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On the intersection motive of certain Shimura varieties: the case of Siegel threefolds

Jörg Wildeshaus

We construct a Hecke-equivariant Chow motive whose realizations equal intersection cohomology of Siegel threefolds with regular algebraic coefficients. As a consequence, we are able to define Grothendieck motives for Siegel modular forms.

0. Introduction

The purpose of this paper is the construction and analysis of the *intersection motive* of Kuga–Sato families over a Siegel threefold relative to its Satake–(Baily–Borel) compactification. As in earlier work on Hilbert–Blumenthal varieties [Wildeshaus 2012b], Picard surfaces [Wildeshaus 2015], and more generally, Picard varieties of arbitrary dimension [Cloître 2017], the use of the formalism of *weight structures* [Bondarko 2010] proves to be successful for dealing with a problem, for which explicit geometrical methods seem inefficient.

However, Siegel threefolds present a characteristic feature different from the cases treated so far: the dimension of the boundary of their Satake–(Baily–Borel) compactification is equal to one. In particular, it is strictly positive.

As a consequence, the context of *geometrical motives*, i.e., motives over a point, is no longer adapted to the problem. Let us explain why.

The present construction, as the preceding ones, depends on *absence of weights* -1 *and* 0 in the *boundary motive*. To prove absence of weights, the idea remains, as previously, to employ *realizations*. But then, realizations need to detect weights (and therefore, their absence). One may expect this to be true in general; let us agree to refer to that principle as *weight conservativity*. To date, weight conservativity is *proved* for the restriction of the (generic) ℓ-adic realization to the category of *motives of abelian type* of characteristic zero [Wildeshaus 2018b].

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However, unless the boundary of the Baily–Borel compactification of a given Shimura variety M is of dimension zero, its boundary motive, as well as the boundary motive of any Kuga–Sato family B over M, is in general not of abelian type; this is in any case true if M is a Siegel threefold. Concretely, this means that even if the realization of the boundary motive were proved to avoid weights -1 and 0, we could not formally conclude that the same is true for the boundary motive itself.

This is where *relative motives*, together with the *formalism of six operations*, enter. Denoting by j the open immersion of M into its Baily–Borel compactification M^*, by i its closed complement, and by $\mathbb{1}_M$ the structural motive over M, there is an exact triangle

$$i_*i^*j_*\mathbb{1}_M[-1] \to j_!\mathbb{1}_M \to j_*\mathbb{1}_M \to i_*i^*j_*\mathbb{1}_M$$

of motives over M^*. The boundary motive of M is isomorphic to the dual of the direct image of $i_*i^*j_*\mathbb{1}_M$ under the structure morphism of M^*. More generally, the boundary motive of B is isomorphic to the dual of the direct image of $i_*i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}})$, where $\pi : \mathrm{B} \to M$ denotes the projection of the Kuga–Sato family B to its base.

It is then true that the relative motive $i_*i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}})$ over M^* *is* of abelian type.

This suggests our strategy of proof. First, identify the ℓ-adic realization of $i_*i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}})$, or more generally, of $i_*i^*j_*\mathcal{V}$, for direct factors $\mathcal{V}$ of $\pi_*(\mathbb{1}_{\mathrm{B}})$; in the cases where weights 0 and 1 are avoided, weight conservativity tells us that $i_*i^*j_*\mathcal{V}$ itself avoids weights 0 and 1. Second, apply the direct image a_* associated to the structure morphism a of M^*. It is proper, therefore, the functor a_* is *weight exact*. In particular, if $i_*i^*j_*\mathcal{V}$ avoids weights 0 and 1, then so does $a_*i_*i^*j_*\mathcal{V}$. The corresponding direct factor of the boundary motive of B thus avoids weights -1 and 0.

It may be useful to remark that if M is a Hilbert–Blumenthal or Picard variety, then there is essentially no difference between $i_*i^*j_*\mathcal{V}$ and its direct image under a, since the latter is of relative dimension zero on the boundary of M^*.

The passage from geometrical motives to relative motives necessitates a certain number of technical adjustments. For better legibility, we decided to separate these from the present text. The result is [Wildeshaus 2018a]; it contains in particular the identification of the boundary motive and the dual of $a_*i_*i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}})$ mentioned above.

Compared to the cases treated earlier, another feature of the boundary of Siegel threefolds is new: its canonical stratification is not reduced to a single type of strata. Indeed, in the boundary, one finds a closed stratum of dimension zero, the so-called *Siegel stratum*, and its complement, the so-called *Klingen stratum*, which is a disjoint union of (open) modular curves. Control of the weights avoided by the restrictions of the ℓ-adic realization $R_\ell(i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}}))$ of $i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}})$ to the two strata is related to but does not a priori determine the weights avoided by $R_\ell(i^*j_*\pi_*(\mathbb{1}_{\mathrm{B}}))$.

In fact, the precise relation is given by a long exact localization sequence. Its control is not obvious. In an earlier attempt, we succeeded to identify sufficiently many terms in this sequence, and (above all) certain morphisms, to prove absence of weights 0 and 1. This approach is technically difficult; moreover, it does not use the auto-duality property of the coefficients. Indeed, the device dual to the localization sequence is the colocalization sequence; even when the coefficients are auto-dual, the two sequences cannot be related. It turns out that both problems admit the same solution. Namely, the theory of *intermediate extensions* allows one to represent $R_\ell(i^* j_* \pi_*(\mathbb{1}_{\mathrm{B}}))$ as an extension of two "halfs", one dual to the other, and both related to the intermediate extension $j_{!*}\pi_*(\mathbb{1}_{\mathrm{B}})$. This observation is equally integrated in [Wildeshaus 2018a]; for our purposes, its concrete interest is to divide by two the number of cohomological degrees for which absence of weights has to be tested, and to reduce the number of morphisms in the localization sequence, which need to be identified, to zero.

The rôle of the intermediate extension is not only technical. It turns out that the dual of its direct image under a is canonically isomorphic to the *interior motive*, which according to [Wildeshaus 2009] can be defined as soon as the boundary motive avoids weights -1 and 0. This motivates the slight change of terminology in the title, as compared to the earlier work mentioned above [Wildeshaus 2012b; 2015; Cloître 2017].

Let us now give a more detailed account of the content of the present article. Section 1 contains the statement of our main result, Theorem 1.6. Denote by $\mathrm{GSp}_{4,\mathbb{Q}}$ the group of symplectic similitudes of a fixed four-dimensional $\mathbb{Q}$-vector space V. As will be recalled, irreducible representations of $\mathrm{GSp}_{4,\mathbb{Q}}$ are indexed by weights $\underline{\alpha}$ depending on three integral parameters: $\underline{\alpha} = \alpha(k_1, k_2, r)$. The weight $\underline{\alpha}$ is dominant if and only if $k_1 \geq k_2 \geq 0$; it is regular if and only if $k_1 > k_2 > 0$. Denote by $V_{\underline{\alpha}}$ the irreducible representation of highest weight $\underline{\alpha}$. According to the main result from [Ancona 2015] (which will be recalled in Theorem 1.4), there is a Chow motive ${}^{\underline{\alpha}}\mathcal{V}$ over the Siegel threefold M whose cohomological (Hodge theoretic or ℓ-adic) realizations equal the classical *canonical construction* $\mu(V_{\underline{\alpha}})$. Part (a) of Theorem 1.6 then states that $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ is of abelian type. Part (b) asserts that if $\underline{\alpha}$ is regular, then $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ avoids weights 0 and 1. It has recently become increasingly important to determine the precise interval containing [0, 1] of weights avoided by $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$. Theorem 1.6(b) gives a complete answer: putting $k := \min(k_1 - k_2, k_2)$, the motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ avoids all the weights between $-k+1$ and k, while both weights $-k$ and $k+1$ *do* occur. Interestingly, this result does not depend on the level of the Siegel threefold. We then list the main consequences of this result (Corollaries 1.7, 1.8, 1.9, 1.11, 1.13), applying the theory developed in [Wildeshaus 2018a].

Section 2 is devoted to the proof of Theorem 1.6. As in previous cases, our control of smooth *toroidal compactifications* of M is sufficiently explicit to verify that, as stated in Theorem 1.6(a), the motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ is indeed of abelian type. Given this result, and weight conservativity of the restriction of the ℓ-adic realization R_ℓ, part (b) of Theorem 1.6 may be checked on the image of $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ under R_ℓ. Given that ${}^{\underline{\alpha}}\mathcal{V}$ realizes to give $\mu(V_{\underline{\alpha}})$, the restriction of $R_\ell(i^* j_* {}^{\underline{\alpha}}\mathcal{V})$ to the (Siegel and Klingen) strata can be computed following a standard pattern, employing Pink's and Kostant's theorems. This computation (Theorem 2.3) is considerably simplified by results of [Lemma 2015]. It remains to glue the information coming from the strata, in order to get control of the weights on the whole boundary. The part of Theorem 1.6(b) asserting that weights $-k$ and $k+1$ occur in $R_\ell(i^* j_* {}^{\underline{\alpha}}\mathcal{V})$ (Proposition 2.9) is the single ingredient requiring a proof longer than any other.

In the final Section 3, we give the necessary ingredients to perform the construction of the Grothendieck motive associated to a (Siegel) automorphic form with coefficients in an irreducible representation with regular highest weight (Definition 3.5). This is the analogue for Siegel threefolds of the main result from [Scholl 1990]. On the level of Galois representations, our definition coincides with Weissauer's [2005, Theorem I]. We also recover Urban's result [2005, Théorème 1] on characteristic polynomials associated to Frobenii (Corollary 3.7).

Conventions. We make use of the triangulated $\mathbb{Q}$-linear categories $DM_{\mathrm{B},c}(X)$ of *constructible Beilinson motives* over X [Cisinski and Déglise 2009, Definition 15.1.1], indexed by schemes X over $\operatorname{Spec}\mathbb{Q}$, which are separated and of finite type. As in [Cisinski and Déglise 2009], the symbol $\mathbb{1}_X$ is used to denote the unit for the tensor product in $DM_{\mathrm{B},c}(X)$. We employ the full formalism of six operations developed in [loc. cit.]. The reader may choose to consult [Hébert 2011, Section 2] or [Wildeshaus 2012a, Section 1] for concise presentations of this formalism.

Beilinson motives can be endowed with a canonical weight structure, thanks to the main results from [Hébert 2011]; see [Bondarko 2010, Proposition 6.5.3] for the case $X = \operatorname{Spec} k$, for a field k of characteristic zero. We refer to it as the *motivic weight structure*. Following [Wildeshaus 2012a, Definition 1.5], the category $CHM(X)_{\mathbb{Q}}$ of *Chow motives* over X is defined as the heart $DM_{\mathrm{B},c}(X)_{w=0}$ of the motivic weight structure on $DM_{\mathrm{B},c}(X)$.

A scheme is said to be *nilregular* if the underlying reduced scheme is regular in the usual sense.

1. Statement of the main result

In order to state our main result (Theorem 1.6), let us introduce the situation we are going to consider. The $\mathbb{Q}$-scheme M^K is a *Siegel threefold*, and the Chow motive ${}^{\underline{\alpha}}\mathcal{V}$ over M^K is associated to a *dominant weight* $\underline{\alpha} = (k_1, k_2, r) \in \mathbb{Z}^3$, $k_1 \geq k_2 \geq 0$ (see below for the precise normalizations). Denote by j the open immersion of M^K into its *Satake–(Baily–Borel) compactification* $(M^K)^*$, and by $i : \partial(M^K)^* \hookrightarrow (M^K)^*$

the immersion of the complement of M^K in $(M^K)^*$ (with the reduced scheme structure, say). Recall the following.

Definition 1.1 (cf. [Wildeshaus 2018a, Definition 2.1(a)]). Let $CHM(M^K)_{\mathbb{Q},\partial w\neq 0,1}$ denote the full subcategory of $CHM(M^K)_{\mathbb{Q}}$ of objects V such that $i^* j_* V$ is without weights 0 and 1.

Theorem 1.6 implies that in our setting, the motive ${}^{\underline{\alpha}}\mathcal{V} \in CHM(M^K)_{\mathbb{Q}}$ belongs to $CHM(M^K)_{\mathbb{Q},\partial w\neq 0,1}$ if and only if $\underline{\alpha}$ is *regular*: $k_1 > k_2 > 0$. More precisely, putting $k := \min(k_1 - k_2, k_2)$, the motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ is without weights $-k+1, -k+2, \dots, k$. The proof of Theorem 1.6 is given in Section 2. It is an application of [Wildeshaus 2018a, Theorem 4.4]; in order to verify the hypotheses of the latter, we heavily rely on results from [Lemma 2015].

Fix a four-dimensional $\mathbb{Q}$-vector space V, together with a $\mathbb{Q}$-valued nondegenerate symplectic bilinear form J.

Definition 1.2. The group scheme G over $\mathbb{Q}$ is defined as the group of symplectic similitudes

$$G := \mathrm{GSp}(V, J) \subset \mathrm{GL}(V).$$

Thus, G is reductive, and for any $\mathbb{Q}$-algebra R, the group $G(R)$ equals

$$\{g \in \mathrm{GL}(V \otimes_{\mathbb{Q}} R) : \exists \lambda(g) \in R^*,\ J(g\,\bullet, g\,\bullet) = \lambda(g) \cdot J(\,\bullet, \bullet)\}.$$

In particular, the similitude norm $\lambda(g)$ defines a canonical morphism

$$\lambda : G \to \mathbb{G}_{m,\mathbb{Q}}\,.$$

The group G is split over $\mathbb{Q}$, and its center $Z(G)$ equals $\mathbb{G}_{m,\mathbb{Q}} \subset \mathrm{GL}(V)$ (inclusion of scalar automorphisms). Maximal $\mathbb{Q}$-split tori, together with an inclusion into a Borel subgroup of G, are in bijection with symplectic $\mathbb{Q}$-bases of V, in which J acquires the 4×4-matrix

$$\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

also denoted by J. Here as in the sequel, we denote by I_2 the 2×2-matrix representing the identity. Fix one such basis (e_1, e_2, e_3, e_4), use it to identify G with the subgroup $\mathrm{GSp}_{4,\mathbb{Q}}$ of $\mathrm{GL}_{4,\mathbb{Q}}$ of matrices g satisfying the relation

$${}^t g J g = \lambda(g) \cdot J,$$

the maximal split torus with the subgroup T of diagonal matrices

$$\{\mathrm{diag}(a, b, a^{-1}q, b^{-1}q) \in \mathrm{GL}_{4,\mathbb{Q}}\},$$

and the Borel subgroup with the subgroup of matrices stabilizing the flag of totally isotropic subspaces $(e_1)_{\mathbb{Q}} \subset (e_1, e_2)_{\mathbb{Q}}$ of V. We consider triplets $(k_1, k_2, r) \in \mathbb{Z}^3$

satisfying the congruence relation

$$r \equiv k_1 + k_2 \mod 2.$$

To such a triplet, let us associate the (representation-theoretic) weight

$$\alpha(k_1, k_2, r) : T \to \mathbb{G}_{m,\mathbb{Q}}, \quad \operatorname{diag}(a, b, a^{-1}q, b^{-1}q) \mapsto a^{k_1} b^{k_2} q^{-(r+k_1+k_2)/2}.$$

Note that restriction of $\alpha(k_1, k_2, r)$ to $T \cap \mathrm{Sp}(V, J)$ corresponds to the projection onto (k_1, k_2). In particular, the weight $\alpha(k_1, k_2, r)$ is dominant if and only if $k_1 \geq k_2 \geq 0$; it is regular if and only if $k_1 > k_2 > 0$. Note also that the composition of $\alpha(k_1, k_2, r)$ with the cocharacter

$$\mathbb{G}_{m,\mathbb{Q}} \to T, \quad x \mapsto \operatorname{diag}(x, x, x, x)$$

equals

$$\mathbb{G}_{m,\mathbb{Q}} \to \mathbb{G}_{m,\mathbb{Q}}, \quad x \mapsto x^{-r}.$$

The character λ on T equals $\alpha(0, 0, -2)$, and $\det = \lambda^2$.

Definition 1.3. The analytic space $\mathcal{H}$ is defined as the subspace of $M_2(\mathbb{C})$ of those complex 2×2-matrices, which are symmetrical, and whose imaginary part is (positive or negative) definite:

$$\mathcal{H} := \{\tau \in M_2(\mathbb{C}) : {}^t\tau = \tau \text{ and } \operatorname{Im}(\tau) \text{ definite}\}.$$

The group of real points $G(\mathbb{R})$ acts on $\mathcal{H}$ by analytical automorphisms [Pink 1989, Example 2.7]. In fact, $(G, \mathcal{H})$ are *pure Shimura data* [Pink 1989, Definition 2.1]. Their *reflex field* [Pink 1989, Section 11.1] equals $\mathbb{Q}$. Given that $Z(G) = \mathbb{G}_{m,\mathbb{Q}}$, the Shimura data $(G, \mathcal{H})$ satisfy condition $(+)$ from [Wildeshaus 2007, Section 5].

Let us now fix additional data:

(A) an open compact subgroup K of $G(\mathbb{A}_f)$ which is neat [Pink 1989, Section 0.6],

(B) a triplet $(k_1, k_2, r) \in \mathbb{Z}^3$ satisfying the congruence

$$r \equiv k_1 + k_2 \mod 2,$$

and in addition,

$$k_1 \geq k_2 \geq 0.$$

In other words, the character $\underline{\alpha} := \alpha(k_1, k_2, r)$ is dominant.

These data are used as follows. The Shimura variety $M^K := M^K(G, \mathcal{H})$ is smooth over $\mathbb{Q}$. This is the Siegel threefold of level K. According to [Pink 1989, Theorem 11.16], it admits an interpretation as modular space of abelian surfaces with additional structures. In particular, there is a universal family B of abelian surfaces over M^K.

The following result holds in the general context of (smooth) Shimura varieties of *PEL*-type.

Theorem 1.4 [Ancona 2015, Théorème 8.6]. *There is a* $\mathbb{Q}$*-linear tensor functor*

$$\widetilde{\mu} : \mathrm{Rep}(G) \to CHM^s(M^K)_{\mathbb{Q}}$$

from the Tannakian category $\mathrm{Rep}(G)$ *of algebraic representations of* G *in finite dimensional* $\mathbb{Q}$*-vector spaces to the* $\mathbb{Q}$*-linear category* $CHM^s(M^K)_{\mathbb{Q}}$ *of* smooth Chow motives *over* M^K (see [Levine 2009, Definition 5.16]). *It has the following properties.*

(a) *The composition of* $\widetilde{\mu}$ *with the cohomological Hodge theoretic realization is isomorphic to the* canonical construction *functor* $\mu_{\boldsymbol{H}}$ (e.g., [Wildeshaus 1997, Theorem 2.2]) *to the category of admissible graded-polarizable variations of Hodge structure on* $M^K_{\mathbb{C}}$.

(b) *The composition of* $\widetilde{\mu}$ *with the cohomological* ℓ*-adic realization is isomorphic to the* canonical construction *functor* μ_ℓ (e.g., [Wildeshaus 1997, Chapter 4]) *to the category of lisse* ℓ*-adic sheaves on* M^K.

(c) *The functor* $\widetilde{\mu}$ *commutes with Tate twists.*

(d) *The functor* $\widetilde{\mu}$ *maps the representation* V *to the dual of the Chow motive* $\pi^1_* \mathbb{1}_{\mathrm{B}}$ *over* M^K.

Here, we denote by $\pi^m_* \mathbb{1}_{\mathrm{B}}$ the m-th *Chow-Künneth component* of the Chow motive $\pi_* \mathbb{1}_{\mathrm{B}}$ over M^K [Deninger and Murre 1991, Theorem 3.1].

Proof. Parts (a), (c) and (d) are identical to [Ancona 2015, Théorème 8.6].

As for part (b), repeat the proof of [loc. cit.], observing that the ℓ-adic analogue of [Ancona 2015, Proposition 8.5] holds (the base change to $\mathbb{Q}_\ell$ of the subgroup G_1 of G coincides with the Lefschetz group). □

Given that the representation on V is faithful, it follows that any object in the image of $\widetilde{\mu}$ is isomorphic to a direct sum of direct factors of Tate twists of the Chow motive $\pi_{n_i,*} \mathbb{1}_{\mathrm{B}^{n_i}}$ associated to B^{n_i}, for suitable $n_i \in \mathbb{N}$, where $\pi_{n_i} : \mathrm{B}^{n_i} \to M^K$ denotes the n_i-fold fibre product of B over M^K.

Definition 1.5. (a) Denote by $V_{\underline{\alpha}} \in \mathrm{Rep}(G)$ the irreducible representation of highest weight $\underline{\alpha}$.

(b) Define ${}^{\underline{\alpha}}\mathcal{V} \in CHM^s(M^K)_{\mathbb{Q}} \subset CHM(M^K)_{\mathbb{Q}}$ as

$${}^{\underline{\alpha}}\mathcal{V} := \widetilde{\mu}(V_{\underline{\alpha}}).$$

Given that $V_{\underline{\alpha}}$ is of weight r, the cohomological realizations of ${}^{\underline{\alpha}}\mathcal{V}$ equal zero in (classical, i.e., nonperverse) degrees $\neq r$, and $\mu_{\boldsymbol{H}}(V_{\underline{\alpha}})$ (in the Hodge theoretic setting) or $\mu_\ell(V_{\underline{\alpha}})$ (in the ℓ-adic setting) in degree r.

Denote by $j : M^K \hookrightarrow (M^K)^*$ the open immersion of M^K into its Satake–(Baily–Borel) compactification, by $i : \partial(M^K)^* \hookrightarrow (M^K)^*$ its complement, and by Φ the natural stratification of $\partial(M^K)^*$ (the latter will be made explicit in the beginning of Section 2). Here is our main result.

Theorem 1.6. (a) *The motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V} \in DM_{B,c}(\partial(M^K)^*)$ is a* Φ-constructible motive of abelian type over $\partial(M^K)^*$ (*see Definition 2.1*).

(b) *The motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ is without weights*

$$-k+1, -k+2, \dots, k,$$

where $k := \min(k_1 - k_2, k_2)$. Both weights $-k$ and $k+1$ do *occur in $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$. In particular, ${}^{\underline{\alpha}}\mathcal{V}$ belongs to the subcategory $CHM(M^K)_{\mathbb{Q},\partial w \neq 0,1}$ of $CHM(M^K)_{\mathbb{Q}}$ if and only if $\underline{\alpha}$ is regular.*

Theorem 1.6 should be compared to [Wildeshaus 2012b, Theorem 3.5], [Wildeshaus 2015, Theorem 3.8], and [Cloître 2017, Theorem 3.6, Proposition 3.8, Proposition 3.9] (see also [Wildeshaus 2018a, Remark 5.8(b)]), which treat the cases of Hilbert–Blumenthal varieties, of Picard surfaces, and of Picard varieties of arbitrary dimension, respectively.

Theorem 1.6 is proved in Section 2. For the rest of the present section, assume that $k = \min(k_1 - k_2, k_2) \geq 1$, i.e., $k_1 > k_2 > 0$. Given that according to Theorem 1.6(b), the motive ${}^{\underline{\alpha}}\mathcal{V}$ belongs to $CHM(M^K)_{\mathbb{Q},\partial w \neq 0,1}$, the *intersection motive of M^K relative to $(M^K)^*$ with coefficients in* ${}^{\underline{\alpha}}\mathcal{V}$ is at our disposal: by [Wildeshaus 2018a, Definition 3.7], it equals

$$a_* j_{!*} {}^{\underline{\alpha}}\mathcal{V} \in CHM(\mathbb{Q})_{\mathbb{Q}},$$

where $a : (M^K)^* \to \operatorname{Spec} \mathbb{Q}$ is the structure morphism of $(M^K)^*$. By abuse of language, let us abbreviate, and refer to $a_* j_{!*} {}^{\underline{\alpha}}\mathcal{V}$ as the *intersection motive with coefficients in* ${}^{\underline{\alpha}}\mathcal{V}$. Let us list the main corollaries of Theorem 1.6.

Corollary 1.7. *Denote by a and $\tilde{a}$ the structure morphisms of $(M^K)^*$ and M^K, respectively, and by m the natural transformation $j_! \to j_*$. Assume $k_1 > k_2 > 0$, i.e., $k \geq 1$.*

(a) *The motive $\tilde{a}_! {}^{\underline{\alpha}}\mathcal{V} \in DM_{B,c}(\mathbb{Q})$ is without weights $-k, -k+1, \dots, -1$, and the motive $\tilde{a}_* {}^{\underline{\alpha}}\mathcal{V} \in DM_{B,c}(\mathbb{Q})$ is without weights $1, 2, \dots, k$. More precisely, the exact triangles*

$$a_* i_* i^* j_{!*} {}^{\underline{\alpha}}\mathcal{V}[-1] \to \tilde{a}_! {}^{\underline{\alpha}}\mathcal{V} \to a_* j_{!*} {}^{\underline{\alpha}}\mathcal{V} \to a_* i_* i^* j_{!*} {}^{\underline{\alpha}}\mathcal{V}$$

and

$$a_* j_{!*} {}^{\underline{\alpha}}\mathcal{V} \to \tilde{a}_* {}^{\underline{\alpha}}\mathcal{V} \to a_* i_* i^! j_{!*} {}^{\underline{\alpha}}\mathcal{V}[1] \to a_* j_{!*} {}^{\underline{\alpha}}\mathcal{V}[1]$$

are weight filtrations (of $\tilde{a}_! {}^{\underline{\alpha}}\mathcal{V}$) avoiding weights $-k, -k+1, \dots, -1$, and (of $\tilde{a}_* {}^{\underline{\alpha}}\mathcal{V}$) avoiding weights $1, 2, \dots, k$, respectively.

(b) *The intersection motive* $a_* j_{!*}{}^{\alpha}\underline{\mathcal{V}} \in CHM(\mathbb{Q})_{\mathbb{Q}}$ *behaves functorially with respect to both* $\tilde{a}_!{}^{\alpha}\underline{\mathcal{V}}$ *and* $\tilde{a}_*{}^{\alpha}\underline{\mathcal{V}}$. *In particular, any endomorphism of* $\tilde{a}_!{}^{\alpha}\underline{\mathcal{V}}$ *or of* $\tilde{a}_*{}^{\alpha}\underline{\mathcal{V}}$ *induces an endomorphism of* $a_* j_{!*}{}^{\alpha}\underline{\mathcal{V}}$.

(c) *Let* $\tilde{a}_!{}^{\alpha}\underline{\mathcal{V}} \to N \to \tilde{a}_*{}^{\alpha}\underline{\mathcal{V}}$ *be a factorization of the morphism* $a_*m : \tilde{a}_!{}^{\alpha}\underline{\mathcal{V}} \to \tilde{a}_*{}^{\alpha}\underline{\mathcal{V}}$ *through a Chow motive* $N \in CHM(\mathbb{Q})_{\mathbb{Q}}$. *Then the intersection motive* $a_* j_{!*}{}^{\alpha}\underline{\mathcal{V}}$ *is canonically identified with a direct factor of* N, *with a canonical direct complement.*

Proof. Given Theorem 1.6, parts (a), (b) and (c) follow from [Wildeshaus 2018a, Theorem 3.4], [Wildeshaus 2018a, Theorem 3.5] and [Wildeshaus 2009, Corollary 2.5], respectively. □

The equivariance statement from Corollary 1.7(b) applies in particular to endomorphisms coming from the *Hecke algebra* $\mathfrak{H}(K, G(\mathbb{A}_f))$ associated to the neat open compact subgroup K of $G(\mathbb{A}_f)$. Recall that by what was said earlier, the relative Chow motive ${}^{\alpha}\underline{\mathcal{V}}$ is a direct factor of a Tate twist of $\pi_{N,*}\mathbb{1}_{\mathrm{B}^N}$, where $\pi_N : \mathrm{B}^N \to M^K$ denotes the N-fold fibre product of the universal abelian scheme B over M^K.

Corollary 1.8. *Assume* $k \geq 1$. *Every element of the Hecke algebra* $\mathfrak{H}(K, G(\mathbb{A}_f))$ *acts naturally on the intersection motive* $a_* j_{!*}{}^{\alpha}\underline{\mathcal{V}}$.

Proof. Let $T \in \mathfrak{H}(K, G(\mathbb{A}_f))$. According to Corollary 1.7(b), it suffices to show that T acts on $\tilde{a}_*{}^{\alpha}\underline{\mathcal{V}}$. To do so, we refer to [Wildeshaus 2017, pp. 591–592]. □

Corollary 1.9. *Assume* $k \geq 1$, *and let* $\widetilde{\mathrm{B}^N}$ *be any smooth compactification of* B^N. *Then the intersection motive* $a_* j_{!*}{}^{\alpha}\underline{\mathcal{V}}$ *is a direct factor of a Tate twist of the Chow motive* $b_*\mathbb{1}_{\widetilde{\mathrm{B}^N}}$ *(*$b :=$ *the structure morphism of the* $\mathbb{Q}$*-scheme* $\widetilde{\mathrm{B}^N}$*).*

Proof. The motive ${}^{\alpha}\underline{\mathcal{V}}$ is a direct factor of a Tate twist of $\pi_{N,*}\mathbb{1}_{\mathrm{B}^N}$:

$$ {}^{\alpha}\underline{\mathcal{V}} \hookrightarrow \pi_{N,*}\mathbb{1}_{\mathrm{B}^N}(\ell)[2\ell] \twoheadrightarrow {}^{\alpha}\underline{\mathcal{V}}, $$

for a suitable integer ℓ. The morphism

$$ a_*m : \tilde{a}_!\pi_{N,*}\mathbb{1}_{\mathrm{B}^N} \to \tilde{a}_*\pi_{N,*}\mathbb{1}_{\mathrm{B}^N} $$

factors through the Chow motive $b_*\mathbb{1}_{\widetilde{\mathrm{B}^N}}$, and hence so does

$$ a_*m : \tilde{a}_!{}^{\alpha}\underline{\mathcal{V}} \to \tilde{a}_*{}^{\alpha}\underline{\mathcal{V}}. $$

Now apply Corollary 1.7(c). □

Remark 1.10. When $r \geq 0$, then according to [Ancona 2017, Lemma 4.13], the Chow motive ${}^{\alpha}\underline{\mathcal{V}}$ is a direct factor of $\pi_{N,*}\mathbb{1}_{\mathrm{B}^N}$ (no Tate twist needed). In this context, let us recall [Wildeshaus 2018a, Corollary 3.10]: the intersection motive $a_* j_{!*}{}^{\alpha}\underline{\mathcal{V}}$ is canonically dual to the $e_{\underline{\alpha}}$-part of the *interior motive* of B^N, where $e_{\underline{\alpha}}$ is the idempotent endomorphism corresponding to the direct factor ${}^{\alpha}\underline{\mathcal{V}}$ of $\pi_{N,*}\mathbb{1}_{\mathrm{B}^N}$.

Corollary 1.11. *Assume $k \geq 1$, i.e., that $\underline{\alpha}$ is regular. Then for all $n \in \mathbb{Z}$, the natural maps*

$$H^n((M^K)^*(\mathbb{C}), j_{!*}\,\mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) \to H^n(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}}))$$

(*in the Hodge theoretic setting*) *and*

$$H^n((M^K)^* \times_{\mathbb{Q}} \overline{\mathbb{Q}}, j_{!*}\,\mu_\ell(V_{\underline{\alpha}})) \to H^n(M^K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}}))$$

(*in the ℓ-adic setting*) *are injective. Dually,*

$$H^n_c(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) \to H^n((M^K)^*(\mathbb{C}), j_{!*}\,\mu_{\boldsymbol{H}}(V_{\underline{\alpha}}))$$

and

$$H^n_c(M^K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})) \to H^n((M^K)^* \times_{\mathbb{Q}} \overline{\mathbb{Q}}, j_{!*}\,\mu_\ell(V_{\underline{\alpha}}))$$

are surjective. In other words, the natural maps from intersection cohomology to cohomology with coefficients in $\mu_{\boldsymbol{H}}(V_{\underline{\alpha}})$ and in $\mu_\ell(V_{\underline{\alpha}})$ identify intersection and interior cohomology, respectively.

Proof. Write ${}^{\underline{\alpha}}\mathcal{V}$ as a direct factor of $\pi_{N,*}\mathbb{1}_{\mathrm{B}^N}(\ell)[2\ell]$, for a suitable integer ℓ. Given Theorem 1.6, we may quote [Wildeshaus 2018a, Remark 3.13(a), (b)] (for $X = (M^K)^*$, $U = M^K$, $C = \mathrm{B}^N$ and $e = e_{\underline{\alpha}}$). □

As pointed out in [Wildeshaus 2018a, Remark 3.13(c)], sheaf theoretic considerations alone suffice to show (without any further reference to geometry) that Theorem 1.6 implies Corollary 1.11.

Corollary 1.11 is already known. Indeed, according to [Mokrane and Tilouine 2002, Proposition 1], the result generalizes to Siegel varieties of arbitrary dimension. (However, the proof of [loc. cit.] is analytic.)

Remark 1.12. By [Wildeshaus 2009, Theorem 4.14], control of the reduction of *some* compactification of B^N implies control of certain properties of the ℓ-adic realization of the intersection motive $a_*\,j_{!*}\,{}^{\underline{\alpha}}\mathcal{V}$. According to [Faltings and Chai 1990, Theorem VI.1.1], there exists a smooth compactification of B^N having good reduction at each prime number p not dividing the level n of K.

Theorem 4.14 of [Wildeshaus 2009] then yields the following:

(a) for all prime numbers p not dividing n, the p-adic realization of $a_*\,j_{!*}\,{}^{\underline{\alpha}}\mathcal{V}$ is crystalline;

(b) if furthermore p and ℓ are different, then the ℓ-adic realization of $a_*\,j_{!*}\,{}^{\underline{\alpha}}\mathcal{V}$ is unramified at p.

Corollary 1.13. *Assume $k \geq 1$. Let p be a prime number not dividing the level of K. Let ℓ be different from p. Then the characteristic polynomials of the following coincide*:

(1) *the action of Frobenius ϕ on the ϕ-filtered module associated to the* (*crystalline*) *p-adic realization of the intersection motive $a_* \, j_{!*}{}^{\alpha}\underline{\mathcal{V}}$,*

(2) *the action of a geometrical Frobenius automorphism at p on the* (*unramified*) *ℓ-adic realization of $a_* \, j_{!*}{}^{\alpha}\underline{\mathcal{V}}$.*

Proof. Fix a smooth compactification $\widetilde{\mathrm{B}^N}$ of B^N with good reduction at p [Faltings and Chai 1990, Theorem VI.1.1]. Thus the $\mathbb{Q}_p$-scheme $\widetilde{\mathrm{B}^N} \times_{\mathbb{Q}} \mathbb{Q}_p$ is the generic fibre of a smooth and proper scheme $\widetilde{\mathcal{B}^N}$ over $\mathbb{Z}_p$. Let us denote by $\widetilde{\mathcal{B}^N_{\mathbb{F}_p}}$ its special fibre.

The ϕ-filtered module associated to p-adic étale cohomology of $\widetilde{\mathrm{B}^N} \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ is first isomorphic to *Hyodo–Kato* cohomology $H^{\bullet}_{HK}(\widetilde{\mathrm{B}^N} \times_{\mathbb{Q}} \overline{\mathbb{Q}}_p)$ [Beilinson 2013, Section 3.2], and this isomorphism can be chosen to be motivic in the sense that it commutes with the action of correspondences in $\widetilde{\mathrm{B}^N} \times_{\mathbb{Q}} \widetilde{\mathrm{B}^N}$ [Déglise and Nizioł 2018, Section 4.15]. By definition, Hyodo–Kato cohomology is log-crystalline cohomology of a log-smooth model; in our case, given good reduction, such a model is given by $\widetilde{\mathcal{B}^N}$ (with divisor equal to zero). In other words, Hyodo–Kato cohomology equals crystalline cohomology of $\widetilde{\mathcal{B}^N}$. This identification commutes with the action of correspondences in $\widetilde{\mathcal{B}^N} \times_{\mathbb{Z}_p} \widetilde{\mathcal{B}^N}$. Finally, crystalline cohomology of $\widetilde{\mathcal{B}^N}$ equals crystalline cohomology of $\widetilde{\mathcal{B}^N_{\mathbb{F}_p}}$.

Altogether, the ϕ-filtered module associated to p-adic étale cohomology of $\widetilde{\mathrm{B}^N} \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ is identified with crystalline cohomology of $\widetilde{\mathcal{B}^N_{\mathbb{F}_p}}$ in a way compatible with the action of correspondences in $\widetilde{\mathcal{B}^N} \times_{\mathbb{Z}_p} \widetilde{\mathcal{B}^N}$. Concretely, this means that given a correspondence e in $\widetilde{\mathcal{B}^N} \times_{\mathbb{Z}_p} \widetilde{\mathcal{B}^N}$, the action of its generic fibre $e_{\mathbb{Q}_p}$ on p-adic étale cohomology is identified with the action of its special fibre $e_{\mathbb{F}_p}$ on crystalline cohomology.

For $\ell \neq p$, smooth and proper base change allows us to identify ℓ-adic cohomology of $\widetilde{\mathrm{B}^N} \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ and ℓ-adic cohomology of $\widetilde{\mathcal{B}^N_{\mathbb{F}_p}} \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p$, again compatibly with correspondences.

According to Corollary 1.9, there is an idempotent endomorphism $e_{\mathbb{Q}}$ of the Chow motive associated to $\widetilde{\mathrm{B}^N}$, or in other words, an idempotent correspondence in $\widetilde{\mathrm{B}^N} \times_{\mathbb{Q}} \widetilde{\mathrm{B}^N}$, whose images in the endomorphism rings of the realizations are projections onto the realizations of $a_* \, j_{!*}{}^{\alpha}\underline{\mathcal{V}}$. We claim that $e_{\mathbb{Q}_p} := e_{\mathbb{Q}} \times_{\mathbb{Q}} \mathbb{Q}_p$ can be extended idempotently to $\widetilde{\mathcal{B}^N} \times_{\mathbb{Z}_p} \widetilde{\mathcal{B}^N}$. Indeed, according to [O'Sullivan 2011, Proposition 5.1.1], the restriction morphism from the endomorphism ring of the Chow motive associated to $\widetilde{\mathcal{B}^N}$ to that of the Chow motive associated to $\widetilde{\mathrm{B}^N}$ is epimorphic, with nilpotent kernel. We now follow a standard line of argument (cf. [Kimura 2005, proof of Corollary 7.8]): let $\mathfrak{e}$ be *any* extension of $e_{\mathbb{Q}_p}$ to $\widetilde{\mathcal{B}^N} \times_{\mathbb{Z}_p} \widetilde{\mathcal{B}^N}$. The difference $\mathfrak{e} - \mathfrak{e}^2$ is nilpotent, say

$$(\mathfrak{e} - \mathfrak{e}^2)^N = 0.$$

But then,

$$e_{\mathbb{Z}_p} := (\mathrm{id}_{\widetilde{\mathcal{B}^N}} - (\mathrm{id}_{\widetilde{\mathcal{B}^N}} - \mathfrak{e})^N)^N$$

equally extends $e_{\mathbb{Q}_p}$ to $\widetilde{\mathcal{B}^N} \times_{\mathbb{Z}_p} \widetilde{\mathcal{B}^N}$, and $e_{\mathbb{Z}_p}$ is idempotent.

Altogether, there is a smooth and proper scheme $\widetilde{\mathcal{B}^N_{\mathbb{F}_p}}$ over $\mathbb{F}_p$, and an idempotent endomorphism $e_{\mathbb{F}_p}$ of the Chow motive associated to $\widetilde{\mathcal{B}^N_{\mathbb{F}_p}}$, whose images in the endomorphism rings of crystalline and ℓ-adic cohomology, respectively, are projections onto the realizations of $a_* j_{!*} {}^{\alpha}\mathcal{V}$. The claim thus follows from [Katz and Messing 1974, Theorem 2(2)]. □

2. Proof of the main result

We keep the notation of the preceding section. In order to prove Theorem 1.6, the idea is to apply the criterion from [Wildeshaus 2018a, Corollary 4.6].

In order to check the hypotheses of [loc. cit.], we first need to fix a finite stratification Φ of $\partial(M^K)^*$ by locally closed subschemes. The canonical choice would be the restriction Φ' to $\partial(M^K)^*$ of the natural (finite) stratification of $(M^K)^*$ from [Pink 1989, Main Theorem 12.3(c)] — in other words, all the strata of $(M^K)^*$ except the open one, i.e., except M^K. According to [Wildeshaus 2017, Lemma 8.2(a)], Φ' is *good*, meaning that the closure of every stratum is a union of strata. Furthermore, by [Wildeshaus 2017, Lemma 8.2(b)], all strata, denoted $i_g(M^{\pi_1(K_1)})$, are smooth over $\mathbb{Q}$ (recall that K is assumed neat, and that $(G, \mathcal{H})$ satisfies condition (+)), hence regular. The same is therefore true for the following coarser stratification $\Phi = \{0, 1\}$ of $\partial(M^K)^*$: denote by $i_0 : Z_0 \hookrightarrow \partial(M^K)^*$ the disjoint union of all closed strata of Φ', and by $i_1 : Z_1 \hookrightarrow \partial(M^K)^*$ the disjoint union of all strata of Φ', which are open in $\partial(M^K)^*$. Indeed, according to [Pink 1989, Section 6.3, Example 4.25 (with $g = 2$)],

$$\partial(M^K)^* = Z_0 \amalg Z_1;$$

more precisely, Z_0 is of dimension zero, and Z_1 of dimension one (hence so is the whole of $\partial(M^K)^*$). Let us refer to Z_0 as the *Siegel stratum*, and to Z_1 as the *Klingen stratum* of $\partial(M^K)^*$. When it is necessary to insist on the structure of stratified scheme of $\partial(M^K)^*$, we write $\partial(M^K)^*(\Phi)$ instead of $\partial(M^K)^*$.

Definition 2.1 [Wildeshaus 2018b, Definitions 3.4 and 3.5]. (a) Let $S(\mathfrak{S}) = \coprod_{\sigma \in \mathfrak{S}} S_\sigma$ be a good stratification of a scheme $S(\mathfrak{S})$. A morphism $\pi : S(\mathfrak{S}) \to \partial(M^K)^*(\Phi)$ is said to be a *morphism of good stratifications* if the preimage of any of the strata Z_0, Z_1 of $\partial(M^K)^*$ is a union of strata S_σ.

(b) A morphism $\pi : S(\mathfrak{S}) \to \partial(M^K)^*(\Phi)$ of good stratifications is said to be *of abelian type* if it is proper, and if the following conditions are satisfied.

(1) All strata S_σ, $\sigma \in \mathfrak{S}$, are nilregular, and for any immersion $i_\tau : S_\tau \hookrightarrow \overline{S_\sigma}$ of a stratum S_τ into the closure $\overline{S_\sigma}$ of a stratum S_σ, the functor $i_\tau^!$ maps $\mathbb{1}_{\overline{S_\sigma}}$ to a *Tate motive over* S_τ [Levine 2010, Section 3.3].

(2) For all $\sigma \in \mathfrak{S}$ such that S_σ is a stratum of $\pi^{-1}(Z_m)$, $m \in \{0, 1\}$, the morphism $\pi_\sigma : S_\sigma \to Z_m$ can be factorized,

$$\pi_\sigma = \pi'_\sigma \circ \pi''_\sigma : S_\sigma \xrightarrow{\pi''_\sigma} B_\sigma \xrightarrow{\pi'_\sigma} Z_m,$$

such that the motive

$$\pi''_{\sigma,*}\mathbb{1}_{S_\sigma} \in DM_{\mathbb{B},c}(B_\sigma)_F$$

belongs to the category $DMT(B_\sigma)_F$ of Tate motives over B_σ, the morphism π'_σ is proper and smooth, and its pull-back to any geometric point of Z_m lying over a generic point is isomorphic to a finite disjoint union of abelian varieties.

(c) An object $V \in DM_{\mathbb{B},c}(\partial(M^K)^*)$ is a Φ-*constructible motive of abelian type over* $\partial(M^K)^*$ if the following holds: the motive V belongs to the strict, full, dense, $\mathbb{Q}$-linear triangulated subcategory $DM^{Ab}_{\mathbb{B},c,\Phi}(\partial(M^K)^*)$ generated by the images under π_* of $\mathfrak{S}$-*constructible Tate motives over* $S(\mathfrak{S})$ [Wildeshaus 2018b, Definition 3.3], where

$$\pi : S(\mathfrak{S}) \to \partial(M^K)^*(\Phi)$$

runs through the morphisms of abelian type with target equal to $\partial(M^K)^*(\Phi)$.

Theorem 2.2. *Let* $\underline{\alpha} = \alpha(k_1, k_2, r)$, *with* $(k_1, k_2, r) \in \mathbb{Z}^3$ *such that*

$$r \equiv k_1 + k_2 \mod 2 \quad \textit{and} \quad k_1 \geq k_2 \geq 0,$$

and consider ${}^{\underline{\alpha}}\mathcal{V} = \widetilde{\mu}(V_{\underline{\alpha}}) \in CHM(M^K)_{\mathbb{Q}}$. *The motive* $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ *belongs to the full subcategory* $DM^{Ab}_{\mathbb{B},c,\Phi}(\partial(M^K)^*)$ *of* $DM_{\mathbb{B},c}(\partial(M^K)^*)$. *In other words, it is a* Φ-*constructible motive of abelian type over* $\partial(M^K)^*$.

Proof. As recalled earlier, the relative Chow motive ${}^{\underline{\alpha}}\mathcal{V}$ belongs to the strict, full, dense, $\mathbb{Q}$-linear triangulated subcategory

$$\pi_{N,*}DMT(\mathrm{B}^N)^\natural_{\mathbb{Q}}$$

of $DM_{\mathbb{B},c}(M^K)$ generated by the images under $\pi_{N,*}$ of the category of Tate motives over B^N. Here as before, $\pi_N : \mathrm{B}^N \to M^K$ denotes the N-fold fibre product of the universal abelian scheme B over M^K.

The latter equals the projection from a *mixed Shimura variety*: indeed [Pink 1989, Example 2.7], the representation V of G is of Hodge type $\{(-1, 0), (0, -1)\}$. The same is then true for the r-th power V^N of V. By [Pink 1989, Proposition 2.17], this allows for the construction of the *unipotent extension* $(P^N, \mathfrak{X}^N)$ of $(G, \mathcal{H})$

by V^N. The pair $(P^N, \mathfrak{X}^N)$ constitute *mixed Shimura data* [Pink 1989, Definition 2.1]. By construction, they come endowed with a morphism of Shimura data $\pi_N : (P^N, \mathfrak{X}^N) \to (G, \mathcal{H})$, identifying $(G, \mathcal{H})$ with the pure Shimura data underlying $(P^N, \mathfrak{X}^N)$. In particular, $(P^N, \mathfrak{X}^N)$ also satisfies condition (+). Now by [Pink 1989, Theorem 11.18] there is an open compact neat subgroup K_N of $P^N(\mathbb{A}_f)$, whose image under π_N equals K, such that B^N is identified with the *mixed Shimura variety* $M^{K_N} := M^{K_N}(P^N, \mathfrak{X}^N)$, and such that the morphism $M^{K_N} \to M^K$ induced by the morphism π_N of Shimura data is identified with the structure morphism of B^N.

Choose a smooth *toroidal compactification* $M^{K_N}(\mathfrak{S}) := M^{K_N}(P^N, \mathfrak{X}^N, \mathfrak{S})$ of M^{K_N}, associated to a K_N*-admissible complete cone decomposition* $\mathfrak{S}$ [Pink 1989, Section 6.4]. Then by [Pink 1989, proof of Theorem 9.21], modulo a suitable refinement of $\mathfrak{S}$, the natural stratification of $M^{K_N}(\mathfrak{S})$, also denoted $\mathfrak{S}$, satisfies the conclusions of [Wildeshaus 2017, Lemma 8.1], i.e., it is good, and the closures of all strata are regular. Note that the unique open stratum equals M^{K_N}. According to [Pink 1989, Section 6.24, Main Theorem 12.4(b)], the morphism $\pi_N : \mathrm{B}^N = M^{K_N} \to M^K$ extends to a proper, surjective morphism $M^{K_N}(\mathfrak{S}) \to (M^K)^*$, still denoted π_N. From the description given in [Pink 1989, Section 7.3], one sees that π_N is a morphism of stratifications.

According to [Wildeshaus 2017, Corollary 4.10(b), Remark 4.7], the category

$$\pi_{N,*} DMT_{\mathfrak{S}}\big(M^{K_N}(\mathfrak{S})\big)^{\natural}_{\mathbb{Q}}$$

is obtained by gluing $\pi_{N,*} DMT(\mathrm{B}^N)^{\natural}_{\mathbb{Q}}$ and $\pi_{N,*} DMT_{\mathfrak{S}}\big(\pi_N^{-1}(\partial(M^K)^*)\big)^{\natural}_{\mathbb{Q}}$. In particular,

$$i^* j_* {}^{\alpha}\mathcal{V} \in \pi_{N,*} DMT_{\mathfrak{S}}\big(\pi_N^{-1}(\partial(M^K)^*)\big)^{\natural}_{\mathbb{Q}}.$$

But π_N is of abelian type [Wildeshaus 2017, Lemma 8.4]; therefore,

$$\pi_{N,*} DMT_{\mathfrak{S}}\big(\pi_N^{-1}(\partial(M^K)^*)\big)^{\natural}_{\mathbb{Q}} \subset DM^{Ab}_{\mathrm{B},c,\Phi}(\partial(M^K)^*)\,. \qquad \square$$

Next, we collect information on the restriction of $i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V})$ to the strata Z_0 and Z_1. The following is essentially due to Lemma [2015, Section 4].

Theorem 2.3. *Let ℓ be a prime number.*

(a) *For all integers $n \le r+2$, the perverse cohomology sheaf*

$$H^n i_0^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V})$$

on Z_0 is of weights $\le n - (k_1 - k_2)$. The perverse cohomology sheaf

$$H^{r+2} i_0^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V})$$

is nonzero, and pure of weight $(r+2) - (k_1 - k_2)$.

(b) *For all integers $n \le r+2$, the perverse cohomology sheaf*

$$H^n i_1^* i^* j_* R_{\ell,M^K}({}^{\alpha}_{-}\mathcal{V})$$

on Z_1 is of weights $\le n-k_2$. The perverse cohomology sheaf

$$H^{r+2} i_1^* i^* j_* R_{\ell,M^K}({}^{\alpha}_{-}\mathcal{V})$$

is nonzero, and pure of weight $(r+2)-k_2$.

The proof of Theorem 2.3 is given after Remark 2.6. In order to prepare it, recall from [Pink 1989, Example 4.25] that Z_0 and Z_1 correspond bijectively to the $G(\mathbb{Q})$-conjugacy classes of proper *rational boundary components* [Pink 1989, Section 4.11] of $(G, \mathcal{H})$. Indeed, the group $G(\mathbb{Q})$ acts transitively on the set of totally isotropic subspaces of V of a given, strictly positive dimension.

We already fixed a basis (e_1, e_2, e_3, e_4) of V, in which our symplectic bilinear form J acquires the 4×4-matrix

$$\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

which we equally denoted by J. The subspaces V_0' and V_1' generated by $\{e_1, e_2\}$ and $\{e_1\}$, respectively, are both totally isotropic.

Following [Pink 1989, Example 4.25], we put $Q_m := \mathrm{Stab}_G(V_m')$, $m = 0, 1$. Let P_m denote the normal subgroup of Q_m underlying the rational boundary component $(P_m, \mathfrak{X}_m)$ giving rise to Z_m [Pink 1989, Section 4.7], and W_m its unipotent radical (which equals the unipotent radical of Q_m). Then, still according to [Pink 1989, Example 4.25],

$$Q_0 = \left\{ \begin{pmatrix} q\cdot A & A\cdot M \\ 0 & {}^tA^{-1} \end{pmatrix} : q \in \mathbb{G}_{m,\mathbb{Q}},\ A \in \mathrm{GL}_{2,\mathbb{Q}},\ {}^tM = M \right\},$$

$$P_0 = \left\{ \begin{pmatrix} q\cdot I_2 & M \\ 0 & I_2 \end{pmatrix} : q \in \mathbb{G}_{m,\mathbb{Q}},\ {}^tM = M \right\},$$

$$W_0 = \left\{ \begin{pmatrix} I_2 & M \\ 0 & I_2 \end{pmatrix} : {}^tM = M \right\},$$

while

$$Q_1 = \left\{ \begin{pmatrix} a & aq^{-1}(bu+dw) & v & aq^{-1}(cu+ew) \\ 0 & b & w & c \\ 0 & 0 & a^{-1}q & 0 \\ 0 & d & -u & e \end{pmatrix} : a,\ be-cd = q \in \mathbb{G}_{m,\mathbb{Q}} \right\},$$

$$P_1 = \left\{ \begin{pmatrix} be-cd & bu+dw & v & cu+ew \\ 0 & b & w & c \\ 0 & 0 & 1 & 0 \\ 0 & d & -u & e \end{pmatrix} : be-cd \in \mathbb{G}_{m,\mathbb{Q}} \right\},$$

$$W_1 = \left\{ \begin{pmatrix} 1 & u & v & w \\ 0 & 1 & w & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u & 1 \end{pmatrix} \right\}.$$

Observe that $Q_0 \cap Q_1$ equals the Borel subgroup of G stabilizing the flag $V_1' \subset V_0'$, and that both Q_0 and Q_1 contain the fixed maximal split torus

$$T = \{\mathrm{diag}(a, b, a^{-1}q, b^{-1}q) : a, b, q \in \mathbb{G}_{m,\mathbb{Q}}\}.$$

In particular, T is canonically identified with a maximal $\mathbb{Q}$-split torus of the reductive group Q_m/W_m, for $m = 0, 1$. Given a (representation-theoretic) weight $\alpha : T \to \mathbb{G}_{m,\mathbb{Q}}$, let us denote by α_m the same application, but with T seen as a subgroup of Q_m/W_m, $m = 0, 1$.

Note that

$$R_{\ell,M^K}({}^{\alpha}\mathcal{V}) = \mu_\ell(V_{\underline{\alpha}})[-r].$$

Recall that we denote by Φ' the natural (finite) stratification of $(M^K)^*$ from [Pink 1989, Main Theorem 12.3(c)], which is finer than Φ. In order to determine the classical cohomology objects $R^n i_m^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$, for $m = 0, 1$, and $n \in \mathbb{Z}$, one applies the following standard strategy.

(1) By Pink's theorem [1992, Theorem (5.3.1)], the restriction of $R^n i_m^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ to any individual stratum Z' of Φ' contributing to Z_m equals

$$R^n i_m^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z'} = \bigoplus_{p+q=n} \mu_{\ell,Z'}\big(H^p(H_C/K_W, H^q(\mathrm{Lie}(W_m), V_{\underline{\alpha}}))\big).$$

Here, H_C/K_W is an arithmetic subgroup (depending on Z') of C_m/W_m [Pink 1992, Section (5.2)], where C_m is the identity component of the Zariski closure of the centralizer in $Q_m(\mathbb{Q})$ of the rational boundary component $(P_m, \mathfrak{X}_m)$ [Pink 1992, Section (3.7)], and $\mu_{\ell,Z'}$ is the canonical construction functor to the category of lisse ℓ-adic sheaves on Z'.

(2) Apply Kostant's theorem [Vogan 1981, Theorem 3.2.3], in order to identify $H^q(\mathrm{Lie}(W_m), V_{\underline{\alpha}})$ as a representation of the reductive group Q_m/W_m; this allows us in particular to obtain its weights, and gives potential information concerning cohomology of H_C/K_W with coefficients in $H^q(\mathrm{Lie}(W_m), V_{\underline{\alpha}})$.

The Hodge theoretic analogue of the above strategy yields the cohomology objects of $i_m^* i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})_{|Z'}$; this was made explicit in [Lemma 2015, Section 4].

Since steps (2) of the ℓ-adic and the Hodge theoretic strategies are identical, we may use the computations from [loc. cit.] in our setting.

Proposition 2.4 [Lemma 2015, Section 4.3]. *Let* $\underline{\alpha} = \alpha(k_1, k_2, r)$ *with* $(k_1, k_2, r) \in \mathbb{Z}^3$ *such that*

$$r \equiv k_1 + k_2 \mod 2 \quad \textit{and} \quad k_1 \geq k_2 \geq 0.$$

(*a*) *For* $m = 0, 1$, *we have*

$$H^q(\mathrm{Lie}(W_m), V_{\underline{\alpha}}) = 0$$

whenever $q < 0$ *or* $q > 3$. *If* $0 \leq q \leq 3$, *the* Q_m/W_m*-representation* $H^q(\mathrm{Lie}(W_1), V_{\underline{\alpha}})$ *is* (*nonzero and*) *irreducible.*

(*b*) *The highest* (*representation-theoretic*) *weight of* $H^q(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$, $0 \leq q \leq 3$, *is*

$$\begin{aligned} \alpha_0(k_1, k_2, r) &\quad \textit{for } q = 0, \\ \alpha_0(k_1, -k_2 - 2, r) &\quad \textit{for } q = 1, \\ \alpha_0(k_2 - 1, -k_1 - 3, r) &\quad \textit{for } q = 2, \\ \alpha_0(-k_2 - 3, -k_1 - 3, r) &\quad \textit{for } q = 3. \end{aligned}$$

(*c*) *The highest* (*representation-theoretic*) *weight of* $H^q(\mathrm{Lie}(W_1), V_{\underline{\alpha}})$, $0 \leq q \leq 3$, *is*

$$\begin{aligned} \alpha_1(k_1, k_2, r) &\quad \textit{for } q = 0, \\ \alpha_1(k_2 - 1, k_1 + 1, r) &\quad \textit{for } q = 1, \\ \alpha_1(-k_2 - 3, k_1 + 1, r) &\quad \textit{for } q = 2, \\ \alpha_1(-k_1 - 4, k_2, r) &\quad \textit{for } q = 3. \end{aligned}$$

Proof. Note that given our normalization, we have

$$\alpha(k_1, k_2, r) = \lambda(k_1, k_2, -r)$$

in the notation of [Lemma 2015, top of p. 87].

Part (a) follows from Kostant's theorem, and from the following fact (see [Lemma 2015, proofs of Lemmas 4.8 and 4.10]), valid for both $m = 0$ and $m = 1$: the set of Weyl representatives for Q_m contains no element of length < 0 or > 3, and exactly one element of respective lengths 0, 1, 2 and 3.

As for part (c), we refer to [Lemma 2015, proof of Lemma 4.10].

[Lemma 2015, proof of Lemma 4.8] contains the complete setting for the application of Kostant's theorem for $m = 0$, but makes it explicit only for $H^2(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$ and $H^3(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$. The reader will have no difficulty filling in the missing information needed for part (b). □

Note that both Q_0/W_0 and Q_1/W_1 are isomorphic to $\mathbb{G}_{m,\mathbb{Q}} \times_{\mathbb{Q}} \mathrm{GL}_{2,\mathbb{Q}}$. More precisely,

$$Q_0/W_0 = P_0/W_0 \times_{\mathbb{Q}} \mathrm{GL}_{2,\mathbb{Q}} = \mathbb{G}_{m,\mathbb{Q}} \times_{\mathbb{Q}} \mathrm{GL}_{2,\mathbb{Q}},$$

the identification given by sending the class of a matrix

$$\begin{pmatrix} q\cdot A & A\cdot M \\ 0 & {}^tA^{-1} \end{pmatrix}$$

to the pair (q, A), and

$$Q_1/W_1 = P_1/W_1 \times_{\mathbb{Q}} \mathbb{G}_{m,\mathbb{Q}} = \mathrm{GL}_{2,\mathbb{Q}} \times_{\mathbb{Q}} \mathbb{G}_{m,\mathbb{Q}},$$

the identification given by sending the class of a matrix

$$\begin{pmatrix} a & aq^{-1}(bu+dw) & v & aq^{-1}(cu+ew) \\ 0 & b & w & c \\ 0 & 0 & a^{-1}q & 0 \\ 0 & d & -u & e \end{pmatrix}$$

to the pair

$$\left(\begin{pmatrix} b & c \\ d & e \end{pmatrix}, aq^{-1}\right).$$

The restriction of the inverse identification to maximal split tori sends

$$\left(q, \begin{pmatrix} x & 0 \\ 0 & x^{-1}y \end{pmatrix}\right) \in P_0/W_0 \times_{\mathbb{Q}} \mathrm{GL}_{2,\mathbb{Q}}$$

to

$$\mathrm{diag}(qx, qx^{-1}y, x^{-1}, xy^{-1}) \in T \subset Q_0/W_0$$

for $m = 0$, and

$$\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1}q \end{pmatrix}, y\right) \in P_1/W_1 \times_{\mathbb{Q}} \mathbb{G}_{m,\mathbb{Q}}$$

to

$$\mathrm{diag}(yq, x, y^{-1}, x^{-1}q) \in T \subset Q_1/W_1$$

for $m = 1$.

In the following, the reader should be particularly careful not to confuse two notions of *weight* associated to representations of reductive groups: the highest weights in the sense of representation theory (e.g., those occurring in Kostant's theorem), when the representation is irreducible, and the weights as determined by the action of the weight cocharacter [Pink 1989, Section 1.3], when the group underlies Shimura data.

Corollary 2.5. (a) *The Q_0/W_0-representations $H^q(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$, $0 \le q \le 2$, are (irreducible and) regular, except when $q = 0$ and $k_1 = k_2$, in which case $H^0(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$ factors through the quotient $\mathbb{G}_{m,\mathbb{Q}} \times_{\mathbb{Q}} \mathbb{G}_{m,\mathbb{Q}}$ of the group*

$$Q_0/W_0 = \mathbb{G}_{m,\mathbb{Q}} \times_{\mathbb{Q}} \mathrm{GL}_{2,\mathbb{Q}}$$

via the determinant on the factor $\mathrm{GL}_{2,\mathbb{Q}}$. *The restriction to* $\mathrm{SL}_{2,\mathbb{Q}} \subset \mathrm{GL}_{2,\mathbb{Q}}$ *of* $H^1(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$ *is of highest (representation-theoretic) weight* k_1+k_2+2. *The restriction to* P_0/W_0 *of* $H^0(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$ *is of weight* $(r+1)-(k_1+k_2)-1$, *and the restriction of* $H^1(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$ *is of weight* $(r+2)-(k_1-k_2)$.

(b) *The restriction to* P_1/W_1 *of* $H^0(\mathrm{Lie}(W_1), V_{\underline{\alpha}})$ *is of weight* $(r+1)-k_1-1$, *and the restriction of* $H^1(\mathrm{Lie}(W_1), V_{\underline{\alpha}})$ *is of weight* $(r+2)-k_2-1$.

Proof. (a): Given the above identifications, the weight $\alpha_0(n_1,n_2,r)$ on T maps

$$\left(q, \begin{pmatrix} x & 0 \\ 0 & x^{-1}y \end{pmatrix}\right) \in P_0/W_0 \times_{\mathbb{Q}} \mathrm{GL}_{2,\mathbb{Q}}$$

to

$$\alpha_0(n_1,n_2,r)(\mathrm{diag}(qx, qx^{-1}y, x^{-1}, xy^{-1})) = x^{n_1-n_2}y^{n_2}q^{-(r-n_1-n_2)/2}.$$

In particular, the restriction of $\alpha_0(n_1,n_2,r)$ to $T \cap \mathrm{SL}_{2,\mathbb{Q}}$ corresponds to the integer n_1-n_2. The first and the second claim thus follow from Proposition 2.4(b).

The weight cocharacter $\mathbb{G}_{m,\mathbb{Q}} \to P_0/W_0 = \mathbb{G}_{m,\mathbb{Q}}$ maps z to z^2 [Pink 1989, Examples 4.25 and 2.8]. Its composition with the inclusion into T, and with $\alpha_0(n_1,n_2,r)$ yields

$$\mathbb{G}_{m,\mathbb{Q}} \to \mathbb{G}_{m,\mathbb{Q}}, \quad z \mapsto z^{-r+n_1+n_2}.$$

The third claim thus follows from Proposition 2.4(b), and from the normalization of weights of representations [Pink 1989, Section 1.3].

(b): The weight cocharacter $\mathbb{G}_{m,\mathbb{Q}} \to P_1/W_1 = \mathrm{GL}_{2,\mathbb{Q}}$ maps z to

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

[Pink 1989, Examples 4.25 and 2.8]. Given the above identifications, its composition with the inclusion into T maps z to $\mathrm{diag}(z^2, z, 1, z)$. Further composition with $\alpha_1(n_1,n_2,r)$ then yields

$$\mathbb{G}_{m,\mathbb{Q}} \to \mathbb{G}_{m,\mathbb{Q}}, \quad z \mapsto z^{-r+n_1}.$$

The claim thus follows from Proposition 2.4(c). □

To complete the ingredients needed for the computation of the $R^n i_m^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ according to the strategy (1), (2) sketched earlier in this section, observe that the group H_C/K_W associated to an individual stratum Z' of Φ' contributing to Z_m is a neat arithmetic subgroup of $\mathrm{GL}_2(\mathbb{Q})$ for $m=0$ [Lemma 2015, proof of Lemma 4.8], and hence of $\mathrm{SL}_2(\mathbb{Q})$. In particular, it is of cohomological dimension one. For $m=1$, the group H_C/K_W, being a neat arithmetic subgroup of $\mathbb{G}_m(\mathbb{Q})$, is trivial [Lemma 2015, proof of Lemma 4.10].

Remark 2.6. When $m = 0$, let V_2 denote the standard representation of $\mathrm{SL}_{2,\mathbb{Q}}$, and $u \in \mathbb{N}$. Then $\mathrm{Sym}^u V_2 \in \mathrm{Rep}(\mathrm{SL}_{2,\mathbb{Q}})$; in fact, $\mathrm{Sym}^u V_2$ is the irreducible representation of highest (representation-theoretic) weight u. Denote by g the genus of the quotient of the upper half space by H_C/K_W, and by $c \geq 1$ the number of its cusps. (Thus, $c \geq 3$ if $g = 0$ since H_C/K_W is neat.) Then $H^1(H_C/K_W, \mathrm{Sym}^u V_2)$ is of dimension $(u+1)(2g-2+c)$ if $u \geq 1$, and of dimension $2g-1+c$ if $u = 0$. In particular,

$$H^1(H_C/K_W, \mathrm{Sym}^u V_2) \neq 0 \quad \text{for all } u \in \mathbb{N}.$$

Proof of Theorem 2.3. (a): According to Corollary 2.5(a) and Proposition 2.4(a),

(o) $0 \neq H^0(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$ is of weight $(r+1)-(k_1+k_2)-1$,

(i) $0 \neq H^1(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$ is of weight $(r+2)-(k_1-k_2)$,

and $H^q(\mathrm{Lie}(W_0), V_{\underline{\alpha}}) = 0$ whenever $q < 0$. The group H_C/K_W associated to a stratum Z' of Z_0 is a neat arithmetic subgroup of $\mathrm{SL}_2(\mathbb{Q})$. It is therefore of cohomological dimension one, and admits no nonzero invariants on regular irreducible representations of $Q_0/W_0 = \mathbb{G}_{m,\mathbb{Q}} \times_{\mathbb{Q}} \mathrm{GL}_{2,\mathbb{Q}}$.

By Proposition 2.4(a) and Corollary 2.5(a), $H^q(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$, $0 \leq q \leq 2$, are irreducible as representations of Q_0/W_0; it is regular unless $q = 0$ and $k_1 = k_2$, in which case $\mathrm{SL}_{2,\mathbb{Q}}$, so H_C/K_W acts trivially. Pink's theorem and [Pink 1992, Proposition (5.5.4)] then tell us that

(o) $R^0 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ is nonzero if and only if $k_1 = k_2$, in which case it is of weight $r-(k_1+k_2)$,

(i) $0 \neq R^1 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ is of weight $(r+1)-(k_1+k_2)-1$,

(ii) $0 \neq R^2 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ is of weight $(r+2)-(k_1-k_2)$,

and that $R^n i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = 0$ whenever $n < 0$ (for the nonvanishing statements in (i), (ii), see Remark 2.6).

The scheme Z_0 is of dimension zero; therefore,

$$H^n i_0^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V}) = H^{n-r} i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = R^{n-r} i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}).$$

From (o), (i), (ii) and the vanishing of $R^n i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = 0$ for $n < 0$, we conclude that

(r) $H^r i_0^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$ is zero if $k_1 > k_2$, and nonzero of weight $r-(k_1+k_2)$ if $k_1 = k_2$,

(r+1) $0 \neq H^{r+1} i_0^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$ is of weight $(r+1)-(k_1+k_2)-1$,

(r+2) $0 \neq H^{r+2} i_0^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$ is of weight $(r+2)-(k_1-k_2)$,

and that $H^n i_0^* i^* j_* R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V}) = 0$ whenever $n \leq r-1$.

(b): According to Corollary 2.5(b) and Proposition 2.4(a),

(o) $0 \neq H^0(\mathrm{Lie}(W_1), V_{\underline{\alpha}})$ is of weight $(r+1) - k_1 - 1$,

(i) $0 \neq H^1(\mathrm{Lie}(W_1), V_{\underline{\alpha}})$ is of weight $(r+2) - k_2 - 1$,

and $H^q(\mathrm{Lie}(W_1), V_{\underline{\alpha}}) = 0$ whenever $q < 0$. The group H_C/K_W associated to a stratum Z' of Z_1 is trivial. Pink's theorem and [Pink 1992, Lemma (5.6.6)] then tell us that

(o) $0 \neq R^0 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ is of weight $(r+1) - k_1 - 1$,

(i) $0 \neq R^1 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ is of weight $(r+2) - k_2 - 1$,

and that $R^n i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = 0$ whenever $n < 0$. Furthermore, Pink's theorem tells us that all classical cohomology objects $R^n i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$, $n \in \mathbb{Z}$, are lisse. The formula

$$H^n i_1^* i^* j_* R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V}) = H^{n-r} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = (R^{n-r-1} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}))[1]$$

is valid: the first equation comes from

$$R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V}) = \mu_\ell(V_{\underline{\alpha}})[-r].$$

As for the second, note that any lisse ℓ-adic sheaf $\mathcal{F}$ on a one-dimensional regular scheme is a perverse sheaf $\mathcal{F}'$ up to a shift by -1:

$$\mathcal{F} = \mathcal{F}'[-1] \quad \text{and} \quad \mathcal{F}' = \mathcal{F}[1].$$

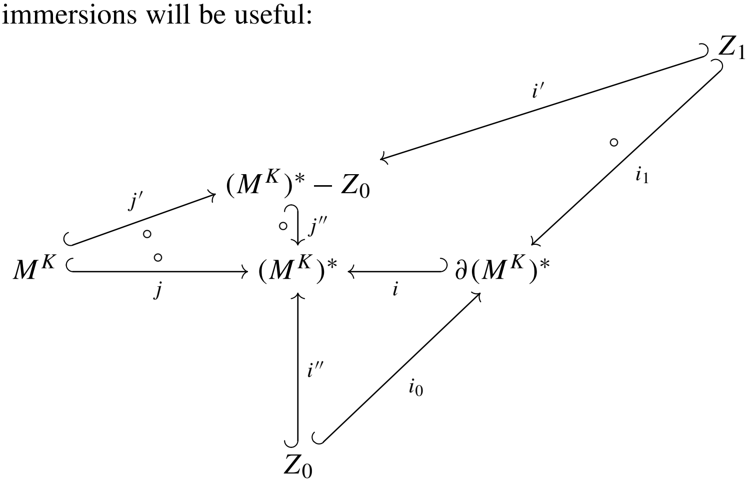
From (o), (i) and the vanishing of $R^n i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = 0$ for $n < 0$, we conclude that

(r+1) $0 \neq H^{r+1} i_1^* i^* j_* R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V})$ is of weight $(r+1) - k_1$,

(r+2) $0 \neq H^{r+2} i_1^* i^* j_* R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V})$ is of weight $(r+2) - k_2$,

and that $H^n i_1^* i^* j_* R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V}) = 0$ whenever $n \leq r$. □

For the final step of the proof of Theorem 1.6, the following commutative diagram of immersions will be useful:



Immersions situated on the same line are complementary to each other (example: j'' and i''), the four immersions marked by "$\circ$" are open (example: i_1), and the other four are closed (example: i').

Remark 2.7. Denote by $\tau_{Z_m}^{t\leq\bullet}$ and $\tau_{Z_m}^{t\geq\bullet}$ the truncation functors with respect to the perverse t-structure on Z_m, $m=0,1$.

(a) The immersions j' and i' being complementary,

$$(i')^* j'_{!*}\mathcal{F}' = \tau_{Z_1}^{t\leq -1}(i')^* j'_*\mathcal{F}'$$

for any perverse sheaf $\mathcal{F}'$ on M^K [Beilinson et al. 1982, Proposition 1.4.23].

(b) The intermediate extension is transitive, i.e.,

$$j_{!*} = j''_{!*}\, j'_{!*}$$

[Beilinson et al. 1982, Corollaire 1.4.24]. Application of the functor $(i'')^* j''_*$ to the exact triangle

$$i'_*\tau_{Z_1}^{t\geq 0}(i')^* j'_*[-1] \to j'_{!*} \to j'_* \to i'_*\tau_{Z_1}^{t\geq 0}(i')^* j'_*$$

of functors on perverse sheaves on M^K (see (a)) yields the exact triangle

$$i_0^* i_{1,*}\tau_{Z_1}^{t\geq 0}(i')^* j'_*[-1] \to (i'')^* j''_*\, j'_{!*} \to i_0^* i^* j_* \to i_0^* i_{1,*}\tau_{Z_1}^{t\geq 0}(i')^* j'_*.$$

The immersions j'' and i'' being complementary, we have as in (a)

$$(i'')^* j''_{!*}\mathcal{F}'' = \tau_{Z_0}^{t\leq -1}(i'')^* j''_*\mathcal{F}''$$

for any perverse sheaf $\mathcal{F}''$ on $(M^K)^* - Z_0$. It follows that for any perverse sheaf $\mathcal{F}'$ on M^K, there are exact sequences of perverse cohomology objects

$$H^{n-1}\big(i_0^* i_{1,*}\tau_{Z_1}^{t\geq 0} i_1^* i^* j_*\mathcal{F}'\big) \to H^n(i_0^* i^* j_{!*}\mathcal{F}') \to H^n(i_0^* i^* j_*\mathcal{F}') \to H^n\big(i_0^* i_{1,*}\tau_{Z_1}^{t\geq 0} i_1^* i^* j_*\mathcal{F}'\big)$$

for $n \leq -1$, while $H^n(i_0^* i^* j_{!*}\mathcal{F}') = 0$ for all $n \geq 0$.

(c) Recall that $R_{\ell,M^K}({}^{\alpha}\mathcal{V}) = \mu_\ell(V_{\underline{\alpha}})[-r]$; the variety M^K being of dimension three, the complex $R_{\ell,M^K}({}^{\alpha}\mathcal{V})$ is therefore concentrated in perverse degree $r+3$. According to our conventions, $i_1^* i^* j_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V}) = (i')^* j'_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V})$ thus equals

$$\big((i')^* j'_{!*}(R_{\ell,M^K}({}^{\alpha}\mathcal{V})[r+3])\big)[-(r+3)].$$

According to (a), we thus have

$$i_1^* i^* j_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V}) = \tau_{Z_1}^{t\leq r+2}(i')^* j'_* R_{\ell,M^K}({}^{\alpha}\mathcal{V}) = \tau_{Z_1}^{t\leq r+2} i_1^* i^* j_* R_{\ell,M^K}({}^{\alpha}\mathcal{V}).$$

Similarly, following (b),

$$H^n i_0^* i^* j_{!*} R_{\ell,M^K}({}^{\alpha}\mathcal{V}) = 0$$

for all $n \geq r+3$, and there are exact sequences

$$H^{n-1} i_0^* i_{1,*} \tau_{Z_1}^{t \geq r+3} i_1^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V}) \to H^n i_0^* i^* j_{!*} R_{\ell, M^K}({}^{\alpha}\mathcal{V}) \to H^n i_0^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V}) \to H^n i_0^* i_{1,*} \tau_{Z_1}^{t \geq r+3} i_1^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V})$$

for $n \leq r+2$.

(d) We claim that

$$H^n i_0^* i_{1,*} \tau_{Z_1}^{t \geq r+3} i_1^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V}) = 0$$

for all $n \leq r+1$. Equivalently,

$$H^n i_0^* i_{1,*} \tau_{Z_1}^{t \geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = 0$$

for all $n \leq 1$. Indeed, by Pink's theorem, the classical cohomology objects of $i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ are all lisse. Applying $\tau_{Z_1}^{t \geq 3}$, we thus get a complex concentrated in classical degrees ≥ 2 (recall that Z_1 is of dimension one). The same is thus true after application of $i_0^* i_{1,*}$ (recall that inverse images are t-exact for the classical t-structure). In other words, the complex

$$i_0^* i_{1,*} \tau_{Z_1}^{t \geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$$

has trivial cohomology (classical or perverse; recall that Z_0 is of dimension zero) in degrees ≤ 1.

(e) From (c) and (d), we deduce that

$$H^n i_0^* i^* j_{!*} R_{\ell, M^K}({}^{\alpha}\mathcal{V}) \xrightarrow{\sim} H^n i_0^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V})$$

for $n \leq r+1$, and that $H^{r+2} i_0^* i^* j_{!*} R_{\ell, M^K}({}^{\alpha}\mathcal{V})$ equals the kernel of

$$H^{r+2} i_0^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V}) \to H^{r+2} i_0^* i_{1,*} \tau_{Z_1}^{t \geq r+3} i_1^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V}).$$

Corollary 2.8. *Let ℓ be a prime number.*

(a) *For all $n \in \mathbb{Z}$,*

$$H^n i_0^* i^* j_{!*} R_{\ell, M^K}({}^{\alpha}\mathcal{V})$$

is of weights $\leq n - (k_1 - k_2)$.

(b) *For all $n \in \mathbb{Z}$,*

$$H^n i_1^* i^* j_{!*} R_{\ell, M^K}({}^{\alpha}\mathcal{V})$$

is of weights $\leq n - k_2$. The perverse cohomology sheaf

$$H^{r+2} i_1^* i^* j_{!*} R_{\ell, M^K}({}^{\alpha}\mathcal{V})$$

is nonzero, and pure of weight $(r+2) - k_2$.

Proof. Part (a) follows from Remark 2.7(c), (e), and from Theorem 2.3(a). Part (b) follows from Remark 2.7(c), and from Theorem 2.3(b). □

Corollary 2.8 suffices to prove the part of Theorem 1.6(b) asserting that regularity of $\underline{\alpha}$ is sufficient for weights 0 and 1 to be avoided by $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$. In order to prove that it is necessary, we need the following statement.

Proposition 2.9. *Let ℓ be a prime number. Then provided that $k_1 \geq 1$, the perverse cohomology sheaf*

$$H^{r+2} i_0^* i^* j_{!*} R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V})$$

is nonzero, and pure of weight $(r+2) - (k_1 - k_2)$.

Proof. According to Remark 2.7(e),

$$H^{r+2} i_0^* i^* j_{!*} R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V})$$

equals the kernel of

$$\mathrm{ad} : H^{r+2} i_0^* i^* j_* R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V}) \to H^{r+2} i_0^* i_{1,*} \tau_{Z_1}^{t \geq r+3} i_1^* i^* j_* R_{\ell, M^K}({}^{\underline{\alpha}}\mathcal{V})$$

— in particular, it is pure of weight $(r+2) - (k_1 - k_2)$ (Theorem 2.3(a)) — i.e., it equals the kernel of

$$H^2 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) \to H^2 i_0^* i_{1,*} \tau_{Z_1}^{t \geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}).$$

Thanks to Pink's theorem, the regularity of $H^2(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$ as a representation of Q_0 / W_0 (Corollary 2.5(a)), and the fact that the group H_C / K_W is of cohomological dimension one, locally on Z_0, the (perverse or classical) sheaf

$$H^2 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = R^2 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$$

equals

$$\mu_{\ell, Z'}\big(H^1(H_C / K_W, H^1(\mathrm{Lie}(W_0), V_{\underline{\alpha}}))\big),$$

for a stratum Z' of Φ' contributing to Z_0. Furthermore, by Corollary 2.5(a), the restriction of $H^1(\mathrm{Lie}(W_0), V_{\underline{\alpha}})$ to H_C / K_W is isomorphic to the (k_1+k_2+2)-nd symmetric power of the standard representation of $\mathrm{SL}_{2,\mathbb{Q}}$. Therefore, by Remark 2.6, $H^2 i_0^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z'}$ is of constant rank $(k_1 + k_2 + 3)(2g - 2 + c)$, where g denotes the genus of H_C / K_W, and c the number of cusps.

We claim that the restriction to the same Z' of

$$H^2 i_0^* i_{1,*} \tau_{Z_1}^{t \geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$$

is of constant rank c. Indeed, according to Remark 2.7(d), the classical cohomology objects of $i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$ are all lisse. Therefore, perverse truncation above degree three equals classical truncation above degree two (recall that Z_1 is of dimension one). The complex

$$i_0^* i_{1,*} \tau_{Z_1}^{t \geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})$$

is concentrated in degrees ≥ 2, and we get

$$H^2 i_0^* i_{1,*} \tau_{Z_1}^{t\geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}) = R^0 i_0^* i_{1,*} R^2 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}}).$$

Restriction to Z' yields

$$H^2 i_0^* i_{1,*} \tau_{Z_1}^{t\geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z'} = \bigoplus_{Z''} (R^0 i_0^* i_{1,*} (R^2 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z''}))_{|Z'},$$

where the direct sum is indexed by all strata Z'' contributing to Z_1, and containing Z' in their closure. For every such Z'',

$$R^2 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z''} = \mu_{\ell,Z''}(H^2(\mathrm{Lie}(W_1), V_{\underline{\alpha}}))$$

according to Pink's theorem (since the group H_C/K_W (for $m = 1$!) is trivial).

Denote by $j_1 : Z_1 \hookrightarrow Z_1^*$ the Baily–Borel compactification, and by $i_{01} : \partial Z_1^* \hookrightarrow Z_1^*$ its complement. The immersion $i_1 : Z_1 \hookrightarrow (M^K)^*$ admits a natural extension $\bar{i}_1 : Z_1^* \to (M^K)^*$ [Pink 1989, Main Theorem 12.3(c), Section 7.6], which is finite. The diagram

$$\begin{array}{ccc} Z_1^* & \xleftarrow{\;i_{01}\;} & \partial Z_1^* \\ {\scriptstyle \bar{i}_1}\downarrow & & \downarrow{\scriptstyle \bar{i}_1} \\ (M^K)^* & \xleftarrow{\;i_0\;} & Z_0 \end{array}$$

is cartesian up to nilpotent elements. Proper base change therefore yields the formula

$$R^0 i_0^* i_{1,*} = R^0 \bar{i}_{1,*} i_{0,1}^* j_{1,*}.$$

The functors $\bar{i}_{1,*}$ and $i_{0,1}^*$ being exact on sheaves, we have

$$R^0 i_0^* i_{1,*} (R^2 i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z''}) = \bar{i}_{1,*} i_{0,1}^* R^0 j_{1,*} \mu_{\ell,Z''}(H^2(\mathrm{Lie}(W_1), V_{\underline{\alpha}})).$$

According to Proposition 2.4(a), $H^2(\mathrm{Lie}(W_1), V_{\underline{\alpha}})$ is irreducible as a representation of Q_1/W_1, and hence of $\mathrm{GL}_{2,\mathbb{Q}}$. Yet another application of Pink's theorem shows that

$$i_{0,1}^* R^0 j_{1,*} \mu_{\ell,Z''}(H^2(\mathrm{Lie}(W_1), V_{\underline{\alpha}}))$$

is of constant rank one on the intersection of ∂Z_1^* with the closure of Z'' in $(Z_1)^*$.

Our claim on the rank of

$$H^2 i_0^* i_{1,*} \tau_{Z_1}^{t\geq 3} i_1^* i^* j_* \mu_\ell(V_{\underline{\alpha}})_{|Z'} = \bar{i}_{1,*} \bigoplus_{Z''} \big(i_{0,1}^* R^0 j_{1,*} \mu_{\ell,Z''}(H^2(\mathrm{Lie}(W_1), V_{\underline{\alpha}}))\big)_{|Z'}$$

is therefore proven as soon as we establish that the number of points in the geometrical fibres of the morphism $\bar{i}_1 : \partial Z_1^* \to Z_0$ above $Z' \subset Z_0$ equals c. This verification can be done on the level of $\mathbb{C}$-valued points, where the adelic description of the situation is at our disposal. More precisely, write $(G_m, \mathcal{H}_m) := (P_m, \mathfrak{X}_m)/W_m$ [Pink 1989, Proposition 2.9], $m = 0, 1$, for the Shimura data contributing to $\partial(M^K)^*$, and

Q_{01} for the Borel subgroup $Q_0 \cap Q_1$ of G. According to [Pink 1989, Section 6.3], the diagram of $\mathbb{C}$-valued points corresponding to the diagram

$$\begin{array}{ccc} Z_1^* & \overset{i_{01}}{\hookleftarrow} & \partial Z_1^* \\ {\scriptstyle \bar{i}_1}\downarrow & & \downarrow{\scriptstyle \bar{i}_1} \\ (M^K)^* & \overset{i_0}{\hookleftarrow} & Z_0 \end{array}$$

equals

$$\begin{array}{ccc} Q_1(\mathbb{Q})\backslash(\mathcal{H}_1^* \times G(\mathbb{A}_f)/K) & \overset{i_{01}}{\hookleftarrow} & Q_{01}(\mathbb{Q})\backslash(\mathcal{H}_0 \times G(\mathbb{A}_f)/K) \\ {\scriptstyle \bar{i}_1}\downarrow & & \downarrow{\scriptstyle \bar{i}_1} \\ G(\mathbb{Q})\backslash(\mathcal{H}^* \times G(\mathbb{A}_f)/K) & \overset{i_0}{\hookleftarrow} & Q_0(\mathbb{Q})\backslash(\mathcal{H}_0 \times G(\mathbb{A}_f)/K) \end{array}$$

where all maps are induced by canonical inclusions of groups and spaces. Indeed, the full group $Q_m(\mathbb{Q})$ (and not only a subgroup of finite index) stabilizes $\mathcal{H}_m$, $m = 0, 1$, and two rational boundary components of $(G_1, \mathcal{H}_1)$ are conjugate under $G_1(\mathbb{Q})$ if and only if they are conjugate under $G(\mathbb{Q})$ (by explicit computation, or [Pink 1989, Remark (iii) on p. 91]). The subscheme $Z' \subset Z_0$ equals the image of a Shimura variety associated to $(G_0, \mathcal{H}_0)$ under a morphism i_g associated to an element $g \in G(\mathbb{A}_f)$ [Pink 1989, Main Theorem 12.3(c)]; given the adelic description of i_g from [Pink 1989, Section 6.3], we see that under the above identification, any $z \in Z'(\mathbb{C})$ equals the class $[h_0, p_0 g]$ in

$$Q_0(\mathbb{Q})\backslash(\mathcal{H}_0 \times G(\mathbb{A}_f)/K)$$

of a pair of the form $(h_0, p_0 g)$, with $h_0 \in \mathcal{H}_0$ and $p_0 \in P_0(\mathbb{A}_f)$. Put

$$Q_0^+(\mathbb{Q}) := \{q_0 \in Q_0(\mathbb{Q}) : \lambda(q_0) > 0\};$$

this group equals the centralizer in $Q_0(\mathbb{Q})$ of h_0, and indeed, of the whole of $\mathcal{H}_0$. Putting

$$H_C' := Q_0^+(\mathbb{Q}) \cap p_0 g K g^{-1} p_0^{-1},$$

we leave it to the reader to verify that the map

$$Q_{01}(\mathbb{Q})\backslash Q_0(\mathbb{Q})/H_C' \to \bar{i}_1^{-1}(z), \quad [q_0] \mapsto q_0[h_0, p_0 g] = [q_0 h_0, q_0 p_0 g]$$

is well-defined, and bijective. By strong approximation,

$$W_0(\mathbb{Q}) \cdot H_C' = Q_0^+(\mathbb{Q}) \cap W_0(\mathbb{A}_f) \cdot p_0 g K g^{-1} p_0^{-1}.$$

But

$$Q_0/W_0 = P_0/W_0 \times_{\mathbb{Q}} \mathrm{GL}_{2,\mathbb{Q}},$$

meaning that modulo W_0, elements in P_0 and in Q_0 commute with each other.

Thus,

$$W_0(\mathbb{Q}) \cdot H'_C = Q_0^+(\mathbb{Q}) \cap W_0(\mathbb{A}_f) \cdot gKg^{-1}.$$

The image of $W_0(\mathbb{Q}) \cdot H'_C$ under the projection $\pi_0 : Q_0 \twoheadrightarrow Q_0/W_0$ coincides with the image of

$$W_0(\mathbb{A}_f) \cdot Q_0^+(\mathbb{Q}) \cap gKg^{-1}$$

(both images equal $\pi_0(Q_0^+(\mathbb{Q})) \cap \pi(gKg^{-1})$). But by definition [Pink 1992, (3.7.4)], $W_0(\mathbb{A}_f) \cdot Q_0^+(\mathbb{Q}) \cap gKg^{-1}$ equals H_C. We thus showed that

$$\pi_0(H'_C) = \pi_0(H_C).$$

Now the quotient morphism $Q_0 \twoheadrightarrow Q_0/P_0$, $q_0 \mapsto \overline{q_0}$ induces an isomorphism

$$Q_{01}(\mathbb{Q}) \backslash Q_0(\mathbb{Q})/H'_C \xrightarrow{\sim} \overline{Q_{01}(\mathbb{Q})} \backslash \overline{Q_0(\mathbb{Q})}/\overline{H'_C} = \overline{Q_{01}(\mathbb{Q})} \backslash \overline{Q_0(\mathbb{Q})}/\overline{H_C}.$$

But $\overline{Q_0(\mathbb{Q})} = \mathrm{GL}_2(\mathbb{Q})$, and under this identification, $\overline{Q_{01}(\mathbb{Q})}$ equals the subgroup of upper triangular matrices, while $\overline{H_C} = H_C/K_W$. In other words,

$$Q_{01}(\mathbb{Q}) \backslash Q_0(\mathbb{Q})/H'_C$$

is identified with the set up cusps of H_C/K_W.

The formula

$$(k_1 + k_2 + 3)(2g - 2 + c) \geq 4(2g - 2 + c) > c$$

(recall that c is greater or equal to 1, and that $c \geq 3$ if $g = 0$) implies that the rank of the source of ad is strictly greater than the rank of its target; the kernel of ad is therefore nontrivial. □

Remark 2.10. (a) As the reader may verify,

$$H^{r+2} i_0^* i_{1,*} \tau_{Z_1}^{t \geq r+3} i_1^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V})$$

is pure of weight $(r+2) - (k_1 - k_2)$, i.e., of the same weight as

$$H^{r+2} i_0^* i^* j_* R_{\ell, M^K}({}^{\alpha}\mathcal{V}).$$

Weight considerations alone therefore do not imply nontriviality of the kernel of the map ad from the proof of Proposition 2.9.

(b) A more conceptual proof of Proposition 2.9 would consist in showing that locally on Z_0, the map ad equals the direct sum over all cusps of H_C/K_W of the residue maps. Identify $H^1(H_C/K_W, H^1(\mathrm{Lie}(W_0), V_{\underline{\alpha}})) \otimes_{\mathbb{Q}} \mathbb{C}$ with the direct sum of the space of modular forms and (the conjugate of) the space of cusp forms for H_C/K_W of weight $k_1 + k_2 + 4 \geq 5$. The kernel of the residues contains the space of cusp forms. Its dimension is computed in [Shimura 1971, Theorems 2.24 and 2.25]; thanks to [Shimura 1971, Proposition 1.40] (always remember that H_C/K_W is neat), this dimension can be seen to be strictly positive.

(c) On the level of geometry of Baily–Borel compactifications, a "strange duality" seems to be involved in the proof of Proposition 2.9: we need to know how many modular curves in the boundary of $(M^K)^*$ contain a given cusp Z' in their closure. The response yields the number of cusps of a "modular curve", which does not explicitly occur in $(M^K)^*$, namely the quotient of the upper half space by H_C/K_W. It would be interesting to see how this phenomenon generalizes to higher dimensional Siegel varieties.

(d) Our computation of the fibres of the morphism $\bar{i}_1 : Z_1^* \to (M^K)^*$ over points of Z_0 is a quantitative version of a classical noninjectivity result of Satake [1958, Exemple on p. 13-06].

Remark 2.11. The Hodge theoretic analogues of Theorem 2.3, Corollary 2.8 and Proposition 2.9 hold. The proofs are identical up to the use of Pink's theorem, which is replaced by [Burgos and Wildeshaus 2004, Theorem 2.9].

Proof of Theorem 1.6. According to Theorem 2.2, $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ is a Φ-constructible motive of abelian type over $\partial(M^K)^*$; this proves part (a) of our claim.

By [Pink 1989, Summary 1.18(d)], there is a perfect pairing

$$V_{\underline{\alpha}} \otimes_{\mathbb{Q}} V_{\underline{\alpha}} \to \mathbb{Q}(-r)$$

in $\mathrm{Rep}(G)$.

Fix a prime ℓ. Applying μ_ℓ, we get a perfect pairing

$$\mu_\ell(V_{\underline{\alpha}}) \otimes_{\mathbb{Q}_\ell} \mu_\ell(V_{\underline{\alpha}}) \to \mathbb{Q}_\ell(-r)$$

of ℓ-adic lisse sheaves on M^K. In terms of local duality, the pairing induces an isomorphism

$$\mathbb{D}_{\ell,M^K}(\mu_\ell(V_{\underline{\alpha}})) \cong \mu_\ell(V_{\underline{\alpha}})(r+3)[6]$$

(M^K is smooth of dimension three). Given $R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V}) = \mu_\ell(V_{\underline{\alpha}})[-r]$, we find that

$$\mathbb{D}_{\ell,M^K}(R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})) \cong R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})(s)[2s],$$

where $s = r+3$.

Corollary 2.8 tells us that for all $n \in \mathbb{Z}$, and $m = 0, 1$,

$$H^n i_m^* i^* j_{!*} R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V})$$

is of weights $\leq n-k$. According to [Wildeshaus 2018a, Corollary 4.6(b)], the motive $i^* j_* {}^{\underline{\alpha}}\mathcal{V}$ therefore avoids weights $-k+1, -k+2, \dots, k$.

In order to conclude the proof of part (b), it remains to show, again thanks to [Wildeshaus 2018a, Corollary 4.6(b)], that for some $n \in \mathbb{Z}$, and $m = 0$ or $m = 1$, weight $n-k$ *does* occur in

$$H^n i_m^* i^* j_{!*} R_{\ell,M^K}({}^{\underline{\alpha}}\mathcal{V}).$$

We take $n = r + 2$, and distinguish two cases. If $k = k_2$, i.e., $k_2 \leq k_1 - k_2$, take $m = 1$; the claim then follows from Corollary 2.8(b). Else, $k_2 > k_1 - k_2$ and $k = k_1 - k_2$. Since $k_1 \geq k_2$, we necessarily have $k_1 \geq 1$. Take $m = 0$ and apply Proposition 2.9. $\square$

Remark 2.12. (a) An element of $H^n(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}}))$ is called a *ghost class* if it lies in the image of

$$H^n(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) \to H^n(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}}))$$

and in the kernel of both restriction maps

$$H^n(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) \to H^n(Z_m(\mathbb{C}), i_m^* i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})),$$

$m = 0, 1$. One of the main results of [Moya Giusti 2018] implies that if $\underline{\alpha}$ is regular, then there are no nonzero ghost classes [Moya Giusti 2018, Theorem 3.1]. This result does not formally imply, nor is it implied by, our Theorem 1.6. Nonetheless, it might be worthwhile to note that the weight arguments that occur in the proofs are quite similar. The most relevant information from Theorem 1.6, as far as [Moya Giusti 2018, Theorem 3.1] is concerned, comes from the weight filtration

$$a_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V} \to \tilde{a}_* {}^{\underline{\alpha}}\mathcal{V} \to a_* i_* i^! \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}[1] \to a_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}[1]$$

avoiding weights $1, 2, \ldots, k$ (Corollary 1.7(a)), and hence avoiding weight 1 if $\underline{\alpha}$ is regular, which we assume in the sequel. This implies that any element of $H^n(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}}))$ not mapping to zero in $H^n(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}}))$, remains nonzero in

$$H^n\big(\partial(M^K)^*(\mathbb{C}), i^! \, j_{!*} \, \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})[1]\big) = H^n\big(\partial(M^K)^*(\mathbb{C}), \tau^{t \geq 3}_{\partial(M^K)^*} i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})\big).$$

In other words, a ghost class vanishing in $H^n\big(\partial(M^K)^*(\mathbb{C}), \tau^{t \geq 3}_{\partial(M^K)^*} i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})\big)$ is zero. The Hodge structure $H^n\big(\partial(M^K)^*(\mathbb{C}), \tau^{t \geq 3}_{\partial(M^K)^*} i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})\big)$ has weights $\geq (r + n) + 2$; the same type of considerations as those leading to Corollary 2.8 then imply that the direct sum of the restriction maps

$$H^n\big(\partial(M^K)^*(\mathbb{C}), \tau^{t \geq 3}_{\partial(M^K)^*} i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})\big) \to H^n\big(Z_m(\mathbb{C}), i_m^* \tau^{t \geq 3}_{\partial(M^K)^*} i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})\big),$$

$m = 0, 1$, is injective.

(b) The above illustrates an observation made by Moya Giusti: *for a class in the cohomology of the boundary whose weight is neither the middle weight nor the middle weight plus one, we can determine exactly whether or not it is in the image of the morphism*

$$H^n(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) \to H^n(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})).$$

In fact, it appears amusing to note that the "middle weight" is relevant in another context than the one studied in the present paper. According to [Moya Giusti 2018, p. 2317, second paragraph], the representation $V_{\underline{\alpha}}$ *satisfies the middle weight property* if the space of ghost classes in $H^n(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}}))$ is pure of weight $r+n$. In particular, [Moya Giusti 2018, Theorem 3.1] implies that for all $\underline{\alpha}$ (regular or not), the representation $V_{\underline{\alpha}}$ does satisfy the middle weight property, while our Theorem 1.6 implies that weights $\{r+n, r+n+1\}$ do not occur at all in $H^n(\partial(M^K)^*(\mathbb{C}), i^* j_* \mu_{\boldsymbol{H}}(V_{\underline{\alpha}}))$, as soon as $\underline{\alpha}$ is regular.

Remark 2.13. Saper's vanishing theorem [2005, Theorem 5] says that if $\underline{\alpha}$ is regular, then the groups $H^n(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}}))$, and hence (by comparison)

$$H^n(M^K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})),$$

vanish for $n < 3 = \dim M^K$. By duality, one obtains that $H_c^n(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) = 0$ and $H_c^n(M^K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})) = 0$ for $n > 3$. It follows that interior cohomology with coefficients in $\mu_{\boldsymbol{H}}(V_{\underline{\alpha}})$, denoted

$$H_!^n(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})),$$

and interior cohomology with coefficients in $\mu_\ell(V_{\underline{\alpha}})$, denoted

$$H_!^n(M^K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})),$$

both vanish for $n \neq 3$, provided that $\underline{\alpha}$ is regular.

3. The motive for an automorphic form

This final section contains the analogues for Siegel threefolds of the main results from [Scholl 1990]. Since we do not restrict ourselves to the case of Hecke eigenforms, our notation becomes a little more technical than in [loc. cit.].

We continue to consider the situation of Sections 1 and 2. In particular, we fix a dominant $\underline{\alpha} = \alpha(k_1, k_2, r)$, which we assume to be regular, i.e., $k_1 > k_2 > 0$. Consider the intersection motive $a_* j_{!*} {}^{\underline{\alpha}}\mathcal{V} \in CHM(\mathbb{Q})_{\mathbb{Q}}$, where $a : (M^K)^* \to \operatorname{Spec} \mathbb{Q}$ again denotes the structure morphism of $(M^K)^*$. According to [Wildeshaus 2018a, Remark 3.13(a)] and Remark 2.13, its Hodge theoretic realization equals

$$H_!^3(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}}))[-(r+3)],$$

and its ℓ-adic realization equals

$$H_!^3(M^K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}}))[-(r+3)].$$

By Corollary 1.8, every element of the Hecke algebra $\mathfrak{H}(K, G(\mathbb{A}_f))$ acts on $a_* j_{!*} {}^{\underline{\alpha}}\mathcal{V}$.

Theorem 3.1 [Harder 2017, Theorem 3.1.1]. *Let L be any field of characteristic zero. Then the $\mathfrak{H}(K, G(\mathbb{A}_f)) \otimes_{\mathbb{Q}} L$-module $H^3_!(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) \otimes_{\mathbb{Q}} L$ is semisimple.*

Note that [Harder 2017, Section 8.1.6, p. 232] gives a proof of Theorem 3.1, while the statement in [Harder 2017, Theorem 3.1.1] is "nonadelic". Denote by $R(\mathfrak{H}) := R(\mathfrak{H}(K, G(\mathbb{A}_f)))$ the image of the Hecke algebra in the endomorphism algebra of $H^3_!(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}}))$.

Corollary 3.2. *Let L be any field of characteristic zero. Then the L-algebra $R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$ is semisimple.*

In particular, the isomorphism classes of simple right $R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$-modules correspond bijectively to isomorphism classes of minimal right ideals.

Fix L, and let Y_{π_f} be such a minimal right ideal of $R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$. There is a (primitive) idempotent $e_{\pi_f} \in R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$ generating Y_{π_f}.

Definition 3.3. (a) The *Hodge structure $W(\pi_f)$ associated to Y_{π_f}* is defined as

$$W(\pi_f) := \mathrm{Hom}_{R(\mathfrak{H}) \otimes_{\mathbb{Q}} L}\big(Y_{\pi_f}, H^3_!(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) \otimes_{\mathbb{Q}} L\big).$$

(b) Let ℓ be a prime number. The *Galois module $W(\pi_f)_\ell$ associated to Y_{π_f}* is defined as

$$W(\pi_f)_\ell := \mathrm{Hom}_{R(\mathfrak{H}) \otimes_{\mathbb{Q}} L}\big(Y_{\pi_f}, H^3_!(M^K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})) \otimes_{\mathbb{Q}} L\big).$$

Definition 3.3(b) should be compared to [Weissauer 2005, Theorem I].

Proposition 3.4. *There is a canonical isomorphism of Hodge structures*

$$W(\pi_f) \xrightarrow{\sim} \big(H^3_!(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) \otimes_{\mathbb{Q}} L\big) \cdot e_{\pi_f},$$

and a canonical isomorphism of Galois modules

$$W(\pi_f)_\ell \xrightarrow{\sim} \big(H^3_!(M^K \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mu_\ell(V_{\underline{\alpha}})) \otimes_{\mathbb{Q}} L\big) \cdot e_{\pi_f}.$$

Proof. We perform the proof for Hodge structures; the one for Galois modules is formally identical. Obviously,

$$\mathrm{Hom}_{R(\mathfrak{H}) \otimes_{\mathbb{Q}} L}\big(R(\mathfrak{H}) \otimes_{\mathbb{Q}} L, H^3_!(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) \otimes_{\mathbb{Q}} L\big)$$

is canonically identified with

$$H^3_!(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) \otimes_{\mathbb{Q}} L$$

by mapping an morphism g to the image of $1 = 1_{R(\mathfrak{H})}$ under g. Inside

$$\mathrm{Hom}_{R(\mathfrak{H}) \otimes_{\mathbb{Q}} L}\big(R(\mathfrak{H}) \otimes_{\mathbb{Q}} L, H^3_!(M^K(\mathbb{C}), \mu_{\boldsymbol{H}}(V_{\underline{\alpha}})) \otimes_{\mathbb{Q}} L\big),$$

the object $W(\pi_f)$ contains precisely those morphisms g vanishing on $1 - e_{\pi_f}$, or in other words, satisfying the relation $g(1) = g(e_{\pi_f}) = g(1) \cdot e_{\pi_f}$. □

Since we do not know whether the Chow motive $a_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}$ is finite dimensional, we cannot apply [Kimura 2005, Corollary 7.8], and therefore do not know whether e_{π_f} can be lifted *idempotently* to the Hecke algebra $\mathfrak{H}(K, G(\mathbb{A}_f))$. This is why we need to descend to the level of *Grothendieck motives*. Denote by $a_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}'$ the Grothendieck motive underlying $a_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}$.

Definition 3.5. Assume $\underline{\alpha} = \alpha(k_1, k_2, r)$ to be regular. Let L be a field of characteristic zero, and Y_{π_f} a minimal right ideal of $R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$. The *motive associated to* Y_{π_f} is defined as

$$\mathcal{W}(\pi_f) := a_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}' \cdot e_{\pi_f}.$$

Definition 3.5 should be compared to [Scholl 1990, Section 4.2.0]. Given our construction, the following is obvious.

Theorem 3.6. *Assume* $\underline{\alpha} = \alpha(k_1, k_2, r)$ *to be regular*, *i.e.*, $k_1 > k_2 > 0$. *Let* L *be a field of characteristic zero*, *and* Y_{π_f} *a minimal right ideal of* $R(\mathfrak{H}) \otimes_F L$. *The realizations of the motive* $\mathcal{W}(\pi_f)$ *associated to* Y_{π_f} *are concentrated in the single cohomological degree* $r+3$, *and they take the values* $W(\pi_f)$ (*in the Hodge theoretic setting*) *and* $W(\pi_f)_\ell$ (*in the* ℓ*-adic setting*).

A special case occurs when Y_{π_f} is of dimension one over L, i.e., corresponds to a nontrivial character of $R(\mathfrak{H})$ with values in L. The automorphic form is then an eigenform for the Hecke algebra. This is the analogue of the situation considered in [Scholl 1990] for elliptic cusp forms.

The motive $\mathcal{W}(\pi_f)$ being a direct factor of $a_* \, j_{!*} {}^{\underline{\alpha}}\mathcal{V}'$, our results on the latter from Section 1 have obvious consequences for the realizations of $\mathcal{W}(\pi_f)$.

Corollary 3.7. *Assume* $\underline{\alpha} = \alpha(k_1, k_2, r)$ *to be regular. Let* L *be a field of characteristic zero*, *and* Y_{π_f} *a minimal right ideal of* $R(\mathfrak{H}) \otimes_{\mathbb{Q}} L$. *Let* p *be a prime number not dividing the level of* K. *Let* ℓ *be different from* p.

(a) *The* p*-adic realization* $W(\pi_f)_p$ *of* $\mathcal{W}(\pi_f)$ *is crystalline.*

(b) *The* ℓ*-adic realization* $W(\pi_f)_\ell$ *of* $\mathcal{W}(\pi_f)$ *is unramified at* p.

(c) *The characteristic polynomials of the following coincide*: (1) *the action of Frobenius* ϕ *on the* ϕ*-filtered module associated to* $W(\pi_f)_p$; (2) *the action of a geometrical Frobenius automorphism at* p *on* $W(\pi_f)_\ell$.

Proof. Parts (a) and (b) follow from Remark 1.12.

As for (c), in order to apply [Katz and Messing 1974, Theorem 2(2)], use the fact that both realizations are cut out by the *same* cycle from the cohomology of a smooth and proper scheme over the field $\mathbb{F}_p$ (cf. the proof of Corollary 1.13). □

Corollary 3.7 should be compared to [Scholl 1990, Theorem 1.2.4].

Remark 3.8. Corollary 3.7(c) is already contained in [Urban 2005, Théorème 1].

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A Baum–Connes conjecture for singular foliations

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We consider singular foliations whose holonomy groupoid may be nicely decomposed using Lie groupoids (of unequal dimension). We construct a K-theory group and a natural assembly type morphism to the K-theory of the foliation C^*-algebra generalizing to the singular case the Baum–Connes assembly map. This map is shown to be an isomorphism under assumptions of amenability. We examine some simple examples that can be described in this way and make explicit computations of their K-theory.

Introduction

The celebrated Baum–Connes conjecture assigns to geometric objects (e.g., discrete groups, Lie groups, (regular) foliations, Lie groupoids) two K-groups and links them with a morphism, the “assembly map”. The “right-hand side” of the assembly map is the K-theory group of the C^*-algebra associated with the geometric object in hand. The other group, the “left-hand-side”, called the topological K-theory, arises from topological constructions associated with the geometric object in hand, such as classifying spaces.

Although this topological K-theory is often not much easier to calculate than the analytic one, constructing it and the assembly map is really important.

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- First of all, the topological K-groups are important and meaningful groups. In particular, they represent — up to torsion — the correct cohomology of the geometric object.
- Injectivity of the assembly map controls the topological K-theory by the analytic one. It thus has important topological consequences, as the homotopy invariance of higher signature, i.e., Novikov's conjecture and its generalizations to foliations [Baum and Connes 1985].
- Surjectivity controls the analytic K-theory by the topological one. It thus has important consequences like the Kadison–Kaplansky conjecture.
- Even its nonbijectivity has strong consequences by constructing secondary invariants of purely analytic type; see [Piazza and Schick 2007].

Foliations, and in particular singular ones, arise in an abundance of interesting mathematical problems, so the formulation of an assembly map is important in its own right. For instance, Poisson manifolds are completely determined by their symplectic foliation [Vaisman 1994]. In particular, regarding the Lie–Poisson structure [Vaisman 2000] associated with a nilpotent Lie group, formulating the Baum–Connes conjecture of the associated symplectic foliation might give a more insightful understanding of the orbit method [Kirillov 2004]. (In fact, Androulidakis and Higson have work in progress in this direction.)

Let $(M, \mathcal{F})$ be a singular Stefan–Sussmann foliation [Stefan 1974; Sussmann 1973]. We constructed its holonomy groupoid and the foliation C^*-algebra $C^*(M, \mathcal{F})$ in [Androulidakis and Skandalis 2009]. In [Androulidakis and Skandalis 2011a; 2011b] we showed that the K-theory of $C^*(M, \mathcal{F})$ is a receptacle for natural index problems along the leaves. It is then natural to look for a "left-hand side" too and try to construct the corresponding topological K-group and assembly map. In particular, this gives some insight into this K-theory. Of course we cannot hope in general for such a map to be an isomorphism (since it is not always an isomorphism in the regular case, as shown in [Higson et al. 2002]), and it is even hard to believe that the topological K-group could be defined for every kind of singular foliation. However, in this paper we manage to construct such a map for a quite general class of singular foliations.

0.A. ***Some examples.*** In order to formulate the assembly map, let us examine a few natural and quite simple examples. Consider the foliation given by a smooth action of a connected Lie group on a manifold M:

(a) the action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$;

(b) the action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$;

(c) any action of $\mathbb{R}$ (given by a vector field X).

In these three cases, we can compute the K-theory thanks to an exact sequence

$$0 \to C^*(\Omega_0, \mathcal{F}_{|\Omega_0}) \to C^*(M, \mathcal{F}) \to C^*(M, \mathcal{F})|_{Y_1} \to 0.$$

Here Ω_0 corresponds to "most regular points" of the foliation (more precisely, the place where the source fibers of the foliation groupoid are of lowest dimension) and $Y_1 = M \setminus \Omega_0$: in example (a), $\Omega_0 = \mathbb{R}^3 \setminus \{0\}$, in example (b), $\Omega_0 = \mathbb{R}^2 \setminus \{0\}$ and in example (c), Ω_0 is the interior of the set of points where X vanishes.

In these examples, the connecting map ∂ of the K-theory exact sequence is easily computed and we can describe precisely $K_*(C^*(M, \mathcal{F}))$.

In other examples that we discuss here, the "regularity" of points varies even more. For instance:

(d) The action on $\mathbb{R}^n$ of a parabolic subgroup G of $\mathrm{GL}(n, \mathbb{R})$; e.g., the minimal parabolic subgroup of upper triangular matrices.

(e) The action of $PG = G/\mathbb{R}^*$ on $\mathbb{R}P^{n-1}$.

(f) The action of $G \times G$ by left and right multiplication on $\mathrm{GL}_n(\mathbb{R})$. (Orbits give the well-known Bruhat decomposition.)

In the last three cases, the computation becomes harder since we obtain a longer sequence of ideals — and therefore spectral sequences instead of short exact sequences. We do not explicitly compute the K-theory in these cases. On the other hand, in all cases, the holonomy groupoid nicely decomposes in locally closed subsets where the source fibers have fixed dimension. We use this decomposition in order to construct the topological K-group and the assembly map.

0.B. ***Nicely decomposable foliations and the height of a nice decomposition.*** Let $(M, \mathcal{F})$ be a singular foliation. Its holonomy groupoid may be very singular. On the other hand, this singularity gives rise to open subsets which are *saturated* for $\mathcal{F}$ (i.e., a union of leaves of $\mathcal{F}$). We thus obtain ideals of $C^*(M, \mathcal{F})$ that we may use to compute the K-theory.

For instance, recall that the source fibers of the holonomy groupoid of the foliation as defined in [Androulidakis and Skandalis 2009] were shown in [Debord 2013] to be smooth manifolds. On the other hand, the dimension of these manifolds varies. Let us denote by $\ell_0 < \ell_1 < \cdots < \ell_k$ the various dimensions occurring (note that k may be infinite, as shown in [Androulidakis and Zambon 2013]). Let Ω_j denote the set of points with source fiber dimension $\leq \ell_j$. We find an ascending sequence $\Omega_0 \subseteq \Omega_1 \subseteq \cdots \subseteq \Omega_{k-1} \subseteq \Omega_k = M$ of saturated open subsets of M. This decomposition yields a sequence of two-sided ideals $J_j = C^*(\Omega_j, \mathcal{F}_{|\Omega_j})$ of $C^*(M, \mathcal{F})$. The quotient C^*-algebra J_j/J_{j-1} is the C^*-algebra of the restriction of the holonomy groupoid $H(\mathcal{F})$ to the locally closed saturated set $Y_j = \Omega_j \setminus \Omega_{j-1}$. The module $\mathcal{F}$, when restricted to Y_j, is finitely generated and projective, and the restriction of $H(\mathcal{F})$ to Y_j is a Lie groupoid (when Y_j is a submanifold) so that we may expect a Baum–Connes map for it.

Our computation of the K-theory is based on an ingredient which we add as an extra assumption (it is satisfied in the above examples).

Let $(M, \mathcal{F})$ a singular foliation. We say that $(M, \mathcal{F})$ is *nicely decomposable with height k* if there is a cover of M by open subsets $(W_j)_{j\in\mathbb{N},\, j\leq k}$, such that for every $j \in \mathbb{N}$ with $j \leq k$, the restriction of the foliation $\mathcal{F}$ to each W_j is defined by a Hausdorff Lie groupoid $\mathcal{G}_j$, the open subset $\Omega_j = \bigcup_{i\leq j} W_i$ is saturated and $\mathcal{G}_j$ coincides with the holonomy groupoid $H(\mathcal{F})$ on the (locally closed) set $Y_j = \Omega_j \setminus \Omega_{j-1}$ (we set $Y_0 = W_0 = \Omega_0$). Moreover, we assume that the groupoids $\mathcal{G}_j$ are linked via morphisms which are submersions

$$\mathcal{G}_j|_{\Omega_{j-1}\cap W_j} \to \mathcal{G}_{j-1}.$$

If $(M, \mathcal{F})$ is nicely decomposable, the quotients J_j/J_{j-1} are given by (restriction to closed sets of) Lie groupoids, for which a Baum–Connes conjecture does exist. This makes the calculation of the K-theory of $C^*(M, \mathcal{F})$ possible, at least in terms of a spectral sequence.

- Singularity height 0 corresponds to foliations whose holonomy groupoid is a Lie groupoid, and there already is a topological K-theory and a Baum–Connes assembly map for Lie groupoids; see [Tu 2000].
- Examples (a), (b), (c) are all of singularity height 1. We will use the decomposition given by the dimensions of the fibers. In examples (a) and (b), the dimensions of the fibers are $\ell_0 = 2$ and $\ell_1 = 3$; in example (c), these dimensions are $\ell_0 = 0$ and $\ell_1 = 1$. For the singularity height 1 case, the topological K-theory can be constructed using the exact sequence of C^*-algebras and a mapping cone construction.
- A new difficulty in the construction of the topological K-theory arises when we have higher singularity height, as in examples (d), (e) and (f). We use here a telescope construction.

0.C. ***The topological K-theory and the assembly map.*** We construct the topological K-theory and the assembly map in two steps:

- The first step consists of replacing the holonomy groupoid $H(\mathcal{F})$ by a slightly more regular one G whose (full) C^*-algebra is E-equivalent to the foliation one. This groupoid is constructed via a mapping cone construction in the height 1 case and via a telescope construction in the higher singularity case.
- In the second step we construct a topological K-theory and the assembly map for the "telescopic groupoid" G which is the K-theory of a proper G-algebra in a generalized sense, together with a Dirac type construction.

0.C.1. *A telescopic construction.*

A mapping cone construction in the height 1 case. Let us explain our strategy more explicitly in the case of a foliation admitting a singularity height 1 decomposition. In this case, we obtain a diagram of *full* C^*-algebras (with $\mathcal{G} = \mathcal{G}_1$):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(\mathcal{G}_{\Omega_0}) & \xrightarrow{i_{\mathcal{G}}} & C^*(\mathcal{G}) & \longrightarrow & C^*(\mathcal{G}_{Y_1}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_{\Omega_0}} & & \downarrow{\scriptstyle \pi} & & \| & & \\ 0 & \longrightarrow & C^*(\Omega_0, \mathcal{F}_{|\Omega_0}) & \xrightarrow{i_{\mathcal{F}}} & C^*(M, \mathcal{F}) & \longrightarrow & C^*(M, \mathcal{F})|_{Y_1} & \longrightarrow & 0 \end{array}$$

The singularity height 1 assumption means that the holonomy groupoid of the restriction $\mathcal{F}_{|\Omega_0}$ of the foliation $\mathcal{F}$ to Ω_0 is a Lie groupoid $\mathcal{G}_0$ and $C^*(\Omega_0, \mathcal{F}_{|\Omega_0}) = C^*(\mathcal{G}_0)$. The lines of this diagram are exact at the level of *full* C^*-algebras.

Since $\mathcal{G}$ defines $\mathcal{F}$, it is an atlas in the sense of [Androulidakis and Skandalis 2009], so $H(\mathcal{F})$ is a quotient of $\mathcal{G}$. Hence the two extensions are connected by the map π and its restriction π_{Ω_0}, which is integration along the fibers of this quotient map $\mathcal{G} \to H(\mathcal{F})$. From this diagram, we conclude that the algebra $C^*(M, \mathcal{F})$ is equivalent in E-theory (up to a shift of degree) with the mapping cone of the morphism

$$(i_{\mathcal{G}}, \pi_{\Omega_0}) : C^*(\mathcal{G}_{\Omega_0}) \to C^*(\mathcal{G}) \oplus C^*(\Omega_0, \mathcal{F}_{|\Omega_0}).$$

Foliations of height ≥ 2. As far as singular foliations with nice decompositions of arbitrary (bounded or not) singularity height are concerned, we show that the strategy developed for the singularity height 1 case can be generalized. In particular, $C^*(M, \mathcal{F})$ is E-equivalent to a "telescopic" C^*-algebra whose components are Lie groupoids. In fact, we see that these telescopes can just be treated as mapping cones.

Now let us see how the above apparatus can be used to formulate the Baum–Connes assembly map for singular foliations. It suffices to explain the idea for the height 1 case.

Longitudinally smooth groupoids. The above mapping cone and the telescopic algebra constructed here are based on morphisms of Lie groupoids which are smooth submersions and open inclusions at the level of objects. These C^*-algebras are immediately seen to be the C^*-algebras of a kind of groupoids which generalize both Lie groupoids and singular foliation groupoids: *longitudinally smooth groupoids.*

0.C.2. *A topological K-theory group for the telescopic groupoid.*

Setting of the problem. Before we outline our construction of a topological K-theory group, let us make a remark. Recall that Jean-Louis Tu [2000] defined a topological K-theory group and a Baum–Connes morphism for Lie groupoid C^*-algebras of the form $K_*^{\mathrm{top}}(\mathcal{G}) \to K_*(C^*(\mathcal{G}))$. In order to construct a topological

K-theory group for this mapping cone, we need to find a "left-hand side" for the morphism $(i_{\mathcal{G}}, \pi_{\Omega_0})$. In fact we not only need it as a morphism at the level of groups K_*^{top}, but we really need to construct it as a KK-element.

The difficulty lies with the understanding of the topological K-theory of the mapping cone of the surjective homomorphism $\pi_{\Omega_0} : C^*(\mathcal{G}_{\Omega_0}) \to C^*(\Omega_0, \mathcal{F}_{|\Omega_0})$. We treat this by deploying the Baum–Douglas formulation given in [Baum and Connes 2000; Baum and Douglas 1982a; 1982b]. At this point we will need further assumptions on the groupoids $\mathcal{G}$ and $\mathcal{G}_1$, namely that their classifying spaces of proper actions are smooth manifolds, to make sure that the Baum–Connes morphisms are naturally given by KK-elements. (In the Appendix we show how this assumption can be weakened.)

Actions of the telescopic groupoid. In order to define the topological K-theory group for the telescopic groupoid, we follow the Lie groupoid case:

- For every longitudinally smooth groupoid G, one defines G-algebras very much in the spirit of [Androulidakis and Skandalis 2009]: algebraic conditions are stated at the level of the groupoid, topological ones at the level of bisubmersions which can be thought of as "smooth local covers" of G (cf. [Androulidakis and Skandalis 2009]). We define the (full and reduced) crossed product for every G-algebra.

- One may define a generalized notion of *"proper G-algebra"*: a G-algebra is said to be "proper" if its restriction to the groupoids corresponding to the various strata is proper in the usual sense. In particular, one may define actions on spaces and "proper" actions on spaces. Of course, they are not proper in the usual sense! But from the point of view of the Baum–Connes conjecture they are as good, since the Baum–Connes conjecture is compatible with extensions (in the amenable case).

- We define Le Gall's equivariant KK-theory [1999] in the context of longitudinally smooth G-algebras, despite the topological pathology of the holonomy groupoid G. We extend the equivariant Kasparov product to this case.

- We may then construct the topological K-theory group and the assembly map for the telescopic algebras of a nice decomposition of a singular foliation. To that end we still need to assume for $(M, \mathcal{F})$ that the Lie groupoids of its decomposition admit smooth manifolds as classifying spaces for proper actions.

- Actually, this point of view allows one to construct a Baum–Connes map *with coefficients* for every G algebra. It is easily seen that, in the case of nicely decomposable foliations, our Baum–Connes map with coefficients in "proper" spaces or algebras is an isomorphism.

The main result. We show then that in cases as above the Baum–Connes map can be constructed canonically. Namely, we prove the following:

Theorem 0.1. (i) *If $(M, \mathcal{F})$ admits a nice decomposition by Lie groupoids whose classifying space for proper actions is a manifold, then there is a well-defined topological K-group and one may construct a Baum–Connes assembly map.*

(ii) *If moreover the groupoids of the nice decomposition are amenable and Hausdorff, then the Baum–Connes map is an isomorphism.*

Note that examples (a) and (c) above are amenable; although example (b) is not, it is "strongly K-amenable" and the Baum–Connes conjecture (for the *full* version) holds for it.

Note also that example (c) is not exactly covered by our theorem since the groupoid $\mathcal{G}_0$ is not assumed to be Hausdorff. However, the Baum–Connes conjecture holds also in this case

For the examples of larger singularity height described in examples (d), (e) and (f), note that, as the minimal parabolic subgroup of $\mathrm{GL}(n, \mathbb{R})$ is amenable, Theorem 0.1 implies that the Baum–Connes conjecture holds.

Let us point out that our constructions of the equivariant KK-theory could in a way be bypassed, but may have its own interest. In particular, we give a simple quite general formulation and proof for the existence of the Kasparov product, which applies in all known equivariant contexts: groups, group actions [Kasparov 1988], groupoids [Le Gall 1999], Hopf algebras [Baaj and Skandalis 1989].

Trying to weaken our assumptions. The assumption on the classifying spaces is quite natural. All the groupoids given by Lie group actions admit manifolds as classifying spaces for proper actions, and this assumption is stable by Morita equivalence. In this way it is satisfied by all the (Hausdorff) groupoids that appear in the examples that we discuss in this work. Nevertheless, it is quite tempting to try to get rid of it. In the Appendix we explain how it can be replaced by a quite weaker, rather technical one: Assumption A.1, which could be true in general, i.e., for every longitudinally smooth groupoid.

Structure of the paper.

• In Section 1 we introduce the notion of singularity height for a singular foliation and define nicely decomposable foliations. We also explain the examples mentioned in the beginning of this introduction.

• Section 2 focuses on nicely decomposable foliations with singularity height 1. We give the construction of the associated mapping cone C^*-algebra and prove that it is E-equivalent to the foliation C^*-algebra. We give there the explicit calculation of the K-theory for examples (a), (b) and (c).

• In Section 3 we extend this construction and result to foliations of arbitrary singularity height, replacing mapping cones with telescopes.

• Section 4 defines longitudinally smooth groupoids and their actions and constructs the associated KK-theory.

• The crucial section is Section 5, where we formulate the Baum–Connes conjecture (topological K-theory and Baum–Connes map) for the telescopic algebra, assuming the classifying spaces of proper actions of the groupoids associated with the nice decomposition of $(M, \mathcal{F})$ are smooth manifolds. The proof of Theorem 0.1 can be found there.

• Finally, in the Appendix we explain how to remove the assumption that the classifying spaces of proper actions are smooth manifolds.

Notation 0.2. Let $(M, \mathcal{F})$ be a foliation. We denote the (minimal, i.e., the groupoid associated with the path holonomy atlas — cf. [Androulidakis and Skandalis 2009]) holonomy groupoid by $H(\mathcal{F})$ (or $H(M, \mathcal{F})$ when needed). We denote by $C^*(M, \mathcal{F})$ and $C^*_{\mathrm{red}}(M, \mathcal{F})$ its *full* and *reduced* C^*-algebras.

We mainly use the *full* C^*-algebra. This is justified by the two following reasons:

• Constructing a Baum–Connes map for the full foliation algebra automatically gives the one for the reduced version. Recall that the Baum–Connes map, in the regular case, factors through the full version of the foliation algebra.

• All our constructions are based on sequences of groupoid C^*-algebras, which are always exact at the full C^*-algebra level, and may fail to be exact at the reduced level (see Section 2.B).

1. Nicely decomposable foliations

1.A. ***Notations and remarks.*** Let M be a smooth manifold and $\mathcal{X}_c(M)$ the $C^\infty(M)$-module of compactly supported vector fields. In [Androulidakis and Skandalis 2009], we defined a singular foliation on M to be a $C^\infty(M)$-submodule $\mathcal{F}$ of $\mathcal{X}_c(M)$ which is locally finitely generated and satisfies $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$.

Given a point $x \in M$ let $I_x = \{f \in C^\infty(M) : f(x) = 0\}$ and recall from [Androulidakis and Skandalis 2009] the fiber $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. The map $M \ni x \mapsto \dim(\mathcal{F}_x)$ is upper semicontinuous [Androulidakis and Skandalis 2009, Proposition 1.5].

When this dimension is constant (continuous if M is not assumed to be connected), i.e., when the module $\mathcal{F}$ is projective, the foliation is said to be *almost regular* and the holonomy groupoid $H(\mathcal{F})$ was proved to be a Lie groupoid in [Debord 2001].

In the present paper, we deal with cases where the dimension of $\mathcal{F}_x$ is not constant. The number of possible dimensions measures the singularity of the foliation. We give a definition of this *singularity height* more appropriate for our purposes in Definition 1.4.

By semicontinuity, the subsets $O_\ell = \{x \in M : \dim(\mathcal{F}_x) \le \ell\}$ are open. They are *saturated*, i.e., unions of leaves of $\mathcal{F}$.

We deal with restrictions of the foliation to open sets. We use the following remark:

Remark 1.1. Let $(M, \mathcal{F})$ be a foliation. Let V be an open subset of M.

(i) The holonomy groupoid of the restriction $\mathcal{F}_{|V}$ to V is the s-connected component of the restriction $H(\mathcal{F})_V^V = \{z \in H(\mathcal{F}) : t(z) \in V \text{ and } s(z) \in V\}$ to V.

(ii) If V is saturated, then $H(\mathcal{F}_{|V}) = H(\mathcal{F})_V^V$.

Actually an analogous statement holds for the pull-back foliation $f^{-1}(\mathcal{F})$ by a smooth map $f : V \to M$ *transverse to* $\mathcal{F}$ [Androulidakis and Skandalis 2009, §1.2.3]: $H(f^{-1}(\mathcal{F}))$ is the s-connected component of

$$H(\mathcal{F})_f^f = \{(v, z, w) \in V \times H(\mathcal{F}) \times V : t(z) = f(v) \text{ and } s(z) = f(w)\}.$$

If moreover f is a submersion whose image is saturated with connected fibers, then $H(f^{-1}(\mathcal{F})) = H(\mathcal{F})_f^f$.

Now let us discuss the notation for C^*-algebras used in the sequel as far as restrictions are concerned. If $\mathcal{G}$ is a locally compact groupoid (with Haar measure) and Y is a locally closed saturated subset of $\mathcal{G}_0$, then $\mathcal{G}_Y = \{x \in \mathcal{G} : s(x) \in Y\}$ is also a locally closed groupoid and we can define its C^*-algebra. We put $C^*(\mathcal{G})_{|Y} = C^*(\mathcal{G}_Y)$. The same construction for foliation algebras is useful in our context:

Notation 1.2. Let $(M, \mathcal{F})$ be a (singular) foliation.

(a) Let $\Omega \subset M$ be a *saturated* open subset. Then

$$C^*(M, \mathcal{F})|_\Omega := C_0(\Omega) C^*(M, \mathcal{F}) = C^*(\Omega, \mathcal{F}_{|\Omega})$$

is the foliation C^*-algebra of the restriction of $\mathcal{F}$ to Ω. The same holds for the reduced C^*-algebras.

(b) If $Y \subset M$ is a saturated closed subset then the *full* $C^*(M, \mathcal{F})|_Y$ is the quotient of $C^*(M, \mathcal{F})$ by $C^*(M, \mathcal{F})|_{M \setminus Y}$.

Note that the natural definition for the reduced one is to take the quotient of $C^*(M, \mathcal{F})$ corresponding to the regular representations at points of Y, i.e., the representations on $L^2(H(M, \mathcal{F})_y)$ for $y \in Y$.

(c) If $Y \subset M$ is a saturated locally closed subset then Y is open in its closure $\overline{Y}$ and the closed subset $\overline{Y} \setminus Y$ is saturated. Let $U = M \setminus (\overline{Y} \setminus Y)$. We denote by $C^*(M, \mathcal{F})|_Y$ the quotient of $C_0(U) C^*(M, \mathcal{F})$ by $C^*(M, \mathcal{F})|_{M \setminus \overline{Y}}$. In other words, $C^*(M, \mathcal{F})_{|Y} = (C^*(M, \mathcal{F})_{|U})_{|Y}$.

1.B. ***Foliations associated with Lie groupoids.*** In the sequel we consider foliations defined from Lie groupoids (at least locally — cf. Section 1.C). Let us make a few observations regarding singular foliations defined by Lie groupoids.

Every Lie algebroid A with base M, and thus every Lie groupoid $(t,s):\mathcal{G}\rightrightarrows M$, defines a foliation. Indeed, the anchor map $\sharp : A \to TM$ is a morphism of Lie algebroids, whence $\sharp(\Gamma_c A) \subset \mathcal{X}_c(M)$ is a singular foliation.

Let $\mathcal{G}$ be a (locally Hausdorff) Lie groupoid over a manifold M and $\mathcal{F}$ the associated foliation. Up to replacing $\mathcal{G}$ by its s-connected component (which is an open subgroupoid of $\mathcal{G}$ with the same algebroid, and thus defines the same foliation on M) we may assume that $\mathcal{G}$ is s-connected, i.e., the fibers of the source map $s:\mathcal{G}\to M$ are connected. Then the groupoid $\mathcal{G}$ is an atlas for our foliation, in the sense of [Androulidakis and Skandalis 2009, Definition 3.1]. As $\mathcal{G}$ is assumed s-connected, it defines the path holonomy atlas [Androulidakis and Skandalis 2009, Example 3.4.3]. The holonomy groupoid $H(M,\mathcal{F})$ is a quotient of $\mathcal{G}$ by the equivalence relation defined in [Androulidakis and Skandalis 2009, Proposition 3.4.2]. Put $q:\mathcal{G}\to H(M,\mathcal{F})$ the associated quotient map.

In order to compute this quotient, we use a lemma from [Androulidakis and Zambon 2013].

Let $\gamma \in \mathcal{G}$ and write $x = s(\gamma)$. Note that if $q(\gamma)$ is a unit, then $t(\gamma) = x$. Choosing a bisection through γ we obtain a local diffeomorphism g of M which acts on the tangent bundle T_xM and fixes the tangent to the leaf F_x. It therefore acts on $N_x = T_xM/F_x$. This action only depends on γ. Denote it by $\nu(\gamma)\in \mathrm{GL}(N_x)$.

Now, it was shown in [Androulidakis and Zambon 2013] that there is an action of $H(\mathcal{F})$ on this “bundle” of normal spaces. As an immediate consequence, we find the following:

Lemma 1.3. *If $q(\gamma)$ is a unit, then $\nu(\gamma)=\mathrm{id}_{N_x}$.* □

1.C. ***Nicely decomposable foliations.*** We now present the constraints that we put on our foliations. We say that the foliation is *nicely decomposable* if it admits a nice decomposition in the following sense.

Definition 1.4. Let $(M,\mathcal{F})$ be a singular foliation and let $k\in\mathbb{N}\cup\{+\infty\}$. A *nice decomposition* of $(M,\mathcal{F})$ of *singularity height* k is given by

(a) a sequence $(W_j)_{0\le j<k+1}$ of open sets of M such that the open set $\Omega_j=\bigcup_{\ell\le j} W_\ell$ is saturated and $\bigcup_{j<k+1} W_j = M$ (with the convention $+\infty+1=+\infty$);

(b) a sequence of Lie groupoids $\mathcal{G}_j \rightrightarrows W_j$ defining the restriction of $\mathcal{F}$ to W_j, and such that $\mathcal{G}_j|_{Y_j} = H(\mathcal{F})|_{Y_j}$, where $Y_0=\Omega_0$ and, for $j\ge 1$, $Y_j=\Omega_j\setminus\Omega_{j-1}$;

(c) morphisms of Lie groupoids $q_j:\mathcal{G}_j|_{\Omega_{j-1}\cap W_j}\to\mathcal{G}_{j-1}$ (for $j>0$) which are submersions, and which at the level of objects are just the inclusion $\Omega_{j-1}\cap W_j\to W_{j-1}$.

Remarks 1.5. (a) If $(M, \mathcal{F})$ is an almost regular foliation then $H(\mathcal{F})$ is a Lie groupoid as shown in [Androulidakis and Skandalis 2009] (it coincides with the one constructed in [Debord 2001]). In our current context, the decomposition sequence of such a foliation has singularity height zero; its realization is $H(\mathcal{F})$ itself. We will not be concerned with such situations in the sequel. Truly singular examples of nicely decomposable singular foliations arise when the singularity height of the decomposition is 1 or larger.

(b) By definition $W_0 = \Omega_0$ and the restriction of $H(\mathcal{F})$ coincides with $\mathcal{G}_0$. It follows that the restriction of $\mathcal{F}$ to Ω_0 is almost regular, which means that Ω_0 is contained in the (open) set of points where $\dim \mathcal{F}$ is continuous and, since $\dim \mathcal{F}$ is upper semicontinuous, these are the places where it has a local minimum.

(c) Such a decomposition need not be unique. In all our examples, $W_j = \Omega_j$ and Ω_j may be constructed using the dimension of the fibers.

For $\ell \in \mathbb{N}$, put

$$O_\ell = \{x \in M : \dim(\mathcal{F}_x) \leq \ell\}.$$

Denote by $\ell_0 < \ell_1 < \cdots < \ell_j$ for $j < k+1$ the various possible dimensions. For $j = 0, 1, \ldots, k$ put $\Omega_j = O_{\ell_j}$.

Note that an example is given in [Androulidakis and Zambon 2013] of a foliation where this k is infinite.

1.D. ***Examples of nicely decomposable foliations.*** We now give a few examples of nice decompositions of foliations.

1.D.1. *Examples of height 1.*

Remark 1.6. In the case of height 1, we have $W_0 = \Omega_0$ and $\mathcal{G}_0$ is the holonomy groupoid of the restriction of $\mathcal{F}$ to Ω_0. We therefore just need to specify the set Ω_0 and the Lie groupoid $\mathcal{G}_1 \rightrightarrows W_1$ defining the foliation $\mathcal{F}$ on an open subset W_1 containing the complement $Y_1 = M \setminus \Omega_0$ of Ω_0 and such that the restriction of $\mathcal{G}$ to Y_1 coincides with that of $H(\mathcal{F})$.

Actually, in our examples $W_1 = M$.

Examples 1.7. We give here examples of singularity height 1 associated with Lie group actions. Some examples of larger singularity height are computed in forthcoming work of Androulidakis and Higson. In this paper, we calculate the associated K-theory explicitly for the following examples.

(a) Let $M = \mathbb{R}^3$ and consider the foliation $\mathcal{F}$ defined by the image of the (infinitesimal) action of SO(3) on $\mathbb{R}^3$ by rotations. The leaves are concentric spheres in $\mathbb{R}^3$ with one singularity at $\{0\}$. Let $\mathcal{G}$ be the action groupoid $\mathbb{R}^3 \rtimes \mathrm{SO}(3) \rightrightarrows \mathbb{R}^3$. Since SO(3) is simple, the restriction of $H(\mathcal{F})$ to 0, which is a quotient of SO(3), has to be SO(3) (we may also use Lemma 1.3 to prove this result). The restriction of $\mathcal{F}$

to $\mathbb{R}^3 \setminus \{0\}$ is really a regular foliation — and in fact the fibration $S^2 \times \mathbb{R}^*_+ \to \mathbb{R}^*_+$, whence the holonomy groupoid of $\mathcal{F}$, is

$$H(\mathcal{F}) = (S^2 \times S^2 \times \mathbb{R}^*_+) \cup \{0\} \times \mathrm{SO}(3).$$

It follows that the foliation has a nice decomposition of singularity height 1, namely $W_1 = \mathbb{R}^3$, $\mathcal{G}_1 = \mathcal{G}$ and $\Omega_0 = \mathbb{R}^3 \setminus \{0\}$.

(b) Let $M = \mathbb{R}^2$ and consider the action of $\mathrm{SL}(2, \mathbb{R})$. It has two leaves, namely $\{0\}$ and $\mathbb{R}^2 \setminus \{0\}$. Using again Lemma 1.3, the associated holonomy groupoid is seen to be

$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2 \setminus \{0\}) \cup \{0\} \times \mathrm{SL}(2, \mathbb{R}).$$

Considering the action groupoid $\mathcal{G} = \mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R})$, we obtain the singularity height 1 nice decomposition $\Omega_1 = \mathbb{R}^2$, $\mathcal{G}_1 = \mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ and $\Omega_0 = \mathbb{R}^2 \setminus \{0\}$, $\mathcal{G}_0 = \Omega_0 \times \Omega_0$.

(c) There are many singular foliations of singularity height 1 arising from group actions which have nice decompositions. For instance, take $n \geq 4$ instead of 3 in example (a) or $n \geq 3$ instead of 2 in example (b).

We may also consider the action of $\mathrm{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$. The associated holonomy groupoid is

$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2 \setminus \{0\}) \cup \{0\} \times \mathrm{GL}^+(2, \mathbb{R}),$$

where $\mathrm{GL}^+(2, \mathbb{R})$ denotes $2{\times}2$ matrices with positive determinant. Considering the action groupoid $\mathcal{G} = \mathbb{R}^2 \rtimes \mathrm{GL}^+(2, \mathbb{R})$, we obtain $\Omega_1 = \mathbb{R}^2$ and $\Omega_0 = \mathbb{R}^2 \setminus \{0\}$. We can of course replace 2 by n also in this situation.

Another example as such comes from the action of $\mathrm{SL}(n, \mathbb{C})$ on $\mathbb{C}^n$. Its holonomy groupoid is

$$H(\mathcal{F}) = (\mathbb{C}^n \setminus \{0\} \times \mathbb{C}^n \setminus \{0\}) \cup \{0\} \times \mathrm{SL}(n, \mathbb{C}).$$

Considering the action groupoid $\mathcal{G} = \mathbb{C}^n \rtimes \mathrm{SL}(n, \mathbb{C})$ we have $\Omega_1 = \mathbb{C}^n$, $\Omega_0 = \mathbb{C}^n \setminus \{0\}$.

(d) We end with an example of a quite different flavor.

Let M be a manifold endowed with a smooth action α of $\mathbb{R}$. Let $\mathcal{G}_1 = M \rtimes_\alpha \mathbb{R}$ be the associated action groupoid, and $\mathcal{F}$ the associated foliation.

Denote by $\mathrm{Fix}(\alpha)$ the set of fixed points of α, by $W = \mathrm{Int}(\mathrm{Fix}(\alpha))$ its interior and by $V = M \setminus \mathrm{Fix}(\alpha)$ its complement. Let $x \in M$.

- If $x \in W$, then $\mathcal{F}_x = 0$.
- For $x \in V$, the dimension of $\mathcal{F}_x$ is 1. By semicontinuity, $\dim \mathcal{F}_x = 1$ for $x \in \overline{V}$.

Let Ω_0 be the set of continuity points of $\dim \mathcal{F}$. Its complement Y_1 is the boundary ∂W of W. The restriction of $\mathcal{F}$ to the open set Ω_0 is almost regular.

We show that the morphism $M \rtimes_\alpha \mathbb{R} \to H(\mathcal{F})$ is injective over Y_1. We thus have a nice decomposition $H(\Omega_0, \mathcal{F}_{|\Omega_0}) \rightrightarrows \Omega_0$, and $M \rtimes_\alpha \mathbb{R} \rightrightarrows M$.

This is done using classical facts based on the period bounding lemma (see [Abraham and Robbin 1967]), which we recall here:

Lemma 1.8 (period bounding). *Let X be a compactly supported C^r-vector field on a C^r-manifold M with $r \geq 2$. There is a real number $\eta > 0$ such that, for any $x \in M$, either $X(x) = 0$ or the prime period τ_x of the integral curve of X passing through x is $\tau_x > \eta$.* □

Put $P = \{(x, u) \in M \times \mathbb{R} : \alpha_u(x) = x\}$. It is obviously a closed subset of $M \times \mathbb{R}$ and the restrictions of the source and target maps to P coincide. By definition of the holonomy groupoid, an element $(x, u) \in \mathcal{G}_1 = M \rtimes \mathbb{R}$ is a trivial element in $H(\mathcal{F})$ if and only if there is an identity bisection through it, i.e., if there exists an open neighborhood U of x and a smooth function $f : U \to \mathbb{R}$ such that $f(x) = u$ and $(z, f(z)) \in P$ for all $z \in U$.

Let $\mathrm{Per}(\alpha)$ be the set of *stably periodic points*, i.e., the set of $x \in M$ such that there exists an open neighborhood U of x and a smooth function $f : U \to \mathbb{R}^*$ such that $(y, f(y)) \in P$ for all $y \in U$. It is the set of $x \in M$ such that

$$\{(x, u) : u \in \mathbb{R}\} \to H(\mathcal{F})$$

is not injective.

Obviously $W \subseteq \mathrm{Per}(\alpha)$.

Proposition 1.9. *The set $Y_1 \cap \mathrm{Per}(\alpha)$ is empty.*

Proof. Let $x \in \overline{W} \cap \mathrm{Per}(\alpha)$. We need to show that $x \notin Y_1$, i.e., that $x \in W$. Up to changing X far from x, we may assume that X has compact support.

Since $x \in \overline{W}$, it follows that X as well as all its derivatives vanish at x. We may then write $X = qY$, where q is a smooth nonnegative function such that $q(x) = 0$ and Y is a smooth vector field with compact support (take for instance q to be a smooth function which coincides near x to the square of the distance to x for some riemannian metric). Let then U be an open relatively compact neighborhood of x and $f : U \to \mathbb{R}^*$ a smooth bounded function such that $(y, f(y)) \in P$ for all $y \in U$. It follows that all the points in U are periodic for X and therefore for Y. When $y \to x$, $f(y) \to f(x)$, so the Y period of y tends to 0. By the period bounding lemma, it follows that any y close enough to x satisfies $Y(y) = 0$, whence $x \in W$. □

It follows that $(H(\Omega_0) \rightrightarrows \Omega_0, M \rtimes_\alpha \mathbb{R} \rightrightarrows M)$ is a nice decomposition for $\mathcal{F}$.

It is worth noticing that the holonomy groupoid $\mathcal{G}_0 = H(\Omega_0, \mathcal{F}_{|\Omega_0})$ is a disjoint union of clopen subgroupoids $W \sqcup H(V', \mathcal{F}_{|V'})$, where V' is the interior of $\overline{V}$, and that its C^*-algebra $C^*(\Omega_0, \mathcal{F}_{|\Omega_0})$ is a direct sum $C_0(W) \oplus C^*(V', \mathcal{F}_{|V'})$.

Note that, in the presence of periodic points, the groupoid $H(V, \mathcal{F}_{|V})$ and therefore $\mathcal{G}_0$ need not be Hausdorff.

Let us also remark that, in the computation above, we could as well have chosen to take Ω_0 to be the set where the fibers are of dimension 0, i.e., the set W.

1.D.2. *An example of larger singularity height.* We start by giving a natural family of examples of nicely decomposable foliations with singularity height larger than 1. Some of them will be studied in forthcoming work of Androulidakis and Higson.

If a subgroup $G \subset \mathrm{GL}_n(\mathbb{R})$ has more than two orbits in its action on $\mathbb{R}^n$, then the transformation groupoid $\mathbb{R}^n \rtimes G$ may give rise to interesting nicely decomposable foliations of singularity height ≥ 2.

A typical example is given by a parabolic subgroup of $\mathrm{GL}(n, \mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$: given a flag $\{0\} = E_k \subset E_{k-1} \subset E_{k-2} \subset \cdots \subset E_1 \subset E_0 = \mathbb{K}^n$ (with $k \leq n$ and E_k pairwise different), let G be the group of (positive if $\mathbb{K} = \mathbb{R}$) automorphisms of this flag, i.e., G is the subgroup of $\mathrm{GL}(n, \mathbb{K})$ of elements fixing the spaces E_k; if $\mathbb{K} = \mathbb{R}$ we further impose that their restriction to E_j has positive determinant (in order to fulfill connectedness).

For $0 \leq j \leq k$, let $\Omega_j = \mathbb{K}^n \setminus E_{j+1}$ and $Y_j = E_j \setminus E_{j+1}$ (with the convention $E_{k+1} = \varnothing$). The set Y_j consists of one or two G orbits (depending on whether $\dim E_j \geq 2 + \dim E_{j+1}$ or $\dim E_j = 1 + \dim E_{j+1}$ — in the complex case the Y_j consists of a single orbit).

For every $j \in \{0, \dots, k\}$, let F_j be the quotient space $F_j = \mathbb{K}^n / E_j$ endowed with the flag $\{0\} \subset E_{j-1}/E_j \subset \cdots \subset E_0/E_j$ and let G_j be the group of positive automorphisms of this flag. The quotient map $\mathbb{K}^n \to F_j$ induces a group homomorphism $q_j : G \to G_j$.

Let also $p_j : \Omega_j \to F_j$ be the restriction of the quotient map to Ω_j. Let then $\widetilde{\mathcal{G}}_j$ be the pull-back groupoid of $F_j \rtimes G_j$ by the map p_j. In other words

$$\widetilde{\mathcal{G}}_j = \{(x, g, y) \in \Omega_j \times G_j \times \Omega_j : p_j(x) = g p_j(y)\}.$$

The map $(x, g, y) \mapsto (x, q_j(g), y)$ is a submersion and a groupoid morphism from $\Omega_j \rtimes G = \{(x, g, y) \in \Omega_j \times G \times \Omega_j : x = gy\}$ into $\widetilde{\mathcal{G}}_j$. Its image is the s-connected component $\mathcal{G}_j$ of $\widetilde{\mathcal{G}}_j$.

It follows from the following obvious lemma that $\mathcal{G}_j \rightrightarrows \mathbb{K}^n$ is a bisubmersion.

Lemma 1.10. *Let M, U, V be manifolds, $(M, \mathcal{F})$ a foliation, $p : U \to V$ a surjective submersion and $t_V, s_V : V \rightrightarrows M$ two submersions. Then $(U, t_V \circ p, s_V \circ p)$ is a bisubmersion for $\mathcal{F}$ if and only if (V, t_V, s_V) is a bisubmersion for $\mathcal{F}$.* □

It follows then from Lemma 1.3 that $H(\mathcal{F})|_{Y_j} = (\mathcal{G}_j)|_{Y_j}$. We deduce:

Proposition 1.11. *The foliation of $\mathbb{K}^n$ by the action of G is nicely decomposed by the groupoids $\mathcal{G}_j \rightrightarrows \Omega_j$. Its holonomy groupoid is a union $\coprod_{j=0}^{k} (\mathcal{G}_j)|_{Y_j}$.*

Remarks 1.12. (a) One may write a projective analogue of this example: let PG be the projective analogue of G acting on $\mathbb{K}P^{n-1}$, namely PG is the quotient of G by its center, the group of similarities in G. It has k orbits: the images $Y_j = PE_j \setminus PE_{j+1}$ of $E_j \setminus E_{j+1}$ by the quotient map $p : \mathbb{K}^n \setminus \{0\} \to \mathbb{K}P^{n-1}$ (for $j > 0$). This foliation is nicely decomposed by the projective analogues $P\mathcal{G}_j$ of the $\mathcal{G}_j$. Note that the map $p : E_j \setminus \{0\} \to p(E_j)$ induces a morphism $p_j : \mathcal{G}_j \to P\mathcal{G}_j$ which is a Morita equivalence in the complex case. In the real case, it is almost a Morita equivalence: the morphism p_j induces an isomorphism of the stabilizer of $x \in E_j \setminus \{0\}$ in $\mathcal{G}_j$ with the stabilizer of $p(x) \in PE_j$ in $P\mathcal{G}_j$, but for $0 < i \leq j$ and $\dim(E_i) = \dim(E_{k+1}) + 1$, the set $E_i \setminus E_{i-1}$ consists of two orbits of the groupoid $\mathcal{G}_j$ which become equivalent in $P\mathcal{G}_j$. The corresponding foliation C^*-algebra is (almost) Morita equivalent to $C^*(\Omega_1, \mathcal{F})$.

(b) There are many other interesting examples of the same flavor. A typical one is given in the following way: let $P_1, P_2 \subset \mathrm{GL}(n, \mathbb{K})$ be two parabolic subgroups, and let $P_1 \times P_2$ act on $\mathrm{GL}(n, \mathbb{K})$ by left and right multiplication. If $P_1 = P_2$ is the minimal parabolic subgroup consisting of upper triangular matrices, the orbits of this action are labeled by the symmetric group $\mathfrak{S}_n$ (Bruhat decomposition). In this example, the decomposition to be taken into account is more complicated than just the dimension of the fibers. One may need to use the partial ordering of the orbits given by the inclusion of the closures.

2. Foliations with singularity height 1

Let $(M, \mathcal{F})$ be a foliation admitting a nice decomposition of height 1. In this section our purpose is to show that the full foliation C^*-algebra $C^*(M, \mathcal{F})$ can be replaced by a mapping cone of Lie groupoid C^*-algebras associated with a nice decomposition of $\mathcal{F}$. We generalize this construction to higher length in the next section, but before this, we make some comments on the difficulties with dealing with reduced C^*-algebras.

2.A. ***A mapping cone construction.*** In the length 1 case, as noted in Remark 1.6, we just need to specify the saturated open subset $\Omega = \Omega_0$ and the Lie groupoid $\mathcal{G} = \mathcal{G}_1 \rightrightarrows W_1 = W$ which defines the foliation on an open set W containing $Y = M \setminus \Omega$ and whose restriction to Y coincides with that of $H(\mathcal{F})$.

The open subset Ω gives rise (at the level of the *full* C^*-algebras) to a short exact sequence

$$0 \to C^*(\Omega, \mathcal{F}_{|\Omega}) \xrightarrow{\iota_{\mathcal{F}}} C^*(M, \mathcal{F}) \xrightarrow{\pi_{\mathcal{F}}} C^*(M, \mathcal{F})|_Y \to 0$$

which in principle allows us to compute its K-theory. This is actually the case in our examples (Sections 2.C and 2.D).

In order to only use Lie groupoids (note that Y need not be a manifold), and also to be able to extend our construction to a more general setting (see Section 3), we also make use of the somewhat more elaborate diagram which appears in Figure 1.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(\mathcal{G}_{W\cap\Omega}) & \xrightarrow{\iota_{\mathcal{G}}} & C^*(\mathcal{G}) & \xrightarrow{p_{\mathcal{G}}} & C^*(\mathcal{G}_Y) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_\Omega} & & \downarrow{\scriptstyle \pi_M} & & \| & & \\ 0 & \longrightarrow & C^*(\Omega, \mathcal{F}_{|\Omega}) & \xrightarrow{\iota_{\mathcal{F}}} & C^*(M, \mathcal{F}) & \xrightarrow{p_{\mathcal{F}}} & C^*(M, \mathcal{F})_{|Y} & \longrightarrow & 0 \end{array}$$

Figure 1. Exact sequences for a nicely decomposable foliation of singularity height 1.

Restricting $\mathcal{G}$ to the open subset $W \cap \Omega$ and $H(\mathcal{F})$ to the open subset Ω, the integration along fibers (see [Androulidakis and Skandalis 2009]) of the quotient map $\mathcal{G} \to H(\mathcal{F})$ induces the diagram of half-exact sequences of *full* C^*-algebras shown in Figure 1.

Let $\mathcal{F}$ be a nicely decomposable foliation of singularity height one. We may use the diagram in Figure 1 in order to compute the K-theory of $C^*(M, \mathcal{F})$ via a Mayer–Vietoris exact sequence.

We explain here how one may replace $C^*(M, \mathcal{F})$ by a mapping cone of Lie groupoid C^*-algebras. We use the following notation:

- For any C^*-algebra Z and a locally compact space X put $Z(X) = C_0(X; Z)$.
- Recall that the mapping cone of a morphism $u : A \to B$ of C^*-algebras is

$$\mathcal{C}_u = \{(a, \phi) \in A \times B([0, 1)) : \phi(0) = u(a)\}.$$

With the notation of the diagram in Figure 1, consider the morphism of C^*-algebras

$$(i_{\mathcal{G}}, \pi_\Omega) : C^*(\mathcal{G}_{W\cap\Omega}) \to C^*(\mathcal{G}) \oplus C^*(W \cap \Omega, \mathcal{F}_{|\Omega}).$$

Proposition 2.1. *With the notation of Figure 1, the (full) foliation C^*-algebra $C^*(M, \mathcal{F})$ is canonically E^1-equivalent to the mapping cone $\mathcal{C}_{(\iota_{\mathcal{G}}, \pi_\Omega)}$.*

Proof. We show that given a diagram of exact sequences of C^*-algebras and morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{i} & B_1 & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi} & & \downarrow & & \| & & \\ 0 & \longrightarrow & B_0 & \xrightarrow{i'} & A & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

the mapping cone $\mathcal{C}_{(i,\pi)}$ of the map $(\pi, i) : I \to B_0 \oplus B_1$ is canonically E^1-equivalent to A.

Indeed, we have canonical morphisms $\mathcal{C}_i \to \mathcal{C}_{i'} \to Q(0, 1)$. Since $\mathcal{C}_i \to Q(0, 1)$ and $\mathcal{C}_{i'} \to Q(0, 1)$ are both onto with contractible kernels ($I[0, 1)$ and $B_0[0, 1)$, respectively), it follows that the morphism $\mathcal{C}_i \to \mathcal{C}_{i'}$ induces an equivalence in

E-theory. Now, using the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_0(0,1) & \longrightarrow & \mathcal{C}_{(i,\pi)} & \longrightarrow & \mathcal{C}_i & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_0(0,1) & \longrightarrow & \mathcal{C}_{(i',\mathrm{id}_{B_0})} & \longrightarrow & \mathcal{C}_{i'} & \longrightarrow & 0 \end{array}$$

we find that the morphism $\mathcal{C}_{(i,\pi)} \to \mathcal{C}_{(i',\mathrm{id}_{B_0})}$ induces an equivalence in E-theory. Finally the (split) exact sequence

$$0 \longrightarrow A(0,1) \longrightarrow \mathcal{C}_{(i',\mathrm{id}_{B_0})} \longrightarrow B_0[0,1) \longrightarrow 0$$

yields the desired E^1-equivalence. □

Remark 2.2. We may note that we have just shown that the morphism

$$\mathcal{C}_{(i,\pi)} \to \mathcal{C}_{(\mathrm{id}_A,\mathrm{id}_A)} \simeq A(0,1)$$

is invertible in E-theory.

2.B. ***Difficulties at the level of reduced C*-algebra.*** Let us discuss the reduced version of the diagram in Figure 1:

- If the restriction $\mathcal{G}_{|Y}$ is an amenable groupoid we also have horizontal exactness at the level of reduced C^*-algebras.
- If $\mathcal{G}_{|W\cap\Omega}$ is not amenable then the integration along fibers may not exist at the level of the kernels. We discuss such an example in Example 2.4.

In view of Examples 1.7 we focus now on foliations $(M,\mathcal{F})$ arising from an action of a Lie group G on a manifold M. We assume that $W = M$, the action groupoid $\mathcal{G} = M \rtimes G$ realizes a nice decomposition of singularity height 1 for $(M,\mathcal{F})$ and the complementary set Y is a point.

If the group G is amenable then integration along fibers of the quotient map $\mathcal{G} \to H(\mathcal{F})$ gives the diagram in Figure 2.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(\Omega)\rtimes G & \xrightarrow{\iota_{\mathcal{G}}} & C_0(M)\rtimes G & \xrightarrow{\pi_{\mathcal{G}}} & C^*(G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_\Omega} & & \downarrow{\scriptstyle \pi} & & \| & & \\ 0 & \longrightarrow & C^*(\Omega,\mathcal{F}_{|\Omega}) & \xrightarrow{\iota_{\mathcal{F}}} & C^*(M,\mathcal{F}) & \xrightarrow{\pi_{\mathcal{F}}} & C^*(G) & \longrightarrow & 0 \end{array}$$

Figure 2. Exact sequences for a nicely decomposable foliation of singularity height 1 arising from the action of an amenable Lie group.

If G is not amenable, the sequences are exact at the level of full C^*-algebras. At the reduced C^*-algebra level,

- the sequences need not be exact;
- the morphism $C_0(\Omega) \rtimes G \to C_r^*(M, \mathcal{F})$ obtained as a composition of π with the morphism $C^*(M, \mathcal{F}) \to C_r^*(M, \mathcal{F})$ doesn't need to pass to the quotient $C_0(\Omega) \rtimes_r G$ of $C_0(\Omega) \rtimes G$.

Note however that

- in most cases that we consider, the top sequence in Figure 2 is exact since the groups we consider are exact;
- we always have some completely positive splittings (see Proposition 2.3);
- in the example of the action of $\mathrm{GL}(2, \mathbb{R})$ on $\mathbb{R}^2$, since the stabilizers are amenable, the morphism $\pi_\Omega : C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{GL}(2, \mathbb{R}) \to \mathcal{K}$ is defined at the reduced C^*-algebra level. As the group $\mathrm{GL}(2, \mathbb{R})$ is K-amenable, we find that in this case the full and reduced C^*-algebra of $\mathcal{F}$ are KK-equivalent.

Proposition 2.3. *Let $\mathcal{G}$ be the action groupoid in Figure 2. Then the morphisms $C_r^*(\mathcal{G}) \to C_r^*(G)$, $C^*(\mathcal{G}) \to C^*(G)$ and $C^*(M, \mathcal{F}) \to C^*(G)$ have completely positive splittings.*

Proof. This is due to the fact that $C^*(G)$ sits in the multiplier algebra of a crossed product $A \rtimes G$ — and the same for reduced ones:

We construct a completely positive splitting for the map $C^*(\mathcal{G}) \to C^*(G)$. Take a function $f \in C_0(M)$ such that $\|f\| = 1$ and $f(x_0) = 1$. Given $\zeta \in C^*(G)$ put $\sigma(\zeta) = f^* \zeta f$. This is obviously a completely positive (and contractive) splitting of the top sequence. (The same is true for the reduced algebra and crossed products.)

Composing the completely positive splitting $C^*(G) \to C^*(\mathcal{G})$ with the morphism $\pi : C^*(\mathcal{G}) \to C^*(M, \mathcal{F})$ (given by integration along the fibers) we obtain a completely positive splitting of the second sequence. □

We now give an example where the morphism π_Ω is not defined at the reduced C^*-algebra level:

Example 2.4. Consider the action of $G = \mathrm{SL}(n, \mathbb{R})$ on $\mathbb{R}^n$ for $n \geq 3$. This action has two orbits: $\{0\}$ and $\Omega = \mathbb{R}^n \setminus \{0\}$. The stabilizer of a nonzero point for this action is isomorphic to $H = \mathbb{R}^{n-1} \rtimes \mathrm{SL}(n-1, \mathbb{R})$, which is not amenable if $n \geq 3$. The full crossed product $C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathrm{SL}(n, \mathbb{R})$ is Morita equivalent to $C^*(H)$. Therefore, the full C^*-algebra of this foliation is the quotient of $C_0(\mathbb{R}^n) \rtimes \mathrm{SL}(n, \mathbb{R})$

sitting in a diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & C_0(\Omega)\rtimes G \simeq \mathcal{K}\otimes C^*(H) & \longrightarrow & C_0(\mathbb{R}^n)\rtimes G & \xrightarrow{\pi_{\mathcal{G}}} & C^*(G) & \longrightarrow 0 \\
& \Big\downarrow{\scriptstyle \mathrm{id}_{\mathcal{K}}\otimes\varepsilon_H} & & \Big\downarrow{\scriptstyle \pi} & & \Big\| & \\
0 \longrightarrow & \mathcal{K} & \longrightarrow & C^*(\mathbb{R}^n,\mathcal{F}) & \xrightarrow{\pi_{\mathcal{F}}} & C^*(G) & \longrightarrow 0
\end{array}$$

where ε_H denotes the trivial representation of $H = \mathbb{R}^{n-1} \rtimes \mathrm{SL}(n-1,\mathbb{R})$. The reduced crossed product $C_0(\mathbb{R}^n \setminus \{0\}) \rtimes_r \mathrm{SL}(n,\mathbb{R})$ is Morita equivalent to $C^*_r(H)$.

Note that the trivial representation $C^*(H) \to \mathbb{C}$ is not defined at the level of $C^*_r(H)$ when the group H is not amenable.

The reduced C^*-algebra $C^*_r(\mathbb{R}^n,\mathcal{F})$ of this foliation is the quotient of $C_0(\mathbb{R}^n)\rtimes G$ corresponding to the sum of the two covariant representations on $L^2(\Omega) = L^2(\mathbb{R}^n)$ and $\{0\}\times G$.

Remark 2.5. In the sequel we use (almost) only the full C^*-algebra to ensure that our sequences are exact and the trivial representation exists. This is legitimate from the point of view of the Baum–Connes conjecture, since the assembly map factors through the K-theory of the full C^*-algebra anyway.

2.C. ***Two examples of foliations of singularity height 1 given by linear actions.*** In this section we compute the K-theory for two simple examples of foliations of singularity height 1 coming from linear actions. In the height 1 case, this can be done rather easily, using six-term exact sequences of K-theory groups and standard K-theory results. The (nonlinear) examples of $\mathbb{R}$-actions (see Example 1.7(d)) are discussed in Section 2.D.

2.C.1. *The* SO(3)*-action.* In this section we consider the foliation $(\mathbb{R}^3,\mathcal{F})$ defined by the action of SO(3) on $\mathbb{R}^3$ (see Example 1.7(a)).

Holonomy groupoid and exact sequences. As discussed in Examples 1.7, $H(\mathcal{F}) = (\mathrm{SO}(3)\times\{0\}) \sqcup (\mathbb{R}^*_+ \times S^2 \times S^2)$ and $\mathcal{F}$ is nicely decomposable, in the sense of Definition 1.4 with $\Omega_0 = \mathbb{R}^3\setminus\{0\}$ and $\mathcal{G}_1 = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$.

Note that SO(3) is compact and therefore amenable, so the reduced and full crossed product C^*-algebras coincide. The (full and reduced) C^*-algebra of $\mathcal{G}_0$ is the crossed product $C_0(\mathbb{R}^3)\rtimes \mathrm{SO}(3)$.

Writing $\mathbb{R}^3\setminus\{0\} = \mathbb{R}^*_+ \times S^2$, we find that $C^*(\mathcal{G}|_{\Omega_0}) = C_0(\mathbb{R}^*_+)\otimes(C(S^2)\rtimes\mathrm{SO}(3))$ and $C^*(M,\mathcal{F})|_{\Omega_0} = C_0(\mathbb{R}^*_+)\otimes\mathcal{K}(L^2(S^2))$. Now Figure 1 reads as in Figure 3. Here $\hat{q} = \mathrm{id}_{C_0(\mathbb{R}^*_+)}\otimes q$, where $q : C(S^2)\rtimes\mathrm{SO}(3) \to \mathcal{K}(L^2(S^2))$ is obtained by integration along the fibers of the groupoid morphism $(t,s) : S^2\rtimes\mathrm{SO}(3) \to S^2\times S^2$.

Calculation of K-theory with mapping cones. To describe the foliation C^*-algebra we give an interpretation of Figure 3 using mapping cones.

$$\begin{array}{ccccccccc}
0 \longrightarrow & C_0(\mathbb{R}_+^*)\otimes(C(S^2)\rtimes \mathrm{SO}(3)) & \xrightarrow{i} & C_0(\mathbb{R}^3)\rtimes \mathrm{SO}(3) & \longrightarrow & C^*(\mathrm{SO}(3)) & \longrightarrow 0 & & \text{(ES1)} \\
 & \downarrow \hat{q} & & \downarrow \pi & & \| & & & \\
0 \longrightarrow & C_0(\mathbb{R}_*^+)\otimes\mathcal{K}(L^2(S^2)) & \longrightarrow & C^*(\mathbb{R}^3, \mathcal{F}) & \longrightarrow & C^*(\mathrm{SO}(3)) & \longrightarrow 0 & & \text{(ES2)}
\end{array}$$

Figure 3. Exact sequences for the SO(3) action.

$$\begin{array}{ccc}
C^*(\mathrm{SO}(3)) & \xrightarrow{j} & C(S^2)\rtimes \mathrm{SO}(3) \\
 & \searrow^{\rho} & \downarrow q \\
 & & \mathcal{K}(L^2(S^2))
\end{array}$$

Figure 4. Mapping cones for the SO(3) action.

Let $\rho : C^*(\mathrm{SO}(3)) \to \mathcal{K}(L^2(S^2))$ be the natural representation of SO(3) on $L^2(S^2)$. We thus have the diagram in Figure 4, where $j : C^*(\mathrm{SO}(3)) \to C(S^2) \rtimes \mathrm{SO}(3)$ is the morphism induced by the unital inclusion $\mathbb{C} \to C(S^2)$.

Identify $C_0(\mathbb{R}^3)$ with the mapping cone of $\mathbb{C} \to C(S^2)$. Taking crossed products by the action of SO(3) and using the diagram in Figure 3, we find:

- The crossed product C^*-algebra $C_0(\mathbb{R}^3) \rtimes \mathrm{SO}(3)$ in extension (ES1) is the mapping cone $\mathcal{C}_p$, where p is the map $j : C^*(\mathrm{SO}(3)) \to C(S^2) \rtimes \mathrm{SO}(3)$.
- The foliation C^*-algebra $C^*(\mathbb{R}^3, \mathcal{F})$ in extension (ES2) is the mapping cone $\mathcal{C}_\rho$.

To describe $C^*(\mathcal{F})$, it suffices to describe the representation

$$\rho : C^*(\mathrm{SO}(3)) \to \mathcal{K}(L^2(S^2)).$$

It follows from the Peter–Weyl theorem that $C^*(\mathrm{SO}(3)) = \bigoplus_{m\in\mathbb{N}} M_{2m+1}(\mathbb{C})$ and $K_0(C^*(\mathrm{SO}(3))) = \mathbb{Z}^{(\mathbb{N})}$ (and $K_1(C^*(\mathrm{SO}(3))) = \{0\}$).

In order to compute the map $\rho_* : K_0(C^*(\mathrm{SO}(3))) \to \mathbb{Z}$, we have to understand how many times the representation σ_m (of dimension $2m+1$) appears in ρ, i.e., count the dimension of $\mathrm{Hom}_{\mathrm{SO}(3)}(\sigma_m, \rho)$.

Since $S^2 = \mathrm{SO}(3)/S^1$, the representation ρ is the representation $\mathrm{Ind}_{S^1}^{\mathrm{SO}(3)}(\varepsilon)$ induced by the trivial representation ε of S^1. Using the Frobenius reciprocity theorem, we know $\dim(\mathrm{Hom}_{\mathrm{SO}(3)}(\sigma_m, \rho)) = \dim(\mathrm{Hom}_{S^1}(\sigma_m, \varepsilon)) = 1$.

It follows that the map $\rho_* : K_0(C^*(\mathrm{SO}(3))) \to \mathbb{Z}$ is the map which sends each generator $[\sigma_m]$ of $K_0(C^*(\mathrm{SO}(3)))$ to 1. We immediately deduce:

Proposition 2.6. *We have* $K_0(C^*(\mathcal{F})) = \ker \rho_* \simeq \mathbb{Z}^{(\mathbb{N})}$ *and* $K_1(C^*(\mathcal{F})) = 0$. □

Remark 2.7. In the same way, one may easily compute

$$j_* : K_0(C^*(\mathrm{SO}(3)) \to K_0(C(S^2) \rtimes C^*(\mathrm{SO}(3)) \quad \text{and} \quad K_*(C_0(\mathbb{R}^3) \rtimes \mathrm{SO}(3)).$$

In fact, this is a classical result, which states that the algebra $C(S^2) \rtimes C^*(\mathrm{SO}(3))$ is Morita equivalent to $C^*(S^1)$ and the morphism $j_* : K_0(C^*(\mathrm{SO}(3))) \to K_0(C^*(S^1))$ is the restriction morphism $R(\mathrm{SO}(3)) \to R(S^1)$, where $R(G) = K_0(C^*(G))$ is the representation ring of a compact group G; see [Rieffel 1976; Julg 1982].

It follows that $j_*([\sigma_m]) = \sum_{k=-m}^{m} [\chi_k]$, where the $(\chi_k)_{k\in\mathbb{Z}}$ are the characters of S^1. The morphism j_* is therefore (split) injective, and we find

$$K_0(C_0(\mathbb{R}^3) \rtimes \mathrm{SO}(3)) = 0,$$
$$K_1(C_0(\mathbb{R}^3) \rtimes \mathrm{SO}(3)) \simeq \mathbb{Z}^{(\mathbb{N})}.$$

2.C.2. *The* SL(2, ℝ)*-action.* We consider the foliation on $\mathbb{R}^2$ induced by the action of SL(2, ℝ). Recall the following:

(a) SL(2, ℝ) is not compact and not amenable, but it was shown in [Kasparov 1984] to be KK-amenable.

(b) Its maximal compact is S^1.

(c) The action of SL(2, ℝ) on $\mathbb{R}^2 \setminus \{0\}$ is transitive and the stabilizer of the point $(1, 0)$ is the set of matrices of the form $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$. Hence the action groupoid $(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2, \mathbb{R})$ is Morita equivalent to the group $\mathbb{R}$. So the crossed product $C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2, \mathbb{R})$ is Morita equivalent to the group C^*-algebra $C^*(\mathbb{R})$.

(d) It follows as above from Lemma 1.3 (see also [Androulidakis and Skandalis 2009, Example 3.7]) that the associated holonomy groupoid is

$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2 \setminus \{0\}) \cup \{0\} \times \mathrm{SL}(2, \mathbb{R}).$$

It follows that this foliation is nicely decomposable of singularity height 1 with $\mathcal{G} = \mathcal{G}_0 = \mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R})$ (see Remark 1.6). Here the diagram of Figure 1 reads as in Figure 5. Recall that the C^*-algebras involved are full C^*-algebras.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (C_0(\mathbb{R}^2 \setminus \{0\})) \rtimes \mathrm{SL}(2,\mathbb{R}) & \xrightarrow{\iota} & C_0(\mathbb{R}^2) \rtimes \mathrm{SL}(2,\mathbb{R}) & \longrightarrow & C^*(\mathrm{SL}(2,\mathbb{R})) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} & & \| & & \\
0 & \longrightarrow & \mathcal{K}(L^2(\mathbb{R}^2 \setminus \{0\})) & \longrightarrow & C^*(\mathcal{F}) & \longrightarrow & C^*(\mathrm{SL}(2,\mathbb{R})) & \longrightarrow & 0
\end{array}$$

Figure 5. Exact sequences for the SL(2, ℝ) action.

Direct Calculation of K-theory. The short exact sequence

$$0 \to \mathcal{K}(L^2(\mathbb{R}^2 \setminus \{0\})) \to C^*(\mathcal{F}) \to C^*(\mathrm{SL}(2,\mathbb{R})) \to 0$$

gives the 6-term exact sequence

$$\begin{array}{ccccc} K_0(\mathcal{K}(L^2(\mathbb{R}^2 \setminus \{0\}))) & \longrightarrow & K_0(C^*(\mathcal{F})) & \longrightarrow & K_0(C^*(\mathrm{SL}(2,\mathbb{R}))) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\mathrm{SL}(2,\mathbb{R}))) & \longleftarrow & K_1(C^*(\mathcal{F})) & \longleftarrow & K_1(\mathcal{K}(L^2(\mathbb{R}^2 \setminus \{0\}))) \end{array}$$

We have $K_0(\mathcal{K}(L^2(\mathbb{R}^2 \setminus \{0\}))) = \mathbb{Z}$ and $K_1(\mathcal{K}(L^2(\mathbb{R}^2 \setminus \{0\}))) = 0$. On the other hand, using the Connes–Kasparov conjecture proved in [Kasparov 1984; Wassermann 1987], we have $K_1(C^*(\mathrm{SL}(2,\mathbb{R}))) = 0$. We conclude that

$$K_1(C^*(\mathbb{R}^2,\mathcal{F})) = 0 \quad \text{and} \quad K_0(C^*(\mathbb{R}^2,\mathcal{F})) = \mathbb{Z} \oplus K_0(C^*(\mathrm{SL}(2,\mathbb{R}))) = \mathbb{Z} \oplus \mathbb{Z}^{(\mathbb{Z})}.$$

Calculation of K-theory with mapping cones. Although the above construction is quite direct, it may be worth examining a construction following the general procedure of Section 2.A (Proposition 2.1).

To apply the mapping cones approach we gave in Section 2.A, we need the following result, which follows from [Kasparov 1984; 1988].

Proposition 2.8. *Let* $\mathrm{SL}(2,\mathbb{R})$ *act on a* C^**-algebra* A *by automorphisms. The algebras* $A \rtimes \mathrm{SL}(2,\mathbb{R})$ *and* $A \rtimes S^1$ *are KK-equivalent.*

Proof. The Lie group S^1 is a maximal compact subgroup of $\mathrm{SL}(2,\mathbb{R})$. Note also that $\mathrm{SL}(2,\mathbb{R})/S^1$ is the Poincaré half plane and therefore admits a complex structure, and hence an $\mathrm{SL}(2,\mathbb{R})$-invariant spin^c structure. The result follows from [Kasparov 1984]. □

It follows in fact from [Kasparov 1984] that the exact sequences

$$0 \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2,\mathbb{R}) \to C_0(\mathbb{R}^2) \rtimes \mathrm{SL}(2,\mathbb{R}) \to C^*(\mathrm{SL}(2,\mathbb{R})) \to 0$$

and

$$0 \to C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes S^1 \to C_0(\mathbb{R}^2) \rtimes S^1 \to C^*(S^1) \to 0$$

are KK-equivalent. We note the following:

- $K_0(C^*(\mathrm{SL}(2,\mathbb{R}))) = K_0(C^*(S^1)) = \mathbb{Z}^{(\mathbb{Z})}$, and $K_1 = 0$.
- Since S^1 acts freely on $\mathbb{R}^2 \setminus \{0\}$ with quotient $\mathbb{R}_+^*$, it follows that $C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes S^1$ is Morita equivalent to $C_0(\mathbb{R}_+^*)$; also since $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{0\}$ with stabilizers isomorphic to $\mathbb{R}$, it follows that $C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2,\mathbb{R})$ is Morita equivalent to $C^*(\mathbb{R})$. It follows that $K_1(C(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2,\mathbb{R})) = K_1(C(\mathbb{R}^2 \setminus \{0\}) \rtimes S^1) = \mathbb{Z}$ and $K_0 = 0$.

- Using the complex structure of $\mathbb{R}^2$, we have a Bott isomorphism between $K_*(C_0(\mathbb{R}^2) \rtimes S^1)$ and $K_*(C^*(S^1))$. It follows that $K_0(C(\mathbb{R}^2) \rtimes \mathrm{SL}(2, \mathbb{R})) = K_0(C(\mathbb{R}^2) \rtimes S^1) = \mathbb{Z}^{(\mathbb{Z})}$, and $K_1 = 0$.

From this discussion, it follows that the morphism

$$\iota : C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2, \mathbb{R}) \to C_0(\mathbb{R}^2) \rtimes \mathrm{SL}(2, \mathbb{R})$$

induces the 0 map in K-theory, and so does the map $\pi : C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}(2, \mathbb{R}) \to \mathcal{K}$.

Remark 2.9. Denoting by $(\chi_n)_{n\in\mathbb{Z}}$ the characters of S^1, for each n the image of $[\chi_n] \in K_*(C_0(\mathbb{R}^2) \rtimes S^1)$ by $C_0(\mathbb{R}^2) \rtimes S^1 \to C^*(S^1)$ (evaluation at 0) is $[\chi_n]-[\chi_{n+1}]$. This morphism is one to one and its image is the set of elements in $R(S^1)$ of dimension 0.

As the maps ι and π induce the 0 map in K-theory, we find as above from Proposition 2.1.

Proposition 2.10. *Let $\mathcal{F}$ be the foliation defined by the action of* $\mathrm{SL}(2, \mathbb{R})$ *on* $\mathbb{R}^2$. *We have*

$$K_0(\mathcal{F}) \simeq K_0(C_0(\mathbb{R}^2) \rtimes \mathrm{SL}^2(\mathbb{R})) \oplus K_0(\mathcal{K}) \oplus K_1(C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}^2(\mathbb{R})) \simeq \mathbb{Z}^{(\mathbb{N})} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

and $K_1(\mathcal{F}) = 0$. □

Note that we have a split short exact sequence $0 \to K_0(C_0(\mathbb{R}^2) \rtimes \mathrm{SL}^2(\mathbb{R})) \to K_0(C^*(\mathrm{SL}^2(\mathbb{R}))) \to K_1(C_0(\mathbb{R}^2 \setminus \{0\}) \rtimes \mathrm{SL}^2(\mathbb{R})) \to 0$, and thus the results of Proposition 2.10 and the direct calculation (Section 2.C.2) are coherent.

2.C.3. *Generalizations.* The examples introduced above can be extended to the action of $\mathrm{SO}(n)$ or $\mathrm{SL}(n, \mathbb{R})$ on $\mathbb{R}^n$. Let us discuss here a slightly more general situation which still gives singularity height 1 foliations.

Subgroups of $\mathrm{SO}(n)$. Let G be a connected closed subgroup of $\mathrm{SO}(n)$. Assume that its action on S^{n-1} is transitive, and let $H \subset G$ be the stabilizer of a point in S^{n-1}. Denote by $\mathcal{F}$ the foliation of $\mathbb{R}^n$ associated with the action of G. Exactly as in the case of the action of $\mathrm{SO}(3) \in \mathbb{R}^3$, we find that

- $H(\mathcal{F}) = (G \times \{0\}) \sqcup (\mathbb{R}_+^* \times S^{n-1} \times S^{n-1})$;
- $C^*(\mathbb{R}^n, \mathcal{F})$ is the mapping cone of the morphism $C^*(G) \to \mathcal{K}(S^{n-1})$.
- The map $R(G) \to \mathbb{Z}$ corresponding to this morphism associates to a (virtual) representation σ the (virtual) dimension of its H fixed points. It is onto, and therefore $K_0(C^*(\mathbb{R}^n, \mathcal{F})) = \mathbb{Z}^{(\mathbb{N})}$ and $K_1(C^*(\mathbb{R}^n, \mathcal{F})) = 0$.

Subgroups of GL_n. Now let G be a closed connected subgroup of $\mathrm{GL}(n,\mathbb{R})$. Assume that its action on $\mathbb{R}^n \setminus \{0\}$ is transitive, and let $H \subset G$ be the stabilizer of a nonzero point in $\mathbb{R}^n$. As for the case of $\mathrm{SL}(2,\mathbb{R})$ acting on $\mathbb{R}^2$, we have:

- The holonomy groupoid is $H(\mathcal{F}) = ((\mathbb{R}^n \setminus \{0\}) \times (\mathbb{R}^n \setminus \{0\})) \sqcup (G \times \{0\})$.
- We have an exact sequence of full C^*-algebras

$$0 \to \mathcal{K}(L^2(\mathbb{R}^n \setminus \{0\})) \to C^*(\mathbb{R}^n, \mathcal{F}) \to C^*(G) \to 0,$$

and therefore an exact sequence

$$0 \to K_1(C^*(\mathbb{R}^n,\mathcal{F})) \to K_1(C^*(G)) \xrightarrow{\partial} \mathbb{Z} \to K_0(C^*(\mathbb{R}^n,\mathcal{F})) \to K_0(C^*(G)) \to 0.$$

In order to try and compute the connecting map ∂, we may use the diagram of Figure 1. Note that the groupoid $(\mathbb{R}^n \setminus \{0\}) \rtimes G$ is Morita equivalent to the group H. Following this diagram, the connecting map ∂ is the composition of the trivial representation of H of with the connecting map

$$\partial' : K_1(C^*(G)) \to K_0(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes G) \simeq K_0(C^*(H)).$$

An example of this kind is of course $\mathrm{SL}(n,\mathbb{C}) \subset \mathrm{SL}(2n,\mathbb{R})$. The stabilizer group of a $z \in \mathbb{C}^n \setminus \{0\}$, say $z = (1, 0, \dots, 0)$, is the group of matrices in $\mathrm{SL}(n,\mathbb{C})$ whose first row is z. That is $\mathbb{C}^{n-1} \rtimes \mathrm{SL}(n-1,\mathbb{C})$.

Another example is given by $G = G_1 \times \mathbb{R}_+^*$, where G_1 is a connected closed subgroup of $\mathrm{SO}(N)$ whose action on S^{n-1} is transitive, and $\mathbb{R}_+^*$ acts by similarities. Note that if $\mathcal{F}_1$ is the foliation defined by the action of G_1, there is a natural action of $\mathbb{R}_+^*$ on $H(\mathcal{F}_1)$ and $H(\mathcal{F})$ is a semidirect product $H(\mathcal{F}_1) \rtimes \mathbb{R}_+^*$; we find $C^*(\mathbb{R}^n,\mathcal{F}) = C^*(\mathbb{R}^n,\mathcal{F}_1) \rtimes \mathbb{R}_+^*$. Thanks to the Connes–Thom isomorphism, the algebras $C^*(\mathbb{R}^n,\mathcal{F})$ and $C^*(\mathbb{R}^n,\mathcal{F}_1)$ have the same K-theory up to a shift of dimension.

2.D. ***Actions of*** $\mathbb{R}$ ***on manifolds.*** Now we come to Example 1.7(d), which also belongs to the case of height 1 foliations. Let M be a manifold endowed with a smooth action α of $\mathbb{R}$. Let $\mathcal{F}$ be the foliation associated with this action — i.e., with the groupoid $M \rtimes_\alpha \mathbb{R}$. We keep the notation of Example 1.7(d). There are several papers concerned with actions of $\mathbb{R}$ and the computation of the associated C^*-algebra; see [Torpe 1985; Wang 1987; Hirsch and Wang 1987]. The particular difficulty with the general case we examine here comes from the (interior of the) set where the vector field vanishes, and partly also from those points where the vector field is periodic.

We showed that $\mathcal{F}$ is nicely decomposable in the sense of Definition 1.4. Here we compute the K-theory using an exact sequence. Note that in this example, in the presence of periodic points, the groupoid $\mathcal{G}_0$ is not always Hausdorff and its classifying space for proper actions is not a manifold. Therefore Theorem 5.13 does not apply directly.

From Proposition 1.9 we deduce that the groupoid $\mathcal{G}_1' \rightrightarrows M$ which coincides with $H(\mathcal{F})$ on the complement of W and with $W \times \mathbb{R}$ on W is a (not necessarily Hausdorff) Lie groupoid and gives rise to the nice decomposition ($W \rightrightarrows W$, $\mathcal{G}_1' \rightrightarrows M$) of $\mathcal{F}$. We exploit this one in the computations below.

Put also $Y = M \setminus W$.

2.D.1. *Exact sequence of fixed points.*

Proposition 2.11. *The KK^1-element associated with the exact sequence*

$$0 \to C_0(W) \to C^*(M, \mathcal{F}) \to C^*(M, \mathcal{F})|_Y \to 0 \tag{ES3}$$

is 0.

Proof. The corresponding exact sequence for the groupoid $\mathcal{G}_1'$ givers rise to the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0(W) & \longrightarrow & C^*(M, \mathcal{F}) & \longrightarrow & C^*(M, \mathcal{F})|_Y & \longrightarrow & 0 \quad \text{(ES3)} \\ & & \uparrow{\scriptstyle \mathrm{ev}_0} & & \uparrow & & \| & & \\ 0 & \longrightarrow & C_0(W \times \mathbb{R}) & \longrightarrow & C^*(\mathcal{G}_1') & \longrightarrow & C^*(\mathcal{G}_1')|_Y & \longrightarrow & 0 \quad \text{(ES4)} \end{array}$$

Denote by z_1, z_2 the KK^1 elements associated with the exact sequences (ES3) and (ES4). We have $z_1 = (\mathrm{ev}_0)_*(z_2)$. But (ev_0), which is the map induced by the inclusion $x \mapsto (x, 0)$ from W to $W \times \mathbb{R}$, is the 0 element in KK, whence $z_1 = 0$ as claimed. □

We immediately deduce:

Corollary 2.12. *We have $K_*(C^*(M, \mathcal{F})) = K_*(C_0(W)) \oplus K_*(C^*(M, \mathcal{F})|_Y)$.* □

If all periodic points are in fact fixed, i.e., if $\mathrm{Per}(\alpha) = W$, then, using Connes' Thom isomorphism [Connes 1981], this computation yields:

Corollary 2.13. *Assume that all the periodic points are in fact fixed. The K-theory group of $C^*(M, \mathcal{F})$ is*

$$\begin{aligned} K_*(C^*(M, \mathcal{F})) &= K_*(C_0(W)) \oplus K_*(C_0(Y) \rtimes \mathbb{R}) \\ &= K_*(C_0(W)) \oplus K_{1-*}(C_0(Y)). \end{aligned}$$ □

Remark 2.14. Corollary 2.13 can be interpreted by saying that, when there are no nontrivial stably periodic points, the classifying space of proper actions of the holonomy groupoid is $W \sqcup Y \times \mathbb{R}$. The associated assembly map is an isomorphism.

2.D.2. *The stably periodic points.* In the presence of nontrivial stable periodic points, the complete computation of the K-theory is not so simple. Even in the regular case, this computation is quite hard. See, e.g., [Torpe 1985].

As a consequence of Proposition 1.9 we find:

Proposition 2.15. *The set* $\widehat{\mathrm{Per}}(\alpha) = \mathrm{Per}(\alpha) \setminus W$ *of nontrivial stably periodic points is open.*

Proof. By Proposition 1.9, the set W is closed in $\mathrm{Per}(\alpha)$, whence its complement is open in $\mathrm{Per}(\alpha)$ — and therefore in M, since $\mathrm{Per}(\alpha)$ is open. □

For $x \in \widehat{\mathrm{Per}}(\alpha)$, let $p(x) \in \mathbb{R}_+$ be the infimum of the set of $t > 0$ such that $(x, t) \in M \times \mathbb{R}$ is the trivial element in $H(\mathcal{F})$. By [Debord 2013] it follows that $p(x) > 0$ and $(x, p(x))$ is the trivial element in $H(\mathcal{F})$.

Proposition 2.16. *The map* $p : \widehat{\mathrm{Per}}(\alpha) \to \mathbb{R}_+$ *is smooth.*

Proof. Since $(x, p(x))$ is the trivial element in $H(\mathcal{F})$, there exists an open neighborhood $U \subset \widehat{\mathrm{Per}}(\alpha)$ of x and a bounded (below and above) smooth function $f : U \to \mathbb{R}_+^*$ such that $(y, f(y)) \in P$ for all $y \in U$ and $p(x) = f(x)$. For $y \in U$, since U is a neighborhood of y, it follows that $f(y)$ is a multiple of $p(y)$. We consider two cases.

• Assume $X(x) = 0$. Let $m \in \mathbb{R}_+$ be such that $f(y) \le m$ for all $y \in U$. Put $V = \{x \in U : \forall t \in [0, m],\ \alpha_t(x) \in U\}$; by compactness of $[0, m]$ it is an open subset of U. Then by periodicity, V is invariant by α_t, $t \in \mathbb{R}$. For $y \in V$ and $t \in [0, T]$, as $f(\alpha_t(y))$ is a multiple of $p(\alpha_t(y)) = p(y)$, it follows by continuity of f that $f(\alpha_t(y)) = f(y)$. Replacing X by $(1/f)X$, we get an action of S^1 on V.

Since S^1 is compact, Bochner's linearization theorem [1945] says that in an open and S^1-equivariant neighborhood U' of x the S^1-action is actually a linear representation of S^1, which is faithful since $f(x) = p(x)$. It follows that $p(y) = f(y)$ for all $y \in U'$.

• Assume $X(x) \neq 0$. Then x is periodic of period $p(x)/k$ with $k \in \mathbb{N}^*$. Now choose a transversal T at x; we get an action of $\mathbb{Z}/k\mathbb{Z}$, and applying Bochner's linearization theorem again, we conclude $f(y) = p(y)$ in a neighborhood of x. □

When restricting to $\widehat{\mathrm{Per}}(\alpha)$, we may therefore replace X by $\frac{1}{p}X$ and obtain an action of S^1. The foliation groupoid is then $W \sqcup \widehat{\mathrm{Per}}(\alpha) \rtimes S^1 \sqcup (M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R}$.

Remarks 2.17. (a) The building blocks of $C^*(M, \mathcal{F})$ are the algebras $C_0(W)$, $C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1$ and $C_0(M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R}$. For each of them there is of course a topological K-theory and a Baum–Connes map. Actually, since the first two are given by compact group actions, they are their own "left-hand side"! The "left-hand side" for $C_0(M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R}$ given by Connes' Thom isomorphism is $(M \setminus \mathrm{Per}(\alpha)) \times \mathbb{R}$.

(b) We already noticed that $K_*(C^*(M,\mathcal{F})) = K_*(W) \oplus K_*(C^*(M,\mathcal{F})|_Y)$. To compute $K_*(C^*(M,\mathcal{F})|_Y)$ we may use the exact sequence

$$0 \to C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1 \to C^*(M,\mathcal{F})|_Y \to C_0(M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R} \to 0 \qquad \text{(ES5)}$$

In order to compute the connecting map of this sequence we note that we have a diagram:

$$\begin{array}{ccccccccc} 0 \longrightarrow & C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1 & \longrightarrow & C^*(\mathcal{F})|_Y & \longrightarrow & C^*(M,\mathcal{F})|_{M\setminus \mathrm{Per}(\alpha)} & \longrightarrow 0 & & \text{(ES5)} \\ & q\uparrow & & \uparrow & & \| & & & \\ 0 \longrightarrow & C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes \mathbb{R} & \longrightarrow & C_0(Y) \rtimes \mathbb{R} & \longrightarrow & C_0(M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R} & \longrightarrow 0 & & \text{(ES6)} \end{array}$$

Denote by z_3, z_4 the KK^1 elements associated with the exact sequences (ES5) and (ES6). We have $z_3 = q_*(z_4) = z_4 \otimes [q]$. To compute z_3 we then remark:

- Through Connes' Thom isomorphism

$$KK^1(C_0(M \setminus \mathrm{Per}(\alpha)) \rtimes \mathbb{R}, C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes \mathbb{R}) = KK^1(C_0(M \setminus \mathrm{Per}(\alpha)), C_0(\widehat{\mathrm{Per}}(\alpha)))$$

the element $[z_4]$ corresponds to the exact sequence of commutative algebras

$$0 \to C_0(\widehat{\mathrm{Per}}(\alpha)) \to C_0(Y) \to C_0(M \setminus \mathrm{Per}(\alpha)) \to 0$$

- Under the Takesaki–Takai isomorphism $C_0(\widehat{\mathrm{Per}}(\alpha)) \otimes \mathcal{K} \simeq (C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1) \rtimes \mathbb{Z}$ the element $[q]$ in

$$\begin{aligned} &KK(C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes \mathbb{R}, C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1) \\ &\qquad = KK^1(C_0(\widehat{\mathrm{Per}}(\alpha)), C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1) \\ &\qquad = KK^1\big((C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1) \rtimes \mathbb{Z}, C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1\big) \end{aligned}$$

is the one associated to the Pimsner–Voiculescu exact sequence

$$0 \to B \otimes \mathcal{K} \to \mathfrak{T} \to B \rtimes \mathbb{Z} \to 0$$

(here $B = C_0(\widehat{\mathrm{Per}}(\alpha)) \rtimes S^1$).

3. Larger singularity height and telescope

In this section we extend the constructions of Section 2 to singular foliations of arbitrary singularity height. The mapping cone of Section 2 is replaced by a telescope. We start by recalling telescope constructions.

3.A. *Mapping telescopes.* Let us recall the following construction of C^*-algebras:

Definition 3.1. Let $n \in \mathbb{N} \cup \{+\infty\}$. Given C^*-algebras $(B_k)_{0 \le k < n}$ and $(I_k)_{1 \le k < n}$ and morphisms $\alpha_k : I_k \to B_{k-1}$ and $\beta_k : I_k \to B_k$, we define the associated *telescopic C^*-algebra*

$$\mathcal{T}((\alpha_k)_{1 \le k < n}, (\beta_k)_{1 \le k < n})$$

to be the C^*-algebra comprising

$$((\phi_k)_{0 \le k < n}, (x_k)_{1 \le k < n}) \in \prod_{0 \le k < n} B_k[k, k+1] \times \prod_{1 \le k < n} I_k$$

such that

- for $1 \le k < n$ we have $\phi_k(k) = \beta_k(x_k)$ and $\phi_{k-1}(k) = \alpha_{k-1}(x_k)$,
- $\phi_0(0) = 0$,
- $\begin{cases} \phi_{n-1}(n) = 0 & \text{if } n \ne +\infty, \\ \lim_{k \to +\infty} \|\phi_k\| = \lim_{k \to +\infty} \|x_k\| = 0 & \text{if } n = +\infty. \end{cases}$

Remark 3.2. A particular case of a telescope is when $I_k = B_{k-1}$ and $\alpha_k = \mathrm{id}_{I_k}$. We denote just by $\mathcal{T}(\beta)$ the associated mapping telescope $\mathcal{T}(\mathrm{id}, \beta)$. In that case, if $n = \infty$, let us also denote by B_∞ the inductive limit of the system (B_k, β_k). We then have an exact sequence

$$0 \to \mathcal{T}(\beta) \to \mathcal{T}'(\beta) \to B_\infty \to 0, \tag{3.3}$$

where $\mathcal{T}'(\beta)$ is the set of elements that have a limit at ∞: it is the inductive limit of $\mathcal{T}'_k(\beta)$ (i.e., the closure in $\mathcal{M}(\mathcal{T}(\beta))$ of the increasing union of $\mathcal{T}'_k(\beta)$) where $\mathcal{T}'_k(\beta)$ is the algebra of functions that become constant after k, i.e., such that ϕ_ℓ is constant for $\ell \ge k$ — and of course equal to the image in B_ℓ of the element $\phi_k(k) \in B_k$. Note that we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}(\beta) & \longrightarrow & \mathcal{T}'(\beta) & \longrightarrow & B_\infty & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B_\infty(0, +\infty) & \longrightarrow & B_\infty(0, +\infty] & \longrightarrow & B_\infty & \longrightarrow & 0 \end{array}$$

It follows that the composition of the element in $E^1(B_\infty, \mathcal{T}(\beta))$ given by the exact sequence (3.3) with the morphism $\mathcal{T}(\beta) \to B_\infty(0, +\infty)$ is the unit element of $E^1(B_\infty, B_\infty(0, +\infty)) = E(B_\infty, B_\infty)$.

Using this remark, one obtains the following results (cf. [Rosenberg and Schochet 1987]):

Proposition 3.4. (a) *If I_k and B_k are E-contractible, then $\mathcal{T}(\alpha, \beta)$ is also E-contractible.*

(b) *If (B_k, β_k) is an inductive system of E-contractible C^*-algebras, then their inductive limit B_∞ is E-contractible.*

(c) *If* (B_k, β_k) *is an inductive system of* C^**-algebras then* $\mathcal{T}'(\beta)$ *is* E*-contractible and the element in* $E^1(B_\infty, \mathcal{T}(\beta))$ *given by the exact sequence* (3.3) *is invertible.*

Proof. (a) Indeed, we have a unital ring morphism

$$\prod E(B_k, B_k) \to E\Big(\bigoplus B_k, \bigoplus B_k\Big)$$

and it follows that if the B_k are E-contractible, then $\bigoplus B_k$ is E-contractible. Since $\bigoplus B_k$ and $\bigoplus I_k$ are E-contractible, then by the exact sequence

$$0 \to \bigoplus B_k(0, 1) \to \mathcal{T}(\alpha, \beta) \to \bigoplus I_k \to 0,$$

the telescope $\mathcal{T}(\alpha, \beta)$ is E-contractible.

(b) Since the telescope $\mathcal{T}(\beta)$ is E-contractible, the algebra B_∞ is E-contractible since, by Remark 3.2, it is E-subequivalent to $\mathcal{T}(\beta)$.

(c) We have (split) exact sequences $0 \to B_k(k, k+1] \to \mathcal{T}'_{k+1}(\beta) \to \mathcal{T}'_k(\beta) \to 0$ and it follows by induction that, for all k, $\mathcal{T}'_k(\beta)$ is KK-contractible, and therefore E-contractible (note that $\mathcal{T}'_0(\beta) = 0$). It follows that the inductive limit $\mathcal{T}'(\beta)$ is E-contractible and therefore the exact sequence (3.3) induces an E^1-equivalence. □

In fact a telescope can be expressed as a mapping torus:

Remarks 3.5. (a) Recall that given C^*-algebras A, B and morphisms $u, v : A \to B$ the torus C^*-algebra $\mathfrak{T}(u, v)$ is

$$\{(a, \phi) \in A \times B[0, 1] : u(a) = \phi(0),\ v(a) = \phi(1)\}.$$

In fact the telescopic C^*-algebra $\mathcal{T}(\alpha, \beta)$ identifies with the torus C^*-algebra $\mathfrak{T}(\breve{\alpha}, \breve{\beta})$ of the morphisms $\breve{\alpha}, \breve{\beta} : \bigoplus_{k=1}^n I_k \to \bigoplus_{k=0}^n B_k$ defined by

$$\breve{\alpha}((x_k)_k) = (0, \alpha_1(x_1), \dots, \alpha_k(x_k), \dots)$$

and

$$\breve{\beta}((x_k)_k) = \begin{cases} (\beta_0(x_1), \dots, \beta_{n-1}(x_n), 0) & \text{if } n \in \mathbb{N}, \\ (\beta_k(x_{k+1}))_{k \in \mathbb{N}} & \text{if } n = +\infty. \end{cases}$$

(b) In turn, a mapping torus is easily seen to be K-equivalent to a mapping cone. Let A, B be C^*-algebras and $j_\pm : A \to B$ $*$-homomorphisms. Let $j : A(\mathbb{R}_+^*) \to B(\mathbb{R})$ be the $*$-homomorphism defined by

$$j(\phi)(t) = \begin{cases} j_+(\phi(t)) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ j_-(\phi(-t)) & \text{if } t < 0. \end{cases}$$

Then $\mathfrak{T}(j_+, j_-)(\mathbb{R}_+^*)$ is canonically isomorphic with C_j.

Indeed,

$$\mathfrak{T}(j_+, j_-)(\mathbb{R}_+^*) \\ = \big\{(\phi, \psi) \in A(\mathbb{R}_+^*) \times B(\mathbb{R}_+^* \times [0,1]) : \psi(t,0) = j_+(\phi(t)),\ \psi(t,1) = j_-(\phi(t))\big\}$$

and

$$C_j = \left\{(\phi,\psi)\in A(\mathbb{R}_+^*)\times B(\mathbb{R}\times\mathbb{R}_+) : \psi(t,0) = \begin{cases} j_+(\phi(t)) & \text{if } t>0, \\ 0 & \text{if } t=0, \\ j_-(\phi(-t)) & \text{if } t<0, \end{cases}\right\}$$
$$= \left\{(\phi,\psi)\in A(\mathbb{R}_+^*)\times B(\mathbb{R}\times\mathbb{R}_+\setminus\{(0,0)\}) : \psi(t,0) = \begin{cases} j_+(\phi(t)) & \text{if } t>0, \\ j_-(\phi(-t)) & \text{if } t<0 \end{cases}\right\}$$

are isomorphic through the homeomorphism $(r,\theta) \mapsto (r\cos\pi\theta, r\sin\pi\theta)$ from $\mathbb{R}_+^* \times [0,1]$ onto $(\mathbb{R}\times\mathbb{R}_+) \setminus \{(0,0)\}$.

3.B. ***The telescope of nicely decomposable foliations.*** Let $(M,\mathcal{F})$ be a *nicely decomposable* foliation of height $n \in \mathbb{N}\cup\{\infty\}$ in the sense of Definition 1.4. Generalizing the case $n = 1$, we construct a C^*-algebra which is E-equivalent with the (full) foliation C^*-algebra. We are thus given open subsets $(W_k)_{k<n+1}$, groupoids $\mathcal{G}_k \rightrightarrows W_k$ and morphisms $\mathcal{G}_k|_{W_k\cap W_{k-1}} \to \mathcal{G}_{k-1}$ satisfying the conditions of Definition 1.4.

Let $\Omega_k = \bigcup_{j\le k} W_j$ be the sequence of strata of this decomposition and $Y_k = \Omega_k \setminus \Omega_{k-1}$. Since $\mathcal{F}$ is assumed to be nicely decomposable, we are given Lie groupoids $\mathcal{G}_k \rightrightarrows W_k$ and morphisms of Lie groupoids $q_k : \mathcal{G}_k|_{\Omega_{k-1}} \to \mathcal{G}_{k-1}$ such that $\mathcal{G}_k|_{Y_k} = H(Y_k,\mathcal{F})$.

For every $0 \le k < n+1$ consider the full C^*-algebras $A_k = C^*(\Omega_k,\mathcal{F})$ and $B_k = C^*(\mathcal{G}_k)$ and the morphism obtained by integration along the fibers $p_k : B_k \to A_k$. Put also $Q_k = C^*(\mathcal{G}_k|_{Y_k})$. We have the diagram in Figure 6. Here the map q_k is integration along the fibers of the groupoid morphism $q_k : \mathcal{G}_{k+1}|_{\Omega_k} \to \mathcal{G}_k$ and $\pi_k = p_k \circ q_k : I_k \to A_k$. The quotient algebras B_k/I_{k-1} and A_k/A_{k-1} coincide (with Q_k).

Let $\tilde{\imath}_k : A_k \to C^*(M,\mathcal{F}) = A_n$ be the inclusion. As for every $k < n$ we have $\tilde{\imath}_k\circ\pi_k = \tilde{\imath}_k\circ p_k\circ q_k = \tilde{\imath}_{k+1}\circ p_{k+1}\circ j_{k+1}$, we get a morphism $\Psi : \mathcal{T}(q,j) \to A_n(0,n+1)$ defined by $\Psi((\phi_k)_{0\le k<n+1}, (x_k)_{1\le k<n+1})(t) = \tilde{\imath}_k \circ p_k(\phi_k(t))$ for $t \in [k,k+1]$.

Theorem 3.6. *With the above notation, the class in $E(\mathcal{T}(q,j), A_n(0,n+1))$ of the morphism Ψ is invertible.*

Proof. Let $\mathcal{B} = \{f \in A_n(0,n+1] : \forall t \in \mathbb{R}_+^*, \forall k \in \mathbb{N},\ t-1 \le k \le n \Rightarrow f(t) \in A_k\}$ and put $\mathcal{J} = \{f \in B : f(n+1) = 0\}$.

The inclusion $\mathcal{J} \to A_n(0,n+1)$ is an E-equivalence (cf. [Rosenberg and Schochet 1987] — if $n < +\infty$, it is a KK-equivalence). Its inverse is given by the exact sequence $0 \to \mathcal{J} \to \mathcal{B} \to A_n \to 0$.

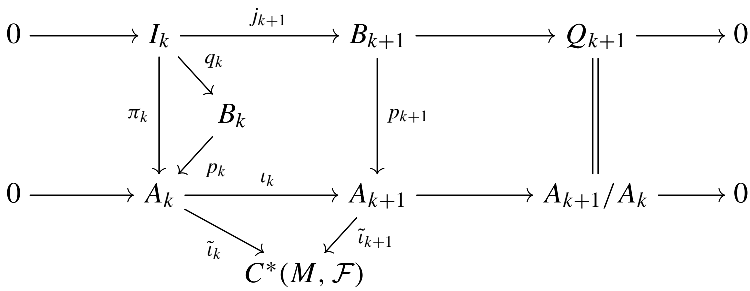

Figure 6. Short exact sequences of strata.

For $\ell \in \mathbb{N}$, $\ell \leq n$, put $\mathcal{J}_\ell = \{f \in \mathcal{J} : f(t) = 0 \text{ if } t \geq \ell + 1\}$ and let $\mathcal{T}_\ell$ be the ideal

$$\mathcal{T}_\ell = \{((\phi_k)_{0 \leq k < n+1}, (x_k)_{1 \leq k < n+1}) \in \mathcal{T} : \forall k > n,\ \phi_k = 0 \text{ and } x_k = 0\}.$$

Let us show by induction that the morphism $\Psi_\ell : \mathcal{T}_\ell \to \mathcal{J}_\ell$ induced by Ψ is an E-equivalence:

- Ψ_0 is an isomorphism (and the case $\ell = 1$ follows from the proof of Proposition 2.1).
- We have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}_{\ell-1} & \longrightarrow & \mathcal{T}_\ell & \longrightarrow & \mathcal{C}_{j_\ell} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Psi_{\ell-1}} & & \downarrow{\scriptstyle \Psi_\ell} & & \downarrow{\scriptstyle \tilde{p}_\ell} & & \\ 0 & \longrightarrow & \mathcal{J}_{\ell-1} & \longrightarrow & \mathcal{J}_\ell & \longrightarrow & \mathcal{C}_{\iota_{\ell-1}} & \longrightarrow & 0 \end{array}$$

where $\tilde{p}_\ell : \mathcal{C}_{j_\ell} \to \mathcal{C}_{\iota_{\ell-1}}$ is the morphism induced by $p_\ell : B_\ell \to A_\ell$ at the cone level.

Examining Figure 6, as j_ℓ and $\iota_{\ell-1}$ are inclusions of ideals and p_ℓ induces an isomorphism $B_\ell / I_\ell \to A_\ell / A_{\ell-1}$, we deduce that $\tilde{p}_\ell$ is E-invertible. We thus obtain the induction step.

If n is finite, the proof is complete.

If $n = +\infty$, the mapping cones $\mathcal{C}_{\Psi_\ell}$ are E-contractible for all ℓ and it follows that their inductive limit $\mathcal{C}_\Psi$ is E-contractible. □

4. Longitudinally smooth groupoids and equivariant *KK*-theory

We proved in Theorem 3.6 that the telescopic algebra $\mathcal{T}(q, j)$ has the same K-theory as $C^*(M, \mathcal{F})$. In the next section, we will build a Baum–Connes map for the telescopic algebra $\mathcal{T}(q, j)$, which will give us a Baum–Connes map for $C^*(M, \mathcal{F})$.

The telescopic algebra $\mathcal{T}(q, j)$ associated with a nicely decomposable foliation, as well as the foliation algebra itself (thanks to [Debord 2013]), is the C^*-algebra of a *longitudinally smooth groupoid* in a sense that we briefly describe here.

Note that all the constructions we give below generalize easily to groupoids that are covered by $C^{\infty,0}$ manifolds or even locally compact spaces with Haar measures.

4.A. ***Some "classical" constructions with Lie groupoids.*** Before proceeding to explain this construction, we recall some constructions based on Lie groupoids that we use.

Pull-back groupoid. Let $G \overset{t,s}{\rightrightarrows} G^{(0)}$ be a Lie groupoid, M a smooth manifold and $q : M \to G^{(0)}$ a smooth submersion. The pull-back groupoid G_q^q is a subgroupoid

$$G_q^q = \{(x, \gamma, y) \in M \times G \times M : q(x) = t(\gamma) \text{ and } q(y) = s(\gamma)\}$$

of the product groupoid of G with the pair groupoid $M \times M$. As q is supposed to be a submersion, G_q^q is a Lie groupoid (actually, a transversality condition suffices). If $q(M)$ meets all the G-orbits, the groupoids G and G_q^q are canonically Morita equivalent.

Actions on spaces. Recall that an action of a groupoid $G \overset{t,s}{\rightrightarrows} G^{(0)}$ on a space X is given by a map $p : X \to G^{(0)}$ and the action $G \times_{s,p} X \to X$ denoted by $(\gamma, x) \mapsto \gamma.x$ with the requirements $p(\gamma.x) = t(\gamma)$, $\gamma.(\gamma'.x) = (\gamma\gamma').x$ and $u.x = x$ if $u = p(x)$.

Semidirect product. If a groupoid G acts on a space X, we may form the semidirect product groupoid $X \rtimes G$:

- As a set $X \rtimes G = X \times_t G = \{(x, \gamma) \in X \times G : t(\gamma) = x\}$.
- $(X \rtimes G)^{(0)} = X$; we have $t(x, \gamma) = x$ and $s(x, \gamma) = \gamma^{-1}.x$.
- The elements (x, γ) and (y, γ') are composable if $x = \gamma y$; the composition is then $(x, \gamma)(y, \gamma') = (x, \gamma\gamma')$.

When p is a submersion, $X \rtimes G$ is a Lie groupoid: it is the closed subgroupoid $\{(x, \gamma, y) \in G_p^p : x = \gamma.y\}$ of G_p^p.

Actions on groupoids. This construction can be generalized. If $X \overset{t_X,s_X}{\rightrightarrows} X^{(0)}$ is a groupoid, we say that the action is by groupoid automorphisms [Brown 1972] if G acts on $X^{(0)}$ through a map $p_0 : X^{(0)} \to G^{(0)}$, we have $p = p_0 \circ t_X = p_0 \circ s_X$ and $\gamma.(xy) = (\gamma.x)(\gamma.y)$. There is a semidirect product construction in this generalized setting.

4.B. ***Longitudinally smooth groupoids.*** A *longitudinally smooth groupoid* is a groupoid $G \overset{t,s}{\rightrightarrows} G^{(0)}$ such that

- its set of objects is endowed with a structure of smooth manifold (possibly with boundary or corners);
- for every $x \in G^{(0)}$, the set $G^x = t^{-1}(\{x\})$ also carries a smooth structure (without boundary) and the source map $s : G^x \to G^{(0)}$ is smooth with (locally) constant rank;

- the "smooth structure" of G itself is given by an *atlas*, which is a family of smooth (Hausdorff) manifolds $(U_i)_{i\in I}$ (possibly with boundary or corners) and maps $q_i : U_i \to G$.

We assume that these smooth structures satisfy the following conditions:

Compatibility. For every $i \in I$, the maps $t \circ q_i$ and $s \circ q_i$ are smooth submersions; for every $i \in I$ and every $x \in G^{(0)}$ the map q_i induces a smooth submersion $q_i^{-1}(G^x) \to G^x$.

Minimal elements. For every $\gamma \in G$, there exists $i \in I$ and $z \in U_i$ such that $q_i(z) = \gamma$ and the map $q_i^{-1}(G^{t(\gamma)}) \to G^{t(\gamma)}$ is a local diffeomorphism near z. If $j \in I$ and $z' \in U_j$ are such that $q_j(z') = \gamma$, then there is an open neighborhood $V' \subset U_j$ of z' and a submersion $\varphi : V' \to U_i$ such that $q_i \circ \varphi = (q_j)_{|V'}$.

Inverse is smooth. For every $i \in I$, there exists $j \in I$ and a diffeomorphism $\kappa : U_i \to U_j$ such that $q_j \circ \kappa(z) = q_i(z)^{-1}$ for every $z \in U_i$.

Composition is smooth. For every $i, j \in I$, let $U_i \circ U_j$ be the fibered product $U_i \times_{s\circ q_i, t\circ q_j} U_j$. For every $(z_i, z_j) \in U_i \circ U_j$, there is a $k \in I$, a neighborhood W of (z_i, z_j) in $U_i \circ U_j$ and a submersion $\varphi : W \to U_k$ such that for all $(w_i, w_j) \in W$ we have $q_i(w_i)q_j(w_j) = q_k \circ \varphi(w_i, w_j)$.

Exactly as in [Androulidakis and Skandalis 2009], we may associate to a longitudinally smooth groupoid a C^*-algebra $C^*(G)$ (as well as a reduced one, since the s-fibers are assumed to be manifolds).

Examples 4.1. (a) A Lie groupoid is of course a longitudinally smooth groupoid. The atlas is the groupoid itself!

(b) The holonomy groupoid of a singular foliation is such a longitudinally smooth groupoid; the atlas is given by bisubmersions [Androulidakis and Skandalis 2009].

(c) The telescopic algebra $\mathcal{T}(q, j)$ constructed in the previous section is associated with the groupoid $G = \bigcup_{k=0}^{n} \mathcal{G}_k \times (k, k+1) \cup \bigcup_{k=1}^{n} (\mathcal{G}_k)_{\Omega_{k-1}\cap W_k} \times \{k\}$. Its set of objects is the open subset

$$\left(\bigcup_{k=0}^{n} W_k \times (k, k+1)\right) \cup \left(\bigcup_{k=1}^{n} (\Omega_{k-1} \cap W_k) \times (k-1, k+1)\right)$$

of $M \times \mathbb{R}_+^*$.

It is endowed with the atlas formed by the Lie groupoids $(\mathcal{G}_k \times (k, k+1))_{k\in\mathbb{N},\, k\le n}$ and $((\mathcal{G}_k)_{\Omega_{k-1}\cap W_k} \times (k-1, k+1))_{k\in\mathbb{N},\, 1\le k\le n}$.

4.C. ***Action of a longitudinally smooth groupoid on a C*-algebra.*** We now fix a longitudinally smooth groupoid $G \overset{t,s}{\rightrightarrows} M$ with an atlas $(U_i, q_i)_{i\in I}$. We put $s_i = s \circ q_i$ and $t_i = t \circ q_i$.

4.C.1. *Action of a locally compact groupoid on a C^*-algebra.* For the convenience of the reader we recall some definitions on $C(X)$-algebras and actions of locally compact groupoids from [Kasparov 1988; Le Gall 1999]. In this section, all spaces — and groupoids — are assumed to be Hausdorff.

Let M be a locally compact space.

(a) A $C_0(M)$-algebra is a pair (A, θ), where A is a C^*-algebra and θ is a $*$-homomorphism from $C_0(M)$ to the center $\mathcal{Z}\mathcal{M}(A)$ of the multiplier algebra of A such that $\theta(C_0(M))A = A$.

(b) Put $A_b = \{a \in \mathcal{M}(A) : \phi a \in A \text{ for all } \phi \in C_0(X)\}$ and $A_c = C_c(X)A$.

(c) Let A, B be $C_0(M)$-algebras. A homomorphism of $C_0(M)$-algebras $\phi : A \to B$ is a $C_0(M)$-linear homomorphism of C^*-algebras.

(d) Let N be a locally compact space and $p : N \to M$ a continuous map. Then $C_0(N)$ is a $C_0(M)$-algebra via the map $\theta = p^* : C_0(M) \to C_b(N) = \mathcal{M}(C_0(M))$.

Let A be a $C_0(M)$-algebra.

(e) For every $x \in M$ there is a fiber $A_x = A/C_x A$, where $C_x = \{h \in C_0(M) : h(x) = 0\}$. The natural map $A \to \prod_{x \in M} A_x$ induced by the quotient maps $\pi_x : A \to A_x$ is injective. For instance, in (d), given $x \in M$ the fiber $C_0(N)_x$ is $C_0(N_x)$, where $N_x = p^{-1}(x)$.

(f) A homomorphism of $C_0(M)$-algebras $\phi : A \to B$ induces a homomorphism of C^*-algebras $(\phi_x)_{x \in M} : \prod_{x \in M} A_x \to \prod_{x \in M} B_x$. The homomorphism ϕ is injective (surjective) if and only if ϕ_x is injective (surjective) for every $x \in M$.

(g) There are natural operations of restriction to open and closed subsets of M. If U is an open subset of M and $F = X \setminus U$, the algebra $C_0(U)$ identifies with the ideal $C_0(U) = \{f \in C_0(M) : f(y) = 0 \text{ for all } y \in F\}$ of $C_0(M)$. Then A_U denotes the $C_0(U)$-algebra $C_0(U)A$ and A_F the $C_0(F)$-algebra A/A_U. If $Y \subset X$ is locally closed, then Y is open in $\overline{Y}$ and A_Y denotes the $C_0(Y)$-algebra $(A_{\overline{Y}})_Y$.

(h) Let A be a $C_0(M)$ and B a $C_0(N)$-algebra. Then $A \otimes_{\max} B$ is a $C_0(M \times N)$-algebra. When $M = N$, the restriction of $A \otimes_{\max} B$ to the diagonal $\{(x, x) : x \in M\}$ (which is a closed subset of $M \times M$) is a $C_0(M)$-algebra denoted $A \otimes_{C_0(M)} B$.

(i) Again let A be a $C_0(M)$-algebra and consider a smooth map $p : N \to M$. We denote by p^*A the $C_0(N)$-algebra obtained by restricting $A \otimes C_0(N)$ to the graph $\{(p(y), y) : y \in N\}$, which is a closed subset of $M \times N$ (here $C_0(N)$ is regarded as a $C_0(N)$-algebra). It is easy to see that this construction has the following properties:

- $(p^*A)_y = A_{p(y)}$ for every $y \in N$;
- if A, B are $C_0(M)$-algebras then $p^*A \otimes_{C_0(Y)} p^*B = p^*(A \otimes_{C_0(M)} B)$;
- if $q : Z \to N$ is a smooth map then $q^*(p^*A) = (p \circ q)^*A$.

(j) With the previous notation, for every $a \in A$ we put $p^*a = a \otimes 1 \in (p^*A)_b$. If $\phi : A \to B$ is a homomorphism of $C_0(M)$-algebras we put

$$p^*\phi = \phi \otimes \mathrm{id}_{C_0(N)} : p^*A \to p^*B.$$

(k) An action of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on a $C_0(M)$-algebra A is defined in [Le Gall 1999] by an isomorphism α of $C_0(\mathcal{G})$-algebras $s^*A \to t^*A$. This isomorphism is given by a family of isomorphisms $\alpha_\gamma : A_{s(\gamma)} \to A_{t(\gamma)}$ for $\gamma \in \mathcal{G}$. The isomorphism α is required to be a representation of $\mathcal{G}$, i.e., to satisfy $\alpha_{\gamma\circ\gamma'} = \alpha_\gamma \circ \alpha_{\gamma'}$ for all $(\gamma, \gamma') \in \mathcal{G}^{(2)} = \mathcal{G} \times_{s,t} \mathcal{G}$.

4.C.2. *Action of a longitudinally smooth groupoid.* Let $G \overset{t,s}{\rightrightarrows} M$ be a longitudinally smooth groupoid with an atlas $(U_i, q_i)_{i\in I}$. We put $s_i = s \circ q_i$ and $t_i = t \circ q_i$.

Definition 4.2. A G-algebra is a $C_0(M)$-algebra A together with an isomorphism of $C_0(U_i)$-algebras $\alpha^i : s_i^*A \to t_i^*A$ for every $i \in I$.

(a) The isomorphism α^i is a family $(\alpha^i_u)_{i\in I}$ of isomorphisms $\alpha^i_u : A_{s_i(u)} \to A_{t_i(u)}$. We require that if $\gamma \in G$ is represented by two elements $u_i \in U_i$ and $u_j \in U_j$ (with $i, j \in I$), then $\alpha^i_{u_i} = \alpha^j_{u_j}$.

(b) By (a), we get a well-defined isomorphism $\alpha_\gamma : A_{s(\gamma)} \to A_{t(\gamma)}$. We require that for every composable $\gamma, \gamma' \in G$, we have $\alpha_{\gamma\gamma'} = \alpha_\gamma \circ \alpha_{\gamma'}$.

Definition 4.3. Let (A, α) and (B, β) be G-algebras.

(a) A morphism $\phi : A \to B$ is said to be G-equivariant if it is $C_0(M)$-linear and for every $\gamma \in G$ we have $\phi_{t(\gamma)} \circ \alpha(\gamma) = \beta(\gamma) \circ \phi_{s(\gamma)}$.

(b) More generally, let $\phi : A \to \mathcal{M}(B)$ be a morphism. Let

$$D = G(\phi) \oplus (0 \oplus B) \subset A \oplus \mathcal{M}(B),$$

where $G(\phi) = \{(x, \phi(x)) : x \in A\}$ is the graph of ϕ. We say that ϕ is equivariant if there is an action of G on D such that the inclusions $A \to D$ and $B \to D$ are equivariant.

Examples 4.4. (a) The algebra $C_0(M)$ is a G-algebra. For every $i \in I$, we have $t^*(C_0(M)) = C_0(U) = s^*(C_0(M))$; the action α is the identity. For $i \in I$, then at every $u \in U_i$ we associate the identity map $(\mathbb{C} \to \mathbb{C})$. In a sense, this corresponds to the trivial representation.

(b) More generally, let $Y \subset M$ be a locally closed saturated subset (i.e., such that for every $\gamma \in G$ we have $t(\gamma) \in Y$ if and only if $s(\gamma) \in Y$). Then $C_0(Y)$ is an $H(\mathcal{F})$-algebra. In that case, for every $i \in I$, we have $t^{-1}(Y) = s^{-1}(Y)$ since Y is saturated and $t^*(C_0(Y)) = C_0(t^{-1}(Y)) = C_0(s^{-1}(Y)) = s^*(C_0(Y))$. Again, the action α is the identity.

4.C.3. *Covariant representations and full crossed products.* Let us very briefly extend some constructions of [Androulidakis and Skandalis 2009, §4 and 5] to the more general case of our longitudinally smooth groupoid $G \rightrightarrows M$ with atlas $(U_i, q_i)_{i\in I}$.

• When $f : N \to M$ is a smooth submersion of manifolds, we may define a Hilbert $C_0(M)$-module $\mathcal{E}_f$ obtained by completion of the space $C_c(N; \Omega^{1/2}\ker(df))$ with respect to the $C_0(M)$-valued inner product defined by $\langle\xi, \eta\rangle(x) = \int_{z\in f^{-1}(x)} \overline{\xi(z)}\eta(z)$. This Hilbert module is endowed with an action of $C_0(N)$.

• Let $i \in I$. We may then construct two Hilbert C^*-modules $\mathcal{E}_{t_i}$ and $\mathcal{E}_{s_i}$ over $C_0(M)$.

• As $C_0(M)$ sits in the multiplier algebra of $C^*(G)$, every representation π_G of $C^*(G)$ on a Hilbert space $\mathcal{H}$ gives rise to a representation π_M of $C_0(M)$.

• The representation π is then characterized by π_M and, for every $i \in I$, a unitary $V_i \in \mathcal{L}(\mathcal{E}_{s_i} \otimes_{C_0(M)} \mathcal{H}, \mathcal{E}_{t_i} \otimes_{C_0(M)} \mathcal{H})$ intertwining the representations of $C_0(U)$. It therefore defines a measurable family of unitaries $U_u : H_{s_i(u)} \to H_{t_i(u)}$. We require that U_u only depends on the class of $q_i(u)$ in G (almost everywhere) and that (almost everywhere) it determines a representation of the groupoid G. See [Androulidakis and Skandalis 2009, §5.2] for the details.

Let G act on a C^*-algebra A and let π_A be a representation of A on a Hilbert space $\mathcal{H}$. Using the morphism from $C_0(M)$ to the multiplier algebra of A, we obtain a representation of $C_0(M)$ to $\mathcal{L}(\mathcal{H})$. For every $i \in I$, as the image of $C_0(M)$ sits in the center of the multiplier algebra of A, we have representations

$$\pi_A^{s_i} : s_i^*(A) \to \mathcal{L}(\mathcal{E}_{s_i} \otimes_{C_0(M)} \mathcal{H}) \quad \text{and} \quad \pi_A^{t_i} : t_i^*(A) \to \mathcal{L}(\mathcal{E}_{t_i} \otimes_{C_0(M)} \mathcal{H}).$$

A *covariant representation* of G and A is given by a representation of π_G of $C^*(G)$ and a representation π_A of A in the same Hilbert space $\mathcal{H}$ such that the two representations of $C_0(M)$ agree and, for every $i \in I$, the unitary V_i intertwines $\pi_A^{s_i} \circ \alpha^i$ with $\pi_A^{t_i}$.

Then the closed linear span of $\pi_A(a)\pi_G(x)$, where a runs over A and x over $C^*(G)$, is a $*$-subalgebra of $\mathcal{L}(\mathcal{H})$.

Definition 4.5. The *full crossed product* $A \rtimes G$ is the completion of this linear span with respect to the supremum norm over all covariant representations.

Using the "regular representations" on $L^2(G_x)$, one may also construct a natural reduced crossed product.

4.C.4. *Actions of a longitudinally smooth groupoid on Hilbert modules.* Let G, $(U_i)_{i\in I}$, s_i, t_i be as above.

Let (A, α) be a G-algebra and $\mathcal{E}$ a Hilbert module over A. As usual, we may define an action of G on $\mathcal{E}$ by saying that it is just given by an action of G on the C^*-algebra $\mathcal{K}(\mathcal{E} \oplus A)$ in such a way that the natural morphism $A \to \mathcal{K}(\mathcal{E} \oplus A)$ is equivariant. This amounts to giving, for any $i \in I$, an isomorphism

$$\widetilde{\alpha} : \mathcal{E} \otimes_A s_i^* A \to \mathcal{E} \otimes_A t_i^* A$$

of Banach spaces, which corresponds to a family of isomorphisms $\widetilde{\alpha}_u : \mathcal{E}_{s_i(u)} \to \mathcal{E}_{t_i(u)}$. We need compatibility with α, which means that for every $x \in A_{s_i(u)}$ and $\xi, \zeta \in \mathcal{E}_{s_i(u)}$, we have $\widetilde{\alpha}_u(\xi x) = \widetilde{\alpha}_u(\xi)\alpha_u(x)$ and $\alpha_u(\langle \xi | \zeta \rangle) = \langle \widetilde{\alpha}_u(\xi) | \widetilde{\alpha}_u(\zeta) \rangle$.

As above, we require that $\widetilde{\alpha}_u$ only depends on the class of u in G and that the so defined $\widetilde{\alpha}_\gamma : \mathcal{E}_{s(\gamma)} \to \mathcal{E}_{t(\gamma)}$ for $\gamma \in H(\mathcal{F})$ defines a morphism of groupoids, which means that $\widetilde{\alpha}_{\gamma\gamma'} = \widetilde{\alpha}_\gamma \tilde{\alpha}_{\gamma'}$. Note also that, given an action of G on a $\mathcal{E}$, we obtain for any $i \in I$ an isomorphism of $C_0(U_i)$-algebras $\mathcal{K}(\mathcal{E} \otimes_A s_i^* A) \to \mathcal{K}(\mathcal{E} \otimes_A t_i^* A)$ and of their multipliers $\check{\alpha}_U : \mathcal{L}(\mathcal{E} \otimes_A s_i^* A) \to \mathcal{L}(\mathcal{E} \otimes_A t_i^* A)$.

4.D. ***G-equivariant KK-theory.*** Let $G \overset{t,s}{\rightrightarrows} M$ be a longitudinally smooth groupoid with an atlas $(U_i, q_i)_{i \in I}$. We put $s_i = s \circ q_i$ and $t_i = t \circ q_i$.

Here we use the apparatus developed in the previous sections to construct the topological K-theory at the left-hand side of the Baum–Connes conjecture in classical terms (e.g., as in [Le Gall 1999]). Namely we define the groups $KK_G(A, B)$ in Section 4.D.1. The difficulty is to construct the Kasparov product; we do this in Section 4.D.3.

4.D.1. *Equivariant Kasparov cycles.* We may of course define graded G-algebras, graded Hilbert modules, etc.

In what follows, all algebras are $\mathbb{Z}/2\mathbb{Z}$-graded and all commutators are graded ones. Also, all the C^*-algebras and Hilbert C^*-modules that we consider are supposed to be separable. Recall the following from [Kasparov 1980]:

- Let A, B be graded C^*-algebras. An (A, B) bimodule is a pair $(\mathcal{E}, \pi_A)$, where $\mathcal{E}$ is a B-Hilbert C^*-module and $\pi_A : A \to \mathcal{L}(\mathcal{E})$ a representation which preserves the degree. For every $\xi \in \mathcal{E}$ and $a \in A$ we denote $a\xi = \pi_A(a)(\xi)$.
- A Kasparov (A, B) bimodule is a triple $(\mathcal{E}, \pi_A, F)$, where $(\mathcal{E}, \pi_A)$ is an (A, B) bimodule and $F \in \mathcal{L}(\mathcal{E})$ is of degree 1 (for the grading) and for all $a \in A$, the elements $[F, \pi_A(a)]$, $(F - F^*)\pi_A(a)$ and $(1 - F^2)\pi_A(a)$ are all in $\mathcal{K}(\mathcal{E})$.

Definition 4.6. Let (A, B) be G-algebras. A *G-equivariant Kasparov (A, B) bimodule* is a Kasparov (A, B) bimodule $(\mathcal{E}, \pi_A, F)$ with the following properties.

(a) $\mathcal{E}$ is endowed with an action of G (see Section 4.C.4) and the representation $\pi_A : A \to \mathcal{L}(\mathcal{E}) = \mathcal{M}(\mathcal{K}(\mathcal{E}))$ is G-equivariant (in the sense of Definition 4.3(b)).

(b) For every $i \in I$ and $h \in C_0(U_i)$, we have $(\check{\alpha}_i(F \otimes 1) - F \otimes 1)h \in \mathcal{K}(\mathcal{E} \otimes_A t_i^* A)$.

Two G-equivariant Kasparov bimodules $(\mathcal{E}, \pi_A, F)$, $(\mathcal{E}', \pi'_A, F')$ are *unitarily equivalent* if there exists a G-equivariant unitary $U \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ of degree 0 which satisfies $UFU^* = F'$ and $U\pi_A(a)U^* = \pi'_A(a)$ for all $a \in A$.

Denote by $E_G(A, B)$ the set of equivalence classes of G-equivariant Kasparov bimodules. A homotopy in $E_G(A, B)$ is an element of $E_G(A, B[0, 1])$. We define $KK_G(A, B)$ to be the set of homotopy classes of elements of $E_G(A, B)$.

The direct sum of Kasparov bimodules induces an abelian group structure in $KK_G(A, B)$. We define the unit element $1_A \in KK_G(A, A)$ as the class of $(A, \iota_A, 0)$, where $\iota_A(a) = a \in \mathcal{K}(A)$ for all $a \in A$, where the action of G on the C^*-module A is the action of G on the C^*-algebra A.

4.D.2. *Kasparov's descent morphism.* Given an equivariant Hilbert B module $\mathcal{E}$, we may define the crossed product $\mathcal{E} \rtimes G = \mathcal{E} \otimes_B B \rtimes G$ — and the same for the reduced crossed product. If we have an equivariant action $A \to \mathcal{L}(E)$, we naturally obtain an action $A \rtimes G \to \mathcal{L}(\mathcal{E} \rtimes G)$.

Let $(\mathcal{E}, F)$ be an equivariant Kasparov (A, B) bimodule. Let

$$F \widehat{\otimes} 1 \in \mathcal{L}(\mathcal{E} \otimes_B B \rtimes G) = \mathcal{L}(\mathcal{E} \rtimes G).$$

We check as in [Le Gall 1999] that $(\mathcal{E} \rtimes G, F \widehat{\otimes} 1)$ is a Kasparov $(A \rtimes G, B \rtimes G)$ bimodule. This construction gives a well-defined *descent morphism*

$$j_G : KK_G(A, B) \to KK(A \rtimes G, B \rtimes G).$$

In the same way we also obtain a *reduced descent morphism.*

4.D.3. *Kasparov product — a general approach.* In order to define the Kasparov product in this equivariant context, we need first to understand the analogue of Kasparov's "technical theorem" [1980, §3, Theorem 4]. It turns out that, in a sense, the original theorem actually applies when formulated in a slightly different way. In turn, this formulation contains many equivariant generalizations.

We start by recalling Voiculescu's theorem on quasicentral approximate units [Voiculescu 1990; Arveson 1977].

Lemma 4.7. *Let D_1 be a C^*-algebra and $D_2 \subset D_1$ be a closed essential two-sided ideal. Let $h \in D_1$ be a strictly positive element with $\|h\| \le 1$. Let $b \in D_1$ and let $K \subset \mathcal{M}(D_2)$ be a (norm) compact subset such that $[h, k] \in D_1$ for all $k \in K$; let $\varepsilon > 0$. Let $f_0 : [0, 1] \to [0, 1]$ be a continuous function such that $f(0) = 0$. Then there exists continuous $f : [0, 1] \to [0, 1]$ such that $f(0) = 0$, $f_0 \le f$, $\|b - f(h)b\| < \varepsilon$ and $\|[f(h), k]\| < \varepsilon$ for all $k \in K$.* □

The following result is in fact proved in [Kasparov 1980, §3, Theorem 4]. Formulated in this way, it further contains many generalizations of the Kasparov product [Kasparov 1988; Baaj and Skandalis 1989; Le Gall 1999]. One immediately sees that Higson's proof [1987] applies, so we omit it.

If J is a closed two-sided ideal in a C^*-algebra B, then

$$\mathcal{M}(B; J) = \{x \in \mathcal{M}(B) : xB \subset J\}.$$

Theorem 4.8. (cf. [Kasparov 1980, §3, Theorem 4]) *Let D_1 be a separable graded C^*-algebra and D_2 a graded closed essential two-sided ideal in D_1. Consider $b \in \mathcal{M}(D_1; D_2)_+$. Let also A_1 be a graded C^*-subalgebra of D_1 containing a strictly positive element of D_1 and such that $A_2 = A_1 \cap D_2$ contains a strictly positive element of D_2. Let $K \subset \mathcal{M}(D_2)$ be a compact subset such that, for every $x \in A_1$ and every $k \in K$, we have $[x, k] \in D_1$. Then there exists $M \in \mathcal{M}(A_1; A_2)^{(0)}$ such that $0 \le M \le 1$, $(1 - M)b \in D_2$ and $[M, K] \subset D_2$.* □

One obtains easily a formulation which encodes many equivariant formulations of the product.

Notation 4.9. Let D be a separable graded C^*-algebra and $A \subset D$ a subalgebra containing a strictly positive element of $D^{(0)}$.

Let $\mathcal{I}$ denote the set of graded closed two-sided essential ideals I of D such that $I \cap A^{(0)}$ contains a strictly positive element of I.

Let $D_1, D_2 \in \mathcal{I}$ be such that $D_2 \subset D_1$. Put $A_i = D_i \cap A$. Denote by $\mathbb{E}_A(D_1, D_2)$ the set of $F \in \mathcal{M}(A_2)^{(1)}$ such that for all $x \in D_1$, we have

$$x(1 - F^2) \in D_2, \quad x(F - F^*) \in D_2, \quad [x, F] \in D_2.$$

In other words (A_2, F) is a Kasparov (A_1, A_2) bimodule and (D_2, F) is a Kasparov (D_1, D_2) bimodule.

Theorem 4.10. *Let $D_0, D_1, D_2 \in \mathcal{I}$ such that $D_2 \subset D_1 \subset D_0$. Let $F_1 \in \mathbb{E}_A(D_0, D_1)$ and $F_2 \in \mathbb{E}_A(D_1, D_2)$. Let*

$$F_1 \sharp F_2 = \{F \in \mathbb{E}_A(D_0, D_2) : F - F_2 \in \mathcal{M}(A_1; A_2),\ [F, F_1] \in \mathcal{M}(A_2)_+ + A_2\}.$$

(a) *For every $F_1 \in \mathbb{E}_A(D_0, D_1)$ and $F_2 \in \mathbb{E}_A(D_1, D_2)$ the set $F_1 \sharp F_2$ is nonempty and path connected.*

(b) *The path connected component of $F \in F_1 \sharp F_2$ in $\mathbb{E}_A(D_0, D_2)$ only depends on the path connected components of $F_1 \in \mathbb{E}_A(D_0, D_1)$ and of $F_2 \in \mathbb{E}_A(D_1, D_2)$.*

(c) *(Associativity). Let $D_3 \in \mathcal{I}$ with $D_3 \subset D_2$ and $F_3 \in \mathbb{E}_A(D_2, D_3)$. Let $F_1' \in F_1 \sharp F_2$ and $F_2' \in F_2 \sharp F_3$. Then $F_1' \sharp F_3$ and $F_1 \sharp F_2'$ are contained in the same path connected component of $\mathbb{E}_A(D_0, D_3)$.*

Proof. The proof is exactly the same as in the "classical" case (cf. [Kasparov 1980; Connes and Skandalis 1984; Skandalis 1984b]).

For instance, to establish that $F_1 \sharp F_2$ is nonempty, we take $Q = C^*(D_0, F_1, F_2)$. Let K be a compact subset of Q generating Q as a closed space and let b be a strictly positive element of $Q \cap \mathcal{M}(D_1; D_2)$. Apply then Theorem 4.8, and put $F = M^{1/2} F_1 + (1 - M)^{1/2} F_2$.

If we start with paths $F_1^t \in \mathbb{E}_A(A_0, A_1)$, $F_2^t \in \mathbb{E}_A(A_1, A_2)$, we just take a bigger algebra: $Q = C^*(A_0, \{F_1^t, F_2^t : t \in [0, 1]\})$. The associativity is proved exactly as Lemma 22 in [Skandalis 1984b]. □

We now introduce further notation in order to relate this theorem with equivariant KK-theory.

Notation 4.11. Let $A, \mathbb{A}$ be separable graded C^*-algebras. Let $\varphi, \psi : \mathcal{M}(A) \to \mathcal{M}(\mathbb{A})$ be two grading-preserving strictly continuous morphisms.

Let $\mathcal{J}$ denote the set of closed two-sided essential ideals I of A such that $\varphi(I)\mathbb{A} = \psi(I)\mathbb{A}$.

Let $A_1, A_2 \in \mathcal{J}$ be such that $A_2 \subset A_1$. Put $\mathbb{A}_i = \varphi(A_i)\mathbb{A}$. Denote by $\mathbb{E}_{\varphi,\psi}(A_1, A_2)$ the set of $F \in \mathcal{M}(A_2)^{(1)}$ such that

(a) for all $x \in A_1$, we have $x(1 - F^2) \in A_2$, $x(F - F^*) \in A_2$, $[x, F] \in A_2$ (in other words (A_2, F) is a Kasparov (A_1, A_2) bimodule);

(b) $(\varphi - \psi)(F) \in \mathcal{M}(\mathbb{A}_1; \mathbb{A}_2)$ (the "equivariance property").

As an immediate consequence of Theorem 4.10, we have:

Corollary 4.12. (a) *For every* $F_1 \in \mathbb{E}_{\varphi,\psi}(A_0, A_1)$ *and* $F_2 \in \mathbb{E}_{\varphi,\psi}(A_1, A_2)$ *the set* $F_1 \sharp F_2$ *is nonempty and path connected.*

(b) *The path connected component of* $F \in F_1 \sharp F_2$ *only depends on the path connected components of* $F_1 \in \mathbb{E}_{\varphi,\psi}(A_0, A_1)$ *and of* $F_2 \in \mathbb{E}_{\varphi,\psi}(A_1, A_2)$.

(c) (*Associativity*). *Let* $A_3 \in \mathcal{I}$ *with* $A_3 \subset A_2$ *and* $F_3 \in \mathbb{E}_{\varphi,\psi}(A_2, A_3)$. *Let* $F_1' \in F_1 \sharp F_2$ *and* $F_2' \in F_2 \sharp F_3$. *Then* $F_1' \sharp F_3$ *and* $F_1 \sharp F_2'$ *are contained in the same path connected component of* $\mathbb{E}_{\varphi,\psi}(A_0, A_3)$.

Proof. Let $\chi : \mathcal{M}(A) \to \mathcal{M}(A \oplus M_2(\mathbb{A}))$ be the morphism

$$x \mapsto x \oplus \begin{pmatrix} \varphi(x) & 0 \\ 0 & \psi(x) \end{pmatrix}$$

and put $D = \chi(A) + (0 \oplus M_2(\mathbb{A})) \subset \mathcal{M}(A \oplus M_2(\mathbb{A}))$.

Let $A_1, A_2 \in \mathcal{J}$ be such that $A_2 \subset A_1$. Put

$$\mathbb{A}_i = \varphi(A_i)\mathbb{A} \quad \text{and} \quad D_i = \chi(A_i) + (0 \oplus M_2(\mathbb{A}_i)) \subset D.$$

We obviously have $\mathbb{E}_{\varphi,\psi}(A_1, A_2) = \mathbb{E}_{\chi(A)}(D_1, D_2)$. Therefore, Theorem 4.10 immediately applies. □

Examples 4.13. It is very easy to apply this abstract theorem (Corollary 4.12) to many equivariant situations.

(a) (see [Kasparov 1988]) If a second countable locally compact group G acts on separable C^*-algebras A and B, an equivariant Kasparov (A, B) bimodule is then a pair $(\mathcal{E}, F)$ where:

1. $\mathcal{E}$ is an (A, B)-equivariant Hilbert bimodule;
2. $F \in \mathbb{E}_{\varphi,\psi}(A_1, A_2)$, where we have put
   - $A_2 = \mathcal{K}(\mathcal{E})$ and $A_1 = A + A_2$,
   - $\mathbb{A}_i = C_0(G; A)$,
   - $\varphi, \psi : A_i \to C_b(G; A_i) \subset \mathcal{M}(\mathbb{A}_i)$ defined by $\varphi(a)(g) = a$ and $\psi(a)(g) = g.a$.

(b) (see [Baaj and Skandalis 1989]) Exactly in the same way, if S is a separable Hopf algebra, given a C^*-algebra A with an action $\alpha : A \to \mathcal{M}(A \otimes S)$ of S, we just put $\mathbb{A} = A \otimes S$ and let $\varphi : a \mapsto a \otimes 1$ and $\psi = \alpha$.

(c) (see [Le Gall 1999]) If $G \overset{s,t}{\rightrightarrows} G^{(0)}$ is a second countable locally compact groupoid, given a C^*-algebra A with an action $\alpha : s^*A \to t^*A$ of G, we just put $\mathbb{A} = t^*A$ and let $\varphi : a \mapsto t^*a \in \mathcal{M}(t^*A)$ and $\psi(a) = \alpha(s^*a)$.

4.D.4. *Kasparov product in KK_G.* Let G be a longitudinally smooth groupoid with atlas $(U_i)_{i \in I}$. We assume that I is countable.

Let A_0, A_1, A_2 be G-algebras. Let $(\mathcal{E}_1, F_1)$ and $(\mathcal{E}_2, F_2)$ be equivariant (A_0, A_1) and (A_1, A_2) cycles. Put $\mathcal{E} = \mathcal{E}_1 \widehat{\otimes}_{A_1} \mathcal{E}_2$. Put $F_1' = F_1 \widehat{\otimes} 1$ and let F_2' be an F_2 connection. Put $\check{A}_2 = \mathcal{K}(\mathcal{E})$, $\check{A}_1 = \mathcal{K}(\mathcal{E}_1) \widehat{\otimes} 1 + \check{A}_2$ and $\check{A}_0 = A_0 + \check{A}_1$ (where we denote by A_0 its image in $\mathcal{L}(\mathcal{E})$).

The algebras $\check{A}_i$ are G algebras, the inclusions $\check{A}_2 \subset \check{A}_1 \subset \check{A}_0$ are equivariant and the pairs $(\check{A}_1, F_1')$ and $(\check{A}_2, F_2')$ are equivariant $(\check{A}_0, \check{A}_1)$ and $(\check{A}_1, \check{A}_2)$ cycles.

Let
$$(U, q) = \coprod_{j \in I} (U_j, q_j).$$

Let $\hat{t} = t \circ q$ and $\hat{s} = s \circ q$. The action of G on $\check{A}_0$ gives a map $\alpha : \hat{s}^*(\check{A}_0) \to \hat{t}^*(\check{A}_0)$. Put $\mathbb{A}_i = \hat{t}^*(\check{A}_i)$. Let $\varphi : \check{A}_0 \to \mathcal{M}(\mathbb{A}_0)$ be the natural map $\hat{t}^*$ defined by $\varphi(x)_u = x_{\hat{t}(u)}$ for all $u \in U$. Let $\psi : \check{A}_0 \to \mathcal{M}(\mathbb{A}_0)$ be the composition of α with the map $\hat{s}^*$. In other words, $\psi(x)_u = \alpha_u(x_{\hat{s}(u)})$. Let also $q \in C_0(U)$ be a strictly positive function.

The equivariance condition means exactly that $(\varphi - \psi)(F_i') \in \mathcal{M}(\mathbb{A}_{i-1}; \mathbb{A}_i)$. We thus may apply Theorem 4.10 and obtain the existence of the Kasparov product in KK_G with the usual properties:

Theorem 4.14. *There is a well-defined bilinear product*
$$KK_G(A_0, A_1) \times KK_G(A_1, A_2) \to KK_G(A_0, A_2)$$
which is natural in all A_i's and associative. The element 1 *acts as a unit element. Moreover, the Kasparov product is compatible with the descent morphisms.* □

5. A Baum–Connes assembly map for the telescopic algebra

In this section, we construct the Baum–Connes map for the telescopic algebra of a nicely decomposable singular foliation.

5.A. *An abstract construction.*

5.A.1. *Setting of the problem.* Let $\mathcal{F}$ be a nicely decomposable foliation. We keep the notation of Section 3.B. We put $\mathcal{G}'_k = (\mathcal{G}_k)_{|\Omega_{k-1}\cap W_k}$.

There is a priori a topological K-group of the Lie groupoid $\mathcal{G}_k$ and $\mathcal{G}'_k$. In order to construct a topological K-group and a Baum–Connes map for $\mathcal{T}(q, j)$, we first wish to understand the morphisms and mapping cones associated with the morphisms j_k and q_k at the "left-hand side" level.

Let us first note that the morphism j_k is just the inclusion $\mathcal{G}'_k \subset \mathcal{G}_k$ of the restriction of $\mathcal{G}_k$ to the (saturated) open subset $\Omega_{k-1} \cap W_k$ of W_k. The mapping cone of such a morphism is just the C^*-algebra of a Lie groupoid (restriction of $\mathcal{G}_k \times [0, 1)$ to the open subset $(\Omega_{k-1} \cap W_k) \times [0, 1) \cup W_k \times (0, 1)$). We may then very easily construct a topological K-group for it.

On the other hand, the morphism q_k corresponds to a groupoid homomorphism which is the identity at the level of objects (W_k) and a surjective submersion at the level of arrows. The corresponding map at the level of topological K-groups is not as easy. Let us also note that, even knowing the map $(q_k)_*^{\mathrm{top}}$ at the level of K_*^{top}, we need more in order to construct the topological K-group for the mapping cone: this morphism only gives a short exact sequence

$$0 \to \operatorname{coker}(q_k)_*^{\mathrm{top}} \to K_*^{\mathrm{top}}(\mathcal{F}) \to \ker(q_k)_*^{\mathrm{top}} \to 0,$$

which is not sufficient in order to determine the group $K_*^{\mathrm{top}}(\mathcal{F})$ that we seek.

In order to understand the K-theory of this mapping cone, one needs in fact to construct $(q_k)_*^{\mathrm{top}}$ as a KK-element. To do so, we need to write explicitly the topological K-groups as K-groups of C^*-algebras and the Baum–Connes maps as KK-elements. To that end we assume:

(i) The Lie groupoids $\mathcal{G}_k$ are Hausdorff.

(ii) The classifying spaces for proper actions of these groupoids are smooth manifolds. This is always the case when the groupoids $\mathcal{G}_k$ are given by (connected) Lie group actions — or are Morita equivalent to those. This is indeed the case in most singularity height 1 examples in Examples 1.7 above — in fact, also in the examples of higher height given in Section 1.D.2.

It turns out that condition (ii) can be somewhat bypassed, thanks to the Baum–Douglas presentation of K^*_{top} [Baum and Douglas 1982a; 1982b; Baum and Connes 2000; Tu 2000]. We discuss this in the Appendix.

5.A.2. *The Baum–Connes map for groupoids.* Let us recall some facts about the Baum–Connes map for groupoids. (See [Baum and Connes 2000; Tu 2000].)

Let G be a Lie groupoid. If the classifying space for proper actions is a manifold M, then there is no inductive limit to be taken, and replacing if necessary M

by the total space of the vector bundle $(\ker dp)^*$, we may assume that the equivariant submersion $p : M \to G^{(0)}$ is K-oriented and then the topological K-group is $K_*(C_0(M) \rtimes G)$ and the Baum–Connes map is just the wrong-way functoriality element $\widehat{p_!} \in KK(C_0(M) \rtimes G, C^*(G))$ constructed in [Connes and Skandalis 1984; Hilsum and Skandalis 1987].

In Le Gall's equivariant KK_G theory and terminology [1999] (see also [Kasparov 1988]), the Baum–Connes assembly map is the element $\widehat{p_!} = j_G(p_!)$, where $p_!$ is the element of $KK_G(C_0(M), C_0(G^{(0)}))$ associated with the G-equivariant K-oriented smooth map p.

This statement is just a Poincaré duality. One easily adapts the constructions in [Connes and Skandalis 1984]. Indeed, the groupoids G and $M \rtimes G$ have the same classifying space for proper actions (namely M). If X is a G-invariant, G-compact subspace of M, by properness we find that the forgetful map

$$KK_{M \rtimes G}(C_0(X), C_0(M)) \to KK_G(C_0(X), C_0(M))$$

is an isomorphism.

Using again properness of X, we see that $p! \in KK_G(C_0(M), C_0(G^{(0)}))$ induces an isomorphism $KK_G(C_0(X), C_0(M)) \to KK_G(C_0(X), C_0(G^{(0)}))$. The inverse of this morphism is the composition of the "induction" construction

$$KK_G(C_0(X), C_0(G^{(0)})) \to KK_{M \rtimes G}(C_0(X \times_{G^{(0)}} M), C_0(M))$$

and the wrong way functoriality element $j_!$ associated with the inclusion map $j : X \to X \times_{G^{(0)}} M$. In other words, the groupoids G and $M \rtimes G$ have the same topological K-groups. Moreover, the groupoid $M \rtimes G$ is proper, and therefore the Baum–Connes conjecture holds for it [Julg 1998] (see also [Tu 1999], since proper groupoids are amenable).

5.A.3. *Submersions of Lie groupoids and "left-hand sides".* Before we proceed and construct a topological K-group for the telescopic groupoid, we examine the case of a morphism $\pi : G_0 \to G_1$ of Hausdorff Lie groupoids $G_i \overset{t_i, s_i}{\rightrightarrows} G_i^{(0)}$ $(i = 0, 1)$. We assume that π is a submersion and that it is an inclusion of an open subset $\pi : G_0^{(0)} \subset G_1^{(0)}$ at the level of units.

Let $p_i : M_i \to G_i^{(0)}$ be smooth manifolds which are classifying spaces for proper actions for G_i. We assume further that the p_i's are K-oriented submersions and that the dimensions of the fibers are even.

- Let $W = M_0 \times_{p_0, t_1} G_1$. The groupoid G_0 acts properly on W; we thus obtain a Hausdorff locally compact quotient $W/G_0 = M_0 \times_{G_0} G_1$. Note that $x \mapsto (x, \pi(p_0(x)))$ defines a continuous map from M_0 to W and therefore $M_0 \to M_0 \times_{G_0} G_1$.
- The groupoid G_1 acts properly on the quotient space $M_0 \times_{G_0} G_1$. Since M_1 is universal, we get a G_1-equivariant map $M_0 \times_{G_0} G_1 \to M_1$. Hence, by composition

we have a G_0-equivariant map $q : M_0 \to M_1$. As $p_1 \circ q = p_0$, we obtain a morphism of proper groupoids

$$q : M_0 \rtimes G_0 \to M_1 \rtimes G_1.$$

- The map q is naturally K-oriented, so it induces an element

$$q_! \in KK_{G_0}(C_0(M_0), C_0(M_1)).$$

Applying the descent map j_{G_0} we obtain an element

$$\widehat{q_!} = \tilde{\pi}_*(j_{G_0}(q_!)) \quad \text{in } KK(C_0(M_0) \rtimes G_0, C_0(M_1) \rtimes G_1),$$

where $\tilde{\pi}$ is the morphism $C_0(M_1) \rtimes G_0 \to C_0(M_1) \rtimes G_1$ induced by the morphism π.

Proposition 5.1. *The morphism $\pi : C^*(G_0) \to C^*(G_1)$ corresponds at the level of topological K-theory to the element $\widehat{q_!}$. More precisely, we have*

$$\pi_*(\widehat{(p_0)_!}) = \widehat{q_!} \otimes \widehat{(p_1)_!}.$$

Proof. The morphism p_1, being G_1-equivariant, is also G_0-equivariant (where G_0 acts through the morphism π). It gives rise to an element

$$\widetilde{(p_1)_!} \in KK(C_0(M_1) \rtimes G_0, C^*(G_0)).$$

The elements $\widetilde{(p_1)_!}$ and $\widehat{(p_1)_!}$ correspond to each other via the morphism $\pi : G_0 \to G_1$, i.e., we have $\pi_*(\widetilde{(p_1)_!}) = \tilde{\pi}^*(\widehat{(p_1)_!})$. In other words, denoting by

$$[\tilde{\pi}] \in KK(C^*(C_0(M_1) \rtimes G_0, C_0(M_1) \rtimes G_1)) \quad \text{and} \quad [\pi] \in KK(C^*(G_0), C^*(G_1))$$

the KK-elements associated with the morphisms $\tilde{\pi}$ and π, respectively, we have $\widetilde{(p_1)_!} \otimes [\pi] = [\tilde{\pi}] \otimes \widehat{(p_1)_!}$. We find

$$\begin{aligned}\widehat{q_!} \otimes \widehat{(p_1)_!} &= j_{G_0}(q_!) \otimes [\tilde{\pi}] \otimes (\widehat{(p_1)_!}) = j_{G_0}(q_!) \otimes \widetilde{(p_1)_!} \otimes [\pi] \\ &= j_{G_0}(q_!) \otimes j_{G_0}((p_1)_!) \otimes [\pi] = j_{G_0}(q_! \otimes (p_1)_!) \otimes [\pi] \\ &= \pi_*(j_{G_0}(q_! \otimes (p_1)_!)) = \pi_*(j_{G_0}((p_0)_!)).\end{aligned}$$

Here, the fourth equality follows from naturality of j_G [Kasparov 1980; 1988; Le Gall 1999], and the last equality from the wrong way functoriality [Connes and Skandalis 1984; Hilsum and Skandalis 1987]. Note that, since the groupoid $M_0 \rtimes G_0$ is proper, the γ obstruction appearing in this computation in [Hilsum and Skandalis 1987] vanishes. □

5.A.4. *Abstract "left-hand sides" for mapping cones.* Next, we wish to construct in a natural way the topological K-group for the mapping cone of the morphism $\pi_{C^*} : C^*(G_0) \to C^*(G_1)$. Proposition 5.1 states that the relative topological K-group of π is an element in $KK(C_0(M_0) \rtimes G_0, C_0(M_1) \rtimes G_1)$. The topological K-group of the cone of π should be a kind of "mapping cone of this KK-element".

In this section, we abstractly construct this mapping cone up to KK-equivalence. We give an explicit description of this topological K-group (Section 5.B) and of the Baum–Connes assembly map (Section 5.B.5) below.

Recall that a KK-element $x \in KK(A, B)$ can be given as a composition

$$x = [f]^{-1} \otimes [g] \qquad (\spadesuit)$$

of a morphism $g : D \to B$ with the KK-inverse $[f]^{-1}$ of a morphism $f : D \to A$ which is invertible in KK-theory (see [Lafforgue 2007, Appendix A]). We may then wish to define (up to KK-equivalence) the cone of x as being the cone of g.

Next, in order to understand the Baum–Connes map, we should construct a KK-element associated with a map between mapping cones. We use the next lemma.

Lemma 5.2. *Let $f_i : A_i \to B_i$ be morphisms of C^*-algebras* ($i = 0$ *or* 1). *Denote by $p_i : C_{f_i} \to A_i$ and $j_i : B_i(0, 1) \to C_{f_i}$ the natural maps* ($p_i(a_i, \phi) = a_i$ *and* $j_i(\phi) = (0, \phi)$). *Let $x \in KK(A_0, A_1)$ and $y \in KK(B_0, B_1)$ satisfy $(f_1)_*(x) = f_0^*(y)$.*

(a) *There is $z \in KK(C_{f_0}, C_{f_1})$ such that $(p_1)_*(z) = p_0^*(x)$ and $(j_1)_*(Sy) = j_0^*(z)$, where $Sy \in KK(B_0(0, 1), B_1(0, 1))$ is deduced from y.*

(b) *If x and y are invertible*, *then so is z.*

In the language of [Meyer and Nest 2006], Lemma 5.2 is one of the axioms of a triangulated category. Although it is proved in [Meyer and Nest 2006], we include a proof for the reader's convenience.

Proof. (a) Note that z is not a priori unique. To construct it, one needs in fact to be more specific. Fix Kasparov bimodules (E_A, F_A) representing x and (E_B, F_B) representing y; a Kasparov $(A_0, B_1[0, 1])$ bimodule (E', F') realizing a homotopy between $(E_A \otimes_{A_1} B_1, F_A \otimes 1)$ and $f_0^*(E_B, F_B)$ gives rise to a Kasparov (A_0, Z_{f_1}) bimodule, where $Z_{f_1} = \{(a_1, \phi) \in A_1 \otimes B_1[0, 1] : f_1(a_1) = \phi(0)\}$ is the mapping cylinder of f_1, which can be glued with $(E_B, F_B)[0, 1)$ to give rise to the desired element in $KK(C_{f_0}, C_{f_1})$.

(b) By (a) applied to x^{-1} and y^{-1}, there exists $z' \in KK(C_{f_1}, C_{f_0})$ such that $(p_1)_*(z') = p_1^*(x^{-1})$ and $(j_0)_*(Sy^{-1}) = j_1^*(z')$. The Kasparov products $u_0 = z \otimes z'$ and $u_1 = z' \otimes z$ are elements in $KK(C_{f_i}, C_{f_i})$ such that $(p_i)_*(1 - u_i) = 0$ and $j_i^*(1 - u_i) = 0$. From the first equality and the mapping cone exact sequence, it follows that there exists $d_i \in KK(C_{f_i}, B_i(0, 1))$ such that $1 - u_i = (j_i)_*(d)$, and it follows that

$$(1 - u_i)^2 = (j_i)_*(d) \otimes (1 - u_i) = d \otimes j_i^*(1 - u_i) = 0,$$

whence u_i is invertible. □

Remark 5.3. Note also that we have a diagram

$$\begin{array}{ccccccccc} K_i(A_0) & \longrightarrow & K_i(B_0) & \longrightarrow & K_{1-i}(C_{f_0}) & \longrightarrow & K_{1-i}(A_0) & \longrightarrow & K_{1-i}(B_0) \\ \downarrow{\scriptstyle \otimes x} & & \downarrow{\scriptstyle \otimes y} & & \downarrow{\scriptstyle \otimes z} & & \downarrow{\scriptstyle \otimes x} & & \downarrow{\scriptstyle \otimes y} \\ K_i(A_1) & \longrightarrow & K_i(B_1) & \longrightarrow & K_{1-i}(C_{f_1}) & \longrightarrow & K_{1-i}(A_1) & \longrightarrow & K_{1-i}(B_1) \end{array}$$

where the lines are exact (Puppe sequences) and the squares commute. It follows that if x and y induce isomorphisms in K-theory, then the same holds for z.

Remarks 5.4. (a) It follows easily from this construction that given an element $x \in KK(A, B)$ the mapping cone C_g does not depend on the decomposition (♠) up to KK-equivalence.

(b) An alternative (and equivalent) way to construct the K-theory of the mapping cone of the KK-element x is to write x as an extension

$$0 \to SB \otimes \mathcal{K} \to D \to A \to 0$$

and define this K-theory as being $K_*(D)$.

(c) One can also define the KK-theory of this mapping cone as a relative KK-group [Skandalis 1984a, Remark 3.7(c)].

5.B. ***Baum–Connes map for mapping cones of submersions of Lie groupoids.*** Let us come back to our morphism $\pi : G_0 \to G_1$ of Hausdorff Lie groupoids, which is assumed to be a submersion and an open inclusion at the level of objects. We assume that the classifying spaces for proper maps of G_i are manifolds M_i. In Section 5.A.3, we explained how to construct an equivariant map $q : M_0 \to M_1$ that can be assumed to be a smooth submersion (up to replacing M_0 by a homotopy equivalent manifold).

As a consequence of Lemma 5.2 and Proposition 5.1, we see that, in order to construct the topological K-group we need to give an explicit construction of the wrong-way functoriality element $\widetilde{\pi}_*(j_{G_0})(q_!) \in KK(C^*(M_0 \rtimes G_0), C^*(M_1 \rtimes G_1))$. Here, using a double deformation longitudinally smooth groupoid we give a groupoid $\mathbb{H}$ which is a family over $[0, 1] \times [0, 1]$, whose vertical lines $\{i\} \times [0, 1]$ can be interpreted as the Baum–Connes maps for the groupoid G_i and whose horizontal lines $[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$ are $q!$ and $[\pi]$, respectively.

We then may define the relative topological K-group of π as the groupoid $\mathbb{H}$ restricted to $[0, 1) \times \{0\}$ and construct the Baum–Connes map using the groupoid $\mathbb{H}$ (restricted to $[0, 1) \times [0, 1]$).

In order to have a "ready to glue" groupoid, in view of the case of the telescopic algebra (Section 5.C), we are lead to perform a slightly more complicated construction.

5.B.1. *Deformation groupoids.*

Deformation to the normal cone. The adiabatic deformation of a Lie groupoid G with Lie algebroid $\mathfrak{G}$ was defined by Alain Connes in the particular case of the pair groupoid [Connes 1994] and generalized by various authors (e.g., [Hilsum and Skandalis 1987; Monthubert and Pierrot 1997; Nistor et al. 1999]). This is based on the notion of deformation to the normal cone, which we briefly recall; see also [Carrillo Rouse 2008; Debord and Skandalis 2014].

Let X be a submanifold of a manifold Y. Denote by N_X^Y the total space of the normal bundle to X in Y. There is a natural way to put a manifold structure to $Y \times \mathbb{R}^* \cup N_X^Y \times \{0\}$; denote this manifold by $DNC(Y, X)$.

The map $p : DNC(Y, X) \to \mathbb{R}$ defined by $p(y, t) = t$ for $(y, t) \in Y \times \mathbb{R}^*$ and $p(\xi, 0) = 0$ for $\xi \in N_X^Y$ is a smooth submersion. For $J \subset \mathbb{R}$, we write $DNC_J(Y, X)$ for $p^{-1}(J)$.

This construction is functorial. Given a commutative diagram of smooth maps

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\ X' & \hookrightarrow & Y' \end{array}$$

where the horizontal arrows are inclusions of submanifolds, we naturally obtain a smooth map $DNC(f) : DNC(Y, X) \to DNC(Y', X')$. If f_Y is a submersion and $X = X' \times_{Y'} Y$ then $DNC(f)$ is a submersion.

Double deformations to the normal cone. Let Z be a smooth manifold, Y a (locally) closed submanifold of Z and X a (locally) closed submanifold of Y. Then $DNC(Y, X)$ is a (locally) closed submanifold of $DNC(Z, X)$. Put then

$$DNC^2(Z, Y, X) = DNC(DNC(Z, X), DNC(Y, X)).$$

We have a submersion $p_2 : DNC^2(Z, Y, X) \to \mathbb{R}^2$. For every subset L of $\mathbb{R}^2$, we put

$$DNC^2_L(Z, Y, X) = p_2^{-1}(L).$$

By definition of the deformation to the normal cone,

$$DNC^2_{\mathbb{R}\times\mathbb{R}^*}(Z, Y, X) = DNC(Z, X) \times \mathbb{R}^*.$$

By functoriality of the DNC construction,

$$DNC^2_{\mathbb{R}^*\times\mathbb{R}}(Z, Y, X) = DNC(Z \times \mathbb{R}^*, Y \times \mathbb{R}^*) \simeq DNC(Z, Y) \times \mathbb{R}^*.$$

Deformation groupoids, adiabatic groupoids. From naturality, it follows that if Y is a Lie groupoid and X is a Lie subgroupoid of Y, then $DNC(Y, X)$ is naturally endowed with a Lie groupoid structure — with objects $DNC(Y^{(0)}, X^{(0)})$, and target and source maps $DNC(t)$ and $DNC(s)$. Of course, if in the above diagram all the maps are groupoid morphisms, then $DNC(f)$ is a morphism of groupoids too.

The *adiabatic groupoid* of a Lie groupoid G is just $G_{\mathrm{ad}} = DNC_{[0,1)}(G, G^{(0)}) = \mathfrak{G} \times \{0\} \cup G \times (0, 1)$ (with base manifold $G^{(0)} \times [0, 1)$). Note that the normal bundle $N^G_{G^{(0)}}$ is, by definition, the Lie algebroid $\mathfrak{G}$ of G. It follows that Z is a Lie groupoid, Y is a Lie subgroupoid of Z and X is a Lie subgroupoid of Y. Then $DNC^2(Z, Y, X)$ is a Lie groupoid.

5.B.2. *The Baum–Connes map of a Lie groupoid via deformation groupoids.* Let G be a Lie groupoid and let M be a smooth manifold on which G acts via a smooth onto submersion $p : M \to G^{(0)}$. We do not assume that p is K oriented but rather consider the total space of $(\ker dp)^*$. Note that if M is a classifying space for proper actions of G, then $(\ker dp)^* \to G^{(0)}$ is also a classifying space of proper actions, and it moreover carries a canonical K-orientation. So we can replace M with $(\ker dp)^*$.

Put then $\Gamma_p = DNC(G^p_p, M \rtimes G)$. As p is supposed to be a surjective submersion, the groupoid G^p_p is Morita equivalent to G. There is a canonical Morita equivalence bimodule $\mathcal{E}$ of the C^*-algebras $C^*(G^p_p)$ and $C^*(G)$.

We have an exact sequence of C^*-algebras:

$$0 \to C^*(G^p_p \times (0, 1]) \to C^*((\Gamma_p)_{[0,1]}) \xrightarrow{\mathrm{ev}_0} C^*(\ker(dp) \rtimes G) \to 0.$$

Note that $C^*(G^p_p \times (0, 1])$ is contractible. It follows that ev_0 is invertible in E-theory. We may then observe the diagram

$$C_0((\ker dp)^*) \rtimes G \xleftarrow{\mathrm{ev}_0} C^*((\Gamma_p)_{[0,1]}) \xrightarrow{\mathrm{ev}_1} C^*(G^p_p) \overset{\mathcal{E}}{\text{——}} C^*(G).$$

We thus obtain an element

$$\mu_M = [\mathrm{ev}_0]^{-1} \otimes [\mathrm{ev}_1] \otimes [\mathcal{E}] \in E(C_0((\ker dp)^*) \rtimes G, C^*(G)).$$

Note that this E-theory coincides with KK-theory if the action of G on M is assumed to be amenable — and in particular, if it is proper.

If M is the classifying space for proper algebras, the morphism on K-groups defined by μ_M is the Baum–Connes map.

5.B.3. *A double deformation construction.* Now let G_0 and G_1 be Lie groupoids and let $\pi : G_0 \to G_1$ be a groupoid morphism which is a smooth submersion whose restriction $\pi^{(0)} : G_0 \to G_1$ is the inclusion of an open subset. Let M_i be manifolds with actions of G_i. We assume that the maps $p_i : M_i \to G_i^{(0)}$ defining these actions are smooth submersions. Let also $q : M_0 \to M_1$ be a smooth submersion which is equivariant, i.e., $q(\gamma.x) = \pi(\gamma)q(x)$ for every $(x, \gamma) \in M_0 \times_s G_0$. In other words, we assume that we have a morphism of semidirect products $\widehat{\pi} : M_0 \rtimes G_0 \to M_1 \rtimes G_1$ defined by $\widehat{\pi}(x, \gamma) = (q(x), \pi(\gamma))$.

The groupoid G_0 acts on the open subspace $M_1' = q(M_0)$ of M_1 through the morphism π: just put $\gamma.q(x) = q(\gamma.x) = \pi(\gamma).q(x)$ for $x \in M_0$ and $\gamma \in G_0$ with $s(x) = p_0(x) = p_1(q(x))$.

We have inclusions of groupoids $M_0 \rtimes G_0 \subset (M_1' \rtimes G_0)_q^q \subset (G_0)_{p_0}^{p_0}$. Indeed,

$$M_0 \rtimes G_0 = \big\{(x, \gamma, y) \in (G_0)_{p_0}^{p_0} : x = \gamma.y\big\},$$
$$(M_1' \rtimes G_0)_q^q = \big\{(x, \gamma, y) \in (G_0)_{p_0}^{p_0} : q(x) = q(\gamma.y)\big\}.$$

Let then $\mathcal{H}_0$ be the double deformation Lie groupoid:

$$\mathcal{H}_0 = DNC^2\big((G_0)_{p_0}^{p_0}, (M_1' \rtimes G_0)_q^q, M_0 \rtimes G_0\big).$$

The groupoid $\mathcal{H}_0$ is a family of groupoids indexed by $\mathbb{R}^2$. For every locally closed subset L of $\mathbb{R}^2$, we may form the locally compact groupoid $(\mathcal{H}_0)_L$.

Let $q' : M_0 \sqcup M_1 \to M_1$ be the map which coincides with q on M_0 and the identity on M_1 and $p' : M_0 \sqcup M_1 \to G_1^{(0)}$, $p' = p_1 \circ q'$. Define the groupoid

$$\mathcal{H}_1 = DNC\big((G_1)_{p'}^{p'} \times \mathbb{R}^*, (M_1 \rtimes G_1)_{q'}^{q'} \times \mathbb{R}^*\big) \simeq DNC\big((G_1)_{p'}^{p'}, (M_1 \rtimes G_1)_{q'}^{q'}\big) \times \mathbb{R}^*$$

(with objects $(M_0 \sqcup M_1) \times \mathbb{R}^* \times \mathbb{R}$).

For every locally closed subset $Y \subset \mathbb{R}^* \times \mathbb{R}$ we denote by $(\mathcal{H}_1)_Y$ the restriction of $\mathcal{H}_1$ to its saturated subset $(M_0 \sqcup M_1) \times Y$.

5.B.4. *A longitudinally smooth groupoid.* Note that

$$(M_1' \rtimes G_0)_q^q = \big\{(x, \gamma, y) \in (G_0)_p^p : q(x) = \pi(\gamma)q(y)\big\}.$$

In other words, $(M_1' \rtimes G_0)_q^q$ is the fibered product $(G_0)_{p_0}^{p_0} \times_{(G_1)_{p_0}^{p_0}} (M_1 \rtimes G_1)_q^q$. We therefore have a commutative diagram

$$\begin{array}{ccc} (M_1' \rtimes G_0)_q^q & \hookrightarrow & (G_0)_{p_0}^{p_0} \\ \downarrow & & \downarrow \\ (M_1 \rtimes G_1)_q^q & \hookrightarrow & (G_1)_{p_0}^{p_0} \end{array}$$

which gives rise to a morphism

$$DNC\big((G_0)_{p_0}^{p_0}, (M_1' \rtimes G_0)_q^q\big) \to DNC\big((G_1)_{p_0}^{p_0}, (M_1 \rtimes G_1)_q^q\big)$$

which is a groupoid morphism, a submersion and the identity at the level of objects $(M_0 \times \mathbb{R})$. We thus obtain a morphism of groupoids

$$\pi : (\mathcal{H}_0)_{\mathbb{R}^* \times \mathbb{R}} \to \mathcal{H}_1$$

which is a submersion. At the level of objects it is the inclusion $M_0 \times \mathbb{R}^* \times \mathbb{R} \to (M_0 \sqcup M_1) \times \mathbb{R}^* \times \mathbb{R}$.

Let $Z_0 = [0, 1) \times \left[0, \frac{1}{2}\right]$, $Q = \left\{(u, v) \in \mathbb{R}^2 : 0 \le u \le v \le \frac{1}{2}\right\}$ and $Z_1 = Z_0 \setminus Q = \{(u, v) \in Z_0 : u > v\}$. We may then construct a longitudinally smooth groupoid $\mathbb{H} = (\mathcal{H}_0)_Q \cup (\mathcal{H}_1)_{Z_1}$ with atlas formed by $(\mathcal{H}_0)_{Z_0}$ and $(\mathcal{H}_1)_{Z_1}$, using the morphism π in order to map $(\mathcal{H}_0)_{Z_1}$ to $(\mathcal{H}_1)_{Z_1}$. We have $\mathbb{H}^{(0)} = (M_0 \times Z_0) \sqcup (M_1 \times Z_1)$.

In the same way as above, for every locally closed subset $Y \subset Z_0$ we denote by $\mathbb{H}_Y$ the restriction of $\mathbb{H}$ to its saturated subset $M_0 \times Y \cup M_1 \times (Y \cap Z_1)$.

Remarks 5.5. (a) It is worthwhile to note that the groupoid $\mathbb{H}$ only depends on $\pi : G_0 \to G_1$, the (proper) actions of G_i on M_i and the submersion q. Also, the restriction $\mathbb{H}_{\{1/2\}\times[0,1/2]}$ is nothing else than $(M_0 \times_{p_0} M_0)_{\mathrm{ad}} \rtimes G_0$ (restricted to $\left[0, \frac{1}{2}\right]$). It does not depend on G_1, M_1, q.

(b) Note $\mathcal{H}_{0,0}$ is isomorphic to the direct sum of vector bundles $\ker dq \oplus q^*(\ker dp_1)$.

5.B.5. *Baum–Connes map for a mapping cone.* Set $F_0 = [0, 1) \times \{0\} \cup \{0\} \times \left[0, \frac{1}{2}\right]$. Note that, since the action of G_i on M_i is proper, the groupoid

$$\mathbb{H}_{F_0} = (\ker dp_0) \rtimes G_0 \times \left[0, \tfrac{1}{2}\right] \cup ((\ker dp_1) \rtimes G_1)_{q'}^{q'} \times (0, 1)$$

is amenable and we have a semisplit exact sequence

$$0 \to C^*(\mathbb{H}_{Z_0 \setminus F_0}) \to C^*(\mathbb{H}) \xrightarrow{\sigma_0} C^*(\mathbb{H}_{F_0}) \to 0.$$

Proposition 5.6. *The homomorphism σ_0 is invertible in KK-theory.*

Proof. We have a semisplit exact sequence

$$0 \to C^*(\mathbb{H}_{Z_1 \setminus F_0}) \to C^*(\mathbb{H}_{Z_0 \setminus F_0}) \to C^*(\mathbb{H}_{Q \setminus F_0}) \to 0.$$

Note that the groupoid $\mathbb{H}$ is constant over the sets $Z_1 \setminus F_0$ and $Q \setminus F_0$:

$$\mathbb{H}_{(u,v)} = (G_1)_{p_1 \circ q'}^{p_1 \circ q'} \quad \text{for } (u, v) \in Z_1 \setminus F_0,$$
$$\mathbb{H}_{(u,v)} = (G_0)_{p_0}^{p_0} \quad \text{for } (u, v) \in Q \setminus F_0.$$

The sets

$$Z_1 \setminus F_0 = \left\{(u, v) : 0 < v < u < 1 \text{ and } v \le \tfrac{1}{2}\right\},$$
$$Q \setminus F_0 = \left\{(u, v) : 0 < u \le v \le \tfrac{1}{2}\right\}$$

are contractible (more precisely, their one point compactification contracts to this point) and it follows that the C^*-algebras $C^*(\mathbb{H}_{Z_1 \setminus F_0})$ and $C^*(\mathbb{H}_{Q \setminus F_0})$ are contractible. It follows that $C^*(\mathbb{H}_{Z_0 \setminus F_0})$ is KK-contractible (it is actually contractible). We deduce that $[\sigma_0]$ is a KK-equivalence. □

Set also $F_1 = \left\{\frac{1}{2}\right\} \times \left[\frac{1}{2}, 1\right)$. One sees that $\mathbb{H}_{F_1}$ is isomorphic to the groupoid $\mathcal{C}_\pi = G_0 \times \{0\} \cup G_1 \times (0, 1)$ pulled back by

$$q'' : (M_0 \times [0, 1)) \sqcup (M_1 \times (0, 1)) \to G_1^{(0)} \times [0, 1)$$

(recall that $G_0^{(0)}$ is an open subset of $G_1^{(0)}$).

Corollary 5.7. *The algebra $C^*(\mathbb{H}_{F_1})$ is canonically Morita equivalent to the mapping cone of $h_{C^*}: C^*(G_0) \to C^*(G_1)$.*

Denote by $\mathcal{E}$ the Morita $(C^*(\mathbb{H}_{F_1}), C^*(\mathcal{C}_h))$ bimodule and $[\mathcal{E}]$ its KK-class. Let $[\sigma_1]: C^*(\mathbb{H}) \to C^*(\mathbb{H}_{F_1})$ be the evaluation.

Definition 5.8. Assume further that the manifolds M_i are classifying spaces for the proper actions of G_i. With the notation above, the topological K-theory of the groupoid $\mathcal{C}_\pi$ is $K_*(C^*(\mathbb{H}_{F_0}))$ and the Baum–Connes morphism is the composition $[\sigma_0]^{-1} \otimes [\sigma_1] \otimes [\mathcal{E}]$.

5.B.6. *Justifying why this is a Baum–Connes map.* Let us explain why this is a "good" definition. First of all, for $v \in \left[0, \frac{1}{2}\right]$, the K-theory of the C^*-algebra $C^*(\mathbb{H}_{(0,v)}) = C^*(\ker p_0 \rtimes G_0) = C_0((\ker p_0)^*) \rtimes G_0$ is the topological K-group for G_0. Also, for $u \in (0, 1)$, the C^*-algebra $C^*(\mathbb{H}_{(u,0)}) = C^*\big((\ker p_1 \rtimes G_1)_{q'}^{q'}\big)$ is Morita equivalent to $C^*(\ker p_1 \rtimes G_1) = C_0((\ker p_1)^*) \rtimes G_1$, whose K-theory is the topological K-theory for G_1.

We may then write a diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^*(\mathbb{H}_{Z_1 \cap F_0}) & \longrightarrow & C^*(\mathbb{H}_{F_0}) & \longrightarrow & C^*(\mathbb{H}_{Q \cap F_0}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \sigma_{1,0}} & & \uparrow{\scriptstyle \sigma_0} & & \uparrow{\scriptstyle \sigma_{0,0}} & & \\
0 & \longrightarrow & C^*(\mathbb{H}_{Z_1}) & \longrightarrow & C^*(\mathbb{H}) & \longrightarrow & C^*(\mathbb{H}_Q) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_{1,1}} & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_{0,1}} & & \\
0 & \longrightarrow & C^*(\mathbb{H}_{Z_1 \cap F_1}) & \longrightarrow & C^*(\mathbb{H}_{F_1}) & \longrightarrow & C^*(\mathbb{H}_{Q \cap F_1}) & \longrightarrow & 0 \\
 & & \Big| \, \mathcal{E}_1 & & \Big| \, \mathcal{E} & & \Big| \, \mathcal{E}_0 & & \\
0 & \longrightarrow & C^*(G_1)(0,1) & \longrightarrow & \mathcal{C}_{h_{C^*}} & \longrightarrow & C^*(G_0) & \longrightarrow & 0
\end{array}$$

In this diagram all sequences are semisplit, the morphisms $\sigma_0, \sigma_{i,0}$ are KK-equivalences and the compositions $[\sigma_{i,0}]^{-1} \otimes [\sigma_{i,1}] \otimes [\mathcal{E}_i]$ are indeed the Baum–Connes maps for $G_1 \times (0, 1)$ and G_0.

It follows also that the class in $KK^1(C^*(\mathbb{H}_{Q \cap F_0}), C^*(\mathbb{H}_{Z_1 \cap F_0}))$ for the first sequence corresponds to the class of $[h_{C^*}] \in KK(C^*(G_0), C^*(G_1))$.

From the discussion in Section 5.A.4, it follows that the K-theory of $C^*(\mathbb{H}_{F_0})$ and the morphism is indeed the right one, and that the composition $[\sigma_0]^{-1} \otimes [\sigma_1] \otimes [\mathcal{E}]$ is indeed a Baum–Connes map.

Remark 5.9. The groupoid $\mathbb{H}_{F_0}$ is a semidirect product $\Lambda \rtimes \mathcal{C}_\pi$, where Λ is a groupoid obtained by gluing $DNC_{[0,1)}\big(M_0, (\ker dp_1)_{q'}^{q'}\big)$ with $\ker dp_0 \times \left[0, \frac{1}{2}\right]$.

One may give a generalized notion of proper algebras on a longitudinally smooth groupoid G by saying that $G^{(0)}$ is an increasing union $\bigcup \Omega_k$ of saturated open subsets such that the restriction of G to $\Omega_k \setminus \Omega_{k-1}$ is Hausdorff. We may say that an action of G on an algebra A is proper if its restriction to each $\Omega_k \setminus \Omega_{k-1}$ is proper.

In this generalized sense, the $\mathcal{C}_\pi$-algebra $C^*(\Lambda)$ is a proper $\mathcal{C}_\pi$-algebra. Its restriction to $Q \cap F_0$ is indeed a proper G_0-algebra and its restriction to $Z_1 \cap F_0$ is a proper $G_1 \times (0, 1)$-algebra.

It may be interesting to look for a way to say that the $C^*\big(DNC_{[0,1)}(M_0, (M_1)_{q'}^{q'})\big)$ is somehow a universal proper algebra.

5.C. ***Baum–Connes map for the telescopic algebra.*** Since a mapping telescope is a mapping cylinder which, in turn, is a mapping cone (cf. Remarks 3.5) we can just proceed and construct the "left-hand side" for the telescopic algebra — and therefore for the foliation one.

We are given a nicely decomposable foliation $(M, \mathcal{F})$, a decomposition given by an increasing sequence Ω_k of saturated sets — we put $Y_k = \Omega_k \setminus \Omega_0$, a sequence of Lie groupoids $\mathcal{G}_k \rightrightarrows W_k \subset \Omega_k$ such that $Y_k \subset W_k$ and $W_k \cap \Omega_{k-1} \subset W_{k-1}$; we put $\mathcal{G}'_k = (\mathcal{G}_k)_{|W_k \cap \Omega_{k-1}}$ and assume that we have a groupoid morphism which is a submersion $\pi_k : \mathcal{G}'_k \to \mathcal{G}_{k-1}$.

We further assume that we have submersions of manifolds $p_k : M_k \to W_k$ which are classifying spaces for proper actions of $\mathcal{G}_k$. For $k \geq 1$, the restriction $p_k^{-1}(\Omega_{k-1})$ of M_k is a classifying space for $\mathcal{G}'_k$ but we may need to modify it: we choose a classifying space given by a submersion $p'_k : M'_k \to \Omega_{k-1} \subset W_{k-1}$ in such a way that the maps $q_k : M'_k \to M_{k-1}$ and $\hat{q}_k : M'_k \to M_k$ are submersions.

We then construct the classifying groupoids

- $\mathbb{H}_k$ associated to the morphism $\pi_k : \mathcal{G}'_k \to \mathcal{G}_{k-1}$ and the submersion $q_k : M'_k \to M_{k-1}$ of classifying spaces;
- $\widehat{\mathbb{H}}_k$ associated to the morphism $j_k : \mathcal{G}'_k \to \mathcal{G}_k$ and the submersion $\hat{q}_k : M'_k \to M_k$ of classifying spaces.

We then glue the groupoids $\mathbb{H}_k$ and $\widehat{\mathbb{H}}_k$ in their common part $(\mathbb{H}_k)_{\{1/2\}\times[0,1/2]} = (\widehat{\mathbb{H}}_k)_{\{1/2\}\times[0,1/2]}$ (cf. Remarks 5.5) and obtain a groupoid $\widetilde{\mathbb{H}}_k$.

For a locally closed part Y of $Z_0 = [0, 1)] \times \big[0, \frac{1}{2}\big]$ we set $(\widetilde{\mathbb{H}}_k)_Y = (\mathbb{H}_k)_Y \cup (\widehat{\mathbb{H}}_k)_Y$.

Recall that $Q = \big\{(u, v) : 0 \leq u \leq v \leq \frac{1}{2}\big\}$ and $Z_1 = Z_0 \setminus Q$. We define diffeomorphisms $\vartheta_k : Z_1 \to [0, 1] \times (k-1, k)$ and $\hat{\vartheta}_k : Z_1 \to [0, 1] \times (k, k+1)$ by setting

$$\vartheta_k(u, v) = \Big(2v, k - \frac{u-v}{1-v}\Big) \quad \text{and} \quad \hat{\vartheta}_k(u, v) = \Big(2v, k + \frac{u-v}{1-v}\Big),$$

respectively. Thanks to this diffeomorphism, we obtain identifications of

- $\Theta_k : (\mathbb{H}_k)_{Z_1} \xrightarrow{\sim} \Big(DNC\big((\mathcal{G}_{k-1})_{p_{k-1}}^{p_{k-1}}, M_{k-1} \rtimes \mathcal{G}_{k-1}\big)_{[0,1]}\Big)_{q'_k}^{q'_k} \times (k-1, k)$
  where $q'_k : M_{k-1} \sqcup M'_k \to M_{k-1}$ is the identity on M_{k-1} and q_k on M'_k;
- $\widehat{\Theta}_k : (\widehat{\mathbb{H}}_k)_{Z_1} \xrightarrow{\sim} \Big(DNC\big((\mathcal{G}_k)_{p_k}^{p_k}, M_k \rtimes \mathcal{G}_k\big)_{[0,1]}\Big)_{\hat{q}'_k}^{\hat{q}'_k} \times (k, k+1)$
  where $\hat{q}'_k : M_k \sqcup M'_k \to M_k$ is the identity on M_k and $\hat{q}_k$ on M'_k.

Define $q_k'' : M_k \sqcup M_k' \sqcup M_{k+1}' \to M_k$ to be the map coinciding with the identity over M_k, $\hat{q}_k$ on M_k' and q_{k+1} on M_{k+1}' — with the convention $M_0' = \varnothing$ and, if $n \neq +\infty$, $M_{n+1}' = \varnothing$.

Definition 5.10. We define the *adiabatic telescopic groupoid* $\mathbb{G}$ to be the union $\bigcup_{k=1}^{n} (\widetilde{\mathbb{H}}_k)_Q \times \{k\}$ with $\bigcup_{k=0}^{n} \big(DNC((\mathcal{G}_k)_{p_k}^{p_k}, M_k \rtimes \mathcal{G}_k)_{[0,1]}\big)_{q_k''}^{q_k''} \times (k, k+1)$. The gluing is obtained by mapping $\mathbb{H}_k \to \mathbb{G}$:

- We map $(\widetilde{\mathbb{H}}_k)_Q$ to $\mathbb{G}$ by the map $\gamma \mapsto (\gamma, k) \in \mathbb{G}$.
- Using Θ_{k+1} we map $(\mathbb{H}_{k+1})_{Z_1}$ to
$$\bigcup_{k=0}^{n} \big(DNC((\mathcal{G}_k)_{p_k}^{p_k}, M_k \rtimes \mathcal{G}_k)_{[0,1]}\big)_{q_k'}^{q_k'} \times (k, k+1),$$
which is a subset of
$$\bigcup_{k=0}^{n} \big(DNC((\mathcal{G}_k)_{p_k}^{p_k}, M_k \rtimes \mathcal{G}_k)_{[0,1]}\big)_{q_k''}^{q_k''} \times (k, k+1) \subset \mathbb{G}.$$
- Using $\widehat{\Theta}_k$ we map $(\widehat{\mathbb{H}}_k)_{Z_1}$ to
$$((M_k \times_{p_k} M_k)_{\overline{\mathrm{ad}}} \rtimes \mathcal{G}_k)_{\hat{q}_k'}^{\hat{q}_k'} \times (k, k+1),$$
which is a subset of
$$((M_k \times_{p_k} M_k)_{\overline{\mathrm{ad}}} \rtimes \mathcal{G}_k)_{q_k''}^{q_k''} \times (k, k+1) \subset \mathbb{G}.$$

We define the obvious map $\chi : \mathbb{G}^{(0)} \to (0, n+1)$ (using the convention $+\infty + 1 = +\infty$ of course). Thanks to χ, the (full) C^*-algebra $C^*(\mathbb{G})$ is a $C_0(0, n+1)$-algebra.

Define a map $\xi : Z \to [0,1]$ by setting $\xi(u, v) = 2\min(u, v)$. Let $\hat{\xi} : \mathbb{G}^{(0)} \to [0,1]$ be defined as the composition
$$\mathbb{G}^{(0)} \to Q \xrightarrow{\xi} [0, 1]$$
on $(\widetilde{\mathbb{H}}_k)_Q \times \{k\}$ and let $\hat{\xi}$ to be the parameter in the adiabatic deformation $(M_k \times_{p_k} M_k)_{\overline{\mathrm{ad}}}$ on
$$\bigcup_{k=0}^{n} ((M_k \times_{p_k} M_k)_{\overline{\mathrm{ad}}} \rtimes \mathcal{G}_k)_{q_k''}^{q_k''} \times (k, k+1).$$

We then define the subgroupoids $\mathbb{G}_0$ and $\mathbb{G}_1$ of $\mathbb{G}$, restrictions of $\mathbb{G}$ to the closed saturated set $\hat{\xi}^{-1}(\{i\})$. We then have:

Proposition 5.11. (a) *The algebra $C^*(\mathbb{G}_0)$ is nuclear.*

(b) *The kernel of the evaluation $\rho_0 : C^*(\mathbb{G}) \to C^*(\mathbb{G}_0)$ is KK-contractible.*

(c) *The algebra $C^*(\mathbb{G}_1)$ is Morita equivalent to the telescopic algebra.*

Proof. (a) In fact $C^*(\mathbb{G}_0)$ sits in an exact sequence

$$0 \to \bigoplus_{k=0}^{n} C^*\big(((\ker p_k)^* \rtimes \mathcal{G}_k)_{q'_k}^{q'_k} \times (0,1)\big) \to C^*(\mathbb{G}_0) \to \bigoplus_{k=1}^{n} C^*((\ker p'_k)^* \rtimes \mathcal{G}'_k \times [0,1]) \to 0$$

and the Lie groupoids $((\ker p_k)^* \rtimes \mathcal{G}_k)_{q'_k}^{q'_k}$ and $(\ker p'_k)^* \rtimes \mathcal{G}'_k$ are proper. It follows that $C^*(\mathbb{G}_0)$ is in fact a type I algebra.

(b) We have a semisplit exact sequence

$$0 \to C_0((0,1] \times (0,1)) \otimes B \to \ker \rho_0 \to C_0(Q \setminus F_0) \otimes B' \to 0 \to 0,$$

where

$$B = \bigoplus_{k=0}^{n} C^*\big(((M_k \times_{p_k} M_k) \rtimes \mathcal{G}_k)_{q''_k}^{q''_k}\big) \quad \text{and} \quad B' = \bigoplus_{k=1}^{n} C^*((M'_k \times_{p'_k} M'_k) \rtimes \mathcal{G}'_k).$$

The algebras $C_0((0,1] \times (0,1))$ and $C_0(Q \setminus F_0)$ are contractible.

(c) Actually the groupoid $\mathbb{G}_1$ is Morita equivalent to the telescopic groupoid. □

Definition 5.12. Let $(M, \mathcal{F})$ be a nicely decomposable foliation. Assume that the classifying spaces of all the groupoids $\mathcal{G}_k \rightrightarrows W_k$ involved in this decomposition are manifolds. With the above construction,

- we define the "left-hand side", i.e., the topological K-theory (of this decomposition) to be the K-theory of $C^*(\mathbb{G}_0)$;
- we define the Baum–Connes map for the telescope to be the composition $[\rho_0]^{-1} \otimes [\rho_1] \otimes [\mathbb{E}]$;
- we define the Baum–Connes map for $(M, \mathcal{F})$ to be the Baum–Connes map for the telescope composed with the isomorphism $K_*(\mathcal{T}) \to K_{*+1}(C^*(M, \mathcal{F}))$.

Let $\rho_1 : C^*(\mathbb{G}) \to C^*(\mathbb{G}_1)$ be evaluation. The kernel of ρ_1 is a $C_0(0, n+1)$-algebra. It follows from the inductive limit construction that if $(\ker \rho_1)_{(k,k+1)}$ and $(\ker \rho_1)_k$ are E-contractible for all k, then so is $\ker \rho_1$. We thus obtain:

Theorem 5.13. *Let $(M, \mathcal{F})$ be a nicely decomposable foliation such that the classifying spaces of all the groupoids $\mathcal{G}_k \rightrightarrows W_k$ involved in this decomposition are manifolds. If the* full *Baum–Connes conjecture holds for all of them, then the* full *Baum–Connes map of Definition 5.12 is an isomorphism.*

Corollary 5.14. *Let $(M, \mathcal{F})$ be a nicely decomposable foliation. If all the groupoids $\mathcal{G}_k \rightrightarrows W_k$ involved in this decomposition are amenable and their classifying spaces are manifolds, then the Baum–Connes map is an isomorphism.*

Appendix: When the classifying spaces are not manifolds

We finally explain how one should be able to get rid of the assumption on the classifying spaces: we just assume that the foliation $\mathcal{F}$ has a nice decomposition with Hausdorff Lie groupoids $\mathcal{G}_i$ but the classifying spaces E_i for proper actions are not manifolds.

To construct a topological K-theory and a Baum–Connes map for $C^*(M, \mathcal{F})$, we just need to construct a topological K-theory for a mapping cone of a morphism $\pi : G \to G'$ of Hausdorff Lie groupoids which is a submersion and the identity at the level of objects. As in the particular cases considered here, we then may construct topological K-theory for mapping tori and then of telescopic algebras.

In fact, given such a morphism $\pi : G \to G'$ we just have to show that

(i) we may express the topological K-theory of G and G' as the K-theory of C^*-algebras T and T';

(ii) the Baum–Connes maps are given by elements μ and μ' in $KK(T, C^*(G))$ and $KK(T', C^*(G'))$, respectively;

(iii) we may construct an element $x \in KK(T, T')$ such that $\pi_*(\mu) = x \otimes \mu'$.

We then write $x = [f]^{-1} \otimes [g]$, where $f : D \to T$ and $g : D \to T'$ are morphisms with f a K-equivalence. A topological K-theory for C_π is then the cone C_g of g. As $f^*(\pi_*(\mu)) = [f] \otimes \mu \otimes [\pi] = [g] \otimes \mu'$, we may construct an element $\widetilde{\mu} \in KK(C_g, C_\pi)$ as in Lemma 5.2 which defines the desired Baum–Connes map.

To do so, recall that if G is a Hausdorff Lie groupoid, then the topological K-theory for the Baum–Connes map can be described in the Baum–Douglas way [Baum and Douglas 1982a; 1982b; Baum and Connes 2000; Baum et al. 1994; Tu 1999; 2000]: there is an inductive limit of manifolds $(M_k)_{k\in\mathbb{N}}$ with maps $h_k : M_k \to M_{k+1}$ forming a sequence of approximations of E. We may assume that the maps $q_k : M_k \to G^{(0)}$ are K-oriented in a G-equivariant way, and therefore so are the maps h_k. We also assume that the dimensions of all the M_k are equal modulo 2. Then the topological K-theory $K_*^{\mathrm{top}}(G)$ is the inductive limit $\varinjlim_k (K_*(C_0(M_k) \rtimes G), (h_k)!)$.

The Baum–Connes map on the image of $K_*(C_0(M_k) \rtimes G)$ is given by the element $(q_k)!$. Put $A_k = C_0(M_k) \rtimes G$. The same construction is then given for the groupoid G', yielding proper G'-manifolds M'_k, maps $h'_k : M'_k \to M'_{k+1}$, algebras $A'_k = C_0(M'_k) \rtimes G'$, etc.

We may (and do) also assume that $h_{k+1}(M_k)/G$ is relatively compact in M_{k+1}/G. As in Section 5.A.3, let $\Gamma = \ker \pi$. As G' acts properly on M_{k+1}/Γ and by the relative compactness assumption, we may embed $h_k(M_{k-1})/\Gamma$ in a manifold approximating the classifying space for proper actions E' of G'. Using a subsequence of the M'_k we may assume that we are given equivariant smooth maps $\ell_k : M_k \to M'_k$.

Up to taking again a subsequence, we may further assume that the maps $h'_k \circ \ell_k$ and $\ell_{k+1} \circ h_k$ are homotopic (where $h'_k : M'_k \to M'_{k+1}$). Note that the maps ℓ_k are automatically K-oriented, and thus we obtain KK-elements $(\ell_k)! \in KK(A_k, A'_k) = KK(C_0(M_k) \rtimes G, C_0(M'_k) \rtimes G')$ satisfying $(\ell_k)! \otimes (h'_k)! = (h_k)! \otimes (\ell_{k+1})!$.

Now, using [Lafforgue 2007, Appendix A], we find (explicit) algebras D_k and morphisms $f_k : D_k \to A_k$ which are K-equivalences and $g_k : D_k \to A'_k$ such that $(\ell_k)! = [f_k]^{-1} \otimes [g_k]$.

Put then $x_k = [f_k] \otimes (h_k)! \otimes [f_{k+1}]^{-1} \in KK(D_k, D_{k+1})$. We find

$$\begin{aligned}(g_{k+1})_*(x_k) &= [f_k] \otimes (h_k)! \otimes [f_{k+1}]^{-1} \otimes [g_{k+1}] \\ &= [f_k] \otimes (h_k)! \otimes (\ell_{k+1})! = [f_k] \otimes (\ell_k)! \otimes (h'_k)! \\ &= [g_k] \otimes (h'_k)!.\end{aligned}$$

As shown in Section 5.B, using precise homotopies between Kasparov bimodules representing these elements, we can then construct elements $y_k \in KK(C_{g_k}, C_{g_{k+1}})$. Note also, that we have the equalities $\pi_*((q_k)_!) = (\ell_k)! \otimes (q'_k)! \in KK(A_k, C^*(G'))$ as in Proposition 5.1, yielding an element $z_k \in KK(C_{g_k}, C_\pi)$.

In order to construct the topological K-theory for the mapping cone we need to make the following assumption — which could be true in general:

Assumption A.1. We assume that the homotopies used in the constructions of y_k and z_k are well matching, so that we have the equality $y_k \otimes z_{k+1} = z_k$.

We can then construct, for each k, C^*-algebras B_k and B'_k, morphisms $u_k : B_k \to D_k$ and $u'_k : B_k \to A'_k$ which are KK-equivalences and $v_k : B_k \to D_{k+1}$ and $v'_k : B_k \to D_{k+1}$ such that $x_k = [u_k]^{-1} \otimes [v_k]$ and $(h'_k)! = [u'_k]^{-1} \otimes [v'_k]$ (using [Lafforgue 2007, Appendix A]).

For the topological K-theory of G and G' (up to a shift of dimension by 1) we can then use the infinite telescopic algebras $T = T(v, u)$ and $T' = T(v', u')$. These algebras are mapping tori $\mathcal{T}(\check{u}, \check{v})$ and $\mathcal{T}(\check{u}', \check{v}')$, where

$$\check{u}, \check{v} : \check{B} = \bigoplus_{k=1}^{+\infty} B_k \to \check{D} = \bigoplus_{k=0}^{+\infty} D_k \quad \text{and} \quad \check{u}', \check{v}' : \check{B}' = \bigoplus_{k=1}^{+\infty} B'_k \to \check{A}' = \bigoplus_{k=0}^{+\infty} A'_k$$

are the maps given by

$$\begin{aligned}\check{u}(x_1, \dots, x_k, \dots) &= (0, u_1(x_1), \dots, u_k(x_k), \dots), \\ \check{v}(x_1, \dots, x_k, \dots) &= (v_1(x_1), \dots, v_k(x_k), \dots),\end{aligned}$$

and analogous formulae for $\check{u}'$ and $\check{v}'$.

The families of $(q_k)!$ and $(q'_k)!$ give elements $\check{q}!$ and $\check{q}'!$ in $KK(\check{D}, C^*(G))$ and $KK(\check{A}', C^*(G'))$, respectively.

The homotopy between $[\check{u}] \otimes q!$ and $[\check{v}] \otimes q!$ (resp. $[\check{u}'] \otimes q'!$ and $[\check{v}'] \otimes q'!$) gives rise to the element $\mu_G \in KK(T, C^*(G))$ (resp. $\mu'_G \in KK(T', C^*(G')))$.

We may now do the same construction at the mapping cone level: writing $y_k = [\alpha_k]^{-1} \otimes [\beta_k]$, where $\alpha_k : V_k \to C_{g_k}$ and $\beta_k : V_k \to C_{g_{k+1}}$ are morphisms, we may consider the infinite telescope $T(\beta, \alpha) = \mathcal{T}(\check{\alpha}, \check{\beta})$ as a topological K-theory $K_{*,\mathrm{top}}(\pi)$ for C_π. The element $\check{z}$ defined by the z_k's gives an element of $KK\big(\bigoplus C_{g_k}, C_\pi\big)$; a homotopy between $[\check{\alpha}] \otimes \check{z}$ and $[\check{\beta}] \otimes \check{z}$ (based on our assumption) gives rise to the Baum–Connes element $\mu_\pi \in KK(T(\beta, \alpha), C_\pi)$ and thus a morphism $\mu_\pi : K_{*,\mathrm{top}}(\pi) \to K_*(C_\pi)$.

Remark A.2. One may push a little further the above calculations. Indeed one needs to check that we have an exact sequence

$$\begin{array}{ccc} K_{*,\mathrm{top}}(G) & \xrightarrow{\quad \pi_* \quad} & K_{*,\mathrm{top}}(G') \\ & \nwarrow \qquad \swarrow & \\ & K_{*,\mathrm{top}}(\pi) & \end{array}$$

compatible with the mapping cone exact sequence. It then follows that if G and G' satisfy the (full version of the) Baum–Connes conjecture, then so does C_π.

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Witt groups of abelian categories and perverse sheaves

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We study the Witt groups $W_{\pm}(\mathrm{Perv} X)$ of perverse sheaves on a finite-dimensional topologically stratified space X with even-dimensional strata. We show that $W_{\pm}(\mathrm{Perv} X)$ has a canonical decomposition as a direct sum of the Witt groups of shifted local systems on strata. We compare this with another "splitting decomposition" for Witt classes of perverse sheaves obtained inductively from our main new tool, a "splitting relation" which is a generalisation of isotropic reduction.

The Witt groups $W_{\pm}(\mathrm{Perv} X)$ are identified with the (nontrivial) Balmer–Witt groups of the constructible derived category $\mathrm{D}^b_c(X)$ of sheaves on X, and also with the corresponding cobordism groups defined by Youssin.

Our methods are primarily algebraic and apply more widely. The general context in which we work is that of a triangulated category with duality, equipped with a self-dual t-structure with noetherian heart, glued from self-dual t-structures on a thick subcategory and its quotient.

1. Introduction

The signature of a compact, oriented manifold is a basic topological invariant. It is an obstruction to the existence of a null-bordism, and plays a key role in surgery theory and the classification of manifolds. The signature can be extended to singular spaces by using intersection cohomology — a compact Witt space W is a space whose rational intersection cohomology satisfies Poincaré duality and $\sigma(W)$ is defined to be the signature of the associated intersection form. For example, any irreducible complex analytic or algebraic variety is a Witt space. A more refined invariant is the Witt class $w(W)$ of the intersection form in the rational Witt group $\mathrm{W}(\mathbb{Q})$. This determines the signature but also contains torsion information which is localised on the singularities of the space. For manifolds, and more generally for spaces with integral Poincaré duality such as integral homology manifolds and intersection Poincaré spaces [Pardon 1990], this torsion information vanishes and the Witt class is simply the signature. The Witt class is the obstruction to the existence of a Witt null-bordism [Siegel 1983]. It plays an analogous role in

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stratified surgery theory and the classification of stratified spaces to that played by the signature for manifolds.

In this paper we study the Witt group $W(\mathrm{Perv}(X))$ of perverse sheaves. Here X is a finite-dimensional topologically stratified space with even-dimensional strata, and $\mathrm{Perv}(X)$ the category of perverse sheaves, constructible with respect to the stratification, with rational coefficients. A proper stratified map $f : W \to X$ from a Witt space W with $\dim W \equiv 0\,(4)$ determines a class $[f_* I_W] \in W(\mathrm{Perv}(X))$ whose pushforward to $W(\mathrm{Perv}(\mathrm{pt})) \cong W(\mathbb{Q})$ for X compact is $w(W)$. Here

$$I_W : \mathcal{IC}(W) \to D\mathcal{IC}(W)$$

is the symmetric intersection form of the corresponding intersection cohomology complex of W, with D the Verdier duality for constructible sheaf complexes. Thus $W(\mathrm{Perv}(X))$ is the natural home for relative invariants of spaces over X.

The category $\mathrm{Perv}(X)$ is constructed by "gluing together" categories of shifted local systems on the strata of X. As a consequence $W(\mathrm{Perv}(X))$ decomposes as a direct sum of the Witt groups of shifted local systems — see Corollary 3.2. We refer to the associated decomposition of a class as the *canonical decomposition*. In Section 3 we give an algorithm, starting from a top-dimensional open stratum (see (3.9)) for computing the canonical decomposition of a class. The algorithm relies on the ability to identify maximal isotropic subobjects of forms on local systems, so its feasibility depends on the complexity of the fundamental groups of the strata of X. We are also interested in the structure of the Witt group $W(\mathrm{Perv}(X))$ itself, which by the above can be reduced to the simpler and more classical case of Witt groups of local systems; for instance, see Example 3.4 for the case of real coefficients and all strata orientable. If for example all strata S of X are simply connected, therefore orientable, then Corollary 3.2 implies

$$W(\mathrm{Perv}(X)) \cong \bigoplus_{S:\dim S \equiv 0\,(4)} W(\mathbb{Q}). \tag{1.1}$$

In particular $W(\mathrm{Perv}(X)) = 0$ when all strata S of X are simply connected with $\dim S \equiv 2\,(4)$. In the above mapping situation this implies $[f_* I_W] = 0$ and hence $w(W) = 0 \in W(\mathbb{Q})$. See [Cappell and Shaneson 1991a, Theorem 6.1] for the corresponding vanishing of the signature $\sigma(W)$.

Cappell and Shaneson [1991b, Theorem 2.1] state an expression for a Witt class as a sum of classes of forms on intersection cohomology complexes; see [Banagl 2007, Chapter 8]. To be a little more precise, they obtain a decomposition for a class in their cobordism group $\Omega_{CS}(X)$ of symmetric self-dual complexes, but we show the latter is isomorphic to the Witt group of perverse sheaves — see Proposition 2.14 and Corollary 2.15:

$$W(\mathrm{Perv}(X)) \cong \Omega_{CS}(X).$$

They view this decomposition as an up-to-cobordism topological analogue of the following famous decomposition theorem:

Theorem 1.2. *Let $f : W \to X$ be a proper stratified morphism of complex algebraic varieties, with W irreducible and all strata S of X also complex algebraic.*

(1) *Decomposition: $Rf_* \mathcal{IC}(W) \cong \bigoplus_i {}^p R^i f_* \mathcal{IC}(W)$ is isomorphic to the direct sum of the corresponding perverse direct image sheaves.*

(2) *Strict support: Each perverse direct image sheaf for $i \in \mathbb{Z}$ is a direct sum ${}^p R^i f_* \mathcal{IC}(W) \cong \bigoplus_S \mathcal{IC}(\bar{S}; \mathcal{L}_{i,S})$ of twisted intersection cohomology sheaf complexes on the closures $\bar{S}$ of the strata S.*

(3) *Semisimplicity: The local system $\mathcal{L}_{i,S}$ on S is semisimple for all i and S.*

Remark 1.3. This decomposition theorem is due to Beilinson, Bernstein, Deligne and Gabber [Beilinson et al. 1982, Théorème 6.2.5] via arithmetic techniques and results for perverse sheaves on schemes over a finite base field. Another proof and far reaching extension, even applying for a projective morphism of complex analytic varieties, was given by M. Saito [1988; 1990] via his theory of pure and mixed Hodge modules. Finally, in the complex algebraic context a more geometric proof was found by de Cataldo and Migliorini [2005]. We refer to the beautiful survey [de Cataldo and Migliorini 2009] for more details, as well as to [Budur and Wang 2017, Introduction] for a short overview of the recent extension of the decomposition theorem to semisimple perverse sheaf complexes.

In our topological context we obtain, in analogy to (2) and (3) above, a decomposition up to isomorphism for anisotropic forms on perverse sheaves, but only up to Witt equivalence in general. In fact the perverse sheaves underlying an anisotropic form are semisimple (see Corollary 2.12 for the corresponding algebraic result in a noetherian abelian category with duality). The perverse sheaves underlying pure algebraic Hodge modules automatically carry anisotropic forms coming from polarisations [Saito 1988, §5.2]. Similarly, polarisations of Hodge structures for suitable topological intersection pairings appear inductively in the proof of [de Cataldo and Migliorini 2005]. This explains why one has a stronger result when working in the algebraic as opposed to in our topological context.

In our notation the *Cappell–Shaneson decomposition* is (1.6) below. Since intersection cohomology complexes are precisely the intermediate extensions of local systems on the strata it makes sense to compare the canonical and Cappell–Shaneson decompositions. Before doing so though, we should mention that there is an error in their proof, and (1.6) needs correcting for stratifications of depth greater than or equal to two. The depth one results cited in [Brasselet et al. 2010, Theorem 4.2] and [Levikov 2011] are correct. An explicit counterexample for a depth two stratification is provided in Section 3C using a quiver description for perverse sheaves on rank stratifications [Braden and Grinberg 1999].

Using a different method of proof we obtain a new, more complicated, expression (3.10) which reduces to Cappell and Shaneson's in certain cases, e.g., for anisotropic forms on perverse sheaves (see Proposition 3.11). The key ingredient in the proof is the following "splitting relation" for Witt classes: Let $\imath : Y \hookrightarrow X$ be the inclusion of a closed stratified subspace, in other words Y is a closed union of strata of X, with $\jmath : U = X - Y \hookrightarrow X$ the complementary open inclusion. Suppose $\beta : \mathcal{B} \to D\mathcal{B}$ is a nondegenerate symmetric form in $\mathrm{Perv}(X)$. Then

$$[\beta] = [\imath_* \imath^{!*} \beta] + [\jmath_{!*} \jmath^* \beta] \tag{1.4}$$

in the Witt group $W(\mathrm{Perv}(X))$. Here

$$\jmath_{!*} = \mathrm{im}({}^p\jmath_! \to {}^p\jmath_*) \quad \text{and} \quad \imath^{!*} = \mathrm{im}({}^p\imath^! \to {}^p\imath^*)$$

are the intermediate extension and restriction, respectively. In the depth one case in which Y and U are topological manifolds and the strata are their connected components, this reduces to the Cappell–Shaneson decomposition (1.6) below. In this case the perverse truncation used to define the intermediate restriction

$$\imath^{!*} = \mathrm{im}({}^p\imath^! \to {}^p\imath^*)$$

is just truncation of a sheaf complex with respect to the standard t-structure. In general however, the intermediate restriction uses the more complicated perverse truncation, which cannot be expressed easily in geometric terms. By iterated application of the "splitting relation" (1.4) we end up with our new decomposition (3.10), which we refer to as the *splitting decomposition*. This involves iterated intermediate restrictions. It turns out that the splitting decomposition (3.10) is *not* the canonical decomposition in general. Moreover, it can depend upon the choice of representative for the Witt class and on a choice of ordering of the strata of X. The reason for these negative results is that intermediate extension is not an exact functor. When it is, one obtains stronger results. In particular, we have the following (see also Corollary 3.16).

Corollary 1.5. *If each stratum has finite fundamental group, or if certain (twisted) intersection cohomology groups of links vanish, then the splitting decomposition (3.10) is the canonical one. Moreover, under the second vanishing condition it simplifies to Cappell and Shaneson's decomposition*

$$[\beta] = \sum_{\text{strata } S} [\imath_{S*} \jmath_{S!*} \jmath_S^* \imath_S^{!*} \beta], \tag{1.6}$$

where $\imath_S : \bar{S} \hookrightarrow X$ and $\jmath_S : S \hookrightarrow \bar{S}$ are the inclusions.

Remark 1.7. In the complex algebraic context the results of this paper don't contribute any new information to the decomposition theorem, except that the decomposition of ${}^pR^0 f_*\mathcal{IC}(W)$ in Theorem 1.2 fits with the Cappell–Shaneson decomposition as well as with our canonical and splitting decompositions, because the induced form f_*I_W on ${}^pR^0 f_*\mathcal{IC}(W)$ is anisotropic.

In our topological context the canonical decomposition comes from the direct sum decomposition of $W(\mathrm{Perv}(X))$, and is very helpful for understanding the structure of this Witt group. However, the canonical decomposition of f_*I_W in the stratified mapping situation $f : W \to X$ for a Witt space W is very difficult to understand in terms of the geometry of f, since one has to find an anisotropic representative in the Witt class $[f_*I_W]$.

Cappell and Shaneson [1991b] give a nice geometric interpretation of their decomposition (1.6), but this may differ from our canonical decomposition and only holds under additional assumptions. Our splitting decomposition can be viewed as a technical tool to relate the Cappell–Shaneson and canonical decompositions, when the former holds.

Remark 1.8. Cappell and Shaneson introduce the notion of a "locally nonsingular" self-dual perverse sheaf and show in [Cappell and Shaneson 1991b, Theorem 3.2] that such a "locally nonsingular" self-dual perverse sheaf is isometric to an orthogonal direct sum of forms on twisted intersection cohomology complexes $\mathcal{IC}(\bar{S}; \mathcal{L}_S)$ as in part (2) of Theorem 1.2. This result can also be shown by induction (starting from a closed stratum of smallest dimension) via the "splitting criterion" of [de Cataldo and Migliorini 2005, Lemma 4.1.3 and Remark 4.1.2] as in the approach of de Cataldo and Migliorini to the decomposition theorem. This corresponds to a decomposition of a perverse sheaf as a direct sum of twisted intersection cohomology complexes $\mathcal{IC}(\bar{S}; \mathcal{L}_S)$, similar to the "strict support decomposition" of pure Hodge modules in [Saito 1988, (5.1.3.5) and Lemma 5.1.4]. It implies the Cappell–Shaneson decomposition (1.6) in the Witt group, but it need not correspond to the canonical decomposition, because here one doesn't require the local systems $\mathcal{L}_S$ to be semisimple; cf. Example 2.21 and (2.25) for abstract algebraic counterparts. In particular the notion of a "locally nonsingular" self-dual perverse sheaf is weaker than that of an anisotropic form on a perverse sheaf.

For the purposes of this introduction we have framed the above results in a geometric context. However, our methods are primarily algebraic and apply more widely; see Examples 2.16 and 2.17. The general context in which we work is that of a triangulated category with duality, and a self-dual t-structure glued from self-dual t-structures on a thick subcategory and its quotient. Our first main result, Proposition 2.14, identifies $W(\mathrm{Perv}(X))$ with the zeroth Balmer–Witt group of the constructible derived category $\mathrm{D}^b_c(X)$ of sheaves on X:

$$W(\mathrm{Perv}(X)) \cong W_0(\mathrm{D}_c^b(X)). \tag{1.9}$$

This implies many functorial properties of the Witt group $W(\mathrm{Perv}(X))$ of perverse sheaves. For a stratified map $f : W \to X$ from a compact Witt space W it implies

$$[f_* I_W] = f_*[I_W] \in W(\mathrm{Perv}(X)) \cong W_0(\mathrm{D}_c^b(X))$$

is the direct image of the symmetric intersection form

$$[I_W] \in W_0(\mathrm{D}_c^b(W))$$

under the pushforward $f_* = f_!$, which commutes with Verdier duality. In a sense this is the substitute for part (1) of Theorem 1.2 in our topological context.

When X is compact and admits a triangulation compatible with the stratification, for instance when X is a compact Whitney or subanalytic stratified space, then we can pass to the zeroth Balmer–Witt group of the PL-constructible derived category. With $\mathbb{Q}$ coefficients, these Witt groups form a generalised homology theory isomorphic to symmetric L-theory [Woolf 2008, Corollary 4.10]. Our splitting decomposition therefore induces formulæ for the L-theoretic fundamental classes $[\beta]_{\mathbb{L}}$ of self-dual perverse sheaves

$$W(\mathrm{Perv}(X)) \cong W_0(\mathrm{D}_c^b(X)) \to W_0(\mathrm{D}_{pl-c}^b(X)) : [\beta] \mapsto [\beta]_{\mathbb{L}}$$

as sums of forms on simple perverse sheaves. In our approach it is important to start with the constructible derived category with respect to a fixed stratification, with its self-dual perverse t-structure, since the latter is not visible in the PL context. Such formulæ for L-theoretic fundamental classes of self-dual perverse sheaves were foreseen in [Cappell and Shaneson 1991b] as natural improvements of their formulæ for homological L-classes of self-dual perverse sheaves. This simple definition of the L-theoretic fundamental classes of self-dual perverse sheaves needs the identification (1.9) with Balmer-Witt groups, and not just the cobordism groups $\Omega_{CS}(X)$ of [Cappell and Shaneson 1991b] (or [Youssin 1997]).

Pushing forward to a point one obtains corresponding formulæ for signatures and Witt classes of self-dual perverse sheaves. These generalise the classical Chern–Hirzebruch–Serre formula for the signature of a smooth fibre bundle to singular spaces and perverse sheaves on them. Since this is not the subject of this paper, we only illustrate it by the following simple example of a compact oriented base manifold X as a one stratum space. Let $f : W \to X$ be a proper stratified map from a Witt space W with $\dim W \equiv 0\,(4)$ to a compact oriented manifold X of even dimension. The fibre F of f is also a Witt space. Assume ${}^pR^0 f_*\mathcal{IC}(W)$ is a constant local system on X, i.e., $\pi_1 X$ acts trivially on the middle-dimensional intersection cohomology $IH^{(\dim W - \dim X)/2}(F)$. Then

$$w(W) = \sigma(X) \cdot w(F) \in W(\mathbb{Q}).$$

The main tool we use is the aforementioned "splitting relation" (Theorem 2.19), which is a generalisation of isotropic reduction. This is expressed most naturally in terms of degenerate forms, and so in Section 2 we review the construction of the Witt group of an abelian category explaining how to treat degenerate forms on an equal footing with nondegenerate ones. The Witt class of a degenerate form is the class of the induced nondegenerate form on its image; for this reason it is essential that we work with abelian categories rather than in the broader context of exact categories, where there is no notion of image.

Our main results are consequences of the splitting relation. Firstly, it implies

$$\mathrm{W}(\mathrm{A}) \cong \bigoplus_{[s \cong Ds]} \mathrm{W}(\langle s \rangle),$$

where A is a noetherian abelian category, $\langle s \rangle$ is the full Serre subcategory generated by the self-dual simple object $s \cong Ds$, and the sum is over isomorphism classes of such objects. This is well-known; see for example [Quebbemann et al. 1979, §6] or [Sheiham 2001, Chapter 5], although the usual proof uses Hermitian dévissage rather than our splitting relation. See also [Youssin 1997, Corollary 4.13], but note that the $\mathrm{W}(\langle s \rangle)$ need not be freely generated as claimed there. Secondly, when

$$\mathrm{A} \xrightarrow{\imath_*} \mathrm{B} \xrightarrow{\jmath^*} \mathrm{C}$$

is an exact triple of triangulated categories with duality and the self-dual t-structure on B is glued from t-structures on A and C, the splitting relation yields a formula

$$[\beta] = [\imath_* \imath^{!*} \beta] + [\jmath_{!*} \jmath^* \beta]$$

in $W(\mathrm{B}^0)$, where B^0 is the self-dual heart of the t-structure. In general this formula depends upon the representative form β.

In Section 3 we apply these algebraic results to categories of perverse sheaves on a topologically stratified space with finitely many strata. The splitting decomposition (3.10) is obtained by iteratively applying the splitting relation: we choose an ordering of the strata and split off terms on an open stratum one-by-one. In Sections 3C and 3D we provide some explicit examples and counterexamples using the quiver description of perverse sheaves on a rank stratification given in [Braden and Grinberg 1999] and on Schubert-stratified projective spaces given in [Braden 2002].

In the final section we turn our attention to algebraically constructible perverse sheaves $\mathrm{Perv}_{\mathrm{alg}}(X)$ on a complex algebraic variety X. If $f : X \to \mathbb{C}$ is an algebraic map then the unipotent nearby and vanishing cycles formalism of [Beilinson 1987] provides an equivalence between this and a "gluing category" built from $\mathrm{Perv}_{\mathrm{alg}}(f^{-1}(0))$ and $\mathrm{Perv}_{\mathrm{alg}}(X - f^{-1}(0))$. In this situation too the Witt group decomposes as a direct sum

$$W(\mathrm{Perv}_{\mathrm{alg}}(X)) \cong W(\mathrm{Perv}_{\mathrm{alg}}(X - f^{-1}(0))) \oplus W(\mathrm{Perv}_{\mathrm{alg}}(f^{-1}(0))).$$

The projection is given by restriction along $\jmath : X - f^{-1}(0) \hookrightarrow X$ and the perverse unipotent vanishing cycles functor Φ_f^{un}, and the inclusions are given by the maximal extension functor Ξ_f^{un} and extension by zero along $\imath : f^{-1}(0) \hookrightarrow X$. Corollary 3.34 relates this decomposition to the terms in the splitting formula, specifically

$$[\imath^{!*}\beta] = \Phi_f^{\mathrm{un}}[\beta] - [\Psi_f^{\mathrm{un}}(\jmath^*\beta) \circ N],$$
$$[\jmath_{!*}\gamma] = \Xi_f^{\mathrm{un}}[\gamma] + \imath_*[\Psi_f^{\mathrm{un}}\gamma \circ N],$$

where Ψ_f^{un} is the perverse unipotent nearby cycles functor, and $N : \Psi_f^{\mathrm{un}} \to \Psi_f^{\mathrm{un}}(-1)$ is, up to a Tate twist, the logarithm of the monodromy μ acting on Ψ_f^{un}.

2. Witt groups

2A. *Categories with duality.* A *category with duality* is a triple (A, D, χ) in which A is a category, D is a functor $\mathsf{A}^{\mathrm{op}} \to \mathsf{A}$, and χ is a natural isomorphism $\mathrm{id} \to D^2$ such that the morphisms

$$Da \xrightarrow{\chi_{Da}} D^3 a \quad \text{and} \quad D^3 a \xrightarrow{D\chi_a} Da$$

are mutually inverse for any object $a \in \mathsf{A}$.

Examples 2.1. We are principally interested in abelian categories with duality. These arise in many contexts in topology, geometry and representation theory, usually related to finite-dimensional representations of some (graded) algebra with involution. Prominent examples include

(1) local systems on a topological manifold M (in the connected case these are modules over the group ring of the fundamental group $\pi_1 M$, with involution induced by the group inverse);

(2) finite-dimensional representations of a quiver with involution (as in [Young 2016, §3.2]);

(3) finitely generated torsion modules over a Dedekind ring R.

In each case the duality is given by morphisms into a dualising object: in the first case this is the orientation sheaf or_M of M — if M is connected and oriented this is the trivial representation of the fundamental group; in the second case it is the constant one-dimensional representation; in the third case it is $Q(R)/R$, where $Q(R)$ is the quotient field.

A *bilinear form* on an object $a \in \mathrm{A}$ is a morphism $\alpha : a \to Da$. A form is *nondegenerate* if α is an isomorphism, and it is ϵ*-symmetric*, where ϵ is either $+1$ or -1, if the diagram

$$\begin{array}{ccc} a & \xrightarrow{\alpha} & Da \\ & {\scriptstyle\chi(a)}\searrow \quad \nearrow{\scriptstyle\epsilon D\alpha} & \\ & D^2 a & \end{array}$$

commutes. To make sense of ϵ-symmetry we need A to be additive. In fact it suffices to consider the case $\epsilon = 1$ since we may always absorb the sign into the definition of the natural transformation χ, i.e., antisymmetric forms are symmetric forms for a different duality.

Fix a bilinear form $\beta : b \to Db$. Given a morphism $f : a \to b$, the *restriction* $\beta|_f$ is the composite $Df \circ \beta \circ f$ on a. When f is a monomorphism we often abuse notation and denote the restriction by $\beta|_a$. The restriction $\beta|_f$ is symmetric whenever β is.

Bilinear forms α and β are *isometric*, written $\alpha \cong \beta$, if there is an isomorphism $f : a \to b$ such that $\alpha = \beta|_f$. For example, when $\alpha : a \to Da$ is nondegenerate then $(D\alpha)^{-1} = \epsilon\chi(a)\alpha^{-1}$ is a symmetric form and is isometric to α because

$$\begin{CD} a @>{\alpha}>> Da \\ @V{\alpha}VV @VV{(D\alpha)^{-1}}V \\ Da @<<{D\alpha}< D^2 a \end{CD}$$

commutes. Isometry is an equivalence relation which preserves nondegeneracy and symmetry. The *Witt monoid of degenerate forms* $\widetilde{\mathrm{MW}}(\mathrm{A})$ is the set of isometry classes of symmetric forms under direct sum. The nondegenerate symmetric forms constitute a submonoid, the *Witt monoid* $\mathrm{MW}(\mathrm{A})$.

Suppose $(\mathrm{A}, D_{\mathrm{A}}, \chi_{\mathrm{A}})$ and $(\mathrm{B}, D_{\mathrm{B}}, \chi_{\mathrm{B}})$ are categories with duality, and $F : \mathrm{A} \to \mathrm{B}$ a functor. We say that F *commutes with duality* if there is a natural isomorphism $\eta : FD_{\mathrm{A}} \to D_{\mathrm{B}}F$ such that

$$\begin{CD} F @>{F\chi}>> FD_{\mathrm{A}}^2 \\ @V{\chi F}VV @VV{\eta D_{\mathrm{A}}}V \\ D_{\mathrm{B}}^2 F @>>{D_{\mathrm{B}}\eta}> D_{\mathrm{B}}FD_{\mathrm{A}} \end{CD}$$

commutes. This ensures that $\eta_a F\alpha$ is symmetric for D_{B} whenever $\alpha : a \to Da$ is symmetric for D_{A}. Such a functor induces a morphism $\widetilde{\mathrm{MW}}(\mathrm{A}) \to \widetilde{\mathrm{MW}}(\mathrm{B})$ which restricts to a morphism between the submonoids of nondegenerate forms. We suppress the natural transformation η_a and simply write $F\alpha$ for the image form.

2B. ***Witt groups of abelian categories.*** Suppose that A is an abelian category with exact duality D. It follows that if $\ker f \hookrightarrow a$ is a kernel of $f : a \to b$ then

$Da \twoheadrightarrow D\ker f$ is a cokernel of $Df : Db \to Da$. Therefore there is a canonical isomorphism $D\ker f \cong \operatorname{coker} Df$, and similarly $D\operatorname{coker} f \cong \ker Df$. In practice we suppress these identifications.

Fix a symmetric form $\beta : b \to Db$. A subobject $\imath : a \hookrightarrow b$ is

(1) *β-isotropic* if the restriction $\beta|_\imath$ is 0;

(2) *β-lagrangian* if the sequence $0 \to a \xrightarrow{\imath} b \xrightarrow{D\imath\beta} Da \to 0$ is exact;

(3) and *β-null* if $\beta \circ \imath = 0$.

When the form β is understood we suppress it from the notation. Null and lagrangian subobjects are isotropic, but not necessarily *vice versa*. Isotropic subobjects are also known, for instance in [Balmer 2005], as *sublagrangians* because any subobject of a lagrangian is isotropic. If a form has no nonzero isotropic subobjects we say it is *anisotropic*.

The *orthogonal complement* of a subobject $\imath : a \hookrightarrow b$ is defined to be the subobject

$$a^\beta = \ker(D\imath\beta).$$

A subobject $\imath : a \hookrightarrow b$ is isotropic if and only if it factors through a^β, lagrangian if and only if the factorisation is an isomorphism $a \cong a^\beta$ and null if and only if the inclusion is an isomorphism $a^\beta \cong b$.

A nondegenerate form η which has a lagrangian is called *metabolic*. Nondegenerate forms β_0 and β_1 are *Witt-equivalent* if they are stably isometric by metabolic forms, i.e., if there exist metabolic forms η_0 and η_1 such that

$$\beta_0 \oplus \eta_0 \cong \beta_1 \oplus \eta_1.$$

This defines an equivalence relation on $\mathrm{MW}(\mathrm{A})$.

Definition 2.2. The *Witt group* $\mathrm{W}(\mathrm{A})$ of A is the set of Witt-equivalence classes in $\mathrm{MW}(\mathrm{A})$ under $\oplus$. This is a group, not just a monoid, because $\beta \oplus -\beta$ is Witt equivalent to 0. The class of a nondegenerate symmetric form β is denoted $[\beta]$.

Remark 2.3. Making the analogous definitions with antisymmetric forms in place of symmetric ones or, as explained above, working with symmetric forms in the category with duality $(\mathrm{A}, D, -\chi)$, we obtain the Witt group $\mathrm{W}_-(\mathrm{A})$ of antisymmetric forms.

If $F : \mathrm{A} \to \mathrm{B}$ is an exact functor which commutes with duality then it preserves metabolic forms and so induces homomorphisms

$$\mathrm{W}_\pm(F) : \mathrm{W}_\pm(\mathrm{A}) \to \mathrm{W}_\pm(\mathrm{B}).$$

We will see shortly that in some cases we can weaken the requirement that F is exact.

2C. *Isotropic reduction.* Fix a symmetric form $\beta : b \to Db$. Given a null subobject $\imath : a \hookrightarrow b$ there is an induced symmetric form on the cokernel of $\imath$ such that

$$\begin{array}{ccc} b & \twoheadrightarrow & \operatorname{coker}\imath \\ \downarrow{\scriptstyle\beta} & & \downarrow \\ Db & \hookleftarrow & \ker D\imath \end{array}$$

commutes; symmetry follows from the uniqueness of the induced morphism. In particular, $\ker\beta$ is always null and the symmetric form $\bar{\beta} : \operatorname{im}\beta \to \operatorname{coim} D\beta$ is nondegenerate.

This is a special case of a more general construction starting from an isotropic subobject $\imath : a \hookrightarrow b$. Note that the factorisation $a \hookrightarrow a^{\beta}$ is always null for the restriction $\beta|_{a^{\beta}}$ because

$$D(a^{\beta}) = D\ker(D\imath\beta) \cong \operatorname{coker}(D\beta D^{2}\imath) \cong \operatorname{coker}(\beta\imath).$$

It is a kernel of $\beta|_{a^{\beta}}$ when β is nondegenerate. The *isotropic reduction* $\beta \lhd a$ is defined to be the induced symmetric form on the cokernel of $a \hookrightarrow a^{\beta}$. We note some special cases: when β is nondegenerate $\beta \lhd a = \overline{\beta|_{a^{\beta}}}$, when a is a null subobject $\beta \lhd a$ is the induced symmetric form on the quotient, and in particular $\beta \lhd \ker\beta = \bar{\beta}$. The isotropic reduction is the zero form on the zero object if, and only if, $\imath : a \to b$ is lagrangian. If β is nondegenerate then so is any reduction of β (but not *vice versa*).

Isotropic reduction is compatible with restriction to a subobject in the following sense.

Lemma 2.4. *Suppose we have a commutative diagram*

$$\begin{array}{ccccc} a & \xhookrightarrow{\imath} & b & \xhookrightarrow{\jmath} & c \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ Da & \underset{D\imath}{\twoheadleftarrow} & Db & \underset{D\jmath}{\twoheadleftarrow} & Dc \end{array}$$

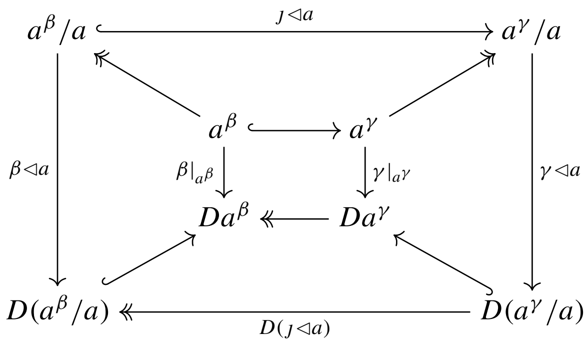
in which $\gamma : c \to Dc$ is symmetric (so that $\beta = \gamma|_{\jmath}$ and a is an isotropic subobject of both β and γ). Then there is a monomorphism $\jmath \lhd a : a^{\beta}/a \to a^{\gamma}/a$ such that

$$(\gamma|_{\jmath}) \lhd a = (\gamma \lhd a)|_{\jmath \lhd a}.$$

Proof. Taking successive pullbacks we obtain a commutative diagram:

$$\begin{array}{ccccc} a & \hookrightarrow & a^{\beta} & \hookrightarrow & a^{\gamma} \\ \| & & \downarrow & & \downarrow \\ a & \xhookrightarrow{\imath} & b & \xhookrightarrow{\jmath} & c \end{array}$$

Hence there is an induced monomorphism $\jmath \lhd a : a^\beta/a \to a^\gamma/a$ such that the top (and dual bottom) inner squares of the following diagram commute:



The remaining internal squares commute by definition. Hence the outer square commutes. □

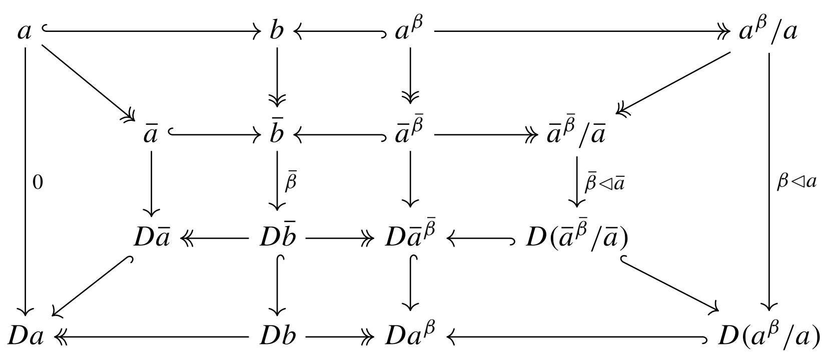
Reduction by the kernel of a degenerate form is compatible with isotropic reductions in the following sense.

Lemma 2.5. *Suppose $\iota : a \hookrightarrow b$ is isotropic for symmetric $\beta : b \to Db$. Then*

$$\overline{\beta \lhd a} \cong \bar{\beta} \lhd \bar{a},$$

where $\bar{a}$ is the image of $a \hookrightarrow b \twoheadrightarrow \operatorname{im}\beta$.

Proof. Let $\bar{b} = \operatorname{im}\beta$ and $\bar{a} = \operatorname{im}(a \hookrightarrow b \twoheadrightarrow \operatorname{im}\beta)$. Then there is a commutative diagram



The result follows by considering the right-hand square and recalling that $\bar{\beta} \lhd \bar{a}$ is an isomorphism. □

Loosely we can say that "reduction by the kernel commutes with all other isotropic reductions".

The proof of the next lemma is an elementary diagram chase, which we omit.

Lemma 2.6. *Let $\gamma : c \to Dc$ be symmetric and $a \hookrightarrow c$ isotropic. Then quotienting by a induces a one-to-one correspondence between factorisations $a \hookrightarrow b \hookrightarrow c$ with b isotropic and isotropic subobjects of the reduced form $\gamma \lhd a$. Furthermore,*

$$\gamma \lhd b \cong (\gamma \lhd a) \lhd (b/a).$$

Since the reduction of a nondegenerate form is nondegenerate, isotropic reduction generates an equivalence relation on $\mathrm{MW}(\mathrm{A})$.

Theorem 2.7 (see, e.g., [Balmer 2005, Theorem 1.1.32 and Remark 1.1.33]). *The equivalence relation on* $\mathrm{MW}(\mathrm{A})$ *generated by isotropic reduction is Witt-equivalence. Hence the set of equivalence classes is* $\mathrm{W}(\mathrm{A})$.

Although isotropic reduction is often only considered for nondegenerate forms, it is a natural operation on degenerate forms too. Let $\widetilde{\mathrm{W}}(\mathrm{A})$ be the set of equivalence classes of the relation generated by isotropic reduction on $\widetilde{\mathrm{MW}}(\mathrm{A})$. Reduction by the kernel defines a map of monoids

$$\widetilde{\mathrm{MW}}(\mathrm{A}) \to \mathrm{MW}(\mathrm{A}) : \beta \mapsto \bar{\beta}.$$

By Lemma 2.5 this map preserves the equivalence relation generated by isotropic reduction. Hence there are maps

$$\mathrm{W}(\mathrm{A}) \to \widetilde{\mathrm{W}}(\mathrm{A}) \to \mathrm{W}(\mathrm{A})$$

induced by $\mathrm{MW}(\mathrm{A}) \hookrightarrow \widetilde{\mathrm{MW}}(\mathrm{A})$ and reduction by the kernel, respectively.

Corollary 2.8. *These maps are inverse to one another. Hence* $\widetilde{\mathrm{W}}(\mathrm{A})$ *is also a group under* $\oplus$ *and it is isomorphic to the Witt group* $\mathrm{W}(\mathrm{A})$.

Proof. In one direction the composition is the identity on representatives, and in the other it is isotropic reduction by the kernel. Both induce the identity on Witt groups. □

Thus one can define the Witt group by using the isotropic reduction relation on either degenerate or on nondegenerate forms.

2D. *The splitting relation.* In this section we introduce a more general relation which allows us to split forms into two pieces. Isotropic reduction corresponds to the special case when one of these pieces is trivial.

Proposition 2.9. *Suppose $\beta : b \to Db$ is a nondegenerate symmetric form and that $0 \to a \xrightarrow{\iota} b \xrightarrow{q} c \to 0$ is a short exact sequence in* A. *Then there are induced symmetric forms $\alpha = \beta|_\iota : a \to Da$ and $\gamma = \beta|_{\beta^{-1}Dq} : Dc \to D^2c$ and*

$$[\beta] = [\overline{\alpha}] + [\overline{\gamma}]$$

in the Witt group $\mathrm{W}(\mathrm{A})$.

Proof. There is a unique isomorphism $f : \ker \alpha \to \ker \gamma$ such that

$$\begin{array}{ccccc}
\ker\alpha & & \overset{f}{\dashrightarrow} & & \ker\gamma \\
\big\downarrow & & & & \big\downarrow \\
a & \overset{\iota}{\hookrightarrow} & b & \overset{\beta^{-1}Dq}{\hookleftarrow} & Dc \\
\big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \gamma} \\
Da & \underset{D\iota}{\twoheadleftarrow} & Db & \underset{\chi(c)q\beta^{-1}}{\twoheadrightarrow} & D^2c
\end{array}$$

commutes. Let $k = \ker\alpha \cong \ker\gamma$. We can apply Lemma 2.4 simultaneously to both lower squares of the above diagram to obtain a new diagram

$$\begin{array}{ccccc}
\operatorname{im}\alpha & \hookrightarrow & k^\beta/k & \hookleftarrow & \operatorname{im}\gamma \\
\big\downarrow{\scriptstyle \overline{\alpha}} & & \big\downarrow{\scriptstyle \beta\lhd k} & & \big\downarrow{\scriptstyle \overline{\gamma}} \\
\operatorname{coim} D\alpha & \twoheadleftarrow & D(k^\beta/k) & \twoheadrightarrow & \operatorname{coim} D\gamma
\end{array}$$

in which the vertical arrows are isomorphisms. Furthermore we can check from the construction of Lemma 2.4 that the diagonal

$$\operatorname{im}\alpha \hookrightarrow k^\beta/k \twoheadrightarrow \operatorname{coim} D\gamma$$

of this new diagram is still short exact (and the other diagonal is the dual short exact sequence).

Thus we can reduce to the special case in which α and γ are nondegenerate. In this case $(\,\iota \;\; \beta^{-1}Dq\,) : a \oplus Dc \to b$ is an isomorphism and

$$\begin{pmatrix} D\iota \\ D^2q\,D\beta^{-1} \end{pmatrix} \beta (\,\iota \;\; \beta^{-1}Dq\,) = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}.$$

So $\beta \cong \alpha \oplus \gamma$ and $[\beta] = [\alpha] + [\gamma]$. More generally this argument shows that $[\beta] = [\beta \lhd k] = [\overline{\alpha}] + [\overline{\gamma}]$. □

Remarks 2.10. (1) In the situation of the above lemma $a^\beta \cong Dc$ and the restricted form γ is isometric to $\beta|_{a^\beta}$. Hence the splitting relation can be written in $\widetilde{\mathrm{W}}(\mathrm{A})$ as

$$[\beta] = [\beta|_a] + [\beta|_{a^\beta}].$$

(2) The proposition shows that the splitting relation holds in W(A). Conversely we could define W(A) using the splitting relation, for both the relation of isometry and that arising from isotropic reduction are special cases obtained by putting $c = 0$ and $\alpha = 0$, respectively.

(3) If β is anisotropic then the proof provides an isometry $\beta \cong \alpha \oplus \gamma$; in particular α and γ are also nondegenerate and anisotropic.

The following result is a minor generalisation of the splitting relation.

Corollary 2.11. *Suppose $\beta : b \to Db$ is a nondegenerate symmetric form and that $a \xrightarrow{f} b \xrightarrow{g} c$ is exact at the middle term. Then there are induced symmetric forms $\alpha = \beta|_f : a \to Da$ and $\gamma = \beta|_{\beta^{-1}Dg} : Dc \to D^2c$ such that $[\beta] = [\overline{\alpha}] + [\overline{\gamma}]$ in the Witt group* W(A).

Proof. Replacing α by $\alpha \triangleleft \ker f$ and γ by $\gamma \triangleleft \ker Dg$, we are in the situation of Proposition 2.9. Hence, using Lemma 2.5,

$$\begin{aligned}[\beta] &= \big[\, \overline{\alpha \triangleleft \ker f}\,\big] + \big[\, \overline{\gamma \triangleleft \ker Dg}\,\big] \\ &= \big[\, \overline{\alpha} \triangleleft \overline{\ker f}\,\big] + \big[\, \overline{\gamma} \triangleleft \overline{\ker Dg}\,\big] = [\overline{\alpha}] + [\overline{\gamma}].\end{aligned}$$ □

In the presence of an exact duality the following are equivalent (the last two by the Jordan–Hölder theorem):

(1) A is noetherian;

(2) A is artinian;

(3) A is artinian and noetherian;

(4) A is a length category, i.e., each object has a finite composition series with simple factors.

Under these conditions the Witt group has a more explicit description.

Corollary 2.12. *Suppose* A *is noetherian. Then the Witt group* W(A) *is the set of isometry classes of anisotropic forms. The group operation is given by choosing an anisotropic representative for the direct sum. Any anisotropic form is isometric to a direct sum of nondegenerate symmetric forms on simple objects of* A. *In particular the Witt group is generated by forms on simple objects.*

Proof. If $\beta : b \to Db$ is a symmetric form then Lemma 2.6 and the noetherian property guarantee that there is a maximal isotropic subobject $a \hookrightarrow b$. The reduction $\beta \triangleleft a$ is thus an anisotropic representative for $[\beta]$. Youssin [1997, Theorem 4.9] shows that anisotropic forms represent the same Witt class if and only if they are isometric. (In other words, even though the Witt cancellation theorem may not hold, its conclusion remains true for anisotropic forms.) Finally, by the third part of Remarks 2.10, and another application of the noetherian property, we can write an anisotropic form as a finite direct sum of forms on simple objects. □

The Witt group is not necessarily freely generated by forms on simple objects (as claimed in [Youssin 1997]) as can be seen by considering, for example, the categories of vector spaces over $\mathbb{Q}$ or $\mathbb{C}$ whose Witt groups have torsion. However, it does have a canonical direct sum decomposition into Witt groups of the Serre subcategories generated by self-dual simple objects. This is well-known; see for example [Quebbemann et al. 1979, §6] or [Sheiham 2001, Chapter 5], although the usual proof uses Hermitian dévissage rather than our splitting relation.

Corollary 2.13. *Suppose* A *is noetherian. Then there is an isomorphism*

$$\mathrm{W}(\mathrm{A}) \cong \bigoplus_{[s \cong Ds]} \mathrm{W}(\langle s \rangle)$$

where the direct sum is over isomorphism classes of self-dual simple objects and $\langle s \rangle$ *denotes the full Serre subcategory generated by self-extensions of* s.

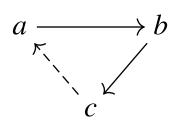
Proof. Suppose s is a self-dual simple object. Then the duality D restricts to a duality on the full Serre subcategory $\langle s \rangle$ and the inclusion ι_s is an exact functor which commutes with duality. Hence there are induced maps $\mathrm{W}(\iota_s) : \mathrm{W}(\langle s \rangle) \to \mathrm{W}(\mathrm{A})$ and combining these a map

$$\bigoplus_{[s \cong Ds]} \mathrm{W}(\langle s \rangle) \to \mathrm{W}(\mathrm{A}).$$

It is surjective by the last part of Corollary 2.12. Moreover, the description of the Witt group as isometry classes of anisotropic forms shows that it is injective; an isometry must preserve the summand consisting of forms on self-extensions of a given simple object. ☐

2E. ***Balmer–Witt groups of triangulated categories.*** A triangulated category B with duality has 4-periodic Balmer–Witt groups. Proposition 2.14 below expresses the Witt groups of the abelian heart of a self-dual t-structure on B in terms of the Balmer–Witt groups of B. This is closely related to [Balmer 2001, Theorem 4.3], which treats the special case in which the triangulated category is the bounded derived category of the heart (but which works in the more general setting of the derived category of an exact category). See also [Youssin 1997, Theorem 7.4], where the analogous result is proved for a slightly different definition of triangulated Witt group.

Suppose B is triangulated with shift functor [1]. Exact triangles in B are denoted either by $a \to b \to c \to a[1]$ or by a diagram



where the dotted arrow denotes a map $c \to a[1]$. In order that the Balmer–Witt groups of B are defined and well-behaved we always assume that

(1) B is essentially small, so that isomorphism classes of objects form a set;

(2) B satisfies the *enriched octahedral axiom*;

(3) 2 is invertible in B, i.e., given $\alpha \in \mathrm{Hom}(a, b)$ there exists α' with $\alpha = 2\alpha'$.

As noted in [Beilinson et al. 1982, Remarque 1.1.13] and [Balmer 2000], the second property is satisfied by all commonly met triangulated categories, in particular by derived categories. It also passes to triangulated subcategories and to localisations.

Suppose D is a triangulated duality on B with natural transformation $\chi : \mathrm{id} \to D^2$. Then one can define Balmer–Witt groups $W_i(\mathrm{B})$ for $i \in \mathbb{Z}$; see [Balmer 2000] but note that we use homological indexing rather than cohomological so that our $W_i(\mathrm{B})$ corresponds to Balmer's $W^{-i}(\mathrm{B})$. The group $W_0(\mathrm{B})$ is the quotient of the Witt monoid by the submonoid generated by *metabolic* forms (or *neutral* forms in the terminology of [Balmer 2000]), i.e., nondegenerate forms $\beta : b \to Db$ for which there is a *lagrangian* $\alpha : a \to b$ such that the triangle

$$a \xrightarrow{\alpha} b \xrightarrow{D\alpha\cdot\beta} Da \xrightarrow{\gamma} a[1]$$

is exact and γ is symmetric, i.e., $(D\gamma)[1] = \chi_{a[1]}\gamma$. The group $W_i(\mathrm{B})$ is defined similarly but using the shifted duality $c \mapsto (Dc)[-i]$ with natural isomorphism

$$(-1)^{i(i-1)/2}\chi : \mathrm{id} \to (D[-i])^2.$$

Although the shifted duality is not triangulated when i is odd, it is still a δ-functor, and this suffices for the construction. In contrast to the abelian case, $[\beta] = 0$ if and only if β is metabolic [Balmer 2000, Theorem 3.5]. There are natural isomorphisms $W_i(\mathrm{B}) \cong W_{i+4}(\mathrm{B})$ given by $[\beta] \mapsto [\beta[-2]]$ so that the groups are 4-periodic. The Balmer–Witt groups are functorial under triangulated functors which commute with duality since these preserve metabolic forms.

Recall that a *t-structure* on B is a strict, full subcategory $\mathrm{B}^{\leq 0} \subset \mathrm{B}$ such that $\mathrm{B}^{\leq 0}[1] \subset \mathrm{B}^{\leq 0}$ and for each $c \in \mathrm{B}$ there is an exact triangle

$$\tau^{\leq 0}c \to c \to \tau^{>0}c \to \tau^{\leq 0}c[1]$$

with $\tau^{\leq 0}c \in \mathrm{B}^{\leq 0}$ and $\tau^{>0}c \in \mathrm{B}^{>0}$, where the latter is the full subcategory on those objects c such that $\mathrm{Hom}(b, c) = 0$ for all $b \in \mathrm{B}^{\leq 0}$. Indeed, the existence of these triangles implies that $\mathrm{B}^{\leq 0}$ is right admissible with right adjoint $\tau^{\leq 0}$ to its inclusion, and that $\mathrm{B}^{>0}$ is left admissible with left adjoint $\tau^{>0}$ to its inclusion. These adjoints are referred to as truncation functors. The exact triangle associated to an object c is unique (up to isomorphism) and the first two maps in it come respectively from the counit and unit of the adjunctions.

Let $\mathrm{B}^{\leq n} = \mathrm{B}^{\leq 0}[-n]$ with left adjoint $\tau^{\leq n}$ to its inclusion, and define $\mathrm{B}^{\geq n}$ and the right adjoint $\tau^{\geq n}$ to its inclusion similarly. The subcategory $\mathrm{B}^0 = \mathrm{B}^{\leq 0} \cap \mathrm{B}^{\geq 0}$ is abelian [Beilinson et al. 1982, Théorème 1.3.6] and is known as the *heart* of the t-structure. The functor $H^0 = \tau^{\leq 0}\tau^{\geq 0} : \mathrm{B} \to \mathrm{B}^0$ is cohomological, i.e., takes exact triangles to long exact sequences.

A triangulated functor $F : \mathrm{B} \to \mathrm{C}$ between categories with respective t-structures $\mathrm{B}^{\leq 0}$ and $\mathrm{C}^{\leq 0}$ is *left t-exact* if $F\mathrm{B}^{\geq 0} \subset \mathrm{C}^{\geq 0}$, *right t-exact* if $F\mathrm{B}^{\leq 0} \subset \mathrm{C}^{\leq 0}$ and *t-exact* if it is both left and right t-exact. The induced functor ${}^pF := H^0F$ between the abelian hearts — the peculiar notation arises from the original occurrence in [Beilinson et al. 1982] of these notions in the context of perverse sheaves — is respectively left exact, right exact and exact accordingly.

If D is an exact duality on B then one can check that $D(\mathrm{B}^{\geq 0})$ is also a t-structure. We refer to this as the dual t-structure, and say a t-structure is *self-dual* if $\mathrm{B}^{\leq 0} = D(\mathrm{B}^{\geq 0})$. The duality D restricts to an exact duality on the heart of a self-dual t-structure. Conversely, if the t-structure is bounded and the heart is invariant under duality then the t-structure is self-dual. If in addition the heart is a length category then this is equivalent to the set of simple objects being invariant under duality.

Proposition 2.14. *Suppose* B *is a triangulated category with exact duality D and* B^0 *is the heart of a self-dual t-structure on* B. *Then*

$$W_i(\mathrm{B}) \cong \begin{cases} W(\mathrm{B}^0) & \text{if } i = 0 \pmod 4, \\ W_-(\mathrm{B}^0) & \text{if } i = 2 \pmod 4, \\ 0 & \text{if } i = 1 \pmod 2. \end{cases}$$

Proof. We treat the case $i = 0$ first. The inclusion $\mathrm{B}^0 \hookrightarrow \mathrm{B}$ commutes with duality and preserves metabolic forms. Therefore it induces a map $W(\mathrm{B}^0) \to W_0(\mathrm{B})$. The functor $H^0 : \mathrm{B} \to \mathrm{B}^0$ also commutes with duality. Whenever $\alpha : a \to b$ is a lagrangian for β there is an exact sequence

$$\cdots \to H^0a \to H^0b \to H^0Da \to \cdots .$$

Hence we can apply Corollary 2.11 to deduce that $[H^0\beta] = 0 \in W(\mathrm{B}^0)$. Therefore there is an induced map $W_0(\mathrm{B}) \to W(\mathrm{B}^0) : [\beta] \mapsto [H^0\beta]$. It is clear that the composite

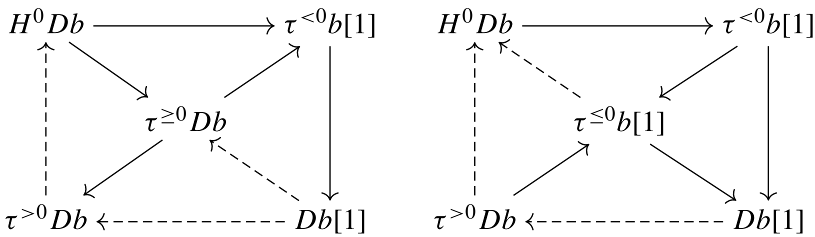
$$W(\mathrm{B}^0) \to W_0(\mathrm{B}) \to W(\mathrm{B}^0)$$

is the identity (in fact on representatives). We also claim that $[\beta] = [H^0\beta]$ in $W(\mathrm{B})$, from which it follows immediately that $W(\mathrm{B}^0) \cong W_0(\mathrm{B})$. To establish the claim we use the sublagrangian construction of [Balmer 2000, §4], which is an analogue of isotropic reduction for the triangulated setting.

Given a nondegenerate symmetric form $\beta : b \to Db$ in B we observe that $\imath : \tau^{<0}b \to b$ is isotropic (or, in the terminology of [Balmer 2000], sublagrangian) because $D\imath\beta\imath = 0$. Furthermore, the natural morphism $\jmath : \tau^{<0}b \to \tau^{\leq 0}b$ is a "good morphism" in the sense of [Balmer 2000, Definition 4.3] because there exist morphisms q and r such that the diagram (in which we omit some natural morphisms)

$$\begin{array}{ccccccc}
\tau^{<0}b & \xrightarrow{\ \iota\ } & b & \xrightarrow{\ q\ } & \tau^{\geq 0}Db & \xrightarrow{\ r[1]\ } & \tau^{<0}b[1] \\
\big\downarrow{\scriptstyle \jmath} & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle D\jmath} & & \big\downarrow{\scriptstyle \jmath[1]} \\
\tau^{\leq 0}b & \xrightarrow{\ Dq\ } & Db & \xrightarrow{\ D\iota\ } & \tau^{>0}Db & \xrightarrow{\ Dr\ } & \tau^{\leq 0}b[1]
\end{array}$$

is commutative. Indeed by applying the enhanced octahedral axiom to the octahedron below (which for ease of reading we draw as upper and lower halves with dotted arrows indicating the boundary morphisms of the exact triangles and labels on morphisms omitted)



we obtain the required triangle which shows that $\jmath$ is a "very good morphism" in the sense of [Balmer 2000, Definition 4.11]. Applying [Balmer 2000, Theorem 4.20] we deduce that $[\beta] = [H^0\beta]$ as required.

The other cases follow more easily: the group $W_1(\mathrm{B})$ vanishes because each representative $b \to (Db)[-1]$ has a lagrangian, namely $\tau^{\leq 0}b \to b$; the case $i = 2$ is similar to $i = 0$, but with symmetric forms replaced by antisymmetric ones, and the case $i = 3$ is similar to $i = 1$. We omit the details. □

Corollary 2.15. *Suppose* B *is a triangulated category with exact duality D and* B^0 *is the heart of a self-dual t-structure on* B. *Then there are canonical isomorphisms*

$$W_0(\mathrm{B}) \cong \Omega_+(\mathrm{B}) \quad \textit{and} \quad W_2(\mathrm{B}) \cong \Omega_-(\mathrm{B})$$

between the nonvanishing Balmer–Witt groups and the Youssin cobordism groups $\Omega_\pm(\mathrm{B})$ *of symmetric and antisymmetric self-dual complexes* (introduced in [Youssin 1997]).

Proof. There are canonical surjective homomorphisms

$$W_0(\mathrm{B}) \to \Omega_+(\mathrm{B}) \quad \text{and} \quad W_2(\mathrm{B}) \to \Omega_-(\mathrm{B});$$

see [Brasselet et al. 2010, p. 31]. These are compatible with the isomorphisms to $W_\pm(\mathrm{B}^0)$ provided by Proposition 2.14 and [Youssin 1997, Theorem 7.4]. □

Examples 2.16. The results above apply in various interesting examples:

(1) The bounded derived category of an abelian category with duality, with its evident induced duality and standard t-structure.

(2) The constructible derived category of sheaves of vector spaces on a finite-dimensional topologically stratified space, with only even-dimensional strata, equipped with Verdier duality and the self-dual perverse t-structure; see for example [Beilinson et al. 1982, §2; Schürmann 2003, §4.2]. Here, by topologically stratified space we mean a locally cone-like stratified space in the sense of Siebenmann; see for example [Schürmann 2003, §4.2].

(3) The constructible derived category of sheaves of torsion modules over a Dedekind ring R on a finite-dimensional topologically stratified space, with only even-dimensional strata, as studied in [Cappell and Shaneson 1991a]. The torsion condition is preserved by push-forward along open inclusions because the stalks of the push-forward can be expressed in terms of the compact link [Schürmann 2003, Remark 4.4.2], and by the Künneth formula they vanish after tensoring with $Q(R)$. It follows from [Beilinson et al. 1982, §3.3] that the perverse t-structure is self-dual for shifted Verdier duality.

(4) Let $f : X \to Y$ be a proper morphism of complex algebraic varieties, of fibre dimension at most 1 and with $Rf_*\mathcal{O}_X = \mathcal{O}_Y$. Then the standard t-structure restricts to a t-structure on the null category C_f, i.e., the full category of $D^b\,\mathrm{Coh}(X)$ on objects $\mathcal{E}$ with $Rf_*\mathcal{E} = 0$; see [Bridgeland 2002, Lemma 3.1]. If in addition f is an isomorphism outside a subvariety of dimension 0 then the heart $C_f \cap \mathrm{Coh}(X)$ is stable under shifted Grothendieck duality by [Bodzenta and Bondal 2015, Proposition 9.7 and Theorem 9.8].

2F. *Gluing and splitting.* Suppose that $\mathrm{A} \xrightarrow{\imath_*} \mathrm{B} \xrightarrow{\jmath^*} \mathrm{C}$ is an exact triple of triangulated categories, i.e., $\imath_*$ is the inclusion of a full, thick triangulated subcategory A of a triangulated category B, and C is the quotient category obtained by localising at all morphisms in A. If B has a triangulated duality which preserves the subcategory A then both A and C inherit triangulated dualities such that the inclusion $\imath_*$ and quotient $\jmath^*$ commute with duality. Theorem 6.2 of [Balmer 2000] states that there is then a long exact sequence

$$\cdots \to W_i(\mathrm{A}) \to W_i(\mathrm{B}) \to W_i(\mathrm{C}) \to W_{i-1}(\mathrm{A}) \to \cdots$$

of Balmer–Witt groups in which the first two maps are induced from $\imath_*$ and $\jmath^*$, respectively (the hypothesis in [Balmer 2000] that C is weakly cancellative is unnecessary; see [Balmer and Walter 2002, Theorem 2.1]).

For the remainder of this section we suppose further that

(1) $\imath_*$ has respective left and right adjoints $\imath^*$ and $\imath^!$;

(2) $\jmath^*$ has respective left and right adjoints $\jmath_!$ and $\jmath_*$;

(3) there exist natural transformations $\imath_*\imath^* \to \jmath_!\jmath^*[1]$ and $\jmath_*\jmath^* \to \imath_*\imath^![1]$ such that there are natural exact triangles

$$\imath_*\imath^! \to \mathrm{id} \to \jmath_*\jmath^* \to \imath_*\imath^![1] \quad \text{and} \quad \jmath_!\jmath^* \to \mathrm{id} \to \imath_*\imath^* \to \jmath_!\jmath^*[1]$$

whose other morphisms are units or counits of the relevant adjunctions;

(4) the units of the adjunctions $\imath_* \dashv \imath^!$ and $\jmath_! \dashv \jmath^*$ are isomorphisms, as are the counits of $\imath^* \dashv \imath_*$ and $\jmath^* \dashv \jmath_*$.

Some of these conditions are redundant: the existence of any one of the adjoints guarantees the existence of the other three, the triangles in the third condition are dual to one another and the final condition follows from the fact that $\imath_*$, $\jmath_*$ and $\jmath_!$ are fully faithful.

Under these conditions [Beilinson et al. 1982, Théorème 1.4.10] states that one can glue given t-structures $\mathrm{A}^{\leq 0} \subset \mathrm{A}$ and $\mathrm{C}^{\leq 0} \subset \mathrm{C}$ to obtain a t-structure

$$\mathrm{B}^{\leq 0} = \langle b \in \mathrm{B} \mid \imath^* b \in \mathrm{A}^{\leq 0},\, \jmath^* b \in \mathrm{C}^{\leq 0} \rangle$$

on B. This glued t-structure is self-dual whenever the given ones on A and C are so. With respect to these t-structures

(1) $\imath_*$ and $\jmath^*$ are t-exact;

(2) $\imath^!$ and $\jmath_*$ are left t-exact;

(3) $\imath^*$ and $\jmath_!$ are right t-exact.

The adjunctions give rise to natural morphisms $\jmath_! \to \jmath_*$ and $\imath^! \to \imath^*$. The *intermediate extension* is defined to be the functor $\jmath_{!*} = \mathrm{im}({}^p\jmath_! \to {}^p\jmath_*)$ and similarly the *intermediate restriction* is defined to be $\imath^{!*} = \mathrm{im}({}^p\imath^! \to {}^p\imath^*)$. By construction both $\jmath_{!*}$ and $\imath^{!*}$ commute with duality.

If $\jmath$ and k are composable quotient functors then $(\jmath k)_{!*} = \jmath_{!*}k_{!*}$; see [Beilinson et al. 1982, 2.1.7.1]. The analogue for composable inclusion functors is false in general; see Example 3.23. Intermediate extensions are neither left nor right exact, but do preserve injections, surjections and images; intermediate restrictions need not have any exactness properties. Finally, intermediate extensions are fully faithful.

Examples 2.17. There are various examples of this gluing situation:

(1) The bounded derived category of a highest weight category B^0 with duality in the sense of [Cline et al. 1989]. The simple objects of B^0 are the elements of a poset; each is fixed by duality. The functor $\imath_*$ is given by the inclusion of $D^b(\mathrm{A}^0)$, where A^0 is the Serre subcategory generated by a downward-closed subset of simple objects. The functor $\jmath^*$ is the induced quotient $D^b(\mathrm{B}^0) \to D^b(\mathrm{B}^0/\mathrm{A}^0)$.

(2) The constructible derived category of sheaves of vector spaces on a finite-dimensional topologically stratified space, with only even-dimensional strata, equipped with Verdier duality and the self-dual perverse t-structure. In this case $\imath$ is the inclusion of a closed union of strata and $\jmath$ the complementary

inclusion of an open union of strata, and $\imath_*$ and $\jmath^*$ the respective induced functors.

The example of the constructible derived category of sheaves of torsion modules over a Dedekind ring R works similarly.

(3) Let $g : X \to Z$ and $h : Z \to Y$ be proper birational morphisms of smooth complex algebraic surfaces, and let $f = g \circ h$. Then the exact triple of null categories, as defined in Example 2.16(4) above,

$$C_g \xrightarrow{\imath_*} C_f \xrightarrow{Rg_*} C_h,$$

where the first functor is the inclusion of the full subcategory C_g into C_f, extends to gluing data by [Bodzenta and Bondal 2018, Proposition 3.5]. This data is compatible as above with shifted Grothendieck duality since we have restricted to surfaces.

Proposition 2.18. *Consider as before a gluing* $\mathrm{A} \xrightarrow{\imath_*} \mathrm{B} \xrightarrow{\jmath^*} \mathrm{C}$ *of self-dual t-structures, with induced dualities on their hearts* A^0, B^0 *and* C^0. *Suppose* B^0 *is noetherian, or equivalently that both* A^0 *and* C^0 *are noetherian. Then* $\mathrm{W}(\mathrm{B}^0) \cong \mathrm{W}(\mathrm{A}^0) \oplus \mathrm{W}(\mathrm{C}^0)$. *The analogue for antisymmetric forms also holds.*

Proof. By [Beilinson et al. 1982, Proposition 1.4.26] each simple object of the heart B^0 is either of the form $\imath_* a$ for simple $a \in \mathrm{A}^0$, or $\jmath_{!*} c$ for simple $c \in \mathrm{C}^0$. Furthermore, duality preserves these two classes. Hence by the same argument as in the proof of Corollary 2.13 we have

$$\mathrm{W}(\mathrm{B}^0) \cong \mathrm{W}(\langle \imath_* a \mid \text{simple } a \in \mathrm{A}^0 \rangle) \oplus \mathrm{W}(\langle \jmath_{!*} c \mid \text{simple } c \in \mathrm{C}^0 \rangle).$$

It is immediate that $\imath_*$ induces an isomorphism $\mathrm{W}(\mathrm{A}^0) \cong \mathrm{W}(\langle \imath_* a \mid \text{simple } a \in \mathrm{A}^0 \rangle)$. It follows from the fact that $\jmath_{!*}$ is fully faithful that $\jmath^*$ induces an isomorphism

$$\mathrm{W}(\langle \jmath_{!*} c \mid \text{simple } c \in \mathrm{C}^0 \rangle) \cong \mathrm{W}(\mathrm{C}^0). \qquad \square$$

Together with Proposition 2.14 this provides an independent proof of the existence of the long exact sequence of Balmer–Witt groups in the case when B^0 is noetherian, and furthermore shows that it splits in this case.

The proof shows that the inclusion $\mathrm{W}(\mathrm{A}^0) \hookrightarrow \mathrm{W}(\mathrm{B}^0)$ is induced by $\imath_*$ and the projection $\mathrm{W}(\mathrm{B}^0) \twoheadrightarrow \mathrm{W}(\mathrm{C}^0)$ by $\jmath^*$. It is harder to obtain explicit descriptions of the other inclusion and projection.

Theorem 2.19. *Suppose* $\beta : b \to Db$ *is a nondegenerate symmetric form in* B. *Then*

$$[\beta] = [\imath_* \imath^{!*} \beta] + [\jmath_{!*} \jmath^* \beta] \tag{2.20}$$

in the Witt group $\mathrm{W}(\mathrm{B}^0) \cong \mathrm{W}(\mathrm{B})$.

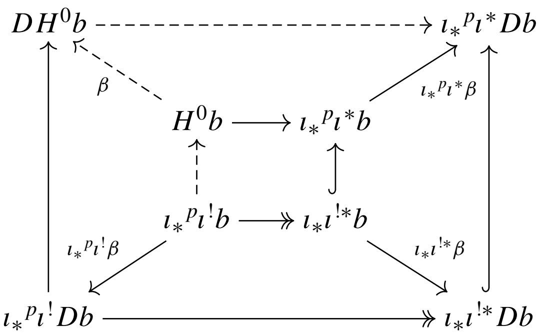
Proof. There is an exact triangle $\imath_*\imath^!b \to b \to \jmath_*\jmath^*b \to \imath_*\imath^!b[1]$ which gives rise to a long exact sequence

$$\cdots \to \imath_*{}^p\imath^!b \to H^0b \to {}^p\jmath_*\jmath^*b \to \cdots$$

in the heart B^0. We apply Corollary 2.11 to write $[H^0\beta] = [\beta]$ as a sum of two terms in the Witt group $\mathrm{W}(\mathrm{B}^0) \cong \mathrm{W}(\mathrm{B})$. The requisite two terms are the induced forms on the images of α and γ in the diagram

$$\begin{array}{ccccccc} & & \imath_*{}^p\imath^!b & \longrightarrow & H^0b & \longleftarrow & {}^p\jmath_!\jmath^*Db & & \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ \imath_*{}^p\imath^*Db & = & D\imath_*{}^p\imath^!b & \longleftarrow & DH^0b & \longrightarrow & D{}^p\jmath_!\jmath^*Db & = & {}^p\jmath_*\jmath^*D^2b \end{array}$$

To identify α consider the commutative diagram below in which α is the composite of the dashed arrows:



The top and left squares arise from the natural morphisms $\mathrm{id} \to \imath_*{}^p\imath^*$ and $\imath_*{}^p\imath^! \to \mathrm{id}$, respectively, and the bottom and right squares from the definition of $\imath^{!*}$. The central square commutes because the composite $\imath_*{}^p\imath^! \to \mathrm{id} \to \imath_*{}^p\imath^*$ is the natural morphism $\imath_*({}^p\imath^! \to {}^p\imath^*)$; see [Beilinson et al. 1982, 1.4.21.1]. It follows that $\bar{\alpha} \cong \imath_*\imath^{!*}\beta$. A similar, slightly more involved, argument shows that $\bar{\gamma} \cong \jmath_{!*}\jmath^*\beta$. □

Example 2.21. Assume the nondegenerate symmetric form $\beta : b \to Db$ in B^0 is a direct sum $\beta = \imath_*\alpha \oplus \jmath_{!*}\gamma$, with nondegenerate symmetric forms $\alpha : a \to Da$ in A^0 and $\gamma : c \to Dc$ in C^0. Then $\imath^{!*}\beta \cong \alpha$ and $\jmath^*\beta \cong \gamma$, so that (2.20) is the image of this decomposition of β in the Witt group.

It is natural to assume that, when B^0 is noetherian, the sum in (2.20) corresponds to the direct sum decomposition of Proposition 2.18. This is false in general. The individual terms depend upon the choice of representative β, not just on the class $[\beta]$ (see Example 3.19). When $\jmath_{!*}$ is exact then it induces a map of Witt groups splitting $\jmath^*$, and moreover $\jmath_{!*}\gamma$ is Witt equivalent to a sum of forms on intermediate extensions of simple objects in C^0. Since $\imath_*$ is a monomorphism it follows that the class $[\imath^{!*}\beta]$ is also well-defined, and so $\imath^{!*}$ induces a map of Witt groups too, splitting $\imath_*$. The next corollary summarises these observations.

Corollary 2.22. *Suppose* B^0 *is noetherian and the intermediate extension* $\jmath_{!*}$ *is exact. Then the direct sum decomposition of Proposition 2.18 is given by the maps* $[\beta] \mapsto ([\imath^{!*}\beta], \jmath^*[\beta])$ *and* $([\alpha], [\gamma]) \mapsto \imath_*[\alpha] + [\jmath_{!*}\gamma]$.

Another important case is when the form β is anisotropic.

Corollary 2.23. *Suppose* B^0 *is noetherian and* $\beta : b \to Db$ *is a nondegenerate anisotropic symmetric form in* B^0. *Then there is an isometry* $\beta \cong \imath_*\imath^{!*}\beta \oplus \jmath_{!*}\jmath^*\beta$ *and (2.20) corresponds to the direct sum decomposition of Proposition 2.18.*

Proof. The existence of the isometry follows from Remarks 2.10. As a consequence $\imath_*\imath^{!*}\beta$ and $\jmath_{!*}\jmath^*\beta$ are also anisotropic. Hence each is a direct sum of nondegenerate forms on simple objects. It is clear that $\imath_*\imath^{!*}b$ has only factors of the form $\imath_*a$ for simple $a \in \mathrm{A}^0$. Since the intermediate extension $\jmath_{!*}\jmath^*b$ cannot have subobjects of the form $\imath_*a$ it follows that no such objects can appear when we write it as a direct sum of simple objects. Hence,

$$\imath_*\imath^{!*}b \in \langle \imath_*a \mid \text{simple } a \in \mathrm{A}^0\rangle \quad \text{and} \quad \jmath_{!*}\jmath^*b \in \langle \jmath_{!*}c \mid \text{simple } c \in \mathrm{C}^0\rangle.$$

The result follows. □

In particular, it follows that the classes $[\imath^{!*}\beta']$ and $[\jmath_{!*}\gamma']$ are well-defined independent of the choice of *anisotropic* representatives β' for $[\beta]$ and γ' for $[\gamma]$. Thus we can define homomorphisms

$$\imath^{!*} : \mathrm{W}(\mathrm{B}^0) \to \mathrm{W}(\mathrm{A}^0) : [\beta] \mapsto [\imath^{!*}\beta'],$$
$$\jmath_{!*} : \mathrm{W}(\mathrm{C}^0) \to \mathrm{W}(\mathrm{B}^0) : [\gamma] \mapsto [\jmath_{!*}\gamma'],$$

where β' and γ' are (choices of) anisotropic representatives. The projections and inclusions of the direct sum decomposition are then the homomorphisms

$$\mathrm{W}(\mathrm{A}^0) \underset{\imath^{!*}}{\overset{\imath_*}{\rightleftarrows}} \mathrm{W}(\mathrm{B}^0) \underset{\jmath_{!*}}{\overset{\jmath^*}{\rightleftarrows}} \mathrm{W}(\mathrm{C}^0). \tag{2.24}$$

In practice it may be difficult to identify maximal isotropic subobjects in B^0, but easier to do so in C^0. For instance in the next section B^0 will be a category of perverse sheaves and C^0 a category of local systems on a stratum. The following approach allows one to compute the canonical direct sum decomposition of Proposition 2.18 provided one can find maximal isotropic subobjects in C^0. Let $c \hookrightarrow \jmath^*b$ be a maximal isotropic subobject of $\jmath^*\beta$. Then $\jmath_{!*}c \hookrightarrow \jmath_{!*}\jmath^*b$ is isotropic for $\jmath_{!*}\jmath^*\beta$. Let $\beta' = \jmath_{!*}\jmath^*\beta \lhd \jmath_{!*}c$ be the reduction. Apply Theorem 2.19 to β' to obtain

$$[\jmath_{!*}\jmath^*\beta] = [\beta'] = [\imath_*\imath^{!*}\beta'] + [\jmath_{!*}\jmath^*\beta']$$

and note that $\jmath^*\beta' = \jmath^*\beta \lhd c$ is anisotropic. It follows that this is the canonical decomposition of $[\jmath_{!*}\jmath^*\beta]$. Hence the canonical decomposition of $[\beta]$ is

$$[\beta] = \big([\imath_*\imath^{!*}\beta] + [\imath_*\imath^{!*}\beta']\big) + [\jmath_{!*}\jmath^*\beta']. \tag{2.25}$$

It is clear that $\imath_*$ and $\jmath^*$ preserve anisotropy; the same holds for intermediate extension and restriction:

Lemma 2.26. *If $\beta : b \to Db$ and $\gamma : c \to Dc$ are anisotropic symmetric forms in* B *and* C, *respectively, then $\imath^{!*}\beta$ and $\jmath_{!*}\gamma$ are also anisotropic.*

Proof. For intermediate restrictions this follows from [Beilinson et al. 1982, Proposition 1.4.17] and the fact that $\imath_*$ is t-exact, which together imply that $\imath_*{}^p\imath^! b \to b$ is a monomorphism. For intermediate extension we note that if $c' \hookrightarrow \jmath_{!*}c$ is an isotropic subobject then $\jmath^*c' = 0$; otherwise it would be an isotropic subobject of c. Hence $c' \cong \imath_*\imath^! c'$. But this is impossible unless $c' = 0$, as intermediate extensions cannot have nonzero subobjects of this form [Beilinson et al. 1982, Corollaire 1.4.25]. □

Remark 2.27. The results of this subsection also hold, with essentially the same proofs, in the context of gluing of abelian categories in the sense of [Franjou and Pirashvili 2004]. In this context one has the same six functor formalism, but with exact sequences

$$0 \to \imath_*\imath^! \to \mathrm{id} \to \jmath_*\jmath^* \quad \text{and} \quad \jmath_!\jmath^* \to \mathrm{id} \to \imath_*\imath^* \to 0$$

replacing the corresponding exact triangles; see [Franjou and Pirashvili 2004, Proposition 4.2]. As above, the simple objects of the glued abelian category have either the form $\imath_*a$ or $\jmath_{!*}c$; see [Berest et al. 2008, Lemma 2]. Since we only use exactness in the middle in the proof of Theorem 2.19, everything works as before.

This is a more general context; there are abelian gluing examples which do not come from gluing of triangulated categories. In particular, Examples 2.17 (1) and (3) can be generalised; see [Krause 2017, Lemma 2.5] and [Bodzenta and Bondal 2018, Proposition 3.11], respectively.

3. Application to stratified spaces

3A. ***Witt groups of local systems.*** Let X be a locally connected topological space, and let $\mathrm{Loc}(X)$ be the category of local systems on X with coefficients in a field $\mathbb{F}$. When X is connected this category is equivalent to the category of $\mathbb{F}$-representations of the fundamental group $\pi_1 X$. A representation $\rho : \pi_1 X \to \mathrm{GL}(V)$ has a dual representation on the vector space dual V^* given by $g \mapsto \rho(g)^{-*} := \rho(g^{-1})^*$. There is an induced duality on local systems which we denote by $\mathcal{L} \mapsto \mathcal{L}^\vee$. Let

$W(\mathrm{Loc}(X))$ be the associated Witt group. It is a ring under the tensor product of local systems, and is covariantly functorial under continuous maps (see also [Bunke and Ma 2004]). If X is a topological manifold, then there is also a second duality $\mathcal{L} \mapsto \mathcal{L}^\vee \otimes \mathrm{or}_X$ obtained by in addition twisting with the orientation sheaf or_X of X. Let $W(\mathrm{Loc}(X), \mathrm{or}_X)$ be the associated Witt group, which agrees with $W(\mathrm{Loc}(X))$ when X is oriented, i.e., when an isomorphism $\mathrm{or}_X \cong \mathbb{F}_X$ has been chosen.

3B. ***Witt groups of perverse sheaves.*** Let X be a finite-dimensional topologically stratified space, i.e., a locally cone-like stratified space. Let $\mathrm{D}^b_c(X)$ be the bounded derived category of constructible sheaves of $\mathbb{F}$-vector spaces on X for a field $\mathbb{F}$. The Poincaré–Verdier dual D makes this into a category with duality. Suppose that X has only even-dimensional strata. Then there is a self-dual perversity $p(S) = -\dim S/2$ which defines a t-structure ${}^pD^{\leq 0}(X)$ on $\mathrm{D}^b_c(X)$ whose heart is the category $\mathrm{Perv}(X)$ of perverse sheaves. This is the full subcategory of $\mathrm{D}^b_c(X)$ whose objects obey the vanishing conditions

$$H^j(k_S^* \mathcal{A}) = 0 \quad \text{for } j > -\dim S/2 \qquad \text{and} \qquad H^j(k_S^! \mathcal{A}) = 0 \quad \text{for } j < -\dim S/2,$$

where $k_S : S \hookrightarrow X$ is the inclusion of a stratum. The category $\mathrm{Perv}(X)$ is both artinian and noetherian.

It follows from the fact that the above vanishing conditions are local on X that tensoring with a local system $\mathcal{L}$ is an exact functor

$$- \otimes \mathcal{L} : \mathrm{Perv}(X) \to \mathrm{Perv}(X).$$

Moreover, Verdier duality and the duality on local systems are related by

$$D(\mathcal{A} \otimes \mathcal{L}) \cong D\mathcal{A} \otimes \mathcal{L}^\vee,$$

where $\mathcal{A}$ is a perverse sheaf and $\mathcal{L}$ a local system. Combining these facts we obtain the following lemma:

Lemma 3.1. *Tensor product makes the Witt group* $W(\mathrm{Perv}(X))$ *of perverse sheaves into a module over the Witt group* $W(\mathrm{Loc}(X))$ *of local systems.*

Let $\imath : Y \hookrightarrow X$ be the inclusion of a closed stratified subspace, in other words Y is a closed union of strata of X. Let $\jmath : U = X - Y \hookrightarrow X$ be the complementary open inclusion. Then

$$\mathrm{D}^b_c(Y) \xrightarrow{\imath_*} \mathrm{D}^b_c(X) \xrightarrow{\jmath_*} \mathrm{D}^b_c(U)$$

is an exact triple of triangulated categories satisfying the conditions of Section 2F. The perverse t-structure on $\mathrm{D}^b_c(X)$ is glued from the perverse t-structures on $\mathrm{D}^b_c(Y)$ and $\mathrm{D}^b_c(U)$. For the remainder of this section we assume that the stratified space X has only finitely many strata, which is the case, for instance, if it is compact. For ease of reading we suppress extensions by zero from closed unions of strata.

Corollary 3.2. *There is a direct sum decomposition*

$$\mathrm{W}(\mathrm{Perv}(X)) \cong \bigoplus_{S\subset X} \mathrm{W}_{\epsilon_S}(\mathrm{Loc}(S), \mathrm{or}_S), \tag{3.3}$$

where $\epsilon_S = (-1)^{\dim S/2}$.

Proof. The decomposition is obtained by applying Proposition 2.18 repeatedly to obtain

$$\mathrm{W}(\mathrm{Perv}(X)) \cong \bigoplus_{S\subset X} \mathrm{W}(\mathrm{Perv}(S)).$$

The decomposition in the statement is equivalent: a perverse sheaf on S is a local system shifted in degree by $\dim S/2$, and this accounts for the signs ϵ_S because odd shifts switch symmetric and antisymmetric forms; see [Balmer 2000, Remark 2.16]. Moreover, Verdier duality corresponds under this identification to the duality of local systems twisted by the orientation sheaf or_S of the stratum S. □

Example 3.4. Assume all strata S are orientable and consider the coefficient field $\mathbb{F} = \mathbb{R}$. Then $\mathrm{W}_{\epsilon_S}(\mathrm{Loc}(S), \mathrm{or}_S) \cong \mathrm{W}_{\epsilon_S}(\mathrm{Loc}(S))$ is by [Bunke and Ma 2004] a direct sum of a free $\mathbb{Z}$-module and a torsion module whose elements are all of order two. So the same is true for the Witt group $\mathrm{W}(\mathrm{Perv}(X))$ of perverse sheaves.

We now discuss how to compute the associated "canonical decomposition" of a class in $\mathrm{W}(\mathrm{Perv}(X))$ into classes of forms on local systems on the strata, or equivalently on their intermediate extensions. Let $\imath_S : \bar{S} \hookrightarrow X$ and $\jmath_S : S \hookrightarrow \bar{S}$ be the inclusions, so that $k_S = \imath_S \circ \jmath_S$. Let $\beta|_S$ be the restricted form

$$\jmath_S^{*}\,{}^{p}\imath_S{}^{!}\mathcal{B} \to \jmath_S^{*}\,{}^{p}\imath_S{}^{!}D\mathcal{B} \cong \jmath_S^{*}D({}^{p}\imath_S{}^{*}\mathcal{B})$$

induced by a symmetric form $\beta : \mathcal{B} \to D\mathcal{B}$. This restricted form may be degenerate; the associated nondegenerate form is, by definition, $\jmath_S^{*}\imath_S^{!*}\beta$.

Lemma 3.5. *Suppose* $\beta : \mathcal{B} \to D\mathcal{B}$ *is nondegenerate and anisotropic. Then there is an isometry*

$$\beta \cong \sum_{S\subset X} \jmath_{S!*}\jmath_S^{*}\imath_S^{!*}\beta, \tag{3.6}$$

and passing to the Witt group we obtain the canonical decomposition of $[\beta]$.

Proof. The existence of the isometry and the fact that it corresponds to the direct sum decomposition follow from Corollary 2.23: applying it first to $\imath_S$ and the complementary open inclusion, and then to $\jmath_S$ and the complementary closed inclusion yields an isometry

$$\beta \cong \beta' \oplus \jmath_{S!*}\jmath_S^{*}\imath_S^{!*}\beta \oplus \beta'',$$

where the middle term is the summand associated to the stratum S. □

The next lemma reduces the problem of identifying an anisotropic form on a perverse sheaf to the analogous question for local systems.

Lemma 3.7. *A symmetric form $\beta : \mathcal{B} \to D\mathcal{B}$ is anisotropic if and only if for each stratum S the restriction $\beta|_S$ is anisotropic.*

Proof. Suppose $\mathcal{A} \hookrightarrow \mathcal{B}$ is a nonzero isotropic subobject for β. Let S be a maximal stratum for which $\mathcal{A}|_S \neq 0$. Then $k_S^* \mathcal{A} = \jmath_S^* {}^p\imath_S^! \mathcal{A} \hookrightarrow \jmath_S^* {}^p\imath_S^! \mathcal{B}$ is a nonzero isotropic subobject for the restriction $\beta|_S$.

In the other direction, if $\mathcal{C} \hookrightarrow \jmath_S^* {}^p\imath_S^! \mathcal{B}$ is a nonzero isotropic subobject for $\beta|_S$ then the image of the composite

$$ {}^p\jmath_{S!}\mathcal{C} \to {}^p\imath_S^! \mathcal{B} \to \mathcal{B} $$

is a nonzero isotropic subobject for β. □

We now describe an inductive procedure for computing the canonical decomposition of a general class in $W(\mathrm{Perv}(X))$. In order to do so we extend the partial order $S \leq T \iff S \subset \overline{T}$ on the strata of X to a total order, and label the strata so that $S_1 > \cdots > S_n$. For $1 \leq k < n$ let

$$ \imath_k : S_{k+1} \cup \cdots \cup S_n \hookrightarrow S_k \cup \cdots \cup S_n $$

be the closed inclusion, and for $1 \leq k \leq n$ let

$$ \jmath_k : S_k \hookrightarrow S_k \cup \cdots \cup S_n $$

be the (complementary) open inclusion, in particular with $\jmath_n : S_n \to S_n$ the identity. Let $\tilde{\imath}_k = \imath_1 \imath_2 \cdots \imath_k : S_{k+1} \cup \cdots \cup S_n \hookrightarrow X$ be the composite.

Lemma 3.8. *Suppose $\beta : \mathcal{B} \to D\mathcal{B}$ is a nondegenerate symmetric form in* $\mathrm{Perv}(X)$ *such that $\beta|_{S_1 \cup \cdots \cup S_{k-1}}$ is anisotropic. Then β has an isotropic subobject such that the reduction by it*, *say $\beta' : \mathcal{B}' \to D\mathcal{B}'$*, *satisfies*

(1) $\beta'|_{S_1 \cup \cdots \cup S_{k-1}} = \beta|_{S_1 \cup \cdots \cup S_{k-1}}$;

(2) *$\beta'|_{S_k}$ is the reduction of $\beta|_{S_k}$ by a maximal isotropic subobject.*

Note that Lemma 3.7 then implies that $\beta'|_{S_1 \cup \cdots \cup S_k}$ is anisotropic.

Proof. Let $\mathcal{A} \hookrightarrow \jmath_k^* {}^p\tilde{\imath}_{k-1}^! \mathcal{B}$ be a maximal isotropic subobject for $\beta|_{S_k}$. Then the image of the composite ${}^p\jmath_{k!}\mathcal{A} \to {}^p\tilde{\imath}_{k-1}^! \mathcal{B} \to \mathcal{B}$ is isotropic for β. Let β' be the reduction. Since ${}^p\jmath_{k!}\mathcal{A}$ is supported on $S_k \cup \cdots \cup S_n$ the first condition

$$ \beta'|_{S_1 \cup \cdots \cup S_{k-1}} = \beta|_{S_1 \cup \cdots \cup S_{k-1}} $$

is satisfied.

By construction $\beta'|_{S_k}$ is anisotropic. Since $\mathcal{A}$ was chosen to be a maximal isotropic subobject $\beta'|_{S_k}$ is isometric to the reduction of $\beta|_{S_k}$ by $\mathcal{A}$. □

The procedure for constructing an anisotropic representative, and for computing the canonical decomposition is as follows. Set $\beta_0 = \beta$. Using Lemma 3.8 we construct, by successive isotropic reductions, forms $\beta_1, \ldots, \beta_n$ such that

(1) $\beta_k|_{S_1\cup\cdots\cup S_k}$ is anisotropic;

(2) $\beta_k|_{S_1\cup\cdots\cup S_{k-1}} = \beta_{k-1}|_{S_1\cup\cdots\cup S_{k-1}}$;

(3) $\beta_k|_{S_k}$ is the reduction of $\beta_{k-1}|_{S_k}$ by a maximal isotropic subobject.

In particular, β_n is an anisotropic representative for $[\beta]$, and the canonical decomposition is

$$[\beta] = \sum_{k=1}^{n} [\jmath_{k\,!*}(\beta_k|_{S_k})]. \tag{3.9}$$

The (anti)symmetric local systems of (3.3) are obtained from the $\beta_k|_{S_k}$ by shifting by $\dim S_k/2$. We now investigate circumstances in which it is possible to find explicit expressions for the $\beta_k|_{S_k}$ in terms of β.

Applying Theorem 2.19 inductively, starting with the complementary inclusions $(\imath_1, \jmath_1)$, one obtains a formula

$$[\beta] = [\jmath_{1\,!*}\jmath_1^*\beta] + \sum_{k=2}^{n} [\jmath_{k\,!*}\jmath_k^*\imath_{k-1}^{!*}\cdots\imath_1^{!*}\beta]. \tag{3.10}$$

In general this is not the above canonical decomposition. There are many similar formulæ, corresponding to different ways of splitting off strata. These formulæ may differ from one another, and each may depend on β, not merely its class $[\beta]$.

When β is anisotropic, Corollary 2.23 guarantees that (3.10) is the canonical decomposition, and so must agree with (3.6). In fact we can verify that the given representatives of terms in (3.10) are isometric to those in (3.6), not merely Witt-equivalent.

Proposition 3.11. *Suppose $\beta : \mathcal{B} \to D\mathcal{B}$ is nondegenerate and anisotropic. Then there are isometries*

$$\jmath_{k\,!*}\jmath_k^*\imath_{k-1}^{!*}\cdots\imath_1^{!*}\beta \cong \jmath_{S_k\,!*}\jmath_{S_k}^*\imath_{S_k}^{!*}\beta$$

for each $k = 2, \dots, n$ so that (3.10) is the image

$$[\beta] = \sum_{S\subset X} [\jmath_{S\,!*}\jmath_S^*\imath_S^{!*}\beta] \tag{3.12}$$

of the isometry (3.6) in the Witt group.

Proof. Let $\imath : Y \hookrightarrow X$ be the inclusion of a closed union of strata. Then it follows from [Beilinson et al. 1982, Proposition 1.4.17] that the (dual) natural morphisms $\imath_*{}^p\imath^!\mathcal{B} \to \mathcal{B}$ and $D\mathcal{B} \to \imath_*{}^p\imath^*D\mathcal{B}$ are respectively monomorphic and epimorphic. Hence there is a commutative diagram

$$\begin{array}{ccccc}
\ker\alpha & \hookrightarrow & \imath_*{}^p\imath^!\mathcal{B} & \hookrightarrow & \mathcal{B} \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
D(\ker\alpha) & \twoheadleftarrow & \imath_*{}^p\imath^*D\mathcal{B} & \twoheadleftarrow & D\mathcal{B}
\end{array}$$

where α is the restriction of β. As β is anisotropic we deduce that $\ker\alpha = 0$, and hence also $\operatorname{coker}\alpha \cong D(\ker D\alpha) \cong D(\ker\alpha) = 0$. Therefore α is an isomorphism and ${}^p\imath^!\mathcal{B} \cong \imath^{!*}\mathcal{B} \cong {}^p\imath^*\mathcal{B}$. By [Beilinson et al. 1982, Proposition 1.3.17] ${}^p\imath_r^!\,{}^p\imath_{r-1}^! \cong {}^p(\imath_{r-1}\imath_r)^!$ and similarly ${}^p\imath_r^*\,{}^p\imath_{r-1}^* \cong {}^p(\imath_{r-1}\imath_r)^*$. Combining these we see that $\imath_r^{!*}\imath_{r-1}^{!*}\beta \cong (\imath_{r-1}\imath_r)^{!*}\beta$ (as forms not merely as Witt classes). By induction $\imath_r^{!*}\cdots\imath_1^{!*}\beta = (\imath_1\cdots\imath_r)^{!*}\beta$.

One can then check that $\jmath_{k!*}J_k^*(\imath_1\cdots\imath_{k-1})^{!*} \cong \jmath_{S_k!*}J_{S_k}^*\imath_{S_k}^{!*}$ are naturally isomorphic, so (3.10) becomes (3.12). □

For applications it is more useful to identify geometric conditions under which (3.10) is the canonical decomposition, and hence is independent of the representative β and choice of ordering of the strata. We approach this by identifying conditions under which intermediate extensions are exact, and then using Corollary 2.22.

Lemma 3.13. *Suppose that S is a stratum with finite fundamental group, and that the characteristic of $\mathbb{F}$ does not divide the order of $\pi_1 S$. Then the intermediate extension $\jmath_{S!*}$ is exact.*

Proof. If $\pi_1 S$ is finite then $\operatorname{Perv}(S)$ is semisimple by Maschke's theorem. The result follows because $\jmath_{S!*}$ is additive. □

Lemma 3.14. *Let $S > T$ be strata in X, and let L be the link of T in $\overline{S}$. Suppose that the intersection cohomology group $IH^{(\dim L-1)/2}(L;\mathcal{L}) = 0$ for any local system $\mathcal{L}$ on the link. Then*

$$\operatorname{Ext}^1(\mathcal{A},\mathcal{B}) = 0 = \operatorname{Ext}^1(\mathcal{B},\mathcal{A}),$$

where $\mathcal{A} = \jmath_{S!*}\mathcal{M}[\dim S/2]$ and $\mathcal{B} = \jmath_{T!*}\mathcal{N}[\dim T/2]$ are the intermediate extensions of (shifted) local systems respectively on S and on T. In fact this holds if the above intersection cohomology group vanishes for those local systems which arise as the restriction of a local system on S.

Proof. It suffices to prove that $\operatorname{Ext}^1(\mathcal{A},\mathcal{B}) = 0$ for any such $\mathcal{A}$ and $\mathcal{B}$, since then by duality $\operatorname{Ext}^1(\mathcal{B},\mathcal{A}) \cong \operatorname{Ext}^1(D\mathcal{A}, D\mathcal{B}) = 0$. By adjunction and the fact that $\mathcal{B} \in {}^pD^0(\overline{T})$ and $\imath_T{}^*\mathcal{A} \in {}^pD^{<0}(\overline{T})$, we have

$$\operatorname{Ext}^1(\mathcal{A},\mathcal{B}) \cong \operatorname{Ext}^1(\imath_T{}^*\mathcal{A},\mathcal{B}) \cong \operatorname{Hom}(H^{-1}(\imath_T{}^*\mathcal{A}),\mathcal{B}),$$

where H^{-1} is cohomology with respect to the standard, not the perverse, t-structure. Since $\mathcal{B}$ has no subobjects supported on $\overline{T} - T$, the right-hand group vanishes, for any such $\mathcal{B}$, if $H^{-1}(\imath_T^*\mathcal{A})$ is supported on $\overline{T} - T$. This is equivalent to the vanishing of the stalk of $H^{-1}(\imath_T^*\mathcal{A})$ at some, hence at all, $x \in T$. This stalk is $IH^{(\dim L-1)/2}(L;\mathcal{M}|_L)$. The result follows. □

The conditions of this lemma are satisfied if, for instance, X is Whitney stratified, all strata have smooth closures — so that all links of pairs of strata are spheres —

and all such links have dimension ≥ 3. In particular it holds for subspace arrangements where the dimension of pairs of subspaces differ by at least 3.

Corollary 3.15. *Suppose that for each pair of strata $S > T$ in X and local system $\mathcal{L}$ on the link L of T in $\bar{S}$ the intersection cohomology group $IH^{(\dim L-1)/2}(L;\mathcal{L})$ is 0. Let $\jmath : Y \hookrightarrow \overline{Y}$ be the inclusion of a locally closed union of strata Y in its closure. Then the intermediate extension $\jmath_{!*} : \mathrm{Perv}(Y) \to \mathrm{Perv}(\overline{Y})$ is exact. In fact it suffices for the above intersection cohomology group to vanish only for those local systems $\mathcal{L}$ which arise as the restriction of a local system on S.*

Proof. The intermediate extension is exact if, and only if, for each $\mathcal{A} \in \mathrm{Perv}(Y)$ the composition series of $\jmath_{!*}\mathcal{A}$ has no factors supported on $\overline{Y} - Y$. It is well-known that the intermediate extension has no nonzero subobjects (or quotients) supported on $\overline{Y} - Y$. However, Lemma 3.14 implies that any factor supported on $\overline{Y} - Y$ would appear as a factor of a subobject supported on $\overline{Y} - Y$. Hence there are no nonzero such factors and the intermediate extension is exact. □

Corollary 3.16. *Suppose that either*

(1) *each stratum S_k has finite fundamental group, or*

(2) *$IH^{(\dim L-1)/2}(L;\mathcal{L}) = 0$ for each link L and each local system $\mathcal{L}$ on an open stratum of the link.*

Then (3.10) is the canonical decomposition of the class $[\beta]$. Moreover, under the second condition, (3.10) can be more simply written as (3.12).

Proof. By Lemma 3.13 and Corollary 3.15 either one of the conditions implies that each $\jmath_{k!*}$ is exact. Then Corollary 2.22 implies that (3.10) is the canonical decomposition, and hence independent of the choice of representative β.

Now suppose the second condition holds, so that the intermediate extension along the inclusion of *any* locally closed union of strata is exact. Choosing an anisotropic representative β', Proposition 3.11 implies that

$$[\beta] = [\beta'] = \sum_{S \subset X} [\jmath_{S!*}\jmath_S^* \imath_S^{!*} \beta'].$$

Then $[\jmath_{S!*}\jmath_S^*\imath_S^{!*}\beta'] = \jmath_{S!*}\jmath_S^*[\imath_S^{!*}\beta']$ because $\jmath_{S!*}$ is exact, and

$$[\imath_S^{!*}\beta'] = [\beta'] - \jmath_{!*}\jmath^*[\beta'] = [\beta] - \jmath_{!*}\jmath^*[\beta] = [\imath_S^{!*}\beta],$$

where $\jmath : X - \bar{S} \hookrightarrow X$ because $\jmath_{!*}$ is exact. The result follows. □

The conditions of this corollary are strong, but strong conditions are necessary. Example 3.23 in the next section shows that (3.12) need not hold even when all strata are simply connected.

3C. ***Perverse sheaves on rank stratifications.*** In this section we use quiver descriptions of perverse sheaves on rank stratifications to illustrate some of our results. The main reference is [Braden and Grinberg 1999].

We begin with the simplest nontrivial example, in which already one sees that one needs to take care in describing how Verdier duality translates to quiver descriptions of perverse sheaves. Let $X = \mathbb{C}$ be stratified by the origin and its complement, and let $\jmath : \mathbb{C} - \{0\} \hookrightarrow \mathbb{C} \hookleftarrow \{0\} : \imath$ be the inclusions. Then the category $\mathrm{Perv}(X)$ of perverse sheaves of vector spaces over a field $\mathbb{F}$ of characteristic zero is equivalent to the category of representations of the quiver

$$0 \underset{v}{\overset{c}{\rightleftarrows}} 1, \tag{3.17}$$

where $1 + cv$ and $1 + vc$ are invertible [Verdier 1985a, §4]. Here the vertex 0 corresponds to the perverse nearby cycles Ψ_z and 1 to the perverse vanishing cycles Φ_z, where z is a coordinate on $\mathbb{C}$. The arrow c corresponds to the canonical map and v to a variation map. In order to define the latter one needs to pick an orientation of $\mathbb{C}^*$, or equivalently a generator of the fundamental group of $\mathbb{C}^*$. The restriction of a perverse sheaf $\mathcal{E}$ to $\mathbb{C}^*$ is a shifted local system $\mathcal{L}[1]$, whose stalk $\mathcal{L}_1$ at the basepoint $1 \in \mathbb{C}^*$ is the perverse nearby cycles, and whose monodromy with respect to the chosen generator is $\mu = 1 + vc$. The restriction of the Verdier dual $D\mathcal{E}$ to $\mathbb{C}^*$ is the dual shifted local system $\mathcal{L}^\vee[1]$ whose monodromy with respect to the *reversed* generator is $\mu^* = 1 + c^*v^*$, where $*$ denotes the vector space dual.

In order to give a simple description of duality, sufficient for the following examples, we restrict to the Serre subcategory of perverse sheaves with unipotent monodromy, i.e., for which both $n_\Phi = cv$ and $n_\Psi = vc$ are nilpotent. This allows us to switch to an alternative description in terms of the logarithms N_Φ and N_Ψ of the monodromies, by which we mean

$$1 + n = e^N$$

in each case. We do so by replacing the variation arrow v by $V = f(n_\Psi)v = vf(n_\Phi)$, where

$$f(t) = \frac{\ln(1+t)}{t} = 1 - \frac{t}{2} + \cdots . \tag{3.18}$$

Verdier duality for unipotent perverse sheaves then corresponds to the duality

$$W_0 \underset{V}{\overset{c}{\rightleftarrows}} W_1 \qquad \longmapsto \qquad W_0^* \underset{c^*}{\overset{-V^*}{\rightleftarrows}} W_1^*$$

on quiver descriptions. Here we first switch to the reversed generator of the fundamental group by changing V to $-V$, and then dualise. This fits with the duality of local systems, in which if $\mathcal{L}$ has unipotent monodromy e^N then $\mathcal{L}^\vee$ has monodromy

e^{-N^*} with respect to the *same* generator of the fundamental group. If $\mathbb{F} \subset \mathbb{C}$ then under the Riemann–Hilbert correspondence this agrees (up to a Tate twist) with the description in terms of regular holonomic D-modules as given in [Saito 1989, Theorem 1.6, Remark 1.7 and Theorem 2.2]. Note that the usual biduality isomorphism $\chi : \mathrm{id} \to D^2$ for quiver representations needs to be modified by a sign at the vertex 0, so that a symmetric bilinear form on a perverse sheaf corresponds to a symmetric bilinear form at the vertex 1 and an antisymmetric form at the vertex 0.

Example 3.19. Even in this simple example, intermediate extension does not induce a map of Witt groups. Let $\beta : \mathcal{B} \to D\mathcal{B}$ be the nondegenerate symmetric form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circlearrowright \mathbb{F}^2 \xrightarrow{\beta} \mathbb{F}^2 \circlearrowleft \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

on a two-dimensional shifted local system on $\mathbb{C} - \{0\}$. Clearly β is metabolic with lagrangian the one-dimensional local system $\mathcal{A}$ with trivial monodromy, in particular $[\beta] = 0$. The intermediate extensions of $\mathcal{A}$ and of $\mathcal{B}$ are

$$\mathbb{F} \rightleftarrows 0 \quad \text{and} \quad \mathbb{F}^2 \underset{(1\,0)^*}{\overset{(0\,1)}{\rightleftarrows}} \mathbb{F},$$

respectively. The subobject $\jmath_{!*}\mathcal{A}$ is isotropic for $\jmath_{!*}\beta$ but no longer lagrangian; the reduction $\jmath_{!*}\beta \lhd \jmath_{!*}\mathcal{A}$ is the form [1] on the point 0. Applying (3.10) to $\jmath_{!*}\beta$ we do not obtain the direct sum decomposition (3.3) into forms on simple local systems.

The intermediate extension $\jmath_{!*}$ is not exact, and moreover $[\jmath_{!*}\beta] \neq 0$ since any form Witt-equivalent to $\jmath_{!*}\beta \lhd \jmath_{!*}\mathcal{A}$ must be a form on an object with an odd number of simple factors. Hence the intermediate extension does not induce a map of Witt groups.

The above description can be generalised to perverse sheaves on a complex line bundle L over a connected stratified space X. Stratify L by the preimages of the strata of X intersected with the zero section and its complement. Identify X with the zero section, and let $\imath : X \hookrightarrow L$ be the inclusion and $\jmath : L - X \hookrightarrow L$ its complement. A perverse sheaf with respect to this stratification is automatically *monodromic* in the sense of [Verdier 1985b] in that it is locally constant on the $\mathbb{C}^*$ fibres of the projection $L - X \to X$. The monodromy of such a perverse sheaf is an automorphism determined by a choice of orientation of L, or equivalently of a generator of the fundamental group of $\mathbb{C}^*$. Perverse sheaves on L are equivalent to representations of the quiver (3.17) but with values in the abelian category $\mathrm{Perv}(L - X)$ rather than in vector spaces — see [Verdier 1985b, Proposition 5.5]. When L is a trivial line bundle this description corresponds to the one via perverse nearby and vanishing cycles for the projection $L \cong X \times \mathbb{C} \to \mathbb{C}$. The initial example considered above is the special one in which X is a point.

A perverse sheaf $\mathcal{E}$ on $L - X$ splits as a direct sum $\mathcal{E}^u \oplus \mathcal{E}^{nu}$ of perverse sheaves with unipotent and nonunipotent monodromy, respectively. For the nonunipotent summand,

$$^p j_! \mathcal{E}^{nu} \cong {}^p j_{!*} \mathcal{E}^{nu} \cong {}^p j_* \mathcal{E}^{nu},$$

so that intermediate extension is exact on the full Serre subcategory of perverse sheaves with nonunipotent monodromy. For this reason we focus on the unipotent part. For this part Verdier duality can be described just as above, once a generator for the fundamental group of the $\mathbb{C}^*$ fibres is chosen, and the variation map is renormalised as before.

Now we consider perverse sheaves on the rank stratification as in [Braden and Grinberg 1999]. Let V be an n-dimensional complex vector space. Stratify $\mathrm{End}(V)$ by the subspaces of endomorphisms of equal rank — for $k = 0, \dots, n$ let

$$S_k = \{x \in \mathrm{End}(V) \mid \mathrm{rank}(x) = n - k\}.$$

Then $\mathrm{codim}\, S_k = k^2$ and each S_k is connected with $\pi_1 S_0 \cong \mathbb{Z}$ and all other strata simply connected. For $n > 0$, [Braden and Grinberg 1999, Theorem 4.6] implies that the category $\mathrm{Perv}(X_n)$ of perverse sheaves on the hypersurface

$$X_n = \{x \in \mathrm{End}(V) \mid \mathrm{rank}(x) < n\} = S_1 \cup \cdots \cup S_n$$

of singular endomorphisms, constructible with respect to this stratification and with coefficients in $\mathbb{F}$, is equivalent to the category of finite-dimensional $\mathbb{F}$-representations of the quiver with relations

$$\mathbb{A}_n = \left(1 \underset{v_1}{\overset{c_1}{\rightleftarrows}} \cdots \underset{v_{n-1}}{\overset{c_{n-1}}{\rightleftarrows}} n \;\middle|\; \begin{array}{l} v_1 c_1 = 0, \\ c_k v_k = v_{k+1} c_{k+1} \ \text{ for } k = 1, \dots, n-2 \end{array} \right).$$

We write $\mu_k = 1 + v_k c_k$ for $k = 1, \dots, n-1$ and $\mu_n = 1 + c_{n-1} v_{n-1}$, and refer to these as the monodromies of the representation. By the conditions above each monodromy is unipotent, with $\mu_1 = 1$. For $n = 1$ the quiver $\mathbb{A}_1$ corresponding to $\mathrm{Perv}(\mathrm{pt})$ has just one vertex and no arrows.

This equivalence between perverse sheaves and quiver representations is obtained in two steps. First one maps a perverse sheaf to its stratified Morse data, a vector space associated to each stratum S_i together with the (microlocal) monodromy μ_i; see [Braden and Grinberg 1999, Theorem 4.6 and Proposition 4.7]. This monodromy depends on a choice of generator of the microlocal fundamental group, which in each of these cases is infinite cyclic. The theory of microlocal perverse sheaves is then used in order to obtain the arrows in the quiver description, ultimately by reducing to considering monodromic perverse sheaves on line bundles; see [Braden and Grinberg 1999, Proposition 2.8]. Since all microlocal monodromies are unipotent, we can renormalise the variation arrows as above, thereby modifying the identification of perverse sheaves with representations of the same quiver with relations $\mathbb{A}_n$. Now the microlocal monodromies of the perverse sheaves correspond to the following monodromies of the representation: $\mu_k = e^{N_k}$ with $N_k = V_k c_k$ for $k = 1, \dots, n-1$ and $\mu_n = e^{N_n}$ with $N_n = c_{n-1} V_{n-1}$.

For the remainder of this section we work with this modified identification. In this, Verdier duality on $\mathrm{Perv}(X_n)$ corresponds to the functor mapping a representation

$$\mathcal{A} = \left(A_1 \underset{V_1}{\overset{c_1}{\rightleftarrows}} \cdots \underset{V_{n-1}}{\overset{c_{n-1}}{\rightleftarrows}} A_n \right)$$

to the "dual" representation

$$\mathcal{A}^* = \left(A_1^* \underset{c_1^*}{\overset{-V_1^*}{\rightleftarrows}} \cdots \underset{c_{n-1}^*}{\overset{-V_{n-1}^*}{\rightleftarrows}} A_n^* \right).$$

The usual biduality isomorphism $\chi : \mathrm{id} \to D^2$ for quiver representations needs to be modified by a sign $(-1)^{n^2-k^2} = (-1)^{n-k}$ at the vertex k, as for quivers with an involution [Young 2016, §3.2], since the vector space associated to each stratum S_k is given as a normal Morse datum shifted by the complex dimension $n^2 - k^2$ of the stratum; see [Schürmann 2003, Corollary 5.1.4]. Then a symmetric bilinear form on a perverse sheaf corresponds to a symmetric bilinear form at the vertex k when $n-k$ is even and an antisymmetric form when $n-k$ is odd.

This differs from the description of duality in [Braden and Grinberg 1999, Proposition 4.8], where the reversal of the generator of the fundamental group is overlooked (as one can check in the $n = 1$ case, which is the example considered at the beginning of this section). A further difference is that [Braden and Grinberg 1999] state their results for perverse sheaves of complex vector spaces, however their methods apply more generally to perverse sheaves of vector spaces over any field $\mathbb{F}$. For the theory of microlocal perverse sheaves in this generality see [Waschkies 2004]. We must restrict to fields of characteristic zero in order to take logarithms of unipotent monodromies.

We now explain how to understand intermediate extension and restriction in the quiver description of $\mathrm{Perv}(X_n)$.

Lemma 3.20. *Under the above identification the diagram*

$$\begin{array}{ccc} \mathrm{Perv}(S_k \cup \cdots \cup S_n) & \longrightarrow & \mathrm{Perv}(X_n) \\ \downarrow & & \downarrow \\ \mathrm{Perv}(S_k \cup \cdots \cup S_l) & \longrightarrow & \mathrm{Perv}(S_1 \cup \cdots \cup S_l) \end{array}$$

in which horizontal arrows are extensions by zero from closed unions of strata and vertical ones restriction to open unions, corresponds to the diagram

$$\begin{array}{ccc}\langle \mathcal{A}\in \mathrm{Rep}(\mathbb{A}_n) \mid A_i=0 \textit{ for } i=1,\dots,k-1\rangle & \longrightarrow & \mathrm{Rep}(\mathbb{A}_n)\\ \downarrow & & \downarrow \\ \langle \mathcal{A}\in \mathrm{Rep}(\mathbb{A}_l) \mid A_i=0 \textit{ for } i=1,\dots,k-1\rangle & \longrightarrow & \mathrm{Rep}(\mathbb{A}_l)\end{array}$$

for $0<k\le l\le n$. Here the horizontal arrows are inclusions of full subcategories of quiver representations and the vertical ones arise from restricting a representation to the subquiver on vertices $1,\dots,l$.

Proof. By [Braden and Grinberg 1999, Proposition 4.8] the restriction of a perverse sheaf to a normal slice to the stratum S_l (and shifted by the complex dimension of S_l) corresponds under the equivalence to the restriction of a representation of $\mathbb{A}_n$ to the subquiver on the vertices $1,\dots,l$. In particular perverse sheaves on a normal slice can be identified with perverse sheaves on X_l. This remains the same under our modified identification. Perverse sheaves on the union $S_1\cup\cdots\cup S_l$ can also be identified with those on a normal slice to S_l — both are naturally equivalent to the category obtained by quotienting $\mathrm{Perv}(X_n)$ by the Serre subcategory of perverse sheaves with vanishing Morse data on the strata $S_1,\dots,S_l$. These correspond to representations $\mathcal{A}$ of the quiver $\mathbb{A}_n$ with $A_i=0$ for $i=1,\dots,l$. The latter subcategory of perverse sheaves is the image of the extension by zero along the closed inclusion $S_{l+1}\cup\cdots\cup S_n\hookrightarrow X$. The result follows. □

Lemma 3.21. *Let $\imath : S_k\cup\cdots\cup S_l\hookrightarrow S_1\cup\cdots\cup S_l$ and $\jmath : S_k\cup\cdots\cup S_l\hookrightarrow S_k\cup\cdots\cup S_n$ be the inclusions, where $0<k\le l\le n$. Identify the representation*

$$\mathcal{A}=\left(A_1 \underset{V_1}{\overset{c_1}{\rightleftarrows}} A_2 \underset{V_2}{\overset{c_2}{\rightleftarrows}} \cdots \underset{V_l}{\overset{c_l}{\rightleftarrows}} A_l \right)$$

with a perverse sheaf on $S_1\cup\cdots\cup S_l$, and the representation

$$\mathcal{B}=\left(B_k \underset{V_{k+1}}{\overset{c_{k+1}}{\rightleftarrows}} B_{k+1} \underset{V_{k+2}}{\overset{c_{k+2}}{\rightleftarrows}} \cdots \underset{V_l}{\overset{c_l}{\rightleftarrows}} B_l \right)$$

with a perverse sheaf on $S_k\cup\cdots\cup S_l$. Then

(1) $^{p}\imath^{!}\mathcal{A}=\left(\ker V_k \rightleftarrows \cdots \rightleftarrows \ker(V_k\cdots V_l)\right)$,

(2) $^{p}\imath^{*}\mathcal{A}=\left(\operatorname{coker} c_k \rightleftarrows \cdots \rightleftarrows \operatorname{coker}(c_l\cdots c_k)\right)$,

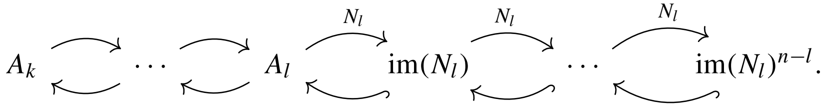
(3) $^{p}\jmath_{!}\mathcal{B}=\left(B_k \rightleftarrows \cdots \rightleftarrows B_l \underset{N_l}{\overset{=}{\rightleftarrows}} \cdots \underset{N_l}{\overset{=}{\rightleftarrows}} B_l\right)$,

(4) $^{p}\jmath_{*}\mathcal{B}=\left(B_k \rightleftarrows \cdots \rightleftarrows B_l \underset{=}{\overset{N_l}{\rightleftarrows}} \cdots \underset{=}{\overset{N_l}{\rightleftarrows}} B_l\right)$.

In each case the unlabelled upper and lower arrows are naturally induced from the c_i *and the* V_i, *respectively. The natural morphisms* ${}^p\imath^!\mathcal{A} \to \mathcal{A} \to {}^p\imath^*\mathcal{A}$ *are given respectively by the evident inclusions and quotients, and the natural morphism* ${}^p\jmath_!\mathcal{B} \to {}^p\jmath_*\mathcal{B}$ *by the identity maps. The intermediate restriction therefore has*

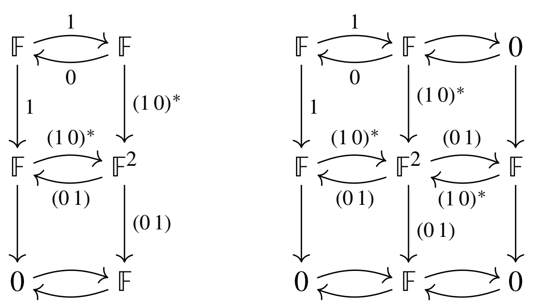
$$(\imath^{!*}\mathcal{A})_{k+i} = \operatorname{im}\big(\ker(V_k \cdots V_{k+i}) \to A_{k+i} \to \operatorname{coker}(c_{k+i} \cdots c_k)\big)$$

for $i = 0, \ldots, l-k$, *and the intermediate extension* $\jmath_{!*}\mathcal{A}$ *is the representation*



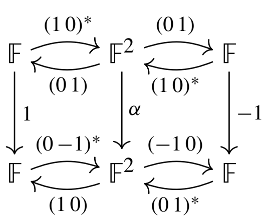
Proof. These results follow from the description of ${}^p\imath^!\mathcal{A}$ and ${}^p\imath^*\mathcal{A}$ as the maximal subobject and quotient of $\mathcal{A}$ supported on $S_k \cup \cdots \cup S_l$, respectively, and of ${}^p\jmath_!\mathcal{B}$ and ${}^p\jmath_*\mathcal{B}$ as initial and terminal objects amongst all extensions of $\mathcal{B}$ to a perverse sheaf on $S_k \cup \cdots \cup S_n$, respectively. □

Example 3.22. Intermediate extension from a union of strata need not be an exact functor, even when all strata are simply connected. Consider the rank stratification for $n = 3$, and specifically the inclusion $\jmath : S_1 \cup S_2 \hookrightarrow S_1 \cup S_2 \cup S_3$. On the left below is a short exact sequence in $\operatorname{Perv}(S_1 \cup S_2)$, and on the right is the result of applying the intermediate extension $\jmath_{!*}$ to it:



It is evident from the final column that the sequence on the right is no longer exact in the middle.

Example 3.23. Let $\mathcal{B}$ be the perverse sheaf on $S_1 \cup S_2 \cup S_3$ in the middle row of the right-hand diagram of Example 3.22 above, and let $\beta : \mathcal{B} \to D\mathcal{B}$ be the nondegenerate symmetric form



where $\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

This has an evident isotropic subobject given by the simple object supported on the middle vertex. The corresponding isotropic reduction is the direct sum

$$I_{S_1 \cup S_2 \cup S_3} \oplus -I_{S_3} = I_{X_3} \oplus -I_{\mathrm{pt}},$$

where we identify perverse sheaves and quiver descriptions, and write I_Y for the intersection form on the intersection cohomology complex, with coefficients in $\mathbb{F}$, of a stratified space Y.

As before, let $S_3 \xrightarrow{\imath_2} S_2 \cup S_3 \xrightarrow{\imath_1} S_1 \cup S_2 \cup S_3$ be the closed inclusions, and let $\jmath_1 : S_1 \hookrightarrow S_1 \cup S_2 \cup S_3$ and $\jmath_2 : S_2 \hookrightarrow S_2 \cup S_3$ be the complementary open inclusions, and $\jmath_3 : S_3 \to S_3$ be the identity. Then

$$\imath_1^{!*}\beta = \begin{pmatrix} 0 & \rightleftarrows & \mathbb{F} \\ \downarrow & & \downarrow_{-1} \\ 0 & \rightleftarrows & \mathbb{F} \end{pmatrix} = -I_{S_3},$$

so that $\imath_2^{!*}\imath_1^{!*}\beta = -I_{S_3}$ too. In contrast, $(\imath_1 \circ \imath_2)^{!*}\beta = 0$. Hence (3.10) is

$$\begin{aligned} [\beta] &= [\jmath_{1!*}\jmath_1^*\beta] + [\jmath_{2!*}\jmath_2^*\imath_1^{!*}\beta] + [\jmath_{3!*}\jmath_3^*\imath_2^{!*}\imath_1^{!*}\beta] \\ &= [I_{S_1 \cup S_2 \cup S_3}] + [0] + [-I_{S_3}] \\ &= [I_{X_3}] - [I_{\mathrm{pt}}], \end{aligned}$$

in agreement with the isotropic reduction. However, the formula (3.12) is false in this case:

$$\sum_{S \subset X} [\jmath_{S!*}\jmath_S^*\imath_S^{!*}\beta] = [I_{X_3}]$$

because $\imath_{S_3}^{!*}\beta = (\imath_1 \circ \imath_2)^{!*}\beta = 0$ and $\imath_{S_2}^{!*}\beta = (\imath_1 \circ \jmath_2)^{!*}\beta = \jmath_2^*\imath_1^{!*}\beta = 0$ by the above calculations. Since $[I_{\mathrm{pt}}] \neq 0$ this does not agree with $[\beta] = [I_{X_3}] - [I_{\mathrm{pt}}]$. Therefore (3.12) does not hold without further assumptions on the form, for instance that it is anisotropic.

3D. ***Perverse sheaves on Schubert-stratified projective spaces.*** We consider a similar example but where the total space and the closures of each stratum are smooth. The main reference is [Braden 2002], although we consider only the special case of projective spaces rather than all Grassmannians. The quiver description of perverse sheaves on Schubert-stratified projective spaces is well-known in the literature — e.g., [Khovanov and Seidel 2002, alternative proof of Proposition 2.9; Stroppel 2006, Example 1.1] — however, we need Braden's geometric approach in order to identify the action of Verdier duality.

Let W be an n-dimensional complex vector space with a complete flag

$$W_0 \subset \cdots \subset W_n = W$$

of linear subspaces where $\dim_{\mathbb{C}} W_i = i$. Let $X = P(W)$ be the corresponding $(n-1)$-dimensional complex projective space with the Schubert stratification with strata $S_i = P(W_{n-i+1}) - P(W_{n-i}) \cong \mathbb{C}^{n-i}$ for $i = 1, \ldots, n$.

The category $\mathrm{Perv}(X)$ of perverse sheaves with coefficients in the field $\mathbb{F}$ constructible with respect to the Schubert stratification is equivalent to the category of finite-dimensional $\mathbb{F}$-representations of the quiver with relations

$$\mathbb{A}'_n = \left(1 \underset{v_1}{\overset{c_1}{\rightleftarrows}} \cdots \underset{v_{n-1}}{\overset{c_{n-1}}{\rightleftarrows}} n \;\middle|\; \begin{array}{ll} v_1 c_1 = 0, & \\ c_k v_k = v_{k+1} c_{k+1} & \text{for } k = 1, \ldots, n-2, \\ c_k c_{k-1} = 0 = v_{k-1} v_k & \text{for } k = 2, \ldots, n-1 \end{array} \right).$$

We write $\mu_k = 1 + v_k c_k$ for $k = 1, \ldots, n-1$ and $\mu_n = 1 + c_{n-1} v_{n-1}$, and refer to these as the monodromies of the representation. By the conditions above, each monodromy is unipotent, with $\mu_1 = 1$. This equivalence is a special case of [Braden 2002, Theorem 1.4.1]. The general quiver description for Grassmannians given in [Braden 2002, §1.3] reduces to the description above in our case.

The equivalence is obtained by a similar procedure as for the rank stratification case, except that one has to consider microlocal perverse sheaves through codimension 0, 1 and now also 2. The argument for codimensions 0 and 1 is as before: first one maps a perverse sheaf to its stratified Morse data, a vector space at each stratum S_i together with the (microlocal) monodromy μ_i; see [Braden 2002, Proposition 4.3.1]. This monodromy depends on a choice of generator of the microlocal fundamental group, which in each of these cases is again infinite cyclic. There is an arrow between vertices i and j if and only if the conormal spaces of S_i and S_j intersect in codimension 1 [Braden 2002, Corollary 2.5.2], which for us is if and only if $|i - j| = 1$.

The relations $v_1 c_1 = 0$ and $c_k v_k = v_{k+1} c_{k+1}$ for $k = 1, \ldots, n-2$ are deduced in the same way as for the rank stratification discussed in the previous section. The key technique is again to reduce to considering monodromic perverse sheaves on line bundles; see [Braden 2002, Lemma 3.4.1]. The third type of relations $c_k c_{k-1} = 0 = v_{k-1} v_k$ for $k = 2, \ldots, n-1$ — see [Braden 2002, (4) on p. 497] — are obtained by considering codimension 2 intersections of conormal spaces, i.e., for strata S_i and S_j with $|i - j| = 2$ [Braden 2002, Proposition 2.6.2].

In order to describe Verdier duality in the quiver description we renormalise as in the rank stratification example. This is possible because as before all microlocal monodromies are unipotent. The quiver with relations $\mathbb{A}'_n$ is unchanged, however the description of the (microlocal) monodromies is now $\mu_k = e^{N_k}$ with $N_k = V_k c_k$ for $k = 1, \ldots, n-1$ and $\mu_n = e^{N_n}$ with $N_n = c_{n-1} V_{n-1}$. Note that as before the usual biduality isomorphism $\chi : \mathrm{id} \to D^2$ for quiver representations needs to be modified by a sign $(-1)^{n-k}$ at the vertex k corresponding to a stratum S_k of complex dimension $n - k$. The descriptions of the six functors and of the intermediate extension and restriction remain the same because, as before, the description is compatible with restriction to a normal slice; see [Braden 2002, §4.2]. We now work only with this modified identification.

Example 3.23 transfers without change to this example in the $n = 3$ case, i.e., for the Schubert stratification of $\mathbb{CP}^2$. Here, not only are all three strata contractible but also their closures are smooth and simply connected. Even under these strong conditions (3.12) does not hold.

Remark 3.24. The path algebras of the quivers with relations $\mathbb{A}_n$ and $\mathbb{A}'_n$, or their representation categories, appear in various other contexts:

(1) as the Auslander algebra of $\mathbb{C}[x]/\langle x^n \rangle$ [Hille and Ploog 2017];

(2) in the braid group actions on categories studied in [Khovanov and Seidel 2002];

(3) as convolution algebras related to hyperplane arrangements [Braden et al. 2010, Example 4.6 and Theorem 4.8];

(4) as "hypertoric enveloping algebras" [Braden et al. 2012, Example 4.11].

As explained in these references, the representation categories of $\mathbb{A}_n$ and $\mathbb{A}'_n$ are Koszul dual. The Koszul grading for $\mathbb{A}'_n$, and more generally for Braden's quiver description of perverse sheaves on Grassmannians, becomes visible only after a renormalisation similar to the one we use to understand Verdier duality, but this time using the square root of the power series (3.18); see [Stroppel 2009, §5.7].

3E. *Relation to Cappell and Shaneson's work.* The paper [Cappell and Shaneson 1991b] introduces a notion of cobordism of self-dual complexes of sheaves of vector spaces, that is, of objects $\mathcal{B} \in \mathrm{D}^b_c(X)$ equipped with an isomorphism $\beta : \mathcal{B} \to D\mathcal{B}$, which is not assumed to have any symmetry properties. (Their definition of self-dual isomorphism involves a shift by $[\dim X]$, but we omit this because we are using the conventions of [Beilinson et al. 1982] for indexing perverse sheaves rather than those of [Goresky and MacPherson 1983].) Let $\Omega^{\pm}_{\mathrm{CS}}(X)$ denote the set of cobordism classes of constructible sheaf complexes with an (anti)symmetric self-duality. The cobordism relation is generated by "elementary cobordisms" which arise from isotropic morphisms $\iota : \mathcal{A} \to \mathcal{B}$. In the special case in which $\mathcal{A}, \mathcal{B} \in \mathrm{Perv}(X)$ and ι is a monomorphism, β and $\beta \lhd \mathcal{A}$ are elementarily cobordant. Thus there is a homomorphism

$$\mathrm{W}_{\pm}(\mathrm{Perv}(X)) \to \Omega^{\pm}_{\mathrm{CS}}(X).$$

(Cappell and Shaneson do not discuss the structure of the set of cobordism classes, but [Yokura 1995] shows that it is an abelian group under direct sum.) Moreover, the homomorphism above is an isomorphism by [Youssin 1997, Theorem 7.4]. This understood, their Theorem 2.1 states that the image of (3.12) holds in $\Omega_{\mathrm{CS}}(X)$.

Example 3.23 above shows that this is incorrect — in that case there is a missing term corresponding to the class of the intersection form of a point — and therefore that further conditions are required for their result. (Cappell and Shaneson work with compact spaces, so to be absolutely precise one should use the counterpart of Example 3.23 for Schubert-stratified projective spaces.) On [Cappell and Shaneson 1991b, p. 534], in order to apply their (1.3), Cappell and Shaneson assume that ${}^p\imath_k^!\mathcal{A} = 0$ implies ${}^p\imath_k^*\imath_{k+1}^!\mathcal{A} = 0$. It is this which fails in Example 3.23.

Cappell and Shaneson's decomposition is valid, and even lifts to the Witt group of perverse sheaves, when the form β is anisotropic. It is also valid for any form β on a sufficiently nice space X, for instance when the second condition of Corollary 3.16 is satisfied. Another case in which it is valid is when the depth of X is one, although in this case it may not correspond to the canonical decomposition.

Let us suppose that we are in one of these "good" cases in which (3.12) holds. Suppose further that $f : Y \to X$ is a proper stratified map — i.e., a proper map such that the preimage of any stratum is a union of strata, and the restriction $f|_S : S \to f(S)$ to any stratum is a locally trivial fibre bundle — of Whitney stratified spaces with only even-dimensional strata. Assume Y has a dense top-dimensional stratum which is oriented. Then the intersection form $I_Y : \mathcal{IC}(Y) \to D\mathcal{IC}(Y)$ of the corresponding intersection cohomology complex is nondegenerate in $\mathrm{Perv}(Y)$ and is symmetric for $\dim Y \equiv 0\,(4)$ and antisymmetric for $\dim Y \equiv 2\,(4)$. Proper push-forward $f_* = f_!$ commutes with duality and so induces a map of Witt groups $\mathrm{W}_\pm(\mathrm{Perv}(Y)) \to \mathrm{W}_\pm(\mathrm{Perv}(X))$. Hence (3.12) yields

$$[f_* I_Y] = \sum_S [\jmath_{S!*}\jmath_S^*\imath_S^{!*} f_* I_Y].$$

By proper base change, $\imath_S^! f_* = f_*\ell_S^!$ and $\imath_S^* f_* = f_*\ell_S^*$, where $\ell_S : f^{-1}\overline{S} \hookrightarrow X$. Hence

$$\imath_S^{!*} f_* = \mathrm{im}(H^0 f_*\ell_S^! I_Y \to H^0 f_*\ell_S^* I_Y).$$

Section 4 of [Cappell and Shaneson 1991b] uses this identification to interpret the local system $\jmath_S^*\imath_S^{!*} f_* I_Y$ on S geometrically. The stalk is the middle-dimensional intersection cohomology of $f^{-1}N_x/f^{-1}L_x$, where N_x is a normal slice to S at $x \in X$ and $L_x = \partial N_x$ is the link. In this way one can obtain formulæ for the Witt class, and thence the signature and L-class, of Y as a sum of terms indexed by the strata of X, each with a natural geometric interpretation.

3F. *Families of stratifications.* We make some brief remarks about Witt groups of perverse sheaves constructible with respect to a family of stratifications, rather than a fixed one. Let $\mathbb{S}$ be a collection of stratifications of X with only even-dimensional strata, and such that any two stratifications admit a common refinement in $\mathbb{S}$. Let $\mathrm{Perv}_{\mathbb{S}}(X)$ be the category of $\mathbb{S}$-constructible perverse sheaves of $\mathbb{F}$-vector spaces on X. For example, $\mathbb{S}$ might consist of a single stratification, or more interestingly X might be a complex algebraic or analytic variety and $\mathbb{S}$ the collection of all algebraic, respectively analytic, stratifications.

We make $\mathbb{S}$ into a poset with the ordering $\mathcal{S} \leq \mathcal{S}' \iff \mathcal{S}'$ is a refinement of $\mathcal{S}$, i.e., the strata of $\mathcal{S}$ are unions of strata of $\mathcal{S}'$. There is a fully faithful inclusion

$$\mathrm{Perv}_{\mathcal{S}}(X) \hookrightarrow \mathrm{Perv}_{\mathcal{S}'}(X)$$

whenever $\mathcal{S} \leq \mathcal{S}'$. Moreover this inclusion commutes with duality and so induces a map of Witt groups.

Proposition 3.25. *Elements of the Witt group of* $\mathbb{S}$*-constructible perverse sheaves* $W(\mathrm{Perv}_{\mathbb{S}}(X))$ *are represented by elements of* $W(\mathrm{Perv}_{\mathcal{S}}(X))$ *for some stratification* $\mathcal{S} \in \mathbb{S}$; *two such represent the same element if and only if they agree in the Witt group of perverse sheaves constructible with respect to a common refinement. In other words*,

$$W(\mathrm{Perv}_{\mathbb{S}}(X)) \cong \mathrm{colim}_{\mathcal{S}\in\mathbb{S}}\, W(\mathrm{Perv}_{\mathcal{S}}(X)).$$

Proof. The universal property of the colimit induces a map

$$\mathrm{colim}_{\mathcal{S}\in\mathbb{S}}\, W(\mathrm{Perv}_{\mathcal{S}}(X)) \to W(\mathrm{Perv}_{\mathbb{S}}(X)).$$

It is surjective since each class in $W(\mathrm{Perv}_{\mathbb{S}}(X))$ is represented by a form on a perverse sheaf which is constructible with respect to some particular stratification in $\mathbb{S}$. If two such forms are equivalent, then the equivalence is realised by a finite sequence of isotropic reductions. So the forms are already equivalent in the Witt group of perverse sheaves constructible with respect to any sufficiently refined stratification for which all objects in this sequence are constructible. Hence the map is also injective. □

Say that $\mathbb{S}$ is *artinian* if the poset of closed unions of strata, considered as subspaces of X ordered by inclusion, of all stratifications in $\mathbb{S}$ is artinian. For example this holds if we work in the complex algebraic (respectively analytic) context with the collection of all algebraic (respectively analytic) Whitney stratifications (on a compact analytic space). When this is the case the category $\mathrm{Perv}_{\mathbb{S}}(X)$ is both artinian and noetherian — for algebraic stratifications this is [Beilinson et al. 1982, Théorème 4.3.1], and the general case is proved in a similar fashion. A simple object is an intermediate extension of an irreducible local system $\mathcal{L}$ on a stratum S. Two such, $\mathcal{L}$ on S, and $\mathcal{L}'$ on S' are isomorphic if and only if there is a stratum S'', dense and open in both S and S', such that $\mathcal{L}|_{S''} \cong \mathcal{L}'|_{S''}$. Applying Corollary 2.12 we obtain another corollary.

Corollary 3.26. *If* $\mathbb{S}$ *is artinian then each class in* $W(\mathrm{Perv}_{\mathbb{S}}(X))$ *has a decomposition into a sum of classes represented by forms on simple objects. The sum* *of terms represented by forms on a given isomorphism class of simple objects is well-defined.*

Irrespective of whether $\mathbb{S}$ is artinian or not, one can apply Theorem 2.19 inductively to obtain formulæ like (3.10). If one decomposes in this way according to a stratification with respect to which a representative for the class is constructible, then the summands will be represented by forms on intermediate extensions of local systems. In "good" cases (anisotropic forms or exact intermediate extensions) this sum will correspond to the canonical decomposition of the above corollary.

3G. ***Unipotent nearby and vanishing cycles.*** Let X be a complex algebraic variety. Let $\mathrm{Perv}_{\mathrm{alg}}(X)$ denote the algebraically constructible perverse sheaves on X. Fix an algebraic map $f : X \to \mathbb{C}$ and let $\imath : Y = f^{-1}(0) \hookrightarrow X$ and $\jmath : U = X - Y \hookrightarrow X$ be the inclusions. An important feature of this situation is that the open inclusion $\jmath : U \hookrightarrow X$ is an affine morphism, which implies that ${}^p\jmath_! = \jmath_!$ and ${}^p\jmath_* = \jmath_*$.

There are exact functors

$$\begin{array}{ccc} \mathrm{Perv}_{\mathrm{alg}}(U) & \xrightarrow{\quad \Xi_f^{\mathrm{un}} \quad} & \mathrm{Perv}_{\mathrm{alg}}(X) \\ & {\scriptstyle \Psi_f^{\mathrm{un}}} \searrow \quad \swarrow {\scriptstyle \Phi_f^{\mathrm{un}}} & \\ & \mathrm{Perv}_{\mathrm{alg}}(Y) & \end{array}$$

constructed in [Beilinson 1987]; see also the notes [Reich 2010]. The functor Ξ_f^{un} is the *maximal extension*, Ψ_f^{un} the *unipotent nearby cycles* and Φ_f^{un} the *unipotent vanishing cycles*. Here we follow the presentation of [Morel 2018], which is better adapted for the discussion of Verdier duality

Remark 3.27. In this section we work in the complex algebraic context without fixing a complex algebraic Whitney stratification. However, all results apply to the case of a fixed Whitney stratification of X in the complex algebraic or analytic context (with the same arguments), if $Y = f^{-1}(0)$ is a closed union of strata. In that situation the corresponding constructible derived categories as well as the categories of perverse sheaves are stable under the functors ${}^p\jmath_! = \jmath_!$ and ${}^p\jmath_* = \jmath_*$, as well as under Ψ_f^{un} and Φ_f^{un}; see [Schürmann 2003, §4.2.2 and §6.0.4].

Let $\mathbb{Z}(1)$ denote the orientation sheaf $\mathrm{or}_{\mathbb{C}^*}$ of $\mathbb{C}^*$ and, by abuse of notation, also its stalk $\mathrm{or}_{\mathbb{C}^*,1} \cong 2\pi i\mathbb{Z}$ at the chosen base point $1 \in \mathbb{C}^*$. There is a natural representation t of $\pi_1(\mathbb{C}^*, 1)$ on $\mathbb{Z}(1)$. A choice of orientation of $\mathbb{C}^*$, equivalently of a generator $g \in \pi_1(\mathbb{C}^*, 1)$, identifies $\mathbb{Z}(1) \cong \mathbb{Z}$ with the constant sheaf of integers with $t(g) = 1$. As previously discussed, Verdier duality switches the chosen orientation to the opposite one with $t(g^{-1}) = -t(g)$. In the following we therefore want to work without choosing an orientation.

For $n \geq 0$ let $\mathbb{Z}(n) = \mathbb{Z}^{\otimes n}$ and $\mathbb{Z}(-n) = \mathbb{Z}(-1)^{\otimes n}$, where $\mathbb{Z}(-1) = \mathbb{Z}(1)^*$ is the dual local system. Again we use the same notation for their stalks at the base point $1 \in \mathbb{C}^*$, as well as for the corresponding local systems on $U = X - Y$ pulled back via $f : U \to \mathbb{C}^*$. Similarly, $-(n) = - \otimes_{\mathbb{Z}} \mathbb{Z}(n)$ denotes the corresponding Tate-twists of sheaves or stalks of $\mathbb{F}$-vector spaces, with $\mathbb{F}$ our base field of characteristic zero.

Consider for $p \geq 1$ the p-dimensional $\mathbb{F}$-vector space

$$L^p = \mathbb{F} \oplus \mathbb{F}(-1) \oplus \cdots \oplus \mathbb{F}(1-p)$$

together with the nilpotent morphism $N : L^p \to L^p(-1)$ given by the matrix

$$N = \begin{pmatrix} 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \\ & & & \ddots \end{pmatrix}.$$

Let $\mathcal{L}^p$ be the corresponding local system on $\mathbb{C}^*$, with stalk L^p in $1 \in \mathbb{C}^*$, and monodromy action

$$\mu(g) = e^{t(g) \cdot N} : L^p \to L^p$$

for $g \in \pi_1(\mathbb{C}^*, 1)$ any generator. For $p + q = n$ there is a short exact sequence

$$0 \to \mathcal{L}^p \to \mathcal{L}^n \to \mathcal{L}^q(-p) \to 0, \tag{3.28}$$

where the maps are inclusion of the first p coordinates and projection onto the last q coordinates.

The unipotent nearby cycles of a perverse sheaf $\mathcal{A}$ on U are defined by

$$\imath_* \Psi_f^{\mathrm{un}} \mathcal{A} = \lim_{n \to \infty} \ker[\jmath_!(\mathcal{A} \otimes f^* \mathcal{L}^n) \to \jmath_*(\mathcal{A} \otimes f^* \mathcal{L}^n)],$$

where the map on the right-hand side is the natural one. The kernel of this map stabilises for sufficiently large n, and the limit denotes this stable kernel; see [Morel 2018, Corollary 3.2]. The maximal extension of $\mathcal{A}$ is constructed similarly as

$$\Xi_f^{\mathrm{un}} \mathcal{A} = \lim_{n \to \infty} \ker[\jmath_!(\mathcal{A} \otimes f^* \mathcal{L}^n) \to \jmath_*(\mathcal{A} \otimes f^* \mathcal{L}^{n-1})(-1)],$$

where the map on the right is induced from the quotient in (3.28) with $q = n - 1$, and once again the kernel stabilises for sufficiently large n; see [Morel 2018, Proposition 5.1]. The action of $N : \mathcal{L} \to \mathcal{L}(-1)$ induces actions $\Psi_f^{\mathrm{un}} \mathcal{A} \to \Psi_f^{\mathrm{un}} \mathcal{A}(-1)$ and $\Xi_f^{\mathrm{un}} \mathcal{A} \to \Xi_f^{\mathrm{un}} \mathcal{A}(-1)$, respectively, which we also denote by N, and the same holds for the induced monodromy action $\mu(g) = e^{t(g) \cdot N}$ of a generator g of $\pi_1(\mathbb{C}^*, 1)$ on $\Psi_f^{\mathrm{un}} \mathcal{A}$ and $\Xi_f^{\mathrm{un}} \mathcal{A}$.

Whereas the maximal extension functor commutes with Verdier duality [Morel 2018, Corollary 5.4], the unipotent nearby cycle functor commutes with Verdier duality only up to a Tate-twist [Morel 2018, Corollary 4.2]:

$$D(\Psi_f^{\mathrm{un}} \mathcal{A}) \cong \Psi_f^{\mathrm{un}}(D(\mathcal{A}))(-1). \tag{3.29}$$

Moreover, there are two natural short exact sequences

$$0 \longrightarrow \jmath_! \mathcal{A} \xrightarrow{\alpha_-} \Xi_f^{\mathrm{un}} \mathcal{A} \xrightarrow{\beta_-} \imath_* \Psi_f^{\mathrm{un}} \mathcal{A}(-1) \longrightarrow 0 \tag{3.30}$$

and

$$0 \longrightarrow \imath_* \Psi_f^{\mathrm{un}} \mathcal{A} \xrightarrow{\beta_+} \Xi_f^{\mathrm{un}} \mathcal{A} \xrightarrow{\alpha_+} \jmath_* \mathcal{A} \longrightarrow 0 \tag{3.31}$$

which are exchanged by duality [Morel 2018, Proposition 5.1, Corollary 5.4]. The maps are induced from those in (3.28) for $(p, q) = (1, n-1)$ and $(n-1, 1)$. The composite $\alpha_+ \circ \alpha_-$ is the natural map, and $\beta_- \circ \beta_+ = N$ [Morel 2018, Remark 5.6]. In particular the action $N : \Psi_f^{\mathrm{un}} \mathcal{A} \to \Psi_f^{\mathrm{un}} \mathcal{A}(-1)$ commutes with the duality isomorphism above: $D \circ N \cong N \circ D$. This also holds without Tate-twists, if one chooses opposite generators of $\pi_1(\mathbb{C}^*, 1)$ on both sides of this identification. Otherwise a minus sign shows up, e.g., if one chooses on both sides the complex orientation of $\mathbb{C}^*$ as in [Saito 1989].

The perverse unipotent vanishing cycles $\Phi_f^{\mathrm{un}} \mathcal{B}$ of $\mathcal{B} \in \mathrm{Perv}_{\mathrm{alg}}(X)$ are defined to be (the restriction of) the cohomology $\imath^* H^0(-)$ of the complex

$$\jmath_! \jmath^* \mathcal{B} \xrightarrow{(\alpha_-, \gamma_-)^t} \Xi_f^{\mathrm{un}} \jmath^* \mathcal{B} \oplus \mathcal{B} \xrightarrow{(\alpha_+, -\gamma_+)} \jmath_* \jmath^* \mathcal{B}$$

sitting in degrees -1 to 1, where $\gamma_\pm$ are the unit and counit of the adjunctions. Note that the first (respectively last) morphism in this complex is injective (respectively surjective) with its cohomology $H^0(-)$ supported on Y (since its restriction to the complement $X - Y$ is vanishing). That Φ_f^{un} commutes with duality follows from the fact that duality interchanges the above two short exact sequences [Morel 2018, Remark 6.1]. One also gets induced morphisms

$$\Psi_f^{\mathrm{un}} \jmath^* \mathcal{B} \xrightarrow{\mathrm{can}} \Phi_f^{\mathrm{un}} \mathcal{B} \xrightarrow{\mathrm{Var}} \Psi_f^{\mathrm{un}} \jmath^* \mathcal{B}(-1)$$

of perverse sheaves on Y with $N = \mathrm{Var} \circ \mathrm{can}$, so that can and Var are exchanged by duality [Morel 2018, Remark 6.1]. Moreover, the category $\mathrm{Perv}_{\mathrm{alg}}(X)$ can be described in terms of the gluing data [Morel 2018, Theorem 8.1]:

$$\mathcal{B} \mapsto (\jmath^* \mathcal{B}, \Phi_f^{\mathrm{un}} \mathcal{B}, \mathrm{can}, \mathrm{Var}).$$

For example $\jmath_{!*} \mathcal{A}$ has the following gluing data description (see also [Reich 2010, Proposition 4.7]):

$$\big(\mathcal{A}, \mathrm{im}(N : \Psi_f^{\mathrm{un}} \mathcal{A} \to \Psi_f^{\mathrm{un}} \mathcal{A}(-1)), N, \mathrm{incl}\big), \tag{3.32}$$

with $N : \Psi_f^{\mathrm{un}}(\jmath^* \jmath_{!*} \mathcal{A}) = \Psi_f^{\mathrm{un}} \mathcal{A} \to \Psi_f^{\mathrm{un}} \mathcal{A}(-1) = \Psi_f^{\mathrm{un}}(\jmath^* \jmath_{!*} \mathcal{A})(-1)$ factorised as

$$\Psi_f^{\mathrm{un}} \mathcal{A} \xrightarrow{N} \mathrm{im}(N) \xrightarrow{\mathrm{incl}} \Psi_f^{\mathrm{un}} \mathcal{A}(-1).$$

Since the unipotent vanishing cycles and the maximal extension commute with duality they induce maps of Witt groups. We abuse notation by using the functors to denote these induced maps.

Lemma 3.33. *The map* $[\beta] \mapsto (\jmath^*[\beta], \Phi_f^{\mathrm{un}}[\beta])$ *is an isomorphism*

$$W(\mathrm{Perv}_{\mathrm{alg}}(X)) \cong W(\mathrm{Perv}_{\mathrm{alg}}(U)) \oplus W(\mathrm{Perv}_{\mathrm{alg}}(Y))$$

with inverse $([\beta], [\beta']) \mapsto \Xi_f^{\mathrm{un}}[\beta] + \imath_*[\beta']$.

Proof. From the constructions, $\Phi_f^{\mathrm{un}} \circ \imath_*$ and $\jmath^* \circ \Xi_f^{\mathrm{un}}$ are the identity. Therefore

$$\Xi_f^{\mathrm{un}}[\beta] = \imath_*[\beta']$$

implies $[\beta] = \jmath^* \Xi_f^{\mathrm{un}}[\beta] = 0$, and hence $[\beta'] = \Phi_f^{\mathrm{un}} \imath_*[\beta'] = 0$ too.

Given $[\beta] \in W(\mathrm{Perv}_{\mathrm{alg}}(X))$, the form $\beta \oplus \Xi_f^{\mathrm{un}} \jmath^*(-\beta)$ is metabolic when restricted to U. Using $\jmath_!$ we can construct an isotropic subobject for this form from a lagrangian for the restriction. The reduction by this isotropic subobject will be supported on Y, so that

$$[\beta] - \Xi_f^{\mathrm{un}} \jmath^*[\beta] = \imath_*[\beta']$$

for some $[\beta'] \in W(\mathrm{Perv}_{\mathrm{alg}}(Y))$. We now show that $[\beta'] = \Phi_f^{\mathrm{un}}[\beta]$, or equivalently that $\Phi_f^{\mathrm{un}} \circ \Xi_f^{\mathrm{un}} = 0$ on Witt groups. To see this recall that there is a functorial short exact sequence [Morel 2018, Corollary 7.2]

$$0 \to \Psi_f^{\mathrm{un}} \to \Phi_f^{\mathrm{un}} \Xi_f^{\mathrm{un}} \to \Psi_f^{\mathrm{un}}(-1) \to 0,$$

so that the induced form $\Phi_f^{\mathrm{un}} \Xi_f^{\mathrm{un}}[\beta]$ is metabolic. Therefore $\Phi_f^{\mathrm{un}} \Xi_f^{\mathrm{un}}[\beta] = 0$ in the Witt group as claimed. □

We can relate the above decomposition to our earlier splitting results.

Corollary 3.34. *For* $[\beta : \mathcal{A} \to D(\mathcal{A})] \in W(\mathrm{Perv}_{\mathrm{alg}}(U))$ *the composite*

$$\Psi_f^{\mathrm{un}} \beta \circ N : \Psi_f^{\mathrm{un}} \mathcal{A} \to \Psi_f^{\mathrm{un}} \mathcal{A}(-1) \to D\Psi_f^{\mathrm{un}} \mathcal{A}$$

is symmetric and $[\jmath_{!*}\beta] = \Xi_f^{\mathrm{un}}[\beta] + \imath_*[\Psi_f^{\mathrm{un}} \beta \circ N]$. *Similarly, for* $[\beta'] \in W(\mathrm{Perv}_{\mathrm{alg}}(X))$ *we have*

$$[\imath^{!*}\beta'] = \Phi_f^{\mathrm{un}}[\beta'] - [\Psi_f^{\mathrm{un}}(\jmath^*\beta') \circ N].$$

Proof. It is easy to verify that

$$\Psi_f^{\mathrm{un}} \beta \circ N : \Psi_f^{\mathrm{un}} \mathcal{A} \to \Psi_f^{\mathrm{un}} \mathcal{A}(-1) \to \Psi_f^{\mathrm{un}}(D(\mathcal{A}))(-1) \cong D\Psi_f^{\mathrm{un}} \mathcal{A}$$

is symmetric, since $N : \Psi_f^{\mathrm{un}} \mathcal{A} \to \Psi_f^{\mathrm{un}} \mathcal{A}(-1)$ commutes with duality. Moreover, from the description of intermediate extensions in terms of gluing data (3.32), one gets

$$\Phi_f^{\mathrm{un}}[\jmath_{!*}\beta] = [\Psi_f^{\mathrm{un}} \beta \circ N].$$

Hence by Lemma 3.33 $[\jmath_{!*}\beta] = \Xi_f^{\mathrm{un}}[\beta] + \iota_*[\Psi_f^{\mathrm{un}}\beta \circ N]$, so that

$$[\iota^{!*}\beta'] = [\beta'] - [\jmath_{!*}\jmath^*\beta'] = \Phi_f^{\mathrm{un}}[\beta'] - [\Psi_f^{\mathrm{un}}(\jmath^*\beta') \circ N]$$

as claimed. □

Remark 3.35. An alternative method of proof is to verify that the first equation is the splitting relation arising from the short exact sequences (3.30) and (3.31). The second is the splitting relation for the following two exact sequences of perverse sheaves which are exchanged by duality [Morel 2018, Proposition 6.2]:

$$\Psi_f^{\mathrm{un}}(\jmath^*\mathcal{B}) \xrightarrow{\mathrm{can}} \Phi_f^{\mathrm{un}}\mathcal{B} \longrightarrow H^0(\iota^*\mathcal{B}) \longrightarrow 0$$

and

$$0 \longrightarrow H^0(\iota^!\mathcal{B}) \longrightarrow \Phi_f^{\mathrm{un}}\mathcal{B} \xrightarrow{\mathrm{Var}} \Psi_f^{\mathrm{un}}(\jmath^*\mathcal{B})(-1),$$

with H^0 the corresponding perverse cohomology.

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A generalized Vaserstein symbol

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Let R be a commutative ring. For any projective R-module P_0 of constant rank 2 with a trivialization of its determinant, we define a generalized Vaserstein symbol on the orbit space of the set of epimorphisms $P_0 \oplus R \to R$ under the action of the group of elementary automorphisms of $P_0 \oplus R$, which maps into the elementary symplectic Witt group. We give criteria for the surjectivity and injectivity of the generalized Vaserstein symbol and deduce that it is an isomorphism if R is a regular Noetherian ring of dimension 2 or a regular affine algebra of dimension 3 over a perfect field k with c.d.$(k) \leq 1$ and $6 \in k^{\times}$.

1. Introduction

In this paper, we provide a generalized construction of the Vaserstein symbol, which was originally introduced by Andrei Suslin and Leonid Vaserstein in [Vaserstein and Suslin 1976]. We let R be a commutative ring and we let $\mathrm{Um}_n(R)$ denote the set of unimodular rows of length n, i.e., row vectors $(a_1, a_2, , \ldots, a_n)$ such that $\langle a_1, a_2, \ldots, a_n \rangle = R$. Such row vectors obviously correspond to epimorphisms $R^n \to R$. Therefore the group $\mathrm{GL}_n(R)$ of invertible $n \times n$-matrices acts on the right on $\mathrm{Um}_n(R)$ (by precomposition); consequently the same holds for any subgroup of $\mathrm{GL}_n(R)$, e.g., the group $\mathrm{SL}_n(R)$ of invertible $n \times n$-matrices with determinant 1 or its subgroup $E_n(R)$ generated by elementary matrices. Note that the set $\mathrm{Um}_n(R)$ has a canonical basepoint given by the row $e_1 = (1, 0, \ldots, 0)$.

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Now let $n = 3$ and let (a_1, a_2, a_3) be a unimodular row of length 3. By definition, there exist elements $b_1, b_2, b_3 \in R$ such that $\sum_{i=1}^{3} a_i b_i = 1$. Therefore the alternating matrix

$$V(a,b) = \begin{pmatrix} 0 & -a_1 & -a_2 & -a_3 \\ a_1 & 0 & -b_3 & b_2 \\ a_2 & b_3 & 0 & -b_1 \\ a_3 & -b_2 & b_1 & 0 \end{pmatrix}$$

has Pfaffian 1 and represents an element of the so-called elementary symplectic Witt group $W_E(R)$. It was shown in [Vaserstein and Suslin 1976, Lemma 5.1] that this element is independent of the choice of the elements b_1, b_2, b_3. Furthermore, it follows from [Vaserstein and Suslin 1976, Theorem 5.2(a)] that this assignment is invariant under the action of $E_3(R)$ on $\mathrm{Um}_3(R)$. Therefore one obtains a well-defined map

$$V : \mathrm{Um}_3(R)/E_3(R) \to W_E(R)$$

called the Vaserstein symbol. Suslin and Vaserstein also found criteria for this map to be surjective or injective in terms of the right action of $E_n(R)$ on $\mathrm{Um}_n(R)$ mentioned above. More precisely, they proved that the Vaserstein symbol is surjective if $\mathrm{Um}_{2n+1}(R) = e_1 E_{2n+1}(R)$ for $n \geq 2$ [Vaserstein and Suslin 1976, Theorem 5.2(b)] and injective if $e_1 E_{2n} = e_1(E(R) \cap \mathrm{GL}_{2n}(R))$ for $n \geq 3$ and $E(R) \cap \mathrm{GL}_4(R) = E_4(R)$ [Vaserstein and Suslin 1976, Theorem 5.2(c) and proof of Corollary 7.4].

These criteria immediately enabled them to deduce that the Vaserstein symbol is a bijection for a Noetherian ring of Krull dimension 2 [Vaserstein and Suslin 1976, Corollary 7.4]. Using local-global principles, Rao and van der Kallen [1994, Corollary 3.5] proved that the Vaserstein symbol is also a bijection for a 3-dimensional regular affine algebra over a field k with $\mathrm{c.d.}(k) \leq 1$, which is supposed to be perfect if $\mathrm{char}(k) = 2, 3$.

The Vaserstein symbol plays an important role in the study of stably free modules of rank 2 [Bass 1975; Fasel 2011]. Indeed, the orbit space $\mathrm{Um}_3(R)/E_3(R)$ naturally surjects onto the set of isomorphism classes of projective R-modules of rank 2 which become free after adding a free direct summand of rank 1 (see Section 2D). In [Fasel et al. 2012, Theorem 7.5], the Vaserstein symbol was crucially used in order to prove that stably free modules of rank $d-1$ over smooth affine k-algebras of dimension $d \geq 3$ are free whenever k is algebraically closed and $(d-1)! \in k^\times$: By reducing to the case of a threefold and by using the result of Rao and van der Kallen mentioned in the previous paragraph, it was proven that any unimodular row of length d can be transformed via elementary matrices to a row of the form $(a_1, a_2, \ldots, a_d^{(d-1)!})$. Then Suslin's theorem that any such row can be completed to an invertible matrix [Suslin 1977b, Theorem 2] implied the result.

While projective modules of rank $\geq d$ are cancellative in the situation of [Fasel et al. 2012, Theorem 7.5], the same is not true in general for projective modules of rank $d-2$ [Mohan Kumar 1985]. In particular, stably isomorphic projective modules of rank 2 over smooth affine fourfolds over algebraically closed fields need not be isomorphic in general.

Our work on the generalization of the Vaserstein symbol is substantially motivated by the study of projective modules as described in the previous paragraphs: The generalized Vaserstein symbol will lead to a conceptual explanation for the failure of the cancellation property of projective modules of rank 2 with trivial determinant over smooth affine fourfolds over algebraically closed fields. By generalizing the approach in [Fasel et al. 2012], we also foresee that the generalized Vaserstein symbol will be an important tool in order to study the cancellation property of projective modules of rank $d-1$ with trivial determinant over smooth affine algebras of dimension d over an algebraically closed field k with $(d-1)! \in k^{\times}$. To keep the length of this paper reasonable, the discussion of these major applications is deferred to subsequent work. Our results in this paper are as follows.

First, recall from [Vaserstein and Suslin 1976] that the elementary symplectic Witt group $W_E(R)$ is defined as a subgroup of a larger group usually denoted $W'_E(R)$, which we will define in Section 3A. The group $W'_E(R)$ is generated by alternating invertible matrices and $W_E(R)$ then corresponds to its subgroup generated by matrices with Pfaffian 1. It is known that the group $W'_E(R)$ is isomorphic to the higher Grothendieck–Witt group $\mathrm{GW}^3_1(R)$ and also to the group $V(R)$ [Fasel et al. 2012] (see Section 3B below). The latter group is generated by isometry classes of triples (P, g, f), where P is a finitely generated projective R-module and f and g are alternating isomorphisms on P (or, equivalently, nondegenerate alternating forms on P). Under the isomorphism $W'_E(R) \cong V(R)$, the group $W_E(R)$ then corresponds to a subgroup of $V(R)$. We denote this subgroup by $\widetilde{V}(R)$.

Now let P_0 be a finitely generated projective R-module of rank 2 with a fixed trivialization $\theta_0 : R \xrightarrow{\cong} \det(P_0)$ of its determinant. We denote by $\mathrm{Um}(P_0 \oplus R)$ the set of epimorphisms $P_0 \oplus R \to R$ and by $E(P_0 \oplus R)$ the group of elementary automorphisms of $P_0 \oplus R$. Any element $a : P_0 \oplus R \to R$ of $\mathrm{Um}(P_0 \oplus R)$ has a section $s : R \to P_0 \oplus R$, which canonically induces an isomorphism

$$i : P_0 \oplus R \xrightarrow{\cong} P(a) \oplus R,$$

where $P(a) = \ker(a)$. We let χ_0 be the alternating form on P_0 which sends a pair (p, q) to the element $\theta_0^{-1}(p \wedge q)$ of R; similarly, there is an isomorphism $\theta : R \xrightarrow{\cong} \det(P(a))$ obtained as the composite of θ_0 and the isomorphism $\det(P_0) \cong \det(P(a))$ induced by a and s. We then denote by χ_a the alternating form on $P(a)$ which sends (p, q) to the element $\theta^{-1}(p \wedge q)$ of R. We then consider the element

$$V(a) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)]$$

of $V(R)$. Our first result is the following:

Theorem 1 (Theorem 4.1, Lemma 4.2 and Theorem 4.3). *The element $V(a)$ is independent of the choice of a section s of $a \in \mathrm{Um}(P_0 \oplus R)$ and is an element of $\widetilde{V}(R)$. Furthermore, we have $V(a) = V(a\varphi)$ in $V(R)$ for all $a \in \mathrm{Um}(P_0 \oplus R)$ and $\varphi \in E(P_0 \oplus R)$. Thus, the assignment above descends to a well-defined map $V : \mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R) \to \widetilde{V}(R)$, which we call the generalized Vaserstein symbol (associated to the trivialization θ_0 of $\det(P_0)$).*

The terminology is justified by the following observation: If we take $P_0 = R^2$ and let $e_1 = (1, 0)$ and $e_2 = (0, 1)$, then it is well-known that there is a canonical isomorphism $\theta_0 : R \xrightarrow{\cong} \det(R^2)$ given by $1 \mapsto e_1 \wedge e_2$. Then the generalized Vaserstein symbol associated to $-\theta_0$ coincides with the usual Vaserstein symbol via the identification $\widetilde{V}(R) \cong W_E(R)$ mentioned above.

Of course, any two trivializations of $\det(P_0)$ are equal up to multiplication by a unit of R. We actually make precise how the generalized Vaserstein symbol depends on the choice of a trivialization of $\det(P_0)$ by means of a canonical $R^\times$-action on $V(R)$.

We also generalize the criteria found by Suslin and Vaserstein on the injectivity and surjectivity of the Vaserstein symbol. For this, let $P_n = P_0 \oplus R^{n-2}$ for all $n \geq 3$ and let $E_\infty(P_0)$ be the direct limit of the groups $E(P_n)$ for $n \geq 3$. Note that $\mathrm{Um}(P_n)$ has a canonical basepoint given by the projection $\pi_{n,n}$ onto the "last" free direct summand of rank 1. We then prove:

Theorem 2 (Theorems 4.5 and 4.14). *The Vaserstein symbol*

$$V : \mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R) \to \widetilde{V}(R)$$

is surjective if $\mathrm{Um}(P_{2n+1}) = \pi_{2n+1,2n+1}E(P_{2n+1})$ for all $n \geq 2$. Furthermore, it is injective if $\pi_{2n,2n}E(P_{2n}) = \pi_{2n,2n}(E_\infty(P_0) \cap \mathrm{Aut}(P_{2n}))$ for all $n \geq 3$ and $E_\infty(P_0) \cap \mathrm{Aut}(P_4) = E(P_4)$.

Using local-global principles for transvection groups [Bak et al. 2010], we may then prove the following result:

Theorem 3 (Theorems 2.15, 2.16 and 4.15). *The equality*

$$E_\infty(P_0) \cap \mathrm{Aut}(P_4) = E(P_4)$$

holds if R is a 2-dimensional regular Noetherian ring or a 3-dimensional regular affine algebra over a perfect field k such that $\mathrm{c.d.}(k) \leq 1$ and $6 \in k^\times$. In particular, it follows that the generalized Vaserstein symbol $V : \mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R) \to \widetilde{V}(R)$ is a bijection in these cases.

As indicated above, the corresponding result by Rao and van der Kallen for the usual Vaserstein symbol in dimension 3 and Suslin's theorem on the completability of unimodular rows were crucially used in the proof of [Fasel et al. 2012, Theorem 7.5]. As a special case of Suslin's theorem, one obtains that unimodular rows of the form (a_1, a_2, a_3^2) are completable to invertible matrices. An explicit completion of such a unimodular row is given, e.g., in [Krusemeyer 1976]. In fact, we can translate this result to our setting: Any epimorphism $a : P_0 \oplus R \to R$ can be written as (a_0, a_R), where a_0 and a_R are the restrictions of a to P_0 and R, respectively. Then we can generalize Krusemeyer's construction in order to give an explicit completion of an epimorphism of the form $a = (a_0, a_R^2)$ to an automorphism of $P_0 \oplus R$ (see Proposition 4.18).

While it should be possible to define a Vaserstein symbol without the assumption of a trivial determinant of P_0, it is by no means obvious that our methods in this paper can be adjusted in order to prove the same results without this assumption.

The organization of the paper is as follows: In Section 2, we prove the technical ingredients for the proofs of the main results of this paper. In particular, we prove some lemmas on elementary automorphisms of projective modules and use local-global principles for transvection groups in order to derive stability results for automorphism groups of projective modules. Section 3 basically covers the definition of the elementary symplectic Witt group $W_E(R)$ and the identifications of $W'_E(R)$, $V(R)$ and $\mathrm{GW}_1^3(R)$. In Section 4, we motivate and give the definition of the generalized Vaserstein symbol and begin to study its basic properties. We then use all the technical lemmas proven in the previous sections in order to deduce the theorems stated above.

Notation and conventions. In this paper, a ring R is always commutative with unit. If k is a perfect field, we denote by $\mathcal{H}(k)$ the $\mathbb{A}^1$-homotopy category as defined by Morel and Voevodsky and by $\mathcal{H}_\bullet(k)$ its pointed version. If $\mathcal{X}$ and $\mathcal{Y}$ are spaces (resp. pointed spaces), we write $[\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1}$ (resp. $[\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1,\bullet}$) for the set of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ in $\mathcal{H}(k)$ (resp. $\mathcal{H}_\bullet(k)$).

2. Preliminaries on projective modules

In this section, we recall some basic facts on projective modules over commutative rings and prove some technical lemmas on elementary automorphisms which will be crucially used in the proofs of the main results of this paper. We also recall the local-global principle for transvection groups in [Bak et al. 2010] in order to prove a stability result on automorphisms of projective modules. At the end of this section, we briefly recall how projective modules stably isomorphic to a given projective module P can be classified in terms of the orbit space of the set of epimorphisms $P \oplus R \to R$ under the action of the groups of automorphisms of $P \oplus R$.

2A. ***Local trivializations and alternating isomorphisms on projective modules.*** Let R be a commutative ring and P be any projective R-module. For any prime ideal $\mathfrak{p}$ of R, the localized $R_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$ is again projective and therefore free (because projective modules over local rings are free). In this weak sense, projective modules are locally free. If the rank of $P_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module is finite for every prime $\mathfrak{p}$, then we say that P is a projective module of finite rank. In this case, there is a well-defined map $\text{rank}_P : \text{Spec}(R) \to \mathbb{Z}$ which sends a prime ideal $\mathfrak{p}$ of R to the rank of $P_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module. It is not true in general that projective modules of finite rank are finitely generated; nevertheless, this is true if rank_P is a constant map [Weibel 2013, Chapter I, Exercise 2.14]. We say that P is locally free of finite rank (in the strong sense) if it admits elements $f_1, \dots, f_n \in R$ generating the unit ideal such that the localizations P_{f_k} are free R_{f_k}-modules of finite rank. In fact, it is well-known that this is true if and only if P is a finitely generated projective module. The following lemma follows from [Weibel 2013, Chapter I, Lemma 2.4] and [Weibel 2013, Chapter I, Exercise 2.11]:

Lemma 2.1. *Let R be a ring and M be an R-module. Then the following statements are equivalent*:

(a) *M is a finitely generated projective R-module*;

(b) *M is locally free of finite rank* (*in the strong sense*);

(c) *M is a finitely presented R-module and $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$ of R*;

(d) *M is a finitely generated R-module, $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for every prime ideal $\mathfrak{p}$ of R and the induced map $\text{rank}_M : \text{Spec}(R) \to \mathbb{Z}$ is continuous.*

For any projective R-module P of finite rank, there is a canonical isomorphism

$$\text{can} : P \to P^{\vee\vee}, \quad p \mapsto (\text{ev}_p : P^{\vee} \to R, a \mapsto a(p))$$

induced by evaluation. A symmetric isomorphism on P is an isomorphism $f : P \to P^{\vee}$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & P^{\vee} \\ {\scriptstyle \text{can}}\downarrow & & \| {\scriptstyle \text{id}} \\ P^{\vee\vee} & \xrightarrow[f^{\vee}]{} & P^{\vee} \end{array}$$

is commutative. Similarly, a skew-symmetric isomorphism on P is an isomorphism $f : P \to P^{\vee}$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & P^{\vee} \\ {\scriptstyle -\text{can}}\downarrow & & \| {\scriptstyle \text{id}} \\ P^{\vee\vee} & \xrightarrow[f^{\vee}]{} & P^{\vee} \end{array}$$

is commutative. Finally, an alternating isomorphism on P is an isomorphism $f : P \to P^{\vee}$ such that $f(p)(p) = 0$ for all $p \in P$.

A symmetric form on a projective R-module P of finite rank is an R-bilinear map $\chi : P \times P \to R$ such that $\chi(p, q) = \chi(q, p)$ for all $p, q \in P$. Similarly, a skew-symmetric form on a projective R-module P of finite rank is an R-bilinear map $\chi : P \times P \to R$ such that $\chi(p, q) = -\chi(q, p)$ for all $p, q \in P$. Moreover, an alternating form on a projective R-module P of finite rank is an R-bilinear map $\chi : P \times P \to R$ such that $\chi(p, p) = 0$ for all $p \in P$. Note that any alternating form on P is automatically skew-symmetric. If $2 \in R^{\times}$, any skew-symmetric form is alternating as well. A (skew-)symmetric or alternating form χ is nondegenerate if the induced map $P \to P^{\vee}, q \mapsto (p \mapsto \chi(p, q))$ is an isomorphism. Obviously, the data of a nondegenerate (skew-)symmetric form is equivalent to the data of a (skew-)symmetric isomorphism. Analogously, the data of a nondegenerate alternating form is equivalent to the data of an alternating isomorphism.

Now let $\chi : M \times M \to R$ be any R-bilinear form on M. This form induces a homomorphism $M \otimes_R M \to R$. For any prime $\mathfrak{p}$ of R, there is an induced homomorphism $M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \cong (M \otimes_R M)_{\mathfrak{p}} \to R_{\mathfrak{p}}$. This gives an R-bilinear form $\chi_{\mathfrak{p}} : M_{\mathfrak{p}} \times M_{\mathfrak{p}} \to R_{\mathfrak{p}}$ on $M_{\mathfrak{p}}$. The following lemma shows that these localized forms completely determine χ.

Lemma 2.2. *Suppose χ_1 and χ_2 are R-bilinear forms on an R-module M. Then $\chi_1 = \chi_2$ if and only if ${\chi_1}_{\mathfrak{p}} = {\chi_2}_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of R.*

Proof. The forms χ_1 and χ_2 agree if and only if $\chi_1(p, q) - \chi_2(p, q) = 0$ for all $p, q \in M$. Therefore the lemma follows immediately from the fact that being 0 is a local property for elements of any R-module. □

2B. ***Elementary automorphisms and unimodular elements.*** Again, let R be a ring and let $M \cong \bigoplus_{i=1}^{n} M_i$ be an R-module which admits a decomposition into a direct sum of R-modules M_i, $i = 1, \ldots, n$. An elementary automorphism φ of M with respect to the given decomposition is an endomorphism of the form $\varphi_{s_{ij}} = \mathrm{id}_M + s_{ij}$, where $s_{ij} : M_j \to M_i$ is an R-linear homomorphism for some $i \neq j$ [Bass 1968, Chapter IV, §3]. Any such homomorphism automatically is an isomorphism with inverse given by $\varphi_{s_{ij}}^{-1} = \mathrm{id}_M - s_{ij}$. For $M = R^n \cong \bigoplus_{i=1}^{n} R$, one just obtains the automorphisms given by elementary matrices. We denote by $\mathrm{Aut}(M)$ the group of automorphisms of M and by $E(M_1, \ldots, M_n)$ (or simply $E(M)$ if the decomposition is understood) the subgroup of $\mathrm{Aut}(M)$ generated by elementary automorphisms.

The following lemma gives a list of useful formulas, which can be checked easily by direct computation.

Lemma 2.3. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a direct sum of R-modules. Then we have*

(a) $\varphi_{s_{ij}} \varphi_{t_{ij}} = \varphi_{(s_{ij}+t_{ij})}$ *for all* $s_{ij} : M_j \to M_i$, $t_{ij} : M_j \to M_i$ *and* $i \neq j$;

(b) $\varphi_{s_{ij}}\varphi_{s_{kl}} = \varphi_{s_{kl}}\varphi_{s_{ij}}$ *for all* $s_{ij} : M_j \to M_i$, $s_{kl} : M_l \to M_k$, $i \neq j$, $k \neq l$, $j \neq k$ *and* $i \neq l$;

(c) $\varphi_{s_{ij}}\varphi_{s_{jk}}\varphi_{-s_{ij}}\varphi_{-s_{jk}} = \varphi_{(s_{ij}s_{jk})}$ *for all* $s_{ij} : M_j \to M_i$, $s_{jk} : M_k \to M_j$ *and distinct* i, j, k;

(d) $\varphi_{s_{ij}}\varphi_{s_{ki}}\varphi_{-s_{ij}}\varphi_{-s_{ki}} = \varphi_{(-s_{ki}s_{ij})}$ *for all* $s_{ij} : M_j \to M_i$, $s_{ki} : M_i \to M_k$ *and distinct* i, j, k.

If we restrict to the case $M_i = M_n$ for $i \geq 2$, we obtain the following result on $E(M)$:

Corollary 2.4. *If* $M_i = M_n$ *for* $i \geq 2$, *then the group* $E(M)$ *is generated by the elementary automorphisms of the form* $\varphi_s = \mathrm{id}_M + s$, *where* s *is an* R*-linear map* $M_i \to M_n$ *or* $M_n \to M_i$ *for some* $i \neq n$. *The same statement holds if one replaces* n *by any other* $k \geq 2$.

Proof. Since $M_i = M_n$ for all $i \geq 2$, we have identities $\mathrm{id}_{in} : M_n \to M_i$ and $\mathrm{id}_{ni} : M_i \to M_n$ for all $i \geq 2$. Let $s_{ij} : M_j \to M_i$ be a morphism with $i \neq j$ and therefore either $i \geq 2$ or $j \geq 2$. We may assume that i, j, n are distinct. If $i \geq 2$, then

$$\varphi_{s_{ij}} = \varphi_{\mathrm{id}_{in}}\varphi_{\mathrm{id}_{ni}\, s_{ij}}\varphi_{-\,\mathrm{id}_{in}}\varphi_{(-\mathrm{id}_{ni}s_{ij})}$$

by the third formula in Lemma 2.3. If $j \geq 2$, then

$$\varphi_{s_{ij}} = \varphi_{(s_{ij}\mathrm{id}_{jn})}\varphi_{\mathrm{id}_{nj}}\varphi_{(-s_{ij}\mathrm{id}_{jn})}\varphi_{-\,\mathrm{id}_{nj}}.$$

by the third formula in Lemma 2.3. This proves the first part of the corollary. The last part follows in the same way if n is replaced by $k \geq 2$. □

The proof of Corollary 2.4 also shows:

Corollary 2.5. *Let* $M = \bigoplus_{i=1}^{n} M_i$ *be a direct sum of* R*-modules and also let* $s : M_j \to M_i$, $i \neq j$, *be an* R*-linear map. Assume that there is* $k \neq i$ *with* $M_k = M_i$ *or* $k \neq j$ *with* $M_k = M_j$. *Then the induced elementary automorphism* φ_s *is a commutator.*

The following lemma is a version of Whitehead's lemma in our general setting:

Lemma 2.6. *Let* $M = M_1 \oplus M_2$ *and let* $f : M_1 \to M_2$, $g : M_2 \to M_1$ *be morphisms. Assume that* $\mathrm{id}_{M_1} + gf$ *is an automorphism of* M_1. *Then*

- $\mathrm{id}_{M_2} + fg$ *is an automorphism of* M_2 *and*
- $(\mathrm{id}_{M_1} + gf) \oplus (\mathrm{id}_{M_2} + fg)^{-1}$ *is an element of* $E(M_1 \oplus M_2)$.

Proof. We have $\mathrm{id}_{M_1} \oplus (\mathrm{id}_{M_2} + fg) = \varphi_{-f}\varphi_{-g}((\mathrm{id}_{M_1} + gf) \oplus \mathrm{id}_{M_2})\varphi_f\varphi_g$. This shows the first statement. For the second statement one checks that

$$(\mathrm{id}_{M_1} + gf) \oplus (\mathrm{id}_{M_2} + fg)^{-1} = \varphi_{-g}\varphi_{-f}\varphi_g\varphi_{(\mathrm{id}_{M_1} + gf)^{-1}g - g}\varphi_{fgf+f}.$$

So $(\mathrm{id}_{M_1} + gf) \oplus (\mathrm{id}_{M_2} + fg)^{-1}$ lies in $E(M_1 \oplus M_2)$. □

Now let P be a finitely generated projective R-module. We denote by $\mathrm{Um}(P)$ the set of epimorphisms $P \to R$. The group $\mathrm{Aut}(P)$ of automorphisms of P then acts on the right on $\mathrm{Um}(P)$; consequently, the same holds for any subgroup of $\mathrm{Aut}(P)$. In particular, it holds for the subgroup $\mathrm{SL}(P)$ of automorphisms of determinant 1 and, if we fix a decomposition $P \cong \bigoplus_{i=1}^{n} P_i$, for the group $E(P) = E(P_1, \dots, P_n)$ as well.

An element $p \in P$ is called unimodular if there is an $a \in \mathrm{Um}(P)$ such that $a(p) = 1$; this means that the morphism $R \to P$, $1 \mapsto p$ defines a section for the epimorphism a. We denote by $\mathrm{Unim.El.}(P)$ the set of unimodular elements of P. Note that the group $\mathrm{Aut}(P)$ and hence also $\mathrm{SL}(P)$ and $E(P)$ act on the left on P; these actions restrict to actions on $\mathrm{Unim.El.}(P)$.

The canonical isomorphism $\mathrm{can} : P \to P^{\vee\vee}$ identifies the set of unimodular elements $\mathrm{Unim.El.}(P)$ of P with the set $\mathrm{Um}(P^\vee)$ of epimorphisms $P^\vee \to R$, i.e., an element $p \in P$ is unimodular if and only if $\mathrm{ev}_p : P^\vee \to R$ is an epimorphism. Furthermore, if p and q are unimodular elements of P and $\varphi \in \mathrm{Aut}(P)$ with $\varphi(p) = q$, then $\mathrm{ev}_p \varphi^\vee = \mathrm{ev}_q : P^\vee \to R$.

We therefore obtain a well-defined map

$$\mathrm{Unim.El.}(P)/\mathrm{Aut}(P) \to \mathrm{Um}(P^\vee)/\mathrm{Aut}(P^\vee).$$

Let us show that this map is actually a bijection. Since the map is automatically surjective, it only remains to show that it is injective. So let $\psi \in \mathrm{Aut}(P^\vee)$ such that $\mathrm{ev}_p \psi = \mathrm{ev}_q$. One can easily check that the map $\mathrm{Aut}(P) \to \mathrm{Aut}(P^\vee)$, $\varphi \mapsto \varphi^\vee$, is bijective; hence $\psi = \varphi^\vee$ for some $\varphi \in \mathrm{Aut}(P)$. Thus, we obtain $\mathrm{ev}_q = \mathrm{ev}_p \varphi^\vee = \mathrm{ev}_{\varphi(p)}$ and therefore $\varphi(p) = q$, because $\mathrm{can} : P \to P^{\vee\vee}$ is injective. Altogether, we obtain a bijection

$$\mathrm{Unim.El.}(P)/\mathrm{Aut}(P) \xrightarrow{\cong} \mathrm{Um}(P^\vee)/\mathrm{Aut}(P^\vee).$$

In particular, if $P \cong P^\vee$, then $\mathrm{Unim.El.}(P)/\mathrm{Aut}(P) \cong \mathrm{Um}(P)/\mathrm{Aut}(P)$.

We introduce some notation. Let P_0 be a finitely generated projective R-module. For any $n \geq 3$, let $P_n = P_0 \oplus Re_3 \oplus \cdots \oplus Re_n$ be the direct sum of P_0 and free R-modules Re_i, $3 \leq i \leq n$, of rank 1 with explicit generators e_i. We denote by $\pi_{k,n} : P_n \to R$ the projections onto the free direct summands of rank 1 with index $k = 3, \dots, n$. For any nondegenerate alternating form χ on P_{2n}, $n \geq 2$, we define $\mathrm{Sp}(\chi) = \{\varphi \in \mathrm{Aut}(P_{2n}) \mid \varphi^t \chi \varphi = \chi\}$.

For $n \geq 3$, we have embeddings $\mathrm{Aut}(P_n) \to \mathrm{Aut}(P_{n+1})$ and $E(P_n) \to E(P_{n+1})$. We denote by $\mathrm{Aut}_\infty(P_0)$ (resp. $E_\infty(P_0)$) the direct limits of the groups $\mathrm{Aut}(P_n)$ (resp. $E(P_n)$) via these embeddings.

In the following lemmas, we denote by ψ_2 the alternating form on R^2 given by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus, for any nondegenerate alternating form χ on P_{2n} for some $n \geq 2$, we obtain a nondegenerate alternating form on P_{2n+2} given by the orthogonal sum $\chi \perp \psi_2$.

With this notation in mind, we may now state and prove a few lemmas which provide the technical groundwork in the proofs of the main results in this paper.

Lemma 2.7. *Let χ be a nondegenerate alternating form on P_{2n} for some $n \geq 2$. Let $p \in P_{2n-1}$ and $a : P_{2n-1} \to R$. Then there are $\varphi, \psi \in \mathrm{Aut}(P_{2n-1})$ such that*

- *the morphism $(\varphi \oplus 1)(\mathrm{id}_{P_{2n}} + p\pi_{2n,2n})$ is an element of $E(P_{2n}) \cap \mathrm{Sp}(\chi)$ and*
- *the morphism $(\psi \oplus 1)(\mathrm{id}_{P_{2n}} + ae_{2n})$ is an element of $E(P_{2n}) \cap \mathrm{Sp}(\chi)$.*

Proof. We let $\Phi : P_{2n} \to P_{2n}^{\vee}$ be the alternating isomorphism induced by χ and Φ^{-1} be its inverse.

For the first part, we introduce the following homomorphisms: Let $d : R \to P_{2n-1}$ be the morphism which sends 1 to $\Phi^{-1}(\pi_{2n,2n})$ (note that because $\Phi^{-1}(\pi_{2n,2n})$ satisfies $\pi_{2n,2n}(\Phi^{-1}(\pi_{2n,2n})) = \chi(\Phi^{-1}(\pi_{2n,2n}), \Phi^{-1}(\pi_{2n,2n})) = 0$, it can be considered an element of P_{2n-1}). Furthermore, let $\nu = \chi(p, -) : P_{2n-1} \to R$. We observe that $\nu d = 0$. By Lemma 2.6, the morphism $\varphi = \mathrm{id}_{P_{2n-1}} - d\nu$ is an automorphism and $\varphi \oplus 1$ is an elementary automorphism. In particular, $(\varphi \oplus 1)(\mathrm{id}_{P_{2n}} + p\pi_{2n,2n})$ is an elementary automorphism. In light of the proof of [Vaserstein and Suslin 1976, Lemma 5.4] and Lemma 2.2, one can check locally that it also lies in $\mathrm{Sp}(\chi)$.

For the second part, we introduce the following homomorphisms: We denote $c = \chi(-, e_{2n}) : P_{2n-1} \to R$. Furthermore, we let $a \oplus 0 : P_{2n} \to R$ be the extension of a to P_{2n} which sends e_{2n} to 0; then we denote by ϑ the homomorphism $R \to P_{2n-1}$ which sends 1 to $\pi\Phi^{-1}(a \oplus 0)$, where $\pi : P_{2n} \to P_{2n-1}$ is the projection. Note that $c\vartheta = 0$. Again by Lemma 2.6, the morphism $\psi = \mathrm{id}_{P_{2n-1}} - \vartheta c$ is an automorphism and $\psi \oplus 1$ is an elementary automorphism. In particular, $(\psi \oplus 1)(\mathrm{id}_{P_{2n}} + ae_{2n})$ is an elementary automorphism as well. Again, in light of the proof of [Vaserstein and Suslin 1976, Lemma 5.4] and Lemma 2.2, one can check locally that it also lies in $\mathrm{Sp}(\chi)$. ☐

Lemma 2.8. *Let χ be a nondegenerate alternating form on the module P_{2n} for some $n \geq 2$. Then $E(P_{2n})e_{2n} = (E(P_{2n}) \cap \mathrm{Sp}(\chi))e_{2n}$.*

Proof. Let $p \in E(P_{2n})e_{2n}$. By Corollary 2.4, the group $E(P_{2n})$ is generated by automorphisms of the form $\mathrm{id}_{P_{2n}} + s$, where s is a morphism $P_{2n-1} \to Re_{2n}$ or $Re_{2n} \to P_{2n-1}$. Hence we can write $(\alpha_1 \cdots \alpha_r)(p) = e_{2n}$, where each α_i is one of these generators. We show by induction on r that $p \in (E(P_{2n}) \cap \mathrm{Sp}(\chi))e_{2n}$. If $r = 0$, there is nothing to show. So let $r \geq 1$. Lemma 2.7 shows that there is $\gamma \in \mathrm{Aut}(P_{2n-1})$ such that $(\gamma \oplus 1)\alpha_r$ lies in $E(P_{2n}) \cap \mathrm{Sp}(\chi)$. We set $\beta_i = (\gamma \oplus 1)\alpha_i(\gamma^{-1} \oplus 1)$ for each $i < r$. Each of the β_i lies in $E(P_{2n})$ and is again one of the generators of $E(P_{2n})$ given above. By construction, we furthermore have $(\beta_1 \cdots \beta_{r-1}(\gamma \oplus 1)\alpha_r)(p) = e_{2n}$. This enables us to conclude by induction. ☐

Lemma 2.9. *Let χ_1 and χ_2 be nondegenerate alternating forms on P_{2n} such that $\varphi^t(\chi_1 \perp \psi_2)\varphi = \chi_2 \perp \psi_2$ for some $\varphi \in E_\infty(P_0) \cap \mathrm{Aut}(P_{2n+2})$. Now let $\chi = \chi_1 \perp \psi_2$. If $(E_\infty(P_0) \cap \mathrm{Aut}(P_{2n+2}))e_{2n+2} = (E_\infty(P_0) \cap \mathrm{Sp}(\chi))e_{2n+2}$ holds, then one has $\psi^t \chi_2 \psi = \chi_1$ for some $\psi \in E_\infty(P_0) \cap \mathrm{Aut}(P_{2n})$.*

Proof. Let $\psi'' e_{2n+2} = \varphi e_{2n+2}$ for some $\psi'' \in E_\infty(P_0) \cap \mathrm{Sp}(\chi)$. Then we simply define $\psi' = (\psi'')^{-1}\varphi$. Since $(\psi')^t(\chi_1 \perp \psi_2)\psi' = \chi_2 \perp \psi_2$, the composite $\psi : P_{2n} \xrightarrow{\psi'} P_{2n+2} \to P_{2n}$ and ψ' satisfy the conditions

- $\psi^t \chi_1 \psi = \chi_2$,
- $\psi'(e_{2n+2}) = e_{2n+2}$,
- $\pi_{2n+1,2n+2}\psi' = \pi_{2n+1,2n+2}$.

The last two conditions imply that ψ equals ψ' up to elementary automorphisms and $\psi \in E_\infty(P_0) \cap \mathrm{Aut}(P_{2n})$, which finishes the proof. □

Lemma 2.10. *Assume that $\pi_{2n+1,2n+1}(E_\infty(P_0) \cap \mathrm{Aut}(P_{2n+1})) = \mathrm{Um}(P_{2n+1})$ holds for some $n \in \mathbb{N}$. Then for any nondegenerate alternating form χ on P_{2n+2} there is an automorphism $\varphi \in E_\infty(P_0) \cap \mathrm{Aut}(P_{2n+2})$ such that $\varphi^t \chi \varphi = \psi \perp \psi_2$ for some nondegenerate alternating form ψ on P_{2n}.*

Proof. Let $d = \chi(-, e_{2n+2}) : P_{2n+1} \to R$. Since d can be locally checked to be an epimorphism, there is an automorphism $\varphi' \in E_\infty(P_0) \cap \mathrm{Aut}(P_{2n+1})$ such that $d\varphi' = \pi_{2n+1,2n+1}$. Then the alternating form $\chi' = (\varphi'^t \oplus 1)\chi(\varphi' \oplus 1)$ satisfies that $\chi'(-, e_{2n+2}) : P_{2n+1} \to R$ is just $\pi_{2n+1,2n+1}$. Now we simply define the morphism $c = \chi'(-, e_{2n+1}) : P_{2n+1} \to R$ and let $\varphi_c = \mathrm{id}_{P_{2n+2}} + ce_{2n+2}$ be the elementary automorphism on P_{2n+2} induced by c; then ${\varphi_c}^t \chi' \varphi_c = \psi \perp \psi_2$ for some nondegenerate alternating form ψ on P_{2n}, as desired. □

Lemma 2.11. *Let P_0 be a finitely generated projective R-module of rank 2. Then we have $E(P_0 \oplus R) \subset \mathrm{SL}(P_0 \oplus R)$. Furthermore, if $\varphi \in \mathrm{SL}(P_0 \oplus R)$, then the induced morphism $\varphi_* : \det(P_0 \oplus R) \to \det(P_0 \oplus R)$ is the identity on $\det(P_0 \oplus R)$.*

Proof. Use that these properties are local and check them when R is local. □

2C. ***The local-global principle for transvection groups.*** We now briefly review the local-global principle for transvection groups proven in [Bak et al. 2010], and use it in order to deduce stability results for stably elementary automorphisms of $P_0 \oplus R^2$. For this, we only have to assume that R is an arbitrary commutative ring with unit.

First of all, let P be a finitely generated projective R-module and $q \in P$, $\varphi \in P^\vee$ such that $\varphi(q) = 0$. This data naturally determines a homomorphism $\varphi_q : P \to P$ by $\varphi_q(p) = \varphi(p)q$ for all $p \in P$. An automorphism of the form $\mathrm{id}_P + \varphi_q$ is called a transvection if either $q \in \mathrm{Unim.El.}(P)$ or $\varphi \in \mathrm{Um}(P)$. We denote by $T(P)$ the subgroup of $\mathrm{Aut}(P)$ generated by transvections.

Now let $Q = P \oplus R$ be a direct sum of a finitely generated projective R-module P of rank ≥ 2 and the free R-module of rank 1. Then the elementary automorphisms of $P \oplus R$ are easily seen to be transvections and are also called elementary transvections. Consequently, we have $E(Q) \subset T(Q) \subset \mathrm{Aut}(Q)$.

In the theorem stated below, we denote by $R[X]$ the polynomial ring in one variable over R and let $Q[X] = Q \otimes_R R[X]$. The evaluation homomorphisms $\mathrm{ev}_0, \mathrm{ev}_1 : R[X] \to R$ induce maps $\mathrm{Aut}(Q[X]) \to \mathrm{Aut}(Q)$. If $\alpha(X) \in \mathrm{Aut}(Q[X])$, then we denote its images under these maps by $\alpha(0)$ and $\alpha(1)$, respectively. Similarly, the localization homomorphism $R \to R_{\mathfrak{m}}$ at any maximal ideal $\mathfrak{m}$ of R induces a map $\mathrm{Aut}(Q[X]) \to \mathrm{Aut}(Q_{\mathfrak{m}}[X])$, where $Q_{\mathfrak{m}}[X] = Q[X] \otimes_{R[X]} R_{\mathfrak{m}}[X]$; if $\alpha(X) \in \mathrm{Aut}(Q[X])$, we denote its image under this map by $\alpha_{\mathfrak{m}}(X)$.

We will use the following result proven by Bak, Basu and Rao (see [Bak et al. 2010, Theorems 3.6 and 3.10]):

Theorem 2.12. *The inclusion $E(Q) \subset T(Q)$ is an equality. If $\alpha(X) \in \mathrm{Aut}(Q[X])$ satisfies $\alpha(0) = \mathrm{id}_Q \in \mathrm{Aut}(Q)$ and $\alpha_{\mathfrak{m}}(X) \in E(Q_{\mathfrak{m}}[X])$ for all maximal ideals $\mathfrak{m}$ of R, then $\alpha(X) \in E(Q[X])$.*

In order to prove the desired stability results, we introduce the following property: Let $\mathfrak{C}$ be either the class of Noetherian rings or the class of affine k-algebras over a fixed field k. Furthermore, let $d \geq 1$ be an integer and $m \in \mathbb{N}$. We say that $\mathfrak{C}$ has the property $\mathcal{P}(d, m)$ if for R in $\mathfrak{C}$ of dimension d and for any finitely generated projective R-module P of rank m the group $\mathrm{SL}(P \oplus R^n)$ acts transitively on $\mathrm{Um}(P \oplus R^n)$ for all $n \geq 2$.

If k is a field, we simply say that k has the property $\mathcal{P}(d, m)$ if the class of affine k-algebras has the property $\mathcal{P}(d, m)$.

Of course, if the class of Noetherian rings has the property $\mathcal{P}(d, m)$, then the same holds for every field. The class of Noetherian rings has the property $\mathcal{P}(d, m)$ for $m \geq d$. Furthermore, it follows from [Bhatwadekar 2003] that any perfect field k of cohomological dimension ≤ 1 satisfies property $\mathcal{P}(d, d-1)$ if $d! \in k^{\times}$.

In the remainder of this section, we let π be the canonical projection $P \oplus R^n \to R$ onto the "last" free direct summand of R^n.

Lemma 2.13. *Let $\mathfrak{C}$ be the class of Noetherian rings or affine k-algebras over a fixed field k. Assume that $\mathfrak{C}$ has the property $\mathcal{P}(d, m)$. Let R be a d-dimensional ring in $\mathfrak{C}$, P a projective R-module of rank m and $a \in \mathrm{Um}(P \oplus R^n)$ for some $n \geq 2$. Moreover, assume that there is an element $t \in R$ and a homomorphism $w : P \oplus R^n \to R$ such that $a - \pi = tw$. Then there is $\varphi \in \mathrm{SL}(P \oplus R^n)$ such that $a = \pi\varphi$ and $\varphi(x) \equiv \mathrm{id}_{P \oplus R^n}(x)$ modulo (t) for all x.*

Proof. We set $B = R[X]/\langle X^2 - tX \rangle$. By assumption, we have $a = \pi + tw$. We lift it to $a(X) = \pi + Xw : (P \oplus R^n) \otimes_R B \to B$, which can be checked to be an epimorphism (as in the proof of [Rao and van der Kallen 1994, Proposition 3.3]).

Therefore we have $a(X) \in \mathrm{Um}((P \oplus R^n) \otimes_R B)$. Since B still is a ring in $\mathfrak{C}$ of dimension d, property $\mathcal{P}(d, m)$ now gives an element $\varphi(X) \in \mathrm{SL}((P \oplus R^n) \otimes_R B)$ with $a(X) = \pi\varphi(X)$. Then $\varphi = \varphi(0)^{-1}\varphi(t)$ is the desired automorphism. □

For any $n \geq 2$, we say that two automorphisms $\varphi, \psi \in \mathrm{SL}(P \oplus R^n)$ are isotopic if there is an automorphism $\tau(X)$ of $(P \oplus R^n) \otimes_R R[X]$ with determinant 1 such that $\tau(0) = \varphi$ and $\tau(1) = \psi$.

Theorem 2.14. *Let $\mathfrak{C}$ be the class of Noetherian rings or affine k-algebras over a fixed field k. Assume that $\mathfrak{C}$ has the property $\mathcal{P}(d+1, m+1)$. Let R be a d-dimensional ring in $\mathfrak{C}$, P a projective R-module of rank m and $\sigma \in \mathrm{Aut}(P \oplus R^n)$ for some $n \geq 2$. Assume that $\sigma \oplus 1 \in E(P \oplus R^{n+1})$. Then σ is isotopic to $\mathrm{id}_{P \oplus R^n}$.*

Proof. Since $\sigma \oplus 1 \in E(P \oplus R^{n+1})$, it is clear that there is a natural isotopy $\tau(X) \in E((P \oplus R^{n+1}) \otimes_R R[X])$ with $\tau(0) = \mathrm{id}_{P \oplus R^{n+1}}$ and $\tau(1) = \sigma \oplus 1$. Now apply the previous lemma to $R[X]$, $X^2 - X$ and $a = \pi\tau(X)$ in order to obtain an automorphism $\chi(X) \in \mathrm{SL}((P \oplus R^{n+1}) \otimes_R R[X])$ with $\pi\chi(X) = a$ such that $\chi(X)(x) \equiv x$ modulo $\langle X^2 - X \rangle$. Thus, $\pi\tau(X)\chi(X)^{-1} = \pi$. Therefore $\tau(X)\chi(X)^{-1}$ equals $\rho(X) \oplus 1$ for some $\rho(X) \in \mathrm{SL}((P \oplus R^n) \otimes_R R[X])$ up to elementary automorphisms. But then $\rho(X)$ is an isotopy from $\mathrm{id}_{P \oplus R^n}$ to σ. □

We can now use Theorem 2.14 in order to deduce the following stability results:

Theorem 2.15. *With the notation of Section 2B, we further assume that P_0 has rank 2. If R is a regular Noetherian ring of dimension 2, then there is an equality $E_\infty(P_0) \cap \mathrm{Aut}(P_4) = E(P_4)$.*

Proof. If $\sigma \in \mathrm{SL}(P_4)$ is stably elementary, then $\sigma \in E(P_{n+1})$ for some $n \geq 4$. We can now apply Theorem 2.14 to $P = P_0$ and deduce that there is an isotopy $\rho(X) \in \mathrm{SL}(P_n[X])$ from id_{P_n} to σ.

But since R is regular, we know that $\rho_{\mathfrak{m}}(X)$ is stably elementary (for any maximal ideal $\mathfrak{m}$ of R). In fact, we can deduce that $\rho_{\mathfrak{m}}(X)$ is elementary: Since $\dim(R) = 2$, the spectrum of the 3-dimensional ring $R_{\mathfrak{m}}[X]$ is the union of a finite number of subspaces of dimension ≤ 2 (see the last paragraph of [Rao 1988, Section 1.1]). Hence it follows from [Vaserstein and Suslin 1976, Lemma 7.5] that the stable rank of $R_{\mathfrak{m}}[X]$ is at most 3. In particular, $\mathrm{SL}_n(R_{\mathfrak{m}}[X]) \cap E(R_{\mathfrak{m}}[X]) = E_n(R_{\mathfrak{m}}[X])$ and $\rho_{\mathfrak{m}}(X)$ is elementary.

Then Theorem 2.12 implies that $\rho(X) \in E(P_n[X])$ and hence $\sigma = \rho(1) \in E(P_n)$. The theorem now follows by inductively repeating this argument and deducing that $\sigma \in E(P_4)$. □

Theorem 2.16. *With the notation of Section 2B, we further assume that P_0 has rank 2. Let k be a perfect field with $\mathcal{P}(4, 3)$. If R is a regular affine k-algebra of dimension 3, then $E_\infty(P_0) \cap \mathrm{Aut}(P_4) = E(P_4)$.*

Proof. By a famous theorem of Vorst [1981], we know that there is an equality $\mathrm{SL}_N(R_{\mathfrak{p}}[X]) = E_N(R_{\mathfrak{p}}[X])$ for any prime $\mathfrak{p}$ of R and $N \geq 4$. We can thus argue as in the proof of Theorem 2.15. □

Theorem 2.17. *With the notation of Section 2B, we further assume that P_0 has rank 2. Let k be a perfect field with $\mathcal{P}(5,3)$. If R is a regular affine k-algebra of dimension 4, then $E_\infty(P_0) \cap \mathrm{Aut}(P_4) = E(P_4)$.*

Proof. We can argue as in the proof of Theorem 2.16. □

2D. ***Classification of stably isomorphic projective modules.*** We consider the map

$$\phi_n : \mathcal{V}_n(R) \to \mathcal{V}_{n+1}(R), \quad [P] \mapsto [P \oplus R],$$

from isomorphism classes of rank n projective modules to rank $n+1$ projective modules, and fix a projective module $P \oplus R$ representing an element of $\mathcal{V}_{n+1}(R)$ in the image of this map. An element $[P']$ of $\mathcal{V}_n(R)$ lies in the fiber over $[P \oplus R]$ if and only if there is an isomorphism $i : P' \oplus R \xrightarrow{\cong} P \oplus R$. Any such isomorphism yields an element of $\mathrm{Um}(P \oplus R)$ given by the composite

$$a(i) : P \oplus R \xrightarrow{i^{-1}} P' \oplus R \xrightarrow{\pi_R} R.$$

Note that if one chooses another module P'' representing the isomorphism class of P' and any isomorphism $j : P'' \oplus R \xrightarrow{\cong} P \oplus R$, the resulting element $a(j)$ of $\mathrm{Um}(P \oplus R)$ still lies in the same orbit of $\mathrm{Um}(P \oplus R)/\mathrm{Aut}(P \oplus R)$: If we choose an isomorphism $k : P' \xrightarrow{\cong} P''$, then we have an equality

$$a(i) = a(j) \circ (j(k \oplus \mathrm{id}_R)i^{-1}).$$

Thus, we obtain a well-defined map

$$\phi_n^{-1}([P \oplus R]) \to \mathrm{Um}(P \oplus R)/\mathrm{Aut}(P \oplus R).$$

Conversely, any element $a \in \mathrm{Um}(P \oplus R)$ gives an element of $\mathcal{V}_n(R)$ lying over $[P \oplus R]$, namely $[P'] = [\ker(a)]$. Note that the kernels of two epimorphisms $P \oplus R \to R$ are isomorphic if these epimorphisms are in the same orbit in $\mathrm{Um}(P \oplus R)/\mathrm{Aut}(P \oplus R)$. Thus, we also obtain a well-defined map

$$\mathrm{Um}(P \oplus R)/\mathrm{Aut}(P \oplus R) \to \phi_n^{-1}([P \oplus R]).$$

One can easily check that the maps $\phi_n^{-1}([P \oplus R]) \to \mathrm{Um}(P \oplus R)/\mathrm{Aut}(P \oplus R)$ and $\mathrm{Um}(P \oplus R)/\mathrm{Aut}(P \oplus R) \to \phi_n^{-1}([P \oplus R])$ are inverse to each other. Note that $[P]$ corresponds to the class represented by the canonical projection $\pi_R : P \oplus R \to R$ under these bijections. In conclusion, we have a pointed bijection between the sets $\mathrm{Um}(P \oplus R)/\mathrm{Aut}(P \oplus R)$ and $\phi_n^{-1}([P \oplus R])$ equipped with $[\pi_R]$ and $[P]$ as their respective basepoints. Moreover, we also obtain a (pointed) surjection $\mathrm{Um}(P \oplus R)/E(P \oplus R) \to \phi_n^{-1}([P \oplus R])$.

3. The elementary symplectic Witt group

In this section, we briefly recall the definition of the so-called elementary symplectic Witt group $W_E(R)$. Primarily, it appears as the kernel of a homomorphism $W_E'(R) \to R^\times$ induced by the Pfaffian of alternating invertible matrices. As we will discuss, the group $W_E'(R)$ itself can be identified with a group denoted $V(R)$ and with $\mathrm{GW}_1^3(R)$, a higher Grothendieck–Witt group of R. We will also prove some lemmas on the group $V(R)$, which will be used to prove the main results of this paper. Furthermore, we introduce a canonical $R^\times$-action on $V(R)$ and identify this action with an action of $R^\times$ on $\mathrm{GW}_1^3(R)$ coming from the multiplicative structure of higher Grothendieck–Witt groups.

3A. ***The group $W_E'(R)$.*** Let R be a commutative ring. For any $n \in \mathbb{N}$, we denote by $A_{2n}(R)$ the set of alternating invertible matrices of rank $2n$. We inductively define an element $\psi_{2n} \in A_{2n}(R)$ by setting

$$\psi_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\psi_{2n+2} = \psi_{2n} \perp \psi_2$. For any $m < n$, there is an embedding of $A_{2m}(R)$ into $A_{2n}(R)$ given by $M \mapsto M \perp \psi_{2n-2m}$. We denote by $A(R)$ the direct limit of the sets $A_{2n}(R)$ under these embeddings. Two alternating invertible matrices $M \in A_{2m}(R)$ and $N \in A_{2n}(R)$ are called equivalent, $M \sim N$, if there is an integer $s \in \mathbb{N}$ and a matrix $E \in E_{2n+2m+2s}$ such that

$$M \perp \psi_{2n+2s} = E^t (N \perp \psi_{2m+2s}) E.$$

The set of equivalence classes $A(R)/{\sim}$ is denoted $W_E'(R)$. Since

$$\begin{pmatrix} 0 & \mathrm{id}_s \\ \mathrm{id}_r & 0 \end{pmatrix} \in E_{r+s}(R)$$

for even rs, it follows that the orthogonal sum equips $W_E'(R)$ with the structure of an abelian monoid. As it is shown in [Vaserstein and Suslin 1976], this abelian monoid is actually an abelian group. An inverse for an element of $W_E'(R)$ represented by a matrix $N \in A_{2n}(R)$ is given by the element represented by the matrix $\sigma_{2n} N^{-1} \sigma_{2n}$, where the matrices σ_{2n} are inductively defined by

$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_{2n+2} = \sigma_{2n} \perp \sigma_2.$$

Now recall that one can assign to any alternating invertible matrix M an element $\mathrm{Pf}(M)$ of $R^\times$ called the Pfaffian of M. The Pfaffian satisfies the following formulas:

- $\mathrm{Pf}(M \perp N) = \mathrm{Pf}(M)\,\mathrm{Pf}(N)$ for all $M \in A_{2m}(R)$ and $N \in A_{2n}(R)$;

- $\mathrm{Pf}(G^tNG) = \det(G)\,\mathrm{Pf}(N)$ for all $G \in \mathrm{GL}_{2n}(R)$ and $N \in A_{2n}(R)$;
- $\mathrm{Pf}(N)^2 = \det(N)$ for all $N \in A_{2n}(R)$;
- $\mathrm{Pf}(\psi_{2n}) = 1$ for all $n \in \mathbb{N}$.

Therefore the Pfaffian determines a group homomorphism $\mathrm{Pf} : W'_E(R) \to R^\times$; its kernel is denoted $W_E(R)$ and is called the elementary symplectic Witt group of R. Note that the homomorphism $\mathrm{Pf} : W'_E(R) \to R^\times$ is split by the homomorphism $R^\times \to W'_E(R)$, which assigns to any $t \in R^\times$ the class in $W'_E(R)$ represented by the matrix

$$\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}.$$

Hence $W'_E(R) \cong W_E(R) \oplus R^\times$.

3B. ***The group V(R).*** Again, let R be a commutative ring. Consider the set of triples (P, g, f), where P is a finitely generated projective R-module and f, g are alternating isomorphisms on P. Two such triples (P, f_0, f_1) and (P', f'_0, f'_1) are called isometric if there is an isomorphism $h : P \to P'$ such that $f_i = h^\vee f'_i h$ for $i = 0, 1$. We denote by $[P, g, f]$ the isometry class of the triple (P, g, f).

Let $V(R)$ be the quotient of the free abelian group on isometry classes of triples as above by the subgroup generated by the relations

- $[P \oplus P', g \perp g', f \perp f'] = [P, g, f] + [P', g', f']$ for alternating isomorphisms f, g on P and f', g' on P',
- $[P, f_0, f_1] + [P, f_1, f_2] = [P, f_0, f_2]$ for alternating isomorphisms f_0, f_1, f_2 on P.

Note that these relations yield the useful identities:

- $[P, f, f] = 0$ in $V(R)$ for any alternating isomorphism f on P,
- $[P, g, f] = -[P, f, g]$ in $V(R)$ for alternating isomorphisms f, g on P,
- $[P, g, \beta^\vee \alpha^\vee f \alpha \beta] = [P, f, \alpha^\vee f \alpha] + [P, g, \beta^\vee f \beta]$ in $V(R)$ for all automorphisms α, β of P and alternating isomorphisms f, g on P.

We may also restrict this construction to free R-modules of finite rank. The corresponding group is denoted $V_{\mathrm{free}}(R)$. Note that there is an obvious group homomorphism $V_{\mathrm{free}}(R) \to V(R)$.

This homomorphism can be seen to be an isomorphism as follows: For any finitely generated projective R-module P, we call

$$H_P = \begin{pmatrix} 0 & \mathrm{id}_{P^\vee} \\ -\mathrm{can} & 0 \end{pmatrix} : P \oplus P^\vee \to P^\vee \oplus P^{\vee\vee}$$

the hyperbolic isomorphism on P.

Now let (P, g, f) be a triple as above. Since P is a finitely generated projective R-module, there is another R-module Q such that $P \oplus Q \cong R^n$ for some $n \in \mathbb{N}$. In particular, $P \oplus P^\vee \oplus Q \oplus Q^\vee$ is free of rank $2n$. Therefore the triple

$$(P \oplus P^\vee \oplus Q \oplus Q^\vee, g \perp \text{can } g^{-1} \perp H_Q, f \perp \text{can } g^{-1} \perp H_Q)$$

represents an element of $V_{\text{free}}(R)$; this element is independent of the choice of Q. It follows that the assignment

$$(P, g, f) \mapsto (P \oplus P^\vee \oplus Q \oplus Q^\vee, g \perp \text{can } g^{-1} \perp H_Q, f \perp \text{can } g^{-1} \perp H_Q)$$

induces a well-defined group homomorphism

$$V(R) \to V_{\text{free}}(R).$$

Since

$$[P, g, f] = [P \oplus P^\vee \oplus Q \oplus Q^\vee, g \perp \text{can } g^{-1} \perp H_Q, f \perp \text{can } g^{-1} \perp H_Q]$$

in $V(R)$, this homomorphism is inverse to the canonical morphism $V_{\text{free}}(R) \to V(R)$. Thus, $V_{\text{free}}(R) \cong V(R)$.

In order to discuss the identification of $V(R)$ with the group $W'_E(R)$ described in the previous section, we first need to prove Lemma 3.1 and Corollaries 3.2 and 3.3 below. They are also used in the proofs of the main results of this paper.

Lemma 3.1. *Let $P = \bigoplus_{i=1}^n P_i$ be a finitely generated projective module and f_i alternating isomorphisms on P_i, $i = 1, \ldots, n$. Let $f = f_1 \perp \cdots \perp f_n$. Then $[P, f, \varphi^\vee f \varphi] = 0$ in $V(R)$ for any element φ of the commutator subgroup of* $\operatorname{Aut}(P)$. *In particular, the same holds for every element of $E(P)$ with respect to the given decomposition.*

Proof. By the third of the useful identities listed above, we have

$$[P, f, \varphi_2^\vee \varphi_1^\vee f \varphi_1 \varphi_2] = [P, f, \varphi_1^\vee f \varphi_1] + [P, f, \varphi_2^\vee f \varphi_2].$$

Therefore, we only have to prove the first statement for commutators. Now if $\varphi = \varphi_1 \varphi_2 \varphi_1^{-1} \varphi_2^{-1}$ is a commutator, then the formula above yields

$$\begin{aligned}[P, f, \varphi^\vee f \varphi] = [P, f, \varphi_1^\vee f \varphi_1] + [P, f, \varphi_2^\vee f \varphi_2] \\ + [P, f, (\varphi_1^{-1})^\vee f \varphi_1^{-1}] + [P, f, (\varphi_2^{-1})^\vee f \varphi_2^{-1}] = 0,\end{aligned}$$

which proves the first part of the lemma.

For the second part, observe that by the formula above we only need to prove the statement for elementary automorphisms. So let φ_s be the elementary automorphism induced by $s : P_j \to P_i$. Since we can add the summand $[P_i, f_i, f_i] = 0$, we may assume that we are in the situation of Corollary 2.5. Therefore we may assume that φ_s is a commutator and the second statement then follows from the first part of the lemma. □

Corollary 3.2. *Let P be a finitely generated projective R-module and χ an alternating isomorphism on P. Then $[P \oplus R^{2n}, \chi \perp \psi_{2n}, \varphi^\vee(\chi \perp \psi_{2n})\varphi] = 0$ in $V(R)$ for any elementary automorphism φ of $P \oplus R^{2n}$. In particular, if f is any alternating isomorphism on $P \oplus R^{2n}$, it follows that there is an equality $[P \oplus R^{2n}, \chi \perp \psi_{2n}, \varphi^\vee f \varphi] = [P \oplus R^{2n}, \chi \perp \psi_{2n}, f]$ in $V(R)$.*

Proof. The first part follows directly from the previous lemma. The second part is then a direct consequence of the second relation given in the definition of the group $V(R)$. □

Corollary 3.3. *Let E be an arbitrary elementary matrix in $E_{2n}(R)$. Then we have $[R^{2n}, \psi_{2n}, E^t \psi_{2n} E] = 0$ in $V(R)$. In particular, for any alternating matrix $N \in \mathrm{GL}_{2n}(R)$, we have $[R^{2n}, \psi_{2n}, N] = [R^{2n}, \psi_{2n}, E^t N E]$ in $V(R)$.*

Using the previous corollary, the group $V_{\text{free}}(R)$ can be identified with $W'_E(R)$ as follows. If $M \in A_{2m}(R)$ represents an element of $W'_E(R)$, then we assign to it the class in $V_{\text{free}}(R)$ represented by $[R^{2m}, \psi_{2m}, M]$. By Corollary 3.3, this assignment descends to a well-defined homomorphism $\nu : W'_E(R) \to V_{\text{free}}(R)$.

Now let us describe the inverse $\xi : V_{\text{free}}(R) \to W'_E(R)$ to this homomorphism. Let (L, g, f) be a triple with L free and g, f alternating isomorphisms on L. We can choose an isomorphism $\alpha : R^{2n} \xrightarrow{\cong} L$ and consider the alternating isomorphism

$$(\alpha^\vee f \alpha) \perp \sigma_{2n}(\alpha^\vee g \alpha)^{-1}\sigma_{2n}^\vee : R^{2n} \oplus (R^{2n})^\vee \to (R^{2n})^\vee \oplus R^{2n}.$$

With respect to the standard basis of R^{2n} and its dual basis of $(R^{2n})^\vee$, we may interpret this alternating isomorphism as an element of $A_{4n}(R)$ and then consider its class $\xi([L, g, f])$ in $W'_E(R)$. In fact, this class is independent of the choice of the isomorphism $\alpha : R^{2n} \xrightarrow{\cong} L$. If $\beta : R^{2n} \xrightarrow{\cong} L$ is another isomorphism, then it suffices to show that the alternating matrix M corresponding to $\alpha^\vee f \alpha \perp \beta^\vee g \beta$ is equivalent in $W'_E(R)$ to the alternating matrix corresponding to $\beta^\vee f \beta \perp \alpha^\vee g \alpha$. But there is an isometry $\gamma = (\alpha^{-1}\beta) \perp (\beta^{-1}\alpha)$ from $\alpha^\vee f \alpha \perp \beta^\vee g \beta$ to $\beta^\vee f \beta \perp \alpha^\vee g \alpha$, which is an elementary automorphism by Whitehead's lemma. One then also checks easily that the defining relations of $V_{\text{free}}(R)$ are also satisfied by the assignment above. Hence it follows that this assignment induces a well-defined homomorphism $\xi : V_{\text{free}}(R) \to W'_E(R)$. By construction, ν and ξ are obviously inverse to each other and therefore identify $W'_E(R)$ with $V_{\text{free}}(R)$.

To conclude this section, we now describe some group actions on $V(R)$. For any finitely generated projective R-module P, alternating isomorphism $\chi : P \to P^\vee$ and $u \in R^\times$, the morphism $u \cdot \chi : P \to P^\vee$ is again an alternating isomorphism on P. Note that $u \cdot \chi$ is canonically isometric to the alternating isomorphism $u \otimes \chi : R \otimes P \xrightarrow{u \otimes \chi} R \otimes P^\vee \cong (R \otimes P)^\vee$, and we therefore have an equality

$$[P, u \cdot \chi_2, u \cdot \chi_1] = [R \otimes P, u \otimes \chi_2, u \otimes \chi_1] \quad \text{in } V(R)$$

for all χ_1, χ_2. One can check easily that the assignment

$$(u, (P, \chi_2, \chi_1)) \mapsto (P, u \cdot \chi_2, u \cdot \chi_1)$$

descends to a well-defined action of $R^\times$ on $V(R)$.

Now let us assume that $2 \in R^\times$ and, furthermore, let $\varphi : Q \to Q^\vee$ be a symmetric isomorphism on a finitely generated projective R-module Q. Then, for any skew-symmetric isomorphism χ on a finitely generated projective R-module P as above, the homomorphism $\varphi \otimes \chi : Q \otimes P \to Q^\vee \otimes P^\vee \cong (Q \otimes P)^\vee$ is again a skew-symmetric isomorphism on $Q \otimes P$. One can check easily that the assignment

$$((Q, \varphi), (P, \chi_2, \chi_1)) \mapsto (Q \otimes P, \varphi \otimes \chi_2, \varphi \otimes \chi_1)$$

induces a well-defined action of the Grothendieck–Witt group $\mathrm{GW}(R) = \mathrm{GW}_0^0(R)$ of R on $V(R)$.

3C. *Grothendieck–Witt groups.* In this section, we recall some basics about the theory of higher Grothendieck–Witt groups, which are a modern version of Hermitian K-theory. The general references of the modern theory are [Schlichting 2010a; 2010b; 2017].

We assume R to be a ring such that $2 \in R^\times$. Then we consider the category $P(R)$ of finitely generated projective R-modules and the category $C^b(R)$ of bounded complexes of objects in $P(R)$. The category $C^b(R)$ inherits a natural structure of an exact category from $P(R)$ by declaring $C'_\bullet \to C_\bullet \to C''_\bullet$ exact if and only if $C'_n \to C_n \to C''_n$ is exact for all n. The duality $\mathrm{Hom}_R(-, L)$ associated to any line bundle L induces a duality $\#_L$ on $C^b(R)$ and the identification of a finitely generated projective R-module with its double dual induces a natural isomorphism of functors $\varpi_L : \mathrm{id} \xrightarrow{\sim} \#_L\#_L$ on $C^b(R)$. Moreover, the translation functor $T : C^b(R) \to C^b(R)$ yields new dualities $\#_L^j = T^j \#_L$ and natural isomorphisms $\varpi_L^j = (-1)^{j(j+1)/2}\varpi_L$. We say that a morphism in $C^b(R)$ is a weak equivalence if and only if it is a quasi-isomorphism, and we denote by qis the class of quasi-isomorphisms. For all j, the quadruple $(C^b(R), \mathrm{qis}, \#_L^j, \varpi_L^j)$ is an exact category with weak equivalences and strong duality [Schlichting 2010b, §2.3].

Following [Schlichting 2010b], one can associate a Grothendieck–Witt space $\mathcal{GW}$ to any exact category with weak equivalences and strong duality. The (higher) Grothendieck–Witt groups are then defined to be its homotopy groups:

Definition 3.4. For any $i \geq 0$, let $\mathcal{GW}(C^b(R), \mathrm{qis}, \#_L^j, \varpi_L^j)$ be the Grothendieck–Witt space associated to the quadruple $(C^b(R), \mathrm{qis}, \#_L^j, \varpi_L^j)$ as above. Then we define

$$\mathrm{GW}_i^j(R, L) = \pi_i \mathcal{GW}(C^b(R), \mathrm{qis}, \#_L^j, \varpi_L^j).$$

If $L = R$, then we set $\mathrm{GW}_i^j(R) = \mathrm{GW}_i^j(R, L)$.

The groups $\mathrm{GW}_i^j(R, L)$ are 4-periodic in j and coincide with Hermitian K-theory and U-theory as defined by Karoubi [1973; 1980], at least if $2 \in R^\times$ (see [Schlichting 2010a, Remark 4.13; 2017, Theorems 6.1–2). In particular, we have isomorphisms $K_i O(R) = \mathrm{GW}_i^0(R)$, ${}_{-1}U_i(R) = \mathrm{GW}_i^1(R)$, $K_i \operatorname{Sp}(R) = \mathrm{GW}_i^2(R)$ and ${}_i(R) = \mathrm{GW}_i^3(R)$.

The group of our particular interest is $\mathrm{GW}_1^3(R) = U_1(R)$. Indeed, it is argued in [Fasel et al. 2012] that there is a natural isomorphism $GW_1^3(R) \cong V_{\mathrm{free}}(R) \cong V(R)$.

The Grothendieck–Witt groups defined as above carry a multiplicative structure. Indeed, the tensor product of complexes induces product maps

$$\mathrm{GW}_i^j(R, L_1) \times \mathrm{GW}_r^s(R, L_2) \to \mathrm{GW}_{i+r}^{j+s}(R, L_1 \otimes L_2)$$

for all i, j, r, s and line bundles L_1, L_2 [Schlichting 2017, §9.2]. In general, it is (probably) difficult to give explicit descriptions of this multiplicative structure; nevertheless, if we restrict ourselves to smooth algebras over a perfect field k (with $\operatorname{char}(k) \neq 2$), then it is known (see [Hornbostel 2005, Theorem 3.1]) that Grothendieck–Witt groups are representable in the (pointed) $\mathbb{A}^1$-homotopy category $\mathcal{H}_\bullet(k)$ as defined by Morel and Voevodsky. As a matter of fact, if we let R be a smooth affine k-algebra over a perfect field k with $\operatorname{char}(k) \neq 2$ and $X = \operatorname{Spec}(R)$, it is shown that there are spaces $\mathcal{GW}^j$ such that

$$[S_s^i \wedge X_+, \mathcal{GW}^j]_{\mathbb{A}^1,\bullet} = \mathrm{GW}_i^j(R),$$

i.e., the spaces $\mathcal{GW}^j$ represent the higher Grothendieck–Witt groups. In order to make these spaces more explicit, we consider for all $n \in \mathbb{N}$ the closed embeddings $\mathrm{GL}_n \to O_{2n}$ and $\mathrm{GL}_n \to \operatorname{Sp}_{2n}$ induced by

$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & (M^{-1})^t \end{pmatrix}.$$

These closed embeddings are compatible with the standard stabilization embeddings $\mathrm{GL}_n \to \mathrm{GL}_{n+1}$, $O_{2n} \to O_{2n+2}$ and $\operatorname{Sp}_{2n} \to \operatorname{Sp}_{2n+2}$. Taking direct limits over all n with respect to the induced maps $O_{2n}/\mathrm{GL}_n \to O_{2n+2}/\mathrm{GL}_{n+1}$ and $\operatorname{Sp}_{2n}/\mathrm{GL}_n \to \operatorname{Sp}_{2n+2}/\mathrm{GL}_{n+1}$, we obtain spaces O/GL and $\operatorname{Sp}/\mathrm{GL}$. Similarly, the natural inclusions $\operatorname{Sp}_{2n} \to \mathrm{GL}_{2n}$ are compatible with the standard stabilization embeddings and we obtain a space $\mathrm{GL}/\operatorname{Sp} = \operatorname{colim}_n \mathrm{GL}_{2n}/\operatorname{Sp}_{2n}$. As proven in [Schlichting and Tripathi 2015, Theorems 8.2 and 8.4], there are canonical $\mathbb{A}^1$-weak equivalences

$$\mathcal{GW}^j \cong \begin{cases} \mathbb{Z} \times \mathrm{OGr} & \text{if } j \equiv 0 \bmod 4, \\ \operatorname{Sp}/\mathrm{GL} & \text{if } j \equiv 1 \bmod 4, \\ \mathbb{Z} \times \mathrm{HGr} & \text{if } j \equiv 2 \bmod 4, \\ O/\mathrm{GL} & \text{if } j \equiv 3 \bmod 4, \end{cases}$$

and

$$\mathcal{R}\Omega^1_s O/\operatorname{GL} \cong \operatorname{GL}/\operatorname{Sp},$$

where OGr is an "infinite orthogonal Grassmannian" and HGr is an "infinite symplectic Grassmannian". As a consequence of all this, there is an isomorphism $[X, \operatorname{GL}/\operatorname{Sp}]_{\mathbb{A}^1} = \operatorname{GW}^3_1(R)$. It is argued in [Asok and Fasel 2017] that the morphisms of schemes $\operatorname{GL}_{2n} \to A_{2n}$, $M \mapsto M^t \psi_{2n} M$ induce an isomorphism $\operatorname{GL}/\operatorname{Sp} \cong A$ of Nisnevich sheaves, where $A = \operatorname{colim}_n A_{2n}$ (the transition maps are given by adding ψ_2). Altogether, we obtain an isomorphism $[X, A]_{\mathbb{A}^1} = \operatorname{GW}^3_1(R)$ and $[X, A]_{\mathbb{A}^1}$ is precisely $A(R)/{\sim} = W'_E(R)$.

We describe an action of $\mathbb{G}_m$ on $\operatorname{GL}/\operatorname{Sp}$. For any ring R and any unit $u \in R^\times$, we denote by $\gamma_{2n,u}$ the invertible $2n \times 2n$-matrix inductively defined by

$$\gamma_{2,u} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

and $\gamma_{2n+2,u} = \gamma_{2n,u} \perp \gamma_{2,u}$. Conjugation by $\gamma^{-1}_{2n,u}$ induces an action of $\mathbb{G}_m$ on GL_{2n} for all n. As Sp_{2n} is preserved by this action, there is an induced action on $\operatorname{GL}_{2n}/\operatorname{Sp}_{2n}$. Since all the morphisms $\operatorname{GL}_{2n}/\operatorname{Sp}_{2n} \to \operatorname{GL}_{2n+2}/\operatorname{Sp}_{2n+2}$ are equivariant for this action, we obtain an action of $\mathbb{G}_m$ on $\operatorname{GL}/\operatorname{Sp}$.

Using the isomorphism $\operatorname{GL}/\operatorname{Sp} \xrightarrow{\simeq} A$ described above, there is an induced action of $R^\times = \mathbb{G}_m(R)$ on $\operatorname{GW}^3_1(R) \cong A(R)/{\sim} = W'_E(R)$ for any smooth k-algebra R by taking $\mathbb{A}^1$-homotopy classes of morphisms. If $M \in \operatorname{GL}_{2n}(R)$ represents a morphism $\operatorname{Spec}(R) \to \operatorname{GL}_{2n}$ and u is a unit of R, note that the conjugation of M by $\gamma^{-1}_{2n,u}$ is sent via the morphism $\operatorname{GL}_{2n} \to \operatorname{GL}_{2n}/\operatorname{Sp}_{2n} \xrightarrow{\cong} A_{2n}$ to

$$\gamma^{-1}_{2n,u} M^t \gamma_{2n,u} \psi_{2n} \gamma_{2n,u} M \gamma^{-1}_{2n,u} = \gamma^{-1}_{2n,u} M^t (u \cdot \psi_{2n}) M \gamma^{-1}_{2n,u}.$$

Note that the isometry induced by the matrix $\gamma_{2n,u}$ yields an equality

$$[R^{2n}, \psi_{2n}, \gamma^{-1}_{2n,u} M^t (u \cdot \psi_{2n}) M \gamma^{-1}_{2n,u}] = [R^{2n}, u \cdot \psi_{2n}, M^t (u \cdot \psi_{2n}) M]$$

in $V(R)$. As a consequence, the action of $R^\times$ on $\operatorname{GW}^3_1(R)$ can be described via the isomorphism $GW^3_1(R) \cong V(R)$ as follows: If (P, g, f) is a triple as in the definition of the group $V(R)$ and $u \in R^\times$, then the action is given by

$$(u, (P, g, f)) \mapsto (P, u \cdot g, u \cdot f).$$

Following [Asok and Fasel 2017], we refer to this action as the conjugation action of $R^\times$ on $\operatorname{GW}^3_1(R) \cong V(R)$. Recall the we have already defined an action of $R^\times$ on $V(R)$ for any commutative ring R in Section 3B. The conjugation action is thus a homotopy-theoretic interpretation of this action in case of a smooth algebra over a perfect field of characteristic $\neq 2$.

Now let us examine the product map

$$\mathrm{GW}_0^0(R) \times \mathrm{GW}_1^3(R) \to \mathrm{GW}_1^3(R)$$

for smooth k-algebras. As described above, there is a canonical isomorphism $\mathrm{GW}_0^0 = K_0 O(R) = \mathrm{GW}(R)$, where $\mathrm{GW}(R)$ is the Grothendieck completion of the abelian monoid of nondegenerate symmetric bilinear forms over R. Furthermore, there is a canonical map

$$R^\times \to \mathrm{GW}(R), \quad u \mapsto (R \times R \to R, (x, y) \mapsto uxy),$$

which induces an action of $R^\times = \mathbb{G}_m(R)$ on $\mathrm{GW}_1^3(R)$ via the product map mentioned above. Again following [Asok and Fasel 2017], we refer to this action as the multiplicative action of $R^\times = \mathbb{G}_m(R)$ on $\mathrm{GW}_1^3(R) \cong V(R)$. It follows from the proof of [Asok and Fasel 2015, Proposition 3.5.1] that the multiplicative action coincides with the conjugation action. Therefore we obtain another interpretation of the $R^\times$-action on $V(R)$ given in Section 3B via the multiplicative structure of higher Grothendieck–Witt groups.

4. Main results

We finally give the definition of the generalized Vaserstein symbol in this section. As a first step, we recall the definition of the usual Vaserstein symbol introduced in [Vaserstein and Suslin 1976] and reinterpret it by means of the isomorphism $W'_E(R) \cong V(R)$ discussed in the previous section. Then we define the generalized symbol and study its basic properties. In particular, we find criteria for the generalized Vaserstein symbol to be injective and surjective (onto the subgroup $\widetilde{V}(R)$ of $V(R)$ corresponding to $W_E(R)$), which are the natural generalizations of the criteria found in [Vaserstein and Suslin 1976, Theorem 5.2]. These criteria enable us to prove that the generalized Vaserstein symbol is a bijection, e.g., for 2-dimensional regular Noetherian rings and for 3-dimensional regular affine algebras over algebraically closed fields such that $6 \in k^\times$.

4A. ***The Vaserstein symbol for unimodular rows.*** For the rest of the section, let R be a commutative ring. Let $\mathrm{Um}_3(R)$ be its set of unimodular rows of length 3, i.e., triples $a = (a_1, a_2, a_3)$ of elements in R such that there are elements $b_1, b_2, b_3 \in R$ with $\sum_{i=1}^3 a_i b_i = 1$. This data determines an exact sequence of the form

$$0 \to P(a) \to R^3 \xrightarrow{a} R \to 0,$$

where $P(a) = \ker(a)$. The triple $b = (b_1, b_2, b_3)$ gives a section to the epimorphism $a : R^3 \to R$ and induces a retraction $r : R^3 \to P(a)$, $e_i \mapsto e_i - a_i b$, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. One then obtains an isomorphism $i = r + a : R^3 \to P(a) \oplus R$, which induces an isomorphism $\det(R^3) \to \det(P(a) \oplus R)$.

Finally, by composing with the canonical isomorphisms $\det(P(a)\oplus R)\cong\det(P(a))$ and $R\to\det(R^3)$, $1\mapsto e_1\wedge e_2\wedge e_3$, one obtains an isomorphism $\theta : R\to\det(P(a))$.

The matrix

$$V(a,b)=\begin{pmatrix}0 & -a_1 & -a_2 & -a_3\\ a_1 & 0 & -b_3 & b_2\\ a_2 & b_3 & 0 & -b_1\\ a_3 & -b_2 & b_1 & 0\end{pmatrix}$$

has Pfaffian 1 and its image in $W_E(R)$ does not depend on the choice of the section b. We therefore obtain a well-defined map $V : \mathrm{Um}_3(R)\to W_E(R)$ called the Vaserstein symbol.

Now let us reinterpret the Vaserstein symbol map in light of the isomorphism $W'_E(R)\cong V(R)$ discussed in Section 3B. The symbol $V(a)$ is sent to the element of $V(R)$ represented by the isometry class $[R^4, \psi_4, V(a,b)]$. If we denote by χ_a the alternating form $P(a)\times P(a)\to R$, $(p,q)\mapsto\theta^{-1}(p\wedge q)$, we obtain an alternating form on R^4 given by $(i\oplus 1)^t(\chi_a\perp\psi_2)(i\oplus 1)$. If we set

$$\sigma=\begin{pmatrix}0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\in E_4(R),$$

then we can check that the form $(i\oplus 1)^t(\chi_a\perp\psi_2)(i\oplus 1)$ is given by the matrix $\sigma^t V(a,b)^t\sigma$. In particular, if we let $M : \mathrm{Um}_3(R)\to\mathrm{Um}_3(R)$ be the map which sends a unimodular row $a=(a_1,a_2,a_3)$ to $M(a)=(-a_1,-a_2,-a_3)$, then the map $\nu\circ V\circ M$ is given by $a\mapsto[R^4,\psi_4,(i\oplus 1)^t(\chi_a\perp\psi_2)(i\oplus 1)]$. Since both M and ν are bijections, these considerations lead to a generalization of the Vaserstein symbol.

4B. ***The generalized Vaserstein symbol.*** Let P_0 be a projective R-module of rank 2. We use the notation of Section 2B. For $n\geq 3$, let $P_n=P_0\oplus Re_3\oplus\cdots\oplus Re_n$ be the direct sum of P_0 and free R-modules Re_i, $3\leq i\leq n$, of rank 1 with explicit generators e_i. We sometimes omit these explicit generators in the notation. We denote by $\pi_{k,n} : P_n\to R$ the projections onto the free direct summands of rank 1 with index $k=3,\ldots,n$.

We assume that P_0 admits a trivialization $\theta_0 : R\to\det(P_0)$ of its determinant. Then we denote by χ_0 the nondegenerate alternating form on P_0 given by $P_0\times P_0\to R$, $(p,q)\mapsto\theta_0^{-1}(p\wedge q)$.

Now let $\mathrm{Um}(P_0\oplus R)$ be the set of epimorphisms $P_0\oplus R\to R$. Any element a of $\mathrm{Um}(P_0\oplus R)$ gives rise to an exact sequence of the form

$$0\to P(a)\to P_0\oplus R\xrightarrow{a} R\to 0,$$

where $P(a) = \ker(a)$. Any section $s : R \to P_0 \oplus R$ of a determines a canonical retraction $r : P_0 \oplus R \to P(a)$ given by $r(p) = p - sa(p)$ and an isomorphism $i : P_0 \oplus R \to P(a) \oplus R$ given by $i(p) = a(p) + r(p)$.

The exact sequence above yields an isomorphism $\det(P_0) \cong \det(P(a))$ and therefore an isomorphism $\theta : R \to \det(P(a))$ obtained by composing with θ_0. We denote by χ_a the nondegenerate alternating form on $P(a)$ given by $P(a) \times P(a) \to R$, $(p, q) \mapsto \theta^{-1}(p \wedge q)$.

We now want to define the generalized Vaserstein symbol

$$V_{\theta_0} : \mathrm{Um}(P_0 \oplus R) \to V(R)$$

associated to P_0 and the fixed trivialization θ_0 of $\det(P_0)$ by

$$V_{\theta_0}(a) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)].$$

If there is no ambiguity, we usually suppress the fixed trivialization θ_0 and denote V_{θ_0} simply by V in order to simplify our notation. In order to prove that this generalized symbol is well-defined, one has to show that our definition is independent of a section of a:

Theorem 4.1. *The generalized Vaserstein symbol is well-defined*, *i.e.*, *the element* $V(a)$ *defined as above is independent of the choice of a section of* a.

Proof. Let $a \in \mathrm{Um}(P_0 \oplus R)$ with two sections $s, t : R \to P_0 \oplus R$. We denote by i_s and i_t the isomorphisms $P_0 \oplus R \cong P(a) \oplus R$ induced by the sections s and t, respectively. Since the isomorphism $\det(P(a)) \cong \det(P_0)$ does not depend on the choice of a section (because the difference of two sections maps R into $P(a)$), the form χ_a is independent of the choice of a section as well. Thus we have to show that the elements $V(a, s) = [P_0 \oplus R^2, (\chi_0 \perp \psi_2), (i_s \oplus 1)^t (\chi_a \perp \psi_2)(i_s \oplus 1)]$ and $V(a, t) = [P_0 \oplus R^2, (\chi_0 \perp \psi_2), (i_t \oplus 1)^t (\chi_a \perp \psi_2)(i_t \oplus 1)]$ are equal in $V(R)$.

We do this in the following three steps:

(1) We define a map $d : P_0 \oplus R \to R$. We get a corresponding automorphism $\varphi \in E(P_0 \oplus R^2)$ defined by $\varphi = \mathrm{id}_{P_0 \oplus R^2} - de_4$.

(2) We show that $\varphi^t (i_s \oplus 1)^t (\chi_a \perp \psi_2)(i_s \oplus 1)\varphi = (i_t \oplus 1)^t (\chi_a \perp \psi_2)(i_t \oplus 1)$.

(3) Using Corollary 3.2, we conclude that $V(a, s) = V(a, t)$.

For step (1), first define a map $d' : P_0 \oplus R \to \det(P_0 \oplus R)$ by $p \mapsto s(1) \wedge t(1) \wedge p$. Then $d : P_0 \oplus R \to R$ is the map obtained from d' by composing with the isomorphisms $\det(P_0 \oplus R) \cong \det(P_0) \cong R$. Let d_0 and d_R be its restrictions to P_0 and R, respectively. Furthermore, let $\varphi_0 = \mathrm{id}_{P_0 \oplus R^2} - d_0 e_4$ and $\varphi_R = \mathrm{id}_{P_0 \oplus R^2} - d_R e_4$ be the elementary automorphisms of $P_0 \oplus R^2$ defined by $-d_0$ and $-d_R$, respectively. Moreover, let $\varphi = \mathrm{id}_{P_0 \oplus R^2} - de_4$. Note that $\varphi = \varphi_0 \varphi_R = \varphi_R \varphi_0 \in E(P_0 \oplus R^2)$.

Now let us conduct step (2). By Lemma 2.2, we can check the desired equality locally. So let $\mathfrak{p}$ be a prime ideal of R and $(e_1^{\mathfrak{p}}, e_2^{\mathfrak{p}})$ a basis of the free $R_{\mathfrak{p}}$-module $(P_0)_{\mathfrak{p}}$ of rank 2. We may further assume $(\theta_0^{-1})_{\mathfrak{p}}(e_1^{\mathfrak{p}} \wedge e_2^{\mathfrak{p}}) = 1$. With respect to the basis $(e_1^{\mathfrak{p}}, e_2^{\mathfrak{p}}, e_3)$ of $(P_0)_{\mathfrak{p}} \oplus R_{\mathfrak{p}}$, the epimorphism $a_{\mathfrak{p}}$ can be represented by the unimodular row $(a_1^{\mathfrak{p}}, a_2^{\mathfrak{p}}, a_3^{\mathfrak{p}})$ and both sections $s_{\mathfrak{p}}$ and $t_{\mathfrak{p}}$ can be represented by the columns $(s_1^{\mathfrak{p}}, s_2^{\mathfrak{p}}, s_3^{\mathfrak{p}})^t$ and $(t_1^{\mathfrak{p}}, t_2^{\mathfrak{p}}, t_3^{\mathfrak{p}})^t$. Using the basis $(e_1^{\mathfrak{p}}, e_2^{\mathfrak{p}}, e_3, e_4)$ of $(P_0)_{\mathfrak{p}} \oplus R_{\mathfrak{p}}^2$, we can check the desired equality locally: If we let $d_1^{\mathfrak{p}} = t_3^{\mathfrak{p}} s_2^{\mathfrak{p}} - t_2^{\mathfrak{p}} s_3^{\mathfrak{p}}$, $d_2^{\mathfrak{p}} = t_1^{\mathfrak{p}} s_3^{\mathfrak{p}} - t_3^{\mathfrak{p}} s_1^{\mathfrak{p}}$ and $d_3^{\mathfrak{p}} = t_2^{\mathfrak{p}} s_1^{\mathfrak{p}} - t_1^{\mathfrak{p}} s_2^{\mathfrak{p}}$ and

$$M_{\mathfrak{p}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -d_1^{\mathfrak{p}} & -d_2^{\mathfrak{p}} & -d_3^{\mathfrak{p}} & 1 \end{pmatrix},$$

this amounts to verifying the equality

$$M_{\mathfrak{p}}^t \begin{pmatrix} 0 & s_3^{\mathfrak{p}} & -s_2^{\mathfrak{p}} & a_1^{\mathfrak{p}} \\ -s_3^{\mathfrak{p}} & 0 & s_1^{\mathfrak{p}} & a_2^{\mathfrak{p}} \\ s_2^{\mathfrak{p}} & -s_1^{\mathfrak{p}} & 0 & a_3^{\mathfrak{p}} \\ -a_1^{\mathfrak{p}} & -a_2^{\mathfrak{p}} & -a_3^{\mathfrak{p}} & 0 \end{pmatrix} M_{\mathfrak{p}} = \begin{pmatrix} 0 & t_3^{\mathfrak{p}} & -t_2^{\mathfrak{p}} & a_1^{\mathfrak{p}} \\ -t_3^{\mathfrak{p}} & 0 & t_1^{\mathfrak{p}} & a_2^{\mathfrak{p}} \\ t_2^{\mathfrak{p}} & -t_1^{\mathfrak{p}} & 0 & a_3^{\mathfrak{p}} \\ -a_1^{\mathfrak{p}} & -a_2^{\mathfrak{p}} & -a_3^{\mathfrak{p}} & 0 \end{pmatrix}.$$

But this follows from the proof of [Vaserstein and Suslin 1976, Lemma 5.1].

Finally, we conclude by Corollary 3.2: Since φ_0 and φ_R are elementary automorphisms of $P_0 \oplus R^2$, the automorphism $\varphi = \varphi_0 \varphi_R$ is an element of $E(P_0 \oplus R^2)$. By Corollary 3.2, we deduce that

$$\begin{aligned} V(a, s) &= [P_0 \oplus R^2, \chi_0 \perp \psi_2, (i_s \oplus 1)^t (\chi_a \perp \psi_2)(i_s \oplus 1)] \\ &= [P_0 \oplus R^2, \chi_0 \perp \psi_2, \varphi^t (i_s \oplus 1)^t (\chi_a \perp \psi_2)(i_s \oplus 1)\varphi]. \end{aligned}$$

But by step (2), we also know that

$$\begin{aligned} &[P_0 \oplus R^2, \chi_0 \perp \psi_2, \varphi^t (i_s \oplus 1)^t (\chi_a \perp \psi_2)(i_s \oplus 1)\varphi] \\ &\qquad = [P_0 \oplus R^2, \chi_0 \perp \psi_2, (i_t \oplus 1)^t (\chi_a \perp \psi_2)(i_t \oplus 1)] = V(a, t). \end{aligned}$$

This finishes the proof. □

We note that there is a homomorphism $\overline{\mathrm{Pf}} : V(R) \to R^\times$ obtained as the composite $V(R) \xrightarrow{\cong} V_{\mathrm{free}}(R) \xrightarrow{\xi} W'_E(R) \xrightarrow{\mathrm{Pf}} R^\times$. We denote its kernel by $\widetilde{V}(R)$. Of course, the isomorphism $V(R) \cong W'_E(R)$ induces an isomorphism $\widetilde{V}(R) \cong W_E(R)$.

As stated in the previous section, the usual Vaserstein symbol of a unimodular row is an element of $W_E(R)$ and is invariant under elementary transformations. We now prove that the analogous statements also hold for the generalized Vaserstein symbol:

Lemma 4.2. *The generalized Vaserstein symbol* $V : \mathrm{Um}(P_0 \oplus R) \to V(R)$ *maps* $\mathrm{Um}(P_0 \oplus R)$ *into* $\widetilde{V}(R)$.

Proof. For this, we note that the Pfaffian of an element of $V(R)$ is completely determined by the Pfaffians of all its images under the maps $V(R) \to V(R_{\mathfrak{p}})$ induced by localization at any prime ideal $\mathfrak{p}$. But the localization $(P_0)_{\mathfrak{p}}$ at any prime $\mathfrak{p}$ is a free $R_{\mathfrak{p}}$-module of rank 2; choosing a basis $(e_1^{\mathfrak{p}}, e_2^{\mathfrak{p}})$ of $(P_0)_{\mathfrak{p}}$ such that $(\theta_0^{-1})_{\mathfrak{p}}(e_1^{\mathfrak{p}} \wedge e_2^{\mathfrak{p}}) = 1$ as in the proof of Theorem 4.1, we may calculate the Pfaffian of any Vaserstein symbol by the usual formula for the Pfaffian of an alternating 4×4-matrix. The lemma then follows immediately. □

Theorem 4.3. *Let* φ *be an elementary automorphism of* $P_0 \oplus R$. *Then we have* $V(a) = V(a\varphi)$ *for any* $a \in \mathrm{Um}(P_0 \oplus R)$. *In particular, we obtain a well-defined map* $V : \mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R) \to \widetilde{V}(R)$.

Proof. Let φ be an elementary automorphism of $P_0 \oplus R$, $a \in \mathrm{Um}(P_0 \oplus R)$ and $s : R \to P_0 \oplus R$ a section of a. Then $\varphi^{-1}s$ is a section of $a\varphi$. Let $i : P_0 \oplus R \to P(a) \oplus R$ and $j : P_0 \oplus R \to P(a\varphi) \oplus R$ be the isomorphisms induced by the sections s and $\varphi^{-1}s$. We show that

$$(\varphi \oplus 1)^t (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)(\varphi \oplus 1) = (j \oplus 1)^t (\chi_{(a\varphi)} \perp \psi_2)(j \oplus 1).$$

The theorem then follows from Corollary 3.2.

So let us show the equality above. Directly from the definitions, one checks that $(i \oplus 1)(\varphi \oplus 1) = ((\varphi \oplus 1) \oplus 1)(j \oplus 1)$, where by abuse of notation we understand φ as the induced isomorphism $P(a\varphi) \to P(a)$. Altogether, it only remains to show that $\varphi^t \chi_a \varphi = \chi_{a\varphi}$.

For this, let (p, q) be a pair of elements in $P(a\varphi)$; by definition, $\chi_{a\varphi}$ sends these elements to the image of $p \wedge q$ under the isomorphism $\det(P(a\varphi)) \cong R$. This element can also be described as the image of $p \wedge q \wedge \varphi^{-1}s(1)$ under the isomorphism $\det(P_0 \oplus R) \cong R$.

Analogously, the alternating form $\varphi^t \chi_a \varphi$ sends (p, q) to the image of the element $\varphi(p) \wedge \varphi(q) \wedge s(1)$ under the isomorphism $\det(P_0 \oplus R) \cong R$. Therefore Lemma 2.11 allows us to conclude as desired, which finishes the proof of the theorem. □

Note that if we equip the set $\mathrm{Um}(P_0 \oplus R)$ with the projection $\pi_R : P_0 \oplus R \to R$ onto R as a basepoint, then the generalized Vaserstein symbol is a map of pointed sets, because $V(\pi_R) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi_0 \perp \psi_2] = 0$.

Let us briefly discuss how the generalized Vaserstein symbol depends on the choice of the trivialization θ_0 of the determinant of P_0. For this, recall that we have defined an action of $R^\times$ on $V(R)$ in Section 3B. In case of a smooth algebra over a perfect field of characteristic $\neq 2$, we saw in Section 3C that this action can be identified with the multiplicative action induced by a product map in the theory of higher Grothendieck–Witt groups.

Now let P_0 be a projective R-module of rank 2 which admits a trivialization θ_0 of its determinant. Furthermore, let $a \in \mathrm{Um}(P_0 \oplus R)$ with section s and let i, χ_0, χ_a be as in the definition of the generalized Vaserstein symbol. We consider another trivialization θ_0' of $\det(P_0)$ and let χ_0' and χ_a' be the corresponding alternating forms on P_0 and $P(a)$. Obviously, there is a unit $u \in R^\times$ such that $\theta_0 = u \cdot \theta_0'$; in particular, we have $u \cdot \chi_0 = \chi_0'$ and $u \cdot \chi_a = \chi_a'$. Thus, if we denote the Vaserstein symbol associated to θ_0' by $V_{\theta_0'}$, then

$$V_{\theta_0'} = [P_0 \oplus R^2, (u \cdot \chi_0) \perp \psi_2, (i \oplus 1)^t((u \cdot \chi_a) \perp \psi_2)(i \oplus 1)].$$

Finally, the isometry given by $P_0 \oplus R^2 \xrightarrow{\mathrm{id}_{P_0} \oplus 1 \oplus u} P_0 \oplus R^2$ yields an equality

$$[P_0 \oplus R^2, (u \cdot \chi_0) \perp \psi_2, (i \oplus 1)^t((u \cdot \chi_a) \perp \psi_2)(i \oplus 1)] \\ = [P_0 \oplus R^2, u \cdot (\chi_0 \perp \psi_2), u \cdot (i \oplus 1)^t(\chi_a \perp \psi_2)(i \oplus 1)].$$

Thus, if we denote the Vaserstein symbol associated to θ_0 by V_{θ_0}, then

$$V_{\theta_0'} = u \cdot V_{\theta_0}.$$

In particular, the property of the generalized Vaserstein symbol to be injective, surjective or bijective onto $\widetilde{V}(R)$ does not depend on the choice of θ_0.

There is another immediate consequence of this: If we let $P_0 = R^2$ be the free R-module of rank 2 and $e_1 = (1, 0)$, $e_2 = (0, 1) \in R^2$, then there is a canonical isomorphism $\theta_0 : R \xrightarrow{\cong} \det(R^2)$ given by $1 \mapsto e_1 \wedge e_2$. Then recall that the usual Vaserstein symbol can be described as $V_{\theta_0} \circ M$ (up to the identification $W_E(R) \cong \widetilde{V}(R)$). Now let a be a unimodular row of length 3 over R with section b and $V(a, b)$ the associated matrix mentioned in Section 4A. By the formula above, it follows that $V_{-\theta_0}(a)$ is given by $[R^4, -\psi_4, V(a, b)]$. But the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

lies in $E_4(R)$ and gives an isometry between ψ_4 and $-\psi_4$. Hence the generalized Vaserstein symbol $V_{-\theta_0}$ associated to the trivialization $-\theta_0$ coincides with the usual Vaserstein symbol via the identification $\widetilde{V}(R) \cong W_E(R)$ mentioned above.

4C. ***Criteria for surjectivity and injectivity of the generalized Vaserstein symbol.*** The main purpose of this section is to find some criteria for the generalized Vaserstein symbol to be surjective onto $\widetilde{V}(R)$ or injective. We have already seen that these properties are independent of the choice of a trivialization of $\det(P_0)$. So let us fix such a trivialization $\theta_0 : R \xrightarrow{\cong} \det(P_0)$.

Recall that a unimodular row of length n is an n-tuple $a = (a_1, \dots, a_n)$ of elements in R such that there are elements $b_1, \dots, b_n \in R$ with $\sum_{i=1}^{n} a_i b_i = 1$. We denote by $\mathrm{Um}_n(R)$ the set of unimodular rows of length n. For any $n \geq 3$, there are obvious maps $U_n : \mathrm{Um}_{n-2}(R) \to \mathrm{Um}(P_n)$.

As a first step towards our criterion for the surjectivity of the generalized Vaserstein symbol (see Theorem 4.5 below), we prove the following statement:

Lemma 4.4. *Any element of the form* $[P_4, \chi_0 \perp \psi_2, \chi] \in \widetilde{V}(R)$ *for a nondegenerate alternating form* χ *on* P_4 *is in the image of the generalized Vaserstein symbol.*

Proof. First of all, we set $a = \chi(-, e_4) : P_0 \oplus Re_3 \to R$. Since χ is nondegenerate, there is an element $p \in P_4$ such that $\chi(-, p) : P_4 \to R$ is just $-\pi_{4,4}$. In fact, since $\chi(p, p) = 0$, it immediately follows that $p \in P_3$. But then $a(p) = \chi(p, e_4) = -\chi(e_4, p) = 1$. Hence p defines a section $s : R \to P_3$, $1 \mapsto p$, of $a : P_0 \oplus Re_3 \to R$.

The generalized Vaserstein symbol of a may thus be computed by means of this section: As in the definition of the generalized Vaserstein symbol, we obtain an isomorphism $i : P_0 \oplus R \to P(a) \oplus R$ and an alternating form χ_a on $P(a) = \ker(a)$ induced by a and its section s. The generalized Vaserstein symbol of a is then given by $[P_0 \oplus R^2, \chi_0 \perp \psi_2, (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)]$. But one can check easily that the form $(i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)$ locally coincides with χ by construction. By Lemma 2.2, it thus also coincides with χ globally. Therefore we obtain the desired equality $V(a) = [P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi]$. □

Using Lemma 4.4 and the technical lemmas proven in previous sections, we may now prove the following criterion for the surjectivity of the generalized Vaserstein symbol:

Theorem 4.5. *Let* $N \in \mathbb{N}$. *Assume that an element* β *of* $\widetilde{V}(R)$ *is of the form* $[P_{2N+2}, \chi_0 \perp \psi_{2N}, \chi]$ *for some nondegenerate alternating form on* P_{2N+2}. *Moreover, assume that* $\pi_{2n+1,2n+1}(E_\infty(P_0) \cap \mathrm{Aut}(P_{2n+1})) = \mathrm{Um}(P_{2n+1})$ *for any* $n \in \mathbb{N}$ *with* $1 < n \leq N$. *Then* β *lies in the image of the generalized Vaserstein symbol. Thus, the generalized Vaserstein symbol* $V : \mathrm{Um}(P_0 \oplus R) \to \widetilde{V}(R)$ *is surjective if* $\pi_{2n+1,2n+1}(E_\infty(P_0) \cap \mathrm{Aut}(P_{2n+1})) = \mathrm{Um}(P_{2n+1})$ *for all* $n \geq 2$.

Proof. By assumption, $\beta \in \widetilde{V}(R)$ has the form $\beta = [P_{2N+2}, \chi_0 \perp \psi_{2N}, \chi]$ for some nondegenerate alternating form on P_{2N+2}. Furthermore, we may inductively apply Lemma 2.10 (because of the second assumption) in order to deduce that there is an elementary automorphism φ on P_{2N+2} such that $\varphi^t \chi \varphi = \psi \perp \psi_{2N-2}$ for some nondegenerate alternating form ψ on P_4. In particular, $\beta = [P_4, \chi_0 \perp \psi_2, \psi]$ by Corollary 3.2. Finally, any element of this form is in the image of the generalized Vaserstein symbol by Lemma 4.4. So β is in the image of the generalized Vaserstein symbol.

For the last statement, note that any element of $\widetilde{V}(R)$ is of the form $[R^{2n}, \psi_{2n}, \chi]$ for some nondegenerate alternating form on R^{2n} (because of the isomorphism $\widetilde{V}(R) \cong W_E(R)$). We may then artificially add a trivial summand $[P_0, \chi_0, \chi_0]$; hence any element of $\widetilde{V}(R)$ is of the form $[P_{2n+2}, \chi_0 \perp \psi_{2n}, \chi_0 \perp \chi]$ for some nondegenerate alternating form on R^{2n}. We can then conclude by the previous paragraph. □

Theorem 4.6. *Let $N \in \mathbb{N}$. Assume that the following conditions are satisfied*:

- *Every element of $\widetilde{V}(R)$ is of the form $[R^{2N}, \psi_{2N}, \chi]$ for some nondegenerate alternating form on R^{2N}.*
- *One has $\pi_{2n+1,2n+1}(E_\infty(P_0) \cap \mathrm{Aut}(P_{2n+1})) = \mathrm{Um}(P_{2n+1})$ for any $n \in \mathbb{N}$ with $1 < n < N$ and $U_{2N+1}(\mathrm{Um}_{2N-1}(R)) \subset \pi_{2N+1,2N+1}E(P_{2N+1})$.*

Then the generalized Vaserstein symbol $V : \mathrm{Um}(P_0 \oplus R) \to \widetilde{V}(R)$ is surjective.

Proof. We proceed as in the proof of Theorem 4.5. By the first assumption, any element of $\widetilde{V}(R)$ is of the form $[R^{2N}, \psi_{2N}, \chi]$ for some nondegenerate alternating form on R^{2N}. Again adding a trivial summand $[P_0, \chi_0, \chi_0]$, we see that any element of $\widetilde{V}(R)$ is of the form $[P_{2N+2}, \chi_0 \perp \psi_{2N}, \chi_0 \perp \chi]$ for some nondegenerate alternating form on R^{2N}. As in the proof of Theorem 4.5, it then follows inductively from Lemma 2.10 that any element of $\widetilde{V}(R)$ is of the form $[P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi]$ for some nondegenerate alternating form χ on $P_0 \oplus R^2$. The generalized Vaserstein symbol is then surjective by Lemma 4.4. Note that the condition $\pi_{2N+1,2N+1}E(P_{2N+1}) = \mathrm{Um}(P_{2N+1})$ can be replaced by the weaker condition $U_{2N+1}(\mathrm{Um}_{2N-1}(R)) \subset \pi_{2N+1,2N+1}E(P_{2N+1})$ in our situation. □

Corollary 4.7. *Assume that the following conditions are satisfied*:

- *The usual Vaserstein symbol $V : \mathrm{Um}_3(R) \to W_E(R)$ is surjective.*
- $U_5(\mathrm{Um}_3(R)) \subset \pi_{5,5}(E_\infty(P_0) \cap \mathrm{Aut}(P_5))$.

Then the generalized Vaserstein symbol $V : \mathrm{Um}(P_0 \oplus R) \to \widetilde{V}(R)$ is surjective.

Proof. The surjectivity of the usual Vaserstein symbol means that any element of $\widetilde{V}(R)$ is of the form $[R^4, \psi_4, \chi]$ for some nondegenerate alternating form on R^4. Now the corollary follows from Theorem 4.6. □

In order to prove our criterion for the injectivity of the generalized Vaserstein symbol, we introduce the following condition: We say that P_0 satisfies condition $(*)$ if $[P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi_1] = [P_0 \oplus R^2, \chi_0 \perp \psi_2, \chi_2] \in \widetilde{V}(R)$ for alternating forms χ_1, χ_2 on $P_0 \oplus R^2$ implies $\alpha^t(\chi_1 \perp \psi_{2n})\alpha = \chi_2 \perp \psi_{2n}$ for some automorphism $\alpha \in E_\infty(P_0) \cap \mathrm{Aut}(P_{2n+4})$.

If P_0 is a free R-module, condition $(*)$ is satisfied, which basically follows from the isomorphism $V(R) \cong W'_E(R)$. Furthermore, using the isomorphisms

$V(R) \cong V_{\mathrm{free}}(R) \cong W'_E(R)$, we see that it is possible to prove that condition $(*)$ is always satisfied (see Lemma 4.9).

As a first step towards Lemma 4.9, we observe:

Lemma 4.8. *Let χ be a nondegenerate alternating form on a finitely generated projective R-module P. Then there exists a finitely generated projective R-module P' with a nondegenerate alternating form χ' on P' and an isomorphism $\tau: R^{2n} \xrightarrow{\cong} P \oplus P'$ such that $\tau^t(\chi \perp \chi')\tau = \psi_{2n}$.*

Proof. Let Q be a finitely generated projective R-module such that $P \oplus Q$ is free. Then, for $Q_1 = P^\vee \oplus Q \oplus Q^\vee$, one has $P \oplus Q_1 \cong R^{2m}$ for some $m \geq 0$. Moreover, for $\phi_1 = \mathrm{can}\,\chi^{-1} \perp H_Q$, the form $\chi \perp \phi_1$ is hence isometric to a form ϕ_2 on R^{2m}. Now let ϕ_3 be a form on R^{2s} for some $s \geq 0$ which represents the inverse of ϕ_2 in $W'_E(R)$. Then $\phi_2 \perp \phi_3 \perp \psi_{2t}$ is isometric to $\psi_{2m+2s+2t}$ for some $t \geq 0$. We set $P' = Q_1 \oplus R^{2s+2t}$ and $\chi' = \phi_1 \perp \phi_3 \perp \psi_{2t}$. Then there is an isometry $\tau : R^{2m+2s+2n} \to P'$ between $\psi_{2m+2s+2t}$ and $\chi \perp \chi'$, as desired. □

Using Lemma 4.8, we may prove:

Lemma 4.9. *Any P_0 satisfies condition $(*)$.*

Proof. We prove Lemma 4.10 below, which obviously implies Lemma 4.9 for $P = P_0 \oplus R^2$ and $\chi = \chi_0 \perp \psi_2$. □

Lemma 4.10. *If $[P, \chi, \chi_1] = [P, \chi, \chi_2] \in V(R)$ for nondegenerate alternating forms χ, χ_1 and χ_2 on a finitely generated projective R-module P, then we have an equality $\alpha^t(\chi_1 \perp \psi_{2n})\alpha = \chi_2 \perp \psi_{2n}$ for some $n \in \mathbb{N}$ and some automorphism $\alpha \in E(P \oplus R^{2n})$.*

Proof. The equality $[P, \chi, \chi_1] = [P, \chi, \chi_2]$ means that $[P, \chi_1, \chi_2] = 0$. By Lemma 4.8, it follows that there is a finitely generated projective R-module P_1 with a nondegenerate alternating form χ' on P_1 and, moreover, with an isomorphism $\tau : R^{2m} \xrightarrow{\cong} P \oplus P_1$ such that $\tau^t(\chi_1 \perp \chi')\tau = \psi_{2m}$. In particular, one has $0 = [P, \chi_1, \chi_2] = [R^{2m}, \psi_{2m}, \tau^t(\chi_2 \perp \chi')\tau] \in V(R)$. Therefore the class of $\tau^t(\chi_2 \perp \chi')\tau$ in $W'_E(R)$ is trivial and there exist $u \geq 1$ and $\zeta \in E(R^{2m+2u})$ such that $\zeta^t((\tau^t(\chi_2 \perp \chi')\tau) \perp \psi_{2u})\zeta = \psi_{2m+2u}$. Note that ζ lies in the commutator subgroup of $\mathrm{Aut}(R^{2m+2u})$.

Again by Lemma 4.8, there exists a finitely generated projective R-module P_2 with a nondegenerate alternating form χ'' on P_2 and with an isomorphism $\beta : R^{2v} \xrightarrow{\cong} P_1 \oplus R^{2u} \oplus P_2$ such that $\beta^t(\chi' \perp \psi_{2u} \perp \chi'')\beta = \psi_{2v}$.

But then the composite

$$\xi = (\mathrm{id}_P \oplus \beta^{-1})(\tau \oplus \mathrm{id}_{R^{2u}} \oplus \mathrm{id}_{P_2})(\zeta^{-1} \oplus \mathrm{id}_{P_2})(\tau^{-1} \oplus \mathrm{id}_{R^{2u}} \oplus \mathrm{id}_{P_2})(\mathrm{id}_P \oplus \beta)$$

is an isometry from $\chi_1 \perp \psi_{2v}$ to $\chi_2 \perp \psi_{2v}$ and lies in the commutator subgroup of $\mathrm{Aut}(P \oplus R^{2v})$ because it is a conjugate of $\zeta^{-1} \perp \mathrm{id}_{P_2}$. In particular, it follows that

$\xi \perp \mathrm{id}_{R^{2w}} \in E(P \oplus R^{2v+2w})$ for some $w \geq 0$. Finally, if we then set $\alpha = \xi \perp \mathrm{id}_{R^{2w}}$ and $n = v + w$, the lemma is proven. □

Now that we have proven that condition $(*)$ is always satisfied, we can find conditions which imply that two elements $a, b \in \mathrm{Um}(P_0 \oplus R)$ with the same Vaserstein symbol are equal up to a stably elementary automorphism of $P_0 \oplus R$:

Theorem 4.11. *Assume that $E(P_{2n})e_{2n} = (E_\infty(P_0) \cap \mathrm{Aut}(P_{2n}))e_{2n}$ for $n \geq 2$. Then the equality $V(a) = V(b)$ for $a, b \in \mathrm{Um}(P_0 \oplus R)$ implies that $b = a\varphi$ for some $\varphi \in E_\infty(P_0) \cap \mathrm{Aut}(P_3)$.*

Proof. Let a and b be elements of $\mathrm{Um}(P_0 \oplus R)$ with respective sections s and t and let $i : P_0 \oplus R \to P(a) \oplus R$ and $j : P_0 \oplus R \to P(a) \oplus R$ be the isomorphisms induced by these sections. Furthermore, we let $V(a, s) = (i \oplus 1)^t (\chi_a \perp \psi_2)(i \oplus 1)$ and $V(b, t) = (j \oplus 1)^t (\chi_b \perp \psi_2)(j \oplus 1)$ be the nondegenerate alternating forms on $P_0 \oplus R^2$ appearing in the definition of the generalized Vaserstein symbols of a and b, respectively. Now assume that $V(a) = V(b)$. Since P_0 satisfies condition $(*)$, there exist $n \in \mathbb{N}$ and an automorphism $\alpha \in E_\infty(P_0) \cap \mathrm{Aut}(P_{2n+4})$ such that $\alpha^t(V(a, s) \perp \psi_{2n})\alpha = V(b, t) \perp \psi_{2n}$. Using Lemma 2.9, we inductively deduce that $\beta^t V(a, s)\beta = V(b, t)$ for some $\beta \in E_\infty(P_0) \cap \mathrm{Aut}(P_0 \oplus R^2)$. Now by Lemma 2.8 and the second assumption in the theorem, there exists an automorphism $\gamma \in E(P_0 \oplus R^2) \cap \mathrm{Sp}(V(a, s))$ such that $\beta e_4 = \gamma e_4$.

We now define $\delta : P_0 \oplus R \to P_0 \oplus R$ as the composite

$$P_0 \oplus Re_3 \to P_0 \oplus Re_3 \oplus Re_4 \xrightarrow{\gamma^{-1}\beta} P_0 \oplus Re_3 \oplus Re_4 \to P_0 \oplus Re_3.$$

One can then check that δ is an element of $E_\infty(P_0) \cap \mathrm{Aut}(P_0 \oplus R)$. Moreover, we have

$$\beta^t (\gamma^{-1})^t V(a, s)\gamma^{-1}\beta = V(b, t)$$

and in particular $a\delta = b$, as desired. □

Corollary 4.12. *Under the hypotheses of Theorem 4.11, furthermore assume that $a(E_\infty(P_0) \cap \mathrm{Aut}(P_0 \oplus R)) = aE(P_0 \oplus R)$ for all $a \in \mathrm{Um}(P_0 \oplus R)$. Then the generalized Vaserstein symbol $V : \mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R) \to \widetilde{V}(R)$ is injective.*

Proof. By Theorem 4.11, we have that $V(a) = V(b)$ implies $b = a\varphi'$ for some $\varphi' \in E_\infty(P_0) \cap \mathrm{Aut}(P_0 \oplus R)$. Now by the additional assumption, there also exists an elementary automorphism φ of $P_0 \oplus R$ such that $b = a\varphi$. So the generalized Vaserstein symbol is injective. □

Regarding the additional assumption in Corollary 4.12, it is possible to adapt the arguments in the proof of [Vaserstein and Suslin 1976, Corollary 7.4] to show that the desired equality $a(E_\infty(P_0) \cap \mathrm{Aut}(P_0 \oplus R)) = aE(P_0 \oplus R)$ holds for all $a \in \mathrm{Um}(P_0 \oplus R)$ if $E_\infty(P_0) \cap \mathrm{Aut}(P_0 \oplus R^2) = E(P_4)$:

Lemma 4.13. *If $E_\infty(P_0) \cap \mathrm{Aut}(P_0 \oplus R^2) = E(P_4)$, then we have an equality $a(E_\infty(P_0) \cap \mathrm{Aut}(P_0 \oplus R)) = aE(P_0 \oplus R)$ for all $a \in \mathrm{Um}(P_0 \oplus R)$.*

Proof. Let $a \in \mathrm{Um}(P_0 \oplus R)$ with section s and let $\varphi \in E_\infty(P_0) \cap \mathrm{Aut}(P_0 \oplus R)$. If we let $V(a, s)$ be the alternating form from the definition of the generalized Vaserstein symbol, then it follows from the proof of Lemma 4.4 that

$$(\varphi \oplus 1)^t V(a, s)(\varphi \oplus 1) = V(a', s')$$

for some $a' \in \mathrm{Um}(P_0 \oplus R)$ with section s'. By assumption, the automorphism $\varphi \oplus 1$ of P_4 is an elementary automorphism. Moreover, by Corollary 2.4, the group $E(P_4)$ is generated by elementary automorphisms $\varphi_g = \mathrm{id}_{P_4} + g$, where g is a homomorphism $Re_3 \to P_0$, $P_0 \to Re_3$, $Re_3 \to Re_4$ or $Re_4 \to Re_3$. It therefore suffices to show the following: If $\varphi_g^t V(a, s)\varphi_g = V(a', s')$ for some such g, then $a' = a\psi$ for some $\psi \in E(P_0 \oplus R)$. The only nontrivial case is the last one, i.e., if g is a homomorphism $Re_4 \to Re_3$.

So let $g : Re_4 \to Re_3$ and let φ_g be the induced elementary automorphism of P_4. As explained above, we assume that $\varphi_g^t V(a, s)\varphi_g = V(a', s')$ for some epimorphism $a' : P_0 \oplus Re_3 \to R$ with section s'. Write $a = (a_0, a_R)$, where a_0 and a_R are the restrictions of a to P_0 and Re_3, respectively. Furthermore, let $p = \pi_{P_0}(s(1))$. From now on, we interpret the alternating form χ_0 in the definition of the generalized Vaserstein symbol as an alternating isomorphism $\chi_0 : P \to P^\vee$. Then one can check locally that

$$a' = (a_0 - g(1) \cdot \chi_0(p), a_R).$$

Let us define an elementary automorphism ψ as follows: We first define an endomorphism of P_0 by

$$\psi_0 = \mathrm{id}_{P_0} - g(1) \cdot \pi_{P_0} \circ s \circ \chi_0(p) : P_0 \to P_0$$

and we also define a morphism $P_0 \to Re_3$ by

$$\psi_R = -g(1) \cdot \pi_R \circ s \circ \chi_0(p) : P_0 \to R.$$

Then we consider the endomorphism of $P_0 \oplus R$ given by

$$\psi = \begin{pmatrix} \psi_0 & 0 \\ \psi_R & \mathrm{id}_R \end{pmatrix}.$$

First of all, this endomorphism coincides up to an elementary automorphism with

$$\begin{pmatrix} \psi_0 & 0 \\ 0 & \mathrm{id}_R \end{pmatrix}.$$

Since $\chi_0(p) \circ \pi_{P_0} \circ s = 0$, this endomorphism is an element of $E(P_0 \oplus R)$ by Lemma 2.6. Hence the same holds for ψ. Finally, one can check easily that $a\psi = a'$ by construction. □

As an immediate consequence, we can finally deduce our criterion for the injectivity of the generalized Vaserstein symbol:

Theorem 4.14. *Assume that $E(P_{2n})e_{2n} = (E_\infty(P_0) \cap \mathrm{Aut}(P_{2n}))e_{2n}$ for all $n \geq 3$ and that $E_\infty(P_0) \cap \mathrm{Aut}(P_4) = E(P_4)$. Then the generalized Vaserstein symbol $V : \mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R) \to \widetilde{V}(R)$ is injective.*

Proof. Combine Corollary 4.12 and Lemma 4.13. □

4D. ***The bijectivity of the generalized Vaserstein symbol in dimension 2 and 3.*** Let us now study the criteria for the surjectivity and injectivity of the generalized Vaserstein symbol. In [Bass 1968] the conditions of Theorem 4.5 and Theorem 4.14 are studied in a very general framework. If R is a Noetherian ring of Krull dimension d, it follows from [Bass 1968, Chapter IV, Theorem 3.4] that actually $\mathrm{Unim.El.}(P_n) = E(P_n)e_n$ for all $n \geq d+2$ (or $\mathrm{Um}(P_n) = \pi_{n,n}E(P_n)$ for all $n \geq d+2$). In particular, if $\dim(R) \leq 4$, then the generalized Vaserstein symbol is injective as soon as $E_\infty(P_0) \cap \mathrm{Aut}(P_4) = E(P_4)$; if $\dim(R) \leq 3$, it is surjective. Hence the following results are immediate consequences of our stability results in Section 2C:

Theorem 4.15. *Assume R is either a regular Noetherian ring of dimension 2 or a regular affine algebra of dimension 3 over a perfect field k with $\mathrm{c.d.}(k) \leq 1$ and $6 \in k^\times$. Then the generalized Vaserstein symbol $V : \mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R) \to \widetilde{V}(R)$ is a bijection.*

Theorem 4.16. *Let R be a 4-dimensional regular affine algebra over a perfect field k satisfying the property $\mathcal{P}(5, 3)$ (see Section 2C). Then the generalized Vaserstein symbol $V : \mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R) \to \widetilde{V}(R)$ is injective.*

Because of the pointed surjection $\mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R) \to \phi_2^{-1}([P_0 \oplus R])$, the bijectivity of the generalized Vaserstein symbol always gives rise to a surjection $W_E(R) \to \phi_2^{-1}([P_0 \oplus R])$; in this case, it seems that the group structure of $W_E(R) \cong \mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R)$ essentially governs the structure of the fiber $\phi_2^{-1}([P_0 \oplus R])$.

The following application follows, to some degree, the pattern of the proof of [Fasel et al. 2012, Theorem 7.5] and illustrates the previous paragraph:

Theorem 4.17. *Let R be a ring and P_0 be a projective R-module of rank 2 which admits a trivialization θ_0 of its determinant. Assume the following conditions are satisfied*:

(a) *The generalized Vaserstein symbol $V : \mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R) \to \widetilde{V}(R)$ induced by θ_0 is a bijection.*

(b) $2V(a_0, a_R) = V(a_0, a_R^2)$ *for* $(a_0, a_R) \in \mathrm{Um}(P_0 \oplus R)$.

(c) *The group $W_E(R)$ is 2-divisible.*

Then $\phi_2^{-1}([P_0 \oplus R])$ is trivial.

Proof. Assume $P' \oplus R \cong P_0 \oplus R$. As we have seen in Section 2D, P' has an associated element of $\mathrm{Um}(P_0 \oplus R)/\operatorname{Aut}(P_0 \oplus R)$. We lift this element to an element $[b]$ of $\mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R)$, where $[b]$ denotes the class of $b \in \mathrm{Um}(P_0 \oplus R)$. Since the generalized Vaserstein symbol is a bijection and $W_E(R)$ is a 2-divisible group by assumption, we get that $[b] = 2[a]$, where $[a]$ denotes the class of an element $a = (a_0, a_R)$ of $\mathrm{Um}(P_0 \oplus R)$ in the orbit space $\mathrm{Um}(P_0 \oplus R)/E(P_0 \oplus R)$. But then the second assumption shows that $2[a] = [(a_0, a_R^2)]$. It follows from [Bhatwadekar 2003, Proposition 2.7] or [Suslin 1977a, Lemma 2] that any element of $\mathrm{Um}(P_0 \oplus R)$ of the form (a_0, a_R^2) is completable to an automorphism of $P_0 \oplus R$, i.e., $\pi_R = a\varphi$ for some automorphism φ of $P_0 \oplus R$. Altogether, π_R and b therefore lie in the same orbit under the action of $\operatorname{Aut}(P_0 \oplus R)$, and hence $P' \cong P$. Thus, $\phi_2^{-1}([P_0 \oplus R])$ is trivial. □

As mentioned in the proof of Theorem 4.16, any element $a \in \mathrm{Um}(P_0 \oplus R)$ of the form $a = (a_0, a_R^2)$ is completable to an automorphism of $P_0 \oplus R$. This follows directly from [Bhatwadekar 2003, Proposition 2.7] or [Suslin 1977a, Lemma 2], because P_0 has a trivial determinant. We now construct a more concrete completion of $a = (a_0, a_R^2)$. For this, let us first look at the case $P_0 \cong R^2$. If (b, c, a^2) is a unimodular row and $qb + rc + ap = 1$, then it follows from [Krusemeyer 1976] that the matrix

$$\begin{pmatrix} -p-qr & q^2 & -c+2aq \\ -r^2 & -p+qr & b+2ar \\ b & c & a^2 \end{pmatrix}$$

is a completion of (b, c, a^2) with determinant 1. We observe that

$$\begin{pmatrix} -qr & q^2 \\ -r^2 & qr \end{pmatrix} = \begin{pmatrix} q \\ r \end{pmatrix} \begin{pmatrix} -r & q \end{pmatrix}$$

and also

$$\begin{pmatrix} -c \\ b \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}.$$

This shows how to generalize the construction of this explicit completion. We denote by $\chi_0 : P_0 \to P_0^\vee$ the alternating isomorphism from the definition of the generalized Vaserstein symbol (we now interpret it as an alternating isomorphism and not as a nondegenerate alternating form). If $a = (a_0, a_R)$ is an element of $\mathrm{Um}(P_0 \oplus R)$ with a section s uniquely given by the element $s(1) = (q, p) \in P_0 \oplus R$, we consider the following morphisms: We define an endomorphism of P_0 by

$$\varphi_0 = -(\pi_{P_0} s) \circ \chi_0(q) - p \cdot \mathrm{id}_{P_0} : P_0 \to P_0$$

and we also define a morphism $R \to P_0$ by

$$\varphi_R : R \to P_0, \quad 1 \mapsto 2a_R(1) \cdot q + \chi_0^{-1}(a_0).$$

Then we consider the endomorphism of $\varphi : P_0 \oplus R$ given by

$$\begin{pmatrix} \varphi_0 & \varphi_R \\ a_0 & a_R^2 \end{pmatrix}.$$

Essentially by construction, φ is a completion of (a_0, a_R^2):

Proposition 4.18. *The endomorphism φ of $P_0 \oplus R$ defined above is an automorphism of $P_0 \oplus R$ of determinant 1 such that $\pi_R \varphi = (a_0, a_R^2)$.*

Proof. Choosing locally a free basis $(e_1^{\mathfrak{p}}, e_2^{\mathfrak{p}})$ of $(P_0)_{\mathfrak{p}}$ at any prime $\mathfrak{p}$ such that $(\theta_0^{-1})_{\mathfrak{p}}(e_1^{\mathfrak{p}} \wedge e_2^{\mathfrak{p}}) = 1$, we can check locally that this endomorphism is an automorphism of determinant 1 (because locally it coincides with the completion given in [Krusemeyer 1976]); by definition, we also have $\pi_R \varphi = (a_0, a_R^2)$. Thus, φ has the desired properties and generalizes Krusemeyer's explicit completion. □

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A Dolbeault–Hilbert complex for a variety with isolated singular points

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Given a compact Hermitian complex space with isolated singular points, we construct a Dolbeault-type Hilbert complex whose cohomology is isomorphic to the cohomology of the structure sheaf. We show that the corresponding K-homology class coincides with the one constructed by Baum, Fulton and MacPherson.

1. Introduction

The program of doing index theory, or more generally elliptic theory, on singular varieties goes back at least to [Singer 1971, §4]. This program takes various directions — for example, the relation between L^2-cohomology and intersection homology. In this paper we consider a somewhat different direction, which is related to the arithmetic genus. This is motivated by work of Baum, Fulton and MacPherson [Baum et al. 1975; 1979].

Let X be a projective complex algebraic variety and let $\mathcal{S}$ be a coherent sheaf on X. In [Baum et al. 1979], the authors associated to $\mathcal{S}$ an element $[\mathcal{S}]_{\mathrm{BFM}} \in \mathrm{K}_0(X)$ of the topological K-homology of X. This class enters into their Riemann–Roch theorem for singular varieties. In particular, under the map $p : X \to \mathrm{pt}$, the image $p_*[\mathcal{S}]_{\mathrm{BFM}} \in \mathrm{K}_0(\mathrm{pt}) \cong \mathbb{Z}$ is expressed in terms of sheaf cohomology by $\sum_i (-1)^i \dim(\mathrm{H}^i(X; \mathcal{S}))$.

In view of the isomorphism between topological K-homology and analytic K-homology [Baum and Douglas 1982; Baum et al. 2007], the class $[\mathcal{S}]_{\mathrm{BFM}}$ can be represented by an "abstract elliptic operator" in the sense of [Atiyah 1970]. This raised the question of how to find an explicit cycle in analytic K-homology, even if X is singular, that represents $[\mathcal{S}]_{\mathrm{BFM}}$. The most basic case is when $\mathcal{S}$ is the structure sheaf $\mathcal{O}_X$. If X is smooth then the operator representing $[\mathcal{O}_X]_{\mathrm{BFM}}$ is $\bar{\partial} + \bar{\partial}^*$. Hence we are looking for the right analog of this operator when X may be singular.

A second related question is to find a Hilbert complex, in the sense of [Brüning and Lesch 1992], whose cohomology is isomorphic to $\mathrm{H}^\star(X; \mathcal{O}_X)$. We want the

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complex to be intrinsic to X. Also, if X is smooth then we want to recover the $\bar\partial$-complex on $(0,\star)$-forms.

In this paper, we answer these questions when X has isolated singular points. To see the nature of the problem, suppose that X is a complex curve, whose normalization has genus g. In this case, the Riemann–Roch theorem says

$$\dim(\mathrm{H}^0(X;\mathcal{O}_X)) - \dim(\mathrm{H}^1(X;\mathcal{O}_X)) = 1 - g - \sum_{x\in X_{\mathrm{sing}}} \delta_x, \tag{1.1}$$

where δ_x is a certain positive integer attached to the singular point x [Hartshorne 1977, p. 298]. To find the appropriate Hilbert complex, it is natural to start with the Dolbeault complex $\Omega_c^{0,0}(X_{\mathrm{reg}}) \xrightarrow{\bar\partial} \Omega_c^{0,1}(X_{\mathrm{reg}})$ of smooth compactly supported forms on X_{reg} and look for a closed operator extension, where X_{reg} is endowed with the induced Riemannian metric from its projective embedding. For the minimal closure $\bar\partial_s$, one finds $\mathrm{Index}(\bar\partial_s) = 1 - g$. Taking a different closure can only make the index go up [Brüning et al. 1990], whereas in view of (1.1) we want the index to go down. (Considering complete Riemannian metrics on X_{reg} does not help.) However, on the level of indices, we can get the right answer by enhancing the codomain by $\bigoplus_{x\in X_{\mathrm{sing}}} \mathbb{C}^{\delta_x}$.

Now let X be a compact Hermitian complex space of pure dimension n. For technical reasons, we assume that the singular set X_{sing} consists of isolated singularities. (In the bulk of the paper we allow coupling to a holomorphic vector bundle, but in this introduction we only discuss the case when the vector bundle is trivial.) Let $\bar\partial_s$ be the minimal closed extension of the $\bar\partial$-operator on $X_{\mathrm{reg}} = X - X_{\mathrm{sing}}$. Its domain $\mathrm{Dom}(\bar\partial_s^{0,\star})$ can be localized to a complex of sheaves $\underline{\mathrm{Dom}}(\bar\partial_s^{0,\star})$. Let $\underline{\mathrm{H}}^{0,\star}(\bar\partial_s)$ denote the cohomology, a sum of skyscraper sheaves on X if $\star > 0$. We write $\mathcal{O}_s$ for $\underline{\mathrm{H}}^{0,0}(\bar\partial_s)$, which is the sheaf of germs of weakly holomorphic functions on X, the latter being in the sense of [Whitney 1972, Section 4.3]. Then $\mathcal{O}_s/\mathcal{O}_X$ is also a sum of skyscraper sheaves on X. Its vector space of global sections will be written as $(\mathcal{O}_s/\mathcal{O}_X)(X)$. Both $\underline{\mathrm{H}}^{0,\star}(\bar\partial_s)$ and $\mathcal{O}_s/\mathcal{O}_X$ can be computed using a resolution of X [Ruppenthal 2018, Corollary 1.2].

Define vector spaces T^* by

$$\begin{aligned} T^0 &= \mathrm{Dom}(\bar\partial_s^{0,0}),\\ T^1 &= \mathrm{Dom}(\bar\partial_s^{0,1}) \oplus (\mathcal{O}_s/\mathcal{O}_X)(X),\\ T^\star &= \mathrm{Dom}(\bar\partial_s^{0,\star}) \oplus (\underline{\mathrm{H}}^{0,\star-1}(\bar\partial_s))(X), \quad \text{if } 2 \le \star \le n. \end{aligned} \tag{1.2}$$

To define a differential on T^*, let $\triangle_s^{0,\star}$ be the Laplacian associated to $\bar\partial_s$. Let $P_{\mathrm{Ker}(\triangle_s^{0,\star})}$ be orthogonal projection onto the kernel of $\triangle_s^{0,\star}$. As elements of $\mathrm{Ker}(\triangle_s^{0,\star})$ are $\bar\partial_s$-closed, for each $x\in X_{\mathrm{sing}}$ there is a well-defined map $\mathrm{Ker}(\triangle_s^{0,\star}) \to (\underline{\mathrm{H}}^{0,\star}(\bar\partial_s))_x$ to the stalk of $\underline{\mathrm{H}}^{0,\star}(\bar\partial_s)$ at x. For $\star > 0$, putting these together for all $x \in X_{\mathrm{sing}}$, and precomposing with $P_{\mathrm{Ker}(\triangle_s^{0,\star})}$, gives a linear map $\gamma : \mathrm{Dom}(\bar\partial_s^{0,\star}) \to (\underline{\mathrm{H}}^{0,\star}(\bar\partial_s))(X)$. For $\star = 0$, we similarly define $\gamma : \mathrm{Dom}(\bar\partial_s^{0,0}) \to (\mathcal{O}_s/\mathcal{O}_X)(X)$. Define a differential $d : T^* \to T^{*+1}$ by

$$\begin{aligned} d(\omega) &= (\bar\partial_s\omega, \gamma(\omega)), \quad \text{if } \star = 0, \\ d(\omega, a) &= (\bar\partial_s\omega, \gamma(\omega)), \quad \text{if } \star > 0. \end{aligned} \tag{1.3}$$

Theorem 1.4. *The cohomology of* (T, d) *is isomorphic to* $\mathrm{H}^*(X; \mathcal{O}_X)$.

Theorem 1.4 can be seen as an extension of [Ruppenthal 2018, Corollary 1.3], which implies the result when X is normal and has rational singularities. To prove Theorem 1.4, we construct a certain resolution of $\mathcal{O}_X$ by fine sheaves. The cohomology of the complex $(\widetilde{T}, \tilde{d})$ of global sections is then isomorphic to $\mathrm{H}^*(X; \mathcal{O}_X)$. The complex $(\widetilde{T}, \tilde{d})$ is not quite the same as (T, d) but we show that they are cochain-equivalent, from which the theorem follows.

The spectral triple $(C(X), T, d + d^*)$ defines an element $[\mathcal{O}_X]_{\mathrm{an}} \in \mathrm{K}_0(X)$ of the analytic K-homology of X.

Theorem 1.5. *If X is a projective algebraic variety with isolated singularities then* $[\mathcal{O}_X]_{\mathrm{an}} = [\mathcal{O}_X]_{\mathrm{BFM}}$ *in* $\mathrm{K}_0(X)$.

There has been some interesting earlier work on the questions addressed in this paper. Ancona and Gaveau [1994] gave a resolution of the structure sheaf of a normal complex space X, assuming that the singular locus is smooth, in terms of differential forms on a resolution of X. The construction depended on the choice of resolution. Fox and Haskell [2000] discussed using a perturbed Dolbeault operator on an ambient manifold to represent the K-homology class of the structure sheaf. Andersson and Samuelsson [2012] gave a resolution of the structure sheaf by certain currents on X that are smooth on X_{reg}. After this paper was written, Bei and Piazza [2019] posted a preprint which also has a proof of Proposition 5.1.

The structure of the paper is the following. In Section 2, given a holomorphic vector bundle V on X, we recall the definition of the minimal closure $\bar\partial_{V,s}$ and show that $\bar\partial_{V,s} + \bar\partial_{V,s}^*$ gives an element of the analytic K-homology group $\mathrm{K}_0(X)$, in the unbounded formalism for the Kasparov KK-group $\mathrm{KK}(C(X); \mathbb{C})$. In Section 3 we construct a resolution of the sheaf $\underline{V}$ by fine sheaves. Their global sections give a Hilbert complex. In Section 4 we deform this to the complex (T_V, d_V). Section 5 has the proof of Theorem 1.5. More detailed descriptions appear at the beginning of the sections.

2. Minimal closure and compact resolvent

In this section we consider a holomorphic vector bundle V on a compact complex space X with isolated singularities. We define the minimal closure $\bar\partial_{V,s}$. We show that the spectral triple $(C(X), \bar\partial_{V,s} + \bar\partial^*_{V,s}, \Omega^{0,*}_{L^2}(X_{\mathrm{reg}}; V))$ gives a well-defined element of the analytic K-homology group $\mathrm{K}_0(X)$, in the unbounded formalism. The main issue is to show that $\bar\partial_{V,s} + \bar\partial^*_{V,s}$ has a compact resolvent. When V is trivial, this was shown in [Øvrelid and Ruppenthal 2014].

Let X be a reduced compact complex space of pure dimension n. For each $x \in X$, there is a neighborhood U of x with an embedding of U into some domain $U' \subset \mathbb{C}^N$, as the zero set of a finite number of holomorphic functions on U'.

Let $\mathcal{O}_X$ be the analytic structure sheaf of X. Let X_{sing} be the set of singular points of X and put $X_{\mathrm{reg}} = X - X_{\mathrm{sing}}$.

We equip X with a Hermitian metric g on X_{reg} which satisfies the property that for each $x \in X$, there are U and U' as above, along with a smooth Hermitian metric G on U', so that $g|_{X_{\mathrm{reg}} \cap U} = G|_{X_{\mathrm{reg}} \cap U}$.

Let V be a finite dimensional holomorphic vector bundle on X or, equivalently, a locally free sheaf $\underline{V}$ of $\mathcal{O}_X$-modules. For each $x \in X$, there are U and U' as above so that $V|_U$ is the restriction of a trivial holomorphic bundle $U' \times \mathbb{C}^N$ on U'. Let h be a Hermitian inner product on $V|_{X_{\mathrm{reg}}}$ which satisfies the property that for each $x \in X$, there are such U and U' so that $h|_{X_{\mathrm{reg}} \cap U}$ is the restriction of a smooth Hermitian metric on $U' \times \mathbb{C}^N$.

Let $\bar\partial_{V,s}$ be the minimal closed extension of the $\bar\partial_V$-operator on X_{reg}. That is, the domain of $\bar\partial_{V,s}$ is the set of $\omega \in \Omega^{0,*}_{L^2}(X_{\mathrm{reg}}; V)$ so that there are a sequence of compactly supported smooth forms $\omega_i \in \Omega^{0,*}(X_{\mathrm{reg}}; V)$ on X_{reg} and some $\eta \in \Omega^{0,*+1}_{L^2}(X_{\mathrm{reg}}; V)$ such that

$$\lim_{i\to\infty} \omega_i = \omega \quad \text{in } \Omega^{0,*}_{L^2}(X_{\mathrm{reg}}; V),$$
$$\lim_{i\to\infty} \bar\partial_{V,s}\omega_i = \eta \quad \text{in } \Omega^{0,*+1}_{L^2}(X_{\mathrm{reg}}; V).$$

We then put $\bar\partial_{V,s}\omega = \eta$, which is uniquely defined.

Hereafter we assume that X_{sing} is finite.

Proposition 2.1. *The spectral triple* $(C(X), \bar\partial_{V,s} + \bar\partial^*_{V,s}, \Omega^{0,*}_{L^2}(X_{\mathrm{reg}}; V))$ *gives a well-defined element of the analytic K-homology group* $\mathrm{K}_0(X)$.

Proof. Put $D_V = \bar\partial_{V,s} + \bar\partial^*_{V,s}$, with dense domain $\mathrm{Dom}(\bar\partial_{V,s}) \cap \mathrm{Dom}(\bar\partial^*_{V,s})$. Put $D = \bar\partial_s + \bar\partial^*_s$, the case when V is the trivial complex line bundle. Put

$$\mathcal{A} = \{f \in C(X) : f(\mathrm{Dom}(D_V)) \subset \mathrm{Dom}(D_V) \text{ and } [D_V, f] \text{ is bounded}\}. \tag{2.2}$$

Using the local trivializations of V, it follows that

$$\mathcal{A} = \{f \in C(X) : f(\mathrm{Dom}(D)) \subset \mathrm{Dom}(D) \text{ and } [D, f] \text{ is bounded}\}. \tag{2.3}$$

To satisfy the definitions of unbounded analytic K-homology [Baaj and Julg 1983; Forsyth et al. 2014; Kaad 2019], we first need to show that $\mathcal{A}$ is dense in $C(X)$.

Given $F \in C(X)$ and $\epsilon > 0$, we can construct $f \in C(X)$ such that

- for each $x_j \in X_{\mathrm{sing}}$, there is a neighborhood $U_j \subset X$ of x_j on which f is constant, with $f(x_j) = F(x_j)$;
- f is smooth on X_{reg};
- $\sup_{x \in X} |f(x) - F(x)| < \epsilon$.

Then $f(\mathrm{Dom}(D)) \subset \mathrm{Dom}(D)$ and $\|[D, f]\| \le \text{const.}\, \|\nabla_h f\|_\infty < \infty$. It follows that $\mathcal{A}$ is dense in $C(X)$.

To prove the proposition, it now suffices to prove the following lemma.

Lemma 2.4. *The operator $(D_V + i)^{-1}$ is compact.*

Proof. If V is trivial then the lemma is true [Øvrelid and Ruppenthal 2014]. We use a parametrix construction to prove it for general V.

Let us first prove the lemma for a special inner product h' on V. We write $X_{\mathrm{sing}} = \{x_j\}_{j=1}^r$. For each j, let U_j be a neighborhood of x_j on which V is trivialized as above, with $\overline{U_j} \cap \overline{U_k} = \varnothing$ for $j \neq k$. Choose open sets with smooth boundary $x_j \in Z_j \subset Y_j \subset W_j \subset U_j$, with $\overline{Z_j} \subset Y_j$, $\overline{Y_j} \subset W_j$ and $\overline{W_j} \subset U_j$. Let $\phi_j \in C(X)$ be identically 1 on Y_j, with support in W_j, and smooth on $U_j - Y_j$. Let $\eta_j \in C(X)$ be identically 1 on W_j, with support in U_j, and smooth on $U_j - Y_j$, so that η_j is 1 on the support of ϕ_j.

Define an inner product h' on V by first taking it to be a trivial inner product on each U_j, in terms of our given trivializations, and then extending it smoothly to the rest of X_{reg}. Let V_j be the extension of the trivialization $U_j \times \mathbb{C}^N$ to a product bundle on $X \times \mathbb{C}^N$ on X, as a smooth vector bundle with trivial inner product. Let $D_{V_j} = D \otimes I_N$ be the corresponding operator. As $(D+i)^{-1}$ is compact [Øvrelid and Ruppenthal 2014], the same is true for D_{V_j}. Let D_{APS} be the operator $\bar{\partial}_V + \bar{\partial}_V^*$ on $X - \bigcup_j Z_j$, with Atiyah–Patodi–Singer boundary conditions [Atiyah et al. 1973]. (The paper [Atiyah et al. 1973] assumes a product structure near the boundary, but this is not necessary.) Then $(D_{\mathrm{APS}} + i)^{-1}$ is compact. Put $\phi_0 = 1 - \sum_j \phi_j$, with support in $X - \bigcup_j \overline{Z_j}$. Pick $\eta_0 \in C(X)$ with support in $X - \bigcup_j \overline{Z_j}$, and smooth on X_{reg}, such that η_0 is one on the support of ϕ_0.

For $\omega \in \Omega^{0,*}_{L^2}(X_{\mathrm{reg}}; V)$, put

$$Q\omega = \eta_0 (D_{\mathrm{APS}} + i)^{-1}(\phi_0 \omega) + \sum_j \eta_j (D_{V_j} + i)^{-1}(\phi_j \omega). \tag{2.5}$$

Then Q is compact and

$$(D_V + i) Q\omega = \omega + [D, \eta_0](D_{\mathrm{APS}} + i)^{-1}(\phi_0 \omega) + \sum_j [D, \eta_j](D_{V_j} + i)^{-1}(\phi_j \omega), \tag{2.6}$$

so

$$(D_V+i)^{-1} = Q-(D_V+i)^{-1}\Big([D,\eta_0](D_{\mathrm{APS}}+i)^{-1}\phi_0+\sum_j [D,\eta_j](D_{V_j}+i)^{-1}\phi_j\Big). \quad (2.7)$$

As $[D, \eta_0]$, $[D, \eta_j]$ and $(D_V + i)^{-1}$ are bounded, it follows that $(D_V + i)^{-1}$ is compact.

As $(D_V + i)^{-1}$ (for the inner product h') is compact, the spectral theorem for compact operators and the functional calculus imply that $(I + D_V^2)^{-1}$ is compact. Writing $\triangle_{V,s} = D_V^2$, there is then a Hodge decomposition

$$\Omega^{0,*}_{L^2}(X_{\mathrm{reg}}; V) = \mathrm{Ker}(\triangle^{0,\star}_{V,s}) \oplus \mathrm{Im}(\bar\partial_{V,s}) \oplus \mathrm{Im}(\bar\partial^*_{V,s}), \quad (2.8)$$

where the right-hand side is a sum of orthogonal closed subspaces. In particular,

(1) $\mathrm{Im}(\bar\partial_{V,s})$ is closed,

(2) $\mathrm{Ker}(\bar\partial_{V,s})/\,\mathrm{Im}(\bar\partial_{V,s})$ is finite dimensional and

(3) the map $\bar\partial_{V,s} : \Omega^{0,*}_{L^2}(X_{\mathrm{reg}}; V)/\,\mathrm{Ker}(\bar\partial_{V,s}) \to \mathrm{Im}(\bar\partial_{V,s})$ is invertible and the inverse is compact, i.e., sends bounded sets to precompact sets.

(The inverse map $\mathrm{Im}(\bar\partial_{V,s}) \to \Omega^{0,*}_{L^2}(X_{\mathrm{reg}}; V)/\,\mathrm{Ker}(\bar\partial_{V,s}) \cong \mathrm{Im}(\bar\partial^*_{V,s})$ is DG, where G is the Green's operator for $\triangle_{V,s}$.) As the L^2-inner products on $\Omega^{0,*}_{L^2}(X_{\mathrm{reg}}; V)$ coming from h' and h are relatively bounded, the above three properties also hold for h. It follows that there is a Hodge decomposition relative to the inner product h, and $(I + D_V^2)^{-1}$ is compact. Hence $(D_V + i)^{-1}$ is compact.

This completes the proof of the lemma, and hence the proposition. □

3. Resolution

In this section we construct a certain resolution of the sheaf of holomorphic sections of a holomorphic vector bundle V on X. To begin, we define a sheaf $\underline{\mathrm{Dom}}(\bar\partial^{0,\star}_{V,s})$ on X, following [Ruppenthal 2018, Section 2.1].

Given an open set $U \subset X$ and a compact subset $K \subset U$, we write U_{reg} for $U \cap X_{\mathrm{reg}}$ and K_{reg} for $K \cap X_{\mathrm{reg}}$.

Let V be a finite dimensional holomorphic vector bundle on X equipped with a Hermitian metric, in the sense of Section 2. There is a sheaf $\underline{\Omega}^{0,\star}_{V,L^2_{\mathrm{loc}}}$ on X whose sections over an open set $U \subset X$ are the locally square integrable V-valued forms of degree $(0, \star)$ on U_{reg}, i.e., they are square integrable on K_{reg} for any compact set $K \subset U$. Convergence means L^2-convergence on each such K_{reg}. By definition, the sections of $\underline{\mathrm{Dom}}(\bar\partial^{0,\star}_{V,s})$ over U are the elements $\omega \in \Omega^{0,\star}_{L^2_{\mathrm{loc}}}(U_{\mathrm{reg}}; V)$ so that there are

- a sequence $f_i \in \Omega^{0,\star}_{C^\infty_c}(U_{\mathrm{reg}}; V)$ and
- some $\eta \in \Omega^{0,\star+1}_{L^2_{\mathrm{loc}}}(U_{\mathrm{reg}}; V)$

such that for any compact $K \subset U$, we have

- $\lim_{i\to\infty} f_i = \omega$ in $\Omega^{0,\star}_{L^2}(K_{\mathrm{reg}}; V)$ and
- $\lim_{i\to\infty} \bar\partial_V f_i = \eta$ in $\Omega^{0,\star+1}_{L^2}(K_{\mathrm{reg}}; V)$.

Then we put $\bar\partial_V \omega = \eta$.

This gives a complex of fine sheaves

$$\dots \xrightarrow{\bar\partial_V} \underline{\mathrm{Dom}}(\bar\partial^{0,\star-1}_{V,s}) \xrightarrow{\bar\partial_V} \underline{\mathrm{Dom}}(\bar\partial^{0,\star}_{V,s}) \xrightarrow{\bar\partial_V} \underline{\mathrm{Dom}}(\bar\partial^{0,\star+1}_{V,s}) \xrightarrow{\bar\partial_V} \cdots . \tag{3.1}$$

The cohomology of the complex is the sheaf $\underline{\mathrm{H}}^{0,\star}(\bar\partial_{V,s})$. For $\star > 0$, it is a direct sum of skyscraper sheaves, with support in X_{sing}. We write $\underline{V}_s$ for $\underline{\mathrm{H}}^{0,0}(\bar\partial_{V,s})$, i.e., the kernel of $\bar\partial_V$ acting on $\underline{\mathrm{Dom}}(\bar\partial^{0,0}_{V,s})$. Then $\underline{V}_s/\underline{V}$ is also a direct sum of skyscraper sheaves with support in X_{sing}.

Although we do not need it here, there is a description of these skyscraper sheaves in terms of a resolution of X. Suppose that $\pi : M \to X$ is a resolution. From [Ruppenthal 2018, Corollary 1.2], if $x \in X$ then we can identify the stalk $(\underline{\mathrm{H}}^{0,q}(\bar\partial_{V,s}))_x$ with $V_x \otimes (R^q\pi_*\mathcal{O}_M)_x$. In particular, we can identify $\underline{V}_s$ with $\underline{V} \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}_M$ or, more intrinsically, with the sheaf of weakly holomorphic sections of V, i.e., bounded holomorphic sections of $V|_{X_{\mathrm{reg}}}$.

There is a quotient morphism of sheaves:

$$q : \underline{\mathrm{Ker}}(\bar\partial^{0,\star}_{V,s}) \to \underline{\mathrm{H}}^{0,\star}(\bar\partial_{V,s}).$$

As $\underline{\mathrm{H}}^{0,\star}(\bar\partial_{V,s})$ is an injective sheaf for $\star > 0$, we can extend q to a morphism $\alpha : \underline{\mathrm{Dom}}(\bar\partial^{0,\star}_{V,s}) \to \underline{\mathrm{H}}^{0,\star}(\bar\partial_{V,s})$. More specifically, if x is a singular point then the stalk $(\underline{\mathrm{H}}^{0,\star}(\bar\partial_{V,s}))_x$ is a finite dimensional complex vector space, so we are extending the quotient map $q_x : (\underline{\mathrm{Ker}}(\bar\partial^{0,\star}_{V,s}))_x \to (\underline{\mathrm{H}}^{0,\star}(\bar\partial_{V,s}))_x$ from the germs of $\bar\partial_V$-closed V-valued forms at x, to the germs of forms in the domain of $\bar\partial_{V,s}$.

Considering $\underline{\mathrm{H}}^{0,\star}(\bar\partial_{V,s})$ to be a complex of sheaves with zero differential, α is a morphism of complexes that is an isomorphism on cohomology in degree $\star > 0$ by construction. Let $\underline{\mathrm{cone}}(\alpha_V)$ be the mapping cone of α_V, with $\underline{\mathrm{cone}}^{0,\star}(\alpha_V) = \underline{\mathrm{Dom}}(\bar\partial^{0,\star}_{V,s}) \oplus \underline{\mathrm{H}}^{0,\star-1}(\bar\partial_{V,s})$ and differential $d_{\mathrm{cone}}(\omega, h) = (\bar\partial_V\omega, \alpha_V(\omega))$. It has vanishing cohomology in degree $\star > 1$. Define a complex of sheaves $\underline{\mathcal{C}}^{0,\star}_V$ by

$$\underline{\mathcal{C}}^{0,\star}_V = \begin{cases} \underline{\mathrm{Dom}}(\bar\partial^{0,0}_{V,s}), & \star = 0, \\ \underline{\mathrm{Dom}}(\bar\partial^{0,1}_{V,s}), & \star = 1, \\ \underline{\mathrm{Dom}}(\bar\partial^{0,\star}_{V,s}) \oplus \underline{\mathrm{H}}^{0,\star-1}(\bar\partial_{V,s}), & \star > 1, \end{cases} \tag{3.2}$$

where the differential in degree $\star = 0$ is $\bar\partial_V$, the differential in degree $\star = 1$ is $(\bar\partial_V, \alpha_V)$, and the differential in degrees $\star > 1$ is d_{cone}. Then $\underline{\mathcal{C}}_V$ is a resolution of $\underline{V}_s$ by fine sheaves.

There is a short exact sequence of sheaves

$$0 \to \underline{V} \to \underline{V}_s \to \underline{V}_s/\underline{V} \to 0. \tag{3.3}$$

We can think of $\underline{V}_s/\underline{V}$ as a resolution of itself, when concentrated in degree zero. Together with the resolution of $\underline{V}_s$ from (3.2), we can construct a resolution of $\underline{V}$ as follows. As $\underline{V}_s/\underline{V}$ is a finite sum of skyscraper sheaves, we can extend the quotient map $\underline{V}_s \to \underline{V}_s/\underline{V}$ to a morphism $\beta_V : \underline{\mathrm{Dom}}(\bar{\partial}^{0,0}_{V,s}) \to \underline{V}_s/\underline{V}$. Define a complex of sheaves $\widetilde{\underline{\mathcal{C}}}_V$ by

$$\widetilde{\underline{\mathcal{C}}}^{0,\star}_V = \begin{cases} \underline{\mathrm{Dom}}(\bar{\partial}^{0,0}_{V,s}), & \star = 0, \\ \underline{\mathrm{Dom}}(\bar{\partial}^{0,1}_{V,s}) \oplus \underline{V}_s/\underline{V}, & \star = 1, \\ \underline{\mathrm{Dom}}(\bar{\partial}^{0,\star}_{V,s}) \oplus \underline{\mathrm{H}}^{0,\star-1}(\bar{\partial}_{V,s}), & \star > 1, \end{cases} \tag{3.4}$$

where the differential in degree $\star = 0$ is $(\bar{\partial}_V, \beta_V)$, the differential in degree $\star = 1$ sends (ω, v) to $(\bar{\partial}_V \omega, \alpha_V(\omega))$, and the differential in degrees $\star > 1$ is d_{cone}. Then $\widetilde{\underline{\mathcal{C}}}_V$ is a resolution of $\underline{V}$ by fine sheaves; see [Iversen 1986, proof of Proposition I.6.10].

Taking global sections of $\widetilde{\underline{\mathcal{C}}}^{0,\star}_V$ gives a cochain complex $(\widetilde{T}_V, \tilde{d}_V)$:

$$\begin{aligned} 0 \to \mathrm{Dom}(\bar{\partial}^{0,0}_{V,s}) \to \mathrm{Dom}(\bar{\partial}^{0,1}_{V,s}) \oplus (\underline{V}_s/\underline{V})(X) \\ \to \mathrm{Dom}(\bar{\partial}^{0,2}_{V,s}) \oplus (\underline{\mathrm{H}}^{0,1}(\bar{\partial}_{V,s}))(X) \to \cdots \\ \to \mathrm{Dom}(\bar{\partial}^{0,n}_{V,s}) \oplus (\underline{\mathrm{H}}^{0,n-1}(\bar{\partial}_{V,s}))(X) \to 0. \end{aligned} \tag{3.5}$$

For the last term, we use the fact that in terms of a resolution $\pi : M \to X$, we have $(\underline{\mathrm{H}}^{0,n}(\bar{\partial}_{V,s}))_x = V_x \otimes (R^n \pi_* \mathcal{O}_M)_x = 0$.

Proposition 3.6. *The cohomology of* $(\widetilde{T}_V, \tilde{d}_V)$ *is isomorphic to* $\mathrm{H}^*(X; \underline{V})$.

Proof. This holds because $\widetilde{\underline{\mathcal{C}}}_V$ is a resolution of $\underline{V}$ by fine sheaves. □

Put arbitrary inner products on the finite dimensional vector spaces $(\underline{V}_s/\underline{V})(X)$ and $(\underline{\mathrm{H}}^{0,*}(\bar{\partial}_{V,s}))(X)$.

4. Hilbert complex

The differential $\tilde{d}_V$ in the Hilbert complex $(\widetilde{T}_V, \tilde{d}_V)$ of the previous section involved somewhat arbitrary choices of α_V and β_V. In this section we replace $(\widetilde{T}_V, \tilde{d}_V)$ by a more canonical Hilbert complex (T_V, d_V).

For brevity of notation, we put

$$A^*_V = \begin{cases} (\underline{V}_s/\underline{V})(X), & \star = 0, \\ (\underline{\mathrm{H}}^{0,\star}(\bar{\partial}_{V,s}))(X), & \star > 0. \end{cases} \tag{4.1}$$

Then the complex $\widetilde{T}_V$ has entries $\widetilde{T}_V^{0,\star} = \mathrm{Dom}(\bar{\partial}_{V,s}^{0,\star}) \oplus A_V^{\star-1}$. Combining α_V and β_V, we have constructed a linear map $\gamma_V : \mathrm{Dom}(\bar{\partial}_{V,s}^{0,\star}) \to A_V^{\star}$ so that the differential of $\widetilde{T}_V$ is given by

$$\tilde{d}_V(\omega, a) = (\partial_V \omega, \gamma_V(\omega)). \tag{4.2}$$

Note that $\gamma_V \circ \bar{\partial}_{V,s} = 0$.

Let $P_{\mathrm{Ker}(\triangle_{V,s}^{0,\star})}$ be orthogonal projection onto $\mathrm{Ker}(\triangle_{V,s}^{0,\star}) \subset \Omega_{L^2}^{0,\star}(X_{\mathrm{reg}}; V)$. Define a new differential d_V on $\widetilde{T}_V$ by

$$d_V(\omega, a) = (\partial_V \omega, \gamma_V(P_{\mathrm{Ker}(\triangle_{V,s}^{0,\star})}\omega)). \tag{4.3}$$

Call the resulting cochain complex (T_V, d_V).

As in (2.8), there is a Hodge decomposition

$$\mathrm{Dom}(\bar{\partial}_{V,s}^{0,\star}) = \mathrm{Ker}(\triangle_{V,s}^{0,\star}) \oplus \mathrm{Im}(\bar{\partial}_{V,s}) \oplus \mathrm{Im}(\bar{\partial}_{V,s}^{*}). \tag{4.4}$$

Here the terms on the right-hand side of (4.4) are the intersections of $\mathrm{Dom}(\bar{\partial}_{V,s}^{0,\star})$ with the corresponding terms in (2.8). In particular, $\mathrm{Ker}(\triangle_{V,s}^{0,\star})$ and $\mathrm{Im}(\bar{\partial}_{V,s})$ are the same, while the elements of $\mathrm{Im}(\bar{\partial}_{V,s}^{*})$ now lie in an H^1-space. Put

$$\mathcal{I}_V = \bar{\partial}_{V,s}|_{\mathrm{Im}(\bar{\partial}_{V,s}^{*})} : \mathrm{Im}(\bar{\partial}_{V,s}^{*}) \to \mathrm{Im}(\bar{\partial}_{V,s}), \tag{4.5}$$

an isomorphism.

Define a linear map $m_V : \mathrm{Dom}(\bar{\partial}_{V,s}^{0,\star}) \oplus A_V^{\star-1} \to \mathrm{Dom}(\bar{\partial}_{V,s}^{0,\star}) \oplus A_V^{\star-1}$ by saying that if

$$(h, \omega_1, \omega_2, a) \in \mathrm{Ker}(\triangle_{V,s}^{0,\star}) \oplus \mathrm{Im}(\bar{\partial}_{V,s}) \oplus \mathrm{Im}(\bar{\partial}_{V,s}^{*}) \oplus A_V^{\star-1}, \tag{4.6}$$

then

$$m_V(h, \omega_1, \omega_2, a) = (h, \omega_1, \omega_2, a + \gamma_V(\mathcal{I}^{-1}(\omega_1))). \tag{4.7}$$

Its inverse is given by

$$m_V^{-1}(h, \omega_1, \omega_2, a) = (h, \omega_1, \omega_2, a - \gamma_V(\mathcal{I}^{-1}(\omega_1))). \tag{4.8}$$

Proposition 4.9. *The linear maps m_V and m_V^{-1} are chain maps between (T_V, d_V) and $(T_V, \tilde{d}_V)$, i.e., $m_V \circ d_V = \tilde{d}_V \circ m_V$ and $m_V^{-1} \circ \tilde{d}_V = d_V \circ m_V^{-1}$.*

Proof. We check that $m_V \circ d_V = \tilde{d}_V \circ m_V$; the proof that $m_V^{-1} \circ \tilde{d}_V = d_V \circ m_V^{-1}$ is similar. Given $(h, \omega_1, \omega_2, a)$ as in (4.6), we have

$$\begin{aligned} d_V(h, \omega_1, \omega_2, a) &= (0, \bar{\partial}_V \omega_2, 0, \gamma_V(h)), \\ m_V(d_V(h, \omega_1, \omega_2, a)) &= (0, \bar{\partial}_V \omega_2, 0, \gamma_V(h) + \gamma_V(\omega_2)), \\ m_V(h, \omega_1, \omega_2, a) &= (h, \omega_1, \omega_2, a + \gamma_V(\mathcal{I}^{-1}(\omega_1))), \\ \tilde{d}_V(m_V(h, \omega_1, \omega_2, a)) &= (0, \bar{\partial}_V \omega_2, 0, \gamma_V(h) + \gamma_V(\omega_2)). \end{aligned} \tag{4.10}$$

This proves the proposition. □

Theorem 4.11. *The cohomology of (T_V, d_V) is isomorphic to $\mathrm{H}^*(X; \underline{V})$.*

Proof. This follows from Propositions 3.6 and 4.9. □

We can now reprove a result from [Fulton 1998, Example 18.3.3 on p. 362].

Proposition 4.12. *In terms of a resolution $\pi : M \to X$, we have*

$$\sum_{i=0}^{n}(-1)^i \dim(\mathrm{H}^i(X; \mathcal{O}_X)) = \int_M \mathrm{Td}(TM) - \dim((\pi_*\mathcal{O}_M/\mathcal{O}_X)(X)) + \sum_{i=1}^{n}(-1)^{i-1}\dim((R^i\pi_*\mathcal{O}_M)(X)). \quad (4.13)$$

Proof. Let (T_1, d_1) denote the complex (T_V, d_V) when the vector bundle V is the trivial bundle. From Theorem 4.11, the left-hand side of (4.13) is the index of $d_1 + d_1^*$. We can deform the chain complex (T_1, d_1) to make the differential equal to $\bar{\partial}_s \oplus 0$ without changing the index. The new index is

$$\sum_{i=0}^{n}(-1)^i \dim(\mathrm{H}^i(\bar{\partial}_s)) - \dim((\mathcal{O}_s/\mathcal{O}_X)(X)) + \sum_{i=1}^{n-1}(-1)^{i-1}\dim((\underline{\mathrm{H}}^{0,i}(\bar{\partial}_s))(X)). \quad (4.14)$$

From [Pardon and Stern 1991], we have $\mathrm{H}^i(\bar{\partial}_s) \cong \mathrm{H}^{0,i}(M)$, so

$$\sum_{i=0}^{n}(-1)^i \dim(\mathrm{H}^i(\bar{\partial}_s)) = \sum_{i=0}^{n}(-1)^i \dim(\mathrm{H}^{0,i}(M)) = \int_M \mathrm{Td}(TM). \quad (4.15)$$

From [Ruppenthal 2018, Corollary 1.2], $\mathcal{O}_s \cong \pi_*\mathcal{O}_M$ and $\underline{\mathrm{H}}^{0,i}(\bar{\partial}_s) \cong R^i\pi_*\mathcal{O}_M$. The proposition follows. □

Remark 4.16. We can write $\int_M \mathrm{Td}(TM) = \int_X \pi_* \mathrm{Td}(TM)$, where we are integrating a top-degree form on X_{reg}. It is not so clear what the relevant theory of characteristic classes on X should be, for which this would be an example. We have in mind a Chern–Weil theory on X_{reg} with control on how the forms behave near X_{sing}. We note that there is a rational homology class $\pi_*(PD[\mathrm{Td}(TM)])$ on X, where $PD[\mathrm{Td}(TM)] \in \mathrm{H}_{\mathrm{even}}(M; \mathbb{Q})$ is the Poincaré dual of $[\mathrm{Td}(TM)] \in \mathrm{H}^{\mathrm{even}}(M; \mathbb{Q})$, and if X is connected then $\int_M \mathrm{Td}(TM)$ can be identified with the degree-zero component of $\pi_*(PD[\mathrm{Td}(TM)])$.

5. K-homology class

In this section we prove Theorem 1.5. We first show that if $\pi : M \to X$ is a resolution of singularities, with a simple normal crossing divisor, then the K-homology class $[\bar{\partial}_s + \bar{\partial}_s^*] \in \mathrm{K}_0(X)$ from Proposition 2.1, with V trivial, equals the pushforward $\pi_*[\bar{\partial}_M + \bar{\partial}_M^*]$. We then prove Theorem 1.5.

Proposition 5.1. *Let $\pi : M \to X$ be a resolution of singularities, with $\pi^{-1}(X_{\mathrm{sing}})$ being a simple normal crossing divisor. Then $[\bar{\partial}_s + \bar{\partial}_s^*] = \pi_*[\bar{\partial}_M + \bar{\partial}_M^*]$.*

Proof. The method of proof comes from [Haskell 1987]. Consider the following part of the K-homology exact sequence for the pair (X, X_{sing}):

$$\mathrm{K}_0(X_{\text{sing}}) \xrightarrow{\alpha} \mathrm{K}_0(X) \xrightarrow{\beta} \mathrm{K}_0(X, X_{\text{sing}}). \tag{5.2}$$

Lemma 5.3. *We have* $\beta([\bar\partial_s + \bar\partial_s^*]) = \beta(\pi_*[\bar\partial_M + \bar\partial_M^*])$ *in* $\mathrm{K}_0(X, X_{\text{sing}})$.

Proof. Put $D = \pi^{-1}(X_{\text{sing}}) \subset M$. Since it has simple normal crossings, there is a small regular neighborhood of D whose closure C' is homotopy equivalent to D. We can also assume that $C = \pi(C')$ is homotopy equivalent to X_{sing} [Milnor 1968, Theorem 2.10]. As $[\bar\partial_M + \bar\partial_M^*]$ is independent of the choice of Hermitian metric on M, we can choose a Hermitian metric on M so that π restricts to an isometry from $M - C'$ to $X - C$.

Consider the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{K}_0(M) & \longrightarrow & \mathrm{K}_0(M, D) & \xrightarrow{\cong} & \mathrm{K}_0(M, C') & \xrightarrow{\cong} & \mathrm{KK}(C_0(M - C'); \mathbb{C}) \\
\downarrow{\scriptstyle \pi_*} & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{K}_0(X) & \xrightarrow{\beta} & \mathrm{K}_0(X, X_{\text{sing}}) & \xrightarrow{\cong} & \mathrm{K}_0(X, C) & \xrightarrow{\cong} & \mathrm{KK}(C_0(X - C); \mathbb{C})
\end{array} \tag{5.4}$$

Starting with $[\bar\partial_M + \bar\partial_M^*] \in \mathrm{K}_0(M)$ and going along the top row, its image in $\mathrm{KK}(C_0(M - C'); \mathbb{C})$ is the restriction of the analytic K-homology class, i.e., one only acts by functions that vanish on C'. The right vertical arrow of the diagram is an isomorphism coming from the bijection between $M - C'$ and $X - C$. By the commutativity of the diagram, we now know what $\beta(\pi_*[\bar\partial_M + \bar\partial_M^*])$ is as an element of $\mathrm{KK}(C_0(X - C); \mathbb{C})$. However, this is isomorphic to the restriction of $[\bar\partial_s + \bar\partial_s^*] \in \mathrm{K}_0(X)$ to an element of $\mathrm{KK}(C_0(X - C); \mathbb{C})$ (since π gives an isometry between $M - C'$ and $X - C$). The latter restriction is the same as $\beta([\bar\partial_s + \bar\partial_s^*])$. This proves the lemma. □

Returning to the proof of Proposition 5.1, we know now that $[\bar\partial_s + \bar\partial_s^*] - \pi_*[\bar\partial_M + \bar\partial_M^*]$ lies in the kernel of β, and so lies in the image of α. For the purpose of the proof, we can assume that X is connected. Let $a : \mathrm{pt} \to X$ be an arbitrary fixed embedding and let $a_* : \mathrm{K}_0(\mathrm{pt}) \to \mathrm{K}_0(X)$ be the induced homomorphism. The connectedness of X implies that $\mathrm{Im}(\alpha) = \mathrm{Im}(a_*)$. Let $b : X \to \mathrm{pt}$ be the unique point map. Consider $\mathrm{pt} \xrightarrow{a} X \xrightarrow{b} \mathrm{pt}$ and the induced homomorphisms $\mathrm{K}_0(\mathrm{pt}) \xrightarrow{a_*} \mathrm{K}_0(X) \xrightarrow{b_*} \mathrm{K}_0(\mathrm{pt})$. Then the map b_* restricts to an isomorphism between $\mathrm{Im}(a_*)$ and $\mathrm{K}_0(\mathrm{pt})$. Hence, to prove the proposition, it suffices to show that $b_*[\bar\partial_s + \bar\partial_s^*] = b_*(\pi_*[\bar\partial_M + \bar\partial_M^*])$ in $\mathrm{K}_0(\mathrm{pt}) \cong \mathbb{Z}$.

Now $b_*[\bar\partial_s + \bar\partial_s^*]$ is the index of $\bar\partial_s + \bar\partial_s^*$, i.e., $\sum_{i=0}^n (-1)^i \dim(\mathrm{H}^i(\bar\partial_s))$, while $b_*(\pi_*[\bar\partial_M + \bar\partial_M^*])$ is the index of $\bar\partial_M + \bar\partial_M^*$, i.e., $\sum_{i=0}^n (-1)^i \dim(\mathrm{H}^i(\bar\partial_M))$. From [Pardon and Stern 1991], these are equal term-by-term. This proves the proposition. □

Proof of Theorem 1.5. Suppose that X is a connected projective algebraic variety. In terms of the resolution $\pi : M \to X$, it was pointed out in [Baum et al. 1975, p. 104] that there is an identity in $\mathrm{K}_0(X)$:

$$[\mathcal{O}_X]_{\mathrm{BFM}} - \pi_*[\mathcal{O}_M]_{\mathrm{BFM}} = \sum_j n_j [\mathcal{O}_{V_j}]_{\mathrm{BFM}}. \tag{5.5}$$

Here the n_j are certain integers and the V_j are irreducible subvarieties of the singular locus of X. In our case of isolated singularities, the V_j are just the points x_j in X_{sing}. As $[\mathcal{O}_M]_{\mathrm{BFM}} = [\bar{\partial}_M + \bar{\partial}_M^*]$, Proposition 5.1 implies that

$$[\mathcal{O}_X]_{\mathrm{BFM}} = [\bar{\partial}_s + \bar{\partial}_s^*] + \sum_j n_j [\mathcal{O}_{V_j}]_{\mathrm{BFM}}. \tag{5.6}$$

Let (T_1, d_1) denote the complex (T_V, d_V) when the vector bundle V is the trivial bundle. Let $[\mathcal{O}_X]_{\mathrm{an}} \in \mathrm{K}_0(X)$ be the K-homology class coming from the operator $d_1 + d_1^*$. We can deform the chain complex (T_1, d_1) to make the differential equal to $\bar{\partial}_s \oplus 0$ without changing the K-homology class arising from the complex. Then (5.6) implies that $[\mathcal{O}_X]_{\mathrm{an}}$ and $[\mathcal{O}_X]_{\mathrm{BFM}}$ have the same image in $\mathrm{K}_0(X, X_{\mathrm{sing}})$; cf. the proof of Lemma 5.3. Let $b : X \to \mathrm{pt}$ be the unique point map. As in the proof of Proposition 5.1, to conclude that $[\mathcal{O}_X]_{\mathrm{an}} = [\mathcal{O}_X]_{\mathrm{BFM}}$ in $\mathrm{K}_0(X)$, it now suffices to show that $b_*[\mathcal{O}_X]_{\mathrm{an}} = b_*[\mathcal{O}_X]_{\mathrm{BFM}}$ in $\mathrm{K}_0(\mathrm{pt}) \cong \mathbb{Z}$. Now $b_*[\mathcal{O}_X]_{\mathrm{an}}$ is the index of $d_1 + d_1^*$ which, from Theorem 4.11, equals the arithmetic genus $\sum_{i=0}^n (-1)^i \dim(\mathrm{H}^i(X; \mathcal{O}_X))$. On the other hand, from [Baum et al. 1979, Section 3], we also have $b_*[\mathcal{O}_X]_{\mathrm{BFM}} = \sum_{i=0}^n (-1)^i \dim(\mathrm{H}^i(X; \mathcal{O}_X))$. This proves the theorem. □

Remark 5.7. We mention some of the issues involved in extending the present paper to nonisolated singularities. First, it seems to be open whether $\bar{\partial}_s + \bar{\partial}_s^*$ has compact resolvent, so the unbounded KK-formalism may not be applicable. However, it is known that the unreduced cohomology of the $\bar{\partial}_s$-complex is finite dimensional, being isomorphic to the cohomology of a resolution [Pardon and Stern 1991]. Hence the $\bar{\partial}_s$-complex is Fredholm and one could use the bounded KK-description of K-homology, although it would be more cumbersome.

We expect that Proposition 5.1 still holds if X has nonisolated singularities. It is known that taking resolutions $\pi : M \to X$, the pushforward $\pi_*[\bar{\partial}_M + \bar{\partial}_M^*] \in \mathrm{K}_0(X)$ is independent of the choice of resolution [Hilsum 2018].

One could ask for an extension of Theorem 4.11 to the case of nonisolated singularities. As an indication, one would expect that taking products of complex spaces would lead to tensor products of the cochain complexes. In particular, suppose that Z is a smooth Hermitian manifold and X has isolated singular points. Then the cochain complex for $Z \times X$ would have contributions from differential forms along the singular locus.

In a related vein, in principle one can apply (5.5) inductively to get an expression for $[\mathcal{O}_X]_{\mathrm{BFM}}$.

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