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A Baum–Connes conjecture for singular foliations

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We consider singular foliations whose holonomy groupoid may be nicely decomposed using Lie groupoids (of unequal dimension). We construct a K-theory group and a natural assembly type morphism to the K-theory of the foliation C^*-algebra generalizing to the singular case the Baum–Connes assembly map. This map is shown to be an isomorphism under assumptions of amenability. We examine some simple examples that can be described in this way and make explicit computations of their K-theory.

Introduction

The celebrated Baum–Connes conjecture assigns to geometric objects (e.g., discrete groups, Lie groups, (regular) foliations, Lie groupoids) two K-groups and links them with a morphism, the “assembly map”. The “right-hand side” of the assembly map is the K-theory group of the C^*-algebra associated with the geometric object in hand. The other group, the “left-hand-side”, called the topological K-theory, arises from topological constructions associated with the geometric object in hand, such as classifying spaces.

Although this topological K-theory is often not much easier to calculate than the analytic one, constructing it and the assembly map is really important.

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- First of all, the topological K-groups are important and meaningful groups. In particular, they represent — up to torsion — the correct cohomology of the geometric object.
- Injectivity of the assembly map controls the topological K-theory by the analytic one. It thus has important topological consequences, as the homotopy invariance of higher signature, i.e., Novikov's conjecture and its generalizations to foliations [Baum and Connes 1985].
- Surjectivity controls the analytic K-theory by the topological one. It thus has important consequences like the Kadison–Kaplansky conjecture.
- Even its nonbijectivity has strong consequences by constructing secondary invariants of purely analytic type; see [Piazza and Schick 2007].

Foliations, and in particular singular ones, arise in an abundance of interesting mathematical problems, so the formulation of an assembly map is important in its own right. For instance, Poisson manifolds are completely determined by their symplectic foliation [Vaisman 1994]. In particular, regarding the Lie–Poisson structure [Vaisman 2000] associated with a nilpotent Lie group, formulating the Baum–Connes conjecture of the associated symplectic foliation might give a more insightful understanding of the orbit method [Kirillov 2004]. (In fact, Androulidakis and Higson have work in progress in this direction.)

Let $(M, \mathcal{F})$ be a singular Stefan–Sussmann foliation [Stefan 1974; Sussmann 1973]. We constructed its holonomy groupoid and the foliation C^*-algebra $C^*(M, \mathcal{F})$ in [Androulidakis and Skandalis 2009]. In [Androulidakis and Skandalis 2011a; 2011b] we showed that the K-theory of $C^*(M, \mathcal{F})$ is a receptacle for natural index problems along the leaves. It is then natural to look for a "left-hand side" too and try to construct the corresponding topological K-group and assembly map. In particular, this gives some insight into this K-theory. Of course we cannot hope in general for such a map to be an isomorphism (since it is not always an isomorphism in the regular case, as shown in [Higson et al. 2002]), and it is even hard to believe that the topological K-group could be defined for every kind of singular foliation. However, in this paper we manage to construct such a map for a quite general class of singular foliations.

0.A. ***Some examples.*** In order to formulate the assembly map, let us examine a few natural and quite simple examples. Consider the foliation given by a smooth action of a connected Lie group on a manifold M:

(a) the action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$;

(b) the action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$;

(c) any action of $\mathbb{R}$ (given by a vector field X).

In these three cases, we can compute the K-theory thanks to an exact sequence

$$0 \to C^*(\Omega_0, \mathcal{F}_{|\Omega_0}) \to C^*(M, \mathcal{F}) \to C^*(M, \mathcal{F})|_{Y_1} \to 0.$$

Here Ω_0 corresponds to "most regular points" of the foliation (more precisely, the place where the source fibers of the foliation groupoid are of lowest dimension) and $Y_1 = M \setminus \Omega_0$: in example (a), $\Omega_0 = \mathbb{R}^3 \setminus \{0\}$, in example (b), $\Omega_0 = \mathbb{R}^2 \setminus \{0\}$ and in example (c), Ω_0 is the interior of the set of points where X vanishes.

In these examples, the connecting map ∂ of the K-theory exact sequence is easily computed and we can describe precisely $K_*(C^*(M, \mathcal{F}))$.

In other examples that we discuss here, the "regularity" of points varies even more. For instance:

(d) The action on $\mathbb{R}^n$ of a parabolic subgroup G of $\mathrm{GL}(n, \mathbb{R})$; e.g., the minimal parabolic subgroup of upper triangular matrices.

(e) The action of $PG = G/\mathbb{R}^*$ on $\mathbb{R}P^{n-1}$.

(f) The action of $G \times G$ by left and right multiplication on $\mathrm{GL}_n(\mathbb{R})$. (Orbits give the well-known Bruhat decomposition.)

In the last three cases, the computation becomes harder since we obtain a longer sequence of ideals — and therefore spectral sequences instead of short exact sequences. We do not explicitly compute the K-theory in these cases. On the other hand, in all cases, the holonomy groupoid nicely decomposes in locally closed subsets where the source fibers have fixed dimension. We use this decomposition in order to construct the topological K-group and the assembly map.

0.B. ***Nicely decomposable foliations and the height of a nice decomposition.*** Let $(M, \mathcal{F})$ be a singular foliation. Its holonomy groupoid may be very singular. On the other hand, this singularity gives rise to open subsets which are *saturated* for $\mathcal{F}$ (i.e., a union of leaves of $\mathcal{F}$). We thus obtain ideals of $C^*(M, \mathcal{F})$ that we may use to compute the K-theory.

For instance, recall that the source fibers of the holonomy groupoid of the foliation as defined in [Androulidakis and Skandalis 2009] were shown in [Debord 2013] to be smooth manifolds. On the other hand, the dimension of these manifolds varies. Let us denote by $\ell_0 < \ell_1 < \cdots < \ell_k$ the various dimensions occurring (note that k may be infinite, as shown in [Androulidakis and Zambon 2013]). Let Ω_j denote the set of points with source fiber dimension $\leq \ell_j$. We find an ascending sequence $\Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega_{k-1} \subseteq \Omega_k = M$ of saturated open subsets of M. This decomposition yields a sequence of two-sided ideals $J_j = C^*(\Omega_j, \mathcal{F}_{|\Omega_j})$ of $C^*(M, \mathcal{F})$. The quotient C^*-algebra J_j/J_{j-1} is the C^*-algebra of the restriction of the holonomy groupoid $H(\mathcal{F})$ to the locally closed saturated set $Y_j = \Omega_j \setminus \Omega_{j-1}$. The module $\mathcal{F}$, when restricted to Y_j, is finitely generated and projective, and the restriction of $H(\mathcal{F})$ to Y_j is a Lie groupoid (when Y_j is a submanifold) so that we may expect a Baum–Connes map for it.

Our computation of the K-theory is based on an ingredient which we add as an extra assumption (it is satisfied in the above examples).

Let $(M, \mathcal{F})$ a singular foliation. We say that $(M, \mathcal{F})$ is *nicely decomposable with height* k if there is a cover of M by open subsets $(W_j)_{j\in\mathbb{N},\, j\le k}$, such that for every $j \in \mathbb{N}$ with $j \le k$, the restriction of the foliation $\mathcal{F}$ to each W_j is defined by a Hausdorff Lie groupoid $\mathcal{G}_j$, the open subset $\Omega_j = \bigcup_{i\le j} W_i$ is saturated and $\mathcal{G}_j$ coincides with the holonomy groupoid $H(\mathcal{F})$ on the (locally closed) set $Y_j = \Omega_j \setminus \Omega_{j-1}$ (we set $Y_0 = W_0 = \Omega_0$). Moreover, we assume that the groupoids $\mathcal{G}_j$ are linked via morphisms which are submersions

$$\mathcal{G}_j|_{\Omega_{j-1}\cap W_j} \to \mathcal{G}_{j-1}.$$

If $(M, \mathcal{F})$ is nicely decomposable, the quotients J_j/J_{j-1} are given by (restriction to closed sets of) Lie groupoids, for which a Baum–Connes conjecture does exist. This makes the calculation of the K-theory of $C^*(M, \mathcal{F})$ possible, at least in terms of a spectral sequence.

- Singularity height 0 corresponds to foliations whose holonomy groupoid is a Lie groupoid, and there already is a topological K-theory and a Baum–Connes assembly map for Lie groupoids; see [Tu 2000].
- Examples (a), (b), (c) are all of singularity height 1. We will use the decomposition given by the dimensions of the fibers. In examples (a) and (b), the dimensions of the fibers are $\ell_0 = 2$ and $\ell_1 = 3$; in example (c), these dimensions are $\ell_0 = 0$ and $\ell_1 = 1$. For the singularity height 1 case, the topological K-theory can be constructed using the exact sequence of C^*-algebras and a mapping cone construction.
- A new difficulty in the construction of the topological K-theory arises when we have higher singularity height, as in examples (d), (e) and (f). We use here a telescope construction.

0.C. ***The topological K-theory and the assembly map.*** We construct the topological K-theory and the assembly map in two steps:

- The first step consists of replacing the holonomy groupoid $H(\mathcal{F})$ by a slightly more regular one G whose (full) C^*-algebra is E-equivalent to the foliation one. This groupoid is constructed via a mapping cone construction in the height 1 case and via a telescope construction in the higher singularity case.
- In the second step we construct a topological K-theory and the assembly map for the "telescopic groupoid" G which is the K-theory of a proper G-algebra in a generalized sense, together with a Dirac type construction.

0.C.1. *A telescopic construction.*

A mapping cone construction in the height 1 case. Let us explain our strategy more explicitly in the case of a foliation admitting a singularity height 1 decomposition. In this case, we obtain a diagram of *full* C^*-algebras (with $\mathcal{G} = \mathcal{G}_1$):

$$\begin{tikzcd} 0 \arrow[r] & C^*(\mathcal{G}_{\Omega_0}) \arrow[r, "i_{\mathcal{G}}"] \arrow[d, "\pi_{\Omega_0}"'] & C^*(\mathcal{G}) \arrow[r] \arrow[d, "\pi"'] & C^*(\mathcal{G}_{Y_1}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C^*(\Omega_0, \mathcal{F}_{|\Omega_0}) \arrow[r, "i_{\mathcal{F}}"] & C^*(M, \mathcal{F}) \arrow[r] & C^*(M, \mathcal{F})|_{Y_1} \arrow[r] & 0 \end{tikzcd}$$

The singularity height 1 assumption means that the holonomy groupoid of the restriction $\mathcal{F}_{|\Omega_0}$ of the foliation $\mathcal{F}$ to Ω_0 is a Lie groupoid $\mathcal{G}_0$ and $C^*(\Omega_0, \mathcal{F}_{|\Omega_0}) = C^*(\mathcal{G}_0)$. The lines of this diagram are exact at the level of *full* C^*-algebras.

Since $\mathcal{G}$ defines $\mathcal{F}$, it is an atlas in the sense of [Androulidakis and Skandalis 2009], so $H(\mathcal{F})$ is a quotient of $\mathcal{G}$. Hence the two extensions are connected by the map π and its restriction π_{Ω_0}, which is integration along the fibers of this quotient map $\mathcal{G} \to H(\mathcal{F})$. From this diagram, we conclude that the algebra $C^*(M, \mathcal{F})$ is equivalent in E-theory (up to a shift of degree) with the mapping cone of the morphism

$$(i_{\mathcal{G}}, \pi_{\Omega_0}) : C^*(\mathcal{G}_{\Omega_0}) \to C^*(\mathcal{G}) \oplus C^*(\Omega_0, \mathcal{F}_{|\Omega_0}).$$

Foliations of height ≥ 2. As far as singular foliations with nice decompositions of arbitrary (bounded or not) singularity height are concerned, we show that the strategy developed for the singularity height 1 case can be generalized. In particular, $C^*(M, \mathcal{F})$ is E-equivalent to a "telescopic" C^*-algebra whose components are Lie groupoids. In fact, we see that these telescopes can just be treated as mapping cones.

Now let us see how the above apparatus can be used to formulate the Baum–Connes assembly map for singular foliations. It suffices to explain the idea for the height 1 case.

Longitudinally smooth groupoids. The above mapping cone and the telescopic algebra constructed here are based on morphisms of Lie groupoids which are smooth submersions and open inclusions at the level of objects. These C^*-algebras are immediately seen to be the C^*-algebras of a kind of groupoids which generalize both Lie groupoids and singular foliation groupoids: *longitudinally smooth groupoids.*

0.C.2. *A topological K-theory group for the telescopic groupoid.*

Setting of the problem. Before we outline our construction of a topological K-theory group, let us make a remark. Recall that Jean-Louis Tu [2000] defined a topological K-theory group and a Baum–Connes morphism for Lie groupoid C^*-algebras of the form $K_*^{\mathrm{top}}(\mathcal{G}) \to K_*(C^*(\mathcal{G}))$. In order to construct a topological

K-theory group for this mapping cone, we need to find a "left-hand side" for the morphism $(i_{\mathcal{G}}, \pi_{\Omega_0})$. In fact we not only need it as a morphism at the level of groups K_*^{top}, but we really need to construct it as a KK-element.

The difficulty lies with the understanding of the topological K-theory of the mapping cone of the surjective homomorphism $\pi_{\Omega_0} : C^*(\mathcal{G}_{\Omega_0}) \to C^*(\Omega_0, \mathcal{F}_{|\Omega_0})$. We treat this by deploying the Baum–Douglas formulation given in [Baum and Connes 2000; Baum and Douglas 1982a; 1982b]. At this point we will need further assumptions on the groupoids $\mathcal{G}$ and $\mathcal{G}_1$, namely that their classifying spaces of proper actions are smooth manifolds, to make sure that the Baum–Connes morphisms are naturally given by KK-elements. (In the Appendix we show how this assumption can be weakened.)

Actions of the telescopic groupoid. In order to define the topological K-theory group for the telescopic groupoid, we follow the Lie groupoid case:

• For every longitudinally smooth groupoid G, one defines G-algebras very much in the spirit of [Androulidakis and Skandalis 2009]: algebraic conditions are stated at the level of the groupoid, topological ones at the level of bisubmersions which can be thought of as "smooth local covers" of G (cf. [Androulidakis and Skandalis 2009]). We define the (full and reduced) crossed product for every G-algebra.

• One may define a generalized notion of *"proper G-algebra"*: a G-algebra is said to be "proper" if its restriction to the groupoids corresponding to the various strata is proper in the usual sense. In particular, one may define actions on spaces and "proper" actions on spaces. Of course, they are not proper in the usual sense! But from the point of view of the Baum–Connes conjecture they are as good, since the Baum–Connes conjecture is compatible with extensions (in the amenable case).

• We define Le Gall's equivariant KK-theory [1999] in the context of longitudinally smooth G-algebras, despite the topological pathology of the holonomy groupoid G. We extend the equivariant Kasparov product to this case.

• We may then construct the topological K-theory group and the assembly map for the telescopic algebras of a nice decomposition of a singular foliation. To that end we still need to assume for $(M, \mathcal{F})$ that the Lie groupoids of its decomposition admit smooth manifolds as classifying spaces for proper actions.

• Actually, this point of view allows one to construct a Baum–Connes map *with coefficients* for every G algebra. It is easily seen that, in the case of nicely decomposable foliations, our Baum–Connes map with coefficients in "proper" spaces or algebras is an isomorphism.

The main result. We show then that in cases as above the Baum–Connes map can be constructed canonically. Namely, we prove the following:

Theorem 0.1. (i) *If $(M, \mathcal{F})$ admits a nice decomposition by Lie groupoids whose classifying space for proper actions is a manifold, then there is a well-defined topological K-group and one may construct a Baum–Connes assembly map.*

(ii) *If moreover the groupoids of the nice decomposition are amenable and Hausdorff, then the Baum–Connes map is an isomorphism.*

Note that examples (a) and (c) above are amenable; although example (b) is not, it is "strongly K-amenable" and the Baum–Connes conjecture (for the *full* version) holds for it.

Note also that example (c) is not exactly covered by our theorem since the groupoid $\mathcal{G}_0$ is not assumed to be Hausdorff. However, the Baum–Connes conjecture holds also in this case

For the examples of larger singularity height described in examples (d), (e) and (f), note that, as the minimal parabolic subgroup of $\mathrm{GL}(n, \mathbb{R})$ is amenable, Theorem 0.1 implies that the Baum–Connes conjecture holds.

Let us point out that our constructions of the equivariant KK-theory could in a way be bypassed, but may have its own interest. In particular, we give a simple quite general formulation and proof for the existence of the Kasparov product, which applies in all known equivariant contexts: groups, group actions [Kasparov 1988], groupoids [Le Gall 1999], Hopf algebras [Baaj and Skandalis 1989].

Trying to weaken our assumptions. The assumption on the classifying spaces is quite natural. All the groupoids given by Lie group actions admit manifolds as classifying spaces for proper actions, and this assumption is stable by Morita equivalence. In this way it is satisfied by all the (Hausdorff) groupoids that appear in the examples that we discuss in this work. Nevertheless, it is quite tempting to try to get rid of it. In the Appendix we explain how it can be replaced by a quite weaker, rather technical one: Assumption A.1, which could be true in general, i.e., for every longitudinally smooth groupoid.

Structure of the paper.

- In Section 1 we introduce the notion of singularity height for a singular foliation and define nicely decomposable foliations. We also explain the examples mentioned in the beginning of this introduction.

- Section 2 focuses on nicely decomposable foliations with singularity height 1. We give the construction of the associated mapping cone C^*-algebra and prove that it is E-equivalent to the foliation C^*-algebra. We give there the explicit calculation of the K-theory for examples (a), (b) and (c).

- In Section 3 we extend this construction and result to foliations of arbitrary singularity height, replacing mapping cones with telescopes.

• Section 4 defines longitudinally smooth groupoids and their actions and constructs the associated KK-theory.

• The crucial section is Section 5, where we formulate the Baum–Connes conjecture (topological K-theory and Baum–Connes map) for the telescopic algebra, assuming the classifying spaces of proper actions of the groupoids associated with the nice decomposition of $(M, \mathcal{F})$ are smooth manifolds. The proof of Theorem 0.1 can be found there.

• Finally, in the Appendix we explain how to remove the assumption that the classifying spaces of proper actions are smooth manifolds.

Notation 0.2. Let $(M, \mathcal{F})$ be a foliation. We denote the (minimal, i.e., the groupoid associated with the path holonomy atlas — cf. [Androulidakis and Skandalis 2009]) holonomy groupoid by $H(\mathcal{F})$ (or $H(M, \mathcal{F})$ when needed). We denote by $C^*(M, \mathcal{F})$ and $C^*_{\mathrm{red}}(M, \mathcal{F})$ its *full* and *reduced* C^*-algebras.

We mainly use the *full* C^*-algebra. This is justified by the two following reasons:

• Constructing a Baum–Connes map for the full foliation algebra automatically gives the one for the reduced version. Recall that the Baum–Connes map, in the regular case, factors through the full version of the foliation algebra.

• All our constructions are based on sequences of groupoid C^*-algebras, which are always exact at the full C^*-algebra level, and may fail to be exact at the reduced level (see Section 2.B).

1. Nicely decomposable foliations

1.A. ***Notations and remarks.*** Let M be a smooth manifold and $\mathcal{X}_c(M)$ the $C^\infty(M)$-module of compactly supported vector fields. In [Androulidakis and Skandalis 2009], we defined a singular foliation on M to be a $C^\infty(M)$-submodule $\mathcal{F}$ of $\mathcal{X}_c(M)$ which is locally finitely generated and satisfies $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$.

Given a point $x \in M$ let $I_x = \{f \in C^\infty(M) : f(x) = 0\}$ and recall from [Androulidakis and Skandalis 2009] the fiber $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. The map $M \ni x \mapsto \dim(\mathcal{F}_x)$ is upper semicontinuous [Androulidakis and Skandalis 2009, Proposition 1.5].

When this dimension is constant (continuous if M is not assumed to be connected), i.e., when the module $\mathcal{F}$ is projective, the foliation is said to be *almost regular* and the holonomy groupoid $H(\mathcal{F})$ was proved to be a Lie groupoid in [Debord 2001].

In the present paper, we deal with cases where the dimension of $\mathcal{F}_x$ is not constant. The number of possible dimensions measures the singularity of the foliation. We give a definition of this *singularity height* more appropriate for our purposes in Definition 1.4.

By semicontinuity, the subsets $O_\ell = \{x \in M : \dim(\mathcal{F}_x) \le \ell\}$ are open. They are *saturated*, i.e., unions of leaves of $\mathcal{F}$.

We deal with restrictions of the foliation to open sets. We use the following remark:

Remark 1.1. Let $(M, \mathcal{F})$ be a foliation. Let V be an open subset of M.

(i) The holonomy groupoid of the restriction $\mathcal{F}_{|V}$ to V is the s-connected component of the restriction $H(\mathcal{F})_V^V = \{z \in H(\mathcal{F}) : t(z) \in V \text{ and } s(z) \in V\}$ to V.

(ii) If V is saturated, then $H(\mathcal{F}_{|V}) = H(\mathcal{F})_V^V$.

Actually an analogous statement holds for the pull-back foliation $f^{-1}(\mathcal{F})$ by a smooth map $f : V \to M$ *transverse to* $\mathcal{F}$ [Androulidakis and Skandalis 2009, §1.2.3]: $H(f^{-1}(\mathcal{F}))$ is the s-connected component of

$$H(\mathcal{F})_f^f = \{(v, z, w) \in V \times H(\mathcal{F}) \times V : t(z) = f(v) \text{ and } s(z) = f(w)\}.$$

If moreover f is a submersion whose image is saturated with connected fibers, then $H(f^{-1}(\mathcal{F})) = H(\mathcal{F})_f^f$.

Now let us discuss the notation for C^*-algebras used in the sequel as far as restrictions are concerned. If $\mathcal{G}$ is a locally compact groupoid (with Haar measure) and Y is a locally closed saturated subset of $\mathcal{G}_0$, then $\mathcal{G}_Y = \{x \in \mathcal{G} : s(x) \in Y\}$ is also a locally closed groupoid and we can define its C^*-algebra. We put $C^*(\mathcal{G})_{|Y} = C^*(\mathcal{G}_Y)$. The same construction for foliation algebras is useful in our context:

Notation 1.2. Let $(M, \mathcal{F})$ be a (singular) foliation.

(a) Let $\Omega \subset M$ be a *saturated* open subset. Then

$$C^*(M, \mathcal{F})|_\Omega := C_0(\Omega) C^*(M, \mathcal{F}) = C^*(\Omega, \mathcal{F}_{|\Omega})$$

is the foliation C^*-algebra of the restriction of $\mathcal{F}$ to Ω. The same holds for the reduced C^*-algebras.

(b) If $Y \subset M$ is a saturated closed subset then the *full* $C^*(M, \mathcal{F})|_Y$ is the quotient of $C^*(M, \mathcal{F})$ by $C^*(M, \mathcal{F})|_{M \setminus Y}$.

Note that the natural definition for the reduced one is to take the quotient of $C^*(M, \mathcal{F})$ corresponding to the regular representations at points of Y, i.e., the representations on $L^2(H(M, \mathcal{F})_y)$ for $y \in Y$.

(c) If $Y \subset M$ is a saturated locally closed subset then Y is open in its closure $\overline{Y}$ and the closed subset $\overline{Y} \setminus Y$ is saturated. Let $U = M \setminus (\overline{Y} \setminus Y)$. We denote by $C^*(M, \mathcal{F})|_Y$ the quotient of $C_0(U) C^*(M, \mathcal{F})$ by $C^*(M, \mathcal{F})|_{M \setminus \overline{Y}}$. In other words, $C^*(M, \mathcal{F})_{|Y} = (C^*(M, \mathcal{F})_{|U})_{|Y}$.

1.B. ***Foliations associated with Lie groupoids.*** In the sequel we consider foliations defined from Lie groupoids (at least locally — cf. Section 1.C). Let us make a few observations regarding singular foliations defined by Lie groupoids.

Every Lie algebroid A with base M, and thus every Lie groupoid $(t,s):\mathcal{G}\rightrightarrows M$, defines a foliation. Indeed, the anchor map $\sharp : A \to TM$ is a morphism of Lie algebroids, whence $\sharp(\Gamma_c A) \subset \mathcal{X}_c(M)$ is a singular foliation.

Let $\mathcal{G}$ be a (locally Hausdorff) Lie groupoid over a manifold M and $\mathcal{F}$ the associated foliation. Up to replacing $\mathcal{G}$ by its s-connected component (which is an open subgroupoid of $\mathcal{G}$ with the same algebroid, and thus defines the same foliation on M) we may assume that $\mathcal{G}$ is s-connected, i.e., the fibers of the source map $s:\mathcal{G}\to M$ are connected. Then the groupoid $\mathcal{G}$ is an atlas for our foliation, in the sense of [Androulidakis and Skandalis 2009, Definition 3.1]. As $\mathcal{G}$ is assumed s-connected, it defines the path holonomy atlas [Androulidakis and Skandalis 2009, Example 3.4.3]. The holonomy groupoid $H(M,\mathcal{F})$ is a quotient of $\mathcal{G}$ by the equivalence relation defined in [Androulidakis and Skandalis 2009, Proposition 3.4.2]. Put $q:\mathcal{G}\to H(M,\mathcal{F})$ the associated quotient map.

In order to compute this quotient, we use a lemma from [Androulidakis and Zambon 2013].

Let $\gamma\in\mathcal{G}$ and write $x=s(\gamma)$. Note that if $q(\gamma)$ is a unit, then $t(\gamma)=x$. Choosing a bisection through γ we obtain a local diffeomorphism g of M which acts on the tangent bundle T_xM and fixes the tangent to the leaf F_x. It therefore acts on $N_x=T_xM/F_x$. This action only depends on γ. Denote it by $\nu(\gamma)\in \mathrm{GL}(N_x)$.

Now, it was shown in [Androulidakis and Zambon 2013] that there is an action of $H(\mathcal{F})$ on this "bundle" of normal spaces. As an immediate consequence, we find the following:

Lemma 1.3. *If $q(\gamma)$ is a unit, then $\nu(\gamma)=\mathrm{id}_{N_x}$.* □

1.C. ***Nicely decomposable foliations.*** We now present the constraints that we put on our foliations. We say that the foliation is *nicely decomposable* if it admits a nice decomposition in the following sense.

Definition 1.4. Let $(M,\mathcal{F})$ be a singular foliation and let $k\in\mathbb{N}\cup\{+\infty\}$. A *nice decomposition* of $(M,\mathcal{F})$ of *singularity height k* is given by

(a) a sequence $(W_j)_{0\le j<k+1}$ of open sets of M such that the open set $\Omega_j=\bigcup_{\ell\le j}W_\ell$ is saturated and $\bigcup_{j<k+1}W_j=M$ (with the convention $+\infty+1=+\infty$);

(b) a sequence of Lie groupoids $\mathcal{G}_j \rightrightarrows W_j$ defining the restriction of $\mathcal{F}$ to W_j, and such that $\mathcal{G}_j|_{Y_j}=H(\mathcal{F})|_{Y_j}$, where $Y_0=\Omega_0$ and, for $j\ge 1$, $Y_j=\Omega_j\setminus\Omega_{j-1}$;

(c) morphisms of Lie groupoids $q_j:\mathcal{G}_j|_{\Omega_{j-1}\cap W_j}\to\mathcal{G}_{j-1}$ (for $j>0$) which are submersions, and which at the level of objects are just the inclusion $\Omega_{j-1}\cap W_j\to W_{j-1}$.

Remarks 1.5. (a) If $(M, \mathcal{F})$ is an almost regular foliation then $H(\mathcal{F})$ is a Lie groupoid as shown in [Androulidakis and Skandalis 2009] (it coincides with the one constructed in [Debord 2001]). In our current context, the decomposition sequence of such a foliation has singularity height zero; its realization is $H(\mathcal{F})$ itself. We will not be concerned with such situations in the sequel. Truly singular examples of nicely decomposable singular foliations arise when the singularity height of the decomposition is 1 or larger.

(b) By definition $W_0 = \Omega_0$ and the restriction of $H(\mathcal{F})$ coincides with $\mathcal{G}_0$. It follows that the restriction of $\mathcal{F}$ to Ω_0 is almost regular, which means that Ω_0 is contained in the (open) set of points where $\dim \mathcal{F}$ is continuous and, since $\dim \mathcal{F}$ is upper semicontinuous, these are the places where it has a local minimum.

(c) Such a decomposition need not be unique. In all our examples, $W_j = \Omega_j$ and Ω_j may be constructed using the dimension of the fibers.

For $\ell \in \mathbb{N}$, put

$$O_\ell = \{x \in M : \dim(\mathcal{F}_x) \leq \ell\}.$$

Denote by $\ell_0 < \ell_1 < \cdots < \ell_j$ for $j < k+1$ the various possible dimensions. For $j = 0, 1, \ldots, k$ put $\Omega_j = O_{\ell_j}$.

Note that an example is given in [Androulidakis and Zambon 2013] of a foliation where this k is infinite.

1.D. ***Examples of nicely decomposable foliations.*** We now give a few examples of nice decompositions of foliations.

1.D.1. *Examples of height 1.*

Remark 1.6. In the case of height 1, we have $W_0 = \Omega_0$ and $\mathcal{G}_0$ is the holonomy groupoid of the restriction of $\mathcal{F}$ to Ω_0. We therefore just need to specify the set Ω_0 and the Lie groupoid $\mathcal{G}_1 \rightrightarrows W_1$ defining the foliation $\mathcal{F}$ on an open subset W_1 containing the complement $Y_1 = M \setminus \Omega_0$ of Ω_0 and such that the restriction of $\mathcal{G}$ to Y_1 coincides with that of $H(\mathcal{F})$.

Actually, in our examples $W_1 = M$.

Examples 1.7. We give here examples of singularity height 1 associated with Lie group actions. Some examples of larger singularity height are computed in forthcoming work of Androulidakis and Higson. In this paper, we calculate the associated K-theory explicitly for the following examples.

(a) Let $M = \mathbb{R}^3$ and consider the foliation $\mathcal{F}$ defined by the image of the (infinitesimal) action of SO(3) on $\mathbb{R}^3$ by rotations. The leaves are concentric spheres in $\mathbb{R}^3$ with one singularity at $\{0\}$. Let $\mathcal{G}$ be the action groupoid $\mathbb{R}^3 \rtimes \mathrm{SO}(3) \rightrightarrows \mathbb{R}^3$. Since SO(3) is simple, the restriction of $H(\mathcal{F})$ to 0, which is a quotient of SO(3), has to be SO(3) (we may also use Lemma 1.3 to prove this result). The restriction of $\mathcal{F}$

to $\mathbb{R}^3 \setminus \{0\}$ is really a regular foliation — and in fact the fibration $S^2 \times \mathbb{R}_+^* \to \mathbb{R}_+^*$, whence the holonomy groupoid of $\mathcal{F}$, is

$$H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}_+^*) \cup \{0\} \times \mathrm{SO}(3).$$

It follows that the foliation has a nice decomposition of singularity height 1, namely $W_1 = \mathbb{R}^3$, $\mathcal{G}_1 = \mathcal{G}$ and $\Omega_0 = \mathbb{R}^3 \setminus \{0\}$.

(b) Let $M = \mathbb{R}^2$ and consider the action of $\mathrm{SL}(2, \mathbb{R})$. It has two leaves, namely $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. Using again Lemma 1.3, the associated holonomy groupoid is seen to be

$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2 \setminus \{0\}) \cup \{0\} \times \mathrm{SL}(2, \mathbb{R}).$$

Considering the action groupoid $\mathcal{G} = \mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R})$, we obtain the singularity height 1 nice decomposition $\Omega_1 = \mathbb{R}^2$, $\mathcal{G}_1 = \mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ and $\Omega_0 = \mathbb{R}^2 \setminus \{0\}$, $\mathcal{G}_0 = \Omega_0 \times \Omega_0$.

(c) There are many singular foliations of singularity height 1 arising from group actions which have nice decompositions. For instance, take $n \geq 4$ instead of 3 in example (a) or $n \geq 3$ instead of 2 in example (b).

We may also consider the action of $\mathrm{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$. The associated holonomy groupoid is

$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2 \setminus \{0\}) \cup \{0\} \times \mathrm{GL}^+(2, \mathbb{R}),$$

where $\mathrm{GL}^+(2, \mathbb{R})$ denotes $2{\times}2$ matrices with positive determinant. Considering the action groupoid $\mathcal{G} = \mathbb{R}^2 \rtimes \mathrm{GL}^+(2, \mathbb{R})$, we obtain $\Omega_1 = \mathbb{R}^2$ and $\Omega_0 = \mathbb{R}^2 \setminus \{0\}$. We can of course replace 2 by n also in this situation.

Another example as such comes from the action of $\mathrm{SL}(n, \mathbb{C})$ on $\mathbb{C}^n$. Its holonomy groupoid is

$$H(\mathcal{F}) = (\mathbb{C}^n \setminus \{0\} \times \mathbb{C}^n \setminus \{0\}) \cup \{0\} \times \mathrm{SL}(n, \mathbb{C}).$$

Considering the action groupoid $\mathcal{G} = \mathbb{C}^n \rtimes \mathrm{SL}(n, \mathbb{C})$ we have $\Omega_1 = \mathbb{C}^n$, $\Omega_0 = \mathbb{C}^n \setminus \{0\}$.

(d) We end with an example of a quite different flavor.

Let M be a manifold endowed with a smooth action α of $\mathbb{R}$. Let $\mathcal{G}_1 = M \rtimes_\alpha \mathbb{R}$ be the associated action groupoid, and $\mathcal{F}$ the associated foliation.

Denote by $\mathrm{Fix}(\alpha)$ the set of fixed points of α, by $W = \mathrm{Int}(\mathrm{Fix}(\alpha))$ its interior and by $V = M \setminus \mathrm{Fix}(\alpha)$ its complement. Let $x \in M$.

- If $x \in W$, then $\mathcal{F}_x = 0$.
- For $x \in V$, the dimension of $\mathcal{F}_x$ is 1. By semicontinuity, $\dim \mathcal{F}_x = 1$ for $x \in \overline{V}$.

Let Ω_0 be the set of continuity points of $\dim \mathcal{F}$. Its complement Y_1 is the boundary ∂W of W. The restriction of $\mathcal{F}$ to the open set Ω_0 is almost regular.

We show that the morphism $M \rtimes_\alpha \mathbb{R} \to H(\mathcal{F})$ is injective over Y_1. We thus have a nice decomposition $H(\Omega_0, \mathcal{F}_{|\Omega_0}) \rightrightarrows \Omega_0$, and $M \rtimes_\alpha \mathbb{R} \rightrightarrows M$.

This is done using classical facts based on the period bounding lemma (see [Abraham and Robbin 1967]), which we recall here:

Lemma 1.8 (period bounding). *Let X be a compactly supported C^r-vector field on a C^r-manifold M with $r \geq 2$. There is a real number $\eta > 0$ such that, for any $x \in M$, either $X(x) = 0$ or the prime period τ_x of the integral curve of X passing through x is $\tau_x > \eta$.* □

Put $P = \{(x, u) \in M \times \mathbb{R} : \alpha_u(x) = x\}$. It is obviously a closed subset of $M \times \mathbb{R}$ and the restrictions of the source and target maps to P coincide. By definition of the holonomy groupoid, an element $(x, u) \in \mathcal{G}_1 = M \rtimes \mathbb{R}$ is a trivial element in $H(\mathcal{F})$ if and only if there is an identity bisection through it, i.e., if there exists an open neighborhood U of x and a smooth function $f : U \to \mathbb{R}$ such that $f(x) = u$ and $(z, f(z)) \in P$ for all $z \in U$.

Let $\mathrm{Per}(\alpha)$ be the set of *stably periodic points*, i.e., the set of $x \in M$ such that there exists an open neighborhood U of x and a smooth function $f : U \to \mathbb{R}^*$ such that $(y, f(y)) \in P$ for all $y \in U$. It is the set of $x \in M$ such that

$$\{(x, u) : u \in \mathbb{R}\} \to H(\mathcal{F})$$

is not injective.

Obviously $W \subseteq \mathrm{Per}(\alpha)$.

Proposition 1.9. *The set $Y_1 \cap \mathrm{Per}(\alpha)$ is empty.*

Proof. Let $x \in \overline{W} \cap \mathrm{Per}(\alpha)$. We need to show that $x \notin Y_1$, i.e., that $x \in W$. Up to changing X far from x, we may assume that X has compact support.

Since $x \in \overline{W}$, it follows that X as well as all its derivatives vanish at x. We may then write $X = qY$, where q is a smooth nonnegative function such that $q(x) = 0$ and Y is a smooth vector field with compact support (take for instance q to be a smooth function which coincides near x to the square of the distance to x for some riemannian metric). Let then U be an open relatively compact neighborhood of x and $f : U \to \mathbb{R}^*$ a smooth bounded function such that $(y, f(y)) \in P$ for all $y \in U$. It follows that all the points in U are periodic for X and therefore for Y. When $y \to x$, $f(y) \to f(x)$, so the Y period of y tends to 0. By the period bounding lemma, it follows that any y close enough to x satisfies $Y(y) = 0$, whence $x \in W$. □

It follows that $(H(\Omega_0) \rightrightarrows \Omega_0, M \rtimes_\alpha \mathbb{R} \rightrightarrows M)$ is a nice decomposition for $\mathcal{F}$.

It is worth noticing that the holonomy groupoid $\mathcal{G}_0 = H(\Omega_0, \mathcal{F}_{|\Omega_0})$ is a disjoint union of clopen subgroupoids $W \sqcup H(V', \mathcal{F}_{|V'})$, where V' is the interior of $\overline{V}$, and that its C^*-algebra $C^*(\Omega_0, \mathcal{F}_{|\Omega_0})$ is a direct sum $C_0(W) \oplus C^*(V', \mathcal{F}_{|V'})$.

Note that, in the presence of periodic points, the groupoid $H(V, \mathcal{F}_{|V})$ and therefore $\mathcal{G}_0$ need not be Hausdorff.

Let us also remark that, in the computation above, we could as well have chosen to take Ω_0 to be the set where the fibers are of dimension 0, i.e., the set W.

1.D.2. *An example of larger singularity height.* We start by giving a natural family of examples of nicely decomposable foliations with singularity height larger than 1. Some of them will be studied in forthcoming work of Androulidakis and Higson.

If a subgroup $G \subset \mathrm{GL}_n(\mathbb{R})$ has more than two orbits in its action on $\mathbb{R}^n$, then the transformation groupoid $\mathbb{R}^n \rtimes G$ may give rise to interesting nicely decomposable foliations of singularity height ≥ 2.

A typical example is given by a parabolic subgroup of $\mathrm{GL}(n, \mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$: given a flag $\{0\} = E_k \subset E_{k-1} \subset E_{k-2} \subset \cdots \subset E_1 \subset E_0 = \mathbb{K}^n$ (with $k \leq n$ and E_k pairwise different), let G be the group of (positive if $\mathbb{K} = \mathbb{R}$) automorphisms of this flag, i.e., G is the subgroup of $\mathrm{GL}(n, \mathbb{K})$ of elements fixing the spaces E_k; if $\mathbb{K} = \mathbb{R}$ we further impose that their restriction to E_j has positive determinant (in order to fulfill connectedness).

For $0 \leq j \leq k$, let $\Omega_j = \mathbb{K}^n \setminus E_{j+1}$ and $Y_j = E_j \setminus E_{j+1}$ (with the convention $E_{k+1} = \varnothing$). The set Y_j consists of one or two G orbits (depending on whether $\dim E_j \geq 2 + \dim E_{j+1}$ or $\dim E_j = 1 + \dim E_{j+1}$ — in the complex case the Y_j consists of a single orbit).

For every $j \in \{0, \ldots, k\}$, let F_j be the quotient space $F_j = \mathbb{K}^n / E_j$ endowed with the flag $\{0\} \subset E_{j-1}/E_j \subset \cdots \subset E_0/E_j$ and let G_j be the group of positive automorphisms of this flag. The quotient map $\mathbb{K}^n \to F_j$ induces a group homomorphism $q_j : G \to G_j$.

Let also $p_j : \Omega_j \to F_j$ be the restriction of the quotient map to Ω_j. Let then $\widetilde{\mathcal{G}}_j$ be the pull-back groupoid of $F_j \rtimes G_j$ by the map p_j. In other words

$$\widetilde{\mathcal{G}}_j = \{(x, g, y) \in \Omega_j \times G_j \times \Omega_j : p_j(x) = g p_j(y)\}.$$

The map $(x, g, y) \mapsto (x, q_j(g), y)$ is a submersion and a groupoid morphism from $\Omega_j \rtimes G = \{(x, g, y) \in \Omega_j \times G \times \Omega_j : x = gy\}$ into $\widetilde{\mathcal{G}}_j$. Its image is the s-connected component $\mathcal{G}_j$ of $\widetilde{\mathcal{G}}_j$.

It follows from the following obvious lemma that $\mathcal{G}_j \rightrightarrows \mathbb{K}^n$ is a bisubmersion.

Lemma 1.10. *Let M, U, V be manifolds, $(M, \mathcal{F})$ a foliation, $p : U \to V$ a surjective submersion and $t_V, s_V : V \rightrightarrows M$ two submersions. Then $(U, t_V \circ p, s_V \circ p)$ is a bisubmersion for $\mathcal{F}$ if and only if (V, t_V, s_V) is a bisubmersion for $\mathcal{F}$.* $\square$

It follows then from Lemma 1.3 that $H(\mathcal{F})|_{Y_j} = (\mathcal{G}_j)|_{Y_j}$. We deduce:

Proposition 1.11. *The foliation of $\mathbb{K}^n$ by the action of G is nicely decomposed by the groupoids $\mathcal{G}_j \rightrightarrows \Omega_j$. Its holonomy groupoid is a union $\coprod_{j=0}^{k} (\mathcal{G}_j)|_{Y_j}$.*

Remarks 1.12. (a) One may write a projective analogue of this example: let PG be the projective analogue of G acting on $\mathbb{K}P^{n-1}$, namely PG is the quotient of G by its center, the group of similarities in G. It has k orbits: the images $Y_j = PE_j \setminus PE_{j+1}$ of $E_j \setminus E_{j+1}$ by the quotient map $p : \mathbb{K}^n \setminus \{0\} \to \mathbb{K}P^{n-1}$ (for $j > 0$). This foliation is nicely decomposed by the projective analogues $P\mathcal{G}_j$ of the $\mathcal{G}_j$. Note that the map $p : E_j \setminus \{0\} \to p(E_j)$ induces a morphism $p_j : \mathcal{G}_j \to P\mathcal{G}_j$ which is a Morita equivalence in the complex case. In the real case, it is almost a Morita equivalence: the morphism p_j induces an isomorphism of the stabilizer of $x \in E_j \setminus \{0\}$ in $\mathcal{G}_j$ with the stabilizer of $p(x) \in PE_j$ in $P\mathcal{G}_j$, but for $0 < i \le j$ and $\dim(E_i) = \dim(E_{k+1}) + 1$, the set $E_i \setminus E_{i-1}$ consists of two orbits of the groupoid $\mathcal{G}_j$ which become equivalent in $P\mathcal{G}_j$. The corresponding foliation C^*-algebra is (almost) Morita equivalent to $C^*(\Omega_1, \mathcal{F})$.

(b) There are many other interesting examples of the same flavor. A typical one is given in the following way: let $P_1, P_2 \subset \mathrm{GL}(n, \mathbb{K})$ be two parabolic subgroups, and let $P_1 \times P_2$ act on $\mathrm{GL}(n, \mathbb{K})$ by left and right multiplication. If $P_1 = P_2$ is the minimal parabolic subgroup consisting of upper triangular matrices, the orbits of this action are labeled by the symmetric group $\mathfrak{S}_n$ (Bruhat decomposition). In this example, the decomposition to be taken into account is more complicated than just the dimension of the fibers. One may need to use the partial ordering of the orbits given by the inclusion of the closures.

2. Foliations with singularity height 1

Let $(M, \mathcal{F})$ be a foliation admitting a nice decomposition of height 1. In this section our purpose is to show that the full foliation C^*-algebra $C^*(M, \mathcal{F})$ can be replaced by a mapping cone of Lie groupoid C^*-algebras associated with a nice decomposition of $\mathcal{F}$. We generalize this construction to higher length in the next section, but before this, we make some comments on the difficulties with dealing with reduced C^*-algebras.

2.A. ***A mapping cone construction.*** In the length 1 case, as noted in Remark 1.6, we just need to specify the saturated open subset $\Omega = \Omega_0$ and the Lie groupoid $\mathcal{G} = \mathcal{G}_1 \rightrightarrows W_1 = W$ which defines the foliation on an open set W containing $Y = M \setminus \Omega$ and whose restriction to Y coincides with that of $H(\mathcal{F})$.

The open subset Ω gives rise (at the level of the *full* C^*-algebras) to a short exact sequence

$$0 \to C^*(\Omega, \mathcal{F}_{|\Omega}) \xrightarrow{\iota_\mathcal{F}} C^*(M, \mathcal{F}) \xrightarrow{\pi_\mathcal{F}} C^*(M, \mathcal{F})|_Y \to 0$$

which in principle allows us to compute its K-theory. This is actually the case in our examples (Sections 2.C and 2.D).

In order to only use Lie groupoids (note that Y need not be a manifold), and also to be able to extend our construction to a more general setting (see Section 3), we also make use of the somewhat more elaborate diagram which appears in Figure 1.

$$\begin{tikzcd} 0 \arrow[r] & C^*(\mathcal{G}_{W\cap\Omega}) \arrow[r, "\iota_\mathcal{G}"] \arrow[d, "\pi_\Omega"'] & C^*(\mathcal{G}) \arrow[r, "p_\mathcal{G}"] \arrow[d, "\pi_M"'] & C^*(\mathcal{G}_Y) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C^*(\Omega, \mathcal{F}_{|\Omega}) \arrow[r, "\iota_\mathcal{F}"] & C^*(M, \mathcal{F}) \arrow[r, "p_\mathcal{F}"] & C^*(M, \mathcal{F})_{|Y} \arrow[r] & 0 \end{tikzcd}$$

Figure 1. Exact sequences for a nicely decomposable foliation of singularity height 1.

Restricting $\mathcal{G}$ to the open subset $W \cap \Omega$ and $H(\mathcal{F})$ to the open subset Ω, the integration along fibers (see [Androulidakis and Skandalis 2009]) of the quotient map $\mathcal{G} \to H(\mathcal{F})$ induces the diagram of half-exact sequences of *full* C^*-algebras shown in Figure 1.

Let $\mathcal{F}$ be a nicely decomposable foliation of singularity height one. We may use the diagram in Figure 1 in order to compute the K-theory of $C^*(M, \mathcal{F})$ via a Mayer–Vietoris exact sequence.

We explain here how one may replace $C^*(M, \mathcal{F})$ by a mapping cone of Lie groupoid C^*-algebras. We use the following notation:

- For any C^*-algebra Z and a locally compact space X put $Z(X) = C_0(X; Z)$.
- Recall that the mapping cone of a morphism $u : A \to B$ of C^*-algebras is

$$\mathcal{C}_u = \{(a, \phi) \in A \times B([0, 1)) : \phi(0) = u(a)\}.$$

With the notation of the diagram in Figure 1, consider the morphism of C^*-algebras

$$(i_\mathcal{G}, \pi_\Omega) : C^*(\mathcal{G}_{W\cap\Omega}) \to C^*(\mathcal{G}) \oplus C^*(W \cap \Omega, \mathcal{F}_{|\Omega}).$$

Proposition 2.1. *With the notation of Figure 1, the (full) foliation C^*-algebra $C^*(M, \mathcal{F})$ is canonically E^1-equivalent to the mapping cone $\mathcal{C}_{(\iota_\mathcal{G}, \pi_\Omega)}$.*

Proof. We show that given a diagram of exact sequences of C^*-algebras and morphisms

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "\pi"] & B_1 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B_0 \arrow[r, "i'"] & A \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

the mapping cone $\mathcal{C}_{(i,\pi)}$ of the map $(\pi, i) : I \to B_0 \oplus B_1$ is canonically E^1-equivalent to A.

Indeed, we have canonical morphisms $\mathcal{C}_i \to \mathcal{C}_{i'} \to Q(0, 1)$. Since $\mathcal{C}_i \to Q(0, 1)$ and $\mathcal{C}_{i'} \to Q(0, 1)$ are both onto with contractible kernels ($I[0, 1)$ and $B_0[0, 1)$, respectively), it follows that the morphism $\mathcal{C}_i \to \mathcal{C}_{i'}$ induces an equivalence in

E-theory. Now, using the diagram

$$\begin{tikzcd} 0 \arrow[r] & B_0(0,1) \arrow[r] \arrow[d, equal] & \mathcal{C}_{(i,\pi)} \arrow[r] \arrow[d] & \mathcal{C}_i \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B_0(0,1) \arrow[r] & \mathcal{C}_{(i',\mathrm{id}_{B_0})} \arrow[r] & \mathcal{C}_{i'} \arrow[r] & 0 \end{tikzcd}$$

we find that the morphism $\mathcal{C}_{(i,\pi)} \to \mathcal{C}_{(i',\mathrm{id}_{B_0})}$ induces an equivalence in E-theory. Finally the (split) exact sequence

$$0 \longrightarrow A(0,1) \longrightarrow \mathcal{C}_{(i',\mathrm{id}_{B_0})} \longrightarrow B_0[0,1) \longrightarrow 0$$

yields the desired E^1-equivalence. □

Remark 2.2. We may note that we have just shown that the morphism

$$\mathcal{C}_{(i,\pi)} \to \mathcal{C}_{(\mathrm{id}_A,\mathrm{id}_A)} \simeq A(0,1)$$

is invertible in E-theory.

2.B. ***Difficulties at the level of reduced C*-algebra.*** Let us discuss the reduced version of the diagram in Figure 1:

- If the restriction $\mathcal{G}_{|Y}$ is an amenable groupoid we also have horizontal exactness at the level of reduced C^*-algebras.
- If $\mathcal{G}_{|W\cap\Omega}$ is not amenable then the integration along fibers may not exist at the level of the kernels. We discuss such an example in Example 2.4.

In view of Examples 1.7 we focus now on foliations $(M,\mathcal{F})$ arising from an action of a Lie group G on a manifold M. We assume that $W = M$, the action groupoid $\mathcal{G} = M \rtimes G$ realizes a nice decomposition of singularity height 1 for $(M,\mathcal{F})$ and the complementary set Y is a point.

If the group G is amenable then integration along fibers of the quotient map $\mathcal{G} \to H(\mathcal{F})$ gives the diagram in Figure 2.

$$\begin{tikzcd} 0 \arrow[r] & C_0(\Omega) \rtimes G \arrow[r, "\iota_{\mathcal{G}}"] \arrow[d, "\pi_\Omega"'] & C_0(M) \rtimes G \arrow[r, "\pi_{\mathcal{G}}"] \arrow[d, "\pi"'] & C^*(G) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & C^*(\Omega, \mathcal{F}_{|\Omega}) \arrow[r, "\iota_{\mathcal{F}}"] & C^*(M,\mathcal{F}) \arrow[r, "\pi_{\mathcal{F}}"] & C^*(G) \arrow[r] & 0 \end{tikzcd}$$

Figure 2. Exact sequences for a nicely decomposable foliation of singularity height 1 arising from the action of an amenable Lie group.

If G is not amenable, the sequences are exact at the level of full C^*-algebras. At the reduced C^*-algebra level,

- the sequences need not be exact;
- the morphism $C_0(\Omega) \rtimes G \to C_r^*(M, \mathcal{F})$ obtained as a composition of π with the morphism $C^*(M, \mathcal{F}) \to C_r^*(M, \mathcal{F})$ doesn't need to pass to the quotient $C_0(\Omega) \rtimes_r G$ of $C_0(\Omega) \rtimes G$.

Note however that

- in most cases that we consider, the top sequence in Figure 2 is exact since the groups we consider are exact;
- we always have some completely positive splittings (see Proposition 2.3);
- in the example of the action of $\mathrm{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$, since the stabilizers are amenable, the morphism $\pi_\Omega : C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{GL}(2, \mathbb{R}) \to \mathcal{K}$ is defined at the reduced C^*-algebra level. As the group $\mathrm{GL}(2, \mathbb{R})$ is K-amenable, we find that in this case the full and reduced C^*-algebra of $\mathcal{F}$ are KK-equivalent.

Proposition 2.3. *Let $\mathcal{G}$ be the action groupoid in Figure 2. Then the morphisms $C_r^*(\mathcal{G}) \to C_r^*(G)$, $C^*(\mathcal{G}) \to C^*(G)$ and $C^*(M, \mathcal{F}) \to C^*(G)$ have completely positive splittings.*

Proof. This is due to the fact that $C^*(G)$ sits in the multiplier algebra of a crossed product $A \rtimes G$ — and the same for reduced ones:

We construct a completely positive splitting for the map $C^*(\mathcal{G}) \to C^*(G)$. Take a function $f \in C_0(M)$ such that $\|f\| = 1$ and $f(x_0) = 1$. Given $\zeta \in C^*(G)$ put $\sigma(\zeta) = f^* \zeta f$. This is obviously a completely positive (and contractive) splitting of the top sequence. (The same is true for the reduced algebra and crossed products.)

Composing the completely positive splitting $C^*(G) \to C^*(\mathcal{G})$ with the morphism $\pi : C^*(\mathcal{G}) \to C^*(M, \mathcal{F})$ (given by integration along the fibers) we obtain a completely positive splitting of the second sequence. □

We now give an example where the morphism π_Ω is not defined at the reduced C^*-algebra level:

Example 2.4. Consider the action of $G = \mathrm{SL}(n, \mathbb{R})$ on $\mathbb{R}^n$ for $n \geq 3$. This action has two orbits: $\{0\}$ and $\Omega = \mathbb{R}^n \setminus \{0\}$. The stabilizer of a nonzero point for this action is isomorphic to $H = \mathbb{R}^{n-1} \rtimes \mathrm{SL}(n-1, \mathbb{R})$, which is not amenable if $n \geq 3$. The full crossed product $C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathrm{SL}(n, \mathbb{R})$ is Morita equivalent to $C^*(H)$. Therefore, the full C^*-algebra of this foliation is the quotient of $C_0(\mathbb{R}^n) \rtimes \mathrm{SL}(n, \mathbb{R})$

sitting in a diagram

$$\begin{tikzcd}
0 \arrow[r] & C_0(\Omega) \rtimes G \simeq \mathcal{K} \otimes C^*(H) \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{K}} \otimes \varepsilon_H"'] & C_0(\mathbb{R}^n) \rtimes G \arrow[r, "\pi_{\mathcal{G}}"] \arrow[d, "\pi"'] & C^*(G) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] & C^*(\mathbb{R}^n, \mathcal{F}) \arrow[r, "\pi_{\mathcal{F}}"] & C^*(G) \arrow[r] & 0
\end{tikzcd}$$

where ε_H denotes the trivial representation of $H = \mathbb{R}^{n-1} \rtimes \mathrm{SL}(n-1, \mathbb{R})$. The reduced crossed product $C_0(\mathbb{R}^n \setminus \{0\}) \rtimes_r \mathrm{SL}(n, \mathbb{R})$ is Morita equivalent to $C_r^*(H)$.

Note that the trivial representation $C^*(H) \to \mathbb{C}$ is not defined at the level of $C_r^*(H)$ when the group H is not amenable.

The reduced C^*-algebra $C_r^*(\mathbb{R}^n, \mathcal{F})$ of this foliation is the quotient of $C_0(\mathbb{R}^n) \rtimes G$ corresponding to the sum of the two covariant representations on $L^2(\Omega) = L^2(\mathbb{R}^n)$ and $\{0\} \times G$.

Remark 2.5. In the sequel we use (almost) only the full C^*-algebra to ensure that our sequences are exact and the trivial representation exists. This is legitimate from the point of view of the Baum–Connes conjecture, since the assembly map factors through the K-theory of the full C^*-algebra anyway.

2.C. ***Two examples of foliations of singularity height 1 given by linear actions.*** In this section we compute the K-theory for two simple examples of foliations of singularity height 1 coming from linear actions. In the height 1 case, this can be done rather easily, using six-term exact sequences of K-theory groups and standard K-theory results. The (nonlinear) examples of $\mathbb{R}$-actions (see Example 1.7(d)) are discussed in Section 2.D.

2.C.1. *The* $\mathrm{SO}(3)$*-action.* In this section we consider the foliation $(\mathbb{R}^3, \mathcal{F})$ defined by the action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$ (see Example 1.7(a)).

Holonomy groupoid and exact sequences. As discussed in Examples 1.7, $H(\mathcal{F}) = (\mathrm{SO}(3) \times \{0\}) \sqcup (\mathbb{R}_+^* \times S^2 \times S^2)$ and $\mathcal{F}$ is nicely decomposable, in the sense of Definition 1.4 with $\Omega_0 = \mathbb{R}^3 \setminus \{0\}$ and $\mathcal{G}_1 = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$.

Note that $\mathrm{SO}(3)$ is compact and therefore amenable, so the reduced and full crossed product C^*-algebras coincide. The (full and reduced) C^*-algebra of $\mathcal{G}_0$ is the crossed product $C_0(\mathbb{R}^3) \rtimes \mathrm{SO}(3)$.

Writing $\mathbb{R}^3 \setminus \{0\} = \mathbb{R}_+^* \times S^2$, we find that $C^*(\mathcal{G}|_{\Omega_0}) = C_0(\mathbb{R}_+^*) \otimes (C(S^2) \rtimes \mathrm{SO}(3))$ and $C^*(M, \mathcal{F})|_{\Omega_0} = C_0(\mathbb{R}_+^*) \otimes \mathcal{K}(L^2(S^2))$. Now Figure 1 reads as in Figure 3. Here $\hat{q} = \mathrm{id}_{C_0(\mathbb{R}_+^*)} \otimes q$, where $q : C(S^2) \rtimes \mathrm{SO}(3) \to \mathcal{K}(L^2(S^2))$ is obtained by integration along the fibers of the groupoid morphism $(t, s) : S^2 \rtimes \mathrm{SO}(3) \to S^2 \times S^2$.

Calculation of K-theory with mapping cones. To describe the foliation C^*-algebra we give an interpretation of Figure 3 using mapping cones.

$$\begin{tikzcd}
0 \arrow[r] & C_0(\mathbb{R}_+^*)\otimes(C(S^2)\rtimes \mathrm{SO}(3)) \arrow[r, "i"] \arrow[d, "\hat{q}"] & C_0(\mathbb{R}^3)\rtimes \mathrm{SO}(3) \arrow[r] \arrow[d, "\pi"] & C^*(\mathrm{SO}(3)) \arrow[r] \arrow[d, equal] & 0 & \text{(ES1)} \\
0 \arrow[r] & C_0(\mathbb{R}_*^+)\otimes\mathcal{K}(L^2(S^2)) \arrow[r] & C^*(\mathbb{R}^3, \mathcal{F}) \arrow[r] & C^*(\mathrm{SO}(3)) \arrow[r] & 0 & \text{(ES2)}
\end{tikzcd}$$

Figure 3. Exact sequences for the SO(3) action.

$$\begin{tikzcd}
C^*(\mathrm{SO}(3)) \arrow[r, "j"] \arrow[dr, "\rho"'] & C(S^2) \rtimes \mathrm{SO}(3) \arrow[d, "q"] \\
& \mathcal{K}(L^2(S^2))
\end{tikzcd}$$

Figure 4. Mapping cones for the SO(3) action.

Let $\rho : C^*(\mathrm{SO}(3)) \to \mathcal{K}(L^2(S^2))$ be the natural representation of SO(3) on $L^2(S^2)$. We thus have the diagram in Figure 4, where $j : C^*(\mathrm{SO}(3)) \to C(S^2) \rtimes \mathrm{SO}(3)$ is the morphism induced by the unital inclusion $\mathbb{C} \to C(S^2)$.

Identify $C_0(\mathbb{R}^3)$ with the mapping cone of $\mathbb{C} \to C(S^2)$. Taking crossed products by the action of SO(3) and using the diagram in Figure 3, we find:

- The crossed product C^*-algebra $C_0(\mathbb{R}^3) \rtimes \mathrm{SO}(3)$ in extension (ES1) is the mapping cone $\mathcal{C}_p$, where p is the map $j : C^*(\mathrm{SO}(3)) \to C(S^2) \rtimes \mathrm{SO}(3)$.
- The foliation C^*-algebra $C^*(\mathbb{R}^3, \mathcal{F})$ in extension (ES2) is the mapping cone $\mathcal{C}_\rho$.

To describe $C^*(\mathcal{F})$, it suffices to describe the representation

$$\rho : C^*(\mathrm{SO}(3)) \to \mathcal{K}(L^2(S^2)).$$

It follows from the Peter–Weyl theorem that $C^*(\mathrm{SO}(3)) = \bigoplus_{m\in\mathbb{N}} M_{2m+1}(\mathbb{C})$ and $K_0(C^*(\mathrm{SO}(3))) = \mathbb{Z}^{(\mathbb{N})}$ (and $K_1(C^*(\mathrm{SO}(3))) = \{0\}$).

In order to compute the map $\rho_* : K_0(C^*(\mathrm{SO}(3))) \to \mathbb{Z}$, we have to understand how many times the representation σ_m (of dimension $2m+1$) appears in ρ, i.e., count the dimension of $\mathrm{Hom}_{\mathrm{SO}(3)}(\sigma_m, \rho)$.

Since $S^2 = \mathrm{SO}(3)/S^1$, the representation ρ is the representation $\mathrm{Ind}_{S^1}^{\mathrm{SO}(3)}(\varepsilon)$ induced by the trivial representation ε of S^1. Using the Frobenius reciprocity theorem, we know $\dim(\mathrm{Hom}_{\mathrm{SO}(3)}(\sigma_m, \rho)) = \dim(\mathrm{Hom}_{S^1}(\sigma_m, \varepsilon)) = 1$.

It follows that the map $\rho_* : K_0(C^*(\mathrm{SO}(3))) \to \mathbb{Z}$ is the map which sends each generator $[\sigma_m]$ of $K_0(C^*(\mathrm{SO}(3)))$ to 1. We immediately deduce:

Proposition 2.6. *We have $K_0(C^*(\mathcal{F})) = \ker \rho_* \simeq \mathbb{Z}^{(\mathbb{N})}$ and $K_1(C^*(\mathcal{F})) = 0$.* □

Remark 2.7. In the same way, one may easily compute

$$j_* : K_0(C^*(\mathrm{SO}(3)) \to K_0(C(S^2) \rtimes C^*(\mathrm{SO}(3)) \quad \text{and} \quad K_*(C_0(\mathbb{R}^3) \rtimes \mathrm{SO}(3)).$$

In fact, this is a classical result, which states that the algebra $C(S^2) \rtimes C^*(\mathrm{SO}(3))$ is Morita equivalent to $C^*(S^1)$ and the morphism $j_* : K_0(C^*(\mathrm{SO}(3))) \to K_0(C^*(S^1))$ is the restriction morphism $R(\mathrm{SO}(3)) \to R(S^1)$, where $R(G) = K_0(C^*(G))$ is the representation ring of a compact group G; see [Rieffel 1976; Julg 1982].

It follows that $j_*([\sigma_m]) = \sum_{k=-m}^{m} [\chi_k]$, where the $(\chi_k)_{k\in\mathbb{Z}}$ are the characters of S^1. The morphism j_* is therefore (split) injective, and we find

$$\begin{aligned} K_0(C_0(\mathbb{R}^3) \rtimes \mathrm{SO}(3)) &= 0, \\ K_1(C_0(\mathbb{R}^3) \rtimes \mathrm{SO}(3)) &\simeq \mathbb{Z}^{(\mathbb{N})}. \end{aligned}$$

2.C.2. *The* $\mathrm{SL}(2, \mathbb{R})$*-action.* We consider the foliation on $\mathbb{R}^2$ induced by the action of $\mathrm{SL}(2, \mathbb{R})$. Recall the following:

(a) $\mathrm{SL}(2, \mathbb{R})$ is not compact and not amenable, but it was shown in [Kasparov 1984] to be KK-amenable.

(b) Its maximal compact is S^1.

(c) The action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2 \setminus \{0\}$ is transitive and the stabilizer of the point $(1, 0)$ is the set of matrices of the form $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. Hence the action groupoid $(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2, \mathbb{R})$ is Morita equivalent to the group $\mathbb{R}$. So the crossed product $C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2, \mathbb{R})$ is Morita equivalent to the group C^*-algebra $C^*(\mathbb{R})$.

(d) It follows as above from Lemma 1.3 (see also [Androulidakis and Skandalis 2009, Example 3.7]) that the associated holonomy groupoid is

$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2 \setminus \{0\}) \cup \{0\} \times \mathrm{SL}(2, \mathbb{R}).$$

It follows that this foliation is nicely decomposable of singularity height 1 with $\mathcal{G} = \mathcal{G}_0 = \mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R})$ (see Remark 1.6). Here the diagram of Figure 1 reads as in Figure 5. Recall that the C^*-algebras involved are full C^*-algebras.

$$\begin{tikzcd} 0 \arrow[r] & (C_0(\mathbb{R}^2 \setminus \{0\})) \rtimes \mathrm{SL}(2, \mathbb{R}) \arrow[r, "\iota"] \arrow[d, "\pi_1"] & C_0(\mathbb{R}^2) \rtimes \mathrm{SL}(2, \mathbb{R}) \arrow[r] \arrow[d, "\pi"] & C^*(\mathrm{SL}(2, \mathbb{R})) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{K}(L^2(\mathbb{R}^2 \setminus \{0\})) \arrow[r] & C^*(\mathcal{F}) \arrow[r] & C^*(\mathrm{SL}(2, \mathbb{R})) \arrow[r] & 0 \end{tikzcd}$$

Figure 5. Exact sequences for the $\mathrm{SL}(2, \mathbb{R})$ action.

Direct Calculation of K-theory. The short exact sequence

$$0 \to \mathcal{K}(L^2(\mathbb{R}^2 \setminus \{0\})) \to C^*(\mathcal{F}) \to C^*(\mathrm{SL}(2,\mathbb{R})) \to 0$$

gives the 6-term exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{K}(L^2(\mathbb{R}^2 \setminus \{0\}))) & \longrightarrow & K_0(C^*(\mathcal{F})) & \longrightarrow & K_0(C^*(\mathrm{SL}(2,\mathbb{R}))) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\mathrm{SL}(2,\mathbb{R}))) & \longleftarrow & K_1(C^*(\mathcal{F})) & \longleftarrow & K_1(\mathcal{K}(L^2(\mathbb{R}^2 \setminus \{0\}))) \end{array}$$

We have $K_0(\mathcal{K}(L^2(\mathbb{R}^2\setminus\{0\}))) = \mathbb{Z}$ and $K_1(\mathcal{K}(L^2(\mathbb{R}^2\setminus\{0\}))) = 0$. On the other hand, using the Connes–Kasparov conjecture proved in [Kasparov 1984; Wassermann 1987], we have $K_1(C^*(\mathrm{SL}(2,\mathbb{R}))) = 0$. We conclude that

$$K_1(C^*(\mathbb{R}^2,\mathcal{F})) = 0 \quad \text{and} \quad K_0(C^*(\mathbb{R}^2,\mathcal{F})) = \mathbb{Z} \oplus K_0(C^*(\mathrm{SL}(2,\mathbb{R}))) = \mathbb{Z} \oplus \mathbb{Z}^{(\mathbb{Z})}.$$

Calculation of K-theory with mapping cones. Although the above construction is quite direct, it may be worth examining a construction following the general procedure of Section 2.A (Proposition 2.1).

To apply the mapping cones approach we gave in Section 2.A, we need the following result, which follows from [Kasparov 1984; 1988].

Proposition 2.8. *Let* $\mathrm{SL}(2,\mathbb{R})$ *act on a* C^**-algebra* A *by automorphisms. The algebras* $A \rtimes \mathrm{SL}(2,\mathbb{R})$ *and* $A \rtimes S^1$ *are KK-equivalent.*

Proof. The Lie group S^1 is a maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$. Note also that $\mathrm{SL}(2,\mathbb{R})/S^1$ is the Poincaré half plane and therefore admits a complex structure, and hence an $\mathrm{SL}(2,\mathbb{R})$-invariant spin^c structure. The result follows from [Kasparov 1984]. ☐

It follows in fact from [Kasparov 1984] that the exact sequences

$$0 \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2,\mathbb{R}) \to C_0(\mathbb{R}^2) \rtimes \mathrm{SL}(2,\mathbb{R}) \to C^*(\mathrm{SL}(2,\mathbb{R})) \to 0$$

and

$$0 \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes S^1 \to C_0(\mathbb{R}^2) \rtimes S^1 \to C^*(S^1) \to 0$$

are KK-equivalent. We note the following:

- $K_0(C^*(\mathrm{SL}(2,\mathbb{R}))) = K_0(C^*(S^1)) = \mathbb{Z}^{(\mathbb{Z})}$, and $K_1 = 0$.
- Since S^1 acts freely on $\mathbb{R}^2\setminus\{0\}$ with quotient $\mathbb{R}_+^*$, it follows that $C_0(\mathbb{R}^2\setminus\{0\}) \rtimes S^1$ is Morita equivalent to $C_0(\mathbb{R}_+^*)$; also since $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{R}^2\setminus\{0\}$ with stabilizers isomorphic to $\mathbb{R}$, it follows that $C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2,\mathbb{R})$ is Morita equivalent to $C^*(\mathbb{R})$. It follows that $K_1(C(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2,\mathbb{R})) = K_1(C(\mathbb{R}^2 \setminus \{0\}) \rtimes S^1) = \mathbb{Z}$ and $K_0 = 0$.

- Using the complex structure of $\mathbb{R}^2$, we have a Bott isomorphism between $K_*(C_0(\mathbb{R}^2) \rtimes S^1)$ and $K_*(C^*(S^1))$. It follows that $K_0(C(\mathbb{R}^2) \rtimes \mathrm{SL}(2, \mathbb{R})) = K_0(C(\mathbb{R}^2) \rtimes S^1) = \mathbb{Z}^{(\mathbb{Z})}$, and $K_1 = 0$.

From this discussion, it follows that the morphism

$$\iota : C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2, \mathbb{R}) \to C_0(\mathbb{R}^2) \rtimes \mathrm{SL}(2, \mathbb{R})$$

induces the 0 map in K-theory, and so does the map $\pi : C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2, \mathbb{R}) \to \mathcal{K}$.

Remark 2.9. Denoting by $(\chi_n)_{n\in\mathbb{Z}}$ the characters of S^1, for each n the image of $[\chi_n] \in K_*(C_0(\mathbb{R}^2) \rtimes S^1)$ by $C_0(\mathbb{R}^2) \rtimes S^1 \to C^*(S^1)$ (evaluation at 0) is $[\chi_n] - [\chi_{n+1}]$. This morphism is one to one and its image is the set of elements in $R(S^1)$ of dimension 0.

As the maps ι and π induce the 0 map in K-theory, we find as above from Proposition 2.1.

Proposition 2.10. *Let $\mathcal{F}$ be the foliation defined by the action of* $\mathrm{SL}(2, \mathbb{R})$ *on* $\mathbb{R}^2$. *We have*

$$K_0(\mathcal{F}) \simeq K_0(C_0(\mathbb{R}^2) \rtimes \mathrm{SL}^2(\mathbb{R})) \oplus K_0(\mathcal{K}) \oplus K_1(C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}^2(\mathbb{R})) \simeq \mathbb{Z}^{(\mathbb{N})} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

and $K_1(\mathcal{F}) = 0$. □

Note that we have a split short exact sequence $0 \to K_0(C_0(\mathbb{R}^2) \rtimes \mathrm{SL}^2(\mathbb{R})) \to K_0(C^*(\mathrm{SL}^2(\mathbb{R}))) \to K_1(C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}^2(\mathbb{R})) \to 0$, and thus the results of Proposition 2.10 and the direct calculation (Section 2.C.2) are coherent.

2.C.3. *Generalizations.* The examples introduced above can be extended to the action of $\mathrm{SO}(n)$ or $\mathrm{SL}(n, \mathbb{R})$ on $\mathbb{R}^n$. Let us discuss here a slightly more general situation which still gives singularity height 1 foliations.

Subgroups of $\mathrm{SO}(n)$. Let G be a connected closed subgroup of $\mathrm{SO}(n)$. Assume that its action on S^{n-1} is transitive, and let $H \subset G$ be the stabilizer of a point in S^{n-1}. Denote by $\mathcal{F}$ the foliation of $\mathbb{R}^n$ associated with the action of G. Exactly as in the case of the action of $\mathrm{SO}(3) \in \mathbb{R}^3$, we find that

- $H(\mathcal{F}) = (G \times \{0\}) \sqcup (\mathbb{R}_+^* \times S^{n-1} \times S^{n-1})$;
- $C^*(\mathbb{R}^n, \mathcal{F})$ is the mapping cone of the morphism $C^*(G) \to \mathcal{K}(S^{n-1})$.
- The map $R(G) \to \mathbb{Z}$ corresponding to this morphism associates to a (virtual) representation σ the (virtual) dimension of its H fixed points. It is onto, and therefore $K_0(C^*(\mathbb{R}^n, \mathcal{F})) = \mathbb{Z}^{(\mathbb{N})}$ and $K_1(C^*(\mathbb{R}^n, \mathcal{F})) = 0$.

Subgroups of GL_n. Now let G be a closed connected subgroup of $\mathrm{GL}(n, \mathbb{R})$. Assume that its action on $\mathbb{R}^n \setminus \{0\}$ is transitive, and let $H \subset G$ be the stabilizer of a nonzero point in $\mathbb{R}^n$. As for the case of $\mathrm{SL}(2, \mathbb{R})$ acting on $\mathbb{R}^2$, we have:

- The holonomy groupoid is $H(\mathcal{F}) = ((\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})) \sqcup (G \times \{0\})$.
- We have an exact sequence of full C^*-algebras

$$0 \to \mathcal{K}(L^2(\mathbb{R}^n \setminus \{0\})) \to C^*(\mathbb{R}^n, \mathcal{F}) \to C^*(G) \to 0,$$

and therefore an exact sequence

$$0 \to K_1(C^*(\mathbb{R}^n, \mathcal{F})) \to K_1(C^*(G)) \xrightarrow{\partial} \mathbb{Z} \to K_0(C^*(\mathbb{R}^n, \mathcal{F})) \to K_0(C^*(G)) \to 0.$$

In order to try and compute the connecting map ∂, we may use the diagram of Figure 1. Note that the groupoid $(\mathbb{R}^n \setminus \{0\}) \rtimes G$ is Morita equivalent to the group H. Following this diagram, the connecting map ∂ is the composition of the trivial representation of H of with the connecting map

$$\partial' : K_1(C^*(G)) \to K_0(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes G) \simeq K_0(C^*(H)).$$

An example of this kind is of course $\mathrm{SL}(n, \mathbb{C}) \subset \mathrm{SL}(2n, \mathbb{R})$. The stabilizer group of a $z \in \mathbb{C}^n \setminus \{0\}$, say $z = (1, 0, \ldots, 0)$, is the group of matrices in $\mathrm{SL}(n, \mathbb{C})$ whose first row is z. That is $\mathbb{C}^{n-1} \rtimes \mathrm{SL}(n-1, \mathbb{C})$.

Another example is given by $G = G_1 \times \mathbb{R}_+^*$, where G_1 is a connected closed subgroup of $\mathrm{SO}(N)$ whose action on S^{n-1} is transitive, and $\mathbb{R}_+^*$ acts by similarities. Note that if $\mathcal{F}_1$ is the foliation defined by the action of G_1, there is a natural action of $\mathbb{R}_+^*$ on $H(\mathcal{F}_1)$ and $H(\mathcal{F})$ is a semidirect product $H(\mathcal{F}_1) \rtimes \mathbb{R}_+^*$; we find $C^*(\mathbb{R}^n, \mathcal{F}) = C^*(\mathbb{R}^n, \mathcal{F}_1) \rtimes \mathbb{R}_+^*$. Thanks to the Connes–Thom isomorphism, the algebras $C^*(\mathbb{R}^n, \mathcal{F})$ and $C^*(\mathbb{R}^n, \mathcal{F}_1)$ have the same K-theory up to a shift of dimension.

2.D. ***Actions of*** $\mathbb{R}$ ***on manifolds.*** Now we come to Example 1.7(d), which also belongs to the case of height 1 foliations. Let M be a manifold endowed with a smooth action α of $\mathbb{R}$. Let $\mathcal{F}$ be the foliation associated with this action — i.e., with the groupoid $M \rtimes_\alpha \mathbb{R}$. We keep the notation of Example 1.7(d). There are several papers concerned with actions of $\mathbb{R}$ and the computation of the associated C^*-algebra; see [Torpe 1985; Wang 1987; Hirsch and Wang 1987]. The particular difficulty with the general case we examine here comes from the (interior of the) set where the vector field vanishes, and partly also from those points where the vector field is periodic.

We showed that $\mathcal{F}$ is nicely decomposable in the sense of Definition 1.4. Here we compute the K-theory using an exact sequence. Note that in this example, in the presence of periodic points, the groupoid $\mathcal{G}_0$ is not always Hausdorff and its classifying space for proper actions is not a manifold. Therefore Theorem 5.13 does not apply directly.

From Proposition 1.9 we deduce that the groupoid $\mathcal{G}_1' \rightrightarrows M$ which coincides with $H(\mathcal{F})$ on the complement of W and with $W \times \mathbb{R}$ on W is a (not necessarily Hausdorff) Lie groupoid and gives rise to the nice decomposition ($W \rightrightarrows W$, $\mathcal{G}_1' \rightrightarrows M$) of $\mathcal{F}$. We exploit this one in the computations below.

Put also $Y = M \setminus W$.

2.D.1. *Exact sequence of fixed points.*

Proposition 2.11. *The KK^1-element associated with the exact sequence*

$$0 \to C_0(W) \to C^*(M, \mathcal{F}) \to C^*(M, \mathcal{F})|_Y \to 0 \tag{ES3}$$

is 0.

Proof. The corresponding exact sequence for the groupoid $\mathcal{G}_1'$ givers rise to the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(W) & \longrightarrow & C^*(M,\mathcal{F}) & \longrightarrow & C^*(M,\mathcal{F})|_Y & \longrightarrow & 0 \qquad \text{(ES3)} \\ & & \uparrow{\scriptstyle \mathrm{ev}_0} & & \uparrow & & \| & & \\ 0 & \longrightarrow & C_0(W \times \mathbb{R}) & \longrightarrow & C^*(\mathcal{G}_1') & \longrightarrow & C^*(\mathcal{G}_1')|_Y & \longrightarrow & 0 \qquad \text{(ES4)} \end{array}$$

Denote by z_1, z_2 the KK^1 elements associated with the exact sequences (ES3) and (ES4). We have $z_1 = (\mathrm{ev}_0)_*(z_2)$. But (ev_0), which is the map induced by the inclusion $x \mapsto (x, 0)$ from W to $W \times \mathbb{R}$, is the 0 element in KK, whence $z_1 = 0$ as claimed. ☐

We immediately deduce:

Corollary 2.12. *We have $K_*(C^*(M, \mathcal{F})) = K_*(C_0(W)) \oplus K_*(C^*(M, \mathcal{F})|_Y)$.* ☐

If all periodic points are in fact fixed, i.e., if $\mathrm{Per}(\alpha) = W$, then, using Connes' Thom isomorphism [Connes 1981], this computation yields:

Corollary 2.13. *Assume that all the periodic points are in fact fixed. The K-theory group of $C^*(M, \mathcal{F})$ is*

$$\begin{aligned} K_*(C^*(M, \mathcal{F})) &= K_*(C_0(W)) \oplus K_*(C_0(Y) \rtimes \mathbb{R}) \\ &= K_*(C_0(W)) \oplus K_{1-*}(C_0(Y)). \end{aligned}$$ ☐

Remark 2.14. Corollary 2.13 can be interpreted by saying that, when there are no nontrivial stably periodic points, the classifying space of proper actions of the holonomy groupoid is $W \sqcup Y \times \mathbb{R}$. The associated assembly map is an isomorphism.

2.D.2. *The stably periodic points.* In the presence of nontrivial stable periodic points, the complete computation of the K-theory is not so simple. Even in the regular case, this computation is quite hard. See, e.g., [Torpe 1985].

As a consequence of Proposition 1.9 we find:

Proposition 2.15. *The set* $\widehat{\mathrm{Per}}(\alpha) = \mathrm{Per}(\alpha) \setminus W$ *of nontrivial stably periodic points is open.*

Proof. By Proposition 1.9, the set W is closed in $\mathrm{Per}(\alpha)$, whence its complement is open in $\mathrm{Per}(\alpha)$ — and therefore in M, since $\mathrm{Per}(\alpha)$ is open. □

For $x \in \widehat{\mathrm{Per}}(\alpha)$, let $p(x) \in \mathbb{R}_+$ be the infimum of the set of $t > 0$ such that $(x, t) \in M \times \mathbb{R}$ is the trivial element in $H(\mathcal{F})$. By [Debord 2013] it follows that $p(x) > 0$ and $(x, p(x))$ is the trivial element in $H(\mathcal{F})$.

Proposition 2.16. *The map* $p : \widehat{\mathrm{Per}}(\alpha) \to \mathbb{R}_+$ *is smooth.*

Proof. Since $(x, p(x))$ is the trivial element in $H(\mathcal{F})$, there exists an open neighborhood $U \subset \widehat{\mathrm{Per}}(\alpha)$ of x and a bounded (below and above) smooth function $f : U \to \mathbb{R}_+^*$ such that $(y, f(y)) \in P$ for all $y \in U$ and $p(x) = f(x)$. For $y \in U$, since U is a neighborhood of y, it follows that $f(y)$ is a multiple of $p(y)$. We consider two cases.

• Assume $X(x) = 0$. Let $m \in \mathbb{R}_+$ be such that $f(y) \le m$ for all $y \in U$. Put $V = \{x \in U : \forall t \in [0, m],\ \alpha_t(x) \in U\}$; by compactness of $[0, m]$ it is an open subset of U. Then by periodicity, V is invariant by α_t, $t \in \mathbb{R}$. For $y \in V$ and $t \in [0, T]$, as $f(\alpha_t(y))$ is a multiple of $p(\alpha_t(y)) = p(y)$, it follows by continuity of f that $f(\alpha_t(y)) = f(y)$. Replacing X by $(1/f)X$, we get an action of S^1 on V.

Since S^1 is compact, Bochner's linearization theorem [1945] says that in an open and S^1-equivariant neighborhood U' of x the S^1-action is actually a linear representation of S^1, which is faithful since $f(x) = p(x)$. It follows that $p(y) = f(y)$ for all $y \in U'$.

• Assume $X(x) \neq 0$. Then x is periodic of period $p(x)/k$ with $k \in \mathbb{N}^*$. Now choose a transversal T at x; we get an action of $\mathbb{Z}/k\mathbb{Z}$, and applying Bochner's linearization theorem again, we conclude $f(y) = p(y)$ in a neighborhood of x. □

When restricting to $\widehat{\mathrm{Per}}(\alpha)$, we may therefore replace X by $\frac{1}{p}X$ and obtain an action of S^1. The foliation groupoid is then $W \sqcup \widehat{\mathrm{Per}}(\alpha) \rtimes S^1 \sqcup (M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R}$.

Remarks 2.17. (a) The building blocks of $C^*(M, \mathcal{F})$ are the algebras $C_0(W)$, $C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1$ and $C_0(M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R}$. For each of them there is of course a topological K-theory and a Baum–Connes map. Actually, since the first two are given by compact group actions, they are their own "left-hand side"! The "left-hand side" for $C_0(M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R}$ given by Connes' Thom isomorphism is $(M \setminus \mathrm{Per}(\alpha)) \times \mathbb{R}$.

(b) We already noticed that $K_*(C^*(M,\mathcal{F})) = K_*(W) \oplus K_*(C^*(M,\mathcal{F})|_Y)$. To compute $K_*(C^*(M,\mathcal{F})|_Y)$ we may use the exact sequence

$$0 \to C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1 \to C^*(M,\mathcal{F})|_Y \to C_0(M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R} \to 0 \qquad \text{(ES5)}$$

In order to compute the connecting map of this sequence we note that we have a diagram:

$$\begin{tikzcd}
0 \arrow[r] & C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1 \arrow[r] & C^*(\mathcal{F})|_Y \arrow[r] & C^*(M,\mathcal{F})|_{M\setminus \mathrm{Per}(\alpha)} \arrow[r] & 0 & \text{(ES5)} \\
0 \arrow[r] & C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes \mathbb{R} \arrow[r] \arrow[u, "q"] & C_0(Y) \rtimes \mathbb{R} \arrow[r] \arrow[u] & C_0(M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R} \arrow[r] \arrow[u, equal] & 0 & \text{(ES6)}
\end{tikzcd}$$

Denote by z_3, z_4 the KK^1 elements associated with the exact sequences (ES5) and (ES6). We have $z_3 = q_*(z_4) = z_4 \otimes [q]$. To compute z_3 we then remark:

- Through Connes' Thom isomorphism

$$KK^1(C_0(M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R}, C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes \mathbb{R}) = KK^1(C_0(M \setminus \mathrm{Per}(\alpha)), C_0(\widehat{\mathrm{Per}}(\alpha)))$$

the element $[z_4]$ corresponds to the exact sequence of commutative algebras

$$0 \to C_0(\widehat{\mathrm{Per}}(\alpha)) \to C_0(Y) \to C_0(M \setminus \mathrm{Per}(\alpha)) \to 0$$

- Under the Takesaki–Takai isomorphism $C_0(\widehat{\mathrm{Per}}(\alpha)) \otimes \mathcal{K} \simeq (C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1) \rtimes \mathbb{Z}$ the element $[q]$ in

$$\begin{aligned}
KK(C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes \mathbb{R}, C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1)& \\
&= KK^1(C_0(\widehat{\mathrm{Per}}(\alpha)), C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1) \\
&= KK^1\big((C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1) \rtimes \mathbb{Z}, C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1\big)
\end{aligned}$$

is the one associated to the Pimsner–Voiculescu exact sequence

$$0 \to B \otimes \mathcal{K} \to \mathfrak{T} \to B \rtimes \mathbb{Z} \to 0$$

(here $B = C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1$).

3. Larger singularity height and telescope

In this section we extend the constructions of Section 2 to singular foliations of arbitrary singularity height. The mapping cone of Section 2 is replaced by a telescope. We start by recalling telescope constructions.

3.A. ***Mapping telescopes.*** Let us recall the following construction of C^*-algebras:

Definition 3.1. Let $n \in \mathbb{N} \cup \{+\infty\}$. Given C^*-algebras $(B_k)_{0\le k<n}$ and $(I_k)_{1\le k<n}$ and morphisms $\alpha_k : I_k \to B_{k-1}$ and $\beta_k : I_k \to B_k$, we define the associated *telescopic C^*-algebra*

$$\mathcal{T}((\alpha_k)_{1\le k<n}, (\beta_k)_{1\le k<n})$$

to be the C^*-algebra comprising

$$((\phi_k)_{0\le k<n}, (x_k)_{1\le k<n}) \in \prod_{0\le k<n} B_k[k, k+1] \times \prod_{1\le k<n} I_k$$

such that

- for $1 \le k < n$ we have $\phi_k(k) = \beta_k(x_k)$ and $\phi_{k-1}(k) = \alpha_{k-1}(x_k)$,
- $\phi_0(0) = 0$,
- $\begin{cases} \phi_{n-1}(n) = 0 & \text{if } n \neq +\infty, \\ \lim_{k\to+\infty} \|\phi_k\| = \lim_{k\to+\infty} \|x_k\| = 0 & \text{if } n = +\infty. \end{cases}$

Remark 3.2. A particular case of a telescope is when $I_k = B_{k-1}$ and $\alpha_k = \mathrm{id}_{I_k}$. We denote just by $\mathcal{T}(\beta)$ the associated mapping telescope $\mathcal{T}(\mathrm{id}, \beta)$. In that case, if $n = \infty$, let us also denote by B_∞ the inductive limit of the system (B_k, β_k). We then have an exact sequence

$$0 \to \mathcal{T}(\beta) \to \mathcal{T}'(\beta) \to B_\infty \to 0, \tag{3.3}$$

where $\mathcal{T}'(\beta)$ is the set of elements that have a limit at ∞: it is the inductive limit of $\mathcal{T}'_k(\beta)$ (i.e., the closure in $\mathcal{M}(\mathcal{T}(\beta))$ of the increasing union of $\mathcal{T}'_k(\beta)$) where $\mathcal{T}'_k(\beta)$ is the algebra of functions that become constant after k, i.e., such that ϕ_ℓ is constant for $\ell \ge k$ — and of course equal to the image in B_ℓ of the element $\phi_k(k) \in B_k$. Note that we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}(\beta) & \longrightarrow & \mathcal{T}'(\beta) & \longrightarrow & B_\infty & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B_\infty(0,+\infty) & \longrightarrow & B_\infty(0,+\infty] & \longrightarrow & B_\infty & \longrightarrow & 0 \end{array}$$

It follows that the composition of the element in $E^1(B_\infty, \mathcal{T}(\beta))$ given by the exact sequence (3.3) with the morphism $\mathcal{T}(\beta) \to B_\infty(0, +\infty)$ is the unit element of $E^1(B_\infty, B_\infty(0, +\infty)) = E(B_\infty, B_\infty)$.

Using this remark, one obtains the following results (cf. [Rosenberg and Schochet 1987]):

Proposition 3.4. (a) *If I_k and B_k are E-contractible, then $\mathcal{T}(\alpha, \beta)$ is also E-contractible.*

(b) *If (B_k, β_k) is an inductive system of E-contractible C^*-algebras, then their inductive limit B_∞ is E-contractible.*

(c) *If* (B_k, β_k) *is an inductive system of* C^**-algebras then* $\mathcal{T}'(\beta)$ *is* E*-contractible and the element in* $E^1(B_\infty, \mathcal{T}(\beta))$ *given by the exact sequence* (3.3) *is invertible.*

Proof. (a) Indeed, we have a unital ring morphism

$$\prod E(B_k, B_k) \to E\Big(\bigoplus B_k, \bigoplus B_k\Big)$$

and it follows that if the B_k are E-contractible, then $\bigoplus B_k$ is E-contractible. Since $\bigoplus B_k$ and $\bigoplus I_k$ are E-contractible, then by the exact sequence

$$0 \to \bigoplus B_k(0,1) \to \mathcal{T}(\alpha,\beta) \to \bigoplus I_k \to 0,$$

the telescope $\mathcal{T}(\alpha,\beta)$ is E-contractible.

(b) Since the telescope $\mathcal{T}(\beta)$ is E-contractible, the algebra B_∞ is E-contractible since, by Remark 3.2, it is E-subequivalent to $\mathcal{T}(\beta)$.

(c) We have (split) exact sequences $0 \to B_k(k, k+1] \to \mathcal{T}'_{k+1}(\beta) \to \mathcal{T}'_k(\beta) \to 0$ and it follows by induction that, for all k, $\mathcal{T}'_k(\beta)$ is KK-contractible, and therefore E-contractible (note that $\mathcal{T}'_0(\beta) = 0$). It follows that the inductive limit $\mathcal{T}'(\beta)$ is E-contractible and therefore the exact sequence (3.3) induces an E^1-equivalence. □

In fact a telescope can be expressed as a mapping torus:

Remarks 3.5. (a) Recall that given C^*-algebras A, B and morphisms $u, v : A \to B$ the torus C^*-algebra $\mathfrak{T}(u,v)$ is

$$\{(a,\phi) \in A \times B[0,1] : u(a) = \phi(0),\ v(a) = \phi(1)\}.$$

In fact the telescopic C^*-algebra $\mathcal{T}(\alpha,\beta)$ identifies with the torus C^*-algebra $\mathfrak{T}(\breve{\alpha},\breve{\beta})$ of the morphisms $\breve{\alpha},\breve{\beta} : \bigoplus_{k=1}^n I_k \to \bigoplus_{k=0}^n B_k$ defined by

$$\breve{\alpha}((x_k)_k) = (0, \alpha_1(x_1), \ldots, \alpha_k(x_k), \ldots)$$

and

$$\breve{\beta}((x_k)_k) = \begin{cases} (\beta_0(x_1), \ldots, \beta_{n-1}(x_n), 0) & \text{if } n \in \mathbb{N}, \\ (\beta_k(x_{k+1}))_{k\in\mathbb{N}} & \text{if } n = +\infty. \end{cases}$$

(b) In turn, a mapping torus is easily seen to be K-equivalent to a mapping cone. Let A, B be C^*-algebras and $j_\pm : A \to B$ $*$-homomorphisms. Let $j : A(\mathbb{R}_+^*) \to B(\mathbb{R})$ be the $*$-homomorphism defined by

$$j(\phi)(t) = \begin{cases} j_+(\phi(t)) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ j_-(\phi(-t)) & \text{if } t < 0. \end{cases}$$

Then $\mathfrak{T}(j_+, j_-)(\mathbb{R}_+^*)$ is canonically isomorphic with C_j.

Indeed,

$$\begin{aligned}&\mathfrak{T}(j_+, j_-)(\mathbb{R}_+^*)\\&\quad=\big\{(\phi,\psi)\in A(\mathbb{R}_+^*)\times B(\mathbb{R}_+^*\times[0,1]):\psi(t,0)=j_+(\phi(t)),\ \psi(t,1)=j_-(\phi(t))\big\}\end{aligned}$$

and

$$\begin{aligned}C_j&=\left\{(\phi,\psi)\in A(\mathbb{R}_+^*)\times B(\mathbb{R}\times\mathbb{R}_+):\psi(t,0)=\begin{cases}j_+(\phi(t)) & \text{if } t>0,\\ 0 & \text{if } t=0,\\ j_-(\phi(-t)) & \text{if } t<0,\end{cases}\right\}\\&=\left\{(\phi,\psi)\in A(\mathbb{R}_+^*)\times B(\mathbb{R}\times\mathbb{R}_+\setminus\{(0,0)\}):\psi(t,0)=\begin{cases}j_+(\phi(t)) & \text{if } t>0,\\ j_-(\phi(-t)) & \text{if } t<0\end{cases}\right\}\end{aligned}$$

are isomorphic through the homeomorphism $(r,\theta)\mapsto(r\cos\pi\theta, r\sin\pi\theta)$ from $\mathbb{R}_+^*\times[0,1]$ onto $(\mathbb{R}\times\mathbb{R}_+)\setminus\{(0,0)\}$.

3.B. ***The telescope of nicely decomposable foliations.*** Let $(M,\mathcal{F})$ be a *nicely decomposable* foliation of height $n\in\mathbb{N}\cup\{\infty\}$ in the sense of Definition 1.4. Generalizing the case $n=1$, we construct a C^*-algebra which is E-equivalent with the (full) foliation C^*-algebra. We are thus given open subsets $(W_k)_{k<n+1}$, groupoids $\mathcal{G}_k \rightrightarrows W_k$ and morphisms $\mathcal{G}_k|_{W_k\cap W_{k-1}}\to\mathcal{G}_{k-1}$ satisfying the conditions of Definition 1.4.

Let $\Omega_k=\bigcup_{j\le k}W_j$ be the sequence of strata of this decomposition and $Y_k=\Omega_k\setminus\Omega_{k-1}$. Since $\mathcal{F}$ is assumed to be nicely decomposable, we are given Lie groupoids $\mathcal{G}_k \rightrightarrows W_k$ and morphisms of Lie groupoids $q_k:\mathcal{G}_k|_{\Omega_{k-1}}\to\mathcal{G}_{k-1}$ such that $\mathcal{G}_k|_{Y_k}=H(Y_k,\mathcal{F})$.

For every $0\le k<n+1$ consider the full C^*-algebras $A_k=C^*(\Omega_k,\mathcal{F})$ and $B_k=C^*(\mathcal{G}_k)$ and the morphism obtained by integration along the fibers $p_k:B_k\to A_k$. Put also $Q_k=C^*(\mathcal{G}_k|_{Y_k})$. We have the diagram in Figure 6. Here the map q_k is integration along the fibers of the groupoid morphism $q_k:\mathcal{G}_{k+1}|_{\Omega_k}\to\mathcal{G}_k$ and $\pi_k=p_k\circ q_k:I_k\to A_k$. The quotient algebras B_k/I_{k-1} and A_k/A_{k-1} coincide (with Q_k).

Let $\tilde{\iota}_k:A_k\to C^*(M,\mathcal{F})=A_n$ be the inclusion. As for every $k<n$ we have $\tilde{\iota}_k\circ\pi_k=\tilde{\iota}_k\circ p_k\circ q_k=\tilde{\iota}_{k+1}\circ p_{k+1}\circ j_{k+1}$, we get a morphism $\Psi:\mathcal{T}(q,j)\to A_n(0,n+1)$ defined by $\Psi((\phi_k)_{0\le k<n+1},(x_k)_{1\le k<n+1})(t)=\tilde{\iota}_k\circ p_k(\phi_k(t))$ for $t\in[k,k+1]$.

Theorem 3.6. *With the above notation, the class in $E(\mathcal{T}(q,j),A_n(0,n+1))$ of the morphism Ψ is invertible.*

Proof. Let $\mathcal{B}=\{f\in A_n(0,n+1]:\forall t\in\mathbb{R}_+^*,\forall k\in\mathbb{N},\ t-1\le k\le n\Rightarrow f(t)\in A_k\}$ and put $\mathcal{J}=\{f\in B: f(n+1)=0\}$.

The inclusion $\mathcal{J}\to A_n(0,n+1)$ is an E-equivalence (cf. [Rosenberg and Schochet 1987] — if $n<+\infty$, it is a KK-equivalence). Its inverse is given by the exact sequence $0\to\mathcal{J}\to\mathcal{B}\to A_n\to 0$.

$$
\begin{tikzcd}
0 \arrow[r] & I_k \arrow[rr, "j_{k+1}"] \arrow[dd, "\pi_k"'] \arrow[dr, "q_k"] & & B_{k+1} \arrow[r] \arrow[dd, "p_{k+1}"] & Q_{k+1} \arrow[r] \arrow[dd, equal] & 0 \\
& & B_k \arrow[dl, "p_k"] & & & \\
0 \arrow[r] & A_k \arrow[rr, "\iota_k"] \arrow[dr, "\tilde{\iota}_k"'] & & A_{k+1} \arrow[r] \arrow[dl, "\tilde{\iota}_{k+1}"] & A_{k+1}/A_k \arrow[r] & 0 \\
& & C^*(M, \mathcal{F}) & & &
\end{tikzcd}
$$

Figure 6. Short exact sequences of strata.

For $\ell \in \mathbb{N}$, $\ell \leq n$, put $\mathcal{J}_\ell = \{f \in \mathcal{J} : f(t) = 0 \text{ if } t \geq \ell + 1\}$ and let $\mathcal{T}_\ell$ be the ideal

$$\mathcal{T}_\ell = \{((\phi_k)_{0 \leq k < n+1}, (x_k)_{1 \leq k < n+1}) \in \mathcal{T} : \forall k > n,\ \phi_k = 0 \text{ and } x_k = 0\}.$$

Let us show by induction that the morphism $\Psi_\ell : \mathcal{T}_\ell \to \mathcal{J}_\ell$ induced by Ψ is an E-equivalence:

- Ψ_0 is an isomorphism (and the case $\ell = 1$ follows from the proof of Proposition 2.1).
- We have an exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_{\ell-1} \arrow[r] \arrow[d, "\Psi_{\ell-1}"] & \mathcal{T}_\ell \arrow[r] \arrow[d, "\Psi_\ell"] & \mathcal{C}_{j_\ell} \arrow[r] \arrow[d, "\tilde{p}_\ell"] & 0 \\
0 \arrow[r] & \mathcal{J}_{\ell-1} \arrow[r] & \mathcal{J}_\ell \arrow[r] & \mathcal{C}_{\iota_{\ell-1}} \arrow[r] & 0
\end{tikzcd}
$$

where $\tilde{p}_\ell : \mathcal{C}_{j_\ell} \to \mathcal{C}_{\iota_{\ell-1}}$ is the morphism induced by $p_\ell : B_\ell \to A_\ell$ at the cone level.

Examining Figure 6, as j_ℓ and $\iota_{\ell-1}$ are inclusions of ideals and p_ℓ induces an isomorphism $B_\ell/I_\ell \to A_\ell/A_{\ell-1}$, we deduce that $\tilde{p}_\ell$ is E-invertible. We thus obtain the induction step.

If n is finite, the proof is complete.

If $n = +\infty$, the mapping cones $\mathcal{C}_{\Psi_\ell}$ are E-contractible for all ℓ and it follows that their inductive limit $\mathcal{C}_\Psi$ is E-contractible. □

4. Longitudinally smooth groupoids and equivariant *KK*-theory

We proved in Theorem 3.6 that the telescopic algebra $\mathcal{T}(q, j)$ has the same K-theory as $C^*(M, \mathcal{F})$. In the next section, we will build a Baum–Connes map for the telescopic algebra $\mathcal{T}(q, j)$, which will give us a Baum–Connes map for $C^*(M, \mathcal{F})$.

The telescopic algebra $\mathcal{T}(q, j)$ associated with a nicely decomposable foliation, as well as the foliation algebra itself (thanks to [Debord 2013]), is the C^*-algebra of a *longitudinally smooth groupoid* in a sense that we briefly describe here.

Note that all the constructions we give below generalize easily to groupoids that are covered by $C^{\infty,0}$ manifolds or even locally compact spaces with Haar measures.

4.A. ***Some "classical" constructions with Lie groupoids.*** Before proceeding to explain this construction, we recall some constructions based on Lie groupoids that we use.

Pull-back groupoid. Let $G \overset{t,s}{\rightrightarrows} G^{(0)}$ be a Lie groupoid, M a smooth manifold and $q : M \to G^{(0)}$ a smooth submersion. The pull-back groupoid G_q^q is a subgroupoid

$$G_q^q = \{(x, \gamma, y) \in M \times G \times M : q(x) = t(\gamma) \text{ and } q(y) = s(\gamma)\}$$

of the product groupoid of G with the pair groupoid $M \times M$. As q is supposed to be a submersion, G_q^q is a Lie groupoid (actually, a transversality condition suffices). If $q(M)$ meets all the G-orbits, the groupoids G and G_q^q are canonically Morita equivalent.

Actions on spaces. Recall that an action of a groupoid $G \overset{t,s}{\rightrightarrows} G^{(0)}$ on a space X is given by a map $p : X \to G^{(0)}$ and the action $G \times_{s,p} X \to X$ denoted by $(\gamma, x) \mapsto \gamma.x$ with the requirements $p(\gamma.x) = t(\gamma)$, $\gamma.(\gamma'.x) = (\gamma\gamma').x$ and $u.x = x$ if $u = p(x)$.

Semidirect product. If a groupoid G acts on a space X, we may form the semidirect product groupoid $X \rtimes G$:

- As a set $X \rtimes G = X \times_t G = \{(x, \gamma) \in X \times G : t(\gamma) = x\}$.
- $(X \rtimes G)^{(0)} = X$; we have $t(x, \gamma) = x$ and $s(x, \gamma) = \gamma^{-1}.x$.
- The elements (x, γ) and (y, γ') are composable if $x = \gamma y$; the composition is then $(x, \gamma)(y, \gamma') = (x, \gamma\gamma')$.

When p is a submersion, $X \rtimes G$ is a Lie groupoid: it is the closed subgroupoid $\{(x, \gamma, y) \in G_p^p : x = \gamma.y\}$ of G_p^p.

Actions on groupoids. This construction can be generalized. If $X \overset{t_X,s_X}{\rightrightarrows} X^{(0)}$ is a groupoid, we say that the action is by groupoid automorphisms [Brown 1972] if G acts on $X^{(0)}$ through a map $p_0 : X^{(0)} \to G^{(0)}$, we have $p = p_0 \circ t_X = p_0 \circ s_X$ and $\gamma.(xy) = (\gamma.x)(\gamma.y)$. There is a semidirect product construction in this generalized setting.

4.B. ***Longitudinally smooth groupoids.*** A *longitudinally smooth groupoid* is a groupoid $G \overset{t,s}{\rightrightarrows} G^{(0)}$ such that

- its set of objects is endowed with a structure of smooth manifold (possibly with boundary or corners);
- for every $x \in G^{(0)}$, the set $G^x = t^{-1}(\{x\})$ also carries a smooth structure (without boundary) and the source map $s : G^x \to G^{(0)}$ is smooth with (locally) constant rank;

- the "smooth structure" of G itself is given by an *atlas*, which is a family of smooth (Hausdorff) manifolds $(U_i)_{i\in I}$ (possibly with boundary or corners) and maps $q_i : U_i \to G$.

We assume that these smooth structures satisfy the following conditions:

Compatibility. For every $i \in I$, the maps $t \circ q_i$ and $s \circ q_i$ are smooth submersions; for every $i \in I$ and every $x \in G^{(0)}$ the map q_i induces a smooth submersion $q_i^{-1}(G^x) \to G^x$.

Minimal elements. For every $\gamma \in G$, there exists $i \in I$ and $z \in U_i$ such that $q_i(z) = \gamma$ and the map $q_i^{-1}(G^{t(\gamma)}) \to G^{t(\gamma)}$ is a local diffeomorphism near z. If $j \in I$ and $z' \in U_j$ are such that $q_j(z') = \gamma$, then there is an open neighborhood $V' \subset U_j$ of z' and a submersion $\varphi : V' \to U_i$ such that $q_i \circ \varphi = (q_j)_{|V'}$.

Inverse is smooth. For every $i \in I$, there exists $j \in I$ and a diffeomorphism $\kappa : U_i \to U_j$ such that $q_j \circ \kappa(z) = q_i(z)^{-1}$ for every $z \in U_i$.

Composition is smooth. For every $i, j \in I$, let $U_i \circ U_j$ be the fibered product $U_i \times_{s\circ q_i, t\circ q_j} U_j$. For every $(z_i, z_j) \in U_i \circ U_j$, there is a $k \in I$, a neighborhood W of (z_i, z_j) in $U_i \circ U_j$ and a submersion $\varphi : W \to U_k$ such that for all $(w_i, w_j) \in W$ we have $q_i(w_i)q_j(w_j) = q_k \circ \varphi(w_i, w_j)$.

Exactly as in [Androulidakis and Skandalis 2009], we may associate to a longitudinally smooth groupoid a C^*-algebra $C^*(G)$ (as well as a reduced one, since the s-fibers are assumed to be manifolds).

Examples 4.1. (a) A Lie groupoid is of course a longitudinally smooth groupoid. The atlas is the groupoid itself!

(b) The holonomy groupoid of a singular foliation is such a longitudinally smooth groupoid; the atlas is given by bisubmersions [Androulidakis and Skandalis 2009].

(c) The telescopic algebra $\mathcal{T}(q, j)$ constructed in the previous section is associated with the groupoid $G = \bigcup_{k=0}^{n} \mathcal{G}_k \times (k, k+1) \cup \bigcup_{k=1}^{n} (\mathcal{G}_k)_{\Omega_{k-1}\cap W_k} \times \{k\}$. Its set of objects is the open subset

$$\left(\bigcup_{k=0}^{n} W_k \times (k, k+1)\right) \cup \left(\bigcup_{k=1}^{n} (\Omega_{k-1} \cap W_k) \times (k-1, k+1)\right)$$

of $M \times \mathbb{R}_+^*$.

It is endowed with the atlas formed by the Lie groupoids $(\mathcal{G}_k \times (k, k+1))_{k\in\mathbb{N},\, k\le n}$ and $((\mathcal{G}_k)_{\Omega_{k-1}\cap W_k} \times (k-1, k+1))_{k\in\mathbb{N},\, 1\le k\le n}$.

4.C. ***Action of a longitudinally smooth groupoid on a C*-algebra.*** We now fix a longitudinally smooth groupoid $G \overset{t,s}{\rightrightarrows} M$ with an atlas $(U_i, q_i)_{i\in I}$. We put $s_i = s \circ q_i$ and $t_i = t \circ q_i$.

4.C.1. *Action of a locally compact groupoid on a C^*-algebra.* For the convenience of the reader we recall some definitions on $C(X)$-algebras and actions of locally compact groupoids from [Kasparov 1988; Le Gall 1999]. In this section, all spaces — and groupoids — are assumed to be Hausdorff.

Let M be a locally compact space.

(a) A $C_0(M)$-algebra is a pair (A, θ), where A is a C^*-algebra and θ is a $*$-homomorphism from $C_0(M)$ to the center $\mathcal{Z}\mathcal{M}(A)$ of the multiplier algebra of A such that $\theta(C_0(M))A = A$.

(b) Put $A_b = \{a \in \mathcal{M}(A) : \phi a \in A \text{ for all } \phi \in C_0(X)\}$ and $A_c = C_c(X)A$.

(c) Let A, B be $C_0(M)$-algebras. A homomorphism of $C_0(M)$-algebras $\phi : A \to B$ is a $C_0(M)$-linear homomorphism of C^*-algebras.

(d) Let N be a locally compact space and $p : N \to M$ a continuous map. Then $C_0(N)$ is a $C_0(M)$-algebra via the map $\theta = p^* : C_0(M) \to C_b(N) = \mathcal{M}(C_0(M))$.

Let A be a $C_0(M)$-algebra.

(e) For every $x \in M$ there is a fiber $A_x = A/C_x A$, where $C_x = \{h \in C_0(M) : h(x) = 0\}$. The natural map $A \to \prod_{x \in M} A_x$ induced by the quotient maps $\pi_x : A \to A_x$ is injective. For instance, in (d), given $x \in M$ the fiber $C_0(N)_x$ is $C_0(N_x)$, where $N_x = p^{-1}(x)$.

(f) A homomorphism of $C_0(M)$-algebras $\phi : A \to B$ induces a homomorphism of C^*-algebras $(\phi_x)_{x \in M} : \prod_{x \in M} A_x \to \prod_{x \in M} B_x$. The homomorphism ϕ is injective (surjective) if and only if ϕ_x is injective (surjective) for every $x \in M$.

(g) There are natural operations of restriction to open and closed subsets of M. If U is an open subset of M and $F = X \setminus U$, the algebra $C_0(U)$ identifies with the ideal $C_0(U) = \{f \in C_0(M) : f(y) = 0 \text{ for all } y \in F\}$ of $C_0(M)$. Then A_U denotes the $C_0(U)$-algebra $C_0(U)A$ and A_F the $C_0(F)$-algebra A/A_U. If $Y \subset X$ is locally closed, then Y is open in $\overline{Y}$ and A_Y denotes the $C_0(Y)$-algebra $(A_{\overline{Y}})_Y$.

(h) Let A be a $C_0(M)$ and B a $C_0(N)$-algebra. Then $A \otimes_{\max} B$ is a $C_0(M \times N)$-algebra. When $M = N$, the restriction of $A \otimes_{\max} B$ to the diagonal $\{(x, x) : x \in M\}$ (which is a closed subset of $M \times M$) is a $C_0(M)$-algebra denoted $A \otimes_{C_0(M)} B$.

(i) Again let A be a $C_0(M)$-algebra and consider a smooth map $p : N \to M$. We denote by p^*A the $C_0(N)$-algebra obtained by restricting $A \otimes C_0(N)$ to the graph $\{(p(y), y) : y \in N\}$, which is a closed subset of $M \times N$ (here $C_0(N)$ is regarded as a $C_0(N)$-algebra). It is easy to see that this construction has the following properties:

- $(p^*A)_y = A_{p(y)}$ for every $y \in N$;
- if A, B are $C_0(M)$-algebras then $p^*A \otimes_{C_0(Y)} p^*B = p^*(A \otimes_{C_0(M)} B)$;
- if $q : Z \to N$ is a smooth map then $q^*(p^*A) = (p \circ q)^*A$.

(j) With the previous notation, for every $a \in A$ we put $p^*a = a \otimes 1 \in (p^*A)_b$. If $\phi : A \to B$ is a homomorphism of $C_0(M)$-algebras we put

$$p^*\phi = \phi \otimes \mathrm{id}_{C_0(N)} : p^*A \to p^*B.$$

(k) An action of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on a $C_0(M)$-algebra A is defined in [Le Gall 1999] by an isomorphism α of $C_0(\mathcal{G})$-algebras $s^*A \to t^*A$. This isomorphism is given by a family of isomorphisms $\alpha_\gamma : A_{s(\gamma)} \to A_{t(\gamma)}$ for $\gamma \in \mathcal{G}$. The isomorphism α is required to be a representation of $\mathcal{G}$, i.e., to satisfy $\alpha_{\gamma\circ\gamma'} = \alpha_\gamma \circ \alpha_{\gamma'}$ for all $(\gamma, \gamma') \in \mathcal{G}^{(2)} = \mathcal{G} \times_{s,t} \mathcal{G}$.

4.C.2. *Action of a longitudinally smooth groupoid.* Let $G \overset{t,s}{\rightrightarrows} M$ be a longitudinally smooth groupoid with an atlas $(U_i, q_i)_{i\in I}$. We put $s_i = s \circ q_i$ and $t_i = t \circ q_i$.

Definition 4.2. A G-algebra is a $C_0(M)$-algebra A together with an isomorphism of $C_0(U_i)$-algebras $\alpha^i : s_i^*A \to t_i^*A$ for every $i \in I$.

(a) The isomorphism α^i is a family $(\alpha^i_u)_{i\in I}$ of isomorphisms $\alpha^i_u : A_{s_i(u)} \to A_{t_i(u)}$. We require that if $\gamma \in G$ is represented by two elements $u_i \in U_i$ and $u_j \in U_j$ (with $i, j \in I$), then $\alpha^i_{u_i} = \alpha^j_{u_j}$.

(b) By (a), we get a well-defined isomorphism $\alpha_\gamma : A_{s(\gamma)} \to A_{t(\gamma)}$. We require that for every composable $\gamma, \gamma' \in G$, we have $\alpha_{\gamma\gamma'} = \alpha_\gamma \circ \alpha_{\gamma'}$.

Definition 4.3. Let (A, α) and (B, β) be G-algebras.

(a) A morphism $\phi : A \to B$ is said to be G-equivariant if it is $C_0(M)$-linear and for every $\gamma \in G$ we have $\phi_{t(\gamma)} \circ \alpha(\gamma) = \beta(\gamma) \circ \phi_{s(\gamma)}$.

(b) More generally, let $\phi : A \to \mathcal{M}(B)$ be a morphism. Let

$$D = G(\phi) \oplus (0 \oplus B) \subset A \oplus \mathcal{M}(B),$$

where $G(\phi) = \{(x, \phi(x)) : x \in A\}$ is the graph of ϕ. We say that ϕ is equivariant if there is an action of G on D such that the inclusions $A \to D$ and $B \to D$ are equivariant.

Examples 4.4. (a) The algebra $C_0(M)$ is a G-algebra. For every $i \in I$, we have $t^*(C_0(M)) = C_0(U) = s^*(C_0(M))$; the action α is the identity. For $i \in I$, then at every $u \in U_i$ we associate the identity map $(\mathbb{C} \to \mathbb{C})$. In a sense, this corresponds to the trivial representation.

(b) More generally, let $Y \subset M$ be a locally closed saturated subset (i.e., such that for every $\gamma \in G$ we have $t(\gamma) \in Y$ if and only if $s(\gamma) \in Y$). Then $C_0(Y)$ is an $H(\mathcal{F})$-algebra. In that case, for every $i \in I$, we have $t^{-1}(Y) = s^{-1}(Y)$ since Y is saturated and $t^*(C_0(Y)) = C_0(t^{-1}(Y)) = C_0(s^{-1}(Y)) = s^*(C_0(Y))$. Again, the action α is the identity.

4.C.3. *Covariant representations and full crossed products.* Let us very briefly extend some constructions of [Androulidakis and Skandalis 2009, §4 and 5] to the more general case of our longitudinally smooth groupoid $G \rightrightarrows M$ with atlas $(U_i, q_i)_{i\in I}$.

• When $f : N \to M$ is a smooth submersion of manifolds, we may define a Hilbert $C_0(M)$-module $\mathcal{E}_f$ obtained by completion of the space $C_c(N; \Omega^{1/2}\ker(df))$ with respect to the $C_0(M)$-valued inner product defined by $\langle \xi, \eta\rangle(x) = \int_{z\in f^{-1}(x)} \overline{\xi(z)}\eta(z)$. This Hilbert module is endowed with an action of $C_0(N)$.

• Let $i \in I$. We may then construct two Hilbert C^*-modules $\mathcal{E}_{t_i}$ and $\mathcal{E}_{s_i}$ over $C_0(M)$.

• As $C_0(M)$ sits in the multiplier algebra of $C^*(G)$, every representation π_G of $C^*(G)$ on a Hilbert space $\mathcal{H}$ gives rise to a representation π_M of $C_0(M)$.

• The representation π is then characterized by π_M and, for every $i \in I$, a unitary $V_i \in \mathcal{L}(\mathcal{E}_{s_i} \otimes_{C_0(M)} \mathcal{H}, \mathcal{E}_{t_i} \otimes_{C_0(M)} \mathcal{H})$ intertwining the representations of $C_0(U)$. It therefore defines a measurable family of unitaries $U_u : H_{s_i(u)} \to H_{t_i(u)}$. We require that U_u only depends on the class of $q_i(u)$ in G (almost everywhere) and that (almost everywhere) it determines a representation of the groupoid G. See [Androulidakis and Skandalis 2009, §5.2] for the details.

Let G act on a C^*-algebra A and let π_A be a representation of A on a Hilbert space $\mathcal{H}$. Using the morphism from $C_0(M)$ to the multiplier algebra of A, we obtain a representation of $C_0(M)$ to $\mathcal{L}(\mathcal{H})$. For every $i \in I$, as the image of $C_0(M)$ sits in the center of the multiplier algebra of A, we have representations

$$\pi_A^{s_i} : s_i^*(A) \to \mathcal{L}(\mathcal{E}_{s_i} \otimes_{C_0(M)} \mathcal{H}) \quad \text{and} \quad \pi_A^{t_i} : t_i^*(A) \to \mathcal{L}(\mathcal{E}_{t_i} \otimes_{C_0(M)} \mathcal{H}).$$

A *covariant representation* of G and A is given by a representation of π_G of $C^*(G)$ and a representation π_A of A in the same Hilbert space $\mathcal{H}$ such that the two representations of $C_0(M)$ agree and, for every $i \in I$, the unitary V_i intertwines $\pi_A^{s_i} \circ \alpha^i$ with $\pi_A^{t_i}$.

Then the closed linear span of $\pi_A(a)\pi_G(x)$, where a runs over A and x over $C^*(G)$, is a $*$-subalgebra of $\mathcal{L}(\mathcal{H})$.

Definition 4.5. The *full crossed product* $A \rtimes G$ is the completion of this linear span with respect to the supremum norm over all covariant representations.

Using the "regular representations" on $L^2(G_x)$, one may also construct a natural reduced crossed product.

4.C.4. *Actions of a longitudinally smooth groupoid on Hilbert modules.* Let G, $(U_i)_{i\in I}$, s_i, t_i be as above.

Let (A, α) be a G-algebra and $\mathcal{E}$ a Hilbert module over A. As usual, we may define an action of G on $\mathcal{E}$ by saying that it is just given by an action of G on the C^*-algebra $\mathcal{K}(\mathcal{E} \oplus A)$ in such a way that the natural morphism $A \to \mathcal{K}(\mathcal{E} \oplus A)$ is equivariant. This amounts to giving, for any $i \in I$, an isomorphism

$$\widetilde{\alpha} : \mathcal{E} \otimes_A s_i^* A \to \mathcal{E} \otimes_A t_i^* A$$

of Banach spaces, which corresponds to a family of isomorphisms $\widetilde{\alpha}_u : \mathcal{E}_{s_i(u)} \to \mathcal{E}_{t_i(u)}$. We need compatibility with α, which means that for every $x \in A_{s_i(u)}$ and $\xi, \zeta \in \mathcal{E}_{s_i(u)}$, we have $\widetilde{\alpha}_u(\xi x) = \widetilde{\alpha}_u(\xi)\alpha_u(x)$ and $\alpha_u(\langle \xi | \zeta \rangle) = \langle \widetilde{\alpha}_u(\xi) | \widetilde{\alpha}_u(\zeta) \rangle$.

As above, we require that $\widetilde{\alpha}_u$ only depends on the class of u in G and that the so defined $\widetilde{\alpha}_\gamma : \mathcal{E}_{s(\gamma)} \to \mathcal{E}_{t(\gamma)}$ for $\gamma \in H(\mathcal{F})$ defines a morphism of groupoids, which means that $\widetilde{\alpha}_{\gamma\gamma'} = \widetilde{\alpha}_\gamma \tilde{\alpha}_{\gamma'}$. Note also that, given an action of G on a $\mathcal{E}$, we obtain for any $i \in I$ an isomorphism of $C_0(U_i)$-algebras $\mathcal{K}(\mathcal{E} \otimes_A s_i^* A) \to \mathcal{K}(\mathcal{E} \otimes_A t_i^* A)$ and of their multipliers $\check{\alpha}_U : \mathcal{L}(\mathcal{E} \otimes_A s_i^* A) \to \mathcal{L}(\mathcal{E} \otimes_A t_i^* A)$.

4.D. ***G-equivariant KK-theory.*** Let $G \overset{t,s}{\rightrightarrows} M$ be a longitudinally smooth groupoid with an atlas $(U_i, q_i)_{i \in I}$. We put $s_i = s \circ q_i$ and $t_i = t \circ q_i$.

Here we use the apparatus developed in the previous sections to construct the topological K-theory at the left-hand side of the Baum–Connes conjecture in classical terms (e.g., as in [Le Gall 1999]). Namely we define the groups $KK_G(A, B)$ in Section 4.D.1. The difficulty is to construct the Kasparov product; we do this in Section 4.D.3.

4.D.1. *Equivariant Kasparov cycles.* We may of course define graded G-algebras, graded Hilbert modules, etc.

In what follows, all algebras are $\mathbb{Z}/2\mathbb{Z}$-graded and all commutators are graded ones. Also, all the C^*-algebras and Hilbert C^*-modules that we consider are supposed to be separable. Recall the following from [Kasparov 1980]:

- Let A, B be graded C^*-algebras. An (A, B) bimodule is a pair $(\mathcal{E}, \pi_A)$, where $\mathcal{E}$ is a B-Hilbert C^*-module and $\pi_A : A \to \mathcal{L}(\mathcal{E})$ a representation which preserves the degree. For every $\xi \in \mathcal{E}$ and $a \in A$ we denote $a\xi = \pi_A(a)(\xi)$.
- A Kasparov (A, B) bimodule is a triple $(\mathcal{E}, \pi_A, F)$, where $(\mathcal{E}, \pi_A)$ is an (A, B) bimodule and $F \in \mathcal{L}(\mathcal{E})$ is of degree 1 (for the grading) and for all $a \in A$, the elements $[F, \pi_A(a)]$, $(F - F^*)\pi_A(a)$ and $(1 - F^2)\pi_A(a)$ are all in $\mathcal{K}(\mathcal{E})$.

Definition 4.6. Let (A, B) be G-algebras. A *G-equivariant Kasparov (A, B) bimodule* is a Kasparov (A, B) bimodule $(\mathcal{E}, \pi_A, F)$ with the following properties.

(a) $\mathcal{E}$ is endowed with an action of G (see Section 4.C.4) and the representation $\pi_A : A \to \mathcal{L}(\mathcal{E}) = \mathcal{M}(\mathcal{K}(\mathcal{E}))$ is G-equivariant (in the sense of Definition 4.3(b)).

(b) For every $i \in I$ and $h \in C_0(U_i)$, we have $(\check{\alpha}_i(F \otimes 1) - F \otimes 1)h \in \mathcal{K}(\mathcal{E} \otimes_A t_i^* A)$.

Two G-equivariant Kasparov bimodules $(\mathcal{E}, \pi_A, F)$, $(\mathcal{E}', \pi'_A, F')$ are *unitarily equivalent* if there exists a G-equivariant unitary $U \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ of degree 0 which satisfies $UFU^* = F'$ and $U\pi_A(a)U^* = \pi'_A(a)$ for all $a \in A$.

Denote by $E_G(A, B)$ the set of equivalence classes of G-equivariant Kasparov bimodules. A homotopy in $E_G(A, B)$ is an element of $E_G(A, B[0, 1])$. We define $KK_G(A, B)$ to be the set of homotopy classes of elements of $E_G(A, B)$.

The direct sum of Kasparov bimodules induces an abelian group structure in $KK_G(A, B)$. We define the unit element $1_A \in KK_G(A, A)$ as the class of $(A, \iota_A, 0)$, where $\iota_A(a) = a \in \mathcal{K}(A)$ for all $a \in A$, where the action of G on the C^*-module A is the action of G on the C^*-algebra A.

4.D.2. *Kasparov's descent morphism.* Given an equivariant Hilbert B module $\mathcal{E}$, we may define the crossed product $\mathcal{E} \rtimes G = \mathcal{E} \otimes_B B \rtimes G$ — and the same for the reduced crossed product. If we have an equivariant action $A \to \mathcal{L}(E)$, we naturally obtain an action $A \rtimes G \to \mathcal{L}(\mathcal{E} \rtimes G)$.

Let $(\mathcal{E}, F)$ be an equivariant Kasparov (A, B) bimodule. Let

$$F \widehat{\otimes} 1 \in \mathcal{L}(\mathcal{E} \otimes_B B \rtimes G) = \mathcal{L}(\mathcal{E} \rtimes G).$$

We check as in [Le Gall 1999] that $(\mathcal{E} \rtimes G, F \widehat{\otimes} 1)$ is a Kasparov $(A \rtimes G, B \rtimes G)$ bimodule. This construction gives a well-defined *descent morphism*

$$j_G : KK_G(A, B) \to KK(A \rtimes G, B \rtimes G).$$

In the same way we also obtain a *reduced descent morphism.*

4.D.3. *Kasparov product — a general approach.* In order to define the Kasparov product in this equivariant context, we need first to understand the analogue of Kasparov's "technical theorem" [1980, §3, Theorem 4]. It turns out that, in a sense, the original theorem actually applies when formulated in a slightly different way. In turn, this formulation contains many equivariant generalizations.

We start by recalling Voiculescu's theorem on quasicentral approximate units [Voiculescu 1990; Arveson 1977].

Lemma 4.7. *Let D_1 be a C^*-algebra and $D_2 \subset D_1$ be a closed essential two-sided ideal. Let $h \in D_1$ be a strictly positive element with $\|h\| \leq 1$. Let $b \in D_1$ and let $K \subset \mathcal{M}(D_2)$ be a (norm) compact subset such that $[h, k] \in D_1$ for all $k \in K$; let $\varepsilon > 0$. Let $f_0 : [0, 1] \to [0, 1]$ be a continuous function such that $f(0) = 0$. Then there exists continuous $f : [0, 1] \to [0, 1]$ such that $f(0) = 0$, $f_0 \leq f$, $\|b - f(h)b\| < \varepsilon$ and $\|[f(h), k]\| < \varepsilon$ for all $k \in K$.* □

The following result is in fact proved in [Kasparov 1980, §3, Theorem 4]. Formulated in this way, it further contains many generalizations of the Kasparov product [Kasparov 1988; Baaj and Skandalis 1989; Le Gall 1999]. One immediately sees that Higson's proof [1987] applies, so we omit it.

If J is a closed two-sided ideal in a C^*-algebra B, then

$$\mathcal{M}(B; J) = \{x \in \mathcal{M}(B) : xB \subset J\}.$$

Theorem 4.8. (cf. [Kasparov 1980, §3, Theorem 4]) *Let D_1 be a separable graded C^*-algebra and D_2 a graded closed essential two-sided ideal in D_1. Consider $b \in \mathcal{M}(D_1; D_2)_+$. Let also A_1 be a graded C^*-subalgebra of D_1 containing a strictly positive element of D_1 and such that $A_2 = A_1 \cap D_2$ contains a strictly positive element of D_2. Let $K \subset \mathcal{M}(D_2)$ be a compact subset such that, for every $x \in A_1$ and every $k \in K$, we have $[x, k] \in D_1$. Then there exists $M \in \mathcal{M}(A_1; A_2)^{(0)}$ such that $0 \le M \le 1$, $(1 - M)b \in D_2$ and $[M, K] \subset D_2$.* □

One obtains easily a formulation which encodes many equivariant formulations of the product.

Notation 4.9. Let D be a separable graded C^*-algebra and $A \subset D$ a subalgebra containing a strictly positive element of $D^{(0)}$.

Let $\mathcal{I}$ denote the set of graded closed two-sided essential ideals I of D such that $I \cap A^{(0)}$ contains a strictly positive element of I.

Let $D_1, D_2 \in \mathcal{I}$ be such that $D_2 \subset D_1$. Put $A_i = D_i \cap A$. Denote by $\mathbb{E}_A(D_1, D_2)$ the set of $F \in \mathcal{M}(A_2)^{(1)}$ such that for all $x \in D_1$, we have

$$x(1 - F^2) \in D_2, \quad x(F - F^*) \in D_2, \quad [x, F] \in D_2.$$

In other words (A_2, F) is a Kasparov (A_1, A_2) bimodule and (D_2, F) is a Kasparov (D_1, D_2) bimodule.

Theorem 4.10. *Let $D_0, D_1, D_2 \in \mathcal{I}$ such that $D_2 \subset D_1 \subset D_0$. Let $F_1 \in \mathbb{E}_A(D_0, D_1)$ and $F_2 \in \mathbb{E}_A(D_1, D_2)$. Let*

$$F_1 \sharp F_2 = \{F \in \mathbb{E}_A(D_0, D_2) : F - F_2 \in \mathcal{M}(A_1; A_2),\ [F, F_1] \in \mathcal{M}(A_2)_+ + A_2\}.$$

(a) *For every $F_1 \in \mathbb{E}_A(D_0, D_1)$ and $F_2 \in \mathbb{E}_A(D_1, D_2)$ the set $F_1 \sharp F_2$ is nonempty and path connected.*

(b) *The path connected component of $F \in F_1 \sharp F_2$ in $\mathbb{E}_A(D_0, D_2)$ only depends on the path connected components of $F_1 \in \mathbb{E}_A(D_0, D_1)$ and of $F_2 \in \mathbb{E}_A(D_1, D_2)$.*

(c) *(Associativity). Let $D_3 \in \mathcal{I}$ with $D_3 \subset D_2$ and $F_3 \in \mathbb{E}_A(D_2, D_3)$. Let $F_1' \in F_1 \sharp F_2$ and $F_2' \in F_2 \sharp F_3$. Then $F_1' \sharp F_3$ and $F_1 \sharp F_2'$ are contained in the same path connected component of $\mathbb{E}_A(D_0, D_3)$.*

Proof. The proof is exactly the same as in the "classical" case (cf. [Kasparov 1980; Connes and Skandalis 1984; Skandalis 1984b]).

For instance, to establish that $F_1 \sharp F_2$ is nonempty, we take $Q = C^*(D_0, F_1, F_2)$. Let K be a compact subset of Q generating Q as a closed space and let b be a strictly positive element of $Q \cap \mathcal{M}(D_1; D_2)$. Apply then Theorem 4.8, and put $F = M^{1/2} F_1 + (1 - M)^{1/2} F_2$.

If we start with paths $F_1^t \in \mathbb{E}_A(A_0, A_1)$, $F_2^t \in \mathbb{E}_A(A_1, A_2)$, we just take a bigger algebra: $Q = C^*(A_0, \{F_1^t, F_2^t : t \in [0, 1]\})$. The associativity is proved exactly as Lemma 22 in [Skandalis 1984b]. □

We now introduce further notation in order to relate this theorem with equivariant KK-theory.

Notation 4.11. Let $A, \mathbb{A}$ be separable graded C^*-algebras. Let $\varphi, \psi : \mathcal{M}(A) \to \mathcal{M}(\mathbb{A})$ be two grading-preserving strictly continuous morphisms.

Let $\mathcal{J}$ denote the set of closed two-sided essential ideals I of A such that $\varphi(I)\mathbb{A} = \psi(I)\mathbb{A}$.

Let $A_1, A_2 \in \mathcal{J}$ be such that $A_2 \subset A_1$. Put $\mathbb{A}_i = \varphi(A_i)\mathbb{A}$. Denote by $\mathbb{E}_{\varphi,\psi}(A_1, A_2)$ the set of $F \in \mathcal{M}(A_2)^{(1)}$ such that

(a) for all $x \in A_1$, we have $x(1 - F^2) \in A_2$, $x(F - F^*) \in A_2$, $[x, F] \in A_2$ (in other words (A_2, F) is a Kasparov (A_1, A_2) bimodule);

(b) $(\varphi - \psi)(F) \in \mathcal{M}(\mathbb{A}_1; \mathbb{A}_2)$ (the "equivariance property").

As an immediate consequence of Theorem 4.10, we have:

Corollary 4.12. (a) *For every $F_1 \in \mathbb{E}_{\varphi,\psi}(A_0, A_1)$ and $F_2 \in \mathbb{E}_{\varphi,\psi}(A_1, A_2)$ the set $F_1 \sharp F_2$ is nonempty and path connected.*

(b) *The path connected component of $F \in F_1 \sharp F_2$ only depends on the path connected components of $F_1 \in \mathbb{E}_{\varphi,\psi}(A_0, A_1)$ and of $F_2 \in \mathbb{E}_{\varphi,\psi}(A_1, A_2)$.*

(c) (*Associativity*). *Let $A_3 \in \mathcal{I}$ with $A_3 \subset A_2$ and $F_3 \in \mathbb{E}_{\varphi,\psi}(A_2, A_3)$. Let $F_1' \in F_1 \sharp F_2$ and $F_2' \in F_2 \sharp F_3$. Then $F_1' \sharp F_3$ and $F_1 \sharp F_2'$ are contained in the same path connected component of $\mathbb{E}_{\varphi,\psi}(A_0, A_3)$.*

Proof. Let $\chi : \mathcal{M}(A) \to \mathcal{M}(A \oplus M_2(\mathbb{A}))$ be the morphism

$$x \mapsto x \oplus \begin{pmatrix} \varphi(x) & 0 \\ 0 & \psi(x) \end{pmatrix}$$

and put $D = \chi(A) + (0 \oplus M_2(\mathbb{A})) \subset \mathcal{M}(A \oplus M_2(\mathbb{A}))$.

Let $A_1, A_2 \in \mathcal{J}$ be such that $A_2 \subset A_1$. Put

$$\mathbb{A}_i = \varphi(A_i)\mathbb{A} \quad \text{and} \quad D_i = \chi(A_i) + (0 \oplus M_2(\mathbb{A}_i)) \subset D.$$

We obviously have $\mathbb{E}_{\varphi,\psi}(A_1, A_2) = \mathbb{E}_{\chi(A)}(D_1, D_2)$. Therefore, Theorem 4.10 immediately applies. □

Examples 4.13. It is very easy to apply this abstract theorem (Corollary 4.12) to many equivariant situations.

(a) (see [Kasparov 1988]) If a second countable locally compact group G acts on separable C^*-algebras A and B, an equivariant Kasparov (A, B) bimodule is then a pair $(\mathcal{E}, F)$ where:

1. $\mathcal{E}$ is an (A, B)-equivariant Hilbert bimodule;
2. $F \in \mathbb{E}_{\varphi,\psi}(A_1, A_2)$, where we have put
   - $A_2 = \mathcal{K}(\mathcal{E})$ and $A_1 = A + A_2$,
   - $\mathbb{A}_i = C_0(G; A)$,
   - $\varphi, \psi : A_i \to C_b(G; A_i) \subset \mathcal{M}(\mathbb{A}_i)$ defined by $\varphi(a)(g) = a$ and $\psi(a)(g) = g.a$.

(b) (see [Baaj and Skandalis 1989]) Exactly in the same way, if S is a separable Hopf algebra, given a C^*-algebra A with an action $\alpha : A \to \mathcal{M}(A \otimes S)$ of S, we just put $\mathbb{A} = A \otimes S$ and let $\varphi : a \mapsto a \otimes 1$ and $\psi = \alpha$.

(c) (see [Le Gall 1999]) If $G \overset{s,t}{\rightrightarrows} G^{(0)}$ is a second countable locally compact groupoid, given a C^*-algebra A with an action $\alpha : s^*A \to t^*A$ of G, we just put $\mathbb{A} = t^*A$ and let $\varphi : a \mapsto t^*a \in \mathcal{M}(t^*A)$ and $\psi(a) = \alpha(s^*a)$.

4.D.4. *Kasparov product in KK_G.* Let G be a longitudinally smooth groupoid with atlas $(U_i)_{i \in I}$. We assume that I is countable.

Let A_0, A_1, A_2 be G-algebras. Let $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$ be equivariant (A_0, A_1) and (A_1, A_2) cycles. Put $\mathcal{E} = \mathcal{E}_1 \widehat{\otimes}_{A_1} \mathcal{E}_2$. Put $F_1' = F_1 \widehat{\otimes} 1$ and let F_2' be an F_2 connection. Put $\check{A}_2 = \mathcal{K}(\mathcal{E})$, $\check{A}_1 = \mathcal{K}(\mathcal{E}_1) \widehat{\otimes} 1 + \check{A}_2$ and $\check{A}_0 = A_0 + \check{A}_1$ (where we denote by A_0 its image in $\mathcal{L}(\mathcal{E})$).

The algebras $\check{A}_i$ are G algebras, the inclusions $\check{A}_2 \subset \check{A}_1 \subset \check{A}_0$ are equivariant and the pairs $(\check{A}_1, F_1')$ and $(\check{A}_2, F_2')$ are equivariant $(\check{A}_0, \check{A}_1)$ and $(\check{A}_1, \check{A}_2)$ cycles.

Let
$$(U, q) = \coprod_{j \in I} (U_j, q_j).$$
Let $\hat{t} = t \circ q$ and $\hat{s} = s \circ q$. The action of G on $\check{A}_0$ gives a map $\alpha : \hat{s}^*(\check{A}_0) \to \hat{t}^*(\check{A}_0)$. Put $\mathbb{A}_i = \hat{t}^*(\check{A}_i)$. Let $\varphi : \check{A}_0 \to \mathcal{M}(\mathbb{A}_0)$ be the natural map $\hat{t}^*$ defined by $\varphi(x)_u = x_{\hat{t}(u)}$ for all $u \in U$. Let $\psi : \check{A}_0 \to \mathcal{M}(\mathbb{A}_0)$ be the composition of α with the map $\hat{s}^*$. In other words, $\psi(x)_u = \alpha_u(x_{\hat{s}(u)})$. Let also $q \in C_0(U)$ be a strictly positive function.

The equivariance condition means exactly that $(\varphi - \psi)(F_i') \in \mathcal{M}(\mathbb{A}_{i-1}; \mathbb{A}_i)$. We thus may apply Theorem 4.10 and obtain the existence of the Kasparov product in KK_G with the usual properties:

Theorem 4.14. *There is a well-defined bilinear product*
$$KK_G(A_0, A_1) \times KK_G(A_1, A_2) \to KK_G(A_0, A_2)$$
which is natural in all A_i's and associative. The element 1 *acts as a unit element. Moreover, the Kasparov product is compatible with the descent morphisms.* □

5. A Baum–Connes assembly map for the telescopic algebra

In this section, we construct the Baum–Connes map for the telescopic algebra of a nicely decomposable singular foliation.

5.A. ***An abstract construction.***

5.A.1. *Setting of the problem.* Let $\mathcal{F}$ be a nicely decomposable foliation. We keep the notation of Section 3.B. We put $\mathcal{G}'_k = (\mathcal{G}_k)_{|\Omega_{k-1}\cap W_k}$.

There is a priori a topological K-group of the Lie groupoid $\mathcal{G}_k$ and $\mathcal{G}'_k$. In order to construct a topological K-group and a Baum–Connes map for $\mathcal{T}(q, j)$, we first wish to understand the morphisms and mapping cones associated with the morphisms j_k and q_k at the "left-hand side" level.

Let us first note that the morphism j_k is just the inclusion $\mathcal{G}'_k \subset \mathcal{G}_k$ of the restriction of $\mathcal{G}_k$ to the (saturated) open subset $\Omega_{k-1} \cap W_k$ of W_k. The mapping cone of such a morphism is just the C^*-algebra of a Lie groupoid (restriction of $\mathcal{G}_k \times [0, 1)$ to the open subset $(\Omega_{k-1} \cap W_k) \times [0, 1) \cup W_k \times (0, 1)$). We may then very easily construct a topological K-group for it.

On the other hand, the morphism q_k corresponds to a groupoid homomorphism which is the identity at the level of objects (W_k) and a surjective submersion at the level of arrows. The corresponding map at the level of topological K-groups is not as easy. Let us also note that, even knowing the map $(q_k)_*^{\mathrm{top}}$ at the level of K_*^{top}, we need more in order to construct the topological K-group for the mapping cone: this morphism only gives a short exact sequence

$$0 \to \operatorname{coker}(q_k)_*^{\mathrm{top}} \to K_*^{\mathrm{top}}(\mathcal{F}) \to \ker(q_k)_*^{\mathrm{top}} \to 0,$$

which is not sufficient in order to determine the group $K_*^{\mathrm{top}}(\mathcal{F})$ that we seek.

In order to understand the K-theory of this mapping cone, one needs in fact to construct $(q_k)_*^{\mathrm{top}}$ as a KK-element. To do so, we need to write explicitly the topological K-groups as K-groups of C^*-algebras and the Baum–Connes maps as KK-elements. To that end we assume:

(i) The Lie groupoids $\mathcal{G}_k$ are Hausdorff.

(ii) The classifying spaces for proper actions of these groupoids are smooth manifolds. This is always the case when the groupoids $\mathcal{G}_k$ are given by (connected) Lie group actions — or are Morita equivalent to those. This is indeed the case in most singularity height 1 examples in Examples 1.7 above — in fact, also in the examples of higher height given in Section 1.D.2.

It turns out that condition (ii) can be somewhat bypassed, thanks to the Baum–Douglas presentation of K^*_{top} [Baum and Douglas 1982a; 1982b; Baum and Connes 2000; Tu 2000]. We discuss this in the Appendix.

5.A.2. *The Baum–Connes map for groupoids.* Let us recall some facts about the Baum–Connes map for groupoids. (See [Baum and Connes 2000; Tu 2000].)

Let G be a Lie groupoid. If the classifying space for proper actions is a manifold M, then there is no inductive limit to be taken, and replacing if necessary M

by the total space of the vector bundle $(\ker dp)^*$, we may assume that the equivariant submersion $p : M \to G^{(0)}$ is K-oriented and then the topological K-group is $K_*(C_0(M) \rtimes G)$ and the Baum–Connes map is just the wrong-way functoriality element $\widehat{p_!} \in KK(C_0(M) \rtimes G, C^*(G))$ constructed in [Connes and Skandalis 1984; Hilsum and Skandalis 1987].

In Le Gall's equivariant KK_G theory and terminology [1999] (see also [Kasparov 1988]), the Baum–Connes assembly map is the element $\widehat{p_!} = j_G(p_!)$, where $p_!$ is the element of $KK_G(C_0(M), C_0(G^{(0)}))$ associated with the G-equivariant K-oriented smooth map p.

This statement is just a Poincaré duality. One easily adapts the constructions in [Connes and Skandalis 1984]. Indeed, the groupoids G and $M \rtimes G$ have the same classifying space for proper actions (namely M). If X is a G-invariant, G-compact subspace of M, by properness we find that the forgetful map

$$KK_{M\rtimes G}(C_0(X), C_0(M)) \to KK_G(C_0(X), C_0(M))$$

is an isomorphism.

Using again properness of X, we see that $p! \in KK_G(C_0(M), C_0(G^{(0)}))$ induces an isomorphism $KK_G(C_0(X), C_0(M)) \to KK_G(C_0(X), C_0(G^{(0)}))$. The inverse of this morphism is the composition of the "induction" construction

$$KK_G(C_0(X), C_0(G^{(0)})) \to KK_{M\rtimes G}(C_0(X \times_{G^{(0)}} M), C_0(M))$$

and the wrong way functoriality element $j_!$ associated with the inclusion map $j : X \to X \times_{G^{(0)}} M$. In other words, the groupoids G and $M \rtimes G$ have the same topological K-groups. Moreover, the groupoid $M \rtimes G$ is proper, and therefore the Baum–Connes conjecture holds for it [Julg 1998] (see also [Tu 1999], since proper groupoids are amenable).

5.A.3. *Submersions of Lie groupoids and "left-hand sides".* Before we proceed and construct a topological K-group for the telescopic groupoid, we examine the case of a morphism $\pi : G_0 \to G_1$ of Hausdorff Lie groupoids $G_i \overset{t_i, s_i}{\rightrightarrows} G_i^{(0)}$ $(i = 0, 1)$. We assume that π is a submersion and that it is an inclusion of an open subset $\pi : G_0^{(0)} \subset G_1^{(0)}$ at the level of units.

Let $p_i : M_i \to G_i^{(0)}$ be smooth manifolds which are classifying spaces for proper actions for G_i. We assume further that the p_i's are K-oriented submersions and that the dimensions of the fibers are even.

- Let $W = M_0 \times_{p_0, t_1} G_1$. The groupoid G_0 acts properly on W; we thus obtain a Hausdorff locally compact quotient $W/G_0 = M_0 \times_{G_0} G_1$. Note that $x \mapsto (x, \pi(p_0(x))$ defines a continuous map from M_0 to W and therefore $M_0 \to M_0 \times_{G_0} G_1$.

- The groupoid G_1 acts properly on the quotient space $M_0 \times_{G_0} G_1$. Since M_1 is universal, we get a G_1-equivariant map $M_0 \times_{G_0} G_1 \to M_1$. Hence, by composition we have a G_0-equivariant map $q : M_0 \to M_1$. As $p_1 \circ q = p_0$, we obtain a morphism of proper groupoids

$$q : M_0 \rtimes G_0 \to M_1 \rtimes G_1.$$

- The map q is naturally K-oriented, so it induces an element

$$q_! \in KK_{G_0}(C_0(M_0), C_0(M_1)).$$

Applying the descent map j_{G_0} we obtain an element

$$\widehat{q_!} = \widetilde{\pi}_*(j_{G_0}(q_!)) \quad \text{in } KK(C_0(M_0) \rtimes G_0, C_0(M_1) \rtimes G_1),$$

where $\widetilde{\pi}$ is the morphism $C_0(M_1) \rtimes G_0 \to C_0(M_1) \rtimes G_1$ induced by the morphism π.

Proposition 5.1. *The morphism $\pi : C^*(G_0) \to C^*(G_1)$ corresponds at the level of topological K-theory to the element $\widehat{q_!}$. More precisely, we have*

$$\pi_*(\widehat{(p_0)_!}) = \widehat{q_!} \otimes \widehat{(p_1)_!}.$$

Proof. The morphism p_1, being G_1-equivariant, is also G_0-equivariant (where G_0 acts through the morphism π). It gives rise to an element

$$\widetilde{(p_1)_!} \in KK(C_0(M_1) \rtimes G_0, C^*(G_0)).$$

The elements $\widetilde{(p_1)_!}$ and $\widehat{(p_1)_!}$ correspond to each other via the morphism $\pi : G_0 \to G_1$, i.e., we have $\pi_*(\widetilde{(p_1)_!}) = \widetilde{\pi}^*(\widehat{(p_1)_!})$. In other words, denoting by

$$[\widetilde{\pi}] \in KK(C^*(C_0(M_1) \rtimes G_0, C_0(M_1) \rtimes G_1)) \quad \text{and} \quad [\pi] \in KK(C^*(G_0), C^*(G_1))$$

the KK-elements associated with the morphisms $\widetilde{\pi}$ and π, respectively, we have $\widetilde{(p_1)_!} \otimes [\pi] = [\widetilde{\pi}] \otimes \widehat{(p_1)_!}$. We find

$$\begin{aligned} \widehat{q_!} \otimes \widehat{(p_1)_!} &= j_{G_0}(q_!) \otimes [\tilde{\pi}] \otimes (\widehat{(p_1)_!}) = j_{G_0}(q_!) \otimes \widetilde{(p_1)_!} \otimes [\pi] \\ &= j_{G_0}(q_!) \otimes j_{G_0}((p_1)_!) \otimes [\pi] = j_{G_0}(q_! \otimes (p_1)_!) \otimes [\pi] \\ &= \pi_*(j_{G_0}(q_! \otimes (p_1)_!)) = \pi_*(j_{G_0}((p_0)_!)). \end{aligned}$$

Here, the fourth equality follows from naturality of j_G [Kasparov 1980; 1988; Le Gall 1999], and the last equality from the wrong way functoriality [Connes and Skandalis 1984; Hilsum and Skandalis 1987]. Note that, since the groupoid $M_0 \rtimes G_0$ is proper, the γ obstruction appearing in this computation in [Hilsum and Skandalis 1987] vanishes. □

5.A.4. *Abstract "left-hand sides" for mapping cones.* Next, we wish to construct in a natural way the topological K-group for the mapping cone of the morphism $\pi_{C^*} : C^*(G_0) \to C^*(G_1)$. Proposition 5.1 states that the relative topological K-group of π is an element in $KK(C_0(M_0) \rtimes G_0, C_0(M_1) \rtimes G_1)$. The topological K-group of the cone of π should be a kind of "mapping cone of this KK-element".

In this section, we abstractly construct this mapping cone up to KK-equivalence. We give an explicit description of this topological K-group (Section 5.B) and of the Baum–Connes assembly map (Section 5.B.5) below.

Recall that a KK-element $x \in KK(A, B)$ can be given as a composition

$$x = [f]^{-1} \otimes [g] \tag{♠}$$

of a morphism $g : D \to B$ with the KK-inverse $[f]^{-1}$ of a morphism $f : D \to A$ which is invertible in KK-theory (see [Lafforgue 2007, Appendix A]). We may then wish to define (up to KK-equivalence) the cone of x as being the cone of g.

Next, in order to understand the Baum–Connes map, we should construct a KK-element associated with a map between mapping cones. We use the next lemma.

Lemma 5.2. *Let $f_i : A_i \to B_i$ be morphisms of C^*-algebras ($i = 0$ or 1). Denote by $p_i : C_{f_i} \to A_i$ and $j_i : B_i(0, 1) \to C_{f_i}$ the natural maps ($p_i(a_i, \phi) = a_i$ and $j_i(\phi) = (0, \phi)$). Let $x \in KK(A_0, A_1)$ and $y \in KK(B_0, B_1)$ satisfy $(f_1)_*(x) = f_0^*(y)$.*

(a) *There is $z \in KK(C_{f_0}, C_{f_1})$ such that $(p_1)_*(z) = p_0^*(x)$ and $(j_1)_*(Sy) = j_0^*(z)$, where $Sy \in KK(B_0(0, 1), B_1(0, 1))$ is deduced from y.*

(b) *If x and y are invertible, then so is z.*

In the language of [Meyer and Nest 2006], Lemma 5.2 is one of the axioms of a triangulated category. Although it is proved in [Meyer and Nest 2006], we include a proof for the reader's convenience.

Proof. (a) Note that z is not a priori unique. To construct it, one needs in fact to be more specific. Fix Kasparov bimodules (E_A, F_A) representing x and (E_B, F_B) representing y; a Kasparov $(A_0, B_1[0, 1])$ bimodule (E', F') realizing a homotopy between $(E_A \otimes_{A_1} B_1, F_A \otimes 1)$ and $f_0^*(E_B, F_B)$ gives rise to a Kasparov (A_0, Z_{f_1}) bimodule, where $Z_{f_1} = \{(a_1, \phi) \in A_1 \otimes B_1[0, 1] : f_1(a_1) = \phi(0)\}$ is the mapping cylinder of f_1, which can be glued with $(E_B, F_B)[0, 1)$ to give rise to the desired element in $KK(C_{f_0}, C_{f_1})$.

(b) By (a) applied to x^{-1} and y^{-1}, there exists $z' \in KK(C_{f_1}, C_{f_0})$ such that $(p_1)_*(z') = p_1^*(x^{-1})$ and $(j_0)_*(Sy^{-1}) = j_1^*(z')$. The Kasparov products $u_0 = z \otimes z'$ and $u_1 = z' \otimes z$ are elements in $KK(C_{f_i}, C_{f_i})$ such that $(p_i)_*(1 - u_i) = 0$ and $j_i^*(1 - u_i) = 0$. From the first equality and the mapping cone exact sequence, it follows that there exists $d_i \in KK(C_{f_i}, B_i(0, 1))$ such that $1 - u_i = (j_i)_*(d)$, and it follows that

$$(1 - u_i)^2 = (j_i)_*(d) \otimes (1 - u_i) = d \otimes j_i^*(1 - u_i) = 0,$$

whence u_i is invertible. □

Remark 5.3. Note also that we have a diagram

$$\begin{array}{ccccccccc}
K_i(A_0) & \longrightarrow & K_i(B_0) & \longrightarrow & K_{1-i}(C_{f_0}) & \longrightarrow & K_{1-i}(A_0) & \longrightarrow & K_{1-i}(B_0) \\
\downarrow{\scriptstyle \otimes x} & & \downarrow{\scriptstyle \otimes y} & & \downarrow{\scriptstyle \otimes z} & & \downarrow{\scriptstyle \otimes x} & & \downarrow{\scriptstyle \otimes y} \\
K_i(A_1) & \longrightarrow & K_i(B_1) & \longrightarrow & K_{1-i}(C_{f_1}) & \longrightarrow & K_{1-i}(A_1) & \longrightarrow & K_{1-i}(B_1)
\end{array}$$

where the lines are exact (Puppe sequences) and the squares commute. It follows that if x and y induce isomorphisms in K-theory, then the same holds for z.

Remarks 5.4. (a) It follows easily from this construction that given an element $x \in KK(A, B)$ the mapping cone C_g does not depend on the decomposition (♠) up to KK-equivalence.

(b) An alternative (and equivalent) way to construct the K-theory of the mapping cone of the KK-element x is to write x as an extension

$$0 \to SB \otimes \mathcal{K} \to D \to A \to 0$$

and define this K-theory as being $K_*(D)$.

(c) One can also define the KK-theory of this mapping cone as a relative KK-group [Skandalis 1984a, Remark 3.7(c)].

5.B. *Baum–Connes map for mapping cones of submersions of Lie groupoids.* Let us come back to our morphism $\pi : G_0 \to G_1$ of Hausdorff Lie groupoids, which is assumed to be a submersion and an open inclusion at the level of objects. We assume that the classifying spaces for proper maps of G_i are manifolds M_i. In Section 5.A.3, we explained how to construct an equivariant map $q : M_0 \to M_1$ that can be assumed to be a smooth submersion (up to replacing M_0 by a homotopy equivalent manifold).

As a consequence of Lemma 5.2 and Proposition 5.1, we see that, in order to construct the topological K-group we need to give an explicit construction of the wrong-way functoriality element $\widetilde{\pi}_*(j_{G_0})(q_!) \in KK(C^*(M_0 \rtimes G_0), C^*(M_1 \rtimes G_1))$. Here, using a double deformation longitudinally smooth groupoid we give a groupoid $\mathbb{H}$ which is a family over $[0, 1] \times [0, 1]$, whose vertical lines $\{i\} \times [0, 1]$ can be interpreted as the Baum–Connes maps for the groupoid G_i and whose horizontal lines $[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$ are $q!$ and $[\pi]$, respectively.

We then may define the relative topological K-group of π as the groupoid $\mathbb{H}$ restricted to $[0, 1) \times \{0\}$ and construct the Baum–Connes map using the groupoid $\mathbb{H}$ (restricted to $[0, 1) \times [0, 1]$).

In order to have a "ready to glue" groupoid, in view of the case of the telescopic algebra (Section 5.C), we are lead to perform a slightly more complicated construction.

5.B.1. *Deformation groupoids.*

Deformation to the normal cone. The adiabatic deformation of a Lie groupoid G with Lie algebroid $\mathfrak{G}$ was defined by Alain Connes in the particular case of the pair groupoid [Connes 1994] and generalized by various authors (e.g., [Hilsum and Skandalis 1987; Monthubert and Pierrot 1997; Nistor et al. 1999]). This is based on the notion of deformation to the normal cone, which we briefly recall; see also [Carrillo Rouse 2008; Debord and Skandalis 2014].

Let X be a submanifold of a manifold Y. Denote by N_X^Y the total space of the normal bundle to X in Y. There is a natural way to put a manifold structure to $Y \times \mathbb{R}^* \cup N_X^Y \times \{0\}$; denote this manifold by $DNC(Y, X)$.

The map $p : DNC(Y, X) \to \mathbb{R}$ defined by $p(y, t) = t$ for $(y, t) \in Y \times \mathbb{R}^*$ and $p(\xi, 0) = 0$ for $\xi \in N_X^Y$ is a smooth submersion. For $J \subset \mathbb{R}$, we write $DNC_J(Y, X)$ for $p^{-1}(J)$.

This construction is functorial. Given a commutative diagram of smooth maps

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f_X"'] & Y \arrow[d, "f_Y"] \\ X' \arrow[r, hook] & Y' \end{tikzcd}$$

where the horizontal arrows are inclusions of submanifolds, we naturally obtain a smooth map $DNC(f) : DNC(Y, X) \to DNC(Y', X')$. If f_Y is a submersion and $X = X' \times_{Y'} Y$ then $DNC(f)$ is a submersion.

Double deformations to the normal cone. Let Z be a smooth manifold, Y a (locally) closed submanifold of Z and X a (locally) closed submanifold of Y. Then $DNC(Y, X)$ is a (locally) closed submanifold of $DNC(Z, X)$. Put then

$$DNC^2(Z, Y, X) = DNC(DNC(Z, X), DNC(Y, X)).$$

We have a submersion $p_2 : DNC^2(Z, Y, X) \to \mathbb{R}^2$. For every subset L of $\mathbb{R}^2$, we put

$$DNC^2_L(Z, Y, X) = p_2^{-1}(L).$$

By definition of the deformation to the normal cone,

$$DNC^2_{\mathbb{R}\times\mathbb{R}^*}(Z, Y, X) = DNC(Z, X) \times \mathbb{R}^*.$$

By functoriality of the DNC construction,

$$DNC^2_{\mathbb{R}^*\times\mathbb{R}}(Z, Y, X) = DNC(Z \times \mathbb{R}^*, Y \times \mathbb{R}^*) \simeq DNC(Z, Y) \times \mathbb{R}^*.$$

Deformation groupoids, adiabatic groupoids. From naturality, it follows that if Y is a Lie groupoid and X is a Lie subgroupoid of Y, then $DNC(Y, X)$ is naturally endowed with a Lie groupoid structure — with objects $DNC(Y^{(0)}, X^{(0)})$, and target and source maps $DNC(t)$ and $DNC(s)$. Of course, if in the above diagram all the maps are groupoid morphisms, then $DNC(f)$ is a morphism of groupoids too.

The *adiabatic groupoid* of a Lie groupoid G is just $G_{\mathrm{ad}} = DNC_{[0,1)}(G, G^{(0)}) = \mathfrak{G} \times \{0\} \cup G \times (0, 1)$ (with base manifold $G^{(0)} \times [0, 1)$). Note that the normal bundle $N^G_{G^{(0)}}$ is, by definition, the Lie algebroid $\mathfrak{G}$ of G. It follows that Z is a Lie groupoid, Y is a Lie subgroupoid of Z and X is a Lie subgroupoid of Y. Then $DNC^2(Z, Y, X)$ is a Lie groupoid.

5.B.2. *The Baum–Connes map of a Lie groupoid via deformation groupoids.* Let G be a Lie groupoid and let M be a smooth manifold on which G acts via a smooth onto submersion $p : M \to G^{(0)}$. We do not assume that p is K oriented but rather consider the total space of $(\ker dp)^*$. Note that if M is a classifying space for proper actions of G, then $(\ker dp)^* \to G^{(0)}$ is also a classifying space of proper actions, and it moreover carries a canonical K-orientation. So we can replace M with $(\ker dp)^*$.

Put then $\Gamma_p = DNC(G^p_p, M \rtimes G)$. As p is supposed to be a surjective submersion, the groupoid G^p_p is Morita equivalent to G. There is a canonical Morita equivalence bimodule $\mathcal{E}$ of the C^*-algebras $C^*(G^p_p)$ and $C^*(G)$.

We have an exact sequence of C^*-algebras:

$$0 \to C^*(G^p_p \times (0, 1]) \to C^*((\Gamma_p)_{[0,1]}) \xrightarrow{\mathrm{ev}_0} C^*(\ker(dp) \rtimes G) \to 0.$$

Note that $C^*(G^p_p \times (0, 1])$ is contractible. It follows that ev_0 is invertible in E-theory. We may then observe the diagram

$$C_0((\ker dp)^*) \rtimes G \xleftarrow{\mathrm{ev}_0} C^*((\Gamma_p)_{[0,1]}) \xrightarrow{\mathrm{ev}_1} C^*(G^p_p) \overset{\mathcal{E}}{\longrightarrow} C^*(G).$$

We thus obtain an element

$$\mu_M = [\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_1] \otimes [\mathcal{E}] \in E(C_0((\ker dp)^*) \rtimes G, C^*(G)).$$

Note that this E-theory coincides with KK-theory if the action of G on M is assumed to be amenable — and in particular, if it is proper.

If M is the classifying space for proper algebras, the morphism on K-groups defined by μ_M is the Baum–Connes map.

5.B.3. *A double deformation construction.* Now let G_0 and G_1 be Lie groupoids and let $\pi : G_0 \to G_1$ be a groupoid morphism which is a smooth submersion whose restriction $\pi^{(0)} : G_0 \to G_1$ is the inclusion of an open subset. Let M_i be manifolds with actions of G_i. We assume that the maps $p_i : M_i \to G_i^{(0)}$ defining these actions are smooth submersions. Let also $q : M_0 \to M_1$ be a smooth submersion which is equivariant, i.e., $q(\gamma.x) = \pi(\gamma)q(x)$ for every $(x, \gamma) \in M_0 \times_s G_0$. In other words, we assume that we have a morphism of semidirect products $\widehat{\pi} : M_0 \rtimes G_0 \to M_1 \rtimes G_1$ defined by $\widehat{\pi}(x, \gamma) = (q(x), \pi(\gamma))$.

The groupoid G_0 acts on the open subspace $M_1' = q(M_0)$ of M_1 through the morphism π: just put $\gamma.q(x) = q(\gamma.x) = \pi(\gamma).q(x)$ for $x \in M_0$ and $\gamma \in G_0$ with $s(x) = p_0(x) = p_1(q(x))$.

We have inclusions of groupoids $M_0 \rtimes G_0 \subset (M_1' \rtimes G_0)_q^q \subset (G_0)_{p_0}^{p_0}$. Indeed,

$$M_0 \rtimes G_0 = \big\{(x, \gamma, y) \in (G_0)_{p_0}^{p_0} : x = \gamma.y\big\},$$
$$(M_1' \rtimes G_0)_q^q = \big\{(x, \gamma, y) \in (G_0)_{p_0}^{p_0} : q(x) = q(\gamma.y)\big\}.$$

Let then $\mathcal{H}_0$ be the double deformation Lie groupoid:

$$\mathcal{H}_0 = DNC^2\big((G_0)_{p_0}^{p_0}, (M_1' \rtimes G_0)_q^q, M_0 \rtimes G_0\big).$$

The groupoid $\mathcal{H}_0$ is a family of groupoids indexed by $\mathbb{R}^2$. For every locally closed subset L of $\mathbb{R}^2$, we may form the locally compact groupoid $(\mathcal{H}_0)_L$.

Let $q' : M_0 \sqcup M_1 \to M_1$ be the map which coincides with q on M_0 and the identity on M_1 and $p' : M_0 \sqcup M_1 \to G_1^{(0)}$, $p' = p_1 \circ q'$. Define the groupoid

$$\mathcal{H}_1 = DNC\big((G_1)_{p'}^{p'} \times \mathbb{R}^*, (M_1 \rtimes G_1)_{q'}^{q'} \times \mathbb{R}^*\big) \simeq DNC\big((G_1)_{p'}^{p'}, (M_1 \rtimes G_1)_{q'}^{q'}\big) \times \mathbb{R}^*$$

(with objects $(M_0 \sqcup M_1) \times \mathbb{R}^* \times \mathbb{R}$).

For every locally closed subset $Y \subset \mathbb{R}^* \times \mathbb{R}$ we denote by $(\mathcal{H}_1)_Y$ the restriction of $\mathcal{H}_1$ to its saturated subset $(M_0 \sqcup M_1) \times Y$.

5.B.4. *A longitudinally smooth groupoid.* Note that

$$(M_1' \rtimes G_0)_q^q = \big\{(x, \gamma, y) \in (G_0)_p^p : q(x) = \pi(\gamma) q(y)\big\}.$$

In other words, $(M_1' \rtimes G_0)_q^q$ is the fibered product $(G_0)_{p_0}^{p_0} \times_{(G_1)_{p_0}^{p_0}} (M_1 \rtimes G_1)_q^q$. We therefore have a commutative diagram

$$\begin{array}{ccc} (M_1' \rtimes G_0)_q^q & \hookrightarrow & (G_0)_{p_0}^{p_0} \\ \downarrow & & \downarrow \\ (M_1 \rtimes G_1)_q^q & \hookrightarrow & (G_1)_{p_0}^{p_0} \end{array}$$

which gives rise to a morphism

$$DNC\big((G_0)_{p_0}^{p_0}, (M_1' \rtimes G_0)_q^q\big) \to DNC\big((G_1)_{p_0}^{p_0}, (M_1 \rtimes G_1)_q^q\big)$$

which is a groupoid morphism, a submersion and the identity at the level of objects $(M_0 \times \mathbb{R})$. We thus obtain a morphism of groupoids

$$\pi : (\mathcal{H}_0)_{\mathbb{R}^* \times \mathbb{R}} \to \mathcal{H}_1$$

which is a submersion. At the level of objects it is the inclusion $M_0 \times \mathbb{R}^* \times \mathbb{R} \to (M_0 \sqcup M_1) \times \mathbb{R}^* \times \mathbb{R}$.

Let $Z_0 = [0, 1) \times \left[0, \frac{1}{2}\right]$, $Q = \left\{(u, v) \in \mathbb{R}^2 : 0 \le u \le v \le \frac{1}{2}\right\}$ and $Z_1 = Z_0 \setminus Q = \{(u, v) \in Z_0 : u > v\}$. We may then construct a longitudinally smooth groupoid $\mathbb{H} = (\mathcal{H}_0)_Q \cup (\mathcal{H}_1)_{Z_1}$ with atlas formed by $(\mathcal{H}_0)_{Z_0}$ and $(\mathcal{H}_1)_{Z_1}$, using the morphism π in order to map $(\mathcal{H}_0)_{Z_1}$ to $(\mathcal{H}_1)_{Z_1}$. We have $\mathbb{H}^{(0)} = (M_0 \times Z_0) \sqcup (M_1 \times Z_1)$.

In the same way as above, for every locally closed subset $Y \subset Z_0$ we denote by $\mathbb{H}_Y$ the restriction of $\mathbb{H}$ to its saturated subset $M_0 \times Y \cup M_1 \times (Y \cap Z_1)$.

Remarks 5.5. (a) It is worthwhile to note that the groupoid $\mathbb{H}$ only depends on $\pi : G_0 \to G_1$, the (proper) actions of G_i on M_i and the submersion q. Also, the restriction $\mathbb{H}_{\{1/2\}\times[0,1/2]}$ is nothing else than $(M_0 \times_{p_0} M_0)_{\mathrm{ad}} \rtimes G_0$ (restricted to $\left[0, \frac{1}{2}\right]$). It does not depend on G_1, M_1, q.

(b) Note $\mathcal{H}_{0,0}$ is isomorphic to the direct sum of vector bundles $\ker dq \oplus q^*(\ker dp_1)$.

5.B.5. *Baum–Connes map for a mapping cone.* Set $F_0 = [0, 1) \times \{0\} \cup \{0\} \times \left[0, \frac{1}{2}\right]$. Note that, since the action of G_i on M_i is proper, the groupoid

$$\mathbb{H}_{F_0} = (\ker dp_0) \rtimes G_0 \times \left[0, \tfrac{1}{2}\right] \cup ((\ker dp_1) \rtimes G_1)_{q'}^{q'} \times (0, 1)$$

is amenable and we have a semisplit exact sequence

$$0 \to C^*(\mathbb{H}_{Z_0 \setminus F_0}) \to C^*(\mathbb{H}) \xrightarrow{\sigma_0} C^*(\mathbb{H}_{F_0}) \to 0.$$

Proposition 5.6. *The homomorphism σ_0 is invertible in KK-theory.*

Proof. We have a semisplit exact sequence

$$0 \to C^*(\mathbb{H}_{Z_1 \setminus F_0}) \to C^*(\mathbb{H}_{Z_0 \setminus F_0}) \to C^*(\mathbb{H}_{Q \setminus F_0}) \to 0.$$

Note that the groupoid $\mathbb{H}$ is constant over the sets $Z_1 \setminus F_0$ and $Q \setminus F_0$:

$$\mathbb{H}_{(u,v)} = (G_1)_{p_1 \circ q'}^{p_1 \circ q'} \quad \text{for } (u, v) \in Z_1 \setminus F_0,$$
$$\mathbb{H}_{(u,v)} = (G_0)_{p_0}^{p_0} \quad \text{for } (u, v) \in Q \setminus F_0.$$

The sets

$$Z_1 \setminus F_0 = \left\{(u, v) : 0 < v < u < 1 \text{ and } v \le \tfrac{1}{2}\right\},$$
$$Q \setminus F_0 = \left\{(u, v) : 0 < u \le v \le \tfrac{1}{2}\right\}$$

are contractible (more precisely, their one point compactification contracts to this point) and it follows that the C^*-algebras $C^*(\mathbb{H}_{Z_1 \setminus F_0})$ and $C^*(\mathbb{H}_{Q \setminus F_0})$ are contractible. It follows that $C^*(\mathbb{H}_{Z_0 \setminus F_0})$ is KK-contractible (it is actually contractible). We deduce that $[\sigma_0]$ is a KK-equivalence. □

Set also $F_1 = \left\{\frac{1}{2}\right\} \times \left[\frac{1}{2}, 1\right)$. One sees that $\mathbb{H}_{F_1}$ is isomorphic to the groupoid $\mathcal{C}_\pi = G_0 \times \{0\} \cup G_1 \times (0, 1)$ pulled back by

$$q'' : (M_0 \times [0, 1)) \sqcup (M_1 \times (0, 1)) \to G_1^{(0)} \times [0, 1)$$

(recall that $G_0^{(0)}$ is an open subset of $G_1^{(0)}$).

Corollary 5.7. *The algebra $C^*(\mathbb{H}_{F_1})$ is canonically Morita equivalent to the mapping cone of $h_{C^*} : C^*(G_0) \to C^*(G_1)$.*

Denote by $\mathcal{E}$ the Morita $(C^*(\mathbb{H}_{F_1}), C^*(\mathcal{C}_h))$ bimodule and $[\mathcal{E}]$ its KK-class. Let $[\sigma_1] : C^*(\mathbb{H}) \to C^*(\mathbb{H}_{F_1})$ be the evaluation.

Definition 5.8. Assume further that the manifolds M_i are classifying spaces for the proper actions of G_i. With the notation above, the topological K-theory of the groupoid $\mathcal{C}_\pi$ is $K_*(C^*(\mathbb{H}_{F_0}))$ and the Baum–Connes morphism is the composition $[\sigma_0]^{-1} \otimes [\sigma_1] \otimes [\mathcal{E}]$.

5.B.6. *Justifying why this is a Baum–Connes map.* Let us explain why this is a "good" definition. First of all, for $v \in \left[0, \frac{1}{2}\right]$, the K-theory of the C^*-algebra $C^*(\mathbb{H}_{(0,v)}) = C^*(\ker p_0 \rtimes G_0) = C_0((\ker p_0)^*) \rtimes G_0$ is the topological K-group for G_0. Also, for $u \in (0, 1)$, the C^*-algebra $C^*(\mathbb{H}_{(u,0)}) = C^*\big((\ker p_1 \rtimes G_1)_{q'}^{q'}\big)$ is Morita equivalent to $C^*(\ker p_1 \rtimes G_1) = C_0((\ker p_1)^*) \rtimes G_1$, whose K-theory is the topological K-theory for G_1.

We may then write a diagram:

$$\begin{tikzcd}
0 \arrow[r] & C^*(\mathbb{H}_{Z_1 \cap F_0}) \arrow[r] & C^*(\mathbb{H}_{F_0}) \arrow[r] & C^*(\mathbb{H}_{Q \cap F_0}) \arrow[r] & 0 \\
0 \arrow[r] & C^*(\mathbb{H}_{Z_1}) \arrow[r] \arrow[u, "\sigma_{1,0}"] \arrow[d, "\sigma_{1,1}"'] & C^*(\mathbb{H}) \arrow[r] \arrow[u, "\sigma_0"] \arrow[d, "\sigma_1"'] & C^*(\mathbb{H}_Q) \arrow[r] \arrow[u, "\sigma_{0,0}"] \arrow[d, "\sigma_{0,1}"'] & 0 \\
0 \arrow[r] & C^*(\mathbb{H}_{Z_1 \cap F_1}) \arrow[r] \arrow[d, dash, "\mathcal{E}_1"] & C^*(\mathbb{H}_{F_1}) \arrow[r] \arrow[d, dash, "\mathcal{E}"] & C^*(\mathbb{H}_{Q \cap F_1}) \arrow[r] \arrow[d, dash, "\mathcal{E}_0"] & 0 \\
0 \arrow[r] & C^*(G_1)(0,1) \arrow[r] & \mathcal{C}_{h_{C^*}} \arrow[r] & C^*(G_0) \arrow[r] & 0
\end{tikzcd}$$

In this diagram all sequences are semisplit, the morphisms $\sigma_0, \sigma_{i,0}$ are KK-equivalences and the compositions $[\sigma_{i,0}]^{-1} \otimes [\sigma_{i,1}] \otimes [\mathcal{E}_i]$ are indeed the Baum–Connes maps for $G_1 \times (0, 1)$ and G_0.

It follows also that the class in $KK^1(C^*(\mathbb{H}_{Q \cap F_0}), C^*(\mathbb{H}_{Z_1 \cap F_0}))$ for the first sequence corresponds to the class of $[h_{C^*}] \in KK(C^*(G_0), C^*(G_1))$.

From the discussion in Section 5.A.4, it follows that the K-theory of $C^*(\mathbb{H}_{F_0})$ and the morphism is indeed the right one, and that the composition $[\sigma_0]^{-1} \otimes [\sigma_1] \otimes [\mathcal{E}]$ is indeed a Baum–Connes map.

Remark 5.9. The groupoid $\mathbb{H}_{F_0}$ is a semidirect product $\Lambda \rtimes \mathcal{C}_\pi$, where Λ is a groupoid obtained by gluing $DNC_{[0,1)}\big(M_0, (\ker dp_1)_{q'}^{q'}\big)$ with $\ker dp_0 \times \left[0, \frac{1}{2}\right]$.

One may give a generalized notion of proper algebras on a longitudinally smooth groupoid G by saying that $G^{(0)}$ is an increasing union $\bigcup \Omega_k$ of saturated open subsets such that the restriction of G to $\Omega_k \setminus \Omega_{k-1}$ is Hausdorff. We may say that an action of G on an algebra A is proper if its restriction to each $\Omega_k \setminus \Omega_{k-1}$ is proper.

In this generalized sense, the $\mathcal{C}_\pi$-algebra $C^*(\Lambda)$ is a proper $\mathcal{C}_\pi$-algebra. Its restriction to $Q \cap F_0$ is indeed a proper G_0-algebra and its restriction to $Z_1 \cap F_0$ is a proper $G_1 \times (0, 1)$-algebra.

It may be interesting to look for a way to say that the $C^*\big(DNC_{[0,1)}(M_0, (M_1)_{q'}^{q'})\big)$ is somehow a universal proper algebra.

5.C. ***Baum–Connes map for the telescopic algebra.*** Since a mapping telescope is a mapping cylinder which, in turn, is a mapping cone (cf. Remarks 3.5) we can just proceed and construct the "left-hand side" for the telescopic algebra — and therefore for the foliation one.

We are given a nicely decomposable foliation $(M, \mathcal{F})$, a decomposition given by an increasing sequence Ω_k of saturated sets — we put $Y_k = \Omega_k \setminus \Omega_0$, a sequence of Lie groupoids $\mathcal{G}_k \rightrightarrows W_k \subset \Omega_k$ such that $Y_k \subset W_k$ and $W_k \cap \Omega_{k-1} \subset W_{k-1}$; we put $\mathcal{G}'_k = (\mathcal{G}_k)_{|W_k \cap \Omega_{k-1}}$ and assume that we have a groupoid morphism which is a submersion $\pi_k : \mathcal{G}'_k \to \mathcal{G}_{k-1}$.

We further assume that we have submersions of manifolds $p_k : M_k \to W_k$ which are classifying spaces for proper actions of $\mathcal{G}_k$. For $k \geq 1$, the restriction $p_k^{-1}(\Omega_{k-1})$ of M_k is a classifying space for $\mathcal{G}'_k$ but we may need to modify it: we choose a classifying space given by a submersion $p'_k : M'_k \to \Omega_{k-1} \subset W_{k-1}$ in such a way that the maps $q_k : M'_k \to M_{k-1}$ and $\hat{q}_k : M'_k \to M_k$ are submersions.

We then construct the classifying groupoids

- $\mathbb{H}_k$ associated to the morphism $\pi_k : \mathcal{G}'_k \to \mathcal{G}_{k-1}$ and the submersion $q_k : M'_k \to M_{k-1}$ of classifying spaces;
- $\widehat{\mathbb{H}}_k$ associated to the morphism $j_k : \mathcal{G}'_k \to \mathcal{G}_k$ and the submersion $\hat{q}_k : M'_k \to M_k$ of classifying spaces.

We then glue the groupoids $\mathbb{H}_k$ and $\widehat{\mathbb{H}}_k$ in their common part $(\mathbb{H}_k)_{\{1/2\} \times [0,1/2]} = (\widehat{\mathbb{H}}_k)_{\{1/2\} \times [0,1/2]}$ (cf. Remarks 5.5) and obtain a groupoid $\widetilde{\mathbb{H}}_k$.

For a locally closed part Y of $Z_0 = [0, 1)] \times \big[0, \frac{1}{2}\big]$ we set $(\widetilde{\mathbb{H}}_k)_Y = (\mathbb{H}_k)_Y \cup (\widehat{\mathbb{H}}_k)_Y$.

Recall that $Q = \big\{(u, v) : 0 \leq u \leq v \leq \frac{1}{2}\big\}$ and $Z_1 = Z_0 \setminus Q$. We define diffeomorphisms $\vartheta_k : Z_1 \to [0, 1] \times (k-1, k)$ and $\hat{\vartheta}_k : Z_1 \to [0, 1] \times (k, k+1)$ by setting

$$\vartheta_k(u, v) = \Big(2v, k - \frac{u-v}{1-v}\Big) \quad \text{and} \quad \hat{\vartheta}_k(u, v) = \Big(2v, k + \frac{u-v}{1-v}\Big),$$

respectively. Thanks to this diffeomorphism, we obtain identifications of

- $\Theta_k : (\mathbb{H}_k)_{Z_1} \xrightarrow{\sim} \big(DNC\big((\mathcal{G}_{k-1})_{p_{k-1}}^{p_{k-1}}, M_{k-1} \rtimes \mathcal{G}_{k-1}\big)_{[0,1]}\big)_{q'_k}^{q'_k} \times (k-1, k)$
  where $q'_k : M_{k-1} \sqcup M'_k \to M_{k-1}$ is the identity on M_{k-1} and q_k on M'_k;
- $\widehat{\Theta}_k : (\widehat{\mathbb{H}}_k)_{Z_1} \xrightarrow{\sim} \big(DNC\big((\mathcal{G}_k)_{p_k}^{p_k}, M_k \rtimes \mathcal{G}_k\big)_{[0,1]}\big)_{\hat{q}'_k}^{\hat{q}'_k} \times (k, k+1)$
  where $\hat{q}'_k : M_k \sqcup M'_k \to M_k$ is the identity on M_k and $\hat{q}_k$ on M'_k.

Define $q''_k : M_k \sqcup M'_k \sqcup M'_{k+1} \to M_k$ to be the map coinciding with the identity over M_k, $\hat{q}_k$ on M'_k and q_{k+1} on M'_{k+1} — with the convention $M'_0 = \varnothing$ and, if $n \neq +\infty$, $M'_{n+1} = \varnothing$.

Definition 5.10. We define the *adiabatic telescopic groupoid* $\mathbb{G}$ to be the union $\bigcup_{k=1}^{n} (\widetilde{\mathbb{H}}_k)_Q \times \{k\}$ with $\bigcup_{k=0}^{n} \big(DNC((\mathcal{G}_k)_{p_k}^{p_k}, M_k \rtimes \mathcal{G}_k)_{[0,1]}\big)_{q''_k}^{q''_k} \times (k, k+1)$. The gluing is obtained by mapping $\mathbb{H}_k \to \mathbb{G}$:

- We map $(\widetilde{\mathbb{H}}_k)_Q$ to $\mathbb{G}$ by the map $\gamma \mapsto (\gamma, k) \in \mathbb{G}$.
- Using Θ_{k+1} we map $(\mathbb{H}_{k+1})_{Z_1}$ to
$$\bigcup_{k=0}^{n} \big(DNC((\mathcal{G}_k)_{p_k}^{p_k}, M_k \rtimes \mathcal{G}_k)_{[0,1]}\big)_{q'_k}^{q'_k} \times (k, k+1),$$
which is a subset of
$$\bigcup_{k=0}^{n} \big(DNC((\mathcal{G}_k)_{p_k}^{p_k}, M_k \rtimes \mathcal{G}_k)_{[0,1]}\big)_{q''_k}^{q''_k} \times (k, k+1) \subset \mathbb{G}.$$
- Using $\widehat{\Theta}_k$ we map $(\widehat{\mathbb{H}}_k)_{Z_1}$ to
$$((M_k \times_{p_k} M_k)_{\overline{\mathrm{ad}}} \rtimes \mathcal{G}_k)_{\hat{q}'_k}^{\hat{q}'_k} \times (k, k+1),$$
which is a subset of
$$((M_k \times_{p_k} M_k)_{\overline{\mathrm{ad}}} \rtimes \mathcal{G}_k)_{q''_k}^{q''_k} \times (k, k+1) \subset \mathbb{G}.$$

We define the obvious map $\chi : \mathbb{G}^{(0)} \to (0, n+1)$ (using the convention $+\infty+1 = +\infty$ of course). Thanks to χ, the (full) C^*-algebra $C^*(\mathbb{G})$ is a $C_0(0, n+1)$-algebra.

Define a map $\xi : Z \to [0,1]$ by setting $\xi(u,v) = 2\min(u,v)$. Let $\hat{\xi} : \mathbb{G}^{(0)} \to [0,1]$ be defined as the composition
$$\mathbb{G}^{(0)} \to Q \xrightarrow{\xi} [0,1]$$
on $(\widetilde{\mathbb{H}}_k)_Q \times \{k\}$ and let $\hat{\xi}$ to be the parameter in the adiabatic deformation $(M_k \times_{p_k} M_k)_{\overline{\mathrm{ad}}}$ on
$$\bigcup_{k=0}^{n} ((M_k \times_{p_k} M_k)_{\overline{\mathrm{ad}}} \rtimes \mathcal{G}_k)_{q''_k}^{q''_k} \times (k, k+1).$$

We then define the subgroupoids $\mathbb{G}_0$ and $\mathbb{G}_1$ of $\mathbb{G}$, restrictions of $\mathbb{G}$ to the closed saturated set $\hat{\xi}^{-1}(\{i\})$. We then have:

Proposition 5.11. (a) *The algebra $C^*(\mathbb{G}_0)$ is nuclear.*

(b) *The kernel of the evaluation $\rho_0 : C^*(\mathbb{G}) \to C^*(\mathbb{G}_0)$ is KK-contractible.*

(c) *The algebra $C^*(\mathbb{G}_1)$ is Morita equivalent to the telescopic algebra.*

Proof. (a) In fact $C^*(\mathbb{G}_0)$ sits in an exact sequence

$$0 \to \bigoplus_{k=0}^{n} C^*\big(((\ker p_k)^* \rtimes \mathcal{G}_k)_{q'_k}^{q'_k} \times (0,1)\big) \to C^*(\mathbb{G}_0) \to \bigoplus_{k=1}^{n} C^*((\ker p'_k)^* \rtimes \mathcal{G}'_k \times [0,1]) \to 0$$

and the Lie groupoids $((\ker p_k)^* \rtimes \mathcal{G}_k)_{q'_k}^{q'_k}$ and $(\ker p'_k)^* \rtimes \mathcal{G}'_k$ are proper. It follows that $C^*(\mathbb{G}_0)$ is in fact a type I algebra.

(b) We have a semisplit exact sequence

$$0 \to C_0((0,1] \times (0,1)) \otimes B \to \ker \rho_0 \to C_0(Q \setminus F_0) \otimes B' \to 0 \to 0,$$

where

$$B = \bigoplus_{k=0}^{n} C^*\big(((M_k \times_{p_k} M_k) \rtimes \mathcal{G}_k)_{q''_k}^{q''_k}\big) \quad \text{and} \quad B' = \bigoplus_{k=1}^{n} C^*((M'_k \times_{p'_k} M'_k) \rtimes \mathcal{G}'_k).$$

The algebras $C_0((0,1] \times (0,1))$ and $C_0(Q \setminus F_0)$ are contractible.

(c) Actually the groupoid $\mathbb{G}_1$ is Morita equivalent to the telescopic groupoid. □

Definition 5.12. Let $(M, \mathcal{F})$ be a nicely decomposable foliation. Assume that the classifying spaces of all the groupoids $\mathcal{G}_k \rightrightarrows W_k$ involved in this decomposition are manifolds. With the above construction,

- we define the "left-hand side", i.e., the topological K-theory (of this decomposition) to be the K-theory of $C^*(\mathbb{G}_0)$;
- we define the Baum–Connes map for the telescope to be the composition $[\rho_0]^{-1} \otimes [\rho_1] \otimes [\mathbb{E}]$;
- we define the Baum–Connes map for $(M, \mathcal{F})$ to be the Baum–Connes map for the telescope composed with the isomorphism $K_*(\mathcal{T}) \to K_{*+1}(C^*(M, \mathcal{F}))$.

Let $\rho_1 : C^*(\mathbb{G}) \to C^*(\mathbb{G}_1)$ be evaluation. The kernel of ρ_1 is a $C_0(0, n+1)$-algebra. It follows from the inductive limit construction that if $(\ker \rho_1)_{(k,k+1)}$ and $(\ker \rho_1)_k$ are E-contractible for all k, then so is $\ker \rho_1$. We thus obtain:

Theorem 5.13. *Let $(M, \mathcal{F})$ be a nicely decomposable foliation such that the classifying spaces of all the groupoids $\mathcal{G}_k \rightrightarrows W_k$ involved in this decomposition are manifolds. If the* full *Baum–Connes conjecture holds for all of them, then the* full *Baum–Connes map of Definition 5.12 is an isomorphism.*

Corollary 5.14. *Let $(M, \mathcal{F})$ be a nicely decomposable foliation. If all the groupoids $\mathcal{G}_k \rightrightarrows W_k$ involved in this decomposition are amenable and their classifying spaces are manifolds, then the Baum–Connes map is an isomorphism.*

Appendix: When the classifying spaces are not manifolds

We finally explain how one should be able to get rid of the assumption on the classifying spaces: we just assume that the foliation $\mathcal{F}$ has a nice decomposition with Hausdorff Lie groupoids $\mathcal{G}_i$ but the classifying spaces E_i for proper actions are not manifolds.

To construct a topological K-theory and a Baum–Connes map for $C^*(M, \mathcal{F})$, we just need to construct a topological K-theory for a mapping cone of a morphism $\pi : G \to G'$ of Hausdorff Lie groupoids which is a submersion and the identity at the level of objects. As in the particular cases considered here, we then may construct topological K-theory for mapping tori and then of telescopic algebras.

In fact, given such a morphism $\pi : G \to G'$ we just have to show that

(i) we may express the topological K-theory of G and G' as the K-theory of C^*-algebras T and T';

(ii) the Baum–Connes maps are given by elements μ and μ' in $KK(T, C^*(G))$ and $KK(T', C^*(G'))$, respectively;

(iii) we may construct an element $x \in KK(T, T')$ such that $\pi_*(\mu) = x \otimes \mu'$.

We then write $x = [f]^{-1} \otimes [g]$, where $f : D \to T$ and $g : D \to T'$ are morphisms with f a K-equivalence. A topological K-theory for C_π is then the cone C_g of g. As $f^*(\pi_*(\mu)) = [f] \otimes \mu \otimes [\pi] = [g] \otimes \mu'$, we may construct an element $\widetilde{\mu} \in KK(C_g, C_\pi)$ as in Lemma 5.2 which defines the desired Baum–Connes map.

To do so, recall that if G is a Hausdorff Lie groupoid, then the topological K-theory for the Baum–Connes map can be described in the Baum–Douglas way [Baum and Douglas 1982a; 1982b; Baum and Connes 2000; Baum et al. 1994; Tu 1999; 2000]: there is an inductive limit of manifolds $(M_k)_{k\in\mathbb{N}}$ with maps $h_k : M_k \to M_{k+1}$ forming a sequence of approximations of E. We may assume that the maps $q_k : M_k \to G^{(0)}$ are K-oriented in a G-equivariant way, and therefore so are the maps h_k. We also assume that the dimensions of all the M_k are equal modulo 2. Then the topological K-theory $K_*^{\mathrm{top}}(G)$ is the inductive limit $\varinjlim_k (K_*(C_0(M_k) \rtimes G), (h_k)!)$.

The Baum–Connes map on the image of $K_*(C_0(M_k) \rtimes G)$ is given by the element $(q_k)!$. Put $A_k = C_0(M_k) \rtimes G$. The same construction is then given for the groupoid G', yielding proper G'-manifolds M'_k, maps $h'_k : M'_k \to M'_{k+1}$, algebras $A'_k = C_0(M'_k) \rtimes G'$, etc.

We may (and do) also assume that $h_{k+1}(M_k)/G$ is relatively compact in M_{k+1}/G. As in Section 5.A.3, let $\Gamma = \ker \pi$. As G' acts properly on M_{k+1}/Γ and by the relative compactness assumption, we may embed $h_k(M_{k-1})/\Gamma$ in a manifold approximating the classifying space for proper actions E' of G'. Using a subsequence of the M'_k we may assume that we are given equivariant smooth maps $\ell_k : M_k \to M'_k$.

Up to taking again a subsequence, we may further assume that the maps $h'_k \circ \ell_k$ and $\ell_{k+1} \circ h_k$ are homotopic (where $h'_k : M'_k \to M'_{k+1}$). Note that the maps ℓ_k are automatically K-oriented, and thus we obtain KK-elements $(\ell_k)! \in KK(A_k, A'_k) = KK(C_0(M_k) \rtimes G, C_0(M'_k) \rtimes G')$ satisfying $(\ell_k)! \otimes (h'_k)! = (h_k)! \otimes (\ell_{k+1})!$.

Now, using [Lafforgue 2007, Appendix A], we find (explicit) algebras D_k and morphisms $f_k : D_k \to A_k$ which are K-equivalences and $g_k : D_k \to A'_k$ such that $(\ell_k)! = [f_k]^{-1} \otimes [g_k]$.

Put then $x_k = [f_k] \otimes (h_k)! \otimes [f_{k+1}]^{-1} \in KK(D_k, D_{k+1})$. We find

$$\begin{aligned}(g_{k+1})_*(x_k) &= [f_k] \otimes (h_k)! \otimes [f_{k+1}]^{-1} \otimes [g_{k+1}]\\ &= [f_k] \otimes (h_k)! \otimes (\ell_{k+1})! = [f_k] \otimes (\ell_k)! \otimes (h'_k)!\\ &= [g_k] \otimes (h'_k)!.\end{aligned}$$

As shown in Section 5.B, using precise homotopies between Kasparov bimodules representing these elements, we can then construct elements $y_k \in KK(C_{g_k}, C_{g_{k+1}})$. Note also, that we have the equalities $\pi_*((q_k)_!) = (\ell_k)! \otimes (q'_k)! \in KK(A_k, C^*(G'))$ as in Proposition 5.1, yielding an element $z_k \in KK(C_{g_k}, C_\pi)$.

In order to construct the topological K-theory for the mapping cone we need to make the following assumption — which could be true in general:

Assumption A.1. We assume that the homotopies used in the constructions of y_k and z_k are well matching, so that we have the equality $y_k \otimes z_{k+1} = z_k$.

We can then construct, for each k, C^*-algebras B_k and B'_k, morphisms $u_k : B_k \to D_k$ and $u'_k : B_k \to A'_k$ which are KK-equivalences and $v_k : B_k \to D_{k+1}$ and $v'_k : B_k \to D_{k+1}$ such that $x_k = [u_k]^{-1} \otimes [v_k]$ and $(h'_k)! = [u'_k]^{-1} \otimes [v'_k]$ (using [Lafforgue 2007, Appendix A]).

For the topological K-theory of G and G' (up to a shift of dimension by 1) we can then use the infinite telescopic algebras $T = T(v, u)$ and $T' = T(v', u')$. These algebras are mapping tori $\mathcal{T}(\check{u}, \check{v})$ and $\mathcal{T}(\check{u}', \check{v}')$, where

$$\check{u}, \check{v} : \check{B} = \bigoplus_{k=1}^{+\infty} B_k \to \check{D} = \bigoplus_{k=0}^{+\infty} D_k \quad \text{and} \quad \check{u}', \check{v}' : \check{B}' = \bigoplus_{k=1}^{+\infty} B'_k \to \check{A}' = \bigoplus_{k=0}^{+\infty} A'_k$$

are the maps given by

$$\begin{aligned}\check{u}(x_1, \ldots, x_k, \ldots) &= (0, u_1(x_1), \ldots, u_k(x_k), \ldots),\\ \check{v}(x_1, \ldots, x_k, \ldots) &= (v_1(x_1), \ldots, v_k(x_k), \ldots),\end{aligned}$$

and analogous formulae for $\check{u}'$ and $\check{v}'$.

The families of $(q_k)!$ and $(q'_k)!$ give elements $\check{q}!$ and $\check{q}'!$ in $KK(\check{D}, C^*(G))$ and $KK(\check{A}', C^*(G'))$, respectively.

The homotopy between $[\check{u}] \otimes q!$ and $[\check{v}] \otimes q!$ (resp. $[\check{u}'] \otimes q'!$ and $[\check{v}'] \otimes q'!$) gives rise to the element $\mu_G \in KK(T, C^*(G))$ (resp. $\mu'_G \in KK(T', C^*(G')))$.

We may now do the same construction at the mapping cone level: writing $y_k = [\alpha_k]^{-1} \otimes [\beta_k]$, where $\alpha_k : V_k \to C_{g_k}$ and $\beta_k : V_k \to C_{g_{k+1}}$ are morphisms, we may consider the infinite telescope $T(\beta, \alpha) = \mathcal{T}(\check{\alpha}, \check{\beta})$ as a topological K-theory $K_{*,\mathrm{top}}(\pi)$ for C_π. The element $\check{z}$ defined by the z_k's gives an element of $KK\big(\bigoplus C_{g_k}, C_\pi\big)$; a homotopy between $[\check{\alpha}] \otimes \check{z}$ and $[\check{\beta}] \otimes \check{z}$ (based on our assumption) gives rise to the Baum–Connes element $\mu_\pi \in KK(T(\beta, \alpha), C_\pi)$ and thus a morphism $\mu_\pi : K_{*,\mathrm{top}}(\pi) \to K_*(C_\pi)$.

Remark A.2. One may push a little further the above calculations. Indeed one needs to check that we have an exact sequence

$$\begin{tikzcd} K_{*,\mathrm{top}}(G) \arrow[rr, "\pi_*"] & & K_{*,\mathrm{top}}(G') \arrow[dl] \\ & K_{*,\mathrm{top}}(\pi) \arrow[ul] & \end{tikzcd}$$

compatible with the mapping cone exact sequence. It then follows that if G and G' satisfy the (full version of the) Baum–Connes conjecture, then so does C_π.

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