *The restriction of the ring homomorphism (5.3) to* $\lambda^{-1}\mathsf{MGLR}_{*(1+\sigma\alpha)}$ *induced by the topological realization functor* $t^{\mathbb{C}}$ *sends the formal group law on* $\lambda^{-1}\mathsf{MGLR}_\star$ *to the formal group law on* $\mathsf{MR}_\star$.

Proof. This is clear since $t^{\mathbb{C}}(B\mathbb{G}_m^{1/z}) = BS^\sigma$. □

Since MGLR is an E_∞-ring spectrum, we may apply constructions as in [Elmendorf et al. 1997]. In particular, we may "kill" or "invert" the image of any sequence of elements of L in the spectrum $\lambda^{-1}\mathsf{MGLR}$. The ring $\mathsf{MGL}_{*(1+\alpha)} = \mathsf{MU}_{2*}$ is the universal formal group law, and so the generator x_i of degree $i(1+\alpha)$ is sent to an element of degree $i(1+\sigma\alpha)$.

Theorem 5.5. *The spectrum* $\Phi^{C_2}_{\text{ét}}(\lambda^{-1}\mathsf{MGLR})$ *is equivalent to* $\theta^{-1}\mathsf{MGLO}$.

Proof. Recall that λ is the map

$$\begin{aligned} S^{2+2\sigma\alpha} \simeq \widetilde{\mathbb{G}_m^{1/z}} \wedge \widetilde{\mathbb{G}_m^{1/z}} \to \widetilde{\mathbb{G}_m^{1/z}} \times \widetilde{\mathbb{G}_m^{1/z}} \to B(\mathbb{G}_m^{1/z} \times \mathbb{G}_m^{1/z}) \to \mathrm{BGL}_2 \\ \to \mathsf{MGLR}(1) \to \Sigma^{1+\sigma+\sigma\alpha+\alpha}\mathsf{MGLR}. \end{aligned}$$

After taking geometric fixed points, this becomes a map

$$S^2 \simeq S^1 \wedge S^1 \to S^1 \times S^1 \to B(\mathbb{Z}/2 \times \mathbb{Z}/2) \to \mathrm{BO}_2 \to \mathsf{MGLO}(1) \to \Sigma^{1+\alpha}\mathsf{MGLO}.$$

This map is nonzero, and it realizes as an element of degree $1-\alpha$ in $\pi_\star(\mathsf{MGLO})$. Notice that there exists exactly one element in $\pi_\star(\mathsf{MGLO})$ of degree $1-\alpha$, the Tate twist. Therefore, the coefficients of $\Phi^{C_2}(\lambda^{-1}\mathsf{MGLR})$ are

$$\pi_\star(\theta^{-1}\mathsf{MGLO}) \cong \pi_*(\mathsf{MO})[\theta^{\pm 1}]. \qquad \square$$

Corollary 5.6. *The spectrum* MGLR *is not equivalent to* $\lambda^{-1}\mathsf{MGLR}$.

Proof. Since MGLR and $\lambda^{-1}\mathsf{MGLR}$ are not equal on geometric fixed points, they cannot possibly be equal equivariantly. □

It is interesting to note that while inverting λ has the effect of inverting the Tate twist θ under the geometric fixed points map, it is not the case that θ is inverted under the forgetful map $\mathsf{MGLR} \to \mathsf{MGL}$, which thinks of the structure nonequivariantly. The reason for this is the forgetful map sends σ and $\sigma\alpha$ grading to 1 and α, respectively. Therefore, λ gets sent to the unit under this map. The next theorem gives more detail.

Theorem 5.7. *Nonequivariantly*, $\lambda^{-1}\mathsf{MGLR} \simeq \mathsf{MGL}$.

Proof. Notice that nonequivariantly, λ realizes as

$$\begin{aligned} S^{2+2\alpha} \simeq \Sigma\mathbb{G}_m \wedge \Sigma\mathbb{G}_m \to \Sigma\mathbb{G}_m \times \Sigma\mathbb{G}_m \to B(\mathbb{G}_m \times \mathbb{G}_m) \\ \to \mathrm{BGL}_2 \to \mathsf{MGL}(1) \to \Sigma^{2+2\alpha}\mathsf{MGL}. \end{aligned}$$

Notice that this map is clearly nonzero, and represents an element in $\pi_\star(\mathsf{MGL})$ of degree 0. Notice that the only nonzero element in $\pi_\star(\mathsf{MGL})$ of degree 0 is the identity element. Therefore, $\lambda^{-1}\mathsf{MGLR}$ is nonequivariantly equivalent to MGL. □

Theorem 5.8. *Localizing at* $p = 2$, *we have that*

$$\mathsf{MGL} = \bigvee_{m_i} \Sigma^{m_i(1+\alpha)}\mathsf{BPGL}$$

for integers m_i. *There exists a spectrum* BPGLR *such that*

$$\mathsf{MGLR} = \bigvee_{m_i} \Sigma^{m_i(1+\sigma\alpha)}\mathsf{BPGLR}.$$

Furthermore, $\Phi^{C_2}_{\text{ét}}(\mathsf{BPGLR}) = \theta^{-1}H\mathbb{Z}/2$.

6. Calculating the coefficients of $\theta^{-1}\lambda^{-1}\mathsf{MGLR}$

The main difficulty in computing the coefficients of MGLR is the lack of a Tate diagram. One would like to use simplicial EC_2 to get a motivic Tate diagram,

$$EC_{2+} \wedge \mathsf{MGLR} \to \mathsf{MGLR} \to \widetilde{EC}_2 \wedge \mathsf{MGLR}.$$

However, the C_2 fixed points of $\widetilde{EC}_2 \wedge \mathsf{MGLR}$ is not the geometric fixed points of MGLR in the sense of [Lewis et al. 1986, Chapter 2, Definition 9.7]. In other words, taking C_2 fixed points of MGLR at the level of prespectra does not form a nonequivariant spectrum equivalent to

$$(\widetilde{EC}_2 \wedge \mathsf{MGLR})^{C_2}.$$

To fix this, we need to use a different model of EC_2. The model we use is

$$\mathbf{E}C_2 := \varinjlim \mathbb{A}(n\sigma) \smallsetminus 0,$$

where $\mathbb{A}(n\sigma) \smallsetminus 0$ denotes $\mathbb{A}^n \smallsetminus 0$ with a C_2 action $z \mapsto -z$. This gives us cofibrations

$$\mathbb{A}(n\sigma) \smallsetminus 0_+ \to S^0 \to S^{n\sigma + n\sigma\alpha}.$$

These piece together to give us a cofibration

$$\mathbf{E}C_{2+} \to S^0 \to \widetilde{\mathbf{E}C}_2.$$

The space $\widetilde{\mathbf{E}C}_2$ takes into account the entire equivariant grading in the C_2 equivariant stable category, and so we have the following.

Theorem 6.1. $\Phi^{C_2}_{\acute{e}t}(\mathsf{MGLR}) := (\widetilde{\mathbf{E}C}_2 \wedge \mathsf{MGLR})^{C_2} \simeq \mathsf{MGLO}.$

Proof. By construction, the n-th term of the prespectrum defining MGLO is equal to the C_2-fixed points of the n-th term of the prespectrum defining MGLR [Hu et al. 2011, Section 6]. Let $\mathsf{MGLR}(n)$ denote the n-th term of the prespectrum defining MGLR. Notice that $(\mathsf{MGLR} \wedge \widetilde{\mathbf{E}C}_2)^{C_2}$ is a nonequivariant spectrum with prespectrum $(\mathsf{MGLR}(1))^{C_2}$, $(\mathsf{MGLR}(2))^{C_2}, \ldots,$ and connecting maps given by

$$\mathbb{P}^1 \wedge (\mathsf{MGLR}(n))^{C_2} \to (\mathsf{MGLR}(n+1))^{C_2}.$$

The claim follows. ∎

Corollary 6.2. MGLO *is a motivic* E_∞*-ring spectrum.*

Proof. In [Hu et al. 2011, Section 6] it is proved that MGLR is a C_2-equivariant motivic E_∞-ring spectrum. Being an E_∞-ring spectrum is preserved by smashing with $S^{\infty\sigma+\infty\sigma\alpha}$ and taking C_2 fixed points. ∎

The author would like to acknowledge the work of the authors of [Heller et al. 2019], who are the first to have written about the geometric classifying space $\mathbf{E}C_2$ in the context of C_2-equivariant motivic spectra. The unfortunate reality is that calculating $F(\mathbf{E}C_{2+}, \mathsf{MGLR})$ via a Borel cohomology spectral sequence involves developing new tools which do not currently exist. The solution presented in this paper, however, is to restrict to a field k of characteristic 0, for which all elements in k are squares. Then after completing at the prime $p = 2$ and inverting two twists in MGLR, we can show

$$(\widetilde{\mathbf{E}C}_2 \wedge \mathsf{MGLR})^{C_2} \simeq (\widetilde{EC}_2 \wedge \mathsf{MGLR})^{C_2}.$$

We can then apply the tools of [Hu and Kriz 2001].

Proposition 6.3. *There exists an element θ of order $1 - \alpha$ in the Borel cohomology and the Tate cohomology of $\lambda^{-1}\mathsf{MGLR}$.*

Proof. Using simplicial EC_2, we can set up a Borel cohomology spectral sequence for $\lambda^{-1}\mathsf{MGLR}$ as follows. First we note that since we have inverted λ, we can choose to ignore all $\sigma\alpha$ grading, and instead only consider the grading $* + *'\sigma + *''\alpha$. Moreover, we filter by α twists. In other words, we consider the grading $* + *'\sigma + k\alpha$ for fixed k. Now for each $k \leq 0$, we have a bijection between the motivic Borel cohomology spectral sequence of $\lambda^{-1}\mathsf{MGLR}$ and the classical Borel cohomology spectral sequence of MR. This is true since $\lambda^{-1}\mathsf{MGLR}$ is nonequivariantly MGL, and over $\mathbb{C}$, there is a bijection between $\pi_{*+k\alpha}(\mathsf{MGL})$ and $\pi_*(\mathsf{MU})$. It follows that the motivic Borel cohomology spectral sequence associated to $\lambda^{-1}\mathsf{MGLR}_{*+*'\sigma+*''\alpha}$, where $*, *' \in \mathbb{Z}$ and $*'' \in \mathbb{Z}^{\leq 0}$, converges to $\pi_{*+*'\sigma+*''\alpha}(F(EC_{2+}, \lambda^{-1}\mathsf{MGLR})) \cong \pi_\star(\mathsf{MR})[\theta]$. It follows that $\theta \in \lambda^{-1}\mathsf{MGLR}$. The same argument works for the Tate cohomology of $\lambda^{-1}\mathsf{MGLR}$. □

Corollary 6.4. *There exists an element, again denoted θ, of degree $1 - \alpha$ in the coefficients of $\lambda^{-1}\mathsf{MGLR}$.*

Proof. This follows by considering the following square originating from the Tate diagram:

$$\begin{tikzcd}
\lambda^{-1}\mathsf{MGLR} \arrow[r] \arrow[d] & S^{\infty\sigma} \wedge \lambda^{-1}\mathsf{MGLR} \arrow[d] \\
F(EC_{2+}, \lambda^{-1}\mathsf{MGLR}) \arrow[r] & S^{\infty\sigma} \wedge F(EC_{2+}, \lambda^{-1}\mathsf{MGLR})
\end{tikzcd}$$

It is easy to see that the element $\theta \in \pi_\star(F(EC_{2+}, \lambda^{-1}\mathsf{MGLR}))$ is mapped to $\theta \in \pi_\star(S^{\infty\sigma} \wedge F(EC_{2+}, \lambda^{-1}\mathsf{MGLR}))$. This is true since the topological realization of θ is just 1, and since the Borel and Tate cohomology spectral sequences of $\lambda^{-1}\mathsf{MGLR}$ and MR are isomorphisms for a fixed alpha twist $k\alpha, k \leq 0$. Now, notice that there is an easily described twist in $\pi_\star(S^{\infty\sigma} \wedge \lambda^{-1}\mathsf{MGLR})$ of degree $1 - \alpha$, which we also call θ. If s is the Euler class $s \in \pi_{-\sigma}(\mathsf{MGLR})$, and t is the Euler class $t \in \pi_{-\sigma\alpha}(\mathsf{MGLR})$, then $\theta \in \pi_{1-\alpha}(S^{\infty\sigma} \wedge \lambda^{-1}\mathsf{MGLR})$ is given by $\lambda s^{-1}t$. By comparison with topology, and in view of the fact that the topological realization of θ is 1, it follows that $\theta \in \pi_{1-\alpha}(S^{\infty\sigma} \wedge \lambda^{-1}\mathsf{MGLR})$ is mapped to $\theta \in \pi_{1-\alpha}(S^{\infty\sigma} \wedge F(EC_{2+}, \lambda^{-1}\mathsf{MGLR}))$. Therefore, the element named θ commutes in the bottom row and rightmost column of the above diagram. Since that diagram is a pullback, there must exist an element $\theta \in \pi_\star(\lambda^{-1}\mathsf{MGLR})$ which is sent to $\theta \in \pi_\star(F(EC_{2+}, \lambda^{-1}\mathsf{MGLR}))$. □

As we inverted $\lambda \in \pi_{1-\sigma+\sigma\alpha-\alpha}(\mathsf{MGLR})$, so too can we invert $\theta \in \pi_{1-\alpha}(\mathsf{MGLR})$. This gives us a spectrum $\theta^{-1}\lambda^{-1}\mathsf{MGLR}$. In its coefficients, the element $\lambda^{-1}\theta$ has degree $\sigma - \sigma\alpha$ and is invertible.

Proposition 6.5. $(S^{\infty\sigma+\infty\sigma\alpha} \wedge \theta^{-1}\lambda^{-1}\mathsf{MGLR})^{C_2} \simeq (S^{\infty\sigma} \wedge \theta^{-1}\lambda^{-1}\mathsf{MGLR})^{C_2}$.

Proof. To simplify notation, we write

$$E := S^{\infty\sigma} \wedge \theta^{-1}\lambda^{-1}\mathsf{MGLR},\ F := S^{\infty\sigma+\infty\sigma\alpha} \wedge \theta^{-1}\lambda^{-1}\mathsf{MGLR}.$$

Notice that $\Sigma^{\sigma\alpha-\sigma}E \simeq E$ since $\theta\lambda^{-1} \in \pi_{\sigma-\sigma\alpha}(E)$ is invertible. Also, it is clear that $\Sigma^{\sigma}E \simeq E$. Putting this together, we have that $\Sigma^{\sigma\alpha}E \simeq E$. Therefore, it follows that $F = \Sigma^{\infty\sigma\alpha}E \simeq E$. □

Theorem 6.6. *We have* $\pi_\star(\theta^{-1}\mathsf{BPGLR}) = \pi_\star(\mathsf{BPR})[\lambda^{\pm1}, \theta^{\pm1}]$. *Here,* $\pi_\star(\mathsf{BPR})$ *is*

$$\mathbb{Z}_{(2)}[v_{n,l}, a \mid n \geq 0, l \in \mathbb{Z}] \Big/ \left(v_{0,0} = 2,\ a^{2^{n+1}-1}v_{n,l} = 0 \mid \textit{for } n \leq m,\ v_{m,k} \cdot v_{n,l2^{m-n}} = v_{m,k+l} \cdot v_{n,0}\right),$$

$|a| = -\sigma$, *and* $|v_{n,l}| = (2^n - 1)(1+\sigma) + l2^{n+1}(\sigma - 1)$.

Proof. The claim is clear by comparison with topology [Hu and Kriz 2001]. In more detail, considering the commutative square of Corollary 6.4, the C_2 fixed points of the top right corner is easily seen to be equal to $\pi_*(\mathsf{MO})[\theta^{\pm1}]$. The bottom right corner is calculated by comparing the Tate cohomology spectral sequence for $\theta^{-1}\lambda^{-1}\mathsf{MGLR}$ to topology. One deduces from the calculation that the C_2 fixed points of the top and bottom right-hand column are equal. From this it follows that $\theta^{-1}\lambda^{-1}\mathsf{MGLR}$ is equal to its Borel cohomology. By comparing with topology, the claim follows. □

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