ANNALS OF K-THEORY

vol. 4 no. 3 2019

Periodic cyclic homology and derived de Rham cohomology

Benjamin Antieau

msp

A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 4, No. 3, 2019
dx.doi.org/10.2140/akt.2019.4.505

Periodic cyclic homology and derived de Rham cohomology

Benjamin Antieau

We use the Beilinson t-structure on filtered complexes and the Hochschild–Kostant–Rosenberg theorem to construct filtrations on the negative cyclic and periodic cyclic homologies of a scheme X with graded pieces given by the Hodge completion of the derived de Rham cohomology of X. Such filtrations have previously been constructed by Loday in characteristic zero and by Bhatt–Morrow–Scholze for p-complete negative cyclic and periodic cyclic homology in the quasisyntomic case.

1. Introduction

Let k be a quasisyntomic ring and $k \to R$ a quasisyntomic k-algebra. Bhatt, Morrow, and Scholze construct in [Bhatt et al. 2019, Theorem 1.17] a functorial complete exhaustive decreasing multiplicative $\mathbb{Z}$-indexed filtration $\mathrm{F}^{\star}_{\mathrm{BMS}}\mathrm{HP}(R/k;\mathbb{Z}_p)$ on the p-adic completion $\mathrm{HP}(R/k;\mathbb{Z}_p)$ of periodic cyclic homology with graded pieces $\mathrm{gr}^n_{\mathrm{BMS}}\mathrm{HP}(-/k;\mathbb{Z}_p) \simeq \overline{\mathrm{L}\Omega}_{R/k}[2n]$, where $\mathrm{L}\Omega_{R/k}$ is the derived de Rham complex and $\overline{\mathrm{L}\Omega}_{R/k}$ is the p-adic completion of the Hodge completion of this complex. The Hodge filtration $\Omega^{\geqslant n}_{R/k}$ for smooth algebras induces a Hodge filtration $\mathrm{L}\Omega^{\geqslant n}_{R/k}$ on the derived de Rham complex and its completed variants. There is a corresponding filtration on negative cyclic homology, with graded pieces given by $\overline{\mathrm{L}\Omega}^{\geqslant n}_{R/k}[2n]$, the p-completion of the Hodge completion of $\mathrm{L}\Omega^{\geqslant n}_{R/k}[2n]$.

For smooth $\mathbb{Q}$-algebras, a similar statement goes back to Loday [1992, 5.1.12]. One can also derive very general results along these lines in characteristic zero from [Toën and Vezzosi 2011]. Related results in the context of commutative differential graded algebras were obtained using explicit mixed complexes by Cortiñas [1999]. The authors of [Bhatt et al. 2019] suggest that such a filtration should exist outside the p-complete setting. In this note, we use the Beilinson t-structure on filtered complexes [Beilinson 1987] to prove that this is indeed the case.

MSC2010: 13D03, 14F40.

Keywords: negative cyclic homology, periodic cyclic homology, derived de Rham cohomology, t-structures, filtered complexes.

Theorem 1.1. *If k is a commutative ring and X is a quasicompact quasiseparated k-scheme, then there are functorial complete decreasing multiplicative $\mathbb{Z}$-indexed filtrations $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(X/k)$ and $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(X/k)$ on negative cyclic homology and periodic cyclic homology, respectively. These filtrations satisfy the following properties.*

(a) *There are natural equivalences*

$$\mathrm{gr}^n_{\mathrm{B}}\mathrm{HC}^-(X/k) \simeq \mathrm{R}\Gamma(X, \widehat{\mathrm{L}\Omega}^{\geqslant n}_{-/k}[2n]),$$
$$\mathrm{gr}^n_{\mathrm{B}}\mathrm{HP}(X/k) \simeq \mathrm{R}\Gamma(X, \widehat{\mathrm{L}\Omega}_{-/k}[2n]),$$

where $\widehat{\mathrm{L}\Omega}_{-/k}$ is the Hodge completion of the derived de Rham complex and $\widehat{\mathrm{L}\Omega}^{\geqslant n}_{-/k}$ is the n-th term in the Hodge filtration on $\widehat{\mathrm{L}\Omega}_{-/k}$.

(b) *The filtered pieces $\mathrm{F}^n_{\mathrm{B}}\mathrm{HC}^-(X/k)$ and $\mathrm{F}^n_{\mathrm{B}}\mathrm{HP}(X/k)$ are equipped with compatible decreasing filtrations which induce the Hodge filtration on $\mathrm{gr}^n_{\mathrm{B}}\mathrm{HC}^-(X/k)$ and $\mathrm{gr}^n_{\mathrm{B}}\mathrm{HP}(X/k)$ under the equivalences of part (a).*

(c) *If X/k is quasi-lci,*[1] *then the filtrations $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(X/k)$ and $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(X/k)$ are exhaustive.*

Negative cyclic homology and periodic cyclic homology satisfy fpqc descent by [Bhatt et al. 2019, Corollary 3.4] as a consequence of the fact that the derived exterior powers $\Lambda^i\mathrm{L}_{-/k}$ of the cotangent complex are fpqc sheaves by [Bhatt et al. 2019, Theorem 3.1]. Since $\widehat{\mathrm{L}\Omega}^{\geqslant n}_{-/k}$ has by definition a complete exhaustive decreasing $\mathbb{N}$-indexed filtration with graded pieces $\Lambda^i\mathrm{L}_{-/k}$, it follows that the Hodge-truncated Hodge-completed derived de Rham complexes $\widehat{\mathrm{L}\Omega}^{\geqslant n}_{-/k}$ are also fpqc sheaves. Thus, to prove the theorem, it suffices to handle the affine case.

Theorem 1.1 follows from a much more general theorem, Theorem 4.6, which states that in a suitable ∞-category of bicomplete bifiltered complexes, the Beilinson filtrations are exhaustive for any quasicompact quasiseparated k-scheme X.

Remark 1.2. (i) In case both are defined, the p-adic completion of the filtration of Theorem 1.1 agrees with the filtration of [Bhatt et al. 2019, Theorem 1.17]. This follows in the smooth case by examining the proofs of each theorem and in general by mapping the left Kan extension of our proof to the filtration obtained by quasisyntomic descent in their proof.

(ii) In [Antieau and Nikolaus 2018], we introduce a t-structure on cyclotomic spectra. As one application of the t-structure, we show using calculations of Hesselholt [1996] that the methods of this paper can be used to construct a filtration $\mathrm{F}^\star_{\mathrm{B}}\mathrm{TP}(X)$ on topological periodic cyclic homology $\mathrm{TP}(X)$ when X is a smooth scheme over a perfect field with graded pieces given by (shifted) crystalline cohomology $\mathrm{gr}^n_{\mathrm{B}}\mathrm{TP}(X) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X/W(k))[2n]$. When $X = \mathrm{Spec}\, R$

[1] We say that a k-scheme X is *quasi-lci* if $\mathrm{L}_{X/k}$ has Tor-amplitude contained in $[0, 1]$.

is smooth and affine, then in fact $\mathrm{gr}^n_{\mathrm{B}}\mathrm{TP}(X)$ is given canonically by $\mathrm{W}\Omega^{\bullet}_R[2n]$, the shifted de Rham–Witt complex. This recovers several parts of [Bhatt et al. 2019, Theorems 1.10, 1.12, and 1.15] in the case of a smooth scheme over a perfect field.

Outline. In Section 2, we outline the theory of filtrations we need. We explain the smooth affine case in Section 3. In Section 4, we give the full proof, which follows from the smooth case by taking nonabelian derived functors in an appropriate ∞-category of bifiltrations.

Conventions and notation. We work with ∞-categories throughout, following the conventions of [Lurie 2009; 2017]. Hochschild homology $\mathrm{HH}(R/k)$ and its relatives are viewed as objects in the derived ∞-category $\mathrm{D}(k)$, possibly with additional structure. Typically, we view objects of $\mathrm{D}(k)$ as being given by chain complexes up to quasi-isomorphism, but several constructions lead us to cochain complexes as well. Given an object $X \in \mathrm{D}(k)$, we write $\mathrm{H}_* X$ for its homology groups. We write $X^{\bullet}$ for a given cochain complex model for X. Thus, $X^{\bullet}$ is an object of the category $\mathrm{Ch}(k)$ of cochain complexes of k-modules. The main example is the de Rham complex $\Omega^{\bullet}_{R/k}$ for a smooth commutative k-algebra R.

2. Background on filtrations

Throughout this section, fix a commutative ring k. Let $\mathrm{D}(k)$ be the derived ∞-category of k, a stable presentable ∞-categorical enhancement of the derived category of unbounded chain complexes of k-modules. The derived tensor product of chain complexes makes $\mathrm{D}(k)$ into a presentably symmetric monoidal stable ∞-category, meaning that $\mathrm{D}(k)$ is a symmetric monoidal presentable ∞-category in which the tensor product commutes with colimits in each variable.

The *filtered derived ∞-category* of k is $\mathrm{DF}(k) = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{D}(k))$, the ∞-category of sequences

$$X(\star) : \cdots \to X(n+1) \to X(n) \to \cdots$$

in $\mathrm{D}(k)$. Write $X(\infty) = \lim_n X(n) \simeq \lim(\cdots \to X(n+1) \to X(n) \to \cdots)$ for the limit of the filtration. A filtered complex $X(\star) \in \mathrm{DF}(k)$ is *complete* if $X(\infty) \simeq 0$. Similarly, write $X(-\infty)$ for $\mathrm{colim}_n X(n) \simeq \mathrm{colim}(\cdots \to X(n+1) \to X(n) \to \cdots)$. Given a map $X(-\infty) \to Y$, we say that $X(\star)$ is a filtration on Y; if the map is an equivalence, we say that $X(\star)$ is an *exhaustive* filtration on Y.

We refer to general objects $X(\star)$ of $\mathrm{DF}(k)$ as *decreasing $\mathbb{Z}$-indexed filtrations*. We write $\mathrm{gr}^n X$ for the cofiber of $X(n+1) \to X(n)$, the n-th graded piece of the filtration. Several filtrations of interest in this paper are in fact $\mathbb{N}$-*indexed*, meaning that $X(0) \simeq X(-1) \simeq X(-2) \simeq \cdots$, or equivalently that $\mathrm{gr}^n X \simeq 0$ for $n < 0$.

Day convolution (using the additive symmetric monoidal structure of $\mathbb{Z}^{\mathrm{op}}$) makes $\mathrm{DF}(k)$ into a presentably symmetric monoidal stable ∞-category. The Day convolution symmetric monoidal structure has the property that if $X(\star)$ and $Y(\star)$ are filtered objects of $\mathrm{D}(k)$, then $(X \otimes_k Y)(\star)$ is a filtered spectrum with graded pieces $\mathrm{gr}^n(X \otimes_k Y) \simeq \bigoplus_{i+j=n} \mathrm{gr}^i X \otimes_k \mathrm{gr}^j Y$.

A filtration $X(\star)$ equipped with the structure of a commutative algebra object (or $\mathbb{E}_\infty$-algebra object) in $\mathrm{DF}(k)$ is called a *multiplicative* filtration.

One source of decreasing filtrations is via the Whitehead tower[2] induced from some t-structure on $\mathrm{D}(k)$. We use the standard t-structure, which has $\mathrm{D}(k)_{\geqslant 0} \subseteq \mathrm{D}(k)$, the full subcategory of $\mathrm{D}(k)$ consisting of X such that $\mathrm{H}_n(X) = 0$ for $n < 0$. Similarly, $\mathrm{D}(k)_{\leqslant 0}$ is the full subcategory of $\mathrm{D}(k)$ consisting of X such that $\mathrm{H}_n(X) = 0$ for $n > 0$. Given an object X, its n-connective cover $\tau_{\geqslant n} X \to X$ has $\mathrm{H}_i(\tau_{\geqslant n} X) \cong \mathrm{H}_i(X)$ for $i \geqslant n$ and $\mathrm{H}_i(\tau_{\geqslant n} X) = 0$ for $i < n$.

Example 2.1. If R is a connective commutative algebra object in $\mathrm{D}(k)$, then the Whitehead tower $\tau_{\geqslant \star} R$ is a complete exhaustive decreasing multiplicative $\mathbb{N}$-indexed filtration on R with $\mathrm{gr}^n \tau_{\geqslant \star} R \simeq \mathrm{H}_n(R)[n]$.

For details and proofs of the statements above, see [Gwilliam and Pavlov 2018]. For more background, see [Bhatt et al. 2019, Section 5]. Now we introduce the Beilinson t-structure on $\mathrm{DF}(k)$.

Definition 2.2. Let $\mathrm{DF}(k)_{\geqslant i} \subseteq \mathrm{DF}(k)$ be the full subcategory of those filtered objects $X(\star)$ such that $\mathrm{gr}^n X \in \mathrm{D}(k)_{\geqslant i-n}$, and $\mathrm{DF}(k)_{\leqslant i} \subseteq \mathrm{DF}(k)$ be the full subcategory of those filtered objects $X(\star)$ such that $X(n) \in \mathrm{D}(k)_{\leqslant i-n}$.

Note the asymmetry in the definition. The pair $(\mathrm{DF}(k)_{\geqslant 0}, \mathrm{DF}(k)_{\leqslant 0})$ defines a t-structure on $\mathrm{DF}(k)$ by [Beilinson 1987]; see also [Bhatt et al. 2019, Theorem 5.4] for a proof. We write $\tau^{\mathrm{B}}_{\leqslant n}$, $\tau^{\mathrm{B}}_{\geqslant n}$, π^{B}_n for the truncation and homotopy object functors in the Beilinson t-structure.

The connective objects $\mathrm{DF}(k)_{\geqslant 0}$ are closed under the tensor product on $\mathrm{DF}(k)$, and hence the natural map $\pi^{\mathrm{B}}_0 : \mathrm{DF}(k)_{\geqslant 0} \to \mathrm{DF}(k)^{\heartsuit}$ is symmetric monoidal. The heart is the abelian category of cochain complexes of k-modules equipped with the usual tensor product of cochain complexes.

Remark 2.3. The Beilinson Whitehead tower $\tau^{\mathrm{B}}_{\geqslant \star} X$ is most naturally a bifiltered object, since each $\tau^{\mathrm{B}}_{\geqslant n+1} X \to \tau^{\mathrm{B}}_{\geqslant n} X$ is a map of objects of $\mathrm{DF}(k)$. If we forget the residual filtration on $\tau^{\mathrm{B}}_{\geqslant \star} X$ (by taking the colimit), then we obtain a new filtration on $X(-\infty)$. In this paper, we need this only for $\mathbb{N}$-indexed filtrations. In this case,

[2]The Whitehead tower of an object X in a stable ∞-category D with a t-structure is the tower

$$\cdots \to \tau_{\geqslant n+1} X \to \tau_{\geqslant n} X \to \cdots,$$

where $\tau_{\geqslant n} X$ denotes truncation with respect to the t-structure.

each n-connective cover $\tau^{\mathrm{B}}_{\geqslant n}X$ is also $\mathbb{N}$-indexed, and we can view the resulting filtration $(\tau^{\mathrm{B}}_{\geqslant n}X)(0)$ as a new filtration on $X(0)$.[3] If X is a commutative algebra object of $\mathrm{DF}(k)$, then the Beilinson Whitehead tower $\tau^{\mathrm{B}}_{\geqslant\star}X$ is a new multiplicative filtration on X.

For our purposes, it is most important to understand the n-connective cover functors. Given $X(\star)\in\mathrm{DF}(k)$, the n-connective cover in the Beilinson t-structure $\tau^{\mathrm{B}}_{\geqslant n}X\to X(\star)$ induces equivalences

$$\mathrm{gr}^i\tau^{\mathrm{B}}_{\geqslant n}X\simeq\tau_{\geqslant n-i}\mathrm{gr}^i X$$

[Bhatt et al. 2019, Theorem 5.4]. From this, we see that $\mathrm{gr}^i\pi^{\mathrm{B}}_n X\simeq(\mathrm{H}_{n-i}(\mathrm{gr}^i X))[-i]$. The cochain complex corresponding to $\pi^{\mathrm{B}}_n X$ is of the form

$$\cdots\to\mathrm{H}_n(\mathrm{gr}^0X)\to\mathrm{H}_{n-1}(\mathrm{gr}^1X)\to\mathrm{H}_{n-2}(\mathrm{gr}^2X)\cdots,$$

where $\mathrm{H}_n(\mathrm{gr}^0X)$ is in cohomological degree 0 and where the differentials are induced from the boundary maps in homology coming from the cofiber sequences $\mathrm{gr}^{i+1}X\to X(i)/X(i+2)\to\mathrm{gr}^i X$. See [Bhatt et al. 2019, Theorem 5.4(3)] for details.

The next example illustrates our main idea in a general setting.

Example 2.4. Let $X\in\mathrm{D}(k)$ be an object equipped with an S^1-action. The Whitehead tower $\tau_{\geqslant\star}X$ defines a complete exhaustive S^1-equivariant $\mathbb{Z}$-indexed filtration $\mathrm{F}^\star_{\mathrm{P}}X$ on X with graded pieces $\mathrm{gr}^n_{\mathrm{P}}X\simeq\mathrm{H}_n(X)[n]$, equipped with the trivial S^1-action. Applying homotopy S^1-fixed points, we obtain a complete $\mathbb{Z}$-indexed filtration $\mathrm{F}^\star_{\mathrm{P}}X^{hS^1}$ on X^{hS^1} with graded pieces $\mathrm{gr}^n_{\mathrm{P}}X^{hS^1}\simeq(\mathrm{H}_n(X)[n])^{hS^1}$. Let $\mathrm{F}^\star_{\mathrm{B}}X^{hS^1}$ be the double-speed Whitehead tower of $\mathrm{F}^\star_{\mathrm{P}}X^{hS^1}$ in the Beilinson t-structure on $\mathrm{DF}(k)$, so that $\mathrm{F}^n_{\mathrm{B}}X^{hS^1}=\tau^{\mathrm{B}}_{\geqslant 2n}\mathrm{F}^\star X^{hS^1}$. By definition, $\mathrm{F}^n_{\mathrm{B}}X^{hS^1}$ is a filtered spectrum with

$$\mathrm{gr}^i\mathrm{F}^n_{\mathrm{B}}X^{hS^1}\simeq\tau_{\geqslant 2n-i}\mathrm{gr}^i_{\mathrm{P}}X^{hS^1}\simeq\tau_{\geqslant 2n-i}(\mathrm{H}_i(X)[i])^{hS^1}.$$

Hence,

$$\mathrm{gr}^i\,\mathrm{gr}^n_{\mathrm{B}}X^{hS^1}\simeq\begin{cases}\mathrm{H}_i(X)[2n-i] & \text{if } n\leqslant i,\\ 0 & \text{otherwise.}\end{cases}$$

This shows in fact that $\mathrm{gr}^n_{\mathrm{B}}X^{hS^1}[-2n]\simeq\pi^{\mathrm{B}}_{2n}\mathrm{F}^\star_{\mathrm{P}}X^{hS^1}$ and hence it is in $\mathrm{DF}(k)^\heartsuit$, the abelian category of cochain complexes, and is represented by a cochain complex

$$0\to\mathrm{H}_n(X)\to\mathrm{H}_{n+1}(X)\to\mathrm{H}_{n+2}(X)\to\cdots,$$

where $\mathrm{H}_n(X)$ is in cohomological degree n. The differential is given by the Connes–Tsygan B-operator. An object $X\in\mathrm{D}(k)$ with an S^1-action is the same as a dg module over $\mathrm{C}_\bullet(S^1,k)$, the dg algebra of chains on S^1. The fundamental class B

[3] Note that this is not an idempotent operation: applying the Beilinson Whitehead tower to $\tau^{\mathrm{B}}_{\geqslant\star}X(0)$ typically produces yet another filtration on $X(0)$.

of the circle defines a k-module generator of $\mathrm{H}_1(S^1, k)$ and $B^2 = 0$. The differential in the cochain complex above is given by the action of B. Hence, we have obtained a filtration $\mathrm{F}^\star_{\mathrm{B}} X^{hS^1}$ on X^{hS^1} with graded pieces given by $(\mathrm{H}_{\bullet \geqslant n}(X), B)[2n]$.

Remark 2.5. The same argument shows that there is a filtration $\mathrm{F}^\star_{\mathrm{B}} X^{tS^1}$ on the S^1-Tate construction X^{tS^1} with graded pieces $\mathrm{gr}^n_{\mathrm{B}} X^{tS^1} \simeq (\mathrm{H}_\bullet(X), B)[2n]$. We ignore for the time being any convergence issues.

3. The smooth case

The Hochschild–Kostant–Rosenberg theorem [Hochschild et al. 1962] implies that there are canonical isomorphisms $\Omega^n_{R/k} \cong \mathrm{HH}_n(R/k)$ when R is a smooth commutative k-algebra. In particular, letting $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R/k)$ denote the usual Whitehead tower, given by the good truncations $\tau_{\geqslant \star}\mathrm{HH}(R/k)$, we see that there are natural equivalences $\mathrm{gr}^n_{\mathrm{HKR}}\mathrm{HH}(R/k) \simeq \Omega^n_{R/k}[n]$ for all $n \geqslant 0$. Applying homotopy S^1-fixed points, we obtain a complete exhaustive decreasing multiplicative $\mathbb{N}$-indexed filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}^-(R/k)$ on $\mathrm{HC}^-(R/k)$.

Definition 3.1. Let $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k)$ be the double-speed Beilinson Whitehead tower for the filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}^-(R/k)$, so that $\mathrm{F}^n_{\mathrm{B}}\mathrm{HC}^-(R/k) = \tau^{\mathrm{B}}_{\geqslant 2n}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}^-(R/k)$. For a picture of this filtration, see Figure 1.

Example 2.4 implies that this filtration is a multiplicative $\mathbb{N}$-indexed filtration on $\mathrm{HC}^-(R/k)$; each graded piece $\pi^{\mathrm{B}}_n \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}^-(R/k) \simeq \mathrm{gr}^n_{\mathrm{B}}\mathrm{HC}^-(R/k)[-2n]$ in $\mathrm{DF}(k)^\heartsuit$ is given by a cochain complex of the form

$$\cdots \to 0 \to \Omega^n_{R/k} \to \Omega^{n+1}_{R/k} \to \cdots,$$

where $\Omega^n_{R/k}$ is in cohomological degree n. It is verified in [Loday 1992, Corollary 2.3.3] that the differential is indeed the de Rham differential. This can also be checked by hand in the case of $k[x]$ to which the general case reduces. It follows that $\mathrm{gr}^n_{\mathrm{B}}\mathrm{HC}^-(R/k) \simeq \Omega^{\bullet \geqslant n}_{R/k}[2n]$. The additional filtration on $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k)$ reduces to the Hodge filtration on $\Omega^{\bullet \geqslant n}_{R/k}[2n]$. The exhaustiveness and completeness of $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k)$ follows from Lemma 3.2 below. The case of $\mathrm{HP}(R/k)$ is similar. This proves Theorem 1.1 in the case of smooth algebras.[4]

We needed the following lemma in the proof.

Lemma 3.2. *Let $X(\star)$ be a complete $\mathbb{N}$-indexed filtration on $X = X(0)$ and let $\tau^{\mathrm{B}}_{\geqslant \star} X$ be the associated Beilinson Whitehead tower in* $\mathrm{DF}(k)$.

(i) *The truncations $\tau^{\mathrm{B}}_{\geqslant n} X$ and $\tau^{\mathrm{B}}_{\leqslant n-1} X$ are complete for all $n \in \mathbb{Z}$.*

(ii) *The filtration $(\tau^{\mathrm{B}}_{\geqslant \star} X)(0)$ on $X \simeq X(0)$ is complete and exhaustive.*

[4]Note that for R a smooth k-algebra, the de Rham complex $\Omega^\bullet_{R/k}$ is already Hodge-complete.

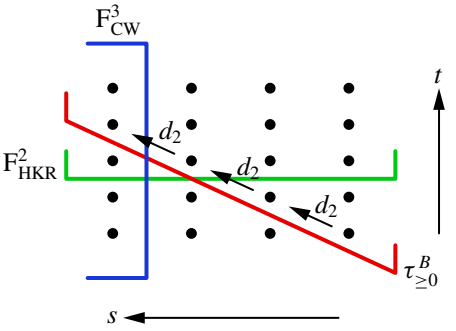

Figure 1. The Beilinson filtration. The figure shows the E_2-page of the spectral sequence $E_2^{s,t} = H^s(BS^1, HH_t(R/k)) \Rightarrow HC^-_{t-s}(R/k)$ and which parts of $HC^-(R/k)$ are cut out by $\tau^{\mathrm{B}}_{\geqslant 0}HC^-(R/k)$, $F^2_{HKR}HC^-(R/k)$, and $F^3_{CW}HC^-(R/k)$, respectively. For the definition of the CW filtration, see Section 4.

Proof. Since the full subcategory $\widehat{\mathrm{DF}}(k) \subseteq \mathrm{DF}(k)$ of complete filtrations is stable, to prove part (i) it is enough to show that $\tau^{\mathrm{B}}_{\leqslant n-1}X$ is complete for all n. However, $(\tau^{\mathrm{B}}_{\leqslant n-1}X)(i) \in \mathrm{D}(k)_{\leqslant n-1-i}$. We find that $\lim_i (\tau^{\mathrm{B}}_{\leqslant n-1}X)(i)$ is in $\mathrm{D}(k)_{\leqslant -\infty} \simeq 0$. This proves (i). It follows from (i) and the fact that complete filtered spectra are closed under colimits that we can view $\lim_n \tau^{\mathrm{B}}_{\geqslant n}X$ as a complete filtered spectrum $Y(\star)$ with graded pieces

$$\mathrm{gr}^i Y \simeq \lim_n \mathrm{gr}^i \tau^{\mathrm{B}}_{\geqslant n}X \simeq \lim_n \tau_{\geqslant n-i}\mathrm{gr}^i X.$$

Hence, each $\mathrm{gr}^i Y$ is ∞-connective. Thus, $\mathrm{gr}^i Y \simeq 0$ for all i and hence $Y(\star) \simeq 0$ as it is complete. This proves the completeness in (ii). Finally, $(\tau^{\mathrm{B}}_{\leqslant n-1}X)(0) \in \mathrm{D}(k)_{\leqslant n-1}$. It follows that $(\tau^{\mathrm{B}}_{\geqslant n}X)(0) \to X(0) \simeq X$ is an n-equivalence, and exhaustiveness follows by letting $n \to -\infty$. □

4. The general case

Our general strategy for the proof of Theorem 1.1 is to left Kan extend from the case of smooth algebras. Because of convergence issues, we are forced to Kan extend in an ∞-category which keeps track of multiple filtrations.

Let k be a commutative ring, sCAlg_k the ∞-category of simplicial k-algebras, and $\mathrm{CAlg}_k^{\mathrm{poly}} \subseteq \mathrm{sCAlg}_k$ the full subcategory of finitely generated polynomial k-algebras. This embedding admits a universal property: given any ∞-category $\mathscr{C}$ which admits sifted colimits, the forgetful functor

$$\mathrm{Fun}'(\mathrm{sCAlg}_k, \mathscr{C}) \to \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{poly}}, \mathscr{C})$$

is an equivalence, where $\mathrm{Fun}'(\mathrm{sCAlg}_k, \mathscr{C})$ is the ∞-category of sifted colimit-preserving functors $\mathrm{sCAlg}_k \to \mathscr{C}$. Given $F : \mathrm{CAlg}_k^{\mathrm{poly}} \to \mathscr{C}$, we call the corresponding sifted colimit-preserving functor $\mathrm{d}F : \mathrm{sCAlg}_k \to \mathscr{C}$ the *left Kan extension* or the *nonabelian derived functor* of F. For details, see [Lurie 2009, Section 5.5.9].

Let $R \in \mathrm{sCAlg}_k$ and fix $F : \mathrm{CAlg}_k^{\mathrm{poly}} \to \mathscr{C}$. Then one extends F to all polynomial rings by taking filtered colimits in $\mathscr{C}$. To compute the value of the left Kan extension $\mathrm{d}F$ of F on R, one takes a simplicial resolution $|P_\bullet| \simeq R$, where each $P_\bullet$ is polynomial (but not necessarily finitely generated), and computes $|F(P_\bullet)|$ in $\mathscr{C}$.

Let k be a commutative ring, and let R be a simplicial commutative k-algebra. Then, $\mathrm{HH}(R/k)$ is a connective commutative algebra object in $\mathrm{D}(k)^{BS^1}$, the ∞-category of complexes of k-modules equipped with an S^1-action. We could apply Example 2.4 to obtain a filtration on $\mathrm{HC}^-(R/k) = \mathrm{HH}(R/k)^{hS^1}$ with graded pieces truncations of the cochain complex $(\mathrm{HH}_*(R/k), B)$. However, in the nonsmooth case, this does not capture derived de Rham cohomology.

We use the fact that Hochschild homology commutes with sifted colimits (see for example [Bhatt et al. 2019, Remark 2.3]) to Kan extend the HKR filtration of [Hochschild et al. 1962] from finitely generated polynomial algebras to all simplicial commutative k-algebras. This gives a functorial complete exhaustive decreasing multiplicative $\mathbb{N}$-indexed S^1-equivariant multiplicative filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R/k)$ on $\mathrm{HH}(R/k)$ with graded pieces $\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{HH}(R/k) \simeq \Lambda^t \mathrm{L}_{R/k}[t]$ with the trivial S^1-action, where $\mathrm{L}_{R/k}$ denotes the cotangent complex and $\Lambda^t \mathrm{L}_{R/k}$ is the t-th derived exterior power of the cotangent complex. Since $\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k)$ is t-connective for all t, it follows that the HKR filtration is complete.

Applying homotopy S^1-fixed points or Tate, we obtain decreasing multiplicative $\mathbb{N}$-indexed filtrations $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}^-(R/k)$ and $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k)$ on negative cyclic homology

$$\mathrm{HC}^-(R/k) = \mathrm{HH}(R/k)^{hS^1}$$

and periodic cyclic homology

$$\mathrm{HP}(R/k) = \mathrm{HH}(R/k)^{tS^1}.$$

These filtrations are both complete. To see this, note first that the induced HKR filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}(R/k)$ on cyclic homology $\mathrm{HC}(R/k) = \mathrm{HH}(R/k)_{hS^1}$ is complete since $\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HC}(R/k) \simeq (\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k))_{hS^1}$ is t-connective. Thus, since we have a cofiber sequence

$$\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}(R/k)[1] \to \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}^-(R/k) \to \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k)$$

in $\mathrm{DF}(k)$, it suffices to see that the HKR filtration on $\mathrm{HC}^-(R/k)$ is complete. But this follows from the fact that $(-)^{hS^1}$ commutes with limits.

Negative cyclic homology admits a second filtration, coming from the standard cell structure $\mathbb{CP}^0 \subseteq \mathbb{CP}^1 \subseteq \cdots$ on $BS^1 \simeq \mathbb{CP}^\infty$. This second filtration is compatible with the HKR filtration since on Hochschild homology the HKR filtration is S^1-equivariant. To be precise, we consider the double filtration

$$\mathrm{F}^t_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) = \mathrm{fib}\big((\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k))^{hS^1} \to (\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k))^{h\Omega\mathbb{CP}^{s-1}}\big),$$

which has graded pieces

$$\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq \Lambda^t \mathrm{L}_{R/k}[t-2s].$$

This bifiltration is multiplicative in the natural sense with respect to the Day convolution symmetric monoidal structure on $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}, \mathrm{D}(k))$, where we give $\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}$ the symmetric monoidal structure coming from (the opposite of) addition in the monoid $\mathbb{N} \times \mathbb{N}$.

We let $\mathrm{DBF}(k)$ denote the ∞-category $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}, \mathrm{D}(k))$ of $\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}$-indexed *bifiltered complexes* of k-modules and we denote by $\widehat{\mathrm{DBF}}(k)$ the full subcategory of $\mathrm{DBF}(k)$ on those *bicomplete* bifiltered complexes, i.e., those $X(\star, \star)$ such that for each s one has $\lim_t X(s,t) \simeq 0$ and for each t one has $\lim_s X(s,t) \simeq 0$. Note that either condition implies that $X(\star,\star)$ is complete in the weaker sense that $\lim_{s,t} X(s,t) \simeq 0$.

Remark 4.1. Bicomplete bifiltered objects are the same as complete filtered objects in the complete filtered derived category.

Lemma 4.2. *For any simplicial commutative k-algebra R, the filtration*

$$\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(R/k)$$

is bicomplete.

Proof. Fix s. We have

$$\lim_t \mathrm{F}^t_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq 0$$

as both $(-)^{hS^1}$ and $(-)^{h\Omega\mathbb{CP}^{s-1}}$ commute with limits. Now fix t. Then we want to show that

$$\lim_s \mathrm{F}^t_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq \mathrm{fib}\big((\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k))^{hS^1} \to \lim_s (\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k))^{h\Omega\mathbb{CP}^{s-1}}\big) \simeq 0.$$

However, for any bounded below spectrum with an S^1-action X, the natural map $X^{hS^1} \to \lim_s X^{h\Omega\mathbb{CP}^{s-1}}$ is an equivalence. Indeed, this follows by a computation if X has a single nonzero homotopy group, and then it follows for all homologically bounded complexes by induction. Then it follows in the limit up the Postnikov tower since both $(-)^{hS^1}$ and $\lim_s (-)^{h\Omega\mathbb{CP}^{s-1}}$ commute with limits. □

We can Kan extend $\mathrm{HC}^-(-/k)$ with its bifiltration from finitely generated polynomial k-algebras to all simplicial commutative k-algebras to obtain a bifiltration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{dHC}^-(R/k)$ on derived negative cyclic homology. Let $\widehat{\mathrm{dHC}}^-(R/k)$ denote bicompleted derived negative cyclic homology and let $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\widehat{\mathrm{dHC}}^-(R/k)$ be the bicomplete bifiltration on bicompleted derived negative cyclic homology, which is the Kan extension of $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(-/k)$ as a functor $\mathrm{CAlg}_k^{\mathrm{poly}} \to \widehat{\mathrm{DBF}}(k)$ to all simplicial commutative k-algebras.

Lemma 4.3. *For any $R \in \mathrm{sCAlg}_k$, the natural map*

$$\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\widehat{\mathrm{dHC}}^-(R/k) \to \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(R/k)$$

is an equivalence in $\widehat{\mathrm{DBF}}(k)$.

Proof. Since both bifiltered objects are bicomplete, it is enough to check on graded pieces. Since the graded pieces functors $\mathrm{gr}^t\mathrm{gr}^s : \widehat{\mathrm{DBF}}(k) \to \mathrm{D}(k)$ commute with colimits, $\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\widehat{\mathrm{dHC}}^-(R/k)$ is the left Kan extension of $R \mapsto \Omega^t_{R/k}[t-2s]$ from finitely generated polynomial algebras to all simplicial commutative k-algebras, which is precisely $\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq \Lambda^t\mathrm{L}_{R/k}[t-2s]$. □

Remark 4.4. The lemma says that even though $\mathrm{HC}^-(-/k)$ does not commute with sifted colimits as a functor $\mathrm{sCAlg}_k \to \mathrm{D}(k)$, it does commute with sifted colimits as a functor $\mathrm{sCAlg}_k \to \widehat{\mathrm{DBF}}(k)$ when equipped with its skeletal and HKR filtrations. In particular, we can compute $\mathrm{HC}^-(R/k)$ by left Kan extending from finitely generated polynomial algebras and then bicompleting.

Fix s and consider the Whitehead tower

$$\cdots \to \tau^{\mathrm{B}}_{\geqslant r}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \to \tau^{\mathrm{B}}_{\geqslant r-1}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \to \cdots$$

in the Beilinson t-structure on filtered complexes, where we are taking Beilinson connective covers in the HKR-direction. Recall that

$$\begin{aligned}\mathrm{gr}^t\tau^{\mathrm{B}}_{\geqslant r}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) &\simeq \tau_{\geqslant -t+r}\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{TC}^-(R/k)\\ &\simeq \tau_{\geqslant -t+r}\mathrm{fib}\big((\Lambda^t\mathrm{L}_{R/k}[t])^{hS^1} \to (\Lambda^t\mathrm{L}_{R/K}[t])^{h\Omega\mathbb{CP}^{s-1}}\big) \quad (4.5)\end{aligned}$$

and hence that

$$\begin{aligned}\mathrm{gr}^t\pi^{\mathrm{B}}_r\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) &\simeq \big(\pi_{-t+r}\mathrm{fib}\big((\Lambda^t\mathrm{L}_{R/k}[t])^{hS^1} \to (\Lambda^t\mathrm{L}_{R/k}[t])^{h\Omega\mathbb{CP}^{s-1}}\big)\big)[-t+r].\end{aligned}$$

Here, the notation implies that we view π_{-t+r} of the object on the right as a complex concentrated in degree $-t+r$. If R/k is smooth, we have $\Lambda^t\mathrm{L}_{R/k} \simeq \Omega^t_{R/k}$. In

particular, in this case, we see that

$$\mathrm{gr}^t \pi_r^{\mathrm{B}} \mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^s_{\mathrm{CW}} \mathrm{HC}^-(R/k) \simeq \begin{cases} \Omega^t_{R/k}[-t+r] & \text{if } r \text{ is even and } r \leqslant 2t-2s, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.6. *There is a complete exhaustive multiplicative decreasing $\mathbb{Z}$-indexed filtration $\mathrm{F}^\star_{\mathrm{B}}$ on the bicomplete bifiltered complex $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(R/k)$. The graded piece $\mathrm{gr}^u_{\mathrm{B}}\mathrm{HC}^-(R/k)$ is naturally equivalent to the Hodge-complete derived de Rham cohomology $\widehat{\mathrm{L}\Omega}^{\geqslant u}_{R/k}[2u]$ of R, naively truncated. Moreover, the remaining HKR and CW filtrations on $\mathrm{gr}^u_{\mathrm{B}}\mathrm{HC}^-(R/k)$ both coincide with the Hodge filtration. Finally, the underlying filtration $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k)$ in the sense of Remark 2.3 is a complete filtration of $\mathrm{HC}^-(R/k)$; if $\mathrm{L}_{R/k}$ has Tor-amplitude contained in $[0,1]$, then the filtration is exhaustive.*

Proof. When R/k is a finitely generated polynomial algebra, we take as our filtration $\mathrm{F}^\star_{\mathrm{B}}$ the double-speed Whitehead filtration $\tau^{\mathrm{B}}_{\geqslant 2\star}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k)$ in the Beilinson t-structure. By definition of the Beilinson t-structure and the analysis in the paragraph above, $\pi^{\mathrm{B}}_{2u}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k)$ is a chain complex of the form

$$0 \to \Omega^{u+s}_{R/k} \to \Omega^{u+s+1}_{R/k} \to \cdots,$$

where $\Omega^{u+s}_{R/k}$ sits in homological degree $u-s$. Thus, as in Section 3, for R smooth,

$$\mathrm{gr}^u_{\mathrm{B}}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq \pi^{\mathrm{B}}_{2u}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq \Omega^{\bullet\geqslant u+s}_{R/k}[2u].$$

Both the CW filtration and the HKR filtration induce the Hodge filtration on this graded piece.

We claim that for R/k a finitely generated polynomial algebra on d variables, for each u, the bifiltered spectrum $\mathrm{F}^u_{\mathrm{B}}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(R/k)$ is bicomplete. For each s, this follows from Lemma 3.2. In the other direction, as soon as $2s > 2d-u$, (4.5) shows that $\mathrm{F}^u_{\mathrm{B}}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq 0$, so completeness in the CW-direction is immediate.

We now view the filtration $\mathrm{F}^\star_{\mathrm{B}}$ as giving a functor $\mathrm{CAlg}^{\mathrm{poly}}_k \to \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \widehat{\mathrm{DBF}}(k))$, which we left Kan extend to a functor $\mathrm{sCAlg}_k \to \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \widehat{\mathrm{DBF}}(k))$. We verify the necessary properties in a series of lemmas.

Lemma 4.7. *For any $R \in \mathrm{sCAlg}_k$,*

$$\operatorname*{colim}_{u\to-\infty} \mathrm{F}^u_{\mathrm{B}}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(R/k),$$

where the colimit is computed in $\widehat{\mathrm{DBF}}(k)$.

Proof. The colimit functor $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \widehat{\mathrm{DBF}}(k)) \to \widehat{\mathrm{DBF}}(k)$ commutes with colimits, so this follows from Lemma 4.3 once we show that the filtration $\mathrm{F}^u_{\mathrm{B}}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star\mathrm{HC}^-(R/k)$ is exhaustive on $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(R/k)$ for R a finitely generated polynomial ring. This follows from Lemma 3.2. □

Lemma 4.8. *We have* $\lim_u \mathrm{F}^u_{\mathrm{B}}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq 0$, *where the limit is computed in* $\widehat{\mathrm{DBF}}(k)$.

Proof. By conservativity of the limit-preserving functors $\mathrm{gr}^t\mathrm{gr}^s : \widehat{\mathrm{DBF}}(k) \to \mathrm{D}(k)$, it is enough to see that

$$\lim_u \mathrm{gr}^t \tau^{\mathrm{B}}_{\geqslant 2u}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq 0$$

for all pairs (s,t). But this object is $(2u-t)$-connective by definition of the Beilinson t-structure and because of the fact that colimits of $(2s-t)$-connective objects are $(2s-t)$-connective. Thus, the limit vanishes. □

Lemma 4.9. *The graded piece* $\mathrm{gr}^u_{\mathrm{B}}\mathrm{HC}^-(R/k)$ *is the bicomplete bifiltered object obtained by left Kan extending* $R \mapsto \Omega^{\geqslant u}[2u]$ *to all simplicial commutative rings, where the filtration is given by* $\mathrm{F}^{(s,t)}\Omega^{\geqslant u}[2u] \simeq \Omega^{\geqslant u+\max(s-u,t-u,0)}[2u]$.

Proof. Indeed, this is clear on finitely generated polynomial algebras by Section 3 so this follows by Kan extension using the fact that $\mathrm{gr}^u : \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \widehat{\mathrm{DBF}}(k)) \to \widehat{\mathrm{DBF}}(k)$ commutes with colimits. □

Thus, we have proved the theorem except for the last sentence. Now we examine the underlying filtration $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k)$ on $\mathrm{HC}^-(R/k)$ given by forgetting the HKR and CW filtrations.

Lemma 4.10. *Let* $\widehat{\mathrm{DBF}}(k) \to \mathrm{D}(k)$ *be the functor that sends a bicomplete* $\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}$*-index bifiltered spectrum* $X(\star,\star)$ *to* $X(0,0)$. *This functor preserves limits.*

Proof. The functor is the composition of the inclusion functor $\widehat{\mathrm{DBF}}(k) \to \mathrm{DBF}(k)$ (a right adjoint) and the limit preserving evaluation functor $X(\star,\star) \mapsto X(0,0)$ on $\mathrm{DBF}(k)$. □

From Lemmas 4.8 and 4.10, it follows that the filtration $\mathrm{F}^u_{\mathrm{B}}\mathrm{HC}^-(R/k)$ is a complete filtration on $\mathrm{HC}^-(R/k)$. Exhaustiveness is somewhat subtle.

Lemma 4.11. *If* $\mathrm{L}_{R/k}$ *has Tor-amplitude contained in* $[0,1]$, *then the filtration* $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k)$ *on* $\mathrm{HC}^-(R/k)$ *is exhaustive.*

Proof. Consider the cofiber C^u of $\mathrm{F}^u_{\mathrm{B}}\mathrm{HC}^-(R/k) \to \mathrm{HC}^-(R/k)$ in $\widehat{\mathrm{DBF}}(k)$. We find that

$$\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\mathrm{F}^u_{\mathrm{B}}\mathrm{HC}^-(R/k) \simeq \begin{cases} 0 & \text{if } u > t-s, \\ \Lambda^t\mathrm{L}_{R/k}[t-2s] & \text{otherwise.} \end{cases}$$

Similarly, $\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq \Lambda^t\mathrm{L}_{R/k}[t-2s]$. It follows that

$$\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}C^u \simeq \begin{cases} \Lambda^t\mathrm{L}_{R/k}[t-2s] & \text{if } t-s < u, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{L}_{R/k}$ has Tor-amplitude contained in $[0, 1]$, it follows that $\Lambda^t \mathrm{L}_{R/k}$ has Tor-amplitude contained in $[0, t]$,[5] and hence $\Lambda^t \mathrm{L}_{R/k}[t-2s]$ has Tor-amplitude contained in $[t-2s, 2t-2s]$. In particular, we see that C^u has a complete filtration with graded pieces having Tor-amplitude in $[t-2s, 2t-2s]$ for $t-s < u$. In particular, since R is discrete, the graded pieces are $2u$-coconnected. Since C^u is a limit of $2u$-coconnected objects, it follows that $\pi_i C^u = 0$ for $i \geqslant 2u$. In particular, $\mathrm{colim}_{u \to -\infty} C^u = 0$ and the filtration is exhaustive as claimed. □

This completes the proof of Theorem 4.6. □

Now we give the argument for $\mathrm{HP}(R/k)$.

Corollary 4.12. *There is a complete filtration* $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(R/k)$ *on* $\mathrm{HP}(R/k)$ *with*

$$\mathrm{gr}^u_{\mathrm{B}}\mathrm{HP}(R/k) \simeq \widehat{\mathrm{L}\Omega}_{R/k}[2u].$$

If R/k is quasi-lci, the filtration is exhaustive.

Proof. We use the cofiber sequence $\mathrm{HC}(R/k)[1] \to \mathrm{HC}^-(R/k) \to \mathrm{HP}(R/k)$. Note that $\mathrm{HC}(-/k) = \mathrm{HH}(R/k)_{hS^1}$ preserves colimits. The Kan extension of the HKR filtration on $\mathrm{HC}(-/k)[1]$ from finitely generated polynomial k-algebras to all simplicial commutative k-algebras thus equips $\mathrm{HC}(-/k)[1]$ with an $\mathbb{N}$-indexed filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}(-/k)[1]$ with graded pieces

$$\mathrm{gr}^n_{\mathrm{HKR}}\mathrm{HC}(-/k)[1] \simeq \Lambda^n \mathrm{L}_{R/k}[n+1].$$

Moreover, since $\mathrm{F}^n_{\mathrm{HKR}}\mathrm{HC}(-/k)[1]$ is n-connective, the filtration is complete. By Lemma 3.2, the double-speed Beilinson Whitehead tower induces a complete exhaustive decreasing $\mathbb{Z}$-indexed filtration $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}(-/k)[1]$ on $\mathrm{HC}^-(-/k)[1]$. A straightforward check implies that the graded pieces are

$$\mathrm{gr}^u_{\mathrm{B}}\mathrm{HC}(-/k)[1] \simeq \mathrm{L}\Omega^{\leqslant u-1}_{R/k}[2u-1].$$

Here, it makes no difference whether we take the Hodge-completed derived de Rham complex or the non-Hodge-completed derived de Rham complex, as the Hodge filtration on $\mathrm{L}\Omega^{\leqslant u-1}_{R/k}$ is finite. Now we have a cofiber sequence

$$\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}(-/k)[1] \to \mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k) \to \mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(R/k).$$

Since the filtrations on $\mathrm{HC}(-/k)$ and $\mathrm{HC}^-(R/k)$ are complete, so is the induced filtration on $\mathrm{HP}(R/k)$. When R/k is quasi-lci, Theorem 4.6 implies that the filtration on $\mathrm{HC}^-(R/k)$ is exhaustive. We have already noted that the filtration on $\mathrm{HC}(R/k)$

[5] Use the fact that $\mathrm{L}_{R/k}$ is quasi-isomorphic to a complex $M_0 \leftarrow M_1$, where M_0, M_1 are flat, the fact that flats are filtered colimits of finitely generated projectives, the standard filtration on $\Lambda^t \mathrm{L}_{R/k}$ with graded pieces $\Lambda^j M_0 \otimes_R \Lambda^{t-j}(M_1[1])$, and the fact that $\Lambda^{t-j}(M_1[1]) \simeq (\Gamma^{t-j} M_1)[t-j]$, where Γ^{t-j} is the divided power functor on flats.

is exhaustive. Hence, the filtration on $\mathrm{HP}(R/k)$ is exhaustive. The graded pieces $\mathrm{gr}^u_{\mathrm{B}}\mathrm{HP}(R/k)$ fit into cofiber sequences

$$\widehat{\mathrm{L}\Omega}^{\geqslant u}_{R/k}[2u] \to \mathrm{gr}^u_{\mathrm{B}}\mathrm{HP}(R/k) \to \mathrm{L}\Omega^{\leqslant u-1}_{R/k}[2u].$$

One finds using the remaining HKR filtration that in the smooth case the graded piece $\mathrm{gr}^u_{\mathrm{B}}\mathrm{HP}(R/k)[-2u]$ is a chain complex (it is in the heart of the Beilinson t-structure) and that this sequence is equivalent to the canonical stupid filtration sequence

$$0 \to \Omega^{\bullet\geqslant u}_{R/k} \to \Omega^{\bullet}_{R/k} \to \Omega^{\bullet\leqslant u-1}_{R/k} \to 0.$$

This completes the proof, since now we see in general that

$$\mathrm{gr}^u_{\mathrm{B}}\mathrm{HP}(R/k) \simeq \widehat{\mathrm{L}\Omega}_{R/k}[2u]. \qquad \square$$

Proof of Theorem 1.1. Theorem 4.6 and Corollary 4.12 establish the theorem for affine k-schemes. It follows for general quasicompact separated schemes because everything in sight is then computed from a finite limit of affine schemes, and the conditions of being complete or exhaustive are stable under finite limits. Finally, it follows for a quasicompact quasiseparated scheme X by induction on the number of affines needed to cover X. $\square$

Acknowledgments

We thank Thomas Nikolaus for explaining the interaction of the S^1-action and the Hopf element η and Peter Scholze for bringing this problem to our attention. We also benefited from conversations with Dmitry Kaledin and Akhil Mathew about the Beilinson t-structure. Finally, Elden Elmanto generously provided detailed comments on a draft of the paper. This work was supported by NSF Grant DMS-1552766.

References

[Antieau and Nikolaus 2018] B. Antieau and T. Nikolaus, "Cartier modules and cyclotomic spectra", preprint, 2018. arXiv

[Beilinson 1987] A. A. Beĭlinson, "On the derived category of perverse sheaves", pp. 27–41 in *K-theory, arithmetic and geometry* (Moscow, 1984–1986), edited by Y. I. Manin, Lecture Notes in Math. **1289**, Springer, 1987. MR Zbl

[Bhatt et al. 2019] B. Bhatt, M. Morrow, and P. Scholze, "Topological Hochschild homology and integral p-adic Hodge theory", *Publ. Math. Inst. Hautes Études Sci.* **129** (2019), 199–310. MR Zbl

[Cortiñas 1999] G. Cortiñas, "On the cyclic homology of commutative algebras over arbitrary ground rings", *Comm. Algebra* **27**:3 (1999), 1403–1412. MR Zbl

[Gwilliam and Pavlov 2018] O. Gwilliam and D. Pavlov, "Enhancing the filtered derived category", *J. Pure Appl. Algebra* **222**:11 (2018), 3621–3674. MR Zbl

[Hesselholt 1996] L. Hesselholt, "On the p-typical curves in Quillen's K-theory", *Acta Math.* **177**:1 (1996), 1–53. MR Zbl

[Hochschild et al. 1962] G. Hochschild, B. Kostant, and A. Rosenberg, "Differential forms on regular affine algebras", *Trans. Amer. Math. Soc.* **102** (1962), 383–408. MR Zbl

[Loday 1992] J.-L. Loday, *Cyclic homology*, Grundlehren der Math. Wissenschaften **301**, Springer, 1992. MR Zbl

[Lurie 2009] J. Lurie, *Higher topos theory*, Annals of Math. Studies **170**, Princeton University Press, 2009. MR Zbl

[Lurie 2017] J. Lurie, "Higher algebra", preprint, 2017, available at http://www.math.harvard.edu/~lurie/papers/HA.pdf.

[Toën and Vezzosi 2011] B. Toën and G. Vezzosi, "Algèbres simpliciales S^1-équivariantes, théorie de de Rham et théorèmes HKR multiplicatifs", *Compos. Math.* **147**:6 (2011), 1979–2000. MR Zbl

Received 9 Oct 2018. Revised 22 Mar 2019. Accepted 10 Apr 2019.

BENJAMIN ANTIEAU: benjamin.antieau@gmail.com
Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL, United States

ANNALS OF K-THEORY

msp.org/akt

EDITORIAL BOARD

Joseph Ayoub — ETH Zürich, Switzerland
joseph.ayoub@math.uzh.ch

Paul Balmer — University of California, Los Angeles, USA
balmer@math.ucla.edu

Guillermo Cortiñas — Universidad de Buenos Aires and CONICET, Argentina
gcorti@dm.uba.ar

Hélène Esnault — Freie Universität Berlin, Germany
liveesnault@math.fu-berlin.de

Eric Friedlander — University of Southern California, USA
ericmf@usc.edu

Max Karoubi — Institut de Mathématiques de Jussieu – Paris Rive Gauche, France
max.karoubi@imj-prg.fr

Moritz Kerz — Universität Regensburg, Germany
moritz.kerz@mathematik.uni-regensburg.de

Huaxin Lin — University of Oregon, USA
livehlin@uoregon.edu

Alexander Merkurjev — University of California, Los Angeles, USA
merkurev@math.ucla.edu

Birgit Richter — Universität Hamburg, Germany
birgit.richter@uni-hamburg.de

Jonathan Rosenberg — (Managing Editor)
University of Maryland, USA
jmr@math.umd.edu

Marco Schlichting — University of Warwick, UK
schlichting@warwick.ac.uk

Charles Weibel — (Managing Editor)
Rutgers University, USA
weibel@math.rutgers.edu

Guoliang Yu — Texas A&M University, USA
guoliangyu@math.tamu.edu

PRODUCTION

Silvio Levy — (Scientific Editor)
production@msp.org

Annals of K-Theory is a journal of the K-Theory Foundation (ktheoryfoundation.org). The K-Theory Foundation acknowledges the precious support of Foundation Compositio Mathematica, whose help has been instrumental in the launch of the Annals of K-Theory.

See inside back cover or msp.org/akt for submission instructions.

The subscription price for 2019 is US $490/year for the electronic version, and $550/year (+$25, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Annals of K-Theory (ISSN 2379-1681 electronic, 2379-1683 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

AKT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2019 Mathematical Sciences Publishers

ANNALS OF K-THEORY

2019 vol. 4 no. 3

Motivic analogues of MO and MSO 345
Dondi Ellis

The IA-congruence kernel of high rank free metabelian groups 383
David El-Chai Ben-Ezra

Vanishing theorems for the negative K-theory of stacks 439
Marc Hoyois and Amalendu Krishna

Higher genera for proper actions of Lie groups 473
Paolo Piazza and Hessel B. Posthuma

Periodic cyclic homology and derived de Rham cohomology 505
Benjamin Antieau

Linkage of Pfister forms over $\mathbb{C}(x_1, \ldots, x_n)$ 521
Adam Chapman and Jean-Pierre Tignol