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Motivic analogues of MO and MSO

Dondi Ellis

We construct algebraic unoriented and oriented cobordism, named MGLO and MSLO, respectively. MGLO is defined and its homotopy groups are explicitly computed, giving an answer to a question of Jack Morava. MSLO is also defined and its coefficients are explicitly computed after completing at a prime p. Similarly to MSO, the homotopy type of MSLO depends on whether the prime p is even or odd. Finally, a computation of a localization of the homotopy groups of MGLR is given.

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1. Introduction

Motivic homotopy theory on smooth schemes over a field was introduced by Morel and Voevodsky [1999] with the purpose of proving the Bloch–Kato conjecture, which was accomplished by Voevodsky [2003a]. Motivic analogues of well known spectra of algebraic topology carry additional deep algebraic information. For example, motivic "ordinary" homology computes Bloch's higher Chow groups, motivic K-theory is algebraic K-theory, and motivic cobordism has a geometric interpretation as algebraic cobordism [Levine and Morel 2007].

In [Hu et al. 2011], Hu, Kriz, and Ormsby (following notes of Deligne [2009]) introduced equivariant stable motivic homotopy theory, and motivic real K-theory (an analogue of Atiyah's KR) to solve Thomason's homotopy limit problem on

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algebraic Hermitian K-theory. A follow-up paper [Berrick et al. 2015] generalized their result.

The authors of [Hu et al. 2011] introduced a motivic analogue of MR, which they denote by MGLR. The computation of the coefficients of MGLR remains a difficult problem.

In this paper we introduce a nonequivariant motivic spectrum MGLO which is related to MGLR and analogous to unoriented (topological) cobordism MO. We prove MGLO is a wedge of suspensions of ordinary motivic homology with coefficients in C_2. Although our result is similar to the analogous result for MO, the pattern of suspensions in MGLO is more subtle due to the Tate twist. Our result is stated in Theorem 3.12. This answers a question of Jack Morava.

An important subtlety arises in the construction of MGLO, and a new concept is developed in the process. The point is that in topology MO can be obtained from MR by a construction called geometric fixed points [Lewis et al. 1986, Chapter 2, Definition 9.7]. In more detail, let EC_2 be a free contractible C_2-equivariant CW complex. Then we have a cofiber sequence

$$EC_{2+} \to S^0 \to \widetilde{EC}_2.$$

For a C_2-equivariant topological spectrum E, we define the geometric fixed points of E as

$$\Phi^{C_2}(\mathsf{E}) = (\widetilde{EC}_2 \wedge \mathsf{E})^{C_2}.$$

In particular,

$$\mathsf{MO} = \Phi^{C_2}(\mathsf{MR}),$$

and this is the spectrum we compute with. A similar point is also relevant in [Hill et al. 2016]. There is also a motivic geometric fixed point functor $\Phi^{C_2}_{\text{ét}}$ (see Section 6). Applying this functor to MGLR gives

$$\mathsf{MGLO} = \Phi^{C_2}_{\text{ét}}(\mathsf{MGLR}).$$

Extending this construction, we also define a motivic analogue of oriented cobordism MSO, which we denote by MSLO. In Theorem 4.12 we compute the coefficients of MSLO completed at an odd prime, and in Theorems 4.25 and 4.23 we show that the 2-completion of MSLO splits as a wedge sum of copies of motivic homology.

We would like to point out that the spectrum MSLO is not the same as the spectrum MSL defined by Panin and Walter [2010]. The topological realization of MSL is MSU, the special unitary cobordism spectrum. The topological counterparts of MSLO and MSL (i.e., MSO and MSU, respectively) are discussed in [Pengelley 1982]. Furthermore, using almost the same construction used to form MGLR [Hu et al. 2011], one can form a spectrum MSLR, which we call special hermitian algebraic cobordism. The underlying nonequivariant spectrum of MSLR is MSL, and the underlying geometric fixed points spectrum of MSL is MSLO.

In Sections 5 and 6 we use our computation of MGLO to obtain some results on the coefficients of MGLR. In particular, we compute the coefficients of 2-completed MGLR localized at two elements θ and λ in Theorem 6.6. In Theorem 5.5 and Corollary 5.6 we show that MGLR is not motivically real-oriented, solving a question asked in [Hu et al. 2011].

Notation and conventions. Throughout the paper, k is a field of characteristic 0. The stable motivic homotopy category of Morel and Voevodsky, as constructed in [Morel and Voevodsky 1999], is denoted by $\mathrm{SH}(\mathsf{k})$. An important feature of motivic homotopy theory is that we have two circles. These we denote as S^1 and S^α, as opposed to the other common notation of $S^{1,0}$ and $S^{1,1}$, respectively. The topological circle S^1 is formed in the usual way as $\Delta^1/\partial\Delta^1$, which we point at 1. The geometric sphere S^α is $\mathbb{G}_m \simeq \mathrm{Spec}(\mathsf{k}[z, z^{-1}])$ pointed at 1.

For a finite group G, let $G\mathrm{Sm}/\mathsf{k}$ denote the category of smooth schemes of finite type over k with left G actions and equivariant maps. The construction of the stable G-equivariant motivic homotopy category $\mathrm{SH}_G(\mathsf{k})$ can be found in [Hu et al. 2011]. We write $[-,-]_G$ for maps in $\mathrm{SH}_G(\mathsf{k})$. An important feature of the C_2-equivariant motivic homotopy category $\mathrm{SH}_{C_2}(\mathsf{k})$ is that we have four circles. These are denoted S^1, S^α, S^σ, and $S^{\sigma\alpha}$. The topological sphere S^1 is the usual simplicial sphere and S^σ the simplicial sphere with action $z \to -z$. The geometric sphere S^α is the pointed scheme $(\mathbb{G}_m, 1)$ equipped with trivial action and $S^{\sigma\alpha}$ is the pointed scheme $(\mathbb{G}_m, 1)$ equipped with the inversion action $z \mapsto z^{-1}$. For this reason we often use the notation $\mathbb{G}_m^{1/z}$ instead of $S^{\sigma\alpha}$.

We adopt the convention that $*$ refers to an integer grading of homotopy or (co)homology groups while $\star$ refers to multidimensional grading. In more detail, $\star$ grading refers to either $\mathbb{Z}^2$ grading in the cases of $\mathrm{SH}(\mathsf{k})$ and the classical stable C_2-equivariant category, or to $\mathbb{Z}^4$ grading in the case of $\mathrm{SH}_{C_2}(\mathsf{k})$.

2. A motivic analogue of MO

In this section, we give a detailed account of how to construct a motivic analogue of the unoriented cobordism spectrum MO. In Section 3, we give a full computation of the coefficients of this spectrum, which we call MGLO, up to knowledge of the coefficients of motivic $H\mathbb{Z}/2$. In particular, one can compute the coefficients explicitly for the fields $\mathbb{R}$ and $\mathbb{C}$. Moreover, the topological realization of MGLO over the field $\mathbb{C}$ is MO.

***The construction of* MGLO.** The idea behind our definition of MGLO is that, just as the geometric fixed points of MO is MR, the geometric fixed points of MGLR should be MGLO. The definition presented in this paper is different from the definition given in [Hu et al. 2011]. Using simplicial EC_2, the authors of [Hu et al. 2011] define MGLO as

$$(\widetilde{EC}_2 \wedge \mathsf{MGLR})^{C_2}. \tag{2.1}$$

However, the functor

$$(\widetilde{EC}_2 \wedge (-))^{C_2}$$

in (2.1) fails to satisfy a crucial property for general motivic spectra. Topologically, given a G-equivariant spectrum E, the functor

$$\Phi^G(-) := (\widetilde{EG} \wedge (-))^G$$

applied to E produces a nonequivariant spectrum $\Phi^G(\mathsf{E})$, which is equivalent to forgetting E to the prespectrum level and then simultaneously taking G-fixed points of the spaces making up the prespectrum of E and the connecting maps to form a nonequivariant prespectrum. One can then promote this to a nonequivariant spectrum in the usual way. Similarly, in our definition, MGLO is defined by forgetting MGLR to the level of prespectra and then taking C_2-fixed points of the spaces and connecting maps to form a nonequivariant prespectrum. Promoting this to a spectrum defines MGLO.

We suspect this alternative definition of MGLO to be different than (2.1), the reason being that simplicial $\widetilde{EC}_2$ is a model for $S^{\infty\sigma}$. This only takes into account the σ grading. However, we need to also take into account the $\sigma\alpha$ grading. In other words, our $\widetilde{EC}_2$ should really be a model of $S^{\infty\sigma+\infty\sigma\alpha}$. It turns out that there is an alternative version of $\widetilde{EC}_2$, whose definition was originally given in [Morel and Voevodsky 1999, Chapter 4.2], and which we redefine in Section 6. We refer to this alternative as the *geometric* model. Our primary definition for MGLO is Definition 2.19. By Theorem 6.1 our primary definition of MGLO is equivalent to

$$(\mathsf{MGLR} \wedge S^{\infty\sigma+\infty\sigma\alpha})^{C_2}. \tag{2.2}$$

While we do not have a proof that (2.2) and (2.1) are different spectra, the nonequivalence of the geometric and simplicial classifying spaces for C_2 imply a general nonequivalence of (2.2) and (2.1) whenever MGLR is replaced by a general C_2-equivariant motivic spectrum E. For this reason, we do not assume an equivalence between (2.2) and (2.1) in this paper. For more detail, see Section 6.

Quadratic forms. The classical Milnor spectrum MO has as its prespectrum the Thom spaces, defined as the quotient $\mathrm{BO}_n \,/\, \mathrm{BO}_{n-1}$ induced by inclusion into the zero section. This is well defined because of the well known equivalence of the geometric realization of the two-sided bar construction

$$|B(*, O_n, S^{n-1})| \simeq |B(*, O_{n-1}, *)| \simeq \mathrm{BO}_{n-1}\,.$$

In other words, a key ingredient in the construction of MO is the orthogonal groups O_n along with their associated transitive action on an appropriate model of a sphere. It is well known that the classical orthogonal group O_n is a special case of a generalized class of orthogonal groups which are defined in terms of symmetric bilinear forms. In more detail, given a symmetric bilinear form $b : k^n \times k^n \to k$, we can define the transpose of a matrix $A \in \mathrm{GL}_n(\mathsf{k})$ to be the unique matrix A^{T_b} such that

$$b(Ax, y) = b(x, A^{T_b} y) \quad \forall x, y \in \mathsf{k}^n.$$

Using this, we can then define the group of orthogonal matrices by

$$O_n^b := \{A \in \mathrm{GL}_n(\mathsf{k}) \mid AA^{T_b} = I\}.$$

We often suppress b in our notation whenever the underlying symmetric bilinear form b is understood from context.

While MGLO is supposed to be a motivic version of MO, it is also supposed to be the geometric fixed points of the C_2-motivic spectrum MGLR, which in turn is a motivic version of the C_2-equivariant spectrum MR. The C_2 action on MR comes from an action on the group $\mathrm{GL}_n(\mathbb{C})$ given by complex conjugation,

$$A \leftrightarrow \bar{A}.$$

However, complex conjugation is trivial over fields which do not contain $\sqrt{-1}$. This motivates the discussion which follows.

Following [Hu et al. 2011, Section 6.1], we instead consider the hyperbolic quadratic form on k^{2n}:

$$q(x_1, \dots, x_{2n}) = x_1 x_2 + \cdots + x_{2n-1} x_{2n}.$$

The associated symmetric bilinear form is

$$b((x_1, \dots, x_{2n}), (y_1, \dots, y_{2n})) = \sum_{i=1}^{n} x_{2i} y_{2i-1} + x_{2i-1} y_{2i}.$$

The b-adjoint of a matrix $A = (a_{i,j})_{i,j=0}^{2n}$ is a $2n \times 2n$ matrix A^{T_b} such that

$$b(Ax, y) = b(x, A^{T_b} y). \tag{2.3}$$

Explicitly, putting $A^{T_b} = (b_{i,j=1}^{2n})$, one has

$$\begin{aligned} b_{2i,2j} &= a_{2j-1,2i-1}, & b_{2i,2j-1} &= a_{2j,2i-1}, \\ b_{2i-1,2j-1} &= a_{2j,2i}, & b_{2i-1,2j} &= a_{2j-1,2i}. \end{aligned}$$

Notice that there is a C_2 action on the quadric

$$\mathcal{Q}_n := \mathbb{V}(x, y \mid b(x, y) = 1)$$

given by

$$x \leftrightarrow y,$$

where $\mathbb{V}(x_i \mid E)$ (sometimes abbreviated to $\mathbb{V}(E)$) denotes the locus of the equations E in the variables x_i.

Taking C_2 fixed points of the quadric under this action, we have

$$\begin{aligned}(\mathcal{Q}_n)^{C_2} &= \mathbb{V}(x, y \mid b(x, y) = 1, x = y)\\ &= \mathbb{V}\bigg(\sum_{i=1}^{n} x_{2i}y_{2i-1} + x_{2i-1}y_{2i} - 1, x = y\bigg).\end{aligned} \tag{2.4}$$

The projection from (2.4) onto the x coordinate scaled by a factor of 2 gives an equivalence to $Q_{2n-1} := \mathbb{V}(x \in \mathsf{k}^{2n} \mid x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n} - 1)$. But the projection from (2.4) onto the x-axis gives the same thing as projecting $\mathcal{Q}_n$ onto the x-axis. So long as $x \neq 0$ there exists a y such that $b(x, y) = 1$. But this means that the image of the projection map is $\mathbb{A}^{2n} \smallsetminus 0$. It is a standard result that $\mathbb{A}^{2n} \smallsetminus 0$ has the homotopy type of $S^{2n-1,n} = S^{n-1+n\alpha}$. Using (2.3) we can define the even-dimensional orthogonal groups by

$$O_{2n} := \{A \in \mathrm{GL}_{2n}(k) \mid AA^{T_b} = I\}.$$

The group O_{2n} acts on the quadric Q_{2n-1} in a natural way. We can write Q_{2n-1} as

$$\mathbb{V}\big(\tfrac{1}{2}b(x, x) - 1\big).$$

The action on Q_{2n-1} is given elementwise by $A \cdot x = Ax$. Notice that

$$b(Ax, Ax) = b(x, A^{T_b}Ax) = b(x, x).$$

Therefore we have defined an O_{2n} action on Q_{2n-1}. We define O_{2n-1} to be

$$O_{2n-1} := \{A \in O_{2n} \mid A(1, 1, 0, \ldots, 0) = (1, 1, 0, \ldots, 0)\}.$$

Lemma 2.5. *For each positive integer n, the even-dimensional orthogonal group O_{2n} acts transitively on the motivic sphere Q_{2n-1}.*

Proof. The quadratic form

$$q(x) = \sum_{i=1}^{n} x_{2i-1}x_{2i}$$

is uniquely defined by a $2n\times 2n$ symmetric matrix A consisting of all zeros, except for n copies of the 2×2 matrix

$$\begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix} \tag{2.6}$$

along its diagonal. The matrix A is in turn congruent to the matrix B consisting of all zeros except for n copies of the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2.7}$$

along its diagonal. Therefore, the claim about transitivity is equivalent to proving transitivity with respect to the orthogonal group and sphere induced from the symmetric bilinear form induced by the matrix B. The symmetric bilinear form represented by B is given by,

$$\sum_{i=1}^{n} x_{2i-1}y_{2i-1} - x_{2i}y_{2i}.$$

Under this symmetric bilinear form b_B, orthogonal matrices consist of a set of vectors $\mathcal{B} = \{b_i\}_{i=1}^{2n}$ such that

$$b_B(b_i, b_j) = \begin{cases} \pm 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Under our equivalent symmetric bilinear form b_B, our sphere is given by

$$Q_{2n-1}^{B} := \mathbb{V}\left(x \in \mathsf{k}^{2n} \;\middle|\; -1 + \sum_{i=1}^{n}(x_{2i-1}^2 - x_{2i}^2)\right).$$

Now, to prove our claim about transitivity, let $\{e_i\}_{i=1}^{2n}$ denote the standard basis for k^{2n}. Proving transitivity is equivalent to proving that for any point $p \in Q_{2n-1}^{B}$ there exists a matrix M such that $Me_1 = p$. Under this assumption, the set of vectors $\mathcal{A} = \{p\} \cup \{p + e_i\}_{i=1}^{2n-1}$ are linearly independent. Using the Gram–Schmidt process with respect to the inner product induced by b_B, we can form an orthonormal set of vectors with respect to the basis $\mathcal{A} = \{p\} \cup \{p + e_i\}_{i=1}^{2n-1}$. The basis will become the rows of M, and our claim will be proven. However, we need to show that the points obtained from the Gram–Schmidt process still live inside of k^{2n}, rather than some potentially bigger field $\mathsf{k}' \supset \mathsf{k}$. To this end, note that

$$\operatorname{proj}_a(a + e_i) = a - \frac{b(a + e_i, a)}{b(a, a)}a = (1 - b(e_i, a))a.$$

Therefore,

$$b_B(\operatorname{proj}_a(a + e_i), \operatorname{proj}_a(a + e_i)) = (1 - b(e_i, a))^2 \cdot b_B(a, a) = (1 - b(e_i, a))^2.$$

This proves the claim. $\square$

Definition 2.8. The odd-dimensional orthogonal groups O_{2n-1} are defined to be the stabilizer of the point $(1, 1, 0, \ldots, 0)$.

Next we define the even-dimensional quadrics as

$$\begin{aligned} Q_{2n-2} &:= \mathbb{V}(x \in \mathsf{k}^{2n} \mid b(x, x^0), b(x,x)+1) \\ &= \{x \in \mathsf{k}^{2n} \mid x_1x_2 + \cdots + x_{2n-1}x_{2n} + 1 = x_1 + x_2 = 0\}. \end{aligned}$$

We would like to make analogous statements to Lemma 2.5 for O_{2n-1} and Q_{2n-2}, but first we show that Q_{2n-2} is homotopy equivalent to a familiar space.

Lemma 2.9. *The motivic space Q_{2n-2} is homotopy equivalent to the motivic sphere $S^{n-1+(n-1)\alpha}$.*

Proof. We have that

$$Q_{2n-2} = \mathbb{V}(x \in \mathsf{k}^{2n} \mid x_1x_2 + \cdots + x_{2n-1}x_{2n} + 1, x_1 + x_2).$$

We note that this space is homotopy equivalent to

$$\mathbb{V}((y, x_3, x_4, \ldots, x_{2n}) \in \mathsf{k}^{2n-1} \mid -y^2 + x_3x_4 + \cdots + x_{2n-1}x_{2n} + 1).$$

But this is easily seen to be equivalent to

$$\operatorname{Spec}(\mathsf{k}[y, x_3, x_4, \ldots, x_{2n-1}, x_{2n}]/((1-y)(1+y) + x_3x_4 + \cdots + x_{2n-1}x_{2n})).$$

Now, by [Asok et al. 2017, Theorem 2], we notice that

$$S^{n-1+(n-1)\alpha} \simeq \operatorname{Spec}(\mathsf{k}[z, a_3, a_4, \ldots, a_{2n-1}, a_{2n}]/(a_3a_4 + \cdots + a_{2n-1}a_{2n} - z(1+z)).$$

Using the change of variables $z \mapsto -\frac{1}{2}(1+y)$, $a_i \mapsto \frac{1}{2}x_i$, we have that

$$\begin{aligned} &\operatorname{Spec}\big(\mathsf{k}[z, a_3, a_4, \ldots, a_{2n-1}, a_{2n}]/(a_3a_4 + \cdots + a_{2n-1}a_{2n} - z(1+z))\big) \\ &\qquad \simeq \operatorname{Spec}\big(\mathsf{k}\big[-\tfrac{1}{2}(1+y), \tfrac{1}{2}x_3, \tfrac{1}{2}x_4, \ldots, \tfrac{1}{2}x_{2n-1}, \tfrac{1}{2}x_{2n}\big] \\ &\qquad\qquad\qquad /\big(\tfrac{1}{4}(x_3x_4 + \cdots + x_{2n-1}x_{2n} + (1-y)(1+y))\big)\big) \\ &\qquad \simeq \operatorname{Spec}\big(\mathsf{k}[y, x_3, x_4, \ldots, x_{2n-1}, x_{2n}] \\ &\qquad\qquad\qquad /(x_3x_4 + \cdots + x_{2n-1}x_{2n} + (1-y)(1+y))\big). \end{aligned}$$

□

The O_{2n} action on Q_{2n-1} induces an O_{2n-1} action on Q_{2n-2}, which we prove presently. Recall that O_{2n-1} acts pointwise on the quadric Q_{2n-2} by $A \cdot x \mapsto Ax$. Notice that Q_{2n-2} is induced from the form $b_{2n}(x, y)$, and $x \in Q_{2n-2}$ implies that $\frac{1}{2}b_{2n}(x, x) = -1$. Since

$$b(Ax, Ax) = b(x, A^{T_b}Ax) = b(x, x),$$

it only remains to show that if $x_1 = -x_2$ and $y = (y_1, y_2, \ldots, y_{2n})$ is the image of x, then $y_1 = -y_2$. But notice that for $x \in Q_{2n-2}$ we have that $b(x, (1, 1, 0, \ldots, 0)) = 0$.

Let $A \in O_{2n-1}$ and let $y = (y_1, y_2, \dots, y_{2n})$ be the image of x. Then

$$\begin{aligned} y_1 + y_2 &= b(y, (1, 1, 0, \dots, 0)) = b(Ax, (1, 1, 0, \dots, 0)) \\ &= b(x, A^{T_b}(1, 1, 0, \dots, 0)) = b(x, (1, 1, 0, \dots, 0)) = x_1 + x_2 = 0. \end{aligned}$$

This proves that O_{2n-1} acts on the quadric Q_{2n-2}.

Lemma 2.10. *O_{2n-1} acts transitively on Q_{2n-2}, and the fixed point subgroup of $y^0 = (1, -1, 0, \dots, 0)$ can be naturally identified with O_{2n-2}.*

Proof. We prove the transitivity claim in a similar manner to Lemma 2.5. It is enough to show that for any $x \in Q_{2n-2}$ there is a matrix $A \in O_{2n-1}$ such that $Ax = y^0$.

Notice that technically our O_{2n-1} lives inside of O_{2n}. We choose orthonormal bases

$$\mathcal{B}_1 = \left\{ \frac{x^0}{\|x^0\|}, \frac{y^0}{\|y^0\|}, e_3, \dots, e_{2n} \right\},$$
$$\mathcal{B}_2 = \left\{ \frac{x^0}{\|x^0\|}, \frac{x}{\|x\|}, v_3, \dots, v_{2n} \right\}.$$

Notice there exists a change of basis matrix P from $\mathcal{B}_2$ to $\mathcal{B}_1$ which sends x^0 to x^0 and x to $y^0/\|y^0\|$.

This implies that for $x \in Q_{2n-2}$ we have that $Px = \lambda y_0$. We have that

$$-1 = \tfrac{1}{2}b(x, x) = \tfrac{1}{2}b(Px, Px) = \tfrac{1}{2}b(\lambda y^0, \lambda y^0) = \tfrac{1}{2}\lambda^2 b(y^0, y^0) = -\lambda^2 \Longrightarrow \lambda = \pm 1.$$

If $\lambda = 1$ then we are done. If $\lambda = -1$ then we have that $(-P)x = y^0$. This proves the transitivity claim.

Now notice that the subgroup of O_{2n-1} which fixes $y^0 = (1, -1, 0, \dots, 0) \in \mathsf{k}^{2n}$ is

$$\{A \in O_{2n} \mid Ax^0 = x^0 \text{ and } Ay^0 = y^0\} = \{A \in O_{2n-1} \mid Ae_1 = e_1 \text{ and } Ae_2 = e_2\}.$$

But this is just matrices $A \in O_{2n}$ of the form

$$A = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & x_{3,3} & \dots & x_{3,2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & x_{2n,3} & \dots & x_{2n,2n} \end{bmatrix}.$$

This shows that O_{2n-2} can be naturally identified with the subgroup of O_{2n-1}, which fixes the point y^0. □

Cellularity. The following definition is due to [Dugger and Isaksen 2005, Definition 2.1]. Let $\mathcal{M}$ be a pointed model category, and let $\mathcal{A}$ be a set of objects in $\mathcal{M}$.

Definition 2.11. The class of $\mathcal{A}$-cellular objects is the smallest class of objects of $\mathcal{M}$ such that

(1) every object of $\mathcal{A}$ is $\mathcal{A}$-cellular;

(2) if X is weakly equivalent to an $\mathcal{A}$-cellular object, then X is cellular;

(3) if $\mathcal{D} : I \to \mathcal{M}$ is a diagram such that $\mathcal{D}$ is $\mathcal{A}$-cellular, then so is $\operatorname{hocolim} \mathcal{D}$.

Choosing $\mathcal{M}$ to be the stable motivic homotopy category, and choosing $\mathcal{A}$ to be

$$\{S^{m+n\alpha} \mid m, n \in \mathbb{Z}\},$$

we obtain the cellular stable motivic homotopy category.

Adapting the proof of [Dugger and Isaksen 2005, Proposition 4.1], we prove the following.

Proposition 2.12. *The variety O_n is stably cellular for every $n \geq 1$.*

Proof. We first suppose that $n = 2k$. Let $x = (1, 1, 0, \dots, 0)$. Now consider the fiber bundle $O_n \to \mathbb{P}^{n-1}$ given by

$$O_n \xrightarrow{m_x} \mathbb{A}^n \to \mathbb{A}^n / \mathbb{A}^n \smallsetminus 0 \simeq \mathbb{P}^{n-1}.$$

Here m_x denotes the map $A \mapsto Ax$. Notice that m_x induces a transitive action of O_n on the motivic sphere Q_{n-1}. The fiber over the point $(1, 0, 0, \dots, 0)$ consists of all $A \in O_n$ such that $a_{11} \neq 1$, and $a_{j1} = 0$ for $j \geq 2$. Recall that

$$O_{n-1} \cong \{A \in O_n \mid A(1, 0, 0, \dots) = (1, 0, 0, \dots)\}.$$

But this is just $\{A \in m_x^{-1}((1, 0, 0, \dots)) \mid a_{11} = 1\}$. Since

$$\det(AA^T) = \det(A)\det(A^T) = \det(A)^2 = 1,$$

it follows that $a_{11} = \pm 1$, and so $m_x^{-1}((1, 0, 0, \dots)) = O_{n-1} \times \{\pm 1\}$. As a scheme, but not as a group, this is isomorphic to

$$\{\pm 1\} \times \mathbb{A}^{n-1} \times O_{n-1},$$

which is stably cellular by induction and [Dugger and Isaksen 2005, Lemma 3.4]. The usual cover of $\mathbb{P}^n$ by affines is a completely trivializing cover for the bundle, so [Dugger and Isaksen 2005, Lemma 3.8] applies. $\square$

Two-sided bar construction. Recall that we have the following equivalences,

$$Q_n \simeq \begin{cases} S^{k+k\alpha} & \text{if } n = 2k, \\ S^{k-1+k\alpha} & \text{if } n = 2k-1. \end{cases}$$

The groups O_n act on the quadrics Q_{n-1}, allowing us to form the two-sided bar construction, which we now discuss.

Let G be a finite group and X and Y motivic spaces. If $X \times G \to X$ is a right G action and $G \times Y \to Y$ is a left G action, then we form the two-sided bar construction $B(X, G, Y)$ as the left derived functor of the coequalizer of $X \times G \times Y \rightrightarrows X \times Y$. We denote the geometric realization of $B(X, G, Y)$ by $|B(X, G, Y)|$.

Definition 2.13. In the special case $X = Y = *$, we define $BG := |B(*, G, *)|$.

Lemma 2.14. *The geometric realization of $B(O_n, O_{n-1}, *)$ is homotopy equivalent to Q_{n-1}.*

Proof. It is well known for $H \hookrightarrow G$ an inclusion of groups that the left coset G/H is isomorphic to $|B(G, H, *)|$. Taking $G = O_n$ and $H = O_{n-1}$, this gives

$$O_n/O_{n-1} \cong |B(O_n, O_{n-1}, *)|.$$

Notice that by the above discussion, O_n acts on Q_{n-1}, and the stabilizer of a point is O_{n-1}. This induces an isomorphism between O_n/O_{n-1} and Q_{n-1}, proving that

$$Q_{n-1} \cong |B(O_n, O_{n-1}, *)|. \qquad \square$$

Lemma 2.15. *The geometric realization of the two-sided bar $B(G, G, *)$ is contractible. In particular, we have $|B(O_n, O_n, *)| \simeq *$.*

Proof. Notice that $* \cong G/G \cong |B(G, G, *)|$. $\square$

Proposition 2.16. *The geometric realization of the two-sided bar construction $B(*, O_n, Q_{n-1})$ is homotopy equivalent to* BO_{n-1}.

Proof. We have that

$$\begin{aligned} |B(*, O_n, Q_{n-1})| &\simeq \big|B\big(*, O_n, |B(O_n, O_{n-1}, *)|\big)\big| \\ &\simeq \big|B\big(|B(*, O_n, O_n)|, O_{n-1}, *\big)\big| \simeq |B(*, O_{n-1}, *)|. \end{aligned} \qquad \square$$

The prespectrum for **MGLO.** We define a motivic prespectrum as follows.

Definition 2.17. A motivic prespectrum E is defined to be a collection of based spaces $E_1, E_2, \ldots$ equipped with connecting maps $S^{1+\alpha} \wedge E_n \xrightarrow{\sigma_n} E_{n+1}$. If the adjoint maps $E_n \xrightarrow{\tilde{\sigma}_n} [S^{1+\alpha}, E_{n+1}]$ are homotopy equivalences, then we say that E is a motivic spectrum.

The identifications from Proposition 2.16 imply that we have a canonical map

$$\mathrm{BO}_{n-1+} \xrightarrow{\pi} \mathrm{BO}_{n+} \tag{2.18}$$

which is defined by projection maps $(\mathrm{BO}_{n-1})_m \xrightarrow{\pi_m} (\mathrm{BO}_n)_m$ given by

$$\underbrace{O_n \times O_n \times \cdots \times O_n}_{m \text{ times}} \times Q_{n-1} \mapsto \underbrace{O_n \times O_n \times \cdots \times O_n}_{m \text{ times}}.$$

Therefore, we can think of (2.18) as a sphere bundle. This allows us to define Thom space-like objects as the homotopy cofiber of π. The Thom space of BO_n, which we denote as $\mathrm{Thom}(\mathrm{BO}_n)$, is defined to be the pushout of the diagram

$$\begin{tikzcd} \mathrm{BO}_{n-1+} \arrow[r] \arrow[d] & \mathrm{BO}_{n+} \arrow[d] \\ * \arrow[r] & \mathrm{Thom}(\mathrm{BO}_n) \end{tikzcd}$$

The spaces $\mathrm{Thom}(\mathrm{BO}_{2n})$ form the spaces for the prespectrum of MGLO.

Definition 2.19 (MGLO). At the level of prespectra, MGLO is defined by

$$(\mathsf{MGLO})_n := \mathrm{Thom}(\mathrm{BO}_{2n}).$$

Notice the natural inclusions $O_{n-1} \times O_{m-1} \subset O_n \times O_m$ induce maps

$$B(O_{n-1} \times O_{m-1}) \to B(O_n \times O_m).$$

We define $\mathrm{Thom}(B(O_{2n} \times O_{2m}))$ to be

$$B(O_{2n} \times O_{2m})/B(O_{2n-1} \times O_{2m-1}) \simeq \mathrm{Thom}(\mathrm{BO}_{2n}) \wedge \mathrm{Thom}(\mathrm{BO}_{2m}).$$

The structure maps

$$S^{1+\alpha} \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) \xrightarrow{\sigma_n} \mathrm{Thom}(\mathrm{BO}_{2n+2})$$

are defined by

$$\begin{aligned} &S^{1+\alpha} \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) \simeq \Sigma\mathbb{G}_m \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) \\ &\quad \to |B(*, O_2, \mathbb{G}_m)|_+ \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) \to \mathrm{BO}_{2+} \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) \\ &\quad \to \mathrm{Thom}(\mathrm{BO}_2) \wedge \mathrm{Thom}(\mathrm{BO}_{2n}) \xrightarrow{\simeq} \mathrm{Thom}(\mathrm{BO}_2 \times \mathrm{BO}_{2n}) \to \mathrm{Thom}(\mathrm{BO}_{2n+2}). \end{aligned}$$

This defines the prespectrum MGLO and we can promote it to a spectrum in the usual way.

Notice that since the orthogonal groups are stably cellular by Proposition 2.12, it follows that the classifying spaces BO_n are also stably cellular. Since each of the Thom spaces $\mathrm{Thom}(\mathrm{BO}_n)$ is constructed as the homotopy cofiber of the inclusion $\mathrm{BO}_{n-1} \to \mathrm{BO}_n$, it follows that the spaces $\mathrm{Thom}(\mathrm{BO}_n)$ are also cellular. Since these are the spaces defining the prespectrum of MGLO, it follows that MGLO is cellular.

3. Computing the coefficients of MGLO

Combining Proposition 2.16 with a Mayer–Vietoris argument as in [Milnor and Stasheff 1974] gives us two Thom isomorphisms in motivic $H\mathbb{Z}/2$ (co)homology:

$$H^{\star}(\mathrm{BO}_{n+}) \cong H^{\star+\omega_n}(\mathrm{Thom}(\mathrm{BO}_n)),$$
$$H_{\star}(\mathrm{BO}_{n+}) \cong H_{\star+\omega_n}(\mathrm{Thom}(\mathrm{BO}_n)).$$

Here $\omega_{2k} := k + k\alpha$ and $\omega_{2k+1} := k + 1 + k\alpha$.

For each space BO_n, we get a unique Thom class $\mathrm{Thom}(\mathrm{BO}_n) \xrightarrow{w_n} \Sigma^{\omega_n} H\mathbb{Z}/2$. Composing w_n with the homotopy cofiber of the map $\mathrm{BO}_{n-1+} \to \mathrm{BO}_{n+}$, we get a class $w_n \in H^{\omega_n}(\mathrm{BO}_{n+})$. The following theorem has essentially been proved by A. Smirnov and A. Vishik [2014, Theorem 3.1.1] using different language from the present paper. The biggest difference between [Smirnov and Vishik 2014] and the theorem presented here is that the former only applies to fields of characteristic 0 for which $\sqrt{-1} \in \mathsf{k}$, whereas the present theorem holds for any field k of characteristic 0.

Theorem 3.1. *There is a unique set of classes $w_1, w_2, \dots, w_n$ belonging to motivic $\mathbb{Z}/2$ cohomology for which*

$$H^{\star}(\mathrm{BO}_{n+}) \cong H^{\star}[w_1, \dots, w_n].$$

Here $\deg(w_{2i}) = i + i\alpha$ *and* $\deg(w_{2i+1}) = i + 1 + i\alpha$.

Proof. Notice that the cofibration $\mathrm{BO}_{n-1+} \to \mathrm{BO}_{n+} \to \mathrm{Thom}(\mathrm{BO}_n)$ induces a long exact sequence in cohomology given by

$$\cdots \to H^{\star}(\mathrm{Thom}(\mathrm{BO}_n)) \to H^{\star}(\mathrm{BO}_{n+}) \to H^{\star}(\mathrm{BO}_{n-1+}) \to H^{\star+1}(\mathrm{Thom}(\mathrm{BO}_n)) \to \cdots.$$

Using the Thom isomorphism $H^{\star}(\mathrm{BO}_{n+}) \xrightarrow{\cong} H^{\star+\omega_n}(\mathrm{Thom}(\mathrm{BO}_n))$ we get the long exact sequence

$$\cdots \to H^{\star}(\mathrm{BO}_{n+}) \xrightarrow{f_n^*} H^{\star+\omega_n}(\mathrm{BO}_{n+}) \xrightarrow{g_n^*} H^{\star+\omega_n}(\mathrm{BO}_{n-1+}) \xrightarrow{h_n^*} H^{\star+1}(\mathrm{BO}_{n+}) \to \cdots.$$

Notice that f_n^* is multiplication by some nonzero class w_n. By induction,

$$H^{\star}(\mathrm{BO}_{n-1+}) = H^{\star}[w_1, \dots, w_{n-1}].$$

Since BO_n is cellular, we have that $H^{p+q\alpha}(\mathrm{BO}_{n+}) = 0$ for $q < 0$. It is also clear that the map f_n^* is injective on $\mathbb{Z}/2 \cong H^0(\mathrm{BO}_{n+})$. We can start with the case $n = 0$ by identifying BO_0 with $|B(*, O_1, Q_0)|$, which is contractible. Therefore, we have that $h_n^*(w_i) = 0$ for $i = 0, \dots, n-1$. It follows that each of the w_i can be uniquely lifted to $H^{\star}(\mathrm{BO}_{n+})$.

Moreover, since $h_n^*(w_i) = 0$ for $i = 0, \dots, n-1$, it follows that $h_n^* = 0$. Thus, the long exact sequence splits and we get the short exact sequence

$$0 \to H^\star(\mathrm{BO}_{n+}) \xrightarrow{f_n^*} H^{\star+\omega_n}(\mathrm{BO}_{n+}) \xrightarrow{g_n^*} H^{\star+\omega_n}(\mathrm{BO}_{n-1_+}) \to 0.$$

The key point is that f_n^* is multiplication by the cohomology class $w_n \in H^{\omega_n}(\mathrm{BO}_{n+})$. In other words, $f_n^* = \smile w_n$.

From this the claim follows. We have

$$\begin{aligned} H^\star(\mathrm{BO}_{n+}) &\cong H^\star[w_1, \dots, w_{n-1}] \oplus H^\star[w_1, \dots, w_{n-1}] \smile w_n \\ &\cong H^\star[w_1, \dots, w_n]. \end{aligned}$$ □

A quick word is in order. We have a Thom isomorphism in (co)homology. We have computed the cohomology of BO_n, but there is a motivic universal coefficient theorem, and so the (co)homologies are essentially the same and there is a duality between the (co)homology classes. Motivically, this is not always the case. However, $\mathrm{BO}_{n+} \wedge H\mathbb{Z}/2$ is a wedge sum of suspensions of $H\mathbb{Z}/2$ of dimensions $p + q\alpha$ with $p \geq q$ and so we can show that the (co)homology classes are dual to one another [Hoyois 2015, Section 5.1]. This gives us the following theorem.

Theorem 3.2. *There is a unique set of classes $w_1, w_2, \dots, w_n$ belonging to motivic $\mathbb{Z}/2$ homology for which*

$$H_\star(\mathrm{BO}_{n+}) \cong H_\star[w_1, \dots, w_n].$$

Here $\deg(w_{2i}) = i + i\alpha$ *and* $\deg(w_{2i+1}) = i + 1 + i\alpha$.

Using analogous arguments to those found in [Milnor and Stasheff 1974], we get the corollary below.

Corollary 3.3. *We have*

$$H_\star(\mathsf{MGLO}) \cong H_\star[w_1, w_2, \dots].$$

Here $\deg(w_{2i}) = i + i\alpha$ *and* $\deg(w_{2i+1}) = i + 1 + i\alpha$.

Since $\mathsf{MGLO} \wedge H\mathbb{Z}/2$ is a wedge sum of suspensions of $H\mathbb{Z}/2$ of dimensions $p + q\alpha$ with $p \geq q$, it follows that the (co)homology classes are dual to one another

Dual motivic Steenrod algebra. We review some results on the dual motivic Steenrod algebra. These results can be found in [Kylling 2017].

The dual motivic Steenrod algebra $\mathcal{A}^\vee_{\mathrm{Mot}}$ is defined to be $H\mathbb{Z}/2 \wedge H\mathbb{Z}/2$. As an $H_\star$-algebra, the coefficients of $\mathcal{A}^\vee_{\mathrm{Mot}}$ are given by

$$H_\star[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]/(\tau_i^2 - \tau\xi_{i+1} - \rho\tau_{i+1} - \rho\tau_0\xi_{i+1}). \tag{3.4}$$

Here $|\xi_i| = (2^i - 1)(1 + \alpha)$, $|\tau_i| = (2^i - 1)(1 + \alpha) + 1$, τ is the generator of $H_{1-\alpha} \cong \mu_2(\mathsf{k})$, and ρ is the class of -1 in $H_{-\alpha} \cong \mathsf{k}^\times/(\mathsf{k}^\times)^2$. Let $E = (\epsilon_0, \dots, \epsilon_n)$,

$\epsilon_i \in \{0, 1\}$, and $R = (r_1, \dots, r_m)$. The dual motivic Steenrod algebra is a free $H_\star$-module with basis consisting of the monomials,

$$\tau(E)\xi(R) := \prod_E \tau_i^{\epsilon_i} \prod_R \xi_i^{r_i}.$$

By comparing the $H_\star$-module basis for the coefficients of $\mathsf{MGLO} \wedge H\mathbb{Z}/2$ and $\mathcal{A}^\vee_{\text{Mot}}$, we see that $\mathsf{MGLO} \wedge H\mathbb{Z}/2$ is a wedge sum of suspensions of $\mathcal{A}^\vee_{\text{Mot}}$. Consider the submodule $\mathcal{M}$ of $H_\star(\mathsf{MGLO})$ obtained by deleting all generators of degree $|\xi_i|$ and squaring all generators of degree $|\tau_i|$. Let $\mathfrak{M}$ be an $H_\star$-module basis for this submodule. Then

$$\mathsf{MGLO} \wedge H\mathbb{Z}/2 \simeq \bigvee_{m_i \in \mathfrak{M}} \Sigma^{|m_i|} \mathcal{A}^\vee_{\text{Mot}}.$$

***The equivalence between* MGLO *and* MGLO/(2, η).** We begin with the definitions of the mod 2 Moore spectrum, and the motivic Hopf map.

Definition 3.5. The mod 2 Moore spectrum is defined to be the stable cofiber $\mathsf{M}(2)$ of the following map induced by multiplication by 2:

$$S^0 \xrightarrow{2} S^0 \to \mathsf{M}(2).$$

Notice that the map 2 is induced by the stable homotopy class represented by $2 \in \mathbb{Z} \subseteq \pi_0(S^0)$, where $2 = 1 + 1$ and 1 is the class representing the unit map.

It is a well known fact that $H\mathbb{Z} \wedge \mathsf{M}(2) \simeq H\mathbb{Z}/2$. Recall that classically $2 = 0$ in the coefficients of MO. The analogous statement is true for MGLO.

Proposition 3.6. *We have* $2 = 0$ *in the coefficients of* MGLO.

Proof. We have a map

$$\pi_{1+\alpha}(\mathrm{BO}_2) \to \pi_{1+\alpha}(\mathrm{Thom}(\mathrm{BO}_2)) \to \pi_0(\mathsf{MGLO}).$$

The unit is the image of $1 \in \pi_{1+\alpha}(B\mathbb{G}_m)$ via the map

$$h : \pi_{1+\alpha}(B\mathbb{G}_m) \to \pi_{1+\alpha}(\mathrm{BO}_2).$$

The map $z \mapsto z^{-1}$ sends $1 \mapsto -1 \in \pi_{1+\alpha}(B\mathbb{G}_m)$, but becomes identified with the identity under h. Thus, $1 = -1 \in \pi_{1+\alpha}(\mathrm{BO}_2)$. □

Consider the Hopf map given by the projection $h : \mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$. Recall that $\mathbb{A}^2 \setminus 0 \simeq S^{1+2\alpha}$ and $\mathbb{P}^1 \simeq S^{1+\alpha}$. It follows that h induces a stable map $\eta : \Sigma^\alpha S^0 \to S^0$. We denote the cokernel of this map by S^0/η. For a general spectrum E, we denote the cokernel of the map $\eta \wedge \mathsf{E}$ by E/η.

Proposition 3.7. *We have* $\eta = 0$ *in the coefficients of* MGLO.

Proof. It is well known that $\eta = 0$ in the coefficients of MGL. We stably prove MGLO is an E_∞-ring spectrum in Corollary 6.2. Therefore, it is enough to produce a map of ring spectra from MGL to MGLO. We accomplish this by producing a surjective map $\mathrm{GL}_n \to O_{2n}$. This map is given by

$$A \mapsto A \oplus (A^{T_b})^{-1}.$$

This in turn induces a map $\mathsf{MGL} \to \mathsf{MGLO}$ as desired. □

Applying the motivic Hurewicz theorem. We use a modified version of the motivic Hurewicz theorem of [Bachmann 2018]. We recall what it means to be $(n-1)$-connected in the motivic sense.

Definition 3.8. A motivic spectrum E is *finite type* $(n-1)$-connected if the following hold:

(1) $\pi_{i+j\alpha}(\mathsf{E}) = 0$ for all $0 < i < n$.

(2) For each fixed $i \in \mathbb{Z}$, $\pi_{i+j\alpha}(\mathsf{E}) = 0$ for all but at most a finite number of $j \in \mathbb{Z}$.

Theorem 3.9. *Let* k *have characteristic* 0, *and suppose that* E *is a* finite type $(n-1)$*-connected cellular stable motivic spectrum for which* 2 *and* η *are* 0. *Then*

$$H_{n+*\alpha}(\mathsf{E}; \mathbb{Z}/2) \cong \pi_{n+*\alpha}(\mathsf{E}).$$

Proof. This follows from [Bachmann 2018, Theorem 3]. □

Consider the basis elements $v_i \in \mathfrak{M} \subset H_\star(\mathsf{MGLO})$. Then each of the v_i is dual to a cohomology class $c_i \in H^\star(\mathsf{MGLO})$, and so there exists a map

$$\mathsf{MGLO} \xrightarrow{f} \bigvee_{m_i \in \mathfrak{M}} \Sigma^{|m_i|} H\mathbb{Z}/2 \tag{3.10}$$

which induces an equivalence on homology.

Theorem 3.11. *The map* f *in* (3.10) *is a homotopy equivalence, and so* MGLO *splits as a wedge sum of* $H\mathbb{Z}/2$.

Proof. Taking the cofiber of the map f we obtain a cofibration

$$\mathsf{MGLO} \xrightarrow{f} \bigvee_{m_i \in \mathfrak{M}} \Sigma^{|m_i|} H\mathbb{Z}/2 \to \mathsf{F}.$$

The idea is that we know that F is cellular, and the coefficients of $\mathsf{F} \wedge H\mathbb{Z}/2$ are 0 by construction. Since 2 and η are 0 in $\bigvee_{m_i \in \mathfrak{M}} \Sigma^{|m_i|} H\mathbb{Z}/2$, it follows that 2^2 and η^2 are 0 in F. Then the motivic Hurewicz theorem combined with the Nakayama lemma implies that $\mathsf{F} = 0$, and so f is an equivalence. □

$\mathsf{MGLO}_\star$ *and a comparison with* MO_*. Combining everything, we have:

Theorem 3.12. *As an $H_\star$ algebra,*

$$\mathsf{MGLO}_\star \cong H_\star[u_{n+n\alpha}, u_{n+1+n\alpha}, u_{(2^i-1)(1+\alpha)+2} \mid n, i \in \mathbb{Z}^{\geq 0}, n \neq 2^i - 1].$$

Let $t^{\mathbb{C}}$ denote the complex topological realization functor. Then

$$t^{\mathbb{C}}(S^1) = S^1, \qquad t^{\mathbb{C}}(S^\alpha) = S^1, \qquad t^{\mathbb{C}}(H\mathbb{Z}/2_{\mathrm{Mot}}) = H\mathbb{Z}/2.$$

From this it follows that $t^{\mathbb{C}}(\mathsf{MGLO}) = \mathsf{MO}$. Over $\mathsf{k} = \mathbb{C}$, we have that

$$\begin{aligned}\mathsf{MGLO}_\star &= H\mathbb{Z}/2_{\mathrm{Mot}\star}[x_2, x_{2+2\alpha}, x_{3+2\alpha}, u_{4+2\alpha}, u_{4+4\alpha}, u_{5+4\alpha}, x_{5+5\alpha}, \dots] \\ &= \mathbb{Z}/2[\theta][u_2, u_{2+2\alpha}, x_{3+2\alpha}, u_{4+2\alpha}, u_{4+4\alpha}, u_{5+4\alpha}, u_{5+5\alpha}, \dots] \\ &= \mathbb{Z}/2[\theta, u_2, u_{2+2\alpha}, u_{3+2\alpha}, u_{4+2\alpha}, u_{4+4\alpha}, u_{5+4\alpha}, u_{5+5\alpha}, \dots].\end{aligned}$$

Recall that

$$\mathsf{MO}_* = \mathbb{Z}/2[a_2, a_4, a_5, a_6, a_8, a_9, a_{10}, \dots]. \tag{3.13}$$

So the generators of MO_* correspond to generators in $\mathsf{MGLO}_\star$ twisted by powers of θ.

4. A motivic analogue of MSO

Recall that the classical oriented cobordism spectrum MSO is closely related to MO. Similarly to MO, the spectrum MSO can be constructed from the Thom spaces of the classifying spaces of SO_n, which we denote by BSO_n. Recall that the group SO_n is defined as

$$\{A \in O_n \mid \det(A) = 1\}.$$

Although many results found in the this section can be generalized to more general fields, many of the proofs rely on the coefficients of the motivic $\mathbb{Z}/p$ cohomology of the mod p Eilenberg–Mac Lane spectrum being equal to $\mathbb{Z}/p[\tau]$, where τ denotes the Tate twist of degree $\alpha - 1$. Therefore, for the entirety of Section 4, the reader should always assume that k is a field of characteristic 0 containing all p-th roots of unity, and for which all its elements are p-th powers.

***Computing the coefficients of* MSLO.** Having constructed a motivic analogue of MO, it becomes apparent that it would be possible to construct a motivic analogue of MSO by mimicking the construction of MGLO. The simple observation is that we can again consider the quadratic form,

$$q(x_1, x_2, \dots, x_{2n}) = x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n}.$$

To this we can associate a unique orthogonal group O_{2n}. Since the determinant function is algebraic, we can define the $2n$-dimensional special orthogonal groups as

$$\mathrm{SO}_{2n} := \{A \in O_{2n} \mid \det A = 1\}.$$

Again, for $n \geq 1$ we get a transitive group action of SO_{2n} on

$$Q_{2n-1} := \mathbb{V}(x \in \mathsf{k}^{2n} \mid q(x) - 1) \simeq S^{n-1+n\alpha}.$$

Letting $x^0 = (1, 1, 0, \dots, 0)$, the stabilizer of x^0 with respect to the group action of SO_{2n} on Q_{2n-1} is defined to be SO_{2n-1}. One easily sees that this is exactly equal to

$$\{A \in O_{2n-1} \mid \det(A) = 1\}.$$

Defining, as before,

$$Q_{2n-2} := \mathbb{V}(x \in \mathsf{k}^{2n} \mid q(x) + 1, x_1 + x_2) \simeq S^{n-1+(n-1)\alpha},$$

we get a group action of SO_{2n-1} on Q_{2n-2}. This action is transitive, and defining $y^0 \in \mathsf{k}^{2n}$ to be $(1, -1, 0, \dots, 0)$, we can show that the stabilizer of y^0 is SO_{2n-2}.

In the lower-dimensional cases, we note that $\mathrm{SO}_2 \simeq \mathbb{G}_m$, and $\mathrm{SO}_1 \simeq *$. The later equivalence is obvious. For the former, we have to do a bit of work.

Proposition 4.1. $$\mathrm{SO}_2 \simeq \mathbb{G}_m.$$

Proof. We consider the symmetric bilinear form $b((x_1, x_2), (y_1, y_2))$ to see how A is related to A^T. Recall that A^T is defined to be the unique matrix $A \in \mathrm{GL}_2(\mathsf{k})$ for which $b(Ax, y) = b(x, A^T y)$. We write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad A^T = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \qquad x = (x_1, x_2), \qquad y = (y_1, y_2). \tag{4.2}$$

Recall that $b(x, y) = x_1 y_2 + x_2 y_1$. Therefore,

$$b(Ax, y) = a x_1 y_2 + b x_2 y_2 + c x_1 y_1 + d x_2 y_1$$

and

$$b(x, A^T y) = c' x_1 y_1 + d' x_1 y_2 + a' x_2 y_1 + b' x_2 y_2.$$

Comparing, we see that

$$A^T = \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

Now notice that we have the further relations $\det(A) = 1$ and $AA^T = I$. Explicitly multiplying the matrices, we see that

$$AA^T = \begin{bmatrix} ad + bc & 2ab \\ 2cd & ad + bc \end{bmatrix}.$$

Since $\det(A) = ad - bc = 1$, we have that $ad + bc = (ad - bc) + 2bc = 1 + 2bc$. Therefore, we get the relations $2bc = 2ab = 2cd = 0$. It follows, from these relations alone, that either $a = c = 0$, $b = c = 0$, or $b = d = 0$. But we also have the relation $ad - bc = 1$. Therefore, it must be the case that $b = c = 0$. Therefore,

$$\mathrm{SO}_2 = \{(a, b, c, d) \in \mathsf{k}^4 \mid b = c = 0, ad = 1\} \simeq \{(v, w) \in \mathsf{k}^2 \mid vw = 1\} \simeq \mathbb{G}_m. \qquad \square$$

Using a two-sided bar construction as before, we have

$$|B(\mathrm{SO}_n, \mathrm{SO}_{n-1}, *)| \simeq Q_{n-1}.$$

Moreover, we are able to show that

$$|B(*, \mathrm{SO}_n, Q_{n-1})| \simeq \mathrm{BSO}_{n-1}\,.$$

Definition 4.3 (MSLO). The n-th Thom space defining the prespectrum for MSLO is given by the homotopy cofiber of the map

$$\mathrm{BSO}_{n-1+} \to \mathrm{BSO}_{n+}.$$

Notice that in particular we have the following lemma.

Lemma 4.4. $$\mathbb{P}^\infty \simeq B\mathbb{G}_m \simeq \mathrm{BSO}_2 \simeq \mathrm{Thom}(\mathrm{BSO}_2).$$

Proof. Since $\mathrm{SO}_1 \simeq *$, we have $\mathrm{BSO}_1 \simeq *$. By definition of $\mathrm{Thom}(\mathrm{BSO}_2)$, the statement follows. □

***Calculating the C_2 cohomology of* MSLO.** The goal of this section is to calculate the motivic C_2 cohomology of MSLO. To do this, we first note that O_n acts on the unit sphere $S^0 \simeq \{\pm 1\}$ by $A \cdot g \mapsto (\det(A))g$ for $A \in O_n$, $g \in \{\pm 1\}$. This action is easily seen to be transitive, and the stabilizer of $1 \in S^0$ is

$$\{A \in O_n \mid \det(A) = 1\} = \mathrm{SO}_n\,.$$

It follows that $|B(*, O_n, S^0)| \simeq \mathrm{BSO}_n$. As before, we get a Thom isomorphism

$$H^\star(\mathrm{BO}_{n+}) \cong H^{\star+1}(\mathrm{BO}_n\,/\,\mathrm{BSO}_n).$$

We can use this to get a Gysin sequence. We consider the long exact sequence

$$\cdots \to H^\star(\mathrm{BO}_n/\mathrm{BSO}_n) \to H^\star(\mathrm{BO}_{n+}) \to H^\star(\mathrm{BSO}_{n+}) \to H^{\star+1}(\mathrm{BO}_n/\mathrm{BSO}_n) \to \cdots.$$

Substituting in the Thom isomorphism gives us

$$\begin{aligned} \cdots \to H^{\star-1}(\mathrm{BO}_{n+}) \to H^\star(\mathrm{BO}_{n+}) \to H^\star(\mathrm{BSO}_{n+}) \\ \to H^\star(\mathrm{BO}_{n+}) \to H^{\star+1}(\mathrm{BO}_{n+}) \to \cdots\,. \end{aligned}$$

Proposition 4.5. *There exists a surjective map*

$$H^\star(\mathrm{BO}_{n+}) \to H^\star(\mathrm{BSO}_{n+}),$$

with kernel generated by w_1 as an $H^\star$-module. Hence,

$$H^\star(\mathrm{BSO}_{n+}) \cong H^\star[w_2, w_3, \ldots, w_n]$$

with $|w_{2i}| = i + i\alpha$, and $|w_{2i+1}| = i + 1 + i\alpha$.

Proof. Let $x \in H^1(\mathrm{BO}_{n+})$ be the composition of the Thom class $u \in H^1(\mathrm{BO}_n/\mathrm{BSO}_n)$ with the homotopy cofiber of the map

$$\mathrm{BSO}_{n+} \to \mathrm{BO}_{n+}.$$

This gives a nonzero class $x \in H^1(\mathrm{BO}_{n+})$. Since there is only one nonzero class $H^\star(\mathrm{BO}_{n+})$ of degree 1, it is clear that x is the same class as $w_1 \in H^1(\mathrm{BO}_{n+})$ from Theorem 3.1.

Thus, we can write

$$\cdots \to H^\star(\mathrm{BO}_{n+}) \to H^\star(\mathrm{BSO}_{n+}) \to H^\star(\mathrm{BO}_{n+}) \xrightarrow{\smile w_1} H^{\star+1}(\mathrm{BO}_{n+}) \to \cdots .$$

Since $H^\star(\mathrm{BO}_{n+}) = H^\star[w_1, \dots, w_n]$, the map $\smile w_1$ is injective in all dimensions, and so the Gysin sequence breaks up into short exact sequences

$$0 \to H^{r+s\alpha-1}(\mathrm{BO}_{n+}) \xrightarrow{\smile w_1} H^{r+s\alpha}(\mathrm{BO}_{n+}) \to H^{r+s\alpha}(\mathrm{BSO}_{n+}) \to 0.$$

The conclusion follows. □

Calculating the $\mathbb{Z}/p$ cohomology of* MSLO *for p an odd prime.

Definition 4.6. The Euler class $x_n \in H^{\omega_n}(\mathrm{BSO}_{n+})$ is defined to be the composition of the Thom class $c \in H^{\omega_n}(\mathrm{Thom}(\mathrm{BSO}_n))$ with the homotopy cofiber f of

$$\mathrm{BSO}_{n-1+} \to \mathrm{BSO}_{n+} \xrightarrow{f} \mathrm{Thom}(\mathrm{BSO}_n).$$

Theorem 4.7. *$H^\star(\mathrm{BSO}_{n+}; \mathbb{Z}/p)$ is the polynomial ring $H\mathbb{Z}/p^\star[x_1^2, \dots, x_k^2]$ for $n = 2k+1$ and $H\mathbb{Z}/p^\star[x_1^2, \dots, x_{k-1}^2, x_k]$ for $n = 2k$.*

Proof. The sphere bundle $S(n-1) \to \mathrm{BSO}_{n-1} \to \mathrm{BSO}_n$ induces a Gysin sequence with $\mathbb{Z}/p$ coefficients

$$\cdots \to H^i(\mathrm{BSO}_{n+}) \xrightarrow{\smile x_n} H^{i+\omega_n}(\mathrm{BSO}_{n+}) \xrightarrow{g_n^*} H^{i+\omega_n}(\mathrm{BSO}_{n-1+}) \xrightarrow{h_n^*} H^{i+1}(\mathrm{BSO}_{n+}) \to \cdots .$$

Now, if $n = 2k$, then by induction we have that $H^\star(\mathrm{BSO}_{n-1+}) \cong H^\star[x_1^2, \dots, x_{k-1}^2]$. Recall that by [Voevodsky 1999], $H\mathbb{Z}/p_\star^{m+n\alpha}(\mathrm{BO}_{n+}) = 0$ for $n < 0$. Using the fact that $\smile x_n$ is an isomorphism on $H^0(\mathrm{BSO}_{n+}) \cong \mathbb{Z}/p$, we see that $h_n^* = 0$ and so g_n^* is surjective and the map breaks into short exact sequences. The proof then follows that of Theorem 3.1.

If $n = 2k+1$, then x_n is zero in $H^{\omega_n}(\mathrm{BSO}_{n+})$ since it has order 2. To see that x_n has order 2, we note that x_n is the element corresponding to $x_n \smile x_n$ under the Thom isomorphism. Therefore, $x_n \smile x_n = -x_n \smile x_n$ by the commutativity relation of the cup product. It follows that $\smile x_n = 0$, and so the Gysin sequence splits into short exact sequences

$$0 \to H^{i+\omega_n}(\mathrm{BSO}_{n+}) \xrightarrow{g_n^*} H^{i+\omega_n}(\mathrm{BSO}_{n-1+}) \xrightarrow{h_n^*} H^{i+1}(\mathrm{BSO}_{n+}) \to 0.$$

Therefore g_n^* injects $H^\star(\mathrm{BSO}_{n+})$ as a subring of

$$H^\star(\mathrm{BSO}_{n-1+}) \cong H^\star[w_1^2, \dots, w_{k-1}^2, w_k].$$

The subring $\mathrm{im}(g_n^*)$ contains $H^\star[x_1^2, \dots, x_k^2]$, and we can show it equals this ring by comparing ranks in each dimension. □

Calculating the coefficients of MSLO_p for p an odd prime. Recall that the computation of MSO at an odd prime is more or less the same as the computation of complex cobordism MU. Similarly, the computation of MSLO will be no harder than the computation of MGL.

We denote the Milnor primitives by $Q_i \in \mathcal{A}^\star$, $|Q_i| = p^i(1+\alpha) - \alpha$. Recall that if p is odd, then the mod p motivic cohomology of MSLO is generated by classes x_i of degree $2(1+\alpha)i$ as a free $H^\star$-module.

The following proof is based on the proof of a similar result due to S. Borghesi [2003, Proposition 6].

Theorem 4.8. *Let p be an odd prime. The mod p cohomology of* MSLO *takes the form*

$$H^\star(\mathsf{MSLO}) = (\mathcal{A}^\star/(Q_0, Q_1, \dots))[m_i \mid i \neq p^n - 1]$$

as an $\mathcal{A}^\star$-module, where $|m_i| = 2i(1+\alpha)$.

Proof. For c a cohomology class of degree $p + q\alpha$, we define $\|c\| := p - q$. We call the number $\|c\|$ the invariance of the cohomology class c. Now note that the motivic Steenrod algebra $\mathcal{A}^\star$ acts on the cohomology of MSLO. Let Q_i denote the Milnor primitives in degree $2^i(1+\alpha) - \alpha$. Notice that $\|Q_i\| = 1$. Recall that as an $H^\star$ module, the cohomology of MSLO has a basis in monomials whose invariance is equal to 0. Call this basis $\mathfrak{M}$. Therefore, $\|Q_i c\| = 1$ implies that $Q_i c = 0$. The reason is because for any $x \in H^\star$, $\|x\| \leq 0$. Putting this together, we have that if $m \in \mathfrak{M}$, and y is a basis element of $\mathcal{A}^\star$ as an $H^\star$ module, then the action of y on m sends m to a sum of elements in $\mathfrak{M}$ with coefficients in $\mathbb{Z}/2$. Now, since $Q_i c = 0$ for all $c \in \mathfrak{M}$, it follows that the action of $\mathcal{A}^\star$ on $H^\star(\mathsf{MSLO})$ factors through $\mathcal{A}^\star/(Q_0, Q_1, \dots)$. By discussion of the $\mathcal{A}^\star$ action on the cohomology of MSLO, it now follows that the action produces an $H^\star$ linear map in which there is no interplay between the $H^\star$ coefficients. Therefore, any dependencies must be topologically induced. But topologically, there are no dependencies, and so the theorem is proved. □

Corollary 4.9. *Let p be an odd prime. The mod p cohomology of* MSLO *takes the form*

$$H^\star(\mathsf{MSLO}) = H^\star(\mathsf{BPGL})[m_i \mid i \neq p^n - 1]$$

as an $\mathcal{A}^\star$-module, where $|m_i| = 2i(1+\alpha)$.

For the remainder of this subsection, we work over the field $\mathsf{k} = \mathbb{C}$. By [Stahn 2016], we know that over $\mathbb{C}$, the motivic $\mathbb{Z}/p$ cohomology of a point is equal to $\mathbb{Z}/p[\tau]$, where $|\tau| = \alpha - 1$. Dually, the motivic $\mathbb{Z}/p$ homology of a point is equal to $\mathbb{Z}/p[\theta]$, where $|\theta| = 1 - \alpha$. Furthermore, we have that $\mathcal{A}_\star \cong \mathcal{A}_\star^{\mathrm{top}} \otimes_{\mathbb{Z}/p} \mathbb{Z}/p[\theta]$.

Definition 4.10. Let $\mathcal{E}(n)$, $0 \le n < \infty$, denote the quotient Hopf algebroid

$$\mathcal{E}(n) := \mathcal{A}_\star//(\xi_1, \xi_2, \ldots, \tau_{n+1}, \tau_{n+2}, \ldots) = H_\star[\tau_0, \ldots, \tau_n]/(\tau_i^2 \mid 0 \le i \le n).$$

If $n = \infty$, let

$$\mathcal{E}(\infty) := \mathcal{A}_\star//(\xi_1, \xi_2, \ldots) = H_\star[\tau_0, \tau_1, \ldots]/(\tau_i^2 \mid 0 \le i).$$

There is a way of switching between $\mathcal{A}^\star$ structures on cohomology and $\mathcal{A}_\star$ structures on homology. In our case we have the following.

Proposition 4.11. *As an $\mathcal{A}_\star$-comodule algebra, $H_\star\mathsf{BPGL} = \mathcal{A}_\star \,\square_{\mathcal{E}(\infty)}\, H_\star$.*

Using a change of rings isomorphism, we have

$$\mathrm{Ext}_{\mathcal{A}_\star}(H_\star, H_\star(\mathsf{BPGL})) \cong \mathrm{Ext}_{\mathcal{A}_\star}(H_\star, \mathcal{A}_\star \,\square_{\mathcal{E}(\infty)}\, H_\star) \cong \mathrm{Ext}_{\mathcal{E}(\infty)}(H_\star, H_\star).$$

If we let $\mathcal{E}(\infty)^{\mathrm{top}}$ and $H_\star^{\mathrm{top}}$ denote the topological analogues of $\mathcal{E}(\infty)$ and $H_\star$, respectively, then it follows that over $\mathsf{k} = \mathbb{C}$,

$$\mathrm{Ext}_{\mathcal{E}(\infty)}(H_\star, H_\star) \cong \mathrm{Ext}_{\mathcal{E}(\infty)^{\mathrm{top}}}(H_\star^{\mathrm{top}}, H_\star^{\mathrm{top}}) \otimes_{\mathbb{Z}/p} \mathbb{Z}/p[\theta].$$

From here the proof proceeds classically, and so we have the following theorem.

Theorem 4.12. *After completing at an odd prime p, the coefficients of MSLO are given by*

$$\pi_\star(\mathsf{MSLO}\widehat{_p}) \cong \mathbb{Z}_{(p)}[\theta, x_1, x_2, x_3, \ldots],$$

where $|x_i| = 2i(1+\alpha)$.

$H\mathbb{Z}/2_\star$-algebra structure of $H_\star(H\mathbb{Z}; \mathbb{Z}/2)$. By [Voevodsky 2003b], the map

$$\psi_\star : \mathcal{A}_\star \to \mathcal{A}_\star \otimes_{H_\star} \mathcal{A}_\star$$

is given by

$$\psi_\star(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i, \qquad \psi_\star(\tau_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \tau_i + \tau_k \otimes 1.$$

As in [Milnor 1958], we define the conjugates of ξ_i and τ_i inductively as

$$\sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes c(\xi_i) = 0 \quad \text{and} \quad \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes c(\tau_i) + \tau_k \otimes 1 = 0,$$

respectively.

This gives us

$$c(\xi_k) = -\xi_k - c(\xi_1)\xi_{k-1}^2 - \cdots - c(\xi_{k-1})\xi_1^{2^{k-1}},$$
$$c(\tau_k) = -\tau_k - c(\tau_0)\xi_k - c(\tau_1)\xi_{k-1}^2 - \cdots - c(\tau_{k-1})\xi_1^{2^{k-1}},$$

respectively.

As in topology, motivically we have a cofibration

$$H\mathbb{Z} \xrightarrow{2} H\mathbb{Z} \xrightarrow{\text{mod } 2} H\mathbb{Z}/2$$

induced from the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \to 0.$$

Taking motivic $H\mathbb{Z}/2$ homology of $H\mathbb{Z}$, we get a long exact sequence

$$\cdots \to H^\star(H\mathbb{Z}) \xrightarrow{2} H^\star(H\mathbb{Z}) \xrightarrow{\text{mod } 2} H^\star(H\mathbb{Z}/2) \xrightarrow{\partial} \cdots .$$

This gives us an exact couple and so induces a Bockstein spectral sequence. In particular, we get the diagram

$$\begin{tikzcd}
H_\star(H\mathbb{Z}) \arrow[rr, "\beta"] & & H_\star(H\mathbb{Z}) \arrow[rr, "\beta"] \arrow[dl, "\text{mod } 2"] & & H_\star(H\mathbb{Z}) \arrow[dl, "\text{mod } 2"] \\
& H_\star(H\mathbb{Z}/2) \arrow[ul, "\partial"] \arrow[rr, "d"] & & H_\star(H\mathbb{Z}/2) \arrow[ul, "\partial"] &
\end{tikzcd}$$

Notice that $2 = 0$ in $H_\star(H\mathbb{Z})$, and so we have that

$$H_\star(H\mathbb{Z}) \xrightarrow{\text{mod } 2} H_\star(H\mathbb{Z}/2)$$

is injective. Thus we have a short exact sequence

$$0 \to H_\star(H\mathbb{Z}) \xrightarrow{\text{mod } 2} H_\star(H\mathbb{Z}/2) \xrightarrow{d} H_\star(H\mathbb{Z}/2) \to 0.$$

Here d is the dual of the Steenrod operation Sq^1. Notice that $H_\star(H\mathbb{Z}) = \ker(d)$.

Lemma 4.13. *The motivic cohomology of $H_\star(H\mathbb{Z})$ over $\mathsf{k} = \mathbb{C}$ is isomorphic to*

$$\mathbb{Z}/2[\theta, \tau_1, \tau_2, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 - \theta\xi_{i+1}).$$

Proof. First, one observes that $d(\tau_0) = 1$ and $d(\tau_i) = \xi_i$ for $i \in \mathbb{Z}^{>0}$. Next, one observes that since d commutes with the Tate twist θ, and since $\tau_i^2 = \theta\xi_{i+1}$, we have

$$0 = 2\tau_i d(\tau_i) = d(\tau_i^2) = \theta d(\xi_{i+1}).$$

Therefore $d(\xi_{i+1}) = 0$. Now, as a $\mathbb{Z}/2[\theta]$-algebra, the classes $\{\xi_i\}_{i=1}^\infty$ and the classes $\{c(\xi_i)\}_{i=1}^\infty$ both generate the same algebra. Looking now at the inductive formula for the conjugate of τ_i, and acknowledging that $2 = 0$ in the coefficients, we have

$$c(\tau_k) = \tau_k + c(\tau_0)\xi_k + c(\tau_1)\xi_{k-1}^2 + \cdots + c(\tau_{k-1})\xi_1^{2^{k-1}}.$$

First we notice that $c(\tau_0)=\tau_0$, and so $d(c(\tau_0))=1$. We claim that $d(c(\tau_i))=0$ for $i\in\mathbb{Z}^{>0}$. For τ_1, we have that $c(\tau_1)=\tau_1+\tau_0\xi_1$. Taking the differential of each side, we have that

$$d(c(\tau_1))=d(\tau_1)+\tau_0 d(\xi_1)+\xi_1 d(\tau_0)=d(\tau_1)+\xi_1=\xi_1+\xi_1=0.$$

Now, by induction we can assume $d(c(\tau_{n-1}))=0$. Therefore,

$$\begin{aligned} d(c(\tau_n)) &= d(\tau_n)+d(c(\tau_0)\xi_n)+d(c(\tau_1)\xi_{n-1}^2)+\cdots+d(c(\tau_{n-1})\xi_1^{2^{n-1}}) \\ &= d(\tau_n)+d(c(\tau_0)\xi_n)=d(\tau_n)+\xi_n=\xi_n+\xi_n=0. \end{aligned}$$

Thus,

$$\ker(d)=\mathbb{Z}/2[\theta, c(\tau_1), c(\tau_2),\dots, c(\xi_1), c(\xi_2),\dots].$$

One can show that $c(\tau_i)^2=\theta c(\xi_{i+1})$. This proves the claim. □

The* $\mathbf{Sq^1}$ *cohomology. Notice that the motivic Steenrod operation Sq^1 has the property that $\mathrm{Sq}^1\circ\mathrm{Sq}^1=0$. Therefore, we can think of Sq^1 as a differential of $H^\star(\mathsf{MSLO})$. We use the notation $H^\star(M;\mathrm{Sq}^1)$ to denote the cohomology of the $\mathcal{A}^\star$ module M with respect to the differential Sq^1.

Following [Voevodsky 2003b], let $I=(\epsilon_0, s_1,\epsilon_1, s_2,\dots, s_k,\epsilon_k)$ be a sequence where $\epsilon_i\in\{0,1\}$ and s_i are nonnegative integers. Denote by P^I the product

$$P^I=\beta^{\epsilon_0}P^{s_1}\cdots P^{s_k}\beta^{\epsilon_k}.$$

A sequence I is called admissible if $s_i\geq 2s_{i+1}+\epsilon_i$. Monomials P^I corresponding to admissible sequences are called admissible monomials. Here $\beta=\mathrm{Sq}^1$.

Lemma 4.14. *Admissible monomials generate $\mathcal{A}^\star$ as a left $H^\star$-module.*

Proof. See [Voevodsky 2003b]. □

Lemma 4.15. *Suppose that $I=(0,s_1,\dots,s_k,0)$ and $J=(0,t_1,\dots,t_r,0)$ with $s_1,\dots,s_k,t_1,\dots,t_r\in\mathbb{Z}^{>0}$. Then $\beta P^I\neq P^J\beta$. Also, $\beta P^s\neq P^t\beta$ for $s,t\in\mathbb{Z}^{>0}$.*

Proof. This follows immediately from Lemma 4.14. □

Lemma 4.16. $H^\star(\mathcal{A}^\star;\mathrm{Sq}^1)=0$ *and* $H^\star(\mathcal{A}^\star/\mathcal{A}^\star\,\mathrm{Sq}^1;\mathrm{Sq}^1)=H^\star$.

Proof. To prove the first statement, note that $\mathrm{im}(\mathrm{Sq}^1)=\ker(\mathrm{Sq}^1)=\mathrm{Sq}^1\mathcal{A}^\star$. For the second statement, we notice that $\mathrm{im}(\mathrm{Sq}^1)=\mathrm{Sq}^1\mathcal{A}^\star/\mathcal{A}^\star\,\mathrm{Sq}^1$. Since $\mathrm{Sq}^1\mathcal{A}^\star/\mathcal{A}^\star\,\mathrm{Sq}^1$ is clearly in both the kernel and image of Sq^1, and using Lemma 4.15, we know that if $I=(0,s_1,\dots,s_k,0)$ with $s_1,\dots,s_k\in\mathbb{Z}^{>0}$ or $I=(s)$ with $s\in\mathbb{Z}^{>0}$, then $\mathrm{Sq}^1P^I\notin\mathcal{A}^\star\,\mathrm{Sq}^1$. We have shown what happens to admissible monomials. We only have to look at what happens to elements of $H^\star$. Clearly these elements get sent to zero since they commute with the Sq^1 operation. Since elements of $H^\star$ are clearly not in the image of Sq^1, it follows that $H^\star(\mathcal{A}^\star/\mathcal{A}^\star\,\mathrm{Sq}^1;\mathrm{Sq}^1)=H^\star$. □

We need the following proposition from [Smirnov and Vishik 2014].

Proposition 4.17. *Recall that* $H^\star(\mathrm{BO}_{n+}) \cong H^\star[w_1, \dots, w_n]$ *as an* $H^\star$*-module. If* -1 *is a square in* k*, then*

$$\mathrm{Sq}^k(w_m) = \sum_{j=0}^{k} \binom{m-k}{j} w_{k-j} w_{m+j}.$$

The Cartan formula over $\mathsf{k} = \mathbb{C}$ gives the following.

Proposition 4.18. *Let* τ *be the Tate twist of degree* $\alpha - 1$ *in* $H^\star$*, and suppose* $H^\star(\mathrm{BO}_{n+}) \cong H^\star[w_1, \dots, w_n]$*. We define*

$$\epsilon_{i,j} = \begin{cases} 1 & \text{if } k \text{ is even and } i, j \text{ are odd,} \\ 0 & \text{otherwise.} \end{cases}$$

If -1 *is a square in* k*, then*

$$\mathrm{Sq}^k(w_r w_s) = \sum_{i+j=k} \tau^{\epsilon_{i,j}} \, \mathrm{Sq}^i(w_r) \, \mathrm{Sq}^j(w_s).$$

Proof. This follows from the formulas given in [Voevodsky 2003b], along with relations between the geometric and simplicial classifying spaces of O_n found in [Smirnov and Vishik 2014]. □

Lemma 4.19. $\mathrm{Sq}^1 t_n = 0$, *where* $t_n \in H^\star(\mathrm{Thom}(\mathrm{BSO}_n))$ *is the Thom class.*

Proof. Let $H^\star(\mathrm{BO}_{n+}) = H^\star[w_1, \dots, w_n]$. Recall that by Proposition 4.5, we can identify $H^\star(\mathrm{BSO}_{n+})$ with $H^\star[w_2, w_3, \dots, w_n] \subset H^\star(\mathrm{BO}_{n+})$. Recall also that there is a Thom isomorphism

$$H^\star(\mathrm{BSO}_{n+}) \smile w_n \cong H^\star(\mathrm{Thom}(\mathrm{BSO}_{n+}). \tag{4.20}$$

Therefore, $\mathrm{Sq}^1(t_n)$ can be identified with $\mathrm{Sq}^1(w_n)$ under (4.20) and so we can work out the Steenrod operation on $H^\star(\mathrm{Thom}(\mathrm{BSO}_n))$ by comparison with $H^\star(\mathrm{BO}_{n+})$. In particular, $\mathrm{Sq}^1(w_n) = w_n w_1$. Since $w_1 = 0$ in $H^\star(\mathrm{BSO}_{n+})$, the claim follows. □

Since $H^\star(\mathsf{MSLO})$ is an $\mathcal{A}^\star$ module, we can compute its Sq^1 cohomology.

Proposition 4.21. $H^\star(H^\star(\mathsf{MSLO}); \mathrm{Sq}^1) = H^\star[u_2^2, u_4^2, u_6^2, \dots]$.

Proof. By Lemma 4.19, Sq^1 commutes with the Thom isomorphism. Therefore, it is enough to show that $H^\star(H^\star(\mathsf{BSO}); \mathrm{Sq}^1) = H^\star[w_2^2, w_4^2, w_6^2, \dots]$. We note that $\mathrm{Sq}^1(w_{2n}) = w_{2n+1}$. From this it follows that $H^\star[u_3, u_5, u_7, \dots] \subset \mathrm{im}(\mathrm{Sq}^1)$. This implies that the only elements which can be in the kernel but not in the image of Sq^1 are $H^\star[w_2^2, w_4^2, w_6^2, \dots] \subset H^\star(\mathsf{BSO})$. Noting that $\mathrm{Sq}^1(w_{2n}^2) = 0$ for all n, the claim follows. □

A motivic version of Wall's theorem.

Lemma 4.22. *The morphism of $\mathcal{A}^\star$-modules*

$$\mathcal{A}^\star \to H^\star(\mathsf{MSLO})$$

given by $a \mapsto a \cdot 1$, where 1 denotes the Thom class $t_0 \in H^{0,0}(\mathsf{MSLO})$, has kernel $J = \mathcal{A}^\star \operatorname{Sq}^1$.

Proof. To simplify notation, we write $\mathcal{A}^\star/\beta := \mathcal{A}^\star/\mathcal{A}^\star \operatorname{Sq}^1$.

First, it is clear that $\operatorname{Sq}^i(w_j) = 0$ if $i > j$ by Proposition 4.17. If $i \leq j$, then $\operatorname{Sq}^1(w_j)$ is a sum of monomials $w_k w_l$ with $k, l < 2j$. The monomials $\operatorname{Sq}^{i_n} \cdots \operatorname{Sq}^{i_1}$ with $i_n \geq 2i_{n-1}$ and $i_1 > 1$ form an $H^\star$-module basis for $\mathcal{A}^\star/\beta$. Therefore, it is enough to show that the polynomials $\operatorname{Sq}^{i_n} \cdots \operatorname{Sq}^{i_1}(t)$ are linearly independent in $H^\star(\mathsf{MSLO})$. Let $I = (i_k, \dots, i_1)$ with $i_s \geq 2i_{s-1}$ and $i_1 > 1$. We order the monomials $w^I = w^{i_k} w^{i_{k-1}} \cdots w^{i_1}$ lexicographically. For example, $w_8 w_4$ is of higher order than $w_4 w_2$ and $w_8 w_2$, but lower order than $w_8 w_4 w_2$ and $w_{10} w_2$. By induction, we assume that $\operatorname{Sq}^{i_{n-1}} \cdots \operatorname{Sq}^{i_1}(t) = w_{i_{n-1}} \cdots w_{i_1} t + \text{lower order terms}$.

Now suppose $w_{j_{n-1}} \cdots w_{j_1} t \in H^\star(\mathsf{MSLO})$ is such that $j_{n-1} \geq j_{n-1} \geq \cdots \geq j_1$. If $i \geq 2j_{n-1}$, then we show $\operatorname{Sq}^i(w_{j_{n-1}} \cdots w_{j_1} t) = w_i w_{j_{n-1}} \cdots w_{j_1} t + \text{lower order terms}$. Using the Cartan formula, we have

$$\begin{aligned}\operatorname{Sq}^i(w_{j_{n-1}} \cdots w_{j_1} t) &= \operatorname{Sq}^i(t) \cdot w_{j_{n-1}} \cdots w_{j_1} + \text{lower order terms} \\ &= w_i w_{j_{n-1}} \cdots w_{j_1} t + \text{lower order terms}.\end{aligned}$$

This proves the lemma. □

Theorem 4.23. *Over $\mathsf{k} = \mathbb{C}$, $H^\star(\mathsf{MSLO})$ is a wedge sum of suspensions of $\mathcal{A}^\star$ and $\mathcal{A}^\star/\mathcal{A}^\star \operatorname{Sq}^1$.*

Proof. Our approach is to define a map from a wedge sum of suspensions of $\mathcal{A}^\star/\beta$ to M which induce an isomorphism in Sq^1 cohomology.

Choose classes $\{x_\alpha\}_{\alpha \in I} \in M$ whose images in $H^\star(M; \operatorname{Sq}^1)$ form a basis of $H^\star(M; \operatorname{Sq}^1)$ as a $\mathbb{Z}/2[\theta]$-module. By Proposition 4.21, we can choose the classes $u_2^2, u_4^2, \dots \in H^\star(\mathsf{MSLO}) \cong H^\star[u_2, u_3, u_4, \dots]$. The x_α are killed by Sq^1 and so we can define a map

$$\phi_1 : \bigoplus_{\alpha \in I} \mathcal{A}^\star/\beta[-\deg(x_\alpha)] \to M.$$

Next, we define

$$\overline{\mathcal{A}^\star} := \{\text{admissible monomials } x \in \mathcal{A}^\star \mid |x| > 0\}.$$

Using this definition, we define

$$\overline{M} := M/\overline{\mathcal{A}^\star} M.$$

Notice that $\bigoplus_{\alpha\in I}\mathcal{A}^\star/\beta[-\deg(x_\alpha)]\cong\mathcal{A}^\star/\beta\otimes_{H^\star}C$ for $C=\mathbb{Z}/2[\theta][u_2^2,u_4^2,\dots]$. We consider the projection map

$$M\xrightarrow{\pi}\overline{M}.$$

We then choose a $\mathbb{Z}/2[\theta]$-submodule $Z\subset M$ such that $\pi_{|Z}$ is injective, and

$$\overline{M}\cong\pi(\phi_1(\mathcal{A}^\star/\beta\otimes_{H^\star}C))\oplus\pi(Z).$$

Now set

$$N=\mathcal{A}^\star/\beta\otimes_{H^\star}C\oplus\mathcal{A}^\star\otimes_{H^\star}Z.$$

The natural map

$$\phi_2:\mathcal{A}^\star\otimes_{H^\star}Z\to M$$

gives a map

$$\Phi:=\phi_1\oplus\phi_2:N\to M.$$

Writing $N=\mathcal{A}^\star/\beta\otimes_{H^\star}C\oplus\mathcal{A}^\star\otimes_{H^\star}Z$, we let N_i denote the $\mathcal{A}^\star$-submodule of N given by $N_i=\mathcal{A}^\star/\beta\otimes_{H^\star}C_i\oplus\mathcal{A}^\star\otimes_{H^\star}Z_i$. Here C_i and Z_i denote all elements in C and Z, respectively, of total degree i. We say the class x with degree $n+m\alpha$ has total degree $n+m$. We define M_i to be the image of N_i under the map Φ. We then define $N^{(n)}$ and $M^{(n)}$ to be $\bigoplus_{i\le n}N_i$ and $\bigoplus_{i\le n}\Phi(N_i)$, respectively.

We show by induction that the map $\Phi:N^{(n)}\to M^{(n)}$ is an isomorphism. Starting with $n=0$, $N^{(0)}=\mathcal{A}^\star/\beta$ and $M^{(0)}=\mathcal{A}^\star\cdot t$, where t is the Thom class. By Lemma 4.22 this map is an isomorphism.

Suppose we have proved $\Phi:N^{(n-1)}\to M^{(n-1)}$ is an isomorphism and let $\lambda:N/N^{(n-1)}\to M/M^{(n-1)}$ be the map induced by Φ. We show $\lambda_{|(N^{(n)}/N^{(n-1)})}$ is injective. Let P be the $\mathbb{Z}/2[\theta]$-module generated by elements of the form c, z, $\mathrm{Sq}^1(z)$ for $c\in C_n$, $z\in Z_n$. We can regard P as a $\mathbb{Z}/2[\theta]$-submodule of the $\mathbb{Z}/2[\theta]$-module $N/N^{(n-1)}$.

We first prove that $\lambda_{|P}$ is injective. Notice that since $H^\star(\mathcal{A}^\star;\mathrm{Sq}^1)=0$, the map

$$\Phi^*:H^\star(N;\mathrm{Sq}^1)\to H^\star(M;\mathrm{Sq}^1)$$

is still an isomorphism. Since

$$\Phi:N^{(n-1)}\to M^{(n-1)}$$

is an isomorphism by induction, it follows that

$$\lambda^*:H^\star(N/N^{(n-1)};\mathrm{Sq}^1)\to H^\star(M/M^{(n-1)};\mathrm{Sq}^1)$$

is also an isomorphism.

Suppose $v\in P$ and $\lambda(v)=0$. Notice that the total dimension of v is n or $n+1$. We consider the two cases separately. If the total dimension of v is n, then $v=c+z$ for $c\in C_n$, $z\in Z_n$. Now $\lambda(v)=0$ implies $\Phi(c+z)\in M^{(n-1)}$ for $\Phi:N^{(n)}\to M^{(n)}$. However, by choice of Z, $\lambda(z)\in M_n$, and so $z=0$. Then $v=c$, and so $\lambda(c)=0$.

Since λ^* is an isomorphism, it follows that $\mathrm{Sq}^1(c) = 0$, and $c = \mathrm{Sq}^1(c')$ for some $c' \in N/N^{(n-1)}$ with total degree n. But every element in $N/N^{(n-1)}$ has total degree $\geq n$ (or = 0), and so $c' = 0$, which implies $c = 0$.

Now, suppose that the total dimension of v is $n+1$. Then $v = \mathrm{Sq}^1(z)$ for some $z \in Z_n$. Suppose $\lambda(v) = 0$. By definition of v, it follows that $\lambda(v) = \lambda(\mathrm{Sq}^1(z)) = 0$. Since Sq^1 commutes with λ, it follows that $\mathrm{Sq}^1(\lambda(z)) = 0$. Now, notice that in Sq^1 homology, $\mathrm{Sq}^1(\lambda(z)) = 0$. But this means $\lambda(z) = \lambda(c) + \mathrm{Sq}^1(z')$ for some $c \in C_n$, and $z' \in (M/M^{(n-1)})$ of total degree $n-1$. Thus $z' = 0$, and we reduce to the previous case.

Now, returning to the induction step, we have that the multiplication map

$$\mu : \mathsf{MSLO} \wedge \mathsf{MSLO} \to \mathsf{MSLO}$$

induces a coproduct map

$$\mu^* : H^\star(\mathsf{MSLO}) \to H^\star(\mathsf{MSLO}) \otimes_{H^\star} H^\star(\mathsf{MSLO}).$$

We define a projection map

$$p : M \to M/M^{(n-1)}.$$

Let $u \in N^{(n)}$, and $\Phi : N^{(n)} \to M^{(n)}$. Then

$$\mu^*\Phi(u) = 1 \otimes_{H^\star} \Phi(u) \quad \text{modulo } M \otimes_{H^\star} M^{(n-1)}.$$

Therefore, for any $v \in P$ we have

$$(1 \otimes_{H^\star} p)\mu^*\Phi(v) = 1 \otimes_{H^\star} \lambda(v).$$

Now choose a $\mathbb{Z}/2[\theta]$-basis $c_1, c_2, \ldots, c_r$ for C_n, and $z_1, z_2, \ldots, z_s$ for Z_n. Then we can give P a $\mathbb{Z}/2[\theta]$-basis

$$\{v_i\} = \{c_1, \ldots, c_r, z_1, z_2, \ldots, z_s, \mathrm{Sq}^1(z_1), \mathrm{Sq}^1(z_2), \ldots, \mathrm{Sq}^1(z_s)\}.$$

Any $v \in N^{(n)}/N^{(n-1)} = N_n$ then has a unique expression in the form $v = \sum_i a_i v_i$ for $a_i \in \mathcal{A}^\star \backslash \mathcal{A}^\star \mathrm{Sq}^1 \cup \{0\}$. Now, we let m denote the maximum total dimension of all of the a_i. Next, let $\{a_{m_1}, a_{m_2}, \ldots, a_{m_v}\}$ denote all of the a_i of total dimension m.

Notice that if $\lambda(v) = 0$, then $\Phi(v) \in M^{(n-1)}$, and hence

$$0 = (1 \otimes_{H^\star} p)\mu^*\Phi(v) = \sum a_{m_j} \cdot 1 \otimes_{H^\star} \lambda(v_{m_j}) + \sum b_k \cdot 1 \otimes_{H^\star} m_k$$

for some $m_k \in M$, $b_k \in \mathcal{A}^\star$ with total dim $b_k < m$.

The fact $\sum a_{m_j} \cdot 1 \otimes_{H^\star} \lambda(v_{m_j}) = \sum b_k \cdot 1 \otimes_{H^\star} m_k$ implies $\sum a_{m_j} \cdot 1 \otimes_{H^\star} \lambda(v_{m_j}) = 0$. However, we showed that $\lambda_{|P}$ is injective, and so the $\lambda(v_{i_j})$ are linearly independent. This then implies $a_{m_j} \cdot 1 = 0$ for all j. But then $a_{m_j} \in \mathcal{A}^\star \mathrm{Sq}^1$, which is a contradiction, and so $\lambda(v) = 0$ implies $v = 0$. $\square$

Corollary 4.24. *Over the field* $\mathsf{k} = \mathbb{C}$,

$$H_\star(\mathsf{MSLO}) \cong H_\star(H\mathbb{Z}/2) \otimes_{\mathbb{Z}/2[\theta]} C \oplus \mathcal{A}_\star \otimes_{\mathbb{Z}/2[\theta]} Z.$$

Here C *is the algebra* $\mathbb{Z}/2[\theta, x_4, x_8, \dots]$, *where the* x_{4i} *are generators of degree* $2(1+\alpha)i$. Z *is a* $\mathbb{Z}/2[\theta]$ *polynomial algebra.*

***The homotopy type of* MSLO.**

Theorem 4.25. *For* $\mathsf{k} = \mathbb{C}$, 2*-completed* MSLO *splits as a wedge sum of suspensions of motivic homology with coefficients in* $\mathbb{Z}/2$ *and* $\mathbb{Z}_2$.

Proof. Once we know that the motivic $\mathbb{Z}/2$ homology of MSLO is a wedge sum of suspensions of $\mathcal{A}^\star$ and $\mathcal{A}^\star/\mathcal{A}^\star \mathrm{Sq}^1$, we can again construct a map

$$\mathsf{MSLO} \to \bigvee_{i\in I} H\mathbb{Z}/2[r_i] \vee \bigvee_{j\in J} H\mathbb{Z}[s_j]$$

which is an equivalence on motivic $\mathbb{Z}/2$ homology. Then, by applying the Nakayama lemma and the motivic Hurewicz theorem [Bachmann 2018], one can show that the map is a homotopy equivalence. □

The dimension of the $H\mathbb{Z}/2$ suspensions. We already showed in Corollary 4.24 that the $H\mathbb{Z}$ suspensions of MSLO must live in degrees generated by monomials x_{4i} of degrees $2i(1+\alpha)$. It remains to describe the degrees of the $H\mathbb{Z}/2$ suspensions. To answer this question we use well known combinatorial counting techniques, as this question very much resembles the coin change problem well known to combinatorists [Harris et al. 2008, Section 2.6.3] and computer scientists [Abelson et al. 1996, Section 1.2.2] alike.

Definition 4.26. Let M be a bigraded module with basis $\mathfrak{B}$. Let $\mathfrak{B}_{n,m}$ denote all elements of $\mathfrak{B}$ with bidegree (n, m). The basis $\mathfrak{B}$ is said to be a *special* basis if the following conditions hold:

(1) $\mathfrak{B}_{n,m} = \{\,\}$ if $n < 0$.

(2) $\mathfrak{B}_{n,m} = \{\,\}$ if $m < 0$.

(3) The size of the set $\mathfrak{B}_{n,m}$ is finite for all $(n, m) \in \mathbb{Z} \times \mathbb{Z}$.

(4) $\mathfrak{B} = \bigcup_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} \mathfrak{B}_{n,m}$.

Clearly $H_\star(\mathsf{MSLO})$, $H\mathbb{Z}/2_\star(H\mathbb{Z}/2)$, and $H\mathbb{Z}/2_\star(H\mathbb{Z})$ each have a special basis under their induced $n + m\alpha$ grading.

We can associate a unique polynomial $f_{\mathfrak{B}}(x, y) = \sum c_{n,m} x^n y^m \in \mathbb{Z}[[x, y]]$ to any special basis $\mathfrak{B}$. Here $c_{n,m}$ represents the number of elements in $\mathfrak{B}$ of bidegree (n, m). Notice that we can order the words $x^n y^m$ by the length of the word followed by the alphabetical order of the word. For example, $x^2 y = xxy$ comes before $xy^2 = xyy$, and $y^2 = yy$ comes before $x^4 = xxxx$. Let $\mathfrak{B}_{H_\star \mathsf{MSLO}}$ be an $H_\star$ basis for $H_\star(\mathsf{MSLO})$, $\mathfrak{B}_{H_\star H\mathbb{Z}}$ an $H_\star$ basis for $H_\star(H\mathbb{Z})$, and $\mathfrak{B}_{H_\star H\mathbb{Z}/2}$ an $H_\star$ basis for $H_\star(H\mathbb{Z}/2)$.

Proposition 4.27. *Let $f_{H_\star \mathsf{MSLO}}(x, y)$, $f_{H_\star H\mathbb{Z}}(x, y)$, and $f_{H_\star H\mathbb{Z}/2}(x, y)$ be the associated formal polynomials for the special bases $\mathfrak{B}_{H_\star \mathsf{MSLO}}$, $\mathfrak{B}_{H_\star H\mathbb{Z}}$, and $\mathfrak{B}_{H_\star H\mathbb{Z}/2}$, respectively. The number of $H\mathbb{Z}/2$ suspensions of $H_\star(\mathsf{MSLO})$ in dimension $n + m\alpha$ is given by the coefficient $c_{n,m}$ in*

$$g(x, y) = \sum c_{n,m} x^n y^m = \frac{f_{H_\star \mathsf{MSLO}}(x, y) - f_{H_\star H\mathbb{Z}}(x, y) \prod_{i=0}^{\infty}(1 - (xy)^{2i})^{-1}}{f_{H_\star H\mathbb{Z}/2}(x, y)}.$$

Proof. The function $a(x, y) = f_{H_\star \mathsf{MSLO}}(x, y)$ represents the number of basis elements of $H_\star\mathsf{MSLO}$ in each degree. The function

$$b(x, y) = f_{H_\star H\mathbb{Z}}(x, y) \prod_{i=0}^{\infty}(1 - (xy)^{2i})^{-1}$$

represents all elements in $a(x, y)$ generated by an $H\mathbb{Z}$ suspension. Therefore, $a(x, y) - b(x, y)$ represents all basis elements of $H_\star\mathsf{MSLO}$ generated by $H\mathbb{Z}/2$ suspensions. Thus, dividing by $f_{H_\star H\mathbb{Z}/2}(x, y)$ gives the number of $H\mathbb{Z}/2$ suspensions in each degree after applying a Taylor expansion around the point (0, 0). □

5. MGLR, an analogue of MR

There is a C_2-equivariant spectrum belonging to classical topology, which was constructed by Landweber. The coefficients of this spectrum were computed by Hu and Kriz [2001]. The coefficients are bigraded. While the bigrading given in [Hu and Kriz 2001] is $\mathsf{MR}_{*+*'\alpha}$, we use σ grading instead of α. The reason for this is that Hu and Kriz used the α to signify the relationship between motivic homotopy theory and classical C_2-equivariant homotopy theory. The topological realization functor over $\mathbb{R}$ sends motivic α grading to the C_2 grading. However, in the present case, we want to stress the relationship between C_2 motivic homotopy theory and C_2 classical homotopy theory using the topological realization over $\mathbb{C}$.

In this section we discuss a C_2-equivariant motivic spectrum MGLR which was constructed by Hu, Kriz, and Ormsby [Hu et al. 2011]. There is a complex topological realization functor $t^{\mathbb{C}}_{C_2}$ for C_2-equivariant motivic spectra, and $t^{\mathbb{C}}_{C_2}(\mathsf{MGLR}) = \mathsf{MR}$.

One should think of MGLR as a motivic analogue of MR. Roughly speaking, the spectrum MR can be thought of as complex cobordism MU endowed with a C_2 action. At its heart, MU is built from the classifying spaces BU_n, where U_n denotes the n-dimensional unitary group. We get an involution on this group given by $A \leftrightarrow \bar{A}^T$. The groups U_n equipped with this involution action determine the construction of MR. If one wanted to mimic this construction motivically, one would immediately be faced with a problem: complex conjugation is not algebraic. A priori this means that the groups U_n are not definable; however, it turns out that over the complex numbers, $U_n \cong \mathrm{GL}_n(\mathbb{C})$. In fact, the motivic analogue of MU is the well known algebraic cobordism MGL.

In analogy with MR, MGLR should be thought of as algebraic cobordism MGL endowed with a C_2 action. Consider the symmetric bilinear form

$$b((x_1, \dots, x_{2n}), (y_1, \dots, y_{2n})) = \sum_{i=1}^{n} x_{2i} y_{2i-1} + x_{2i-1} y_{2i}.$$

For any $A \in \mathrm{GL}_{2n}(\mathsf{k})$, there is a unique matrix A^{T_b} for which $b(Ax, y) = b(x, A^{T_b} y)$ for all $x, y \in \mathsf{k}^{2n}$. The C_2 action of MGLR is induced from the involution action $A \leftrightarrow (A^{T_b})^{-1}$.

The λ twist. In [Hu and Kriz 2001], the authors show that MR completed at 2 splits as a wedge sum of suspensions of a spectrum BPR whose suspensions are in degrees $m_i(1+\sigma)$ for $m_i \neq 2^{i+1} - 1$, $\Phi^{C_2}(\mathsf{BPR}) = H\mathbb{Z}/2$, and nonequivariantly $\mathsf{BPR} = \mathsf{BP}$. This splitting comes from applying the Quillen idempotent to the formal group law on $\mathsf{MR}_{*(1+\sigma)}$. From this, it follows that $\mathsf{MR}_\star$ is freely generated by generators x_n of degree $n(1+\sigma)$ for $n \neq 2^{i+1} - 1$ as a $\mathsf{BPR}_\star$ algebra. One could ask whether MGLR splits as a wedge sum of suspensions of BPGLR, with $\Phi^{C_2}(\mathsf{BPGLR}) = H\mathbb{Z}/2$ and $\mathsf{BPGLR} = \mathsf{BPGL}$ nonequivariantly, in such a way that $\mathsf{MGLR}_\star$ is free as a $\mathsf{BPGLR}_\star$ algebra. Unfortunately, there does not appear to be any way to construct such a splitting. However, there exists an element $\lambda \in \pi_{1-\sigma+\sigma\alpha-\alpha}(\mathsf{MGLR})$. If we invert this element, then we get a formal group law and we can use the Quillen idempotent construction to get a splitting. First, let us elaborate on this mysterious element λ.

In the topological setting there is the notion of real-oriented spectra and it turns out that MR is universal among real-oriented spectra. There is also a notion of real orientation found in [Hu et al. 2011]. Following that paper's notation, we define $\widetilde{X}$ to be the functorial fibrant replacement of $\overline{X}$, the reduced suspension of X.

Definition 5.1. A C_2-equivariant ring spectrum E is real-oriented if the following two conditions are satisfied. Here MGLR(1) denotes the first term of the prespectrum defining MGLR.

(1) The unit in $E^\star(S^{1+\sigma\alpha+\sigma+\alpha})$ restricts to the unit ϕ_E of $E^\star(\mathsf{MGLR}(1))$.

(2) The map

$$S^{2+2\sigma\alpha} \simeq \widetilde{\mathbb{G}_m^{1/z}} \wedge \widetilde{\mathbb{G}_m^{1/z}} \to \widetilde{\mathbb{G}_m^{1/z}} \times \widetilde{\mathbb{G}_m^{1/z}} \to B(\mathbb{G}_m^{1/z} \times \mathbb{G}_m^{1/z}) \to \mathrm{BGL}_2 \to \mathsf{MGLR}(1),$$

with representative $\omega \in \pi_{2+2\sigma\alpha}$, composes with ϕ_E to give a unit λ_E.

Whenever this is satisfied we get many results analogous to those found in [Hu and Kriz 2001].

Theorem 5.2. *If the C_2-equivariant ring spectrum E is real-oriented, then we have* $E^\star(B\mathbb{G}_m^{1/z}) = E^\star[u]$, *where* $\deg(u) = -(1+\sigma\alpha)$.

Unfortunately, it is not clear whether or not MGLR satisfies Definition 5.1. Clearly MGLR satisfies condition (1) of Definition 5.1. However, it is not clear that λ_{MGLR} is invertible. Using the methods of [Elmendorf et al. 1997] we can "invert" λ_{MGLR} to construct a spectrum $\lambda^{-1}\mathsf{MGLR}$ satisfying both conditions of Definition 5.1. The formal group law of Theorem 5.2 then gives a canonical map

$$\mathsf{L} \to \lambda^{-1}\mathsf{MGLR}_{*(1+\sigma\alpha)}.$$

Here L denotes the Lazard ring.

Notice that the topological realization functor over $\mathbb{C}$, which we denote by $t^{\mathbb{C}}$, is a symmetric monoidal functor, and so if it is applied to the spectrum MGLR, we get a ring homomorphism

$$\mathsf{MGLR}_\star \to \mathsf{MR}_\star.$$

One can show that λ_{MGLR} is sent to the unit 1 under this ring homomorphism, and so we get a ring homomorphism

$$\lambda^{-1}\mathsf{MGLR}_\star \to \mathsf{MR}_\star. \tag{5.3}$$

Since the homomorphism $t^{\mathbb{C}}$ sends $1+\sigma\alpha$ grading to $1+\sigma$ grading, and since $\lambda^{-1}\mathsf{MGLR}_{*(1+\sigma\alpha)} \subset \lambda^{-1}\mathsf{MGLR}_\star$ and $\mathsf{MR}_{*(1+\sigma)} \subset \mathsf{MR}_\star$ are commutative rings, we have the following result.

Lemma 5.4. *The restriction of the ring homomorphism (5.3) to* $\lambda^{-1}\mathsf{MGLR}_{*(1+\sigma\alpha)}$ *induced by the topological realization functor* $t^{\mathbb{C}}$ *sends the formal group law on* $\lambda^{-1}\mathsf{MGLR}_\star$ *to the formal group law on* $\mathsf{MR}_\star$.

Proof. This is clear since $t^{\mathbb{C}}(B\mathbb{G}_m^{1/z}) = BS^\sigma$. □

Since MGLR is an E_∞-ring spectrum, we may apply constructions as in [Elmendorf et al. 1997]. In particular, we may "kill" or "invert" the image of any sequence of elements of L in the spectrum $\lambda^{-1}\mathsf{MGLR}$. The ring $\mathsf{MGL}_{*(1+\alpha)} = \mathsf{MU}_{2*}$ is the universal formal group law, and so the generator x_i of degree $i(1+\alpha)$ is sent to an element of degree $i(1+\sigma\alpha)$.

Theorem 5.5. *The spectrum* $\Phi^{C_2}_{\text{ét}}(\lambda^{-1}\mathsf{MGLR})$ *is equivalent to* $\theta^{-1}\mathsf{MGLO}$.

Proof. Recall that λ is the map

$$S^{2+2\sigma\alpha} \simeq \widetilde{\mathbb{G}_m^{1/z}} \wedge \widetilde{\mathbb{G}_m^{1/z}} \to \widetilde{\mathbb{G}_m^{1/z}} \times \widetilde{\mathbb{G}_m^{1/z}} \to B(\mathbb{G}_m^{1/z} \times \mathbb{G}_m^{1/z}) \to \mathrm{BGL}_2 \\ \to \mathsf{MGLR}(1) \to \Sigma^{1+\sigma+\sigma\alpha+\alpha}\mathsf{MGLR}.$$

After taking geometric fixed points, this becomes a map

$$S^2 \simeq S^1 \wedge S^1 \to S^1 \times S^1 \to B(\mathbb{Z}/2 \times \mathbb{Z}/2) \to \mathrm{BO}_2 \to \mathsf{MGLO}(1) \to \Sigma^{1+\alpha}\mathsf{MGLO}.$$

This map is nonzero, and it realizes as an element of degree $1-\alpha$ in $\pi_\star(\mathsf{MGLO})$. Notice that there exists exactly one element in $\pi_\star(\mathsf{MGLO})$ of degree $1-\alpha$, the Tate twist. Therefore, the coefficients of $\Phi^{C_2}(\lambda^{-1}\mathsf{MGLR})$ are

$$\pi_\star(\theta^{-1}\mathsf{MGLO}) \cong \pi_*(\mathsf{MO})[\theta^{\pm 1}]. \qquad \square$$

Corollary 5.6. *The spectrum* MGLR *is not equivalent to* $\lambda^{-1}\mathsf{MGLR}$.

Proof. Since MGLR and $\lambda^{-1}\mathsf{MGLR}$ are not equal on geometric fixed points, they cannot possibly be equal equivariantly. $\square$

It is interesting to note that while inverting λ has the effect of inverting the Tate twist θ under the geometric fixed points map, it is not the case that θ is inverted under the forgetful map $\mathsf{MGLR} \to \mathsf{MGL}$, which thinks of the structure nonequivariantly. The reason for this is the forgetful map sends σ and $\sigma\alpha$ grading to 1 and α, respectively. Therefore, λ gets sent to the unit under this map. The next theorem gives more detail.

Theorem 5.7. *Nonequivariantly*, $\lambda^{-1}\mathsf{MGLR} \simeq \mathsf{MGL}$.

Proof. Notice that nonequivariantly, λ realizes as

$$\begin{aligned} S^{2+2\alpha} \simeq \Sigma\mathbb{G}_m \wedge \Sigma\mathbb{G}_m \to \Sigma\mathbb{G}_m \times \Sigma\mathbb{G}_m \to B(\mathbb{G}_m \times \mathbb{G}_m) \\ \to \mathrm{BGL}_2 \to \mathsf{MGL}(1) \to \Sigma^{2+2\alpha}\mathsf{MGL}. \end{aligned}$$

Notice that this map is clearly nonzero, and represents an element in $\pi_\star(\mathsf{MGL})$ of degree 0. Notice that the only nonzero element in $\pi_\star(\mathsf{MGL})$ of degree 0 is the identity element. Therefore, $\lambda^{-1}\mathsf{MGLR}$ is nonequivariantly equivalent to MGL. $\square$

Theorem 5.8. *Localizing at* $p = 2$, *we have that*

$$\mathsf{MGL} = \bigvee_{m_i} \Sigma^{m_i(1+\alpha)}\mathsf{BPGL}$$

for integers m_i. *There exists a spectrum* BPGLR *such that*

$$\mathsf{MGLR} = \bigvee_{m_i} \Sigma^{m_i(1+\sigma\alpha)}\mathsf{BPGLR}.$$

Furthermore, $\Phi^{C_2}_{\text{ét}}(\mathsf{BPGLR}) = \theta^{-1}H\mathbb{Z}/2$.

6. Calculating the coefficients of $\theta^{-1}\lambda^{-1}\mathsf{MGLR}$

The main difficulty in computing the coefficients of MGLR is the lack of a Tate diagram. One would like to use simplicial EC_2 to get a motivic Tate diagram,

$$EC_{2+} \wedge \mathsf{MGLR} \to \mathsf{MGLR} \to \widetilde{EC}_2 \wedge \mathsf{MGLR}.$$

However, the C_2 fixed points of $\widetilde{EC}_2 \wedge \mathsf{MGLR}$ is not the geometric fixed points of MGLR in the sense of [Lewis et al. 1986, Chapter 2, Definition 9.7]. In other words, taking C_2 fixed points of MGLR at the level of prespectra does not form a nonequivariant spectrum equivalent to

$$(\widetilde{EC}_2 \wedge \mathsf{MGLR})^{C_2}.$$

To fix this, we need to use a different model of EC_2. The model we use is

$$\mathbf{E}C_2 := \varinjlim \mathbb{A}(n\sigma) \smallsetminus 0,$$

where $\mathbb{A}(n\sigma) \smallsetminus 0$ denotes $\mathbb{A}^n \smallsetminus 0$ with a C_2 action $z \mapsto -z$. This gives us cofibrations

$$\mathbb{A}(n\sigma) \smallsetminus 0_+ \to S^0 \to S^{n\sigma + n\sigma\alpha}.$$

These piece together to give us a cofibration

$$\mathbf{E}C_{2+} \to S^0 \to \widetilde{\mathbf{E}C}_2.$$

The space $\widetilde{\mathbf{E}C}_2$ takes into account the entire equivariant grading in the C_2 equivariant stable category, and so we have the following.

Theorem 6.1. $$\Phi^{C_2}_{ét}(\mathsf{MGLR}) := (\widetilde{\mathbf{E}C}_2 \wedge \mathsf{MGLR})^{C_2} \simeq \mathsf{MGLO}.$$

Proof. By construction, the n-th term of the prespectrum defining MGLO is equal to the C_2-fixed points of the n-th term of the prespectrum defining MGLR [Hu et al. 2011, Section 6]. Let $\mathsf{MGLR}(n)$ denote the n-th term of the prespectrum defining MGLR. Notice that $(\mathsf{MGLR} \wedge \widetilde{\mathbf{E}C}_2)^{C_2}$ is a nonequivariant spectrum with prespectrum $(\mathsf{MGLR}(1))^{C_2}$, $(\mathsf{MGLR}(2))^{C_2}, \dots$, and connecting maps given by

$$\mathbb{P}^1 \wedge (\mathsf{MGLR}(n))^{C_2} \to (\mathsf{MGLR}(n+1))^{C_2}.$$

The claim follows. □

Corollary 6.2. MGLO *is a motivic* E_∞*-ring spectrum.*

Proof. In [Hu et al. 2011, Section 6] it is proved that MGLR is a C_2-equivariant motivic E_∞-ring spectrum. Being an E_∞-ring spectrum is preserved by smashing with $S^{\infty\sigma+\infty\sigma\alpha}$ and taking C_2 fixed points. □

The author would like to acknowledge the work of the authors of [Heller et al. 2019], who are the first to have written about the geometric classifying space $\mathbf{E}C_2$ in the context of C_2-equivariant motivic spectra. The unfortunate reality is that calculating $F(\mathbf{E}C_{2+}, \mathsf{MGLR})$ via a Borel cohomology spectral sequence involves developing new tools which do not currently exist. The solution presented in this paper, however, is to restrict to a field k of characteristic 0, for which all elements in k are squares. Then after completing at the prime $p = 2$ and inverting two twists in MGLR, we can show

$$(\widetilde{\mathbf{E}C}_2 \wedge \mathsf{MGLR})^{C_2} \simeq (\widetilde{EC}_2 \wedge \mathsf{MGLR})^{C_2}.$$

We can then apply the tools of [Hu and Kriz 2001].

Proposition 6.3. *There exists an element θ of order $1-\alpha$ in the Borel cohomology and the Tate cohomology of $\lambda^{-1}\mathsf{MGLR}$.*

Proof. Using simplicial EC_2, we can set up a Borel cohomology spectral sequence for $\lambda^{-1}\mathsf{MGLR}$ as follows. First we note that since we have inverted λ, we can choose to ignore all $\sigma\alpha$ grading, and instead only consider the grading $* + *'\sigma + *''\alpha$. Moreover, we filter by α twists. In other words, we consider the grading $* + *'\sigma + k\alpha$ for fixed k. Now for each $k \le 0$, we have a bijection between the motivic Borel cohomology spectral sequence of $\lambda^{-1}\mathsf{MGLR}$ and the classical Borel cohomology spectral sequence of MR. This is true since $\lambda^{-1}\mathsf{MGLR}$ is nonequivariantly MGL, and over $\mathbb{C}$, there is a bijection between $\pi_{*+k\alpha}(\mathsf{MGL})$ and $\pi_*(\mathsf{MU})$. It follows that the motivic Borel cohomology spectral sequence associated to $\lambda^{-1}\mathsf{MGLR}_{*+*'\sigma+*''\alpha}$, where $*, *' \in \mathbb{Z}$ and $*'' \in \mathbb{Z}^{\le 0}$, converges to $\pi_{*+*'\sigma+*''\alpha}(F(EC_{2+}, \lambda^{-1}\mathsf{MGLR})) \cong \pi_\star(\mathsf{MR})[\theta]$. It follows that $\theta \in \lambda^{-1}\mathsf{MGLR}$. The same argument works for the Tate cohomology of $\lambda^{-1}\mathsf{MGLR}$. □

Corollary 6.4. *There exists an element, again denoted θ, of degree $1-\alpha$ in the coefficients of $\lambda^{-1}\mathsf{MGLR}$.*

Proof. This follows by considering the following square originating from the Tate diagram:

$$\begin{tikzcd}
\lambda^{-1}\mathsf{MGLR} \arrow[r] \arrow[d] & S^{\infty\sigma} \wedge \lambda^{-1}\mathsf{MGLR} \arrow[d] \\
F(EC_{2+}, \lambda^{-1}\mathsf{MGLR}) \arrow[r] & S^{\infty\sigma} \wedge F(EC_{2+}, \lambda^{-1}\mathsf{MGLR})
\end{tikzcd}$$

It is easy to see that the element $\theta \in \pi_\star(F(EC_{2+}, \lambda^{-1}\mathsf{MGLR}))$ is mapped to $\theta \in \pi_\star(S^{\infty\sigma} \wedge F(EC_{2+}, \lambda^{-1}\mathsf{MGLR}))$. This is true since the topological realization of θ is just 1, and since the Borel and Tate cohomology spectral sequences of $\lambda^{-1}\mathsf{MGLR}$ and MR are isomorphisms for a fixed alpha twist $k\alpha, k \le 0$. Now, notice that there is an easily described twist in $\pi_\star(S^{\infty\sigma} \wedge \lambda^{-1}\mathsf{MGLR})$ of degree $1-\alpha$, which we also call θ. If s is the Euler class $s \in \pi_{-\sigma}(\mathsf{MGLR})$, and t is the Euler class $t \in \pi_{-\sigma\alpha}(\mathsf{MGLR})$, then $\theta \in \pi_{1-\alpha}(S^{\infty\sigma} \wedge \lambda^{-1}\mathsf{MGLR})$ is given by $\lambda s^{-1} t$. By comparison with topology, and in view of the fact that the topological realization of θ is 1, it follows that $\theta \in \pi_{1-\alpha}(S^{\infty\sigma} \wedge \lambda^{-1}\mathsf{MGLR})$ is mapped to $\theta \in \pi_{1-\alpha}(S^{\infty\sigma} \wedge F(EC_{2+}, \lambda^{-1}\mathsf{MGLR}))$. Therefore, the element named θ commutes in the bottom row and rightmost column of the above diagram. Since that diagram is a pullback, there must exist an element $\theta \in \pi_\star(\lambda^{-1}\mathsf{MGLR})$ which is sent to $\theta \in \pi_\star(F(EC_{2+}, \lambda^{-1}\mathsf{MGLR}))$. □

As we inverted $\lambda \in \pi_{1-\sigma+\sigma\alpha-\alpha}(\mathsf{MGLR})$, so too can we invert $\theta \in \pi_{1-\alpha}(\mathsf{MGLR})$. This gives us a spectrum $\theta^{-1}\lambda^{-1}\mathsf{MGLR}$. In its coefficients, the element $\lambda^{-1}\theta$ has degree $\sigma - \sigma\alpha$ and is invertible.

Proposition 6.5. $(S^{\infty\sigma+\infty\sigma\alpha} \wedge \theta^{-1}\lambda^{-1}\mathsf{MGLR})^{C_2} \simeq (S^{\infty\sigma} \wedge \theta^{-1}\lambda^{-1}\mathsf{MGLR})^{C_2}$.

Proof. To simplify notation, we write

$$E := S^{\infty\sigma} \wedge \theta^{-1}\lambda^{-1}\mathsf{MGLR},\ F := S^{\infty\sigma+\infty\sigma\alpha} \wedge \theta^{-1}\lambda^{-1}\mathsf{MGLR}.$$

Notice that $\Sigma^{\sigma\alpha-\sigma}E \simeq E$ since $\theta\lambda^{-1} \in \pi_{\sigma-\sigma\alpha}(E)$ is invertible. Also, it is clear that $\Sigma^{\sigma}E \simeq E$. Putting this together, we have that $\Sigma^{\sigma\alpha}E \simeq E$. Therefore, it follows that $F = \Sigma^{\infty\sigma\alpha}E \simeq E$. □

Theorem 6.6. *We have* $\pi_\star(\theta^{-1}\mathsf{BPGLR}) = \pi_\star(\mathsf{BPR})[\lambda^{\pm 1}, \theta^{\pm 1}]$. *Here*, $\pi_\star(\mathsf{BPR})$ *is*

$$\mathbb{Z}_{(2)}[v_{n,l}, a \mid n \geq 0, l \in \mathbb{Z}] \Big/ \big(v_{0,0} = 2,\ a^{2^{n+1}-1}v_{n,l} = 0 \mid \textit{for } n \leq m,\ v_{m,k} \cdot v_{n,l2^{m-n}} = v_{m,k+l} \cdot v_{n,0}\big),$$

$|a| = -\sigma$, *and* $|v_{n,l}| = (2^n - 1)(1+\sigma) + l2^{n+1}(\sigma - 1)$.

Proof. The claim is clear by comparison with topology [Hu and Kriz 2001]. In more detail, considering the commutative square of Corollary 6.4, the C_2 fixed points of the top right corner is easily seen to be equal to $\pi_*(\mathsf{MO})[\theta^{\pm 1}]$. The bottom right corner is calculated by comparing the Tate cohomology spectral sequence for $\theta^{-1}\lambda^{-1}\mathsf{MGLR}$ to topology. One deduces from the calculation that the C_2 fixed points of the top and bottom right-hand column are equal. From this it follows that $\theta^{-1}\lambda^{-1}\mathsf{MGLR}$ is equal to its Borel cohomology. By comparing with topology, the claim follows. □

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The IA-congruence kernel of high rank free metabelian groups

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The congruence subgroup problem for a finitely generated group Γ and $G \leq \mathrm{Aut}(\Gamma)$ asks whether the map $\hat{G} \to \mathrm{Aut}(\hat{\Gamma})$ is injective, or more generally, what its kernel $C(G, \Gamma)$ is. Here $\hat{X}$ denotes the profinite completion of X. In this paper we investigate $C(\mathrm{IA}(\Phi_n), \Phi_n)$, where Φ_n is a free metabelian group on $n \geq 4$ generators, and $\mathrm{IA}(\Phi_n) = \ker(\mathrm{Aut}(\Phi_n) \to \mathrm{GL}_n(\mathbb{Z}))$.

We show that in this case $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is abelian, but not trivial, and not even finitely generated. This behavior is very different from what happens for a free metabelian group on $n = 2$ or 3 generators, or for finitely generated nilpotent groups.

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1. Introduction

The classical congruence subgroup problem (CSP) asks for, say, $G = \mathrm{SL}_n(\mathbb{Z})$ or $G = \mathrm{GL}_n(\mathbb{Z})$, whether every finite index subgroup of G contains a principal congruence subgroup, i.e., a subgroup of the form $G(m) = \ker(G \to \mathrm{GL}_n(\mathbb{Z}/m\mathbb{Z}))$ for some $0 \neq m \in \mathbb{Z}$. It is a classical 19th century result that the answer is negative for $n = 2$. On the other hand, quite surprisingly, it was proved in the sixties by Mennicke [1965] and by Bass, Lazard and Serre [Bass et al. 1964] that for $n \geq 3$

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the answer to the CSP is affirmative. A rich theory of the CSP for more general arithmetic groups has been developed since then.

By the observation $\mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{Aut}(\mathbb{Z}^n)$, the CSP can be generalized to automorphism groups as follows: Let Γ be a group and $G \leq \mathrm{Aut}(\Gamma)$. For a finite index characteristic subgroup $M \leq \Gamma$, define

$$G(M) = \ker(G \to \mathrm{Aut}(\Gamma/M)).$$

A finite index subgroup of G which contains $G(M)$ for some M is called a "congruence subgroup". The CSP for the pair (G, Γ) asks whether every finite index subgroup of G is a congruence subgroup.

One can easily see that the CSP is equivalent to the question: Is the congruence map $\hat{G} = \varprojlim G/U \to \varprojlim G/G(M)$ injective? Here, U ranges over all finite index normal subgroups of G, and M ranges over all finite index characteristic subgroups of Γ. When Γ is finitely generated, it has only finitely many subgroups of given index m, and thus, the characteristic subgroups $M_m = \bigcap\{\Delta \lhd \Gamma \mid [\Gamma : \Delta] \mid m\}$ are of finite index in Γ. Hence, one can write $\hat{\Gamma} = \varprojlim_{m\in\mathbb{N}} \Gamma/M_m$ and have[1]

$$\begin{aligned}\varprojlim G/G(M) = \varprojlim_{m\in\mathbb{N}} G/G(M_m) &\leq \varprojlim_{m\in\mathbb{N}} \mathrm{Aut}(\Gamma/M_m)\\ &\leq \mathrm{Aut}(\varprojlim_{m\in\mathbb{N}}(\Gamma/M_m)) = \mathrm{Aut}(\hat{\Gamma}).\end{aligned}$$

Therefore, when Γ is finitely generated, the CSP is equivalent to the question: Is the congruence map: $\hat{G} \to \mathrm{Aut}(\hat{\Gamma})$ injective? More generally, the CSP asks what is the kernel $C(G, \Gamma)$ of this map. For $G = \mathrm{Aut}(\Gamma)$ we also use the simpler notation $C(\Gamma) = C(G, \Gamma)$. The classical congruence subgroup result mentioned above can therefore be reformulated as $C(\mathbb{Z}^n) = \{e\}$ for $n \geq 3$, and it is also known that $C(\mathbb{Z}^2) = \hat{F}_\omega$, where $\hat{F}_\omega$ is the free nonabelian profinite group on a countable number of generators; see [Melnikov 1976; Lubotzky 1982].

Very few results are known when Γ is nonabelian. Most of the results are related to $\Gamma = \pi(S_{g,n})$, the fundamental group of the closed surface of genus g with n punctures; see [Diaz et al. 1989; McReynolds 2012; Asada 2001; Boggi 2009; 2016]. As observed in [Bux et al. 2011], the result of Asada [2001] actually gives an affirmative solution to the case $\Gamma = F_2$, $G = \mathrm{Aut}(F_2)$; see also [Ben-Ezra and Lubotzky 2018]. Note that for every $n > 0$, one has $\pi(S_{g,n}) \cong F_{2g+n-1} =$ the free group on $2g + n - 1$ generators. Hence, the aforementioned results relate to various subgroups of the automorphism group of finitely generated free groups. However, the CSP for the full $\mathrm{Aut}(F_n)$ when $n \geq 3$ is still unsettled.

[1]By the celebrated theorem of Nikolov and Segal [2003], which asserts that every finite index subgroup of a finitely generated profinite group is open, the second inequality is actually an equality. However, we do not need it.

Denote now the free metabelian group on n generators by $\Phi_n = F_n/F_n''$. Considering the metabelian case, it was shown in [Ben-Ezra and Lubotzky 2018] (see also [Ben-Ezra 2016]) that $C(\Phi_2) = \hat{F}_\omega$. In addition, it was proven there that $C(\Phi_3) \supseteq \hat{F}_\omega$. The basic motivation which led to this paper was to complete the picture in the free metabelian case and investigate $C(\Phi_n)$ for $n \geq 4$. Now, let $\mathrm{IA}(\Phi_n) = \ker(\mathrm{Aut}(\Phi_n) \to \mathrm{GL}_n(\mathbb{Z}))$. Then the commutative exact diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{IA}(\Phi_n) \arrow[r] \arrow[dr] & \mathrm{Aut}(\Phi_n) \arrow[r] \arrow[d] & \mathrm{GL}_n(\mathbb{Z}) \arrow[r] \arrow[d] & 1 \\ & & \mathrm{Aut}(\hat{\Phi}_n) \arrow[r] & \mathrm{GL}_n(\hat{\mathbb{Z}}) & \end{tikzcd}$$

gives rise to the commutative exact diagram (see Lemma 2.1 in [Bux et al. 2011])

$$\begin{tikzcd} \widehat{\mathrm{IA}(\Phi_n)} \arrow[r] \arrow[dr] & \widehat{\mathrm{Aut}(\Phi_n)} \arrow[r] \arrow[d] & \widehat{\mathrm{GL}_n(\mathbb{Z})} \arrow[r] \arrow[d] & 1 \\ & \mathrm{Aut}(\hat{\Phi}_n) \arrow[r] & \mathrm{GL}_n(\hat{\mathbb{Z}}) & \end{tikzcd}$$

Hence, by using the fact that $\widehat{\mathrm{GL}_n(\mathbb{Z})} \to \mathrm{GL}_n(\hat{\mathbb{Z}})$ is injective for $n \geq 3$, one can obtain that $C(\Phi_n)$ is an image of $C(\mathrm{IA}(\Phi_n), \Phi_n)$. Thus, for investigating $C(\Phi_n)$ it seems to be worthwhile to investigate $C(\mathrm{IA}(\Phi_n), \Phi_n)$.

The first goal of the present paper is to prove the following theorem:

Theorem 1.1. *For every $n \geq 4$, the group $C(\mathrm{IA}(\Phi_n), \Phi_n)$ contains a subgroup C which satisfies the following properties*:

- *C is isomorphic to a product $C = \prod_{i=1}^n C_i$ of n copies of*

$$C_i \cong \ker\big(\mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x^{\pm 1}]}) \to \mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x^{\pm 1}]})\big).$$

- *C is a direct factor of $C(\mathrm{IA}(\Phi_n), \Phi_n)$; that is, there is a normal subgroup $N \lhd C(\mathrm{IA}(\Phi_n), \Phi_n)$ such that $C(\mathrm{IA}(\Phi_n), \Phi_n) = N \times C$.*

Using techniques of Kassabov and Nikolov [2006], one can show that the subgroups C_i are not finitely generated. So as an immediate corollary, we obtain the following theorem:

Theorem 1.2. *For every $n \geq 4$, the group $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is not finitely generated.*

It will be shown in an upcoming paper that when Γ is a finitely generated nilpotent group (of any class), then $C(\mathrm{IA}(\Gamma), \Gamma) = \{e\}$ is always trivial. So the free metabelian cases behave completely different from nilpotent cases. This result gives the impression that $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is "big". On the other hand, we have the following theorem (see [Ben-Ezra 2017]):

Theorem 1.3. *For every $n \geq 4$, the group $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is central in $\widehat{\mathrm{IA}(\Phi_n)}$.*

We remark that in the case of arithmetic groups, the congruence kernel is known to have a dichotomous behavior: it is central if and only if it is finite (see [Prasad and Rapinchuk 2010, Theorem 2]). So in some sense, the congruence kernel $C(\mathrm{IA}(\Phi_n), \Phi_n)$ for $n \geq 4$ has an intermediate behavior: central, but not finite. The latter is similar to the behavior of the congruence kernel

$$\ker\bigl(\widehat{\mathrm{SL}_d(\mathbb{Z}[x])} \to \mathrm{SL}_d(\widehat{\mathbb{Z}[x]})\bigr) \quad \text{for } d \geq 3$$

that was investigated in [Kassabov and Nikolov 2006, Theorem 4.1].

Theorem 1.3 has already been stated in [Ben-Ezra 2017]. However, a substantial portion of the proof of Theorem 1.3 appears in this paper — this is the second goal of the present paper. To be more precise, all the steps of the proof of Theorem 1.3 that involve arguments in algebraic K-theory are given in this paper, and in [Ben-Ezra 2017] we describe the structure of the proof, and present all the other steps. As presented in Section 5, the steps that are given in this present paper by themselves are sufficient for showing that the subgroup $C \leq C(\mathrm{IA}(\Phi_n), \Phi_n)$ presented in Theorem 1.1 is contained in the center of $\widehat{\mathrm{IA}(\Phi_n)}$. We remark that the main results in this paper that are used in [Ben-Ezra 2017] in order to prove Theorem 1.3 are Lemma 7.1 and our work in Section 5 (see Remark 5.6 for a more precise description). The following problem is still open:

Problem 1.4. Is $C(\mathrm{IA}(\Phi_n), \Phi_n) = \prod_{i=1}^{n} C_i$ or does it contain more elements?

Remark 1.5. Considering the action of $\mathrm{Aut}(\Phi_n)$ on $\mathrm{IA}(\Phi_n)$ by conjugation, we have a natural map $\mathrm{Aut}(\Phi_n) \to \mathrm{Aut}(\mathrm{IA}(\Phi_n))$ in which the copy of $\mathrm{IA}(\Phi_n)$ in $\mathrm{Aut}(\Phi_n)$ is mapped onto $\mathrm{IA}(\Phi_n) \to \mathrm{Inn}(\mathrm{IA}(\Phi_n))$. Now let

$$\mathrm{IA}_{n,m} = \bigcap \{N \lhd \mathrm{IA}(\Phi_n) \mid [\mathrm{IA}(\Phi_n) : N] \mid m\}.$$

Then as for every $n \geq 4$, the group $\mathrm{IA}(\Phi_n)$ is finitely generated [Bachmuth and Mochizuki 1985], the characteristic subgroups $\mathrm{IA}_{n,m} \leq \mathrm{IA}(\Phi_n)$ are of finite index. Hence $\widehat{\mathrm{IA}(\Phi_n)} = \varprojlim_{m \in \mathbb{N}} (\mathrm{IA}(\Phi_n)/\mathrm{IA}_{n,m})$ and therefore the action of $\mathrm{Aut}(\Phi_n)$ on $\mathrm{IA}(\Phi_n)$ induces an action of $\mathrm{Aut}(\Phi_n)$ on $\widehat{\mathrm{IA}(\Phi_n)}$, so we have a map $\mathrm{Aut}(\Phi_n) \to \varprojlim_{m \in \mathbb{N}} \mathrm{Aut}(\mathrm{IA}(\Phi_n)/\mathrm{IA}_{n,m})) \leq \mathrm{Aut}(\widehat{\mathrm{IA}(\Phi_n)})$. The latter gives rise to a map

$$\widehat{\mathrm{Aut}(\Phi_n)} \to \varprojlim_{m \in \mathbb{N}} \mathrm{Aut}(\mathrm{IA}(\Phi_n)/\mathrm{IA}_{n,m})) \leq \mathrm{Aut}(\widehat{\mathrm{IA}(\Phi_n)})$$

that actually gives an action of $\widehat{\mathrm{Aut}(\Phi_n)}$ on $\widehat{\mathrm{IA}(\Phi_n)}$ such that the closure $\overline{\mathrm{IA}(\Phi_n)}$ of $\mathrm{IA}(\Phi_n)$ in $\widehat{\mathrm{Aut}(\Phi_n)}$ acts trivially on $Z(\widehat{\mathrm{IA}(\Phi_n)})$, the center of $\widehat{\mathrm{IA}(\Phi_n)}$. Thus, as we have $\widehat{\mathrm{Aut}(\Phi_n)}/\overline{\mathrm{IA}(\Phi_n)} = \widehat{\mathrm{GL}_n(\mathbb{Z})}$ we obtain a natural action of $\widehat{\mathrm{GL}_n(\mathbb{Z})}$ on $Z(\widehat{\mathrm{IA}(\Phi_n)})$. It will be clear from the description in the paper that the permutation matrices permute the copies C_i through this natural action.

The aforementioned behavior of $C(\mathrm{IA}(\Phi_n), \Phi_n)$ for $n \geq 4$ is also different from the behavior of $C(\mathrm{IA}(\Phi_n), \Phi_n)$ for $n = 2, 3$. More precisely, as $C(\mathbb{Z}^3) = \{e\}$, similar arguments show that when $n = 3$ the group $C(\Phi_3)$ is an image of $C(\mathrm{IA}(\Phi_3), \Phi_3)$. So as $C(\Phi_3) \supseteq \hat{F}_\omega$ [Ben-Ezra and Lubotzky 2018], we obtain that $C(\mathrm{IA}(\Phi_3), \Phi_3)$ is infinite nonabelian. On the other hand, regarding the case $n = 2$, it is known that $\mathrm{IA}(\Phi_2) = \mathrm{Inn}(\Phi_2)$ (see [Bachmuth 1965]) and it is known that the center of Φ_2 and $\hat{\Phi}_2$ is trivial (see [Ben-Ezra 2016]). It follows that we have a canonical isomorphism

$$\widehat{\mathrm{IA}(\Phi_2)} = \widehat{\mathrm{Inn}(\Phi_2)} \cong \hat{\Phi}_2 \cong \mathrm{Inn}(\hat{\Phi}_2) \leq \mathrm{Aut}(\hat{\Phi}_2),$$

so $C(\mathrm{IA}(\Phi_2), \Phi_2) = \{e\}$ is trivial. Our results show that when $n \geq 4$, the behavior of $C(\mathrm{IA}(\Phi_n), \Phi_n)$ stabilizes and it is abelian, but not trivial.

We also note that considering our basic motivation, as $C(\Phi_n)$ is an image of $C(\mathrm{IA}(\Phi_n), \Phi_n)$ we actually obtain from Theorem 1.3 that when $n \geq 4$, the situation is dramatically different from the cases of $n = 2, 3$ described above:

Theorem 1.6. *For every $n \geq 4$, the group $C(\Phi_n)$ is abelian.*

We remark that despite the result of the latter theorem, we do not know whether $C(\Phi_n)$ is also not finitely generated. In fact we cannot even prove at this point that it is not trivial.

The paper is organized as follows. For a ring R, ideal $H \lhd R$ and $d \in \mathbb{N}$ let

$$\mathrm{GL}_d(R, H) = \ker(\mathrm{GL}_d(R) \to \mathrm{GL}_d(R/H)).$$

For $n \in \mathbb{N}$ define also the ring $R_n = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathbb{Z}[\mathbb{Z}^n]$. Using the Magnus embedding of $\mathrm{IA}(\Phi_n)$, in which $\mathrm{IA}(\Phi_n)$ can be viewed as

$$\mathrm{IA}(\Phi_n) = \left\{ A \in \mathrm{GL}_n(R_n) \;\middle|\; A \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix} = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_n - 1 \end{pmatrix} \right\},$$

we obtain in Section 3, for every $1 \leq i \leq n$, a natural embedding

$$\mathrm{GL}_{n-1}(R_n, (x_i - 1)R_n) \hookrightarrow \mathrm{IA}(\Phi_n)$$

and a surjective natural homomorphism

$$\mathrm{IA}(\Phi_n) \xrightarrow{\rho_i} \mathrm{GL}_{n-1}\big(\mathbb{Z}[x_i^{\pm 1}], (x_i - 1)\mathbb{Z}[x_i^{\pm 1}]\big)$$

in which the obvious copy of the subgroup $\mathrm{GL}_{n-1}\big(\mathbb{Z}[x_i^{\pm 1}], (x_i - 1)\mathbb{Z}[x_i^{\pm 1}]\big)$ of the group $\mathrm{GL}_{n-1}(R_n, (x_i - 1)R_n)$ is mapped onto itself via the composition map (see Proposition 3.7). This description, combined with some classical notions and results from algebraic K-theory presented in Section 2, enables us in Section 4 to show that for every $n \geq 4$ and $1 \leq i \leq n$, the group $C(\mathrm{IA}(\Phi_n), \Phi_n)$ contains a copy of

$$\begin{aligned} C_i &\cong \ker\big(\mathrm{GL}_{n-1}\big(\widehat{\mathbb{Z}[x_i^{\pm 1}], (x_i - 1)\mathbb{Z}[x_i^{\pm 1}]}\big) \to \mathrm{GL}_{n-1}(\widehat{\mathbb{Z}[x_i^{\pm 1}]})\big) \\ &\cong \ker\big(\widehat{\mathrm{SL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}])} \to \mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x_i^{\pm 1}]})\big) \end{aligned} \tag{1.7}$$

such that $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is mapped onto C_i through the map $\hat{\rho}_i$ induced by ρ_i. The second isomorphism in (1.7) is obtained by using some classical results from algebraic K-theory (Propositions 4.5 and 4.6), and the main lemma, Lemma 7.1. The proof of Lemma 7.1 will be postponed until the end of the paper. In particular, we get that for every $1 \leq i \leq n$ one has

$$C(\mathrm{IA}(\Phi_n), \Phi_n) = \big(C(\mathrm{IA}(\Phi_n), \Phi_n) \cap \ker \hat{\rho}_i\big) \rtimes C_i.$$

(see Proposition 4.3). In Section 4 we also show that the copies C_i lie in $\ker \hat{\rho}_j$ whenever $j \neq i$ (Proposition 4.2). In particular, we get that the copies C_i intersect each other trivially. Then, following the techniques of Kassabov and Nikolov [2006] we show that C_i is not finitely generated, and thus deduce that $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is not finitely generated either, i.e., we prove Theorem 1.2 (see the end of Section 4). Then, in Section 5 we show that the copies C_i lie in the center of $\widehat{\mathrm{IA}(\Phi_n)}$, using classical results from algebraic K-theory and Lemma 7.1. In particular, using the aforementioned results, we obtain that

$$C(\mathrm{IA}(\Phi_n), \Phi_n) = \left(C(\mathrm{IA}(\Phi_n), \Phi_n) \cap \bigcap_{i=1}^{n} \ker \hat{\rho}_i\right) \times \prod_{i=1}^{n} C_i.$$

This completes the proof of Theorem 1.1.

After that, we turn to prove Lemma 7.1. In Section 6 we introduce some elements in $\langle \mathrm{IA}(\Phi_n)^m \rangle$ which are needed for the proof of the lemma. In Section 7, using classical results from algebraic K-theory, we conclude the paper by proving Lemma 7.1, which asserts that for every $1 \leq i \leq n$, we have

$$\mathrm{GL}_{n-1}(R_n, (x_i - 1)R_n) \cap E_{n-1}(R_n, H_{n,m^2}) \subseteq \langle \mathrm{IA}(\Phi_n)^m \rangle, \tag{1.8}$$

where

- $\mathrm{GL}_{n-1}(R_n, (x_i - 1)R_n)$ denotes its appropriate copy in $\mathrm{IA}(\Phi_n)$ described above;
- $E_{n-1}(R_n, H_{n,m^2})$ is the subgroup of $E_{n-1}(R_n) = \langle I_{n-1} + rE_{i,j} \mid r \in R_n \rangle$ which is generated as a normal subgroup by the elementary matrices of the form $I_{n-1} + hE_{i,j}$ for $h \in H_{n,m^2} = \ker(R_n \to \mathbb{Z}_{m^2}[\mathbb{Z}_{m^2}^n])$, $1 \leq i \neq j \leq n$. Here, I_{n-1} is the $(n-1) \times (n-1)$ unit matrix and $E_{i,j}$ is the matrix which has 1 in the (i, j)-th entry and 0 elsewhere.
- The intersection in the inclusion (1.8) is obtained by viewing the copy of $\mathrm{GL}_{n-1}(R_n, (x_i - 1)R_n)$ in $\mathrm{IA}(\Phi_n)$ as a subgroup of $\mathrm{GL}_{n-1}(R_n)$.

We note that as described above, Lemma 7.1 is used in two places in the course of the paper. It is used once to prove the second isomorphism in (1.7). The second place is in the proof that the group C lies in the center of $\widehat{\mathrm{IA}(\Phi_n)}$. We also note that almost all the work that we do in order to show that C lies in the center of $\widehat{\mathrm{IA}(\Phi_n)}$, including Lemma 7.1 (but also most of Section 5), is used in [Ben-Ezra 2017] to prove Theorem 1.3 (see Remark 5.6).

2. Some background in algebraic K-theory

In this section we fix some notation and recall some definitions and background in algebraic K-theory which will be used throughout the paper. One can find more general information in the references [Rosenberg 1994; Milnor 1971; Bass 1968]. In this section R always denotes a commutative ring with identity. We start by recalling the following notation. Let R be a commutative ring, $H \lhd R$ an ideal, and $d \in \mathbb{N}$. Then:

- $\mathrm{GL}_d(R) = \{A \in M_n(R) \mid \det(A) \in R^*\}$.
- $\mathrm{SL}_d(R) = \{A \in \mathrm{GL}_d(R) \mid \det(A) = 1\}$.
- $E_d(R) = \langle I_d + rE_{i,j} \mid r \in R,\ 1 \le i \ne j \le d\rangle$.
- $\mathrm{GL}_d(R, H) = \ker(\mathrm{GL}_d(R) \to \mathrm{GL}_d(R/H))$.
- $\mathrm{SL}_d(R, H) = \ker(\mathrm{SL}_d(R) \to \mathrm{SL}_d(R/H))$.
- $E_d(R, H)$ = the normal subgroup of $E_d(R)$, which is generated as a normal subgroup by the elementary matrices of the form $I_d + hE_{i,j}$ for $h \in H$.

For every $d \ge 3$, the subgroup $E_d(R, H)$ is normal in $\mathrm{GL}_d(R)$; see Corollary 1.4 in [Suslin 1977]. Hence, we can consider the groups

$$K_1(R; d) = \mathrm{GL}_d(R)/E_d(R), \qquad K_1(R, H; d) = \mathrm{GL}_d(R, H)/E_d(R, H),$$
$$\mathrm{SK}_1(R; d) = \mathrm{SL}_d(R)/E_d(R), \qquad \mathrm{SK}_1(R, H; d) = \mathrm{SL}_d(R, H)/E_d(R, H).$$

We now go ahead with the following definition:

Definition 2.1. Let R be a commutative ring, and $3 \le d \in \mathbb{N}$. We define the "Steinberg group" $\mathrm{St}_d(R)$ to be the group generated by the elements $x_{i,j}(r)$ for $r \in R$ and $1 \le i \ne j \le d$, under the relations

- $x_{i,j}(r_1) \cdot x_{i,j}(r_2) = x_{i,j}(r_1 + r_2)$,
- $[x_{i,j}(r_1), x_{j,k}(r_2)] = x_{i,k}(r_1 \cdot r_2)$,
- $[x_{i,j}(r_1), x_{k,l}(r_2)] = 1$,

for every distinct $1 \le i, j, k, l \le d$ and every $r_1, r_2 \in R$.

As the elementary matrices $I_d + rE_{i,j}$ satisfy the relations which define $\mathrm{St}_d(R)$, the map $x_{i,j}(r) \mapsto I_d + rE_{i,j}$ defines a natural homomorphism $\phi_d : \mathrm{St}_d(R) \to E_d(R)$. The kernel of this map is denoted by $K_2(R; d) = \ker(\phi_d)$. Now, for two invertible elements $u, v \in R^*$ and $1 \le i \ne j \le d$, define the "Steinberg symbol" by

$$\{u, v\}_{i,j} = h_{i,j}(uv)h_{i,j}(u)^{-1}h_{i,j}(v)^{-1} \in \mathrm{St}_d(R)$$
$$\text{where } h_{i,j}(u) = w_{i,j}(u)w_{i,j}(-1) \text{ and } w_{i,j}(u) = x_{i,j}(u)x_{j,i}(-u^{-1})x_{i,j}(u).$$

One can show that $\{u, v\}_{i,j} \in K_2(R; d)$ and lie in the center of $\mathrm{St}_d(R)$. In addition, for every $3 \le d \in \mathbb{N}$, the Steinberg symbols $\{u, v\}_{i,j}$ do not depend on the indices i, j, so they can be denoted simply by $\{u, v\}$; see [Dennis and Stein 1973]. The Steinberg symbols satisfy many identities. For example,

$$\{uv, w\} = \{u, w\}\{v, w\}, \qquad \{u, vw\} = \{u, v\}\{u, w\}. \tag{2.2}$$

In the semilocal case we have the following:

Theorem 2.3 [Stein and Dennis 1973, Theorem 2.7]. *Let R be a semilocal commutative ring and $d \ge 3$. Then $K_2(R; d)$ is generated by the Steinberg symbols $\{u, v\}$ for $u, v \in R^*$. In particular, $K_2(R; d)$ is central in* $\mathrm{St}_d(R)$.

Now let R be a commutative ring, $H \lhd R$ an ideal and $d \ge 3$. Let $\bar{R} = R/H$. Clearly, there is a natural map $E_d(R) \to E_d(\bar{R})$. It is clear that $E_d(R, H)$ lies in the kernel of the latter map, so we have a map

$$\pi_d : E_d(R)/E_d(R, H) \to E_d(\bar{R}).$$

In addition, it is easy to see that we have a surjective map

$$\psi_d : \mathrm{St}_d(\bar{R}) \twoheadrightarrow E_d(R)/E_d(R, H)$$

defined by $x_{i,j}(\bar{r}) \mapsto I_d + rE_{i,j}$ such that $\phi_d : \mathrm{St}_d(\bar{R}) \to E_d(\bar{R})$ satisfies $\phi_d = \pi_d \circ \psi_d$. Therefore, we obtain the surjective map

$$\begin{aligned} K_2(\bar{R}; d) = \ker(\phi_d) \xrightarrow{\psi_d}\!\!\!\!\rightarrow \ker(\pi_d) &= (E_d(R) \cap \mathrm{SL}_d(R, H))/E_d(R, H) \\ &\le \mathrm{SK}_1(R, H; d). \end{aligned}$$

In particular, it implies that if $E_d(R) = \mathrm{SL}_d(R)$, then we have a natural surjective map

$$K_2(R/H; d) \twoheadrightarrow \mathrm{SK}_1(R, H; d).$$

From this one can easily deduce the following corollary, which will be needed later in the paper.

Corollary 2.4. *Let R be a commutative ring, $H \lhd R$ an ideal of finite index and $d \ge 3$. Assume also that $E_d(R) = \mathrm{SL}_d(R)$.*

(1) $\mathrm{SK}_1(R, H; d)$ *is a finite group.*

(2) $\mathrm{SK}_1(R, H; d)$ *is central in* $\mathrm{GL}_d(R)/E_d(R, H)$.

(3) *Every element of* $\mathrm{SK}_1(R, H; d)$ *has a representative in* $\mathrm{SL}_d(R, H)$ *of the form*

$$\begin{pmatrix} A & 0 \\ 0 & I_{d-2} \end{pmatrix}$$

such that $A \in \mathrm{SL}_2(R, H)$.

Proof. The ring $\bar{R} = R/H$ is finite. In particular, $\bar{R}$ is Artinian and hence semilocal. Thus, by Theorem 2.3, $K_2(\bar{R}; d)$ is an abelian group which is generated by the Steinberg symbols $\{u, v\}$ for $u, v \in \bar{R}^*$. As $\bar{R}$ is finite, so is the number of the Steinberg symbols. From (2.2) we obtain that the order of any Steinberg symbol is finite. So $K_2(\bar{R}; d)$ is a finitely generated abelian group whose generators are of finite order. Thus, $K_2(\bar{R}; d)$ is finite. Moreover, as $\bar{R}$ is semilocal, Theorem 2.3 implies that $K_2(\bar{R}; d)$ is central $\mathrm{St}_d(\bar{R})$. Now, as we assume that $E_d(R) = \mathrm{SL}_d(R)$, we obtain that $\mathrm{SK}_1(R, H; d)$ is the image of $K_2(\bar{R}; d)$ under the surjective map

$$\mathrm{St}_d(\bar{R}) \twoheadrightarrow E_d(R)/E_d(R, H) = \mathrm{SL}_d(R)/E_d(R, H).$$

This implies part (1) and that $\mathrm{SK}_1(R, H; d)$ is central in $\mathrm{SL}_d(R)/E_d(R, H)$.

Moreover, as $d \geq 3$, we have $\{u, v\} = \{u, v\}_{1,2}$ for every $u, v \in \bar{R}^*$. Now, it is easy to check from the definition of the Steinberg symbols that the image of $\{u, v\}_{1,2}$ under the map $\mathrm{St}_d(\bar{R}) \twoheadrightarrow \mathrm{SL}_d(R)/E_d(R, H)$ is of the form

$$\begin{pmatrix} A & 0 \\ 0 & I_{d-2} \end{pmatrix} \cdot E_d(R, H) \tag{2.5}$$

for some $A \in \mathrm{SL}_2(R, H)$. So as $\mathrm{SK}_1(R, H; d)$ is generated by the images of the Steinberg symbols, the same holds for every element in $\mathrm{SK}_1(R, H; d)$. So we obtain part (3). Now, as $d \geq 3$ we can write

$$\mathrm{GL}_d(R) = \mathrm{SL}_d(R) \cdot \{I_d + (r-1)E_{3,3} \mid r \in R^*\}.$$

Observe also that mod $E_d(R, H)$, all the elements of the form $I_d + (r-1)E_{3,3}$ for $r \in R^*$ commute with all the elements of the form (2.5). Hence, the centrality of $\mathrm{SK}_1(R, H; d)$ in $\mathrm{SL}_d(R)/E_d(R, H)$ shows that actually $\mathrm{SK}_1(R, H; d)$ is central in $\mathrm{GL}_d(R)/E_d(R, H)$, as required in part (2). □

3. IA(Φ_n) and its subgroups

We start our discussion of the IA-automorphism group of the free metabelian group, $G = \mathrm{IA}(\Phi_n) = \ker\big(\mathrm{Aut}(\Phi_n) \to \mathrm{Aut}(\Phi_n/\Phi_n') = \mathrm{GL}_n(\mathbb{Z})\big)$, by presenting some of its properties and subgroups. We begin with the following notation:

- $\Phi = \Phi_n = F_n/F_n'' =$ the free metabelian group on n elements. Here F_n'' denotes the second derivative of F_n, the free group on n elements.

- $\Psi_m = \Phi/M_m$, where $M_m = (\Phi'\Phi^m)'(\Phi'\Phi^m)^m$.
- $\mathrm{IG}_m = G(M_m) = \ker(\mathrm{IA}(\Phi) \to \mathrm{Aut}(\Psi_m))$.
- $\mathrm{IA}_m = \bigcap\{N \lhd \mathrm{IA}(\Phi) \mid [\mathrm{IA}(\Phi) : N] \mid m\}$.
- $R_n = \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, where $x_1, \dots, x_n$ are the generators of $\mathbb{Z}^n$.
- $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$.
- $\sigma_i = x_i - 1$ for $1 \le i \le n$.
- $\vec{\sigma} =$ the column vector which has σ_i in its i-th entry.
- $\mathfrak{A} = \sum_{i=1}^n \sigma_i R_n =$ the augmentation ideal of R_n.
- $H_m = \ker\big(R_n \to \mathbb{Z}_m[\mathbb{Z}_m^n]\big) = \sum_{i=1}^n (x_i^m - 1)R_n + mR_n$.

By the well-known Magnus embedding [Birman 1974; Remeslennikov and Sokolov 1970; Magnus 1939], one can identify Φ with the matrix group

$$\Phi = \left\{ \begin{pmatrix} g & a_1t_1 + \cdots + a_nt_n \\ 0 & 1 \end{pmatrix} \;\middle|\; g \in \mathbb{Z}^n,\ a_i \in R_n,\ g - 1 = \sum_{i=1}^n a_i\sigma_i \right\},$$

where t_i is a free basis for an R_n-module, under the identification of the generators of Φ with the matrices

$$\begin{pmatrix} x_i & t_i \\ 0 & 1 \end{pmatrix} \quad \text{for } 1 \le i \le n.$$

Moreover, for every $\alpha \in \mathrm{IA}(\Phi)$, one can describe α by its action on the generators of Φ by

$$\alpha : \begin{pmatrix} x_i & t_i \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} x_i & a_{i,1}t_1 + \cdots + a_{i,n}t_n \\ 0 & 1 \end{pmatrix}.$$

This description gives an injective homomorphism (see [Bachmuth 1965; Birman 1974])

$$\mathrm{IA}(\Phi) \hookrightarrow \mathrm{GL}_n(R_n), \qquad \alpha \mapsto \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix},$$

which gives an identification of $\mathrm{IA}(\Phi)$ with the group

$$\mathrm{IA}(\Phi) = \{A \in \mathrm{GL}_n(R_n) \mid A\vec{\sigma} = \vec{\sigma}\} = \{I_n + A \in \mathrm{GL}_n(R_n) \mid A\vec{\sigma} = \vec{0}\}.$$

Consider now the map

$$\begin{array}{c} \Phi = \left\{ \begin{pmatrix} g & a_1t_1 + \cdots + a_nt_n \\ 0 & 1 \end{pmatrix} \;\middle|\; g \in \mathbb{Z}^n,\ a_i \in R_n,\ g - 1 = \sum_{i=1}^n a_i\sigma_i \right\} \\ \downarrow \\ \left\{ \begin{pmatrix} g & a_1t_1 + \cdots + a_nt_n \\ 0 & 1 \end{pmatrix} \;\middle|\; g \in \mathbb{Z}_m^n,\ a_i \in \mathbb{Z}_m[\mathbb{Z}_m^n],\ g - 1 = \sum_{i=1}^n a_i\sigma_i \right\} \end{array}$$

which is induced by the projections $\mathbb{Z}^n \to \mathbb{Z}_m^n$, $R_n = \mathbb{Z}[\mathbb{Z}^n] \to \mathbb{Z}_m[\mathbb{Z}_m^n]$. Using a result of Romanovskii [1999], it is shown in [Ben-Ezra 2016] that this map is surjective and that Ψ_m is canonically isomorphic to its image. Therefore, we can identify IG_m, the principal congruence subgroup of $\mathrm{IA}(\Phi)$, with

$$\begin{aligned}\mathrm{IG}_m &= \left\{A \in \ker\bigl(\mathrm{GL}_n(R_n) \to \mathrm{GL}_n(\mathbb{Z}_m[\mathbb{Z}_m^n])\bigr) \mid A\vec{\sigma} = \vec{\sigma}\right\},\\ &= \left\{I_n + A \in \mathrm{GL}_n(R_n, H_m) \mid A\vec{\sigma} = \vec{0}\right\}.\end{aligned}$$

Proposition 3.1. *Let $I_n + A \in \mathrm{IA}(\Phi)$ and denote the entries of A by $a_{k,l}$ for $1 \le k, l \le n$. Then for every $1 \le k, l \le n$, we have $a_{k,l} \in \sum_{l\neq i=1}^{n} \sigma_i R_n \subseteq \mathfrak{A}$.*

Proof. For a given $1 \le k \le n$, the condition $A\vec{\sigma} = \vec{0}$ gives the equality

$$0 = a_{k,1}\sigma_1 + a_{k,2}\sigma_2 + \cdots + a_{k,n}\sigma_n.$$

Thus, for a given $1 \le l \le n$, the map $R_n \to S_l = \mathbb{Z}[x_l^{\pm 1}]$ defined by $x_i \mapsto 1$ for every $i \neq l$ maps

$$0 = a_{k,1}\sigma_1 + a_{k,2}\sigma_2 + \cdots + a_{k,n}\sigma_n \mapsto \bar{a}_{k,l}\sigma_l \in \mathbb{Z}[x_l^{\pm 1}].$$

Hence, as $\mathbb{Z}[x_l^{\pm 1}]$ is a domain, $\bar{a}_{k,l} = 0 \in \mathbb{Z}[x_l^{\pm 1}]$. Thus $a_{k,l} \in \sum_{l\neq i=1}^{n} \sigma_i R_n \subseteq \mathfrak{A}$, as required. □

Proposition 3.2. *Let $I_n + A \in \mathrm{IA}(\Phi)$. Then $\det(I_n + A)$ is of the form*

$$\det(I_n + A) = \prod_{r=1}^{n} x_r^{s_r} \quad \textit{for some } s_r \in \mathbb{Z}.$$

Proof. The invertible elements in R_n are the elements of the form $\pm \prod_{i=1}^{n} x_r^{s_r}$; see [Crowell and Fox 1963, Chapter 8]. Thus, as $I_n + A \in \mathrm{GL}_n(R_n)$ we have $\det(I_n + A) = \pm \prod_{i=1}^{n} x_r^{s_r}$. However, according to Proposition 3.1, for every entry $a_{k,l}$ of A we have $a_{k,l} \in \mathfrak{A}$. Hence, under the projection $x_i \mapsto 1$ for every $1 \le i \le n$, one has $I_n + A \mapsto I_n$, and thus, $\pm \prod_{i=1}^{n} x_r^{s_r} = \det(I_n + A) \mapsto \det(I_n) = 1$. Therefore, the option $\det(I_n + A) = -\prod_{i=1}^{n} x_r^{s_r}$ is impossible, as required. □

Let us step forward with the following definition:

Definition 3.3. Let $A \in \mathrm{GL}_n(R_n)$, and for $1 \le i \le n$, denote by $A_{i,i}$ the minor which is obtained from A by erasing its i-th row and i-th column. Now, for every $1 \le i \le n$, define the subgroup $\mathrm{IGL}_{n-1,i} \le \mathrm{IA}(\Phi)$ by

$$\mathrm{IGL}_{n-1,i} = \left\{ I_n + A \in \mathrm{IA}(\Phi) \;\middle|\; \begin{array}{l} \text{the } i\text{-th row of } A \text{ is } 0, \\ \qquad I_{n-1} + A_{i,i} \in \mathrm{GL}_{n-1}(R_n, \sigma_i R_n) \end{array} \right\}.$$

Proposition 3.4. *For every $1 \le i \le n$, we have $\mathrm{IGL}_{n-1,i} \cong \mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$.*

Proof. The definition of $\mathrm{IGL}_{n-1,i}$ gives us a natural projection

$$\mathrm{IGL}_{n-1,i} \to \mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$$

which maps an element $I_n + A \in \mathrm{IGL}_{n-1,i}$ to $I_{n-1} + A_{i,i} \in \mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$. Thus, all we need is to explain why this map is injective and surjective.

Injectivity: Here, it is enough to show that given an element $I_n + A \in \mathrm{IA}(\Phi)$, every entry in the i-th column is determined uniquely by the other entries in its row. Indeed, as A satisfies the condition $A\vec{\sigma} = \vec{0}$, for every $1 \le k \le n$ we have

$$a_{k,1}\sigma_1 + a_{k,2}\sigma_2 + \cdots + a_{k,n}\sigma_n = 0 \quad \Rightarrow \quad a_{k,i} = \frac{-\sum_{i\neq l=1}^{n} a_{k,l}\sigma_l}{\sigma_i}, \tag{3.5}$$

i.e., we have a formula for $a_{k,i}$ in terms of the other entries in its row.

Surjectivity: Without loss of generality we assume $i = n$. Let $I_{n-1} + \sigma_n B$ be in $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$, and denote by $\vec{b}_l$ the column vectors of B. Define

$$\begin{pmatrix} I_{n-1} + \sigma_n B & -\sum_{l=1}^{n-1} \sigma_l \vec{b}_l \\ 0 & 1 \end{pmatrix} \in \mathrm{IGL}_{n-1,n}\,.$$

This is clearly a preimage of $I_{n-1} + \sigma_n B$. □

Under the above identification of $\mathrm{IGL}_{n-1,i}$ with $\mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$, we will use throughout the paper the following notation:

Definition 3.6. Let $H \lhd R_n$. We define

$$\begin{aligned} \mathrm{ISL}_{n-1,i}(H) &= \mathrm{IGL}_{n-1,i} \cap \mathrm{SL}_{n-1}(R_n, H), \\ \mathrm{IE}_{n-1,i}(H) &= \mathrm{IGL}_{n-1,i} \cap E_{n-1}(R_n, H) \le \mathrm{ISL}_{n-1,i}(H). \end{aligned}$$

Observe that as for every $1 \le i \le n$ we have

$$\mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}]) \le \mathrm{GL}_{n-1}(R_n, \sigma_i R_n),$$

the isomorphism $\mathrm{GL}_{n-1}(R_n, \sigma_i R_n) \cong \mathrm{IGL}_{n-1,i} \le \mathrm{IA}(\Phi)$ gives also a natural embedding of $\mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}])$ as a subgroup of $\mathrm{IA}(\Phi)$.

Proposition 3.7. *For every $1 \le i \le n$, there is a canonical surjective homomorphism*

$$\rho_i : \mathrm{IA}(\Phi) \twoheadrightarrow \mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}])$$

such that the following composition map is the identity:

$$\mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}]) \hookrightarrow \mathrm{IA}(\Phi) \xrightarrow{\rho_i} \mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}]).$$

Hence $\mathrm{IA}(\Phi) = \ker \rho_i \rtimes \mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}])$.

Proof. Without loss of generality we assume $i = n$. First, consider the homomorphism $\mathrm{IA}(\Phi) \to \mathrm{GL}_n(\mathbb{Z}[x_n^{\pm 1}])$, which is induced by the projection $R_n \to \mathbb{Z}[x_n^{\pm 1}]$ that is defined by $x_j \mapsto 1$ for every $j \neq n$. By Proposition 3.1, given $I_n + A \in \mathrm{IA}(\Phi)$, all the entries of the n-th column of A are in $\sum_{j=1}^{n-1} \sigma_j R_n$. Hence, the above map $\mathrm{IA}(\Phi) \to \mathrm{GL}_n(\mathbb{Z}[x_n^{\pm 1}])$ is actually a map

$$\mathrm{IA}(\Phi) \to \left\{ I_n + \bar{A} \in \mathrm{GL}_n(\mathbb{Z}[x_n^{\pm 1}]) \mid \text{the } n\text{-th column of } \bar{A} \text{ is } \vec{0} \right\}.$$

Observe now that the right side of the above map can be mapped naturally to $\mathrm{GL}_{n-1}(\mathbb{Z}[x_n^{\pm 1}])$ by erasing the n-th column and the n-th row from every element. Hence we obtain a map

$$\mathrm{IA}(\Phi) \to \mathrm{GL}_{n-1}(\mathbb{Z}[x_n^{\pm 1}]).$$

Now, by Proposition 3.1, every entry of A such that $I_n + A \in \mathrm{IA}(\Phi)$ is in $\mathfrak{A}$. Thus, the entries of every $\bar{A}$ such that $I_{n-1} + \bar{A} \in \mathrm{GL}_{n-1}(\mathbb{Z}[x_n^{\pm 1}])$ is an image of $I_n + A \in \mathrm{IA}(\Phi)$ are all in $\sigma_n \mathbb{Z}[x_n^{\pm 1}]$. Hence, we actually obtain a homomorphism

$$\rho_n : \mathrm{IA}(\Phi) \to \mathrm{GL}_{n-1}(\mathbb{Z}[x_n^{\pm 1}], \sigma_n \mathbb{Z}[x_n^{\pm 1}]).$$

We conclude by observing that the copy of $\mathrm{GL}_{n-1}(\mathbb{Z}[x_n^{\pm 1}], \sigma_n \mathbb{Z}[x_n^{\pm 1}])$ in $\mathrm{IGL}_{n-1,n}$ is mapped isomorphically to itself by ρ_n. $\square$

Proposition 3.8. *Write $S_i = \mathbb{Z}[x_i^{\pm 1}] \subseteq R_n$ and $J_{i,m} = (x_i^m - 1)S_i + mS_i \subseteq H_m$ for $1 \le i \le n$. Then, by identifying*

$$\mathrm{Im}(\rho_i) \cong \mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i \mathbb{Z}[x_i^{\pm 1}]) = \mathrm{GL}_{n-1}(S_i, \sigma_i S_i),$$

for every $m \in \mathbb{N}$ one has

$$\mathrm{Im}(\rho_i) \cap \mathrm{IG}_m = \mathrm{GL}_{n-1}(S_i, \sigma_i J_{i,m}).$$

Proof. By the identification

$$\mathrm{IG}_m = \{ I_n + A \in \mathrm{GL}_n(R_n, H_m) \mid A\vec{\sigma} = \vec{0} \}$$

and by applying the formula of (3.5) to the i-th column of elements in $\mathrm{IGL}_{n-1,i}$, it is easy to see that the elements of $\mathrm{IGL}_{n-1,i}$ which correspond to the elements of $\mathrm{GL}_{n-1}(S_i, \sigma_i J_{i,m})$ are clearly in $\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m$. For the opposite inclusion, without loss of generality assume that $i = n$, and let $I_n + A \in \mathrm{Im}\,\rho_n \cap \mathrm{IG}_m$. Then $I_n + A$ has the form

$$\begin{pmatrix} I_{n-1} + \sigma_n B & -\sum_{l=1}^{n-1} \sigma_l \vec{b}_l \\ 0 & 1 \end{pmatrix} \in \mathrm{IGL}_{n-1,n},$$

where the entries of B satisfy $b_{k,l} \in S_n$ and $\sum_{j=1}^{n-1} \sigma_j b_{k,j} \in H_m$. Notice now that for every $l \neq n$, by projecting $\sigma_j \mapsto 0$ for $j \neq l, n$, we see that actually $\sigma_l b_{k,l} \in H_m$.

From here it is easy to see that we necessarily have $b_{k,l} \in H_m$, i.e.,

$$b_{k,l} \in H_m \cap S_n = (x_n^m - 1)S_n + mS_n = J_{n,m},$$

and the claim follows. $\square$

Proposition 3.9. *For every* $1 \le i \le n$ *and* $m \in \mathbb{N}$ *one has*

$$\rho_i(\mathrm{IG}_{m^2}) \subseteq \mathrm{Im}(\rho_i) \cap \mathrm{IG}_m \subseteq \rho_i(\mathrm{IG}_m).$$

Proof. As every element in $\mathrm{Im}\,\rho_i$ is mapped to itself via ρ_i we clearly have

$$\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m = \rho_i(\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m) \subseteq \rho_i(\mathrm{IG}_m).$$

On the other hand, if $I_n + A \in \mathrm{IG}_{m^2}$ then viewing $\mathrm{Im}\,\rho_i \cong \mathrm{GL}_{n-1}(S_i, \sigma_i S_i)$ for $S_i = \mathbb{Z}[x_i^{\pm 1}]$, the entries of $\rho_i(I_n + A) = I_{n-1} + B$ belong to $(x_i^{m^2} - 1)S_i + m^2\sigma_i S_i$. Observe now that we have $\sum_{r=0}^{m-1} x_i^{mr} \subseteq (x_i^m - 1)S_i + mS_i = J_{i,m}$. Hence

$$x_i^{m^2} - 1 = \sigma_i \sum_{r=1}^{m^2-1} x_i^r = \sigma_i \sum_{r=0}^{m-1} x_i^r \sum_{r=0}^{m-1} x_i^{mr} \in \sigma_i J_{i,m}. \tag{3.10}$$

So by Proposition 3.8, $\rho_i(I_n + A) \in \mathrm{Im}\,\rho_i \cap \mathrm{IG}_m$, as required. $\square$

Proposition 3.11. *For every* $m \in \mathbb{N}$ *and* $1 \le i \le n$ *one has*

$$\rho_i(\mathrm{IA}_m) = \mathrm{Im}(\rho_i) \cap \mathrm{IA}_m,$$

where $\mathrm{IA}_m = \bigcap \{N \lhd \mathrm{IA}(\Phi) \mid [\mathrm{IA}(\Phi) : N] \mid m\}$.

Proof. As every element in $\mathrm{Im}\,\rho_i$ is mapped to itself via ρ_i, we clearly have $\mathrm{Im}\,\rho_i \cap \mathrm{IA}_m = \rho_i(\mathrm{Im}\,\rho_i \cap \mathrm{IA}_m) \subseteq \rho_i(\mathrm{IA}_m)$. For the opposite, assume that $\alpha \in \mathrm{IA}_m$, and let $\rho_i(\alpha) = \beta \in \mathrm{Im}\,\rho_i$. We want to show that $\beta \in \mathrm{IA}_m$. So let $N \lhd \mathrm{IA}(\Phi)$ such that $[\mathrm{IA}(\Phi) : N] \mid m$. Then obviously $[\mathrm{Im}\,\rho_i : (N \cap \mathrm{Im}\,\rho_i)] \mid m$. Thus, as ρ_i is surjective, $[\mathrm{IA}(\Phi) : \rho_i^{-1}(N \cap \mathrm{Im}\,\rho_i)] \mid m$ so $\alpha \in \rho_i^{-1}(N \cap \mathrm{Im}\,\rho_i)$ and hence $\beta = \rho_i(\alpha) \in N \cap \mathrm{Im}\,\rho_i \le N$. As this is valid for every such N, we have $\beta \in \mathrm{IA}_m$, as required. $\square$

We close this section with the following definition:

Definition 3.12. For every $1 \le i \le n$, define

$$\mathrm{IGL}'_{n-1,i} = \{I_n + A \in \mathrm{IA}(\Phi) \mid \text{the } i\text{-th row of } A \text{ is } 0\}.$$

Obviously, $\mathrm{IGL}_{n-1,i} \le \mathrm{IGL}'_{n-1,i}$, and by the same injectivity argument as in the proof of Proposition 3.4, one can deduce the next proposition:

Proposition 3.13. *The subgroup* $\mathrm{IGL}'_{n-1,i} \le \mathrm{IA}(\Phi)$ *is canonically embedded in* $\mathrm{GL}_{n-1}(R_n)$ *by the map* $I_n + A \mapsto I_{n-1} + A_{i,i}$.

Remark 3.14. Note that in general $\mathrm{IGL}_{n-1,i} \lneqq \mathrm{IGL}'_{n-1,i}$. For example,

$$I_4 + \sigma_3 E_{1,2} - \sigma_2 E_{1,3} \in \mathrm{IGL}'_{3,4} \setminus \mathrm{IGL}_{3,4}\,.$$

4. The subgroups C_i

In this section we define the subgroups $C_i \le C(\mathrm{IA}(\Phi_n), \Phi_n)$, and we show that for each i we can view $C(\mathrm{IA}(\Phi_n), \Phi_n)$ as a semidirect product of C_i with another subgroup. We also show that when $n \ge 4$,

$$C_i \cong \ker\bigl(\widehat{\mathrm{SL}_{n-1}(\mathbb{Z}[x^{\pm 1}])} \to \mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x^{\pm 1}]})\bigr)$$

and use it to show that $C(\mathrm{IA}(\Phi_n), \Phi_n)$ is not finitely generated. Recall the notation

- $\Phi = \Phi_n$,
- $\Psi_m = \Phi/M_m$, where $M_m = (\Phi'\Phi^m)'(\Phi'\Phi^m)^m$,
- $\mathrm{IG}_m = G(M_m) = \ker(\mathrm{IA}(\Phi) \to \mathrm{Aut}(\Psi_m))$,
- $\mathrm{IA}_m = \bigcap\bigl\{N \lhd \mathrm{IA}(\Phi) \mid [\mathrm{IA}(\Phi) : N] \mid m\bigr\}$.

It is proven in [Ben-Ezra 2016] that $\hat{\Phi} = \varprojlim \Psi_m$. So, as for every $m \in \mathbb{N}$ the group $\ker(\Phi \to \Psi_m)$ is characteristic in Φ, we can write explicitly

$$\begin{aligned} C(\mathrm{IA}(\Phi), \Phi) &= \ker\bigl(\widehat{\mathrm{IA}(\Phi)} \to \mathrm{Aut}(\hat{\Phi})\bigr) \\ &= \ker\bigl(\widehat{\mathrm{IA}(\Phi)} \to \varprojlim \mathrm{Aut}(\Psi_m)\bigr) \\ &= \ker\bigl(\widehat{\mathrm{IA}(\Phi)} \to \varprojlim(\mathrm{IA}(\Phi)/\mathrm{IG}_m)\bigr). \end{aligned}$$

Now, as for every $n \ge 4$ we know that $\mathrm{IA}(\Phi)$ is finitely generated (see [Bachmuth and Mochizuki 1985]), as explained in Remark 1.5, we have

$$\widehat{\mathrm{IA}(\Phi)} = \varprojlim(\mathrm{IA}(\Phi)/\mathrm{IA}_m).$$

Hence

$$\begin{aligned} C(\mathrm{IA}(\Phi), \Phi) &= \ker\bigl(\varprojlim(\mathrm{IA}(\Phi)/\mathrm{IA}_m) \to \varprojlim(\mathrm{IA}(\Phi)/\mathrm{IG}_m)\bigr) \\ &= \ker\bigl(\varprojlim(\mathrm{IA}(\Phi)/\mathrm{IA}_m) \to \varprojlim(\mathrm{IA}(\Phi)/\mathrm{IG}_m \cdot \mathrm{IA}_m)\bigr) \\ &= \varprojlim(\mathrm{IA}_m \cdot \mathrm{IG}_m / \mathrm{IA}_m). \end{aligned}$$

Similarly, we can write $C(\mathrm{IA}(\Phi), \Phi) = \varprojlim(\mathrm{IA}_m \cdot \mathrm{IG}_{m^2} / \mathrm{IA}_m)$.

Remember now that for every $1 \le i \le n$ the composition map

$$\mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}]) \hookrightarrow \mathrm{IA}(\Phi) \xrightarrow{\rho_i} \mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}])$$

is the identity on $\mathrm{GL}_{n-1}(\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}])$. Hence, the induced composition map of the profinite completions

$$\mathrm{GL}_{n-1}\bigl(\widehat{\mathbb{Z}[x_i^{\pm 1}], \sigma_i}\mathbb{Z}[x_i^{\pm 1}]\bigr) \xrightarrow{\hat{\varrho}} \widehat{\mathrm{IA}(\Phi)} \xrightarrow{\hat{\rho}_i} \mathrm{GL}_{n-1}\bigl(\widehat{\mathbb{Z}[x_i^{\pm 1}], \sigma_i}\mathbb{Z}[x_i^{\pm 1}]\bigr)$$

is the identity on $\mathrm{GL}_{n-1}\big(\widehat{\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}]}\big)$. In particular, the map $\hat{\varrho}$ is injective, so we can write

$$\mathrm{GL}_{n-1}\big(\widehat{\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}]}\big) \hookrightarrow \widehat{\mathrm{IA}(\Phi)} \xrightarrow{\hat{\rho}_i} \mathrm{GL}_{n-1}\big(\widehat{\mathbb{Z}[x_i^{\pm 1}], \sigma_i\mathbb{Z}[x_i^{\pm 1}]}\big).$$

This enables us to write $\mathrm{IA}(\Phi) = \ker \rho_i \rtimes \mathrm{Im}\, \rho_i$ and $\widehat{\mathrm{IA}(\Phi)} = \ker \hat{\rho}_i \rtimes \mathrm{Im}\, \hat{\rho}_i$.

Definition 4.1. We define

$$C_i = C(\mathrm{IA}(\Phi), \Phi) \cap \mathrm{Im}\, \hat{\rho}_i = \ker(\mathrm{Im}\, \hat{\rho}_i \to \mathrm{Aut}(\hat{\Phi})).$$

Proposition 4.2. *If $1 \le i \ne j \le n$, then $C_i \subseteq \ker \hat{\rho}_j$. In particular, for every $i \ne j$ we have $C_i \cap C_j = \{e\}$.*

Proof. By the explicit description $\widehat{\mathrm{IA}(\Phi)} = \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IA}_m)$, one can write

$$\begin{aligned}
C_i &= \ker(\mathrm{Im}\, \hat{\rho}_i \to \mathrm{Aut}(\hat{\Phi}))\\
&= \ker\big(\varprojlim(\mathrm{IA}_m \cdot \mathrm{Im}\, \rho_i /\, \mathrm{IA}_m) \to \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IG}_m)\big)\\
&= \ker\big(\varprojlim(\mathrm{IA}_m \cdot \mathrm{Im}\, \rho_i /\, \mathrm{IA}_m) \to \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IG}_m \cdot \mathrm{IA}_m)\big)\\
&= \varprojlim((\mathrm{IA}_m \cdot \mathrm{Im}\, \rho_i) \cap (\mathrm{IA}_m \cdot \mathrm{IG}_m))/\,\mathrm{IA}_m,
\end{aligned}$$

and similarly, $C_i = \varprojlim((\mathrm{IA}_m \cdot \mathrm{Im}\, \rho_i) \cap (\mathrm{IA}_m \cdot \mathrm{IG}_{m^2}))/\,\mathrm{IA}_m$. We claim now that

$$\begin{aligned}
(\mathrm{IA}_m \cdot \mathrm{Im}\, \rho_i) \cap (\mathrm{IA}_m \cdot \mathrm{IG}_{m^2}) &\subseteq \mathrm{IA}_m \cdot (\mathrm{Im}\, \rho_i \cap \mathrm{IG}_m)\\
&\subseteq (\mathrm{IA}_m \cdot \mathrm{Im}\, \rho_i) \cap (\mathrm{IA}_m \cdot \mathrm{IG}_m).
\end{aligned}$$

The second inclusion is obvious. For the first one, we have to show that if $ar = bs$ such that $a, b \in \mathrm{IA}_m$, $r \in \mathrm{Im}\, \rho_i$ and $s \in \mathrm{IG}_{m^2}$, then there exist $c \in \mathrm{IA}_m$ and $t \in \mathrm{Im}\, \rho_i \cap \mathrm{IG}_m$ such that $ar = bs = ct$. Indeed, write $\mathrm{Im}\, \rho_i \ni r = a^{-1}bs$. Then $r = \rho_i(r) = \rho_i(a^{-1}b)\rho_i(s)$, and by Propositions 3.9 and 3.11,

$$\rho_i(a^{-1}b) \in \rho_i(\mathrm{IA}_m) = \mathrm{Im}\, \rho_i \cap \mathrm{IA}_m, \qquad \rho_i(s) \in \rho_i(\mathrm{IG}_{m^2}) \subseteq \mathrm{Im}\, \rho_i \cap \mathrm{IG}_m\,.$$

Therefore, by defining $c = a \cdot \rho_i(a^{-1}b)$ and $t = \rho_i(s)$ we get the required inclusion. Thus, for $j \ne i$ we have

$$C_i = \varprojlim(\mathrm{IA}_m \cdot (\mathrm{Im}\, \rho_i \cap \mathrm{IG}_m)/\,\mathrm{IA}_m) \xrightarrow{\hat{\rho}_j} \varprojlim \rho_j(\mathrm{IA}_m) \cdot \rho_j(\mathrm{Im}\, \rho_i \cap \mathrm{IG}_m)/\rho_j(\mathrm{IA}_m).$$

Using the definition of ρ_j, it is not difficult to show that

$$\begin{aligned}
\rho_j(\mathrm{Im}\, \rho_i \cap \mathrm{IG}_m) &= \big\langle I_n + m(\sigma_i E_{k,j} - \sigma_j E_{k,i}) \mid k \ne i, j\big\rangle\\
&= \rho_j\big(\big\langle I_n + m(\sigma_i E_{k,j} - \sigma_j E_{k,i}) \mid k \ne i, j\big\rangle\big)\\
&= \rho_j\big(\big\langle I_n + \sigma_i E_{k,j} - \sigma_j E_{k,i} \mid k \ne i, j\big\rangle^m\big) \subseteq \rho_j(\mathrm{IA}_m).
\end{aligned}$$

Hence, $C_i \subseteq \ker \hat{\rho}_j$, as required. $\square$

We can now prove the following proposition:

Proposition 4.3. *For every* $1 \le i \le n$ *we have*

$$C_i \hookrightarrow C(\mathrm{IA}(\Phi), \Phi) \xrightarrow{\hat{\rho}_i} C_i.$$

In particular, $C(\mathrm{IA}(\Phi), \Phi) = (\ker \hat{\rho}_i \cap C(\mathrm{IA}(\Phi), \Phi)) \rtimes C_i$.

Proof. In the proof of Proposition 4.2 we saw that

$$C_i = \varprojlim(\mathrm{IA}_m \cdot (\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m)/\,\mathrm{IA}_m).$$

Similarly, $C_i = \varprojlim(\mathrm{IA}_m \cdot (\mathrm{Im}\,\rho_i \cap \mathrm{IG}_{m^2})/\,\mathrm{IA}_m)$. We recall that by Propositions 3.9 and 3.11, we have

$$\rho_i(\mathrm{IG}_{m^2}) \subseteq \mathrm{Im}\,\rho_i \cap \mathrm{IG}_m \subseteq \rho_i(\mathrm{IG}_m), \qquad \rho_i(\mathrm{IA}_m) = \mathrm{Im}\,\rho_i \cap \mathrm{IA}_m\,.$$

Therefore, we have

$$\begin{aligned} C_i &= \varprojlim \mathrm{IA}_m \cdot (\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m)/\,\mathrm{IA}_m = \varprojlim \mathrm{IA}_m \cdot (\mathrm{Im}\,\rho_i \cap \mathrm{IG}_{m^2})/\,\mathrm{IA}_m \\ &\hookrightarrow \varprojlim \mathrm{IA}_m \cdot \mathrm{IG}_m\,/\,\mathrm{IA}_m = \varprojlim \mathrm{IA}_m \cdot \mathrm{IG}_{m^2}\,/\,\mathrm{IA}_m = C(\mathrm{IA}(\Phi), \Phi) \\ &\xrightarrow{\hat{\rho}_i} \varprojlim \rho_i(\mathrm{IA}_m) \cdot \rho_i(\mathrm{IG}_m)/\rho_i(\mathrm{IA}_m) = \varprojlim \rho_i(\mathrm{IA}_m) \cdot \rho_i(\mathrm{IG}_{m^2})/\rho_i(\mathrm{IA}_m) \\ &= \varprojlim(\mathrm{Im}\,\rho_i \cap \mathrm{IA}_m) \cdot (\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m)/(\mathrm{Im}\,\rho_i \cap \mathrm{IA}_m) \\ &= \varprojlim \mathrm{IA}_m \cdot (\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m)/\,\mathrm{IA}_m = C_i. \end{aligned}$$

The latter equality follows from the inclusion $\mathrm{Im}\,\rho_i \cap \mathrm{IG}_m \subseteq \mathrm{Im}\,\rho_i$. □

Computing C_i. We turn now to the computation of C_i. We are going to show that the C_i are canonically isomorphic to

$$\ker\bigl(\mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x^{\pm 1}]}) \to \mathrm{SL}_{n-1}(\widehat{\mathbb{Z}[x^{\pm 1}]})\bigr)$$

and then use that fact in order to show that $C(\mathrm{IA}(\Phi), \Phi)$ is not finitely generated. So fix $n \ge 4$, $1 \le i_0 \le n$, and let

- $x = x_{i_0}$,
- $\sigma = \sigma_{i_0} = x_{i_0} - 1$,
- $\mathrm{IGL}_{n-1} = \mathrm{IGL}_{n-1,i_0}$,
- $\mathrm{IE}_{n-1}(H) = \mathrm{IE}_{n-1,i_0}(H)$,
- $S = \mathbb{Z}[x^{\pm 1}] = \mathbb{Z}[x_{i_0}^{\pm 1}]$,
- $J_m = (x^m - 1)S + mS$ for $m \in \mathbb{N}$,
- $\rho = \rho_{i_0} : \mathrm{IA}(\Phi) \twoheadrightarrow \mathrm{GL}_{n-1}(S, \sigma S)$,
- $\hat{\rho} = \hat{\rho}_{i_0} : \widehat{\mathrm{IA}(\Phi)} \twoheadrightarrow \mathrm{GL}_{n-1}\widehat{(S, \sigma S)}$.

Now, write

$$\begin{aligned}
C_{i_0} &= \ker(\operatorname{Im}\hat{\rho} \to \operatorname{Aut}(\hat{\Phi})) \\
&= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \operatorname{Aut}(\hat{\Phi})\big) \\
&= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim(\mathrm{IA}(\Phi)/\,\mathrm{IG}_m)\big) \\
&= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim(\mathrm{GL}_{n-1}(S,\sigma S)\cdot \mathrm{IG}_m\,/\,\mathrm{IG}_m)\big) \\
&= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim \mathrm{GL}_{n-1}(S,\sigma S)/(\mathrm{GL}_{n-1}(S,\sigma S)\cap \mathrm{IG}_m)\big) \\
&= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim \mathrm{GL}_{n-1}(S,\sigma S)/\,\mathrm{GL}_{n-1}(S,\sigma J_m)\big)
\end{aligned}$$

(the last equality is by Proposition 3.8). Now, by the same computation as in Proposition 3.9 one can show that for every $m \in \mathbb{N}$ we have

$$(J_{m^2}\cap \sigma S) \subseteq \sigma J_m \subseteq (J_m \cap \sigma S),$$

so the latter is equal to

$$\begin{aligned}
&\ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim \mathrm{GL}_{n-1}(S,\sigma S)/(\mathrm{GL}_{n-1}(S,\sigma S)\cap \mathrm{GL}_{n-1}(S,\,J_m))\big) \\
&\quad= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim(\mathrm{GL}_{n-1}(S,\sigma S)\cdot \mathrm{GL}_{n-1}(S,\,J_m))/\,\mathrm{GL}_{n-1}(S,\,J_m)\big) \\
&\quad= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim \mathrm{GL}_{n-1}(S)/\,\mathrm{GL}_{n-1}(S,\,J_m)\big) \\
&\quad= \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \varprojlim \mathrm{GL}_{n-1}(S/J_m)\big).
\end{aligned}$$

Now, if $\bar{S}$ is a finite quotient of S, then as x is invertible in S, its image $\bar{x} \in \bar{S}$ is invertible in $\bar{S}$. Thus, there exists $r \in \mathbb{N}$ such that $\bar{x}^r = 1_{\bar{S}}$. In addition, there exists $t \in \mathbb{N}$ such that

$$\underbrace{1_{\bar{S}} + \cdots + 1_{\bar{S}}}_{t} = 0_{\bar{S}}.$$

Therefore, for $m = r\cdot t$ the map $S \to \bar{S}$ factorizes through $\mathbb{Z}_m[\mathbb{Z}_m] \cong S/J_m$. Thus, we have $\hat{S} = \varprojlim(S/J_m)$, which implies that $\mathrm{GL}_{n-1}(\hat{S}) = \varprojlim \mathrm{GL}_{n-1}(S/J_m)$. Therefore,

$$C_{i_0} = \ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \mathrm{GL}_{n-1}(\hat{S})\big).$$

Now, the short exact sequence

$$1 \to \mathrm{GL}_{n-1}(S,\sigma S) \to \mathrm{GL}_{n-1}(S) \to \mathrm{GL}_{n-1}(\mathbb{Z}) \to 1$$

gives rise to the exact sequence (see [Bux et al. 2011, Lemma 2.1])

$$\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \widehat{\mathrm{GL}_{n-1}(S)} \to \widehat{\mathrm{GL}_{n-1}(\mathbb{Z})} \to 1,$$

which gives rise to the commutative diagram

$$\begin{tikzcd}
\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \arrow[r] \arrow[dr] & \widehat{\mathrm{GL}_{n-1}(S)} \arrow[r] \arrow[d] & \widehat{\mathrm{GL}_{n-1}(\mathbb{Z})} \arrow[r] \arrow[d] & 1 \\
& \mathrm{GL}_{n-1}(\hat{S}) \arrow[r] & \mathrm{GL}_{n-1}(\hat{\mathbb{Z}}) \arrow[r] & 1
\end{tikzcd}$$

Assuming that $n \geq 4$ and using the affirmative answer to the classical congruence subgroup problem [Mennicke 1965; Bass et al. 1964], we have that the map $\widehat{\mathrm{GL}_{n-1}(\mathbb{Z})} \to \mathrm{GL}_{n-1}(\hat{\mathbb{Z}})$ is injective. Thus, by diagram chasing we obtain that $\ker\big(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)} \to \mathrm{GL}_{n-1}(\hat{S})\big)$ is mapped onto $\ker\big(\widehat{\mathrm{GL}_{n-1}(S)} \to \mathrm{GL}_{n-1}(\hat{S})\big)$. In order to proceed from here we need the following lemma.

Lemma 4.4. *Let $d \geq 3$ and $D_m = \{I_d + (x^{k\cdot m} - 1)E_{1,1} \mid k \in \mathbb{Z}\}$ for $m \in \mathbb{N}$. Then*

$$\widehat{\mathrm{GL}_d(S)} = \varprojlim\big(\mathrm{GL}_d(S)/(D_m E_d(S, J_m))\big),$$

$$\widehat{\mathrm{SL}_d(S)} = \varprojlim\big(\mathrm{SL}_d(S)/E_d(S, J_m)\big).$$

Proof. We prove the first statement; the second is similar but easier. We first claim that $D_m E_d(S, J_m)$ is a finite index normal subgroup of $\mathrm{GL}_d(S)$. Indeed, by a well-known result of Suslin [1977], $\mathrm{SL}_d(S) = E_d(S)$. Thus, by Corollary 2.4, $\mathrm{SK}_1(S, J_m; d) = \mathrm{SL}_d(S, J_m)/E_d(S, J_m)$ is finite. As the subgroup $\mathrm{SL}_d(S, J_m)$ is of finite index in $\mathrm{SL}_d(S)$, so is $E_d(S, J_m)$. Now, it is not difficult to see that the group of invertible elements of S is equal to $S^* = \{\pm x^k \mid k \in \mathbb{Z}\}$ (see [Crowell and Fox 1963, Chapter 8]). So as $\{x^{k\cdot m} \mid k \in \mathbb{Z}\}$ is of finite index in S^*, the subgroup $D_m\,\mathrm{SL}_d(S)$ is of finite index in $\mathrm{GL}_d(S)$. We deduce that also $D_m E_d(S, J_m)$ is of finite index in $\mathrm{GL}_d(S)$. It remains to show that $D_m E_d(S, J_m)$ is normal in $\mathrm{GL}_d(S)$.

We already stated previously (see Section 2) that $E_d(S, J_m)$ is normal in $\mathrm{GL}_d(S)$. Thus, noticing the group identity

$$gheg^{-1} = h(h^{-1}ghg^{-1})(geg^{-1}),$$

it is enough to show that the commutators of the elements of D_m with any set of generators of $\mathrm{GL}_d(S)$ are in $E_d(S, J_m)$. By the aforementioned result of Suslin and as $S^* = \{\pm x^r \mid r \in \mathbb{Z}\}$, the group $\mathrm{GL}_d(S)$ is generated by the elements of the forms

1. $I_d + (\pm x - 1)E_{1,1}$,
2. $I_d + rE_{i,j}$, $\quad r \in S,\ 2 \leq i \neq j \leq d$,
3. $I_d + rE_{1,j}$, $\quad r \in S,\ 2 \leq j \leq d$,
4. $I_d + rE_{i,1}$, $\quad r \in S,\ 2 \leq i \leq d$.

Now, obviously, the elements of D_m commute with the elements of the forms 1 and 2. In addition, for the elements of the forms 3 and 4, one can easily compute that

$$\begin{aligned}\left[I_d+(x^{k\cdot m}-1)E_{1,1},\, I_d+rE_{1,j}\right]&=I_d+r(x^{k\cdot m}-1)E_{1,j}\in E_d(S,\,J_m),\\ \left[I_d+(x^{k\cdot m}-1)E_{1,1},\, I_d+rE_{i,1}\right]&=I_d+r(x^{-k\cdot m}-1)E_{i,1}\in E_d(S,\,J_m)\end{aligned}$$

for every $2\le i,\, j\le d$, as required.

Now, clearly, every finite index normal subgroup of $\mathrm{GL}_d(S)$ contains D_m for some $m\in\mathbb{N}$. In addition, it is not hard to show that when $d\ge 3$, every finite index normal subgroup $N\lhd \mathrm{GL}_d(S)$ contains $E_d(S,\,J)$ for some finite index ideal $J\lhd S$; see [Kassabov and Nikolov 2006, Section 1]. Thus, as we saw previously that every finite index ideal $J\lhd S_n$ contains J_m for some m, we obtain that $\widehat{\mathrm{GL}_d(S)}=\varprojlim\bigl(\mathrm{GL}_d(S)/(D_mE_d(S,\,J_m))\bigr)$, as required. □

In order to prove the following proposition, we are going to use Lemma 7.1, the proof of which is left to the last section of the paper.

Proposition 4.5. *Let $n\ge 4$. Then the map $\widehat{\mathrm{GL}_{n-1}(S,\sigma S)}\to\widehat{\mathrm{GL}_{n-1}(S)}$ is injective. Hence, the surjective map*

$$C_{i_0}=\ker\bigl(\widehat{\mathrm{GL}_{n-1}(S,\sigma S)}\to\mathrm{GL}_{n-1}(\hat S)\bigr)\twoheadrightarrow\ker\bigl(\widehat{\mathrm{GL}_{n-1}(S)}\to\mathrm{GL}_{n-1}(\hat S)\bigr)$$

is an isomorphism.

Proof. We showed in the previous lemma that

$$\widehat{\mathrm{GL}_{n-1}(S)}=\varprojlim \mathrm{GL}_{n-1}(S)/(D_mE_{n-1}(S,\,J_m)),$$

where $D_m=\{I_{n-1}+(x^{k\cdot m}-1)E_{1,1}\mid k\in\mathbb{Z}\}$ and $J_m=(x^m-1)S+mS$. Hence, the image of $\widehat{\mathrm{GL}_{n-1}(S,\sigma S)}$ in $\widehat{\mathrm{GL}_{n-1}(S)}$ is

$$\begin{aligned}&\varprojlim(\mathrm{GL}_{n-1}(S,\sigma S)\cdot D_mE_{n-1}(S,\,J_m))/(D_mE_{n-1}(S,\,J_m))\\ &\qquad=\varprojlim \mathrm{GL}_{n-1}(S,\sigma S)/(\mathrm{GL}_{n-1}(S,\sigma S)\cap D_mE_{n-1}(S,\,J_m)).\end{aligned}$$

Using the fact that $D_m\subseteq \mathrm{GL}_{n-1}(S,\sigma S)$, one can see that the latter equals

$$\varprojlim \mathrm{GL}_{n-1}(S,\sigma S)/\bigl(D_m(\mathrm{GL}_{n-1}(S,\sigma S)\cap E_{n-1}(S,\,J_m))\bigr).$$

Recall now the following notation:

- $R_n=\mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1}]$.
- $H_m=\sum_{i=1}^n(x_i^m-1)R_n+mR_n\lhd R_n$.
- $\mathrm{IE}_{n-1}(H_m)=\mathrm{IGL}_{n-1}\cap E_{n-1}(R_n,\,H_m)$ under the identification of IGL_{n-1} with $\mathrm{GL}_{n-1}(R_n,\sigma R_n)$.

Then, following the definition of the map $\rho:\mathrm{IA}(\Phi)\twoheadrightarrow\mathrm{GL}_{n-1}(S,\sigma S)$, we have

$$\begin{aligned}\langle\mathrm{IA}(\Phi)^m\rangle&\overset{\rho}{\twoheadrightarrow}\langle\mathrm{GL}_{n-1}(S,\sigma S)^m\rangle,\\ \mathrm{IE}_{n-1}(H_m)&\overset{\rho}{\twoheadrightarrow}\mathrm{GL}_{n-1}(S,\sigma S)\cap E_{n-1}(S,\,J_m).\end{aligned}$$

So, since by the main lemma (Lemma 7.1) we have $\mathrm{IE}_{n-1}(H_{m^2}) \subseteq \langle \mathrm{IA}(\Phi)^m \rangle$, we have also

$$\mathrm{GL}_{n-1}(S, \sigma S) \cap E_{n-1}(S, J_{m^2}) \subseteq \langle \mathrm{GL}_{n-1}(S, \sigma S)^m \rangle.$$

As obviously $D_{m^2} \subseteq \langle \mathrm{GL}_{n-1}(S, \sigma S)^m \rangle$, we deduce the following natural surjective maps:

$$\begin{aligned}
&\varprojlim \mathrm{GL}_{n-1}(S, \sigma S)/\big(D_m(\mathrm{GL}_{n-1}(S, \sigma S) \cap E_{n-1}(S, J_m))\big) \\
&\qquad = \varprojlim \mathrm{GL}_{n-1}(S, \sigma S)/\big(D_{m^2}(\mathrm{GL}_{n-1}(S, \sigma S) \cap E_{n-1}(S, J_{m^2}))\big) \\
&\qquad\qquad \twoheadrightarrow \varprojlim \mathrm{GL}_{n-1}(S, \sigma S)/\langle \mathrm{GL}_{n-1}(S, \sigma S)^m \rangle \\
&\qquad\qquad \twoheadrightarrow \mathrm{GL}_{n-1}\widehat{(S, \sigma} S) \\
&\qquad\qquad \twoheadrightarrow \varprojlim \mathrm{GL}_{n-1}(S, \sigma S)/\big(D_m(\mathrm{GL}_{n-1}(S, \sigma S) \cap E_{n-1}(S, J_m))\big)
\end{aligned}$$

such that the composition gives the identity map. Hence, these maps are also injective, and in particular, the map

$$\mathrm{GL}_{n-1}\widehat{(S, \sigma} S) \twoheadrightarrow \varprojlim \mathrm{GL}_{n-1}(S, \sigma S)/\big(D_m(\mathrm{GL}_{n-1}(S, \sigma S) \cap E_{n-1}(S, J_m))\big)$$

is injective, as required. □

Proposition 4.6. *Let $d \geq 3$. Then the natural embedding $\mathrm{SL}_d(S) \leq \mathrm{GL}_d(S)$ induces a natural isomorphism*

$$\ker(\widehat{\mathrm{GL}_d(S)} \to \mathrm{GL}_d(\hat{S})) \cong \ker(\widehat{\mathrm{SL}_d(S)} \to \mathrm{SL}_d(\hat{S})).$$

Proof. By Lemma 4.4 we have

$$\begin{aligned}
\ker\big(\widehat{\mathrm{GL}_d(S)} \to \mathrm{GL}_d(\hat{S}) = \varprojlim \mathrm{GL}_d(S/J_m)\big) &= \ker(\varprojlim \mathrm{GL}_d(S)/D_m E_d(S, J_m) \\
&\to \varprojlim \mathrm{GL}_d(S)/\mathrm{GL}_d(S, J_m)) = \varprojlim \mathrm{GL}_d(S, J_m)/D_m E_d(S, J_m),
\end{aligned}$$

where $D_m = \{I_{n-1} + (x^{k\cdot m} - 1)E_{1,1} \mid k \in \mathbb{Z}\}$. We claim now that when $m > 2$ then $\mathrm{GL}_d(S, J_m) = D_m \mathrm{SL}_d(S, J_m)$. Indeed, for every $A \in \mathrm{GL}_d(S, J_m)$ we have $\det(A) = \pm x^k$ for some $k \in \mathbb{Z}$. However, as under the map $S \to \mathbb{Z}_m[\mathbb{Z}_m]$ we have $A \mapsto I_d$, the map $S \to \mathbb{Z}_m[\mathbb{Z}_m]$ also implies $\det(A) \mapsto 1$. Hence $\det(A) = \pm x^{k\cdot m}$ for some $k \in \mathbb{Z}$, and when $m > 2$ we even get $\det(A) = x^{k\cdot m}$ for some $k \in \mathbb{Z}$. It follows that $\mathrm{GL}_d(S, J_m) = D_m \mathrm{SL}_d(S, J_m)$. Therefore, since $D_m \cap \mathrm{SL}_d(S, J_m) = \{I_d\}$, we deduce that

$$\begin{aligned}
\ker(\widehat{\mathrm{GL}_d(S)} \to \mathrm{GL}_d(\hat{S})) &= \varprojlim D_m \mathrm{SL}_d(S, J_m)/D_m E_d(S, J_m) \\
&= \varprojlim \mathrm{SL}_d(S, J_m)/E_d(S, J_m) \\
&= \varprojlim \ker(\widehat{\mathrm{SL}_d(S)} \to \mathrm{SL}_d(\hat{S})).
\end{aligned}$$

□

The immediate corollary from Propositions 4.5 and 4.6 is as follows:

Corollary 4.7. *For every $n \geq 4$, we have $C_{i_0} \cong \ker(\widehat{\mathrm{SL}_{n-1}(S)} \to \mathrm{SL}_{n-1}(\hat{S}))$.*

We close the section by showing that $\ker(\widehat{\mathrm{SL}_{n-1}(S)} \to \mathrm{SL}_{n-1}(\hat{S}))$ is not finitely generated, using the techniques in [Kassabov and Nikolov 2006]. It is known that the group ring $S = \mathbb{Z}[x^{\pm 1}] = \mathbb{Z}[\mathbb{Z}]$ is Noetherian; see [Ivanov 1989; Brown et al. 1981]. In addition, it is known that the Krull dimension of $\mathbb{Z}$ is $\dim(\mathbb{Z}) = 1$ and thus $\dim(S) = \dim(\mathbb{Z}[\mathbb{Z}]) = 2$; see [Smith 1972]. Therefore, by Proposition 1.6 in [Suslin 1977], as $n-1 \geq 3$, for every $J \lhd S$, the canonical map

$$\mathrm{SK}_1(S, J; n-1) \to \mathrm{SK}_1(S, J) := \varinjlim_{d\in\mathbb{N}} \mathrm{SK}_1(S, J; d)$$

is surjective. Hence, the canonical map (when $J \lhd S$ ranges over all finite index ideals of S)

$$\begin{aligned}\ker(\widehat{\mathrm{SL}_{n-1}(S)} \to \mathrm{SL}_{n-1}(\hat{S})) &= \varprojlim(\mathrm{SL}_{n-1}(S, J)/E_{n-1}(S, J)) \\ &= \varprojlim \mathrm{SK}_1(S, J; n-1) \to \varprojlim \mathrm{SK}_1(S, J)\end{aligned}$$

is surjective, so it is enough to show that $\varprojlim \mathrm{SK}_1(S, J)$ is not finitely generated.

By a result of Bass [1968, Chapter 5, Corollary 9.3], for every $J \lhd K \lhd S$ of finite index in S, the map $\mathrm{SK}_1(S, J) \to \mathrm{SK}_1(S, K)$ is surjective. Hence, it is enough to show that for every $l \in \mathbb{N}$ there exists a finite index ideal $J \lhd S$ such that $\mathrm{SK}_1(S, J)$ is generated by at least l elements. Now, as $\mathrm{SK}_1(S) = 1$ [Suslin 1977], we obtain the exact sequence

$$K_2(S) \to K_2(S/J) \to \mathrm{SK}_1(S, J) \to \mathrm{SK}_1(S) = 1$$

for every $J \lhd S$ (see Theorem 6.2 in [Milnor 1971]). In addition, by a classical result of Quillen (see [Quillen 1973; Rosenberg 1994, Theorem 5.3.30]), we have

$$K_2(S) = K_2(\mathbb{Z}[x^{\pm 1}]) = K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z}),$$

so by the classical facts $K_2(\mathbb{Z}) = K_1(\mathbb{Z}) = \{\pm 1\}$ (see [Milnor 1971, Chapters 3 and 10]) we deduce that $K_2(S)$ is of order 4. Hence, it is enough to prove that for every $l \in \mathbb{N}$ there exists a finite index ideal $J \lhd S$ such that $K_2(S/J)$ is generated by at least l elements. Following [Kassabov and Nikolov 2006], we state the following proposition (which holds by the proof of Theorem 2.8 in [Stein and Dennis 1973]):

Proposition 4.8. *Let p be a prime, $l \in \mathbb{N}$ and denote by $P \lhd \mathbb{Z}[y]$ the ideal which is generated by p^2 and y^{p^l}. Then for $\bar{S} = \mathbb{Z}[y]/P$, the group $K_2(\bar{S})$ is an elementary abelian p-group of rank $\geq l$.*

Observe now that for every $l \geq 0$,

$$(y+1)^{p^{l+1}} = (y^{p^l} + 1 + p \cdot a(y))^p = 1 \mod P,$$

so $y+1$ is invertible in $\bar{S}$. Therefore we have a well-defined surjective homomorphism $S \to \bar{S}$ which is defined by sending $x \mapsto y+1$. In particular, $J = \ker(S \to \bar{S})$

is a finite index ideal of S which satisfies the above requirements. This shows that indeed $C_{i_0} = \ker(\widehat{\mathrm{SL}_{n-1}(S)} \to \mathrm{SL}_{n-1}(\hat{S}))$ is not finitely generated, and by the description in Proposition 4.3 it follows that $C(\mathrm{IA}(\Phi), \Phi)$ is not finitely generated either.

5. The centrality of C_i

In this section we prove that for every $n \geq 4$, the copies C_i lie in the center of $\widehat{\mathrm{IA}(\Phi)}$. Throughout the section we assume that $n \geq 4$ is constant, and show it for $i = n$. Symmetrically, it is valid for every i. We recall:

- $R_n = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.
- $H_m = \sum_{i=1}^{n} (x_i^m - 1) R_n + m R_n$.
- $\mathrm{IG}_m = \{I_n + A \in \mathrm{GL}_n(R_n, H_m) \mid A\vec{\sigma} = \vec{0}\}$.
- $\mathrm{IA}_m = \bigcap \{N \lhd \mathrm{IA}(\Phi) \mid [\mathrm{IA}(\Phi) : N] \mid m\}$.
- $S = S_n = \mathbb{Z}[x_n^{\pm 1}]$.
- $\mathrm{Im}\,\rho \cap \mathrm{IG}_m = \mathrm{Im}\,\rho_n \cap \mathrm{IG}_m \simeq \mathrm{GL}_{n-1}(S, \sigma_n H_m \cap S))$ (see Proposition 3.8).

We saw in Section 4 that we can write

$$\begin{aligned} C_n &= \varprojlim (\mathrm{IA}_m \cdot (\mathrm{Im}\,\rho \cap \mathrm{IG}_m) / \mathrm{IA}_m) \\ &= \varprojlim (\mathrm{IA}_m \cdot (\mathrm{Im}\,\rho \cap \mathrm{IG}_{m^4}) / \mathrm{IA}_m) \leq \varprojlim (\mathrm{IA}(\Phi) / \mathrm{IA}_m) = \widehat{\mathrm{IA}(\Phi)}. \end{aligned}$$

Hence, if we want to show that C_n lies in the center of $\widehat{\mathrm{IA}(\Phi)}$, it suffices to show that for every $m \in \mathbb{N}$, the group $\mathrm{IA}_m \cdot (\mathrm{Im}\,\rho \cap \mathrm{IG}_{m^4}) / \mathrm{IA}_m$ lies in the center of $\mathrm{IA}(\Phi) / \mathrm{IA}_m$.

We first claim that under the isomorphism $\mathrm{Im}\,\rho \cap \mathrm{IG}_{m^4} \simeq \mathrm{GL}_{n-1}(S, \sigma_n H_{m^4} \cap S)$, one has

$$\mathrm{IA}_m \cdot (\mathrm{Im}\,\rho \cap \mathrm{IG}_{m^4}) / \mathrm{IA}_m \subseteq \mathrm{IA}_m \cdot \mathrm{SL}_{n-1}(S, \sigma_n H_{m^2} \cap S) / \mathrm{IA}_m \,. \tag{5.1}$$

Indeed, if $\alpha \in \mathrm{Im}\,\rho \cap \mathrm{IG}_{m^4}$ then $\det(\alpha) \in 1 + \sigma_n H_{m^4} \cap S \subseteq 1 + H_{m^4} \cap S$. Combining it with Proposition 3.2, $\det(\alpha)$ has the form $\det(\alpha) = x_n^{m^4 t}$ for some $t \in \mathbb{Z}$. Hence

$$\det\bigl((I_n + \sigma_n E_{1,1} - \sigma_1 E_{1,n})^{-m^4 t} \cdot \alpha\bigr) = 1.$$

Now, as we have

$$\begin{aligned} x_n^{m^4 t} &= 1 + (x_n^{m^4 t} - 1) = 1 + \sigma_n \sum_{i=1}^{m^4 - 1} (x_n^t)^i \\ &\in 1 + \sigma_n \bigl((x_n^{m^2 t} - 1) S + m^2 S\bigr) \subseteq 1 + \sigma_n H_{m^2} \cap S \end{aligned}$$

(see the computation in the proof of Proposition 3.9), we obtain that

$$(I_n+\sigma_n E_{1,1}-\sigma_1 E_{1,n})^{m^4 t}\in\langle \mathrm{IA}(\Phi)^m\rangle\cap \mathrm{GL}_{n-1}(S,\sigma_n H_{m^2}\cap S)$$
$$\subseteq \mathrm{IA}_m\cap \mathrm{GL}_{n-1}(S,\sigma_n H_{m^2}\cap S).$$

Therefore, writing $\alpha=(I_n+\sigma_n E_{1,1}-\sigma_1 E_{1,n})^{m^4 t}\cdot\big((I_n+\sigma_n E_{1,1}-\sigma_1 E_{1,n})^{-m^4 t}\cdot\alpha\big)$, we deduce that

$$\operatorname{Im}\rho\cap \mathrm{IG}_{m^4}\subseteq \mathrm{IA}_m\cdot \mathrm{SL}_{n-1}(S,\sigma_n H_{m^2}\cap S)$$

and we get the inclusion (5.1). It follows that if we want to show that C_n lies in the center of $\widehat{\mathrm{IA}(\Phi)}$, it suffices to show that $\mathrm{IA}_m\cdot \mathrm{SL}_{n-1}(S,\sigma_n H_{m^2}\cap S)/\,\mathrm{IA}_m$ lies in the center of $\mathrm{IA}(\Phi)/\,\mathrm{IA}_m$. However, we are going to show even more:

Proposition 5.2. *For every $m\in\mathbb{N}$, the group*

$$\mathrm{IA}_m\cdot \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})/\,\mathrm{IA}_m$$

lies in the center of $\mathrm{IA}(\Phi)/\,\mathrm{IA}_m$.

Let F be the free group on $f_1,\dots,f_n$. It is a classical result by Magnus (see [Magnus et al. 1966, Chapter 3, Theorem N4]) that $\mathrm{IA}(F)$ is generated by the automorphisms of the form

$$\alpha_{r,s,t}=\begin{cases} f_r\mapsto [f_t,f_s]f_r, & \\ f_u\mapsto f_u, & u\neq r,\end{cases}$$

where $[f_t,f_s]=f_t f_s f_t^{-1} f_s^{-1}$ and $1\le r,s\neq t\le n$ (notice that we may have $r=s$). Bachmuth and Mochizuki [1985] show that when $n\ge 4$, the group $\mathrm{IA}(\Phi)$ is generated by the images of these generators under the natural map $\mathrm{Aut}(F)\to\mathrm{Aut}(\Phi)$, i.e., $\mathrm{IA}(\Phi)$ is generated by the elements of the form

$$E_{r,s,t}=I_n+\sigma_t E_{r,s}-\sigma_s E_{r,t},\quad 1\le r,s\neq t\le n.$$

Therefore, to show the centrality of C_n, it is enough to show that given

- an element $\bar\lambda\in \mathrm{IA}_m\cdot \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})/\,\mathrm{IA}_m$,
- and one of the generators $E_{r,s,t}=I_n+\sigma_t E_{r,s}-\sigma_s E_{r,t}$ for $1\le r,s\neq t\le n$,

there exists $\lambda\in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})$, a representative of $\bar\lambda$, such that $[E_{r,s,t},\lambda]\in \mathrm{IA}_m$. So, assume that we have an element $\bar\lambda\in \mathrm{IA}_m\cdot \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})/\,\mathrm{IA}_m$. Then a representative for $\bar\lambda$ has the form

$$\lambda=\begin{pmatrix} I_{n-1}+\sigma_n B & -\sum_{i=1}^{n-1}\sigma_i\vec{b}_i\\ 0 & 1\end{pmatrix}\in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})$$

for some $(n-1)\times(n-1)$ matrix B with entries $b_{i,j}\in H_{m^2}$, and with column vectors denoted by $\vec{b}_i$.

Lemma 5.3. *Let $\bar{\lambda} \in \mathrm{IA}_m \cdot \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2}) / \mathrm{IA}_m$. Then, for every $1 \le l < k \le n-1$, $\bar{\lambda}$ has a representative in $\mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})$ of the following form*:

$$\begin{array}{cc}
\begin{pmatrix}
I_{l-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 1+\sigma_n a & 0 & \sigma_n b & 0 & -\sigma_l a - \sigma_k b \\
0 & 0 & I_{k-l-1} & 0 & 0 & 0 \\
0 & \sigma_n c & 0 & 1+\sigma_n d & 0 & -\sigma_l c - \sigma_k d \\
0 & 0 & 0 & 0 & I_{n-k-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} & \begin{array}{l} \\ \leftarrow l\text{-th row} \\ \\ \leftarrow k\text{-th row} \\ \\ \\ \end{array} \\
\begin{array}{cccccc} & \uparrow & & \uparrow & & \uparrow \\ & l\text{-th column} & & k\text{-th column} & & n\text{-th column} \end{array} &
\end{array} \tag{5.4}$$

for some $a, b, c, d \in H_{m^2}$. (The above notation means that the matrix is similar to the identity matrix, except for the entries in the l-th and k-th rows.)

Proof. We demonstrate the proof in the case $l = 1$, $k = 2$, and symmetrically, the arguments hold for arbitrary $1 \le l < k \le n-1$. Consider an arbitrary representative of $\bar{\lambda}$,

$$\lambda = \begin{pmatrix} I_{n-1} + \sigma_n B & -\sum_{i=1}^{n-1} \sigma_i \vec{b}_i \\ 0 & 1 \end{pmatrix} \in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2}).$$

Then $I_{n-1} + \sigma_n B \in \mathrm{SL}_{n-1}(R_n, \sigma_n H_{m^2})$. Consider now the ideal

$$R_n \rhd H'_{m^2} = \sum_{r=1}^{n-1} (x_r^{m^2} - 1) R_n + \sigma_n (x_n^{m^2} - 1) R_n + m^2 R_n.$$

Observe that $\sigma_n H_{m^2} \lhd H'_{m^2} \lhd H_{m^2} \lhd R_n$ and that $H'_{m^2} \cap \sigma_n R_n = \sigma_n H_{m^2}$. In addition, by similar computations as in the proof of Proposition 3.9, for every $x \in R_n$ we have $x^{m^4} - 1 \in (x-1)(x^{m^2} - 1) R_n + (x-1) m^2 R_n$, and thus $H_{m^4} \subseteq H'_{m^2}$, so H'_{m^2} is of finite index in R_n.

Now, $I_{n-1} + \sigma_n B \in \mathrm{SL}_{n-1}(R_n, \sigma_n H_{m^2}) \subseteq \mathrm{SL}_{n-1}(R_n, H'_{m^2})$. Thus, by the third part of Corollary 2.4, as $H'_{m^2} \lhd R_n$ is an ideal of finite index, $n-1 \ge 3$ and $E_{n-1}(R_n) = \mathrm{SL}_{n-1}(R_n)$ [Suslin 1977], one can write the matrix $I_{n-1} + \sigma_n B$ as

$$I_{n-1} + \sigma_n B = AD \quad \text{when } A = \begin{pmatrix} A' & 0 \\ 0 & I_{n-3} \end{pmatrix}$$

for some $A' \in \mathrm{SL}_2(R_n, H'_{m^2})$ and $D \in E_{n-1}(R_n, H'_{m^2})$. Now consider the images of D and A under the projection $\sigma_n \to 0$, which we denote by $\bar{D}$ and $\bar{A}$. Observe that obviously, $\bar{D} \in E_{n-1}(R_n, H'_{m^2})$. In addition, observe that

$$AD \in \mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \quad \Rightarrow \quad \bar{A}\bar{D} = I_{n-1}.$$

Thus, we have $I_{n-1} + \sigma_n B = A\bar{A}^{-1}\bar{D}^{-1}D$. Therefore, by replacing D by $\bar{D}^{-1}D$

and A by $A\bar{A}^{-1}$ we can assume that

$$I_{n-1}+\sigma_n B = AD \quad \text{for } A=\begin{pmatrix} A' & 0 \\ 0 & I_{n-3}\end{pmatrix},$$

where $A' \in \mathrm{SL}_2(R_n, H'_{m^2}) \cap \mathrm{GL}_2(R_n, \sigma_n R_n) = \mathrm{SL}_2(R_n, \sigma_n H_{m^2})$, and

$$\begin{aligned} D \in E_{n-1}(R_n, H'_{m^2}) \cap \mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \\ \subseteq E_{n-1}(R_n, H_{m^2}) \cap \mathrm{GL}_{n-1}(R_n, \sigma_n R_n) := \mathrm{IE}_{n-1,n}(H_{m^2}). \end{aligned}$$

Now, as we prove in the main lemma (Lemma 7.1) that

$$\mathrm{IE}_{n-1,n}(H_{m^2}) \subseteq \langle \mathrm{IA}(\Phi)^m \rangle \subseteq \mathrm{IA}_m,$$

this argument shows that λ can be replaced by a representative of the form (5.4). □

We now return to our initial mission. Let $\bar{\lambda} \in \mathrm{IA}_m \cdot \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})/\,\mathrm{IA}_m$, and let $E_{r,s,t} = I_n + \sigma_t E_{r,s} - \sigma_s E_{r,t}$ for $1 \le r, s \ne t \le n$ be one of the above generators for $\mathrm{IA}(\Phi)$. We want to show that there exists $\lambda \in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})$, a representative of $\bar{\lambda}$, such that $[E_{r,s,t}, \lambda] \in \mathrm{IA}_m$. We separate the treatment to two cases. We note that Lemma 5.3 is needed only for the second case, which is a bit more delicate.

<u>First Case</u>: $1 \le r \le n-1$.

In this case one can take an arbitrary representative $\lambda \in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2}) \cong \mathrm{SL}_{n-1}(R_n, \sigma_n H_{m^2})$. Considering the embedding of $\mathrm{IGL}'_{n-1,n}$ in $\mathrm{GL}_{n-1}(R_n)$, we have $E_{r,s,t} \in \mathrm{IGL}'_{n-1,n} \subseteq \mathrm{GL}_{n-1}(R_n)$ (see Definition 3.12 and Proposition 3.13). Thus, since by Corollary 2.4

$$\mathrm{SK}_1(R_n, H_{m^2}; n-1) = \mathrm{SL}_{n-1}(R_n, H_{m^2})/E_{n-1}(R_n, H_{m^2})$$

is central in $\mathrm{GL}_{n-1}(R_n)/E_{n-1}(R_n, H_{m^2})$, we have

$$[E_{r,s,t}, \lambda] \in [\mathrm{GL}_{n-1}(R_n), \mathrm{SL}_{n-1}(R_n, \sigma_n H_{m^2})] \subseteq E_{n-1}(R_n, H_{m^2}).$$

In addition, as $\mathrm{SL}_{n-1}(R_n, \sigma_n H_{m^2}) \le \mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$ and $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$ is normal in $\mathrm{GL}_{n-1}(R_n)$, we have

$$[E_{r,s,t}, \lambda] \in [\mathrm{GL}_{n-1}(R_n), \mathrm{GL}_{n-1}(R_n, \sigma_n R_n)] \subseteq \mathrm{GL}_{n-1}(R_n, \sigma_n R_n).$$

Thus, we obtain from Lemma 7.1 that

$$\begin{aligned} [E_{r,s,t}, \lambda] \in E_{n-1}(R_n, H_{m^2}) \cap \mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \\ = \mathrm{IE}_{n-1,n}(H_{m^2}) \subseteq \langle \mathrm{IA}(\Phi)^m \rangle \subseteq \mathrm{IA}_m\,. \end{aligned}$$

<u>Second Case</u>: $r = n$.

This case is a bit more complicated than the previous one, as $E_{r,s,t}$ is not in $\mathrm{IGL}'_{n-1,n}$. Here, by Lemma 5.3 one can choose $\lambda \in \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2})$ whose t-th row equals the standard vector $\vec{e}_t$. As $t \ne r = n$, we thus obtain that both λ

and $E_{r,s,t}$ are in $\mathrm{IGL}'_{n-1,t}$. Considering the embedding $\mathrm{IGL}'_{n-1,t} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$, we have $E_{r,s,t} \in \mathrm{GL}_{n-1}(R_n, \sigma_t R_n)$. In addition, remember that λ has the form

$$\lambda = \begin{pmatrix} I_{n-1} + \sigma_n B & -\sum_{i=1}^{n-1} \sigma_i \vec{b}_i \\ 0 & 1 \end{pmatrix}$$

for $I_{n-1} + \sigma_n B \in \mathrm{SL}_{n-1}(R_n, \sigma_n H_{m^2})$, so that the entries of $\vec{b}_i$ are in H_{m^2}. It follows that regarding the embedding $\mathrm{IGL}'_{n-1,t} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$, we have $\lambda \in \mathrm{SL}_{n-1}(R_n, H_{m^2})$.

Remark 5.5. Note that when considering $\lambda \in \mathrm{IGL}'_{n-1,n} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$, i.e., when considering $\lambda \in \mathrm{GL}_{n-1}(R_n)$ through the embedding of $\mathrm{IGL}'_{n-1,n}$ in $\mathrm{GL}_{n-1}(R_n)$, we have $\lambda \in \mathrm{GL}_{n-1}(R_n, \sigma_n H_{m^2}) \leq \mathrm{GL}_{n-1}(R_n)$. However, when we consider $\lambda \in \mathrm{IGL}'_{n-1,t} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$, we do not necessarily have $\lambda \in \mathrm{GL}_{n-1}(R_n, \sigma_n H_{m^2})$, but we still have $\lambda \in \mathrm{GL}_{n-1}(R_n, H_{m^2})$.

Thus, by similar arguments as in the first case,

$$\begin{aligned} [E_{r,s,t}, \lambda] \in [\mathrm{GL}_{n-1}(R_n, \sigma_t R_n), \mathrm{SL}_{n-1}(R_n, H_{m^2})] & \\ \subseteq E_{n-1}(R_n, H_{m^2}) \cap \mathrm{GL}_{n-1}(R_n, \sigma_t R_n) & \\ = \mathrm{IE}_{n-1,t}(H_{m^2}) \subseteq \langle \mathrm{IA}(\Phi)^m \rangle \subseteq \mathrm{IA}_m . & \end{aligned}$$

This finishes the argument which shows that the C_i are central in $\widehat{\mathrm{IA}(\Phi)}$.

Remark 5.6. One can follow and see that completely similar arguments give that the group

$$\langle \mathrm{IA}(\Phi)^m \rangle \cdot \mathrm{ISL}_{n-1,n}(\sigma_n H_{m^2}) / \langle \mathrm{IA}(\Phi)^m \rangle$$

lies in the center of $\mathrm{IA}(\Phi)/\langle \mathrm{IA}(\Phi)^m \rangle$. The reason is that the only property of IA_m that we used here was that $\langle \mathrm{IA}(\Phi)^m \rangle \subseteq \mathrm{IA}_m$. This claim is used in [Ben-Ezra 2017] to prove Theorem 1.3. We note that in this paper we were careful not to use the subgroups $\langle \mathrm{IA}(\Phi)^m \rangle$ directly as we still didn't show that they are of finite index in $\mathrm{IA}(\Phi)$, and therefore we cannot write $\widehat{\mathrm{IA}(\Phi)} = \varprojlim (\mathrm{IA}(\Phi)/\langle \mathrm{IA}(\Phi)^m \rangle)$. However, on the way to proving Theorem 1.3, we do show that the $\langle \mathrm{IA}(\Phi)^m \rangle$ are of finite index in $\mathrm{IA}(\Phi)$ (provided $n \geq 4$).

6. Some elementary elements of $\langle \mathrm{IA}(\Phi_n)^m \rangle$

In this section we introduce some elements in $\langle \mathrm{IA}(\Phi_n)^m \rangle$ which are needed for the proof of Lemma 7.1. In [Ben-Ezra 2017] we introduce a list of elements in $\langle \mathrm{IA}(\Phi_n)^m \rangle$ that contains the list below (see Propositions 4.1 and 4.2 therein). However, we do not need the whole list of [Ben-Ezra 2017] here, and also do not need all the notation that is used there. Hence, for the convenience of the reader we include here only the list that is needed for the proof of Lemma 7.1, and repeat the arguments that are related to this shorter list.

Proposition 6.1. *Let $n \geq 4$, $1 \leq u \leq n$ and $m \in \mathbb{N}$. Denote by $\vec{e}_i$ the i-th standard row vector. Then the elements of* $\mathrm{IA}(\Phi_n)$ *of the following form lie in* $\langle \mathrm{IA}(\Phi_n)^m \rangle$:

$$\begin{pmatrix} & I_{u-1} & & 0 & & 0 & \\ a_{u,1} & \cdots & a_{u,u-1} & 1 & a_{u,u+1} & \cdots & a_{u,n} \\ & 0 & & 0 & & I_{n-u} & \end{pmatrix} \leftarrow u\text{-th row} \tag{6.2}$$

when $(a_{u,1}, \ldots, a_{u,u-1}, 0, a_{u,u+1}, \ldots, a_{u,n})$ is a linear combination of the vectors

1. $\left\{ m(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \mid i, j \neq u,\ i \neq j \right\}$,
2. $\left\{ (x_k^m - 1)(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \mid i, j, k \neq u,\ i \neq j \right\}$,
3. $\left\{ \sigma_u (x_u^m - 1)(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \mid i, j \neq u,\ i \neq j \right\}$,

with coefficients in R_n. The notation in (6.2) means that the matrix is similar to the identity matrix, except the entries in the u-th row.

Proof. Without loss of generality, we assume that $u = 1$. Observe now that for every $a_i, b_i \in R_n$ for $2 \leq i \leq n$ one has

$$\begin{pmatrix} 1 & a_2 & \cdots & a_n \\ 0 & & I_{n-1} & \end{pmatrix} \begin{pmatrix} 1 & b_2 & \cdots & b_n \\ 0 & & I_{n-1} & \end{pmatrix} = \begin{pmatrix} 1 & a_2 + b_2 & \cdots & a_n + b_n \\ 0 & & I_{n-1} & \end{pmatrix}.$$

Hence, it is enough to prove that the elements of the following forms belong to $\langle \mathrm{IA}(\Phi_n)^m \rangle$ (when we write $a\vec{e}_i$ we mean that the entry of the i-th column in the first row is a):

1. $\begin{pmatrix} 1 & mf(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}$, $\quad i, j \neq 1,\ i \neq j,\ f \in R_n$,
2. $\begin{pmatrix} 1 & (x_k^m - 1) f(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}$, $\quad i, j, k \neq 1,\ i \neq j,\ f \in R_n$,
3. $\begin{pmatrix} 1 & \sigma_1 (x_1^m - 1) f(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}$, $\quad i, j \neq 1,\ i \neq j,\ f \in R_n$.

We start with the elements of Form 1. Here we have

$$\begin{pmatrix} 1 & mf(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & f(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}^m \in \langle \mathrm{IA}(\Phi_n)^m \rangle.$$

We pass to the elements of Form 2. In this case we have

$$\begin{aligned} \langle \mathrm{IA}(\Phi_n)^m \rangle \ni & \left[\begin{pmatrix} 1 & f(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}^{-1}, \begin{pmatrix} x_k & -\sigma_1 \vec{e}_k \\ 0 & I_{n-1} \end{pmatrix}^m \right] \\ & = \begin{pmatrix} 1 & (x_k^m - 1) f(\sigma_i \vec{e}_j - \sigma_j \vec{e}_i) \\ 0 & I_{n-1} \end{pmatrix}. \end{aligned}$$

We finish with the elements of Form 3. The computation here is more complicated than in the previous cases, so we demonstrate it for the special case $n = 4$, $i = 2$, $j = 3$. It is clear that symmetrically, with similar arguments, the same holds in general when $n \geq 4$ for every $i, j \neq 1$, $i \neq j$. By similar arguments as in the previous case we get

$$\langle \mathrm{IA}(\Phi_4)^m \rangle \ni \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \sigma_3(x_1^m - 1)f & -\sigma_2(x_1^m - 1)f & 1 \end{pmatrix}.$$

Therefore, we also have

$$\langle \mathrm{IA}(\Phi_4)^m \rangle \ni \left[\begin{pmatrix} x_4 & 0 & 0 & -\sigma_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \sigma_3(x_1^m - 1)f & -\sigma_2(x_1^m - 1)f & 1 \end{pmatrix} \right]$$
$$= \begin{pmatrix} 1 & -\sigma_3\sigma_1(x_1^m - 1)f & \sigma_2\sigma_1(x_1^m - 1)f & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad \square$$

7. The main lemma

We recall and present some new notation that is used in this section:

- $\mathrm{IA}^m = \langle \mathrm{IA}(\Phi)^m \rangle$, where $\Phi = \Phi_n$.
- $R_n = \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, where $x_1, \dots, x_n$ are the generators of $\mathbb{Z}^n$.
- $\sigma_r = x_r - 1$ for $1 \leq r \leq n$.
- $U_{r,m} = (x_r^m - 1)R_n$ for $1 \leq r \leq n$ and $m \in \mathbb{N}$.
- $O_m = mR_n$.
- $H_m = \sum_{r=1}^n (x_r^m - 1)R_n + mR_n = \sum_{r=1}^n U_{r,m} + O_m$.
- $\mathrm{IE}_{n-1,i}(H) = \mathrm{IGL}_{n-1,i} \cap E_{n-1}(R_n, H) \leq \mathrm{ISL}_{n-1,i}(H)$ for $H \lhd R_n$ under the identification of $\mathrm{IGL}_{n-1,i} \leq \mathrm{IA}(\Phi)$ with $\mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$ (see Proposition 3.4 and Definition 3.6).

In this section, we prove the following main lemma:

Lemma 7.1. *For every $n \geq 4$, $m \in \mathbb{N}$ and $1 \leq i \leq n$, we have*

$$\mathrm{IE}_{n-1,i}(H_{m^2}) \subseteq \mathrm{IA}^m.$$

To simplify the proof and the notation, we prove the lemma for the special case $i = n$, and symmetrically, all the arguments are valid for every $1 \leq i \leq n$.

In addition, using the identification $\mathrm{IGL}_{n-1,n} \cong \mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$, we identify $\mathrm{IGL}_{n-1,n}$ with $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$, and the group $\mathrm{IE}_{n-1,n}(H_m)$ with the group $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, H_m)$. So the goal of this section is proving that

$$\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, H_{m^2}) \subseteq \mathrm{IA}^m .$$

Throughout the proof we use also elements of $\mathrm{IGL}'_{n-1,n}$ (see Definition 3.12). We recall that

$$\mathrm{IE}_{n-1,n}(H_m) \le \mathrm{IGL}_{n-1,n} \le \mathrm{IGL}'_{n-1,n} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$$

(Proposition 3.13), so all the elements that are being used throughout the section are naturally embedded in $\mathrm{GL}_{n-1}(R_n)$. Using this embedding, we do all the computations in $\mathrm{GL}_{n-1}(R_n)$, and make the notation simpler by omitting the n-th row and column from each matrix.

We note that many ideas in the proof of Lemma 7.1 below are based on ideas of the proof of the "main lemma" in [Bachmuth and Mochizuki 1985] (see Section 4 therein). However, our arguments do not rely directly on the arguments in [Bachmuth and Mochizuki 1985], so on the whole we cannot make a formal reference to that work throughout the proof of Lemma 7.1.

Decomposing the proof. In this subsection we start the proof of Lemma 7.1. At the end of the subsection, there will be a few tasks left, which will be accomplished in the forthcoming subsections. We start with the following definition:

Definition 7.2. For every $m \in \mathbb{N}$, define the following ideal of R_n:

$$T_m = \sum_{r=1}^{n} \sigma_r^2 U_{r,m} + \sum_{r=1}^{n} \sigma_r O_m + O_m^2 .$$

Observe that as for every $x \in R_n$ we have $\sum_{j=0}^{m-1} x^j \in (x-1)R_n + mR_n$, one has

$$\begin{aligned} x^{m^2} - 1 = (x-1) \sum_{j=0}^{m^2-1} x^j &= (x-1) \sum_{j=0}^{m-1} x^j \sum_{j=0}^{m-1} x^{jm} \\ &\in (x-1)((x-1)R_n + mR_n)((x^m-1)R_n + mR_n) \\ &\subseteq (x-1)^2(x^m-1)R_n + (x-1)^2 mR_n + (x-1)m^2 R_n . \end{aligned}$$

It follows that $H_{m^2} \subseteq T_m$. Hence, it is enough to prove that

$$\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m) \subseteq \mathrm{IA}^m .$$

Equivalently, it is enough to prove that the group

$$(\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m)) \cdot \mathrm{IA}^m \,/\, \mathrm{IA}^m$$

is trivial. We continue with the following proposition, which is actually a proposition of Suslin [1977, Corollary 1.4] with some elaborations of [Bachmuth and

Mochizuki 1985] (see the remark that follows their Proposition 3.5 and the beginning of the proof of the "main lemma" in Section 4 therein).

Proposition 7.3. *Let R be a commutative ring, $d \geq 3$ and $H \lhd R$ an ideal. Then $E_d(R, H)$ is generated by the matrices of the form*

$$(I_d - fE_{i,j})(I_d + hE_{j,i})(I_d + fE_{i,j}) \tag{7.4}$$

for $h \in H$, $f \in R$ and $1 \leq i \neq j \leq d$.

Proof. In the proof of Corollary 1.4 in [Suslin 1977], Suslin shows that whenever $d \geq 3$, $E_d(R, H)$ is generated by the elements of the form

$$I_d + h\vec{u}^t(u_j\vec{e}_i - u_i\vec{e}_j),$$

where $h \in H$, $i \neq j$ and $\vec{u} = (u_1, u_2, \ldots, u_d) \in R^n$ such that $\vec{u}\cdot\vec{v}^t = 1$ for some $\vec{v} \in R^n$. In the remark which follows Proposition 3.5 in [Bachmuth and Mochizuki 1985], it is observed that

$$\begin{aligned} I_d + h\vec{u}^t(u_j\vec{e}_i - u_i\vec{e}_j) &= (I_d + h(u_i\vec{e}_i + u_j\vec{e}_j)^t(u_j\vec{e}_i - u_i\vec{e}_j)) \\ &\quad \cdot \prod_{l\neq i,j}(I_d + h(u_l\vec{e}_l)^t u_j\vec{e}_i) \cdot \prod_{l\neq i,j}(I_d - h(u_l\vec{e}_l)^t u_i\vec{e}_j). \end{aligned}$$

Hence, by observing that all the factors in the above expression are all of the form

$$I_d + h(f_1\vec{e}_i + f_2\vec{e}_j)^t(f_2\vec{e}_i - f_1\vec{e}_j) \tag{7.5}$$

for some $f_1, f_2 \in R$, $h \in H$ and $1 \leq i \neq j \leq d$, it is enough to show that the matrices of the form (7.5) are generated by the matrices of the form (7.4). We show it for the case $i, j, d = 1, 2, 3$, and it will be clear that the general argument is similar. So we have the matrix

$$I_d + h(f_1\vec{e}_1 + f_2\vec{e}_2)^t(f_2\vec{e}_1 - f_1\vec{e}_2) = \begin{pmatrix} 1 + hf_1f_2 & -hf_1^2 & 0 \\ hf_2^2 & 1 - hf_1f_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $f_1, f_2 \in R$ and $h \in H$, which is equal to

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & -hf_1 \\ 0 & 1 & -hf_2 \\ f_2 & -f_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & hf_1 \\ 0 & 1 & hf_2 \\ -f_2 & f_1 & 1 \end{pmatrix} \\ &\qquad = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_2 & -f_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -hf_1 \\ 0 & 1 & -hf_2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -f_2 & f_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & hf_1 \\ 0 & 1 & hf_2 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

As the matrix

$$\begin{pmatrix} 1 & 0 & hf_1 \\ 0 & 1 & hf_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & hf_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & hf_2 \\ 0 & 0 & 1 \end{pmatrix}$$

is generated by the matrices of the form (7.4), it remains to show that

$$\begin{aligned}\begin{pmatrix}1&0&0\\0&1&0\\f_2&-f_1&1\end{pmatrix}&\begin{pmatrix}1&0&-hf_1\\0&1&-hf_2\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&f_1&1\end{pmatrix}\\&=\begin{pmatrix}1&0&0\\0&1&0\\f_2&-f_1&1\end{pmatrix}\begin{pmatrix}1&0&-hf_1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&f_1&1\end{pmatrix}\\&\qquad\cdot\begin{pmatrix}1&0&0\\0&1&0\\f_2&-f_1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-hf_2\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&f_1&1\end{pmatrix}\end{aligned}$$

is generated by the matrices of the form (7.4). Now

$$\begin{aligned}\begin{pmatrix}1&0&0\\0&1&0\\f_2&-f_1&1\end{pmatrix}&\begin{pmatrix}1&0&-hf_1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&f_1&1\end{pmatrix}\\&=\begin{pmatrix}1&0&0\\0&1&0\\0&-hf_1^2f_2&1\end{pmatrix}\begin{pmatrix}1&-hf_1^2&0\\0&1&0\\0&0&1\end{pmatrix}\\&\qquad\cdot\left[\begin{pmatrix}1&0&0\\0&1&0\\f_2&0&1\end{pmatrix}\begin{pmatrix}1&0&-hf_1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&0&1\end{pmatrix}\right]\end{aligned}$$

is generated by the matrices of the form (7.4), and by a similar computation

$$\begin{pmatrix}1&0&0\\0&1&0\\f_2&-f_1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-hf_2\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f_2&f_1&1\end{pmatrix}$$

is generated by these matrices as well. □

We proceed with the following lemma. Some of the ideas in its proof are based on the proof of Proposition 3.5 in [Bachmuth and Mochizuki 1985].

Lemma 7.6. *Let $n \geq 4$. Recall $U_{r,m} = (x_r^m - 1)R_n$, $O_m = mR_n$, and denote the corresponding ideals of $R_{n-1} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}] \subseteq R_n$ by*

$$\bar{O}_m = mR_{n-1} \subseteq O_m, \qquad \bar{U}_{r,m} = (x_r^m - 1)R_{n-1} \subseteq U_{r,m} \quad \text{for } 1 \leq r \leq n-1.$$

Then every element of $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cap E_{n-1}(R_n, T_m)$ can be decomposed as a product of elements of the following four forms:

1. $A^{-1}(I_{n-1}+hE_{i,j})A$, $\quad h\in\sigma_n O_m$,
2. $A^{-1}(I_{n-1}+hE_{i,j})A$, $\quad h\in\sigma_n^2 U_{n,m}$ *or* $h\in\sigma_n\sigma_r^2 U_{r,m}$ *for* $1\le r\le n-1$,
3. $A^{-1}\big[(I_{n-1}+hE_{i,j}),(I_{n-1}+fE_{j,i})\big]A$, $\quad h\in\overline{O}_m^2,\ f\in\sigma_n R_n$,
4. $A^{-1}\big[(I_{n-1}+hE_{i,j}),(I_{n-1}+fE_{j,i})\big]A$, $\quad h\in\sigma_r^2\overline{U}_{r,m}$ *or* $h\in\sigma_r\overline{O}_m$ *for* $1\le r\le n-1,\ f\in\sigma_n R_n$,

where $A\in\mathrm{GL}_{n-1}(R_n)$ *and* $i\neq j$.

Remark 7.7. Notice that as $\mathrm{GL}_{n-1}(R_n,\sigma_n R_n)$ is normal in $\mathrm{GL}_{n-1}(R_n)$, every element of the above forms is an element of $\mathrm{GL}_{n-1}(R_n,\sigma_n R_n)\cong\mathrm{IGL}_{n-1,n}\le\mathrm{IA}(\Phi)$.

Proof of Lemma 7.6. Let $B\in\mathrm{GL}_{n-1}(R_n,\sigma_n R_n)\cap E_{n-1}(R_n,T_m)$. We first claim that to prove the lemma, it is enough to show that B can be decomposed as a product of the elements in the lemma, and arbitrary elements in $\mathrm{GL}_{n-1}(R_{n-1})$. Indeed, assume that we can write $B=A_1D_1\cdots A_nD_n$ for some D_i of the forms in the lemma and $A_i\in\mathrm{GL}_{n-1}(R_{n-1})$ (notice that A_1 or D_n might be equal to I_{n-1}). Observe now that we can therefore write

$$B=A_1D_1A_1^{-1}\cdots(A_1\cdots A_n)D_n(A_1\cdots A_n)^{-1}(A_1\cdots A_n),$$

and by definition, the conjugations of the D_i can also be considered to be in the forms in the lemma. On the other hand, we have

$$(A_1\cdots A_n)D_n^{-1}(A_1\cdots A_n)^{-1}\cdots A_1D_1^{-1}A_1^{-1}B=A_1\cdots A_n$$

and as the matrices of the forms in the lemma are all in $\mathrm{GL}_{n-1}(R_n,\sigma_n R_n)$ (by Remark 7.7), we deduce that

$$A_1\cdots A_n\in\mathrm{GL}_{n-1}(R_n,\sigma_n R_n)\cap\mathrm{GL}_{n-1}(R_{n-1})=\{I_{n-1}\},$$

i.e., $A_1\cdots A_n=I_{n-1}$. Hence

$$B=A_1D_1A_1^{-1}\cdots(A_1\cdots A_n)D_n(A_1\cdots A_n)^{-1},$$

i.e., B is a product of matrices of the forms in the lemma, as required.

So let $B\in\mathrm{GL}_{n-1}(R_n,\sigma_n R_n)\cap E_{n-1}(R_n,T_m)$. According to Proposition 7.3, as $B\in E_{n-1}(R_n,T_m)$ and $n-1\ge 3$, we can write B as a product of elements of the form

$$(I_{n-1}-fE_{i,j})(I_{n-1}+hE_{j,i})(I_{n-1}+fE_{i,j})$$

for some $f\in R_n$, $h\in T_m=\sum_{r=1}^{n}\sigma_r^2U_{r,m}+\sum_{r=1}^{n}\sigma_r O_m+O_m^2$ and $1\le i\neq j\le n-1$. We show now that every element of the above form can be written as a product of the elements of the forms in the lemma and elements of $\mathrm{GL}_{n-1}(R_{n-1})$.

So let $h \in T$ and $f \in R_n$. Observe first that by division by σ_n (with residue), one has

$$\begin{aligned} T_m &= \sum_{r=1}^{n} \sigma_r^2 U_{r,m} + \sum_{r=1}^{n} \sigma_r O_m + O_m^2 \\ &\subseteq \sigma_n\left(\sum_{r=1}^{n-1} \sigma_r^2 U_{r,m} + \sigma_n U_{n,m} + O_m\right) + \sum_{r=1}^{n-1} \sigma_r^2 \bar{U}_{r,m} + \sum_{r=1}^{n-1} \sigma_r \bar{O}_m + \bar{O}_m^2. \end{aligned}$$

Hence, we can decompose $h = \sigma_n h_1 + h_2$ for some $h_1 \in \sum_{r=1}^{n-1} \sigma_r^2 U_{r,m} + \sigma_n U_{n,m} + O_m$ and $h_2 \in \sum_{r=1}^{n-1} \sigma_r^2 \bar{U}_{r,m} + \sum_{r=1}^{n-1} \sigma_r \bar{O}_m + \bar{O}_m^2$. Therefore, we can write

$$\begin{aligned} &(I_{n-1} - f E_{i,j})(I_{n-1} + h E_{j,i})(I_{n-1} + f E_{i,j}) \\ &\qquad = (I_{n-1} - f E_{i,j})(I_{n-1} + \sigma_n h_1 E_{j,i})(I_{n-1} + f E_{i,j}) \\ &\qquad\qquad \cdot (I_{n-1} - f E_{i,j})(I_{n-1} + h_2 E_{j,i})(I_{n-1} + f E_{i,j}). \end{aligned}$$

Thus, as the matrix $(I_{n-1} - f E_{i,j})(I_{n-1} + \sigma_n h_1 E_{j,i})(I_{n-1} + f E_{i,j})$ is clearly a product of elements of Forms 1 and 2 in the lemma, it is enough to deal with the matrix

$$(I_{n-1} - f E_{i,j})(I_{n-1} + h_2 E_{j,i})(I_{n-1} + f E_{i,j})$$

when $h_2 \in \sum_{r=1}^{n-1} \sigma_r^2 \bar{U}_{r,m} + \sum_{r=1}^{n-1} \sigma_r \bar{O}_m + \bar{O}_m^2$. Let us now write $f = \sigma_n f_1 + f_2$ for some $f_1 \in R_n$ and $f_2 \in R_{n-1}$, and write

$$\begin{aligned} &(I_{n-1} - f E_{i,j})(I_{n-1} + h_2 E_{j,i})(I_{n-1} + f E_{i,j}) \\ &\qquad = (I_{n-1} - f_2 E_{i,j})(I_{n-1} - \sigma_n f_1 E_{i,j}) \\ &\qquad\qquad \cdot (I_{n-1} + h_2 E_{j,i})(I_{n-1} + \sigma_n f_1 E_{i,j})(I_{n-1} + f_2 E_{i,j}). \end{aligned}$$

Now, as $(I_{n-1} \pm f_2 E_{i,j}) \in \mathrm{GL}_{n-1}(R_{n-1})$, it is enough to deal with the element

$$(I_{n-1} - \sigma_n f_1 E_{i,j})(I_{n-1} + h_2 E_{j,i})(I_{n-1} + \sigma_n f_1 E_{i,j}),$$

which can be written as a product of elements of the form

$$\begin{aligned} &(I_{n-1} - \sigma_n f_1 E_{i,j})(I_{n-1} + k E_{j,i})(I_{n-1} + \sigma_n f_1 E_{i,j}) \\ &\qquad\qquad \text{with } k \in \bar{O}_m^2,\ \sigma_r^2 \bar{U}_{r,m},\ \sigma_r \bar{O}_m,\ \text{ for } 1 \le r \le n-1. \end{aligned}$$

Finally, as for every such k one can write

$$\begin{aligned} &(I_{n-1} - \sigma_n f_1 E_{i,j})(I_{n-1} + k E_{j,i})(I_{n-1} + \sigma_n f_1 E_{i,j}) \\ &\qquad = (I_{n-1} + k E_{j,i})[(I_{n-1} - k E_{j,i}), (I_{n-1} - \sigma_n f_1 E_{i,j})], \end{aligned}$$

and $(I_{n-1} + k E_{j,i}) \in \mathrm{GL}_{n-1}(R_{n-1})$, we are actually finished. □

Corollary 7.8. *To prove Lemma 7.1, it is enough to show that every element of the forms in Lemma 7.6 is in* IA^m.

We start here by dealing with the elements of Form 1.

Proposition 7.9. *Recall $O_m = mR_n$. Elements of the following form are in* IA^m:

$$A^{-1}(I_{n-1} + hE_{i,j})A \quad \text{for } A \in \mathrm{GL}_{n-1}(R_n),\ h \in \sigma_n O_m \text{ and } i \neq j.$$

Proof. In this case we can write $h = \sigma_n m h'$ for some $h' \in R_n$. So, as

$$A^{-1}(I_{n-1} + \sigma_n h' E_{i,j})A \in \mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \leq \mathrm{IA}(\Phi),$$

we obtain that

$$\begin{aligned} A^{-1}(I_{n-1} + hE_{i,j})A &= A^{-1}(I_{n-1} + \sigma_n m h' E_{i,j})A \\ &= (A^{-1}(I_{n-1} + \sigma_n h' E_{i,j})A)^m \in \mathrm{IA}^m, \end{aligned}$$

as required. □

We devote the remaining sections to dealing with the elements of the other three forms. In these cases the proof is more difficult, and we will need the help of the computations in the next subsection.

Some auxiliary computations.

Proposition 7.10. *For every $f, g \in R_n$ we have the following equalities*:

$$\begin{aligned}
\begin{pmatrix} 1-fg & -fg & 0 \\ fg & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ fg & 1 & 0 \\ fg^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & -f \\ 0 & fg^2 & 1-fg \end{pmatrix} \begin{pmatrix} 1 & -fg & 0 \\ 0 & 1 & 0 \\ 0 & -fg^2 & 1 \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 1-fg & 0 & f \\ 0 & 1 & 0 \\ -fg^2 & 0 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} && (7.11) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ fg & 1 & 0 \\ -fg^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & f \\ 0 & -fg^2 & 1-fg \end{pmatrix} \begin{pmatrix} 1 & -fg & 0 \\ 0 & 1 & 0 \\ 0 & fg^2 & 1 \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 1-fg & 0 & -f \\ 0 & 1 & 0 \\ fg^2 & 0 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix} && (7.12) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f & f & 1 \end{pmatrix} \begin{pmatrix} 1-fg & 0 & fg^2 \\ 0 & 1 & 0 \\ -f & 0 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ fg & 1 & -fg^2 \\ 0 & 0 & 1 \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & fg^2 \\ 0 & -f & 1-fg \end{pmatrix} \begin{pmatrix} 1 & -fg & -fg^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} && (7.13)
\end{aligned}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -f & -f & 1 \end{pmatrix} \begin{pmatrix} 1-fg & 0 & -fg^2 \\ 0 & 1 & 0 \\ f & 0 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ fg & 1 & fg^2 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & -fg^2 \\ 0 & f & 1-fg \end{pmatrix} \begin{pmatrix} 1 & -fg & fg^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.14}$$

Proof. We use square brackets to help the reader follow the steps of the computation. Here is the computation for (7.11):

$$\begin{aligned}
&\begin{pmatrix} 1-fg & -fg & 0 \\ fg & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & -f \\ g & g & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ -g & -g & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & g & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & -g & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \\
&= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & g & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & -g & 1 \end{pmatrix} \right] \\
&\qquad \cdot \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g & g & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g & -g & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ fg & 1+fg & -f \\ fg^2 & fg^2 & 1-fg \end{pmatrix} \begin{pmatrix} 1-fg & -fg & f \\ 0 & 1 & 0 \\ -fg^2 & -fg^2 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ fg & 1 & 0 \\ fg^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & -f \\ 0 & fg^2 & 1-fg \end{pmatrix} \\
&\qquad \cdot \begin{pmatrix} 1 & -fg & 0 \\ 0 & 1 & 0 \\ 0 & -fg^2 & 1 \end{pmatrix} \begin{pmatrix} 1-fg & 0 & f \\ 0 & 1 & 0 \\ -fg^2 & 0 & 1+fg \end{pmatrix} \begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Line (7.12) is obtained similarly by changing the signs of f and g simultaneously.

Here is the computation for (7.13):

$$\begin{pmatrix} 1-fg & -fg & 0 \\ fg & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & g \\ 0 & 1 & -g \\ f & f & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -g \\ 0 & 1 & g \\ -f & -f & 1 \end{pmatrix}$$

$$= \begin{pmatrix}1&0&0\\0&1&0\\f&f&1\end{pmatrix}\begin{pmatrix}1&0&g\\0&1&-g\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f&-f&1\end{pmatrix}\begin{pmatrix}1&0&-g\\0&1&g\\0&0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0\\0&1&0\\f&f&1\end{pmatrix}\left[\begin{pmatrix}1&0&g\\0&1&-g\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-f&0&1\end{pmatrix}\begin{pmatrix}1&0&-g\\0&1&g\\0&0&1\end{pmatrix}\right]$$

$$\cdot\left[\begin{pmatrix}1&0&g\\0&1&-g\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&-f&1\end{pmatrix}\begin{pmatrix}1&0&-g\\0&1&g\\0&0&1\end{pmatrix}\right]$$

$$= \begin{pmatrix}1&0&0\\0&1&0\\f&f&1\end{pmatrix}\begin{pmatrix}1-fg&0&fg^2\\fg&1&-fg^2\\-f&0&1+fg\end{pmatrix}\begin{pmatrix}1&-fg&-fg^2\\0&1+fg&fg^2\\0&-f&1-fg\end{pmatrix}$$

$$= \begin{pmatrix}1&0&0\\0&1&0\\f&f&1\end{pmatrix}\begin{pmatrix}1-fg&0&fg^2\\0&1&0\\-f&0&1+fg\end{pmatrix}\begin{pmatrix}1&0&0\\fg&1&-fg^2\\0&0&1\end{pmatrix}$$

$$\cdot\begin{pmatrix}1&0&0\\0&1+fg&fg^2\\0&-f&1-fg\end{pmatrix}\begin{pmatrix}1&-fg&-fg^2\\0&1&0\\0&0&1\end{pmatrix}.$$

We obtain (7.14) similarly by changing the signs of f and g simultaneously. □

In the following corollary, a 3×3 matrix $B \in \mathrm{GL}_3(R_n)$ denotes the block matrix

$$\begin{pmatrix}B&0\\0&I_{n-4}\end{pmatrix} \in \mathrm{GL}_{n-1}(R_n).$$

Corollary 7.15. *Let $n \geq 4$, $f \in \sigma_n\big(\sum_{r=1}^{n-1}\sigma_r U_{r,m} + U_{n,m} + O_m\big)$ and $g \in R_n$. Then, mod IA^m we have the following equalities* (*the indices are intended to help us later to recognize forms of matrices*: *form* 7, *form* 12, *etc.*):

$$\begin{pmatrix}1-fg&-fg&0\\fg&1+fg&0\\0&0&1\end{pmatrix}_{13} \equiv \begin{pmatrix}1&0&0\\0&1+fg&-f\\0&fg^2&1-fg\end{pmatrix}_1\begin{pmatrix}1-fg&0&f\\0&1&0\\-fg^2&0&1+fg\end{pmatrix}_2$$

$$\equiv \begin{pmatrix}1&0&0\\0&1+fg&f\\0&-fg^2&1-fg\end{pmatrix}_3\begin{pmatrix}1-fg&0&-f\\0&1&0\\fg^2&0&1+fg\end{pmatrix}_4$$

$$\equiv \begin{pmatrix}1-fg&0&fg^2\\0&1&0\\-f&0&1+fg\end{pmatrix}_5\begin{pmatrix}1&0&0\\0&1+fg&fg^2\\0&-f&1-fg\end{pmatrix}_6$$

$$\equiv \begin{pmatrix} 1-fg & 0 & -fg^2 \\ 0 & 1 & 0 \\ f & 0 & 1+fg \end{pmatrix}_7 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & -fg^2 \\ 0 & f & 1-fg \end{pmatrix}_8 \tag{7.16}$$

and

$$\begin{pmatrix} 1-fg & fg & 0 \\ -fg & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{14} \equiv \begin{pmatrix} 1-fg & 0 & -fg^2 \\ 0 & 1 & 0 \\ f & 0 & 1+fg \end{pmatrix}_7 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fg & fg^2 \\ 0 & -f & 1-fg \end{pmatrix}_6 . \tag{7.17}$$

Moreover, we have (the inverse of a matrix is denoted by the same index — one can observe that the inverse of each matrix in these equations is obtained by changing the sign of f)

$$\begin{aligned} \begin{pmatrix} 1-fg & 0 & -fg \\ 0 & 1 & 0 \\ fg & 0 & 1+fg \end{pmatrix}_{15} &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & fg^2 \\ 0 & -f & 1+fg \end{pmatrix}_8 \begin{pmatrix} 1-fg & f & 0 \\ -fg^2 & 1+fg & 1 \\ 0 & 0 & 0 \end{pmatrix}_9 \\ &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & -fg^2 \\ 0 & f & 1+fg \end{pmatrix}_6 \begin{pmatrix} 1-fg & -f & 0 \\ fg^2 & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{10} \\ &\equiv \begin{pmatrix} 1-fg & fg^2 & 0 \\ -f & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & -f \\ 0 & fg^2 & 1+fg \end{pmatrix}_3 \\ &\equiv \begin{pmatrix} 1-fg & -fg^2 & 0 \\ f & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{12} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & f \\ 0 & -fg^2 & 1+fg \end{pmatrix}_1 \end{aligned} \tag{7.18}$$

and

$$\begin{pmatrix} 1-fg & 0 & fg \\ 0 & 1 & 0 \\ -fg & 0 & 1+fg \end{pmatrix}_{16} \equiv \begin{pmatrix} 1-fg & -fg^2 & 0 \\ f & 1+fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{12} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & -f \\ 0 & fg^2 & 1+fg \end{pmatrix}_3 \tag{7.19}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & -fg \\ 0 & fg & 1+fg \end{pmatrix}_{17} \equiv \begin{pmatrix} 1-fg & 0 & fg^2 \\ 0 & 1 & 0 \\ -f & 0 & 1+fg \end{pmatrix}_5 \begin{pmatrix} 1+fg & -fg^2 & 0 \\ f & 1-fg & 1 \\ 0 & 0 & 0 \end{pmatrix}_{11} \tag{7.20}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-fg & fg \\ 0 & -fg & 1+fg \end{pmatrix}_{18} \equiv \begin{pmatrix} 1+fg & f & 0 \\ -fg^2 & 1-fg & 0 \\ 0 & 0 & 1 \end{pmatrix}_{10} \begin{pmatrix} 1-fg & 0 & -f \\ 0 & 1 & 0 \\ fg^2 & 0 & 1+fg \end{pmatrix}_4 . \tag{7.21}$$

Remark 7.22. We remark that since $f \in \sigma_n R_n$, every matrix which takes part in the above equalities is indeed in $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \cong \mathrm{IGL}_{n-1,n} \leq \mathrm{IA}(\Phi)$.

Proof. As $f \in \sigma_n\left(\sum_{r=1}^{n-1} \sigma_r U_{r,m} + U_{n,m} + O_m\right)$, (7.16) is obtained by applying Proposition 7.10 combined with Proposition 6.1. We obtain (7.17) similarly by transposing all the computations which led to the first part of (7.16). Similarly, by switching the roles of the second row and column with the third row and column, one obtains (7.18) and (7.19). By switching one more time the roles of the first row and column with the second row and column, we obtain (7.20) and (7.21) as well. $\square$

***Elements of Form* 2.**

Proposition 7.23. *Recall $U_{r,m} = (x_r^m - 1)R_n$. The elements of the form*

$$A^{-1}(I_{n-1} + hE_{i,j})A,$$

where $A \in \mathrm{GL}_{n-1}(R_n)$, $h \in \sigma_n\sigma_r^2 U_{r,m}$, $\sigma_n^2 U_{n,m}$ for $1 \leq r \leq n-1$ and $i \neq j$, belong to IA^m.

Notice that for every $n \geq 4$, the groups $E_{n-1}(\sigma_n^2 U_{n,m})$ and $E_{n-1}(\sigma_n\sigma_r^2 U_{r,m})$ for $1 \leq r \leq n-1$ are normal in $\mathrm{GL}_{n-1}(R_n)$, and thus, all the above elements are in $E_{n-1}(\sigma_n^2 U_{n,m})$ and $E_{n-1}(\sigma_n\sigma_r^2 U_{r,m})$. Hence, to prove Proposition 7.23, it is enough to show that for every $1 \leq r \leq n-1$, we have

$$E_{n-1}(\sigma_n^2 U_{n,m}),\ E_{n-1}(\sigma_n\sigma_r^2 U_{r,m}) \subseteq \mathrm{IA}^m .$$

Therefore, by Proposition 7.3, to prove Proposition 7.23, it is enough to show that the elements of the form

$$(I_{n-1} - fE_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} + fE_{j,i})$$

are in IA^m when $h \in \sigma_n\sigma_r^2 U_{r,m}$, $\sigma_n^2 U_{n,m}$ for $1 \leq r \leq n-1$, $f \in R_n$ and $i \neq j$. We prove this in a few stages, starting with the following lemma.

Lemma 7.24. *Let $h \in \sigma_n\sigma_r U_{r,m}$, $\sigma_n U_{n,m}$ for $1 \leq r \leq n-1$ and $f_1, f_2 \in R_n$. Assume that the elements of the forms*

$$(I_{n-1} \pm f_1 E_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} \mp f_1 E_{j,i}),$$
$$(I_{n-1} \pm f_2 E_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} \mp f_2 E_{j,i}),$$

for every $1 \leq i \neq j \leq n-1$, belong to IA^m. Then the elements of the form

$$(I_{n-1} \pm (f_1 + f_2)E_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} \mp (f_1 + f_2)E_{j,i})$$

for $1 \leq i \neq j \leq n-1$ also belong to IA^m.

Proof. Observe first that by Proposition 6.1, all the matrices of the form

$$I_{n-1} + hE_{i,j} \quad \text{for } h \in \sigma_n\sigma_r U_{r,m}, \sigma_n U_{n,m}$$

belong to IA^m. We use this in the following computations. Without loss of generality, under the assumptions of the proposition, we show that for $i, j = 2, 1$ we have

$$(I_{n-1} - (f_1 + f_2)E_{1,2})(I_{n-1} + hE_{2,1})(I_{n-1} + (f_1 + f_2)E_{1,2}) \in \mathrm{IA}^m$$

and the general argument is similar. In the computation, a block matrix of the form

$$\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$$

is denoted by $B \in \mathrm{GL}_3(R_n)$. We use square brackets to help the reader follow the steps of the computation.

So we compute

$$\begin{aligned}
&(I_{n-1} - (f_1 + f_2)E_{1,2})(I_{n-1} + hE_{2,1})(I_{n-1} + (f_1 + f_2)E_{1,2}) \\
&\qquad = \begin{pmatrix} 1 - h(f_1 + f_2) & -h(f_1 + f_2)^2 & 0 \\ h & 1 + h(f_1 + f_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 1 & 0 & -(f_1 + f_2) \\ 0 & 1 & 1 \\ -h & -h(f_1 + f_2) & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & (f_1 + f_2) \\ 0 & 1 & -1 \\ h & h(f_1 + f_2) & 1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -h & -h(f_1 + f_2) & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -(f_1 + f_2) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
&\qquad\qquad\qquad \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & h(f_1 + f_2) & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & (f_1 + f_2) \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \\
&\qquad = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -h & -h(f_1 + f_2) & 1 \end{pmatrix} \\
&\qquad\qquad\qquad \cdot \left[\begin{pmatrix} 1 & 0 & -(f_1 + f_2) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & hf_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & f_1 + f_2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\right] \\
&\qquad\qquad\qquad \cdot \left[\begin{pmatrix} 1 & 0 & -f_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h & hf_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & f_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]
\end{aligned}$$

$$\cdot\left[\begin{pmatrix}1&0&-f_2\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-h&-hf_1&1\end{pmatrix}\begin{pmatrix}1&0&-f_1\\0&1&1\\0&0&1\end{pmatrix}\right.$$

$$\left.\cdot\begin{pmatrix}1&0&0\\0&1&0\\h&hf_1&1\end{pmatrix}\begin{pmatrix}1&0&f_1\\0&1&-1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&f_2\\0&1&0\\0&0&1\end{pmatrix}\right]$$

$$=\begin{pmatrix}1&0&0\\0&1&0\\-h&-h(f_1+f_2)&1\end{pmatrix}$$

$$\cdot\begin{pmatrix}1&0&0\\0&1+hf_2&-hf_2\\0&hf_2&1-hf_2\end{pmatrix}\begin{pmatrix}1&-(f_1+f_2)hf_2&(f_1+f_2)hf_2\\0&1&0\\0&0&1\end{pmatrix}$$

$$\cdot\begin{pmatrix}1&-hf_1f_2&0\\0&1&0\\0&hf_1&1\end{pmatrix}\begin{pmatrix}1-hf_2&0&-hf_2^2\\0&1&0\\h&0&1+hf_2\end{pmatrix}$$

$$\cdot\begin{pmatrix}1&0&-hf_1f_2\\0&1&hf_2\\0&0&1\end{pmatrix}\begin{pmatrix}1-hf_1&-hf_1^2&0\\h&1+hf_1&0\\0&0&1\end{pmatrix}.$$

Notice now that by assumption, and by the remark at the beginning of the proof, the latter expression is congruent mod IA^m to

$$\begin{pmatrix}1&0&0\\0&1+hf_2&-hf_2\\0&hf_2&1-hf_2\end{pmatrix}.$$

Consider now (7.21) in Corollary 7.15, and switch the roles of f and g by $-h$ and f_2, respectively. Using this identity we deduce that, mod IA^m, the latter expression is congruent to

$$\begin{pmatrix}1-hf_2&-h&0\\hf_2^2&1+hf_2&0\\0&0&1\end{pmatrix}\begin{pmatrix}1+hf_2&0&h\\0&1&0\\-hf_2^2&0&1-hf_2\end{pmatrix},$$

which is congruent to I_{n-1} by assumption. This finishes the proof of the lemma. □

We pass to the next stage:

Proposition 7.25. *The elements of the form*

$$(I_{n-1}-fE_{j,i})(I_{n-1}+hE_{i,j})(I_{n-1}+fE_{j,i}),$$

where $h\in\sigma_n\sigma_r^2U_{r,m}$, $\sigma_n^2U_{n,m}$ *for* $1\le r\le n-1$, $\underline{f\in\mathbb{Z}}$ *and* $i\neq j$, *belong to* IA^m.

Remark 7.26. We note that some of the matrices that we use in the following computations lie in $\mathrm{IGL}'_{n-1,n} \hookrightarrow \mathrm{GL}_{n-1}(R_n)$ and not necessarily in $\mathrm{IGL}_{n-1,n}$ (see Definition 3.12 and Proposition 3.13).

Proof of Proposition 7.25. According to Lemma 7.24, it is enough to prove the proposition for $f = \pm 1$. Without loss of generality, we prove the proposition for $r = 1$, i.e., $h \in \sigma_n \sigma_1^2 U_{1,m}$, and symmetrically, the same is valid for every $1 \le r \le n-1$. The case $h \in \sigma_n^2 U_{n,m}$ is considered separately.

So let $h \in \sigma_n \sigma_1^2 U_{1,m}$ and write $h = \sigma_1 u$ for some $u \in \sigma_n \sigma_1 U_{1,m}$. We prove the proposition for $i \neq j \in \{1, 2, 3\}$ — as one can see below, we do it simultaneously for all the options for $i \neq j \in \{1, 2, 3\}$. The treatment in the other cases in which $i \neq j \in \{1, k, l\}$ such that $1 < k \neq l \le n-1$ is obtained symmetrically, so we get that the proposition is valid for every $1 \le i \neq j \le n-1$.

As before, we denote a block matrix of the form

$$\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$$

by $B \in \mathrm{GL}_3(R_n)$. In the following computations, the indices of the matrices are intended to help the reader recognize the corresponding matrix type in Corollary 7.15, as explained below. We recall that the inverse of a matrix is denoted by the same index, and one can observe that the inverse of each indexed matrix is obtained by changing the sign of u. We also recall that $u \in \sigma_n \sigma_1 U_{1,m} \subseteq \sigma_n R_n$. Thus, by Proposition 6.1 we have

$$\begin{pmatrix} 1-\sigma_1 u & -\sigma_1^2 u & 0 \\ u & 1+\sigma_1 u & 0 \\ 0 & 0 & 1 \end{pmatrix}_{12} = \begin{pmatrix} x_2 & -\sigma_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ ux_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2^{-1} & x_2^{-1}\sigma_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{IA}^m,$$

$$\begin{pmatrix} 1-\sigma_1 u & 0 & -\sigma_1^2 u \\ 0 & 1 & 0 \\ u & 0 & 1+\sigma_1 u \end{pmatrix}_{7} = \begin{pmatrix} x_3 & 0 & -\sigma_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ux_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_3^{-1} & 0 & x_3^{-1}\sigma_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{IA}^m,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\sigma_1 u & u \\ 0 & -\sigma_1^2 u & 1-\sigma_1 u \end{pmatrix}_{3} = \begin{pmatrix} 1 & 0 & 0 \\ u\sigma_2 & 1 & 0 \\ -u\sigma_1\sigma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_2 & -\sigma_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sigma_2 & \sigma_1 & 1 \end{pmatrix} \in \mathrm{IA}^m,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\sigma_1 u & -\sigma_1^2 u \\ 0 & u & 1+\sigma_1 u \end{pmatrix}_{6} = \begin{pmatrix} 1 & 0 & 0 \\ -u\sigma_1\sigma_3 & 1 & 0 \\ u\sigma_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \sigma_3 & 1 & -\sigma_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\sigma_3 & 1 & \sigma_1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{IA}^m.$$

By switching the signs of σ_1, σ_2 and σ_3 in the two latter computations we obtain also that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\sigma_1 u & u \\ 0 & -\sigma_1^2 u & 1+\sigma_1 u \end{pmatrix}_{1}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\sigma_1 u & -\sigma_1^2 u \\ 0 & u & 1-\sigma_1 u \end{pmatrix}_{8} \in \mathrm{IA}^m.$$

Consider now the identities which we got in Corollary 7.15, and switch the roles of f, g in the corollary by u, σ_1, respectively. Remember that $u \in \sigma_n \sigma_1 U_{1,m}$. Hence, as by the computations above matrices of Forms 7 and 8 belong to IA^m, we obtain from the last part of (7.16) that also matrices of Form 13 belong to IA^m. Thus, as we showed that Forms 1, 3, 6 also belong to IA^m, (7.16) shows that Forms 2, 4, 5 also belong to IA^m. Similar arguments show that (7.16)–(7.21) give that all the 18 forms belong to IA^m. In particular, the matrices which correspond to Forms 13–18 belong to IA^m, and these matrices (and their inverses) are precisely the matrices of the form

$$(I_{n-1} \pm E_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} \mp E_{j,i}), \quad i \neq j \in \{1,2,3\}$$

(recalling that $h = \sigma_1 u$). Clearly, by similar arguments, the proposition holds for every $1 \le i \neq j \le n-1$ and every $h \in \sigma_n \sigma_r^2 U_{r,m}$ for $1 \le r \le n-1$.

The case $h \in \sigma_n^2 U_{n,m}$ is a bit different, but easier. In this case one can consider the same computations we built for $r = 1$, with the following modifications: Firstly, write $h \in \sigma_n^2 U_{n,m}$ as $h = \sigma_n u$ for some $u \in \sigma_n U_{n,m}$. Secondly, change σ_1 to σ_n, change σ_2, σ_3 to 0 and change x_2, x_3 to 1 in the right side of the above equations. It is easy to see that in this situation we obtain in the left side of the equations the same matrices, just that instead of σ_1 we have σ_n. From here we continue exactly the same. □

Proposition 7.27. *The elements of the following form belong to* IA^m:

$$(I_{n-1} - fE_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} + fE_{j,i}),$$

where $h \in \sigma_n^2 U_{n,m}, \sigma_n \sigma_r^2 U_{r,m}$ *for* $1 \le r \le n-1$, $\underline{f \in \sigma_s R_n}$ *for* $1 \le s \le n$ *and* $i \neq j$.

Proof. We prove it for $s = 1$, $i \neq j \in \{1,2,3\}$, and denote a block matrix of the form

$$\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$$

by $B \in \mathrm{GL}_3(R_n)$. We use again the result of Corollary 7.15, when we switch the roles of f, g in the corollary by $h, \sigma_1 u$, respectively, for some $u \in R_n$.

As $h \in \sigma_n \sigma_r^2 U_{r,m}, \sigma_n^2 U_{n,m}$, we have also $\sigma_1 uh \in \sigma_n \sigma_r^2 U_{r,m}, \sigma_n^2 U_{n,m}$. Hence, we obtain from the previous proposition that the matrices of Forms 13–18 belong to IA^m. In addition,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-u\sigma_1 h & h \\ 0 & -u^2\sigma_1^2 h & 1+u\sigma_1 h \end{pmatrix}_1$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ -hu\sigma_2 & 1 & 0 \\ -hu^2\sigma_1\sigma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -u\sigma_2 & u\sigma_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & h \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u\sigma_2 & -u\sigma_1 & 1 \end{pmatrix} \in \mathrm{IA}^m,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-u\sigma_1 h & -u^2\sigma_1^2 h \\ 0 & h & 1+u\sigma_1 h \end{pmatrix}_6$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ -hu^2\sigma_1\sigma_3 & 1 & 0 \\ hu\sigma_3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ u\sigma_3 & 1 & -u\sigma_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & h & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -u\sigma_3 & 1 & u\sigma_1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{IA}^m,$$

and by switching the signs of u and h simultaneously, we get also Forms 3 and 8. So we easily conclude from Corollary 7.15 ((7.16) and (7.18)) that also the matrices of the other eight forms are in IA^m. In particular, the matrices of the form

$$(I_{n-1} - \sigma_1 u E_{j,i})(I_{n-1} + hE_{i,j})(I_{n-1} + \sigma_1 u E_{j,i}), \quad i \neq j \in \{1, 2, 3\},$$

belong to IA^m. The treatment for every $i \neq j$ and $1 \leq s \leq n-1$ is similar, and the treatment in the case $s = n$ is obtained by replacing σ_1 by σ_n and σ_2, σ_3 by 0 in the above equations. □

Corollary 7.28. *As every $f \in R_n$ can be decomposed as*

$$f = \sum_{s=1}^{n} \sigma_s f_s + f_0$$

for some $f_0 \in \mathbb{Z}$ and $f_i \in R_n$, we obtain from Lemma 7.24 and from the above two propositions that we actually finished the proof of Proposition 7.23.

***Elements of Form* 3.**

Proposition 7.29. *Recall $\bar{O}_m = mR_{n-1}$, where*

$$R_{n-1} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}] \subseteq R_n.$$

Then the elements of the form

$$A^{-1}[(I_{n-1} + hE_{i,j}), (I_{n-1} + fE_{j,i})]A$$

belong to IA^m, *where $A \in \mathrm{GL}_{n-1}(R_n)$, $f \in \sigma_n R_n$, $h \in \bar{O}_m^2$ and $i \neq j$.*

We prove the proposition in the case $i, j = 2, 1$, and the same arguments are valid for arbitrary $i \neq j$. In this case one can write $h = m^2 h'$ for some $h' \in R_{n-1}$, and thus, our element is of the form

$$A^{-1} \begin{pmatrix} 1 - fm^2h' & f & 0 \\ -f(m^2h')^2 & 1 + fm^2h' & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} \begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} A$$

for some $A \in \mathrm{GL}_{n-1}(R_n)$, $f \in \sigma_n R_n$ and $h' \in R_{n-1}$. The proposition follows easily from the following lemma.

Lemma 7.30. *Let $h_1, h_2 \in R_n$, $f \in \sigma_n R_n$ and denote a block matrix of the form*
$$\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$$
by $B \in \mathrm{GL}_3(R_n)$. Then
$$\begin{aligned}
&A^{-1}\begin{pmatrix} 1-fm(h_1+h_2) & f & 0 \\ -f(m(h_1+h_2))^2 & 1+fm(h_1+h_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}A \\
&\qquad\equiv A^{-1}\begin{pmatrix} 1-fmh_1 & f & 0 \\ -f(mh_1)^2 & 1+fmh_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\qquad\quad\cdot\begin{pmatrix} 1-fmh_2 & f & 0 \\ -f(mh_2)^2 & 1+fmh_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}A \mod \mathrm{IA}^m .
\end{aligned}$$

Now, if Lemma 7.30 is proved, one can deduce that for $f \in \sigma_n R_n$ and $h = m^2h'$, $h' \in R_n$, we have
$$\begin{aligned}
&A^{-1}\begin{pmatrix} 1-fm^2h' & f & 0 \\ -f(m^2h')^2 & 1+fm^2h' & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}\begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}A \\
&\qquad\equiv \left[A^{-1}\begin{pmatrix} 1-fmh' & f & 0 \\ -f(mh')^2 & 1+fmh' & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}\begin{pmatrix} 1 & -f & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix}A\right]^m \mod \mathrm{IA}^m
\end{aligned}$$
and as the latter element obviously belongs to IA^m, Proposition 7.29 follows. So it is enough to prove Lemma 7.30.

Proof of Lemma 7.30. Throughout this computation we use the observation that as $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n)$ is normal in $\mathrm{GL}_{n-1}(R_n)$, every conjugate of an element of $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \leq \mathrm{IA}(\Phi)$ by an element of $\mathrm{GL}_{n-1}(R_n)$ is also an element of $\mathrm{GL}_{n-1}(R_n, \sigma_n R_n) \leq \mathrm{IA}(\Phi)$ (as was mentioned in Remark 7.7) — even though $\mathrm{GL}_{n-1}(R_n) \not\leq \mathrm{IA}(\Phi)$. Throughout the computation, we use the below notation:

- A matrix $\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$ is denoted by $B \in \mathrm{GL}_3(R_n)$.
- "$=$" denotes an equality between matrices in $\mathrm{GL}_{n-1}(R_n)$.
- "$\equiv$" denotes an equality in $\mathrm{IA}(\Phi)/\mathrm{IA}^m$.
- We use square brackets to help the reader follow the steps of the computation. Whenever square brackets are used, it is recommended to concentrate on the expression inside them separately in order to follow the transition to the next step.

So, let's compute:

$$A^{-1}\begin{pmatrix}1-fm(h_1+h_2) & f & 0\\ -f(m(h_1+h_2))^2 & 1+fm(h_1+h_2) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -f & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$$=A^{-1}\begin{pmatrix}1 & 0 & -f\\ 0 & 1 & -fm(h_1+h_2)\\ -m(h_1+h_2) & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & f\\ 0 & 1 & fm(h_1+h_2)\\ m(h_1+h_2) & -1 & 1\end{pmatrix}\begin{pmatrix}1 & -f & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$$=A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -m(h_1+h_2) & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -f\\ 0 & 1 & -fm(h_1+h_2)\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ m(h_1+h_2) & -1 & 1\end{pmatrix}$$

$$\cdot\begin{pmatrix}1 & 0 & f\\ 0 & 1 & fm(h_1+h_2)\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -f & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$$=A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -mh_2 & 0 & 1\end{pmatrix}\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -mh_1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -f\\ 0 & 1 & -fmh_1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ mh_1 & -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & f\\ 0 & 1 & fmh_1\\ 0 & 0 & 1\end{pmatrix}\right]$$

$$\cdot\begin{pmatrix}1 & 0 & -f\\ 0 & 1 & -fmh_1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ mh_2 & 0 & 1\end{pmatrix}A$$

$$\cdot\left[A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -m(h_1+h_2) & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -fmh_2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ m(h_1+h_2) & -1 & 1\end{pmatrix}A\right]$$

$$\cdot\left[A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & f(h_1+h_2)\\ 0 & 0 & 1\end{pmatrix}A\right]^m A^{-1}\begin{pmatrix}1 & -f & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$$\equiv A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -mh_2 & 0 & 1\end{pmatrix}\begin{pmatrix}1-fmh_1 & f & 0\\ -f(mh_1)^2 & 1+fmh_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & -f\\ 0 & 1 & -fmh_1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ mh_2 & 0 & 1\end{pmatrix}A$$

$$\cdot\left[A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -m(h_1+h_2) & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -fh_2\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ m(h_1+h_2) & -1 & 1\end{pmatrix}A\right]^m$$

$$\cdot A^{-1}\begin{pmatrix}1 & -f & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$

$$\equiv A^{-1}\left[\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1-fmh_1&f&0\\-f(mh_1)^2&1+fmh_1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}\right]$$

$$\cdot\left[\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1&0&-f\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}\right]$$

$$\cdot\left[\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-fmh_1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}\right]\begin{pmatrix}1&-f&f\\0&1&0\\0&0&1\end{pmatrix}A$$

$$= A^{-1}\begin{pmatrix}1-fmh_1&f&0\\-f(mh_1)^2&1+fmh_1&0\\0&0&1\end{pmatrix}A\left[A^{-1}\begin{pmatrix}1&0&0\\0&1&0\\fmh_1h_2&-fh_2&1\end{pmatrix}A\right]^m$$

$$\cdot A^{-1}\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1&0&-f\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}A$$

$$\cdot\left[A^{-1}\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-fh_1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}A\right]^m A^{-1}\begin{pmatrix}1&-f&f\\0&1&0\\0&0&1\end{pmatrix}A$$

$$\equiv A^{-1}\begin{pmatrix}1-fmh_1&f&0\\-f(mh_1)^2&1+fmh_1&0\\0&0&1\end{pmatrix}$$

$$\cdot\left[\begin{pmatrix}1&0&0\\0&1&0\\-mh_2&0&1\end{pmatrix}\begin{pmatrix}1&0&-f\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\mh_2&0&1\end{pmatrix}\right]\begin{pmatrix}1&-f&f\\0&1&0\\0&0&1\end{pmatrix}A$$

$$= A^{-1}\begin{pmatrix}1-fmh_1&f&0\\-f(mh_1)^2&1+fmh_1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-f&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\cdot\left[\begin{pmatrix}1&f&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1-mfh_2&0&-f\\0&1&0\\f(mh_2)^2&0&1+mfh_2\end{pmatrix}\begin{pmatrix}1&-f&0\\0&1&0\\0&0&1\end{pmatrix}\right]\begin{pmatrix}1&0&f\\0&1&0\\0&0&1\end{pmatrix}A$$

$$= A^{-1}\begin{pmatrix}1-fmh_1&f&0\\-f(mh_1)^2&1+fmh_1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&-f&0\\0&1&0\\0&0&1\end{pmatrix}A\left[A^{-1}\begin{pmatrix}1&f^2h_2&0\\0&1&0\\0&-m(fh_2)^2&1\end{pmatrix}A\right]^m A^{-1}$$

$$\cdot\begin{pmatrix}1-mfh_2&0&-f\\0&1&0\\f(mh_2)^2&0&1+mfh_2\end{pmatrix}\begin{pmatrix}1&0&f\\0&1&0\\0&0&1\end{pmatrix}A$$

$$\equiv A^{-1}\begin{pmatrix}1-fmh_1 & f & 0\\ -f(mh_1)^2 & 1+fmh_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -f & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$
$$\cdot\begin{pmatrix}1-fmh_2 & 0 & -f\\ 0 & 1 & 0\\ f(mh_2)^2 & 0 & 1+fmh_2\end{pmatrix}\begin{pmatrix}1 & 0 & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A.$$

So it remains to show that

$$A^{-1}\begin{pmatrix}1-fmh_2 & 0 & -f\\ 0 & 1 & 0\\ f(mh_2)^2 & 0 & 1+fmh_2\end{pmatrix}\begin{pmatrix}1 & 0 & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A$$
$$\equiv A^{-1}\begin{pmatrix}1-fmh_2 & f & 0\\ -f(mh_2)^2 & 1+fmh_2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & -f & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A. \quad (7.31)$$

By a similar computation as for (7.14), switching the roles of f, g in the equation by f, mh_2, respectively, and then switching the roles of the first row and column with the third row and column, we have

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1+fmh_2 & fmh_2\\ 0 & -fmh_2 & 1-fmh_2\end{pmatrix}$$
$$=\begin{pmatrix}1 & -f & -f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1+fmh_2 & 0 & f\\ 0 & 1 & 0\\ -f(mh_2)^2 & 0 & 1-fmh_2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ f(mh_2)^2 & 1 & fmh_2\\ 0 & 0 & 1\end{pmatrix}$$
$$\cdot\begin{pmatrix}1-fmh_2 & f & 0\\ -f(mh_2)^2 & 1+fmh_2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ f(mh_2)^2 & -fmh_2 & 1\end{pmatrix}.$$

Therefore, using Proposition 7.9 and the observation

$$A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1+fmh_2 & fmh_2\\ 0 & -fmh_2 & 1-fmh_2\end{pmatrix}A=\left[A^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & 1+fh_2 & fh_2\\ 0 & -fh_2 & 1-fh_2\end{pmatrix}A\right]^m\in \mathrm{IA}^m,$$

we obtain that mod IA^m we have

$$A^{-1}\begin{pmatrix}1+fmh_2 & 0 & f\\ 0 & 1 & 0\\ -f(mh_2)^2 & 0 & 1-fmh_2\end{pmatrix}\begin{pmatrix}1-fmh_2 & f & 0\\ -f(mh_2)^2 & 1+fmh_2 & 0\\ 0 & 0 & 1\end{pmatrix}A$$
$$\equiv A^{-1}\begin{pmatrix}1 & f & f\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}A. \quad (7.32)$$

From here, we easily get (7.31) by noticing that the inverse of every matrix in (7.32) is obtained by replacing f by $-f$. This finishes the proof of the lemma, and hence, also the proof of Proposition 7.29. $\square$

***Elements of Form* 4.**

Proposition 7.33. *Recall $\overline{O}_m = mR_{n-1}$ and $\overline{U}_{r,m} = (x_r^m - 1)R_{n-1}$ for $1 \le r \le n-1$, where $R_{n-1} = \mathbb{Z}[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}] \subseteq R_n$. The elements of the form*

$$A^{-1}[(I_{n-1} + hE_{i,j}), (I_{n-1} + fE_{j,i})]A$$

where $A \in \mathrm{GL}_{n-1}(R_n)$, $f \in \sigma_n R_n$, $h \in \sigma_r^2 \overline{U}_{r,m}, \sigma_r \overline{O}_m$ for $1 \le r \le n-1$ and $i \neq j$, belong to IA^m.

As before, throughout the subsection we denote a block matrix of the form

$$\begin{pmatrix} B & 0 \\ 0 & I_{n-4} \end{pmatrix} \in \mathrm{GL}_{n-1}(R_n)$$

by $B \in \mathrm{GL}_3(R_n)$. We start the proof of this proposition with the following lemma.

Lemma 7.34. *Let $f, h \in R_n$ and $A \in \mathrm{GL}_{n-1}(R_n)$. Then*

$$\begin{aligned}
&A^{-1}\begin{pmatrix} 1-fh & -fh & 0 \\ fh & 1+fh & 0 \\ 0 & 0 & 1 \end{pmatrix}A \\
&\quad = A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ fh & 1 & 0 \\ fh^2 & 0 & 1 \end{pmatrix}AA^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fh & -f \\ 0 & fh^2 & 1-fh \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}A \\
&\qquad \cdot A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix}AA^{-1}\begin{pmatrix} 1 & -fh & 0 \\ 0 & 1 & 0 \\ 0 & -fh^2 & 1 \end{pmatrix}AA^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}A \\
&\qquad \cdot A^{-1}\begin{pmatrix} 1-fh & 0 & f \\ 0 & 1 & 0 \\ -fh^2 & 0 & 1+fh \end{pmatrix}AA^{-1}\begin{pmatrix} 1 & 0 & 0 \\ f^2h^2 & 1 & -f^2h \\ 0 & 0 & 1 \end{pmatrix}AA^{-1}\begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}A.
\end{aligned}$$

Proof. The lemma follows from Proposition 7.10, line (7.11), by substituting g with h, combined with verifying the identity

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1-fh & 0 & f \\ 0 & 1 & 0 \\ -fh^2 & 0 & 1+fh \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \\
&\qquad\qquad = \begin{pmatrix} 1-fh & 0 & f \\ 0 & 1 & 0 \\ -fh^2 & 0 & 1+fh \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ f^2h^2 & 1 & -f^2h \\ 0 & 0 & 1 \end{pmatrix}. \quad \square
\end{aligned}$$

Observe now that if we have $f \in \sigma_n R_n$ and $h \in \sigma_r^2 \overline{U}_{r,m}, \sigma_r \overline{O}_m$ for $1 \le r \le n-1$, then by Propositions 7.9 and 7.23, we have

$$A^{-1}\begin{pmatrix} 1-fh & -fh & 0 \\ fh & 1+fh & 0 \\ 0 & 0 & 1 \end{pmatrix} A = A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -fh & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A \in \mathrm{IA}^m .$$

So, by Propositions 7.9 and 7.23 and the previous lemma, for every $A \in \mathrm{GL}_{n-1}(R_n)$ we have the following equality mod IA^m:

$$\begin{aligned} &A^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+fh & -f \\ 0 & fh^2 & 1-fh \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} A \\ &\qquad \equiv A^{-1}\left[\begin{pmatrix} 1-fh & 0 & f \\ 0 & 1 & 0 \\ -fh^2 & 0 & 1+fh \end{pmatrix}\begin{pmatrix} 1 & 0 & -f \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]^{-1} A. \end{aligned}$$

We thus have the following corollary (notice that we switched the sign of f).

Corollary 7.35. *For every $h \in \sigma_r^2 \overline{U}_{r,m}, \sigma_r \overline{O}_m$ for $1 \le r \le n-1$, $f \in \sigma_n R_n$ and $A \in \mathrm{GL}_{n-1}(R_n)$, the following elements are congruent mod* IA^m:

$$A^{-1}[(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})]A \equiv A^{-1}[(I_{n-1} - fE_{1,3}), (I_{n-1} + hE_{3,1})]A.$$

We proceed with the following proposition:

Proposition 7.36. *Let $h \in \sigma_1^2 \overline{U}_{1,m}, \sigma_1 \overline{O}_m$ and $f \in \sigma_n R_n$. Then*

$$[(I_{n-1} + hE_{3,2}), (I_{n-1} + fE_{2,3})] \in \mathrm{IA}^m .$$

Proof. Let $h = \sigma_1 u$ for some $u \in \sigma_1 \overline{U}_{1,m}, \overline{O}_m$. By Proposition 6.1, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sigma_2 u & \sigma_1 u & 1 \end{pmatrix} \in \mathrm{IA}^m$$

and hence

$$\begin{aligned} \mathrm{IA}^m \ni &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\sigma_2 u & \sigma_1 u & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_2 u & -\sigma_1 u & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ f\sigma_2 u & 1 & 0 \\ \sigma_1\sigma_2 u^2 f & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sigma_1 u & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\sigma_1 u & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

As by Proposition 6.1 the first matrix in the right-hand side is also in IA^m, we obtain that

$$[(I_{n-1}+hE_{3,2}),(I_{n-1}+fE_{2,3})]=\left[\begin{pmatrix}1&0&0\\0&1&0\\0&h&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&f\\0&0&1\end{pmatrix}\right]\in\mathrm{IA}^m$$

as required. □

We can now pass to the following proposition.

Proposition 7.37. *Let $h\in\sigma_1^2\overline{U}_{1,m},\sigma_1\overline{O}_m$, $f\in\sigma_n R_n$ and $A\in\mathrm{GL}_{n-1}(R_n)$. Then*

$$A^{-1}[(I_{n-1}+hE_{3,2}),(I_{n-1}+fE_{2,3})]A\in\mathrm{IA}^m\,.$$

Proof. We prove the proposition by induction. By a result of Suslin [1977], as $n-1\geq 3$, the group $\mathrm{SL}_{n-1}(R_n)$ is generated by the elementary matrices of the form

$$I_{n-1}+rE_{l,k}\quad\text{for } r\in R_n \text{ and } 1\leq l\neq k\leq n-1.$$

So as the invertible elements of R_n are the elements of the form

$$\pm\prod_{i=1}^{n}x_i^{s_i}\quad\text{for } s_i\in\mathbb{Z}$$

(see [Crowell and Fox 1963, Chapter 8]), $\mathrm{GL}_{n-1}(R_n)$ is generated by the elementary matrices and the matrices of the form

$$I_{n-1}+(\pm x_i-1)E_{1,1}\quad\text{for } 1\leq i\leq n.$$

Therefore, by the previous proposition it is enough to show that if

$$A^{-1}[(I_{n-1}+hE_{3,2}),(I_{n-1}+fE_{2,3})]A\in\mathrm{IA}^m$$

and E is one of the above generators, then mod IA^m we have

$$A^{-1}E^{-1}[(I_{n-1}+hE_{3,2}),(I_{n-1}+fE_{2,3})]EA \equiv A^{-1}[(I_{n-1}+hE_{3,2}),(I_{n-1}+fE_{2,3})]A. \quad (7.38)$$

So if E is of the form $I_{n-1}+(\pm x_i-1)E_{1,1}$, we obviously have Property (7.38). If E is an elementary matrix of the form $I_{n-1}+rE_{l,k}$ such that $l,k\notin\{2,3\}$, then we also have Property (7.38) in an obvious way. Consider now the case $l,k=2,3$. In this case, by Corollary 7.35 we have the following mod IA^m:

$$\begin{aligned}
A^{-1}E^{-1}&\left[\begin{pmatrix}1&0&0\\0&1&0\\0&h&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&f\\0&0&1\end{pmatrix}\right]EA\\
&\equiv A^{-1}\begin{pmatrix}1&0&0\\0&1&-r\\0&0&1\end{pmatrix}\left[\begin{pmatrix}1&0&-f\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\h&0&1\end{pmatrix}\right]\begin{pmatrix}1&0&0\\0&1&r\\0&0&1\end{pmatrix}A\\
&=A^{-1}\begin{pmatrix}1&0&0\\0&1&-r\\0&0&1\end{pmatrix}\begin{pmatrix}1-hf+h^2f^2&0&-hf^2\\0&1&0\\-h^2f&0&1+hf\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&r\\0&0&1\end{pmatrix}A\\
&=A^{-1}\begin{pmatrix}1-hf+h^2f^2&0&-hf^2\\0&1&0\\-h^2f&0&1+hf\end{pmatrix}AA^{-1}\begin{pmatrix}1&0&0\\rh^2f&1&-rhf\\0&0&1\end{pmatrix}A\\
&=A^{-1}\left[\begin{pmatrix}1&0&-f\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\h&0&1\end{pmatrix}\right]AA^{-1}\begin{pmatrix}1&0&0\\rh^2f&1&-rhf\\0&0&1\end{pmatrix}A.
\end{aligned}$$

So by applying Propositions 7.9 and 7.23 and Corollary 7.35 once again in the opposite way, we obtain Property (7.38). The other cases for l,k are treated by similar arguments: if $l,k=3,2$ we do exactly the same, and if l or k are different from 2 and 3, then the situation is easier — we use similar arguments, but without passing to $[(I_{n-1}-fE_{1,3}),(I_{n-1}+hE_{3,1})]$ through Corollary 7.35. □

Corollary 7.39. *Let $h\in\sigma_1^2\overline{U}_{1,m},\sigma_1\overline{O}_m$, $f\in\sigma_nR$ and $A\in\mathrm{GL}_{n-1}(R_n)$. Then for every $i\neq j$, we have*

$$A^{-1}[(I_{n-1}+hE_{i,j}),(I_{n-1}+fE_{j,i})]A\in\mathrm{IA}^m.$$

Proof. Denote a permutation matrix, such that its action on $\mathrm{GL}_{n-1}(R_n)$ by conjugation moves $2\mapsto j$ and $3\mapsto i$, by P. Then, by the previous proposition, we have

$$\begin{aligned}A^{-1}&[(I_{n-1}+hE_{i,j}),(I_{n-1}+fE_{j,i})]A\\
&=A^{-1}P^{-1}[(I_{n-1}+hE_{3,2}),(I_{n-1}+fE_{2,3})]PA\in\mathrm{IA}^m.\end{aligned}$$ □

Now, since one can see that symmetrically, the above corollary is valid for every $h\in\sigma_r^2\overline{U}_{r,m},\sigma_r\overline{O}_m$ for $1\le r\le n-1$, we have actually finished the proof of Proposition 7.33.

8. Index of notation

For convenience, we gather here some notation that plays a role in the paper, and mention the section where they appear for the first time:

- $F_n =$ the free group on n elements (Section 3).
- $\Phi = \Phi_n = F_n/F_n'' =$ the free metabelian group on n elements (Section 3).
- $\Psi_m = \Phi/M_m$ where $M_m = (\Phi'\Phi^m)'(\Phi'\Phi^m)^m$ (Section 3).
- $\mathrm{IA}(\Phi) = \ker(\mathrm{Aut}(\Phi) \to \mathrm{Aut}(\Phi/\Phi'))$ (Section 3).
- $\mathrm{IG}_m = G(M_m) = \ker(\mathrm{IA}(\Phi) \to \mathrm{Aut}(\Psi_m))$ (Section 3).
- $\mathrm{IA}^m = \langle \mathrm{IA}(\Phi)^m \rangle$ (Section 7).
- $\mathrm{IA}_m = \bigcap\{N \lhd \mathrm{IA}(\Phi) \mid [\mathrm{IA}(\Phi) : N] \mid m\}$ (Section 3).
- $R_n = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, where the $x_1, \ldots, x_n$ are free commutative variables (Section 3).
- $R_{n-1} = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$ (Section 7).
- $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ (Section 3).
- $\sigma_i = x_i - 1$ for $1 \le i \le n$ (Section 3).
- $\vec{\sigma} =$ the column vector which has σ_i in its i-th entry (Section 3).
- $\mathfrak{A} = \sum_{i=1}^n \sigma_i R_n \lhd R_n =$ the augmentation ideal of R_n (Section 3).
- $O_m = mR_n \lhd R_n$ (Section 7).
- $\bar{O}_m = mR_{n-1} \lhd R_{n-1}$ (Section 7).
- $U_{r,m} = (x_r^m - 1)R_n \lhd R_n$ for $1 \le r \le n$ (Section 7).
- $\bar{U}_{r,m} = (x_r^m - 1)R_{n-1} \lhd R_{n-1}$ for $1 \le r \le n$ (Section 7).
- $H_m = \sum_{i=1}^n (x_i^m - 1)R_n + mR_n \lhd R_n$ (Section 3).
- $S = \mathbb{Z}[x^{\pm 1}]$ (Section 4).
- $J_m = (x^m - 1)S + mS \lhd S$ (Section 4).
- $E_d(R) = \langle I_d + rE_{i,j} \mid r \in R,\ 1 \le i \ne j \le d \rangle \le \mathrm{SL}_d(R)$, where R is a ring and $E_{i,j}$ is the matrix that has 1 in its (i, j)-th entry and 0 elsewhere (Section 2).
- $\mathrm{SL}_d(R, H) = \ker(\mathrm{SL}_d(R) \to \mathrm{SL}_d(R/H))$, where R is a ring and $H \lhd R$ (Section 2).
- $\mathrm{GL}_d(R, H) = \ker(\mathrm{GL}_d(R) \to \mathrm{GL}_d(R/H))$, where R is a ring and $H \lhd R$ (Section 2).
- $E_d(R, H) =$ the normal subgroup of $E_d(R)$, generated as a normal subgroup by the matrices of the form $I_d + hE_{i,j}$ for $h \in H$ (Section 2).
- $\mathrm{IGL}_{n-1,i} = \left\{ I_n + A \in \mathrm{IA}(\Phi) \;\middle|\; \begin{array}{l} \text{the } i\text{-th row of } A \text{ is } 0, \\ \qquad I_{n-1} + A_{i,i} \in \mathrm{GL}_{n-1}(R_n, \sigma_i R_n) \end{array} \right\}$, for $1 \le i \le n$ (Section 3).
- $\mathrm{ISL}_{n-1,i}(H) = \mathrm{IGL}_{n-1,i} \cap \mathrm{SL}_{n-1}(R_n, H)$, under the identification of $\mathrm{IGL}_{n-1,i}$ with $\mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$ (Section 3).

- $\mathrm{IE}_{n-1,i}(H) = \mathrm{IGL}_{n-1,i} \cap E_{n-1}(R_n, H)$, under the identification of the group $\mathrm{IGL}_{n-1,i}$ with $\mathrm{GL}_{n-1}(R_n, \sigma_i R_n)$ (Section 3).
- $\mathrm{IGL}'_{n-1,i} = \{I_n + A \in \mathrm{IA}(\Phi) \mid \text{the } i\text{-th row of } A \text{ is } 0\}$ for $1 \le i \le n$ (Section 3).

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Vanishing theorems for the negative K-theory of stacks

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We prove that the homotopy algebraic K-theory of tame quasi-DM stacks satisfies cdh-descent. We apply this descent result to prove that if $\mathcal{X}$ is a Noetherian tame quasi-DM stack and $i < -\dim(\mathcal{X})$, then $K_i(\mathcal{X})[1/n] = 0$ if n is nilpotent on $\mathcal{X}$ and $K_i(\mathcal{X}, \mathbb{Z}/n) = 0$ if n is invertible on $\mathcal{X}$. Our descent and vanishing results apply more generally to certain Artin stacks whose stabilizers are extensions of finite group schemes by group schemes of multiplicative type.

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1. Introduction

The negative K-theory of rings was defined by Bass [1968] and later generalized to all schemes by Thomason and Trobaugh [1990], who established its fundamental properties such as localization, excision, Mayer–Vietoris, and the projective bundle formula.

As explained in [Thomason and Trobaugh 1990], these properties of K-theory give rise to the Bass–Thomason–Trobaugh nonconnective K-theory, or K^B-theory, which is usually nontrivial in negative degrees for singular schemes. A famous conjecture of Weibel asserts that for a Noetherian scheme X of Krull dimension d, the group $K_i(X)$ vanishes for $i < -d$. This conjecture was settled by Weibel [2001] for excellent surfaces, by Cortiñas, Haesemeyer, Schlichting and Weibel [Cortiñas et al. 2008] for schemes essentially of finite type over a field of characteristic zero, and recently by Kerz, Strunk and Tamme [Kerz et al. 2018] for all Noetherian schemes.

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Before a complete proof of Weibel's conjecture for schemes appeared in [Kerz et al. 2018], Kelly [2014] used the alteration methods of de Jong and Gabber to show that the vanishing conjecture for negative K-theory holds in characteristic $p > 0$ if one is allowed to invert p. Later, Kerz and Strunk [2017] gave a different proof of Kelly's theorem by proving Weibel's conjecture for negative homotopy K-theory, or KH-theory, a variant of K-theory introduced by Weibel [1989]. In their proof, Kerz and Strunk used the method of flatification by blow-up instead of alterations.

It is natural to ask for an extension of Weibel's conjecture to algebraic stacks. The algebraic K-theory of quotient stacks was introduced by Thomason [1987a] in order to study algebraic K-theory of a scheme which can be equipped with an action of a group scheme. The localization, excision, and Mayer–Vietoris properties for the algebraic K-theory of tame Deligne–Mumford stacks were proven by the second author and Østvær [Krishna and Østvær 2012], and together with the projective bundle formula they were established for more general quotient stacks by the second author and Ravi [Krishna and Ravi 2018]. The K^B-theory of Bass–Thomason–Trobaugh and the KH-theory of Weibel were also generalized to such quotient stacks in [Krishna and Ravi 2018].

The purpose of this paper is to show that the approach of Kerz and Strunk can be generalized to a large class of algebraic stacks, including all tame Artin stacks in the sense of [Abramovich et al. 2008]. As a consequence, we obtain a generalization of Kelly's vanishing theorem for the negative K-theory of such stacks.

1A. ***Vanishing of negative K-theory of stacks.*** Our main results apply to certain algebraic stacks with finite or multiplicative type stabilizers. More precisely, let $\mathbf{Stk}'$ be the category consisting of the following algebraic stacks:

- stacks with separated diagonal and linearly reductive finite stabilizers;
- stacks with affine diagonal whose stabilizers are extensions of linearly reductive finite groups by groups of multiplicative type.

Note that $\mathbf{Stk}'$ contains tame Artin stacks with separated diagonal in the sense of [Abramovich et al. 2008]. The *blow-up dimension* of a Noetherian stack $\mathcal{X}$ is a modification of the Krull dimension which is invariant under blow-ups (see Definition 7.7); it coincides with the usual dimension when $\mathcal{X}$ is a quasi-DM stack.

Theorem 1.1 (see Theorems 7.10, 7.14, and 7.16). *Let $\mathcal{X}$ be a stack in $\mathbf{Stk}'$ satisfying the resolution property or having finite inertia. Assume that $\mathcal{X}$ is Noetherian of blow-up dimension d. Then the following hold.*

(1) $KH_i(\mathcal{X}) = 0$ *for* $i < -d$.

(2) *If n is nilpotent on $\mathcal{X}$*, $K_i(\mathcal{X})[1/n] = 0$ *for* $i < -d$.

(3) *If n is invertible on $\mathcal{X}$*, $K_i(\mathcal{X}, \mathbb{Z}/n) = 0$ *for* $i < -d$.

1B. ***Cdh-descent for the homotopy K-theory of stacks.*** Cdh-descent plays a key role in all the existing vanishing theorems for negative K-theory. In the recent proof of Weibel's conjecture in [Kerz et al. 2018], the central result is pro-cdh-descent for nonconnective algebraic K-theory. Earlier results towards Weibel's conjecture used instead cdh-descent for homotopy K-theory KH. For schemes over a field of characteristic zero, this descent result was proven by Haesemeyer [2004], and in arbitrary characteristic, it was shown by Cisinski [2013]. For the equivariant KH-theory of quasiprojective schemes acted on by a diagonalizable or finite linearly reductive group over an arbitrary base, cdh-descent was proven by the first author [Hoyois 2016]. A key step in the proof of Theorem 1.1 is a generalization of the latter to more general algebraic stacks:

Theorem 1.2 (see Theorem 6.2). *The presheaf of homotopy K-theory spectra KH satisfies cdh-descent on the category* $\mathbf{Stk}'$.

Cdh-descent is the combination of two descent properties: descent for the Nisnevich topology and descent for abstract blow-ups. Descent for the Nisnevich topology holds much more generally (see Corollary 4.10) and in fact it holds for nonconnective K-theory as well (see Corollary 4.6). Descent for abstract blow-ups is more difficult and uses several nontrivial properties of the category $\mathbf{Stk}'$. The proof ultimately relies on the proper base change theorem in stable equivariant motivic homotopy theory, proved in [Hoyois 2017].

2. Preliminaries on algebraic stacks

A *stack* in this text means a quasicompact and quasiseparated algebraic stack. Note that all morphisms between such stacks are quasicompact and quasiseparated. Similarly, algebraic spaces and schemes are always assumed to be quasicompact and quasiseparated. We say that a morphism of stacks is *representable* if it is representable by algebraic spaces, and *schematic* if it is representable by schemes. Recall that the diagonal of a stack is representable by definition; see [Stacks 2005–, Tag 026N]. If $\mathcal{X}$ is a stack, k is a field, and $x : \mathrm{Spec}(k) \to \mathcal{X}$ is a k-point, then the stabilizer $G_x \to \mathrm{Spec}(k)$ is a flat separated group scheme of finite type [Stacks 2005–, Tag 0B8D].

All group schemes are assumed flat and finitely presented. With this convention, if G is a group scheme over a scheme S, then $\mathcal{B}G = [S/G]$ is a stack. Recall that G is called *linearly reductive* if the pushforward functor $\mathrm{QCoh}(\mathcal{B}G) \to \mathrm{QCoh}(S)$ on quasicoherent sheaves is exact. One knows from [Abramovich et al. 2008, Theorem 2.16] that a finite étale group scheme G over S is linearly reductive if and only if its degree at each point of S is prime to the residual characteristic. Diagonalizable group schemes are also linearly reductive by [SGA 3_{I} 1970, Exposé I, Théorème 5.3.3]. As linear reductivity is an fpqc-local property on S [Abramovich et al. 2008, Proposition 2.4], every group scheme of multiplicative type is linearly reductive.

We say that a group scheme G is *almost multiplicative* if it is an extension of a finite étale group scheme by a group scheme of multiplicative type. Since the class of linearly reductive group schemes is closed under quotients and extensions [Alper 2013, Proposition 12.17], an almost multiplicative group scheme G over S is linearly reductive if and only if, for every $s \in S$, the number of geometric components of G_s is invertible in $\kappa(s)$.

2A. ***Quasiprojective morphisms.*** Recall from [Laumon and Moret-Bailly 2000, §14.3] that if $\mathcal{X}$ is a stack and $\mathcal{A}^\bullet$ is a quasicoherent sheaf of graded $\mathcal{O}_\mathcal{X}$-algebras, then $\operatorname{Proj}(\mathcal{A}^\bullet)$ is a local construction on the fppf site of $\mathcal{X}$ just like for schemes and hence defines a schematic morphism of stacks $q : \operatorname{Proj}(\mathcal{A}^\bullet) \to \mathcal{X}$. A morphism $f : \mathcal{Y} \to \mathcal{X}$ is called *quasiprojective* [Laumon and Moret-Bailly 2000, §14.3.4; Rydh 2016, Theorem 8.6] if there is a finitely generated quasicoherent sheaf $\mathcal{E}$ on $\mathcal{X}$ and a factorization

$$\mathcal{Y} \xhookrightarrow{\iota} \mathbb{P}(\mathcal{E}) \xrightarrow{q} \mathcal{X}$$

of f, where $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}^\bullet(\mathcal{E}))$ and ι is a quasicompact immersion. We say that f is *projective* if it is quasiprojective and proper. It is clear that a quasiprojective morphism of stacks is schematic and hence representable.

Lemma 2.1. *If $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are quasiprojective* (*resp. projective*) *morphisms of stacks*, *then $g \circ f$ is quasiprojective* (*resp. projective*).

Proof. The proof is the same as [Hoyois 2017, Lemma 2.13], the key point being that every quasicoherent sheaf on a quasicompact quasiseparated stack is the colimit of its finitely generated quasicoherent subsheaves [Rydh 2016]. □

If $\mathcal{I} \subset \mathcal{O}_\mathcal{X}$ is a finitely generated quasicoherent sheaf of ideals, defining a finitely presented closed substack $\mathcal{Z} \subset \mathcal{X}$, then $\operatorname{Proj}\big(\bigoplus_{i \geq 0} \mathcal{I}^i\big) = \operatorname{Bl}_\mathcal{Z}(\mathcal{X})$ is called the blow-up of $\mathcal{X}$ with center $\mathcal{Z}$. Note that $\operatorname{Bl}_\mathcal{Z}(\mathcal{X})$ is a closed substack of $\mathbb{P}(\mathcal{I})$. Since $\mathcal{I}$ is finitely generated, it follows that the structure map $\operatorname{Bl}_\mathcal{Z}(\mathcal{X}) \to \mathcal{X}$ is projective. If $\mathcal{U} \subset \mathcal{X}$ is an open substack, we say that a blow-up of $\mathcal{X}$ is *$\mathcal{U}$-admissible* if its center is disjoint from $\mathcal{U}$.

2B. ***Flatification by blow-ups.***

Theorem 2.2 (Rydh). *Let $\mathcal{S}$ be a quasicompact and quasiseparated algebraic stack and let $f : \mathcal{X} \to \mathcal{S}$ be a morphism of finite type. Let $\mathcal{F}$ be a finitely generated quasicoherent $\mathcal{O}_\mathcal{X}$-module. Let $\mathcal{U} \subseteq \mathcal{S}$ be an open substack such that $f|_\mathcal{U}$ is of finite presentation and $\mathcal{F}|_{f^{-1}(\mathcal{U})}$ is of finite presentation and flat over $\mathcal{U}$. Then there exists a sequence of $\mathcal{U}$-admissible blow-ups $\widetilde{\mathcal{S}} \to \mathcal{S}$ such that the strict transform of $\mathcal{F}$ is of finite presentation and flat over $\widetilde{\mathcal{S}}$.*

Proof. This is proved in [Rydh ≥ 2019, Theorem 4.2]. □

Lemma 2.3. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a flat, proper, finitely presented, representable, and birational morphism of stacks. Then f is an isomorphism.*

Proof. We can assume that $\mathcal{X}$ and hence $\mathcal{Y}$ are algebraic spaces. Since f is flat, proper, and finitely presented, its fibers have locally constant dimension [Stacks 2005–, Tag 0D4R]. Since f is birational, its fibers must have dimension 0, so f is quasifinite [Stacks 2005–, Tag 04NV]. By Zariski's main theorem [Stacks 2005–, Tag 082K], we deduce that f is in fact finite. Being finite, flat, and finitely presented, f is locally free, and it must be of rank 0. □

Corollary 2.4 (Rydh). *Let $f : \mathcal{Y} \to \mathcal{X}$ be a proper representable morphism of stacks that is an isomorphism over some quasicompact open substack $\mathcal{U} \subset \mathcal{X}$. Then there exists a projective morphism $g : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ that is an isomorphism over $\mathcal{U}$ such that $f \circ g$ is also projective.*

Proof. By first blowing up a finitely presented complement of $\mathcal{U}$ in $\mathcal{X}$ (which exists by [Rydh 2016, Proposition 8.2]) and replacing $\mathcal{Y}$ by its strict transform, we may assume that $\mathcal{U}$ is dense in $\mathcal{X}$. By Theorem 2.2, we can find a sequence of $\mathcal{U}$-admissible blow-ups $\widetilde{\mathcal{X}} \to \mathcal{X}$ such that the strict transform $\tilde{f} : \widetilde{\mathcal{Y}} \to \widetilde{\mathcal{X}}$ is flat and of finite presentation. Let $g : \widetilde{\mathcal{Y}} \to \mathcal{Y}$ be the induced map:

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook] \arrow[d, equal] & \widetilde{\mathcal{Y}} \arrow[r, "g"] \arrow[d, "\tilde{f}"] & \mathcal{Y} \arrow[d, "f"] \\ \mathcal{U} \arrow[r, hook] & \widetilde{\mathcal{X}} \arrow[r] & \mathcal{X} \end{tikzcd}$$

Then g is a sequence of $\mathcal{U}$-admissible blow-ups and hence it is projective by Lemma 2.1. Moreover, $\tilde{f}$ is flat, proper, finitely presented, representable, and birational, whence an isomorphism (Lemma 2.3). Thus, $f \circ g$ is the composition of an isomorphism and the sequence of blow-ups $\widetilde{\mathcal{X}} \to \mathcal{X}$, so it is projective by Lemma 2.1. □

2C. *Nisnevich coverings of stacks.* The following definition appears in [Hall and Rydh 2018, Definition 3.1] and, for Deligne–Mumford stacks, in [Krishna and Østvær 2012, Definition 6.3].

Definition 2.5. Let $\mathcal{X}$ be a stack. A family of étale morphisms $\{\mathcal{U}_i \to \mathcal{X}\}_{i \in I}$ is called a *Nisnevich covering* if, for every $x \in \mathcal{X}$, there exists $i \in I$ and $u \in \mathcal{U}_i$ above x such that the induced morphism of residual gerbes $\eta_u \to \eta_x$ is an isomorphism.

Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of stacks. A *monomorphic splitting sequence* for f is a sequence of quasicompact open substacks

$$\varnothing = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_n = \mathcal{X}$$

such that f admits a monomorphic section over the reduced substack $\mathcal{U}_i \setminus \mathcal{U}_{i-1}$ for all i. Note that if f is étale, such a section is an open immersion

$$\mathcal{U}_i \setminus \mathcal{U}_{i-1} \hookrightarrow \mathcal{Y} \times_{\mathcal{X}} (\mathcal{U}_i \setminus \mathcal{U}_{i-1}).$$

Proposition 2.6. *Let $\mathcal{X}$ be a stack. A family of étale morphisms $\{\mathcal{U}_i \to \mathcal{X}\}_i$ is a Nisnevich covering if and only if the morphism $\coprod_i \mathcal{U}_i \to \mathcal{X}$ admits a monomorphic splitting sequence.*

Proof. See [Hall and Rydh 2018, Proposition 3.3]. □

Corollary 2.7. *Let $\mathcal{X}$ be a stack and let $\{\mathcal{U}_i \to \mathcal{X}\}_{i\in I}$ be a Nisnevich covering. Then there exists a finite subset $J \subset I$ such that $\{\mathcal{U}_i \to \mathcal{X}\}_{i\in J}$ is a Nisnevich covering.*

Proof. This follows at once from Proposition 2.6. □

A *Nisnevich square* in the category of stacks is a Cartesian square of the form

$$\begin{tikzcd} \mathcal{W} \arrow[r, hook] \arrow[d] & \mathcal{V} \arrow[d, "f"] \\ \mathcal{U} \arrow[r, hook, "e"] & \mathcal{X} \end{tikzcd} \tag{2.8}$$

where f is an étale morphism (not necessarily representable) and e is an open immersion with reduced complement $\mathcal{Z}$ such that the induced map $\mathcal{Z} \times_{\mathcal{X}} \mathcal{V} \to \mathcal{Z}$ is an isomorphism. Nisnevich squares form a cd-structure on the category of stacks, in the sense of [Voevodsky 2010].

Proposition 2.9. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a Nisnevich covering. Then there exist sequences of quasicompact open substacks*

$$\mathcal{Y}_1 \subset \cdots \subset \mathcal{Y}_n \subset \mathcal{Y}, \quad \varnothing = \mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathcal{X}_n = \mathcal{X},$$

such that $f(\mathcal{Y}_i) \subset \mathcal{X}_i$ and such that each square

$$\begin{tikzcd} \mathcal{X}_{i-1} \times_{\mathcal{X}} \mathcal{Y}_i \arrow[r, hook] \arrow[d] & \mathcal{Y}_i \arrow[d, "f"] \\ \mathcal{X}_{i-1} \arrow[r, hook] & \mathcal{X}_i \end{tikzcd}$$

is a Nisnevich square.

Proof. The proof is exactly the same as [Morel and Voevodsky 1999, Proposition 1.4]. Let $\mathcal{X}_0 \subset \cdots \subset \mathcal{X}_n$ be a monomorphic splitting sequence for f (see Proposition 2.6), and $s_i : \mathcal{X}_i \setminus \mathcal{X}_{i-1} \to \mathcal{Y} \times_{\mathcal{X}} (\mathcal{X}_i \setminus \mathcal{X}_{i-1})$ a monomorphic section of the projection. Then s_i is an open immersion, so the complement of the image of s_i is a closed substack $\mathcal{Z}_i \subset \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_i$. We can then take $\mathcal{Y}_i = (\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}_i) \setminus \mathcal{Z}_i$. □

Proposition 2.9 implies that the Grothendieck topology associated with the Nisnevich cd-structure is exactly the topology generated by Nisnevich coverings. The Nisnevich cd-structure on the category of stacks clearly satisfies the assumptions of Voevodsky's descent criterion [Asok et al. 2017, Theorem 3.2.5]. It follows that a presheaf of spaces or spectra $\mathcal{F}$ satisfies descent for Nisnevich coverings if and only if, for every Nisnevich square (2.8), the induced square

$$\begin{tikzcd} \mathcal{F}(\mathcal{X}) \arrow[r, "e^*"] \arrow[d, "f^*"'] & \mathcal{F}(\mathcal{U}) \arrow[d] \\ \mathcal{F}(\mathcal{V}) \arrow[r] & \mathcal{F}(\mathcal{W}) \end{tikzcd}$$

is homotopy Cartesian.

The following recent result of Alper, Hall, and Rydh [Alper et al. $\geq$ 2019] on the Nisnevich-local structure of some stacks plays an important role in the proof of our cdh-descent theorem.

Theorem 2.10 (Alper–Hall–Rydh). *Let $\mathcal{X}$ be a stack, let $x \in \mathcal{X}$ be a point, and let η_x be its residual gerbe. Suppose that the stabilizer of $\mathcal{X}$ at a representative of x is a linearly reductive almost multiplicative group scheme. Then there exists*

- *a morphism of affine schemes $U \to S$,*
- *a linearly reductive almost multiplicative group scheme G over S acting on U,*
- *a commutative diagram of stacks*

$$\begin{tikzcd} \eta_x \arrow[r, hook] \arrow[d, equal] & {[U/G]} \arrow[d, "f"] \\ \eta_x \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

where f is étale.

If $\mathcal{X}$ has affine diagonal, we can moreover choose f affine. If $\mathcal{X}$ has finite inertia and coarse moduli space $\pi : \mathcal{X} \to X$, we can take S to be an étale neighborhood of $\pi(x)$ in X.

Remark 2.11. Linearly reductive almost multiplicative group schemes are called *nice* in [Hall and Rydh 2015] and [Alper et al. $\geq$ 2019], but this terminology is used differently in [Krishna and Ravi 2018], so we avoid using it. □

3. Perfect complexes on algebraic stacks

3A. *Sheaves on stacks.* Let $\mathcal{X}$ be a stack. Let Lis-Ét($\mathcal{X}$) denote the *lisse-étale site* of $\mathcal{X}$. Its objects are smooth morphisms $X \to \mathcal{X}$, where X is a quasicompact quasiseparated scheme. The coverings are generated by the étale covers of schemes. Let Mod($\mathcal{X}$) denote the abelian category of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules, and QCoh($\mathcal{X}$)

that of quasicoherent sheaves, on Lis-Ét$(\mathcal{X})$. It is well known that QCoh$(\mathcal{X})$ and Mod$(\mathcal{X})$ are Grothendieck abelian categories and hence have enough injectives and all limits.

Let Ch$(\mathcal{X})$ denote the category of all (possibly unbounded) chain complexes over Mod$(\mathcal{X})$, and $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ the full subcategory of Ch$(\mathcal{X})$ consisting of those chain complexes whose cohomology lies in QCoh$(\mathcal{X})$. Let $D(\mathcal{X})$ and $D_{\mathrm{qc}}(\mathcal{X})$ denote their corresponding derived categories, obtained by inverting quasi-isomorphisms. If $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed substack with open complement $j : \mathcal{U} \hookrightarrow \mathcal{X}$, we let

$$\mathrm{Ch}_{\mathrm{qc},\mathcal{Z}}(\mathcal{X}) = \{\mathcal{F} \in \mathrm{Ch}_{\mathrm{qc}}(\mathcal{X}) \mid j^*(\mathcal{F}) \text{ is quasi-isomorphic to } 0\}.$$

The derived category of $\mathrm{Ch}_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$ is denoted by $D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$.

Let $j : \mathcal{X} \to \mathcal{Y}$ be a smooth morphism of algebraic stacks. We then have the pullback functor $j^* : \mathrm{Mod}(\mathcal{Y}) \to \mathrm{Mod}(\mathcal{X})$, which preserves quasicoherent sheaves. Since j is smooth, the functor j^* is simply the restriction functor under the inclusion Lis-Ét$(\mathcal{X}) \subset$ Lis-Ét$(\mathcal{Y})$.

Recall from [SGA 6 1971, Definition I.4.2] that a complex of $\mathcal{O}_X$-modules on a scheme X is perfect if it is locally quasi-isomorphic to a bounded complex of locally free sheaves.

Definition 3.1. Let $\mathcal{X}$ be a stack. A chain complex $P \in \mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ is called *perfect* if for any affine scheme $U = \mathrm{Spec}(A)$ with a smooth morphism $s : U \to \mathcal{X}$, the complex of A-modules $s^*(P) \in \mathrm{Ch}(\mathrm{Mod}(A))$ is quasi-isomorphic to a bounded complex of finitely generated projective A-modules. Equivalently, $s^*(P)$ is a perfect complex in Ch(Mod(A)) in the sense of [Thomason and Trobaugh 1990].

It follows from [Krishna and Ravi 2018, Lemma 2.5] that the above definition coincides with that of [Thomason and Trobaugh 1990] if $\mathcal{X}$ is a scheme. We denote the derived category of perfect complexes on $\mathcal{X}$ by $D_{\mathrm{perf}}(\mathcal{X})$. The derived category of perfect complexes on $\mathcal{X}$ whose cohomology is supported on a closed substack $\mathcal{Z}$ is denoted by $D_{\mathrm{perf},\mathcal{Z}}(\mathcal{X})$.

We also need to use the canonical dg-enhancements of the triangulated categories $D_{\mathrm{qc}}(\mathcal{X})$ and $D_{\mathrm{perf}}(\mathcal{X})$, denoted by $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ and $\mathsf{D}_{\mathrm{perf}}(\mathcal{X})$, respectively, whose construction we now recall. If $\mathcal{X}$ is an affine scheme, $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ is the usual symmetric monoidal derived dg-category of $\mathcal{O}(\mathcal{X})$. The 2-category of stacks embeds fully faithfully in the 2-category of presheaves of groupoids on affine schemes, which further embeds in the ∞-category sPre(**Aff**) of simplicial presheaves on affine schemes. Then one defines D_{qc} as a presheaf of symmetric monoidal dg-categories on sPre(**Aff**) to be the homotopy right Kan extension of $\mathsf{D}_{\mathrm{qc}}|_{\mathbf{Aff}}$; see [Lurie 2018, §6.2]. In other words, it is the unique extension of $\mathsf{D}_{\mathrm{qc}}|_{\mathbf{Aff}}$ that transforms homotopy colimits into homotopy limits. One can show that D_{qc} satisfies descent for the fpqc topology on sPre(**Aff**) [Lurie 2018, Proposition 6.2.3.1]. For $\mathcal{X}$ a stack, the homotopy category of $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ is then equivalent to $D_{\mathrm{qc}}(\mathcal{X})$. If $\mathcal{X}$ is an algebraic space (or more generally a Deligne–Mumford stack), this is proved in [Lurie 2018, Proposition 6.2.4.1]. In general, this follows from the description of $D_{\mathrm{qc}}(\mathcal{X})$ in terms of a smooth representable cover of $\mathcal{X}$ by an algebraic space; see for instance [Hall and Rydh 2017, §1.1]. Finally, $\mathsf{D}_{\mathrm{perf}} \subset \mathsf{D}_{\mathrm{qc}}$ is the full symmetric monoidal dg-subcategory spanned by the dualizable objects. Since the process of passing to dualizable objects preserves homotopy limits of dg-categories [Lurie 2017, Proposition 4.6.1.11], $\mathsf{D}_{\mathrm{perf}}$ is similarly the unique extension of $\mathsf{D}_{\mathrm{perf}}|_{\mathbf{Aff}}$ to sPre($\mathbf{Aff}$) that transforms homotopy colimits into homotopy limits, and it satisfies fpqc descent.

Proposition 3.2. *Let $f : \mathcal{X}' \to \mathcal{X}$ be an étale morphism of stacks and let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack with quasicompact open complement such that the projection $\mathcal{Z} \times_{\mathcal{X}} \mathcal{X}' \to \mathcal{Z}$ is an isomorphism of associated reduced stacks. Then the functor*

$$f^* : D_{\mathrm{perf},\mathcal{Z}}(\mathcal{X}) \to D_{\mathrm{perf},\mathcal{Z}\times_{\mathcal{X}}\mathcal{X}'}(\mathcal{X}')$$

is an equivalence of triangulated categories.

Proof. The presheaf of dg-categories $\mathcal{X} \mapsto \mathsf{D}_{\mathrm{perf}}(\mathcal{X})$ satisfies descent for the fpqc topology on stacks. In particular, it satisfies Nisnevich descent, so that the square of dg-categories

$$\begin{tikzcd} \mathsf{D}_{\mathrm{perf}}(\mathcal{X}) \arrow[r] \arrow[d, "f^*"'] & \mathsf{D}_{\mathrm{perf}}(\mathcal{X} \setminus \mathcal{Z}) \arrow[d] \\ \mathsf{D}_{\mathrm{perf}}(\mathcal{X}') \arrow[r] & \mathsf{D}_{\mathrm{perf}}(\mathcal{X}' \setminus (\mathcal{Z} \times_{\mathcal{X}} \mathcal{X}')) \end{tikzcd}$$

is homotopy Cartesian. It follows that f^* induces an equivalence between the kernels of the horizontal functors. □

3B. *Perfect stacks.*

Definition 3.3. Let $\mathcal{X}$ be a stack. We say that $\mathcal{X}$ is *perfect* if the triangulated category $D_{\mathrm{qc}}(\mathcal{X})$ is compactly generated and $\mathcal{O}_{\mathcal{X}}$ is compact in $D_{\mathrm{qc}}(\mathcal{X})$.

If $\mathcal{Z} \subset \mathcal{X}$ is a closed substack with quasicompact open complement, we say that the pair $(\mathcal{X}, \mathcal{Z})$ is *perfect* if $\mathcal{X}$ is perfect and there exists a perfect complex on $\mathcal{X}$ with support $|\mathcal{Z}|$.

We will see in Proposition 3.5 below that our notion of perfect stack agrees with the one introduced in [Ben-Zvi et al. 2010], except that we do not require perfect stacks to have affine diagonal.

Let $f : \mathcal{X}' \to \mathcal{X}$ be a morphism of stacks. We say that f is *concentrated* if for every morphism $g : \mathcal{Z} \to \mathcal{X}$, the morphism $f' : \mathcal{X}' \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Z}$ has finite cohomological dimension for quasicoherent sheaves.

Lemma 3.4. *Let $f : \mathcal{X}' \to \mathcal{X}$ be a representable morphism of stacks. Then f is concentrated. In particular, if $\mathcal{O}_{\mathcal{X}}$ is compact, then $\mathcal{O}_{\mathcal{X}'}$ is compact.*

Proof. Since f is representable, and since concentrated morphisms have faithfully flat descent by [Hall and Rydh 2017, Lemma 2.5(2)], we can assume that f is a morphism of algebraic spaces. Now the result follows because any quasicompact and quasiseparated morphism of algebraic spaces is concentrated [Stacks 2005–, Tag 073G]. For the second statement, it suffices to show using [Neeman 1996, Theorem 5.1] that the right adjoint $f_* : D_{qc}(\mathcal{X}') \to D_{qc}(\mathcal{X})$ of f^* preserves small coproducts. This follows from the first statement and [Hall and Rydh 2017, Theorem 2.6(3)]. □

Proposition 3.5. *Let $(\mathcal{X}, \mathcal{Z})$ be a perfect pair. Then the triangulated category $D_{qc,\mathcal{Z}}(\mathcal{X})$ is compactly generated. Moreover, an object of $D_{qc,\mathcal{Z}}(\mathcal{X})$ is compact if and only if it is perfect.*

Proof. Since $\mathcal{O}_{\mathcal{X}} \in D_{qc}(\mathcal{X})$ is compact and since a perfect complex on $\mathcal{X}$ is dualizable, it follows that every perfect complex on $\mathcal{X}$ is compact. On the other hand, it follows from the proofs of [Krishna and Ravi 2018, Proposition 2.7, Lemma 2.8] that compact objects of $D_{qc}(\mathcal{X})$ and $D_{qc,\mathcal{Z}}(\mathcal{X})$ are perfect. The only remark we need to make here is that the proofs in [loc. cit.] assume that $\mathcal{X}$ is a quotient stack. However, this assumption is used only to ensure that if we choose an atlas $u : U \to \mathcal{X}$, then u has finite cohomological dimension for quasicoherent sheaves. But this follows from Lemma 3.4 because $\mathcal{X}$ has representable diagonal and hence u is representable. Finally, the existence of a perfect complex with support $|\mathcal{Z}|$ implies, by [Hall and Rydh 2017, Lemma 4.10], that $D_{qc,\mathcal{Z}}(\mathcal{X})$ is compactly generated. □

Lemma 3.6. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a schematic morphism of stacks with a relatively ample family of line bundles. If $D_{qc}(\mathcal{X})$ is compactly generated, so is $D_{qc}(\mathcal{Y})$.*

Proof. Let $\{\mathcal{L}_i\}_{i \in I}$ be an f-ample family of line bundles on $\mathcal{Y}$. By Lemma 3.4, f is a concentrated morphism. It follows from [Hall and Rydh 2017, Theorem 2.6(3)] that $f_* : D_{qc}(\mathcal{Y}) \to D_{qc}(\mathcal{X})$ preserves small coproducts, and hence that its left adjoint f^* preserves compact objects. It therefore suffices to show that $D_{qc}(\mathcal{Y})$ is generated by the objects $f^*(\mathcal{F}) \otimes \mathcal{L}_i^{\otimes -n}$, for $\mathcal{F} \in D_{qc}(\mathcal{X})$ compact, $i \in I$, and $n \geq 1$. So let $\mathcal{G} \in D_{qc}(\mathcal{Y})$ be such that $\mathrm{Hom}(f^*(\mathcal{F}) \otimes \mathcal{L}_i^{\otimes -n}, \mathcal{G}) = 0$ for every $\mathcal{F}$ compact, $i \in I$, and $n \geq 1$. By adjunction, we have $\mathrm{Hom}(\mathcal{F}, f_*(\mathcal{G} \otimes \mathcal{L}_i^{\otimes n})) = 0$. Since $D_{qc}(\mathcal{X})$ is compactly generated, it follows that

$$f_*(\mathcal{G} \otimes \mathcal{L}_i^{\otimes n}) = 0 \tag{3.7}$$

for every $i \in I$ and $n \geq 1$.

To show that $\mathcal{G} = 0$ in $D_{qc}(\mathcal{Y})$, we let $u : U \to \mathcal{X}$ be a smooth surjective morphism such that U is affine. This gives rise to a Cartesian square

$$\begin{tikzcd} V \arrow[r, "v"] \arrow[d, "g"'] & \mathcal{Y} \arrow[d, "f"] \\ U \arrow[r, "u"] & \mathcal{X} \end{tikzcd}$$

where V is a scheme. As v is faithfully flat, it suffices to show $v^*(\mathcal{G})=0$. It follows from [Hall and Rydh 2017, Corollary 4.13] and (3.7) that $g_*(v^*\mathcal{G}\otimes v^*(\mathcal{L}_i)^{\otimes n})=0$ for all $i\in I$. Replacing $\mathcal{Y}$ by V and $\mathcal{L}_i$ by $v^*(\mathcal{L}_i)$, we can assume that $\mathcal{X}$ is an affine scheme, so that $\mathcal{Y}$ is a scheme and $\{\mathcal{L}_i\}_{i\in I}$ is an ample family of line bundles on $\mathcal{Y}$. In this case, (3.7) says that $\mathrm{Hom}(\mathcal{L}_i^{\otimes -n}[m],\mathcal{G})=0$ for all $i\in I$, $n\geq 1$, and $m\in\mathbb{Z}$. But this implies that $\mathcal{G}$ is acyclic because $D_{\mathrm{qc}}(\mathcal{Y})$ is generated by $\{\mathcal{L}_i^{\otimes -n}\}_{i\in I,\,n\geq 1}$. Indeed, $D_{\mathrm{qc}}(\mathcal{Y})$ is compactly generated by bounded complexes of vector bundles [Thomason and Trobaugh 1990, Theorem 2.3.1(d)], and every vector bundle admits an epimorphism from a sum of line bundles of the form $\mathcal{L}_i^{\otimes -n}$. □

Proposition 3.8. *Let $(\mathcal{X},\mathcal{Z})$ be a perfect pair.*

(1) *For every algebraic space Y and closed subspace $W\subset Y$ with quasicompact open complement, $(\mathcal{X}\times Y,\mathcal{Z}\times W)$ is perfect.*

(2) *For every schematic morphism $f:\mathcal{Y}\to\mathcal{X}$ with a relatively ample family of line bundles, $(\mathcal{Y},\mathcal{Y}\times_{\mathcal{X}}\mathcal{Z})$ is perfect.*

Proof. Let $\mathcal{P}$ be a perfect complex on $\mathcal{X}$ with support $|\mathcal{Z}|$.

(1) By [Hall and Rydh 2017, Theorem A], there exists a perfect complex $\mathcal{Q}$ on Y with support $|W|$. Then $\pi_1^*(\mathcal{P})\otimes\pi_2^*(\mathcal{Q})$ is a perfect complex on $\mathcal{X}\times Y$ with support $|\mathcal{Z}\times W|$. Since the projection $\pi_1:\mathcal{X}\times Y\to\mathcal{X}$ is representable, $\mathcal{O}_{\mathcal{X}\times Y}$ is compact by Lemma 3.4. It remains to show that $D_{\mathrm{qc}}(\mathcal{X}\times Y)$ is compactly generated. We claim that there is an equivalence of presentable dg-categories

$$\mathsf{D}_{\mathrm{qc}}(\mathcal{X}\times Y)\simeq\mathsf{D}_{\mathrm{qc}}(\mathcal{X})\otimes\mathsf{D}_{\mathrm{qc}}(Y). \tag{3.9}$$

Since the tensor product of compactly generated dg-categories is compactly generated, this will complete the proof. Since Y is a quasicompact and quasiseparated algebraic space, the dg-category $\mathsf{D}_{\mathrm{qc}}(Y)$ is dualizable [Lurie 2018, §9.4], and hence tensoring with $\mathsf{D}_{\mathrm{qc}}(Y)$ preserves homotopy limits. Since $\mathsf{D}_{\mathrm{qc}}(-)$ is the homotopy right Kan extension of its restriction to affine schemes, we are reduced to proving (3.9) when $\mathcal{X}$ is an affine scheme, in which case it is a special case of [Lurie 2018, Corollary 9.4.2.4].

(2) The perfect complex $f^*(\mathcal{P})$ has support $|\mathcal{Y}\times_{\mathcal{X}}\mathcal{Z}|$. By Lemma 3.4, $\mathcal{O}_{\mathcal{Y}}$ is compact. It remains to show that $D_{\mathrm{qc}}(\mathcal{Y})$ is compactly generated, but this follows from Lemma 3.6. □

Proposition 3.10. *Let $(\mathcal{X},\mathcal{Z})$ be a perfect pair and let $j:\mathcal{U}\hookrightarrow\mathcal{X}$ be the open immersion complement to $\mathcal{Z}$. Then*

$$j^*:\frac{D_{\mathrm{perf}}(\mathcal{X})}{D_{\mathrm{perf},\mathcal{Z}}(\mathcal{X})}\to D_{\mathrm{perf}}(\mathcal{U})$$

is an equivalence of triangulated categories, up to direct factors.

Proof. For any pair $(\mathcal{X}, \mathcal{Z})$, we have an equivalence of triangulated categories

$$j^* : \frac{D_{\mathrm{qc}}(\mathcal{X})}{D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})} \to D_{\mathrm{qc}}(\mathcal{U}).$$

Indeed, the functor $j_* : D_{\mathrm{qc}}(\mathcal{U}) \to D_{\mathrm{qc}}(\mathcal{X})$ is fully faithful by flat base change, so $j_* j^*$ is a localization endofunctor of $D_{\mathrm{qc}}(\mathcal{X})$ whose kernel is $D_{\mathrm{qc},\mathcal{Z}}(\mathcal{X})$ by definition. The claim now follows from [Krause 2010, Proposition 4.9.1]. If $(\mathcal{X}, \mathcal{Z})$ is perfect, then $\mathcal{U}$ is also perfect by Proposition 3.8(2). By Proposition 3.5, all three categories are compactly generated and their subcategories of compact and perfect objects coincide. We conclude using [Krause 2010, Theorem 5.6.1]. □

Proposition 3.11. *Suppose that $\mathcal{X}$ is the limit of a filtered diagram $(\mathcal{X}_\alpha)$ of perfect stacks with affine transition morphisms. Then $\mathcal{X}$ is perfect and the canonical map*

$$\operatorname*{hocolim}_{\alpha} \mathsf{D}_{\mathrm{perf}}(\mathcal{X}_\alpha) \to \mathsf{D}_{\mathrm{perf}}(\mathcal{X}) \tag{3.12}$$

is a weak equivalence of dg-categories.

Proof. It follows from Proposition 3.8(2) that $\mathcal{X}$ is perfect. By Proposition 3.5, $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})$ is compactly generated and $\mathsf{D}_{\mathrm{qc}}(\mathcal{X})^c = \mathsf{D}_{\mathrm{perf}}(\mathcal{X})$, and similarly for each $\mathcal{X}_\alpha$. Since the pullback functors $\mathsf{D}_{\mathrm{qc}}(\mathcal{X}_\alpha) \to \mathsf{D}_{\mathrm{qc}}(\mathcal{X}_\beta)$ preserve compact objects, it follows from [Lurie 2009, Propositions 5.5.7.6 and 5.5.7.8] and [Lurie 2017, Lemma 7.3.5.10] that (3.12) is a weak equivalence if and only if the canonical map

$$\mathsf{D}_{\mathrm{qc}}(\mathcal{X}) \to \operatorname*{holim}_{\alpha} \mathsf{D}_{\mathrm{qc}}(\mathcal{X}_\alpha) \tag{3.13}$$

is a weak equivalence. Choosing a smooth hypercover of some $\mathcal{X}_\alpha$ by schemes and using flat base change, we see that the map (3.13) is the homotopy limit of a cosimplicial diagram of similar maps with $\mathcal{X}_\alpha$ replaced by a scheme. Hence, it suffices to prove that (3.12) is a weak equivalence when $\mathcal{X}_\alpha$ is a scheme, but this follows from [Thomason and Trobaugh 1990, Proposition 3.20]. □

We now state the following two results of Hall and Rydh, which provide many examples of perfect stacks.

Theorem 3.14 (Hall–Rydh). *Let $\mathcal{X}$ be a stack satisfying one of the following properties.*

(1) *$\mathcal{X}$ has characteristic zero.*

(2) *$\mathcal{X}$ has linearly reductive almost multiplicative stabilizers.*

(3) *$\mathcal{X}$ has finitely presented inertia and linearly reductive almost multiplicative stabilizers at points of positive characteristic.*

Then $\mathcal{O}_{\mathcal{X}}$ is compact in $D_{\mathrm{qc}}(\mathcal{X})$.

Proof. See [Hall and Rydh 2015, Theorem 2.1]. □

Theorem 3.15 (Hall–Rydh). *Let $\mathcal{X}$ be a stack satisfying the following properties.*

(1) *$\mathcal{O}_{\mathcal{X}}$ is compact in $D_{\mathrm{qc}}(\mathcal{X})$.*

(2) *There exists a faithfully flat, representable, separated, and quasifinite morphism $f : \mathcal{X}' \to \mathcal{X}$ of finite presentation such that $\mathcal{X}'$ has affine stabilizers and satisfies the resolution property.*

Then, for every closed substack $\mathcal{Z} \subset \mathcal{X}$, the pair $(\mathcal{X}, \mathcal{Z})$ is perfect.

Proof. By Lemma 3.4, $\mathcal{O}_{\mathcal{X}'} = f^*(\mathcal{O}_{\mathcal{X}})$ is compact in $D_{\mathrm{qc}}(\mathcal{X}')$. Since $\mathcal{X}'$ has affine stabilizers and satisfies the resolution property, it has affine diagonal by [Gross 2017, Theorem 1.1]. Since moreover $\mathcal{O}_{\mathcal{X}'}$ is compact, it follows from [Hall and Rydh 2017, Proposition 8.4] that $\mathcal{X}'$ is *crisp*. We now apply [Hall and Rydh 2017, Theorem C] to conclude that $\mathcal{X}$ is also crisp. By definition of crispness, this implies that $(\mathcal{X}, \mathcal{Z})$ is perfect. □

Corollary 3.16. *Let $\mathcal{X}$ be a quasi-DM stack with separated diagonal and linearly reductive stabilizers. Then, for every closed substack $\mathcal{Z} \subset \mathcal{X}$, the pair $(\mathcal{X}, \mathcal{Z})$ is perfect.*

Proof. Recall that a quasi-DM stack is a stack whose diagonal is quasifinite. It follows from [Stacks 2005–, Tag 06MC] that a stack $\mathcal{X}$ is quasi-DM if and only if there exists an affine scheme X and a faithfully flat map $f : X \to \mathcal{X}$ of finite presentation which is quasifinite. Since the diagonal of $\mathcal{X}$ is representable and separated, it follows that f is representable and separated. Since X is affine and hence has the resolution property, the corollary follows from Theorems 3.14 and 3.15. □

Corollary 3.17. *Let $\mathcal{X}$ be a stack with affine diagonal and linearly reductive almost multiplicative stabilizers. Then, for every closed substack $\mathcal{Z} \subset \mathcal{X}$, $(\mathcal{X}, \mathcal{Z})$ is perfect.*

Proof. By Theorem 2.10, there exists a Nisnevich covering $\{f_i : [U_i/G_i] \to \mathcal{X}\}_{i\in I}$, where f_i is affine, U_i is affine over an affine scheme S_i, and G_i is a linearly reductive almost multiplicative group scheme over S_i. By taking a further affine Nisnevich covering of S_i, we can ensure that G_i is almost isotrivial and hence that $[U_i/G_i]$ has the resolution property; see [Hoyois 2017, Example 2.8 and Remark 2.9]. By Corollary 2.7, we can also assume that I is finite. Let $\mathcal{X}' = \coprod_i [U_i/G_i]$. Then the induced map $\mathcal{X}' \to \mathcal{X}$ is faithfully flat, quasifinite, and affine. Since $\mathcal{X}'$ has the resolution property, we conclude that $(\mathcal{X}, \mathcal{Z})$ is perfect by Theorems 3.14 and 3.15. □

4. K-theory of perfect stacks

In this section, we establish some descent properties of the K-theory, negative K-theory, and homotopy K-theory of stacks. Special cases of these results were earlier proven in [Krishna and Østvær 2012; Krishna and Ravi 2018; Hoyois 2016].

4A. ***Localization, excision, and continuity.*** Let $\mathcal{X}$ be an algebraic stack. The algebraic K-theory spectrum of $\mathcal{X}$ is defined to be the K-theory spectrum of the complicial bi-Waldhausen category of perfect complexes in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$ in the sense of [Thomason and Trobaugh 1990, §1.5.2]. Here, the complicial bi-Waldhausen category structure is given with respect to the degreewise split monomorphisms as cofibrations and quasi-isomorphisms as weak equivalences. This K-theory spectrum is denoted by $K(\mathcal{X})$. Equivalently, one may define $K(\mathcal{X})$ as the K-theory spectrum of the dg-category $\mathsf{D}_{\mathrm{perf}}(\mathcal{X})$; see [Blumberg et al. 2013, Corollary 7.12]. Note that the negative homotopy groups of $K(\mathcal{X})$ are zero [Thomason and Trobaugh 1990, §1.5.3]. We shall extend this definition to negative integers in the next section. For a closed substack $\mathcal{Z}$ of $\mathcal{X}$, $K(\mathcal{X}\text{ on }\mathcal{Z})$ is the K-theory spectrum of the complicial bi-Waldhausen category of those perfect complexes on $\mathcal{X}$ which are acyclic on $\mathcal{X}\setminus\mathcal{Z}$.

Theorem 4.1 (localization). *Let $(\mathcal{X},\mathcal{Z})$ be a perfect pair and let $j:\mathcal{U}\hookrightarrow\mathcal{X}$ be the open immersion complement to $\mathcal{Z}$. Then the morphisms of spectra*

$$K(\mathcal{X}\text{ on }\mathcal{Z})\to K(\mathcal{X})\xrightarrow{j^*}K(\mathcal{U})$$

induce a long exact sequence

$$\cdots\to K_i(\mathcal{X}\text{ on }\mathcal{Z})\to K_i(\mathcal{X})\to K_i(\mathcal{U})\to K_{i-1}(\mathcal{X}\text{ on }\mathcal{Z})\to\cdots$$
$$\to K_0(\mathcal{X}\text{ on }\mathcal{Z})\to K_0(\mathcal{X})\to K_0(\mathcal{U}).$$

Proof. This follows from Proposition 3.10 as in [Krishna and Ravi 2018, Theorem 3.4]. □

Theorem 4.2 (excision). *Let $f:\mathcal{X}'\to\mathcal{X}$ be an étale morphism of stacks and let $\mathcal{Z}\subset\mathcal{X}$ be a closed substack with quasicompact open complement such that the projection $\mathcal{Z}\times_{\mathcal{X}}\mathcal{X}'\to\mathcal{Z}$ is an isomorphism of associated reduced stacks. Then the map $f^*:K(\mathcal{X}\text{ on }\mathcal{Z})\to K(\mathcal{X}'\text{ on }\mathcal{Z}\times_{\mathcal{X}}\mathcal{X}')$ is a homotopy equivalence of spectra.*

Proof. This follows from Proposition 3.2 using [Thomason and Trobaugh 1990, Theorem 1.9.8]. □

Theorem 4.3 (continuity). *Let $\mathcal{X}$ be the limit of a filtered diagram $(\mathcal{X}_\alpha)$ of perfect stacks with affine transition morphisms. Then the canonical map*

$$\operatorname*{hocolim}_{\alpha}K(\mathcal{X}_\alpha)\to K(\mathcal{X})$$

is a homotopy equivalence.

Proof. This follows from Proposition 3.11 and the fact that K preserves filtered homotopy colimits of dg-categories. □

4B. ***The Bass construction and negative K-theory.*** The nonconnective K-theory spectrum of any stack may be defined from the complicial bi-Waldhausen category of perfect complexes, following [Schlichting 2006], or from the dg-category $\mathsf{D}_{\mathrm{perf}}(\mathcal{X})$, following [Cisinski and Tabuada 2011]. This allows one to define the negative K-theory of stacks.

In this subsection, we will see that for perfect stacks a nonconnective K-theory spectrum K^B can be defined much more explicitly using the construction of Bass–Thomason–Trobaugh. One may prove that this construction agrees with those of Schlichting and Cisinski–Tabuada exactly as in [Krishna and Ravi 2018, Theorem 3.21].

The K^B-theory spectrum $K^B(\mathcal{X})$ was constructed in [Krishna and Ravi 2018, §3E] based on the following two assumptions.

(1) $\mathcal{X}$ is a quotient stack of the form $[X/G]$ over a field, where G is a linearly reductive group scheme.

(2) $\mathcal{X}$ satisfies the resolution property.

Since perfect stacks need not satisfy these conditions, we cannot directly quote the results of [Krishna and Ravi 2018] for the construction of the K^B-theory of stacks. But the proofs are identical to those in [Krishna and Ravi 2018] using Theorems 4.1 and 4.2, so we only give a brief sketch of the construction. The existence of the K^B-theory is based on the following version of the fundamental theorem of Bass.

Theorem 4.4 (Bass fundamental theorem). *Let $\mathcal{X}$ be a perfect stack and let $\mathcal{X}[T]$ denote the stack $\mathcal{X} \times \operatorname{Spec}(\mathbb{Z}[T])$. Then the following hold.*

(1) *For $n \geq 1$, there is an exact sequence*

$$0 \to K_n(\mathcal{X}) \xrightarrow{(p_1^*, -p_2^*)} K_n(\mathcal{X}[T]) \oplus K_n(\mathcal{X}[T^{-1}]) \xrightarrow{(j_1^*, j_2^*)} K_n(\mathcal{X}[T, T^{-1}]) \xrightarrow{\partial_T} K_{n-1}(\mathcal{X}) \to 0.$$

Here p_1^*, p_2^* are induced by the projections $\mathcal{X}[T] \to \mathcal{X}$, etc. and j_1^*, j_2^* are induced by the open immersions $\mathcal{X}[T^{\pm 1}] = \mathcal{X}[T, T^{-1}] \to \mathcal{X}[T]$, etc. The sum of these exact sequences for $n = 1, 2, \ldots$ is an exact sequence of graded $K_*(\mathcal{X})$-modules.

(2) *For $n \geq 0$, $\partial_T : K_{n+1}(\mathcal{X}[T^{\pm 1}]) \to K_n(\mathcal{X})$ is naturally split by a map h_T of $K_*(\mathcal{X})$-modules. Indeed, the cup product with $T \in K_1(\mathbb{Z}[T^{\pm 1}])$ splits ∂_T up to a natural automorphism of $K_n(\mathcal{X})$.*

(3) *There is an exact sequence*

$$0 \to K_0(\mathcal{X}) \xrightarrow{(p_1^*, -p_2^*)} K_0(\mathcal{X}[T]) \oplus K_0(\mathcal{X}[T^{-1}]) \xrightarrow{(j_1^*, j_2^*)} K_0(\mathcal{X}[T^{\pm 1}]).$$

Proof. The proof of this theorem is word for word identical to the proof of its scheme version given in [Thomason and Trobaugh 1990, Theorem 6.1], once we know that the algebraic K-theory spectrum satisfies the following properties:

(1) The projective bundle formula for the projective line $\mathbb{P}^1_{\mathcal{X}}$.

(2) Localization for the pairs $(\mathbb{P}^1_{\mathcal{X}}, \mathcal{X}[T^{-1}])$ and $(\mathcal{X}[T], \mathcal{X}[T^{\pm 1}])$.

(3) Excision.

Property (1) follows from [Krishna and Ravi 2018, Theorem 3.8], which holds for any algebraic stack. Property (2) follows from Proposition 3.8(1) and Theorem 4.1, and property (3) is Theorem 4.2. □

As an immediate consequence of Theorem 4.4, one obtains the following.

Theorem 4.5. *Let $\mathcal{X}$ be a perfect stack. Then there exists a spectrum $K^B(\mathcal{X})$, together with a natural map $K(\mathcal{X}) \to K^B(\mathcal{X})$ of spectra inducing isomorphisms $\pi_i K(\mathcal{X}) \cong \pi_i K^B(\mathcal{X})$ for $i \geq 0$, which satisfies the following properties.*

(1) *Let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack with quasicompact open complement $j : \mathcal{U} \hookrightarrow \mathcal{X}$ such that $(\mathcal{X}, \mathcal{Z})$ is perfect. Then there is a homotopy fiber sequence of spectra*
$$K^B(\mathcal{X} \text{ on } \mathcal{Z}) \to K^B(\mathcal{X}) \xrightarrow{j^*} K^B(\mathcal{U}).$$

(2) *Let $f : \mathcal{Y} \to \mathcal{X}$ be an étale map between perfect stacks such that the projection $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Z}$ is an isomorphism on the associated reduced stacks. Then the map $f^* : K^B(\mathcal{X} \text{ on } \mathcal{Z}) \to K^B(\mathcal{Y} \text{ on } \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y})$ is a homotopy equivalence.*

(3) *Let $\pi : \mathbb{P}(\mathcal{E}) \to \mathcal{X}$ be the projective bundle associated to a vector bundle $\mathcal{E}$ on $\mathcal{X}$ of rank r. Then the map*
$$\prod_0^{r-1} K^B(\mathcal{X}) \to K^B(\mathbb{P}(\mathcal{E}))$$
that sends $(a_0, \ldots, a_{r-1})$ to $\sum_i \mathcal{O}(-i) \otimes \pi^*(a_i)$ is a homotopy equivalence.

(4) *Let $i : \mathcal{Y} \hookrightarrow \mathcal{X}$ be a regular closed immersion and let $p : \mathcal{X}' \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ with center $\mathcal{Y}$. Then the square of spectra*
$$\begin{tikzcd} K^B(\mathcal{X}) \arrow[r, "i^*"] \arrow[d, "p^*"'] & K^B(\mathcal{Y}) \arrow[d] \\ K^B(\mathcal{X}') \arrow[r] & K^B(\mathcal{X}' \times_{\mathcal{X}} \mathcal{Y}) \end{tikzcd}$$
is homotopy Cartesian.

(5) *Suppose $\mathcal{X}$ is the limit of a filtered diagram $(\mathcal{X}_\alpha)$ of perfect stacks with affine transition morphisms. Then the canonical map $\operatorname{hocolim}_\alpha K^B(\mathcal{X}_\alpha) \to K^B(\mathcal{X})$ is a homotopy equivalence.*

Proof. The spectrum $K^B(\mathcal{X})$ is constructed word for word using Theorem 4.4 and the formalism given in (6.2)–(6.4) of [Thomason and Trobaugh 1990] for the case of schemes. The proof of the asserted properties is a standard deduction from the analogous properties of $K(\mathcal{X})$. The sketch of this deduction for (1)–(4) can be found in [Krishna and Ravi 2018, Theorem 3.20]. Note that quasicompact open substacks of $\mathcal{X}$, projective bundles over $\mathcal{X}$, and blow-ups of $\mathcal{X}$ are perfect stacks by Proposition 3.8(2). For (5), it suffices to check $\operatorname{colim}_\alpha \pi_n K^B(\mathcal{X}_\alpha) \cong \pi_n K^B(\mathcal{X})$ for all $n \in \mathbb{Z}$. This follows from Theorem 4.3, since $\pi_{-n} K^B(\mathcal{X})$, for $n > 0$, is a natural retract of $K_0(\mathbb{G}_m^n \times \mathcal{X})$. $\square$

Corollary 4.6. *Let*

$$\begin{tikzcd} \mathcal{W} \arrow[r, hook] \arrow[d] & \mathcal{V} \arrow[d, "f"] \\ \mathcal{U} \arrow[r, hook, "e"] & \mathcal{X} \end{tikzcd}$$

be a Nisnevich square of stacks, and suppose that the pairs $(\mathcal{X}, \mathcal{X}\backslash\mathcal{U})$ and $(\mathcal{V}, \mathcal{V}\backslash\mathcal{W})$ are perfect. Then the induced square of spectra

$$\begin{tikzcd} K^B(\mathcal{X}) \arrow[r, "f^*"] \arrow[d, "e^*"'] & K^B(\mathcal{V}) \arrow[d] \\ K^B(\mathcal{U}) \arrow[r] & K^B(\mathcal{W}) \end{tikzcd}$$

is homotopy Cartesian.

Proof. This follows immediately from Theorem 4.5(1) and (2). $\square$

Remark 4.7. We remark that if $(\mathcal{X}, \mathcal{X}\backslash\mathcal{U})$ is a perfect pair and if the map $f : \mathcal{V} \to \mathcal{X}$ in Corollary 4.6 is representable and separated, then $(\mathcal{V}, \mathcal{V} \setminus \mathcal{W})$ is automatically a perfect pair. The reason is that in this case, f is quasi-affine by Zariski's main theorem for stacks [Laumon and Moret-Bailly 2000, Theorem 16.5] and one can apply Proposition 3.8(2).

Since the homotopy groups of the two spectra $K(\mathcal{X})$ and $K^B(\mathcal{X})$ agree in non-negative degrees by Theorem 4.5, we make the following definition.

Definition 4.8. Let $\mathcal{X}$ be a perfect stack and $i \in \mathbb{Z}$. We let $K_i(\mathcal{X})$ denote the i-th homotopy group of the spectrum $K^B(\mathcal{X})$.

4C. ***The homotopy K-theory of perfect stacks.*** For $n \in \mathbb{N}$, let

$$\Delta^n = \operatorname{Spec}\left(\frac{\mathbb{Z}[t_0, \ldots, t_n]}{(\sum_i t_i - 1)}\right).$$

Recall that $\Delta^\bullet$ is a cosimplicial scheme. For a perfect stack $\mathcal{X}$, the *homotopy*

K-theory of $\mathcal{X}$ is defined as

$$KH(\mathcal{X}) = \operatorname*{hocolim}_{n \in \Delta^{\mathrm{op}}} K^B(\mathcal{X} \times \Delta^n).$$

There is a natural map $K^B(\mathcal{X}) \to KH(\mathcal{X})$ induced by $0 \in \Delta^{\mathrm{op}}$.

Theorem 4.9. *Let $\mathcal{X}$ be a perfect stack.*

(1) *Let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack with quasicompact open complement $j : \mathcal{U} \hookrightarrow \mathcal{X}$ such that $(\mathcal{X}, \mathcal{Z})$ is perfect. Then there is a homotopy fiber sequence of spectra*

$$KH(\mathcal{X} \text{ on } \mathcal{Z}) \to KH(\mathcal{X}) \xrightarrow{j^*} KH(\mathcal{U}).$$

(2) *Let $f : \mathcal{Y} \to \mathcal{X}$ be an étale map between perfect stacks such that the projection $\mathcal{Z} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Z}$ is an isomorphism on the associated reduced stacks. Then the map $f^* : KH(\mathcal{X} \text{ on } \mathcal{Z}) \to KH(\mathcal{Y} \text{ on } \mathcal{Z} \times_{\mathcal{X}} \mathcal{Y})$ is a homotopy equivalence.*

(3) *Let $\pi : \mathbb{P}(\mathcal{E}) \to \mathcal{X}$ be the projective bundle associated to a vector bundle $\mathcal{E}$ on $\mathcal{X}$ of rank r. Then the map*

$$\prod_{0}^{r-1} KH(\mathcal{X}) \to KH(\mathbb{P}(\mathcal{E}))$$

that sends $(a_0, \dots, a_{r-1})$ to $\sum_i \mathcal{O}(-i) \otimes \pi^*(a_i)$ is a homotopy equivalence.

(4) *Let $i : \mathcal{Y} \hookrightarrow \mathcal{X}$ be a regular closed immersion and let $p : \mathcal{X}' \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ with center $\mathcal{Y}$. Then the square of spectra*

$$\begin{tikzcd} KH(\mathcal{X}) \arrow[r, "i^*"] \arrow[d, "p^*"'] & KH(\mathcal{Y}) \arrow[d] \\ KH(\mathcal{X}') \arrow[r] & KH(\mathcal{X}' \times_{\mathcal{X}} \mathcal{Y}) \end{tikzcd}$$

is homotopy Cartesian.

(5) *Suppose that $\mathcal{X}$ is the limit of a filtered diagram $(\mathcal{X}_\alpha)$ of perfect stacks with affine transition morphisms. Then the canonical map*

$$\operatorname*{hocolim}_{\alpha} KH(\mathcal{X}_\alpha) \to KH(\mathcal{X})$$

is a homotopy equivalence.

(6) *Suppose that $u : \mathcal{E} \to \mathcal{X}$ is a vector bundle over $\mathcal{X}$. Then the induced map $u^* : KH(\mathcal{X}) \to KH(\mathcal{E})$ is a homotopy equivalence.*

Proof. Properties (1)–(5) follow immediately from the definition of $KH(\mathcal{X})$ and Theorem 4.5. The proof of (6) for quotient stacks is given in [Krishna and Ravi 2018, Theorem 5.2] and the same proof is valid for perfect stacks. □

Corollary 4.10. *Let*

$$\begin{tikzcd} \mathcal{W} \arrow[r, hook] \arrow[d] & \mathcal{V} \arrow[d, "f"] \\ \mathcal{U} \arrow[r, hook, "e"] & \mathcal{X} \end{tikzcd}$$

be a Nisnevich square of stacks, and suppose that the pairs $(\mathcal{X}, \mathcal{X}\backslash\mathcal{U})$ *and* $(\mathcal{V}, \mathcal{V}\backslash\mathcal{W})$ *are perfect. Then the induced square of spectra*

$$\begin{tikzcd} KH(\mathcal{X}) \arrow[r, "f^*"] \arrow[d, "e^*"'] & KH(\mathcal{V}) \arrow[d] \\ KH(\mathcal{U}) \arrow[r] & KH(\mathcal{W}) \end{tikzcd}$$

is homotopy Cartesian.

Proof. This follows immediately from Theorem 4.9(1) and (2). □

Remark 4.11. In [Hoyois 2016], a potentially different definition of KH is given for certain quotient stacks, which forces KH to be invariant with respect to vector bundle *torsors* and not just vector bundles. The two definitions agree for quotients of schemes by finite or multiplicative type groups, as we will show in Lemma 6.1, but they may differ in general. We do not know if the above definition of KH has good properties for general perfect stacks. □

5. G-theory and the case of regular stacks

Our goal in this section is to show that perfect stacks that are Noetherian and regular have no negative K-groups. We do this by comparing the K-theory and G-theory of such stacks.

Let $\mathcal{X}$ be a stack. Recall that $\mathrm{Mod}(\mathcal{X})$ is the abelian category of $\mathcal{O}_{\mathcal{X}}$-modules on the lisse-étale site of $\mathcal{X}$ and $\mathrm{QCoh}(\mathcal{X}) \subset \mathrm{Mod}(\mathcal{X})$ is the abelian subcategory of quasicoherent sheaves.

Lemma 5.1. *Assume that $\mathcal{X}$ is a Noetherian stack. Then the inclusion*

$$\iota_{\mathcal{X}} : \mathrm{QCoh}(\mathcal{X}) \hookrightarrow \mathrm{Mod}(\mathcal{X})$$

induces an equivalence of the derived categories $D^+(\mathrm{QCoh}(\mathcal{X})) \xrightarrow{\simeq} D^+_{\mathrm{qc}}(\mathcal{X})$.

Proof. To show that $D^+(\mathrm{QCoh}(\mathcal{X})) \to D^+_{\mathrm{qc}}(\mathcal{X})$ is full and faithful, it suffices, using standard reduction, to show that the natural map

$$\mathrm{Ext}^i_{\mathrm{QCoh}(\mathcal{X})}(N, M) \to \mathrm{Ext}^i_{\mathrm{Mod}(\mathcal{X})}(N, M)$$

is an isomorphism for all $i \in \mathbb{Z}$ for $N, M \in \mathrm{QCoh}(\mathcal{X})$. Since this is clearly true for $i \leq 0$, and since $\iota_{\mathcal{X}} : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Mod}(\mathcal{X})$ is exact, it suffices to show that this functor preserves injective objects.

Let $\mathcal{F}$ be an injective quasicoherent sheaf on $\mathcal{X}$. Since a direct summand of an injective object in $\mathrm{Mod}(\mathcal{X})$ is injective and since a quasicoherent sheaf which is injective as a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules is also an injective quasicoherent sheaf, it suffices to show that there is an inclusion $\mathcal{F} \hookrightarrow \mathcal{G}$ in $\mathrm{QCoh}(\mathcal{X})$ such that $\iota_{\mathcal{X}}(\mathcal{G})$ is injective in $\mathrm{Mod}(\mathcal{X})$.

Since $\mathcal{X}$ is Noetherian, we can find a smooth atlas $u : U \to \mathcal{X}$, where U is a Noetherian scheme. We can now find an inclusion $u^*(\mathcal{F}) \hookrightarrow \mathcal{H}$ in $\mathrm{QCoh}(U)$ such that $\mathcal{H}$ is injective as a sheaf of $\mathcal{O}_U$-modules, by [Thomason and Trobaugh 1990, B.4]. We now consider the maps

$$\mathcal{F} \to u_* u^*(\mathcal{F}) \to u_*(\mathcal{H}). \tag{5.2}$$

As U is Noetherian, it is clear that $u_*(\mathcal{H})$ is a quasicoherent sheaf on $\mathcal{X}$. Furthermore, u_* has a left adjoint $u^* : \mathrm{Mod}(\mathcal{X}) \to \mathrm{Mod}(U)$ which preserves quasicoherent sheaves. Since $(u : U \to \mathcal{X})$ is an object of $\text{Lis-Ét}(\mathcal{X})$, it follows that $u^* : \mathrm{Mod}(\mathcal{X}) \to \mathrm{Mod}(U)$ is exact. In particular, $u_* : \mathrm{Mod}(U) \to \mathrm{Mod}(\mathcal{X})$ has an exact left adjoint. This implies that it must preserve injective sheaves. It follows that $u_*(\mathcal{H})$ is a quasicoherent sheaf on $\mathcal{X}$ which is injective as a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules.

Letting $\mathcal{G} = u_*(\mathcal{H})$, we are now left with showing that the two maps in (5.2) are injective. The first map is injective because u is faithfully flat and $\mathcal{F}$ is quasicoherent. The second map is injective because $u_* : \mathrm{QCoh}(U) \to \mathrm{QCoh}(\mathcal{X})$ is left exact and hence preserves injections.

To show that the functor $D^+(\mathrm{QCoh}(\mathcal{X})) \to D^+_{\mathrm{qc}}(\mathcal{X})$ is essentially surjective, we can use its full and faithfulness shown above and an induction on the length to first see that $D^b(\mathrm{QCoh}(\mathcal{X})) \xrightarrow{\simeq} D^b_{\mathrm{qc}}(\mathcal{X})$. Since every object of $D^+_{\mathrm{qc}}(\mathcal{X})$ is a colimit of objects in $D^b_{\mathrm{qc}}(\mathcal{X})$ (using good truncations), a limit argument concludes the proof. □

Lemma 5.3. *Let $\mathcal{X}$ be as in Lemma 5.1 and let $P \in D(\mathrm{QCoh}(\mathcal{X}))$ be a compact object. Then the following hold.*

(1) *There exists an integer $r \geq 0$ such that $\mathrm{Hom}_{D_{\mathrm{qc}}(\mathcal{X})}(P, N[i]) = 0$ for all $N \in \mathrm{QCoh}(\mathcal{X})$ and $i > r$.*

(2) *There exists an integer $r \geq 0$ such that the natural map*

$$\tau^{\geq j}\mathrm{RHom}_{D_{\mathrm{qc}}(\mathcal{X})}(P, M) \to \tau^{\geq j}\mathrm{RHom}_{D_{\mathrm{qc}}(\mathcal{X})}(P, \tau^{\geq j-r} M)$$

is a quasi-isomorphism for all $M \in D_{\mathrm{qc}}(\mathcal{X})$ and integers j.

(3) *There exists an integer $r \geq 0$ such that the natural map*

$$\tau^{\geq j}\mathrm{RHom}_{D(\mathrm{QCoh}(\mathcal{X}))}(P, M) \to \tau^{\geq j}\mathrm{RHom}_{D(\mathrm{QCoh}(\mathcal{X}))}(P, \tau^{\geq j-r} M)$$

is a quasi-isomorphism for all $M \in D(\mathrm{QCoh}(\mathcal{X}))$ and integers j.

Proof. It follows from Lemma 5.1 that $\iota_{\mathcal{X}}$ induces an equivalence between the derived categories of perfect complexes of quasicoherent sheaves and perfect complexes of $\mathcal{O}_{\mathcal{X}}$-modules. Since the compact objects of $D_{\mathrm{qc}}(\mathcal{X})$ are perfect [Krishna and Ravi 2018, Proposition 2.7], it follows that $D(\mathrm{QCoh}(\mathcal{X}))$ and $D_{\mathrm{qc}}(\mathcal{X})$ have equivalent full subcategories of compact objects. The parts (1) and (2) now follow from [Hall and Rydh 2017, Lemma 4.5] and the proof of [Hall et al. 2014, Lemma 2.4] shows that (1) implies (3) for any stack. □

Lemma 5.4. *Let $\mathcal{X}$ be a Noetherian stack such that $D_{\mathrm{qc}}(\mathcal{X})$ is compactly generated. Then $\iota_{\mathcal{X}} : \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Mod}(\mathcal{X})$ induces an equivalence of the unbounded derived categories $D(\mathrm{QCoh}(\mathcal{X})) \xrightarrow{\simeq} D_{\mathrm{qc}}(\mathcal{X})$.*

Proof. Let $\Psi : D(\mathrm{QCoh}(\mathcal{X})) \to D_{\mathrm{qc}}(\mathcal{X})$ denote the derived functor induced by $\iota_{\mathcal{X}}$. We have shown in the proof of Lemma 5.3 that Ψ restricts to an equivalence between the full subcategories of compact objects. Using [Benson et al. 2011, Lemma 4.5], it thus suffices to show that $D(\mathrm{QCoh}(\mathcal{X}))$ is compactly generated.

So let $M \in D(\mathrm{QCoh}(\mathcal{X}))$ be such that $\mathrm{Hom}_{D(\mathrm{QCoh}(\mathcal{X}))}(P, M) = 0$ for every compact object P. We need to show that $M = 0$. Since any compact object of $D(\mathrm{QCoh}(\mathcal{X}))$ is perfect, and Ψ is conservative and induces equivalence of compact objects, it suffices to show that $\mathrm{RHom}(P, M) \xrightarrow{\simeq} \mathrm{RHom}(\Psi(P), \Psi(M))$ for every perfect complex P. Equivalently, we need to show that for every integer j, the map $\tau^{\geq j}\mathrm{RHom}(P, M) \to \tau^{\geq j}\mathrm{RHom}(\Psi(P), \Psi(M))$ is a quasi-isomorphism. Lemma 5.3 now allows us to assume that $M \in D^{+}(\mathrm{QCoh}(\mathcal{X}))$. But in this case, the result follows from Lemma 5.1. □

For a Noetherian stack $\mathcal{X}$, let $G^{\mathrm{naive}}(\mathcal{X})$ denote the K-theory spectrum of the exact category of coherent $\mathcal{O}_{\mathcal{X}}$-modules in the sense of Quillen, and let $G(\mathcal{X})$ be the K-theory spectrum of the complicial bi-Waldhausen category of cohomologically bounded pseudocoherent complexes in $\mathrm{Ch}_{\mathrm{qc}}(\mathcal{X})$, in the sense of [Thomason and Trobaugh 1990, §1.5.2]. We have a natural map of spectra $G^{\mathrm{naive}}(\mathcal{X}) \to G(\mathcal{X})$.

Lemma 5.5. *Let $\mathcal{X}$ be a Noetherian stack such that $D_{\mathrm{qc}}(\mathcal{X})$ is compactly generated. Then the map $G^{\mathrm{naive}}(\mathcal{X}) \to G(\mathcal{X})$ is a homotopy equivalence.*

Proof. It follows from Lemma 5.4 that $G(\mathcal{X})$ is homotopy equivalent to the K-theory of the Waldhausen category $\mathrm{Ch}_{\mathrm{pc}}(\mathrm{QCoh}(\mathcal{X}))$ of cohomologically bounded pseudocoherent chain complexes of quasicoherent sheaves on $\mathcal{X}$. Let $\mathrm{Ch}^{b}(\mathrm{Coh}(\mathcal{X}))$ denote the Waldhausen category of bounded complexes of coherent $\mathcal{O}_{\mathcal{X}}$-modules.

Using the fact that every quasicoherent sheaf on $\mathcal{X}$ is a filtered colimit of coherent subsheaves [Laumon and Moret-Bailly 2000, Proposition 15.4], we can mimic the proof of [Thomason and Trobaugh 1990, Lemma 3.12] to conclude that the inclusion $\mathrm{Ch}^{b}(\mathrm{Coh}(\mathcal{X})) \hookrightarrow \mathrm{Ch}_{\mathrm{pc}}(\mathrm{QCoh}(\mathcal{X}))$ induces a homotopy equivalence between the associated K-theory spectra. By induction on the length of complexes in

$\mathrm{Ch}^b(\mathrm{Coh}(\mathcal{X}))$, the map $G^{\mathrm{naive}}(\mathcal{X}) \to K(\mathrm{Ch}^b(\mathrm{Coh}(\mathcal{X})))$ is also a homotopy equivalence, so we conclude the proof. □

Lemma 5.6. *Let $\mathcal{X}$ be a Noetherian regular stack. Then the canonical map of spectra $K(\mathcal{X}) \to G(\mathcal{X})$ is a homotopy equivalence.*

Proof. As for schemes [Thomason and Trobaugh 1990, Theorem 3.21], it suffices to show that every cohomologically bounded pseudocoherent complex $E^\bullet$ on $\mathcal{X}$ is perfect. Let $u : U \to \mathcal{X}$ be a smooth atlas such that U is affine. Since $(u : U \to \mathcal{X})$ is an object of Lis-Ét$(\mathcal{X})$, the functor $u^* : \mathrm{Mod}(\mathcal{X}) \to \mathrm{Mod}(U)$ is exact and preserves coherent sheaves. It follows that $u^*(E^\bullet)$ is a cohomologically bounded pseudocoherent complex on U. Since U is a regular scheme, we conclude from the proof of [Thomason and Trobaugh 1990, Theorem 3.21] that $u^*(E^\bullet)$ is perfect. But this implies that $E^\bullet$ is perfect on $\mathcal{X}$. □

Theorem 5.7. *Let $\mathcal{X}$ be a Noetherian regular stack such that $D_{\mathrm{qc}}(\mathcal{X})$ is compactly generated. Then the following hold.*

(1) *The canonical maps $K(\mathcal{X}) \to G(\mathcal{X}) \leftarrow G^{\mathrm{naive}}(\mathcal{X})$ are homotopy equivalences.*

(2) *For any vector bundle $\mathcal{E}$ on $\mathcal{X}$ and any $\mathcal{E}$-torsor $\pi : \mathcal{Y} \to \mathcal{X}$, the pullback map $K(\mathcal{X}) \to K(\mathcal{Y})$ is a homotopy equivalence.*

(3) *The canonical morphisms of spectra*
$$K(\mathcal{X}) \to K^B(\mathcal{X}) \to KH(\mathcal{X})$$
are homotopy equivalences. In particular, $K_i(\mathcal{X}) = KH_i(\mathcal{X}) = 0$ for $i < 0$.

Proof. Part (1) of the theorem follows directly from Lemmas 5.5 and 5.6. As shown in [Merkurjev 2005, Theorem 2.11], there exists a short exact sequence of vector bundles
$$0 \to \mathcal{E} \to \mathcal{W} \xrightarrow{\phi} \mathbb{A}^1_{\mathcal{X}} \to 0$$
such that $\mathcal{Y} = \phi^{-1}(1)$. In particular, $\mathcal{Y}$ is the complement of the projective bundle $\mathbb{P}(\mathcal{E})$ in $\mathbb{P}(\mathcal{W})$. It follows from our hypothesis and Lemma 3.6 that $\mathbb{P}(\mathcal{W})$ is a Noetherian regular stack such that $D_{\mathrm{qc}}(\mathbb{P}(\mathcal{W}))$ is compactly generated. The same holds for $\mathbb{P}(\mathcal{E})$ as well. The Quillen localization sequence
$$G^{\mathrm{naive}}(\mathbb{P}(\mathcal{E})) \to G^{\mathrm{naive}}(\mathbb{P}(\mathcal{W})) \to G^{\mathrm{naive}}(\mathcal{Y}) \tag{5.8}$$
and the projective bundle formula [Krishna and Ravi 2018, Theorem 3.8] now prove (2).

By (1) and the Quillen localization sequence for $G^{\mathrm{naive}}(-)$ associated to the inclusions $\mathcal{X}[T] \hookrightarrow \mathbb{P}^1_{\mathcal{X}}$ and $\mathcal{X}[T^{\pm 1}] \hookrightarrow \mathcal{X}[T]$, we see that the Bass fundamental theorem holds for $\mathcal{X}$ and moreover that the sequence (3) of Theorem 4.4 is a short exact sequence. This implies that one can define $K^B(\mathcal{X})$ as in Section 4B, and moreover that $K(\mathcal{X}) \xrightarrow{\simeq} K^B(\mathcal{X})$. On the other hand, it follows from (2) that

$K^B(\mathcal{X}) \xrightarrow{\simeq} K^B(\mathcal{X} \times \Delta^n)$ for every $n \geq 0$, which implies that $K^B(\mathcal{X}) \xrightarrow{\simeq} KH(\mathcal{X})$. The last assertion of (3) holds because $K(\mathcal{X})$ has no negative homotopy groups. □

6. Cdh-descent for homotopy K-theory

We denote by $\mathbf{Stk}'$ the category of stacks $\mathcal{X}$ satisfying one of the following conditions:

- $\mathcal{X}$ has separated diagonal and linearly reductive finite stabilizers.
- $\mathcal{X}$ has affine diagonal and linearly reductive almost multiplicative stabilizers.

By Corollaries 3.16 and 3.17, for every $\mathcal{X} \in \mathbf{Stk}'$ and every closed substack $\mathcal{Z} \subset \mathcal{X}$ with quasicompact open complement, the pair $(\mathcal{X}, \mathcal{Z})$ is perfect. Furthermore, by Theorem 2.10, $\mathcal{X}$ admits a Nisnevich covering by quotient stacks $[U/G]$ where U is affine over an affine scheme S and G is a linearly reductive almost multiplicative group scheme over S

Note that if $\mathcal{X} \in \mathbf{Stk}'$ and $\mathcal{Y} \to \mathcal{X}$ is a representable morphism with affine diagonal, then also $\mathcal{Y} \in \mathbf{Stk}'$, since the stabilizers of $\mathcal{Y}$ are subgroups of the stabilizers of $\mathcal{X}$.

Lemma 6.1. *Let $\mathcal{X}$ be a stack in $\mathbf{Stk}'$ and let $f : \mathcal{Y} \to \mathcal{X}$ be a torsor under a vector bundle. Then*

$$f^* : KH(\mathcal{X}) \to KH(\mathcal{Y})$$

is a homotopy equivalence.

Proof. By Theorem 2.10, there exists a Nisnevich covering $[U/G] \to \mathcal{X}$, where U is affine over an affine scheme S and G is a linearly reductive S-group scheme. By Proposition 2.9 and Corollary 4.10, we are reduced to showing that

$$KH([U/G]) \to KH([U/G] \times_{\mathcal{X}} \mathcal{Y})$$

is a homotopy equivalence. But since U and S are affine and G is linearly reductive, the vector bundle torsor $[U/G] \times_{\mathcal{X}} \mathcal{Y} \to [U/G]$ has a section and hence is a vector bundle. The result now follows from Theorem 4.9(6). □

The following theorem is our cdh-descent result for the homotopy K-theory of stacks.

Theorem 6.2. *Let $\mathcal{X}$ be a stack in $\mathbf{Stk}'$ and let*

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook] \arrow[d] & \mathcal{Y} \arrow[d, "p"] \\ \mathcal{Z} \arrow[r, hook, "e"] & \mathcal{X} \end{tikzcd}$$

be a Cartesian square where p is a proper representable morphism, e is a closed immersion, and p induces an isomorphism $\mathcal{Y} \setminus \mathcal{E} \cong \mathcal{X} \setminus \mathcal{Z}$. Then the induced square *of spectra*

$$\begin{array}{ccc} KH(\mathcal{X}) & \xrightarrow{p^*} & KH(\mathcal{Y}) \\ {\scriptstyle e^*}\downarrow & & \downarrow \\ KH(\mathcal{Z}) & \longrightarrow & KH(\mathcal{E}) \end{array} \tag{6.3}$$

is homotopy Cartesian.

Proof. We proceed in several steps.

Step 1. We prove the result under the assumptions that p is projective, that p and e are of finite presentation, and that $\mathcal{X} = [U/G]$ is quasi-affine over $[S/G]$ for some affine scheme S and some linearly reductive isotrivial almost multiplicative group scheme G over S (note that such a stack belongs to $\mathbf{Stk}'$ and has the resolution property). Since G is finitely presented, we can write U as an inverse limit of quasi-affine G-schemes of finite presentation over S. By Theorem 4.9(5), KH transforms such limits into homotopy colimits. Since homotopy colimits of spectra commute with homotopy pullbacks, we can assume that U is finitely presented over S. We are now in the situation of [Hoyois 2016, Theorem 1.3], and we deduce that (6.3) is a homotopy Cartesian square for the KH-theory defined in [Hoyois 2016] (more precisely, for the presheaf of spectra $KH_{[S/G]}$ defined in [Hoyois 2016, §4]). But the latter agrees with the KH-theory defined in this paper, by Lemma 6.1.

Step 2. We prove the result under the assumption that p is projective and that $\mathcal{X}$ is as in Step 1. Since every quasicoherent sheaf on $\mathcal{X}$ is the union of its finitely generated quasicoherent subsheaves [Rydh 2016], we can write $\mathcal{Z}$ as a filtered intersection of finitely presented closed substacks of $\mathcal{X}$. By continuity of KH (Theorem 4.9(5)), we can therefore assume that e is finitely presented. In particular, $\mathcal{U} = \mathcal{X} \setminus \mathcal{Z}$ is quasicompact. Since $\mathcal{Y}$ is projective over $\mathcal{X}$, it is a closed substack of $\mathbb{P}(\mathcal{F})$ for some finitely generated quasicoherent sheaf $\mathcal{F}$ on $\mathcal{X}$. Since $\mathcal{X}$ has the resolution property and affine stabilizers, we can write $\mathcal{X} = [V/\mathrm{GL}_n]$ for some quasi-affine scheme V [Gross 2017, Theorem 1.1]. On such stacks, it is known that every quasicoherent sheaf is a filtered colimit of finitely presented quasicoherent sheaves [Rydh 2015, Theorem A and Proposition 2.10(iii)]. In particular, $\mathcal{F}$ is a quotient of a finitely presented sheaf, so we can assume without loss of generality that $\mathcal{F}$ is finitely presented. We can again write $\mathcal{Y}$ as a filtered intersection of finitely presented closed substacks $\mathcal{Y}_i \subset \mathbb{P}(\mathcal{F})$. By [Rydh 2015, Theorem C(ii)], the projection $\mathcal{Y}_i \times_{\mathcal{X}} \mathcal{U} \to \mathcal{U}$ is a closed immersion for sufficiently large i. But since it has a section, it must be an isomorphism. By continuity of KH, we can therefore assume that p is finitely presented, and we are thus reduced to Step 1.

Step 3. We prove the result assuming only that p is projective. By Theorem 2.10 and the fact that groups of multiplicative type are isotrivial locally in the Nisnevich topology [Hoyois 2017, Remark 2.9], there exists a Nisnevich covering $\{\mathcal{U}_i \to \mathcal{X}\}$

where each $\mathcal{U}_i$ is as in Step 1. By Proposition 2.9, there is a sequence of quasicompact open substacks $\varnothing = \mathcal{X}_0 \subset \cdots \subset \mathcal{X}_n = \mathcal{X}$ together with Nisnevich squares

$$\begin{tikzcd} \mathcal{W}_j \arrow[r, hook] \arrow[d] & \mathcal{V}_j \arrow[d] \\ \mathcal{X}_{j-1} \arrow[r, hook] & \mathcal{X}_j \end{tikzcd}$$

where each $\mathcal{V}_j$ is a quasicompact open substack of $\coprod_i \mathcal{U}_i$. In particular, each $\mathcal{V}_j$ and each $\mathcal{W}_j$ is as in Step 1. Since KH satisfies Nisnevich descent (Corollary 4.10), we deduce from Step 2 by a straightforward induction on j that (6.3) is a homotopy Cartesian square.

Step 4. We prove the result in general. As in Step 2, we can assume that the complement of e is quasicompact. By Corollary 2.4, there exists a projective morphism $\mathcal{Y}' \to \mathcal{Y}$ which is an isomorphism over the complement of e and such that the composite $\mathcal{Y}' \to \mathcal{Y} \to \mathcal{X}$ is projective. Consider the squares

$$\begin{tikzcd} KH(\mathcal{X}) \arrow[r, "p^*"] \arrow[d, "e^*"'] & KH(\mathcal{Y}) \arrow[r] \arrow[d] & KH(\mathcal{Y}') \arrow[d] \\ KH(\mathcal{Z}) \arrow[r] & KH(\mathcal{E}) \arrow[r] & KH(\mathcal{E}') \end{tikzcd}$$

where $\mathcal{E}' = \mathcal{E} \times_{\mathcal{Y}} \mathcal{Y}'$. The right-hand square and the total square are both homotopy Cartesian by Step 3. Hence, the left-hand square is also homotopy Cartesian, as desired. □

7. The vanishing theorems

Our goal now is to use the cdh-descent for homotopy K-theory to prove the vanishing theorems for negative K-theory. In order to apply cdh-descent, Kerz and Strunk [2017] used the idea of killing classes in the negative K-theory of schemes using Gruson–Raynaud flatification [Raynaud and Gruson 1971]. In Section 7A, we prove an analog of this result for stacks. This is done essentially like in the case of schemes, where we replace Gruson–Raynaud flatification with Rydh's flatification theorem for algebraic stacks (Theorem 2.2). The vanishing results are proven in Sections 7B and 7C.

7A. ***Killing by flatification.*** We need the following two preparatory results about quasicoherent sheaves on stacks.

Lemma 7.1. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a quasi-affine morphism of stacks. If $\mathcal{X}$ satisfies the resolution property, so does $\mathcal{Y}$.*

Proof. This is [Hall and Rydh 2017, Lemma 7.1]. □

Lemma 7.2. *Let $f : \mathcal{Y} \to \mathcal{X}$ be a smooth morphism of Noetherian stacks and let $\mathcal{F}$ be a coherent sheaf on $\mathcal{Y}$ which is flat over $\mathcal{X}$. Then $\mathcal{F}$ has finite tor-dimension over $\mathcal{Y}$.*

Proof. Since the question is smooth-local on $\mathcal{X}$ and $\mathcal{Y}$, we can assume that $\mathcal{X}$ and $\mathcal{Y}$ are Noetherian schemes. In this case, the result is [Kerz and Strunk 2017, Lemma 6]. □

Proposition 7.3 (the killing lemma). *Let $\mathcal{X}$ be a reduced Noetherian stack and let $f : \mathcal{Y} \to \mathcal{X}$ be a smooth morphism of finite type such that $\mathcal{Y}$ is perfect and satisfies the resolution property. Let $n > 0$ be an integer and let $\xi \in K_{-n}(\mathcal{Y})$. Then there exists a sequence of blow-ups $u : \mathcal{X}' \to \mathcal{X}$ with nowhere dense centers such that for the induced map $u_{\mathcal{Y}} : \mathcal{Y}' := \mathcal{X}' \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$, one has $u^*_{\mathcal{Y}}(\xi) = 0$ in $K_{-n}(\mathcal{Y}')$.*

Proof. We repeat the proof of [Kerz and Strunk 2017, Proposition 5] with minor modifications. By the construction of negative K-theory of perfect stacks (see Definition 4.8), there exists a surjection

$$\operatorname{Coker}(K_0(\mathcal{Y} \times \mathbb{A}^n) \to K_0(\mathcal{Y} \times \mathbb{G}_m^n)) \twoheadrightarrow K_{-n}(\mathcal{Y}), \tag{7.4}$$

natural in $\mathcal{Y}$. It therefore suffices to prove that, for any $\xi \in K_0(\mathcal{Y} \times \mathbb{G}_m^n)$, there exists a sequence of blow-ups $u : \mathcal{X}' \to \mathcal{X}$ with nowhere dense centers such that $u^*_{\mathcal{Y} \times \mathbb{G}_m^n}(\xi)$ lies in the image of the restriction map

$$j^* : K_0(\mathcal{Y}' \times \mathbb{A}^n) \to K_0(\mathcal{Y}' \times \mathbb{G}_m^n). \tag{7.5}$$

Since $\mathcal{Y}$ satisfies the resolution property, it follows from Lemma 7.1 that $\mathcal{Y} \times \mathbb{G}_m^n$ also satisfies the resolution property. In particular, $K_0(\mathcal{Y} \times \mathbb{G}_m^n)$ is generated by classes of vector bundles on $\mathcal{Y} \times \mathbb{G}_m^n$. Since any finite collection of sequences of blow-ups of $\mathcal{X}$ can be refined by a single such sequence, we can assume that ξ is represented by a vector bundle $\mathcal{E}$ on $\mathcal{Y} \times \mathbb{G}_m^n$. We can now extend $\mathcal{E}$ to a coherent sheaf $\mathcal{F}$ on $\mathcal{Y} \times \mathbb{A}^n$ by [Gross 2017, Theorem 1.1; Thomason 1987b, Lemma 1.4].

Choose a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

where X and Y are algebraic spaces, p and q are smooth surjective maps, and g is smooth of finite type. By generic flatness [Stacks 2005–, Tag 06QR], we can find a dense open subspace $U \subset X$ such that $(q \times \mathrm{id}_{\mathbb{A}^n})^*(\mathcal{F})$ is flat over U under the composite map $Y \times \mathbb{A}^n \to Y \to X$. Then U induces a dense open substack $\mathcal{U} \subset \mathcal{X}$ such that $\mathcal{F}$ is flat over $\mathcal{U}$. We now apply Theorem 2.2 to find a sequence of blow-ups $u : \mathcal{X}' \to \mathcal{X}$ whose centers are disjoint from $\mathcal{U}$ such that the strict transform $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ on $\mathcal{Y}' \times \mathbb{A}^n$ is flat over $\mathcal{X}'$.

We consider the commutative diagram of Cartesian squares

$$\begin{tikzcd} \mathcal{Y}' \times \mathbb{G}_m^n \arrow[r, "j"] \arrow[d, "q"'] & \mathcal{Y}' \times \mathbb{A}^n \arrow[r, "e'"] \arrow[d, "w"] & \mathcal{Y}' \arrow[r] \arrow[d, "v"] & \mathcal{X}' \arrow[d, "u"] \\ \mathcal{Y} \times \mathbb{G}_m^n \arrow[r] & \mathcal{Y} \times \mathbb{A}^n \arrow[r, "e"] & \mathcal{Y} \arrow[r] & \mathcal{X} \end{tikzcd} \tag{7.6}$$

in which the vertical arrows are blow-ups and the horizontal arrows are smooth. We next recall that the strict transform $\widetilde{\mathcal{F}}$ is defined by the cokernel of the map $\mathcal{H}^0_{E\times\mathbb{A}^n}(w^*(\mathcal{F})) \hookrightarrow w^*(\mathcal{F})$, where $E \hookrightarrow \mathcal{Y}'$ is the exceptional locus of the blow-up and $\mathcal{H}^0_{(-)}$ is the sheaf of sections with support. Since $\mathcal{F}$ restricts to the vector bundle $\mathcal{E}$ over $\mathcal{Y} \times \mathbb{G}_m^n$, which in turn is smooth over $\mathcal{X}$, it follows that $j^*(\widetilde{\mathcal{F}}) = q^*(\mathcal{E})$ by [Stacks 2005–, Tag 080F].

Lemma 7.2 says that $\widetilde{\mathcal{F}}$ has finite tor-dimension over $\mathcal{Y}' \times \mathbb{A}^n$. In particular, it defines a class $[\widetilde{\mathcal{F}}] \in K_0(\mathcal{Y}' \times \mathbb{A}^n)$. Moreover, we have $[q^*(\mathcal{E})] = [j^*(\widetilde{\mathcal{F}})] = j^*([\widetilde{\mathcal{F}}])$. This finishes the proof. □

7B. ***Vanishing of negative homotopy K-theory.*** We now use the techniques of cdh-descent and killing by flatification to prove our main results on the vanishing of negative K-theory of stacks.

Definition 7.7. Let $\mathcal{X}$ be a Noetherian stack.

(1) The *Krull dimension* $\operatorname{Kr}\dim(\mathcal{X}) \in \mathbb{N} \cup \{\pm\infty\}$ is the Krull dimension of the underlying topological space $|\mathcal{X}|$; see [Laumon and Moret-Bailly 2000, Chapter 5] for the definition of $|\mathcal{X}|$.

(2) The *blow-up dimension* $\operatorname{bl}\dim(\mathcal{X}) \in \mathbb{N} \cup \{\pm\infty\}$ is the supremum of the integers $n \geq 0$ such that there exists a sequence $\mathcal{X}_n \to \mathcal{X}_{n-1} \to \cdots \to \mathcal{X}_0 = \mathcal{X}$ of nonempty stacks where each $\mathcal{X}_i$ is a nowhere dense closed substack of an iterated blow-up of $\mathcal{X}_{i-1}$.

(3) The *covering dimension* $\operatorname{cov}\dim(\mathcal{X}) \in \mathbb{N} \cup \{\pm\infty\}$ is the least dimension of a scheme X admitting a faithfully flat finitely presented morphism $X \to \mathcal{X}$.

Lemma 7.8. *Let $\mathcal{X}$ be a Noetherian stack. Then*

$$\operatorname{Kr}\dim(\mathcal{X}) \leq \operatorname{bl}\dim(\mathcal{X}) \leq \operatorname{cov}\dim(\mathcal{X}).$$

If $\mathcal{X}$ is a quasi-DM stack, all three are equal to $\dim(\mathcal{X})$.

Proof. The inequality $\operatorname{Kr}\dim(\mathcal{X}) \leq \operatorname{bl}\dim(\mathcal{X})$ follows directly from the definitions, since $\operatorname{Kr}\dim(\mathcal{X})$ is the supremum of a subset of the set of integers described in Definition 7.7(2). For the inequality $\operatorname{bl}\dim(\mathcal{X}) \leq \operatorname{cov}\dim(\mathcal{X})$, it suffices to prove the following:

(i) If $\mathcal{Y} \to \mathcal{X}$ is a blow-up, then $\operatorname{cov}\dim(\mathcal{Y}) \leq \operatorname{cov}\dim(\mathcal{X})$.

(ii) If $\mathcal{Z} \subset \mathcal{X}$ is a nowhere dense closed substack, then $\operatorname{cov}\dim(\mathcal{Z}) \leq \operatorname{cov}\dim(\mathcal{X}) - 1$.

Let $f : X \to \mathcal{X}$ be an fppf cover, where X is a scheme. Then X is Noetherian and $X \times_{\mathcal{X}} \mathcal{Y} \to X$ is a blow-up of X. It follows that $\dim(X \times_{\mathcal{X}} \mathcal{Y}) \le \dim(X)$, whence (i). By [Stacks 2005–, Tag 04XL], the induced map of topological spaces $|f| : |X| \to |\mathcal{X}|$ is continuous and open. Using [Stacks 2005–, Tag 03HR], we deduce that $X \times_{\mathcal{X}} \mathcal{Z}$ is a nowhere dense closed subscheme of X. It follows that $\dim(X \times_{\mathcal{X}} \mathcal{Z}) \le \dim(X) - 1$, whence (ii).

For the last statement, we prove more generally that the following hold for every faithfully flat representable quasifinite morphism of Noetherian stacks $f : \mathcal{Y} \to \mathcal{X}$:

(i) $\dim(\mathcal{Y}) = \dim(\mathcal{X})$.

(ii) $\operatorname{Kr}\dim(\mathcal{Y}) \le \operatorname{Kr}\dim(\mathcal{X})$.

If $\mathcal{X}$ is quasi-DM, we can take $\mathcal{Y}$ to be a scheme and we deduce $\operatorname{cov}\dim(\mathcal{X}) \le \dim(\mathcal{X})$ and $\dim(\mathcal{X}) \le \operatorname{Kr}\dim(\mathcal{X})$, as desired. To prove (i), by definition of the dimension of a stack [Stacks 2005–, Tag 0AFL], we are immediately reduced to the case where $\mathcal{X}$ is an algebraic space. In this case, the claim follows from [Stacks 2005–, Tags 04NV and 0AFH]. If $Z_0 \subset \cdots \subset Z_n$ is a strictly increasing sequence of irreducible closed subsets of $|\mathcal{Y}|$, then $\overline{f(Z_0)} \subset \cdots \subset \overline{f(Z_n)}$ is a sequence of irreducible closed subsets of $|\mathcal{X}|$. To check that it is strictly increasing, we may again assume that $\mathcal{X}$ is an algebraic space. If the sequence were not strictly increasing, we would have a nontrivial specialization in a fiber of $|f|$, which is a discrete space [Stacks 2005–, Tag 06RW]. This proves (ii). □

Example 7.9. Let k be a field, let $n \ge 1$, and let $\mathcal{X}$ be the stack quotient of $\mathbb{A}^n_k$ by the standard action of the general linear group GL_n. Then $\operatorname{Kr}\dim(\mathcal{X}) = \operatorname{bl}\dim(\mathcal{X}) = 1$, $\operatorname{cov}\dim(\mathcal{X}) = n$, and $\dim(\mathcal{X}) = n - n^2$. We do not know an example where $\operatorname{Kr}\dim(\mathcal{X}) \ne \operatorname{bl}\dim(\mathcal{X})$. □

See Section 6 for the definition of the category $\mathbf{Stk}'$ appearing in the next theorem.

Theorem 7.10. *Let $\mathcal{X}$ be a stack in $\mathbf{Stk}'$ satisfying the resolution property. If $\mathcal{X}$ is Noetherian of blow-up dimension d, then $KH_i(\mathcal{X}) = 0$ for $i < -d$.*

Proof. We prove the theorem by induction on d. Since KH is nil-invariant (take $\mathcal{Y} = \varnothing$ in Theorem 6.2), we can assume that $\mathcal{X}$ is reduced. We can write $KH(\mathcal{X}) = \operatorname{hocolim}_n F_n(\mathcal{X})$, where

$$F_n(\mathcal{X}) = \operatorname*{hocolim}_{\Delta^{\mathrm{op}}_{\le n}} K^B(\mathcal{X} \times \Delta^\bullet).$$

It suffices to show inductively on n that the canonical map $\pi_i F_n(\mathcal{X}) \to KH_i(\mathcal{X})$ is zero for all $i < -d$. This is trivial if $n < 0$, so assume $n \ge 0$.

Let $C_{i,n}(\mathcal{X})$ denote the cokernel of $\pi_i F_{n-1}(\mathcal{X}) \to \pi_i F_n(\mathcal{X})$. Since the cofiber of the map $F_{n-1}(\mathcal{X}) \to F_n(\mathcal{X})$ is canonically a direct summand of $\Sigma^n K^B(\mathcal{X} \times \Delta^n)$ (see for instance [Lurie 2017, Remark 1.2.4.7]), we may identify $C_{i,n}(\mathcal{X})$ with a subgroup of $K_{i-n}(\mathcal{X}\times\Delta^n)$. By the induction hypothesis, the map $\pi_i F_n(\mathcal{X}) \to KH_i(\mathcal{X})$ factors through $C_{i,n}(\mathcal{X})$:

$$\begin{tikzcd}
\pi_i F_{n-1}(\mathcal{X}) \arrow[r] \arrow[dr, "0"'] & \pi_i F_n(\mathcal{X}) \arrow[r, two heads] \arrow[d] & C_{i,n}(\mathcal{X}) \arrow[r, hook] \arrow[dl, dashed, "\phi_{i,n}"] & K_{i-n}(\mathcal{X}\times\Delta^n) \\
& KH_i(\mathcal{X}) & &
\end{tikzcd} \tag{7.11}$$

Hence, it suffices to show that $\phi_{i,n} : C_{i,n}(\mathcal{X}) \to KH_i(\mathcal{X})$ is zero. Let ξ be in $C_{i,n}(\mathcal{X}) \subset K_{i-n}(\mathcal{X}\times\Delta^n)$. By Lemma 7.1, $\mathcal{X}\times\Delta^n$ satisfies the resolution property. By Proposition 7.3, there exists a sequence of blow-ups $u : \mathcal{X}' \to \mathcal{X}$ with nowhere dense centers such that $u^*(\xi) = 0$ in $C_{i,n}(\mathcal{X}') \subset K_{i-n}(\mathcal{X}'\times\Delta^n)$ (note that $i-n<0$). Let $\mathcal{Z} \subset \mathcal{X}$ be a nowhere dense closed substack of $\mathcal{X}$ such that u is an isomorphism over the complement of $\mathcal{Z}$. By Theorem 6.2, we have a long exact sequence

$$\cdots \to KH_{i+1}(u^{-1}(\mathcal{Z})) \to KH_i(\mathcal{X}) \to KH_i(\mathcal{X}') \oplus KH_i(\mathcal{Z}) \to \cdots .$$

Note that both $\mathcal{Z}$ and $u^{-1}(\mathcal{Z})$ have blow-up dimension strictly less than d. By the induction hypothesis, $KH_{i+1}(u^{-1}(\mathcal{Z}))$ and $KH_i(\mathcal{Z})$ are both zero, so the map $u^* : KH_i(\mathcal{X}) \to KH_i(\mathcal{X}')$ is injective. Since $\phi_{i,n}$ is natural in $\mathcal{X}$, we have $u^*\phi_{i,n}(\xi) = \phi_{i,n}u^*(\xi) = 0$, and we conclude that $\phi_{i,n}(\xi) = 0$. This finishes the proof. □

Our next goal is to remove the resolution property assumption from Theorem 7.10. We are able to do so under the additional assumption that $\mathcal{X}$ has finite inertia. If X is a Noetherian algebraic space, we denote by $\mathbf{\acute{E}t}_X$ the category of algebraic spaces over X that are étale, separated, and of finite type. The following lemma is a Nisnevich variant of [Kerz and Strunk 2017, Proposition 3].

Lemma 7.12. *Let X be a Noetherian algebraic space, let $\mathcal{F}$ be a Nisnevich sheaf of abelian groups on $\mathbf{\acute{E}t}_X$, and let r be an integer. Suppose that $\mathcal{F}(\mathcal{O}^h_{Y,y}) = 0$ for every point $y \in Y \in \mathbf{\acute{E}t}_X$ with $\dim\overline{\{y\}} > r$. Then $H^i_{\mathrm{Nis}}(X,\mathcal{F}) = 0$ for all $i > r$.*

Proof. Let $s \in \mathcal{F}(X)$ be a section, and let $i : Z \hookrightarrow X$ be a closed immersion such that the support of s is $|Z|$, i.e., $|Z|$ is the closed subset of points $x \in X$ such that s is nonzero in every open neighborhood of x. We claim that

$$\dim(Z) \le r.$$

Otherwise, let $y \in Z$ be a generic point such that $\dim\overline{\{y\}} > r$. Then $i^*(\mathcal{F})(\mathcal{O}_{Z,y}) \cong \mathcal{F}(\mathcal{O}^h_{X,y}) = 0$, so the section $i^*(s)$ of $i^*(\mathcal{F})$ vanishes on an open neighborhood Y of y in Z. This means that s itself vanishes on an étale neighborhood of Y. Since it also vanishes on $X \setminus Z$ and $\mathcal{F}$ is a Nisnevich sheaf, it follows that s vanishes on the open $(X \setminus Z) \cup Y$, which is a contradiction.

Let S be a finite set of local sections of $\mathcal{F}$, and let $\mathcal{F}_S \subset \mathcal{F}$ be the subsheaf generated by S. Let $i_S : X_S \hookrightarrow X$ be a closed immersion such that $|X_S|$ is the union of the closures of the images of the supports of the sections in S, and let $j_S : X \setminus X_S \hookrightarrow X$ be the complementary open immersion. Then $j_S^*(\mathcal{F}_S) = 0$ since every $s \in S$ is zero over $X \setminus X_S$. Using the gluing short exact sequence

$$0 \to (j_S)_! j_S^*(\mathcal{F}_S) \to \mathcal{F}_S \to (i_S)_* i_S^*(\mathcal{F}_S) \to 0,$$

we deduce $\mathcal{F}_S \cong (i_S)_* i_S^*(\mathcal{F}_S)$. If we now write $\mathcal{F}$ as a filtered colimit $\mathcal{F} \cong \operatorname{colim}_S b\mathcal{F}_S$, we obtain

$$H^i_{\mathrm{Nis}}(X, \mathcal{F}) \cong \operatorname*{colim}_S H^i_{\mathrm{Nis}}(X, \mathcal{F}_S) \cong \operatorname*{colim}_S H^i_{\mathrm{Nis}}(X_S, i_S^*(\mathcal{F}_S)).$$

The last isomorphism holds because $(i_S)_*$ is an exact functor on Nisnevich sheaves of abelian groups. By our preliminary result, $\dim(X_S) \leq r$. Since X_S is a Noetherian algebraic space, its Nisnevich cohomological dimension is bounded by its Krull dimension [Lurie 2018, Theorem 3.7.7.1]. We therefore have $H^i_{\mathrm{Nis}}(X_S, i_S^*(\mathcal{F}_S)) = 0$ for $i > r$, whence $H^i_{\mathrm{Nis}}(X, \mathcal{F}) = 0$ for $i > r$. □

Lemma 7.13. *Let X be a Noetherian algebraic space of finite Krull dimension, let $\mathcal{F}$ be a presheaf of spectra on $\mathbf{\acute{E}t}_X$ satisfying Nisnevich descent, and let n be an integer. Suppose that, for every point $y \in Y \in \mathbf{\acute{E}t}_X$, $\mathcal{F}(\mathcal{O}^h_{Y,y})$ is $(n + \dim \overline{\{y\}})$-connective. Then the spectrum $\mathcal{F}(X)$ is n-connective.*

Proof. We can assume without loss of generality that $n = 0$. Let $\pi_*\mathcal{F}$ denote the Nisnevich sheaves of homotopy groups of $\mathcal{F}$. Since X is a Noetherian algebraic space of finite Krull dimension, its Nisnevich topos has finite homotopy dimension [Lurie 2018, Theorem 3.7.7.1], so that the descent spectral sequence

$$H^p_{\mathrm{Nis}}(X, \pi_q\mathcal{F}) \Rightarrow \pi_{q-p}\mathcal{F}(X)$$

is strongly convergent. Applying Lemma 7.12 to $\pi_q\mathcal{F}$, we deduce that

$$H^p_{\mathrm{Nis}}(X, \pi_q\mathcal{F}) = 0$$

for all $p > q$, and we conclude using the above spectral sequence. □

Theorem 7.14. *Let $\mathcal{X}$ be a stack in $\mathbf{Stk}'$ with finite inertia, e.g., a separated quasi-DM stack with linearly reductive stabilizers. Assume that $\mathcal{X}$ is Noetherian of dimension d. Then $KH_i(\mathcal{X}) = 0$ for $i < -d$.*

Proof. Let X be the coarse moduli space of $\mathcal{X}$. Note that X is a Noetherian algebraic space of dimension d. Let $\mathcal{F}$ be the presheaf of spectra on $\mathbf{\acute{E}t}_X$ defined by

$$\mathcal{F}(Y) = KH(\mathcal{X} \times_X Y).$$

By Corollary 4.10, $\mathcal{F}$ satisfies Nisnevich descent on $\acute{\mathbf{E}}\mathbf{t}_X$. For $y \in Y \in \acute{\mathbf{E}}\mathbf{t}_X$, let $\mathcal{X}_y^h = \mathcal{X} \times_X \mathrm{Spec}(\mathcal{O}_{Y,y}^h)$. By continuity of KH (Theorem 4.9(5)), we have

$$\mathcal{F}(\mathcal{O}_{Y,y}^h) \simeq KH(\mathcal{X}_y^h).$$

By Theorem 2.10, the stack $\mathcal{X}_y^h$ has the form $[U/G]$, where U is affine and G is a finite group scheme over $\mathrm{Spec}(\mathcal{O}_{Y,y}^h)$. In particular, $\mathcal{X}_y^h$ belongs to $\mathbf{Stk}'$ and satisfies the resolution property. Moreover, the dimension of $\mathcal{X}_y^h$ is at most $d - \dim \overline{\{y\}}$, and it equals its blow-up dimension by Lemma 7.8. It follows from Theorem 7.10 that $\mathcal{F}(\mathcal{O}_{Y,y}^h)$ is $(-d + \dim \overline{\{y\}})$-connective. By Lemma 7.13, we deduce that $\mathcal{F}(X)$ is $(-d)$-connective, i.e., that $KH_i(\mathcal{X}) = 0$ for $i < -d$. □

7C. *Vanishing of negative K-theory with coefficients.* Let $\mathcal{X}$ be a perfect stack and let $n \in \mathbb{Z}$. Recall from [Krishna and Ravi 2018, §5C] that the algebraic K-theory of $\mathcal{X}$ with coefficients is defined by

$$\begin{aligned} K^B(\mathcal{X})[1/n] &:= \mathrm{hocolim}(K^B(\mathcal{X}) \xrightarrow{n} K^B(\mathcal{X}) \xrightarrow{n} \cdots), \\ K^B(\mathcal{X}, \mathbb{Z}/n) &:= K^B(\mathcal{X}) \wedge \mathbb{S}/n, \end{aligned}$$

where $\mathbb{S}/n$ is the mod-n Moore spectrum, and similarly for KH.

Proposition 7.15. *Let $\mathcal{X}$ be a perfect stack.*

(1) *If n is nilpotent on $\mathcal{X}$, then the canonical map $K^B(\mathcal{X})[1/n] \to KH(\mathcal{X})[1/n]$ is a homotopy equivalence.*

(2) *If n is invertible on $\mathcal{X}$, then the canonical map $K^B(\mathcal{X}, \mathbb{Z}/n) \to KH(\mathcal{X}, \mathbb{Z}/n)$ is a homotopy equivalence.*

Proof. We have shown in the proof of Proposition 3.8(1) that there is a weak equivalence of dg-categories

$$\mathsf{D}_{\mathrm{perf}}(\mathcal{X} \times \mathbb{A}^1) \simeq \mathsf{D}_{\mathrm{perf}}(\mathcal{X}) \otimes \mathsf{D}_{\mathrm{perf}}(\mathbb{A}^1).$$

Given this, the proposition follows immediately from [Tabuada 2017, Theorem 1.2]. □

Theorem 7.16. *Let $\mathcal{X}$ be a stack in $\mathbf{Stk}'$ satisfying the resolution property or having finite inertia. Assume that $\mathcal{X}$ is Noetherian of blow-up dimension d. Then the following hold.*

(1) *If n is nilpotent on $\mathcal{X}$, then $K_i(\mathcal{X})[1/n] = 0$ for any $i < -d$.*

(2) *If n is invertible on $\mathcal{X}$, then $K_i(\mathcal{X}, \mathbb{Z}/n) = 0$ for any $i < -d$.*

Proof. This follows from Theorems 7.10 and 7.14 and Proposition 7.15. □

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Higher genera for proper actions of Lie groups

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Let G be a Lie group with finitely many connected components and let K be a maximal compact subgroup. We assume that G satisfies the rapid decay (RD) property and that G/K has a nonpositive sectional curvature. As an example, we can take G to be a connected semisimple Lie group. Let M be a G-proper manifold with compact quotient M/G. Building on work by Connes and Moscovici (1990) and Pflaum et al. (2015), we establish index formulae for the C^*-higher indices of a G-equivariant Dirac-type operator on M. We use these formulae to investigate geometric properties of suitably defined higher genera on M. In particular, we establish the G-homotopy invariance of the higher signatures of a G-proper manifold and the vanishing of the $\widehat{A}$-genera of a G-spin G-proper manifold admitting a G-invariant metric of positive scalar curvature.

1. Introduction

The aim of this paper is to introduce certain geometric invariants associated to proper actions of Lie groups, generalizing the (higher) signatures and $\widehat{A}$-genera. Let G be a Lie group satisfying the following assumptions:

- G has finitely many components.
- Because $|\pi_0(G)| < \infty$, G has a maximal compact subgroup K, unique up to conjugation, and we assume that the homogeneous space G/K has nonpositive sectional curvature with respect to the G-invariant metric induced by an Ad_K-invariant inner product $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g}$.
- G satisfies the rapid decay (RD) property.

We explain these last two hypothesis in the course of the paper; it suffices for now to remark that natural examples of groups satisfying our assumptions are given by connected semisimple Lie groups. The homogeneous space G/K is a smooth model for $\underline{E}G$, the classifying space for proper actions of G [Baum et al. 1994]:

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for any smooth proper action of G on a manifold M, there exists a smooth G-equivariant classifying map $\psi_M : M \to G/K$, unique up to G-equivariant homotopy. Assuming in addition that the action is *cocompact*, i.e., that the quotient M/G is compact, we can fix a *cut-off* function χ_M for M. This is a smooth function $\chi_M \in C_c^\infty(M)$ satisfying

$$\int_G \chi_M(g^{-1}x)\, dg = 1 \quad \text{for all } x \in M.$$

For any proper action of G on M, we consider $\Omega^\bullet_{\mathrm{inv}}(M)$, the complex of G-invariant differential forms on M, and its cohomology denoted by $H^\bullet_{\mathrm{inv}}(M)$. In the universal case this cohomology can be identified with the K-relative Lie algebra cohomology of the Lie algebra $\mathfrak{g}$ of G: $H^\bullet_{\mathrm{inv}}(G/K) \cong H^\bullet_{\mathrm{CE}}(\mathfrak{g}; K)$, where CE stands for Chevalley–Eilenberg. For any $\alpha \in \Omega^\bullet_{\mathrm{inv}}(G/K)$, consider its pull-back $\psi_M^*\alpha \in \Omega^\bullet_{\mathrm{inv}}(M)$. The higher signature associated to α is the real number

$$\sigma(M,\alpha) := \int_M \chi_M L(M) \wedge \psi_M^*(\alpha), \tag{1.1}$$

where $L(M)$ is the invariant de Rham form representing the L-class of M. The insertion of the cut-off function χ_M, which has compact support, ensures that the integral is well-defined, and it can be shown that it only depends on the class $[L(M) \wedge \psi_M^*(\alpha)] \in H^\bullet_{\mathrm{inv}}(M)$. The numbers in the collection

$$\{\sigma(M,\alpha) : [\alpha] \in H^\bullet_{\mathrm{inv}}(G/K)\} \tag{1.2}$$

are called the higher signatures of M. Similarly, the higher $\widehat{A}$-genus associated to M and to $[\alpha] \in H^\bullet_{\mathrm{inv}}(G/K)$ is the real number

$$\widehat{A}(M,\alpha) := \int_M \chi_M \widehat{A}(M) \wedge \psi_M^*(\alpha), \tag{1.3}$$

where $\widehat{A}(M)$ is the de Rham class associated to the $\widehat{A}$-differential form for a G-invariant metric. The numbers in the collection

$$\{\widehat{A}(M,\alpha) : \alpha \in H^\bullet_{\mathrm{inv}}(G/K)\} \tag{1.4}$$

are called the higher $\widehat{A}$-genera of M.

In this paper we establish the following result:

Theorem 1.5. *Let G be a Lie group with finitely many connected components satisfying property RD, and such that G/K is of nonpositive sectional curvature for a maximal compact subgroup K. Let M be an orientable manifold with a proper, cocompact action of G. Then the following hold true*:

(i) *each higher signature $\sigma(M,\alpha)$, $\alpha \in H^\bullet_{\mathrm{inv}}(G/K)$, is a G-homotopy invariant of M*;

(ii) *if M admits a G-invariant spin structure and a G-invariant metric of positive scalar curvature, then each higher $\widehat{A}$-genus $\widehat{A}(M,\alpha)$, $\alpha \in H^{\bullet}_{\mathrm{inv}}(G/K)$, vanishes.*

We prove this result by adapting to the G-proper context the seminal paper of Connes and Moscovici on the cyclic cohomological approach to the Novikov conjecture for discrete Gromov hyperbolic groups. Crucial to this program is the proof of a higher index formula for higher indices associated to elements in $H^{\bullet}_{\mathrm{diff}}(G)$ and to the index class $\mathrm{Ind}_{C^*_r(G)}(D) \in K_*(C^*_r(G))$ of a G-equivariant Dirac operator on an even-dimensional M acting on the sections of a complex vector bundle E. Here are the main steps for establishing this result (for this introduction we expunge from the notation the vector bundle E):

(1) First, we remark that for any almost connected Lie group G there is a van Est isomorphism $H^{\bullet}_{\mathrm{diff}}(G) \simeq H^{\bullet}_{\mathrm{inv}}(G/K) \equiv H^{\bullet}_{\mathrm{inv}}(\underline{E}G)$.

(2) Under the assumption of nonpositive sectional curvature for G/K we prove that each $\alpha \in H^{\bullet}_{\mathrm{diff}}(G)$ has a representative cocycle of polynomial growth.

(3) If G is unimodular then for each $\alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G)$ we define a cyclic cocycle τ^G_α for the convolution algebra $C^\infty_c(G)$, and thus a homomorphism
$$\langle \tau^G_\alpha, \cdot\rangle : K_0(C^\infty_c(G)) \to \mathbb{C}.$$

(4) For each $\alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G)$ we also consider a cyclic cocycle τ^M_α for the algebra of G-equivariant smooth kernels of G-compact support $\mathcal{A}^c_G(M)$; this defines a homomorphism $\langle \tau^M_\alpha, \cdot\rangle : K_0(\mathcal{A}^c_G(M)) \to \mathbb{C}$.

(5) We show that if in addition G satisfies the RD property, for example, if G is semisimple connected, then τ^G_α extends to $K_0(C^*_r(G))$ and τ^M_α extends to $K_0(C^*(M)^G)$, with $C^*(M)^G$ denoting the Roe algebra of M.

(6) If D is a G-equivariant Dirac operator we consider its index class $\mathrm{Ind}_{C^*_r(G)}(D)$ in $K_0(C^*_r(G))$ and its Morita equivalent index class $\mathrm{Ind}_{C^*(M)^G}(D)$ in $K_0(C^*(M)^G)$ and show that
$$\langle \tau^G_\alpha, \mathrm{Ind}_{C^*_r(G)}(D)\rangle = \langle \tau^M_\alpha, \mathrm{Ind}_{C^*(M)^G}(D)\rangle.$$

(7) We apply the index theorem of Pflaum, Posthuma and Tang [Pflaum et al. 2015b] in order to compute $\langle \tau^M_\alpha, \mathrm{Ind}_{C^*(M)^G}(D)\rangle$, thus establishing our higher C^*-index formula in the even-dimensional case.

We remark that item (2) above is of independent interest, and should be compared with the literature on bounded cohomology of Lie groups; see [Hartnick and Ott 2012; Kim and Kim 2015]

The geometric applications in Theorem 1.5 are then a direct consequence of the G-homotopy invariance of the signature index class established by Fukumoto

[2017] and, for the higher $\widehat{A}$-genera, of the vanishing of the index class $\mathrm{Ind}_{C_r^*(G)}(\eth) \in K_*(C_r^*(G))$ of the spin Dirac operator $\eth$ of a G-spin G-proper manifold endowed with a G-metric of positive scalar curvature, established by Guo, Mathai and Wang [Guo et al. 2017]. In the odd-dimensional case we argue by suspension. Notice that for (certain) 2-degree classes α, the G-proper homotopy invariance of the higher signatures $\sigma(M, \alpha)$ had already been established by Fukumoto.

2. Preliminaries: proper actions and cohomology

2A. *Proper actions.* In this section we introduce the geometric setting for this paper, and list some basic tools that we will need at several points later on. Let G be a Lie group with finitely many connected components. Recall that a smooth left action of G on a manifold M is called *proper* if the associated map

$$G \times M \to M \times M, \quad (g, x) \mapsto (x, gx), \quad g \in G,\ x \in M,$$

is a proper map. This implies that the stabilizer groups G_x of all points $x \in M$ are compact and that the quotient space M/G is Hausdorff. The action is said to be *cocompact* if the quotient is compact.

The class of manifolds equipped with a proper action of G can be assembled into a category where the morphisms are given by G-equivariant smooth maps. It is a basic fact that this category has a final object $\underline{E}G$, meaning that any proper G-action on M is classified by a G-equivariant map $\psi : M \to \underline{E}G$, unique up to G-equivariant homotopy. This $\underline{E}G$ is called the *classifying space for proper G-actions*, and in fact we can take $\underline{E}G := G/K$, where K is a maximal compact subgroup. Then, by writing $S := \psi^{-1}(eK)$ we see that the S is in fact a global slice: it is a K-stable submanifold for which there is a diffeomorphism

$$G \times_K S \cong M, \quad [g, x] \mapsto gx, \quad g \in G,\ x \in S.$$

The existence of such a global slice for proper Lie group actions with finitely many connected components was first proved in [Abels 1974]. When the action is cocompact, S is compact as well. Closely related to the global slice is the existence of a *cut-off* function. This is a smooth function $\chi \in C^\infty(M)$ satisfying

$$\int_G \chi(g^{-1}x)\, dg = 1 \quad \text{for all } x \in M.$$

Here we have chosen, for the rest of the paper, a Haar measure which we normalized so that the volume of the maximal compact subgroup $K \subset G$ is equal to 1. When the action of G is cocompact, we can even choose χ to have compact support. The cut-off function is constructed from the global slice $S \subset M$ as follows: Choose a smooth function $h \in C^\infty(M)$ which is equal to 1 on S and 0 outside an open neighborhood of S in M. Then the function

$$\chi(x) = \left(\int_G h(g^{-1}x)\, dg \right)^{-1} h(x)$$

is a cut-off function for the action of G.

Choosing a G-invariant Riemannian metric g on M we can refine this construction as follows: Choose the initial function h to have support inside the tube of distance 1 in M around S. Then, rescaling by $\epsilon > 0$ along the radial coordinate near S, we obtain a family of functions h_ϵ satisfying

$$h_\epsilon(x) = \begin{cases} 1 & \text{for } x \in S, \\ 0 & \text{for } d(x, S) > \epsilon. \end{cases}$$

Using this as input for the construction of the cut-off function above gives a family of cut-off functions χ_ϵ approaching χ_S:

Lemma 2.1. *The family of cut-off functions* χ_ϵ, $\epsilon > 0$, *satisfies*

$$\lim_{\epsilon \downarrow 0} \chi_\epsilon = \chi_S,$$

distributionally.

Proof. We begin by remarking that pointwise

$$\lim_{\epsilon \downarrow 0} \chi_\epsilon(x) = \begin{cases} 1 & \text{for } x \in S, \\ 0 & \text{for } x \notin S. \end{cases}$$

This is because for fixed $x \in S$ the family $h_\epsilon(g^{-1}x)$ of functions on G converges pointwise to the characteristic function of $K \subset G$, and therefore by dominated convergence we have

$$\lim_{\epsilon \downarrow 0} \int_G h_\epsilon(g^{-1}x)\, dg = \int_G \lim_{\epsilon \downarrow 0} h_\epsilon(g^{-1}x)\, dg = \int_K dg = 1,$$

by our normalization of the Haar measure on G. With this pointwise limit of $\chi_\epsilon(x)$ we have, once again by dominated convergence, that

$$\lim_{\epsilon \downarrow 0} \int_M \chi_\epsilon(x) f(x)\, dx = \int_M \lim_{\epsilon \downarrow 0} \chi_\epsilon(x) f(x)\, dx = \int_S f(x)\, dx$$

for any test function $f \in C_c^\infty(M)$. □

2B. ***Invariant cohomology and the van Est map.*** The main point of this subsection is to define the van Est map associated to a proper action of a Lie group G on M, and to reinterpret this map as the pull-back in cohomology along the classifying map $\psi_M : M \to G/K$.

Let M be a smooth manifold equipped with a smooth proper action of G. We define

$$\Omega^\bullet_{\mathrm{inv}}(M) := \{\omega \in \Omega^\bullet(M) : g^*\omega = \omega, \text{ for all } g \in G\},$$

the vector space of invariant differential forms. The de Rham differential restricts to this space of invariant forms and its cohomology, called the *invariant cohomology*, is denoted by $H^\bullet_{\mathrm{inv}}(M)$. Taking the invariant cohomology defines a contravariant functor on the category of proper G-manifolds with an equivariant map $f : M \to N$ acting on cohomology by pull-back of differential forms as usual. It is not difficult to see that the induced map $f^* : H^\bullet_{\mathrm{inv}}(N) \to H^\bullet_{\mathrm{inv}}(M)$ depends only on the G-homotopy class it is in. Given the choice of a cut-off function χ, it is shown in [Pflaum et al. 2015b] that for a closed form $\alpha \in \Omega^{\dim(M)}_{\mathrm{inv,cl}}(M)$, the integral

$$\int_M \chi\alpha$$

only depends on the cohomology class $[\alpha] \in H^{\dim(M)}_{\mathrm{inv}}(M)$.

For any manifold M equipped with a proper action of G, the *van Est map* is a map $H^\bullet_{\mathrm{diff}}(G) \to H^\bullet_{\mathrm{inv}}(M)$, where $H^\bullet_{\mathrm{diff}}(G)$ is the so-called *smooth group cohomology* of G. Let us first recall the definition of this smooth group cohomology. For G a Lie group, the space of smooth homogeneous group k-cochains is given by

$$\begin{aligned} C^k_{\mathrm{diff}}(G) := \{c : G^{\times(k+1)} \to \mathbb{C} \text{ smooth},\\ c(gg_0, \dots, gg_k) = c(g_0, \dots, g_k), \text{ for all } g, g_0, \dots, g_k \in G\}. \end{aligned}$$

The differential $\delta : C^k_{\mathrm{diff}}(G) \to C^{k+1}_{\mathrm{diff}}(G)$ is defined as

$$(\delta c)(g_0, \dots, g_{k+1}) := \sum_{i=0}^{k+1} (-1)^i c(g_1, \dots, \hat{g}_i, \dots, g_{k+1}), \tag{2.2}$$

where the ˆ means omission from the argument of the function. The cohomology of the resulting complex is called the smooth group cohomology, written as $H^\bullet_{\mathrm{diff}}(G)$.

With this, the van Est map is constructed as follows: given a smooth group cochain $c \in C^k_{\mathrm{diff}}(G)$, define the differential form

$$\omega^\chi_c := (d_1 \cdots d_k f_c)|_\Delta, \tag{2.3}$$

where d_i means taking the differential with respect to the i-th variable of the function $f_c \in C^\infty(M^{\times(k+1)})$ defined as

$$\begin{aligned} f_c(x_0, \dots, x_k)\\ := \int_{G^{\times(k+1)}} \chi(g_0^{-1}x_0) \cdots \chi(g_k^{-1}x_k) c(g_0, \dots, g_k)\, d\mu(g_0) \cdots d\mu(g_k). \end{aligned} \tag{2.4}$$

Proposition 2.5. *The map $c \mapsto \omega_c^\chi$ defines a morphism of complexes*

$$\Phi_M^\chi : (C^\bullet_{\mathrm{diff}}(G), \delta) \to (\Omega^\bullet_{\mathrm{inv}}(M), d_{dR}).$$

On the level of cohomology, it is independent of the choice of cut-off function χ.

Remark 2.6. Because of this last property, we often omit the superscript χ and write ω_c and Φ_M when the context only refers to the cohomological meaning of the differential form and the van Est map.

Proof of Proposition 2.5. We start by giving the abstract cohomological definition of the map Φ_M following [Crainic 2003] using a double complex, after which we show how to obtain the explicit chain morphism by constructing a splitting of the rows. The double complex is given as follows. We define

$$C^{p,q} := C^\infty(G^{\times(p+1)}, \Omega^q(M)).$$

The vertical differential $\delta_v : C^{p,q} \to C^{p,q+1}$ is simply given by the de Rham differential, leaving the G-variables untouched. As for the horizontal differential $\delta_h : C^{p,q} \to C^{p+1,q}$, this is given by the differential computing the smooth groupoid cohomology of the action groupoid $G \times M \rightrightarrows M$ with coefficients in $\bigwedge^q T^*M$, viewed as a representation of this groupoid. Since the G-action is proper, the groupoid $G \times M \rightrightarrows M$ is proper by definition. Therefore, the vanishing theorem for the groupoid cohomology of proper Lie groupoids in [Crainic 2003] applies, and we see that the rows in this double complex are exact. There are obvious inclusions $C^\bullet_{\mathrm{diff}}(G) \hookrightarrow C^{\bullet,0}$, and $\Omega^\bullet_{\mathrm{inv}}(M) \hookrightarrow C^{0,\bullet}$, and now we see that by finding the appropriate splittings we can "zig-zag" from one end to the other in the double complex:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
\Omega^1_{\mathrm{inv}}(M) \arrow[u, "d"] \arrow[r] & C^{0,1} \arrow[u, "\delta_v"] \arrow[r, "\delta_h"] & C^{1,1} \arrow[u, "\delta_h"] \arrow[r, "\delta_h"] \arrow[l, dashed, bend left, "s_1"] & C^{2,1} \arrow[u, "\delta_h"] \arrow[r, "\delta_h"] & \cdots \\
\Omega^0_{\mathrm{inv}}(M) \arrow[u, "d"] \arrow[r] & C^{0,0} \arrow[u, "\delta_v"] \arrow[r, "\delta_h"] & C^{1,0} \arrow[u, "\delta_h"] \arrow[r, "\delta_h"] & C^{2,0} \arrow[u, "\delta_h"] \arrow[r, "\delta_h"] \arrow[l, dashed, bend left, "s_2"] & \cdots \\
 & C^0_{\mathrm{diff}}(G) \arrow[u] \arrow[r, "\delta"] & C^1_{\mathrm{diff}}(G) \arrow[u] \arrow[r, "\delta"] & C^2_{\mathrm{diff}}(G) \arrow[u] \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

So it remains to find an appropriate splitting $s_p : C^{p,\bullet} \to C^{p+1,\bullet}$. Given a choice of cut-off function χ, the formula

$$(s_p\alpha)(g_0, \dots, g_{p-1}) := \int_G \chi(g^{-1}x_0)\alpha(g, g_0, \dots, g_{p-1})\big|_\Delta, \quad \alpha \in C^{p,q}$$

does the job: a straightforward computation shows that

$$\delta_h \circ s + s \circ \delta_h = \mathrm{id}.$$

With this choice of contraction map, one obtains exactly (2.3) for the invariant differential form associated to a group cochain. The preceding argument therefore shows that the map $c \mapsto \omega_c$ is indeed a morphism of cochain complexes. ☐

Remark 2.7 (the van Est isomorphism). The main theorem of [Crainic 2003] states that if M is cohomologically n-connected, the map Φ_M induces an isomorphism in cohomology in degree $\leq n$ and is injective in degree $n+1$. In the universal case for the action of G on G/K, which is contractible, we therefore find an isomorphism $H^\bullet_{\mathrm{diff}}(G) \cong H^\bullet_{\mathrm{inv}}(G/K)$. This is one version of the classical van Est theorem [1955a; 1955b]. In this case we have by left translation

$$\Omega^\bullet_{\mathrm{inv}}(G/K) \cong \left(\bigwedge^{\bullet}(\mathfrak{g}/\mathfrak{k})^*\right)^K, \tag{2.8}$$

under which the de Rham differential identifies with the Chevalley–Eilenberg differential computing the relative Lie algebra cohomology $H^\bullet_{\mathrm{CE}}(\mathfrak{g};\, K)$. With this, the van Est isomorphism is written as

$$H^\bullet_{\mathrm{diff}}(G) \cong H^\bullet_{\mathrm{CE}}(\mathfrak{g};\, K). \tag{2.9}$$

Proposition 2.10. *Let $f : M \to N$ be an equivariant smooth map between proper G-manifolds. Then the following diagram commutes*:

$$\begin{tikzcd} H^\bullet_{\mathrm{diff}}(G) \arrow[r, "\Phi_N"] \arrow[dr, "\Phi_M"'] & H^\bullet_{\mathrm{inv}}(N) \arrow[d, "f^*"] \\ & H^\bullet_{\mathrm{inv}}(M) \end{tikzcd}$$

Proof. Let χ_M be a cut-off function for the G-action on M. Then the pull-back $f^*\chi_M$ is a cut-off function for the G-action on N. For this cut-off function we obviously have $\omega_c^{f^*\chi_M} = f^*\omega_c^{\chi_M}$. The result now follows from the fact that the van Est map is independent of the choice of cut-off function. ☐

Corollary 2.11. *Under the van Est isomorphism $H^\bullet_{\mathrm{diff}}(G) \cong H^\bullet_{\mathrm{inv}}(G/K)$, the van Est map is identified with the pull-back along the classifying map $\psi_M : M \to G/K$, i.e.,*

$$\Phi_M = \psi_M^*.$$

2C. ***Group cocycles of polynomial growth.*** In a later stage of the paper, in the discussion of the extension properties of cyclic cocycles associated to smooth group cocycles, it will be important to control the growth of these group cocycles. To this end, we shall prove below a criterion guaranteeing that we can represent classes in $H^\bullet_{\mathrm{diff}}(G)$ by cocycles that have at most polynomial growth. For this, we begin by recalling Dupont's inverse [1976] of the van Est map $\Phi_{G/K}$ establishing the isomorphism (2.9). Choose an Ad_K-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$, which, by left translations, induces a G-invariant Riemannian metric on G/K. This metric defines an orthogonal decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ with $\mathfrak{p} \cong T_{eK}(G/K)$. Since K is maximal compact, the (Riemannian) exponential map induces an isomorphism $\mathfrak{p} \cong G/K$ (with inverse denoted by log), and we can define the contraction

$$\varphi_s(x) := \exp(s\log(x))$$

of G/K to its basepoint $eK \in G/K$, i.e., $\varphi_1 = \mathrm{id}_{G/K}$ and $\varphi_0(x) = eK$. Now, given $k+1$ points $g_0K, \dots, g_kK \in G/K$, also denoted $\bar{g}_0, \dots, \bar{g}_k$, we can consider the geodesic simplex $\Delta^k(g_0K, \dots, g_kK) \subset G/K$ defined inductively as the cone over $\Delta^{k-1}(\bar{g}_1, \dots, \bar{g}_k)$ with tip point $\bar{g}_0$. More precisely, define the singular simplex $\sigma^k(\bar{g}_0, \dots, \bar{g}_k) : \Delta^k \to G/K$, where $\Delta^k := \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} : t_i \geq 0, \sum_i t_i = 1\}$, by

$$\begin{aligned}\sigma^k(g_0K, \dots, g_kK)&(t_0, \dots, t_k)\\ &:= g_0\varphi_{t_0}\left(\sigma^{k-1}(g_0^{-1}g_1K, \dots, g_0^{-1}g_kK)\left(\frac{t_1}{1-t_0}, \dots, \frac{t_k}{1-t_0}\right)\right),\end{aligned} \tag{2.12}$$

and $\sigma^0(gK) := gK$. We write $\Delta^k(g_0K, \dots, g_kK)$ for the image of this simplex. By construction, this k-simplex is G-invariant: $g\Delta^k(\bar{g}_0, \dots, \bar{g}_k) = \Delta^k(g\bar{g}_0, \dots, g\bar{g}_k)$. With these simplices we define a map

$$J : \Omega^\bullet_{\mathrm{inv}}(G/K) \to C^\bullet_{\mathrm{diff}}(G), \quad \alpha \mapsto J(\alpha)(g_0, \dots, g_k) := \int_{\Delta^k(g_0K, \dots, g_kK)} \alpha, \tag{2.13}$$

which is easily checked to be a morphism of cochain complexes. Since $\Phi_{G/K} \circ J = \mathrm{id}$, J is a quasi-isomorphism.

Theorem 2.14. *Let G be a Lie group with finitely many connected components. Let K be a maximal compact subgroup and assume that G/K is of nonpositive sectional curvature with respect to the G-invariant metric induced by an* Ad_K-*invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Then the group cocycle associated to a closed $\alpha \in \Omega^k_{\mathrm{inv}}(G/K)$ has polynomial growth. More precisely, if we write $d(g)$ for the distance from eK to gK in G/K, there exists a constant $C > 0$ and a natural number $N \in \mathbb{N}$ such that the following estimate holds true*:

$$|J(\alpha)(g_0, \dots, g_k)| \leq C(1 + d(g_0))^N \cdots (1 + d(g_k))^N.$$

Proof. Denote by $\|\alpha\|$ the norm of the Lie algebra cocycle $\alpha \in C^k_{\mathrm{CE}}(\mathfrak{g}; K) = \Omega^k_{\mathrm{inv}}(G/K)$ defined by the K-invariant metric on the Lie algebra $\mathfrak{g}$ of G that defines the metric on G/K. Since α is a G-invariant differential form we obviously have the inequality

$$|J(\alpha)(g_0, \dots, g_k)| \leq \|\alpha\| \operatorname{Vol}(\Delta^k(\bar{g}_0, \dots, \bar{g}_k)).$$

We now prove that, under the assumptions of the theorem, the volume of the geodesic k-simplex on G/K has at most polynomial growth in the geodesic distance of its vertices, thus completing the proof. For this we adapt an argument from [Inoue and Yano 1982, Proposition 1]; we thank Andrea Sambusetti for very useful discussions on this point and for bringing this article to our attention.

Let $\tau : \Delta^{k-1}(g_1K, \dots, g_kK) \times [0, 1] \to \Delta^k(eK, g_1K, \dots, g_kK)$ be defined by

$$\tau(x, t) := \varphi_{1-t}(x).$$

With this we can write

$$\tau^* \, d\mathrm{vol}_{\Delta^k(eK,g_1K,\dots,g_kK)} = \phi(x, t)\, dt \wedge \pi^* \, d\mathrm{vol}_{\Delta^{k-1}(g_1K,\dots,g_kK)}$$

for some function $\phi(x, t)$, where

$$\pi : \Delta^{k-1}(g_1K, \dots, g_kK) \times [0, 1] \to \Delta^{k-1}(g_1K, \dots, g_kK)$$

is the projection.

Choose $x_0 \in \Delta^{k-1}(g_1K, \dots, g_kK)$ and let $\gamma_{x_0}(t) := \varphi_s(x_0)$ be the geodesic starting in $\gamma_{x_0}(0) = x_0$ and ending in the basepoint $\gamma_{x_0}(1) = eK$. Let $X_0(0), \dots, X_{n-1}(0)$ be an orthonormal frame of $T_{x_0}(G/K)$ such that $X_0(0) = \dot{\gamma}(0)/L$, with $L = d(eK, x_0)$ the length of γ_{x_0}, and such that $X_0(0), \dots, X_{k-1}(0)$ span $T_{x_0}\Delta^k(eK, g_1K, \dots, g_kK)$. We denote by $X_i(t)$ the unique extension to parallel vector fields along $\gamma_{x_0}(t)$.

We now choose local coordinates $(y^1, \dots, y^{k-1})$ on $\Delta^{k-1}(g_1K, \dots, g_kK)$ around x_0 satisfying

$$\frac{\partial}{\partial y^i}(x_0) = X_i(0) + b_i X_0(0), \tag{2.15}$$

for some constants $b_i \in \mathbb{R}$. We then get local coordinates $(y^1, \dots, y^{k-1}, t)$ around the image of γ, such that $(y_0^1, \dots, y_0^{k-1}, 0)$ corresponds to the point x_0. Comparing the vector fields $\partial/\partial y^i$ with X_j defines functions $a_{ij} : [0, 1] \to \mathbb{R}$ by

$$\frac{\partial}{\partial y^i}(\gamma(t)) = \sum_{j=0}^{n-1} a_{ij}(t) X_j(t). \tag{2.16}$$

The normal projection of $\partial/\partial y^i$ along $\gamma_{x_0}(t)$ is then the vector field

$$Y_i(t) := \sum_{j=1}^{n-1} a_{ij}(t) X_j(t), \quad i = 1, \dots, k-1,$$

satisfying $Y_i(0) = X_i(0)$ and $Y_i(1) = 0$. Now note that the vector field $\partial/\partial y^i$ is a Jacobi field along the geodesic $\gamma_{x_0}(t)$, because by its definition we have

$$\frac{\partial}{\partial y^i}(\gamma(t)) = \frac{d}{ds}\gamma_{(y_0^1,\dots,sy_0^i,\dots,y_0^{k-1})}(t)\Big|_{s=0},$$

where $\gamma_{(y^1,\dots,y^{k-1})}(t)$ is the geodesic $\varphi_t(0, y^1, \dots, y^k)$ connecting

$$(y^1, \dots, y^{k-1}) \in \Delta^{k-1}(g_1 K, \dots, g_k K)$$

with eK, and $x_0 = (y_0^1, \dots, y_0^{k-1})$ in local coordinates. It follows that $Y_i(t)$ is also a Jacobi field along $\gamma_{x_0}(t)$ because it is the normal projection of $\partial/\partial y^i$. (The normal and tangential projections of a Jacobi vector field are Jacobi.)

We define the $(k-1) \times (k-1)$ matrix $A(t)$ with entries

$$\langle Y_i(t), Y_j(t)\rangle = \sum_{k=1}^{n-1} a_{ik}(t) a_{kj}(t).$$

Now, computing the inner products of the vector fields $\partial/\partial y^i$ we get from (2.15) and an elementary computation that

$$d\mathrm{vol}_{\Delta^{k-1}(g_1 K,\dots,g_k K)} = \sqrt{\Bigl(1 + \sum_i b_i^2\Bigr)}\, dy^1 \wedge \cdots dy^k,$$

whereas from (2.16) we get

$$d\mathrm{vol}_{\Delta^{k}(eK, g_1 K,\dots,g_k K)} = L\sqrt{\det(A(t))}\, dt \wedge dy^1 \wedge \cdots \wedge dy^k.$$

It follows that

$$\phi(x_0, t) = \frac{L\sqrt{\det(A(t))}}{\sqrt{\bigl(1 + \sum_i b_i^2\bigr)}} \le L\sqrt{\det(A(t))}\,.$$

Consider now the Jacobi field $U(t) = \sum_{i=1}^{k-1} u^i Y_i(t)$, for a vector $u = (u^i)_{i=1}^{k-1} \in \mathbb{R}^{k-1}$. By the Jacobi equation we now have

$$\frac{d^2}{dt^2}\|U(t)\| = 2\|\nabla_{\partial/\partial t} U(t)\|^2 - 2\Bigl\langle R\Bigl(U(t), \frac{\partial}{\partial t}\Bigr)\frac{\partial}{\partial t}, U(t)\Bigr\rangle \ge 0.$$

Together with the fact that $\|U(0)\|^2 = \|u\|^2$ and $\|U(1)\|^2 = 0$, it follows that $\|U(t)\|^2 \le \|u\|^2(1-t)$.

We obviously have

$$\det(A(t)) \le \left(\sup_{u\neq 0} \frac{u^t A(t) u}{\|u\|^2}\right)^{k-1},$$

and $u^t A(t) u = \|U(t)\|^2$, so that we can conclude that

$$\det(A(t)) \le (1-t)^{k-1}.$$

This is the crucial estimate that we use below. Before we complete the proof of the theorem, we prove the following lemma:

Lemma 2.17. *For $x \in \Delta^{k-1}(g_1 K, \dots, g_k K) \subset \Delta^k(g_0 K, \dots, g_k K)$, we have*

$$d(g_0 K, x) \le \max\{d(g_0 K, g_1 K), \dots, d(g_0 K, g_k K)\}.$$

Proof. We prove this by induction. For $k=1$, the statement is obvious. Suppose now that we have proved the lemma for $k-1$. Consider

$$x \in \Delta^{k-1}(g_1K, \dots, g_kK) \subset \Delta^k(g_0K, \dots, g_kK).$$

Let $\gamma(t)$ be the geodesic connecting g_1K and x, but continued until it hits the simplex $\Delta^{k-2}(g_2K, \dots, g_kK)$ in a point that we call y. Using convexity of the distance function on a manifold with nonpositive sectional curvature, we see that $d(g_0K, x) \le \max\{(d(g_0K, g_1K), d(g_0K, y)\}$. To estimate the distance $d(g_0K, y)$, we now consider the geodesic simplex $\Delta^{k-1}(g_0K, g_2K, \dots, g_kK)$ and apply the induction hypothesis. □

The final step in the proof of the theorem is an induction argument: First observe that for $k=1$, the statement of the theorem is obviously true because $\Delta^1(g_0K, g_1K)$ is simply the geodesic connecting g_0K and g_1K. Suppose now that we have proved the statement for $k-1$. Then we compute the volume of the simplex $\Delta^k(eK, g_1K, \dots, g_kK)$ as follows:

$$\begin{aligned}
\mathrm{Vol}(\Delta^k(eK, g_1K, \dots, g_kK))&\\
&= \int_{\Delta^k(eK,g_1K,\dots,g_kK)} d\mathrm{vol}_{\Delta^k(eK,g_1K,\dots,g_kK)}\\
&= \int_{\Delta^{k-1}(g_1K,\dots,g_kK)} \left(\int_0^1 \phi(x,t)\,dt\right) d\mathrm{vol}_{\Delta^{k-1}(g_1K,\dots,g_kK)}(x)\\
&\le \int_{\Delta^{k-1}(g_1K,\dots,g_kK)} L(x)\left(\int_0^1 (1-t)^{(k-1)/2}\,dt\right) d\mathrm{vol}_{\Delta^{k-1}(g_1K,\dots,g_kK)}(x)\\
&\le C\int_{\Delta^{k-1}(g_1K,\dots,g_kK)} L(x)\,d\mathrm{vol}_{\Delta^{k-1}(g_1K,\dots,g_kK)}(x)\\
&\le C\prod_{i=1}^{k}(1+d(g_i))\,\mathrm{Vol}(\Delta^{k-1}(g_1K, \dots, g_kK)).
\end{aligned}$$

By the induction assumption,

$$\mathrm{Vol}(\Delta^{k-1}(g_1K, \dots, g_kK)) \le C' \prod_{i=1}^{k-1}(1+d(g_1K, g_iK)).$$

Together with the inequality $d(g_1K, g_iK) \le d(g_0K, g_1K) + d(g_0K, g_iK)$, this completes the proof of the theorem. □

Example 2.18. As an example, consider the abelian group $G=\mathbb{R}^2$ with maximal compact group given by the trivial group $\{0\} \subset \mathbb{R}^2$. In this abelian case we have that $H^\bullet_{\mathrm{inv}}(\mathbb{R}^2) = \bigwedge^\bullet \mathbb{R}^2$, and a generator in degree 2 is given by the area form $dx \wedge dy$, so that we find

$$J(dx \wedge dy)(x, y, z) = \mathrm{Area}_{\mathbb{R}^2}(\Delta^2(x, y, z)), \tag{2.19}$$

which evidently grows polynomially in the norm of x, y and z.

Remark 2.20. (i) When G is a connected semisimple Lie group, G/K is a noncompact symmetric space and has nonpositive sectional curvature [Helgason 2001]. Therefore the curvature assumptions in the lemma are automatically satisfied in this case. In fact, the conjecture in [Dupont 1979] is that for semisimple Lie groups all these cocycles are bounded. For recent work on this conjecture, see [Hartnick and Ott 2012; Kim and Kim 2015]. In this last reference, different simplices are used, given by the barycentric subdivision of the geodesic ones, to prove boundedness of the top-dimensional cocycle for general connected semisimple Lie groups.

(ii) In general, the polynomial bounds of the lemma above are not sharp, as expected from the conjecture mentioned in (i). For example, when $G = \mathrm{SL}(2, \mathbb{R})$, the maximal compact subgroup is given by $K = \mathrm{SO}(2)$ so that $G/K = \mathbb{H}^2$, the hyperbolic 2-plane. Again, we have $H^2_{\mathrm{inv}}(\mathbb{H}^2) = \mathbb{R}$, with generator the hyperbolic area form. This leads to a smooth group cocycle given by the same formula as (2.19) above, replacing the Euclidean area by the hyperbolic one, but this time the cocycle is bounded because the area of a hyperbolic triangle does not exceed π, confirming the boundedness in top-degree mentioned in (i).

3. Algebras of invariant kernels

3A. ***Smoothing kernels of G-compact support.*** Let M again be a closed smooth manifold carrying a smooth proper action of a Lie group G with $|\pi_0(G)| < \infty$ and with compact quotient. We choose an invariant complete Riemannian metric, denoted h, with associated distance function denoted by $d_M(x, y)$ for $x, y \in M$, and volume form $d\mathrm{vol}(x)$. We fix a left-invariant metric on G and we denote by d_G the associated distance function.

Definition 3.1. Consider a G-equivariant smoothing kernel $k \in C^\infty(M \times M)$; thus k is an element in $C^\infty(M \times M)^G$. We say that k is of G-compact support if the projection of $\mathrm{supp}(k) \subset M \times M$ in $(M \times M)/G$, with G acting diagonally, is compact.

We denote by $\mathcal{A}^c_G(M)$ the set of G-equivariant smoothing kernels of G-compact support. It is well known that $\mathcal{A}^c_G(M)$ has the structure of a Fréchet algebra with respect to the convolution product

$$(k * k')(x, z) = \int_M k(x, y)k'(y, z)\, d\mathrm{vol}(y).$$

It is also well known that each element $k \in \mathcal{A}^c_G(M)$ defines an equivariant linear operator $S_k : C^\infty_c(M) \to C^\infty_c(M)$, the integral operator associated to the kernel k, and that $S_k \circ S_{k'} = S_{k*k'}$. Moreover, S_k extends to an equivariant bounded operator on $L^2(M)$. We have therefore defined a subalgebra of $\mathcal{B}(L^2(M))$, which we denote as $\mathcal{S}^c_G(M)$; by definition,

$$\mathcal{S}^c_G(M) := \{S_k : k \in \mathcal{A}^c_G(M)\}. \tag{3.2}$$

The case in which there is an equivariant vector bundle E on M is similar, in that we start with G-equivariant elements in $C^\infty(M \times M, E \boxtimes E^*)$ and then proceed analogously, defining in this way the Fréchet algebra $\mathcal{A}^c_G(M, E)$ and $\mathcal{S}^c_G(M, E) := \{S_k : k \in \mathcal{A}^c_G(M, E)\}$, a subalgebra of $\mathcal{B}(L^2(M, E))$.

Notation. Keeping with a well-established abuse of notation, we often identify $\mathcal{A}^c_G(M, E)$ with $\mathcal{S}^c_G(M, E)$, thus identifying a smoothing kernel k in $\mathcal{A}^c_G(M, E)$ with the corresponding operator $S_k \in \mathcal{S}^c_G(M, E)$.

3B. ***Holomorphically closed subalgebras.*** Using the remarks at the end of the previous subsection we see that $\mathcal{S}^c_G(M, E)$ is in an obvious way a subalgebra of the reduced Roe C^*-algebra $C^*(M, E)^G$. Recall that $C^*(M, E)^G$ is defined as the norm closure in $\mathcal{B}(L^2(M, E))$ of the algebra $C^*_c(M, E)^G$ of G-equivariant bounded operators of finite propagation and locally compact. In fact, $\mathcal{S}^c_G(M, E) \subset C^*_c(M, E)^G$. The Roe algebra is canonically isomorphic to $\mathbb{K}(\mathcal{E})$, the C^*-algebra of compact operators of the Hilbert $C^*_r(G)$-Hilbert module $\mathcal{E}$ obtained by closing the space $C^\infty_c(M, E)$ of compactly supported sections of E on M, endowed with the $C^*_r G$-valued inner product

$$(e, e')_{C^*_r G}(x) := (e, x \cdot e')_{L^2(M,E)}, \quad e, e' \in C^\infty_c(M, E),\ x \in G. \tag{3.3}$$

See for example [Hochs and Wang 2018], where the Morita isomorphism

$$K_*(\mathbb{K}(\mathcal{E})) = K_*(C^*(M, E)^G) \xrightarrow{\mathcal{M}} K_*(C^*_r G)$$

is explicitly discussed. We shall come back to this important point in a moment. The subalgebra $\mathcal{S}^c_G(M, E)$ is not holomorphically closed in $C^*(M, E)^G$. On the other hand, such a subalgebra of $C^*(M, E)^G$ is implicitly constructed in [Hochs and Wang 2018, Section 3.1] by making use of the slice theorem. We recall the essential ingredients, following [Hochs and Wang 2018, Section 3.1] (we also extend the context slightly for future use).

As already remarked in the previous section, under our assumptions on G, there exists a global slice for the action of G on M. Thus if K is a maximal compact subgroup of G there exists a K-invariant compact submanifold $S \subset M$ such that the action map $[g, s] \mapsto gs$, $g \in G$, $s \in S$, defines a G-equivariant diffeomorphism

$$G \times_K S \xrightarrow{\alpha} M,$$

where S is compact because the action is cocompact. Corresponding to this diffeomorphism we have an isomorphism $E \cong G \times_K (E|_S)$, and thus isomorphisms

$$C_c^\infty(M, E) \cong (C_c^\infty(G) \hat{\otimes} C^\infty(S, E|_S))^K,$$
$$C^\infty(M, E) \cong (C^\infty(G) \hat{\otimes} C^\infty(S, E|_S))^K.$$

See [Hochs and Wang 2018, Section 3.1]. Here we are taking the projective tensor product $\hat{\otimes}_\pi$ of the two Fréchet algebras; however, since $C^\infty(S, E|_S)$ is nuclear, the injective $\hat{\otimes}_\epsilon$ and projective $\hat{\otimes}_\pi$ tensor products coincide, which is why we do not use a subscript. Consider now $\Psi^{-\infty}(S, E|_S)$, also a nuclear Fréchet algebra, and let

$$\widetilde{A}^c_G(M, E) := (C_c^\infty(G) \hat{\otimes} \Psi^{-\infty}(S, E|_S))^{K \times K}.$$

$\widetilde{A}^c_G(M, E)$ is a Fréchet algebra, with product denoted by $*$. Let $\tilde{k} \in \widetilde{A}^c_G(M, E)$ and consider the operator $T_{\tilde{k}}$ on $L^2(M, E)$ given by

$$(T_{\tilde{k}} e)(gs) = \int_G \int_S g\tilde{k}(g^{-1}g', s, s')g'^{-1} e(g's')\, ds'\, dg'. \tag{3.4}$$

This is a bounded G-equivariant operator with smooth G-equivariant Schwartz kernel given by

$$\kappa(gs, g's') = g\tilde{k}(g^{-1}g', s, s')g'^{-1},$$

where the g and g'^{-1} on the right-hand side are used in order to identify fibers on the vector bundle E. The assignment $\tilde{k} \to T_{\tilde{k}}$ is injective and satisfies

$$T_{\tilde{k}} \circ T_{\tilde{k}'} = T_{\tilde{k} * \tilde{k}'}.$$

Consider the subalgebra of the bounded operators on $L^2(M, E)$ given by

$$\{T_{\tilde{k}} : \tilde{k} \in \widetilde{A}^c_G(M, E)\}$$

endowed with the Fréchet algebra structure induced by the injective homomorphism $\tilde{k} \to T_{\tilde{k}}$. It is easy to see that this algebra is precisely equal to the algebra we have considered in the previous subsection, $\mathcal{S}^c_G(M, E) := \{S_k : k \in \mathcal{A}^c_G(M, E)\}$. Thus,

$$\mathcal{S}^c_G(M, E) = \{T_{\tilde{k}} : \tilde{k} \in \widetilde{A}^c_G(M, E)\}. \tag{3.5}$$

In summary, using the slice theorem we have realized $\mathcal{S}^c_G(M, E)$ as a projective tensor product of convolution operators on G and smoothing operators on S. This preliminary result puts us in the position of enlarging the algebra $\mathcal{S}^c_G(M, E)$ and obtaining a subalgebra dense and holomorphically closed in $C^*(M, E)^G$. To this end we give the following definition.

Definition 3.6. Let $\mathcal{A}(G)$ a set of functions on G. We say that $\mathcal{A}(G)$ is admissible if the following properties are satisfied:

(1) $\mathcal{A}(G)$ is a Fréchet space and there are continuous inclusions

$$C_c^\infty(G) \subset \mathcal{A}(G) \subset L^2(G);$$

(2) the action by convolution defines a continuous injective map $\mathcal{A}(G) \hookrightarrow C_r^*(G)$ which makes $\mathcal{A}(G)$ a subalgebra of $C_r^*(G)$;

(3) $\mathcal{A}(G)$ is holomorphically closed in $C_r^*(G)$.

We can then consider

$$\widetilde{A}_G(M, E) := (\mathcal{A}(G) \hat{\otimes} \Psi^{-\infty}(S, E|_S))^{K \times K},$$

a Fréchet algebra, and for $\tilde{k} \in \widetilde{A}_G(M, E)$, the bounded operator $T_{\tilde{k}}$ on $L^2(M, E)$ given by

$$(T_{\tilde{k}} e)(gs) = \int_G \int_S g\tilde{k}(g^{-1}g', s, s')g'^{-1} e(g's')\, ds'\, dg'. \tag{3.7}$$

The operator $T_{\tilde{k}}$ is an integral operator with G-equivariant Schwartz kernel κ given by $\kappa(gs, g's') = g\tilde{k}(g^{-1}g', s, s')g'^{-1}$. Since $\mathcal{A}(G) \hookrightarrow C_r^*(G)$, with $\mathcal{A}(G)$ acting by convolution, we see that $T_{\tilde{k}}$ is L^2-bounded.

Definition 3.8. We define $\mathcal{A}_G(M, E)$ as the subalgebra of the bounded operators on $L^2(M, E)$ given by

$$\mathcal{A}_G(M, E) := \{T_{\tilde{k}} : \tilde{k} \in \widetilde{A}_G(M, E)\}.$$

We endow $\mathcal{A}_G(M, E)$ with the structure of a Fréchet algebra induced by the injective homomorphism $\tilde{k} \to T_{\tilde{k}}$.

Proposition 3.9. *Under the assumptions* (1)–(3) *for* $\mathcal{A}(G)$ *in Definition 3.6, the following hold*:

(i) *We have a continuous inclusion of Fréchet algebras*

$$\mathcal{S}_G^c(M, E) \subset \mathcal{A}_G(M, E). \tag{3.10}$$

(ii) $\mathcal{A}_G(M, E)$ *is a dense subalgebra of* $C^*(M, E)^G$ *and it is holomorphically closed.*

Proof. (i) The continuous inclusion of Fréchet algebras $\mathcal{S}_G^c(M, E) \subset \mathcal{A}_G(M, E)$ follows immediately from (3.5).

(ii) The fact that $\mathcal{A}_G(M, E)$ is a dense subalgebra of $C^*(M, E)^G$ is proved precisely as in [Hochs and Wang 2018, Lemma 3.3]; the property of being holomorphically closed follows easily from the hypothesis that $\mathcal{A}(G)$ is holomorphically closed in $C_r^* G$ and the well-known fact that $\Psi^{-\infty}(S, E|_S)$ is holomorphically closed in the compact operators of $L^2(S, E|_S)$. □

Definition 3.11. Let G be a Lie group and let L be a length function on G. We consider

$$H_L^\infty(G) = \left\{ f \in L^2(G) : \int_G (1+L(x))^{2k} |f(x)|^2 \, dx < +\infty \text{ for all } k \in \mathbb{N} \right\} \tag{3.12}$$

endowed with the Fréchet topology induced by the sequence of seminorms

$$\nu_k(f) := \|(1+L)^k f\|_{L^2}. \tag{3.13}$$

We say that the pair (G, L) satisfies the rapid decay property (RD) if there is a continuous inclusion $H_L^\infty(G) \hookrightarrow C_r^*(G)$.

For conditions equivalent to the one given here, see [Chatterji et al. 2007]. We also recall that if G satisfies (RD) then G is unimodular [Ji and Schweitzer 1996].

Proposition 3.14. *Let G be a Lie group with $|\pi_0(G)| < \infty$; we can and shall choose L to be the length function associated to a left-invariant Riemannian metric. Assume additionally that G satisfies* (*RD*) (*with respect to this L*). *Then*

$$H_L^\infty(G) = \left\{ f \in L^2(G) : \int_G (1+L(x))^{2k} |f(x)|^2 \, dx < +\infty \right\} \tag{3.15}$$

satisfies the properties (1)–(3) *given in Definition 3.6. Consequently*, *for G with $|\pi_0(G)| < \infty$ and with the* (*RD*) *property*, *there exists a subalgebra of $C^*(M, E)^G$*, *denoted $\mathcal{S}_G^\infty(M, E)$*, *which consists of integral operators*, *is dense and holomorphically closed in $C^*(M, E)^G$ and contains $\mathcal{S}_G^c(M, E)$ as a subalgebra.*

Proof. The fact that $H_L^\infty(G)$ is not only contained in $C_r^*(G)$, via convolution, but is in fact a subalgebra of it, follows from [Jolissaint 1990]. Hence $H_L^\infty(G)$ satisfies the properties (1) and (2) given in Definition 3.6. The fact that this subalgebra is holomorphically closed is proved as in [Jolissaint 1989]. The rest of the proposition then follows from Proposition 3.9. □

Example 3.16. Here are two examples of Lie groups that satisfy property (RD), and to which our theory applies:

(1) The abelian group $\mathbb{R}^n$ satisfies (RD). In this case the algebra $H_L^\infty(\mathbb{R}^n)$ associated to the length function defined by the Euclidean metric is the algebra of rapidly decaying functions on $\mathbb{R}^n$.

(2) Connected semisimple Lie groups satisfy property (RD) [Chatterji et al. 2007], for example $G = \mathrm{SL}(2, \mathbb{R})$. In this case the algebra $H_L^\infty(G)$ is closely related to Harish–Chandra's Schwartz algebra $\mathcal{C}(G)$ (see below).

Remark 3.17. We have just seen that for G semisimple, by taking $\mathcal{A}(G) = H_L^\infty(G)$ we obtain a holomorphically closed subalgebra $\mathcal{S}_G^\infty(M, E) \subset C^*(M, E)^G$. Notice that there are other algebras that can be considered. For example, we can consider as in [Hochs and Wang 2018] the Harish-Chandra Schwartz algebra $\mathcal{C}(G) \subset C_r^*(G)$.

This is a holomorphically closed subalgebra of $C_r^*(G)$ [Lafforgue 2002], which is made of smooth functions acting by convolution. The corresponding algebra $\mathcal{C}_G(M,E) \subset C^*(M,E)^G$ is a subalgebra of $C^*(M,E)^G$ with elements that are in fact smoothing operators. One can prove that $\mathcal{C}(G) \subset H_L^\infty(G)$ [Varadarajan 1977, §II.9] and consequently, $\mathcal{C}_G(M,E) \subset \mathcal{S}_G^\infty(M,E)$. Notice that Hochs and Wang have proved that the heat operator $\exp(-tD^2)$ is an element in $\mathcal{C}_G(M,E)$. Hence $\exp(-tD^2) \in \mathcal{S}_G^\infty(M,E)$.

4. Index classes

From now on we make constant use of the identification $\mathcal{A}_G^c(M,E) \equiv \mathcal{S}_G^c(M,E)$.

4A. ***The index class in $K_*(C^*(M,E)^G)$.*** We consider as before a closed even-dimensional manifold M with a proper cocompact action of G. Let D be a G-equivariant odd $\mathbb{Z}_2$-graded Dirac operator. Recall, first of all, the classical Connes–Skandalis idempotent. Let Q_σ be a G-equivariant parametrix of G-compact support with remainders $S_\pm$; here the subscript σ stands for symbolic. Consider the 2×2 matrix

$$P_\sigma := \begin{pmatrix} S_+^2 & S_+(I+S_+)Q \\ S_-D^+ & I-S_-^2 \end{pmatrix}. \tag{4.1}$$

This produces a class

$$\operatorname{Ind}_c(D) := [P_\sigma] - [e_1] \in K_0(\mathcal{A}_G^c(M,E)) \quad \text{with } e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.2}$$

To understand where this definition comes from, see for example [Connes and Moscovici 1990]. Recall now that $\mathcal{A}_G^c(M,E) \subset C^*(M,E)^G$.

Definition 4.3. The C^*-index associated to D is the class

$$\operatorname{Ind}_{C^*(M,E)}(D) \in K_0(C^*(M,E)^G)$$

obtained by taking the image of the Connes–Skandalis projector in $K_0(C^*(M,E)^G)$. Unless absolutely necessary, we denote this index class simply by $\operatorname{Ind}(D)$.

Remark 4.4. If we are in the position of considering a dense holomorphically closed subalgebra $\mathcal{A}_G(M,E)$ of $C^*(M,E)^G$ as in the previous section, then we can equivalently take the image of the Connes–Skandalis projector in $K_0(\mathcal{A}_G(M,E))$ (recall that, by construction, $\mathcal{A}_G^c(M,E) \subset \mathcal{A}_G(M,E) \subset C^*(M,E)^G$). For example, if G satisfies (RD) and $|\pi_0(G)| < \infty$, then we can take the C^*-index class as the image of the Connes–Skandalis projector in $K_0(\mathcal{S}_G^\infty(M,E))$.

Remark 4.5. There are other representatives of $\operatorname{Ind}(D) \in K_0(C^*(M,E)^G)$ that can be of great interest. For example, as in [Connes and Moscovici 1990], we can choose the parametrix (which is not of G-compact support)

$$Q_V := \frac{I - \exp\bigl(-\frac{1}{2}D^- D^+\bigr)}{D^- D^+} D^+,$$

obtaining $I - Q_V D^+ = \exp\bigl(-\frac{1}{2}D^- D^+\bigr)$, $I - D^+ Q_V = \exp\bigl(-\frac{1}{2}D^+ D^-\bigr)$. This particular choice of parametrix produces the idempotent

$$V_D = \begin{pmatrix} e^{-D^- D^+} & e^{-\frac{1}{2}D^- D^+}\bigl(\frac{I - e^{-D^- D^+}}{D^- D^+}\bigr) D^- \\ e^{-\frac{1}{2}D^+ D^-} D^+ & I - e^{-D^+ D^-} \end{pmatrix}. \tag{4.6}$$

We call this the Connes–Moscovici idempotent. One can also consider the graph-projection $[e_D] - [e_1] \in K_0(C^*(M, E)^G)$ with e_D given by

$$e_D = \begin{pmatrix} (I + D^- D^+)^{-1} & (I + D^- D^+)^{-1} D^- \\ D^+ (I + D^- D^+)^{-1} & D^+ (I + D^- D^+)^{-1} D^- \end{pmatrix}. \tag{4.7}$$

Finally, following [Moscovici and Wu 1994], we can consider the projector

$$P(D) := \begin{pmatrix} S_+^2 & S_+ (I + S_+) \mathcal{P} \\ S_- D^+ & I - S_-^2 \end{pmatrix} \tag{4.8}$$

with $\mathcal{P} = \bar{u}(D^- D^+) D^-$, $S_+ = I - \mathcal{P} D^+$, $S_- = I - D^+ \mathcal{P}$ and $\bar{u}(x) := u(x^2)$ with $u \in C^\infty(\mathbb{R})$ an even function with the property that $w(x) = 1 - x^2 u(x)$ is a Schwartz function and w and u have compactly supported Fourier transform. One proves easily that $P(D) \in M_{2\times 2}(\mathcal{A}_G^c(M, E))$ (with the identity adjoined). It is not difficult to prove that

$$\begin{aligned} \operatorname{Ind}(D) &:= [P_\sigma] - [e_1] \\ &= [V_D] - [e_1] = [e_D] - [e_1] = [P(D)] - [e_1] \quad \text{in } K_0(C^*(M, E)^G). \end{aligned}$$

The advantage of using the Connes–Moscovici projection, the graph projection or the Moscovici–Wu projection is that Getzler rescaling can be used in order to prove the corresponding higher index formulae. This is crucial if one wishes to pass, for example, to manifolds with boundary. However, in this paper we concentrate solely on closed manifolds and use the approach to the index theorem given in [Pflaum et al. 2015b]; this employs the algebraic index theorem in a fundamental way.

4B. *The index class in $K_\bullet(C_r^*(G))$.* There is a canonical Morita isomorphism $\mathcal{M}$ between $K_*(C^*(M, E)^G)$ and $K_*(C_r^*(G))$. This is clear once we bear in mind that $C^*(M, E)^G$ is isomorphic to $\mathbb{K}(\mathcal{E})$; however, for reasons connected with the extension of cyclic cocycles, we want to be explicit about this isomorphism. We assume the existence of a dense holomorphically closed subalgebra $\mathcal{A}(G) \subset C_r^*(G)$ and follow [Hochs and Wang 2018]. Let $\mathcal{A}_G(M, E)$ be the dense holomorphically dense subalgebra of $C^*(M, E)^G$ corresponding to $\mathcal{A}(G)$, as defined in Section 3B.

Define a partial trace map $\mathrm{Tr}_S : \mathcal{A}_G(M, E) \to \mathcal{A}(G)$ associated to the slice S as follows: if $f \otimes k \in (\mathcal{A}(G)) \hat{\otimes} \Psi^{-\infty}(S, E|_S))^{K\times K}$ then

$$\mathrm{Tr}_S(f \otimes k) := f\,\mathrm{Tr}(T_k) = f \int_S \mathrm{tr}\, k(s, s)\, ds,$$

with T_k denoting the smoothing operator on S defined by k and $\mathrm{Tr}(T_k)$ its functional analytic trace on $L^2(S, E|_S)$. It is proved in [Hochs and Wang 2018] that this map induces the Morita isomorphism $\mathcal{M}$ between $K_*(C^*(M, E)^G)$ and $K_*(C^*_r(G))$. We denote the image through $\mathcal{M}$ of the index class $\mathrm{Ind}(D) \in K_0(C^*(M)^G)$ in the group $K_0(C^*_r(G))$ by $\mathrm{Ind}_{C^*_r(G)}(D)$. There are other well-known descriptions of the latter index class: one, following [Kasparov 1980], describes the $C^*_r(G)$-index class as the difference of two finitely generated projective $C^*_r(G)$-modules, using the invertibility modulo $C^*_r(G)$-compact operators of (the bounded transform of) D; the other description is via assembly and KK-theory, as in [Baum et al. 1994]. All these descriptions of the class $\mathrm{Ind}_{C^*_r(G)}(D) \in K_0(C^*_r(G))$ are equivalent. See [Roe 2002; Piazza and Schick 2014, Proposition 2.1].

5. Cyclic cocycles and pairings with K-theory

5A. ***Cyclic cohomology.*** In this subsection we briefly review the basic complex computing cyclic cohomology. Let A be a unital algebra. The space of reduced Hochschild cochains is defined as

$$C^\bullet_{\mathrm{red}}(A) := \mathrm{Hom}_{\mathbb{C}}(A \otimes (A/\mathbb{C}1)^\bullet, \mathbb{C})$$

and is equipped with the Hochschild differential $b : C^k_{\mathrm{red}}(A) \to C^{k+1}_{\mathrm{red}}(A)$ given by the standard formula

$$b\tau(a_0, \dots, a_{k+1}) := \sum_{i=0}^{k} (-1)^i \tau(a_0, \dots, a_i a_{i+1}, \dots, a_k) + (-1)^{k+1} \tau(a_k a_0, \dots, a_{k-1}).$$

The cyclic bicomplex is given by

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
C^2_{\mathrm{red}}(A) \arrow[u, "b"] \arrow[r, "B"] & C^1_{\mathrm{red}}(A) \arrow[u, "b"] \arrow[r, "B"] & C^0_{\mathrm{red}}(A) \arrow[u, "b"] \\
C^1_{\mathrm{red}}(A) \arrow[u, "b"] \arrow[r, "B"] & C^0_{\mathrm{red}}(A) \arrow[u, "b"] & \\
C^0_{\mathrm{red}}(A) \arrow[u, "b"] & &
\end{tikzcd}$$

where $B : C^k_{\mathrm{red}}(A) \to C^{k-1}_{\mathrm{red}}(A)$ denotes Connes' cyclic differential

$$B\tau(a_0, \dots, a_{k-1}) := \sum_{i=0}^{k-1} (-1)^{(k-1)i} \tau(1, a_i, \dots, a_{k-1}, a_0, \dots, a_{i-1}).$$

We denote the total complex associated to this double complex by $CC^\bullet(A)$. When A is not unital, we consider the unitization $\widetilde{A} = A \oplus \mathbb{C}$, and compute cyclic cohomology from the complex $CC^\bullet(A) := CC^\bullet(\widetilde{A})/CC^\bullet(\mathbb{C})$.

Finally, let us close by mentioning that the structure underlying the definition of cyclic cohomology is that of a cocyclic object. This is a cosimplicial object $(X^\bullet, \partial^\bullet, \sigma^\bullet)$ equipped with an additional cyclic symmetry $t^n : X^n \to X^n$ of order $n+1$ satisfying well-known compatibility conditions with respect to the coface operators ∂ and degeneracies σ; see [Loday 1998]. For the cyclic cohomology of an algebra the underlying cosimplicial object is given by $X^k = C^k(A)$ with coface and degeneracies controlling the Hochschild complex. The additional cyclic symmetry t underlying cyclic cohomology is simply the operator which in degree k cyclically permutes the $k+1$ entries in a cochain $\tau \in C^k(A)$.

5B. ***The van Est map in cyclic cohomology.*** Let G be a unimodular Lie group with $|\pi_0(G)| < \infty$. In this subsection we describe, following [Pflaum et al. 2015a; 2015b], how to obtain cyclic cocycles from smooth group cocycles. In this, we can work with two algebras: $C^\infty_c(G)$, the convolution algebra of the group, and $\mathcal{A}^c_G(M)$, the algebra of invariant smoothing operators with cocompact support. In order to simplify the notation we take the vector bundle E to be the product bundle of rank 1.

We start by recalling a well-known fact: inspection of the differential (2.2) shows that the cochain complex $(C^\bullet_{\mathrm{diff}}(G), \delta)$ computing smooth group cohomology $H^\bullet_{\mathrm{diff}}(G)$ comes from an underlying cosimplicial structure given by coface maps ∂^i and codegeneracies σ^j defined on the vector space of homogeneous smooth group cochains $C^\bullet_{\mathrm{diff}}(G)$. This simplicial vector space can be upgraded to a cocyclic one by the cyclic operator $t : C^\bullet \to C^\bullet$ given by

$$(tf)(g_0, \dots, g_k) = f(g_k, g_0, \dots, g_{k-1}), \qquad f \in C^k_{\mathrm{diff}}(G).$$

As seen above, the Hochschild theory of this cocyclic complex is just the smooth group cohomology. The associated cyclic theory is given by $\bigoplus_{i\geq 0} H^{\bullet-2i}_{\mathrm{diff}}(G)$.

Let us now describe the associated cyclic cocycles on the convolution algebra $C^\infty_c(G)$. Instead of using the full complex of smooth group cochains, we restrict to the quasi-isomorphic subcomplex $C^\bullet_{\mathrm{diff},\lambda}(G) \subset C^\bullet_{\mathrm{diff}}(G)$ of *cyclic* cochains, i.e., cochains $c \in C^k_{\mathrm{diff}}(G)$ satisfying

$$c(g_0, \dots, g_k) = (-1)^k c(g_k, g_0, \dots, g_{k-1}).$$

Let $c \in C^k_{\mathrm{diff}}(G)$ be a smooth homogeneous group cochain. Define the cyclic cochain $\tau_c \in C^k(C^\infty_c(G))$ by

$$\tau^G_c(a_0, \dots, a_k) := \int_{G^{\times k}} c(e, g_1, g_1g_2, \dots, g_1 \cdots g_k) \cdot a_0((g_1 \cdots g_k)^{-1}) a_1(g_1) \cdots a_k(g_k)\, dg_1 \cdots dg_k. \tag{5.1}$$

Next up is the algebra $\mathcal{A}^c_G(M)$ of invariant smoothing operators with cocompact support. Again given a smooth homogeneous group cochain $c \in C^k_{\mathrm{diff}}(G)$, we now define a cyclic cochain on this algebra by the formula

$$\begin{aligned}\tau^M_c&(k_0, \dots, k_n)\\ &:= \int_{G^{\times k}} \int_{M^{\times(k+1)}} \chi(x_0) \cdots \chi(x_n) k_0(x_0, g_1x_1) \cdots k_n(x_n, (g_1 \cdots g_n)^{-1}x_0)\\ &\qquad\cdot c(e, g_1, g_1g_2, \dots, g_1 \cdots g_n)\, dx_0 \cdots dx_n\, dg_1 \cdots dg_n.\end{aligned} \tag{5.2}$$

Proposition 5.3. (i) *The map $c \mapsto \tau^G_c$ defined above is a morphism of cochain complexes, and therefore induces a map*

$$\Psi_G : H^\bullet_{\mathrm{diff}}(G) \to HC^\bullet(C^\infty_c(G)).$$

(ii) *The map $c \mapsto \tau^M_c$ defined above is a morphism of cocyclic complexes, and therefore induces a map*

$$\Psi_M : H^\bullet_{\mathrm{diff}}(G) \to HC^\bullet(\mathcal{A}^c_G(M)).$$

Proof. Both of the statements are already known: for the first one, see [Pflaum et al. 2015a, §1.3], and for the second, [Pflaum et al. 2015b, §2.2]. ☐

Example 5.4. In Example 2.18 we discussed the smooth group 2-cocycles for $G = \mathbb{R}^2$, $G = \mathrm{SL}(2, \mathbb{R})$, associated to the area forms of the homogeneous space G/K, equal to $\mathbb{R}^2$ and $\mathbb{H}^2$, respectively. Let us now consider the cyclic cocycles defined by these forms via the construction (5.1) above. For $G = \mathrm{SL}(2, \mathbb{R})$ this gives the following cyclic 2-cocycle on $C^\infty_c(\mathrm{SL}(2, \mathbb{R}))$:

$$\tau^{\mathrm{SL}(2,\mathbb{R})}_\omega(f_0, f_1, f_2) := \int_{\mathrm{SL}(2,\mathbb{R})} \int_{\mathrm{SL}(2,\mathbb{R})} f_0((g_1g_2)^{-1}) f_1(g_1) f_2(g_2) \cdot \mathrm{Area}_{\mathbb{H}^2}(\Delta^2(\bar{e}, \bar{g}_1, \bar{g}_2))\, dg_1\, dg_2.$$

This is exactly the cyclic cocycle considered in [Connes 1985, §9]. For $G = \mathbb{R}^2$ we get a cyclic 2-cocycle on $C^\infty_c(\mathbb{R}^2)$ (with convolution product) given by the same formula with the hyperbolic area replaced by the Euclidean area, and integrations being over $\mathbb{R}^2$ instead of $\mathrm{SL}(2, \mathbb{R})$, again considered in [Connes 1985, §9]. After Fourier transform $f \mapsto \hat{f}$ this cocycle takes the usual form

$$\tau_\omega(f_0, f_1, f_2) = \int_{\mathbb{R}^2} \hat{f}_0\, d\hat{f}_1 \wedge d\hat{f}_2 \quad \text{for } f_0, f_1, f_2 \in C^\infty_c(\mathbb{R}^2).$$

5C. *Extension properties.* In the previous subsection we constructed cyclic cocycles τ_c^G on $C_c^\infty(G)$ and τ_c^M on $\mathcal{A}_G^c(M)$ from a homogeneous smooth group cocycle c. (Recall, once again, that for notational convenience we are taking E to be the product rank 1 bundle.) In Section 3B we have given sufficient conditions on G ensuring that these algebras embed into holomorphically closed subalgebras $\mathcal{A}(G)$ and $\mathcal{A}_G(M)$ of the reduced group C^*-algebra and of the Roe algebra. Now we want to discuss the extension properties of these cocycles. Assume, quite generally, that we are given a subalgebra $\mathcal{A}(G)$ as in Definition 3.6, with associated algebra of operators on $L^2(M)$ denoted, as usual, as $\mathcal{A}_G(M)$. First, we have:

Proposition 5.5. *Let $c \in C^k_{\mathrm{diff},\lambda}(G)$ be a smooth group cocycle. Then*

$$\tau_c^G \textit{ extends to } \mathcal{A}(G) \quad \Longrightarrow \quad \tau_c^M \textit{ extends to } \mathcal{A}_G(M).$$

Proof. Recall that the algebra $\mathcal{A}_G(M)$ is constructed from the choice of subset $\mathcal{A}(G) \subset C_r^*(G)$ by the slice theorem: an invariant kernel k belongs to $\mathcal{A}_G(M)$ if the function

$$\tilde{k}(g, s_1, s_2) := k(s_1, gs_2)$$

belongs to

$$(\mathcal{A}(G) \mathbin{\hat{\otimes}} \Psi^{-\infty}(S, E|_S))^{K\times K}.$$

These functions $\tilde{k}_i(g_i, x_i, x_{i+1})$, $i = 0, \ldots, n-1$, and $\tilde{k}_n((g_1 \cdots g_n)^{-1}, x_n, x_0)$ are used in the formula (5.2) for the cocycle τ_c^M. Since the cut-off function χ has compact support, performing the integrations over M in (5.2), we end up with the pairing of an element in $\mathcal{A}(G)^{\otimes(k+1)}$ with the group cocycle c as defined in (5.1). But then it is clear that τ_c^M is well-defined on $\mathcal{A}_G(M)$ if τ_c^G is well-defined on $\mathcal{A}(G)$. □

For the following, recall from Section 2C the explicit form (2.13) of the van Est isomorphism mapping a closed invariant form $\alpha \in \Omega^k_{\mathrm{inv}}(G/K)$ to a smooth group cocycle $J(\alpha) \in C^k_{\mathrm{diff}}(G)$. For notational convenience, we drop the J in the description of the associated cyclic cocycles, writing τ_α^G and τ_α^M instead of $\tau^G_{J(\alpha)}$ and $\tau^M_{J(\alpha)}$.

Proposition 5.6. *Let G be a Lie group with finitely many connected components and satisfying the rapid decay property* (*RD*)*. Assume that G/K is of nonpositive sectional curvature. Then the cocycle τ_α^G associated to a closed invariant differential form $\alpha \in \Omega^k_{\mathrm{inv}}(G/K)$ extends continuously to $H_L^\infty(G)$. Consequently, the cyclic cocycle τ_α^M extends to $\mathcal{S}_G^\infty(M)$.*

Proof. Recall the definition of the smooth group cocycle $J(\alpha) \in C^k_{\mathrm{diff}}(G)$ defined in (2.13), satisfying the polynomial estimates of Theorem 2.14. This, together with the rapid decay property of G, ensures we can follow the line of proof of [Connes and Moscovici 1990, Proposition 6.5], where the analogous extension property is proved for certain discrete groups. To show that the cyclic cocycle τ_α extends continuously to the algebra $H_L^\infty(G)$, we need to show that it is bounded with respect to the seminorm ν_k in (3.13) defining the Fréchèt topology, for some $k \in \mathbb{N}$. Let $a_0, \ldots, a_k \in H_L^\infty(G)$, and write $\tilde{a}_0 := |a_0|$, $\tilde{a}_i(g) := (1+d(g))^k |a_i(g)|$, $i = 1, \ldots, k$. Then we can make the following estimate:

$$\begin{aligned}
|\tau_\alpha^G(a_0, \ldots, a_k)| &\le C \int_{G^{\times k}} (1+d(g_1))^k \cdots (1+d(g_k))^k |a_0((g_1 \cdots g_k)^{-1})| \\
&\qquad\qquad\qquad\qquad \cdot |a_1(g_1)| \cdots |a_k(g_k)| \, dg_1 \cdots dg_k \\
&= C(\tilde{a}_0 * \cdots * \tilde{a}_k)(e) \\
&\le C \|\tilde{a}_0 * \cdots * \tilde{a}_k\|_{C_r^*(G)} \\
&\le C \|\tilde{a}_0\|_{C_r^*(G)} \cdots \|\tilde{a}_k\|_{C_r^*(G)} \\
&\le CD^{k+1} \nu_p(\tilde{a}_0) \cdots \nu_p(\tilde{a}_k) = CD^{k+1} \nu_{p+k}(a_0) \cdots \nu_{p+k}(a_k).
\end{aligned}$$

In this computation we have used the fact that the Plancherel trace $a \mapsto a(e)$ on the convolution algebra has a continuous extension to $C_r^*(G)$, together with the rapid decay property: $\|a\|_{C_r^*(G)} \le D \|(1+d)^p a\|_{L^2}$, for some p. Altogether, this proves the proposition. □

5D. *Pairing with K-theory.* Cyclic cohomology was first developed by Connes to pair with K-theory via the Chern character. Let us recall this construction. Let $\tau = (\tau_0, \tau_2, \ldots, \tau_{2k}) \in CC^{2k}(A)$ be a cyclic cocycle of degree $2k$ on a unital algebra A, and $[p]-[q]$ an element in $K_0(A)$ represented by idempotents $p, q \in M_N(A)$. The number

$$\begin{aligned}
&\langle [p]-[q], \tau \rangle \\
&\quad := \sum_{n=0}^{k} (-1)^n \frac{(2n)!}{n!} \left(\tau_{2n}\left(\mathrm{tr}\left(p - \frac{1}{2}, p, \ldots, p\right)\right) - \tau_{2n}\left(\mathrm{tr}\left(q - \frac{1}{2}, q, \ldots, q\right)\right)\right),
\end{aligned}$$

where $\mathrm{tr} : M_N(A)^{\otimes(n+1)} \to A^{\otimes(n+1)}$ is the generalized matrix trace, is well-defined and depends only on the (periodic) cyclic cohomology class of τ.

Proposition 5.7. *Let c, $\mathcal{A}(G)$ and $\mathcal{A}_G(M)$ be as in Proposition 5.5, and assume that τ_c^G, and therefore τ_c^M, extends. Then, under the Morita isomorphism*

$$\mathcal{M} : K_0(C^*(M, E)^G) \xrightarrow{\cong} K_0(C_r^*(G)),$$

we have the equality

$$\langle [p]-[q], \tau_c^M \rangle = \langle \mathcal{M}([p]-[q]), \tau_c^G \rangle.$$

Proof. Recall that the isomorphism $\mathcal{M} : K(C^*(M, E)^G) \to K(C_r^*(G))$ is implemented by the partial trace map $\mathrm{Tr}_S : \mathcal{A}_G(M, E) \to \mathcal{A}(G)$ on the respective dense subalgebras. By the abstract Morita isomorphism $\mathcal{M}$, it suffices to consider a simple idempotent $e = e_1 \otimes e_2 \in M_n(\mathcal{A}_G(M, E))$, so that $\mathrm{Tr}_S(e) = \mathrm{Tr}_S(e_2)e_1$ yields an idempotent in $M_n(\mathcal{A}(G))$, where we have extended Tr_S to matrix algebras in the usual way by combining with the matrix trace.

Because we know that the cyclic cohomology class of $\tilde{\tau}_c$ is independent of the choice of a cut-off function, the pairing with K-theory does not depend on this choice either, so we can choose the family χ_ϵ constructed in Lemma 2.1 and take the limit as $\epsilon \downarrow 0$:

$$\begin{aligned}
&\langle [e], \tau_c^M \rangle \\
&\quad = \lim_{\epsilon\downarrow 0} \frac{(2k)!}{k!} \int_{G^{\times k}} \int_{M^{\times(k+1)}} \chi_\epsilon(x_0)\cdots\chi_\epsilon(x_n) e(x_0, g_1x_1)\cdots e(x_n, (g_1\cdots g_n)^{-1}x_0) \\
&\qquad\qquad\qquad\qquad \cdot c(e, g_1, g_1g_2, \ldots, g_1\cdots g_n)\, dx_0\cdots dx_n\, dg_1\cdots dg_n \\
&\quad = \frac{(2k)!}{k!} \int_{G^{\times k}} \int_{S^{\times(k+1)}} e(x_0, g_1x_1)\cdots e(x_n, (g_1\cdots g_n)^{-1}x_0) \\
&\qquad\qquad\qquad\qquad \cdot c(e, g_1, g_1g_2, \ldots, g_1\cdots g_n)\, dx_0\cdots dx_n\, dg_1\cdots dg_n \\
&\quad = \frac{(2k)!}{k!} \mathrm{Tr}_S(e_2\cdots e_2) \int_{G^{\times k}} e_1(g_1)\cdots e_1((g_1\cdots g_n)^{-1}) \\
&\qquad\qquad\qquad\qquad \cdot c(e, g_1, g_1g_2, \ldots, g_1\cdots g_n)\, dg_1\cdots dg_n \\
&\quad = \langle [\mathcal{M}(e)], \tau_c^G \rangle,
\end{aligned}$$

where, to go to the last line, we have used the fact that $e_2^2 = e_2$ is an idempotent. This completes the proof. □

6. Higher C^*-indices and geometric applications

6A. ***Higher C^*-indices and the index formula.*** Let M and G be as above, with M even-dimensional. Hence G is a unimodular Lie group with $|\pi_0(G)| < \infty$. (For the time being we do not put additional hypotheses on G.) Let E be an equivariant complex vector bundle. Consider an odd $\mathbb{Z}_2$-graded Dirac type operator D acting on the sections of E. We have then defined the compactly supported index class $\mathrm{Ind}_c(D) \in K_0(\mathcal{A}_G^c(M, E))$. Let $\alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G)$ and let $\Psi_M(\alpha) \in HC^{\mathrm{even}}(\mathcal{A}_G^c(M, E))$ be the cyclic cohomology class corresponding to α. We know that, in general, we have a pairing

$$K_0(\mathcal{A}_G^c(M, E)) \times HC^{\mathrm{even}}(\mathcal{A}_G^c(M, E)) \to \mathbb{C}. \tag{6.1}$$

We thus obtain, through $\Psi_M : H^\bullet_{\mathrm{diff}}(G) \to HC^\bullet(\mathcal{A}_G^c(M, E))$, a pairing

$$K_0(\mathcal{A}_G^c(M, E)) \times H^{\mathrm{even}}_{\mathrm{diff}}(G) \to \mathbb{C}. \tag{6.2}$$

In particular, by pairing $\mathrm{Ind}_c(D) \in K_0(\mathcal{A}_G^c(M, E))$ with $\alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G)$ we obtain the *higher indices*

$$\mathrm{Ind}_{c,\alpha}(D) := \langle \mathrm{Ind}_c(D), \Psi_M(\alpha) \rangle, \quad \alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G).$$

On the other hand, we can also take the image of α through the van Est map $\Phi_M : H^\bullet_{\rm diff}(G) \to H^\bullet_{\rm inv}(M)$; recall that this is nothing but the pull-back through the classifying map $\psi_M : M \to G/K$ once we identify $H^\bullet_{\rm diff}(G)$ with $H^\bullet_{\rm inv}(G/K)$. The following theorem is proved in [Pflaum et al. 2015b]:

Theorem 6.3 (Pflaum–Posthuma–Tang). *Let M, G and D be as above. In particular, M is even-dimensional. Let $\alpha \in H^{\rm even}_{\rm diff}(G)$. Then the identity*

$$\mathrm{Ind}_{c,\alpha}(D) = \int_M \chi_M(m)\, \mathrm{AS}(M) \wedge \Phi_M(\alpha) \tag{6.4}$$

holds true, where $\mathrm{AS}(M)$ is the Atiyah–Singer integrand on M:

$$\mathrm{AS}(M) := \widehat{A}(M, \nabla^M) \wedge \mathrm{Ch}'(E, \nabla^E).$$

Equivalently,

$$\mathrm{Ind}_{c,\alpha}(D) = \int_M \chi_M(m)\, \mathrm{AS}(M) \wedge \psi_M^*(\alpha) \tag{6.5}$$

if we identify $H^\bullet_{\rm diff}(G)$ and $H^\bullet_{\rm inv}(G/K)$ via the van Est isomorphism (see Remark 2.7).

We now make the fundamental assumption that G satisfies the rapid decay property and that G/K is of nonpositive sectional curvature. Consider the dense holomorphically closed subalgebra $\mathcal{S}^\infty_G(M, E) \subset C^*(M, E)^G$ defined by the rapid decay algebra $H^\infty_L(G) \subset C^*_r(G)$. Thanks to the results of the previous section we can extend the pairing (6.2) to a pairing

$$K_0(\mathcal{S}^\infty_G(M, E)) = K_0(C^*(M, E)^G) \times H^{\rm even}_{\rm diff}(G) \to \mathbb{C}, \tag{6.6}$$

obtaining in this way the *higher C^*-indices* of D, denoted $\mathrm{Ind}_\alpha(D)$. These numbers are well-defined and can be computed by choosing a suitable representative of the class $\mathrm{Ind}(D) \in K_0(C^*(M, E)^G)$. Choosing the Connes–Skandalis projector, we can apply again the index formula of Pflaum–Posthuma–Tang, obtaining for each $\alpha \in H^{\rm even}_{\rm diff}(G)$ the C^*-index formula

$$\mathrm{Ind}_\alpha(D) = \int_M \chi_M(m)\, \mathrm{AS}(M) \wedge \Phi_M(\alpha). \tag{6.7}$$

Notice that we also have a pairing

$$K_0(C^\infty_c(G)) \times HC^{\rm even}(C^\infty_c(G)) \to \mathbb{C} \tag{6.8}$$

and thus, through the homomorphism $\Psi_G : H^\bullet_{\rm diff}(G) \to HC^*(C^\infty_c(G))$, a pairing

$$K_0(C^\infty_c(G)) \times H^{\rm even}_{\rm diff}(G) \to \mathbb{C}. \tag{6.9}$$

According to the results of the previous section this pairing extends to a pairing

$$K_0(C^*_r(G)) \times H^{\rm even}_{\rm diff}(G) \to \mathbb{C} \tag{6.10}$$

if G satisfies (RD). In particular, we can define the $C_r^*(G)$-indices $\mathrm{Ind}_{C_r^*(G),\alpha}(D)$ by pairing $\mathrm{Ind}_{C_r^*(G)}(D) \in K_0(C_r^*(G))$ with $\alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G)$. Further, from Proposition 5.7 we get the equality

$$\langle \mathrm{Ind}(D), \Psi_M(\alpha)\rangle = \langle \mathrm{Ind}_{C_r^*(G)}(D), \Psi_G(\alpha)\rangle, \tag{6.11}$$

which means that

$$\mathrm{Ind}_{C_r^*(G),\alpha}(D) = \mathrm{Ind}_\alpha(D) \quad \text{for all } \alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G) \tag{6.12}$$

and thus, thanks to (6.7), we can state the following fundamental result:

Theorem 6.13. *Let G be a Lie group satisfying the properties stated in the introduction*: $|\pi_0(G)| < \infty$, (*RD*) *and $\underline{E}G$ of nonpositive curvature. Let $\alpha \in H^{\mathrm{even}}_{\mathrm{diff}}(G)$. Then there is a well-defined associated higher $C_r^*(G)$-index* $\mathrm{Ind}_{C_r^*(G),\alpha}(D)$, *and the formula*

$$\mathrm{Ind}_{C_r^*(G),\alpha}(D) = \int_M \chi_M(m)\, \mathrm{AS}(M) \wedge \Phi_M(\alpha) \tag{6.14}$$

holds. Equivalently, if we identify $H^\bullet_{\mathrm{diff}}(G)$ and $H^\bullet_{\mathrm{inv}}(G/K) \equiv H^\bullet_{\mathrm{inv}}(\underline{E}G)$ via the van Est isomorphism, then

$$\mathrm{Ind}_{C_r^*(G),\alpha}(D) = \int_M \chi_M(m)\, \mathrm{AS}(M) \wedge \psi_M^*\alpha.$$

For $\alpha = 1$, the associated cyclic cocycle (5.1) is the Plancherel trace $\tau^G(f) = f(e)$ on $C_r^*(G)$, and the theorem reduces to the L^2-index theorem first proved by Wang [2014]. Remark that in this case the trace extends to $C_r^*(G)$ without problems, so the assumptions on the curvature of G/K and property (RD) are unnecessary.

6B. ***Higher signatures and their G-homotopy invariance.*** Let M and N be two orientable G-proper manifolds and let $f : M \to N$ be a G-homotopy equivalence. Let us denote by D_M^{sign} and D_N^{sign} the corresponding signature operators. Then, according to the main result in [Fukumoto 2017] we have that

$$\mathrm{Ind}_{C_r^*(G)}(D_M^{\mathrm{sign}}) = \mathrm{Ind}_{C_r^*(G)}(D_N^{\mathrm{sign}}) \quad \text{in } K_0(C_r^*(G)). \tag{6.15}$$

Consequently, from (6.14), we obtain the following result, stated as item (i) in Theorem 1.5 in the introduction:

Theorem 6.16. *Let G be a Lie group satisfying the properties stated in the introduction*: $|\pi_0(G)| < \infty$, (*RD*) *and $\underline{E}G$ of nonpositive curvature. Let M and N be two orientable G-proper manifolds and assume that there exists an orientation preserving G-homotopy equivalence between M and N. Let us identify $H^\bullet_{\mathrm{diff}}(G)$ and $H^\bullet_{\mathrm{inv}}(G/K) \equiv H^\bullet_{\mathrm{inv}}(\underline{E}G)$ via the van Est isomorphism. Then for each $\alpha \in H^\bullet_{\mathrm{inv}}(\underline{E}G)$,*

$$\int_M \chi_M(m)\, L(M) \wedge \psi_M^*\alpha = \int_N \chi_N(n)\, L(N) \wedge \psi_N^*\alpha.$$

Proof. For even-dimensional manifolds, this follows immediately from the previous discussion. For the odd-dimensional case we argue by suspension. Thus, let M be an orientable odd-dimensional G-proper manifold. We endow M with a G-invariant Riemannian metric g_M. Consider $\mathbb{R}$ and the natural action of $\mathbb{Z}$ on it by translations (this is a free, proper and cocompact action). Taking the product of M and $\mathbb{R}$ we obtain the even-dimensional $(G\times\mathbb{Z})$-proper manifold $M\times\mathbb{R}$; it has compact quotient equal to $M/G\times S^1$. We endow $M\times\mathbb{R}$ with the $(G\times\mathbb{Z})$-invariant metric g_M+dt^2. Consider the dual group $T^1:=\mathrm{Hom}(\mathbb{Z},U(1))$. The signature operator on $M\times\mathbb{R}$ defines an index class in the group $K_0(C^*(M\times\mathbb{R})^{G\times\mathbb{Z}})$, which is isomorphic to $K_0(C^*(G)\mathbin{\hat{\otimes}}C(T^1))$. Consider the generator d' of $H^1(\mathbb{Z};\mathbb{Z})\subset H^*(\mathbb{Z};\mathbb{C})$ and let $d:=\big(\sqrt{-1}/(2\pi)\big)d'\in H^*(\mathbb{Z};\mathbb{C})$. We know that $H^*(\mathbb{Z};\mathbb{C})$ can be identified with $H^*_{\mathbb{Z}}(E\mathbb{Z};\mathbb{C})$ and that $E\mathbb{Z}=\mathbb{R}$; we denote this isomorphism by $\Xi: H^*(\mathbb{Z};\mathbb{C})\to H^*_{\mathbb{Z}}(\mathbb{R};\mathbb{C})=H^1(S^1)$. Consider $\underline{E}G\times E\mathbb{Z}\equiv\underline{E}G\times\mathbb{R}\equiv G/K\times\mathbb{R}$. To $\alpha\in H^{\mathrm{odd}}_{\mathrm{diff}}(G)\equiv H^{\mathrm{odd}}_{\mathrm{inv}}(\underline{E}G)\equiv H^{\mathrm{odd}}_{\mathrm{inv}}(G/K)$ we associate

$$\beta:=\alpha\otimes\Xi(d)\in H^{\mathrm{odd}}_{\mathrm{inv}}(G/K)\otimes H^1_{\mathbb{Z}}(\mathbb{R})=H^{\mathrm{odd}}_{\mathrm{inv}}(G/K)\otimes H^1(S^1).$$

Now, on the one hand, we have natural homomorphisms

$$\Psi_{G\times\mathbb{Z}}: H^{\mathrm{odd}}_{\mathrm{inv}}(G/K)\otimes H^1(S^1)\to HC^{\mathrm{even}}(C^\infty_c(G)\mathbin{\hat{\otimes}}C^\infty(S^1))$$

and

$$\Psi_{M\times\mathbb{R}}: H^{\mathrm{odd}}_{\mathrm{inv}}(G/K)\otimes H^1(S^1)\to HC^{\mathrm{even}}(\mathcal{A}^c_{G\times\mathbb{Z}}(M\times\mathbb{R})),$$

noting that $\mathcal{A}^c_{G\times\mathbb{Z}}(M\times\mathbb{R})=\mathcal{A}^c_G(M)\mathbin{\hat{\otimes}}\mathcal{A}^c_{\mathbb{Z}}(\mathbb{R})$ and $\mathcal{A}^c_{G\times\mathbb{Z}}(M\times\mathbb{R})=C^*_c(M\times\mathbb{R})^{G\times\mathbb{Z}}$. On the other hand, the classifying map ψ_M and the classifying map for the $\mathbb{Z}$-action on $\mathbb{R}$ together give a smooth $(G\times\mathbb{Z})$-equivariant map $\psi_{M\times\mathbb{R}}: M\times\mathbb{R}\to G/K\times\mathbb{R}$. We can apply the Pflaum–Posthuma–Tang index theorem and obtain, for the signature operator,

$$\begin{aligned}\big\langle\mathrm{Ind}_{C^*_c(M\times\mathbb{R})^{G\times\mathbb{Z}}}(D_{M\times\mathbb{R}}),\Psi_{M\times\mathbb{R}}(\beta)\big\rangle&=\int_G\int_{S^1}\chi_M L(M\times\mathbb{R})\psi^*_M(\alpha)\wedge\Xi(d)\\&=\int_G\chi_M L(M)\psi^*_M(\alpha)=\sigma(M,\alpha).\end{aligned}$$

If G satisfies (RD), then this formula remains true for the $C^*(M\times\mathbb{R})^{G\times\mathbb{Z}}$-index, because $\mathcal{S}^\infty_G(M)\mathbin{\hat{\otimes}}\mathcal{S}_{\mathbb{Z}}(\mathbb{R})$, with $\mathcal{S}_{\mathbb{Z}}(\mathbb{R})$ denoting the smooth $\mathbb{Z}$-invariant kernels of $\mathbb{R}\times\mathbb{R}$ of rapid polynomial decay, is a dense holomorphically closed subalgebra of $C^*(M\times\mathbb{R})^{G\times\mathbb{Z}}$ to which the pairing with $\Psi_{M\times\mathbb{R}}(\beta)$ extends. Consequently,

$$\big\langle\mathrm{Ind}_{C^*(G)\hat{\otimes}C(S^1)}(D_{M\times\mathbb{R}}),\Psi_{G\times\mathbb{Z}}(\beta)\big\rangle=\sigma(M,\alpha).$$

Now, if M and N are G-homotopy equivalent, then $M\times\mathbb{R}$ and $N\times\mathbb{R}$ are $G\times\mathbb{Z}$ homotopy equivalent. Hence the corresponding signature index classes in

$K_0(C^*(G) \hat{\otimes} C(T^1))$ are equal; thus

$$\langle \mathrm{Ind}_{C^*(G)\hat{\otimes}C(S^1)}(D_{M\times\mathbb{R}}),\, \Psi_{G\times\mathbb{Z}}(\beta)\rangle = \langle \mathrm{Ind}_{C^*(G)\hat{\otimes}C(S^1)}(D_{N\times\mathbb{R}}),\, \Psi_{G\times\mathbb{Z}}(\beta)\rangle.$$

This gives us

$$\sigma(M,\alpha) = \sigma(N,\alpha),$$

which is what we wanted to prove in odd dimension. □

6C. ***Higher $\widehat{A}$-genera and G-metrics of positive scalar curvature.*** Let S be a compact smooth manifold with an action of a compact Lie group K. In general, the existence of a K-invariant metric of positive scalar curvature on S is a more refined property than the existence of a positive scalar curvature metric on S; indeed, as shown in [Bérard-Bergery 1981], averaging a positive scalar curvature metric on S might destroy the positivity of the scalar curvature. For sufficient conditions on K and S ensuring the existence of such metrics, see [Lawson and Yau 1974; Hanke 2008].

If M is a G-proper manifold we can try to built a G-invariant positive scalar curvature metric on M through a K-invariant positive scalar curvature metric on the slice S. This is precisely what is achieved in [Guo et al. 2017]:

Theorem 6.17 (Guo–Mathai–Wang). *Let G be an almost connected Lie group and let K be a maximal compact subgroup of G. If S is a compact manifold with a K-invariant metric of positive scalar curvature, then the G-proper manifold $G\times_K S$ admits a G-invariant metric of positive scalar curvature.*

This result shows that the space of positive scalar curvature G-metrics on a G-proper manifold can be nonempty.

We can ask for numerical obstructions to the existence of a positive scalar curvature G-metric. Assume that M has a G-equivariant spin structure and let $\eth$ be the associated spin-Dirac operator. Then one can show that

$$\mathrm{Ind}_{C_r^*(G)}(\eth) = 0 \quad \text{in } K_*(C_r^*G); \tag{6.18}$$

see again [Guo et al. 2017]. The following result was item (ii) in Theorem 1.5 in the introduction:

Theorem 6.19. *Let G be a Lie group satisfying the properties stated in the introduction: $|\pi_0(G)| < \infty$, (RD) and $\underline{E}G$ of nonpositive curvature. Let M be a G-proper manifold admitting a G-equivariant spin structure. Let us identify $H^\bullet_{\mathrm{diff}}(G)$ and $H^\bullet_{\mathrm{inv}}(G/K) \equiv H^\bullet_{\mathrm{inv}}(\underline{E}G)$ via the van Est isomorphism. If M admits a G-invariant metric of positive scalar curvature, then*

$$\widehat{A}(M,\alpha) := \int_M \chi_M(m)\,\widehat{A}(M) \wedge \psi_M^*\alpha = 0$$

for each $\alpha \in H^\bullet_{\mathrm{inv}}(\underline{E}G)$.

Proof. The even-dimensional case follows directly from our C^*-index formula and from (6.18). In the odd-dimensional case we argue by suspension, as we did for the signature operator. It suffices to observe that if M is an odd-dimensional G-proper manifold admitting a G-equivariant spin structure and a G-invariant metric of positive scalar curvature g_M, then $M \times \mathbb{R}$ is an even-dimensional $(G \times \mathbb{Z})$-proper manifold with a $(G \times \mathbb{Z})$-equivariant spin structure and with a $(G \times \mathbb{Z})$-invariant metric $g_M + dt^2$ which is of positive scalar curvature too. Consequently, the analogue of (6.18) holds for the spin Dirac operator on $M \times \mathbb{R}$ and so, arguing as for the signature operator, we finally obtain that

$$\widehat{A}(M, \alpha) := \int_M \chi_M(m)\, \widehat{A}(M) \wedge \psi_M^* \alpha = 0,$$

as required. □

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Periodic cyclic homology and derived de Rham cohomology

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We use the Beilinson t-structure on filtered complexes and the Hochschild–Kostant–Rosenberg theorem to construct filtrations on the negative cyclic and periodic cyclic homologies of a scheme X with graded pieces given by the Hodge completion of the derived de Rham cohomology of X. Such filtrations have previously been constructed by Loday in characteristic zero and by Bhatt–Morrow–Scholze for p-complete negative cyclic and periodic cyclic homology in the quasisyntomic case.

1. Introduction

Let k be a quasisyntomic ring and $k \to R$ a quasisyntomic k-algebra. Bhatt, Morrow, and Scholze construct in [Bhatt et al. 2019, Theorem 1.17] a functorial complete exhaustive decreasing multiplicative $\mathbb{Z}$-indexed filtration $\mathrm{F}^{\star}_{\mathrm{BMS}}\mathrm{HP}(R/k;\mathbb{Z}_p)$ on the p-adic completion $\mathrm{HP}(R/k;\mathbb{Z}_p)$ of periodic cyclic homology with graded pieces $\mathrm{gr}^n_{\mathrm{BMS}}\mathrm{HP}(-/k;\mathbb{Z}_p) \simeq \overline{\mathrm{L}\Omega}_{R/k}[2n]$, where $\mathrm{L}\Omega_{R/k}$ is the derived de Rham complex and $\overline{\mathrm{L}\Omega}_{R/k}$ is the p-adic completion of the Hodge completion of this complex. The Hodge filtration $\Omega^{\geqslant n}_{R/k}$ for smooth algebras induces a Hodge filtration $\mathrm{L}\Omega^{\geqslant n}_{R/k}$ on the derived de Rham complex and its completed variants. There is a corresponding filtration on negative cyclic homology, with graded pieces given by $\overline{\mathrm{L}\Omega}^{\geqslant n}_{R/k}[2n]$, the p-completion of the Hodge completion of $\mathrm{L}\Omega^{\geqslant n}_{R/k}[2n]$.

For smooth $\mathbb{Q}$-algebras, a similar statement goes back to Loday [1992, 5.1.12]. One can also derive very general results along these lines in characteristic zero from [Toën and Vezzosi 2011]. Related results in the context of commutative differential graded algebras were obtained using explicit mixed complexes by Cortiñas [1999]. The authors of [Bhatt et al. 2019] suggest that such a filtration should exist outside the p-complete setting. In this note, we use the Beilinson t-structure on filtered complexes [Beilinson 1987] to prove that this is indeed the case.

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Theorem 1.1. *If k is a commutative ring and X is a quasicompact quasiseparated k-scheme, then there are functorial complete decreasing multiplicative $\mathbb{Z}$-indexed filtrations $\mathrm{F}^{\star}_{\mathrm{B}}\mathrm{HC}^{-}(X/k)$ and $\mathrm{F}^{\star}_{\mathrm{B}}\mathrm{HP}(X/k)$ on negative cyclic homology and periodic cyclic homology, respectively. These filtrations satisfy the following properties.*

(a) *There are natural equivalences*

$$\mathrm{gr}^n_{\mathrm{B}}\mathrm{HC}^{-}(X/k) \simeq \mathrm{R}\Gamma(X, \widehat{\mathrm{L}\Omega}^{\geqslant n}_{-/k}[2n]),$$
$$\mathrm{gr}^n_{\mathrm{B}}\mathrm{HP}(X/k) \simeq \mathrm{R}\Gamma(X, \widehat{\mathrm{L}\Omega}_{-/k}[2n]),$$

where $\widehat{\mathrm{L}\Omega}_{-/k}$ is the Hodge completion of the derived de Rham complex and $\widehat{\mathrm{L}\Omega}^{\geqslant n}_{-/k}$ is the n-th term in the Hodge filtration on $\widehat{\mathrm{L}\Omega}_{-/k}$.

(b) *The filtered pieces $\mathrm{F}^n_{\mathrm{B}}\mathrm{HC}^{-}(X/k)$ and $\mathrm{F}^n_{\mathrm{B}}\mathrm{HP}(X/k)$ are equipped with compatible decreasing filtrations which induce the Hodge filtration on $\mathrm{gr}^n_{\mathrm{B}}\mathrm{HC}^{-}(X/k)$ and $\mathrm{gr}^n_{\mathrm{B}}\mathrm{HP}(X/k)$ under the equivalences of part (a).*

(c) *If X/k is quasi-lci,*[1] *then the filtrations $\mathrm{F}^{\star}_{\mathrm{B}}\mathrm{HC}^{-}(X/k)$ and $\mathrm{F}^{\star}_{\mathrm{B}}\mathrm{HP}(X/k)$ are exhaustive.*

Negative cyclic homology and periodic cyclic homology satisfy fpqc descent by [Bhatt et al. 2019, Corollary 3.4] as a consequence of the fact that the derived exterior powers $\Lambda^i \mathrm{L}_{-/k}$ of the cotangent complex are fpqc sheaves by [Bhatt et al. 2019, Theorem 3.1]. Since $\widehat{\mathrm{L}\Omega}^{\geqslant n}_{-/k}$ has by definition a complete exhaustive decreasing $\mathbb{N}$-indexed filtration with graded pieces $\Lambda^i \mathrm{L}_{-/k}$, it follows that the Hodge-truncated Hodge-completed derived de Rham complexes $\widehat{\mathrm{L}\Omega}^{\geqslant n}_{-/k}$ are also fpqc sheaves. Thus, to prove the theorem, it suffices to handle the affine case.

Theorem 1.1 follows from a much more general theorem, Theorem 4.6, which states that in a suitable ∞-category of bicomplete bifiltered complexes, the Beilinson filtrations are exhaustive for any quasicompact quasiseparated k-scheme X.

Remark 1.2. (i) In case both are defined, the p-adic completion of the filtration of Theorem 1.1 agrees with the filtration of [Bhatt et al. 2019, Theorem 1.17]. This follows in the smooth case by examining the proofs of each theorem and in general by mapping the left Kan extension of our proof to the filtration obtained by quasisyntomic descent in their proof.

(ii) In [Antieau and Nikolaus 2018], we introduce a t-structure on cyclotomic spectra. As one application of the t-structure, we show using calculations of Hesselholt [1996] that the methods of this paper can be used to construct a filtration $\mathrm{F}^{\star}_{\mathrm{B}}\mathrm{TP}(X)$ on topological periodic cyclic homology $\mathrm{TP}(X)$ when X is a smooth scheme over a perfect field with graded pieces given by (shifted) crystalline cohomology $\mathrm{gr}^n_{\mathrm{B}}\mathrm{TP}(X) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X/W(k))[2n]$. When $X = \operatorname{Spec} R$

[1] We say that a k-scheme X is *quasi-lci* if $\mathrm{L}_{X/k}$ has Tor-amplitude contained in [0, 1].

is smooth and affine, then in fact $\mathrm{gr}^n_{\mathrm{B}}\mathrm{TP}(X)$ is given canonically by $\mathrm{W}\Omega^\bullet_R[2n]$, the shifted de Rham–Witt complex. This recovers several parts of [Bhatt et al. 2019, Theorems 1.10, 1.12, and 1.15] in the case of a smooth scheme over a perfect field.

Outline. In Section 2, we outline the theory of filtrations we need. We explain the smooth affine case in Section 3. In Section 4, we give the full proof, which follows from the smooth case by taking nonabelian derived functors in an appropriate ∞-category of bifiltrations.

Conventions and notation. We work with ∞-categories throughout, following the conventions of [Lurie 2009; 2017]. Hochschild homology $\mathrm{HH}(R/k)$ and its relatives are viewed as objects in the derived ∞-category $\mathrm{D}(k)$, possibly with additional structure. Typically, we view objects of $\mathrm{D}(k)$ as being given by chain complexes up to quasi-isomorphism, but several constructions lead us to cochain complexes as well. Given an object $X \in \mathrm{D}(k)$, we write H_*X for its homology groups. We write $X^\bullet$ for a given cochain complex model for X. Thus, $X^\bullet$ is an object of the category $\mathrm{Ch}(k)$ of cochain complexes of k-modules. The main example is the de Rham complex $\Omega^\bullet_{R/k}$ for a smooth commutative k-algebra R.

2. Background on filtrations

Throughout this section, fix a commutative ring k. Let $\mathrm{D}(k)$ be the derived ∞-category of k, a stable presentable ∞-categorical enhancement of the derived category of unbounded chain complexes of k-modules. The derived tensor product of chain complexes makes $\mathrm{D}(k)$ into a presentably symmetric monoidal stable ∞-category, meaning that $\mathrm{D}(k)$ is a symmetric monoidal presentable ∞-category in which the tensor product commutes with colimits in each variable.

The *filtered derived ∞-category* of k is $\mathrm{DF}(k) = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{D}(k))$, the ∞-category of sequences

$$X(\star) : \cdots \to X(n+1) \to X(n) \to \cdots$$

in $\mathrm{D}(k)$. Write $X(\infty) = \lim_n X(n) \simeq \lim(\cdots \to X(n+1) \to X(n) \to \cdots)$ for the limit of the filtration. A filtered complex $X(\star) \in \mathrm{DF}(k)$ is *complete* if $X(\infty) \simeq 0$. Similarly, write $X(-\infty)$ for $\mathrm{colim}_n X(n) \simeq \mathrm{colim}(\cdots \to X(n+1) \to X(n) \to \cdots)$. Given a map $X(-\infty) \to Y$, we say that $X(\star)$ is a filtration on Y; if the map is an equivalence, we say that $X(\star)$ is an *exhaustive* filtration on Y.

We refer to general objects $X(\star)$ of $\mathrm{DF}(k)$ as *decreasing $\mathbb{Z}$-indexed filtrations*. We write $\mathrm{gr}^n X$ for the cofiber of $X(n+1) \to X(n)$, the n-th graded piece of the filtration. Several filtrations of interest in this paper are in fact *$\mathbb{N}$-indexed*, meaning that $X(0) \simeq X(-1) \simeq X(-2) \simeq \cdots$, or equivalently that $\mathrm{gr}^n X \simeq 0$ for $n < 0$.

Day convolution (using the additive symmetric monoidal structure of $\mathbb{Z}^{\mathrm{op}}$) makes $\mathrm{DF}(k)$ into a presentably symmetric monoidal stable ∞-category. The Day convolution symmetric monoidal structure has the property that if $X(\star)$ and $Y(\star)$ are filtered objects of $\mathrm{D}(k)$, then $(X \otimes_k Y)(\star)$ is a filtered spectrum with graded pieces $\mathrm{gr}^n(X \otimes_k Y) \simeq \bigoplus_{i+j=n} \mathrm{gr}^i X \otimes_k \mathrm{gr}^j Y$.

A filtration $X(\star)$ equipped with the structure of a commutative algebra object (or $\mathbb{E}_\infty$-algebra object) in $\mathrm{DF}(k)$ is called a *multiplicative* filtration.

One source of decreasing filtrations is via the Whitehead tower[2] induced from some t-structure on $\mathrm{D}(k)$. We use the standard t-structure, which has $\mathrm{D}(k)_{\geqslant 0} \subseteq \mathrm{D}(k)$, the full subcategory of $\mathrm{D}(k)$ consisting of X such that $\mathrm{H}_n(X) = 0$ for $n < 0$. Similarly, $\mathrm{D}(k)_{\leqslant 0}$ is the full subcategory of $\mathrm{D}(k)$ consisting of X such that $\mathrm{H}_n(X) = 0$ for $n > 0$. Given an object X, its n-connective cover $\tau_{\geqslant n} X \to X$ has $\mathrm{H}_i(\tau_{\geqslant n} X) \cong \mathrm{H}_i(X)$ for $i \geqslant n$ and $\mathrm{H}_i(\tau_{\geqslant n} X) = 0$ for $i < n$.

Example 2.1. If R is a connective commutative algebra object in $\mathrm{D}(k)$, then the Whitehead tower $\tau_{\geqslant \star} R$ is a complete exhaustive decreasing multiplicative $\mathbb{N}$-indexed filtration on R with $\mathrm{gr}^n \tau_{\geqslant \star} R \simeq \mathrm{H}_n(R)[n]$.

For details and proofs of the statements above, see [Gwilliam and Pavlov 2018]. For more background, see [Bhatt et al. 2019, Section 5]. Now we introduce the Beilinson t-structure on $\mathrm{DF}(k)$.

Definition 2.2. Let $\mathrm{DF}(k)_{\geqslant i} \subseteq \mathrm{DF}(k)$ be the full subcategory of those filtered objects $X(\star)$ such that $\mathrm{gr}^n X \in \mathrm{D}(k)_{\geqslant i-n}$, and $\mathrm{DF}(k)_{\leqslant i} \subseteq \mathrm{DF}(k)$ be the full subcategory of those filtered objects $X(\star)$ such that $X(n) \in \mathrm{D}(k)_{\leqslant i-n}$.

Note the asymmetry in the definition. The pair $(\mathrm{DF}(k)_{\geqslant 0}, \mathrm{DF}(k)_{\leqslant 0})$ defines a t-structure on $\mathrm{DF}(k)$ by [Beilinson 1987]; see also [Bhatt et al. 2019, Theorem 5.4] for a proof. We write $\tau^{\mathrm{B}}_{\leqslant n}$, $\tau^{\mathrm{B}}_{\geqslant n}$, π^{B}_n for the truncation and homotopy object functors in the Beilinson t-structure.

The connective objects $\mathrm{DF}(k)_{\geqslant 0}$ are closed under the tensor product on $\mathrm{DF}(k)$, and hence the natural map $\pi^{\mathrm{B}}_0 : \mathrm{DF}(k)_{\geqslant 0} \to \mathrm{DF}(k)^{\heartsuit}$ is symmetric monoidal. The heart is the abelian category of cochain complexes of k-modules equipped with the usual tensor product of cochain complexes.

Remark 2.3. The Beilinson Whitehead tower $\tau^{\mathrm{B}}_{\geqslant \star} X$ is most naturally a bifiltered object, since each $\tau^{\mathrm{B}}_{\geqslant n+1} X \to \tau^{\mathrm{B}}_{\geqslant n} X$ is a map of objects of $\mathrm{DF}(k)$. If we forget the residual filtration on $\tau^{\mathrm{B}}_{\geqslant \star} X$ (by taking the colimit), then we obtain a new filtration on $X(-\infty)$. In this paper, we need this only for $\mathbb{N}$-indexed filtrations. In this case,

---

[2]The Whitehead tower of an object X in a stable ∞-category D with a t-structure is the tower

$$\cdots \to \tau_{\geqslant n+1} X \to \tau_{\geqslant n} X \to \cdots,$$

where $\tau_{\geqslant n} X$ denotes truncation with respect to the t-structure.

each n-connective cover $\tau^{\mathrm{B}}_{\geqslant n}X$ is also $\mathbb{N}$-indexed, and we can view the resulting filtration $(\tau^{\mathrm{B}}_{\geqslant n}X)(0)$ as a new filtration on $X(0)$.[3] If X is a commutative algebra object of $\mathrm{DF}(k)$, then the Beilinson Whitehead tower $\tau^{\mathrm{B}}_{\geqslant\star}X$ is a new multiplicative filtration on X.

For our purposes, it is most important to understand the n-connective cover functors. Given $X(\star)\in\mathrm{DF}(k)$, the n-connective cover in the Beilinson t-structure $\tau^{\mathrm{B}}_{\geqslant n}X\to X(\star)$ induces equivalences

$$\mathrm{gr}^i\tau^{\mathrm{B}}_{\geqslant n}X\simeq\tau_{\geqslant n-i}\mathrm{gr}^iX$$

[Bhatt et al. 2019, Theorem 5.4]. From this, we see that $\mathrm{gr}^i\pi^{\mathrm{B}}_nX\simeq(\mathrm{H}_{n-i}(\mathrm{gr}^iX))[-i]$. The cochain complex corresponding to π^{B}_nX is of the form

$$\cdots\to\mathrm{H}_n(\mathrm{gr}^0X)\to\mathrm{H}_{n-1}(\mathrm{gr}^1X)\to\mathrm{H}_{n-2}(\mathrm{gr}^2X)\cdots,$$

where $\mathrm{H}_n(\mathrm{gr}^0X)$ is in cohomological degree 0 and where the differentials are induced from the boundary maps in homology coming from the cofiber sequences $\mathrm{gr}^{i+1}X\to X(i)/X(i+2)\to\mathrm{gr}^iX$. See [Bhatt et al. 2019, Theorem 5.4(3)] for details.

The next example illustrates our main idea in a general setting.

Example 2.4. Let $X\in\mathrm{D}(k)$ be an object equipped with an S^1-action. The Whitehead tower $\tau_{\geqslant\star}X$ defines a complete exhaustive S^1-equivariant $\mathbb{Z}$-indexed filtration $\mathrm{F}^\star_{\mathrm{P}}X$ on X with graded pieces $\mathrm{gr}^n_{\mathrm{P}}X\simeq\mathrm{H}_n(X)[n]$, equipped with the trivial S^1-action. Applying homotopy S^1-fixed points, we obtain a complete $\mathbb{Z}$-indexed filtration $\mathrm{F}^\star_{\mathrm{P}}X^{hS^1}$ on X^{hS^1} with graded pieces $\mathrm{gr}^n_{\mathrm{P}}X^{hS^1}\simeq(\mathrm{H}_n(X)[n])^{hS^1}$. Let $\mathrm{F}^\star_{\mathrm{B}}X^{hS^1}$ be the double-speed Whitehead tower of $\mathrm{F}^\star_{\mathrm{P}}X^{hS^1}$ in the Beilinson t-structure on $\mathrm{DF}(k)$, so that $\mathrm{F}^n_{\mathrm{B}}X^{hS^1}=\tau^{\mathrm{B}}_{\geqslant 2n}\mathrm{F}^\star X^{hS^1}$. By definition, $\mathrm{F}^n_{\mathrm{B}}X^{hS^1}$ is a filtered spectrum with

$$\mathrm{gr}^i\mathrm{F}^n_{\mathrm{B}}X^{hS^1}\simeq\tau_{\geqslant 2n-i}\mathrm{gr}^i_{\mathrm{P}}X^{hS^1}\simeq\tau_{\geqslant 2n-i}(\mathrm{H}_i(X)[i])^{hS^1}.$$

Hence,

$$\mathrm{gr}^i\mathrm{gr}^n_{\mathrm{B}}X^{hS^1}\simeq\begin{cases}\mathrm{H}_i(X)[2n-i] & \text{if } n\leqslant i,\\ 0 & \text{otherwise.}\end{cases}$$

This shows in fact that $\mathrm{gr}^n_{\mathrm{B}}X^{hS^1}[-2n]\simeq\pi^{\mathrm{B}}_{2n}\mathrm{F}^\star_{\mathrm{P}}X^{hS^1}$ and hence it is in $\mathrm{DF}(k)^\heartsuit$, the abelian category of cochain complexes, and is represented by a cochain complex

$$0\to\mathrm{H}_n(X)\to\mathrm{H}_{n+1}(X)\to\mathrm{H}_{n+2}(X)\to\cdots,$$

where $\mathrm{H}_n(X)$ is in cohomological degree n. The differential is given by the Connes–Tsygan B-operator. An object $X\in\mathrm{D}(k)$ with an S^1-action is the same as a dg module over $\mathrm{C}_\bullet(S^1,k)$, the dg algebra of chains on S^1. The fundamental class B

[3] Note that this is not an idempotent operation: applying the Beilinson Whitehead tower to $\tau^{\mathrm{B}}_{\geqslant\star}X(0)$ typically produces yet another filtration on $X(0)$.

of the circle defines a k-module generator of $\mathrm{H}_1(S^1, k)$ and $B^2 = 0$. The differential in the cochain complex above is given by the action of B. Hence, we have obtained a filtration $\mathrm{F}^{\star}_{\mathrm{B}} X^{hS^1}$ on X^{hS^1} with graded pieces given by $(\mathrm{H}_{\bullet\geqslant n}(X), B)[2n]$.

Remark 2.5. The same argument shows that there is a filtration $\mathrm{F}^{\star}_{\mathrm{B}} X^{tS^1}$ on the S^1-Tate construction X^{tS^1} with graded pieces $\mathrm{gr}^n_{\mathrm{B}} X^{tS^1} \simeq (\mathrm{H}_{\bullet}(X), B)[2n]$. We ignore for the time being any convergence issues.

3. The smooth case

The Hochschild–Kostant–Rosenberg theorem [Hochschild et al. 1962] implies that there are canonical isomorphisms $\Omega^n_{R/k} \cong \mathrm{HH}_n(R/k)$ when R is a smooth commutative k-algebra. In particular, letting $\mathrm{F}^{\star}_{\mathrm{HKR}}\mathrm{HH}(R/k)$ denote the usual Whitehead tower, given by the good truncations $\tau_{\geqslant\star}\mathrm{HH}(R/k)$, we see that there are natural equivalences $\mathrm{gr}^n_{\mathrm{HKR}}\mathrm{HH}(R/k) \simeq \Omega^n_{R/k}[n]$ for all $n \geqslant 0$. Applying homotopy S^1-fixed points, we obtain a complete exhaustive decreasing multiplicative $\mathbb{N}$-indexed filtration $\mathrm{F}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(R/k)$ on $\mathrm{HC}^-(R/k)$.

Definition 3.1. Let $\mathrm{F}^{\star}_{\mathrm{B}}\mathrm{HC}^-(R/k)$ be the double-speed Beilinson Whitehead tower for the filtration $\mathrm{F}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(R/k)$, so that $\mathrm{F}^n_{\mathrm{B}}\mathrm{HC}^-(R/k) = \tau^{\mathrm{B}}_{\geqslant 2n}\mathrm{F}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(R/k)$. For a picture of this filtration, see Figure 1.

Example 2.4 implies that this filtration is a multiplicative $\mathbb{N}$-indexed filtration on $\mathrm{HC}^-(R/k)$; each graded piece $\pi^{\mathrm{B}}_n\mathrm{F}^{\star}_{\mathrm{HKR}}\mathrm{HC}^-(R/k) \simeq \mathrm{gr}^n_{\mathrm{B}}\mathrm{HC}^-(R/k)[-2n]$ in $\mathrm{DF}(k)^{\heartsuit}$ is given by a cochain complex of the form

$$\cdots \to 0 \to \Omega^n_{R/k} \to \Omega^{n+1}_{R/k} \to \cdots,$$

where $\Omega^n_{R/k}$ is in cohomological degree n. It is verified in [Loday 1992, Corollary 2.3.3] that the differential is indeed the de Rham differential. This can also be checked by hand in the case of $k[x]$ to which the general case reduces. It follows that $\mathrm{gr}^n_{\mathrm{B}}\mathrm{HC}^-(R/k) \simeq \Omega^{\bullet\geqslant n}_{R/k}[2n]$. The additional filtration on $\mathrm{F}^{\star}_{\mathrm{B}}\mathrm{HC}^-(R/k)$ reduces to the Hodge filtration on $\Omega^{\bullet\geqslant n}_{R/k}[2n]$. The exhaustiveness and completeness of $\mathrm{F}^{\star}_{\mathrm{B}}\mathrm{HC}^-(R/k)$ follows from Lemma 3.2 below. The case of $\mathrm{HP}(R/k)$ is similar. This proves Theorem 1.1 in the case of smooth algebras.[4]

We needed the following lemma in the proof.

Lemma 3.2. *Let $X(\star)$ be a complete $\mathbb{N}$-indexed filtration on $X = X(0)$ and let $\tau^{\mathrm{B}}_{\geqslant\star}X$ be the associated Beilinson Whitehead tower in* $\mathrm{DF}(k)$.

(i) *The truncations $\tau^{\mathrm{B}}_{\geqslant n}X$ and $\tau^{\mathrm{B}}_{\leqslant n-1}X$ are complete for all $n \in \mathbb{Z}$.*

(ii) *The filtration $(\tau^{\mathrm{B}}_{\geqslant\star}X)(0)$ on $X \simeq X(0)$ is complete and exhaustive.*

[4] Note that for R a smooth k-algebra, the de Rham complex $\Omega^{\bullet}_{R/k}$ is already Hodge-complete.

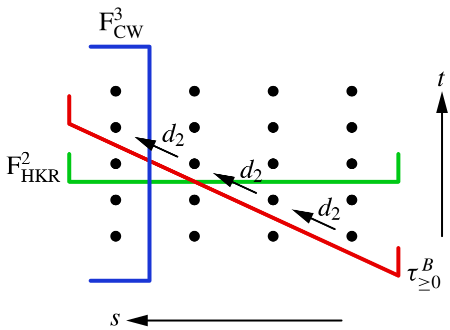

Figure 1. The Beilinson filtration. The figure shows the E_2-page of the spectral sequence $E_2^{s,t} = H^s(BS^1, HH_t(R/k)) \Rightarrow HC^-_{t-s}(R/k)$ and which parts of $HC^-(R/k)$ are cut out by $\tau^B_{\geqslant 0} HC^-(R/k)$, $F^2_{HKR} HC^-(R/k)$, and $F^3_{CW} HC^-(R/k)$, respectively. For the definition of the CW filtration, see Section 4.

Proof. Since the full subcategory $\widehat{DF}(k) \subseteq DF(k)$ of complete filtrations is stable, to prove part (i) it is enough to show that $\tau^B_{\leqslant n-1} X$ is complete for all n. However, $(\tau^B_{\leqslant n-1} X)(i) \in D(k)_{\leqslant n-1-i}$. We find that $\lim_i (\tau^B_{\leqslant n-1} X)(i)$ is in $D(k)_{\leqslant -\infty} \simeq 0$. This proves (i). It follows from (i) and the fact that complete filtered spectra are closed under colimits that we can view $\lim_n \tau^B_{\geqslant n} X$ as a complete filtered spectrum $Y(\star)$ with graded pieces

$$\mathrm{gr}^i Y \simeq \lim_n \mathrm{gr}^i \tau^B_{\geqslant n} X \simeq \lim_n \tau_{\geqslant n-i} \mathrm{gr}^i X.$$

Hence, each $\mathrm{gr}^i Y$ is ∞-connective. Thus, $\mathrm{gr}^i Y \simeq 0$ for all i and hence $Y(\star) \simeq 0$ as it is complete. This proves the completeness in (ii). Finally, $(\tau^B_{\leqslant n-1} X)(0) \in D(k)_{\leqslant n-1}$. It follows that $(\tau^B_{\geqslant n} X)(0) \to X(0) \simeq X$ is an n-equivalence, and exhaustiveness follows by letting $n \to -\infty$. □

4. The general case

Our general strategy for the proof of Theorem 1.1 is to left Kan extend from the case of smooth algebras. Because of convergence issues, we are forced to Kan extend in an ∞-category which keeps track of multiple filtrations.

Let k be a commutative ring, sCAlg_k the ∞-category of simplicial k-algebras, and $\mathrm{CAlg}_k^{\mathrm{poly}} \subseteq \mathrm{sCAlg}_k$ the full subcategory of finitely generated polynomial k-algebras. This embedding admits a universal property: given any ∞-category $\mathscr{C}$ which admits sifted colimits, the forgetful functor

$$\mathrm{Fun}'(\mathrm{sCAlg}_k, \mathscr{C}) \to \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{poly}}, \mathscr{C})$$

is an equivalence, where $\mathrm{Fun}'(\mathrm{sCAlg}_k, \mathscr{C})$ is the ∞-category of sifted colimit-preserving functors $\mathrm{sCAlg}_k \to \mathscr{C}$. Given $F : \mathrm{CAlg}_k^{\mathrm{poly}} \to \mathscr{C}$, we call the corresponding sifted colimit-preserving functor $\mathrm{d}F : \mathrm{sCAlg}_k \to \mathscr{C}$ the *left Kan extension* or the *nonabelian derived functor* of F. For details, see [Lurie 2009, Section 5.5.9].

Let $R \in \mathrm{sCAlg}_k$ and fix $F : \mathrm{CAlg}_k^{\mathrm{poly}} \to \mathscr{C}$. Then one extends F to all polynomial rings by taking filtered colimits in $\mathscr{C}$. To compute the value of the left Kan extension $\mathrm{d}F$ of F on R, one takes a simplicial resolution $|P_\bullet| \simeq R$, where each $P_\bullet$ is polynomial (but not necessarily finitely generated), and computes $|F(P_\bullet)|$ in $\mathscr{C}$.

Let k be a commutative ring, and let R be a simplicial commutative k-algebra. Then, $\mathrm{HH}(R/k)$ is a connective commutative algebra object in $\mathrm{D}(k)^{BS^1}$, the ∞-category of complexes of k-modules equipped with an S^1-action. We could apply Example 2.4 to obtain a filtration on $\mathrm{HC}^-(R/k) = \mathrm{HH}(R/k)^{hS^1}$ with graded pieces truncations of the cochain complex $(\mathrm{HH}_*(R/k), B)$. However, in the nonsmooth case, this does not capture derived de Rham cohomology.

We use the fact that Hochschild homology commutes with sifted colimits (see for example [Bhatt et al. 2019, Remark 2.3]) to Kan extend the HKR filtration of [Hochschild et al. 1962] from finitely generated polynomial algebras to all simplicial commutative k-algebras. This gives a functorial complete exhaustive decreasing multiplicative $\mathbb{N}$-indexed S^1-equivariant multiplicative filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HH}(R/k)$ on $\mathrm{HH}(R/k)$ with graded pieces $\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{HH}(R/k) \simeq \Lambda^t \mathrm{L}_{R/k}[t]$ with the trivial S^1-action, where $\mathrm{L}_{R/k}$ denotes the cotangent complex and $\Lambda^t \mathrm{L}_{R/k}$ is the t-th derived exterior power of the cotangent complex. Since $\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k)$ is t-connective for all t, it follows that the HKR filtration is complete.

Applying homotopy S^1-fixed points or Tate, we obtain decreasing multiplicative $\mathbb{N}$-indexed filtrations $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}^-(R/k)$ and $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k)$ on negative cyclic homology

$$\mathrm{HC}^-(R/k) = \mathrm{HH}(R/k)^{hS^1}$$

and periodic cyclic homology

$$\mathrm{HP}(R/k) = \mathrm{HH}(R/k)^{tS^1}.$$

These filtrations are both complete. To see this, note first that the induced HKR filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}(R/k)$ on cyclic homology $\mathrm{HC}(R/k) = \mathrm{HH}(R/k)_{hS^1}$ is complete since $\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HC}(R/k) \simeq (\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k))_{hS^1}$ is t-connective. Thus, since we have a cofiber sequence

$$\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}(R/k)[1] \to \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}^-(R/k) \to \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HP}(R/k)$$

in $\mathrm{DF}(k)$, it suffices to see that the HKR filtration on $\mathrm{HC}^-(R/k)$ is complete. But this follows from the fact that $(-)^{hS^1}$ commutes with limits.

Negative cyclic homology admits a second filtration, coming from the standard cell structure $\mathbb{CP}^0 \subseteq \mathbb{CP}^1 \subseteq \cdots$ on $BS^1 \simeq \mathbb{CP}^\infty$. This second filtration is compatible with the HKR filtration since on Hochschild homology the HKR filtration is S^1-equivariant. To be precise, we consider the double filtration

$$\mathrm{F}^t_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) = \mathrm{fib}\big((\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k))^{hS^1} \to (\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k))^{h\Omega\mathbb{CP}^{s-1}}\big),$$

which has graded pieces

$$\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq \Lambda^t\mathrm{L}_{R/k}[t-2s].$$

This bifiltration is multiplicative in the natural sense with respect to the Day convolution symmetric monoidal structure on $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}, \mathrm{D}(k))$, where we give $\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}$ the symmetric monoidal structure coming from (the opposite of) addition in the monoid $\mathbb{N} \times \mathbb{N}$.

We let $\mathrm{DBF}(k)$ denote the ∞-category $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}, \mathrm{D}(k))$ of $\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}$-indexed *bifiltered complexes* of k-modules and we denote by $\widehat{\mathrm{DBF}}(k)$ the full subcategory of $\mathrm{DBF}(k)$ on those *bicomplete* bifiltered complexes, i.e., those $X(\star, \star)$ such that for each s one has $\lim_t X(s,t) \simeq 0$ and for each t one has $\lim_s X(s,t) \simeq 0$. Note that either condition implies that $X(\star,\star)$ is complete in the weaker sense that $\lim_{s,t} X(s,t) \simeq 0$.

Remark 4.1. Bicomplete bifiltered objects are the same as complete filtered objects in the complete filtered derived category.

Lemma 4.2. *For any simplicial commutative k-algebra R, the filtration*

$$\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(R/k)$$

is bicomplete.

Proof. Fix s. We have

$$\lim_t \mathrm{F}^t_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq 0$$

as both $(-)^{hS^1}$ and $(-)^{h\Omega\mathbb{CP}^{s-1}}$ commute with limits. Now fix t. Then we want to show that

$$\lim_s \mathrm{F}^t_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq \mathrm{fib}\big((\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k))^{hS^1} \to \lim_s(\mathrm{F}^t_{\mathrm{HKR}}\mathrm{HH}(R/k))^{h\Omega\mathbb{CP}^{s-1}}\big) \simeq 0.$$

However, for any bounded below spectrum with an S^1-action X, the natural map $X^{hS^1} \to \lim_s X^{h\Omega\mathbb{CP}^{s-1}}$ is an equivalence. Indeed, this follows by a computation if X has a single nonzero homotopy group, and then it follows for all homologically bounded complexes by induction. Then it follows in the limit up the Postnikov tower since both $(-)^{hS^1}$ and $\lim_s(-)^{h\Omega\mathbb{CP}^{s-1}}$ commute with limits. □

We can Kan extend $\mathrm{HC}^-(-/k)$ with its bifiltration from finitely generated polynomial k-algebras to all simplicial commutative k-algebras to obtain a bifiltration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{dHC}^-(R/k)$ on derived negative cyclic homology. Let $\widehat{\mathrm{dHC}}^-(R/k)$ denote bicompleted derived negative cyclic homology and let $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\widehat{\mathrm{dHC}}^-(R/k)$ be the bicomplete bifiltration on bicompleted derived negative cyclic homology, which is the Kan extension of $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(-/k)$ as a functor $\mathrm{CAlg}_k^{\mathrm{poly}} \to \widehat{\mathrm{DBF}}(k)$ to all simplicial commutative k-algebras.

Lemma 4.3. *For any $R \in \mathrm{sCAlg}_k$, the natural map*

$$\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\widehat{\mathrm{dHC}}^-(R/k) \to \mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(R/k)$$

is an equivalence in $\widehat{\mathrm{DBF}}(k)$.

Proof. Since both bifiltered objects are bicomplete, it is enough to check on graded pieces. Since the graded pieces functors $\mathrm{gr}^t\mathrm{gr}^s : \widehat{\mathrm{DBF}}(k) \to \mathrm{D}(k)$ commute with colimits, $\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\widehat{\mathrm{dHC}}^-(R/k)$ is the left Kan extension of $R \mapsto \Omega^t_{R/k}[t-2s]$ from finitely generated polynomial algebras to all simplicial commutative k-algebras, which is precisely $\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq \Lambda^t\mathrm{L}_{R/k}[t-2s]$. □

Remark 4.4. The lemma says that even though $\mathrm{HC}^-(-/k)$ does not commute with sifted colimits as a functor $\mathrm{sCAlg}_k \to \mathrm{D}(k)$, it does commute with sifted colimits as a functor $\mathrm{sCAlg}_k \to \widehat{\mathrm{DBF}}(k)$ when equipped with its skeletal and HKR filtrations. In particular, we can compute $\mathrm{HC}^-(R/k)$ by left Kan extending from finitely generated polynomial algebras and then bicompleting.

Fix s and consider the Whitehead tower

$$\cdots \to \tau^{\mathrm{B}}_{\geqslant r}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \to \tau^{\mathrm{B}}_{\geqslant r-1}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \to \cdots$$

in the Beilinson t-structure on filtered complexes, where we are taking Beilinson connective covers in the HKR-direction. Recall that

$$\begin{aligned}\mathrm{gr}^t\tau^{\mathrm{B}}_{\geqslant r}&\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k)\\ &\simeq \tau_{\geqslant -t+r}\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{TC}^-(R/k)\\ &\simeq \tau_{\geqslant -t+r}\mathrm{fib}\big((\Lambda^t\mathrm{L}_{R/k}[t])^{hS^1} \to (\Lambda^t\mathrm{L}_{R/K}[t])^{h\Omega\mathbb{CP}^{s-1}}\big) \qquad (4.5)\end{aligned}$$

and hence that

$$\begin{aligned}\mathrm{gr}^t\pi^{\mathrm{B}}_r&\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k)\\ &\simeq \big(\pi_{-t+r}\mathrm{fib}\big((\Lambda^t\mathrm{L}_{R/k}[t])^{hS^1} \to (\Lambda^t\mathrm{L}_{R/k}[t])^{h\Omega\mathbb{CP}^{s-1}}\big)\big)[-t+r].\end{aligned}$$

Here, the notation implies that we view π_{-t+r} of the object on the right as a complex concentrated in degree $-t+r$. If R/k is smooth, we have $\Lambda^t\mathrm{L}_{R/k} \simeq \Omega^t_{R/k}$. In particular, in this case, we see that

$$\mathrm{gr}^t \pi_r^{\mathrm{B}} \mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^s_{\mathrm{CW}} \mathrm{HC}^-(R/k) \simeq \begin{cases} \Omega^t_{R/k}[-t+r] & \text{if } r \text{ is even and } r \leqslant 2t-2s, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.6. *There is a complete exhaustive multiplicative decreasing $\mathbb{Z}$-indexed filtration $\mathrm{F}^\star_{\mathrm{B}}$ on the bicomplete bifiltered complex $\mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^\star_{\mathrm{CW}} \mathrm{HC}^-(R/k)$. The graded piece $\mathrm{gr}^u_{\mathrm{B}} \mathrm{HC}^-(R/k)$ is naturally equivalent to the Hodge-complete derived de Rham cohomology $\widehat{\mathrm{L}\Omega}^{\geqslant u}_{R/k}[2u]$ of R, naively truncated. Moreover, the remaining HKR and CW filtrations on $\mathrm{gr}^u_{\mathrm{B}} \mathrm{HC}^-(R/k)$ both coincide with the Hodge filtration. Finally, the underlying filtration $\mathrm{F}^\star_{\mathrm{B}} \mathrm{HC}^-(R/k)$ in the sense of Remark 2.3 is a complete filtration of $\mathrm{HC}^-(R/k)$; if $\mathrm{L}_{R/k}$ has Tor-amplitude contained in $[0, 1]$, then the filtration is exhaustive.*

Proof. When R/k is a finitely generated polynomial algebra, we take as our filtration $\mathrm{F}^\star_{\mathrm{B}}$ the double-speed Whitehead filtration $\tau^{\mathrm{B}}_{\geqslant 2\star} \mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^s_{\mathrm{CW}} \mathrm{HC}^-(R/k)$ in the Beilinson t-structure. By definition of the Beilinson t-structure and the analysis in the paragraph above, $\pi^{\mathrm{B}}_{2u} \mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^s_{\mathrm{CW}} \mathrm{HC}^-(R/k)$ is a chain complex of the form

$$0 \to \Omega^{u+s}_{R/k} \to \Omega^{u+s+1}_{R/k} \to \cdots ,$$

where $\Omega^{u+s}_{R/k}$ sits in homological degree $u-s$. Thus, as in Section 3, for R smooth,

$$\mathrm{gr}^u_{\mathrm{B}} \mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^s_{\mathrm{CW}} \mathrm{HC}^-(R/k) \simeq \pi^{\mathrm{B}}_{2u} \mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^s_{\mathrm{CW}} \mathrm{HC}^-(R/k) \simeq \Omega^{\bullet \geqslant u+s}_{R/k}[2u].$$

Both the CW filtration and the HKR filtration induce the Hodge filtration on this graded piece.

We claim that for R/k a finitely generated polynomial algebra on d variables, for each u, the bifiltered spectrum $\mathrm{F}^u_{\mathrm{B}} \mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^\star_{\mathrm{CW}} \mathrm{HC}^-(R/k)$ is bicomplete. For each s, this follows from Lemma 3.2. In the other direction, as soon as $2s > 2d - u$, (4.5) shows that $\mathrm{F}^u_{\mathrm{B}} \mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^s_{\mathrm{CW}} \mathrm{HC}^-(R/k) \simeq 0$, so completeness in the CW-direction is immediate.

We now view the filtration $\mathrm{F}^\star_{\mathrm{B}}$ as giving a functor $\mathrm{CAlg}^{\mathrm{poly}}_k \to \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \widehat{\mathrm{DBF}}(k))$, which we left Kan extend to a functor $\mathrm{sCAlg}_k \to \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \widehat{\mathrm{DBF}}(k))$. We verify the necessary properties in a series of lemmas.

Lemma 4.7. *For any $R \in \mathrm{sCAlg}_k$,*

$$\operatorname*{colim}_{u \to -\infty} \mathrm{F}^u_{\mathrm{B}} \mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^\star_{\mathrm{CW}} \mathrm{HC}^-(R/k) \simeq \mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^\star_{\mathrm{CW}} \mathrm{HC}^-(R/k),$$

where the colimit is computed in $\widehat{\mathrm{DBF}}(k)$.

Proof. The colimit functor $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \widehat{\mathrm{DBF}}(k)) \to \widehat{\mathrm{DBF}}(k)$ commutes with colimits, so this follows from Lemma 4.3 once we show that the filtration $\mathrm{F}^u_{\mathrm{B}} \mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^\star \mathrm{HC}^-(R/k)$ is exhaustive on $\mathrm{F}^\star_{\mathrm{HKR}} \mathrm{F}^\star_{\mathrm{CW}} \mathrm{HC}^-(R/k)$ for R a finitely generated polynomial ring. This follows from Lemma 3.2. □

Lemma 4.8. *We have* $\lim_u \mathrm{F}^u_{\mathrm{B}}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{F}^\star_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq 0$, *where the limit is computed in* $\widehat{\mathrm{DBF}}(k)$.

Proof. By conservativity of the limit-preserving functors $\mathrm{gr}^t\mathrm{gr}^s : \widehat{\mathrm{DBF}}(k) \to \mathrm{D}(k)$, it is enough to see that

$$\lim_u \mathrm{gr}^t \tau^{\mathrm{B}}_{\geqslant 2u}\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq 0$$

for all pairs (s,t). But this object is $(2u-t)$-connective by definition of the Beilinson t-structure and because of the fact that colimits of $(2s-t)$-connective objects are $(2s-t)$-connective. Thus, the limit vanishes. □

Lemma 4.9. *The graded piece* $\mathrm{gr}^u_{\mathrm{B}}\mathrm{HC}^-(R/k)$ *is the bicomplete bifiltered object obtained by left Kan extending* $R \mapsto \Omega^{\geqslant u}[2u]$ *to all simplicial commutative rings, where the filtration is given by* $\mathrm{F}^{(s,t)}\Omega^{\geqslant u}[2u] \simeq \Omega^{\geqslant u+\max(s-u,t-u,0)}[2u]$.

Proof. Indeed, this is clear on finitely generated polynomial algebras by Section 3 so this follows by Kan extension using the fact that $\mathrm{gr}^u : \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \widehat{\mathrm{DBF}}(k)) \to \widehat{\mathrm{DBF}}(k)$ commutes with colimits. □

Thus, we have proved the theorem except for the last sentence. Now we examine the underlying filtration $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k)$ on $\mathrm{HC}^-(R/k)$ given by forgetting the HKR and CW filtrations.

Lemma 4.10. *Let* $\widehat{\mathrm{DBF}}(k) \to \mathrm{D}(k)$ *be the functor that sends a bicomplete* $\mathbb{N}^{\mathrm{op}} \times \mathbb{N}^{\mathrm{op}}$*-index bifiltered spectrum* $X(\star,\star)$ *to* $X(0,0)$. *This functor preserves limits.*

Proof. The functor is the composition of the inclusion functor $\widehat{\mathrm{DBF}}(k) \to \mathrm{DBF}(k)$ (a right adjoint) and the limit preserving evaluation functor $X(\star,\star) \mapsto X(0,0)$ on $\mathrm{DBF}(k)$. □

From Lemmas 4.8 and 4.10, it follows that the filtration $\mathrm{F}^u_{\mathrm{B}}\mathrm{HC}^-(R/k)$ is a complete filtration on $\mathrm{HC}^-(R/k)$. Exhaustiveness is somewhat subtle.

Lemma 4.11. *If* $\mathrm{L}_{R/k}$ *has Tor-amplitude contained in* $[0,1]$, *then the filtration* $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k)$ *on* $\mathrm{HC}^-(R/k)$ *is exhaustive.*

Proof. Consider the cofiber C^u of $\mathrm{F}^u_{\mathrm{B}}\mathrm{HC}^-(R/k) \to \mathrm{HC}^-(R/k)$ in $\widehat{\mathrm{DBF}}(k)$. We find that

$$\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\mathrm{F}^u_{\mathrm{B}}\mathrm{HC}^-(R/k) \simeq \begin{cases} 0 & \text{if } u > t-s, \\ \Lambda^t\mathrm{L}_{R/k}[t-2s] & \text{otherwise.} \end{cases}$$

Similarly, $\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}\mathrm{HC}^-(R/k) \simeq \Lambda^t\mathrm{L}_{R/k}[t-2s]$. It follows that

$$\mathrm{gr}^t_{\mathrm{HKR}}\mathrm{gr}^s_{\mathrm{CW}}C^u \simeq \begin{cases} \Lambda^t\mathrm{L}_{R/k}[t-2s] & \text{if } t-s < u, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{L}_{R/k}$ has Tor-amplitude contained in $[0, 1]$, it follows that $\Lambda^t \mathrm{L}_{R/k}$ has Tor-amplitude contained in $[0, t]$,[5] and hence $\Lambda^t \mathrm{L}_{R/k}[t - 2s]$ has Tor-amplitude contained in $[t - 2s, 2t - 2s]$. In particular, we see that C^u has a complete filtration with graded pieces having Tor-amplitude in $[t - 2s, 2t - 2s]$ for $t - s < u$. In particular, since R is discrete, the graded pieces are $2u$-coconnected. Since C^u is a limit of $2u$-coconnected objects, it follows that $\pi_i C^u = 0$ for $i \geqslant 2u$. In particular, $\operatorname{colim}_{u \to -\infty} C^u = 0$ and the filtration is exhaustive as claimed. □

This completes the proof of Theorem 4.6. □

Now we give the argument for $\mathrm{HP}(R/k)$.

Corollary 4.12. *There is a complete filtration* $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(R/k)$ *on* $\mathrm{HP}(R/k)$ *with*

$$\mathrm{gr}^u_{\mathrm{B}}\mathrm{HP}(R/k) \simeq \widehat{\mathrm{L}\Omega}_{R/k}[2u].$$

If R/k is quasi-lci, the filtration is exhaustive.

Proof. We use the cofiber sequence $\mathrm{HC}(R/k)[1] \to \mathrm{HC}^-(R/k) \to \mathrm{HP}(R/k)$. Note that $\mathrm{HC}(-/k) = \mathrm{HH}(R/k)_{hS^1}$ preserves colimits. The Kan extension of the HKR filtration on $\mathrm{HC}(-/k)[1]$ from finitely generated polynomial k-algebras to all simplicial commutative k-algebras thus equips $\mathrm{HC}(-/k)[1]$ with an $\mathbb{N}$-indexed filtration $\mathrm{F}^\star_{\mathrm{HKR}}\mathrm{HC}(-/k)[1]$ with graded pieces

$$\mathrm{gr}^n_{\mathrm{HKR}}\mathrm{HC}(-/k)[1] \simeq \Lambda^n \mathrm{L}_{R/k}[n+1].$$

Moreover, since $\mathrm{F}^n_{\mathrm{HKR}}\mathrm{HC}(-/k)[1]$ is n-connective, the filtration is complete. By Lemma 3.2, the double-speed Beilinson Whitehead tower induces a complete exhaustive decreasing $\mathbb{Z}$-indexed filtration $\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}(-/k)[1]$ on $\mathrm{HC}^-(-/k)[1]$. A straightforward check implies that the graded pieces are

$$\mathrm{gr}^u_{\mathrm{B}}\mathrm{HC}(-/k)[1] \simeq \mathrm{L}\Omega^{\leqslant u-1}_{R/k}[2u-1].$$

Here, it makes no difference whether we take the Hodge-completed derived de Rham complex or the non-Hodge-completed derived de Rham complex, as the Hodge filtration on $\mathrm{L}\Omega^{\leqslant u-1}_{R/k}$ is finite. Now we have a cofiber sequence

$$\mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}(-/k)[1] \to \mathrm{F}^\star_{\mathrm{B}}\mathrm{HC}^-(R/k) \to \mathrm{F}^\star_{\mathrm{B}}\mathrm{HP}(R/k).$$

Since the filtrations on $\mathrm{HC}(-/k)$ and $\mathrm{HC}^-(R/k)$ are complete, so is the induced filtration on $\mathrm{HP}(R/k)$. When R/k is quasi-lci, Theorem 4.6 implies that the filtration on $\mathrm{HC}^-(R/k)$ is exhaustive. We have already noted that the filtration on $\mathrm{HC}(R/k)$

[5] Use the fact that $\mathrm{L}_{R/k}$ is quasi-isomorphic to a complex $M_0 \leftarrow M_1$, where M_0, M_1 are flat, the fact that flats are filtered colimits of finitely generated projectives, the standard filtration on $\Lambda^t \mathrm{L}_{R/k}$ with graded pieces $\Lambda^j M_0 \otimes_R \Lambda^{t-j}(M_1[1])$, and the fact that $\Lambda^{t-j}(M_1[1]) \simeq (\Gamma^{t-j} M_1)[t-j]$, where Γ^{t-j} is the divided power functor on flats.

is exhaustive. Hence, the filtration on $\mathrm{HP}(R/k)$ is exhaustive. The graded pieces $\mathrm{gr}^u_{\mathrm{B}}\mathrm{HP}(R/k)$ fit into cofiber sequences

$$\widehat{\mathrm{L}\Omega}^{\geqslant u}_{R/k}[2u] \to \mathrm{gr}^u_{\mathrm{B}}\mathrm{HP}(R/k) \to \mathrm{L}\Omega^{\leqslant u-1}_{R/k}[2u].$$

One finds using the remaining HKR filtration that in the smooth case the graded piece $\mathrm{gr}^u_{\mathrm{B}}\mathrm{HP}(R/k)[-2u]$ is a chain complex (it is in the heart of the Beilinson t-structure) and that this sequence is equivalent to the canonical stupid filtration sequence

$$0 \to \Omega^{\bullet \geqslant u}_{R/k} \to \Omega^{\bullet}_{R/k} \to \Omega^{\bullet \leqslant u-1}_{R/k} \to 0.$$

This completes the proof, since now we see in general that

$$\mathrm{gr}^u_{\mathrm{B}}\mathrm{HP}(R/k) \simeq \widehat{\mathrm{L}\Omega}_{R/k}[2u]. \qquad \square$$

Proof of Theorem 1.1. Theorem 4.6 and Corollary 4.12 establish the theorem for affine k-schemes. It follows for general quasicompact separated schemes because everything in sight is then computed from a finite limit of affine schemes, and the conditions of being complete or exhaustive are stable under finite limits. Finally, it follows for a quasicompact quasiseparated scheme X by induction on the number of affines needed to cover X. $\square$

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Linkage of Pfister forms over $\mathbb{C}(x_1, \ldots, x_n)$

Adam Chapman and Jean-Pierre Tignol

We prove the existence of a set of cardinality 2^n of n-fold Pfister forms over $\mathbb{C}(x_1, \ldots, x_n)$ which do not share a common $(n-1)$-fold factor. This gives a negative answer to a question raised by Becher. The main tools are the existence of the dyadic valuation on the complex numbers and recent results on symmetric bilinear forms over fields of characteristic 2.

The field $\mathbb{C}(x_1, x_2)$ of rational functions in two indeterminates over the field of complex numbers is known to be a C_2-field in the sense of Lang; see [Elman et al. 2008, Section 97]. It follows that every quadratic form in five variables over $\mathbb{C}(x_1, x_2)$ is isotropic, which implies that any two quaternion algebras over $\mathbb{C}(x_1, x_2)$ share a common maximal subfield; see [Lam 2005, Theorem X.4.20]. Fields with this property are said to be *linked*. It was noticed by Becher [2018] and by Chapman, Dolphin, and Leep [Chapman et al. 2018, Corollary 5.3] that the following stronger property holds: $\mathbb{C}(x_1, x_2)$ is 3-*linked* in the sense that any *three* quaternion algebras over $\mathbb{C}(x_1, x_2)$ share a common maximal subfield. In contrast, algebraic number fields are known to be m-linked for *every* integer m; this follows from Lenstra's proof that K_2 of global fields consists of symbols [Lenstra 1976, Proposition, p. 70]. We are indebted to an anonymous referee for the following short argument: a common maximal subfield of quaternion algebras $Q_1, \ldots, Q_m$ defined over a number field F is given by $F(\sqrt{d})$, where $d \in F^\times$ is a nonsquare in each of the completions F_p, where p runs through the finitely many primes that are either archimedean or dyadic, or where at least one of the Q_i is nonsplit. Comparison with the case of number fields suggests asking whether there exists an upper bound on the integer m for which $\mathbb{C}(x_1, x_2)$ is m-linked.

Theorem A. *The following quaternion algebras over $\mathbb{C}(x_1, x_2)$ do not share a common maximal subfield*:

$$(x_1, x_2), \qquad (x_1, x_2+1), \qquad (x_2, x_1+1), \qquad (x_2, x_1x_2+1).$$

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The arguments apply to a more general linkage question raised by Becher [2018]. Given a field F, the Witt ring WF of (Witt classes of) symmetric bilinear forms over F has a natural filtration by the powers of the maximal ideal IF of even-dimensional forms:

$$WF \supset IF \supset I^2F \supset \cdots .$$

Each I^nF is generated by (bilinear) n-fold Pfister forms, i.e., forms of the shape

$$\langle\langle \alpha_1, \dots, \alpha_n \rangle\rangle = \langle 1, -\alpha_1 \rangle \otimes \cdots \otimes \langle 1, -\alpha_n \rangle \quad \text{with } \alpha_1, \dots, \alpha_n \in F^\times.$$

For $m, n \geq 2$, we say that I^nF is *m-linked* if any m bilinear n-fold Pfister forms over F share a common $(n-1)$-fold factor. If $\mathrm{char}(F) \neq 2$, quadratic forms can be identified with their symmetric bilinear polar forms, and in particular the 2-fold Pfister forms are the norm forms of quaternion algebras. Hence F is m-linked in the sense discussed above if and only if I^2F is m-linked. Becher raised the following question:

Question [Becher 2018, Question 5.2]. Suppose I^nF is 3-linked for some $n \geq 2$. Does it follow that I^nF is m-linked for every $m \geq 3$?

This question was answered in the negative for fields F of $\mathrm{char}(F) = 2$ in [Chapman 2018]. In this note, we show how Becher's question can be answered also in the case of $\mathrm{char}(F) = 0$ using the main result of [Chapman 2018] on symmetric bilinear forms over fields of characteristic 2 and the existence of a dyadic valuation on $\mathbb{C}$:

Theorem B. *For $F = \mathbb{C}(x_1, \dots, x_n)$ with $n \geq 2$, I^nF is 3-linked but not 2^n-linked.*

Proofs

Notation 1. For a given integer $n \geq 2$, let $\mathbf{2}^n = \{0, 1\}^{\times n}$, and write $\mathbf{0} = (0, \dots, 0) \in \mathbf{2}^n$. Given a sequence $\alpha_1, \dots, \alpha_n$ in the multiplicative group of a field F and $\boldsymbol{d} = (d_1, \dots, d_n) \in \mathbf{2}^n$, let $\alpha^{\boldsymbol{d}} = \prod_{i=1}^n \alpha_i^{d_i} \in F^\times$. If $\boldsymbol{d} \neq \mathbf{0}$ and $1 + \alpha^{\boldsymbol{d}} \neq 0$, let

$$\varphi_{\boldsymbol{d}} = \langle\langle \alpha_1, \dots, \widehat{\alpha_\ell}, \dots, \alpha_n \rangle\rangle \otimes \langle\langle 1 + \alpha^{\boldsymbol{d}} \rangle\rangle,$$

where ℓ is the minimal index in $\{1, \dots, n\}$ for which $d_\ell \neq 0$, and let

$$\varphi_{\mathbf{0}} = \langle\langle \alpha_1, \dots, \alpha_n \rangle\rangle.$$

The following result is from [Chapman 2018, Theorem 3.3]:

Proposition 2. *Suppose* $\mathrm{char}(F) = 2$ *and* $\alpha_1, \dots, \alpha_n$ *are 2-independent in F, which means that $(\alpha^{\boldsymbol{d}})_{\boldsymbol{d} \in \mathbf{2}^n}$ is a linearly independent family in F viewed as an F^2-vector space. Then the forms $\varphi_{\boldsymbol{d}}$ for $\boldsymbol{d} \in \mathbf{2}^n$ are anisotropic and have no common 1-fold factor.*

The main result from which Theorems A and B derive is the following.

Proposition 3. *Let $F = k(x_1, \dots, x_n)$ be the field of rational functions in n indeterminates over an arbitrary field k of characteristic zero, for some $n \geq 2$. Let $\varphi_{\boldsymbol{d}}$ for $\boldsymbol{d} \in \mathbf{2}^n$ be the Pfister forms defined as in Notation 1 with the sequence $x_1, \dots, x_n$ for $\alpha_1, \dots, \alpha_n$. The forms $\varphi_{\boldsymbol{d}}$ do not have a common 1-fold factor.*

Proof. A theorem of Chevalley [Engler and Prestel 2005, Theorem 3.1.2] shows that the 2-adic valuation on $\mathbb{Q}$ extends to a valuation v_0 on k. Let $\bar{k}$ be the residue field of this valuation, which has characteristic 2. The valuation v_0 has a Gauss extension to a valuation v on F such that $v(x_i) = 0$ for $i = 1, \dots, n$ and $\overline{x_1}, \dots, \overline{x_n}$ are algebraically independent over $\bar{k}$; see [Engler and Prestel 2005, Corollary 2.2.2]. The residue field of v is thus $\overline{F} = \bar{k}(\overline{x_1}, \dots, \overline{x_n})$, a field of rational functions in n indeterminates over $\bar{k}$. Since the coefficients of the forms $\{\varphi_{\boldsymbol{d}} : \boldsymbol{d} \in \mathbf{2}^n\}$ are all of value 0, they have residue forms $\{\overline{\varphi}_{\boldsymbol{d}} : \boldsymbol{d} \in \mathbf{2}^n\}$, where the coefficients of $\overline{\varphi}_{\boldsymbol{d}}$ are the residues of the coefficients of $\varphi_{\boldsymbol{d}}$. The forms $\overline{\varphi}_{\boldsymbol{d}}$ are bilinear Pfister forms as defined in Notation 1, with the 2-independent sequence $\overline{x_1}, \dots, \overline{x_n}$ for $\alpha_1, \dots, \alpha_n$.

For $\boldsymbol{d} \in \mathbf{2}^n$, let $\boldsymbol{t_d} = (t_{1,\boldsymbol{d}}, \dots, t_{2^n-1,\boldsymbol{d}})$ be a $(2^n - 1)$-tuple of indeterminates. Suppose the bilinear forms $\varphi_{\boldsymbol{d}}$ have a common factor $\langle\langle\alpha\rangle\rangle$. Then the pure subforms $\varphi'_{\boldsymbol{d}}$ defined by the equation $\varphi_{\boldsymbol{d}} = \langle 1 \rangle \perp \varphi'_{\boldsymbol{d}}$ all represent $-\alpha$. Hence the system of equations

$$\varphi'_{\boldsymbol{d}}(\boldsymbol{t_d}, \boldsymbol{t_d}) = -\alpha \quad \text{for } \boldsymbol{d} \in \mathbf{2}^n$$

has a solution. We may therefore find nontrivial solutions to the system of equations

$$\varphi'_{\boldsymbol{d}}(\boldsymbol{t_d}, \boldsymbol{t_d}) = \varphi'_{\mathbf{0}}(\boldsymbol{t_0}, \boldsymbol{t_0}) \quad \text{for } \boldsymbol{d} \in \mathbf{2}^n \setminus \{\mathbf{0}\}.$$

Since these equations are homogeneous, upon scaling we may find solutions $(\boldsymbol{u_d})_{\boldsymbol{d} \in \mathbf{2}^n}$ such that

$$\min\{v(u_{i,\boldsymbol{d}}) \mid i = 1, \dots, 2^n - 1,\ \boldsymbol{d} \in \mathbf{2}^n\} = 0.$$

Taking residues, we obtain

$$\overline{\varphi}'_{\boldsymbol{d}}(\overline{\boldsymbol{u_d}}, \overline{\boldsymbol{u_d}}) = \overline{\varphi}'_{\mathbf{0}}(\overline{\boldsymbol{u_0}}, \overline{\boldsymbol{u_0}}) \quad \text{for } \boldsymbol{d} \in \mathbf{2}^n \setminus \{\mathbf{0}\}.$$

Since at least one $\overline{u_{i,\boldsymbol{d}}}$ is nonzero and the forms $\overline{\varphi}'_{\boldsymbol{d}}$ are anisotropic, it follows that these forms all represent some $\beta \in \overline{F}^{\times}$. Hence the forms $\overline{\varphi}_{\boldsymbol{d}}$ have a common factor $\langle\langle\beta\rangle\rangle$ by [Elman et al. 2008, Lemma 6.11]. This yields a contradiction to Proposition 2. □

Theorem A readily follows from Proposition 3 with $n = 2$ and $k = \mathbb{C}$, because the forms $\varphi_{\mathbf{0}}$, $\varphi_{(0,1)}$, $\varphi_{(1,0)}$, and $\varphi_{(1,1)}$ are the norm forms of the quaternion algebras (x_1, x_2), $(x_1, x_2 + 1)$, $(x_2, x_1 + 1)$, and $(x_2, x_1x_2 + 1)$, respectively.

Proof of Theorem B. The field $F = \mathbb{C}(x_1, \dots, x_n)$ is a C_n-field, and hence $F(t)$ is a C_{n+1}-field; see [Elman et al. 2008, Corollary 97.6]. In particular, $u(F(t)) = 2^{n+1}$, and it follows from [Becher 2018, Corollary 5.4] that I^nF is 3-linked. Apply

Proposition 3 with $k = \mathbb{C}$ to obtain a set of cardinality 2^n of n-fold Pfister forms that do not have a common 1-fold factor, and hence are not linked. □

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