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Linkage of Pfister forms over $\mathbb{C}(x_1, \dots, x_n)$

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We prove the existence of a set of cardinality 2^n of n-fold Pfister forms over $\mathbb{C}(x_1, \dots, x_n)$ which do not share a common $(n-1)$-fold factor. This gives a negative answer to a question raised by Becher. The main tools are the existence of the dyadic valuation on the complex numbers and recent results on symmetric bilinear forms over fields of characteristic 2.

The field $\mathbb{C}(x_1, x_2)$ of rational functions in two indeterminates over the field of complex numbers is known to be a C_2-field in the sense of Lang; see [Elman et al. 2008, Section 97]. It follows that every quadratic form in five variables over $\mathbb{C}(x_1, x_2)$ is isotropic, which implies that any two quaternion algebras over $\mathbb{C}(x_1, x_2)$ share a common maximal subfield; see [Lam 2005, Theorem X.4.20]. Fields with this property are said to be *linked*. It was noticed by Becher [2018] and by Chapman, Dolphin, and Leep [Chapman et al. 2018, Corollary 5.3] that the following stronger property holds: $\mathbb{C}(x_1, x_2)$ is 3-*linked* in the sense that any *three* quaternion algebras over $\mathbb{C}(x_1, x_2)$ share a common maximal subfield. In contrast, algebraic number fields are known to be m-linked for *every* integer m; this follows from Lenstra's proof that K_2 of global fields consists of symbols [Lenstra 1976, Proposition, p. 70]. We are indebted to an anonymous referee for the following short argument: a common maximal subfield of quaternion algebras $Q_1, \dots, Q_m$ defined over a number field F is given by $F(\sqrt{d})$, where $d \in F^\times$ is a nonsquare in each of the completions F_p, where p runs through the finitely many primes that are either archimedean or dyadic, or where at least one of the Q_i is nonsplit. Comparison with the case of number fields suggests asking whether there exists an upper bound on the integer m for which $\mathbb{C}(x_1, x_2)$ is m-linked.

Theorem A. *The following quaternion algebras over $\mathbb{C}(x_1, x_2)$ do not share a common maximal subfield*:

$$(x_1, x_2), \qquad (x_1, x_2+1), \qquad (x_2, x_1+1), \qquad (x_2, x_1x_2+1).$$

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The arguments apply to a more general linkage question raised by Becher [2018]. Given a field F, the Witt ring WF of (Witt classes of) symmetric bilinear forms over F has a natural filtration by the powers of the maximal ideal IF of even-dimensional forms:

$$WF \supset IF \supset I^2F \supset \cdots .$$

Each I^nF is generated by (bilinear) n-fold Pfister forms, i.e., forms of the shape

$$\langle\langle \alpha_1, \dots, \alpha_n \rangle\rangle = \langle 1, -\alpha_1 \rangle \otimes \cdots \otimes \langle 1, -\alpha_n \rangle \quad \text{with } \alpha_1, \dots, \alpha_n \in F^\times.$$

For $m, n \geq 2$, we say that I^nF is *m-linked* if any m bilinear n-fold Pfister forms over F share a common $(n-1)$-fold factor. If $\mathrm{char}(F) \neq 2$, quadratic forms can be identified with their symmetric bilinear polar forms, and in particular the 2-fold Pfister forms are the norm forms of quaternion algebras. Hence F is m-linked in the sense discussed above if and only if I^2F is m-linked. Becher raised the following question:

Question [Becher 2018, Question 5.2]. Suppose I^nF is 3-linked for some $n \geq 2$. Does it follow that I^nF is m-linked for every $m \geq 3$?

This question was answered in the negative for fields F of $\mathrm{char}(F) = 2$ in [Chapman 2018]. In this note, we show how Becher's question can be answered also in the case of $\mathrm{char}(F) = 0$ using the main result of [Chapman 2018] on symmetric bilinear forms over fields of characteristic 2 and the existence of a dyadic valuation on $\mathbb{C}$:

Theorem B. *For $F = \mathbb{C}(x_1, \dots, x_n)$ with $n \geq 2$, I^nF is 3-linked but not 2^n-linked.*

Proofs

Notation 1. For a given integer $n \geq 2$, let $\mathbf{2}^n = \{0, 1\}^{\times n}$, and write $\mathbf{0} = (0, \dots, 0) \in \mathbf{2}^n$. Given a sequence $\alpha_1, \dots, \alpha_n$ in the multiplicative group of a field F and $\boldsymbol{d} = (d_1, \dots, d_n) \in \mathbf{2}^n$, let $\alpha^{\boldsymbol{d}} = \prod_{i=1}^n \alpha_i^{d_i} \in F^\times$. If $\boldsymbol{d} \neq \mathbf{0}$ and $1 + \alpha^{\boldsymbol{d}} \neq 0$, let

$$\varphi_{\boldsymbol{d}} = \langle\langle \alpha_1, \dots, \widehat{\alpha_\ell}, \dots, \alpha_n \rangle\rangle \otimes \langle\langle 1 + \alpha^{\boldsymbol{d}} \rangle\rangle,$$

where ℓ is the minimal index in $\{1, \dots, n\}$ for which $d_\ell \neq 0$, and let

$$\varphi_{\mathbf{0}} = \langle\langle \alpha_1, \dots, \alpha_n \rangle\rangle.$$

The following result is from [Chapman 2018, Theorem 3.3]:

Proposition 2. *Suppose* $\mathrm{char}(F) = 2$ *and* $\alpha_1, \dots, \alpha_n$ *are* 2*-independent in* F, *which means that* $(\alpha^{\boldsymbol{d}})_{\boldsymbol{d} \in \mathbf{2}^n}$ *is a linearly independent family in* F *viewed as an* F^2*-vector space. Then the forms* $\varphi_{\boldsymbol{d}}$ *for* $\boldsymbol{d} \in \mathbf{2}^n$ *are anisotropic and have no common* 1*-fold factor.*

The main result from which Theorems A and B derive is the following.

Proposition 3. *Let $F = k(x_1,\dots,x_n)$ be the field of rational functions in n indeterminates over an arbitrary field k of characteristic zero, for some $n \geq 2$. Let $\varphi_{\boldsymbol{d}}$ for $\boldsymbol{d} \in \mathbf{2}^n$ be the Pfister forms defined as in Notation 1 with the sequence $x_1,\dots,x_n$ for $\alpha_1,\dots,\alpha_n$. The forms $\varphi_{\boldsymbol{d}}$ do not have a common 1-fold factor.*

Proof. A theorem of Chevalley [Engler and Prestel 2005, Theorem 3.1.2] shows that the 2-adic valuation on $\mathbb{Q}$ extends to a valuation v_0 on k. Let $\bar{k}$ be the residue field of this valuation, which has characteristic 2. The valuation v_0 has a Gauss extension to a valuation v on F such that $v(x_i) = 0$ for $i = 1,\dots,n$ and $\overline{x_1},\dots,\overline{x_n}$ are algebraically independent over $\bar{k}$; see [Engler and Prestel 2005, Corollary 2.2.2]. The residue field of v is thus $\overline{F} = \bar{k}(\overline{x_1},\dots,\overline{x_n})$, a field of rational functions in n indeterminates over $\bar{k}$. Since the coefficients of the forms $\{\varphi_{\boldsymbol{d}} : \boldsymbol{d} \in \mathbf{2}^n\}$ are all of value 0, they have residue forms $\{\overline{\varphi}_{\boldsymbol{d}} : \boldsymbol{d} \in \mathbf{2}^n\}$, where the coefficients of $\overline{\varphi}_{\boldsymbol{d}}$ are the residues of the coefficients of $\varphi_{\boldsymbol{d}}$. The forms $\overline{\varphi}_{\boldsymbol{d}}$ are bilinear Pfister forms as defined in Notation 1, with the 2-independent sequence $\overline{x_1},\dots,\overline{x_n}$ for $\alpha_1,\dots,\alpha_n$.

For $\boldsymbol{d} \in \mathbf{2}^n$, let $\boldsymbol{t_d} = (t_{1,\boldsymbol{d}},\dots,t_{2^n-1,\boldsymbol{d}})$ be a (2^n-1)-tuple of indeterminates. Suppose the bilinear forms $\varphi_{\boldsymbol{d}}$ have a common factor $\langle\langle\alpha\rangle\rangle$. Then the pure subforms $\varphi'_{\boldsymbol{d}}$ defined by the equation $\varphi_{\boldsymbol{d}} = \langle 1\rangle \perp \varphi'_{\boldsymbol{d}}$ all represent $-\alpha$. Hence the system of equations

$$\varphi'_{\boldsymbol{d}}(\boldsymbol{t_d},\boldsymbol{t_d}) = -\alpha \quad \text{for } \boldsymbol{d} \in \mathbf{2}^n$$

has a solution. We may therefore find nontrivial solutions to the system of equations

$$\varphi'_{\boldsymbol{d}}(\boldsymbol{t_d},\boldsymbol{t_d}) = \varphi'_{\mathbf{0}}(\boldsymbol{t_0},\boldsymbol{t_0}) \quad \text{for } \boldsymbol{d} \in \mathbf{2}^n \setminus \{\mathbf{0}\}.$$

Since these equations are homogeneous, upon scaling we may find solutions $(\boldsymbol{u_d})_{\boldsymbol{d}\in\mathbf{2}^n}$ such that

$$\min\{v(u_{i,\boldsymbol{d}}) \mid i = 1,\dots,2^n-1,\ \boldsymbol{d} \in \mathbf{2}^n\} = 0.$$

Taking residues, we obtain

$$\overline{\varphi}'_{\boldsymbol{d}}(\overline{\boldsymbol{u_d}},\overline{\boldsymbol{u_d}}) = \overline{\varphi}'_{\mathbf{0}}(\overline{\boldsymbol{u_0}},\overline{\boldsymbol{u_0}}) \quad \text{for } \boldsymbol{d} \in \mathbf{2}^n \setminus \{\mathbf{0}\}.$$

Since at least one $\overline{u_{i,\boldsymbol{d}}}$ is nonzero and the forms $\overline{\varphi}'_{\boldsymbol{d}}$ are anisotropic, it follows that these forms all represent some $\beta \in \overline{F}^\times$. Hence the forms $\overline{\varphi}_{\boldsymbol{d}}$ have a common factor $\langle\langle\beta\rangle\rangle$ by [Elman et al. 2008, Lemma 6.11]. This yields a contradiction to Proposition 2. □

Theorem A readily follows from Proposition 3 with $n = 2$ and $k = \mathbb{C}$, because the forms $\varphi_{\mathbf{0}}$, $\varphi_{(0,1)}$, $\varphi_{(1,0)}$, and $\varphi_{(1,1)}$ are the norm forms of the quaternion algebras (x_1,x_2), (x_1,x_2+1), (x_2,x_1+1), and (x_2,x_1x_2+1), respectively.

Proof of Theorem B. The field $F = \mathbb{C}(x_1,\dots,x_n)$ is a C_n-field, and hence $F(t)$ is a C_{n+1}-field; see [Elman et al. 2008, Corollary 97.6]. In particular, $u(F(t)) = 2^{n+1}$, and it follows from [Becher 2018, Corollary 5.4] that I^nF is 3-linked. Apply

Proposition 3 with $k = \mathbb{C}$ to obtain a set of cardinality 2^n of n-fold Pfister forms that do not have a common 1-fold factor, and hence are not linked. □

References

[Becher 2018] K. J. Becher, "Triple linkage", *Ann. K-Theory* **3**:3 (2018), 369–378. MR Zbl

[Chapman 2018] A. Chapman, "Common slots of bilinear and quadratic Pfister forms", *Bull. Aust. Math. Soc.* **98**:1 (2018), 38–47. MR Zbl

[Chapman et al. 2018] A. Chapman, A. Dolphin, and D. B. Leep, "Triple linkage of quadratic Pfister forms", *Manuscripta Math.* **157**:3-4 (2018), 435–443. MR Zbl

[Elman et al. 2008] R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications **56**, American Mathematical Society, Providence, RI, 2008. MR Zbl

[Engler and Prestel 2005] A. J. Engler and A. Prestel, *Valued fields*, Springer, 2005. MR Zbl

[Lam 2005] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Math. **67**, American Mathematical Society, Providence, RI, 2005. MR Zbl

[Lenstra 1976] H. W. Lenstra, Jr., "K_2 of a global field consists of symbols", pp. 69–73 in *Algebraic K-theory* (Evanston, IL, 1976), edited by M. R. Stein, Lecture Notes in Math. **551**, 1976. MR Zbl

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