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Higher genera for proper actions of Lie groups

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Let G be a Lie group with finitely many connected components and let K be a maximal compact subgroup. We assume that G satisfies the rapid decay (RD) property and that G/K has a nonpositive sectional curvature. As an example, we can take G to be a connected semisimple Lie group. Let M be a G-proper manifold with compact quotient M/G. Building on work by Connes and Moscovici (1990) and Pflaum et al. (2015), we establish index formulae for the C^*-higher indices of a G-equivariant Dirac-type operator on M. We use these formulae to investigate geometric properties of suitably defined higher genera on M. In particular, we establish the G-homotopy invariance of the higher signatures of a G-proper manifold and the vanishing of the $\widehat{A}$-genera of a G-spin G-proper manifold admitting a G-invariant metric of positive scalar curvature.

1. Introduction

The aim of this paper is to introduce certain geometric invariants associated to proper actions of Lie groups, generalizing the (higher) signatures and $\widehat{A}$-genera. Let G be a Lie group satisfying the following assumptions:

- G has finitely many components.
- Because $|\pi_0(G)| < \infty$, G has a maximal compact subgroup K, unique up to conjugation, and we assume that the homogeneous space G/K has nonpositive sectional curvature with respect to the G-invariant metric induced by an Ad_K-invariant inner product $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g}$.
- G satisfies the rapid decay (RD) property.

We explain these last two hypothesis in the course of the paper; it suffices for now to remark that natural examples of groups satisfying our assumptions are given by connected semisimple Lie groups. The homogeneous space G/K is a smooth model for $\underline{E}G$, the classifying space for proper actions of G [Baum et al. 1994]:

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for any smooth proper action of G on a manifold M, there exists a smooth G-equivariant classifying map $\psi_M : M \to G/K$, unique up to G-equivariant homotopy. Assuming in addition that the action is *cocompact*, i.e., that the quotient M/G is compact, we can fix a *cut-off* function χ_M for M. This is a smooth function $\chi_M \in C_c^\infty(M)$ satisfying

$$\int_G \chi_M(g^{-1}x)\, dg = 1 \quad \text{for all } x \in M.$$

For any proper action of G on M, we consider $\Omega^\bullet_{\mathrm{inv}}(M)$, the complex of G-invariant differential forms on M, and its cohomology denoted by $H^\bullet_{\mathrm{inv}}(M)$. In the universal case this cohomology can be identified with the K-relative Lie algebra cohomology of the Lie algebra $\mathfrak{g}$ of G: $H^\bullet_{\mathrm{inv}}(G/K) \cong H^\bullet_{\mathrm{CE}}(\mathfrak{g}; K)$, where CE stands for Chevalley–Eilenberg. For any $\alpha \in \Omega^\bullet_{\mathrm{inv}}(G/K)$, consider its pull-back $\psi_M^*\alpha \in \Omega^\bullet_{\mathrm{inv}}(M)$. The higher signature associated to α is the real number

$$\sigma(M,\alpha) := \int_M \chi_M L(M) \wedge \psi_M^*(\alpha), \tag{1.1}$$

where $L(M)$ is the invariant de Rham form representing the L-class of M. The insertion of the cut-off function χ_M, which has compact support, ensures that the integral is well-defined, and it can be shown that it only depends on the class $[L(M) \wedge \psi_M^*(\alpha)] \in H^\bullet_{\mathrm{inv}}(M)$. The numbers in the collection

$$\{\sigma(M,\alpha) : [\alpha] \in H^\bullet_{\mathrm{inv}}(G/K)\} \tag{1.2}$$

are called the higher signatures of M. Similarly, the higher $\widehat{A}$-genus associated to M and to $[\alpha] \in H^\bullet_{\mathrm{inv}}(G/K)$ is the real number

$$\widehat{A}(M,\alpha) := \int_M \chi_M \widehat{A}(M) \wedge \psi_M^*(\alpha), \tag{1.3}$$

where $\widehat{A}(M)$ is the de Rham class associated to the $\widehat{A}$-differential form for a G-invariant metric. The numbers in the collection

$$\{\widehat{A}(M,\alpha) : \alpha \in H^\bullet_{\mathrm{inv}}(G/K)\} \tag{1.4}$$

are called the higher $\widehat{A}$-genera of M.

In this paper we establish the following result:

Theorem 1.5. *Let G be a Lie group with finitely many connected components satisfying property RD, and such that G/K is of nonpositive sectional curvature for a maximal compact subgroup K. Let M be an orientable manifold with a proper, cocompact action of G. Then the following hold true*:

(i) *each higher signature $\sigma(M,\alpha)$, $\alpha \in H^\bullet_{\mathrm{inv}}(G/K)$, is a G-homotopy invariant of M*;

(ii) *if M admits a G-invariant spin structure and a G-invariant metric of positive scalar curvature, then each higher $\widehat{A}$-genus $\widehat{A}(M,\alpha)$, $\alpha \in H^{\bullet}_{\mathrm{inv}}(G/K)$, vanishes.*

We prove this result by adapting to the G-proper context the seminal paper of Connes and Moscovici on the cyclic cohomological approach to the Novikov conjecture for discrete Gromov hyperbolic groups. Crucial to this program is the proof of a higher index formula for higher indices associated to elements in $H^{\bullet}_{\mathrm{diff}}(G)$ and to the index class $\mathrm{Ind}_{C^*_r(G)}(D) \in K_*(C^*_r(G))$ of a G-equivariant Dirac operator on an even-dimensional M acting on the sections of a complex vector bundle E. Here are the main steps for establishing this result (for this introduction we expunge from the notation the vector bundle E):

(1) First, we remark that for any almost connected Lie group G there is a van Est isomorphism $H^{\bullet}_{\mathrm{diff}}(G) \simeq H^{\bullet}_{\mathrm{inv}}(G/K) \equiv H^{\bullet}_{\mathrm{inv}}(\underline{E}G)$.

(2) Under the assumption of nonpositive sectional curvature for G/K we prove that each $\alpha \in H^{\bullet}_{\mathrm{diff}}(G)$ has a representative cocycle of polynomial growth.

(3) If G is unimodular then for each $\alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G)$ we define a cyclic cocycle τ^G_α for the convolution algebra $C^\infty_c(G)$, and thus a homomorphism

$$\langle \tau^G_\alpha, \cdot \rangle : K_0(C^\infty_c(G)) \to \mathbb{C}.$$

(4) For each $\alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G)$ we also consider a cyclic cocycle τ^M_α for the algebra of G-equivariant smooth kernels of G-compact support $\mathcal{A}^c_G(M)$; this defines a homomorphism $\langle \tau^M_\alpha, \cdot \rangle : K_0(\mathcal{A}^c_G(M)) \to \mathbb{C}$.

(5) We show that if in addition G satisfies the RD property, for example, if G is semisimple connected, then τ^G_α extends to $K_0(C^*_r(G))$ and τ^M_α extends to $K_0(C^*(M)^G)$, with $C^*(M)^G$ denoting the Roe algebra of M.

(6) If D is a G-equivariant Dirac operator we consider its index class $\mathrm{Ind}_{C^*_r(G)}(D)$ in $K_0(C^*_r(G))$ and its Morita equivalent index class $\mathrm{Ind}_{C^*(M)^G}(D)$ in $K_0(C^*(M)^G)$ and show that

$$\langle \tau^G_\alpha, \mathrm{Ind}_{C^*_r(G)}(D)\rangle = \langle \tau^M_\alpha, \mathrm{Ind}_{C^*(M)^G}(D)\rangle.$$

(7) We apply the index theorem of Pflaum, Posthuma and Tang [Pflaum et al. 2015b] in order to compute $\langle \tau^M_\alpha, \mathrm{Ind}_{C^*(M)^G}(D)\rangle$, thus establishing our higher C^*-index formula in the even-dimensional case.

We remark that item (2) above is of independent interest, and should be compared with the literature on bounded cohomology of Lie groups; see [Hartnick and Ott 2012; Kim and Kim 2015]

The geometric applications in Theorem 1.5 are then a direct consequence of the G-homotopy invariance of the signature index class established by Fukumoto

[2017] and, for the higher $\widehat{A}$-genera, of the vanishing of the index class $\mathrm{Ind}_{C_r^*(G)}(\eth) \in K_*(C_r^*(G))$ of the spin Dirac operator $\eth$ of a G-spin G-proper manifold endowed with a G-metric of positive scalar curvature, established by Guo, Mathai and Wang [Guo et al. 2017]. In the odd-dimensional case we argue by suspension. Notice that for (certain) 2-degree classes α, the G-proper homotopy invariance of the higher signatures $\sigma(M, \alpha)$ had already been established by Fukumoto.

2. Preliminaries: proper actions and cohomology

2A. ***Proper actions.*** In this section we introduce the geometric setting for this paper, and list some basic tools that we will need at several points later on. Let G be a Lie group with finitely many connected components. Recall that a smooth left action of G on a manifold M is called *proper* if the associated map

$$G \times M \to M \times M, \quad (g, x) \mapsto (x, gx), \quad g \in G,\ x \in M,$$

is a proper map. This implies that the stabilizer groups G_x of all points $x \in M$ are compact and that the quotient space M/G is Hausdorff. The action is said to be *cocompact* if the quotient is compact.

The class of manifolds equipped with a proper action of G can be assembled into a category where the morphisms are given by G-equivariant smooth maps. It is a basic fact that this category has a final object $\underline{E}G$, meaning that any proper G-action on M is classified by a G-equivariant map $\psi : M \to \underline{E}G$, unique up to G-equivariant homotopy. This $\underline{E}G$ is called the *classifying space for proper G-actions*, and in fact we can take $\underline{E}G := G/K$, where K is a maximal compact subgroup. Then, by writing $S := \psi^{-1}(eK)$ we see that the S is in fact a global slice: it is a K-stable submanifold for which there is a diffeomorphism

$$G \times_K S \cong M, \quad [g, x] \mapsto gx, \quad g \in G,\ x \in S.$$

The existence of such a global slice for proper Lie group actions with finitely many connected components was first proved in [Abels 1974]. When the action is cocompact, S is compact as well. Closely related to the global slice is the existence of a *cut-off* function. This is a smooth function $\chi \in C^\infty(M)$ satisfying

$$\int_G \chi(g^{-1}x)\, dg = 1 \quad \text{for all } x \in M.$$

Here we have chosen, for the rest of the paper, a Haar measure which we normalized so that the volume of the maximal compact subgroup $K \subset G$ is equal to 1. When the action of G is cocompact, we can even choose χ to have compact support. The cut-off function is constructed from the global slice $S \subset M$ as follows: Choose a smooth function $h \in C^\infty(M)$ which is equal to 1 on S and 0 outside an open neighborhood of S in M. Then the function

$$\chi(x) = \left(\int_G h(g^{-1}x)\, dg \right)^{-1} h(x)$$

is a cut-off function for the action of G.

Choosing a G-invariant Riemannian metric g on M we can refine this construction as follows: Choose the initial function h to have support inside the tube of distance 1 in M around S. Then, rescaling by $\epsilon > 0$ along the radial coordinate near S, we obtain a family of functions h_ϵ satisfying

$$h_\epsilon(x) = \begin{cases} 1 & \text{for } x \in S, \\ 0 & \text{for } d(x, S) > \epsilon. \end{cases}$$

Using this as input for the construction of the cut-off function above gives a family of cut-off functions χ_ϵ approaching χ_S:

Lemma 2.1. *The family of cut-off functions* χ_ϵ, $\epsilon > 0$, *satisfies*

$$\lim_{\epsilon \downarrow 0} \chi_\epsilon = \chi_S,$$

distributionally.

Proof. We begin by remarking that pointwise

$$\lim_{\epsilon \downarrow 0} \chi_\epsilon(x) = \begin{cases} 1 & \text{for } x \in S, \\ 0 & \text{for } x \notin S. \end{cases}$$

This is because for fixed $x \in S$ the family $h_\epsilon(g^{-1}x)$ of functions on G converges pointwise to the characteristic function of $K \subset G$, and therefore by dominated convergence we have

$$\lim_{\epsilon \downarrow 0} \int_G h_\epsilon(g^{-1}x)\, dg = \int_G \lim_{\epsilon \downarrow 0} h_\epsilon(g^{-1}x)\, dg = \int_K dg = 1,$$

by our normalization of the Haar measure on G. With this pointwise limit of $\chi_\epsilon(x)$ we have, once again by dominated convergence, that

$$\lim_{\epsilon \downarrow 0} \int_M \chi_\epsilon(x) f(x)\, dx = \int_M \lim_{\epsilon \downarrow 0} \chi_\epsilon(x) f(x)\, dx = \int_S f(x)\, dx$$

for any test function $f \in C_c^\infty(M)$. □

2B. ***Invariant cohomology and the van Est map.*** The main point of this subsection is to define the van Est map associated to a proper action of a Lie group G on M, and to reinterpret this map as the pull-back in cohomology along the classifying map $\psi_M : M \to G/K$.

Let M be a smooth manifold equipped with a smooth proper action of G. We define

$$\Omega^\bullet_{\mathrm{inv}}(M) := \{\omega \in \Omega^\bullet(M) : g^*\omega = \omega, \text{ for all } g \in G\},$$

the vector space of invariant differential forms. The de Rham differential restricts to this space of invariant forms and its cohomology, called the *invariant cohomology*, is denoted by $H^\bullet_{\mathrm{inv}}(M)$. Taking the invariant cohomology defines a contravariant functor on the category of proper G-manifolds with an equivariant map $f : M \to N$ acting on cohomology by pull-back of differential forms as usual. It is not difficult to see that the induced map $f^* : H^\bullet_{\mathrm{inv}}(N) \to H^\bullet_{\mathrm{inv}}(M)$ depends only on the G-homotopy class it is in. Given the choice of a cut-off function χ, it is shown in [Pflaum et al. 2015b] that for a closed form $\alpha \in \Omega^{\dim(M)}_{\mathrm{inv,cl}}(M)$, the integral

$$\int_M \chi\alpha$$

only depends on the cohomology class $[\alpha] \in H^{\dim(M)}_{\mathrm{inv}}(M)$.

For any manifold M equipped with a proper action of G, the *van Est map* is a map $H^\bullet_{\mathrm{diff}}(G) \to H^\bullet_{\mathrm{inv}}(M)$, where $H^\bullet_{\mathrm{diff}}(G)$ is the so-called *smooth group cohomology* of G. Let us first recall the definition of this smooth group cohomology. For G a Lie group, the space of smooth homogeneous group k-cochains is given by

$$C^k_{\mathrm{diff}}(G) := \{c : G^{\times(k+1)} \to \mathbb{C} \text{ smooth}, \\ c(gg_0, \dots, gg_k) = c(g_0, \dots, g_k), \text{ for all } g, g_0, \dots, g_k \in G\}.$$

The differential $\delta : C^k_{\mathrm{diff}}(G) \to C^{k+1}_{\mathrm{diff}}(G)$ is defined as

$$(\delta c)(g_0, \dots, g_{k+1}) := \sum_{i=0}^{k+1} (-1)^i c(g_1, \dots, \hat{g}_i, \dots, g_{k+1}), \tag{2.2}$$

where the ˆ means omission from the argument of the function. The cohomology of the resulting complex is called the smooth group cohomology, written as $H^\bullet_{\mathrm{diff}}(G)$.

With this, the van Est map is constructed as follows: given a smooth group cochain $c \in C^k_{\mathrm{diff}}(G)$, define the differential form

$$\omega^\chi_c := (d_1 \cdots d_k f_c)|_\Delta, \tag{2.3}$$

where d_i means taking the differential with respect to the i-th variable of the function $f_c \in C^\infty(M^{\times(k+1)})$ defined as

$$f_c(x_0, \dots, x_k) \\ := \int_{G^{\times(k+1)}} \chi(g_0^{-1}x_0) \cdots \chi(g_k^{-1}x_k) c(g_0, \dots, g_k)\, d\mu(g_0) \cdots d\mu(g_k). \tag{2.4}$$

Proposition 2.5. *The map $c \mapsto \omega_c^\chi$ defines a morphism of complexes*

$$\Phi_M^\chi : (C^\bullet_{\mathrm{diff}}(G), \delta) \to (\Omega^\bullet_{\mathrm{inv}}(M), d_{dR}).$$

On the level of cohomology, it is independent of the choice of cut-off function χ.

Remark 2.6. Because of this last property, we often omit the superscript χ and write ω_c and Φ_M when the context only refers to the cohomological meaning of the differential form and the van Est map.

Proof of Proposition 2.5. We start by giving the abstract cohomological definition of the map Φ_M following [Crainic 2003] using a double complex, after which we show how to obtain the explicit chain morphism by constructing a splitting of the rows. The double complex is given as follows. We define

$$C^{p,q} := C^\infty(G^{\times(p+1)}, \Omega^q(M)).$$

The vertical differential $\delta_v : C^{p,q} \to C^{p,q+1}$ is simply given by the de Rham differential, leaving the G-variables untouched. As for the horizontal differential $\delta_h : C^{p,q} \to C^{p+1,q}$, this is given by the differential computing the smooth groupoid cohomology of the action groupoid $G \times M \rightrightarrows M$ with coefficients in $\bigwedge^q T^*M$, viewed as a representation of this groupoid. Since the G-action is proper, the groupoid $G \times M \rightrightarrows M$ is proper by definition. Therefore, the vanishing theorem for the groupoid cohomology of proper Lie groupoids in [Crainic 2003] applies, and we see that the rows in this double complex are exact. There are obvious inclusions $C^\bullet_{\mathrm{diff}}(G) \hookrightarrow C^{\bullet,0}$, and $\Omega^\bullet_{\mathrm{inv}}(M) \hookrightarrow C^{0,\bullet}$, and now we see that by finding the appropriate splittings we can "zig-zag" from one end to the other in the double complex:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow d & & \uparrow \delta_v & & \uparrow \delta_h & & \uparrow \delta_h & & \\
\Omega^1_{\mathrm{inv}}(M) & \longrightarrow & C^{0,1} & \xrightarrow{\delta_h} & C^{1,1} & \xrightarrow{\delta_h} & C^{2,1} & \xrightarrow{\delta_h} & \cdots \\
 & & & \overset{s_1}{\dashleftarrow} & & & & & \\
\uparrow d & & \uparrow \delta_v & & \uparrow \delta_h & & \uparrow \delta_h & & \\
\Omega^0_{\mathrm{inv}}(M) & \longrightarrow & C^{0,0} & \xrightarrow{\delta_h} & C^{1,0} & \xrightarrow{\delta_h} & C^{2,0} & \xrightarrow{\delta_h} & \cdots \\
 & & & & & \overset{s_2}{\dashleftarrow} & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & C^0_{\mathrm{diff}}(G) & \xrightarrow{\delta} & C^1_{\mathrm{diff}}(G) & \xrightarrow{\delta} & C^2_{\mathrm{diff}}(G) & \xrightarrow{\delta} & \cdots
\end{array}$$

So it remains to find an appropriate splitting $s_p : C^{p,\bullet} \to C^{p+1,\bullet}$. Given a choice of cut-off function χ, the formula

$$(s_p\alpha)(g_0, \dots, g_{p-1}) := \int_G \chi(g^{-1}x_0)\alpha(g, g_0, \dots, g_{p-1})\big|_\Delta, \quad \alpha \in C^{p,q}$$

does the job: a straightforward computation shows that

$$\delta_h \circ s + s \circ \delta_h = \mathrm{id}.$$

With this choice of contraction map, one obtains exactly (2.3) for the invariant differential form associated to a group cochain. The preceding argument therefore shows that the map $c \mapsto \omega_c$ is indeed a morphism of cochain complexes. □

Remark 2.7 (the van Est isomorphism). The main theorem of [Crainic 2003] states that if M is cohomologically n-connected, the map Φ_M induces an isomorphism in cohomology in degree $\leq n$ and is injective in degree $n+1$. In the universal case for the action of G on G/K, which is contractible, we therefore find an isomorphism $H^\bullet_{\mathrm{diff}}(G) \cong H^\bullet_{\mathrm{inv}}(G/K)$. This is one version of the classical van Est theorem [1955a; 1955b]. In this case we have by left translation

$$\Omega^\bullet_{\mathrm{inv}}(G/K) \cong \left(\bigwedge^{\bullet} (\mathfrak{g}/\mathfrak{k})^* \right)^K, \tag{2.8}$$

under which the de Rham differential identifies with the Chevalley–Eilenberg differential computing the relative Lie algebra cohomology $H^\bullet_{\mathrm{CE}}(\mathfrak{g}; K)$. With this, the van Est isomorphism is written as

$$H^\bullet_{\mathrm{diff}}(G) \cong H^\bullet_{\mathrm{CE}}(\mathfrak{g}; K). \tag{2.9}$$

Proposition 2.10. *Let $f : M \to N$ be an equivariant smooth map between proper G-manifolds. Then the following diagram commutes*:

$$\begin{array}{ccc} H^\bullet_{\mathrm{diff}}(G) & \xrightarrow{\Phi_N} & H^\bullet_{\mathrm{inv}}(N) \\ & \searrow^{\Phi_M} & \downarrow f^* \\ & & H^\bullet_{\mathrm{inv}}(M) \end{array}$$

Proof. Let χ_M be a cut-off function for the G-action on M. Then the pull-back $f^*\chi_M$ is a cut-off function for the G-action on N. For this cut-off function we obviously have $\omega_c^{f^*\chi_M} = f^*\omega_c^{\chi_M}$. The result now follows from the fact that the van Est map is independent of the choice of cut-off function. □

Corollary 2.11. *Under the van Est isomorphism $H^\bullet_{\mathrm{diff}}(G) \cong H^\bullet_{\mathrm{inv}}(G/K)$, the van Est map is identified with the pull-back along the classifying map $\psi_M : M \to G/K$, i.e.*,

$$\Phi_M = \psi_M^*.$$

2C. ***Group cocycles of polynomial growth.*** In a later stage of the paper, in the discussion of the extension properties of cyclic cocycles associated to smooth group cocycles, it will be important to control the growth of these group cocycles. To this end, we shall prove below a criterion guaranteeing that we can represent classes in $H^\bullet_{\mathrm{diff}}(G)$ by cocycles that have at most polynomial growth. For this, we begin by recalling Dupont's inverse [1976] of the van Est map $\Phi_{G/K}$ establishing the isomorphism (2.9). Choose an Ad_K-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, which, by left translations, induces a G-invariant Riemannian metric on G/K. This metric defines an orthogonal decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ with $\mathfrak{p} \cong T_{eK}(G/K)$. Since K is maximal compact, the (Riemannian) exponential map induces an isomorphism $\mathfrak{p} \cong G/K$ (with inverse denoted by log), and we can define the contraction

$$\varphi_s(x) := \exp(s \log(x))$$

of G/K to its basepoint $eK \in G/K$, i.e., $\varphi_1 = \mathrm{id}_{G/K}$ and $\varphi_0(x) = eK$. Now, given $k+1$ points $g_0K, \dots, g_kK \in G/K$, also denoted $\bar{g}_0, \dots, \bar{g}_k$, we can consider the geodesic simplex $\Delta^k(g_0K, \dots, g_kK) \subset G/K$ defined inductively as the cone over $\Delta^{k-1}(\bar{g}_1, \dots, \bar{g}_k)$ with tip point $\bar{g}_0$. More precisely, define the singular simplex $\sigma^k(\bar{g}_0, \dots, \bar{g}_k) : \Delta^k \to G/K$, where $\Delta^k := \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} : t_i \geq 0, \sum_i t_i = 1\}$, by

$$\begin{aligned}\sigma^k(g_0K, \dots, g_kK)(t_0, \dots, t_k) \\ := g_0\varphi_{t_0}\left(\sigma^{k-1}(g_0^{-1}g_1K, \dots, g_0^{-1}g_kK)\left(\frac{t_1}{1-t_0}, \dots, \frac{t_k}{1-t_0}\right)\right), \quad (2.12)\end{aligned}$$

and $\sigma^0(gK) := gK$. We write $\Delta^k(g_0K, \dots, g_kK)$ for the image of this simplex. By construction, this k-simplex is G-invariant: $g\Delta^k(\bar{g}_0, \dots, \bar{g}_k) = \Delta^k(g\bar{g}_0, \dots, g\bar{g}_k)$. With these simplices we define a map

$$J : \Omega^\bullet_{\mathrm{inv}}(G/K) \to C^\bullet_{\mathrm{diff}}(G), \quad \alpha \mapsto J(\alpha)(g_0, \dots, g_k) := \int_{\Delta^k(g_0K, \dots, g_kK)} \alpha, \quad (2.13)$$

which is easily checked to be a morphism of cochain complexes. Since $\Phi_{G/K} \circ J = \mathrm{id}$, J is a quasi-isomorphism.

Theorem 2.14. *Let G be a Lie group with finitely many connected components. Let K be a maximal compact subgroup and assume that G/K is of nonpositive sectional curvature with respect to the G-invariant metric induced by an* Ad_K-*invariant inner product $\langle\ ,\ \rangle$ on* $\mathfrak{g}$. *Then the group cocycle associated to a closed $\alpha \in \Omega^k_{\mathrm{inv}}(G/K)$ has polynomial growth. More precisely, if we write $d(g)$ for the distance from eK to gK in G/K, there exists a constant $C > 0$ and a natural number $N \in \mathbb{N}$ such that the following estimate holds true*:

$$|J(\alpha)(g_0, \dots, g_k)| \leq C(1 + d(g_0))^N \cdots (1 + d(g_k))^N.$$

Proof. Denote by $\|\alpha\|$ the norm of the Lie algebra cocycle $\alpha \in C^k_{\mathrm{CE}}(\mathfrak{g}; K) = \Omega^k_{\mathrm{inv}}(G/K)$ defined by the K-invariant metric on the Lie algebra $\mathfrak{g}$ of G that defines the metric on G/K. Since α is a G-invariant differential form we obviously have the inequality

$$|J(\alpha)(g_0, \dots, g_k)| \leq \|\alpha\| \operatorname{Vol}(\Delta^k(\bar{g}_0, \dots, \bar{g}_k)).$$

We now prove that, under the assumptions of the theorem, the volume of the geodesic k-simplex on G/K has at most polynomial growth in the geodesic distance of its vertices, thus completing the proof. For this we adapt an argument from [Inoue and Yano 1982, Proposition 1]; we thank Andrea Sambusetti for very useful discussions on this point and for bringing this article to our attention.

Let $\tau : \Delta^{k-1}(g_1K, \dots, g_kK) \times [0, 1] \to \Delta^k(eK, g_1K, \dots, g_kK)$ be defined by

$$\tau(x, t) := \varphi_{1-t}(x).$$

With this we can write

$$\tau^* \, d\mathrm{vol}_{\Delta^k(eK, g_1K, \dots, g_kK)} = \phi(x, t)\, dt \wedge \pi^* \, d\mathrm{vol}_{\Delta^{k-1}(g_1K, \dots, g_kK)}$$

for some function $\phi(x, t)$, where

$$\pi : \Delta^{k-1}(g_1K, \dots, g_kK) \times [0, 1] \to \Delta^{k-1}(g_1K, \dots, g_kK)$$

is the projection.

Choose $x_0 \in \Delta^{k-1}(g_1K, \dots, g_kK)$ and let $\gamma_{x_0}(t) := \varphi_s(x_0)$ be the geodesic starting in $\gamma_{x_0}(0) = x_0$ and ending in the basepoint $\gamma_{x_0}(1) = eK$. Let $X_0(0), \dots, X_{n-1}(0)$ be an orthonormal frame of $T_{x_0}(G/K)$ such that $X_0(0) = \dot\gamma(0)/L$, with $L = d(eK, x_0)$ the length of γ_{x_0}, and such that $X_0(0), \dots, X_{k-1}(0)$ span $T_{x_0}\Delta^k(eK, g_1K, \dots, g_kK)$. We denote by $X_i(t)$ the unique extension to parallel vector fields along $\gamma_{x_0}(t)$.

We now choose local coordinates $(y^1, \dots, y^{k-1})$ on $\Delta^{k-1}(g_1K, \dots, g_kK)$ around x_0 satisfying

$$\frac{\partial}{\partial y^i}(x_0) = X_i(0) + b_i X_0(0), \tag{2.15}$$

for some constants $b_i \in \mathbb{R}$. We then get local coordinates $(y^1, \dots, y^{k-1}, t)$ around the image of γ, such that $(y_0^1, \dots, y_0^{k-1}, 0)$ corresponds to the point x_0. Comparing the vector fields $\partial/\partial y^i$ with X_j defines functions $a_{ij} : [0, 1] \to \mathbb{R}$ by

$$\frac{\partial}{\partial y^i}(\gamma(t)) = \sum_{j=0}^{n-1} a_{ij}(t) X_j(t). \tag{2.16}$$

The normal projection of $\partial/\partial y^i$ along $\gamma_{x_0}(t)$ is then the vector field

$$Y_i(t) := \sum_{j=1}^{n-1} a_{ij}(t) X_j(t), \quad i = 1, \dots, k-1,$$

satisfying $Y_i(0) = X_i(0)$ and $Y_i(1) = 0$. Now note that the vector field $\partial/\partial y^i$ is a Jacobi field along the geodesic $\gamma_{x_0}(t)$, because by its definition we have

$$\frac{\partial}{\partial y^i}(\gamma(t)) = \frac{d}{ds}\gamma_{(y_0^1, \dots, s y_0^i, \dots, y_0^{k-1})}(t)\big|_{s=0},$$

where $\gamma_{(y^1,\dots,y^{k-1})}(t)$ is the geodesic $\varphi_t(0, y^1, \dots, y^k)$ connecting

$$(y^1, \dots, y^{k-1}) \in \Delta^{k-1}(g_1K, \dots, g_kK)$$

with eK, and $x_0 = (y_0^1, \dots, y_0^{k-1})$ in local coordinates. It follows that $Y_i(t)$ is also a Jacobi field along $\gamma_{x_0}(t)$ because it is the normal projection of $\partial/\partial y^i$. (The normal and tangential projections of a Jacobi vector field are Jacobi.)

We define the $(k-1) \times (k-1)$ matrix $A(t)$ with entries

$$\langle Y_i(t), Y_j(t)\rangle = \sum_{k=1}^{n-1} a_{ik}(t)a_{kj}(t).$$

Now, computing the inner products of the vector fields $\partial/\partial y^i$ we get from (2.15) and an elementary computation that

$$d\mathrm{vol}_{\Delta^{k-1}(g_1K,\dots,g_kK)} = \sqrt{\Big(1+\sum_i b_i^2\Big)}\, dy^1 \wedge \cdots dy^k,$$

whereas from (2.16) we get

$$d\mathrm{vol}_{\Delta^{k}(eK,g_1K,\dots,g_kK)} = L\sqrt{\det(A(t))}\, dt \wedge dy^1 \wedge \cdots \wedge dy^k.$$

It follows that

$$\phi(x_0, t) = \frac{L\sqrt{\det(A(t))}}{\sqrt{\big(1+\sum_i b_i^2\big)}} \le L\sqrt{\det(A(t))}\,.$$

Consider now the Jacobi field $U(t) = \sum_{i=1}^{k-1} u^i Y_i(t)$, for a vector $u = (u^i)_{i=1}^{k-1} \in \mathbb{R}^{k-1}$. By the Jacobi equation we now have

$$\frac{d^2}{dt^2}\|U(t)\| = 2\|\nabla_{\partial/\partial t}U(t)\|^2 - 2\Big\langle R\Big(U(t), \frac{\partial}{\partial t}\Big)\frac{\partial}{\partial t}, U(t)\Big\rangle \ge 0.$$

Together with the fact that $\|U(0)\|^2 = \|u\|^2$ and $\|U(1)\|^2 = 0$, it follows that $\|U(t)\|^2 \le \|u\|^2(1-t)$.

We obviously have

$$\det(A(t)) \le \left(\sup_{u\neq 0} \frac{u^t A(t)u}{\|u\|^2}\right)^{k-1},$$

and $u^t A(t)u = \|U(t)\|^2$, so that we can conclude that

$$\det(A(t)) \le (1-t)^{k-1}.$$

This is the crucial estimate that we use below. Before we complete the proof of the theorem, we prove the following lemma:

Lemma 2.17. *For $x \in \Delta^{k-1}(g_1K, \dots, g_kK) \subset \Delta^k(g_0K, \dots, g_kK)$, we have*

$$d(g_0K, x) \le \max\{d(g_0K, g_1K), \dots, d(g_0K, g_kK)\}.$$

Proof. We prove this by induction. For $k = 1$, the statement is obvious. Suppose now that we have proved the lemma for $k - 1$. Consider

$$x \in \Delta^{k-1}(g_1K, \dots, g_kK) \subset \Delta^k(g_0K, \dots, g_kK).$$

Let $\gamma(t)$ be the geodesic connecting g_1K and x, but continued until it hits the simplex $\Delta^{k-2}(g_2K, \dots, g_kK)$ in a point that we call y. Using convexity of the distance function on a manifold with nonpositive sectional curvature, we see that $d(g_0K, x) \le \max\{(d(g_0K, g_1K), d(g_0K, y)\}$. To estimate the distance $d(g_0K, y)$, we now consider the geodesic simplex $\Delta^{k-1}(g_0K, g_2K, \dots, g_kK)$ and apply the induction hypothesis. □

The final step in the proof of the theorem is an induction argument: First observe that for $k=1$, the statement of the theorem is obviously true because $\Delta^1(g_0K, g_1K)$ is simply the geodesic connecting g_0K and g_1K. Suppose now that we have proved the statement for $k - 1$. Then we compute the volume of the simplex $\Delta^k(eK, g_1K, \dots, g_kK)$ as follows:

$$\begin{aligned}
&\mathrm{Vol}(\Delta^k(eK, g_1K, \dots, g_kK)) \\
&\qquad = \int_{\Delta^k(eK,g_1K,\dots,g_kK)} d\mathrm{vol}_{\Delta^k(eK,g_1K,\dots,g_kK)} \\
&\qquad = \int_{\Delta^{k-1}(g_1K,\dots,g_kK)} \left(\int_0^1 \phi(x,t)\,dt \right) d\mathrm{vol}_{\Delta^{k-1}(g_1K,\dots,g_kK)}(x) \\
&\qquad \le \int_{\Delta^{k-1}(g_1K,\dots,g_kK)} L(x) \left(\int_0^1 (1-t)^{(k-1)/2}\,dt \right) d\mathrm{vol}_{\Delta^{k-1}(g_1K,\dots,g_kK)}(x) \\
&\qquad \le C \int_{\Delta^{k-1}(g_1K,\dots,g_kK)} L(x)\, d\mathrm{vol}_{\Delta^{k-1}(g_1K,\dots,g_kK)}(x) \\
&\qquad \le C \prod_{i=1}^{k} (1 + d(g_i))\, \mathrm{Vol}(\Delta^{k-1}(g_1K, \dots, g_kK)).
\end{aligned}$$

By the induction assumption,

$$\mathrm{Vol}(\Delta^{k-1}(g_1K, \dots, g_kK)) \le C' \prod_{i=1}^{k-1} (1 + d(g_1K, g_iK)).$$

Together with the inequality $d(g_1K, g_iK) \le d(g_0K, g_1K) + d(g_0K, g_iK)$, this completes the proof of the theorem. □

Example 2.18. As an example, consider the abelian group $G = \mathbb{R}^2$ with maximal compact group given by the trivial group $\{0\} \subset \mathbb{R}^2$. In this abelian case we have that $H^\bullet_{\mathrm{inv}}(\mathbb{R}^2) = \bigwedge^\bullet \mathbb{R}^2$, and a generator in degree 2 is given by the area form $dx \wedge dy$, so that we find

$$J(dx \wedge dy)(x, y, z) = \mathrm{Area}_{\mathbb{R}^2}(\Delta^2(x, y, z)), \tag{2.19}$$

which evidently grows polynomially in the norm of x, y and z.

Remark 2.20. (i) When G is a connected semisimple Lie group, G/K is a noncompact symmetric space and has nonpositive sectional curvature [Helgason 2001]. Therefore the curvature assumptions in the lemma are automatically satisfied in this case. In fact, the conjecture in [Dupont 1979] is that for semisimple Lie groups all these cocycles are bounded. For recent work on this conjecture, see [Hartnick and Ott 2012; Kim and Kim 2015]. In this last reference, different simplices are used, given by the barycentric subdivision of the geodesic ones, to prove boundedness of the top-dimensional cocycle for general connected semisimple Lie groups.

(ii) In general, the polynomial bounds of the lemma above are not sharp, as expected from the conjecture mentioned in (i). For example, when $G = \mathrm{SL}(2, \mathbb{R})$, the maximal compact subgroup is given by $K = \mathrm{SO}(2)$ so that $G/K = \mathbb{H}^2$, the hyperbolic 2-plane. Again, we have $H^2_{\mathrm{inv}}(\mathbb{H}^2) = \mathbb{R}$, with generator the hyperbolic area form. This leads to a smooth group cocycle given by the same formula as (2.19) above, replacing the Euclidean area by the hyperbolic one, but this time the cocycle is bounded because the area of a hyperbolic triangle does not exceed π, confirming the boundedness in top-degree mentioned in (i).

3. Algebras of invariant kernels

3A. ***Smoothing kernels of G-compact support.*** Let M again be a closed smooth manifold carrying a smooth proper action of a Lie group G with $|\pi_0(G)| < \infty$ and with compact quotient. We choose an invariant complete Riemannian metric, denoted h, with associated distance function denoted by $d_M(x, y)$ for $x, y \in M$, and volume form $d\mathrm{vol}(x)$. We fix a left-invariant metric on G and we denote by d_G the associated distance function.

Definition 3.1. Consider a G-equivariant smoothing kernel $k \in C^\infty(M \times M)$; thus k is an element in $C^\infty(M \times M)^G$. We say that k is of G-compact support if the projection of $\mathrm{supp}(k) \subset M \times M$ in $(M \times M)/G$, with G acting diagonally, is compact.

We denote by $\mathcal{A}^c_G(M)$ the set of G-equivariant smoothing kernels of G-compact support. It is well known that $\mathcal{A}^c_G(M)$ has the structure of a Fréchet algebra with respect to the convolution product

$$(k * k')(x, z) = \int_M k(x, y)k'(y, z)\, d\mathrm{vol}(y).$$

It is also well known that each element $k \in \mathcal{A}^c_G(M)$ defines an equivariant linear operator $S_k : C^\infty_c(M) \to C^\infty_c(M)$, the integral operator associated to the kernel k, and that $S_k \circ S_{k'} = S_{k*k'}$. Moreover, S_k extends to an equivariant bounded operator on $L^2(M)$. We have therefore defined a subalgebra of $\mathcal{B}(L^2(M))$, which we denote as $\mathcal{S}^c_G(M)$; by definition,

$$\mathcal{S}^c_G(M) := \{S_k : k \in \mathcal{A}^c_G(M)\}. \tag{3.2}$$

The case in which there is an equivariant vector bundle E on M is similar, in that we start with G-equivariant elements in $C^\infty(M \times M, E \boxtimes E^*)$ and then proceed analogously, defining in this way the Fréchet algebra $\mathcal{A}^c_G(M, E)$ and $\mathcal{S}^c_G(M, E) := \{S_k : k \in \mathcal{A}^c_G(M, E)\}$, a subalgebra of $\mathcal{B}(L^2(M, E))$.

Notation. Keeping with a well-established abuse of notation, we often identify $\mathcal{A}^c_G(M, E)$ with $\mathcal{S}^c_G(M, E)$, thus identifying a smoothing kernel k in $\mathcal{A}^c_G(M, E)$ with the corresponding operator $S_k \in \mathcal{S}^c_G(M, E)$.

3B. ***Holomorphically closed subalgebras.*** Using the remarks at the end of the previous subsection we see that $\mathcal{S}^c_G(M, E)$ is in an obvious way a subalgebra of the reduced Roe C^*-algebra $C^*(M, E)^G$. Recall that $C^*(M, E)^G$ is defined as the norm closure in $\mathcal{B}(L^2(M, E))$ of the algebra $C^*_c(M, E)^G$ of G-equivariant bounded operators of finite propagation and locally compact. In fact, $\mathcal{S}^c_G(M, E) \subset C^*_c(M, E)^G$. The Roe algebra is canonically isomorphic to $\mathbb{K}(\mathcal{E})$, the C^*-algebra of compact operators of the Hilbert $C^*_r(G)$-Hilbert module $\mathcal{E}$ obtained by closing the space $C^\infty_c(M, E)$ of compactly supported sections of E on M, endowed with the $C^*_r G$-valued inner product

$$(e, e')_{C^*_r G}(x) := (e, x \cdot e')_{L^2(M,E)}, \quad e, e' \in C^\infty_c(M, E),\ x \in G. \tag{3.3}$$

See for example [Hochs and Wang 2018], where the Morita isomorphism

$$K_*(\mathbb{K}(\mathcal{E})) = K_*(C^*(M, E)^G) \xrightarrow{\mathcal{M}} K_*(C^*_r G)$$

is explicitly discussed. We shall come back to this important point in a moment. The subalgebra $\mathcal{S}^c_G(M, E)$ is not holomorphically closed in $C^*(M, E)^G$. On the other hand, such a subalgebra of $C^*(M, E)^G$ is implicitly constructed in [Hochs and Wang 2018, Section 3.1] by making use of the slice theorem. We recall the essential ingredients, following [Hochs and Wang 2018, Section 3.1] (we also extend the context slightly for future use).

As already remarked in the previous section, under our assumptions on G, there exists a global slice for the action of G on M. Thus if K is a maximal compact subgroup of G there exists a K-invariant compact submanifold $S \subset M$ such that the action map $[g, s] \mapsto gs$, $g \in G$, $s \in S$, defines a G-equivariant diffeomorphism

$$G \times_K S \xrightarrow{\alpha} M,$$

where S is compact because the action is cocompact. Corresponding to this diffeomorphism we have an isomorphism $E \cong G \times_K (E|_S)$, and thus isomorphisms

$$C_c^\infty(M, E) \cong (C_c^\infty(G) \hat{\otimes} C^\infty(S, E|_S))^K,$$
$$C^\infty(M, E) \cong (C^\infty(G) \hat{\otimes} C^\infty(S, E|_S))^K.$$

See [Hochs and Wang 2018, Section 3.1]. Here we are taking the projective tensor product $\hat{\otimes}_\pi$ of the two Fréchet algebras; however, since $C^\infty(S, E|_S)$ is nuclear, the injective $\hat{\otimes}_\epsilon$ and projective $\hat{\otimes}_\pi$ tensor products coincide, which is why we do not use a subscript. Consider now $\Psi^{-\infty}(S, E|_S)$, also a nuclear Fréchet algebra, and let

$$\widetilde{A}^c_G(M, E) := (C_c^\infty(G) \hat{\otimes} \Psi^{-\infty}(S, E|_S))^{K \times K}.$$

$\widetilde{A}^c_G(M, E)$ is a Fréchet algebra, with product denoted by $*$. Let $\tilde{k} \in \widetilde{A}^c_G(M, E)$ and consider the operator $T_{\tilde{k}}$ on $L^2(M, E)$ given by

$$(T_{\tilde{k}} e)(gs) = \int_G \int_S g\tilde{k}(g^{-1}g', s, s')g'^{-1} e(g's')\, ds'\, dg'. \tag{3.4}$$

This is a bounded G-equivariant operator with smooth G-equivariant Schwartz kernel given by

$$\kappa(gs, g's') = g\tilde{k}(g^{-1}g', s, s')g'^{-1},$$

where the g and g'^{-1} on the right-hand side are used in order to identify fibers on the vector bundle E. The assignment $\tilde{k} \to T_{\tilde{k}}$ is injective and satisfies

$$T_{\tilde{k}} \circ T_{\tilde{k}'} = T_{\tilde{k} * \tilde{k}'}.$$

Consider the subalgebra of the bounded operators on $L^2(M, E)$ given by

$$\{T_{\tilde{k}} : \tilde{k} \in \widetilde{A}^c_G(M, E)\}$$

endowed with the Fréchet algebra structure induced by the injective homomorphism $\tilde{k} \to T_{\tilde{k}}$. It is easy to see that this algebra is precisely equal to the algebra we have considered in the previous subsection, $\mathcal{S}^c_G(M, E) := \{S_k : k \in \mathcal{A}^c_G(M, E)\}$. Thus,

$$\mathcal{S}^c_G(M, E) = \{T_{\tilde{k}} : \tilde{k} \in \widetilde{A}^c_G(M, E)\}. \tag{3.5}$$

In summary, using the slice theorem we have realized $\mathcal{S}^c_G(M, E)$ as a projective tensor product of convolution operators on G and smoothing operators on S. This preliminary result puts us in the position of enlarging the algebra $\mathcal{S}^c_G(M, E)$ and obtaining a subalgebra dense and holomorphically closed in $C^*(M, E)^G$. To this end we give the following definition.

Definition 3.6. Let $\mathcal{A}(G)$ a set of functions on G. We say that $\mathcal{A}(G)$ is admissible if the following properties are satisfied:

(1) $\mathcal{A}(G)$ is a Fréchet space and there are continuous inclusions

$$C_c^\infty(G) \subset \mathcal{A}(G) \subset L^2(G);$$

(2) the action by convolution defines a continuous injective map $\mathcal{A}(G) \hookrightarrow C_r^*(G)$ which makes $\mathcal{A}(G)$ a subalgebra of $C_r^*(G)$;

(3) $\mathcal{A}(G)$ is holomorphically closed in $C_r^*(G)$.

We can then consider

$$\widetilde{A}_G(M, E) := (\mathcal{A}(G) \hat{\otimes} \Psi^{-\infty}(S, E|_S))^{K \times K},$$

a Fréchet algebra, and for $\tilde{k} \in \widetilde{A}_G(M, E)$, the bounded operator $T_{\tilde{k}}$ on $L^2(M, E)$ given by

$$(T_{\tilde{k}} e)(gs) = \int_G \int_S g\tilde{k}(g^{-1}g', s, s')g'^{-1} e(g's')\, ds'\, dg'. \tag{3.7}$$

The operator $T_{\tilde{k}}$ is an integral operator with G-equivariant Schwartz kernel κ given by $\kappa(gs, g's') = g\tilde{k}(g^{-1}g', s, s')g'^{-1}$. Since $\mathcal{A}(G) \hookrightarrow C_r^*(G)$, with $\mathcal{A}(G)$ acting by convolution, we see that $T_{\tilde{k}}$ is L^2-bounded.

Definition 3.8. We define $\mathcal{A}_G(M, E)$ as the subalgebra of the bounded operators on $L^2(M, E)$ given by

$$\mathcal{A}_G(M, E) := \{T_{\tilde{k}} : \tilde{k} \in \widetilde{A}_G(M, E)\}.$$

We endow $\mathcal{A}_G(M, E)$ with the structure of a Fréchet algebra induced by the injective homomorphism $\tilde{k} \to T_{\tilde{k}}$.

Proposition 3.9. *Under the assumptions* (1)–(3) *for $\mathcal{A}(G)$ in Definition 3.6, the following hold*:

(i) *We have a continuous inclusion of Fréchet algebras*

$$\mathcal{S}_G^c(M, E) \subset \mathcal{A}_G(M, E). \tag{3.10}$$

(ii) *$\mathcal{A}_G(M, E)$ is a dense subalgebra of $C^*(M, E)^G$ and it is holomorphically closed.*

Proof. (i) The continuous inclusion of Fréchet algebras $\mathcal{S}_G^c(M, E) \subset \mathcal{A}_G(M, E)$ follows immediately from (3.5).

(ii) The fact that $\mathcal{A}_G(M, E)$ is a dense subalgebra of $C^*(M, E)^G$ is proved precisely as in [Hochs and Wang 2018, Lemma 3.3]; the property of being holomorphically closed follows easily from the hypothesis that $\mathcal{A}(G)$ is holomorphically closed in $C_r^* G$ and the well-known fact that $\Psi^{-\infty}(S, E|_S)$ is holomorphically closed in the compact operators of $L^2(S, E|_S)$. □

Definition 3.11. Let G be a Lie group and let L be a length function on G. We consider

$$H_L^\infty(G) = \left\{ f \in L^2(G) : \int_G (1+L(x))^{2k} |f(x)|^2 \, dx < +\infty \text{ for all } k \in \mathbb{N} \right\} \quad (3.12)$$

endowed with the Fréchet topology induced by the sequence of seminorms

$$\nu_k(f) := \|(1+L)^k f\|_{L^2}. \quad (3.13)$$

We say that the pair (G, L) satisfies the rapid decay property (RD) if there is a continuous inclusion $H_L^\infty(G) \hookrightarrow C_r^*(G)$.

For conditions equivalent to the one given here, see [Chatterji et al. 2007]. We also recall that if G satisfies (RD) then G is unimodular [Ji and Schweitzer 1996].

Proposition 3.14. *Let G be a Lie group with $|\pi_0(G)| < \infty$; we can and shall choose L to be the length function associated to a left-invariant Riemannian metric. Assume additionally that G satisfies* (*RD*) (*with respect to this L*). *Then*

$$H_L^\infty(G) = \left\{ f \in L^2(G) : \int_G (1+L(x))^{2k} |f(x)|^2 \, dx < +\infty \right\} \quad (3.15)$$

satisfies the properties (1)–(3) *given in Definition 3.6. Consequently*, *for G with $|\pi_0(G)| < \infty$ and with the* (*RD*) *property*, *there exists a subalgebra of $C^*(M, E)^G$*, *denoted $\mathcal{S}_G^\infty(M, E)$*, *which consists of integral operators*, *is dense and holomorphically closed in $C^*(M, E)^G$ and contains $\mathcal{S}_G^c(M, E)$ as a subalgebra.*

Proof. The fact that $H_L^\infty(G)$ is not only contained in $C_r^*(G)$, via convolution, but is in fact a subalgebra of it, follows from [Jolissaint 1990]. Hence $H_L^\infty(G)$ satisfies the properties (1) and (2) given in Definition 3.6. The fact that this subalgebra is holomorphically closed is proved as in [Jolissaint 1989]. The rest of the proposition then follows from Proposition 3.9. □

Example 3.16. Here are two examples of Lie groups that satisfy property (RD), and to which our theory applies:

(1) The abelian group $\mathbb{R}^n$ satisfies (RD). In this case the algebra $H_L^\infty(\mathbb{R}^n)$ associated to the length function defined by the Euclidean metric is the algebra of rapidly decaying functions on $\mathbb{R}^n$.

(2) Connected semisimple Lie groups satisfy property (RD) [Chatterji et al. 2007], for example $G = \mathrm{SL}(2, \mathbb{R})$. In this case the algebra $H_L^\infty(G)$ is closely related to Harish–Chandra's Schwartz algebra $\mathcal{C}(G)$ (see below).

Remark 3.17. We have just seen that for G semisimple, by taking $\mathcal{A}(G) = H_L^\infty(G)$ we obtain a holomorphically closed subalgebra $\mathcal{S}_G^\infty(M, E) \subset C^*(M, E)^G$. Notice that there are other algebras that can be considered. For example, we can consider as in [Hochs and Wang 2018] the Harish-Chandra Schwartz algebra $\mathcal{C}(G) \subset C_r^*(G)$.

This is a holomorphically closed subalgebra of $C^*_r(G)$ [Lafforgue 2002], which is made of smooth functions acting by convolution. The corresponding algebra $\mathcal{C}_G(M, E) \subset C^*(M, E)^G$ is a subalgebra of $C^*(M, E)^G$ with elements that are in fact smoothing operators. One can prove that $\mathcal{C}(G) \subset H^\infty_L(G)$ [Varadarajan 1977, §II.9] and consequently, $\mathcal{C}_G(M, E) \subset \mathcal{S}^\infty_G(M, E)$. Notice that Hochs and Wang have proved that the heat operator $\exp(-tD^2)$ is an element in $\mathcal{C}_G(M, E)$. Hence $\exp(-tD^2) \in \mathcal{S}^\infty_G(M, E)$.

4. Index classes

From now on we make constant use of the identification $\mathcal{A}^c_G(M, E) \equiv \mathcal{S}^c_G(M, E)$.

4A. *The index class in $K_*(C^*(M, E)^G)$.* We consider as before a closed even-dimensional manifold M with a proper cocompact action of G. Let D be a G-equivariant odd $\mathbb{Z}_2$-graded Dirac operator. Recall, first of all, the classical Connes–Skandalis idempotent. Let Q_σ be a G-equivariant parametrix of G-compact support with remainders $S_\pm$; here the subscript σ stands for symbolic. Consider the 2×2 matrix

$$P_\sigma := \begin{pmatrix} S_+^2 & S_+(I+S_+)Q \\ S_-D^+ & I-S_-^2 \end{pmatrix}. \tag{4.1}$$

This produces a class

$$\mathrm{Ind}_c(D) := [P_\sigma] - [e_1] \in K_0(\mathcal{A}^c_G(M, E)) \quad \text{with } e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.2}$$

To understand where this definition comes from, see for example [Connes and Moscovici 1990]. Recall now that $\mathcal{A}^c_G(M, E) \subset C^*(M, E)^G$.

Definition 4.3. The C^*-index associated to D is the class

$$\mathrm{Ind}_{C^*(M,E)}(D) \in K_0(C^*(M, E)^G)$$

obtained by taking the image of the Connes–Skandalis projector in $K_0(C^*(M, E)^G)$. Unless absolutely necessary, we denote this index class simply by $\mathrm{Ind}(D)$.

Remark 4.4. If we are in the position of considering a dense holomorphically closed subalgebra $\mathcal{A}_G(M, E)$ of $C^*(M, E)^G$ as in the previous section, then we can equivalently take the image of the Connes–Skandalis projector in $K_0(\mathcal{A}_G(M, E))$ (recall that, by construction, $\mathcal{A}^c_G(M, E) \subset \mathcal{A}_G(M, E) \subset C^*(M, E)^G$). For example, if G satisfies (RD) and $|\pi_0(G)| < \infty$, then we can take the C^*-index class as the image of the Connes–Skandalis projector in $K_0(\mathcal{S}^\infty_G(M, E))$.

Remark 4.5. There are other representatives of $\mathrm{Ind}(D) \in K_0(C^*(M, E)^G)$ that can be of great interest. For example, as in [Connes and Moscovici 1990], we can choose the parametrix (which is not of G-compact support)

$$Q_V := \frac{I - \exp\left(-\frac{1}{2}D^- D^+\right)}{D^- D^+} D^+,$$

obtaining $I - Q_V D^+ = \exp\left(-\frac{1}{2}D^- D^+\right)$, $I - D^+ Q_V = \exp\left(-\frac{1}{2}D^+ D^-\right)$. This particular choice of parametrix produces the idempotent

$$V_D = \begin{pmatrix} e^{-D^- D^+} & e^{-\frac{1}{2}D^- D^+}\left(\frac{I - e^{-D^- D^+}}{D^- D^+}\right) D^- \\ e^{-\frac{1}{2}D^+ D^-} D^+ & I - e^{-D^+ D^-} \end{pmatrix}. \tag{4.6}$$

We call this the Connes–Moscovici idempotent. One can also consider the graph-projection $[e_D] - [e_1] \in K_0(C^*(M, E)^G)$ with e_D given by

$$e_D = \begin{pmatrix} (I + D^- D^+)^{-1} & (I + D^- D^+)^{-1} D^- \\ D^+(I + D^- D^+)^{-1} & D^+(I + D^- D^+)^{-1} D^- \end{pmatrix}. \tag{4.7}$$

Finally, following [Moscovici and Wu 1994], we can consider the projector

$$P(D) := \begin{pmatrix} S_+^2 & S_+(I + S_+)\mathcal{P} \\ S_- D^+ & I - S_-^2 \end{pmatrix} \tag{4.8}$$

with $\mathcal{P} = \bar{u}(D^- D^+) D^-$, $S_+ = I - \mathcal{P} D^+$, $S_- = I - D^+ \mathcal{P}$ and $\bar{u}(x) := u(x^2)$ with $u \in C^\infty(\mathbb{R})$ an even function with the property that $w(x) = 1 - x^2 u(x)$ is a Schwartz function and w and u have compactly supported Fourier transform. One proves easily that $P(D) \in M_{2\times 2}(\mathcal{A}^c_G(M, E))$ (with the identity adjoined). It is not difficult to prove that

$$\begin{aligned} \operatorname{Ind}(D) &:= [P_\sigma] - [e_1] \\ &= [V_D] - [e_1] = [e_D] - [e_1] = [P(D)] - [e_1] \quad \text{in } K_0(C^*(M, E)^G). \end{aligned}$$

The advantage of using the Connes–Moscovici projection, the graph projection or the Moscovici–Wu projection is that Getzler rescaling can be used in order to prove the corresponding higher index formulae. This is crucial if one wishes to pass, for example, to manifolds with boundary. However, in this paper we concentrate solely on closed manifolds and use the approach to the index theorem given in [Pflaum et al. 2015b]; this employs the algebraic index theorem in a fundamental way.

4B. ***The index class in $K_\bullet(C_r^*(G))$.*** There is a canonical Morita isomorphism $\mathcal{M}$ between $K_*(C^*(M, E)^G)$ and $K_*(C_r^*(G))$. This is clear once we bear in mind that $C^*(M, E)^G$ is isomorphic to $\mathbb{K}(\mathcal{E})$; however, for reasons connected with the extension of cyclic cocycles, we want to be explicit about this isomorphism. We assume the existence of a dense holomorphically closed subalgebra $\mathcal{A}(G) \subset C_r^*(G)$ and follow [Hochs and Wang 2018]. Let $\mathcal{A}_G(M, E)$ be the dense holomorphically dense subalgebra of $C^*(M, E)^G$ corresponding to $\mathcal{A}(G)$, as defined in Section 3B.

Define a partial trace map $\mathrm{Tr}_S : \mathcal{A}_G(M, E) \to \mathcal{A}(G)$ associated to the slice S as follows: if $f \otimes k \in (\mathcal{A}(G)) \hat{\otimes} \Psi^{-\infty}(S, E|_S))^{K\times K}$ then

$$\mathrm{Tr}_S(f \otimes k) := f\,\mathrm{Tr}(T_k) = f \int_S \mathrm{tr}\, k(s, s)\, ds,$$

with T_k denoting the smoothing operator on S defined by k and $\mathrm{Tr}(T_k)$ its functional analytic trace on $L^2(S, E|_S)$. It is proved in [Hochs and Wang 2018] that this map induces the Morita isomorphism $\mathcal{M}$ between $K_*(C^*(M, E)^G)$ and $K_*(C^*_r(G))$. We denote the image through $\mathcal{M}$ of the index class $\mathrm{Ind}(D) \in K_0(C^*(M)^G)$ in the group $K_0(C^*_r(G))$ by $\mathrm{Ind}_{C^*_r(G)}(D)$. There are other well-known descriptions of the latter index class: one, following [Kasparov 1980], describes the $C^*_r(G)$-index class as the difference of two finitely generated projective $C^*_r(G)$-modules, using the invertibility modulo $C^*_r(G)$-compact operators of (the bounded transform of) D; the other description is via assembly and KK-theory, as in [Baum et al. 1994]. All these descriptions of the class $\mathrm{Ind}_{C^*_r(G)}(D) \in K_0(C^*_r(G))$ are equivalent. See [Roe 2002; Piazza and Schick 2014, Proposition 2.1].

5. Cyclic cocycles and pairings with *K*-theory

5A. ***Cyclic cohomology.*** In this subsection we briefly review the basic complex computing cyclic cohomology. Let A be a unital algebra. The space of reduced Hochschild cochains is defined as

$$C^\bullet_{\mathrm{red}}(A) := \mathrm{Hom}_{\mathbb{C}}(A \otimes (A/\mathbb{C}1)^\bullet, \mathbb{C})$$

and is equipped with the Hochschild differential $b : C^k_{\mathrm{red}}(A) \to C^{k+1}_{\mathrm{red}}(A)$ given by the standard formula

$$b\tau(a_0, \ldots, a_{k+1}) := \sum_{i=0}^{k} (-1)^i \tau(a_0, \ldots, a_i a_{i+1}, \ldots, a_k) + (-1)^{k+1} \tau(a_k a_0, \ldots, a_{k-1}).$$

The cyclic bicomplex is given by

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\uparrow b & & \uparrow b & & \uparrow b \\
C^2_{\mathrm{red}}(A) & \xrightarrow{B} & C^1_{\mathrm{red}}(A) & \xrightarrow{B} & C^0_{\mathrm{red}}(A) \\
\uparrow b & & \uparrow b & & \\
C^1_{\mathrm{red}}(A) & \xrightarrow{B} & C^0_{\mathrm{red}}(A) & & \\
\uparrow b & & & & \\
C^0_{\mathrm{red}}(A) & & & &
\end{array}$$

where $B : C^k_{\mathrm{red}}(A) \to C^{k-1}_{\mathrm{red}}(A)$ denotes Connes' cyclic differential

$$B\tau(a_0, \dots, a_{k-1}) := \sum_{i=0}^{k-1} (-1)^{(k-1)i} \tau(1, a_i, \dots, a_{k-1}, a_0, \dots, a_{i-1}).$$

We denote the total complex associated to this double complex by $CC^\bullet(A)$. When A is not unital, we consider the unitization $\widetilde{A} = A \oplus \mathbb{C}$, and compute cyclic cohomology from the complex $CC^\bullet(A) := CC^\bullet(\widetilde{A})/CC^\bullet(\mathbb{C})$.

Finally, let us close by mentioning that the structure underlying the definition of cyclic cohomology is that of a cocyclic object. This is a cosimplicial object $(X^\bullet, \partial^\bullet, \sigma^\bullet)$ equipped with an additional cyclic symmetry $t^n : X^n \to X^n$ of order $n+1$ satisfying well-known compatibility conditions with respect to the coface operators ∂ and degeneracies σ; see [Loday 1998]. For the cyclic cohomology of an algebra the underlying cosimplicial object is given by $X^k = C^k(A)$ with coface and degeneracies controlling the Hochschild complex. The additional cyclic symmetry t underlying cyclic cohomology is simply the operator which in degree k cyclically permutes the $k+1$ entries in a cochain $\tau \in C^k(A)$.

5B. ***The van Est map in cyclic cohomology.*** Let G be a unimodular Lie group with $|\pi_0(G)| < \infty$. In this subsection we describe, following [Pflaum et al. 2015a; 2015b], how to obtain cyclic cocycles from smooth group cocycles. In this, we can work with two algebras: $C^\infty_c(G)$, the convolution algebra of the group, and $\mathcal{A}^c_G(M)$, the algebra of invariant smoothing operators with cocompact support. In order to simplify the notation we take the vector bundle E to be the product bundle of rank 1.

We start by recalling a well-known fact: inspection of the differential (2.2) shows that the cochain complex $(C^\bullet_{\mathrm{diff}}(G), \delta)$ computing smooth group cohomology $H^\bullet_{\mathrm{diff}}(G)$ comes from an underlying cosimplicial structure given by coface maps ∂^i and codegeneracies σ^j defined on the vector space of homogeneous smooth group cochains $C^\bullet_{\mathrm{diff}}(G)$. This simplicial vector space can be upgraded to a cocyclic one by the cyclic operator $t : C^\bullet \to C^\bullet$ given by

$$(tf)(g_0, \dots, g_k) = f(g_k, g_0, \dots, g_{k-1}), \quad f \in C^k_{\mathrm{diff}}(G).$$

As seen above, the Hochschild theory of this cocyclic complex is just the smooth group cohomology. The associated cyclic theory is given by $\bigoplus_{i\geq 0} H^{\bullet-2i}_{\mathrm{diff}}(G)$.

Let us now describe the associated cyclic cocycles on the convolution algebra $C^\infty_c(G)$. Instead of using the full complex of smooth group cochains, we restrict to the quasi-isomorphic subcomplex $C^\bullet_{\mathrm{diff},\lambda}(G) \subset C^\bullet_{\mathrm{diff}}(G)$ of *cyclic* cochains, i.e., cochains $c \in C^k_{\mathrm{diff}}(G)$ satisfying

$$c(g_0, \dots, g_k) = (-1)^k c(g_k, g_0, \dots, g_{k-1}).$$

Let $c \in C^k_{\mathrm{diff}}(G)$ be a smooth homogeneous group cochain. Define the cyclic cochain $\tau_c \in C^k(C^\infty_c(G))$ by

$$\tau^G_c(a_0, \dots, a_k) := \int_{G^{\times k}} c(e, g_1, g_1 g_2, \dots, g_1 \cdots g_k) \cdot a_0((g_1 \cdots g_k)^{-1}) a_1(g_1) \cdots a_k(g_k)\, dg_1 \cdots dg_k. \quad (5.1)$$

Next up is the algebra $\mathcal{A}^c_G(M)$ of invariant smoothing operators with cocompact support. Again given a smooth homogeneous group cochain $c \in C^k_{\mathrm{diff}}(G)$, we now define a cyclic cochain on this algebra by the formula

$$\tau^M_c(k_0, \dots, k_n) := \int_{G^{\times k}} \int_{M^{\times (k+1)}} \chi(x_0) \cdots \chi(x_n) k_0(x_0, g_1 x_1) \cdots k_n(x_n, (g_1 \cdots g_n)^{-1} x_0) \cdot c(e, g_1, g_1 g_2, \dots, g_1 \cdots g_n)\, dx_0 \cdots dx_n\, dg_1 \cdots dg_n. \quad (5.2)$$

Proposition 5.3. (i) *The map $c \mapsto \tau^G_c$ defined above is a morphism of cochain complexes, and therefore induces a map*

$$\Psi_G : H^\bullet_{\mathrm{diff}}(G) \to HC^\bullet(C^\infty_c(G)).$$

(ii) *The map $c \mapsto \tau^M_c$ defined above is a morphism of cocyclic complexes, and therefore induces a map*

$$\Psi_M : H^\bullet_{\mathrm{diff}}(G) \to HC^\bullet(\mathcal{A}^c_G(M)).$$

Proof. Both of the statements are already known: for the first one, see [Pflaum et al. 2015a, §1.3], and for the second, [Pflaum et al. 2015b, §2.2]. □

Example 5.4. In Example 2.18 we discussed the smooth group 2-cocycles for $G = \mathbb{R}^2$, $G = \mathrm{SL}(2, \mathbb{R})$, associated to the area forms of the homogeneous space G/K, equal to $\mathbb{R}^2$ and $\mathbb{H}^2$, respectively. Let us now consider the cyclic cocycles defined by these forms via the construction (5.1) above. For $G = \mathrm{SL}(2, \mathbb{R})$ this gives the following cyclic 2-cocycle on $C^\infty_c(\mathrm{SL}(2, \mathbb{R}))$:

$$\tau^{\mathrm{SL}(2,\mathbb{R})}_\omega(f_0, f_1, f_2) := \int_{\mathrm{SL}(2,\mathbb{R})} \int_{\mathrm{SL}(2,\mathbb{R})} f_0((g_1 g_2)^{-1}) f_1(g_1) f_2(g_2) \cdot \mathrm{Area}_{\mathbb{H}^2}(\Delta^2(\bar{e}, \bar{g}_1, \bar{g}_2))\, dg_1\, dg_2.$$

This is exactly the cyclic cocycle considered in [Connes 1985, §9]. For $G = \mathbb{R}^2$ we get a cyclic 2-cocycle on $C^\infty_c(\mathbb{R}^2)$ (with convolution product) given by the same formula with the hyperbolic area replaced by the Euclidean area, and integrations being over $\mathbb{R}^2$ instead of $\mathrm{SL}(2, \mathbb{R})$, again considered in [Connes 1985, §9]. After Fourier transform $f \mapsto \hat{f}$ this cocycle takes the usual form

$$\tau_\omega(f_0, f_1, f_2) = \int_{\mathbb{R}^2} \hat{f}_0\, d\hat{f}_1 \wedge d\hat{f}_2 \quad \text{for } f_0, f_1, f_2 \in C^\infty_c(\mathbb{R}^2).$$

5C. ***Extension properties.*** In the previous subsection we constructed cyclic cocycles τ_c^G on $C_c^\infty(G)$ and τ_c^M on $\mathcal{A}_G^c(M)$ from a homogeneous smooth group cocycle c. (Recall, once again, that for notational convenience we are taking E to be the product rank 1 bundle.) In Section 3B we have given sufficient conditions on G ensuring that these algebras embed into holomorphically closed subalgebras $\mathcal{A}(G)$ and $\mathcal{A}_G(M)$ of the reduced group C^*-algebra and of the Roe algebra. Now we want to discuss the extension properties of these cocycles. Assume, quite generally, that we are given a subalgebra $\mathcal{A}(G)$ as in Definition 3.6, with associated algebra of operators on $L^2(M)$ denoted, as usual, as $\mathcal{A}_G(M)$. First, we have:

Proposition 5.5. *Let $c \in C^k_{\mathrm{diff},\lambda}(G)$ be a smooth group cocycle. Then*

$$\tau_c^G \text{ extends to } \mathcal{A}(G) \quad\Longrightarrow\quad \tau_c^M \text{ extends to } \mathcal{A}_G(M).$$

Proof. Recall that the algebra $\mathcal{A}_G(M)$ is constructed from the choice of subset $\mathcal{A}(G) \subset C_r^*(G)$ by the slice theorem: an invariant kernel k belongs to $\mathcal{A}_G(M)$ if the function

$$\tilde{k}(g, s_1, s_2) := k(s_1, g s_2)$$

belongs to

$$(\mathcal{A}(G) \hat{\otimes} \Psi^{-\infty}(S, E|_S))^{K\times K}.$$

These functions $\tilde{k}_i(g_i, x_i, x_{i+1})$, $i = 0, \dots, n-1$, and $\tilde{k}_n((g_1 \cdots g_n)^{-1}, x_n, x_0)$ are used in the formula (5.2) for the cocycle τ_c^M. Since the cut-off function χ has compact support, performing the integrations over M in (5.2), we end up with the pairing of an element in $\mathcal{A}(G)^{\otimes(k+1)}$ with the group cocycle c as defined in (5.1). But then it is clear that τ_c^M is well-defined on $\mathcal{A}_G(M)$ if τ_c^G is well-defined on $\mathcal{A}(G)$. □

For the following, recall from Section 2C the explicit form (2.13) of the van Est isomorphism mapping a closed invariant form $\alpha \in \Omega^k_{\mathrm{inv}}(G/K)$ to a smooth group cocycle $J(\alpha) \in C^k_{\mathrm{diff}}(G)$. For notational convenience, we drop the J in the description of the associated cyclic cocycles, writing τ_α^G and τ_α^M instead of $\tau^G_{J(\alpha)}$ and $\tau^M_{J(\alpha)}$.

Proposition 5.6. *Let G be a Lie group with finitely many connected components and satisfying the rapid decay property* (*RD*)*. Assume that G/K is of nonpositive sectional curvature. Then the cocycle τ_α^G associated to a closed invariant differential form $\alpha \in \Omega^k_{\mathrm{inv}}(G/K)$ extends continuously to $H_L^\infty(G)$. Consequently, the cyclic cocycle τ_α^M extends to $\mathcal{S}_G^\infty(M)$.*

Proof. Recall the definition of the smooth group cocycle $J(\alpha) \in C^k_{\mathrm{diff}}(G)$ defined in (2.13), satisfying the polynomial estimates of Theorem 2.14. This, together with the rapid decay property of G, ensures we can follow the line of proof of [Connes and Moscovici 1990, Proposition 6.5], where the analogous extension property is proved for certain discrete groups. To show that the cyclic cocycle τ_α extends continuously to the algebra $H_L^\infty(G)$, we need to show that it is bounded with respect to the seminorm ν_k in (3.13) defining the Fréchèt topology, for some $k \in \mathbb{N}$. Let $a_0, \ldots, a_k \in H_L^\infty(G)$, and write $\tilde{a}_0 := |a_0|$, $\tilde{a}_i(g) := (1+d(g))^k |a_i(g)|$, $i = 1, \ldots, k$. Then we can make the following estimate:

$$\begin{aligned}
|\tau_\alpha^G(a_0, \ldots, a_k)| &\leq C \int_{G^{\times k}} (1+d(g_1))^k \cdots (1+d(g_k))^k |a_0((g_1 \cdots g_k)^{-1})| \\
&\qquad\qquad \cdot |a_1(g_1)| \cdots |a_k(g_k)|\, dg_1 \cdots dg_k \\
&= C(\tilde{a}_0 * \cdots * \tilde{a}_k)(e) \\
&\leq C \|\tilde{a}_0 * \cdots * \tilde{a}_k\|_{C_r^*(G)} \\
&\leq C \|\tilde{a}_0\|_{C_r^*(G)} \cdots \|\tilde{a}_k\|_{C_r^*(G)} \\
&\leq C D^{k+1} \nu_p(\tilde{a}_0) \cdots \nu_p(\tilde{a}_k) = C D^{k+1} \nu_{p+k}(a_0) \cdots \nu_{p+k}(a_k).
\end{aligned}$$

In this computation we have used the fact that the Plancherel trace $a \mapsto a(e)$ on the convolution algebra has a continuous extension to $C_r^*(G)$, together with the rapid decay property: $\|a\|_{C_r^*(G)} \leq D \|(1+d)^p a\|_{L^2}$, for some p. Altogether, this proves the proposition. □

5D. *Pairing with K-theory.* Cyclic cohomology was first developed by Connes to pair with K-theory via the Chern character. Let us recall this construction. Let $\tau = (\tau_0, \tau_2, \ldots, \tau_{2k}) \in CC^{2k}(A)$ be a cyclic cocycle of degree $2k$ on a unital algebra A, and $[p] - [q]$ an element in $K_0(A)$ represented by idempotents $p, q \in M_N(A)$. The number

$$\begin{aligned}
&\langle [p] - [q], \tau \rangle \\
&\quad := \sum_{n=0}^{k} (-1)^n \frac{(2n)!}{n!} \Big(\tau_{2n}\Big(\mathrm{tr}\Big(p - \frac{1}{2}, p, \ldots, p\Big)\Big) - \tau_{2n}\Big(\mathrm{tr}\Big(q - \frac{1}{2}, q, \ldots, q\Big)\Big)\Big),
\end{aligned}$$

where $\mathrm{tr} : M_N(A)^{\otimes(n+1)} \to A^{\otimes(n+1)}$ is the generalized matrix trace, is well-defined and depends only on the (periodic) cyclic cohomology class of τ.

Proposition 5.7. *Let c, $\mathcal{A}(G)$ and $\mathcal{A}_G(M)$ be as in Proposition 5.5, and assume that τ_c^G, and therefore τ_c^M, extends. Then, under the Morita isomorphism*

$$\mathcal{M} : K_0(C^*(M, E)^G) \xrightarrow{\cong} K_0(C_r^*(G)),$$

we have the equality

$$\langle [p] - [q], \tau_c^M \rangle = \langle \mathcal{M}([p] - [q]), \tau_c^G \rangle.$$

Proof. Recall that the isomorphism $\mathcal{M} : K(C^*(M, E)^G) \to K(C_r^*(G))$ is implemented by the partial trace map $\mathrm{Tr}_S : \mathcal{A}_G(M, E) \to \mathcal{A}(G)$ on the respective dense subalgebras. By the abstract Morita isomorphism $\mathcal{M}$, it suffices to consider a simple idempotent $e = e_1 \otimes e_2 \in M_n(\mathcal{A}_G(M, E))$, so that $\mathrm{Tr}_S(e) = \mathrm{Tr}_S(e_2)e_1$ yields an idempotent in $M_n(\mathcal{A}(G))$, where we have extended Tr_S to matrix algebras in the usual way by combining with the matrix trace.

Because we know that the cyclic cohomology class of $\tilde{\tau}_c$ is independent of the choice of a cut-off function, the pairing with K-theory does not depend on this choice either, so we can choose the family χ_ϵ constructed in Lemma 2.1 and take the limit as $\epsilon \downarrow 0$:

$$\begin{aligned}
&\langle [e], \tau_c^M \rangle \\
&\quad = \lim_{\epsilon\downarrow 0} \frac{(2k)!}{k!} \int_{G^{\times k}} \int_{M^{\times(k+1)}} \chi_\epsilon(x_0)\cdots\chi_\epsilon(x_n) e(x_0, g_1x_1)\cdots e(x_n, (g_1\cdots g_n)^{-1}x_0) \\
&\qquad\qquad \cdot c(e, g_1, g_1g_2, \ldots, g_1\cdots g_n)\, dx_0\cdots dx_n\, dg_1\cdots dg_n \\
&\quad = \frac{(2k)!}{k!} \int_{G^{\times k}} \int_{S^{\times(k+1)}} e(x_0, g_1x_1)\cdots e(x_n, (g_1\cdots g_n)^{-1}x_0) \\
&\qquad\qquad \cdot c(e, g_1, g_1g_2, \ldots, g_1\cdots g_n)\, dx_0\cdots dx_n\, dg_1\cdots dg_n \\
&\quad = \frac{(2k)!}{k!} \mathrm{Tr}_S(e_2\cdots e_2) \int_{G^{\times k}} e_1(g_1)\cdots e_1((g_1\cdots g_n)^{-1}) \\
&\qquad\qquad \cdot c(e, g_1, g_1g_2, \ldots, g_1\cdots g_n)\, dg_1\cdots dg_n \\
&\quad = \langle [\mathcal{M}(e)], \tau_c^G \rangle,
\end{aligned}$$

where, to go to the last line, we have used the fact that $e_2^2 = e_2$ is an idempotent. This completes the proof. □

6. Higher C^*-indices and geometric applications

6A. ***Higher C^*-indices and the index formula.*** Let M and G be as above, with M even-dimensional. Hence G is a unimodular Lie group with $|\pi_0(G)| < \infty$. (For the time being we do not put additional hypotheses on G.) Let E be an equivariant complex vector bundle. Consider an odd $\mathbb{Z}_2$-graded Dirac type operator D acting on the sections of E. We have then defined the compactly supported index class $\mathrm{Ind}_c(D) \in K_0(\mathcal{A}_G^c(M, E))$. Let $\alpha \in H_{\mathrm{diff}}^{\mathrm{even}}(G)$ and let $\Psi_M(\alpha) \in HC^{\mathrm{even}}(\mathcal{A}_G^c(M, E))$ be the cyclic cohomology class corresponding to α. We know that, in general, we have a pairing

$$K_0(\mathcal{A}_G^c(M, E)) \times HC^{\mathrm{even}}(\mathcal{A}_G^c(M, E)) \to \mathbb{C}. \tag{6.1}$$

We thus obtain, through $\Psi_M : H_{\mathrm{diff}}^\bullet(G) \to HC^\bullet(\mathcal{A}_G^c(M, E))$, a pairing

$$K_0(\mathcal{A}_G^c(M, E)) \times H_{\mathrm{diff}}^{\mathrm{even}}(G) \to \mathbb{C}. \tag{6.2}$$

In particular, by pairing $\mathrm{Ind}_c(D) \in K_0(\mathcal{A}_G^c(M, E))$ with $\alpha \in H_{\mathrm{diff}}^{\mathrm{even}}(G)$ we obtain the *higher indices*

$$\mathrm{Ind}_{c,\alpha}(D) := \langle \mathrm{Ind}_c(D), \Psi_M(\alpha) \rangle, \quad \alpha \in H_{\mathrm{diff}}^{\mathrm{even}}(G).$$

On the other hand, we can also take the image of α through the van Est map $\Phi_M : H^\bullet_{\text{diff}}(G) \to H^\bullet_{\text{inv}}(M)$; recall that this is nothing but the pull-back through the classifying map $\psi_M : M \to G/K$ once we identify $H^\bullet_{\text{diff}}(G)$ with $H^\bullet_{\text{inv}}(G/K)$. The following theorem is proved in [Pflaum et al. 2015b]:

Theorem 6.3 (Pflaum–Posthuma–Tang). *Let M, G and D be as above. In particular, M is even-dimensional. Let $\alpha \in H^{\text{even}}_{\text{diff}}(G)$. Then the identity*

$$\operatorname{Ind}_{c,\alpha}(D) = \int_M \chi_M(m)\, \operatorname{AS}(M) \wedge \Phi_M(\alpha) \tag{6.4}$$

holds true, where $\operatorname{AS}(M)$ is the Atiyah–Singer integrand on M:

$$\operatorname{AS}(M) := \widehat{A}(M, \nabla^M) \wedge \operatorname{Ch}'(E, \nabla^E).$$

Equivalently,

$$\operatorname{Ind}_{c,\alpha}(D) = \int_M \chi_M(m)\, \operatorname{AS}(M) \wedge \psi_M^*(\alpha) \tag{6.5}$$

if we identify $H^\bullet_{\text{diff}}(G)$ and $H^\bullet_{\text{inv}}(G/K)$ via the van Est isomorphism (see Remark 2.7).

We now make the fundamental assumption that G satisfies the rapid decay property and that G/K is of nonpositive sectional curvature. Consider the dense holomorphically closed subalgebra $\mathcal{S}^\infty_G(M, E) \subset C^*(M, E)^G$ defined by the rapid decay algebra $H^\infty_L(G) \subset C^*_r(G)$. Thanks to the results of the previous section we can extend the pairing (6.2) to a pairing

$$K_0(\mathcal{S}^\infty_G(M, E)) = K_0(C^*(M, E)^G) \times H^{\text{even}}_{\text{diff}}(G) \to \mathbb{C}, \tag{6.6}$$

obtaining in this way the *higher C^*-indices* of D, denoted $\operatorname{Ind}_\alpha(D)$. These numbers are well-defined and can be computed by choosing a suitable representative of the class $\operatorname{Ind}(D) \in K_0(C^*(M, E)^G)$. Choosing the Connes–Skandalis projector, we can apply again the index formula of Pflaum–Posthuma–Tang, obtaining for each $\alpha \in H^{\text{even}}_{\text{diff}}(G)$ the C^*-index formula

$$\operatorname{Ind}_\alpha(D) = \int_M \chi_M(m)\, \operatorname{AS}(M) \wedge \Phi_M(\alpha). \tag{6.7}$$

Notice that we also have a pairing

$$K_0(C^\infty_c(G)) \times HC^{\text{even}}(C^\infty_c(G)) \to \mathbb{C} \tag{6.8}$$

and thus, through the homomorphism $\Psi_G : H^\bullet_{\text{diff}}(G) \to HC^*(C^\infty_c(G))$, a pairing

$$K_0(C^\infty_c(G)) \times H^{\text{even}}_{\text{diff}}(G) \to \mathbb{C}. \tag{6.9}$$

According to the results of the previous section this pairing extends to a pairing

$$K_0(C^*_r(G)) \times H^{\text{even}}_{\text{diff}}(G) \to \mathbb{C} \tag{6.10}$$

if G satisfies (RD). In particular, we can define the $C^*_r(G)$-indices $\mathrm{Ind}_{C^*_r(G),\alpha}(D)$ by pairing $\mathrm{Ind}_{C^*_r(G)}(D) \in K_0(C^*_r(G))$ with $\alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G)$. Further, from Proposition 5.7 we get the equality

$$\langle \mathrm{Ind}(D), \Psi_M(\alpha)\rangle = \langle \mathrm{Ind}_{C^*_r(G)}(D), \Psi_G(\alpha)\rangle, \tag{6.11}$$

which means that

$$\mathrm{Ind}_{C^*_r(G),\alpha}(D) = \mathrm{Ind}_\alpha(D) \quad \text{for all } \alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G) \tag{6.12}$$

and thus, thanks to (6.7), we can state the following fundamental result:

Theorem 6.13. *Let G be a Lie group satisfying the properties stated in the introduction*: $|\pi_0(G)| < \infty$, *(RD) and $\underline{E}G$ of nonpositive curvature. Let $\alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G)$. Then there is a well-defined associated higher $C^*_r(G)$-index* $\mathrm{Ind}_{C^*_r(G),\alpha}(D)$, *and the formula*

$$\mathrm{Ind}_{C^*_r(G),\alpha}(D) = \int_M \chi_M(m)\, \mathrm{AS}(M) \wedge \Phi_M(\alpha) \tag{6.14}$$

holds. Equivalently, if we identify $H^\bullet_{\mathrm{diff}}(G)$ and $H^\bullet_{\mathrm{inv}}(G/K) \equiv H^\bullet_{\mathrm{inv}}(\underline{E}G)$ via the van Est isomorphism, then

$$\mathrm{Ind}_{C^*_r(G),\alpha}(D) = \int_M \chi_M(m)\, \mathrm{AS}(M) \wedge \psi_M^*\alpha.$$

For $\alpha = 1$, the associated cyclic cocycle (5.1) is the Plancherel trace $\tau^G(f) = f(e)$ on $C^*_r(G)$, and the theorem reduces to the L^2-index theorem first proved by Wang [2014]. Remark that in this case the trace extends to $C^*_r(G)$ without problems, so the assumptions on the curvature of G/K and property (RD) are unnecessary.

6B. ***Higher signatures and their G-homotopy invariance.*** Let M and N be two orientable G-proper manifolds and let $f : M \to N$ be a G-homotopy equivalence. Let us denote by D^{sign}_M and D^{sign}_N the corresponding signature operators. Then, according to the main result in [Fukumoto 2017] we have that

$$\mathrm{Ind}_{C^*_r(G)}(D^{\mathrm{sign}}_M) = \mathrm{Ind}_{C^*_r(G)}(D^{\mathrm{sign}}_N) \quad \text{in } K_0(C^*_r(G)). \tag{6.15}$$

Consequently, from (6.14), we obtain the following result, stated as item (i) in Theorem 1.5 in the introduction:

Theorem 6.16. *Let G be a Lie group satisfying the properties stated in the introduction*: $|\pi_0(G)| < \infty$, *(RD) and $\underline{E}G$ of nonpositive curvature. Let M and N be two orientable G-proper manifolds and assume that there exists an orientation preserving G-homotopy equivalence between M and N. Let us identify $H^\bullet_{\mathrm{diff}}(G)$ and $H^\bullet_{\mathrm{inv}}(G/K) \equiv H^\bullet_{\mathrm{inv}}(\underline{E}G)$ via the van Est isomorphism. Then for each $\alpha \in H^\bullet_{\mathrm{inv}}(\underline{E}G)$,*

$$\int_M \chi_M(m)\, L(M) \wedge \psi_M^*\alpha = \int_N \chi_N(n)\, L(N) \wedge \psi_N^*\alpha.$$

Proof. For even-dimensional manifolds, this follows immediately from the previous discussion. For the odd-dimensional case we argue by suspension. Thus, let M be an orientable odd-dimensional G-proper manifold. We endow M with a G-invariant Riemannian metric g_M. Consider $\mathbb{R}$ and the natural action of $\mathbb{Z}$ on it by translations (this is a free, proper and cocompact action). Taking the product of M and $\mathbb{R}$ we obtain the even-dimensional $(G \times \mathbb{Z})$-proper manifold $M \times \mathbb{R}$; it has compact quotient equal to $M/G \times S^1$. We endow $M \times \mathbb{R}$ with the $(G \times \mathbb{Z})$-invariant metric $g_M + dt^2$. Consider the dual group $T^1 := \mathrm{Hom}(\mathbb{Z}, U(1))$. The signature operator on $M \times \mathbb{R}$ defines an index class in the group $K_0(C^*(M \times \mathbb{R})^{G\times\mathbb{Z}})$, which is isomorphic to $K_0(C^*(G) \hat{\otimes} C(T^1))$. Consider the generator d' of $H^1(\mathbb{Z}; \mathbb{Z}) \subset H^*(\mathbb{Z}; \mathbb{C})$ and let $d := \big(\sqrt{-1}/(2\pi)\big)d' \in H^*(\mathbb{Z}; \mathbb{C})$. We know that $H^*(\mathbb{Z}; \mathbb{C})$ can be identified with $H^*_{\mathbb{Z}}(E\mathbb{Z}; \mathbb{C})$ and that $E\mathbb{Z} = \mathbb{R}$; we denote this isomorphism by $\Xi : H^*(\mathbb{Z}; \mathbb{C}) \to H^*_{\mathbb{Z}}(\mathbb{R}; \mathbb{C}) = H^1(S^1)$. Consider $\underline{E}G \times E\mathbb{Z} \equiv \underline{E}G \times \mathbb{R} \equiv G/K \times \mathbb{R}$. To $\alpha \in H^{\mathrm{odd}}_{\mathrm{diff}}(G) \equiv H^{\mathrm{odd}}_{\mathrm{inv}}(\underline{E}G) \equiv H^{\mathrm{odd}}_{\mathrm{inv}}(G/K)$ we associate

$$\beta := \alpha \otimes \Xi(d) \in H^{\mathrm{odd}}_{\mathrm{inv}}(G/K) \otimes H^1_{\mathbb{Z}}(\mathbb{R}) = H^{\mathrm{odd}}_{\mathrm{inv}}(G/K) \otimes H^1(S^1).$$

Now, on the one hand, we have natural homomorphisms

$$\Psi_{G\times\mathbb{Z}} : H^{\mathrm{odd}}_{\mathrm{inv}}(G/K) \otimes H^1(S^1) \to HC^{\mathrm{even}}(C^\infty_c(G) \hat{\otimes} C^\infty(S^1))$$

and

$$\Psi_{M\times\mathbb{R}} : H^{\mathrm{odd}}_{\mathrm{inv}}(G/K) \otimes H^1(S^1) \to HC^{\mathrm{even}}(\mathcal{A}^c_{G\times\mathbb{Z}}(M \times \mathbb{R})),$$

noting that $\mathcal{A}^c_{G\times\mathbb{Z}}(M\times\mathbb{R}) = \mathcal{A}^c_G(M) \hat{\otimes} \mathcal{A}^c_{\mathbb{Z}}(\mathbb{R})$ and $\mathcal{A}^c_{G\times\mathbb{Z}}(M\times\mathbb{R}) = C^*_c(M\times\mathbb{R})^{G\times\mathbb{Z}}$. On the other hand, the classifying map ψ_M and the classifying map for the $\mathbb{Z}$-action on $\mathbb{R}$ together give a smooth $(G\times\mathbb{Z})$-equivariant map $\psi_{M\times\mathbb{R}} : M\times\mathbb{R} \to G/K\times\mathbb{R}$. We can apply the Pflaum–Posthuma–Tang index theorem and obtain, for the signature operator,

$$\begin{aligned}\big\langle \mathrm{Ind}_{C^*_c(M\times\mathbb{R})^{G\times\mathbb{Z}}}(D_{M\times\mathbb{R}}), \Psi_{M\times\mathbb{R}}(\beta)\big\rangle &= \int_G \int_{S^1} \chi_M L(M \times \mathbb{R}) \psi^*_M(\alpha) \wedge \Xi(d) \\ &= \int_G \chi_M L(M) \psi^*_M(\alpha) = \sigma(M, \alpha).\end{aligned}$$

If G satisfies (RD), then this formula remains true for the $C^*(M \times \mathbb{R})^{G\times\mathbb{Z}}$-index, because $\mathcal{S}^\infty_G(M) \hat{\otimes} \mathcal{S}_{\mathbb{Z}}(\mathbb{R})$, with $\mathcal{S}_{\mathbb{Z}}(\mathbb{R})$ denoting the smooth $\mathbb{Z}$-invariant kernels of $\mathbb{R} \times \mathbb{R}$ of rapid polynomial decay, is a dense holomorphically closed subalgebra of $C^*(M \times \mathbb{R})^{G\times\mathbb{Z}}$ to which the pairing with $\Psi_{M\times\mathbb{R}}(\beta)$ extends. Consequently,

$$\big\langle \mathrm{Ind}_{C^*(G)\hat{\otimes}C(S^1)}(D_{M\times\mathbb{R}}), \Psi_{G\times\mathbb{Z}}(\beta)\big\rangle = \sigma(M, \alpha).$$

Now, if M and N are G-homotopy equivalent, then $M \times \mathbb{R}$ and $N \times \mathbb{R}$ are $G \times \mathbb{Z}$ homotopy equivalent. Hence the corresponding signature index classes in

$K_0(C^*(G)\,\hat{\otimes}\,C(T^1))$ are equal; thus

$$\big\langle \mathrm{Ind}_{C^*(G)\hat{\otimes}C(S^1)}(D_{M\times\mathbb{R}}),\, \Psi_{G\times\mathbb{Z}}(\beta)\big\rangle = \big\langle \mathrm{Ind}_{C^*(G)\hat{\otimes}C(S^1)}(D_{N\times\mathbb{R}}),\, \Psi_{G\times\mathbb{Z}}(\beta)\big\rangle.$$

This gives us

$$\sigma(M,\alpha)=\sigma(N,\alpha),$$

which is what we wanted to prove in odd dimension. □

6C. ***Higher $\widehat{A}$-genera and G-metrics of positive scalar curvature.*** Let S be a compact smooth manifold with an action of a compact Lie group K. In general, the existence of a K-invariant metric of positive scalar curvature on S is a more refined property than the existence of a positive scalar curvature metric on S; indeed, as shown in [Bérard-Bergery 1981], averaging a positive scalar curvature metric on S might destroy the positivity of the scalar curvature. For sufficient conditions on K and S ensuring the existence of such metrics, see [Lawson and Yau 1974; Hanke 2008].

If M is a G-proper manifold we can try to built a G-invariant positive scalar curvature metric on M through a K-invariant positive scalar curvature metric on the slice S. This is precisely what is achieved in [Guo et al. 2017]:

Theorem 6.17 (Guo–Mathai–Wang). *Let G be an almost connected Lie group and let K be a maximal compact subgroup of G. If S is a compact manifold with a K-invariant metric of positive scalar curvature, then the G-proper manifold $G\times_K S$ admits a G-invariant metric of positive scalar curvature.*

This result shows that the space of positive scalar curvature G-metrics on a G-proper manifold can be nonempty.

We can ask for numerical obstructions to the existence of a positive scalar curvature G-metric. Assume that M has a G-equivariant spin structure and let $\eth$ be the associated spin-Dirac operator. Then one can show that

$$\mathrm{Ind}_{C_r^*(G)}(\eth)=0 \quad \text{in } K_*(C_r^*G); \tag{6.18}$$

see again [Guo et al. 2017]. The following result was item (ii) in Theorem 1.5 in the introduction:

Theorem 6.19. *Let G be a Lie group satisfying the properties stated in the introduction: $|\pi_0(G)|<\infty$, (RD) and $\underline{E}G$ of nonpositive curvature. Let M be a G-proper manifold admitting a G-equivariant spin structure. Let us identify $H^\bullet_{\mathrm{diff}}(G)$ and $H^\bullet_{\mathrm{inv}}(G/K)\equiv H^\bullet_{\mathrm{inv}}(\underline{E}G)$ via the van Est isomorphism. If M admits a G-invariant metric of positive scalar curvature, then*

$$\widehat{A}(M,\alpha):=\int_M \chi_M(m)\,\widehat{A}(M)\wedge\psi_M^*\alpha=0$$

for each $\alpha\in H^\bullet_{\mathrm{inv}}(\underline{E}G)$.

Proof. The even-dimensional case follows directly from our C^*-index formula and from (6.18). In the odd-dimensional case we argue by suspension, as we did for the signature operator. It suffices to observe that if M is an odd-dimensional G-proper manifold admitting a G-equivariant spin structure and a G-invariant metric of positive scalar curvature g_M, then $M \times \mathbb{R}$ is an even-dimensional $(G \times \mathbb{Z})$-proper manifold with a $(G \times \mathbb{Z})$-equivariant spin structure and with a $(G \times \mathbb{Z})$-invariant metric $g_M + dt^2$ which is of positive scalar curvature too. Consequently, the analogue of (6.18) holds for the spin Dirac operator on $M \times \mathbb{R}$ and so, arguing as for the signature operator, we finally obtain that

$$\widehat{A}(M, \alpha) := \int_M \chi_M(m)\, \widehat{A}(M) \wedge \psi_M^* \alpha = 0,$$

as required. □

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References

[Abels 1974] H. Abels, “Parallelizability of proper actions, global K-slices and maximal compact subgroups”, *Math. Ann.* **212** (1974), 1–19. MR Zbl

[Baum et al. 1994] P. Baum, A. Connes, and N. Higson, “Classifying space for proper actions and K-theory of group C^*-algebras”, pp. 240–291 in *C^*-algebras: 1943–1993* (San Antonio, TX, 1993), edited by R. S. Doran, Contemp. Math. **167**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[Bérard-Bergery 1981] L. Bérard-Bergery, “La courbure scalaire des variétés riemanniennes”, pp. 30–45 in *ERA Conferences*, Inst. Élie Cartan **4**, Univ. Nancy, 1981. MR Zbl

[Chatterji et al. 2007] I. Chatterji, C. Pittet, and L. Saloff-Coste, “Connected Lie groups and property RD”, *Duke Math. J.* **137**:3 (2007), 511–536. MR Zbl

[Connes 1985] A. Connes, “Noncommutative differential geometry”, *Inst. Hautes Études Sci. Publ. Math.* **62** (1985), 257–360. MR Zbl

[Connes and Moscovici 1990] A. Connes and H. Moscovici, “Cyclic cohomology, the Novikov conjecture and hyperbolic groups”, *Topology* **29**:3 (1990), 345–388. MR Zbl

[Crainic 2003] M. Crainic, “Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes”, *Comment. Math. Helv.* **78**:4 (2003), 681–721. MR Zbl

[Dupont 1976] J. L. Dupont, "Simplicial de Rham cohomology and characteristic classes of flat bundles", *Topology* **15**:3 (1976), 233–245. MR Zbl

[Dupont 1979] J. L. Dupont, "Bounds for characteristic numbers of flat bundles", pp. 109–119 in *Algebraic topology* (Aarhus, 1978), edited by J. L. Dupont and I. H. Madsen, Lecture Notes in Math. **763**, Springer, 1979. MR Zbl

[van Est 1955a] W. T. van Est, "On the algebraic cohomology concepts in Lie groups, I", *Indag. Math.* **17** (1955), 225–233. MR Zbl

[van Est 1955b] W. T. van Est, "On the algebraic cohomology concepts in Lie groups, II", *Indag. Math.* **17** (1955), 286–294. MR Zbl

[Fukumoto 2017] Y. Fukumoto, "G-homotopy invariance of the analytic signature of proper co-compact G-manifolds and equivariant Novikov conjecture", preprint, 2017. arXiv

[Guo et al. 2017] H. Guo, V. Mathai, and H. Wang, "Positive scalar curvature and Poincaré duality for proper actions", preprint, 2017. arXiv

[Hanke 2008] B. Hanke, "Positive scalar curvature with symmetry", *J. Reine Angew. Math.* **614** (2008), 73–115. MR Zbl

[Hartnick and Ott 2012] T. Hartnick and A. Ott, "Surjectivity of the comparison map in bounded cohomology for Hermitian Lie groups", *Int. Math. Res. Not.* **2012**:9 (2012), 2068–2093. MR Zbl

[Helgason 2001] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Math. **34**, Amer. Math. Soc., Providence, RI, 2001. Corrected reprint of the 1978 original. MR Zbl

[Hochs and Wang 2018] P. Hochs and H. Wang, "A fixed point formula and Harish-Chandra's character formula", *Proc. Lond. Math. Soc.* (3) **116**:1 (2018), 1–32. MR Zbl

[Inoue and Yano 1982] H. Inoue and K. Yano, "The Gromov invariant of negatively curved manifolds", *Topology* **21**:1 (1982), 83–89. MR Zbl

[Ji and Schweitzer 1996] R. Ji and L. B. Schweitzer, "Spectral invariance of smooth crossed products, and rapid decay locally compact groups", *K-Theory* **10**:3 (1996), 283–305. MR Zbl

[Jolissaint 1989] P. Jolissaint, "K-theory of reduced C^*-algebras and rapidly decreasing functions on groups", *K-Theory* **2**:6 (1989), 723–735. MR Zbl

[Jolissaint 1990] P. Jolissaint, "Rapidly decreasing functions in reduced C^*-algebras of groups", *Trans. Amer. Math. Soc.* **317**:1 (1990), 167–196. MR Zbl

[Kasparov 1980] G. G. Kasparov, "The operator K-functor and extensions of C^*-algebras", *Izv. Akad. Nauk SSSR Ser. Mat.* **44**:3 (1980), 571–636. In Russian; translated in *Math. USSR-Izv.* **16**:3 (1981), 513–572. MR Zbl

[Kim and Kim 2015] S. Kim and I. Kim, "Simplicial volume, barycenter method, and bounded cohomology", preprint, 2015. arXiv

[Lafforgue 2002] V. Lafforgue, "K-théorie bivariante pour les algèbres de Banach et conjecture de Baum–Connes", *Invent. Math.* **149**:1 (2002), 1–95. MR Zbl

[Lawson and Yau 1974] H. B. Lawson, Jr. and S. T. Yau, "Scalar curvature, non-abelian group actions, and the degree of symmetry of exotic spheres", *Comment. Math. Helv.* **49** (1974), 232–244. MR Zbl

[Loday 1998] J.-L. Loday, *Cyclic homology*, 2nd ed., Grundlehren der Math. Wiss. **301**, Springer, 1998. MR Zbl

[Moscovici and Wu 1994] H. Moscovici and F.-B. Wu, "Localization of topological Pontryagin classes via finite propagation speed", *Geom. Funct. Anal.* **4**:1 (1994), 52–92. MR Zbl

[Pflaum et al. 2015a] M. J. Pflaum, H. Posthuma, and X. Tang, "The localized longitudinal index theorem for Lie groupoids and the van Est map", *Adv. Math.* **270** (2015), 223–262. MR Zbl

[Pflaum et al. 2015b] M. J. Pflaum, H. Posthuma, and X. Tang, "The transverse index theorem for proper cocompact actions of Lie groupoids", *J. Differential Geom.* **99**:3 (2015), 443–472. MR Zbl

[Piazza and Schick 2014] P. Piazza and T. Schick, "Rho-classes, index theory and Stolz' positive scalar curvature sequence", *J. Topol.* **7**:4 (2014), 965–1004. MR Zbl

[Roe 2002] J. Roe, "Comparing analytic assembly maps", *Q. J. Math.* **53**:2 (2002), 241–248. MR Zbl

[Varadarajan 1977] V. S. Varadarajan, *Harmonic analysis on real reductive groups*, Lecture Notes in Math. **576**, Springer, 1977. MR Zbl

[Wang 2014] H. Wang, "L^2-index formula for proper cocompact group actions", *J. Noncommut. Geom.* **8**:2 (2014), 393–432. MR Zbl

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