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# Hausdorffified algebraic $K_1$-groups and invariants for $C^*$-algebras with the ideal property

Guihua Gong, Chunlan Jiang and Liangqing Li

*Dedicated to the memory of Professor Ronald G. Douglas*

A $C^*$-algebra $A$ is said to have the ideal property if each closed two-sided ideal of $A$ is generated as a closed two-sided ideal by the projections inside the ideal. $C^*$-algebras with the ideal property are a generalization and unification of real rank zero $C^*$-algebras and unital simple $C^*$-algebras. It was long expected that an invariant that we call $\mathrm{Inv}^0(A)$, consisting of the scaled ordered total $K$-group $(\underline{K}(A); \underline{K}(A)^+; \Sigma A)_\Lambda$ (used in the real rank zero case), along with the tracial state spaces $T(pAp)$ for each cut-down algebra $pAp$, as part of the Elliott invariant of $pAp$ (for each $[p] \in \Sigma A$), with certain compatibility conditions, is the complete invariant for a certain well behaved class of $C^*$-algebras with the ideal property (e.g., $AH$ algebras with no dimension growth). In this paper, we construct two nonisomorphic $A\mathbb{T}$ algebras $A$ and $B$ with the ideal property such that $\mathrm{Inv}^0(A) \cong \mathrm{Inv}^0(B)$, disproving this conjecture. The invariant to distinguish the two algebras is the collection of Hausdorffified algebraic $K_1$-groups $U(pAp)/\overline{DU(pAp)}$ (for each $[p] \in \Sigma A$), along with certain compatibility conditions. We will prove in a separate article that, after adding this new ingredient, the invariant becomes the complete invariant for $AH$ algebras (of no dimension growth) with the ideal property.

## 1. Introduction

A $C^*$-algebra $A$ is called an $AH$ algebra [Blackadar 1993] if it is the inductive limit $C^*$-algebra of

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \longrightarrow \cdots \longrightarrow A_n \longrightarrow \cdots$$

with $A = \lim_{n\to\infty}\big(A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m}\big)$, where $X_{n,i}$ are compact metric spaces, $t_n$ and $[n, i]$ are positive integers, and $P_{n,i} \in M_{[n,i]}(C(X_{n,i}))$ are projections. An $AH$ algebra is called of no dimension growth, if one can choose the spaces $X_{n,i}$ such that $\sup_{n,i} \dim(X_{n,i}) < +\infty$. If all the spaces $X_{n,i}$ can be

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chosen to be the single point space {pt}, then $A$ is called an *AF* algebra. If all the spaces can be chosen to be the interval [0, 1] or circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, then $A$ is called an *AI* algebra or $A\mathbb{T}$ algebra, respectively.

G. Elliott [1993b] initiated the classification program by classifying all real rank zero $A\mathbb{T}$ algebras (without the condition of simplicity), and he conjectured that the scaled ordered $K_*$-group $(K_*(A); K_*(A)^+; \Sigma A)$, where $K_*(A) = K_0(A) \oplus K_1(A)$, is a complete invariant for separable nuclear $C^*$-algebras of real rank zero and stable rank one. Elliott [1993a] also successfully classified all unital simple *AI* algebras by the so called Elliott invariant

$$\mathrm{Ell}(A) = (K_0(A); K_0(A)^+; \Sigma A, K_1(A); T(A); \rho_A),$$

where $T(A)$ is the space of all unital traces on $A$, and $\rho_A$ is the nature map from $K_0(A)$ to $\mathrm{AffT}(A)$ (the ordered Banach space of all affine maps from $T(A)$ to $\mathbb{R}$).

Later, Gong [1998] constructed two nonisomorphic (not simple) real rank zero *AH* algebras (with 2-dimensional local spectra) $A$ and $B$ such that

$$(K_*(A); K_*(A)^+; \Sigma A) \cong (K_*(B); K_*(B)^+; \Sigma B),$$

which disproved the conjecture of Elliott for $C^*$-algebras of real rank zero and stable rank one. This result led to a sequence of research by Dadarlat and Loring [1996a; 1996b] and Eilers [1996] culminating with Dadarlat and Gong's [1997] complete classification of real rank zero *AH* algebras by scaled ordered total $K$-theory $(\underline{K}(A); \underline{K}(A)^+; \Sigma A)_\Lambda$, where $\underline{K}(A) = K_*(A) \oplus \bigoplus_{p=2}^{\infty} K_*(A, \mathbb{Z}/p\mathbb{Z})$ and $\Lambda$ is the system of Bockstein operations; see also [Dadarlat 1995a; 1995b; Elliott and Gong 1996a; 1996b; Elliott et al. 1996; 1998; Gong 1997; 1998; Gong and Lin 2000; Lin 1996; 2001]. Elliott, Gong, and Li [Elliott et al. 2007] completely classified simple *AH* algebras of no dimension growth by Elliott invariant; see also [Elliott 1997; Elliott et al. 2005; 1997; Gong 2002; Li 1997; 1999; Lin 2007; Nielsen and Thomsen 1996; Thomsen 1994; 1997]. A natural generalization and unification of real rank zero $C^*$-algebras and unital simple $C^*$-algebras is the class of $C^*$-algebras with the ideal property: each closed two-sided ideal of the $C^*$-algebra is generated as a closed two-sided ideal by the projections inside the ideal. It was long expected that a combination of scaled ordered total $K$-theory (used in the classification of real rank zero $C^*$-algebras) and the Elliott invariant (used in the classification of simple $C^*$-algebras), including tracial state spaces $T(pAp)$ — part of the Elliott invariant of cut-down algebras $\{pAp\}_{[p]\in\Sigma A}$ with certain compatibility conditions, called $\mathrm{Inv}^0(A)$ (see [Jiang 2011, 2.18]), is a complete invariant for certain well behaved $C^*$-algebras (e.g., *AH* algebras of no dimension growth or $\mathcal{Z}$-stable $C^*$-algebras, where $\mathcal{Z}$ is the Jiang–Su algebra of [Jiang and Su 1999]) with the ideal property; see [Stevens 1998; Pasnicu 2000; Ji and Jiang 2011; Jiang and Wang 2012; Jiang 2011].

The main purpose of this paper is to construct two unital $\mathcal{Z}$-stable $A\mathbb{T}$ algebras $A$ and $B$ with the ideal property such that $\mathrm{Inv}^0(A) \cong \mathrm{Inv}^0(B)$, but $A \ncong B$. The invariant to distinguish these two $C^*$-algebras is the Hausdorffified algebraic $K_1$-groups $U(pAp)/\overline{DU(pAp)}$ of the cut-down algebra $pAp$ (for each element $x \in \Sigma A$, we chose one projection $p \in A$ such that $[p] = x$) along with a certain compatibility condition, where $DU(A)$ is the group generated by commutators $\{uvu^*v^* : u, v \in U(A)\}$. In this paper, we introduce the invariant $\mathrm{Inv}'(A)$ and its simplified version $\mathrm{Inv}(A)$, by adding these new ingredients — the Hausdorffified algebraic $K_1$-groups of cut-down algebras along with certain compatibility conditions — to $\mathrm{Inv}^0(A)$.

In [Gong et al. 2016], we will prove that $\mathrm{Inv}(A)$ is a complete invariant for $AH$ algebras (of no dimension growth) with the ideal property.

Note that for the above $C^*$-algebras $A$ and $B$, we have that $Cu(A) \cong Cu(B)$ and $Cu(A \otimes C(S^1)) \cong Cu(B \otimes C(S^1))$. That is, the new invariant can not be detected by the Cuntz semigroup.

In Section 2, we define $\mathrm{Inv}(A)$ and discuss its properties. These properties will be used in [Gong et al. 2016]. In Section 3, we present the construction of $A\mathbb{T}$ algebras $A$ and $B$ with the ideal property such that $\mathrm{Inv}(A) \ncong \mathrm{Inv}(B)$ but $\mathrm{Inv}^0(A) \cong \mathrm{Inv}^0(B)$.

## 2. The invariant

In this section, we recall the definition of $\mathrm{Inv}^0(A)$ from [Jiang 2011] (also see [Stevens 1998; Ji and Jiang 2011; Jiang and Wang 2012]), and then introduce the invariant $\mathrm{Inv}(A)$. Furthermore, we discuss the properties of $\mathrm{Inv}(A)$ in the context of $AH$ algebras and $A\mathcal{HD}$ algebras (for the definition of $A\mathcal{HD}$ algebras, see 2.3 below), which will be used in [Gong et al. 2016].

**2.1.** In the notation for an inductive limit system $\lim(A_n, \phi_{n,m})$, we understand that

$$\phi_{n,m} = \phi_{m-1,m} \circ \phi_{m-2,m-1} \circ \cdots \circ \phi_{n,n+1},$$

where all $\phi_{n,m} : A_n \to A_m$ are homomorphisms.

We assume that, for any summand $A_n^i$ in the direct sum $A_n = \bigoplus_{i=1}^{t_n} A_n^i$, necessarily $\phi_{n,n+1}(\mathbb{1}_{A_n^i}) \neq 0$, since otherwise, we could simply delete $A_n^i$ from $A_n$ without changing the limit algebra.

If $A_n = \bigoplus_i A_n^i$, $A_m = \bigoplus_j A_m^j$, we use $\phi_{n,m}^{i,j}$ to denote the partial map of $\phi_{n,m}$ from the $i$-th block $A_n^i$ of $A_n$ to the $j$-th block $A_m^j$ of $A_m$. Also, we use $\phi_{n,m}^{-,j}$ to denote the partial map of $\phi_{n,m}$ from $A_n$ to $A_m^j$. That is, $\phi_{n,m}^{-,j} = \bigoplus_i \phi_{n,m}^{i,j} = \pi_j \phi_{n,m}$, where $\pi_j : A_m \to A_m^j$ is the canonical projection. Sometimes, we also use $\phi_{n,m}^{i,-}$ to denote $\phi_{n,m}|_{A_n^i} : A_n^i \to A_m$.

**2.2.** As in [Elliott and Gong 1996b], let $T_{\mathrm{II},k}$ be the 2-dimensional connected simplicial complex with $H^1(T_{\mathrm{II},k}) = 0$ and $H^2(T_{\mathrm{II},k}) = \mathbb{Z}/k\mathbb{Z}$, and let $I_k$ be the subalgebra of $M_k(C[0, 1]) = C([0, 1], M_k(\mathbb{C}))$ consisting of all functions $f$ with the properties $f(0) \in \mathbb{C} \cdot \mathbb{1}_k$ and $f(1) \in \mathbb{C} \cdot \mathbb{1}_k$ (this algebra is called an Elliott dimension drop interval algebra). Denote by $\mathcal{HD}$ the class of algebras consisting of direct sums of the building blocks of the forms $M_l(I_k)$ and $PM_n(C(X))P$, with $X$ being one of the spaces {pt}, [0, 1], $S^1$, and $T_{\mathrm{II},k}$, and with $P \in M_n(C(X))$ being a projection. (In [Dadarlat and Gong 1997], this class is denoted by $SH(2)$, and in [Jiang 2011] by $\mathcal{B}$). We call a $C^*$-algebra an $A\mathcal{HD}$ algebra if it is an inductive limit of the algebras in $\mathcal{HD}$.

For each basic building block $A = PM_n(C(X))P$, where $X = \{\mathrm{pt}\}, [0, 1], S^1, T_{\mathrm{II},k}$, or $A = M_l(I_k)$, we have $K_0(A) = \mathbb{Z}$ or $\mathbb{Z}/k\mathbb{Z}$ (for the case $A = PM_n(C(T_{\mathrm{II},k}))P$). Hence there is a natural map rank : $K_0(A) \to \mathbb{Z}$. This map also gives a map from $\{p \in (M_\infty(A)) : p \text{ is a projection}\}$ to $\mathbb{Z}_+$. For example, if $p \in A = PM_n(C(X))P$, then $\mathrm{rank}(p)$ is the rank of projection $p(x) \in P(x)M_n(\mathbb{C})P(x) \cong M_{\mathrm{rank}(P)}(\mathbb{C})$ for any $x \in X$; and if $p \in A = M_l(I_k)$, then $\mathrm{rank}(p)$ is the rank of projection $p(0) \in M_l(\mathbb{C})$. (Note that we regard $p(0)$ as in $M_l(\mathbb{C}) \cong \mathbb{1}_k \otimes M_l(\mathbb{C})$, not $M_{lk}(\mathbb{C})$.)

**2.3.** By $A\mathcal{HD}$ algebra, we mean the inductive limit of

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \longrightarrow \cdots \longrightarrow \cdots,$$

where $A_n \in \mathcal{HD}$ for each $n$.

For an $A\mathcal{HD}$ inductive limit $A = \lim(A_n, \phi_{nm})$, we write $A_n = \bigoplus_{i=1}^{t_n} A_n^i$, where $A_n^i = P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$ or $A_n^i = M_{[n,i]}(I_{k_{n,i}})$. For convenience, even for a block $A_n^i = M_{[n,i]}(I_{k_{n,i}})$, we still use $X_{n,i}$ for $\mathrm{Sp}(A_n^i) = [0, 1]$ — that is, $A_n^i$ is regarded as a homogeneous algebra or a subhomogeneous algebra over $X_{n,i}$.

**2.4.** In [Gong et al. 2010; 2018], joint with Cornel Pasnicu, the authors proved the reduction theorem for *AH* algebras with the ideal property provided that the inductive limit systems have no dimension growth. That is, if $A$ is an inductive limit of $A_n = \bigoplus A_n^i = \bigoplus P_{n,i}M_{[n,i]}C(X_{n,i})P_{n,i}$ with $\sup_{n,i} \dim(X_{n,i}) < +\infty$, and if we further assume that $A$ has the ideal property, then $A$ can be rewritten as an inductive limit of $B_n = \bigoplus B_n^j = \bigoplus Q_{n,j}M_{\{n,j\}}C(Y_{n,i})Q_{n,j}$, with $Y_{n,i}$ being one of {pt}, [0, 1], $S^1$, $T_{\mathrm{II},k}$, $T_{\mathrm{III},k}$, $S^2$. In turn, Jiang [2017] proved (also see [Li 2006]) that the above inductive limit can be rewritten as the inductive limit of the direct sums of homogeneous algebras over {pt}, [0, 1], $S^1$, $T_{\mathrm{II},k}$ and $M_l(I_k)$. Combining these two results, we know that all *AH* algebras of no dimension growth with the ideal property are $A\mathcal{HD}$ algebras. Let us point out that, as proved in [Dadarlat and Gong 1997], there are real rank zero $A\mathcal{HD}$ algebras which are not *AH* algebras.

**2.5.** Let $A$ be a $C^*$-algebra. Then $K_0(A)^+ \subset K_0(A)$ is defined to be the semigroup of $K_0(A)$ generated by $[p] \in K_0(A)$, where $p \in M_\infty(A)$ are projections. For all

$C^*$-algebras considered in this paper — for example, $A \in \mathcal{HD}$, or $A$ is an $A\mathcal{HD}$ algebra, or $A = B \otimes C(T_{\mathrm{II},k} \times S^1)$, where $B$ is an $\mathcal{HD}$ or $A\mathcal{HD}$ algebra — we always have

$$K_0(A)^+ \cap (-K_0(A)^+) = \{0\} \quad \text{and} \quad K_0(A)^+ - K_0(A)^+ = K_0(A). \qquad (*)$$

Therefore $(K_0(A), K_0(A)^+)$ is an ordered group. Define $\Sigma A \subset K_0(A)^+$ to be

$$\Sigma A = \{[p] \in K_0(A)^+ : p \text{ is a projection in } A\}.$$

Then $(K_0(A), K_0(A)^+, \Sigma A)$ is a scaled ordered group. (Note that for purely infinite $C^*$-algebras or stable projectionless $C^*$-algebras, condition $(*)$ does not hold.)

**2.6.** Let $\underline{K}(A) = K_*(A) \oplus \left(\bigoplus_{k=2}^{+\infty} K_*(A, \mathbb{Z}/k\mathbb{Z})\right)$ be as in [Dadarlat and Gong 1997]. Let $\wedge$ be the Bockstein operation on $\underline{K}(A)$ (see [Dadarlat and Gong 1997, 4.1]). It is well known that $K_*(A, Z \oplus \mathbb{Z}/k\mathbb{Z}) = K_0(A \otimes C(W_k \times S^1))$, where $W_k = T_{\mathrm{II},k}$.

As in [Dadarlat and Gong 1997], let $K_*(A, Z \oplus \mathbb{Z}/k\mathbb{Z})^+ = K_0(A \otimes C(W_k \times S^1)^+)$ and let $\underline{K}(A)^+$ be the semigroup generated by $\{K_*(A, \mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z})^+ : k = 2, 3, \dots\}$.

**2.7.** Let $\mathrm{Hom}_\wedge(\underline{K}(A), \underline{K}(B))$ be the set of homomorphisms between $\underline{K}(A)$ and $\underline{K}(B)$ compatible with the Bockstein operations $\wedge$. There is a surjective map (see [Dadarlat and Gong 1997])

$$\Gamma : KK(A, B) \to \mathrm{Hom}_\wedge(\underline{K}(A), \underline{K}(B)).$$

Following Rørdam [1995], we write $KL(A, B) \triangleq KK(A, B)/\mathrm{Pext}(K_*(A), K_{*+1}(B))$, where $\mathrm{Pext}(K_*(A), K_{*+1}(B))$ is identified with $\ker \Gamma$ by [Dadarlat and Loring 1996b]. The triple $(\underline{K}(A); \underline{K}(A)^+; \Sigma A)$ is part of our invariant. For two $C^*$-algebras $A$ and $B$, by a "homomorphism"

$$\alpha : (\underline{K}(A); \underline{K}(A)^+; \Sigma A) \to (\underline{K}(B); \underline{K}(B)^+; \Sigma B)$$

we mean a system of maps

$$\alpha_k^i : K_i(A, \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z}), \quad i = 0, 1, \ k = 0, 2, 3, \dots$$

which are compatible with the Bockstein operations and $\alpha = \bigoplus_{k,i} \alpha_k^i$ satisfies $\alpha(\underline{K}(A)^+) \subset \underline{K}(B)^+$. And finally, $\alpha_0^0(\Sigma A) \subset \Sigma B$.

**2.8.** For a unital $C^*$-algebra $A$, let $T(A)$ denote the space of tracial states of $A$, i.e., $\tau \in T(A)$ if and only if $\tau$ is a positive linear map from $A$ to $\mathbb{C}$ with $\tau(xy) = \tau(yx)$, and $\tau(\mathbb{1}) = 1$. Endow $T(A)$ with the weak-* topology, that is, for any net $\{\tau_\alpha\}_\alpha \subset T(A)$ and $\tau \in T(A)$, $\tau_\alpha \to \tau$ if and only if $\lim_\alpha \tau_\alpha(x) = \tau(x)$ for any $x \in A$. Then $T(A)$ is a compact Hausdorff space with convex structure, that is, if $\lambda \in [0, 1]$ and $\tau_1, \tau_2 \in T(A)$, then $\lambda\tau_1 + (1 - \lambda)\tau_2 \in T(A)$. $\mathrm{AffT}(A)$ is the collection of all continuous affine maps from $T(A)$ to $\mathbb{R}$, which is a real Banach space with

$\|f\| = \sup_{\tau\in T(A)} |f(\tau)|$. Let $(\mathrm{AffT}(A))_+$ be the subset of $\mathrm{AffT}(A)$ consisting of all nonnegative affine functions. An element $\mathbb{1} \in \mathrm{AffT}(A)$, defined by $\mathbb{1}(\tau) = 1$ for all $\tau \in T(A)$, is called the order unit (or scale) of $\mathrm{AffT}(A)$. Note that any $f \in \mathrm{AffT}(A)$ can be written as $f = f_+ - f_-$ with $f_1, f_2 \in \mathrm{AffT}(A)_+$, $\|f_1\| \le \|f\|$ and $\|f_2\| \le \|f\|$. Therefore $(\mathrm{AffT}(A), \mathrm{AffT}(A)_+, \mathbb{1})$ forms a scaled ordered real Banach space. If $\phi : \mathrm{AffT}(A) \to \mathrm{AffT}(B)$ is a unital positive linear map, then $\phi$ is bounded and therefore continuous.

There is a natural homomorphism $\rho_A : K_0(A) \to \mathrm{AffT}(A)$ defined by setting $\rho_A([p])(\tau) = \sum_{i=1}^{n} \tau(p_{ii})$ for $\tau \in T(A)$ and $[p] \in K_0(A)$ represented by the projection $p = (p_{ij}) \in M_n(A)$.

If $\phi : A \to B$ is a unital homomorphism, then $\phi$ induces a continuous affine map $T\phi : T(B) \to T(A)$, which, in turn, induces a unital positive linear map $\mathrm{AffT}\,\phi : \mathrm{AffT}(A) \to \mathrm{AffT}(B)$.

If $\phi : A \to B$ is not unital, we still use $\mathrm{AffT}\,\phi$ to denote the unital positive linear map

$$\mathrm{AffT}\,\phi : \mathrm{AffT}(A) \to \mathrm{AffT}(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))$$

by regarding $\phi$ as the unital homomorphism from $A$ to $\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A)$ — that is, for any $l \in \mathrm{AffT}(A)$ represented by $x \in A_{s.a}$ as $l(t) = t(x)$ for any $t \in T(A)$, we define

$$((\mathrm{AffT}\,\phi)(l))(\tau) = \tau(\phi(x)) \quad \text{for any } \tau \in T(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A)),$$

where $\phi(x)$ is regarded as an element in $\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A)$. In the above equation, if we regard $\phi(x)$ as element in $B$ (rather than in $\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A)$), the homomorphism $\phi$ also induces a positive linear map, denoted by $\phi_T$ to avoid confusion, from $\mathrm{AffT}(A)$ to $\mathrm{AffT}(B)$ — that is, for the $l$ as above,

$$((\phi_T)(l))(\tau) = \tau(\phi(x)) \quad \text{for any } \tau \in T(B),$$

where $\phi(x)$ is now regarded as an element in $B$. But this map does not preserve the unit $\mathbb{1}$. It has the property that $\phi_T(\mathbb{1}_{\mathrm{AffT}(A)}) \le \mathbb{1}_{\mathrm{AffT}(B)}$.

In this paper, we often use the notation $\phi_T$ for the following situation: if $p_1 < p_2$ are two projections in $A$, and $\phi = \iota : p_1Ap_1 \to p_2Ap_2$ is the inclusion, then $\iota_T$ denotes the (not necessarily unital) map from $\mathrm{AffT}(p_1Ap_1)$ to $\mathrm{AffT}(p_2Ap_2)$ induced by $\iota$.

**2.9.** If $\alpha : (\underline{K}(A); \underline{K}(A)^+; \Sigma A) \to (\underline{K}(B); \underline{K}(B)^+; \Sigma B)$ is a homomorphism as in 2.7, then for each projection $p \in A$, there is a projection $q \in B$ such that $\alpha([p]) = [q]$.

Since $I_k$ has stable rank one and the spaces $X$ involved in the definition of $\mathcal{HD}$ class (see $PM_n(C(X))P$ in 2.2) are of dimension at most two, we know that for all $C^*$-algebras $A$ considered in this paper, $\mathcal{HD}$ class or $A\mathcal{HD}$ algebra, the following statement is true: if $[p_1] = [p_2] \in K_0(A)$, then there is a unitary $u \in A$ such that $up_1u^* = p_2$. Therefore, $\mathrm{AffT}(pAp)$ and $\mathrm{AffT}(qBq)$ depend only on the classes $[p] \in K_0(A)$ and $[q] \in K_0(B)$, respectively. Furthermore, if $[p_1] = [p_2]$, then the identification of $\mathrm{AffT}(p_1Ap_1)$ and $\mathrm{AffT}(p_2Ap_2)$ via the unitary equivalence $up_1u^* = p_2$ is canonical — that is, it does not depend on the choice of unitary $u$. For classes $[p] \in \Sigma A (\subset K_0(A)^+ \subset K_0(A))$, we also take $\mathrm{AffT}(pAp)$ as part of our invariant. We consider a system of unital positive linear maps

$$\xi^{p,q} : \mathrm{AffT}(pAp) \to \mathrm{AffT}(qBq)$$

associated with all pairs of two classes $[p] \in \Sigma A$ and $[q] \in \Sigma B$, with $\alpha([p]) = [q]$. Such a system of maps is said to be compatible if for any $p_1 \le p_2$ with $\alpha([p_1]) = [q_1]$, $\alpha([p_2]) = [q_2]$, and $q_1 \le q_2$, the diagram

$$\begin{array}{ccc} \mathrm{AffT}(p_1Ap_1) & \xrightarrow{\xi^{p_1,q_1}} & \mathrm{AffT}(q_1Bq_1) \\ {\scriptstyle \iota_T}\downarrow & & {\scriptstyle \iota_T}\downarrow \\ \mathrm{AffT}(p_2Ap_2) & \xrightarrow{\xi^{p_2,q_2}} & \mathrm{AffT}(q_2Bq_2) \end{array} \tag{2.10}$$

commutes, where the vertical maps are induced by the inclusions. (See [Ji and Jiang 2011] and [Stevens 1998].)

**2.11.** In this paper, we denote

$$(\underline{K}(A);\ \underline{K}(A)^+;\ \Sigma A;\ \{\mathrm{AffT}(pAp)\}_{[p]\in\Sigma A})$$

by $\mathrm{Inv}^0(A)$, where $\mathrm{AffT}(pAp)$ are scaled ordered Banach spaces as in 2.8. By a map between the invariants $\mathrm{Inv}^0(A)$ and $\mathrm{Inv}^0(B)$, we mean a map

$$\alpha : (\underline{K}(A);\ \underline{K}(A)^+;\ \Sigma A) \to (\underline{K}(B);\ \underline{K}(B)^+;\ \Sigma B)$$

as in 2.7, and for each pair $[p] \in \Sigma A$, $[q] \in \Sigma B$ with $\alpha[p] = [q]$, there is an associated unital positive linear map

$$\xi^{p,q} : \mathrm{AffT}(pAp) \to \mathrm{AffT}(qBq)$$

(which is automatically continuous, as pointed out in 2.8). These maps are compatible in the sense of 2.9 (that is, the diagram (2.10) is commutative for any pair of projections $p_1 \le p_2$).

**2.12.** Let $[p] \in \Sigma A$ be represented by $p \in A$. Let $\alpha([p]) = [q]$ for $q \in B$. Then $\alpha$ induces a map (still denoted by $\alpha$) $\alpha : K_0(pAp) \to K_0(qBq)$. Note that the natural map $\rho := \rho_{pAp} : K_0(pAp) \to \mathrm{AffT}(pAp)$, defined in 2.8, satisfies $\rho(K_0(pAp)^+) \subseteq \mathrm{AffT}(pAp)_+$ and $\rho([p]) = \mathbb{1} \in \mathrm{AffT}(pAp)$. By [Ji and Jiang 2011, 1.20], the compatibility in 2.9 (diagram (2.10)) implies that the following diagram commutes:

$$\begin{array}{ccc} K_0(pAp) & \xrightarrow{\ \rho\ } & \mathrm{AffT}(pAp) \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\xi^{p,q}} \\ K_0(qBq) & \xrightarrow{\ \rho\ } & \mathrm{AffT}(qBq) \end{array} \tag{2.13}$$

For $p = \mathbb{1}_A$, this compatibility (the commutativity of diagram (2.13)) is included as a part of the Elliott invariant for unital simple $C^*$-algebras. But this information is contained in our invariant $\mathrm{Inv}^0(A)$, as pointed out in [Ji and Jiang 2011].

**2.14.** Let $A$ be a unital $C^*$-algebra, $B \in \mathcal{HD}$ and $\{p_i\}_{i=1}^n \subset B$ be mutually orthogonal projections with $\Sigma p_i = \mathbb{1}_B$. Write $B = \bigoplus_{j=1}^m B^j$ with $B^j$ being either $PM_\bullet(C(X))P$ or $M_l(I_k)$, and for any $i = 1, 2, \ldots, n$ write $p_i = \bigoplus_{j=1}^m p_i^j$ with $p_i^j \in B^j$, for $j = 1, 2, \ldots, m$. Note that for all $\tau \in T(B^j)$,

$$\tau(p_i^j) = \frac{\mathrm{rank}(p_i^j)}{\mathrm{rank}(\mathbb{1}_{B^j})}$$

(see 2.2 for the definition of the rank function), which is independent of $\tau \in T(B^j)$.

Let $\xi_i = (\xi_i^1, \xi_i^2, \ldots, \xi_i^m) : \mathrm{AffT}(A) \to \mathrm{AffT}(p_i B p_i) = \bigoplus_{j=1}^m \mathrm{AffT}(p_i^j B^j p_i^j)$ be unital positive linear maps. Then we can define $\xi = (\xi^1, \xi^2, \ldots, \xi^m) : \mathrm{AffT}(A) \to \mathrm{AffT}(B) = \bigoplus_{j=1}^m \mathrm{AffT}(B^j)$ as below:

$$\xi^j(f)(\tau) = \sum_{\{i : \tau(p_i^j) \neq 0\}} \tau(p_i^j) \xi_i^j(f)\left(\frac{\tau|_{p_i^j B^j p_i^j}}{\tau(p_i^j)}\right) \quad \text{for } f \in \mathrm{AffT}(A) \text{ and } \tau \in T(B^j).$$

Note that $\tau|_{p_i^j B^j p_i^j}/\tau(p_i^j) \in T(p_i^j B^j p_i^j)$. So $\xi_i^j(f)$ can evaluate at $\tau|_{p_i^j B^j p_i^j}/\tau(p_i^j)$. Since the value of $\tau(p_i^j)$ is independent of $\tau \in T(B^j)$, it is straightforward to verify that $\xi^j \in \mathrm{AffT}(B^j)$. We denote such $\xi$ by $\bigoplus \xi_i$. (For the case that $B$ is general stably finite unital simple $C^*$-algebras with mutually orthogonal projections $\{p_i\}$ with sum $\mathbb{1}_B$, this kind of construction can be carried out by using of [Lin 2017, Lemma 6.4].)

If $\phi_i : A \to p_i B p_i$ are unital homomorphisms and $\phi = \bigoplus \phi_i : A \to B$, then

$$(\mathrm{AffT}\,\phi)^j(f)(\tau) = \sum_{\{i : \tau(p_i^j) \neq 0\}} \tau(p_i^j)\, \mathrm{AffT}\,\phi_i^j(f)\left(\frac{\tau|_{p_i^j B^j p_i^j}}{\tau(p_i^j)}\right),$$

where $\phi_i^j : A \to p_i^j B^j p_i^j$ is the $j$-th component of $\phi_i$. That is, $\mathrm{AffT}\,\phi = \bigoplus \mathrm{AffT}\,\phi_i$. In particular, if $\|\mathrm{AffT}\,\phi_i(f) - \xi_i(f)\| < \varepsilon$ for all $i$, then

$$\|\mathrm{AffT}\,\phi(f) - \xi(f)\| < \varepsilon.$$

**2.15.** Now we introduce the new ingredient of our invariant, a simplified version of $U(pAp)/\overline{DU(pAp)}$ for any $[p] \in \Sigma A$, where $DU(pAp)$ is the commutator subgroup of $U(pAp)$. Some notations and preliminary results are quoted from [Thomsen 1997; 1995; Nielsen and Thomsen 1996].

**2.16.** Let $A$ be a unital $C^*$-algebra. Let $U(A)$ denote the group of unitaries of $A$ and $U_0(A)$ the connected component of $\mathbb{1}_A$ in $U(A)$. Let $DU(A)$ and $DU_0(A)$ denote the commutator subgroups of $U(A)$ and $U_0(A)$, respectively. (Recall that the commutator subgroup of a group $G$ is the subgroup generated by all elements of the form $aba^{-1}b^{-1}$, where $a, b \in G$.) One can introduce the following metric $D_A$ on $U(A)/\overline{DU(A)}$ (see §3 of [Nielsen and Thomsen 1996]). For $u, v \in U(A)/\overline{DU(A)}$

$$D_A(u, v) = \inf\{\|uv^* - c\| : c \in \overline{DU(A)}\},$$

where, on the right-hand side of the equation, we use $u, v$ to denote any elements in $U(A)$ which represent the elements $u, v \in U(A)/\overline{DU(A)}$.

**Remark 2.17.** Obviously, $D_A(u, v) \le 2$. Also, if $u, v \in U(A)/\overline{DU(A)}$ define two different elements in $K_1(A)$, then $D_A(u, v) = 2$. (This fact follows from the fact that $\|u - v\| < 2$ implies $uv^* \in U_0(A)$.)

**2.18.** Let $A$ be a unital $C^*$-algebra. Let $\mathrm{AffT}(A)$ and $\rho_A : K_0(A) \to \mathrm{AffT}(A)$ be as defined as in 2.8. For simplicity, we use $\rho K_0(A)$ to denote the set $\rho_A(K_0(A))$. The metric $d_A$ on $\mathrm{AffT}(A)/\overline{\rho K_0(A)}$ is defined as follows (see §3 of [Nielsen and Thomsen 1996]).

Let $d'$ denote the quotient metric on $\mathrm{AffT}(A)/\overline{\rho K_0(A)}$. That is, for $f, g$ in $\mathrm{AffT}(A)/\overline{\rho K_0(A)}$, let

$$d'(f, g) = \inf\{\|f - g - h\| : h \in \overline{\rho K_0(A)}\}.$$

Define $d_A$ by

$$d_A(f, g) = \begin{cases} 2 & \text{if } d'(f, g) \ge \frac{1}{2}, \\ |e^{2\pi i d'(f,g)} - 1| & \text{if } d'(f, g) < \frac{1}{2}. \end{cases}$$

Obviously, $d_A(f, g) \le 2\pi d'(f, g)$.

**2.19.** For $A = PM_k(C(X))P$, let $SU(A)$ be the set of unitaries $u \in PM_k(C(X))P$ such that for each $x \in X$, $u(x) \in P(x)M_k(\mathbb{C})P(x) \cong M_{\mathrm{rank}(P)}(\mathbb{C})$ has determinant 1 (note that the determinant of $u(x)$ does not depend on the identification of $P(x)M_k(\mathbb{C})P(x) \cong M_{\mathrm{rank}(P)}(\mathbb{C})$). For $A = M_l(I_k)$, by $u \in SU(A)$ we mean that $u \in SU(M_{lk}(C[0, 1]))$, where we consider $A$ to be a subalgebra of $M_{lk}(C[0, 1])$. For all basic building blocks $A \ne M_l(I_k)$, we have $SU(A) = \overline{DU(A)}$. But for $A = M_l(I_k)$, this is not true (see 2.20 and 2.21 below).

In [Elliott et al. 2007], the authors also defined $SU(A)$ for $A$ a homogeneous algebra and a certain $AH$ inductive limit $C^*$-algebra. This definition cannot be generalized to a more general class of $C^*$-algebras, but we define $\widetilde{SU(A)}$ for any unital $C^*$ algebra $A$. Later, in our definition of $\mathrm{Inv}(A)$, we only make use of $\widetilde{SU(A)}$ (rather than $SU(A)$).

**2.20.** Let $A = I_k$. Then $K_1(A) = \mathbb{Z}/k\mathbb{Z}$, which is generated by $[u]$, where $u$ is the unitary

$$u = \begin{pmatrix} e^{2\pi i t(k-1)/k} & & & \\ & e^{2\pi i(-t/k)} & & \\ & & \ddots & \\ & & & e^{2\pi i(-t/k)} \end{pmatrix} \in I_k.$$

(Note that $u(0) = \mathbb{1}_k$, $u(1) = e^{2\pi i(-1/k)} \cdot \mathbb{1}_k$.)

Note that the above $u$ is in $SU(A)$, but not in $U_0(A)$, and therefore not in $DU(A)$.

**2.21.** By [Thomsen 1995] (or [Gong et al. 2015a]), $u \in M_l(I_k)$ is in $\overline{DU(A)}$ if and only if for any irreducible representation $\pi : M_l(I_k) \to B(H)$ ($\dim H < +\infty$), we have $\det(\pi(u)) = 1$. For the unitary $u$ in 2.20, and irreducible representation $\pi$ corresponding to 1, $\pi(u) = e^{2\pi i(-1/k)}$, whose determinant is $e^{2\pi i(-1/k)} \neq 1$. By [Thomsen 1997, 6.1] one knows that if $A = I_k$, then

$$U_0(A) \cap SU(A) = \{e^{2\pi i(j/k)} : j = 0, 1, \dots, k-1\} \cdot \overline{DU(A)}.$$

If $A = M_l(I_k)$, then for any $j \in \mathbb{Z}$, $e^{2\pi i(j/l)} \cdot \mathbb{1}_A \in \overline{DU(A)}$. Consequently,

$$U_0(A) \cap SU(A) = \{e^{2\pi i(j/kl)} : j = 0, 1, \dots, kl-1\} \cdot \overline{DU(A)}.$$

**2.22.** Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then for any $A \in \mathcal{HD}$, $\mathbb{T} \cdot \overline{DU(A)} \subset U_0(A)$. From 2.19 and 2.21, we have either $SU(A) = \overline{DU(A)}$ or $U_0(A) \cap SU(A) \subset \mathbb{T} \cdot \overline{DU(A)}$.

**Lemma 2.23.** *Let $A = PM_k(C(X))P \in \mathcal{HD}$. For any $u, v \in U(A)$, if $uv^* \in \mathbb{T} \cdot \overline{DU(A)}$ (in particular if both $u$, $v$ are in $\mathbb{T} \cdot \overline{DU(A)}$), then $D_A(u, v) \le 2\pi/\mathrm{rank}(P)$.*

*Let $A = M_l(I_k)$. For any $u$, $v$, if $uv^* \in \mathbb{T} \cdot \overline{DU(A)}$, then $D_A(u, v) \le 2\pi/l$.*

*Proof.* There is $\omega \in \overline{DU(A)}$ such that $uv^* = \lambda\omega$ for some $\lambda \in \mathbb{T}$. Choose $\lambda_0 = e^{2\pi i j/\mathrm{rank}(P)}$, $j \in \mathbb{N}$, such that $|\lambda - \lambda_0| < 2\pi/\mathrm{rank}(P)$. Then $\lambda_0 \cdot P \in PM_k(C(X))P$ has determinant 1 everywhere and is in $\overline{DU(A)}$. And so does $\lambda_0\omega$. Also, we have $|uv^* - \lambda_0\omega| < 2\pi/\mathrm{rank}(P)$.

The case $A = M_l(I_k)$ is similar. □

**2.24.** Let $\mathrm{path}(U(A))$ denote the set of piecewise smooth paths $\xi : [0, 1] \to U(A)$. Recall that the de la Harp–Skandalis determinant $\Delta : \mathrm{path}(U(A)) \to \mathrm{AffT}(A)$ is defined by

$$\Delta(\xi)(\tau) = \frac{1}{2\pi i} \int_0^1 \tau\Big(\frac{d\xi}{dt} \cdot \xi^*\Big)\, dt$$

(see [de la Harpe and Skandalis 1984]). It is proved there (see also [Thomsen 1995]) that $\Delta$ induces a map $\Delta^\circ : \pi_1(U_0(A)) \to \mathrm{AffT}(A)$. For any two paths $\xi_1, \xi_2$ starting at $\xi_1(0) = \xi_2(0) = 1 \in A$ and ending at the same unitary $u = \xi_1(1) = \xi_2(1)$, we have that

$$\Delta(\xi_1) - \Delta(\xi_2) = \Delta(\xi_1 \cdot \xi_2^*) \subset \Delta^\circ(\pi_1(U_0(A))).$$

Consequently, $\Delta$ induces a map

$$\bar{\Delta} : U_0(A) \to \mathrm{AffT}(A)/\Delta^\circ(\pi_1(U_0(A)))$$

(see Section 3 of [Thomsen 1995]). Passing to matrix over $A$, we have a map $\bar{\Delta}_n : U_0(M_n(A)) \to \mathrm{AffT}(A)/\Delta_n^\circ(\pi_1(U_0(M_n(A))))$.

If $1 \le m < n$, then path$(U(M_m(A)))$ (and $U_0(M_m(A))$) can be embedded into path$(U(M_n(A)))$ (and $U_0(M_n(A))$) by sending $u(t)$ to $\mathrm{diag}(u(t), \mathbb{1}_{n-m})$. From the above definition, and the formula

$$\frac{d}{dt}(\mathrm{diag}(u(t), \mathbb{1}_{n-m})) = \mathrm{diag}\Big(\frac{d}{dt}(u(t)), 0_{n-m}\Big),$$

one gets

$$\bar{\Delta}_n|_{U_0(M_m(A))} = \bar{\Delta}_m.$$

Recall that the Bott isomorphism $b : K_0(A) \to K_1(SA)$ is given by the following: for any $x \in K_0(A)$ represented by a projection $p \in M_n(A)$, we have

$$b(x) = [e^{2\pi i t} p + (\mathbb{1}_n - p)] \in K_1(SA).$$

If $\xi(t) = e^{2\pi i t} p + (\mathbb{1}_n - p)$, then

$$\begin{aligned}(\Delta^\circ \xi)(\tau) &= \frac{1}{2\pi i}\int_0^1 \tau\big((2\pi i e^{2\pi i t} p)\cdot(e^{-2\pi i t} p + (1-p))\big)\, dt \\ &= \frac{1}{2\pi i}\int_0^1 \tau(2\pi i p)\, dt = \tau(p).\end{aligned}$$

Since the Bott map is an isomorphism, it follows that each loop in $\pi_1(U_0(A))$ is homotopic to a product of loops of the form $\xi(t)$. Consequently,

$$\Delta^\circ(\pi_1(U_0(M_n(A)))) \subset \rho_A K_0(A).$$

Hence $\bar{\Delta}_n$ can be regarded as a map

$$\bar{\Delta}_n : U_0(M_n(A)) \to \mathrm{AffT}(A)/\overline{\rho_A K_0(A)}.$$

**Proposition 2.25.** *For $A \in \mathcal{HD}$ or $A \in A\mathcal{HD}$, $\overline{DU_0(A)} = \overline{DU(A)}$.*

*Proof.* Let the determinant function

$$\bar{\Delta}_n : U_0(M_n(A)) \to \mathrm{AffT}(A)/\overline{\Delta_n^0(\pi_1 U_0(M_n(A)))}$$

be defined as in §3 of [Thomsen 1995] (see 2.24 above). As observed in [Nielsen and Thomsen 1996, top of p. 33], Lemma 3.1 of [Thomsen 1995] implies that $\overline{DU_0(A)} = U_0(A) \cap \overline{DU(A)}$. For the reader's convenience, we give a brief proof of this fact. Namely, the equation

$$\begin{pmatrix} uvu^{-1}v^{-1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} v & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & v^{-1} \end{pmatrix} \begin{pmatrix} u^{-1} & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} v^{-1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & v \end{pmatrix}$$

implies that $\overline{DU(A)} \subset \overline{DU_0(M_3(A))}$. Therefore by Lemma 3.1 of [Thomsen 1995], $\overline{DU(A)} \subset \ker \bar{\Delta}_3$. If $x \in U_0(A) \cap \overline{DU(A)}$, then $\bar{\Delta}_1$ is defined at $x$. By calculation in 2.24, $\bar{\Delta}_3|_{U_0(A)} = \bar{\Delta}_1$. So we have $\bar{\Delta}_1(x) = 0$, and thus $x \in \overline{DU_0(A)} = \ker \bar{\Delta}_1$, by [Thomsen 1995, Lemma 3.1]. Note if $A \in \mathcal{HD}$ or $A\mathcal{HD}$, then $\overline{DU(A)} \subset U_0(A)$. □

(It is not known to the authors whether it is always true that $\overline{DU_0(A)} = \overline{DU(A)}$.)

**2.26.** There is a natural map $\alpha : \pi_1(U(A)) \to K_0(A)$, or more generally, for each $n \in \mathbb{N}$ a map $\alpha_n : \pi_1(U(M_n(A))) \to K_0(A)$. We need the following notation. For a unital $C^*$-algebra $A$, let $\mathcal{P}_n K_0(A)$ (see [Gong et al. 2015b]) be the subgroup of $K_0(A)$ generated by the formal difference of projections $p, q \in M_n(A)$ (instead of $M_\infty(A)$). Then

$$\mathcal{P}_n K_0(A) \subset \mathrm{image}(\alpha_n).$$

In particular, if $\rho : K_0(A) \to \mathrm{AffT}(A)$ satisfies $\rho(\mathcal{P}_n K_0(A)) = \rho K_0(A)$, then by Theorem 3.2 of [Thomsen 1995],

$$U_0(M_n(A))/\overline{DU_0(M_n(A))} \cong U_0(M_\infty(A))/\overline{DU_0(M_\infty(A))} \cong \mathrm{AffT}(A)/\overline{\rho K_0(A)}.$$

Note that for all $A \in \mathcal{HD}$, we have $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$ (see below). Consequently,

$$U_0(A)/\overline{DU_0(A)} \cong \mathrm{AffT}(A)/\overline{\rho K_0(A)}.$$

If $A$ does not contain building blocks of the form $PM_n(C(T_{\mathrm{II},k}))P$, then such $A$ is the special case of [Thomsen 1997], and the above fact is observed in [Thomsen 1997] (for circle algebras in [Nielsen and Thomsen 1996] earlier) — in this special case, we ever have $\mathcal{P}_1 K_0(A) = K_0(A)$ (as used in [Nielsen and Thomsen 1996] and [Thomsen 1997] in the form of surjectivity of $\alpha : \pi_1(U(A)) \to K_0(A)$). For $A = PM_n(C(T_{\mathrm{II},k}))P$, we do not have the surjectivity of $\alpha : \pi_1(U(A)) \to K_0(A)$ anymore. But $K_0(A) = \mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$ and $\mathrm{image}(\alpha) = \mathcal{P}_1 K_0(A)$ contains at least one element which corresponds to a rank one projection (any bundle over $T_{\mathrm{II},k}$ has a subbundle of rank 1) — that is,

$$\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A) (\subseteq \mathrm{AffT}(A))$$

consisting of all constant functions from $T_{\mathrm{II},k}$ to $(1/\mathrm{rank}(P))\mathbb{Z}$.

As in [Nielsen and Thomsen 1996, Lemma 3.1; Thomsen 1997, Lemma 6.4], the map $\bar{\Delta} : U_0(A) \to \mathrm{AffT}(A)/\overline{\rho_A(K_0(A))}$ (see 2.24) has $\ker \bar{\Delta} = \overline{DU(A)}$ and the following lemma holds.

**Lemma 2.27.** *Suppose that a unital $C^*$-algebra $A$ satisfies $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$ and $\overline{DU_0(A)} = \overline{DU(A)}$* (see 2.26 and 2.25), *and in particular, that $A \in \mathcal{HD}$ or $A \in A\mathcal{HD}$. Then the following hold*:

(1) *There is a split exact sequence*

$$0 \to \mathrm{AffT}(A)/\overline{\rho K_0(A)} \xrightarrow{\lambda_A} U(A)/\overline{DU(A)} \to K_1(A) \to 0.$$

(2) *$\lambda_A$ is an isometry with respect to the metrics $d_A$ and $D_A$.*

**2.28.** Recall from §3 of [Thomsen 1995], the de la Harpe–Skandalis determinant (see [de la Harpe and Skandalis 1984]) can be used to define

$$\bar{\Delta} : U_0(A)/\overline{DU(A)} \to \mathrm{AffT}(A)/\overline{\rho K_0(A)}.$$

With the condition of Lemma 2.27 above, this map is an isometry with respect to the metrics $d_A$ and $D_A$. In fact, the inverse of this map is $\lambda_A$ in Lemma 2.27.

It follows from the definition of $\bar{\Delta}$ [Thomsen 1995, §3] that

$$\bar{\Delta}(e^{2\pi i t p}) = t \cdot \rho([p]) \quad (\mathrm{mod}\ (\overline{\rho K_0(A)})), \tag{2.29}$$

where $[p] \in K_0(A)$ is the element represented by projection $p \in A$.

It is convenient to introduce the extended commutator group $DU^+(A)$, which is generated by $DU(A) \subset U(A)$ and the set

$$\{e^{2\pi i t p} = e^{2\pi i t} p + (\mathbb{1} - p) \in U(A) : t \in \mathbb{R},\ p \in A \text{ is a projection}\}.$$

Let $\widetilde{DU(A)}$ denote the closure of $DU^+(A)$. That is, $\widetilde{DU(A)} = \overline{DU^+(A)}$.

Let us use $\widetilde{\rho K_0(A)}$ to denote the real vector space spanned by $\overline{\rho K_0(A)}$. That is,

$$\widetilde{\rho K_0(A)} := \overline{\{\Sigma \lambda_i \phi_i : \lambda_i \in \mathbb{R}, \phi_i \in \rho K_0(A)\}}.$$

Suppose that $\overline{\rho K_0(A)} = \overline{\rho(\mathcal{P}_1 K_0(A))}$. It follows from (2.29) that the image of $\widetilde{DU(A)}/\overline{DU(A)}$ under the map $\bar{\Delta}$ is exactly $\widetilde{\rho K_0(A)}/\overline{\rho K_0(A)}$. Therefore, $\lambda_A$ takes $\widetilde{\rho K_0(A)}/\overline{\rho K_0(A)}$ to $\widetilde{DU(A)}/\overline{DU(A)}$. Hence $\bar{\Delta} : U_0(A)/\overline{DU(A)} \to \mathrm{AffT}(A)/\overline{\rho K_0(A)}$ also induces a quotient map (still denoted by $\bar{\Delta}$)

$$\bar{\Delta} : U_0(A)/\widetilde{DU(A)} \to \mathrm{AffT}(A)/\widetilde{\rho K_0(A)},$$

which is an isometry using the quotient metrics of $d_A$ and $D_A$. The inverse of this quotient map $\bar{\Delta}$ gives rise to the isometry

$$\tilde{\lambda}_A : \mathrm{AffT}(A)/\widetilde{\rho K_0(A)} \to U_0(A)/\widetilde{DU(A)} \hookrightarrow U(A)/\widetilde{DU(A)},$$

which is an isometry with respect to the quotient metrics $\widetilde{d_A}$ and $\overline{D_A}$ as described below.

For any $u, v \in U(A)/\widetilde{DU(A)}$,

$$\overline{D_A}(u, v) = \inf\{\|uv^* - c\| : c \in \widetilde{DU(A)}\}.$$

Let $\widetilde{d}'$ denote the quotient metric on $\mathrm{AffT}(A)/\widetilde{\rho K_0(A)}$ of $\mathrm{AffT}(A)$, that is,

$$\widetilde{d}'(f,g)=\inf\{\|f-g-h\|:h\in\widetilde{\rho K_0(A)}\}\quad \text{for all } f,g\in \mathrm{AffT}(A)/\widetilde{\rho K_0(A)}.$$

Define $\widetilde{d}_A$ by

$$\widetilde{d}_A(f,g)=\begin{cases}2 & \text{if } \widetilde{d}'(f,g)\geq \frac{1}{2},\\ |e^{2\pi i\widetilde{d}'(f,g)}-1| & \text{if } \widetilde{d}'(f,g)<\frac{1}{2}.\end{cases}$$

The following result is a consequence of Lemma 2.27.

**Lemma 2.30.** *Suppose that a unital $C^*$-algebra $A$ satisfies $\rho(\mathcal{P}_1K_0(A))=\rho K_0(A)$ and $\overline{DU_0(A)}=\overline{DU(A)}$* (see 2.26 and 2.25), *and in particular, that $A\in\mathcal{HD}$ or $A\in A\mathcal{HD}$. Then we have the following*:

(1) *There is a split exact sequence*

$$0\to \mathrm{AffT}(A)/\widetilde{\rho K_0(A)}\xrightarrow{\tilde{\lambda}_A} U(A)/\widetilde{DU(A)}\xrightarrow{\pi_A} K_1(A)\to 0.$$

(2) *$\tilde{\lambda}_A$ is an isometry with respect to $\widetilde{d}_A$ and $\overline{D_A}$.*

*Proof.* As we mentioned in 2.28, the map $\lambda_A$ in Lemma 2.27 takes $\widetilde{\rho K_0(A)}/\overline{\rho K_0(A)}$ to $\widetilde{DU(A)}/\overline{DU(A)}$. From the exact sequence in Lemma 2.27, passing to quotient, one gets the exact sequence in (1).

Note that $\widetilde{d}_A$ on $\mathrm{AffT}(A)/\widetilde{\rho K_0(A)}$ is the quotient metric induced by $d_A$ on $\mathrm{AffT}(A)/\overline{\rho K_0(A)}$ and $\overline{D_A}$ on $U(A)/\widetilde{DU(A)}$ is the quotient metric induced by $D_A$ on $U(A)/\overline{DU(A)}$. Hence $\tilde{\lambda}_A$ is an isometry, since so is $\lambda_A$. □

**2.31.** Instead of $\widetilde{DU(A)}$, we need the group

$$\widetilde{SU(A)}:=\overline{\{x\in U(A):x^n\in\widetilde{DU(A)} \text{ for some } n\in\mathbb{Z}_+\backslash\{0\}\}}.$$

For $A\in\mathcal{HD}$, say $A=PM_l(C(X))P$ ($X=[0,1]$, $S^1$ or $T_{\mathrm{II},k}$) or $A=M_l(I_k)$, $\widetilde{SU(A)}$ is the set of all unitaries $u\in P(M_lC(X))P$ or $u\in M_l(I_k)$ such that the determinant function

$$X\ni x\mapsto \det(u(x))\quad \text{or}\quad (0,1)\ni t\mapsto \det(u(t))$$

is a constant function. Comparing with the set $SU(A)$ in [Elliott et al. 2007] or 2.19 above (which only defines for $\mathcal{HD}$ blocks), where the function will be constant 1, here we allow the function to be an arbitrary constant in $\mathbb{T}$. Hence for a basic building block $A=PM_n(C(X))P\in\mathcal{HD}$ or $A=M_l(I_k)$,

$$\widetilde{SU(A)}=\mathbb{T}\cdot SU(A).$$

The notations $\widetilde{\rho K_0(A)}$, $\widetilde{DU(A)}$ and $\widetilde{SU(A)}$ reflect that they are constructed from $\rho K_0(A)$, $DU(A)$ and $SU(A)$, respectively. *To make the notation simpler, from now on we use $\widetilde{\rho K}_0(A)$ to denote $\widetilde{\rho K_0(A)}=\widetilde{\rho_A(K_0(A))}$, $\widetilde{DU}(A)$ to denote $\widetilde{DU(A)}$, and $\widetilde{SU}(A)$ to denote $\widetilde{SU(A)}$.*

**Lemma 2.32.** *Let* $\alpha, \beta : K_1(A) \to U(A)/\widetilde{DU}(A)$ *be splittings of* $\pi_A$ *in Lemma 2.30. Then*

$$\alpha|_{\mathrm{tor}\, K_1(A)} = \beta|_{\mathrm{tor}\, K_1(A)}$$

*and* $\alpha(\mathrm{tor}\, K_1(A)) \subset \widetilde{SU}(A)/\widetilde{DU}(A)$. *Furthermore*, $\alpha$ *identifies* $\mathrm{tor}(K_1(A))$ *with* $\widetilde{SU}(A)/\widetilde{DU}(A)$.

*Proof.* For any $z \in \mathrm{tor}\, K_1(A)$, with $kz = 0$ for some integer $k > 0$, we have

$$\pi_A \alpha(z) = z = \pi_A \beta(z).$$

By the exactness of the sequence, there is an element $f \in \mathrm{AffT}(A)/\widetilde{\rho K_0}(A)$ such that

$$\alpha(z) - \beta(z) = \tilde{\lambda}_A(f).$$

Since $k\alpha(z) - k\beta(z) = \alpha(kz) - \beta(kz) = 0$, we have $\tilde{\lambda}_A(kf) = 0$. By the injectivity of $\tilde{\lambda}_A$, $kf = 0$. Note that $\mathrm{AffT}(A)/\widetilde{\rho K_0}(A)$ is an $\mathbb{R}$-vector space, $f = 0$. Furthermore, $k\alpha(z) = 0$ in $U(A)/\widetilde{DU}(A)$ implies that

$$\alpha(z) \in \widetilde{SU}(A)/\widetilde{DU}(A).$$

Thus we get $\alpha(\mathrm{tor}\, K_1(A)) \subset \widetilde{SU}(A)$. If $u \in \widetilde{SU}(A)/\widetilde{DU}(A)$ then $\alpha(\pi_A(u)) = u$. $\square$

**2.33.** Let $U_{\mathrm{tor}}(A)$ denote the set of unitaries $u \in A$ such that $[u] \in \mathrm{tor}\, K_1(A)$. For any $C^*$-algebra $A$ we have $\widetilde{SU}(A) \subset U_{\mathrm{tor}}(A)$. If we further assume $\overline{DU_0(A)} = \overline{DU(A)}$, then

$$\widetilde{DU}(A) = U_0(A) \cap \widetilde{SU}(A) \quad \text{and} \quad U_{\mathrm{tor}}(A) = U_0(A) \cdot \widetilde{SU}(A).$$

We have $U_0(A)/\widetilde{DU}(A) \cong U_{\mathrm{tor}}(A)/\widetilde{SU}(A)$. The metric $\overline{D_A}$ on $U(A)/\widetilde{DU}(A)$ induces a metric $\widetilde{D}_A$ on $U(A)/\widetilde{SU}(A)$. And the above identification $U_0(A)/\widetilde{DU}(A)$ with $U_{\mathrm{tor}}(A)/\widetilde{SU}(A)$ is an isometry with respect to $\overline{D_A}$ and $\widetilde{D}_A$. Hence $\tilde{\lambda}_A$ in 2.28 can be regarded as a map (still denoted by $\tilde{\lambda}_A$):

$$\tilde{\lambda}_A : \mathrm{AffT}(A)/\widetilde{\rho K_0}(A) \to U_{\mathrm{tor}}(A)/\widetilde{SU(A)} \hookrightarrow U(A)/\widetilde{SU(A)}.$$

Similar to Lemma 2.30, we have the following.

**Lemma 2.34.** *Suppose that a unital* $C^*$*-algebra* $A$ *satisfies* $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$ *and* $\overline{DU_0(A)} = \overline{DU(A)}$ (see 2.26 and 2.25), *and in particular, that* $A \in \mathcal{HD}$ *or* $A \in A\mathcal{HD}$. *Then the following hold*:

(1) *There is a split exact sequence*

$$0 \to \mathrm{AffT}(A)/\widetilde{\rho K_0}(A) \xrightarrow{\tilde{\lambda}_A} U(A)/\widetilde{SU}(A) \xrightarrow{\pi_A} K_1(A)/\mathrm{tor}\, K_1(A) \to 0.$$

(2) $\tilde{\lambda}_A$ *is an isometry with respect to the metrics* $\widetilde{d}_A$ *and* $\widetilde{D}_A$.

**2.35.** For each pair of projections $p_1, p_2 \in A$ with $p_1 = up_2u^*$,

$$U(p_1Ap_1)/\widetilde{SU}(p_1Ap_1) \cong U(p_2Ap_2)/\widetilde{SU}(p_2Ap_2).$$

Also, since in any unital $C^*$-algebra $A$ and unitaries $u, v \in U(A)$, $v$ and $uvu^*$ represent the same element in $U(A)/\widetilde{SU}(A)$, the above identification does not depend on the choice of $u$ to implement $p_1 = up_2u^*$. That is, for any $[p] \in \Sigma A$, the group $U(pAp)/\widetilde{SU}(pAp)$ is well defined, which does not depend on choice of $p \in [p]$. We include this group (with metric) as part of our invariant. If $[p] \le [q]$, then we can choose $p, q$ such that $p \le q$. In this case, there is a natural inclusion map $\iota : pAp \to qAq$, which induces

$$\iota_* : U(pAp)/\widetilde{SU}(pAp) \to U(qAq)/\widetilde{SU}(qAq),$$

where $\iota_*$ is defined by

$$\iota_*(u) = u \oplus (q-p) \in U(qAq) \quad \text{for all } u \in U(pAp).$$

A unital homomorphism $\phi : A \to B$ induces a contractive group homomorphism

$$\phi^\natural : U(A)/\widetilde{SU}(A) \to U(B)/\widetilde{SU}(B).$$

If $\phi$ is not unital, then the map

$$\phi^\natural : U(A)/\widetilde{SU}(A) \to U(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))/\widetilde{SU}(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))$$

is induced by the corresponding unital homomorphism. In this case, $\phi$ also induces the map $\iota_* \circ \phi^\natural : U(A)/\widetilde{SU}(A) \to U(B)/\widetilde{SU}(B)$, which is denoted by $\phi_*$ to avoid confusion. If $\phi$ is unital, then $\phi^\natural = \phi_*$. If $\phi$ is not unital, then $\phi^\natural$ and $\phi_*$ have different codomains. That is, $\phi^\natural$ has codomain $U(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))/\widetilde{SU}(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))$, but $\phi_*$ has codomain $U(B)/\widetilde{SU}(B)$. (See the last paragraph of 3.8 below for some further explanation with an example.)

Since $U(A)/\widetilde{SU}(A)$ is an abelian group, we call the unit $[\mathbb{1}] \in U(A)/\widetilde{SU}(A)$ the zero element. If $\phi : A \to B$ satisfies $\phi(U(A)) \subset \widetilde{SU}(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))$, then $\phi^\natural = 0$. In particular, if the image of $\phi$ is of finite dimension, then $\phi^\natural = 0$.

**2.36.** In this paper and [Gong et al. 2016], we denote

$$(\underline{K}(A); \underline{K}(A)^+; \Sigma A; \{\mathrm{AffT}(pAp)\}_{[p]\in\Sigma A}; \{U(pAp)/\widetilde{SU}(pAp)\}_{[p]\in\Sigma A})$$

by $\mathrm{Inv}(A)$. By a map from $\mathrm{Inv}(A)$ to $\mathrm{Inv}(B)$, we mean

$$\alpha : (\underline{K}(A); \underline{K}(A)^+; \Sigma A) \to (\underline{K}(B); \underline{K}(B)^+; \Sigma B)$$

as in 2.7, and for each pair $([p], [\bar{p}]) \in \Sigma A \times \Sigma B$ with $\alpha([p]) = [\bar{p}]$, there exist an associate unital positive (continuous) linear map

$$\xi^{p,\bar{p}} : \mathrm{AffT}(pAp) \to \mathrm{AffT}(\bar{p}B\bar{p})$$

and an associate contractive group homomorphism

$$\chi^{p,\bar{p}} : U(pAp)/\widetilde{SU}(pAp) \to U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p})$$

satisfying the following compatibility conditions. (Note that $\chi^{p,\bar{p}}$ is continuous, as it is a contractive group homomorphism from a metric group to another metric group.)

(a) If $p < q$, then the diagrams

$$\begin{array}{ccc} \mathrm{AffT}(pAp) & \xrightarrow{\xi^{p,\bar{p}}} & \mathrm{AffT}(\bar{p}B\bar{p}) \\ {\scriptstyle \iota_T}\downarrow & & {\scriptstyle \iota_T}\downarrow \\ \mathrm{AffT}(qAq) & \xrightarrow{\xi^{q,\bar{q}}} & \mathrm{AffT}(\bar{q}B\bar{q}) \end{array} \tag{I}$$

and

$$\begin{array}{ccc} U(pAp)/\widetilde{SU}(pAp) & \xrightarrow{\chi^{p,\bar{p}}} & U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p}) \\ {\scriptstyle \iota_*}\downarrow & & {\scriptstyle \iota_*}\downarrow \\ U(qAq)/\widetilde{SU}(qAq) & \xrightarrow{\chi^{q,\bar{q}}} & U(\bar{q}B\bar{q})/\widetilde{SU}(\bar{q}B\bar{q}) \end{array} \tag{II}$$

commutes, where the vertical maps are induced by inclusions.

(b) The diagram

$$\begin{array}{ccc} K_0(pAp) & \xrightarrow{\rho} & \mathrm{AffT}(pAp) \\ {\scriptstyle \alpha}\downarrow & & {\scriptstyle \xi^{p,\bar{p}}}\downarrow \\ K_0(\bar{p}B\bar{p}) & \xrightarrow{\rho} & \mathrm{AffT}(\bar{p}B\bar{p}) \end{array} \tag{III}$$

commutes, and therefore $\xi^{p,\bar{p}}$ induces a map (still denoted by $\xi^{p,\bar{p}}$)

$$\xi^{p,\bar{p}} : \mathrm{AffT}(pAp)/\widetilde{\rho K_0}(pAp) \to \mathrm{AffT}(\bar{p}B\bar{p})/\widetilde{\rho K_0}(\bar{p}B\bar{p}).$$

(The commutativity of (III) follows from the commutativity of (I), by [Ji and Jiang 2011, 1.20]. So this is not an extra requirement.)

(c) The diagrams

$$\begin{array}{ccc} \mathrm{AffT}(pAp)/\widetilde{\rho K_0}(pAp) & \longrightarrow & U(pAp)/\widetilde{SU}(pAp) \\ {\scriptstyle \xi^{p,\bar{p}}}\downarrow & & {\scriptstyle \chi^{p,\bar{p}}}\downarrow \\ \mathrm{AffT}(\bar{p}B\bar{p})/\widetilde{\rho K_0}(\bar{p}B\bar{p}) & \longrightarrow & U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p}) \end{array} \tag{IV}$$

and

$$\begin{array}{ccc} U(pAp)/\widetilde{SU}(pAp) & \longrightarrow & K_1(pAp)/\operatorname{tor} K_1(pAp) \\ {\scriptstyle \chi^{p,\bar{p}}}\downarrow & & {\scriptstyle \alpha_1}\downarrow \\ U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p}) & \longrightarrow & K_1(\bar{p}B\bar{p})/\operatorname{tor} K_1(\bar{p}B\bar{p}) \end{array} \tag{V}$$

commute, where $\alpha_1$ is induced by $\alpha$.

We denote the map from $\mathrm{Inv}(A)$ to $\mathrm{Inv}(B)$ by

$$(\alpha,\xi,\chi):\big(\underline{K}(A);\{\mathrm{AffT}(pAp)\}_{[p]\in\Sigma A};\{U(pAp)/\widetilde{SU}(pAp)\}_{[p]\in\Sigma A}\big) \\ \to \big(\underline{K}(B);\{\mathrm{AffT}(\bar{p}B\bar{p})\}_{[\bar{p}]\in\Sigma B};\{U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p})\}_{[\bar{p}]\in\Sigma B}\big).$$

Completely similar to [Nielsen and Thomsen 1996, Lemma 3.2] and [Thomsen 1997, Lemma 6.5], we have the following propositions.

**Proposition 2.37.** *Let unital $C^*$-algebras $A$, $B$ satisfy $\rho(\mathcal{P}_1K_0(A))=\rho K_0(A)$, $\rho(\mathcal{P}_1K_0(B))=\rho K_0(B)$ and $\overline{DU_0(A)}=\overline{DU(A)}$, $\overline{DU_0(B)}=\overline{DU(B)}$. In particular, let $A$, $B\in\mathcal{HD}$ or $A\mathcal{HD}$ be unital $C^*$-algebras. Assume that*

$$\psi_1:K_1(A)\to K_1(B)\quad\text{and}\quad\psi_0:\mathrm{AffT}(A)/\overline{\rho K_0(A)}\to\mathrm{AffT}(B)/\overline{\rho K_0(B)}$$

*are group homomorphisms such that $\psi_0$ is a contraction with respect to $d_A$ and $d_B$. Then there is a group homomorphism*

$$\psi:U(A)/\overline{DU(A)}\to U(B)/\overline{DU(B)}$$

*which is a contraction with respect to $D_A$ and $D_B$ such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{AffT}(A)/\overline{\rho K_0(A)} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) & \longrightarrow & 0 \\ & & \downarrow{\psi_0} & & \downarrow{\psi} & & \downarrow{\psi_1} & & \\ 0 & \longrightarrow & \mathrm{AffT}(A)/\overline{\rho K_0(B)} & \xrightarrow{\lambda_B} & U(B)/\overline{DU(B)} & \xrightarrow{\pi_B} & K_1(B) & \longrightarrow & 0 \end{array}$$

*commutes. If $\psi_0$ is an isometric isomorphism and $\psi_1$ is an isomorphism, then $\psi$ is an isometric isomorphism.*

**Proposition 2.38.** *Let unital $C^*$-algebras $A$, $B$ satisfy $\rho(\mathcal{P}_1K_0(A))=\rho K_0(A)$, $\rho(\mathcal{P}_1K_0(B))=\rho K_0(B)$ and $\overline{DU_0(A)}=\overline{DU(A)}$, $\overline{DU_0(B)}=\overline{DU(B)}$. In particular, let $A$, $B\in\mathcal{HD}$ or $A\mathcal{HD}$ be unital $C^*$-algebras. Assume that*

$$\psi_1:K_1(A)\to K_1(B)\quad\text{and}\quad\psi_0:\mathrm{AffT}(A)/\widetilde{\rho K_0}(A)\to\mathrm{AffT}(B)/\widetilde{\rho K_0}(B)$$

*are group homomorphisms such that $\psi_0$ is a contraction with respect to $\widetilde{d}_A$ and $\widetilde{d}_B$. Then there is a group homomorphism*

$$\psi:U(A)/\widetilde{SU}(A)\to U(B)/\widetilde{SU}(B)$$

*which is a contraction with respect to $\widetilde{D}_A$ and $\widetilde{D}_B$ such that the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{AffT}(A)/\widetilde{\rho K_0}(A) & \xrightarrow{\tilde{\lambda}_A} & U(A)/\widetilde{SU}(A) & \xrightarrow{\widetilde{\pi}_A} & K_1(A)/\operatorname{tor}K_1(A) & \longrightarrow & 0 \\ & & \downarrow{\psi_0} & & \downarrow{\psi} & & \downarrow{\psi_1} & & \\ 0 & \longrightarrow & \mathrm{AffT}(A)/\widetilde{\rho K_0}(B) & \xrightarrow{\tilde{\lambda}_B} & U(B)/\widetilde{SU}(B) & \xrightarrow{\widetilde{\pi}_B} & K_1(B)/\operatorname{tor}K_1(B) & \longrightarrow & 0 \end{array}$$

*commutes. If $\psi_0$ is an isometric isomorphism and $\psi_1$ is an isomorphism, then $\psi$ is an isometric isomorphism.*

**Remark 2.39.** As in Proposition 2.38 (or Proposition 2.37), for each fixed pair $p \in A$, $\bar{p} \in B$ with $\alpha([p]) = [\bar{p}]$, if we have an isometric isomorphism between the quotients $\mathrm{AffT}(pAp)/\widetilde{\rho K_0}(pAp)$ and $\mathrm{AffT}(\bar{p}B\bar{p})/\widetilde{\rho K_0}(\bar{p}B\bar{p})$ (or between $\mathrm{AffT}(pAp)/\overline{\rho K_0(pAp)}$ and $\mathrm{AffT}(\bar{p}B\bar{p})/\overline{\rho K_0(\bar{p}B\bar{p})}$) and an isomorphism between $K_1(pAp)$ and $K_1(pBp)$, then we also have an isometric isomorphism between $U(pAp)/\widetilde{SU}(pAp)$ and $U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p})$ (or between $U(pAp)/\overline{DU(pAp)}$ and $U(\bar{p}B\bar{p})/\overline{DU(\bar{p}B\bar{p})}$) making both diagrams (IV) and (V) commute. This is the reason $U(A)/\overline{DU(A)}$ is not included in the Elliott invariant in the classification of simple $C^*$-algebras. For our setting, even though for each pair of projections $(p, \bar{p})$ with $\alpha([p]) = [\bar{p}]$, we can find an isometric isomorphism between $U(pAp)/\widetilde{SU}(pAp)$ and $U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p})$, provided that the other parts of invariants $\mathrm{Inv}^0(A)$ and $\mathrm{Inv}^0(B)$ are isomorphic, we still cannot make such a system of isometric isomorphisms compatible — that is, we cannot make the diagram (II) commute for $p < q$. We present two nonisomorphic $C^*$-algebras $A$ and $B$ in our class such that $\mathrm{Inv}^0(A) \cong \mathrm{Inv}^0(B)$ in the next section, where $\mathrm{Inv}^0(B)$ is defined in 2.11. Hence it is essential to include $\{U(pAp)/\widetilde{SU}(pAp)\}_{p\in\Sigma}$ with the compatibility as part of $\mathrm{Inv}(A)$.

**2.40.** Replacing $U(pAp)/\widetilde{SU}(pAp)$, one can also use $U(pAp)/\overline{DU(pAp)}$ as the part of the invariant. That is, one can define $\mathrm{Inv}'(A)$ as

$$\big(\underline{K}(A);\ \underline{K}(A)^+;\ \Sigma A;\ \{\mathrm{AffT}(pAp)\}_{[p]\in\Sigma A};\ \{U(pAp)/\overline{DU(pAp)}\}_{[p]\in\Sigma A}\big),$$

with corresponding compatibility condition — one needs to change diagrams (IV) and (V) to the corresponding ones. It is not difficult to see that $\mathrm{Inv}'(A) \cong \mathrm{Inv}'(B)$ implies $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$. We choose the formulation of $\mathrm{Inv}(A)$, since it is much more convenient for the proof of the main theorem in [Gong et al. 2016] and it is formally a weaker requirement than the one to require the isomorphism between $\mathrm{Inv}'(A)$ and $\mathrm{Inv}'(B)$, and the theorem is formally stronger. (Let us point out that, in the construction of the example (and its proof) in Section 3 of this article, $\mathrm{Inv}'(A)$ is as convenient as $\mathrm{Inv}(A)$, and therefore if only for the sake of the example in Section 3 of this paper, it is not necessary to introduce $\widetilde{SU}(A)$.)

Furthermore, it is straightforward to check the following proposition:

**Proposition 2.41.** *Let unital $C^*$-algebras $A$, $B$ satisfy $\rho(\mathcal{P}_1K_0(A)) = \rho K_0(A)$, $\rho(\mathcal{P}_1K_0(B)) = \rho K_0(B)$ and $\overline{DU_0(A)} = \overline{DU(A)}$, $\overline{DU_0(B)} = \overline{DU(B)}$. In particular, let $A$, $B \in \mathcal{HD}$ or $\mathcal{AHD}$ be unital $C^*$-algebras. Suppose that $K_1(A) = \mathrm{tor}(K_1(A))$ and $K_1(B) = \mathrm{tor}(K_1(B))$. Then $\mathrm{Inv}^0(A) \cong \mathrm{Inv}^0(B)$ implies $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$.*

*Proof.* It follows from the fact that any isomorphism

$$\xi^{p,\bar{p}}:\ \mathrm{AffT}(pAp)/\widetilde{\rho K_0}(pAp)\to \mathrm{AffT}(\bar{p}B\bar{p})/\widetilde{\rho K_0}(\bar{p}B\bar{p})$$

induces a unique isomorphism

$$\chi^{p,\bar{p}}: U(pAp)/\widetilde{SU}(pAp)\to U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p}).$$

(Note that by the split exact sequence in Lemma 2.34, $\mathrm{AffT}(pAp)/\widetilde{\rho K_0}(pAp)\cong U(pAp)/\widetilde{SU}(pAp)$.) □

The following calculations and notations will be used in [Gong et al. 2016].

**2.42.** In general, for $A=\bigoplus A^i$, we have

$$\widetilde{SU}(A)=\bigoplus_i \widetilde{SU}(A^i).$$

For $A=PM_l(C(X))P\in\mathcal{HD}$, we have $\widetilde{SU}(A)=\widetilde{DU}(A)$, and for $A=M_l(I_k)$, $\widetilde{SU}(A)=\widetilde{DU}(A)\oplus K_1(A)$. For both cases, $U(A)/\widetilde{SU}(A)$ can be identified with $C_1(X,S^1):=C(X,S^1)/\{\text{constant functions}\}$, or in the case $A=M_l(I_k)$, with $C_1([0,1],S^1)=C([0,1],S^1)/\{\text{constant functions}\}$.

Furthermore, $C_1(X,S^1)$ can be identified as the set of continuous functions from $X$ to $S^1$ such that $f(x_0)=1$ for a certain fixed base point $x_0\in X$. For $X=[0,1]$, we choose 0 to be the base point. For $X=S^1$, we choose $1\in S^1$ to be the base point.

**2.43.** Let $A=\bigoplus_{i=1}^n A^i\in\mathcal{HD}$, $B=\bigoplus_{j=1}^m B^j\in\mathcal{HD}$. In this subsection we discuss some consequences of the compatibility of the maps between AffT spaces. Let

$$p=\bigoplus p^i<q=\bigoplus q^i\in A\quad\text{and}\quad \bar{p}=\bigoplus_{j=1}^m\bar{p}^j<\bar{q}=\bigoplus_{j=1}^m\bar{q}^j\in B$$

be projections satisfying $\alpha([p])=[\bar{p}]$ and $\alpha([q])=[\bar{q}]$. Suppose two unital positive linear maps $\xi_1:\mathrm{AffT}(pAp)\to\mathrm{AffT}(\bar{p}B\bar{p})$ and $\xi_2:\mathrm{AffT}(qAq)\to\mathrm{AffT}(\bar{q}B\bar{q})$ are compatible with $\alpha$ (see diagram (2.13)) and compatible with each other (see diagram (2.10)). Since the (not necessarily unital) maps $\mathrm{AffT}(pAp)\to\mathrm{AffT}(qAq)$ and $\mathrm{AffT}(\bar{p}B\bar{p})\to\mathrm{AffT}(\bar{q}B\bar{q})$ induced by inclusions are injective, we know that the map $\xi_1$ is completely determined by $\xi_2$. Let

$$\xi_2^{i,j}:\mathrm{AffT}(q^iAq^i)\to\mathrm{AffT}(\overline{q^j}B^j\overline{q^j})\quad\text{or}\quad \xi_1^{i,j}:\mathrm{AffT}(p^iAp^i)\to\mathrm{AffT}(\overline{p^j}B^j\overline{p^j}))$$

be the corresponding component of the map $\xi_2$ (or $\xi_1$). If $p^i\neq0$ and $\bar{p}^j\neq0$, then $\xi_1^{i,j}$ is given by the following formula: for any $f\in\mathrm{AffT}(p^iA^ip^i)=C_{\mathbb{R}}(\mathrm{Sp}(A^i))$ ($\cong\mathrm{AffT}(q^iAq^i)$),

$$\xi_1^{i,j}(f)=\frac{\operatorname{rank}\overline{q_j}}{\operatorname{rank}\overline{p_j}}\cdot\frac{\operatorname{rank}\alpha^{i,j}(p^i)}{\operatorname{rank}\alpha^{i,j}(q^i)}\cdot\xi_2^{i,j}(f).$$

In particular, if $q = \mathbb{1}_A$ with $\bar{q} = \alpha_0[\mathbb{1}_A]$, and $\xi_2 = \xi : \mathrm{AffT}(A) \to \mathrm{Aff}\, \alpha_0[\mathbb{1}_A] B \alpha_0[\mathbb{1}_A]$ (note that since $\mathrm{AffT}(QBQ)$ only depends on the unitary equivalence class of $Q$, it is convenient to denote it as $\mathrm{AffT}([Q]B[Q])$), then we denote $\xi_1$ by $\xi|_{([p],\alpha[p])}$. Even for the general case, we can also write $\xi_1 = \xi_2|_{([p],\alpha[p])}$, when $p < q$ as above.

**2.44.** As in 2.43, let $A = \bigoplus_{i=1}^n A^i$, $B = \bigoplus_{j=1}^m B^j$ and $p < q \in A$, $\bar{p} < \bar{q} \in B$, with $\alpha_0[p] = [\bar{p}]$ and $\alpha_0[q] = [\bar{q}]$. If

$$\gamma_1 : U(pAp)/\widetilde{SU}(pAp) \to U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p})$$

is compatible with

$$\gamma_2 : U(qAq)/\widetilde{SU}(qAq) \to U(\bar{q}B\bar{q})/\widetilde{SU}(\bar{q}B\bar{q}),$$

then $\gamma_1$ is completely determined by $\gamma_2$ (since both maps

$$U(pAp)/\widetilde{SU}(pAp) \to U(qAq)/\widetilde{SU}(qAq),$$
$$U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p}) \to U(\bar{q}B\bar{q})/\widetilde{SU}(\bar{q}B\bar{q})$$

are injective). Therefore we can denote $\gamma_1$ by $\gamma_2|_{([p],\alpha[p])}$.

**2.45.** Let us point out that, in 2.43 and 2.44, if $A \in A\mathcal{HD}$ and $B \in A\mathcal{HD}$, $\xi_1$ is not completely determined by $\xi_2$ and $\gamma_1$ is not completely determined by $\gamma_2$.

## 3. The counterexample

**3.1.** In this section, we present an example of $A\mathbb{T}$ algebras to prove that $\mathrm{Inv}'(A)$ or $\mathrm{Inv}(A)$ is not completely determined by $\mathrm{Inv}^0(A)$. That is, the Hausdorffified algebraic $K_1$-groups $\{U(pAp)/\overline{DU(pAp)}\}_{p\in\mathrm{proj}(A)}$ or $\{U(pAp)/\widetilde{SU}(pAp)\}_{p\in\mathrm{proj}(A)}$ with the corresponding compatibilities are indispensable as a part of the invariant for $\mathrm{Inv}'(A)$ or $\mathrm{Inv}(A)$. This is one of the essential differences between the simple $C^*$-algebras and the $C^*$-algebras with the ideal property. In fact, for all the unital $C^*$-algebras $A$ satisfying a reasonable condition (e.g., $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$ and $\overline{DU_0(A)} = \overline{DU(A)}$), we have

$$U(pAp)/\overline{DU(pAp)} \cong \mathrm{AffT}(pAp)/\overline{\rho K_0(pAp)} \oplus K_1(pAp),$$
$$U(pAp)/\widetilde{SU}(pAp) \cong \mathrm{AffT}(pAp)/\widetilde{\rho K_0}(pAp) \oplus K_1(pAp)/\operatorname{tor} K_1(pAp),$$

i.e., the metric groups $U(pAp)/\overline{DU(pAp)}$ and $U(pAp)/\widetilde{SU}(pAp)$ themselves are completely determined by $\mathrm{AffT}(pAp)$ and $K_1(pAp)$, which are included in other parts of the invariants, i.e., they are determined by $\mathrm{Inv}^0(A)$, but the compatibilities make the difference. The point is that the above isomorphisms are not natural and therefore the isomorphisms corresponding to the cutting down algebras $pAp$ and $qAq$ $(p < q)$ may not be chosen to be compatible.

As pointed out in 2.40, $\mathrm{Inv}'(A) \cong \mathrm{Inv}'(B)$ implies $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$. For the $C^*$-algebras $A$ and $B$ constructed in this paper, we only need to prove $\mathrm{Inv}^0(A) \cong \mathrm{Inv}^0(B)$ but $\mathrm{Inv}(A) \ncong \mathrm{Inv}(B)$. Consequently, $\mathrm{Inv}'(A) \ncong \mathrm{Inv}'(B)$.

**3.2.** Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11, \ldots, p_n$ be the first $n$ prime numbers, and let $1 < k_1 < k_2 < k_3 < \cdots$ be a sequence of positive integers. Let

$$\begin{aligned}
&A_1 = B_1 = C(S^1),\\
&A_2 = B_2 = M_{p_1^{k_1}}(C[0,1]) \oplus M_{p_1^{k_1}}(C(S^1)) = A_1^1 \oplus A_1^2 = B_1^1 \oplus B_1^2,\\
&A_3 = B_3 = M_{p_1^{k_1}p_1^{k_2}}(C[0,1]) \oplus M_{p_1^{k_1}p_2^{k_2}}(C[0,1]) \oplus M_{p_1^{k_1}p_2^{k_2}}(C(S^1)),\\
&A_4 = B_4 = M_{p_1^{k_1}p_1^{k_2}p_1^{k_3}}(C[0,1]) \oplus M_{p_1^{k_1}p_2^{k_2}p_2^{k_3}}(C[0,1])\\
&\qquad\qquad \oplus M_{p_1^{k_1}p_2^{k_2}p_3^{k_3}}(C[0,1]) \oplus M_{p_1^{k_1}p_2^{k_2}p_3^{k_3}}(C(S^1)).
\end{aligned}$$

In general, let

$$\begin{aligned}
A_n = B_n &= \bigoplus_{i=1}^{n-1} M_{p_1^{k_1}p_2^{k_2}\cdots p_i^{k_i}p_i^{k_{i+1}}\cdots p_i^{k_{n-1}}}(C[0,1]) \oplus M_{p_1^{k_1}p_2^{k_2}\cdots p_{n-1}^{k_{n-1}}}(C(S^1))\\
&= \bigoplus_{i=1}^{n-1} M_{\prod_{j=1}^{i} p_j^{k_j}\cdot\prod_{j=i+1}^{n-1} p_i^{k_j}}(C[0,1]) \oplus M_{\prod_{i=1}^{n-1} p_i^{k_i}}(C(S^1)).
\end{aligned}$$

For $1 \le i \le n-1$, let $[n,i] = \prod_{j=1}^{i} p_j^{k_j} \cdot \prod_{j=i+1}^{n-1} p_i^{k_j}$ and $[n,n] = [n,n-1]$. Then

$$A_n = B_n = \bigoplus_{i=1}^{n-1} M_{[n,i]}(C[0,1]) \oplus M_{[n,n]}(C(S^1)).$$

(Note that the last two blocks have the same size $[n,n] = [n,n-1]$.)

Note that $[n+1,i] = [n,i] \cdot p_i^{k_n}$ for all $i \in \{1,2,\ldots,n-1\}$ and $[n+1,n+1] = [n+1,n] = [n,n] \cdot p_n^{k_n}$.

**3.3.** Let $\{t_n\}_{n=1}^\infty$ be a dense subset of $[0,1]$ and $\{z_n\}_{n=1}^\infty$ be a dense subset of $S^1$. In this subsection, we define the connecting homomorphisms

$$\phi_{n,n+1} : A_n \to A_{n+1} \quad \text{and} \quad \psi_{n,n+1} : B_n \to B_{n+1}.$$

For $i \le n-1$, define $\phi_{n,n+1}^{i,i} = \psi_{n,n+1}^{i,i} : M_{[n,i]}(C[0,1]) \to M_{[n+1,i]}(C[0,1])$ $(= M_{[n,i]\cdot p_i^{k_n}}(C[0,1]))$ by

$$\begin{aligned}
\phi_{n,n+1}^{i,i}(f)(t) &= \psi_{n,n+1}^{i,i}(f)(t)\\
&= \mathrm{diag}(\underbrace{f(t), f(t), \ldots, f(t)}_{p_i^{k_n}-1}, f(t_n)) \quad \text{for all } f \in M_{[n,i]}(C[0,1]).
\end{aligned}$$

Define $\phi_{n,n+1}^{n,n+1} = \psi_{n,n+1}^{n,n+1} : M_{[n,n]}(C(S^1)) \to M_{[n+1,n+1]}(C(S^1)) = M_{[n,n]\cdot p_n^{k_n}}(C(S^1))$ by

$$\begin{aligned}\phi_{n,n+1}^{n,n+1}(f)(z) &= \psi_{n,n+1}^{n,n+1}(f)(z)\\ &= \operatorname{diag}(f(z), \underbrace{f(z_n), f(z_n), \dots, f(z_n)}_{p_n^{k_n}-1}) \quad \text{for all } f \in M_{[n,n]}(C(S^1)).\end{aligned}$$

But $\phi_{n,n+1}^{n,n}$ and $\psi_{n,n+1}^{n,n}$ are defined differently — this is the only nonequal component of $\phi_{n,n+1}$ and $\psi_{n,n+1}$.

Let $l = p_n^{k_n} - 1$. Then

$$\begin{aligned}\phi_{n,n+1}^{n,n}(f)(t) &= \operatorname{diag}\big(f(e^{2\pi i t}), f(e^{-2\pi i t}), f(e^{2\pi i/l}), \dots, f(e^{2\pi i(l-1)/l})\big),\\ \psi_{n,n+1}^{n,n}(f)(t) &= \operatorname{diag}\big(f(e^{2\pi i l_n t}), f(e^{-2\pi i\cdot 0/l}), f(e^{2\pi i/l}), \dots, f(e^{2\pi i(l-1)/l})\big)\end{aligned}$$

for any $f \in M_{[n,n]}(C(S^1))$, where $l_n = 4^n \cdot [n+1, n] \in \mathbb{N}$.

Let all other parts $\phi_{n,n+1}^{i,j}$, $\psi_{n,n+1}^{i,j}$ of $\phi_{n,n+1}$, $\psi_{n,n+1}$ (except $i = j \le n$ or $i = n$, $j = n+1$, as defined above) be zero. Note that all $\phi_{n,n+1}^{i,j}$, $\psi_{n,n+1}^{i,j}$ are either injective or zero.

Let $A = \lim(A_n, \phi_{n,m})$, $B = \lim(B_n, \psi_{n,m})$. Then it follows from the density of the sets $\{t_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ that both $A$ and $B$ have the ideal property (see the characterization theorem for *AH* algebras with the ideal property [Pasnicu 2000]).

**Proposition 3.4.** *There is an isomorphism between* $\operatorname{Inv}^0(A)$ *and* $\operatorname{Inv}^0(B)$ (see 2.11), *that is, there is an isomorphism*

$$\alpha : (\underline{K}(A); \underline{K}(A)^+; \Sigma A) \to (\underline{K}(B); \underline{K}(B)^+; \Sigma B)$$

*which is compatible with Bockstein operations, and for pairs* $(p, q)$ *with* $p \in \Sigma A$, $q \in \Sigma B$ *and* $\alpha([p]) = [q]$, *there are associated unital positive linear maps*

$$\xi^{p,q} : \operatorname{AffT}(pAp) \to \operatorname{AffT}(qBq)$$

*which are compatible in the sense of 2.9* (see diagram (2.10)).

*Proof.* As $KK(\phi_{n,m}) = KK(\psi_{n,m})$ and $\phi_{n,m} \sim_h \psi_{n,m}$, the identity maps $\eta_n : A_n \to B_n$ induce a shape equivalence between $A = \lim(A_n, \phi_{n,m})$ and $B = \lim(B_n, \psi_{n,m})$, and therefore induce an isomorphism

$$\alpha : (\underline{K}(A); \underline{K}(A)^+; \Sigma A) \to (\underline{K}(B); \underline{K}(B)^+; \Sigma B).$$

Note that $\phi_{n,n+1}^{i,i} = \psi_{n,n+1}^{i,i}$ for $i \le n-1$, $\phi_{n,n+1}^{n,n+1} = \psi_{n,n+1}^{n,n+1}$, and

$$\big\| \operatorname{AffT} \phi_{n,n+1}^{n,n}(f) - \operatorname{AffT} \psi_{n,n+1}^{n,n}(f) \big\| \le \frac{2}{p_n^{k_n}} \| f \|$$

(see the definition of $\phi_{n,n+1}$ and $\psi_{n,n+1}$). Therefore,

$$\mathrm{AffT}\,\eta_n : \mathrm{AffT}(A_n) \to \mathrm{AffT}(B_n) \quad \text{and} \quad \mathrm{AffT}\,\eta_n^{-1} : \mathrm{AffT}(B_n) \to \mathrm{AffT}(A_n)$$

induce the approximately intertwining diagram

$$\begin{array}{ccccccc} \mathrm{AffT}(A_1) & \longrightarrow & \mathrm{AffT}(A_2) & \longrightarrow & \cdots & \longrightarrow & \mathrm{AffT}(A) \\ \downarrow\uparrow & & \downarrow\uparrow & & & & \\ \mathrm{AffT}(B_1) & \longrightarrow & \mathrm{AffT}(B_2) & \longrightarrow & \cdots & \longrightarrow & \mathrm{AffT}(B) \end{array}$$

in the sense of [Elliott 1993b]. Therefore, there is a unital positive isomorphism

$$\xi : \mathrm{AffT}(A) \to \mathrm{AffT}(B).$$

Also, for any projection $[P] \in K_0(A)$, there is a projection $P_n \in A_n = B_n$ (for $n$ large enough) with $P_n^i = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0) \in M_{[n,i]}(C(X_{n,i}))$, where $X_{n,i} = [0, 1]$ for $i \le n-1$, and $X_{n,n} = S^1$, such that $\phi_{n,\infty}([P_n]) = [P] \in K_0(A)$. Note that for any constant functions $f \in A_n^i = B_n^i$ (e.g., $P_n^i$ above) and for any $j$, $\phi_{n,n+1}^{i,j}(f)$ and $\psi_{n,n+1}^{i,j}(f)$ are still constant functions, and $\phi_{n,n+1}^{i,j}(f) = \psi_{n,n+1}^{i,j}(f)$. That is, we have

$$\phi_{n,n+1}(P_n) = \psi_{n,n+1}(P_n) \quad (\text{denoted by } P_{n+1}),$$

$$\phi_{n,m}(P_n) = \psi_{n,m}(P_n) \qquad (\text{denoted by } P_m).$$

Let $P_\infty = \phi_{n,\infty}(P_n)$ and $Q_\infty = \psi_{n,\infty}(P_n)$. Then the identity maps $\{\eta_m\}_{m>n}$ also induce the approximate intertwining diagram

$$\begin{array}{ccccccc} \mathrm{AffT}(P_nA_nP_n) & \longrightarrow & \mathrm{AffT}(P_{n+1}A_{n+1}P_{n+1}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{AffT}(P_\infty A P_\infty) \\ \downarrow\uparrow & & \downarrow\uparrow & & & & \\ \mathrm{AffT}(P_nB_nP_n) & \longrightarrow & \mathrm{AffT}(P_{n+1}B_{n+1}P_{n+1}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{AffT}(Q_\infty B Q_\infty) \end{array}$$

and hence induce a positive linear isomorphism

$$\xi^{[P],\alpha[P]} : \mathrm{AffT}(P_\infty A P_\infty) \to \mathrm{AffT}(Q_\infty B Q_\infty).$$

(Note that $[P_\infty] = [P]$ and $[Q_\infty] = \alpha[P]$ in $K_0(A)$ and $K_0(B)$, respectively.) Evidently those maps are compatible since, they are induced by the same sequence of homomorphisms $\{\eta_n\}$ and $\{\eta_n^{-1}\}$. □

Definition 3.5 and Proposition 3.6 are inspired by [Elliott 1997].

**Definition 3.5.** Let $C = \lim(C_n, \phi_{n,m})$ be an $\mathcal{AHD}$ inductive limit. We say the system $(C_n, \phi_{n,m})$ has the uniformly varied determinant if for any $C_n^i = M_{[n,i]}(C(S^1))$

(that is, $C_n^i$ has spectrum $S^1$), $C_{n+1}^j$, and $f \in C_n^i$ defined by

$$f(z) = \begin{pmatrix} z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{[n,i]\times[n,i]} \qquad \text{for all } z \in S^1,$$

we have either that $\det(\phi_{n,n+1}^{i,j}(f)(x))$ is constant for $x \in \mathrm{Sp}(C_{n+1}^j) \neq S^1$ or that $\det(\phi_{n,n+1}^{i,j}(f)(z)) = \lambda z^k$ ($\lambda \in \mathbb{C}$) for $z \in \mathrm{Sp}(C_{n+1}^j) = S^1$, where $j$ satisfies $\phi_{n,n+1}^{i,j} \neq 0$ and the determinant is taken inside $\phi_{n,n+1}^{i,j}(\mathbb{1}_{C_n^i}) C_{n+1}^j \phi_{n,n+1}^{i,j}(\mathbb{1}_{C_n^i})$.

**Proposition 3.6.** *If the inductive limit system $C = (C_n, \phi_{n,m})$ has the uniformly varied determinant, then for any elements $[p] \in \sum C$, there are splitting maps*

$$K_1(pCp)/\operatorname{tor} K_1(pCp) \xrightarrow{S_{pCp}} U(pCp)/\widetilde{SU}(pCp)$$

*of the exact sequences*

$$0 \to \mathrm{AffT}(pCp)/\rho\widetilde{K}_0(pCp) \to U(pCp)/\widetilde{SU}(pCp) \xrightarrow{\pi_{pCp}} K_1(pCp)/\operatorname{tor} K_1(pCp) \to 0$$

(*that is*, $\pi_{pCp} \circ S_{pCp} = \mathrm{id}$ *on* $K_1(pCp)/\operatorname{tor} K_1(pCp)$) *such that the system of maps* $\{S_{pCp}\}_{[p]\in\sum C}$ *are compatible in the following sense*: *if* $p < q$, *then the diagram*

$$\begin{array}{ccc} K_1(pCp)/\operatorname{tor} K_1(pCp) & \xrightarrow{S_{pCp}} & U(pCp)/\widetilde{SU}(pCp) \\ \downarrow & & \downarrow \\ K_1(qCq)/\operatorname{tor} K_1(qCq) & \xrightarrow{S_{qCq}} & U(qCq)/\widetilde{SU}(qCq) \end{array} \tag{3.7}$$

*commutes*, *where the vertical maps are induced by the inclusions* $pCp \to qCq$.

*Proof.* Fix $p \in C$. Let $x \in K_1(pCp)/\operatorname{tor} K_1(pCp)$. There exist a $C_n$ and $p_n \in C_n$ such that $[\phi_{n,\infty}(p_n)] = [p] \in K_0(C)$. Without lose of generality, we can assume $\phi_{n,\infty}(p_n) = p$. By increasing $n$ if necessary, we can assume that there is an element $x_n \in K_1(p_n C_n p_n)/\operatorname{tor} K_1(p_n C_n p_n)$ such that

$$(\phi_{n,\infty})_*(x_n) = x \in K_1(pCp)/\operatorname{tor} K_1(pCp).$$

Write $p_n C_n p_n = D = \bigoplus D^i$. Let $I = \{i : \mathrm{Sp}(D^i) = S^1\}$. For $i \in I$, $D^i$ can be identified with $M_{l_i}(C(S^1))$. Let $u_i \in D^i$ be defined by

$$u_i(z) = \begin{pmatrix} z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{l_i\times l_i} \qquad \text{for all } z \in S^1,$$

which represents the standard generator of $K_1(D^i)$. Then $x_n$ can be represented by

$$u=\bigoplus_{i\in I}u_i^{k_i}\oplus\bigoplus_{j\notin I}\mathbb{1}_{D^j}\in\bigoplus_{i\in I}D^i\oplus\bigoplus_{j\notin I}D^j=D\subseteq p_nC_np_n.$$

Define $S(x)=[\phi_{n,\infty}(u)]\in U(pCp)/\widetilde{SU}(pCp)$. Note that all unitaries with constant determinants are in $\widetilde{SU}$, and that the inductive system has the uniformly varied determinant. It is routine to verify that $S(x)$ is well defined and the system $\{S_{pCp}\}_{[p]\in\sum C}$ makes the diagram (3.7) commute. □

**3.8.** Let $\mathcal{A}$ be a unital $C^*$-algebra. Then $\mathrm{AffT}(\mathcal{A})$ is a real Banach space with quotient space $\mathrm{AffT}(\mathcal{A})/\widetilde{\rho K_0}(\mathcal{A})$. Let us use $\|\cdot\|^{\sim}$ to denote the quotient norm. Note that $\tilde{\lambda}_{\mathcal{A}}$ identifies $U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ with $\mathrm{AffT}(\mathcal{A})/\widetilde{\rho K_0}(\mathcal{A})$. Thus, $U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ is regarded as a real Banach space, whose norm is also denoted by $\|\cdot\|^{\sim}$. In general, we have

$$U(\mathcal{A})/\widetilde{SU}(\mathcal{A})\cong U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})\times K_1(\mathcal{A})/\operatorname{tor}K_1(\mathcal{A}),$$

but the identification is not canonical. Even though $U(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ is not a Banach space, it is an abelian group: for $[u],[v]\in U(\mathcal{A})/\widetilde{SU}(\mathcal{A})$, define $[u]-[v]=[uv^*]$.

The norm $\|\cdot\|^{\sim}$ is related to the metrics $\widetilde{d}_{\mathcal{A}}$ (on $\mathrm{AffT}(\mathcal{A})/\widetilde{\rho K_0}(\mathcal{A})$; see 2.28) and $\widetilde{D}_{\mathcal{A}}$ (on $U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$; see 2.33) as below. Let $\varepsilon<1$. For $f,g\in\mathrm{AffT}(\mathcal{A})/\widetilde{\rho K_0}(\mathcal{A})$,

$$\|f-g\|^{\sim}<\frac{\varepsilon}{2\pi}\quad\Longrightarrow\quad\widetilde{d}_{\mathcal{A}}(f,g)<\varepsilon\quad\Longrightarrow\quad\|f-g\|^{\sim}<\frac{\varepsilon}{4}.$$

And for any $[u],[v]\in U(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ with $[u]-[v]=[uv^*]\in U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$,

$$\big\|[u]-[v]\big\|^{\sim}<\frac{\varepsilon}{2\pi}\quad\Longrightarrow\quad\widetilde{D}_{\mathcal{A}}([u],[v])<\varepsilon\quad\Longrightarrow\quad\big\|[u]-[v]\big\|^{\sim}<\frac{\varepsilon}{4}.$$

For $\mathcal{A}=PM_l(C(X))P\in\mathcal{HD}$ or $\mathcal{A}=M_l(I_k)$ (in this case we also denote $[0,1]$ by $X$), there are canonical identifications

$$U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})\cong\mathrm{AffT}(\mathcal{A})/\widetilde{\rho K_0}(\mathcal{A})\cong C(X,\mathbb{R})/\{\text{constant functions}\}$$

(see 2.42). Choose a base point $x_0\in X$. Let $C_{x_0}(X,\mathbb{R})$ be the set of functions $f\in C(X,\mathbb{R})$ with $f(x_0)=0$. Then $C(X,\mathbb{R})/\{\text{constant functions}\}\cong C_{x_0}(X,\mathbb{R})$. For $[f]\in\mathrm{AffT}(\mathcal{A})/\widetilde{\rho K_0}(\mathcal{A})$ (or $[f]\in U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$) identified with a function $f\in C_{x_0}(X,\mathbb{R})$, we have

$$\|[f]\|^{\sim}=\frac{1}{2}\big(\max_{x\in X}(f(x))-\min_{x\in X}(f(x))\big)$$

(rather than $\sup_{x\in X}\{|f(x)|\}$).

In the above case, if $p\in\mathcal{A}$ is a nonzero projection, then $U_{\mathrm{tor}}(p\mathcal{A}p)/\widetilde{SU}(p\mathcal{A}p)\cong\mathrm{AffT}(p\mathcal{A}p)/\widetilde{\rho K_0}(p\mathcal{A}p)$ is also identified with $C_{x_0}(X,\mathbb{R})$. Consider the inclusion map $\imath:pAp\to A$. Then the map $\imath_*$ as a map from $U_{\mathrm{tor}}(p\mathcal{A}p)/\widetilde{SU}(p\mathcal{A}p)\cong$

$\mathrm{AffT}(p\mathcal{A}p)/\widetilde{\rho K_0}(p\mathcal{A}p)$ to $U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ can be described as follows: if

$$u \in U_{\mathrm{tor}}(p\mathcal{A}p)/\widetilde{SU}(p\mathcal{A}p) \cong \mathrm{AffT}(p\mathcal{A}p)/\widetilde{\rho K_0}(p\mathcal{A}p)$$

is identified with $f \in C_{x_0}(X, \mathbb{R})$, then $\iota_*(u) \in U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ is identified with

$$\frac{\mathrm{rank}(p)}{\mathrm{rank}(\mathbb{1}_{\mathcal{A}})} f.$$

But $\iota^\natural$ is the identity map from $U_{\mathrm{tor}}(p\mathcal{A}p)/\widetilde{SU}(p\mathcal{A}p) \cong \mathrm{AffT}(p\mathcal{A}p)/\widetilde{\rho K_0}(p\mathcal{A}p)$ to itself (not to $U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$).

**3.9.** It is easy to see that $K_1(A) = K_1(B) = \mathbb{Z}$.

In the definition of $A_n = \bigoplus_{i=1}^n A_n^i$, only one block $A_n^n = M_{[n,n]}(C(S^1))$ has spectrum $S^1$, and only two partial maps $\phi_{n,n+1}^{n,j}$ for $j = n$, $j = n+1$ (of $\phi_{n,n+1}$ from $A_n^n$) are nonzero. Let $f \in A_n^n$ be defined as in Definition 3.5. Then $\det\big(\phi_{n,n+1}^{n,n+1}(f)(z)\big) = z$ and $\det\big(\phi_{n,n+1}^{n,n}(f)(t)\big) = e^{2\pi i t} e^{-2\pi i t} e^{2\pi i/l} e^{2\pi i(2/l)} \cdots e^{2\pi i(l-1)/l} = \pm 1$ (see 3.3). So the inductive limit system $(A_n, \phi_{n,m})$ has the uniformly varied determinant, and therefore the limit algebra $A$ has compatible splitting maps $S_p : K_1(pAp) \to U(pAp)/\widetilde{SU}(pAp)$.

We prove that $B = \lim(B_n, \psi_{n,m})$ does not have such a compatible system of splitting maps $\{K_1(pBp) \to U(pBp)/\widetilde{SU}(pBp)\}_{[p]\in \sum B}$.

Before proving the above fact, let us describe the $K_0$-group of $A$ and $B$. Let

$$G_1 = \left\{ \frac{m}{p_1^l} : m \in \mathbb{Z}, l \in \mathbb{Z}_+ \right\},$$

$$G_2 = \left\{ \frac{m}{p_1^{k_1} p_2^l} : m \in \mathbb{Z}, l \in \mathbb{Z}_+ \right\},$$

$$G_3 = \left\{ \frac{m}{p_1^{k_1} p_2^{k_2} p_3^l} : m \in \mathbb{Z}, l \in \mathbb{Z}_+ \right\},$$

$$\vdots$$

$$G_n = \left\{ \frac{m}{p_1^{k_1} p_2^{k_2} \cdots p_{n-1}^{k_{n-1}} p_n^l} : m \in \mathbb{Z}, l \in \mathbb{Z}_+ \right\},$$

$$G_\infty = \left\{ \frac{m}{p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}} : t \in \mathbb{Z}_+, m \in \mathbb{Z} \right\},$$

where $p_1 = 2$, $p_2 = 3, \ldots, p_i, \ldots$ and $k_1, k_2, \ldots, k_i \ldots$ are defined in 3.2. Then

$$K_0(A) = K_0(B) = \left\{ (a_1, a_2, \ldots, a_n, \ldots) \in \prod_{n=1}^{\infty} G_n : \exists N \text{ such that } a_N = a_{N+1} = \cdots \in \mathbb{Q} \right\} \triangleq \widetilde{G}.$$

Furthermore, their positive cones consist of the elements whose coordinates are nonnegative, and their order units are $[\mathbb{1}_A]=[\mathbb{1}_B]=(1,1,\dots,1,\dots)\in\prod_{n=1}^{\infty}G_n$. Let

$$\begin{aligned}\alpha_0:(K_0(A),K_0(A)^+,[\mathbb{1}_A])&=\big(\widetilde{G},\widetilde{G}^+,(1,1,\dots,1,\dots)\big)\\ &\to(K_0(B),K_0(B)^+,[\mathbb{1}_B])=\big(\widetilde{G},\widetilde{G}^+,(1,1,\dots,1,\dots)\big)\end{aligned}$$

be a scaled ordered isomorphism. Then $\alpha_0((1,1,\dots,1,\dots))=(1,1,\dots,1,\dots)$. Note that an element $x\in\widetilde{G}$ is divisible by power $p_1^n$ (for any $n$) of the first prime number $p_1=2$ if and only if $x=(t,0,0,\dots,0,\dots)\in G_1\subset\widetilde{G}$. Hence $\alpha_0((1,0,0,\dots,0,\dots))=(t,0,0,\dots,0,\dots)$ for some $t\in G_1$ with $t>0$. Hence

$$\alpha_0((0,1,1,\dots,1,\dots))=(1-t,1,1,\dots,1,\dots).$$

Since $\alpha_0$ preserves the positive cone, we have $1-t\geq 0$, which implies $t\leq 1$. On the other hand, $(\alpha_0)^{-1}$ takes $(1,0,0,\dots,0,\dots)$ to $(1/t,0,0,\dots,0,\dots)$. But $(\alpha_0)^{-1}$ also preserves the positive cone. Symmetrically, we get $t\geq 1$. That is, $\alpha_0((1,0,0,\dots,0,\dots))=(1,0,0,\dots,0,\dots)$. Similarly, using the fact that $G_k$ is the subgroup of all elements in $\widetilde{G}$ which can be divisible by any power of $p_k$ — the $k$-th prime number, we can prove that

$$\alpha_0\big((\underbrace{0,\dots,0}_{k-1},1,0,\dots,0,\dots)\big)=\big(\underbrace{0,\dots,0}_{k-1},1,0,\dots,0,\dots\big)\in G_k\subset\widetilde{G}.$$

That is, $\alpha_0$ is the identity on $\widetilde{G}$.

Note that $\mathrm{Sp}(A)=\mathrm{Sp}(B)$ is the one point compactification of $\{1,2,3\dots\}$ — or, in other words, $\{1,2,3\dots,\infty\}$. If we let $I_n$ (or $J_n$) be the primitive ideal $A$ (or $B$) corresponding to $n$ (including $n=\infty$), then

$$K_0(A/I_n)=K_0(B/J_n)=G_n.$$

Note also that if $m'>m>n\in\mathbb{N}$, then $\phi_{m,m'}(A_m^n)\subset A_{m'}^n$ and $\psi_{m,m'}(B_m^n)\subset B_{m'}^n$. Hence $A/I_n=\lim_{n<m\to\infty}(A_m^n,\phi_{m,m'}|_{A_m^n})$ (resp. $B/J_n=\lim_{n<m\to\infty}(B_m^n,\psi_{m,m'}|_{B_m^n})$) are ideals of $A$ (resp. $B$). But $A/I_\infty$ (or $B/J_\infty$) is not an ideal of $A$ (or $B$).

Let $\alpha:(\underline{K}(A),\underline{K}(A)^+,\Sigma A)\to(\underline{K}(B),\underline{K}(B)^+,\Sigma B)$ be an isomorphism. By 3.9 the induced map $\alpha_0$ on $K_0$ group is identity, when both $K_0(A)$ and $K_0(B)$ are identified with $\widetilde{G}$ as scaled ordered groups. That is, $\alpha_0$ is the same as the $\alpha_0$ induced by the shape equivalence in the proof of Proposition 3.4. In particular, if there is an isomorphism $\wedge:A\to B$, then for all $i\leq n-1$, $\wedge_*[(\phi_{n,\infty}(\mathbb{1}_{A_n^i}))]=[\psi_{n,\infty}(\mathbb{1}_{B_n^i})]$. This implies $\wedge(\phi_{n,\infty}(\mathbb{1}_{A_n^i}))=\psi_{n,\infty}(\mathbb{1}_{B_n^i})$, since $\psi_{n,\infty}(\mathbb{1}_{B_n^i})=\mathbb{1}_{B/I_i}$, which is in the center of $B$ (any element in the center of the $C^*$-algebra can only unitary equivalent to itself). Hence it is also true that $\wedge(\phi_{n,\infty}(\mathbb{1}_{A_n^i}))=\psi_{n,\infty}(\mathbb{1}_{B_n^i})$ for $i=n$.

**3.10.** Let $P_1 = \mathbb{1}_B = \psi_{1,\infty}(\mathbb{1}_{B_1})$ and $P_n = \psi_{n,\infty}(\mathbb{1}_{B_n^n})$ for $n > 1$. Then we have $P_1 > P_2 > \cdots > P_n > \cdots$. We prove that there are no splittings

$$K_1(P_n B P_n) \to U(P_n B P_n)/\widetilde{SU}(P_n B P_n)$$

which are compatible for all pairs of projections $P_n > P_m$ (see diagram (3.7)) in the next subsection. Before doing so, we need some preparations.

Set $Q_1 = P_1 - P_2$, $Q_2 = P_2 - P_3, \ldots, Q_n = P_n - P_{n+1}$. Then for each $n$, we have the inductive limit

$$Q_n B Q_n = \lim_{m\to\infty} (B_m^n, \psi_{m,m'}^{n,n})$$

(note that for $m > n$, $\psi_{m,m+1}^{n,j} = 0$ if $j \neq n$), which is the quotient algebra corresponding to the primitive ideal of $n \in \mathrm{Sp}(B) = \{1, 2, 3 \ldots, \infty\}$. Note that $Q_n B Q_n$ is a simple $AI$ algebra. The inductive limit of the $C^*$-algebras

$$B_{n+1}^n \to B_{n+2}^n \to B_{n+3}^n \to \cdots \to Q_n B Q_n$$

induces the inductive limit of the ordered Banach spaces

$$\mathrm{AffT}(B_{n+1}^n) \xrightarrow{\xi_{n+1,n+2}} \mathrm{AffT}(B_{n+2}^n) \xrightarrow{\xi_{n+2,n+3}} \cdots \to \mathrm{AffT}(Q_n B Q_n),$$

whose connecting maps $\xi_{m,m+1} : C_{\mathbb{R}}([0, 1]) \to C_{\mathbb{R}}([0, 1])$ (for $m > n$) satisfy

$$\|\xi_{m,m+1}(f) - f\| \leq \frac{1}{p_n^{k_m}} \|f\| \quad \text{for all } f \in C_{\mathbb{R}}[0, 1],\ m > n.$$

Hence we have the following approximate intertwining diagram:

$$\begin{array}{ccccccccc}
C_{\mathbb{R}}[0,1] & \xrightarrow{\xi_{n,n+1}} & C_{\mathbb{R}}[0,1] & \xrightarrow{\xi_{n+1,n+2}} & C_{\mathbb{R}}[0,1] & \longrightarrow & \cdots & \longrightarrow & \mathrm{AffT}(Q_nBQ_n)\\
\downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & & & \\
C_{\mathbb{R}}[0,1] & \xrightarrow{\mathrm{id}} & C_{\mathbb{R}}[0,1] & \xrightarrow{\mathrm{id}} & C_{\mathbb{R}}[0,1] & \longrightarrow & \cdots & \longrightarrow & C_{\mathbb{R}}[0,1]
\end{array}$$

Consequently, $\mathrm{AffT}(Q_n B Q_n) \cong C_{\mathbb{R}}[0, 1]$, and the maps

$$\xi_{m,\infty} : \mathrm{AffT}(B_m^n) = C_{\mathbb{R}}[0, 1] \to \mathrm{AffT}(Q_n B Q_n) \cong C_{\mathbb{R}}[0, 1]$$

(under the identification) satisfy

$$\|\xi_{m,\infty}(f) - f\| \leq \Big(\frac{1}{p_n^{k_m}} + \frac{1}{p_n^{k_{m+1}}} + \cdots\Big)\|f\| \leq \frac{1}{4}\|f\| \quad \text{for all } f \in C_{\mathbb{R}}[0, 1].$$

Therefore $\|\xi_{m,\infty}(f)\| \geq \frac{3}{4}\|f\|$.

Note that $\rho\widetilde{K}_0(Q_n B Q_n) = \mathbb{R} = \rho\widetilde{K}_0(B_m^n)$ consists of constant functions on $[0, 1]$. Take an element $h \in C_{\mathbb{R}}[0, 1] = \mathrm{AffT}(B_m^n)$. Considering $\xi_{m,\infty}(h)$ as an element of

$\mathrm{AffT}(Q_nBQ_n)/\rho\widetilde{K_0}(Q_nBQ_n)$, we have

$$\|\xi_{m,\infty}(h)\|^{\sim} \geq \frac{1}{2}\cdot\frac{3}{4}\big(\max_{t\in[0,1]} h(t) - \min_{t\in[0,1]} h(t)\big),$$

where $\|\cdot\|^{\sim}$ is defined in 3.8.

**3.11.** We now prove that no compatible splittings

$$S_n : K_1(P_nBP_n) \to U(P_nBP_n)/\widetilde{SU}(P_nBP_n)$$

exist. Suppose such splittings exist. Then consider the generator $x \in K_1(B) = \mathbb{Z}$.

Note that $x \in K_1(P_nBP_n) \cong K_1(B)$ for all $P_n$. Note also that the diagram

$$\begin{array}{ccc} K_1(P_{n+1}BP_{n+1}) & \xrightarrow{S_{n+1}} & U(P_{n+1}BP_{n+1})/\widetilde{SU}(P_{n+1}BP_{n+1}) \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \iota_*} \\ K_1(P_1BP_1) & \xrightarrow{S_1} & U(P_1BP_1)/\widetilde{SU}(P_1BP_1) \end{array}$$

commutes ($P_1BP_1 = B$). The composition

$$U(P_{n+1}BP_{n+1})/\widetilde{SU}(P_{n+1}BP_{n+1}) \xrightarrow{\iota_*} U(P_1BP_1)/\widetilde{SU}(P_1BP_1) \to \bigoplus_{i=1}^{n} U(Q_iBQ_i)/\widetilde{SU}(Q_iBQ_i)$$

is the zero map. (Note that $Q_iBQ_i$ is an ideal of $B$ and is also the quotient $B/J_i$.) Consequently, we have

$$\pi_n^{\natural}(S_1(x)) = \pi_n^{\natural}(\iota_* S_{n+1}(x)) = 0, \qquad (*)$$

where $\pi_n : B \to Q_nBQ_n$ is the quotient map. Let $S_1(x)$ be represented by a unitary $u \in U(B)$. Then there are an $n$ (large enough) and $[u_n] \in U(B_n)/\widetilde{SU}(B_n)$, represented by unitary $u_n \in B_n$, such that

$$\psi^{\natural}_{n,\infty}([u_n]) - S_1(x) \in U_{\mathrm{tor}}(B_n)/\widetilde{SU}(B_n) \quad \text{and} \quad \|\psi^{\natural}_{n,\infty}([u_n]) - S_1(x)\|^{\sim} < \tfrac{1}{16}.$$

Note that

$$(\psi_{n,m})_* : K_1(B_n) \to K_1(B_m)$$

is the identify map from $\mathbb{Z}$ to $\mathbb{Z}$. Let $g \in M_{[n,n]}(C(S^1)) = B_n^n$ be defined by

$$g(z) = \begin{pmatrix} z & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}_{[n,n]\times[n,n]}$$

Then $[g^{-1}u_n] = 0$ in $K_1(B_n)$. By the exactness of the sequence

$$0 \to \mathrm{AffT}(B_n)/\widetilde{\rho K_0}(B_n) \to U(B_n)/\widetilde{SU}(B_n) \to K_1(B_1) \to 0,$$

there is an $h \in \bigoplus_{i=1}^n C_{\mathbb{R}}[0,1] \oplus C_{\mathbb{R}}(S^1) = \mathrm{AffT}(B_n)$ such that

$$[u_n] = [g] \cdot (e^{2\pi i h} \cdot \mathbb{1}_{B_n}) \in U(B_n)/\widetilde{SU}(B_n).$$

Let $\|h\| = M$. Choose $m > n$ such that $4^{m-1} > 8M + 8$.

Consider

$$\psi_{n,m}^{n,m-1} : B_n^n = M_{[n,n]}(C(S^1)) \to B_m^{m-1} = M_{[m,m-1]}(C([0,1])),$$

which is the composition

$$\psi_{m-1,m}^{m-1,m-1} \circ \psi_{n,m-1}^{n,m-1} : M_{[n,n]}(C(S^1)) \to M_{[m-1,m-1]}(C(S^1)) \to M_{[m,m-1]}(C([0,1])).$$

Let $g' = \psi_{n,m}^{n,m-1}(g)$. We know that

$$g'(t) = \psi_{n,m}^{n,m-1}(g)(t) = \begin{pmatrix} e^{2\pi i l_{m-1} t} & & & & \\ & * & & & \\ & & * & & \\ & & & \ddots & \\ & & & & * \end{pmatrix}_{[m,m-1]\times[m,m-1]}$$

where the $*$'s represent constant functions on [0, 1], and therefore

$$g' = e^{2\pi i h'} \quad (\mathrm{mod}\ \widetilde{SU}(B_m^{m-1}))$$

with $h'(t) = \dfrac{l_{m-1}}{[m, m-1]} \cdot t \cdot \mathbb{1}_{[m,m-1]}$. When we identify $U(B_m^{m-1})/\widetilde{SU}(B_m^{m-1})$ with

$$\mathrm{AffT}(B_m^{m-1})/\widetilde{\rho K_0}(B_m^{m-1}) = C_{\mathbb{R}}[0,1]/\{\text{constants}\},$$

$g'$ is identified with $\tilde{h} \in C_{\mathbb{R}}[0,1]$, where

$$\tilde{h}(t) = \frac{l_{m-1}}{[m, m-1]} t.$$

Since $\dfrac{l_{m-1}}{[m, m-1]} \geq 8M + 8$, we have

$$\|\tilde{h}\|^{\sim} = \frac{1}{2}\Big(\max_{t\in[0,1]} \tilde{h}(t) - \min_{t\in[0,1]} \tilde{h}(t)\Big) \geq 4M + 4$$

(see 3.8). On the other hand,

$$[u_n] = [g] + \tilde{\lambda}_{B_n}([h]) \in U(B_n)/\widetilde{SU}(B_n),$$

where $[h] \in \mathrm{AffT}(B_n)/\widetilde{\rho K_0}(B_n)$ is the element defined by $h$, and

$$\tilde{\lambda}_{B_n} : \mathrm{AffT}(B_n)/\widetilde{\rho K_0}(B_n) \to U(B_n)/\widetilde{SU}(B_n)$$

is the map defined in 2.33 (also see 2.28). Consequently,

$$\begin{aligned}(\psi_{n,m}^{n,m-1})^{\natural}(u) &= \mathrm{AffT}\,\psi_{n,m}^{n,m-1}(h) + \tilde{h} \\ &\triangleq \tilde{\tilde{h}} \in \mathrm{AffT}(B_m^{m-1})/\widetilde{\rho K_0}(B_m^{m-1}) \cong U(B_m^{m-1})/\widetilde{SU}(B_m^{m-1})\end{aligned}$$

with

$$\|\tilde{\tilde{h}}\|^{\sim} = \frac{1}{2}\big(\max_{t\in[0,1]} \tilde{\tilde{h}}(t) - \min_{t\in[0,1]} \tilde{\tilde{h}}(t)\big) \geq 4,$$

since $\|h\| \leq M$. Therefore,

$$\begin{aligned}(\pi_{m-1} \circ \psi_{n,\infty})^{\natural}(u) &\in U(Q_{m-1}BQ_{m-1})/\widetilde{SU}(Q_{m-1}BQ_{m-1}) \\ &\cong \mathrm{AffT}(Q_{m-1}BQ_{m-1})/\widetilde{\rho K_0}(Q_{m-1}BQ_{m-1}),\end{aligned}$$

satisfies

$$\begin{aligned}&\|(\pi_{m-1} \circ \psi_{n,\infty})^{\natural}(u)\|^{\sim} \\ &\quad = \frac{1}{2}\big(\max_{t\in[0,1]} (\pi_{m-1} \circ \psi_{n,\infty})^{\natural}(u)(t) - \min_{t\in[0,1]} (\pi_{m-1} \circ \psi_{n,\infty})^{\natural}(u)(t)\big) \geq \frac{3}{4} \cdot 4 = 3,\end{aligned}$$

where $\pi_{m-1} : B \to Q_{m-1}BQ_{m-1}$ is the quotient map. On the other hand,

$$\pi_{m-1}^{\natural}(S_1(x)) = 0$$

as calculated in $(*)$. Recall that

$$\|(\psi_{n,\infty})^{\natural}(u) - S_1(x)\|^{\sim} < \frac{1}{16}.$$

We get

$$\|(\pi_{m-1} \circ \psi_{n,\infty})^{\natural}(u)\|^{\sim} < \frac{1}{16},$$

which is a contradiction. This contradiction proves that such a system of splittings does not exist. Hence $\mathrm{Inv}(A) \ncong \mathrm{Inv}(B)$ and $A \ncong B$.

**3.12.** One can easily verify that

$$\begin{aligned}\mathrm{AffT}(A) &= \mathrm{AffT}(B) \\ &= \Big\{(f_1, f_2, \ldots, f_n, \ldots) \in \prod_{n=1}^{\infty} C_{\mathbb{R}}[0,1] : \exists r \in \mathbb{R} \text{ such that } f_n(x) \text{ converges to } r \text{ uniformly}\Big\},\end{aligned}$$

$$\begin{aligned}&\overline{\rho K_0(A)} (= \overline{\rho K_0(B)}) \\ &\quad = \Big\{(r_1, r_2, \ldots, r_n, \ldots) \in \prod_{n=1}^{\infty} \mathbb{R} : \exists r \in \mathbb{R} \text{ such that } r_n \text{ converges to } r\Big\} \\ &\quad \subset \mathrm{AffT}(A) (= \mathrm{AffT}(B)).\end{aligned}$$

Since $\overline{\rho K_0(A)}(=\overline{\rho K_0(B)})$ is already a vector space, we have $\widetilde{\rho K_0}(A)=\overline{\rho K_0(A)}$ and $\widetilde{\rho K_0}(B)=\overline{\rho K_0(B)}$. Therefore,

$$U_{\mathrm{tor}}(A)/\widetilde{SU}(A)\cong \mathrm{AffT}(A)/\widetilde{\rho K_0}(A)=\mathrm{AffT}(A)/\overline{\rho K_0(A)}\cong U_0(A)/\overline{DU(A)}.$$

On the other hand, $U_{\mathrm{tor}}(A)=U_0(A)$. Hence $\widetilde{SU}(A)=\overline{DU(A)}$. Furthermore, the map $\lambda_A:\mathrm{AffT}(A)/\overline{\rho K_0(A)}\to U(A)/\overline{DU(A)}$ can be identified with the map $\tilde{\lambda}_A:\mathrm{AffT}(A)/\widetilde{\rho K_0}(A)\to U(A)/\widetilde{SU(A)}$. That is, $\mathrm{Inv}'(A)=\mathrm{Inv}(A)$. Similarly, $\mathrm{Inv}(B)=\mathrm{Inv}'(B)$.

**3.13.** A routine calculation (we omit the details) shows that for any finite subset $F\subset A_n$ and $\varepsilon>0$, there is an $m>n$ and two finite dimensional unital sub-$C^*$-algebras $C,D\subset A_m$ with nonabelian central projection such that

$$\big\|[\phi_{n,m}(f),c]\big\|<\varepsilon\|c\| \quad\text{and}\quad \big\|[\psi_{n,m}(f),d]\big\|<\varepsilon\|d\| \quad \text{for all } f\in F, c\in C, d\in D.$$

Consequently, both $C^*$-algebras $A$ and $B$ are approximately divisible in the sense of [Blackadar et al. 1992, Definition 1.2]. By [Toms and Winter 2008, Theorem 2.3], both $A$ and $B$ are $\mathcal{Z}$-stable. That is, $A\otimes\mathcal{Z}\cong A$ and $B\otimes\mathcal{Z}\cong B$, where $\mathcal{Z}$ is the Jiang–Su algebra (see [Jiang and Su 1999]). Furthermore, by using [Tikuisis 2011] (see also [Coward et al. 2008]), one can prove that $Cu(A)\cong Cu(B)$ and $Cu(A\otimes C(S^1))\cong Cu(B\otimes C(S^1))$.

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GUIHUA GONG: guihua.gong@upr.edu
*College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei, China*

and

*Department of Mathematics, University of Puerto Rico, Rio Piedras, PR, United States*

CHUNLAN JIANG: cljiang@hebtu.edu.cn
*College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei, China*

LIANGQING LI: liangqing.li@upr.edu
*Department of Mathematics, University of Puerto Rico, Rio Piedras, PR, United States*

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