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The congruence subgroup problem for a finitely generated group Γ and $G \le \mathrm{Aut}(\Gamma)$ asks whether the map $\hat{G} \to \mathrm{Aut}(\hat{\Gamma})$ is injective, or more generally, what its kernel $C(G, \Gamma)$ is. Here $\hat{X}$ denotes the profinite completion of X. In this paper we investigate $C(\mathrm{IA}(\Phi_n), \Phi_n)$, where Φ_n is a free metabelian group on $n \ge 4$ generators, and $\mathrm{IA}(\Phi_n) = \ker(\mathrm{Aut}(\Phi_n) \to \mathrm{GL}_n(\mathbb{Z}))$.

We show that in this case $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is abelian, but not trivial, and not even finitely generated. This behavior is very different from what happens for a free metabelian group on $n = 2$ or 3 generators, or for finitely generated nilpotent groups.

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1. Introduction

The classical congruence subgroup problem (CSP) asks for, say, $G = \mathrm{SL}_n(\mathbb{Z})$ or $G = \mathrm{GL}_n(\mathbb{Z})$, whether every finite index subgroup of G contains a principal congruence subgroup, i.e., a subgroup of the form $G(m) = \ker(G \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z}))$ for some $0 \ne m \in \mathbb{Z}$. It is a classical 19th century result that the answer is negative for $n = 2$. On the other hand, quite surprisingly, it was proved in the sixties by Mennicke [1965] and by Bass, Lazard and Serre [Bass et al. 1964] that for $n \ge 3$

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the answer to the CSP is affirmative. A rich theory of the CSP for more general arithmetic groups has been developed since then.

By the observation $\mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{Aut}(\mathbb{Z}^n)$, the CSP can be generalized to automorphism groups as follows: Let Γ be a group and $G \le \mathrm{Aut}(\Gamma)$. For a finite index characteristic subgroup $M \le \Gamma$, define

$$G(M) = \ker(G \to \mathrm{Aut}(\Gamma/M)).$$

A finite index subgroup of G which contains $G(M)$ for some M is called a "congruence subgroup". The CSP for the pair (G, Γ) asks whether every finite index subgroup of G is a congruence subgroup.

One can easily see that the CSP is equivalent to the question: Is the congruence map $\hat{G} = \varprojlim G/U \to \varprojlim G/G(M)$ injective? Here, U ranges over all finite index normal subgroups of G, and M ranges over all finite index characteristic subgroups of Γ. When Γ is finitely generated, it has only finitely many subgroups of given index m, and thus, the characteristic subgroups $M_m = \bigcap\{\Delta \lhd \Gamma \mid [\Gamma : \Delta] \mid m\}$ are of finite index in Γ. Hence, one can write $\hat{\Gamma} = \varprojlim_{m\in\mathbb{N}} \Gamma/M_m$ and have[1]

$$\begin{aligned}\varprojlim G/G(M) = \varprojlim_{m\in\mathbb{N}} G/G(M_m) &\le \varprojlim_{m\in\mathbb{N}} \mathrm{Aut}(\Gamma/M_m)\\ &\le \mathrm{Aut}(\varprojlim_{m\in\mathbb{N}}(\Gamma/M_m)) = \mathrm{Aut}(\hat{\Gamma}).\end{aligned}$$

Therefore, when Γ is finitely generated, the CSP is equivalent to the question: Is the congruence map: $\hat{G} \to \mathrm{Aut}(\hat{\Gamma})$ injective? More generally, the CSP asks what is the kernel $C(G, \Gamma)$ of this map. For $G = \mathrm{Aut}(\Gamma)$ we also use the simpler notation $C(\Gamma) = C(G, \Gamma)$. The classical congruence subgroup result mentioned above can therefore be reformulated as $C(\mathbb{Z}^n) = \{e\}$ for $n \ge 3$, and it is also known that $C(\mathbb{Z}^2) = \hat{F}_\omega$, where $\hat{F}_\omega$ is the free nonabelian profinite group on a countable number of generators; see [Melnikov 1976; Lubotzky 1982].

Very few results are known when Γ is nonabelian. Most of the results are related to $\Gamma = \pi(S_{g,n})$, the fundamental group of the closed surface of genus g with n punctures; see [Diaz et al. 1989; McReynolds 2012; Asada 2001; Boggi 2009; 2016]. As observed in [Bux et al. 2011], the result of Asada [2001] actually gives an affirmative solution to the case $\Gamma = F_2$, $G = \mathrm{Aut}(F_2)$; see also [Ben-Ezra and Lubotzky 2018]. Note that for every $n > 0$, one has $\pi(S_{g,n}) \cong F_{2g+n-1}$ = the free group on $2g+n-1$ generators. Hence, the aforementioned results relate to various subgroups of the automorphism group of finitely generated free groups. However, the CSP for the full $\mathrm{Aut}(F_n)$ when $n \ge 3$ is still unsettled.

[1]By the celebrated theorem of Nikolov and Segal [2003], which asserts that every finite index subgroup of a finitely generated profinite group is open, the second inequality is actually an equality. However, we do not need it.

Denote now the free metabelian group on n generators by $\Phi_n = F_n/F_n''$. Considering the metabelian case, it was shown in [Ben-Ezra and Lubotzky 2018] (see also [Ben-Ezra 2016]) that $C(\Phi_2) = \hat{F}_\omega$. In addition, it was proven there that $C(\Phi_3) \supseteq \hat{F}_\omega$. The basic motivation which led to this paper was to complete the picture in the free metabelian case and investigate $C(\Phi_n)$ for $n \geq 4$. Now, let $\mathrm{IA}(\Phi_n) = \ker(\mathrm{Aut}(\Phi_n) \to \mathrm{GL}_n(\mathbb{Z}))$. Then the commutative exact diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{IA}(\Phi_n) & \longrightarrow & \mathrm{Aut}(\Phi_n) & \longrightarrow & \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & 1 \\ & & & \searrow & \downarrow & & \downarrow & & \\ & & & & \mathrm{Aut}(\hat{\Phi}_n) & \longrightarrow & \mathrm{GL}_n(\hat{\mathbb{Z}}) & & \end{array}$$

gives rise to the commutative exact diagram (see Lemma 2.1 in [Bux et al. 2011])

$$\begin{array}{ccccccc} \widehat{\mathrm{IA}(\Phi_n)} & \longrightarrow & \widehat{\mathrm{Aut}(\Phi_n)} & \longrightarrow & \widehat{\mathrm{GL}_n(\mathbb{Z})} & \longrightarrow & 1 \\ & \searrow & \downarrow & & \downarrow & & \\ & & \mathrm{Aut}(\hat{\Phi}_n) & \longrightarrow & \mathrm{GL}_n(\hat{\mathbb{Z}}) & & \end{array}$$

Hence, by using the fact that $\widehat{\mathrm{GL}_n(\mathbb{Z})} \to \mathrm{GL}_n(\hat{\mathbb{Z}})$ is injective for $n \geq 3$, one can obtain that $C(\Phi_n)$ is an image of $C(\mathrm{IA}(\Phi_n), \Phi_n)$. Thus, for investigating $C(\Phi_n)$ it seems to be worthwhile to investigate $C(\mathrm{IA}(\Phi_n), \Phi_n)$.

The first goal of the present paper is to prove the following theorem:

Theorem 1.1. *For every $n \geq 4$, the group $C(\mathrm{IA}(\Phi_n), \Phi_n)$ contains a subgroup C which satisfies the following properties*:

- *C is isomorphic to a product $C = \prod_{i=1}^{n} C_i$ of n copies of*

$$C_i \cong \ker\bigl(\mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x^{\pm 1}]}) \to \mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x^{\pm 1}]})\bigr).$$

- *C is a direct factor of $C(\mathrm{IA}(\Phi_n), \Phi_n)$; that is, there is a normal subgroup $N \lhd C(\mathrm{IA}(\Phi_n), \Phi_n)$ such that $C(\mathrm{IA}(\Phi_n), \Phi_n) = N \times C$.*

Using techniques of Kassabov and Nikolov [2006], one can show that the subgroups C_i are not finitely generated. So as an immediate corollary, we obtain the following theorem:

Theorem 1.2. *For every $n \geq 4$, the group $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is not finitely generated.*

It will be shown in an upcoming paper that when Γ is a finitely generated nilpotent group (of any class), then $C(\mathrm{IA}(\Gamma), \Gamma) = \{e\}$ is always trivial. So the free metabelian cases behave completely different from nilpotent cases. This result gives the impression that $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is "big". On the other hand, we have the following theorem (see [Ben-Ezra 2017]):

Theorem 1.3. *For every $n \geq 4$, the group $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is central in $\widehat{\mathrm{IA}(\Phi_n)}$.*

We remark that in the case of arithmetic groups, the congruence kernel is known to have a dichotomous behavior: it is central if and only if it is finite (see [Prasad and Rapinchuk 2010, Theorem 2]). So in some sense, the congruence kernel $C(\mathrm{IA}(\Phi_n), \Phi_n)$ for $n \geq 4$ has an intermediate behavior: central, but not finite. The latter is similar to the behavior of the congruence kernel

$$\ker\big(\widehat{\mathrm{SL}_d(\mathbb{Z}[x])} \to \mathrm{SL}_d(\widehat{\mathbb{Z}[x]})\big) \quad \text{for } d \geq 3$$

that was investigated in [Kassabov and Nikolov 2006, Theorem 4.1].

Theorem 1.3 has already been stated in [Ben-Ezra 2017]. However, a substantial portion of the proof of Theorem 1.3 appears in this paper — this is the second goal of the present paper. To be more precise, all the steps of the proof of Theorem 1.3 that involve arguments in algebraic K-theory are given in this paper, and in [Ben-Ezra 2017] we describe the structure of the proof, and present all the other steps. As presented in Section 5, the steps that are given in this present paper by themselves are sufficient for showing that the subgroup $C \leq C(\mathrm{IA}(\Phi_n), \Phi_n)$ presented in Theorem 1.1 is contained in the center of $\widehat{\mathrm{IA}(\Phi_n)}$. We remark that the main results in this paper that are used in [Ben-Ezra 2017] in order to prove Theorem 1.3 are Lemma 7.1 and our work in Section 5 (see Remark 5.6 for a more precise description). The following problem is still open:

Problem 1.4. Is $C(\mathrm{IA}(\Phi_n), \Phi_n) = \prod_{i=1}^{n} C_i$ or does it contain more elements?

Remark 1.5. Considering the action of $\mathrm{Aut}(\Phi_n)$ on $\mathrm{IA}(\Phi_n)$ by conjugation, we have a natural map $\mathrm{Aut}(\Phi_n) \to \mathrm{Aut}(\mathrm{IA}(\Phi_n))$ in which the copy of $\mathrm{IA}(\Phi_n)$ in $\mathrm{Aut}(\Phi_n)$ is mapped onto $\mathrm{IA}(\Phi_n) \to \mathrm{Inn}(\mathrm{IA}(\Phi_n))$. Now let

$$\mathrm{IA}_{n,m} = \bigcap \{N \lhd \mathrm{IA}(\Phi_n) \mid [\mathrm{IA}(\Phi_n) : N] \mid m\}.$$

Then as for every $n \geq 4$, the group $\mathrm{IA}(\Phi_n)$ is finitely generated [Bachmuth and Mochizuki 1985], the characteristic subgroups $\mathrm{IA}_{n,m} \leq \mathrm{IA}(\Phi_n)$ are of finite index. Hence $\widehat{\mathrm{IA}(\Phi_n)} = \varprojlim_{m\in\mathbb{N}} (\mathrm{IA}(\Phi_n)/\mathrm{IA}_{n,m})$ and therefore the action of $\mathrm{Aut}(\Phi_n)$ on $\mathrm{IA}(\Phi_n)$ induces an action of $\mathrm{Aut}(\Phi_n)$ on $\widehat{\mathrm{IA}(\Phi_n)}$, so we have a map $\mathrm{Aut}(\Phi_n) \to \varprojlim_{m\in\mathbb{N}} \mathrm{Aut}(\mathrm{IA}(\Phi_n)/\mathrm{IA}_{n,m})) \leq \mathrm{Aut}(\widehat{\mathrm{IA}(\Phi_n)})$. The latter gives rise to a map

$$\widehat{\mathrm{Aut}(\Phi_n)} \to \varprojlim_{m\in\mathbb{N}} \mathrm{Aut}(\mathrm{IA}(\Phi_n)/\mathrm{IA}_{n,m})) \leq \mathrm{Aut}(\widehat{\mathrm{IA}(\Phi_n)})$$

that actually gives an action of $\widehat{\mathrm{Aut}(\Phi_n)}$ on $\widehat{\mathrm{IA}(\Phi_n)}$ such that the closure $\overline{\mathrm{IA}(\Phi_n)}$ of $\mathrm{IA}(\Phi_n)$ in $\widehat{\mathrm{Aut}(\Phi_n)}$ acts trivially on $Z(\widehat{\mathrm{IA}(\Phi_n)})$, the center of $\widehat{\mathrm{IA}(\Phi_n)}$. Thus, as we have $\widehat{\mathrm{Aut}(\Phi_n)}/\overline{\mathrm{IA}(\Phi_n)} = \widehat{\mathrm{GL}_n(\mathbb{Z})}$ we obtain a natural action of $\widehat{\mathrm{GL}_n(\mathbb{Z})}$ on $Z(\widehat{\mathrm{IA}(\Phi_n)})$. It will be clear from the description in the paper that the permutation matrices permute the copies C_i through this natural action.

The aforementioned behavior of $C(\mathrm{IA}(\Phi_n), \Phi_n)$ for $n \geq 4$ is also different from the behavior of $C(\mathrm{IA}(\Phi_n), \Phi_n)$ for $n = 2, 3$. More precisely, as $C(\mathbb{Z}^3) = \{e\}$, similar arguments show that when $n = 3$ the group $C(\Phi_3)$ is an image of $C(\mathrm{IA}(\Phi_3), \Phi_3)$. So as $C(\Phi_3) \supseteq \hat{F}_\omega$ [Ben-Ezra and Lubotzky 2018], we obtain that $C(\mathrm{IA}(\Phi_3), \Phi_3)$ is infinite nonabelian. On the other hand, regarding the case $n = 2$, it is known that $\mathrm{IA}(\Phi_2) = \mathrm{Inn}(\Phi_2)$ (see [Bachmuth 1965]) and it is known that the center of Φ_2 and $\hat{\Phi}_2$ is trivial (see [Ben-Ezra 2016]). It follows that we have a canonical isomorphism

$$\widehat{\mathrm{IA}(\Phi_2)} = \widehat{\mathrm{Inn}(\Phi_2)} \cong \hat{\Phi}_2 \cong \mathrm{Inn}(\hat{\Phi}_2) \leq \mathrm{Aut}(\hat{\Phi}_2),$$

so $C(\mathrm{IA}(\Phi_2), \Phi_2) = \{e\}$ is trivial. Our results show that when $n \geq 4$, the behavior of $C(\mathrm{IA}(\Phi_n), \Phi_n)$ stabilizes and it is abelian, but not trivial.

We also note that considering our basic motivation, as $C(\Phi_n)$ is an image of $C(\mathrm{IA}(\Phi_n), \Phi_n)$ we actually obtain from Theorem 1.3 that when $n \geq 4$, the situation is dramatically different from the cases of $n = 2, 3$ described above:

Theorem 1.6. *For every $n \geq 4$, the group $C(\Phi_n)$ is abelian.*

We remark that despite the result of the latter theorem, we do not know whether $C(\Phi_n)$ is also not finitely generated. In fact we cannot even prove at this point that it is not trivial.

The paper is organized as follows. For a ring R, ideal $H \lhd R$ and $d \in \mathbb{N}$ let

$$\mathrm{GL}_d(R, H) = \ker(\mathrm{GL}_d(R) \to \mathrm{GL}_d(R/H)).$$

For $n \in \mathbb{N}$ define also the ring $R_n = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathbb{Z}[\mathbb{Z}^n]$. Using the Magnus embedding of $\mathrm{IA}(\Phi_n)$, in which $\mathrm{IA}(\Phi_n)$ can be viewed as

$$\mathrm{IA}(\Phi_n) = \left\{ A \in \mathrm{GL}_n(R_n) \;\middle|\; A \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix} = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix} \right\},$$

we obtain in Section 3, for every $1 \leq i \leq n$, a natural embedding

$$\mathrm{GL}_{n-1}(R_n, (x_i - 1)R_n) \hookrightarrow \mathrm{IA}(\Phi_n)$$

and a surjective natural homomorphism

$$\mathrm{IA}(\Phi_n) \xrightarrow{\rho_i} \mathrm{GL}_{n-1}\big(\mathbb{Z}[x_i^{\pm 1}], (x_i - 1)\mathbb{Z}[x_i^{\pm 1}]\big)$$

in which the obvious copy of the subgroup $\mathrm{GL}_{n-1}\big(\mathbb{Z}[x_i^{\pm 1}], (x_i - 1)\mathbb{Z}[x_i^{\pm 1}]\big)$ of the group $\mathrm{GL}_{n-1}(R_n, (x_i - 1)R_n)$ is mapped onto itself via the composition map (see Proposition 3.7). This description, combined with some classical notions and results from algebraic K-theory presented in Section 2, enables us in Section 4 to show that for every $n \geq 4$ and $1 \leq i \leq n$, the group $C(\mathrm{IA}(\Phi_n), \Phi_n)$ contains a copy of

$$C_i \cong \ker\bigl(\mathrm{GL}_{n-1}\bigl(\widehat{\mathbb{Z}[x_i^{\pm 1}], (x_i - 1)}\mathbb{Z}[x_i^{\pm 1}]\bigr) \to \mathrm{GL}_{n-1}(\widehat{\mathbb{Z}[x_i^{\pm 1}]})\bigr)$$
$$\cong \ker\bigl(\widehat{\mathrm{SL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}])} \to \mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x_i^{\pm 1}]})\bigr) \tag{1.7}$$

such that $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is mapped onto C_i through the map $\hat{\rho}_i$ induced by ρ_i. The second isomorphism in (1.7) is obtained by using some classical results from algebraic K-theory (Propositions 4.5 and 4.6), and the main lemma, Lemma 7.1. The proof of Lemma 7.1 will be postponed until the end of the paper. In particular, we get that for every $1 \leq i \leq n$ one has

$$C(\mathrm{IA}(\Phi_n), \Phi_n) = \bigl(C(\mathrm{IA}(\Phi_n), \Phi_n) \cap \ker \hat{\rho}_i\bigr) \rtimes C_i.$$

(see Proposition 4.3). In Section 4 we also show that the copies C_i lie in $\ker \hat{\rho}_j$ whenever $j \neq i$ (Proposition 4.2). In particular, we get that the copies C_i intersect each other trivially. Then, following the techniques of Kassabov and Nikolov [2006] we show that C_i is not finitely generated, and thus deduce that $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is not finitely generated either, i.e., we prove Theorem 1.2 (see the end of Section 4). Then, in Section 5 we show that the copies C_i lie in the center of $\widehat{\mathrm{IA}(\Phi_n)}$, using classical results from algebraic K-theory and Lemma 7.1. In particular, using the aforementioned results, we obtain that

$$C(\mathrm{IA}(\Phi_n), \Phi_n) = \Bigl(C(\mathrm{IA}(\Phi_n), \Phi_n) \cap \bigcap_{i=1}^{n} \ker \hat{\rho}_i\Bigr) \times \prod_{i=1}^{n} C_i.$$

This completes the proof of Theorem 1.1.

After that, we turn to prove Lemma 7.1. In Section 6 we introduce some elements in $\langle \mathrm{IA}(\Phi_n)^m \rangle$ which are needed for the proof of the lemma. In Section 7, using classical results from algebraic K-theory, we conclude the paper by proving Lemma 7.1, which asserts that for every $1 \leq i \leq n$, we have

$$\mathrm{GL}_{n-1}(R_n, (x_i - 1)R_n) \cap E_{n-1}(R_n, H_{n,m^2}) \subseteq \langle \mathrm{IA}(\Phi_n)^m \rangle, \tag{1.8}$$

where

- $\mathrm{GL}_{n-1}(R_n, (x_i - 1)R_n)$ denotes its appropriate copy in $\mathrm{IA}(\Phi_n)$ described above;
- $E_{n-1}(R_n, H_{n,m^2})$ is the subgroup of $E_{n-1}(R_n) = \langle I_{n-1} + rE_{i,j} \mid r \in R_n \rangle$ which is generated as a normal subgroup by the elementary matrices of the form $I_{n-1} + hE_{i,j}$ for $h \in H_{n,m^2} = \ker(R_n \to \mathbb{Z}_{m^2}[\mathbb{Z}_{m^2}^n])$, $1 \leq i \neq j \leq n$. Here, I_{n-1} is the $(n-1) \times (n-1)$ unit matrix and $E_{i,j}$ is the matrix which has 1 in the (i, j)-th entry and 0 elsewhere.
- The intersection in the inclusion (1.8) is obtained by viewing the copy of $\mathrm{GL}_{n-1}(R_n, (x_i - 1)R_n)$ in $\mathrm{IA}(\Phi_n)$ as a subgroup of $\mathrm{GL}_{n-1}(R_n)$.

We note that as described above, Lemma 7.1 is used in two places in the course of the paper. It is used once to prove the second isomorphism in (1.7). The second place is in the proof that the group C lies in the center of $\widehat{\mathrm{IA}(\Phi_n)}$. We also note that almost all the work that we do in order to show that C lies in the center of $\widehat{\mathrm{IA}(\Phi_n)}$, including Lemma 7.1 (but also most of Section 5), is used in [Ben-Ezra 2017] to prove Theorem 1.3 (see Remark 5.6).

2. Some background in algebraic K-theory

In this section we fix some notation and recall some definitions and background in algebraic K-theory which will be used throughout the paper. One can find more general information in the references [Rosenberg 1994; Milnor 1971; Bass 1968]. In this section R always denotes a commutative ring with identity. We start by recalling the following notation. Let R be a commutative ring, $H \lhd R$ an ideal, and $d \in \mathbb{N}$. Then:

- $\mathrm{GL}_d(R) = \{A \in M_n(R) \mid \det(A) \in R^*\}$.
- $\mathrm{SL}_d(R) = \{A \in \mathrm{GL}_d(R) \mid \det(A) = 1\}$.
- $E_d(R) = \langle I_d + rE_{i,j} \mid r \in R,\ 1 \le i \ne j \le d\rangle$.
- $\mathrm{GL}_d(R, H) = \ker(\mathrm{GL}_d(R) \to \mathrm{GL}_d(R/H))$.
- $\mathrm{SL}_d(R, H) = \ker(\mathrm{SL}_d(R) \to \mathrm{SL}_d(R/H))$.
- $E_d(R, H)$ = the normal subgroup of $E_d(R)$, which is generated as a normal subgroup by the elementary matrices of the form $I_d + hE_{i,j}$ for $h \in H$.

For every $d \ge 3$, the subgroup $E_d(R, H)$ is normal in $\mathrm{GL}_d(R)$; see Corollary 1.4 in [Suslin 1977]. Hence, we can consider the groups

$$K_1(R; d) = \mathrm{GL}_d(R)/E_d(R), \qquad K_1(R, H; d) = \mathrm{GL}_d(R, H)/E_d(R, H),$$
$$\mathrm{SK}_1(R; d) = \mathrm{SL}_d(R)/E_d(R), \qquad \mathrm{SK}_1(R, H; d) = \mathrm{SL}_d(R, H)/E_d(R, H).$$

We now go ahead with the following definition:

Definition 2.1. Let R be a commutative ring, and $3 \le d \in \mathbb{N}$. We define the "Steinberg group" $\mathrm{St}_d(R)$ to be the group generated by the elements $x_{i,j}(r)$ for $r \in R$ and $1 \le i \ne j \le d$, under the relations

- $x_{i,j}(r_1) \cdot x_{i,j}(r_2) = x_{i,j}(r_1 + r_2)$,
- $[x_{i,j}(r_1), x_{j,k}(r_2)] = x_{i,k}(r_1 \cdot r_2)$,
- $[x_{i,j}(r_1), x_{k,l}(r_2)] = 1$,

for every distinct $1 \le i, j, k, l \le d$ and every $r_1, r_2 \in R$.

As the elementary matrices $I_d + rE_{i,j}$ satisfy the relations which define $\mathrm{St}_d(R)$, the map $x_{i,j}(r) \mapsto I_d + rE_{i,j}$ defines a natural homomorphism $\phi_d : \mathrm{St}_d(R) \to E_d(R)$. The kernel of this map is denoted by $K_2(R; d) = \ker(\phi_d)$. Now, for two invertible elements $u, v \in R^*$ and $1 \le i \ne j \le d$, define the "Steinberg symbol" by

$$\{u, v\}_{i,j} = h_{i,j}(uv)h_{i,j}(u)^{-1}h_{i,j}(v)^{-1} \in \mathrm{St}_d(R)$$
$$\text{where } h_{i,j}(u) = w_{i,j}(u)w_{i,j}(-1) \text{ and } w_{i,j}(u) = x_{i,j}(u)x_{j,i}(-u^{-1})x_{i,j}(u).$$

One can show that $\{u, v\}_{i,j} \in K_2(R; d)$ and lie in the center of $\mathrm{St}_d(R)$. In addition, for every $3 \le d \in \mathbb{N}$, the Steinberg symbols $\{u, v\}_{i,j}$ do not depend on the indices i, j, so they can be denoted simply by $\{u, v\}$; see [Dennis and Stein 1973]. The Steinberg symbols satisfy many identities. For example,

$$\{uv, w\} = \{u, w\}\{v, w\}, \qquad \{u, vw\} = \{u, v\}\{u, w\}. \tag{2.2}$$

In the semilocal case we have the following:

Theorem 2.3 [Stein and Dennis 1973, Theorem 2.7]. *Let R be a semilocal commutative ring and $d \ge 3$. Then $K_2(R; d)$ is generated by the Steinberg symbols $\{u, v\}$ for $u, v \in R^*$. In particular, $K_2(R; d)$ is central in* $\mathrm{St}_d(R)$.

Now let R be a commutative ring, $H \lhd R$ an ideal and $d \ge 3$. Let $\bar{R} = R/H$. Clearly, there is a natural map $E_d(R) \to E_d(\bar{R})$. It is clear that $E_d(R, H)$ lies in the kernel of the latter map, so we have a map

$$\pi_d : E_d(R)/E_d(R, H) \to E_d(\bar{R}).$$

In addition, it is easy to see that we have a surjective map

$$\psi_d : \mathrm{St}_d(\bar{R}) \twoheadrightarrow E_d(R)/E_d(R, H)$$

defined by $x_{i,j}(\bar{r}) \mapsto I_d + rE_{i,j}$ such that $\phi_d : \mathrm{St}_d(\bar{R}) \to E_d(\bar{R})$ satisfies $\phi_d = \pi_d \circ \psi_d$. Therefore, we obtain the surjective map

$$K_2(\bar{R}; d) = \ker(\phi_d) \overset{\psi_d}{\twoheadrightarrow} \ker(\pi_d) = (E_d(R) \cap \mathrm{SL}_d(R, H))/E_d(R, H)$$
$$\le \mathrm{SK}_1(R, H; d).$$

In particular, it implies that if $E_d(R) = \mathrm{SL}_d(R)$, then we have a natural surjective map

$$K_2(R/H; d) \twoheadrightarrow \mathrm{SK}_1(R, H; d).$$

From this one can easily deduce the following corollary, which will be needed later in the paper.

Corollary 2.4. *Let R be a commutative ring, $H \lhd R$ an ideal of finite index and $d \ge 3$. Assume also that $E_d(R) = \mathrm{SL}_d(R)$.*

(1) $\mathrm{SK}_1(R, H; d)$ *is a finite group.*

(2) $\mathrm{SK}_1(R, H; d)$ *is central in* $\mathrm{GL}_d(R)/E_d(R, H)$.

(3) *Every element of* $\mathrm{SK}_1(R, H; d)$ *has a representative in* $\mathrm{SL}_d(R, H)$ *of the form*

$$\begin{pmatrix} A & 0 \\ 0 & I_{d-2} \end{pmatrix}$$

such that $A \in \mathrm{SL}_2(R, H)$.

Proof. The ring $\bar{R} = R/H$ is finite. In particular, $\bar{R}$ is Artinian and hence semilocal. Thus, by Theorem 2.3, $K_2(\bar{R}; d)$ is an abelian group which is generated by the Steinberg symbols $\{u, v\}$ for $u, v \in \bar{R}^*$. As $\bar{R}$ is finite, so is the number of the Steinberg symbols. From (2.2) we obtain that the order of any Steinberg symbol is finite. So $K_2(\bar{R}; d)$ is a finitely generated abelian group whose generators are of finite order. Thus, $K_2(\bar{R}; d)$ is finite. Moreover, as $\bar{R}$ is semilocal, Theorem 2.3 implies that $K_2(\bar{R}; d)$ is central $\mathrm{St}_d(\bar{R})$. Now, as we assume that $E_d(R) = \mathrm{SL}_d(R)$, we obtain that $\mathrm{SK}_1(R, H; d)$ is the image of $K_2(\bar{R}; d)$ under the surjective map

$$\mathrm{St}_d(\bar{R}) \twoheadrightarrow E_d(R)/E_d(R, H) = \mathrm{SL}_d(R)/E_d(R, H).$$

This implies part (1) and that $\mathrm{SK}_1(R, H; d)$ is central in $\mathrm{SL}_d(R)/E_d(R, H)$.

Moreover, as $d \geq 3$, we have $\{u, v\} = \{u, v\}_{1,2}$ for every $u, v \in \bar{R}^*$. Now, it is easy to check from the definition of the Steinberg symbols that the image of $\{u, v\}_{1,2}$ under the map $\mathrm{St}_d(\bar{R}) \twoheadrightarrow \mathrm{SL}_d(R)/E_d(R, H)$ is of the form

$$\begin{pmatrix} A & 0 \\ 0 & I_{d-2} \end{pmatrix} \cdot E_d(R, H) \tag{2.5}$$

for some $A \in \mathrm{SL}_2(R, H)$. So as $\mathrm{SK}_1(R, H; d)$ is generated by the images of the Steinberg symbols, the same holds for every element in $\mathrm{SK}_1(R, H; d)$. So we obtain part (3). Now, as $d \geq 3$ we can write

$$\mathrm{GL}_d(R) = \mathrm{SL}_d(R) \cdot \{I_d + (r-1)E_{3,3} \mid r \in R^*\}.$$

Observe also that mod $E_d(R, H)$, all the elements of the form $I_d + (r-1)E_{3,3}$ for $r \in R^*$ commute with all the elements of the form (2.5). Hence, the centrality of $\mathrm{SK}_1(R, H; d)$ in $\mathrm{SL}_d(R)/E_d(R, H)$ shows that actually $\mathrm{SK}_1(R, H; d)$ is central in $\mathrm{GL}_d(R)/E_d(R, H)$, as required in part (2). □

3. IA(Φ_n) and its subgroups

We start our discussion of the IA-automorphism group of the free metabelian group, $G = \mathrm{IA}(\Phi_n) = \ker\big(\mathrm{Aut}(\Phi_n) \to \mathrm{Aut}(\Phi_n/\Phi_n') = \mathrm{GL}_n(\mathbb{Z})\big)$, by presenting some of its properties and subgroups. We begin with the following notation:

- $\Phi = \Phi_n = F_n/F_n'' =$ the free metabelian group on n elements. Here F_n'' denotes the second derivative of F_n, the free group on n elements.

- $\Psi_m = \Phi/M_m$, where $M_m = (\Phi'\Phi^m)'(\Phi'\Phi^m)^m$.
- $\mathrm{IG}_m = G(M_m) = \ker(\mathrm{IA}(\Phi) \to \mathrm{Aut}(\Psi_m))$.
- $\mathrm{IA}_m = \bigcap\{N \lhd \mathrm{IA}(\Phi) \mid [\mathrm{IA}(\Phi) : N] \mid m\}$.
- $R_n = \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, where $x_1, \dots, x_n$ are the generators of $\mathbb{Z}^n$.
- $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$.
- $\sigma_i = x_i - 1$ for $1 \le i \le n$.
- $\vec{\sigma}$ = the column vector which has σ_i in its i-th entry.
- $\mathfrak{A} = \sum_{i=1}^n \sigma_i R_n$ = the augmentation ideal of R_n.
- $H_m = \ker\bigl(R_n \to \mathbb{Z}_m[\mathbb{Z}_m^n]\bigr) = \sum_{i=1}^n (x_i^m - 1)R_n + mR_n$.

By the well-known Magnus embedding [Birman 1974; Remeslennikov and Sokolov 1970; Magnus 1939], one can identify Φ with the matrix group

$$\Phi = \left\{ \begin{pmatrix} g & a_1t_1 + \dots + a_nt_n \\ 0 & 1 \end{pmatrix} \;\middle|\; g \in \mathbb{Z}^n,\ a_i \in R_n,\ g - 1 = \sum_{i=1}^n a_i\sigma_i \right\},$$

where t_i is a free basis for an R_n-module, under the identification of the generators of Φ with the matrices

$$\begin{pmatrix} x_i & t_i \\ 0 & 1 \end{pmatrix} \quad \text{for } 1 \le i \le n.$$

Moreover, for every $\alpha \in \mathrm{IA}(\Phi)$, one can describe α by its action on the generators of Φ by

$$\alpha : \begin{pmatrix} x_i & t_i \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} x_i & a_{i,1}t_1 + \dots + a_{i,n}t_n \\ 0 & 1 \end{pmatrix}.$$

This description gives an injective homomorphism (see [Bachmuth 1965; Birman 1974])

$$\mathrm{IA}(\Phi) \hookrightarrow \mathrm{GL}_n(R_n), \qquad \alpha \mapsto \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix},$$

which gives an identification of $\mathrm{IA}(\Phi)$ with the group

$$\mathrm{IA}(\Phi) = \{A \in \mathrm{GL}_n(R_n) \mid A\vec{\sigma} = \vec{\sigma}\} = \{I_n + A \in \mathrm{GL}_n(R_n) \mid A\vec{\sigma} = \vec{0}\}.$$

Consider now the map

$$\begin{array}{c} \Phi = \left\{ \begin{pmatrix} g & a_1t_1 + \dots + a_nt_n \\ 0 & 1 \end{pmatrix} \;\middle|\; g \in \mathbb{Z}^n,\ a_i \in R_n,\ g - 1 = \sum_{i=1}^n a_i\sigma_i \right\} \\ \downarrow \\ \left\{ \begin{pmatrix} g & a_1t_1 + \dots + a_nt_n \\ 0 & 1 \end{pmatrix} \;\middle|\; g \in \mathbb{Z}_m^n,\ a_i \in \mathbb{Z}_m[\mathbb{Z}_m^n],\ g - 1 = \sum_{i=1}^n a_i\sigma_i \right\} \end{array}$$

which is induced by the projections $\mathbb{Z}^n \to \mathbb{Z}_m^n$, $R_n = \mathbb{Z}[\mathbb{Z}^n] \to \mathbb{Z}_m[\mathbb{Z}_m^n]$. Using a result of Romanovskii [1999], it is shown in [Ben-Ezra 2016] that this map is surjective and that Ψ_m is canonically isomorphic to its image. Therefore, we can identify IG_m, the principal congruence subgroup of $\mathrm{IA}(\Phi)$, with

$$\begin{aligned}\mathrm{IG}_m &= \left\{A \in \ker\big(\mathrm{GL}_n(R_n) \to \mathrm{GL}_n(\mathbb{Z}_m[\mathbb{Z}_m^n])\big) \mid A\vec{\sigma} = \vec{\sigma}\right\},\\ &= \left\{I_n + A \in \mathrm{GL}_n(R_n, H_m) \mid A\vec{\sigma} = \vec{0}\right\}.\end{aligned}$$

Proposition 3.1. *Let $I_n + A \in \mathrm{IA}(\Phi)$ and denote the entries of A by $a_{k,l}$ for $1 \le k, l \le n$. Then for every $1 \le k, l \le n$, we have $a_{k,l} \in \sum_{l \ne i=1}^{n} \sigma_i R_n \subseteq \mathfrak{A}$.*

Proof. For a given $1 \le k \le n$, the condition $A\vec{\sigma} = \vec{0}$ gives the equality

$$0 = a_{k,1}\sigma_1 + a_{k,2}\sigma_2 + \cdots + a_{k,n}\sigma_n.$$

Thus, for a given $1 \le l \le n$, the map $R_n \to S_l = \mathbb{Z}[x_l^{\pm 1}]$ defined by $x_i \mapsto 1$ for every $i \ne l$ maps

$$0 = a_{k,1}\sigma_1 + a_{k,2}\sigma_2 + \cdots + a_{k,n}\sigma_n \mapsto \bar{a}_{k,l}\sigma_l \in \mathbb{Z}[x_l^{\pm 1}].$$

Hence, as $\mathbb{Z}[x_l^{\pm 1}]$ is a domain, $\bar{a}_{k,l} = 0 \in \mathbb{Z}[x_l^{\pm 1}]$. Thus $a_{k,l} \in \sum_{l \ne i=1}^{n} \sigma_i R_n \subseteq \mathfrak{A}$, as required. □

Proposition 3.2. *Let $I_n + A \in \mathrm{IA}(\Phi)$. Then $\det(I_n + A)$ is of the form*

$$\det(I_n + A) = \prod_{r=1}^{n} x_r^{s_r} \quad \textit{for some } s_r \in \mathbb{Z}.$$

Proof. The invertible elements in R_n are the elements of the form $\pm \prod_{i=1}^{n} x_r^{s_r}$; see [Crowell and Fox 1963, Chapter 8]. Thus, as $I_n + A \in \mathrm{GL}_n(R_n)$ we have $\det(I_n + A) = \pm \prod_{i=1}^{n} x_r^{s_r}$. However, according to Proposition 3.1, for every entry $a_{k,l}$ of A we have $a_{k,l} \in \mathfrak{A}$. Hence, under the projection $x_i \mapsto 1$ for every $1 \le i \le n$, one has $I_n + A \mapsto I_n$, and thus, $\pm \prod_{i=1}^{n} x_r^{s_r} = \det(I_n + A) \mapsto \det(I_n) = 1$. Therefore, the option $\det(I_n + A) = -\prod_{i=1}^{n} x_r^{s_r}$ is impossible, as required. □

Let us step forward with the following definition:

Definition 3.3. Let $A \in \mathrm{GL}_n(R_n)$, and for $1 \le i \le n$, denote by $A_{i,i}$ the minor which is obtained from A by erasing its i-th row and i-th column. Now, for every $1 \le i \le n$, define the subgroup $\mathrm{IGL}_{n-1,i} \le \mathrm{IA}(\Phi)$ by

$$\mathrm{IGL}_{n-1,i} = \left\{ I_n + A \in \mathrm{IA}(\Phi) \;\middle|\; \begin{array}{l} \text{the } i\text{-th row of } A \text{ is } 0, \\ \qquad I_{n-1} + A_{i,i} \in \mathrm{GL}_{n-1}(R_n, \sigma_i R_n) \end{array} \right\}.$$

Proposition 3.4. *For every $1 \le i \le n$, we have $\mathrm{IGL}_{n-1,i} \cong \mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$.*

Proof. The definition of $\mathrm{IGL}_{n-1,i}$ gives us a natural projection

$$\mathrm{IGL}_{n-1,i} \to \mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$$

which maps an element $I_n + A \in \mathrm{IGL}_{n-1,i}$ to $I_{n-1} + A_{i,i} \in \mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$. Thus, all we need is to explain why this map is injective and surjective.

Injectivity: Here, it is enough to show that given an element $I_n + A \in \mathrm{IA}(\Phi)$, every entry in the i-th column is determined uniquely by the other entries in its row. Indeed, as A satisfies the condition $A\vec{\sigma} = \vec{0}$, for every $1 \le k \le n$ we have

$$a_{k,1}\sigma_1 + a_{k,2}\sigma_2 + \cdots + a_{k,n}\sigma_n = 0 \quad \Rightarrow \quad a_{k,i} = \frac{-\sum_{i\neq l=1}^{n} a_{k,l}\sigma_l}{\sigma_i}, \tag{3.5}$$

i.e., we have a formula for $a_{k,i}$ in terms of the other entries in its row.

Surjectivity: Without loss of generality we assume $i = n$. Let $I_{n-1} + \sigma_n B$ be in $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$, and denote by $\vec{b}_l$ the column vectors of B. Define

$$\begin{pmatrix} I_{n-1} + \sigma_n B & -\sum_{l=1}^{n-1} \sigma_l \vec{b}_l \\ 0 & 1 \end{pmatrix} \in \mathrm{IGL}_{n-1,n}\,.$$

This is clearly a preimage of $I_{n-1} + \sigma_n B$. □

Under the above identification of $\mathrm{IGL}_{n-1,i}$ with $\mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$, we will use throughout the paper the following notation:

Definition 3.6. Let $H \lhd R_n$. We define

$$\mathrm{ISL}_{n-1,i}(H) = \mathrm{IGL}_{n-1,i} \cap \mathrm{SL}_{n-1}(R_n, H),$$
$$\mathrm{IE}_{n-1,i}(H) = \mathrm{IGL}_{n-1,i} \cap E_{n-1}(R_n, H) \le \mathrm{ISL}_{n-1,i}(H).$$

Observe that as for every $1 \le i \le n$ we have

$$\mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}]) \le \mathrm{GL}_{n-1}(R_n, \sigma_i R_n),$$

the isomorphism $\mathrm{GL}_{n-1}(R_n, \sigma_i R_n) \cong \mathrm{IGL}_{n-1,i} \le \mathrm{IA}(\Phi)$ gives also a natural embedding of $\mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}])$ as a subgroup of $\mathrm{IA}(\Phi)$.

Proposition 3.7. *For every $1 \le i \le n$, there is a canonical surjective homomorphism*

$$\rho_i : \mathrm{IA}(\Phi) \twoheadrightarrow \mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}])$$

such that the following composition map is the identity:

$$\mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}]) \hookrightarrow \mathrm{IA}(\Phi) \xrightarrow{\rho_i} \mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}]).$$

Hence $\mathrm{IA}(\Phi) = \ker \rho_i \rtimes \mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}])$.

Proof. Without loss of generality we assume $i = n$. First, consider the homomorphism $\mathrm{IA}(\Phi) \to \mathrm{GL}_n(\mathbb{Z}[x_n^{\pm 1}])$, which is induced by the projection $R_n \to \mathbb{Z}[x_n^{\pm 1}]$ that is defined by $x_j \mapsto 1$ for every $j \neq n$. By Proposition 3.1, given $I_n + A \in \mathrm{IA}(\Phi)$, all the entries of the n-th column of A are in $\sum_{j=1}^{n-1} \sigma_j R_n$. Hence, the above map $\mathrm{IA}(\Phi) \to \mathrm{GL}_n(\mathbb{Z}[x_n^{\pm 1}])$ is actually a map

$$\mathrm{IA}(\Phi) \to \left\{ I_n + \bar{A} \in \mathrm{GL}_n(\mathbb{Z}[x_n^{\pm 1}]) \mid \text{the } n\text{-th column of } \bar{A} \text{ is } \vec{0} \right\}.$$

Observe now that the right side of the above map can be mapped naturally to $\mathrm{GL}_{n-1}(\mathbb{Z}[x_n^{\pm 1}])$ by erasing the n-th column and the n-th row from every element. Hence we obtain a map

$$\mathrm{IA}(\Phi) \to \mathrm{GL}_{n-1}(\mathbb{Z}[x_n^{\pm 1}]).$$

Now, by Proposition 3.1, every entry of A such that $I_n + A \in \mathrm{IA}(\Phi)$ is in $\mathfrak{A}$. Thus, the entries of every $\bar{A}$ such that $I_{n-1} + \bar{A} \in \mathrm{GL}_{n-1}(\mathbb{Z}[x_n^{\pm 1}])$ is an image of $I_n + A \in \mathrm{IA}(\Phi)$ are all in $\sigma_n \mathbb{Z}[x_n^{\pm 1}]$. Hence, we actually obtain a homomorphism

$$\rho_n : \mathrm{IA}(\Phi) \to \mathrm{GL}_{n-1}(\mathbb{Z}[x_n^{\pm 1}], \sigma_n \mathbb{Z}[x_n^{\pm 1}]).$$

We conclude by observing that the copy of $\mathrm{GL}_{n-1}(\mathbb{Z}[x_n^{\pm 1}], \sigma_n \mathbb{Z}[x_n^{\pm 1}])$ in $\mathrm{IGL}_{n-1,n}$ is mapped isomorphically to itself by ρ_n. □

Proposition 3.8. *Write $S_i = \mathbb{Z}[x_i^{\pm 1}] \subseteq R_n$ and $J_{i,m} = (x_i^m - 1)S_i + mS_i \subseteq H_m$ for $1 \leq i \leq n$. Then, by identifying*

$$\mathrm{Im}(\rho_i) \cong \mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}]) = \mathrm{GL}_{n-1}(S_i, \sigma_i S_i),$$

for every $m \in \mathbb{N}$ one has

$$\mathrm{Im}(\rho_i) \cap \mathrm{IG}_m = \mathrm{GL}_{n-1}(S_i, \sigma_i J_{i,m}).$$

Proof. By the identification

$$\mathrm{IG}_m = \{ I_n + A \in \mathrm{GL}_n(R_n, H_m) \mid A\vec{\sigma} = \vec{0} \}$$

and by applying the formula of (3.5) to the i-th column of elements in $\mathrm{IGL}_{n-1,i}$, it is easy to see that the elements of $\mathrm{IGL}_{n-1,i}$ which correspond to the elements of $\mathrm{GL}_{n-1}(S_i, \sigma_i J_{i,m})$ are clearly in $\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m$. For the opposite inclusion, without loss of generality assume that $i = n$, and let $I_n + A \in \mathrm{Im}\,\rho_n \cap \mathrm{IG}_m$. Then $I_n + A$ has the form

$$\begin{pmatrix} I_{n-1} + \sigma_n B & -\sum_{l=1}^{n-1} \sigma_l \vec{b}_l \\ 0 & 1 \end{pmatrix} \in \mathrm{IGL}_{n-1,n},$$

where the entries of B satisfy $b_{k,l} \in S_n$ and $\sum_{j=1}^{n-1} \sigma_j b_{k,j} \in H_m$. Notice now that for every $l \neq n$, by projecting $\sigma_j \mapsto 0$ for $j \neq l, n$, we see that actually $\sigma_l b_{k,l} \in H_m$.

From here it is easy to see that we necessarily have $b_{k,l} \in H_m$, i.e.,

$$b_{k,l} \in H_m \cap S_n = (x_n^m - 1)S_n + mS_n = J_{n,m},$$

and the claim follows. □

Proposition 3.9. *For every* $1 \le i \le n$ *and* $m \in \mathbb{N}$ *one has*

$$\rho_i(\mathrm{IG}_{m^2}) \subseteq \mathrm{Im}(\rho_i) \cap \mathrm{IG}_m \subseteq \rho_i(\mathrm{IG}_m).$$

Proof. As every element in $\mathrm{Im}\,\rho_i$ is mapped to itself via ρ_i we clearly have

$$\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m = \rho_i(\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m) \subseteq \rho_i(\mathrm{IG}_m).$$

On the other hand, if $I_n + A \in \mathrm{IG}_{m^2}$ then viewing $\mathrm{Im}\,\rho_i \cong \mathrm{GL}_{n-1}(S_i, \sigma_i S_i)$ for $S_i = \mathbb{Z}[x_i^{\pm 1}]$, the entries of $\rho_i(I_n + A) = I_{n-1} + B$ belong to $(x_i^{m^2} - 1)S_i + m^2\sigma_i S_i$. Observe now that we have $\sum_{r=0}^{m-1} x_i^{mr} \subseteq (x_i^m - 1)S_i + mS_i = J_{i,m}$. Hence

$$x_i^{m^2} - 1 = \sigma_i \sum_{r=1}^{m^2-1} x_i^r = \sigma_i \sum_{r=0}^{m-1} x_i^r \sum_{r=0}^{m-1} x_i^{mr} \in \sigma_i J_{i,m}. \tag{3.10}$$

So by Proposition 3.8, $\rho_i(I_n + A) \in \mathrm{Im}\,\rho_i \cap \mathrm{IG}_m$, as required. □

Proposition 3.11. *For every* $m \in \mathbb{N}$ *and* $1 \le i \le n$ *one has*

$$\rho_i(\mathrm{IA}_m) = \mathrm{Im}(\rho_i) \cap \mathrm{IA}_m,$$

where $\mathrm{IA}_m = \bigcap \{N \lhd \mathrm{IA}(\Phi) \mid [\mathrm{IA}(\Phi) : N] \mid m\}$.

Proof. As every element in $\mathrm{Im}\,\rho_i$ is mapped to itself via ρ_i, we clearly have $\mathrm{Im}\,\rho_i \cap \mathrm{IA}_m = \rho_i(\mathrm{Im}\,\rho_i \cap \mathrm{IA}_m) \subseteq \rho_i(\mathrm{IA}_m)$. For the opposite, assume that $\alpha \in \mathrm{IA}_m$, and let $\rho_i(\alpha) = \beta \in \mathrm{Im}\,\rho_i$. We want to show that $\beta \in \mathrm{IA}_m$. So let $N \lhd \mathrm{IA}(\Phi)$ such that $[\mathrm{IA}(\Phi) : N] \mid m$. Then obviously $[\mathrm{Im}\,\rho_i : (N \cap \mathrm{Im}\,\rho_i)] \mid m$. Thus, as ρ_i is surjective, $[\mathrm{IA}(\Phi) : \rho_i^{-1}(N \cap \mathrm{Im}\,\rho_i)] \mid m$ so $\alpha \in \rho_i^{-1}(N \cap \mathrm{Im}\,\rho_i)$ and hence $\beta = \rho_i(\alpha) \in N \cap \mathrm{Im}\,\rho_i \le N$. As this is valid for every such N, we have $\beta \in \mathrm{IA}_m$, as required. □

We close this section with the following definition:

Definition 3.12. For every $1 \le i \le n$, define

$$\mathrm{IGL}'_{n-1,i} = \{I_n + A \in \mathrm{IA}(\Phi) \mid \text{the } i\text{-th row of } A \text{ is } 0\}.$$

Obviously, $\mathrm{IGL}_{n-1,i} \le \mathrm{IGL}'_{n-1,i}$, and by the same injectivity argument as in the proof of Proposition 3.4, one can deduce the next proposition:

Proposition 3.13. *The subgroup* $\mathrm{IGL}'_{n-1,i} \le \mathrm{IA}(\Phi)$ *is canonically embedded in* $\mathrm{GL}_{n-1}(R_n)$ *by the map* $I_n + A \mapsto I_{n-1} + A_{i,i}$.

Remark 3.14. Note that in general $\mathrm{IGL}_{n-1,i} \lneqq \mathrm{IGL}'_{n-1,i}$. For example,

$$I_4 + \sigma_3 E_{1,2} - \sigma_2 E_{1,3} \in \mathrm{IGL}'_{3,4} \setminus \mathrm{IGL}_{3,4}\,.$$

4. The subgroups C_i

In this section we define the subgroups $C_i \le C(\mathrm{IA}(\Phi_n), \Phi_n)$, and we show that for each i we can view $C(\mathrm{IA}(\Phi_n), \Phi_n)$ as a semidirect product of C_i with another subgroup. We also show that when $n \ge 4$,

$$C_i \cong \ker\bigl(\mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x^{\pm 1}]}) \to \mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x^{\pm 1}]})\bigr)$$

and use it to show that $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is not finitely generated. Recall the notation

- $\Phi = \Phi_n$,
- $\Psi_m = \Phi/M_m$, where $M_m = (\Phi'\Phi^m)'(\Phi'\Phi^m)^m$,
- $\mathrm{IG}_m = G(M_m) = \ker(\mathrm{IA}(\Phi) \to \mathrm{Aut}(\Psi_m))$,
- $\mathrm{IA}_m = \bigcap\bigl\{N \lhd \mathrm{IA}(\Phi) \mid [\mathrm{IA}(\Phi) : N] \mid m\bigr\}$.

It is proven in [Ben-Ezra 2016] that $\hat{\Phi} = \varprojlim \Psi_m$. So, as for every $m \in \mathbb{N}$ the group $\ker(\Phi \to \Psi_m)$ is characteristic in Φ, we can write explicitly

$$\begin{aligned} C(\mathrm{IA}(\Phi), \Phi) &= \ker\bigl(\widehat{\mathrm{IA}(\Phi)} \to \mathrm{Aut}(\hat{\Phi})\bigr) \\ &= \ker\bigl(\widehat{\mathrm{IA}(\Phi)} \to \varprojlim \mathrm{Aut}(\Psi_m)\bigr) \\ &= \ker\bigl(\widehat{\mathrm{IA}(\Phi)} \to \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IG}_m)\bigr). \end{aligned}$$

Now, as for every $n \ge 4$ we know that $\mathrm{IA}(\Phi)$ is finitely generated (see [Bachmuth and Mochizuki 1985]), as explained in Remark 1.5, we have

$$\widehat{\mathrm{IA}(\Phi)} = \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IA}_m).$$

Hence

$$\begin{aligned} C(\mathrm{IA}(\Phi), \Phi) &= \ker\bigl(\varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IA}_m) \to \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IG}_m)\bigr) \\ &= \ker\bigl(\varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IA}_m) \to \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IG}_m \cdot \mathrm{IA}_m)\bigr) \\ &= \varprojlim(\mathrm{IA}_m \cdot \mathrm{IG}_m \,/\, \mathrm{IA}_m). \end{aligned}$$

Similarly, we can write $C(\mathrm{IA}(\Phi), \Phi) = \varprojlim(\mathrm{IA}_m \cdot \mathrm{IG}_{m^2} \,/\, \mathrm{IA}_m)$.

Remember now that for every $1 \le i \le n$ the composition map

$$\mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}]) \hookrightarrow \mathrm{IA}(\Phi) \xrightarrow{\rho_i} \mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}])$$

is the identity on $\mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}])$. Hence, the induced composition map of the profinite completions

$$\mathrm{GL}_{n-1}\bigl(\widehat{\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}]}\bigr) \xrightarrow{\hat{\varrho}} \widehat{\mathrm{IA}(\Phi)} \xrightarrow{\hat{\rho}_i} \mathrm{GL}_{n-1}\bigl(\widehat{\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}]}\bigr)$$

is the identity on $\mathrm{GL}_{n-1}\big(\widehat{\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}]}\big)$. In particular, the map $\hat{\varrho}$ is injective, so we can write

$$\mathrm{GL}_{n-1}\big(\widehat{\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}]}\big) \hookrightarrow \widehat{\mathrm{IA}(\Phi)} \xrightarrow{\hat{\rho}_i} \mathrm{GL}_{n-1}\big(\widehat{\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}]}\big).$$

This enables us to write $\mathrm{IA}(\Phi) = \ker\rho_i \rtimes \mathrm{Im}\,\rho_i$ and $\widehat{\mathrm{IA}(\Phi)} = \ker\hat{\rho}_i \rtimes \mathrm{Im}\,\hat{\rho}_i$.

Definition 4.1. We define

$$C_i = C(\mathrm{IA}(\Phi), \Phi) \cap \mathrm{Im}\,\hat{\rho}_i = \ker(\mathrm{Im}\,\hat{\rho}_i \to \mathrm{Aut}(\hat{\Phi})).$$

Proposition 4.2. *If* $1 \le i \ne j \le n$, *then* $C_i \subseteq \ker\hat{\rho}_j$. *In particular, for every* $i \ne j$ *we have* $C_i \cap C_j = \{e\}$.

Proof. By the explicit description $\widehat{\mathrm{IA}(\Phi)} = \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IA}_m)$, one can write

$$\begin{aligned} C_i &= \ker(\mathrm{Im}\,\hat{\rho}_i \to \mathrm{Aut}(\hat{\Phi})) \\ &= \ker\big(\varprojlim(\mathrm{IA}_m \cdot \mathrm{Im}\,\rho_i/\,\mathrm{IA}_m) \to \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IG}_m)\big) \\ &= \ker\big(\varprojlim(\mathrm{IA}_m \cdot \mathrm{Im}\,\rho_i/\,\mathrm{IA}_m) \to \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IG}_m \cdot \mathrm{IA}_m)\big) \\ &= \varprojlim((\mathrm{IA}_m \cdot \mathrm{Im}\,\rho_i) \cap (\mathrm{IA}_m \cdot \mathrm{IG}_m))/\,\mathrm{IA}_m, \end{aligned}$$

and similarly, $C_i = \varprojlim((\mathrm{IA}_m \cdot \mathrm{Im}\,\rho_i) \cap (\mathrm{IA}_m \cdot \mathrm{IG}_{m^2}))/\,\mathrm{IA}_m$. We claim now that

$$\begin{aligned} (\mathrm{IA}_m \cdot \mathrm{Im}\,\rho_i) \cap (\mathrm{IA}_m \cdot \mathrm{IG}_{m^2}) &\subseteq \mathrm{IA}_m \cdot (\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m) \\ &\subseteq (\mathrm{IA}_m \cdot \mathrm{Im}\,\rho_i) \cap (\mathrm{IA}_m \cdot \mathrm{IG}_m). \end{aligned}$$

The second inclusion is obvious. For the first one, we have to show that if $ar = bs$ such that $a, b \in \mathrm{IA}_m$, $r \in \mathrm{Im}\,\rho_i$ and $s \in \mathrm{IG}_{m^2}$, then there exist $c \in \mathrm{IA}_m$ and $t \in \mathrm{Im}\,\rho_i \cap \mathrm{IG}_m$ such that $ar = bs = ct$. Indeed, write $\mathrm{Im}\,\rho_i \ni r = a^{-1}bs$. Then $r = \rho_i(r) = \rho_i(a^{-1}b)\rho_i(s)$, and by Propositions 3.9 and 3.11,

$$\rho_i(a^{-1}b) \in \rho_i(\mathrm{IA}_m) = \mathrm{Im}\,\rho_i \cap \mathrm{IA}_m, \qquad \rho_i(s) \in \rho_i(\mathrm{IG}_{m^2}) \subseteq \mathrm{Im}\,\rho_i \cap \mathrm{IG}_m\,.$$

Therefore, by defining $c = a \cdot \rho_i(a^{-1}b)$ and $t = \rho_i(s)$ we get the required inclusion. Thus, for $j \ne i$ we have

$$C_i = \varprojlim(\mathrm{IA}_m \cdot (\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m)/\,\mathrm{IA}_m) \xrightarrow{\hat{\rho}_j} \varprojlim \rho_j(\mathrm{IA}_m) \cdot \rho_j(\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m)/\rho_j(\mathrm{IA}_m).$$

Using the definition of ρ_j, it is not difficult to show that

$$\begin{aligned} \rho_j(\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m) &= \langle I_n + m(\sigma_i E_{k,j} - \sigma_j E_{k,i}) \mid k \ne i, j\rangle \\ &= \rho_j\big(\langle I_n + m(\sigma_i E_{k,j} - \sigma_j E_{k,i}) \mid k \ne i, j\rangle\big) \\ &= \rho_j\big(\langle I_n + \sigma_i E_{k,j} - \sigma_j E_{k,i} \mid k \ne i, j\rangle^m\big) \subseteq \rho_j(\mathrm{IA}_m). \end{aligned}$$

Hence, $C_i \subseteq \ker\hat{\rho}_j$, as required. $\square$

We can now prove the following proposition:

Proposition 4.3. *For every* $1 \le i \le n$ *we have*

$$C_i \hookrightarrow C(\mathrm{IA}(\Phi), \Phi) \xrightarrow{\hat{\rho}_i} C_i.$$

In particular, $C(\mathrm{IA}(\Phi), \Phi) = (\ker \hat{\rho}_i \cap C(\mathrm{IA}(\Phi), \Phi)) \rtimes C_i$.

Proof. In the proof of Proposition 4.2 we saw that

$$C_i = \varprojlim (\mathrm{IA}_m \cdot (\mathrm{Im}\, \rho_i \cap \mathrm{IG}_m)/\, \mathrm{IA}_m).$$

Similarly, $C_i = \varprojlim (\mathrm{IA}_m \cdot (\mathrm{Im}\, \rho_i \cap \mathrm{IG}_{m^2})/\, \mathrm{IA}_m)$. We recall that by Propositions 3.9 and 3.11, we have

$$\rho_i(\mathrm{IG}_{m^2}) \subseteq \mathrm{Im}\, \rho_i \cap \mathrm{IG}_m \subseteq \rho_i(\mathrm{IG}_m), \qquad \rho_i(\mathrm{IA}_m) = \mathrm{Im}\, \rho_i \cap \mathrm{IA}_m\,.$$

Therefore, we have

$$\begin{aligned} C_i &= \varprojlim \mathrm{IA}_m \cdot (\mathrm{Im}\, \rho_i \cap \mathrm{IG}_m)/\, \mathrm{IA}_m = \varprojlim \mathrm{IA}_m \cdot (\mathrm{Im}\, \rho_i \cap \mathrm{IG}_{m^2})/\, \mathrm{IA}_m \\ &\hookrightarrow \varprojlim \mathrm{IA}_m \cdot \mathrm{IG}_m \,/\, \mathrm{IA}_m = \varprojlim \mathrm{IA}_m \cdot \mathrm{IG}_{m^2} \,/\, \mathrm{IA}_m = C(\mathrm{IA}(\Phi), \Phi) \\ &\xrightarrow{\hat{\rho}_i} \varprojlim \rho_i(\mathrm{IA}_m) \cdot \rho_i(\mathrm{IG}_m)/\rho_i(\mathrm{IA}_m) = \varprojlim \rho_i(\mathrm{IA}_m) \cdot \rho_i(\mathrm{IG}_{m^2})/\rho_i(\mathrm{IA}_m) \\ &= \varprojlim (\mathrm{Im}\, \rho_i \cap \mathrm{IA}_m) \cdot (\mathrm{Im}\, \rho_i \cap \mathrm{IG}_m)/(\mathrm{Im}\, \rho_i \cap \mathrm{IA}_m) \\ &= \varprojlim \mathrm{IA}_m \cdot (\mathrm{Im}\, \rho_i \cap \mathrm{IG}_m)/\, \mathrm{IA}_m = C_i. \end{aligned}$$

The latter equality follows from the inclusion $\mathrm{Im}\, \rho_i \cap \mathrm{IG}_m \subseteq \mathrm{Im}\, \rho_i$. □

Computing $\boldsymbol{C_i}$**.** We turn now to the computation of C_i. We are going to show that the C_i are canonically isomorphic to

$$\ker\bigl(\mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x^{\pm 1}]}) \to \mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x^{\pm 1}]})\bigr)$$

and then use that fact in order to show that $C(\mathrm{IA}(\Phi), \Phi)$ is not finitely generated. So fix $n \ge 4$, $1 \le i_0 \le n$, and let

- $x = x_{i_0}$,
- $\sigma = \sigma_{i_0} = x_{i_0} - 1$,
- $\mathrm{IGL}_{n-1} = \mathrm{IGL}_{n-1,i_0}$,
- $\mathrm{IE}_{n-1}(H) = \mathrm{IE}_{n-1,i_0}(H)$,
- $S = \mathbb{Z}[x^{\pm 1}] = \mathbb{Z}[x_{i_0}^{\pm 1}]$,
- $J_m = (x^m - 1)S + mS$ for $m \in \mathbb{N}$,
- $\rho = \rho_{i_0} : \mathrm{IA}(\Phi) \twoheadrightarrow \mathrm{GL}_{n-1}(S, \sigma S)$,
- $\hat{\rho} = \hat{\rho}_{i_0} : \widehat{\mathrm{IA}(\Phi)} \twoheadrightarrow \mathrm{GL}_{n-1}\widehat{(S, \sigma S)}$.

Now, write

$$\begin{aligned} C_{i_0} &= \ker(\operatorname{Im}\hat{\rho} \to \operatorname{Aut}(\hat{\Phi})) \\ &= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \operatorname{Aut}(\hat{\Phi})\big) \\ &= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IG}_m)\big) \\ &= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim(\mathrm{GL}_{n-1}(S,\sigma S)\cdot \mathrm{IG}_m\,/\,\mathrm{IG}_m)\big) \\ &= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim \mathrm{GL}_{n-1}(S,\sigma S)/(\mathrm{GL}_{n-1}(S,\sigma S)\cap \mathrm{IG}_m)\big) \\ &= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim \mathrm{GL}_{n-1}(S,\sigma S)/\,\mathrm{GL}_{n-1}(S,\sigma J_m)\big) \end{aligned}$$

(the last equality is by Proposition 3.8). Now, by the same computation as in Proposition 3.9 one can show that for every $m \in \mathbb{N}$ we have

$$(J_{m^2} \cap \sigma S) \subseteq \sigma J_m \subseteq (J_m \cap \sigma S),$$

so the latter is equal to

$$\begin{aligned} &\ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim \mathrm{GL}_{n-1}(S,\sigma S)/(\mathrm{GL}_{n-1}(S,\sigma S)\cap \mathrm{GL}_{n-1}(S,J_m))\big) \\ &\quad = \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim(\mathrm{GL}_{n-1}(S,\sigma S)\cdot \mathrm{GL}_{n-1}(S,J_m))/\,\mathrm{GL}_{n-1}(S,J_m)\big) \\ &\quad = \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim \mathrm{GL}_{n-1}(S)/\,\mathrm{GL}_{n-1}(S,J_m)\big) \\ &\quad = \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim \mathrm{GL}_{n-1}(S/J_m)\big). \end{aligned}$$

Now, if $\bar{S}$ is a finite quotient of S, then as x is invertible in S, its image $\bar{x} \in \bar{S}$ is invertible in $\bar{S}$. Thus, there exists $r \in \mathbb{N}$ such that $\bar{x}^r = 1_{\bar{S}}$. In addition, there exists $t \in \mathbb{N}$ such that

$$\underbrace{1_{\bar{S}} + \cdots + 1_{\bar{S}}}_{t} = 0_{\bar{S}}.$$

Therefore, for $m = r\cdot t$ the map $S \to \bar{S}$ factorizes through $\mathbb{Z}_m[\mathbb{Z}_m] \cong S/J_m$. Thus, we have $\hat{S} = \varprojlim(S/J_m)$, which implies that $\mathrm{GL}_{n-1}(\hat{S}) = \varprojlim \mathrm{GL}_{n-1}(S/J_m)$. Therefore,

$$C_{i_0} = \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \mathrm{GL}_{n-1}(\hat{S})\big).$$

Now, the short exact sequence

$$1 \to \mathrm{GL}_{n-1}(S,\sigma S) \to \mathrm{GL}_{n-1}(S) \to \mathrm{GL}_{n-1}(\mathbb{Z}) \to 1$$

gives rise to the exact sequence (see [Bux et al. 2011, Lemma 2.1])

$$\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \widehat{\mathrm{GL}_{n-1}(S)} \to \widehat{\mathrm{GL}_{n-1}(\mathbb{Z})} \to 1,$$

which gives rise to the commutative diagram

$$\begin{array}{ccccccc}
\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} & \longrightarrow & \widehat{\mathrm{GL}_{n-1}(S)} & \longrightarrow & \widehat{\mathrm{GL}_{n-1}(\mathbb{Z})} & \longrightarrow & 1 \\
& \searrow & \downarrow & & \downarrow & & \\
& & \mathrm{GL}_{n-1}(\hat{S}) & \longrightarrow & \mathrm{GL}_{n-1}(\hat{\mathbb{Z}}) & \longrightarrow & 1
\end{array}$$

Assuming that $n \geq 4$ and using the affirmative answer to the classical congruence subgroup problem [Mennicke 1965; Bass et al. 1964], we have that the map $\widehat{\mathrm{GL}_{n-1}(\mathbb{Z})} \to \mathrm{GL}_{n-1}(\hat{\mathbb{Z}})$ is injective. Thus, by diagram chasing we obtain that $\ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \mathrm{GL}_{n-1}(\hat{S})\big)$ is mapped onto $\ker\big(\widehat{\mathrm{GL}_{n-1}(S)} \to \mathrm{GL}_{n-1}(\hat{S})\big)$. In order to proceed from here we need the following lemma.

Lemma 4.4. *Let $d \geq 3$ and $D_m = \{I_d + (x^{k\cdot m} - 1)E_{1,1} \mid k \in \mathbb{Z}\}$ for $m \in \mathbb{N}$. Then*

$$\widehat{\mathrm{GL}_d(S)} = \varprojlim\big(\mathrm{GL}_d(S)/(D_m E_d(S, J_m))\big),$$

$$\widehat{\mathrm{SL}_d(S)} = \varprojlim\big(\mathrm{SL}_d(S)/E_d(S, J_m)\big).$$

Proof. We prove the first statement; the second is similar but easier. We first claim that $D_m E_d(S, J_m)$ is a finite index normal subgroup of $\mathrm{GL}_d(S)$. Indeed, by a well-known result of Suslin [1977], $\mathrm{SL}_d(S) = E_d(S)$. Thus, by Corollary 2.4, $\mathrm{SK}_1(S, J_m; d) = \mathrm{SL}_d(S, J_m)/E_d(S, J_m)$ is finite. As the subgroup $\mathrm{SL}_d(S, J_m)$ is of finite index in $\mathrm{SL}_d(S)$, so is $E_d(S, J_m)$. Now, it is not difficult to see that the group of invertible elements of S is equal to $S^* = \{\pm x^k \mid k \in \mathbb{Z}\}$ (see [Crowell and Fox 1963, Chapter 8]). So as $\{x^{k\cdot m} \mid k \in \mathbb{Z}\}$ is of finite index in S^*, the subgroup $D_m \,\mathrm{SL}_d(S)$ is of finite index in $\mathrm{GL}_d(S)$. We deduce that also $D_m E_d(S, J_m)$ is of finite index in $\mathrm{GL}_d(S)$. It remains to show that $D_m E_d(S, J_m)$ is normal in $\mathrm{GL}_d(S)$.

We already stated previously (see Section 2) that $E_d(S, J_m)$ is normal in $\mathrm{GL}_d(S)$. Thus, noticing the group identity

$$gheg^{-1} = h(h^{-1}ghg^{-1})(geg^{-1}),$$

it is enough to show that the commutators of the elements of D_m with any set of generators of $\mathrm{GL}_d(S)$ are in $E_d(S, J_m)$. By the aforementioned result of Suslin and as $S^* = \{\pm x^r \mid r \in \mathbb{Z}\}$, the group $\mathrm{GL}_d(S)$ is generated by the elements of the forms

1. $I_d + (\pm x - 1)E_{1,1}$,
2. $I_d + rE_{i,j}$, $\quad r \in S,\ 2 \leq i \neq j \leq d$,
3. $I_d + rE_{1,j}$, $\quad r \in S,\ 2 \leq j \leq d$,
4. $I_d + rE_{i,1}$, $\quad r \in S,\ 2 \leq i \leq d$.

Now, obviously, the elements of D_m commute with the elements of the forms 1 and 2. In addition, for the elements of the forms 3 and 4, one can easily compute that

$$\left[I_d+(x^{k\cdot m}-1)E_{1,1},\,I_d+rE_{1,j}\right]=I_d+r(x^{k\cdot m}-1)E_{1,j}\in E_d(S,\,J_m),$$
$$\left[I_d+(x^{k\cdot m}-1)E_{1,1},\,I_d+rE_{i,1}\right]=I_d+r(x^{-k\cdot m}-1)E_{i,1}\in E_d(S,\,J_m)$$

for every $2\le i,\,j\le d$, as required.

Now, clearly, every finite index normal subgroup of $\mathrm{GL}_d(S)$ contains D_m for some $m\in\mathbb{N}$. In addition, it is not hard to show that when $d\ge 3$, every finite index normal subgroup $N\lhd\mathrm{GL}_d(S)$ contains $E_d(S,\,J)$ for some finite index ideal $J\lhd S$; see [Kassabov and Nikolov 2006, Section 1]. Thus, as we saw previously that every finite index ideal $J\lhd S_n$ contains J_m for some m, we obtain that $\widehat{\mathrm{GL}_d(S)}=\varprojlim\big(\mathrm{GL}_d(S)/(D_m E_d(S,\,J_m))\big)$, as required. □

In order to prove the following proposition, we are going to use Lemma 7.1, the proof of which is left to the last section of the paper.

Proposition 4.5. *Let* $n\ge 4$. *Then the map* $\widehat{\mathrm{GL}_{n-1}(S,\sigma S)}\to\widehat{\mathrm{GL}_{n-1}(S)}$ *is injective. Hence, the surjective map*

$$C_{i_0}=\ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)}\to\mathrm{GL}_{n-1}(\hat{S})\big)\twoheadrightarrow\ker\big(\widehat{\mathrm{GL}_{n-1}(S)}\to\mathrm{GL}_{n-1}(\hat{S})\big)$$

is an isomorphism.

Proof. We showed in the previous lemma that

$$\widehat{\mathrm{GL}_{n-1}(S)}=\varprojlim\mathrm{GL}_{n-1}(S)/(D_m E_{n-1}(S,\,J_m)),$$

where $D_m=\{I_{n-1}+(x^{k\cdot m}-1)E_{1,1}\mid k\in\mathbb{Z}\}$ and $J_m=(x^m-1)S+mS$. Hence, the image of $\widehat{\mathrm{GL}_{n-1}(S,\sigma S)}$ in $\widehat{\mathrm{GL}_{n-1}(S)}$ is

$$\begin{aligned}&\varprojlim(\mathrm{GL}_{n-1}(S,\sigma S)\cdot D_m E_{n-1}(S,\,J_m))/(D_m E_{n-1}(S,\,J_m))\\&\qquad=\varprojlim\mathrm{GL}_{n-1}(S,\sigma S)/(\mathrm{GL}_{n-1}(S,\sigma S)\cap D_m E_{n-1}(S,\,J_m)).\end{aligned}$$

Using the fact that $D_m\subseteq\mathrm{GL}_{n-1}(S,\sigma S)$, one can see that the latter equals

$$\varprojlim\mathrm{GL}_{n-1}(S,\sigma S)/\big(D_m(\mathrm{GL}_{n-1}(S,\sigma S)\cap E_{n-1}(S,\,J_m))\big).$$

Recall now the following notation:

- $R_n=\mathbb{Z}[x_1^{\pm1},\ldots,x_n^{\pm1}]$.
- $H_m=\sum_{i=1}^n(x_i^m-1)R_n+mR_n\lhd R_n$.
- $\mathrm{IE}_{n-1}(H_m)=\mathrm{IGL}_{n-1}\cap E_{n-1}(R_n,\,H_m)$ under the identification of IGL_{n-1} with $\mathrm{GL}_{n-1}(R_n,\sigma R_n)$.

Then, following the definition of the map $\rho:\mathrm{IA}(\Phi)\twoheadrightarrow\mathrm{GL}_{n-1}(S,\sigma S)$, we have

$$\langle\mathrm{IA}(\Phi)^m\rangle\xrightarrow{\rho}\!\!\!\!\rightarrow\langle\mathrm{GL}_{n-1}(S,\sigma S)^m\rangle,$$
$$\mathrm{IE}_{n-1}(H_m)\xrightarrow{\rho}\!\!\!\!\rightarrow\mathrm{GL}_{n-1}(S,\sigma S)\cap E_{n-1}(S,\,J_m).$$

So, since by the main lemma (Lemma 7.1) we have $\mathrm{IE}_{n-1}(H_{m^2}) \subseteq \langle \mathrm{IA}(\Phi)^m \rangle$, we have also

$$\mathrm{GL}_{n-1}(S, \sigma S) \cap E_{n-1}(S, J_{m^2}) \subseteq \langle \mathrm{GL}_{n-1}(S, \sigma S)^m \rangle.$$

As obviously $D_{m^2} \subseteq \langle \mathrm{GL}_{n-1}(S, \sigma S)^m \rangle$, we deduce the following natural surjective maps:

$$\begin{aligned}
&\varprojlim \mathrm{GL}_{n-1}(S, \sigma S)/\big(D_m(\mathrm{GL}_{n-1}(S, \sigma S) \cap E_{n-1}(S, J_m))\big) \\
&\qquad = \varprojlim \mathrm{GL}_{n-1}(S, \sigma S)/\big(D_{m^2}(\mathrm{GL}_{n-1}(S, \sigma S) \cap E_{n-1}(S, J_{m^2}))\big) \\
&\qquad\qquad \twoheadrightarrow \varprojlim \mathrm{GL}_{n-1}(S, \sigma S)/\langle \mathrm{GL}_{n-1}(S, \sigma S)^m \rangle \\
&\qquad\qquad \twoheadrightarrow \mathrm{GL}_{n-1}\widehat{(S, \sigma S)} \\
&\qquad\qquad \twoheadrightarrow \varprojlim \mathrm{GL}_{n-1}(S, \sigma S)/\big(D_m(\mathrm{GL}_{n-1}(S, \sigma S) \cap E_{n-1}(S, J_m))\big)
\end{aligned}$$

such that the composition gives the identity map. Hence, these maps are also injective, and in particular, the map

$$\mathrm{GL}_{n-1}\widehat{(S, \sigma S)} \twoheadrightarrow \varprojlim \mathrm{GL}_{n-1}(S, \sigma S)/\big(D_m(\mathrm{GL}_{n-1}(S, \sigma S) \cap E_{n-1}(S, J_m))\big)$$

is injective, as required. □

Proposition 4.6. *Let $d \geq 3$. Then the natural embedding $\mathrm{SL}_d(S) \leq \mathrm{GL}_d(S)$ induces a natural isomorphism*

$$\ker(\widehat{\mathrm{GL}_d(S)} \to \mathrm{GL}_d(\hat{S})) \cong \ker(\widehat{\mathrm{SL}_d(S)} \to \mathrm{SL}_d(\hat{S})).$$

Proof. By Lemma 4.4 we have

$$\begin{aligned}
\ker\big(\widehat{\mathrm{GL}_d(S)} \to \mathrm{GL}_d(\hat{S}) = \varprojlim \mathrm{GL}_d(S/J_m)\big) &= \ker(\varprojlim \mathrm{GL}_d(S)/D_m E_d(S, J_m) \\
&\to \varprojlim \mathrm{GL}_d(S)/\mathrm{GL}_d(S, J_m)) = \varprojlim \mathrm{GL}_d(S, J_m)/D_m E_d(S, J_m),
\end{aligned}$$

where $D_m = \{I_{n-1} + (x^{k\cdot m} - 1)E_{1,1} \mid k \in \mathbb{Z}\}$. We claim now that when $m > 2$ then $\mathrm{GL}_d(S, J_m) = D_m \,\mathrm{SL}_d(S, J_m)$. Indeed, for every $A \in \mathrm{GL}_d(S, J_m)$ we have $\det(A) = \pm x^k$ for some $k \in \mathbb{Z}$. However, as under the map $S \to \mathbb{Z}_m[\mathbb{Z}_m]$ we have $A \mapsto I_d$, the map $S \to \mathbb{Z}_m[\mathbb{Z}_m]$ also implies $\det(A) \mapsto 1$. Hence $\det(A) = \pm x^{k\cdot m}$ for some $k \in \mathbb{Z}$, and when $m > 2$ we even get $\det(A) = x^{k\cdot m}$ for some $k \in \mathbb{Z}$. It follows that $\mathrm{GL}_d(S, J_m) = D_m \,\mathrm{SL}_d(S, J_m)$. Therefore, since $D_m \cap \mathrm{SL}_d(S, J_m) = \{I_d\}$, we deduce that

$$\begin{aligned}
\ker(\widehat{\mathrm{GL}_d(S)} \to \mathrm{GL}_d(\hat{S})) &= \varprojlim D_m \,\mathrm{SL}_d(S, J_m)/D_m E_d(S, J_m) \\
&= \varprojlim \mathrm{SL}_d(S, J_m)/E_d(S, J_m) \\
&= \varprojlim \ker(\widehat{\mathrm{SL}_d(S)} \to \mathrm{SL}_d(\hat{S})).
\end{aligned}$$ □

The immediate corollary from Propositions 4.5 and 4.6 is as follows:

Corollary 4.7. *For every $n \geq 4$, we have $C_{i_0} \cong \ker(\widehat{\mathrm{SL}_{n-1}(S)} \to \mathrm{SL}_{n-1}(\hat{S}))$.*

We close the section by showing that $\ker(\widehat{\mathrm{SL}_{n-1}(S)} \to \mathrm{SL}_{n-1}(\hat{S}))$ is not finitely generated, using the techniques in [Kassabov and Nikolov 2006]. It is known that the group ring $S = \mathbb{Z}[x^{\pm 1}] = \mathbb{Z}[\mathbb{Z}]$ is Noetherian; see [Ivanov 1989; Brown et al. 1981]. In addition, it is known that the Krull dimension of $\mathbb{Z}$ is $\dim(\mathbb{Z}) = 1$ and thus $\dim(S) = \dim(\mathbb{Z}[\mathbb{Z}]) = 2$; see [Smith 1972]. Therefore, by Proposition 1.6 in [Suslin 1977], as $n-1 \geq 3$, for every $J \lhd S$, the canonical map

$$\mathrm{SK}_1(S, J; n-1) \to \mathrm{SK}_1(S, J) := \varinjlim_{d \in \mathbb{N}} \mathrm{SK}_1(S, J; d)$$

is surjective. Hence, the canonical map (when $J \lhd S$ ranges over all finite index ideals of S)

$$\begin{aligned}\ker(\widehat{\mathrm{SL}_{n-1}(S)} \to \mathrm{SL}_{n-1}(\hat{S})) &= \varprojlim(\mathrm{SL}_{n-1}(S, J)/E_{n-1}(S, J)) \\ &= \varprojlim \mathrm{SK}_1(S, J; n-1) \to \varprojlim \mathrm{SK}_1(S, J)\end{aligned}$$

is surjective, so it is enough to show that $\varprojlim \mathrm{SK}_1(S, J)$ is not finitely generated.

By a result of Bass [1968, Chapter 5, Corollary 9.3], for every $J \lhd K \lhd S$ of finite index in S, the map $\mathrm{SK}_1(S, J) \to \mathrm{SK}_1(S, K)$ is surjective. Hence, it is enough to show that for every $l \in \mathbb{N}$ there exists a finite index ideal $J \lhd S$ such that $\mathrm{SK}_1(S, J)$ is generated by at least l elements. Now, as $\mathrm{SK}_1(S) = 1$ [Suslin 1977], we obtain the exact sequence

$$K_2(S) \to K_2(S/J) \to \mathrm{SK}_1(S, J) \to \mathrm{SK}_1(S) = 1$$

for every $J \lhd S$ (see Theorem 6.2 in [Milnor 1971]). In addition, by a classical result of Quillen (see [Quillen 1973; Rosenberg 1994, Theorem 5.3.30]), we have

$$K_2(S) = K_2(\mathbb{Z}[x^{\pm 1}]) = K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z}),$$

so by the classical facts $K_2(\mathbb{Z}) = K_1(\mathbb{Z}) = \{\pm 1\}$ (see [Milnor 1971, Chapters 3 and 10]) we deduce that $K_2(S)$ is of order 4. Hence, it is enough to prove that for every $l \in \mathbb{N}$ there exists a finite index ideal $J \lhd S$ such that $K_2(S/J)$ is generated by at least l elements. Following [Kassabov and Nikolov 2006], we state the following proposition (which holds by the proof of Theorem 2.8 in [Stein and Dennis 1973]):

Proposition 4.8. *Let p be a prime, $l \in \mathbb{N}$ and denote by $P \lhd \mathbb{Z}[y]$ the ideal which is generated by p^2 and y^{p^l}. Then for $\bar{S} = \mathbb{Z}[y]/P$, the group $K_2(\bar{S})$ is an elementary abelian p-group of rank $\geq l$.*

Observe now that for every $l \geq 0$,

$$(y+1)^{p^{l+1}} = (y^{p^l} + 1 + p \cdot a(y))^p = 1 \mod P,$$

so $y+1$ is invertible in $\bar{S}$. Therefore we have a well-defined surjective homomorphism $S \to \bar{S}$ which is defined by sending $x \mapsto y+1$. In particular, $J = \ker(S \to \bar{S})$

is a finite index ideal of S which satisfies the above requirements. This shows that indeed $C_{i_0} = \ker(\widehat{\mathrm{SL}_{n-1}(S)} \to \mathrm{SL}_{n-1}(\hat{S}))$ is not finitely generated, and by the description in Proposition 4.3 it follows that $C(\mathrm{IA}(\Phi), \Phi)$ is not finitely generated either.

5. The centrality of C_i

In this section we prove that for every $n \geq 4$, the copies C_i lie in the center of $\widehat{\mathrm{IA}(\Phi)}$. Throughout the section we assume that $n \geq 4$ is constant, and show it for $i = n$. Symmetrically, it is valid for every i. We recall:

- $R_n = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.
- $H_m = \sum_{i=1}^{n} (x_i^m - 1) R_n + m R_n$.
- $\mathrm{IG}_m = \{I_n + A \in \mathrm{GL}_n(R_n, H_m) \mid A\vec{\sigma} = \vec{0}\}$.
- $\mathrm{IA}_m = \bigcap \{N \lhd \mathrm{IA}(\Phi) \mid [\mathrm{IA}(\Phi) : N] \mid m\}$.
- $S = S_n = \mathbb{Z}[x_n^{\pm 1}]$.
- $\mathrm{Im}\,\rho \cap \mathrm{IG}_m = \mathrm{Im}\,\rho_n \cap \mathrm{IG}_m \simeq \mathrm{GL}_{n-1}(S, \sigma_n H_m \cap S))$ (see Proposition 3.8).

We saw in Section 4 that we can write

$$\begin{aligned} C_n &= \varprojlim (\mathrm{IA}_m \cdot (\mathrm{Im}\,\rho \cap \mathrm{IG}_m)/\,\mathrm{IA}_m) \\ &= \varprojlim (\mathrm{IA}_m \cdot (\mathrm{Im}\,\rho \cap \mathrm{IG}_{m^4})/\,\mathrm{IA}_m) \leq \varprojlim (\mathrm{IA}(\Phi)/\,\mathrm{IA}_m) = \widehat{\mathrm{IA}(\Phi)}. \end{aligned}$$

Hence, if we want to show that C_n lies in the center of $\widehat{\mathrm{IA}(\Phi)}$, it suffices to show that for every $m \in \mathbb{N}$, the group $\mathrm{IA}_m \cdot (\mathrm{Im}\,\rho \cap \mathrm{IG}_{m^4})/\,\mathrm{IA}_m$ lies in the center of $\mathrm{IA}(\Phi)/\,\mathrm{IA}_m$.

We first claim that under the isomorphism $\mathrm{Im}\,\rho \cap \mathrm{IG}_{m^4} \simeq \mathrm{GL}_{n-1}(S, \sigma_n H_{m^4} \cap S)$, one has

$$\mathrm{IA}_m \cdot (\mathrm{Im}\,\rho \cap \mathrm{IG}_{m^4})/\,\mathrm{IA}_m \subseteq \mathrm{IA}_m \cdot \mathrm{SL}_{n-1}(S, \sigma_n H_{m^2} \cap S)/\,\mathrm{IA}_m\,. \tag{5.1}$$

Indeed, if $\alpha \in \mathrm{Im}\,\rho \cap \mathrm{IG}_{m^4}$ then $\det(\alpha) \in 1 + \sigma_n H_{m^4} \cap S \subseteq 1 + H_{m^4} \cap S$. Combining it with Proposition 3.2, $\det(\alpha)$ has the form $\det(\alpha) = x_n^{m^4 t}$ for some $t \in \mathbb{Z}$. Hence

$$\det\bigl((I_n + \sigma_n E_{1,1} - \sigma_1 E_{1,n})^{-m^4 t} \cdot \alpha\bigr) = 1.$$

Now, as we have

$$\begin{aligned} x_n^{m^4 t} &= 1 + (x_n^{m^4 t} - 1) = 1 + \sigma_n \sum_{i=1}^{m^4 - 1} (x_n^t)^i \\ &\in 1 + \sigma_n\bigl((x_n^{m^2 t} - 1)S + m^2 S\bigr) \subseteq 1 + \sigma_n H_{m^2} \cap S \end{aligned}$$

(see the computation in the proof of Proposition 3.9), we obtain that

$$(I_n+\sigma_n E_{1,1}-\sigma_1 E_{1,n})^{m^4t} \in \langle \mathrm{IA}(\Phi)^m\rangle \cap \mathrm{GL}_{n-1}(S,\sigma_n H_{m^2}\cap S) \\ \subseteq \mathrm{IA}_m \cap \mathrm{GL}_{n-1}(S,\sigma_n H_{m^2}\cap S).$$

Therefore, writing $\alpha=(I_n+\sigma_n E_{1,1}-\sigma_1 E_{1,n})^{m^4t}\cdot\big((I_n+\sigma_n E_{1,1}-\sigma_1 E_{1,n})^{-m^4t}\cdot\alpha\big)$, we deduce that

$$\mathrm{Im}\,\rho\cap \mathrm{IG}_{m^4}\subseteq \mathrm{IA}_m\cdot \mathrm{SL}_{n-1}(S,\sigma_n H_{m^2}\cap S)$$

and we get the inclusion (5.1). It follows that if we want to show that C_n lies in the center of $\widehat{\mathrm{IA}(\Phi)}$, it suffices to show that $\mathrm{IA}_m\cdot \mathrm{SL}_{n-1}(S,\sigma_n H_{m^2}\cap S)/\,\mathrm{IA}_m$ lies in the center of $\mathrm{IA}(\Phi)/\,\mathrm{IA}_m$. However, we are going to show even more:

Proposition 5.2. *For every $m\in\mathbb{N}$, the group*

$$\mathrm{IA}_m\cdot \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})/\,\mathrm{IA}_m$$

lies in the center of $\mathrm{IA}(\Phi)/\,\mathrm{IA}_m$.

Let F be the free group on $f_1,\dots,f_n$. It is a classical result by Magnus (see [Magnus et al. 1966, Chapter 3, Theorem N4]) that $\mathrm{IA}(F)$ is generated by the automorphisms of the form

$$\alpha_{r,s,t}=\begin{cases} f_r\mapsto [f_t,f_s]f_r, & \\ f_u\mapsto f_u, & u\neq r,\end{cases}$$

where $[f_t,f_s]=f_tf_sf_t^{-1}f_s^{-1}$ and $1\le r,s\neq t\le n$ (notice that we may have $r=s$). Bachmuth and Mochizuki [1985] show that when $n\ge 4$, the group $\mathrm{IA}(\Phi)$ is generated by the images of these generators under the natural map $\mathrm{Aut}(F)\to\mathrm{Aut}(\Phi)$, i.e., $\mathrm{IA}(\Phi)$ is generated by the elements of the form

$$E_{r,s,t}=I_n+\sigma_t E_{r,s}-\sigma_s E_{r,t},\quad 1\le r,s\neq t\le n.$$

Therefore, to show the centrality of C_n, it is enough to show that given

- an element $\bar\lambda\in \mathrm{IA}_m\cdot\mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})/\,\mathrm{IA}_m$,
- and one of the generators $E_{r,s,t}=I_n+\sigma_tE_{r,s}-\sigma_sE_{r,t}$ for $1\le r,s\neq t\le n$,

there exists $\lambda\in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})$, a representative of $\bar\lambda$, such that $[E_{r,s,t},\lambda]\in\mathrm{IA}_m$. So, assume that we have an element $\bar\lambda\in \mathrm{IA}_m\cdot\mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})/\,\mathrm{IA}_m$. Then a representative for $\bar\lambda$ has the form

$$\lambda=\begin{pmatrix} I_{n-1}+\sigma_n B & -\sum_{i=1}^{n-1}\sigma_i\vec{b}_i\\ 0 & 1\end{pmatrix}\in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})$$

for some $(n-1)\times(n-1)$ matrix B with entries $b_{i,j}\in H_{m^2}$, and with column vectors denoted by $\vec{b}_i$.

Lemma 5.3. *Let* $\bar{\lambda} \in \mathrm{IA}_m \cdot \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})/\mathrm{IA}_m$. *Then, for every* $1 \le l < k \le n-1$, $\bar{\lambda}$ *has a representative in* $\mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})$ *of the following form*:

$$\begin{pmatrix} I_{l-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+\sigma_n a & 0 & \sigma_n b & 0 & -\sigma_l a - \sigma_k b \\ 0 & 0 & I_{k-l-1} & 0 & 0 & 0 \\ 0 & \sigma_n c & 0 & 1+\sigma_n d & 0 & -\sigma_l c - \sigma_k d \\ 0 & 0 & 0 & 0 & I_{n-k-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \leftarrow l\text{-th row} \\ \\ \leftarrow k\text{-th row} \\ \\ \\ \end{matrix} \tag{5.4}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ l\text{-th column} & k\text{-th column} & n\text{-th column} \end{matrix}$$

for some $a, b, c, d \in H_{m^2}$. (*The above notation means that the matrix is similar to the identity matrix, except for the entries in the* l*-th and* k*-th rows.*)

Proof. We demonstrate the proof in the case $l = 1$, $k = 2$, and symmetrically, the arguments hold for arbitrary $1 \le l < k \le n-1$. Consider an arbitrary representative of $\bar{\lambda}$,

$$\lambda = \begin{pmatrix} I_{n-1} + \sigma_n B & -\sum_{i=1}^{n-1} \sigma_i \vec{b}_i \\ 0 & 1 \end{pmatrix} \in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2}).$$

Then $I_{n-1} + \sigma_n B \in \mathrm{SL}_{n-1}(R_n, \sigma_n H_{m^2})$. Consider now the ideal

$$R_n \rhd H'_{m^2} = \sum_{r=1}^{n-1} (x_r^{m^2} - 1) R_n + \sigma_n (x_n^{m^2} - 1) R_n + m^2 R_n.$$

Observe that $\sigma_n H_{m^2} \lhd H'_{m^2} \lhd H_{m^2} \lhd R_n$ and that $H'_{m^2} \cap \sigma_n R_n = \sigma_n H_{m^2}$. In addition, by similar computations as in the proof of Proposition 3.9, for every $x \in R_n$ we have $x^{m^4} - 1 \in (x-1)(x^{m^2} - 1) R_n + (x-1) m^2 R_n$, and thus $H_{m^4} \subseteq H'_{m^2}$, so H'_{m^2} is of finite index in R_n.

Now, $I_{n-1} + \sigma_n B \in \mathrm{SL}_{n-1}(R_n, \sigma_n H_{m^2}) \subseteq \mathrm{SL}_{n-1}(R_n, H'_{m^2})$. Thus, by the third part of Corollary 2.4, as $H'_{m^2} \lhd R_n$ is an ideal of finite index, $n - 1 \ge 3$ and $E_{n-1}(R_n) = \mathrm{SL}_{n-1}(R_n)$ [Suslin 1977], one can write the matrix $I_{n-1} + \sigma_n B$ as

$$I_{n-1} + \sigma_n B = AD \quad \text{when } A = \begin{pmatrix} A' & 0 \\ 0 & I_{n-3} \end{pmatrix}$$

for some $A' \in \mathrm{SL}_2(R_n, H'_{m^2})$ and $D \in E_{n-1}(R_n, H'_{m^2})$. Now consider the images of D and A under the projection $\sigma_n \to 0$, which we denote by $\bar{D}$ and $\bar{A}$. Observe that obviously, $\bar{D} \in E_{n-1}(R_n, H'_{m^2})$. In addition, observe that

$$AD \in \mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \quad \Rightarrow \quad \bar{A}\bar{D} = I_{n-1}.$$

Thus, we have $I_{n-1} + \sigma_n B = A \bar{A}^{-1} \bar{D}^{-1} D$. Therefore, by replacing D by $\bar{D}^{-1} D$

and A by $A\bar{A}^{-1}$ we can assume that

$$I_{n-1}+\sigma_n B = AD \quad \text{for } A=\begin{pmatrix} A' & 0 \\ 0 & I_{n-3}\end{pmatrix},$$

where $A'\in \mathrm{SL}_2(R_n, H'_{m^2})\cap \mathrm{GL}_2(R_n,\sigma_n R_n)=\mathrm{SL}_2(R_n,\sigma_n H_{m^2})$, and

$$D\in E_{n-1}(R_n, H'_{m^2})\cap \mathrm{GL}_{n-1}(R_n,\sigma_n R_n) \subseteq E_{n-1}(R_n, H_{m^2})\cap \mathrm{GL}_{n-1}(R_n,\sigma_n R_n):=\mathrm{IE}_{n-1,n}(H_{m^2}).$$

Now, as we prove in the main lemma (Lemma 7.1) that

$$\mathrm{IE}_{n-1,n}(H_{m^2})\subseteq \langle \mathrm{IA}(\Phi)^m\rangle \subseteq \mathrm{IA}_m,$$

this argument shows that λ can be replaced by a representative of the form (5.4). □

We now return to our initial mission. Let $\bar{\lambda}\in \mathrm{IA}_m\cdot \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})/\,\mathrm{IA}_m$, and let $E_{r,s,t}=I_n+\sigma_t E_{r,s}-\sigma_s E_{r,t}$ for $1\le r, s\neq t\le n$ be one of the above generators for $\mathrm{IA}(\Phi)$. We want to show that there exists $\lambda\in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})$, a representative of $\bar{\lambda}$, such that $[E_{r,s,t},\lambda]\in \mathrm{IA}_m$. We separate the treatment to two cases. We note that Lemma 5.3 is needed only for the second case, which is a bit more delicate.

First Case: $1\le r\le n-1$.

In this case one can take an arbitrary representative $\lambda\in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})\cong \mathrm{SL}_{n-1}(R_n,\sigma_n H_{m^2})$. Considering the embedding of $\mathrm{IGL}'_{n-1,n}$ in $\mathrm{GL}_{n-1}(R_n)$, we have $E_{r,s,t}\in \mathrm{IGL}'_{n-1,n}\subseteq \mathrm{GL}_{n-1}(R_n)$ (see Definition 3.12 and Proposition 3.13). Thus, since by Corollary 2.4

$$\mathrm{SK}_1(R_n, H_{m^2}; n-1)=\mathrm{SL}_{n-1}(R_n, H_{m^2})/E_{n-1}(R_n, H_{m^2})$$

is central in $\mathrm{GL}_{n-1}(R_n)/E_{n-1}(R_n, H_{m^2})$, we have

$$[E_{r,s,t},\lambda]\in[\mathrm{GL}_{n-1}(R_n), \mathrm{SL}_{n-1}(R_n,\sigma_n H_{m^2})]\subseteq E_{n-1}(R_n, H_{m^2}).$$

In addition, as $\mathrm{SL}_{n-1}(R_n,\sigma_n H_{m^2})\le \mathrm{GL}_{n-1}(R_n,\sigma_n R_n)$ and $\mathrm{GL}_{n-1}(R_n,\sigma_n R_n)$ is normal in $\mathrm{GL}_{n-1}(R_n)$, we have

$$[E_{r,s,t},\lambda]\in[\mathrm{GL}_{n-1}(R_n), \mathrm{GL}_{n-1}(R_n,\sigma_n R_n)]\subseteq \mathrm{GL}_{n-1}(R_n,\sigma_n R_n).$$

Thus, we obtain from Lemma 7.1 that

$$[E_{r,s,t},\lambda]\in E_{n-1}(R_n, H_{m^2})\cap \mathrm{GL}_{n-1}(R_n,\sigma_n R_n) = \mathrm{IE}_{n-1,n}(H_{m^2})\subseteq \langle \mathrm{IA}(\Phi)^m\rangle\subseteq \mathrm{IA}_m\,.$$

Second Case: $r=n$.

This case is a bit more complicated than the previous one, as $E_{r,s,t}$ is not in $\mathrm{IGL}'_{n-1,n}$. Here, by Lemma 5.3 one can choose $\lambda\in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})$ whose t-th row equals the standard vector $\vec{e}_t$. As $t\neq r=n$, we thus obtain that both λ

and $E_{r,s,t}$ are in $\mathrm{IGL}'_{n-1,t}$. Considering the embedding $\mathrm{IGL}'_{n-1,t} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$, we have $E_{r,s,t} \in \mathrm{GL}_{n-1}(R_n, \sigma_t R_n)$. In addition, remember that λ has the form

$$\lambda = \begin{pmatrix} I_{n-1} + \sigma_n B & -\sum_{i=1}^{n-1} \sigma_i \vec{b}_i \\ 0 & 1 \end{pmatrix}$$

for $I_{n-1} + \sigma_n B \in \mathrm{SL}_{n-1}(R_n, \sigma_n H_{m^2})$, so that the entries of $\vec{b}_i$ are in H_{m^2}. It follows that regarding the embedding $\mathrm{IGL}'_{n-1,t} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$, we have $\lambda \in \mathrm{SL}_{n-1}(R_n, H_{m^2})$.

Remark 5.5. Note that when considering $\lambda \in \mathrm{IGL}'_{n-1,n} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$, i.e., when considering $\lambda \in \mathrm{GL}_{n-1}(R_n)$ through the embedding of $\mathrm{IGL}'_{n-1,n}$ in $\mathrm{GL}_{n-1}(R_n)$, we have $\lambda \in \mathrm{GL}_{n-1}(R_n, \sigma_n H_{m^2}) \leq \mathrm{GL}_{n-1}(R_n)$. However, when we consider $\lambda \in \mathrm{IGL}'_{n-1,t} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$, we do not necessarily have $\lambda \in \mathrm{GL}_{n-1}(R_n, \sigma_n H_{m^2})$, but we still have $\lambda \in \mathrm{GL}_{n-1}(R_n, H_{m^2})$.

Thus, by similar arguments as in the first case,

$$\begin{aligned}[E_{r,s,t}, \lambda] \in [\mathrm{GL}_{n-1}(R_n, \sigma_t R_n), \mathrm{SL}_{n-1}(R_n, H_{m^2})]& \\ \subseteq E_{n-1}(R_n, H_{m^2}) \cap \mathrm{GL}_{n-1}(R_n, \sigma_t R_n)& \\ = \mathrm{IE}_{n-1,t}(H_{m^2}) \subseteq \langle \mathrm{IA}(\Phi)^m \rangle \subseteq \mathrm{IA}_m\,.&\end{aligned}$$

This finishes the argument which shows that the C_i are central in $\widehat{\mathrm{IA}(\Phi)}$.

Remark 5.6. One can follow and see that completely similar arguments give that the group

$$\langle \mathrm{IA}(\Phi)^m \rangle \cdot \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2}) / \langle \mathrm{IA}(\Phi)^m \rangle$$

lies in the center of $\mathrm{IA}(\Phi)/\langle \mathrm{IA}(\Phi)^m \rangle$. The reason is that the only property of IA_m that we used here was that $\langle \mathrm{IA}(\Phi)^m \rangle \subseteq \mathrm{IA}_m$. This claim is used in [Ben-Ezra 2017] to prove Theorem 1.3. We note that in this paper we were careful not to use the subgroups $\langle \mathrm{IA}(\Phi)^m \rangle$ directly as we still didn't show that they are of finite index in $\mathrm{IA}(\Phi)$, and therefore we cannot write $\widehat{\mathrm{IA}(\Phi)} = \varprojlim (\mathrm{IA}(\Phi)/\langle \mathrm{IA}(\Phi)^m \rangle)$. However, on the way to proving Theorem 1.3, we do show that the $\langle \mathrm{IA}(\Phi)^m \rangle$ are of finite index in $\mathrm{IA}(\Phi)$ (provided $n \geq 4$).

6. Some elementary elements of $\langle \mathrm{IA}(\Phi_n)^m \rangle$

In this section we introduce some elements in $\langle \mathrm{IA}(\Phi_n)^m \rangle$ which are needed for the proof of Lemma 7.1. In [Ben-Ezra 2017] we introduce a list of elements in $\langle \mathrm{IA}(\Phi_n)^m \rangle$ that contains the list below (see Propositions 4.1 and 4.2 therein). However, we do not need the whole list of [Ben-Ezra 2017] here, and also do not need all the notation that is used there. Hence, for the convenience of the reader we include here only the list that is needed for the proof of Lemma 7.1, and repeat the arguments that are related to this shorter list.

Proposition 6.1. *Let $n \geq 4$, $1 \leq u \leq n$ and $m \in \mathbb{N}$. Denote by $\vec{e}_i$ the i-th standard row vector. Then the elements of $\mathrm{IA}(\Phi_n)$ of the following form lie in $\langle \mathrm{IA}(\Phi_n)^m \rangle$:*

$$\begin{pmatrix} & I_{u-1} & & 0 & & 0 & \\ a_{u,1} & \cdots & a_{u,u-1} & 1 & a_{u,u+1} & \cdots & a_{u,n} \\ & 0 & & 0 & & I_{n-u} & \end{pmatrix} \leftarrow u\text{-th row} \tag{6.2}$$

when $(a_{u,1}, \ldots, a_{u,u-1}, 0, a_{u,u+1}, \ldots, a_{u,n})$ is a linear combination of the vectors

1. $\left\{ m(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \mid i, j \neq u,\ i \neq j \right\}$,
2. $\left\{ (x_k^m - 1)(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \mid i, j, k \neq u,\ i \neq j \right\}$,
3. $\left\{ \sigma_u (x_u^m - 1)(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \mid i, j \neq u,\ i \neq j \right\}$,

with coefficients in R_n. The notation in (6.2) means that the matrix is similar to the identity matrix, except the entries in the u-th row.

Proof. Without loss of generality, we assume that $u = 1$. Observe now that for every $a_i, b_i \in R_n$ for $2 \leq i \leq n$ one has

$$\begin{pmatrix} 1 & a_2 & \cdots & a_n \\ 0 & & I_{n-1} & \end{pmatrix} \begin{pmatrix} 1 & b_2 & \cdots & b_n \\ 0 & & I_{n-1} & \end{pmatrix} = \begin{pmatrix} 1 & a_2 + b_2 & \cdots & a_n + b_n \\ 0 & & I_{n-1} & \end{pmatrix}.$$

Hence, it is enough to prove that the elements of the following forms belong to $\langle \mathrm{IA}(\Phi_n)^m \rangle$ (when we write $a\vec{e}_i$ we mean that the entry of the i-th column in the first row is a):

1. $\begin{pmatrix} 1 & mf(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}$, $\quad i, j \neq 1,\ i \neq j,\ f \in R_n$,
2. $\begin{pmatrix} 1 & (x_k^m - 1) f(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}$, $\quad i, j, k \neq 1,\ i \neq j,\ f \in R_n$,
3. $\begin{pmatrix} 1 & \sigma_1 (x_1^m - 1) f(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}$, $\quad i, j \neq 1,\ i \neq j,\ f \in R_n$.

We start with the elements of Form 1. Here we have

$$\begin{pmatrix} 1 & mf(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & f(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}^m \in \langle \mathrm{IA}(\Phi_n)^m \rangle.$$

We pass to the elements of Form 2. In this case we have

$$\begin{aligned} \langle \mathrm{IA}(\Phi_n)^m \rangle \ni & \left[\begin{pmatrix} 1 & f(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}^{-1}, \begin{pmatrix} x_k & -\sigma_1 \vec{e}_k \\ 0 & I_{n-1} \end{pmatrix}^m \right] \\ & = \begin{pmatrix} 1 & (x_k^m - 1) f(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}. \end{aligned}$$

We finish with the elements of Form 3. The computation here is more complicated than in the previous cases, so we demonstrate it for the special case $n=4$, $i=2$, $j=3$. It is clear that symmetrically, with similar arguments, the same holds in general when $n \geq 4$ for every $i, j \neq 1$, $i \neq j$. By similar arguments as in the previous case we get

$$\langle \mathrm{IA}(\Phi_4)^m \rangle \ni \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \sigma_3(x_1^m - 1)f & -\sigma_2(x_1^m - 1)f & 1 \end{pmatrix}.$$

Therefore, we also have

$$\langle \mathrm{IA}(\Phi_4)^m \rangle \ni \left[\begin{pmatrix} x_4 & 0 & 0 & -\sigma_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \sigma_3(x_1^m - 1)f & -\sigma_2(x_1^m - 1)f & 1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 & -\sigma_3\sigma_1(x_1^m - 1)f & \sigma_2\sigma_1(x_1^m - 1)f & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad \square$$

7. The main lemma

We recall and present some new notation that is used in this section:

- $\mathrm{IA}^m = \langle \mathrm{IA}(\Phi)^m \rangle$, where $\Phi = \Phi_n$.
- $R_n = \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, where $x_1, \dots, x_n$ are the generators of $\mathbb{Z}^n$.
- $\sigma_r = x_r - 1$ for $1 \leq r \leq n$.
- $U_{r,m} = (x_r^m - 1)R_n$ for $1 \leq r \leq n$ and $m \in \mathbb{N}$.
- $O_m = mR_n$.
- $H_m = \sum_{r=1}^n (x_r^m - 1)R_n + mR_n = \sum_{r=1}^n U_{r,m} + O_m$.
- $\mathrm{IE}_{n-1,i}(H) = \mathrm{IGL}_{n-1,i} \cap E_{n-1}(R_n, H) \leq \mathrm{ISL}_{n-1,i}(H)$ for $H \lhd R_n$ under the identification of $\mathrm{IGL}_{n-1,i} \leq \mathrm{IA}(\Phi)$ with $\mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$ (see Proposition 3.4 and Definition 3.6).

In this section, we prove the following main lemma:

Lemma 7.1. *For every $n \geq 4$, $m \in \mathbb{N}$ and $1 \leq i \leq n$, we have*

$$\mathrm{IE}_{n-1,i}(H_{m^2}) \subseteq \mathrm{IA}^m.$$

To simplify the proof and the notation, we prove the lemma for the special case $i = n$, and symmetrically, all the arguments are valid for every $1 \leq i \leq n$.

In addition, using the identification $\mathrm{IGL}_{n-1,n} \cong \mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$, we identify $\mathrm{IGL}_{n-1,n}$ with $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$, and the group $\mathrm{IE}_{n-1,n}(H_m)$ with the group $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, H_m)$. So the goal of this section is proving that

$$\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, H_{m^2}) \subseteq \mathrm{IA}^m .$$

Throughout the proof we use also elements of $\mathrm{IGL}'_{n-1,n}$ (see Definition 3.12). We recall that

$$\mathrm{IE}_{n-1,n}(H_m) \le \mathrm{IGL}_{n-1,n} \le \mathrm{IGL}'_{n-1,n} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$$

(Proposition 3.13), so all the elements that are being used throughout the section are naturally embedded in $\mathrm{GL}_{n-1}(R_n)$. Using this embedding, we do all the computations in $\mathrm{GL}_{n-1}(R_n)$, and make the notation simpler by omitting the n-th row and column from each matrix.

We note that many ideas in the proof of Lemma 7.1 below are based on ideas of the proof of the "main lemma" in [Bachmuth and Mochizuki 1985] (see Section 4 therein). However, our arguments do not rely directly on the arguments in [Bachmuth and Mochizuki 1985], so on the whole we cannot make a formal reference to that work throughout the proof of Lemma 7.1.

Decomposing the proof. In this subsection we start the proof of Lemma 7.1. At the end of the subsection, there will be a few tasks left, which will be accomplished in the forthcoming subsections. We start with the following definition:

Definition 7.2. For every $m \in \mathbb{N}$, define the following ideal of R_n:

$$T_m = \sum_{r=1}^{n} \sigma_r^2 U_{r,m} + \sum_{r=1}^{n} \sigma_r O_m + O_m^2 .$$

Observe that as for every $x \in R_n$ we have $\sum_{j=0}^{m-1} x^j \in (x-1)R_n + mR_n$, one has

$$\begin{aligned} x^{m^2} - 1 = (x-1)\sum_{j=0}^{m^2-1} x^j &= (x-1)\sum_{j=0}^{m-1} x^j \sum_{j=0}^{m-1} x^{jm} \\ &\in (x-1)((x-1)R_n + mR_n)((x^m-1)R_n + mR_n) \\ &\subseteq (x-1)^2(x^m-1)R_n + (x-1)^2 mR_n + (x-1)m^2 R_n . \end{aligned}$$

It follows that $H_{m^2} \subseteq T_m$. Hence, it is enough to prove that

$$\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m) \subseteq \mathrm{IA}^m .$$

Equivalently, it is enough to prove that the group

$$(\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m)) \cdot \mathrm{IA}^m \,/\, \mathrm{IA}^m$$

is trivial. We continue with the following proposition, which is actually a proposition of Suslin [1977, Corollary 1.4] with some elaborations of [Bachmuth and

Mochizuki 1985] (see the remark that follows their Proposition 3.5 and the beginning of the proof of the "main lemma" in Section 4 therein).

Proposition 7.3. *Let R be a commutative ring, $d \geq 3$ and $H \lhd R$ an ideal. Then $E_d(R, H)$ is generated by the matrices of the form*

$$(I_d - f E_{i,j})(I_d + h E_{j,i})(I_d + f E_{i,j}) \tag{7.4}$$

for $h \in H$, $f \in R$ and $1 \leq i \neq j \leq d$.

Proof. In the proof of Corollary 1.4 in [Suslin 1977], Suslin shows that whenever $d \geq 3$, $E_d(R, H)$ is generated by the elements of the form

$$I_d + h\vec{u}^t(u_j\vec{e}_i - u_i\vec{e}_j),$$

where $h \in H$, $i \neq j$ and $\vec{u} = (u_1, u_2, \ldots, u_d) \in R^n$ such that $\vec{u} \cdot \vec{v}^t = 1$ for some $\vec{v} \in R^n$. In the remark which follows Proposition 3.5 in [Bachmuth and Mochizuki 1985], it is observed that

$$\begin{aligned} I_d + h\vec{u}^t(u_j\vec{e}_i - u_i\vec{e}_j) &= (I_d + h(u_i\vec{e}_i + u_j\vec{e}_j)^t(u_j\vec{e}_i - u_i\vec{e}_j)) \\ &\quad \cdot \prod_{l \neq i,j} (I_d + h(u_l\vec{e}_l)^t u_j\vec{e}_i) \cdot \prod_{l \neq i,j} (I_d - h(u_l\vec{e}_l)^t u_i\vec{e}_j). \end{aligned}$$

Hence, by observing that all the factors in the above expression are all of the form

$$I_d + h(f_1\vec{e}_i + f_2\vec{e}_j)^t(f_2\vec{e}_i - f_1\vec{e}_j) \tag{7.5}$$

for some $f_1, f_2 \in R$, $h \in H$ and $1 \leq i \neq j \leq d$, it is enough to show that the matrices of the form (7.5) are generated by the matrices of the form (7.4). We show it for the case $i, j, d = 1, 2, 3$, and it will be clear that the general argument is similar. So we have the matrix

$$I_d + h(f_1\vec{e}_1 + f_2\vec{e}_2)^t(f_2\vec{e}_1 - f_1\vec{e}_2) = \begin{pmatrix} 1 + hf_1f_2 & -hf_1^2 & 0 \\ hf_2^2 & 1 - hf_1f_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $f_1, f_2 \in R$ and $h \in H$, which is equal to

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & -hf_1 \\ 0 & 1 & -hf_2 \\ f_2 & -f_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & hf_1 \\ 0 & 1 & hf_2 \\ -f_2 & f_1 & 1 \end{pmatrix} \\ &\qquad = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_2 & -f_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -hf_1 \\ 0 & 1 & -hf_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -f_2 & f_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & hf_1 \\ 0 & 1 & hf_2 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

As the matrix

$$\begin{pmatrix} 1 & 0 & hf_1 \\ 0 & 1 & hf_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & hf_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & hf_2 \\ 0 & 0 & 1 \end{pmatrix}$$

is generated by the matrices of the form (7.4), it remains to show that

$$\begin{pmatrix}1&0&0\\0&1&0\\f_2&-f_1&1\end{pmatrix}\begin{pmatrix}1&0&-hf_1\\0&1&-hf_2\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&f_1&1\end{pmatrix}$$
$$=\begin{pmatrix}1&0&0\\0&1&0\\f_2&-f_1&1\end{pmatrix}\begin{pmatrix}1&0&-hf_1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&f_1&1\end{pmatrix}$$
$$\cdot\begin{pmatrix}1&0&0\\0&1&0\\f_2&-f_1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-hf_2\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&f_1&1\end{pmatrix}$$

is generated by the matrices of the form (7.4). Now

$$\begin{pmatrix}1&0&0\\0&1&0\\f_2&-f_1&1\end{pmatrix}\begin{pmatrix}1&0&-hf_1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&f_1&1\end{pmatrix}$$
$$=\begin{pmatrix}1&0&0\\0&1&0\\0&-hf_1^2f_2&1\end{pmatrix}\begin{pmatrix}1&-hf_1^2&0\\0&1&0\\0&0&1\end{pmatrix}$$
$$\cdot\left[\begin{pmatrix}1&0&0\\0&1&0\\f_2&0&1\end{pmatrix}\begin{pmatrix}1&0&-hf_1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&0&1\end{pmatrix}\right]$$

is generated by the matrices of the form (7.4), and by a similar computation

$$\begin{pmatrix}1&0&0\\0&1&0\\f_2&-f_1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-hf_2\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&f_1&1\end{pmatrix}$$

is generated by these matrices as well. □

We proceed with the following lemma. Some of the ideas in its proof are based on the proof of Proposition 3.5 in [Bachmuth and Mochizuki 1985].

Lemma 7.6. *Let $n \geq 4$. Recall $U_{r,m} = (x_r^m - 1)R_n$, $O_m = mR_n$, and denote the corresponding ideals of $R_{n-1} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}] \subseteq R_n$ by*

$$\bar{O}_m = mR_{n-1} \subseteq O_m, \qquad \bar{U}_{r,m} = (x_r^m - 1)R_{n-1} \subseteq U_{r,m} \quad \text{for } 1 \leq r \leq n-1.$$

Then every element of $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m)$ can be decomposed as a product of elements of the following four forms:

1. $A^{-1}(I_{n-1}+hE_{i,j})A,$ $\qquad h \in \sigma_n O_m,$
2. $A^{-1}(I_{n-1}+hE_{i,j})A,$ $\qquad h \in \sigma_n^2 U_{n,m}$ *or* $h \in \sigma_n \sigma_r^2 U_{r,m}$ *for* $1 \le r \le n-1,$
3. $A^{-1}\big[(I_{n-1}+hE_{i,j}),(I_{n-1}+fE_{j,i})\big]A,$ $\qquad h \in \bar{O}_m^2,\ f \in \sigma_n R_n,$
4. $A^{-1}\big[(I_{n-1}+hE_{i,j}),(I_{n-1}+fE_{j,i})\big]A,$ $\qquad h \in \sigma_r^2 \bar{U}_{r,m}$ *or* $h \in \sigma_r \bar{O}_m$ *for* $1 \le r \le n-1,\ f \in \sigma_n R_n,$

where $A \in \mathrm{GL}_{n-1}(R_n)$ *and* $i \neq j$.

Remark 7.7. Notice that as $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$ is normal in $\mathrm{GL}_{n-1}(R_n)$, every element of the above forms is an element of $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cong \mathrm{IGL}_{n-1,n} \le \mathrm{IA}(\Phi)$.

Proof of Lemma 7.6. Let $B \in \mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m)$. We first claim that to prove the lemma, it is enough to show that B can be decomposed as a product of the elements in the lemma, and arbitrary elements in $\mathrm{GL}_{n-1}(R_{n-1})$. Indeed, assume that we can write $B = A_1 D_1 \cdots A_n D_n$ for some D_i of the forms in the lemma and $A_i \in \mathrm{GL}_{n-1}(R_{n-1})$ (notice that A_1 or D_n might be equal to I_{n-1}). Observe now that we can therefore write

$$B = A_1 D_1 A_1^{-1} \cdots (A_1 \cdots A_n) D_n (A_1 \cdots A_n)^{-1} (A_1 \cdots A_n),$$

and by definition, the conjugations of the D_i can also be considered to be in the forms in the lemma. On the other hand, we have

$$(A_1 \cdots A_n) D_n^{-1} (A_1 \cdots A_n)^{-1} \cdots A_1 D_1^{-1} A_1^{-1} B = A_1 \cdots A_n$$

and as the matrices of the forms in the lemma are all in $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$ (by Remark 7.7), we deduce that

$$A_1 \cdots A_n \in \mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap \mathrm{GL}_{n-1}(R_{n-1}) = \{I_{n-1}\},$$

i.e., $A_1 \cdots A_n = I_{n-1}$. Hence

$$B = A_1 D_1 A_1^{-1} \cdots (A_1 \cdots A_n) D_n (A_1 \cdots A_n)^{-1},$$

i.e., B is a product of matrices of the forms in the lemma, as required.

So let $B \in \mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m)$. According to Proposition 7.3, as $B \in E_{n-1}(R_n, T_m)$ and $n-1 \ge 3$, we can write B as a product of elements of the form

$$(I_{n-1} - fE_{i,j})(I_{n-1} + hE_{j,i})(I_{n-1} + fE_{i,j})$$

for some $f \in R_n$, $h \in T_m = \sum_{r=1}^n \sigma_r^2 U_{r,m} + \sum_{r=1}^n \sigma_r O_m + O_m^2$ and $1 \le i \neq j \le n-1$. We show now that every element of the above form can be written as a product of the elements of the forms in the lemma and elements of $\mathrm{GL}_{n-1}(R_{n-1})$.

So let $h \in T$ and $f \in R_n$. Observe first that by division by σ_n (with residue), one has

$$T_m = \sum_{r=1}^{n} \sigma_r^2 U_{r,m} + \sum_{r=1}^{n} \sigma_r O_m + O_m^2$$
$$\subseteq \sigma_n \left(\sum_{r=1}^{n-1} \sigma_r^2 U_{r,m} + \sigma_n U_{n,m} + O_m \right) + \sum_{r=1}^{n-1} \sigma_r^2 \overline{U}_{r,m} + \sum_{r=1}^{n-1} \sigma_r \overline{O}_m + \overline{O}_m^2.$$

Hence, we can decompose $h = \sigma_n h_1 + h_2$ for some $h_1 \in \sum_{r=1}^{n-1} \sigma_r^2 U_{r,m} + \sigma_n U_{n,m} + O_m$ and $h_2 \in \sum_{r=1}^{n-1} \sigma_r^2 \overline{U}_{r,m} + \sum_{r=1}^{n-1} \sigma_r \overline{O}_m + \overline{O}_m^2$. Therefore, we can write

$$(I_{n-1} - f E_{i,j})(I_{n-1} + h E_{j,i})(I_{n-1} + f E_{i,j})$$
$$= (I_{n-1} - f E_{i,j})(I_{n-1} + \sigma_n h_1 E_{j,i})(I_{n-1} + f E_{i,j})$$
$$\cdot (I_{n-1} - f E_{i,j})(I_{n-1} + h_2 E_{j,i})(I_{n-1} + f E_{i,j}).$$

Thus, as the matrix $(I_{n-1} - f E_{i,j})(I_{n-1} + \sigma_n h_1 E_{j,i})(I_{n-1} + f E_{i,j})$ is clearly a product of elements of Forms 1 and 2 in the lemma, it is enough to deal with the matrix

$$(I_{n-1} - f E_{i,j})(I_{n-1} + h_2 E_{j,i})(I_{n-1} + f E_{i,j})$$

when $h_2 \in \sum_{r=1}^{n-1} \sigma_r^2 \overline{U}_{r,m} + \sum_{r=1}^{n-1} \sigma_r \overline{O}_m + \overline{O}_m^2$. Let us now write $f = \sigma_n f_1 + f_2$ for some $f_1 \in R_n$ and $f_2 \in R_{n-1}$, and write

$$(I_{n-1} - f E_{i,j})(I_{n-1} + h_2 E_{j,i})(I_{n-1} + f E_{i,j})$$
$$= (I_{n-1} - f_2 E_{i,j})(I_{n-1} - \sigma_n f_1 E_{i,j})$$
$$\cdot (I_{n-1} + h_2 E_{j,i})(I_{n-1} + \sigma_n f_1 E_{i,j})(I_{n-1} + f_2 E_{i,j}).$$

Now, as $(I_{n-1} \pm f_2 E_{i,j}) \in \mathrm{GL}_{n-1}(R_{n-1})$, it is enough to deal with the element

$$(I_{n-1} - \sigma_n f_1 E_{i,j})(I_{n-1} + h_2 E_{j,i})(I_{n-1} + \sigma_n f_1 E_{i,j}),$$

which can be written as a product of elements of the form

$$(I_{n-1} - \sigma_n f_1 E_{i,j})(I_{n-1} + k E_{j,i})(I_{n-1} + \sigma_n f_1 E_{i,j})$$
$$\text{with } k \in \overline{O}_m^2,\ \sigma_r^2 \overline{U}_{r,m},\ \sigma_r \overline{O}_m,\ \text{ for } 1 \le r \le n-1.$$

Finally, as for every such k one can write

$$(I_{n-1} - \sigma_n f_1 E_{i,j})(I_{n-1} + k E_{j,i})(I_{n-1} + \sigma_n f_1 E_{i,j})$$
$$= (I_{n-1} + k E_{j,i})[(I_{n-1} - k E_{j,i}), (I_{n-1} - \sigma_n f_1 E_{i,j})],$$

and $(I_{n-1} + k E_{j,i}) \in \mathrm{GL}_{n-1}(R_{n-1})$, we are actually finished. □

Corollary 7.8. *To prove Lemma 7.1, it is enough to show that every element of the forms in Lemma 7.6 is in* IA^m.

We start here by dealing with the elements of Form 1.

Proposition 7.9. *Recall $O_m = mR_n$. Elements of the following form are in* IA^m:

$$A^{-1}(I_{n-1}+hE_{i,j})A \quad \text{for } A \in \mathrm{GL}_{n-1}(R_n),\ h \in \sigma_n O_m \text{ and } i \neq j.$$

Proof. In this case we can write $h = \sigma_n m h'$ for some $h' \in R_n$. So, as

$$A^{-1}(I_{n-1}+\sigma_n h' E_{i,j})A \in \mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \leq \mathrm{IA}(\Phi),$$

we obtain that

$$\begin{aligned} A^{-1}(I_{n-1}+hE_{i,j})A &= A^{-1}(I_{n-1}+\sigma_n m h' E_{i,j})A \\ &= (A^{-1}(I_{n-1}+\sigma_n h' E_{i,j})A)^m \in \mathrm{IA}^m, \end{aligned}$$

as required. □

We devote the remaining sections to dealing with the elements of the other three forms. In these cases the proof is more difficult, and we will need the help of the computations in the next subsection.

Some auxiliary computations.

Proposition 7.10. *For every $f, g \in R_n$ we have the following equalities*:

$$\begin{aligned}
\begin{pmatrix} 1-fg & -fg & 0 \\ fg & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ fg & 1 & 0 \\ fg^2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & -f \\ 0 & fg^2 & 1-fg \end{pmatrix}\begin{pmatrix} 1 & -fg & 0 \\ 0 & 1 & 0 \\ 0 & -fg^2 & 1 \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 1-fg & 0 & f \\ 0 & 1 & 0 \\ -fg^2 & 0 & 1+fg \end{pmatrix}\begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \qquad (7.11) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ fg & 1 & 0 \\ -fg^2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & f \\ 0 & -fg^2 & 1-fg \end{pmatrix}\begin{pmatrix} 1 & -fg & 0 \\ 0 & 1 & 0 \\ 0 & fg^2 & 1 \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 1-fg & 0 & -f \\ 0 & 1 & 0 \\ fg^2 & 0 & 1+fg \end{pmatrix}\begin{pmatrix} 1 & 0 & f \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix} \qquad (7.12) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f & f & 1 \end{pmatrix}\begin{pmatrix} 1-fg & 0 & fg^2 \\ 0 & 1 & 0 \\ -f & 0 & 1+fg \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ fg & 1 & -fg^2 \\ 0 & 0 & 1 \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & fg^2 \\ 0 & -f & 1-fg \end{pmatrix}\begin{pmatrix} 1 & -fg & -fg^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (7.13)
\end{aligned}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -f & -f & 1 \end{pmatrix} \begin{pmatrix} 1-fg & 0 & -fg^2 \\ 0 & 1 & 0 \\ f & 0 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ fg & 1 & fg^2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & -fg^2 \\ 0 & f & 1-fg \end{pmatrix} \begin{pmatrix} 1 & -fg & fg^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.14}$$

Proof. We use square brackets to help the reader follow the steps of the computation. Here is the computation for (7.11):

$$\begin{aligned}
&\begin{pmatrix} 1-fg & -fg & 0 \\ fg & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & -f \\ g & g & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ -g & -g & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & g & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & -g & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & g & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & -g & 1 \end{pmatrix} \right] \\
&\qquad \cdot \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & g & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & -g & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ fg & 1+fg & -f \\ fg^2 & fg^2 & 1-fg \end{pmatrix} \begin{pmatrix} 1-fg & -fg & f \\ 0 & 1 & 0 \\ -fg^2 & -fg^2 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ fg & 1 & 0 \\ fg^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & -f \\ 0 & fg^2 & 1-fg \end{pmatrix} \\
&\qquad \cdot \begin{pmatrix} 1 & -fg & 0 \\ 0 & 1 & 0 \\ 0 & -fg^2 & 1 \end{pmatrix} \begin{pmatrix} 1-fg & 0 & f \\ 0 & 1 & 0 \\ -fg^2 & 0 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Line (7.12) is obtained similarly by changing the signs of f and g simultaneously.

Here is the computation for (7.13):

$$\begin{pmatrix} 1-fg & -fg & 0 \\ fg & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & g \\ 0 & 1 & -g \\ f & f & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -g \\ 0 & 1 & g \\ -f & -f & 1 \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix}1&0&0\\0&1&0\\f&f&1\end{pmatrix}\begin{pmatrix}1&0&g\\0&1&-g\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f&-f&1\end{pmatrix}\begin{pmatrix}1&0&-g\\0&1&g\\0&0&1\end{pmatrix}\\
&= \begin{pmatrix}1&0&0\\0&1&0\\f&f&1\end{pmatrix}\left[\begin{pmatrix}1&0&g\\0&1&-g\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f&0&1\end{pmatrix}\begin{pmatrix}1&0&-g\\0&1&g\\0&0&1\end{pmatrix}\right]\\
&\qquad\cdot\left[\begin{pmatrix}1&0&g\\0&1&-g\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&-f&1\end{pmatrix}\begin{pmatrix}1&0&-g\\0&1&g\\0&0&1\end{pmatrix}\right]\\
&= \begin{pmatrix}1&0&0\\0&1&0\\f&f&1\end{pmatrix}\begin{pmatrix}1-fg&0&fg^2\\fg&1&-fg^2\\-f&0&1+fg\end{pmatrix}\begin{pmatrix}1&-fg&-fg^2\\0&1+fg&fg^2\\0&-f&1-fg\end{pmatrix}\\
&= \begin{pmatrix}1&0&0\\0&1&0\\f&f&1\end{pmatrix}\begin{pmatrix}1-fg&0&fg^2\\0&1&0\\-f&0&1+fg\end{pmatrix}\begin{pmatrix}1&0&0\\fg&1&-fg^2\\0&0&1\end{pmatrix}\\
&\qquad\cdot\begin{pmatrix}1&0&0\\0&1+fg&fg^2\\0&-f&1-fg\end{pmatrix}\begin{pmatrix}1&-fg&-fg^2\\0&1&0\\0&0&1\end{pmatrix}.
\end{aligned}
$$

We obtain (7.14) similarly by changing the signs of f and g simultaneously. □

In the following corollary, a 3×3 matrix $B \in \mathrm{GL}_3(R_n)$ denotes the block matrix

$$\begin{pmatrix}B&0\\0&I_{n-4}\end{pmatrix} \in \mathrm{GL}_{n-1}(R_n).$$

Corollary 7.15. *Let $n \geq 4$, $f \in \sigma_n\big(\sum_{r=1}^{n-1}\sigma_r U_{r,m} + U_{n,m} + O_m\big)$ and $g \in R_n$. Then, mod IA^m we have the following equalities (the indices are intended to help us later to recognize forms of matrices: form 7, form 12, etc.):*

$$
\begin{aligned}
\begin{pmatrix}1-fg&-fg&0\\fg&1+fg&0\\0&0&1\end{pmatrix}_{13} &\equiv \begin{pmatrix}1&0&0\\0&1+fg&-f\\0&fg^2&1-fg\end{pmatrix}_1\begin{pmatrix}1-fg&0&f\\0&1&0\\-fg^2&0&1+fg\end{pmatrix}_2\\
&\equiv \begin{pmatrix}1&0&0\\0&1+fg&f\\0&-fg^2&1-fg\end{pmatrix}_3\begin{pmatrix}1-fg&0&-f\\0&1&0\\fg^2&0&1+fg\end{pmatrix}_4\\
&\equiv \begin{pmatrix}1-fg&0&fg^2\\0&1&0\\-f&0&1+fg\end{pmatrix}_5\begin{pmatrix}1&0&0\\0&1+fg&fg^2\\0&-f&1-fg\end{pmatrix}_6
\end{aligned}
$$

$$\equiv \begin{pmatrix} 1-fg & 0 & -fg^2 \\ 0 & 1 & 0 \\ f & 0 & 1+fg \end{pmatrix}_7 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & -fg^2 \\ 0 & f & 1-fg \end{pmatrix}_8 \quad (7.16)$$

and

$$\begin{pmatrix} 1-fg & fg & 0 \\ -fg & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{14} \equiv \begin{pmatrix} 1-fg & 0 & -fg^2 \\ 0 & 1 & 0 \\ f & 0 & 1+fg \end{pmatrix}_7 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & fg^2 \\ 0 & -f & 1-fg \end{pmatrix}_6. \quad (7.17)$$

Moreover, we have (the inverse of a matrix is denoted by the same index — one can observe that the inverse of each matrix in these equations is obtained by changing the sign of f)

$$\begin{aligned} \begin{pmatrix} 1-fg & 0 & -fg \\ 0 & 1 & 0 \\ fg & 0 & 1+fg \end{pmatrix}_{15} &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & fg^2 \\ 0 & -f & 1+fg \end{pmatrix}_8 \begin{pmatrix} 1-fg & f & 0 \\ -fg^2 & 1+fg & 1 \\ 0 & 0 & 0 \end{pmatrix}_9 \\ &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & -fg^2 \\ 0 & f & 1+fg \end{pmatrix}_6 \begin{pmatrix} 1-fg & -f & 0 \\ fg^2 & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{10} \\ &\equiv \begin{pmatrix} 1-fg & fg^2 & 0 \\ -f & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & -f \\ 0 & fg^2 & 1+fg \end{pmatrix}_3 \\ &\equiv \begin{pmatrix} 1-fg & -fg^2 & 0 \\ f & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{12} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & f \\ 0 & -fg^2 & 1+fg \end{pmatrix}_1 \quad (7.18) \end{aligned}$$

and

$$\begin{pmatrix} 1-fg & 0 & fg \\ 0 & 1 & 0 \\ -fg & 0 & 1+fg \end{pmatrix}_{16} \equiv \begin{pmatrix} 1-fg & -fg^2 & 0 \\ f & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{12} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & -f \\ 0 & fg^2 & 1+fg \end{pmatrix}_3 \quad (7.19)$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & -fg \\ 0 & fg & 1+fg \end{pmatrix}_{17} \equiv \begin{pmatrix} 1-fg & 0 & fg^2 \\ 0 & 1 & 0 \\ -f & 0 & 1+fg \end{pmatrix}_5 \begin{pmatrix} 1+fg & -fg^2 & 0 \\ f & 1-fg & 1 \\ 0 & 0 & 0 \end{pmatrix}_{11} \quad (7.20)$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & fg \\ 0 & -fg & 1+fg \end{pmatrix}_{18} \equiv \begin{pmatrix} 1+fg & f & 0 \\ -fg^2 & 1-fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{10} \begin{pmatrix} 1-fg & 0 & -f \\ 0 & 1 & 0 \\ fg^2 & 0 & 1+fg \end{pmatrix}_4. \quad (7.21)$$

Remark 7.22. We remark that since $f \in \sigma_n R_n$, every matrix which takes part in the above equalities is indeed in $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cong \mathrm{IGL}_{n-1,n} \leq \mathrm{IA}(\Phi)$.

Proof. As $f \in \sigma_n\big(\sum_{r=1}^{n-1} \sigma_r U_{r,m} + U_{n,m} + O_m\big)$, (7.16) is obtained by applying Proposition 7.10 combined with Proposition 6.1. We obtain (7.17) similarly by transposing all the computations which led to the first part of (7.16). Similarly, by switching the roles of the second row and column with the third row and column, one obtains (7.18) and (7.19). By switching one more time the roles of the first row and column with the second row and column, we obtain (7.20) and (7.21) as well. □

Elements of Form 2.

Proposition 7.23. *Recall $U_{r,m} = (x_r^m - 1)R_n$. The elements of the form*

$$A^{-1}(I_{n-1} + hE_{i,j})A,$$

where $A \in \mathrm{GL}_{n-1}(R_n)$, $h \in \sigma_n\sigma_r^2 U_{r,m},\ \sigma_n^2 U_{n,m}$ for $1 \leq r \leq n-1$ and $i \neq j$, belong to IA^m.

Notice that for every $n \geq 4$, the groups $E_{n-1}(\sigma_n^2 U_{n,m})$ and $E_{n-1}(\sigma_n\sigma_r^2 U_{r,m})$ for $1 \leq r \leq n-1$ are normal in $\mathrm{GL}_{n-1}(R_n)$, and thus, all the above elements are in $E_{n-1}(\sigma_n^2 U_{n,m})$ and $E_{n-1}(\sigma_n\sigma_r^2 U_{r,m})$. Hence, to prove Proposition 7.23, it is enough to show that for every $1 \leq r \leq n-1$, we have

$$E_{n-1}(\sigma_n^2 U_{n,m}),\ E_{n-1}(\sigma_n\sigma_r^2 U_{r,m}) \subseteq \mathrm{IA}^m\,.$$

Therefore, by Proposition 7.3, to prove Proposition 7.23, it is enough to show that the elements of the form

$$(I_{n-1} - fE_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} + fE_{j,i})$$

are in IA^m when $h \in \sigma_n\sigma_r^2 U_{r,m},\ \sigma_n^2 U_{n,m}$ for $1 \leq r \leq n-1$, $f \in R_n$ and $i \neq j$. We prove this in a few stages, starting with the following lemma.

Lemma 7.24. *Let $h \in \sigma_n\sigma_r U_{r,m},\ \sigma_n U_{n,m}$ for $1 \leq r \leq n-1$ and $f_1, f_2 \in R_n$. Assume that the elements of the forms*

$$(I_{n-1} \pm f_1E_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} \mp f_1E_{j,i}),$$
$$(I_{n-1} \pm f_2E_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} \mp f_2E_{j,i}),$$

for every $1 \leq i \neq j \leq n-1$, belong to IA^m. Then the elements of the form

$$(I_{n-1} \pm (f_1 + f_2)E_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} \mp (f_1 + f_2)E_{j,i})$$

for $1 \leq i \neq j \leq n-1$ also belong to IA^m.

Proof. Observe first that by Proposition 6.1, all the matrices of the form

$$I_{n-1} + hE_{i,j} \quad \text{for } h \in \sigma_n\sigma_r U_{r,m}, \sigma_n U_{n,m}$$

belong to IA^m. We use this in the following computations. Without loss of generality, under the assumptions of the proposition, we show that for $i, j = 2, 1$ we have

$$(I_{n-1} - (f_1 + f_2)E_{1,2})(I_{n-1} + hE_{2,1})(I_{n-1} + (f_1 + f_2)E_{1,2}) \in \mathrm{IA}^m$$

and the general argument is similar. In the computation, a block matrix of the form

$$\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$$

is denoted by $B \in \mathrm{GL}_3(R_n)$. We use square brackets to help the reader follow the steps of the computation.

So we compute

$$\begin{aligned}
&(I_{n-1} - (f_1 + f_2)E_{1,2})(I_{n-1} + hE_{2,1})(I_{n-1} + (f_1 + f_2)E_{1,2}) \\
&\quad = \begin{pmatrix} 1 - h(f_1+f_2) & -h(f_1+f_2)^2 & 0 \\ h & 1 + h(f_1+f_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\quad = \begin{pmatrix} 1 & 0 & -(f_1+f_2) \\ 0 & 1 & 1 \\ -h & -h(f_1+f_2) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & (f_1+f_2) \\ 0 & 1 & -1 \\ h & h(f_1+f_2) & 1 \end{pmatrix} \\
&\quad = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -h & -h(f_1+f_2) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -(f_1+f_2) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
&\qquad\qquad \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & h(f_1+f_2) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & (f_1+f_2) \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\
&\quad = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -h & -h(f_1+f_2) & 1 \end{pmatrix} \\
&\qquad\qquad \cdot \left[\begin{pmatrix} 1 & 0 & -(f_1+f_2) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & hf_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & f_1+f_2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \right] \\
&\qquad\qquad \cdot \left[\begin{pmatrix} 1 & 0 & -f_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & hf_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & f_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]
\end{aligned}$$

$$
\begin{aligned}
&\cdot\left[\begin{pmatrix}1&0&-f_2\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-h&-hf_1&1\end{pmatrix}\begin{pmatrix}1&0&-f_1\\0&1&1\\0&0&1\end{pmatrix}\right.\\
&\left.\cdot\begin{pmatrix}1&0&0\\0&1&0\\h&hf_1&1\end{pmatrix}\begin{pmatrix}1&0&f_1\\0&1&-1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&f_2\\0&1&0\\0&0&1\end{pmatrix}\right]\\
=&\begin{pmatrix}1&0&0\\0&1&0\\-h&-h(f_1+f_2)&1\end{pmatrix}\\
&\cdot\begin{pmatrix}1&0&0\\0&1+hf_2&-hf_2\\0&hf_2&1-hf_2\end{pmatrix}\begin{pmatrix}1&-(f_1+f_2)hf_2&(f_1+f_2)hf_2\\0&1&0\\0&0&1\end{pmatrix}\\
&\cdot\begin{pmatrix}1&-hf_1f_2&0\\0&1&0\\0&hf_1&1\end{pmatrix}\begin{pmatrix}1-hf_2&0&-hf_2^2\\0&1&0\\h&0&1+hf_2\end{pmatrix}\\
&\cdot\begin{pmatrix}1&0&-hf_1f_2\\0&1&hf_2\\0&0&1\end{pmatrix}\begin{pmatrix}1-hf_1&-hf_1^2&0\\h&1+hf_1&0\\0&0&1\end{pmatrix}.
\end{aligned}
$$

Notice now that by assumption, and by the remark at the beginning of the proof, the latter expression is congruent mod IA^m to

$$
\begin{pmatrix}1&0&0\\0&1+hf_2&-hf_2\\0&hf_2&1-hf_2\end{pmatrix}.
$$

Consider now (7.21) in Corollary 7.15, and switch the roles of f and g by $-h$ and f_2, respectively. Using this identity we deduce that, mod IA^m, the latter expression is congruent to

$$
\begin{pmatrix}1-hf_2&-h&0\\hf_2^2&1+hf_2&0\\0&0&1\end{pmatrix}\begin{pmatrix}1+hf_2&0&h\\0&1&0\\-hf_2^2&0&1-hf_2\end{pmatrix},
$$

which is congruent to I_{n-1} by assumption. This finishes the proof of the lemma. □

We pass to the next stage:

Proposition 7.25. *The elements of the form*

$$
(I_{n-1}-fE_{j,i})(I_{n-1}+hE_{i,j})(I_{n-1}+fE_{j,i}),
$$

where $h\in\sigma_n\sigma_r^2U_{r,m}$, $\sigma_n^2U_{n,m}$ *for* $1\le r\le n-1$, $\underline{f\in\mathbb{Z}}$ *and* $i\neq j$, *belong to* IA^m.

Remark 7.26. We note that some of the matrices that we use in the following computations lie in $\mathrm{IGL}'_{n-1,n} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$ and not necessarily in $\mathrm{IGL}_{n-1,n}$ (see Definition 3.12 and Proposition 3.13).

Proof of Proposition 7.25. According to Lemma 7.24, it is enough to prove the proposition for $f = \pm 1$. Without loss of generality, we prove the proposition for $r = 1$, i.e., $h \in \sigma_n \sigma_1^2 U_{1,m}$, and symmetrically, the same is valid for every $1 \le r \le n-1$. The case $h \in \sigma_n^2 U_{n,m}$ is considered separately.

So let $h \in \sigma_n \sigma_1^2 U_{1,m}$ and write $h = \sigma_1 u$ for some $u \in \sigma_n \sigma_1 U_{1,m}$. We prove the proposition for $i \neq j \in \{1, 2, 3\}$ — as one can see below, we do it simultaneously for all the options for $i \neq j \in \{1, 2, 3\}$. The treatment in the other cases in which $i \neq j \in \{1, k, l\}$ such that $1 < k \neq l \le n-1$ is obtained symmetrically, so we get that the proposition is valid for every $1 \le i \neq j \le n-1$.

As before, we denote a block matrix of the form

$$\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$$

by $B \in \mathrm{GL}_3(R_n)$. In the following computations, the indices of the matrices are intended to help the reader recognize the corresponding matrix type in Corollary 7.15, as explained below. We recall that the inverse of a matrix is denoted by the same index, and one can observe that the inverse of each indexed matrix is obtained by changing the sign of u. We also recall that $u \in \sigma_n \sigma_1 U_{1,m} \subseteq \sigma_n R_n$. Thus, by Proposition 6.1 we have

$$\begin{pmatrix} 1-\sigma_1 u & -\sigma_1^2 u & 0 \\ u & 1+\sigma_1 u & 0 \\ 0 & 0 & 1 \end{pmatrix}_{12} = \begin{pmatrix} x_2 & -\sigma_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ u x_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2^{-1} & x_2^{-1}\sigma_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{IA}^m,$$

$$\begin{pmatrix} 1-\sigma_1 u & 0 & -\sigma_1^2 u \\ 0 & 1 & 0 \\ u & 0 & 1+\sigma_1 u \end{pmatrix}_{7} = \begin{pmatrix} x_3 & 0 & -\sigma_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u x_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_3^{-1} & 0 & x_3^{-1}\sigma_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{IA}^m,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\sigma_1 u & u \\ 0 & -\sigma_1^2 u & 1-\sigma_1 u \end{pmatrix}_{3} = \begin{pmatrix} 1 & 0 & 0 \\ u\sigma_2 & 1 & 0 \\ -u\sigma_1\sigma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_2 & -\sigma_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sigma_2 & \sigma_1 & 1 \end{pmatrix} \in \mathrm{IA}^m,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\sigma_1 u & -\sigma_1^2 u \\ 0 & u & 1+\sigma_1 u \end{pmatrix}_{6} = \begin{pmatrix} 1 & 0 & 0 \\ -u\sigma_1\sigma_3 & 1 & 0 \\ u\sigma_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \sigma_3 & 1 & -\sigma_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\sigma_3 & 1 & \sigma_1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{IA}^m.$$

By switching the signs of σ_1, σ_2 and σ_3 in the two latter computations we obtain also that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\sigma_1 u & u \\ 0 & -\sigma_1^2 u & 1+\sigma_1 u \end{pmatrix}_{1}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\sigma_1 u & -\sigma_1^2 u \\ 0 & u & 1-\sigma_1 u \end{pmatrix}_{8} \in \mathrm{IA}^m.$$

Consider now the identities which we got in Corollary 7.15, and switch the roles of f, g in the corollary by u, σ_1, respectively. Remember that $u \in \sigma_n\sigma_1 U_{1,m}$. Hence, as by the computations above matrices of Forms 7 and 8 belong to IA^m, we obtain from the last part of (7.16) that also matrices of Form 13 belong to IA^m. Thus, as we showed that Forms 1, 3, 6 also belong to IA^m, (7.16) shows that Forms 2, 4, 5 also belong to IA^m. Similar arguments show that (7.16)–(7.21) give that all the 18 forms belong to IA^m. In particular, the matrices which correspond to Forms 13–18 belong to IA^m, and these matrices (and their inverses) are precisely the matrices of the form

$$(I_{n-1} \pm E_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} \mp E_{j,i}), \quad i \neq j \in \{1, 2, 3\}$$

(recalling that $h = \sigma_1 u$). Clearly, by similar arguments, the proposition holds for every $1 \le i \neq j \le n-1$ and every $h \in \sigma_n\sigma_r^2 U_{r,m}$ for $1 \le r \le n-1$.

The case $h \in \sigma_n^2 U_{n,m}$ is a bit different, but easier. In this case one can consider the same computations we built for $r = 1$, with the following modifications: Firstly, write $h \in \sigma_n^2 U_{n,m}$ as $h = \sigma_n u$ for some $u \in \sigma_n U_{n,m}$. Secondly, change σ_1 to σ_n, change σ_2, σ_3 to 0 and change x_2, x_3 to 1 in the right side of the above equations. It is easy to see that in this situation we obtain in the left side of the equations the same matrices, just that instead of σ_1 we have σ_n. From here we continue exactly the same. □

Proposition 7.27. *The elements of the following form belong to* IA^m:

$$(I_{n-1} - fE_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} + fE_{j,i}),$$

where $h \in \sigma_n^2 U_{n,m}, \sigma_n\sigma_r^2 U_{r,m}$ *for* $1 \le r \le n-1$, $\underline{f \in \sigma_s R_n}$ *for* $1 \le s \le n$ *and* $i \neq j$.

Proof. We prove it for $s = 1$, $i \neq j \in \{1, 2, 3\}$, and denote a block matrix of the form

$$\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$$

by $B \in \mathrm{GL}_3(R_n)$. We use again the result of Corollary 7.15, when we switch the roles of f, g in the corollary by $h, \sigma_1 u$, respectively, for some $u \in R_n$.

As $h \in \sigma_n\sigma_r^2 U_{r,m}, \sigma_n^2 U_{n,m}$, we have also $\sigma_1 uh \in \sigma_n\sigma_r^2 U_{r,m}, \sigma_n^2 U_{n,m}$. Hence, we obtain from the previous proposition that the matrices of Forms 13–18 belong to IA^m. In addition,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-u\sigma_1 h & h \\ 0 & -u^2\sigma_1^2 h & 1+u\sigma_1 h \end{pmatrix}_1$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -hu\sigma_2 & 1 & 0 \\ -hu^2\sigma_1\sigma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -u\sigma_2 & u\sigma_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & h \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u\sigma_2 & -u\sigma_1 & 1 \end{pmatrix} \in \mathrm{IA}^m,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-u\sigma_1 h & -u^2\sigma_1^2 h \\ 0 & h & 1+u\sigma_1 h \end{pmatrix}_6$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ -hu^2\sigma_1\sigma_3 & 1 & 0 \\ hu\sigma_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ u\sigma_3 & 1 & -u\sigma_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & h & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -u\sigma_3 & 1 & u\sigma_1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{IA}^m,$$

and by switching the signs of u and h simultaneously, we get also Forms 3 and 8. So we easily conclude from Corollary 7.15 ((7.16) and (7.18)) that also the matrices of the other eight forms are in IA^m. In particular, the matrices of the form

$$(I_{n-1} - \sigma_1 u E_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} + \sigma_1 u E_{j,i}), \quad i \neq j \in \{1, 2, 3\},$$

belong to IA^m. The treatment for every $i \neq j$ and $1 \leq s \leq n-1$ is similar, and the treatment in the case $s = n$ is obtained by replacing σ_1 by σ_n and σ_2, σ_3 by 0 in the above equations. □

Corollary 7.28. *As every $f \in R_n$ can be decomposed as*

$$f = \sum_{s=1}^{n} \sigma_s f_s + f_0$$

for some $f_0 \in \mathbb{Z}$ and $f_i \in R_n$, we obtain from Lemma 7.24 and from the above two propositions that we actually finished the proof of Proposition 7.23.

***Elements of Form* 3.**

Proposition 7.29. *Recall $\bar{O}_m = mR_{n-1}$, where*

$$R_{n-1} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}] \subseteq R_n.$$

Then the elements of the form

$$A^{-1}[(I_{n-1} + hE_{i,j}), (I_{n-1} + fE_{j,i})]A$$

belong to IA^m*, where $A \in \mathrm{GL}_{n-1}(R_n)$, $f \in \sigma_n R_n$, $h \in \bar{O}_m^2$ and $i \neq j$.*

We prove the proposition in the case $i, j = 2, 1$, and the same arguments are valid for arbitrary $i \neq j$. In this case one can write $h = m^2 h'$ for some $h' \in R_{n-1}$, and thus, our element is of the form

$$A^{-1} \begin{pmatrix} 1 - fm^2h' & f & 0 \\ -f(m^2h')^2 & 1 + fm^2h' & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} \begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} A$$

for some $A \in \mathrm{GL}_{n-1}(R_n)$, $f \in \sigma_n R_n$ and $h' \in R_{n-1}$. The proposition follows easily from the following lemma.

Lemma 7.30. *Let $h_1, h_2 \in R_n$, $f \in \sigma_n R_n$ and denote a block matrix of the form*

$$\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$$

by $B \in \mathrm{GL}_3(R_n)$. Then

$$A^{-1}\begin{pmatrix} 1-fm(h_1+h_2) & f & 0 \\ -f(m(h_1+h_2))^2 & 1+fm(h_1+h_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}A$$
$$\equiv A^{-1}\begin{pmatrix} 1-fmh_1 & f & 0 \\ -f(mh_1)^2 & 1+fmh_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\cdot\begin{pmatrix} 1-fmh_2 & f & 0 \\ -f(mh_2)^2 & 1+fmh_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}A \mod \mathrm{IA}^m.$$

Now, if Lemma 7.30 is proved, one can deduce that for $f \in \sigma_n R_n$ and $h = m^2h'$, $h' \in R_n$, we have

$$A^{-1}\begin{pmatrix} 1-fm^2h' & f & 0 \\ -f(m^2h')^2 & 1+fm^2h' & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}\begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}A$$
$$\equiv \left[A^{-1}\begin{pmatrix} 1-fmh' & f & 0 \\ -f(mh')^2 & 1+fmh' & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}\begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}A\right]^m \mod \mathrm{IA}^m$$

and as the latter element obviously belongs to IA^m, Proposition 7.29 follows. So it is enough to prove Lemma 7.30.

Proof of Lemma 7.30. Throughout this computation we use the observation that as $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$ is normal in $\mathrm{GL}_{n-1}(R_n)$, every conjugate of an element of $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \leq \mathrm{IA}(\Phi)$ by an element of $\mathrm{GL}_{n-1}(R_n)$ is also an element of $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \leq \mathrm{IA}(\Phi)$ (as was mentioned in Remark 7.7) — even though $\mathrm{GL}_{n-1}(R_n) \not\leq \mathrm{IA}(\Phi)$. Throughout the computation, we use the below notation:

- A matrix $\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$ is denoted by $B \in \mathrm{GL}_3(R_n)$.
- "$=$" denotes an equality between matrices in $\mathrm{GL}_{n-1}(R_n)$.
- "$\equiv$" denotes an equality in $\mathrm{IA}(\Phi)/\mathrm{IA}^m$.
- We use square brackets to help the reader follow the steps of the computation. Whenever square brackets are used, it is recommended to concentrate on the expression inside them separately in order to follow the transition to the next step.

So, let's compute:

$$A^{-1}\begin{pmatrix}1-fm(h_1+h_2) & f & 0\\ -f(m(h_1+h_2))^2 & 1+fm(h_1+h_2) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -f & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$$= A^{-1}\begin{pmatrix}1 & 0 & -f\\ 0 & 1 & -fm(h_1+h_2)\\ -m(h_1+h_2) & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & f\\ 0 & 1 & fm(h_1+h_2)\\ m(h_1+h_2) & -1 & 1\end{pmatrix}\begin{pmatrix}1 & -f & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$$= A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -m(h_1+h_2) & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -f\\ 0 & 1 & -fm(h_1+h_2)\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ m(h_1+h_2) & -1 & 1\end{pmatrix}$$

$$\cdot\begin{pmatrix}1 & 0 & f\\ 0 & 1 & fm(h_1+h_2)\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -f & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$$= A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -mh_2 & 0 & 1\end{pmatrix}\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -mh_1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -f\\ 0 & 1 & -fmh_1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ mh_1 & -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & f\\ 0 & 1 & fmh_1\\ 0 & 0 & 1\end{pmatrix}\right]$$

$$\cdot\begin{pmatrix}1 & 0 & -f\\ 0 & 1 & -fmh_1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ mh_2 & 0 & 1\end{pmatrix}A$$

$$\cdot\left[A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -m(h_1+h_2) & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -fmh_2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ m(h_1+h_2) & -1 & 1\end{pmatrix}A\right]$$

$$\cdot\left[A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & f(h_1+h_2)\\ 0 & 0 & 1\end{pmatrix}A\right]^m A^{-1}\begin{pmatrix}1 & -f & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$$\equiv A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -mh_2 & 0 & 1\end{pmatrix}\begin{pmatrix}1-fmh_1 & f & 0\\ -f(mh_1)^2 & 1+fmh_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -f\\ 0 & 1 & -fmh_1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ mh_2 & 0 & 1\end{pmatrix}A$$

$$\cdot\left[A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -m(h_1+h_2) & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -fh_2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ m(h_1+h_2) & -1 & 1\end{pmatrix}A\right]^m$$

$$\cdot A^{-1}\begin{pmatrix}1 & -f & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$$\equiv A^{-1}\left[\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1-fmh_1&f&0\\-f(mh_1)^2&1+fmh_1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}\right]$$

$$\cdot\left[\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1&0&-f\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}\right]$$

$$\cdot\left[\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-fmh_1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}\right]\begin{pmatrix}1&-f&f\\0&1&0\\0&0&1\end{pmatrix}A$$

$$= A^{-1}\begin{pmatrix}1-fmh_1&f&0\\-f(mh_1)^2&1+fmh_1&0\\0&0&1\end{pmatrix}A\left[A^{-1}\begin{pmatrix}1&0&0\\0&1&0\\fmh_1h_2&-fh_2&1\end{pmatrix}A\right]^m$$

$$\cdot A^{-1}\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1&0&-f\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}A$$

$$\cdot\left[A^{-1}\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-fh_1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}A\right]^m A^{-1}\begin{pmatrix}1&-f&f\\0&1&0\\0&0&1\end{pmatrix}A$$

$$\equiv A^{-1}\begin{pmatrix}1-fmh_1&f&0\\-f(mh_1)^2&1+fmh_1&0\\0&0&1\end{pmatrix}$$

$$\cdot\left[\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1&0&-f\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}\right]\begin{pmatrix}1&-f&f\\0&1&0\\0&0&1\end{pmatrix}A$$

$$= A^{-1}\begin{pmatrix}1-fmh_1&f&0\\-f(mh_1)^2&1+fmh_1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-f&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\cdot\left[\begin{pmatrix}1&f&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1-mfh_2&0&-f\\0&1&0\\f(mh_2)^2&0&1+mfh_2\end{pmatrix}\begin{pmatrix}1&-f&0\\0&1&0\\0&0&1\end{pmatrix}\right]\begin{pmatrix}1&0&f\\0&1&0\\0&0&1\end{pmatrix}A$$

$$= A^{-1}\begin{pmatrix}1-fmh_1&f&0\\-f(mh_1)^2&1+fmh_1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-f&0\\0&1&0\\0&0&1\end{pmatrix}A\left[A^{-1}\begin{pmatrix}1&f^2h_2&0\\0&1&0\\0&-m(fh_2)^2&1\end{pmatrix}A\right]^m A^{-1}$$

$$\cdot\begin{pmatrix}1-mfh_2&0&-f\\0&1&0\\f(mh_2)^2&0&1+mfh_2\end{pmatrix}\begin{pmatrix}1&0&f\\0&1&0\\0&0&1\end{pmatrix}A$$

$$\equiv A^{-1}\begin{pmatrix} 1-fmh_1 & f & 0\\ -f(mh_1)^2 & 1+fmh_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & -f & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$
$$\cdot\begin{pmatrix} 1-fmh_2 & 0 & -f\\ 0 & 1 & 0\\ f(mh_2)^2 & 0 & 1+fmh_2\end{pmatrix}\begin{pmatrix} 1 & 0 & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A.$$

So it remains to show that

$$A^{-1}\begin{pmatrix} 1-fmh_2 & 0 & -f\\ 0 & 1 & 0\\ f(mh_2)^2 & 0 & 1+fmh_2\end{pmatrix}\begin{pmatrix} 1 & 0 & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$
$$\equiv A^{-1}\begin{pmatrix} 1-fmh_2 & f & 0\\ -f(mh_2)^2 & 1+fmh_2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & -f & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A. \quad (7.31)$$

By a similar computation as for (7.14), switching the roles of f, g in the equation by f, mh_2, respectively, and then switching the roles of the first row and column with the third row and column, we have

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 1+fmh_2 & fmh_2\\ 0 & -fmh_2 & 1-fmh_2\end{pmatrix}$$
$$=\begin{pmatrix} 1 & -f & -f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1+fmh_2 & 0 & f\\ 0 & 1 & 0\\ -f(mh_2)^2 & 0 & 1-fmh_2\end{pmatrix}\begin{pmatrix} 1 & 0 & 0\\ f(mh_2)^2 & 1 & fmh_2\\ 0 & 0 & 1\end{pmatrix}$$
$$\cdot\begin{pmatrix} 1-fmh_2 & f & 0\\ -f(mh_2)^2 & 1+fmh_2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ f(mh_2)^2 & -fmh_2 & 1\end{pmatrix}.$$

Therefore, using Proposition 7.9 and the observation

$$A^{-1}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1+fmh_2 & fmh_2\\ 0 & -fmh_2 & 1-fmh_2\end{pmatrix}A=\left[A^{-1}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1+fh_2 & fh_2\\ 0 & -fh_2 & 1-fh_2\end{pmatrix}A\right]^m\in \mathrm{IA}^m,$$

we obtain that mod IA^m we have

$$A^{-1}\begin{pmatrix} 1+fmh_2 & 0 & f\\ 0 & 1 & 0\\ -f(mh_2)^2 & 0 & 1-fmh_2\end{pmatrix}\begin{pmatrix} 1-fmh_2 & f & 0\\ -f(mh_2)^2 & 1+fmh_2 & 0\\ 0 & 0 & 1\end{pmatrix}A$$
$$\equiv A^{-1}\begin{pmatrix} 1 & f & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A. \quad (7.32)$$

From here, we easily get (7.31) by noticing that the inverse of every matrix in (7.32) is obtained by replacing f by $-f$. This finishes the proof of the lemma, and hence, also the proof of Proposition 7.29. □

***Elements of Form* 4.**

Proposition 7.33. *Recall* $\overline{O}_m = mR_{n-1}$ *and* $\overline{U}_{r,m} = (x_r^m - 1)R_{n-1}$ *for* $1 \le r \le n-1$, *where* $R_{n-1} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}] \subseteq R_n$. *The elements of the form*

$$A^{-1}[(I_{n-1} + hE_{i,j}), (I_{n-1} + fE_{j,i})]A$$

where $A \in \mathrm{GL}_{n-1}(R_n)$, $f \in \sigma_n R_n$, $h \in \sigma_r^2 \overline{U}_{r,m}, \sigma_r \overline{O}_m$ *for* $1 \le r \le n-1$ *and* $i \ne j$, *belong to* IA^m.

As before, throughout the subsection we denote a block matrix of the form

$$\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$$

by $B \in \mathrm{GL}_3(R_n)$. We start the proof of this proposition with the following lemma.

Lemma 7.34. *Let* $f, h \in R_n$ *and* $A \in \mathrm{GL}_{n-1}(R_n)$. *Then*

$$\begin{aligned}
A^{-1}&\begin{pmatrix} 1-fh & -fh & 0 \\ fh & 1+fh & 0 \\ 0 & 0 & 1 \end{pmatrix} A \\
&= A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ fh & 1 & 0 \\ fh^2 & 0 & 1 \end{pmatrix} A A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fh & -f \\ 0 & fh^2 & 1-fh \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} A \\
&\quad \cdot A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix} A A^{-1}\begin{pmatrix} 1 & -fh & 0 \\ 0 & 1 & 0 \\ 0 & -fh^2 & 1 \end{pmatrix} A A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} A \\
&\quad \cdot A^{-1}\begin{pmatrix} 1-fh & 0 & f \\ 0 & 1 & 0 \\ -fh^2 & 0 & 1+fh \end{pmatrix} A A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ f^2h^2 & 1 & -f^2h \\ 0 & 0 & 1 \end{pmatrix} A A^{-1}\begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A.
\end{aligned}$$

Proof. The lemma follows from Proposition 7.10, line (7.11), by substituting g with h, combined with verifying the identity

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1-fh & 0 & f \\ 0 & 1 & 0 \\ -fh^2 & 0 & 1+fh \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 1-fh & 0 & f \\ 0 & 1 & 0 \\ -fh^2 & 0 & 1+fh \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ f^2h^2 & 1 & -f^2h \\ 0 & 0 & 1 \end{pmatrix}. \quad \square
\end{aligned}$$

Observe now that if we have $f \in \sigma_n R_n$ and $h \in \sigma_r^2 \overline{U}_{r,m}, \sigma_r \overline{O}_m$ for $1 \le r \le n-1$, then by Propositions 7.9 and 7.23, we have

$$A^{-1}\begin{pmatrix} 1-fh & -fh & 0 \\ fh & 1+fh & 0 \\ 0 & 0 & 1 \end{pmatrix} A = A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -fh & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \in \mathrm{IA}^m .$$

So, by Propositions 7.9 and 7.23 and the previous lemma, for every $A \in \mathrm{GL}_{n-1}(R_n)$ we have the following equality mod IA^m:

$$A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fh & -f \\ 0 & fh^2 & 1-fh \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} A$$
$$\equiv A^{-1}\left[\begin{pmatrix} 1-fh & 0 & f \\ 0 & 1 & 0 \\ -fh^2 & 0 & 1+fh \end{pmatrix}\begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]^{-1} A.$$

We thus have the following corollary (notice that we switched the sign of f).

Corollary 7.35. *For every $h \in \sigma_r^2 \overline{U}_{r,m}, \sigma_r \overline{O}_m$ for $1 \le r \le n-1$, $f \in \sigma_n R_n$ and $A \in \mathrm{GL}_{n-1}(R_n)$, the following elements are congruent mod* IA^m:

$$A^{-1}[(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})]A \equiv A^{-1}[(I_{n-1} - fE_{1,3}), (I_{n-1} + hE_{3,1})]A.$$

We proceed with the following proposition:

Proposition 7.36. *Let $h \in \sigma_1^2 \overline{U}_{1,m}, \sigma_1 \overline{O}_m$ and $f \in \sigma_n R_n$. Then*

$$[(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})] \in \mathrm{IA}^m .$$

Proof. Let $h = \sigma_1 u$ for some $u \in \sigma_1 \overline{U}_{1,m}, \overline{O}_m$. By Proposition 6.1, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sigma_2 u & \sigma_1 u & 1 \end{pmatrix} \in \mathrm{IA}^m$$

and hence

$$\mathrm{IA}^m \ni \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sigma_2 u & \sigma_1 u & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_2 u & -\sigma_1 u & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ f\sigma_2 u & 1 & 0 \\ \sigma_1\sigma_2 u^2 f & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sigma_1 u & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\sigma_1 u & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix}.$$

As by Proposition 6.1 the first matrix in the right-hand side is also in IA^m, we obtain that

$$[(I_{n-1}+hE_{3,2}),(I_{n-1}+fE_{2,3})]=\left[\begin{pmatrix}1&0&0\\0&1&0\\0&h&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&f\\0&0&1\end{pmatrix}\right]\in \mathrm{IA}^m$$

as required. □

We can now pass to the following proposition.

Proposition 7.37. *Let* $h\in\sigma_1^2\overline{U}_{1,m},\sigma_1\overline{O}_m$, $f\in\sigma_nR_n$ *and* $A\in\mathrm{GL}_{n-1}(R_n)$. *Then*

$$A^{-1}[(I_{n-1}+hE_{3,2}),(I_{n-1}+fE_{2,3})]A\in\mathrm{IA}^m\,.$$

Proof. We prove the proposition by induction. By a result of Suslin [1977], as $n-1\geq 3$, the group $\mathrm{SL}_{n-1}(R_n)$ is generated by the elementary matrices of the form

$$I_{n-1}+rE_{l,k}\quad\text{for } r\in R_n \text{ and } 1\leq l\neq k\leq n-1.$$

So as the invertible elements of R_n are the elements of the form

$$\pm\prod_{i=1}^{n}x_i^{s_i}\quad\text{for } s_i\in\mathbb{Z}$$

(see [Crowell and Fox 1963, Chapter 8]), $\mathrm{GL}_{n-1}(R_n)$ is generated by the elementary matrices and the matrices of the form

$$I_{n-1}+(\pm x_i-1)E_{1,1}\quad\text{for } 1\leq i\leq n.$$

Therefore, by the previous proposition it is enough to show that if

$$A^{-1}[(I_{n-1}+hE_{3,2}),(I_{n-1}+fE_{2,3})]A\in\mathrm{IA}^m$$

and E is one of the above generators, then mod IA^m we have

$$A^{-1}E^{-1}[(I_{n-1}+hE_{3,2}),(I_{n-1}+fE_{2,3})]EA \equiv A^{-1}[(I_{n-1}+hE_{3,2}),(I_{n-1}+fE_{2,3})]A.\quad(7.38)$$

So if E is of the form $I_{n-1}+(\pm x_i-1)E_{1,1}$, we obviously have Property (7.38). If E is an elementary matrix of the form $I_{n-1}+rE_{l,k}$ such that $l,k\notin\{2,3\}$, then we also have Property (7.38) in an obvious way. Consider now the case $l,k=2,3$. In this case, by Corollary 7.35 we have the following mod IA^m:

$$A^{-1}E^{-1}\left[\begin{pmatrix}1&0&0\\0&1&0\\0&h&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&f\\0&0&1\end{pmatrix}\right]EA$$
$$\equiv A^{-1}\begin{pmatrix}1&0&0\\0&1&-r\\0&0&1\end{pmatrix}\left[\begin{pmatrix}1&0&-f\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\h&0&1\end{pmatrix}\right]\begin{pmatrix}1&0&0\\0&1&r\\0&0&1\end{pmatrix}A$$
$$=A^{-1}\begin{pmatrix}1&0&0\\0&1&-r\\0&0&1\end{pmatrix}\begin{pmatrix}1-hf+h^2f^2&0&-hf^2\\0&1&0\\-h^2f&0&1+hf\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&r\\0&0&1\end{pmatrix}A$$
$$=A^{-1}\begin{pmatrix}1-hf+h^2f^2&0&-hf^2\\0&1&0\\-h^2f&0&1+hf\end{pmatrix}AA^{-1}\begin{pmatrix}1&0&0\\rh^2f&1&-rhf\\0&0&1\end{pmatrix}A$$
$$=A^{-1}\left[\begin{pmatrix}1&0&-f\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\h&0&1\end{pmatrix}\right]AA^{-1}\begin{pmatrix}1&0&0\\rh^2f&1&-rhf\\0&0&1\end{pmatrix}A.$$

So by applying Propositions 7.9 and 7.23 and Corollary 7.35 once again in the opposite way, we obtain Property (7.38). The other cases for l, k are treated by similar arguments: if $l, k = 3, 2$ we do exactly the same, and if l or k are different from 2 and 3, then the situation is easier — we use similar arguments, but without passing to $[(I_{n-1} - fE_{1,3}), (I_{n-1} + hE_{3,1})]$ through Corollary 7.35. □

Corollary 7.39. *Let $h \in \sigma_1^2\overline{U}_{1,m}, \sigma_1\overline{O}_m$, $f \in \sigma_n R$ and $A \in \mathrm{GL}_{n-1}(R_n)$. Then for every $i \neq j$, we have*

$$A^{-1}[(I_{n-1} + hE_{i,j}), (I_{n-1} + fE_{j,i})]A \in \mathrm{IA}^m.$$

Proof. Denote a permutation matrix, such that its action on $\mathrm{GL}_{n-1}(R_n)$ by conjugation moves $2 \mapsto j$ and $3 \mapsto i$, by P. Then, by the previous proposition, we have

$$A^{-1}[(I_{n-1} + hE_{i,j}), (I_{n-1} + fE_{j,i})]A = A^{-1}P^{-1}[(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})]PA \in \mathrm{IA}^m. \quad \square$$

Now, since one can see that symmetrically, the above corollary is valid for every $h \in \sigma_r^2\overline{U}_{r,m}, \sigma_r\overline{O}_m$ for $1 \le r \le n-1$, we have actually finished the proof of Proposition 7.33.

8. Index of notation

For convenience, we gather here some notation that plays a role in the paper, and mention the section where they appear for the first time:

- $F_n =$ the free group on n elements (Section 3).
- $\Phi = \Phi_n = F_n/F_n''$ = the free metabelian group on n elements (Section 3).
- $\Psi_m = \Phi/M_m$ where $M_m = (\Phi'\Phi^m)'(\Phi'\Phi^m)^m$ (Section 3).
- $\mathrm{IA}(\Phi) = \ker(\mathrm{Aut}(\Phi) \to \mathrm{Aut}(\Phi/\Phi'))$ (Section 3).
- $\mathrm{IG}_m = G(M_m) = \ker(\mathrm{IA}(\Phi) \to \mathrm{Aut}(\Psi_m))$ (Section 3).
- $\mathrm{IA}^m = \langle \mathrm{IA}(\Phi)^m \rangle$ (Section 7).
- $\mathrm{IA}_m = \bigcap \{ N \lhd \mathrm{IA}(\Phi) \mid [\mathrm{IA}(\Phi) : N] \mid m \}$ (Section 3).
- $R_n = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, where the $x_1, \ldots, x_n$ are free commutative variables (Section 3).
- $R_{n-1} = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$ (Section 7).
- $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ (Section 3).
- $\sigma_i = x_i - 1$ for $1 \le i \le n$ (Section 3).
- $\vec{\sigma}$ = the column vector which has σ_i in its i-th entry (Section 3).
- $\mathfrak{A} = \sum_{i=1}^n \sigma_i R_n \lhd R_n$ = the augmentation ideal of R_n (Section 3).
- $O_m = mR_n \lhd R_n$ (Section 7).
- $\bar{O}_m = mR_{n-1} \lhd R_{n-1}$ (Section 7).
- $U_{r,m} = (x_r^m - 1)R_n \lhd R_n$ for $1 \le r \le n$ (Section 7).
- $\bar{U}_{r,m} = (x_r^m - 1)R_{n-1} \lhd R_{n-1}$ for $1 \le r \le n$ (Section 7).
- $H_m = \sum_{i=1}^n (x_i^m - 1)R_n + mR_n \lhd R_n$ (Section 3).
- $S = \mathbb{Z}[x^{\pm 1}]$ (Section 4).
- $J_m = (x^m - 1)S + mS \lhd S$ (Section 4).
- $E_d(R) = \langle I_d + rE_{i,j} \mid r \in R,\ 1 \le i \ne j \le d \rangle \le \mathrm{SL}_d(R)$, where R is a ring and $E_{i,j}$ is the matrix that has 1 in its (i, j)-th entry and 0 elsewhere (Section 2).
- $\mathrm{SL}_d(R, H) = \ker(\mathrm{SL}_d(R) \to \mathrm{SL}_d(R/H))$, where R is a ring and $H \lhd R$ (Section 2).
- $\mathrm{GL}_d(R, H) = \ker(\mathrm{GL}_d(R) \to \mathrm{GL}_d(R/H))$, where R is a ring and $H \lhd R$ (Section 2).
- $E_d(R, H)$ = the normal subgroup of $E_d(R)$, generated as a normal subgroup by the matrices of the form $I_d + hE_{i,j}$ for $h \in H$ (Section 2).
- $\mathrm{IGL}_{n-1,i} = \left\{ I_n + A \in \mathrm{IA}(\Phi) \;\middle|\; \begin{array}{l} \text{the } i\text{-th row of } A \text{ is } 0, \\ \qquad I_{n-1} + A_{i,i} \in \mathrm{GL}_{n-1}(R_n, \sigma_i R_n) \end{array} \right\}$, for $1 \le i \le n$ (Section 3).
- $\mathrm{ISL}_{n-1,i}(H) = \mathrm{IGL}_{n-1,i} \cap \mathrm{SL}_{n-1}(R_n, H)$, under the identification of $\mathrm{IGL}_{n-1,i}$ with $\mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$ (Section 3).

- $\mathrm{IE}_{n-1,i}(H) = \mathrm{IGL}_{n-1,i} \cap E_{n-1}(R_n, H)$, under the identification of the group $\mathrm{IGL}_{n-1,i}$ with $\mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$ (Section 3).
- $\mathrm{IGL}'_{n-1,i} = \{I_n + A \in \mathrm{IA}(\Phi) \mid \text{the } i\text{-th row of } A \text{ is } 0\}$ for $1 \le i \le n$ (Section 3).

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