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Twisted iterated algebraic K-theory and topological T-duality for sphere bundles

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We introduce a periodic form of the iterated algebraic K-theory of ku, the (connective) complex K-theory spectrum, as well as a natural twisting of this cohomology theory by higher gerbes. Furthermore, we prove a form of topological T-duality for sphere bundles oriented with respect to this theory.

Let ku be the connective complex K-theory spectrum. The underlying infinite loop space $\Omega^\infty ku = \mathbb{Z} \times BU$ of ku classifies virtual complex vector bundles. The cohomology theory associated to the algebraic K-theory spectrum $K(ku)$, the subject of much recent research in homotopy theory [Ausoni and Rognes 2002], has a geometric interpretation as a Grothendieck group of 2-vector bundles [Baas et al. 2004; 2011; Lind 2016]. A 2-vector bundle is a bundle whose fiber is a 2-vector space, which is a categorified form of a vector space introduced by Kapranov and Voevodsky [1994]. Forming equivalence classes of 2-vector bundles over X leads to a bimonoidal category $2\mathrm{Vect}(X)$. By [Baas et al. 2011], the Grothendieck group completion of $2\mathrm{Vect}(X)$ is represented by the infinite loop space $\Omega^\infty K(ku)$ underlying the algebraic K-theory of ku.

Applying the functor $K(\cdot)$ again, one is naturally led to imagine that the iterated algebraic K-theory spectrum

$$\mathfrak{a}_n := K^{(n-1)}(ku) = \underbrace{K(K(\cdots K}_{n-1}(ku)\cdots))$$

has an interpretation in terms of categorified bundles native to n-category theory. It is expected that algebraic K-theory in many cases increases chromatic complexity by one, i.e., that it produces a constant "red-shift" by one chromatic layer in stable homotopy theory [Ausoni and Rognes 2002]. In this paper, we study a Bott-periodic form $\mathfrak{A}_n := K^{(n-1)}(ku)[\beta_n^{-1}]$ of iterated algebraic K-theory. While our results do not provide direct evidence either for or against the Ausoni–Rognes red-shift conjectures [2008], our interest in the relationship between the geometric content of iterated algebraic K-theory and chromatic homotopy theory is a primary motivation for the study of T-duality in $K^{(n-1)}(ku)[\beta_n^{-1}]$-theory.

Much as line bundles are the fundamental building blocks of vector bundles, and hence play an essential role in the K-theory of vector bundles, $(n-1)$-gerbes are the simplest forms of n-vector bundles. For a general definition of n-gerbes as n-truncated and n-connected objects, see [Lurie 2009a, §7.2.2]. For example, when $n = 2$, a 1-gerbe (also known as a gerbe with band $U(1)$) gives rise to a rank-one 2-vector bundle. This fact is witnessed at the level of classifying spaces by a map

$$K(\mathbb{Z}, 3) \to B\,\mathrm{GL}_1(ku) \to \Omega^\infty K(ku).$$

This is the 2-categorical analog of the map $\mathbb{C}P^\infty \to \mathbb{Z} \times BU$ representing the inclusion of line bundles into the Grothendieck group of vector bundles. The adjoint map $\Sigma^\infty \mathbb{C}P_+^\infty \to ku$ is a map of E_∞ ring spectra. We study here a family of analogous E_∞ ring maps

$$\Sigma^\infty K(\mathbb{Z}, n+1)_+ \to \mathfrak{a}_n := K^{(n-1)}(ku) \tag{0.1}$$

that we think of as representing the inclusion of $(n-1)$-gerbes into the Grothendieck group of n-vector bundles.

These maps are adjoint (under the adjunction described in [Ando et al. 2014b; May 2009]) to maps of spectra $\Sigma^{n+1} H\mathbb{Z} \to \mathrm{gl}_1(\mathfrak{a}_n)$ or, equivalently, maps of E_∞ spaces $K(\mathbb{Z}, n+1) \to \mathrm{GL}_1(\mathfrak{a}_n)$. Delooping once, we obtain E_∞-twistings of the cohomology theory $\mathfrak{a}_n$ by $(n+2)$-dimensional cohomology[1] classes: each class $H \in H^{n+2}(X; \mathbb{Z})$ in the cohomology of a topological space X gives rise to twisted cohomology groups $\mathfrak{a}_n^*(X; H)$. When $n = 1$, this returns the usual notion of connective complex K-theory twisted by a gerbe (or rather, a representative of its Dixmier–Douady class in H^3). The new twists that we study have the same degree as the twists of Morava K-theory and E-theory studied previously [Sati and Westerland 2015]; we hope that further understanding of the red-shift conjecture will relate the results of that paper and this one.

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[1] A caveat is in order: the twisted cohomology group $\mathfrak{a}_n^*(X; H)$ depends upon the representative map $H : X \to K(\mathbb{Z}, n+2)$. Just as is the case for twisted K-theory, homotopic maps (i.e., cohomologous classes) yield isomorphic twisted cohomology groups; however, the isomorphism is not canonical.

Notice that the element $\beta \in \pi_2 ku$, given as the composite $S^2 \to \mathbb{C}P^\infty \to \mathbb{Z} \times BU$ of the fundamental class of $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ and its inclusion as $BU(1) \times \{1\}$, is the Bott class. Localizing ku at this class yields periodic complex K-theory: $KU := ku[\beta^{-1}]$. Our first goal is an analog of this construction for $\mathfrak{a}_n$:

Theorem 0.2. *The composite of the fundamental class of $K(\mathbb{Z}, n+1)$ with the map in (0.1) yields an essential class $\beta_n \in \pi_{n+1}\mathfrak{a}_n$. Further, when n is odd, β_n is nonnilpotent, and none of its powers are torsion.*

This is proven as Corollary 1.7 and Theorem 1.10, below. For odd n, we will notate the localized spectrum $\mathfrak{a}_n[\beta_n^{-1}]$, i.e., the Bott-inverted iterated algebraic K-theory, by $\mathfrak{A}_n$. Theorem 0.2 suggests that this is a nontrivial object of study. However, it is a consequence of a theorem of Arthan [1983] that $\mathfrak{A}_n$ is a *rational* spectrum whenever $n > 1$. We may regard it as a higher categorical analog of periodic topological K-theory, or perhaps an étale form of iterated K-theory.

The twisting of $\mathfrak{a}_n$ by cohomology classes in degree $n+2$ extends in a natural way to a twisting of $\mathfrak{A}_n$ via the composite

$$\Sigma^{n+1} H\mathbb{Z} \to \mathrm{gl}_1 \, \mathfrak{a}_n \to \mathrm{gl}_1 \, \mathfrak{A}_n,$$

where the second map is induced by the E_∞ ring map given by localization $\mathfrak{a}_n \to \mathfrak{A}_n$. Our second main purpose in this paper is to prove an analog of the topological T-duality isomorphism of [Bouwknegt et al. 2004; Bunke and Schick 2005] in this context.

Given a base space X, Bouwknegt, Evslin, and Mathai give a criterion for two sets of data (E, H) and $(\widehat{E}, \widehat{H})$ to be *T-dual* [Bouwknegt et al. 2004]. Here E and $\widehat{E}$ are principal S^1-bundles over X, and H and $\widehat{H}$ are cocycles representing cohomology classes in $H^3(E)$ and $H^3(\widehat{E})$, respectively. The criterion in [Bouwknegt et al. 2004] is given in terms of relations among various characteristic classes. Bunke and Schick [2005] reinterpret this criterion as amounting to the existence of a Thom class on a certain S^3-bundle over X into which both E and $\widehat{E}$ embed. When (E, H) and $(\widehat{E}, \widehat{H})$ are a T-dual pair, there is an isomorphism of twisted K-theory groups $K^*(E; H) \cong K^{*-1}(\widehat{E}, \widehat{H})$ called the T-duality isomorphism.

Let E and $\widehat{E}$ be fiber bundles over X with fiber the q-sphere S^q, and let H and $\widehat{H}$ be classes in $H^{2q+1}(E)$ and $H^{2q+1}(\widehat{E})$, respectively. In Definition 3.2, we will give a Thom class criterion for (E, H) and $(\widehat{E}, \widehat{H})$ to be T-dual in the higher (dimensional and categorical) context, similar to Bunke and Schick's. We then prove the following extension of the T-duality isomorphism in this setting.

Theorem 0.3. *Let $n = 2q - 1$, and assume that (E, H) and $(\widehat{E}, \widehat{H})$ are a T-dual pair. Let $\tau_E : E \to B\,\mathrm{GL}_1 \, \mathfrak{A}_n$ denote the orientation twisting determined by the*

vertical tangent bundle of $E \to X$. Then there is an isomorphism of twisted cohomology groups

$$T := \hat{p}_! \circ \Lambda \circ p^* : \mathfrak{A}_n^*(E; \tau_E \otimes H) \to \mathfrak{A}_n^*(\widehat{E}; \widehat{H})$$

given in terms of a Fourier–Mukai push-pull construction on the correspondence space $E \times_X \widehat{E}$. Given an $\mathfrak{A}$-orientation of the fiber bundle $E \to X$, the T-duality isomorphism takes the form

$$T : \mathfrak{A}_n^{*+q}(E; H) \to \mathfrak{A}_n^*(\widehat{E}; \widehat{H}).$$

This is proven as Theorem 3.3. In fact, we prove the result for a larger class of cohomology theories R than just $\mathfrak{A}_n$, namely those which may be twisted by n-gerbes in such a fashion that the analog of the class β_n is invertible in $\pi_* R$. One consequence of this invertibility assumption (again via [Arthan 1983]) is that R is a rational spectrum when $n > 1$. We will show in Theorem 3.9 that this is no accident: every cohomology theory R for which the T-duality map T of the previous theorem is an isomorphism must be rational.

Additionally, we study criteria to ensure the orientability requirement of the previous result, and analyze the homotopy type of the classifying spaces for T-dual pairs, much as in [Bunke and Schick 2005] in the case $q = n = 1$.

For $q = 3$, T-duality for S^3-bundles in rational cohomology, and in twisted K-theory under some conditions on the cohomology and dimensions of the underlying manifolds, was considered recently in [Bouwknegt et al. 2015a]. The authors discuss twistings of topological K-theory by a 7-dimensional class. It is known that $B\,\mathrm{GL}_1(ku)$ splits as $B\mathbb{Z}/2 \times K(\mathbb{Z}, 3) \times BB\,\mathrm{SU}_\otimes$. Furthermore, $BB\,\mathrm{SU}_\otimes$ has a 7-dimensional homotopy group isomorphic to $\mathbb{Z}$; this is visible via the k-invariant $BB\,\mathrm{SU}_\otimes\langle 6 \rangle \to K(\mathbb{Z}, 7)$ in a Postnikov tower for $BB\,\mathrm{SU}_\otimes$. An essential map in the other direction — which is necessary for there to be a twist of K-theory by H^7 — does not exist [Antieau et al. 2014]; hence, that degree cannot be "isolated" in general, as highlighted in [Sati 2009] and clarified further in [Sati and Westerland 2015]. The conditions imposed by [Bouwknegt et al. 2015a] evade the obstructions for dimension reasons and allow one to isolate such a twist in special cases. If one rationalizes, as done for the most part in [Bouwknegt et al. 2015a], one gets periodic cohomology, which does have a degree-seven twist. We recover this result and analogs for all sphere bundles as a consequence of the variant of Theorem 0.3 for periodic rational cohomology. Indeed, it is a consequence of Theorem 3.9 that such a T-duality result *must* be rational outside of the case $q = 1$.

Our general result also provides a recipe for determining whether there exists a T-dual for (E, H), where E is an S^q-bundle with structure group $G \to \mathrm{Homeo}(S^q)$ equipped with an n-gerbe H, and whether the T-dual is unique. This is determined by analyzing the connectivity of the Euler class map $BG \to K(\mathbb{Z}, q+1)$. For the

case $q = 1$, this recovers some of the results of Baraglia [2014] and Mathai and Rosenberg [2014] for existence and uniqueness of T-dual bundles for nonprincipal circle bundles.

The paper is organized as follows. In Section 1A, we describe the multiplicative behavior of the algebraic K-theory functor and consider generalities about twisted cohomology theories. In Section 1B, we construct and analyze the twisting of iterated algebraic K-theory $K^{(n-1)}(ku)$ by higher gerbes. We prolong some computations of the homotopy of $K(ku)$ due to [Ausoni et al. 2008] to the higher setting in Section 1C. Then in Section 1D we consider a higher analog of Snaith's construction of KU, which in turn admits the universal periodic twist.

In order to obtain concrete expressions and with an eye for applications to T-duality, we study Chern characters and orientations in Section 2. We introduce two Chern characters associated to the new twisted theories, one in Section 2A and one in Section 2B, and then we describe orientations with respect the new theories in Section 2C.

The general setup and the proof of the main T-duality theorem and its converse are given in Section 3. Section 4 is concerned with the question of the existence and the uniqueness of T-dual pairs, which we analyze homotopy-theoretically in terms of classifying spaces. In particular, in Section 4A we introduce a space $P_n(G)$ which classifies the possible T-dual pairs. Then in Section 4B we classify, via a space $R_n(G)$, bundles and Thom classes that can arise in our context of T-duality. The two constructions are related in Section 4C by constructing a forgetful map from $P_n(G)$ to $R_n(G)$, where we also show how our results reproduce earlier results on T-duality.

The final section, Section 5, is a speculative account of the relationship between the iterated algebraic K-theory considered earlier in the paper and a proposed model for the algebraic K-theory of n-vector spaces defined in terms of enriched higher category theory.

1. Iterated algebraic K-theory and its twistings by n-gerbes

We recall here the multiplicative behavior of the algebraic K-theory functor and describe the twisting of iterated algebraic K-theory by higher gerbes.

1A. *Generalities on twistings of algebraic K-theory.* Given an E_∞ ring spectrum A, the algebraic K-theory spectrum $K(A)$ is once again an E_∞ ring spectrum. There is a natural map $B \operatorname{GL}_1 A \to \Omega^\infty K(A)$ coming from the inclusion of A-lines into all cell A-modules. It is not the case that this is an infinite loop map: the multiplication on $B \operatorname{GL}_1 A$ is by tensor products of A-lines, whereas that in $\Omega^\infty K(A)$ is from the sum of modules. However, this map has image in $\operatorname{GL}_1 K(A)$

(since A-lines are invertible A-modules), and in fact the induced map $B\,\mathrm{GL}_1\,A \to \mathrm{GL}_1\,K(A)$ is an infinite loop map.

We explain the above in detail. Let A be a connective commutative ring spectrum and let $K(A)$ denote the connective algebraic K-theory spectrum of A. The underlying infinite loop space of the spectrum $K(A)$ receives a map

$$B\,\mathrm{GL}_1\,A \to \coprod_{n \geq 0} B\,\mathrm{GL}_n\,A \to \Omega^\infty K(A) \tag{1.1}$$

from the classifying space of the space of units $\mathrm{GL}_1\,A$ via the classifying space of finite-rank free A-module spectra. This is the analog in algebraic K-theory of the map $\mathbb{C}P^\infty \to \mathbb{Z} \times B\,\mathrm{U}$ into topological K-theory classifying the homomorphism from the Picard group into the Grothendieck group of vector bundles. As in the classical case, the source and target both inherit E_∞ space structures from the multiplication on A.

Proposition 1.2. *The map* (1.1) *has image in* $\mathrm{SL}_1\,K(A)$; *the result is a map of* E_∞ *spaces and so lifts to a map of spectra*

$$\mu : \Sigma\,\mathrm{gl}_1\,A \to \mathrm{gl}_1\,K(A).$$

Proof. We employ the ∞-categorical model for algebraic K-theory developed by Gepner, Groth, and Nikolaus [Gepner et al. 2015] using the language of quasi-categories. Let Perf_A denote the stable ∞-category of compact A-module spectra. Let Line_A denote the ∞-category of rank-one A-module spectra. Both of these ∞-categories admit symmetric monoidal structures under the smash product $\wedge_A$ of A-modules. Writing ι_0 for the groupoid core functor which takes a quasicategory to its maximal sub-Kan complex, the inclusion of Line_A into Perf_A induces a symmetric monoidal functor $i : \iota_0\mathrm{Line}_A \to \iota_0\mathrm{Perf}_A$ of symmetric monoidal ∞-groupoids. The ∞-category Perf_A admits an additional symmetric monoidal structure under the coproduct of A-modules and the two monoidal structures combine to give the Kan complex $\iota_0\mathrm{Perf}_A$ the structure of an E_∞ ring space [Gepner et al. 2015, Corollary 8.11]. The algebraic K-theory $K(A)$ is the connective E_∞ ring spectrum for which $\Omega^\infty K(A)$ is the group completion of $\iota_0\mathrm{Perf}_A$ as an E_∞ ring space [Gepner et al. 2015, Definition 8.3 and Theorem 8.6]. The resulting composite

$$\iota_0\mathrm{Line}_A \to \iota_0\mathrm{Perf}_A \to \Omega^\infty K(A)$$

is a map of E_∞ spaces, where we use the multiplicative structure on $\iota_0\mathrm{Perf}_A$ and $\Omega^\infty K(A)$. This map is a model for the map (1.1), as can be verified along the lines of [Ando et al. 2014a, Proposition 2.9], so the map in question is E_∞.

The source of the map is a connected Kan complex and its image lies in the component of the unit object $A \in \iota_0\mathrm{Perf}_A$, hence in the component $\mathrm{SL}_1\,K(A)$ of the

unit in the E_∞ ring space $\Omega^\infty K(A)$. In particular, the map factors as an E_∞ map through the space $\mathrm{GL}_1 K(A)$ of units, which deloops to the map of spectra μ. $\square$

We now recall how to construct twisted forms of the cohomology theory represented by the ring spectrum A, following [Ando et al. 2010; 2018; 2014b]. Given a space X and a map $\omega : X \to B \mathrm{GL}_1 A$, let $E \to X$ be the $\mathrm{GL}_1(A)$-fibration pulled back from $B \mathrm{GL}_1 A$ via ω. Define an A-module spectrum

$$\mathbb{A}(X; \omega) := A \wedge^L_{\Sigma^\infty \mathrm{GL}_1 A_+} \Sigma^\infty E_+.$$

This is the Thom spectrum associated to the parametrized spectrum of A-lines over X classified by ω.

Definition 1.3. *The ω-twisted A-homology and A-cohomology groups of X are* defined by

$$A_q(X; \omega) := \pi_q \mathbb{A}(X; \omega) \quad \text{and} \quad A^q(X; \omega) := \pi_{-q} F_A(\mathbb{A}(X; -\omega), A).$$

Notice that in the definition of twisted cohomology we use the inverse twist $-\omega$ under the monoidal structure induced by the smash product of invertible A-modules. Sometimes the opposite convention is used, but we prefer this choice because it ensures that twisted cohomology agrees with homotopy classes of sections of the parametrized spectrum associated to ω [Lind 2016], and conforms with grading conventions when ω carries a topological dimension, for example when it arises from a classifying map for a virtual vector bundle via the J-homomorphism.

Suppose that h is a spectrum, and that $\tau : h \to \Sigma \, \mathrm{gl}_1 A$ is a map of spectra. We regard the map τ as an E_∞ *twisting of A*, because the infinite loop map $\Omega^\infty \tau : \Omega^\infty h \to B \mathrm{GL}_1 A$ allows us to twist the A-cohomology of a space X by elements $[H] \in h^0(X)$. To do so, we represent the class $[H]$ as a map $H : X \to \Omega^\infty h$ and define the H-twisted cohomology of X to be the twisted A-homology and A-cohomology groups associated to the twist $\Omega^\infty \tau \circ H : X \to B \mathrm{GL}_1 A$, i.e., we make the abbreviation $A^q(X; H) = A^q(X; \Omega^\infty \tau \circ H)$. Different representatives for the same cohomology class give isomorphic twisted cohomology groups, so the twisted cohomology theory associated to H only depends on the underlying cohomology class $[H]$, but only up to noncanonical isomorphism.

We can use the map $\mu : \Sigma \, \mathrm{gl}_1 A \to \mathrm{gl}_1 K(A)$ considered in Proposition 1.2 to construct E_∞ twists of the algebraic K-theory spectrum $K(A)$ from (shifts of) E_∞ twists of A, and we would like to know whether or not the twists constructed in this way are essential, i.e., homotopically nontrivial.

Proposition 1.4. *For every E_∞ twisting $\tau : h \to \Sigma \, \mathrm{gl}_1 A$, the composite*

$$\Sigma(\mu \circ \tau) : \Sigma h \to \Sigma \, \mathrm{gl}_1 K(A)$$

is an E_∞ twisting of $K(A)$. Further, if the rationalization of τ is essential, so too is the rationalization of $\Sigma(\mu \circ \tau)$.

Proof. Only the latter statement needs to be proved. One can see this via the rational determinant $\det_\mathbb{Q}$, defined by Ausoni and Rognes [2012, Proposition 5.4]. This is a map of spaces

$$\det_\mathbb{Q} : \Omega^\infty K(A) \to (B\operatorname{GL}_1 A)_\mathbb{Q}$$

whose composite with $\Omega^\infty \mu$ is the rationalization map of $B\operatorname{GL}_1 A$. Therefore, $\Omega^\infty \Sigma(\mu \circ \tau)$ factorizes the rationalization of $\Omega^\infty \tau$, which yields the result. $\square$

Remark 1.5. There is not an integral determinant map lifting $\det_\mathbb{Q}$. In fact, the obstruction to the existence of a continuous map $\Omega^\infty K(ku) \to B\operatorname{GL}_1(ku)$ with determinant-like properties leads to the notion of oriented 2-vector bundles, and gives rise to an oriented version of K-theory of 2-vector bundles with a lift of the natural map from $K(\mathbb{Z}, 3)$, called the determinant gerbe map [Kragh 2013].

1B. *Twisting the iterated algebraic K-theory of ku.* Our main example arises via connective complex topological K-theory, ku. There is a well known E_∞ twisting of ku by 3-dimensional cohomology classes; the map

$$\tau : \Sigma^3 H\mathbb{Z} \to \Sigma \operatorname{gl}_1 ku$$

is the delooping of the map of E_∞-spaces $\mathbb{C}P^\infty \to \operatorname{GL}_1 ku$ that regards a complex line as an invertible $\mathbb{C}$-module.

Definition 1.6. Write $\mathfrak{a}_n$ for the *iterated algebraic K-theory spectrum* $K^{(n-1)}(ku)$. Let

$$\tau_n : \Sigma^{n+2} H\mathbb{Z} \to \Sigma \operatorname{gl}_1 \mathfrak{a}_n$$

be the E_∞ *twisting of $\mathfrak{a}_n$* obtained by an $(n-1)$-fold iteration of the procedure in Proposition 1.4 applied to $\tau = \tau_1$.

Since the original map τ was rationally essential in homotopy, so too are the maps τ_n:

Corollary 1.7. *The composite of τ_n and the fundamental class ι of $\Sigma^{n+2} H\mathbb{Z}$ defines a nontorsion element $\tau_n \circ \iota \in \pi_{n+2} \Sigma \operatorname{gl}_1 \mathfrak{a}_n$.*

Definition 1.8. We write $\Omega^{\infty+1} \tau_n \circ \iota : S^{n+1} \to \operatorname{GL}_1 \mathfrak{a}_n$ for the map of spaces representing the image of the class $\tau_n \circ \iota$ under the canonical isomorphism $\pi_{n+2} \Sigma \operatorname{gl}_1 \mathfrak{a}_n \cong \pi_{n+1} \operatorname{gl}_1 \mathfrak{a}_n$. The map $\Omega^{\infty+1} \tau_n$ carries S^{n+1} into the component of $\operatorname{GL}_1 \mathfrak{a}_n$ corresponding to $1 \in \pi_0 \mathfrak{a}_n$. Subtracting 1 gives a based map into $\Omega^\infty \mathfrak{a}_n$, and thus a class $\beta_n \in \pi_{n+1} \mathfrak{a}_n$. In other words, we define $\beta_n := [\Omega \tau_n \circ \iota] - 1$, where $[\Omega \tau_n \circ \iota]$ denotes the image of $\Omega \tau_n \circ \iota$ under the isomorphism $\pi_{n+1} \operatorname{gl}_1 \mathfrak{a}_n \cong \pi_{n+1} \mathfrak{a}_n$ induced by the

inclusion of components $\mathrm{GL}_1\,\mathfrak{a}_n \subset \Omega^\infty \mathfrak{a}_n$. Equivalently, β_n is represented by the composite map of spectra

$$\beta_n : S^{n+1} \overset{\iota}{\to} \Sigma^\infty K(\mathbb{Z}, n+1)_+ \overset{t_n}{\to} \mathfrak{a}_n, \tag{1.9}$$

where ι is the fundamental class and t_n is adjoint to $\Omega\tau_n$ in the adjunction

$$[\Sigma^\infty K(\mathbb{Z}, n+1)_+, \mathfrak{a}_n]_{E_\infty \mathrm{ring}} \cong [\Sigma^{n+1} H\mathbb{Z}, \mathrm{gl}_1\,\mathfrak{a}_n]_{\mathrm{Sp}}.$$

Note that $\beta_1 \in \pi_2\mathfrak{a}_1 = \pi_2 ku$ is the usual Bott class which we invert to obtain KU.

Theorem 1.10. *If n is even, $2\beta_n^2 = 0$. However, if n is odd, then β_n is not nilpotent and all powers β_n^m are nontorsion, for any positive integer m.*

Proof. The first claim is simply that β_n is an odd-dimensional element of a graded-commutative ring. For the second, consider the composite map (1.9) and note that the Hurewicz image of $\iota : S^{n+1} \to \Sigma^\infty K(\mathbb{Z}, n+1)_+$ is essential. In fact, whenever n is odd, the homology ring $H_*(K(\mathbb{Z}, n+1); \mathbb{Q})$ is the rational polynomial ring on the Hurewicz image of ι. In particular, all powers of ι are rationally essential.

We recall from [Segal 1973] that the natural map of ring spectra $\Sigma^\infty \mathbb{C}P_+^\infty \to ku$ adjoint to τ is a rational equivalence. Therefore, the iterated Dennis trace map, which is a transformation from algebraic K-theory to topological Hochschild homology THH [Bökstedt et al. 1993; Dundas et al. 2013], followed by rationalization, may be written as

$$\mathfrak{a}_n = K^{(n-1)}(ku) \xrightarrow{\mathrm{tr}^{n-1}} THH^{(n-1)}(ku)$$
$$\to THH^{(n-1)}(ku_\mathbb{Q}) \overset{\simeq}{\leftarrow} THH^{(n-1)}((\Sigma^\infty \mathbb{C}P_+^\infty)_\mathbb{Q}).$$

However, for a loop space G, one has $THH(\Sigma^\infty G_+) \simeq \Sigma^\infty LBG_+$. When G is an m-fold loop space, BG (and hence LBG) is an $(m-1)$-fold loop space. We iteratively observe that

$$THH^{(m)}(\Sigma^\infty G_+) \simeq \Sigma^\infty (LB)^m(G)_+ \simeq \Sigma^\infty \mathrm{Map}(T^m, B^m G)_+$$

if G is connected. In the second equivalence, we employ the fact that for connected G, $LBG \simeq BLG$. The space of constant functions $T^m \to B^m G$ is homeomorphic to $B^m G$, and so $THH^{(m)}(\Sigma^\infty G_+)$ contains a copy of $\Sigma^\infty B^m G_+$ as a wedge summand.

In the case $G = \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$, it follows that $\Sigma^\infty K(\mathbb{Z}, n+1)_+$ is a summand of $THH^{(n-1)}(\Sigma^\infty \mathbb{C}P_+^\infty)$. Since rationalization is a smashing localization,

$$THH(A_\mathbb{Q}) \simeq (THH(A))_\mathbb{Q}$$

for ring spectra A. Thus, the target of the iterated trace map above splits off a wedge factor of $(\Sigma^\infty K(\mathbb{Z}, n+1)_+)_\mathbb{Q}$.

We claim that the composite of t_n with the iterated trace map and the projection to this wedge summand is the map from $\Sigma^\infty K(\mathbb{Z}, n+1)_+$ to its rationalization; then we may conclude that t_n is an injection on rational homotopy. To see the claim, note that if G is an infinite loop space, there is a homotopy commutative diagram

$$\Sigma^\infty BG_+ \longrightarrow \Sigma^\infty B\operatorname{GL}_1(\Sigma^\infty G_+)_+ \xrightarrow{(1.1)} K(\Sigma^\infty G_+) \xrightarrow{\operatorname{tr}} \begin{array}{c} THH(\Sigma^\infty G_+) \\ \simeq \Sigma^\infty LBG_+ \\ \downarrow^{\operatorname{ev}} \\ \Sigma^\infty BG_+ \end{array}$$

$$\text{id}$$

yielding Waldhausen's splitting [1979] of $Q(BG_+)$ off of $A(BG) = \Omega^\infty K(\Sigma^\infty G_+)$. Then the claim follows by iteration (taking $G = K(\mathbb{Z}, m)$ for $m = 1, \ldots, n$) and rationalization (to accommodate the rational Segal equivalence). $\qquad\square$

Definition 1.11. When n is odd, define $\mathfrak{A}_n := \mathfrak{a}_n[\beta_n^{-1}]$.

One could of course make this construction for n even, but the nilpotence of β_n away from the prime 2 in that case will force the resulting spectrum to have 2-torsion homotopy. In fact, these spectra are contractible when n is even; this follows from Theorem 1.14, below.

The E_∞ twisting τ_n of $\mathfrak{a}_n$ induces an E_∞ twisting $K(\mathbb{Z}, n+2) \to B\operatorname{GL}_1 \mathfrak{A}_n$ by composition with the localization map. By the phrase "n-gerbe" we mean a generic term for any geometric structure classified by the Eilenberg–MacLane space $K(\mathbb{Z}, n+2)$ (for a more general discussion, see [Lurie 2009a, Example 7.2.2.2]). Thus, a complex line bundle is a 0-gerbe, and a gerbe with band $U(1)$ is a 1-gerbe. We say that the E_∞ twisting of $\mathfrak{a}_n$ and $\mathfrak{A}_n$ constructed in this section are *twistings by n-gerbes*, since an n-gerbe over X gives rise to a map $H : X \to K(\mathbb{Z}, n+2)$ by which we can twist these theories.

1C. *Some relations in the homotopy of $\mathfrak{a}_n$.* Although the following observations will not be used in our work with the periodic spectrum $\mathfrak{A}_n$, they provide the first known results about the homotopy type of iterated algebraic K-theory of ku. In [Ausoni et al. 2008], it is shown that for $n = 2$ there is a class $\zeta \in \pi_3 K(ku)$ with the property that

$$\beta_2 = 2\zeta - \nu,$$

where ν is the image of the quaternionic Hopf fibration $\nu \in \pi_3(S^0)$ under the unit map to $K(ku) = \mathfrak{a}_2$. One may iteratively prolong this equation to one in $\pi_{n+1}\mathfrak{a}_n$ in a natural fashion, as we now explain. Following the recipe given in Proposition 1.2,

we have a homomorphism[2]

$$\pi_{n+1}\mathfrak{a}_n \cong \pi_{n+1}\,\mathrm{gl}_1\,\mathfrak{a}_n \to \pi_{n+1}\Sigma^{-1}\,\mathrm{gl}_1\,K(\mathfrak{a}_n) \cong \pi_{n+2}\mathfrak{a}_{n+1}$$

for each $n > 0$. By construction, this map carries β_n to β_{n+1}; it also allows us to define classes Z_n and N_n in $\pi_{n+1}\mathfrak{a}_n$ as the iterated images of ζ and ν. Since this is a group homomorphism, the relation $\beta_2 = 2\zeta - \nu$ persists:

Proposition 1.12. *The class β_n is given in terms of the iteratively defined classes Z_n and N_n by*

$$\beta_n = 2Z_n - N_n.$$

Furthermore, since $24\nu = 0$, we also have $24N_n = 0$. Additionally, ν is nilpotent. Since the map above is not a ring homomorphism (in fact, it behaves in some sense more like a derivation), we cannot conclude the same for N_n. However, if N_n is actually nilpotent, then a sufficiently large power of β_n is 2-divisible (at least after inverting 3).

1D. *Higher Snaith spectra.* When $n = 1$, Definition 1.11 tautologically yields periodic complex K-theory: $\mathfrak{A}_1 = KU$. In this case, there is an alternative construction of KU due to Snaith [1981]. The map $\Sigma^\infty K(\mathbb{Z}, 2)_+[\iota^{-1}] \to KU$ induced by the localization of the map $t_1 : \Sigma^\infty K(\mathbb{Z}, 2)_+ \to ku$ studied in the proof of Theorem 1.10 is an equivalence of spectra. In this section, we consider generalizations of this construction using the domain of the map $t_n : \Sigma^\infty K(\mathbb{Z}, n+1)_+ \to \mathfrak{a}_n$ and discuss their relationship with the twistings from the previous section.

Guided by Snaith's theorem, we make the following definition.

Definition 1.13. For n odd, define the *Arthan spectrum* $\mathfrak{S}_n$ to be the localization

$$\mathfrak{S}_n := \Sigma^\infty K(\mathbb{Z}, n+1)_+[\iota^{-1}]$$

of the suspension spectrum of $K(\mathbb{Z}, n+1)$ at its fundamental class.

The E_∞ twisting map $t_n : \Sigma^\infty K(\mathbb{Z}, n+1)_+ \to \mathfrak{a}_n$ carries ι to β_n, and so descends to a map between the periodic spectra $T_n : \mathfrak{S}_n \to \mathfrak{A}_n$. Snaith's theorem indicates that this is an equivalence when $n = 1$. When $n > 1$, these spectra were studied by Arthan [1983], where they were found to display remarkably different behavior:

Theorem 1.14 (Arthan). *If n is even, $\mathfrak{S}_n$ is contractible. If n is odd and greater than 1, $\mathfrak{S}_n$ is $(n+1)$-periodic rational cohomology:*

$$\mathfrak{S}_n \cong H\mathbb{Q}[\iota^{\pm 1}].$$

---

[2]Here we identify $\pi_k R = \pi_k \Omega_0^\infty R$ with $\pi_k\,\mathrm{gl}_1\,R = \pi_k\,\mathrm{GL}_1\,R$ for $k > 0$ via the shift in components by 1.

Let R be an A_∞ ring spectrum which receives an A_∞ ring map $\phi : \mathfrak{S}_n \to R$. The case $R = \mathfrak{A}_n$ is our main example. From ϕ, we obtain a natural composite of maps of spaces

$$K(\mathbb{Z}, n+2) \to B\,\mathrm{GL}_1(\Sigma^\infty K(\mathbb{Z}, n+1)_+) \to B\,\mathrm{GL}_1(\mathfrak{S}_n) \to B\,\mathrm{GL}_1(R),$$

and so, for a topological space X, we may form the twisted cohomology $R^*(X; H)$ associated to a class $[H] \in H^{n+2}(X)$. Furthermore, since ι is invertible in $\mathfrak{S}_n$, the element $\phi(\iota)$ must be invertible in R.

Definition 1.15. We will call the data of the A_∞ map $\phi : \mathfrak{S}_n \to R$ a *periodic twisting of R by n-gerbes*. An equivalent (if less colorful) term is: R is an $\mathfrak{S}_n$-*algebra spectrum*.

The spectrum $R = \mathfrak{S}_n$ obviously admits the universal periodic twisting by n-gerbes. Theorem 1.10 ensures that $\mathfrak{A}_n$ is another nontrivial example. An immediate consequence of being an algebra spectrum over a rational algebra is the following:

Proposition 1.16. *If R admits a periodic twisting by n-gerbes for $n > 1$, then it is rational.*

2. Chern characters, $\mathfrak{A}$-orientations, and Umkehr maps

Many approaches to T-duality involve cohomological expressions. In order to deduce T-duality isomorphisms in ordinary cohomology, we investigate Chern characters and orientations associated with the spectra $\mathfrak{a}_n$ and $\mathfrak{A}_n$ introduced in Section 1B. These are, of course, also interesting in their own right.

2A. *The Chern–Dold character.* The Chern character in complex topological K-theory of a manifold X is a ring homomorphism from $KU^0(X)$ to even rational cohomology $H^{\mathrm{even}}(X; \mathbb{Q})$ and from $KU^1(X)$ to odd rational cohomology $H^{\mathrm{odd}}(X; \mathbb{Q})$. This periodicity can be encoded by the homomorphism of $\mathbb{Z}$-graded rings

$$KU^*(X) \to H^*(X; \mathbb{Q}[u^{\pm 1}]),$$

where the latter is rational cohomology with coefficients in $KU^*(\mathrm{pt}) \otimes \mathbb{Q} \cong \mathbb{Q}[u^{\pm 1}]$, and u^{-1} is the Bott generator. The Chern character may be regarded as a map of spectra

$$\mathrm{ch} : KU \to H\mathbb{Q}[u^{\pm 1}],$$

where $H\mathbb{Q}[u^{\pm 1}]$ is 2-periodic rational cohomology. The target is a priori formal power series over $\mathbb{Q}$, but becomes polynomials when evaluating on finite-dimensional manifolds.

The above Chern character has a generalization to any generalized cohomology theory. The standard Chern–Dold character (see, e.g., [Adams 1972, Chapter 14])

is based upon the Eilenberg–MacLane spectrum for the rationalization of the coefficients of our theory. Let $\mathfrak{a}_n^*(\mathrm{pt}) = \pi_*(\mathfrak{a}_n) = R^* = \sum_j R^j$. Unfortunately, we have little understanding of an explicit formula for these coefficients. Further, the results of [Ausoni 2010] suggest that R^* is most likely a very complicated ring even for $n = 2$. Nonetheless, we may make the following construction.

Definition 2.1. The *Chern–Dold character for the theory* $\mathfrak{a}_n$ is the map of cohomology theories

$$\mathrm{ch}_{\mathfrak{a}_n} : \mathfrak{a}_n^* \to H^*(\,\cdot\,; R^* \otimes \mathbb{Q})$$

induced by the rationalization map

$$\mathrm{ch}_{\mathfrak{a}_n} : \mathfrak{a}_n \to (\mathfrak{a}_n)_{\mathbb{Q}} \simeq H(R_* \otimes \mathbb{Q}),$$

where we identify the rationalization of $\mathfrak{a}_n$ with the generalized Eilenberg–MacLane spectrum associated to the graded ring $R_* \otimes \mathbb{Q}$.

The following properties of the above character are evident from the definition and follow similarly to those of the standard Chern–Dold character [Buhštaber 1970; Adams 1972].

Lemma 2.2 (properties of the Chern–Dold character). (1) *Over a point*, $\mathrm{ch}_{\mathfrak{a}_n}$ *is the canonical homomorphism* $R^* \to R^* \otimes \mathbb{Q}$.

(2) *For a finite complex X the homomorphism*

$$\mathrm{ch}_{\mathfrak{a}_n} \otimes \mathbb{Q} : \mathfrak{a}_n^*(X) \otimes \mathbb{Q} \to H^*(X; R^* \otimes \mathbb{Q})$$

is an isomorphism.

(3) $\mathrm{ch}_{\mathfrak{a}_n}$ *is a ring homomorphism.*

2B. A (twisted) higher Chern character via THH. Although we do not have a good understanding of the ring R^* or its rationalization, there is a stand-in for the Chern–Dold character with target a recognizable generalized Eilenberg–MacLane spectrum.

Let n be odd, and consider the composite $\mathrm{ch}_n^{\geq 0}$:

described in the proof of Theorem 1.10 with ev the basepoint evaluation projection from $THH^{(n-1)}(\Sigma^\infty \mathbb{C}P_+^\infty) = \Sigma^\infty \operatorname{Map}(T^{n-1}, K(\mathbb{Z}, n+1))_+$ to $\Sigma^\infty K(\mathbb{Z}, n+1)_+$. The target of this map is the generalized rational Eilenberg–MacLane ring spectrum $(\Sigma^\infty K(\mathbb{Z}, n+1)_+)_\mathbb{Q} \simeq H\mathbb{Q}[\beta_n]$ whose homotopy is the ring $\mathbb{Q}[\beta_n]$.

Definition 2.3. For n odd, define the *higher Chern character* $\operatorname{ch}_n : \mathfrak{A}_n \to H\mathbb{Q}[\beta_n^{\pm 1}]$ as the localization (at β_n) of the map $\operatorname{ch}_n^{\geq 0} : \mathfrak{a}_n \to H\mathbb{Q}[\beta_n]$.

We note that when $n = 1$, this is the usual Chern character, which may be alternatively described as the map $KU \to KU_\mathbb{Q} \simeq (\Sigma^\infty \mathbb{C}P_+^\infty[\beta^{\pm 1}])_\mathbb{Q}$ induced by rationalization and Snaith's (or Segal's) theorem. More generally, ch_n factors the Chern–Dold character and evidently forgets some information regarding the rational homotopy of $\mathfrak{A}_n = \mathfrak{a}_n[\beta_n^{-1}]$:

$$
\begin{array}{ccc}
\mathfrak{a}_n & \xrightarrow{\operatorname{ch}_{\mathfrak{a}_n}} & H(R_* \otimes \mathbb{Q}) \\
\downarrow & & \downarrow \\
\mathfrak{A}_n & \xrightarrow{\operatorname{ch}_n} & H\mathbb{Q}[\beta_n^{\pm 1}]
\end{array}
$$

(the right vertical map is induced by $\operatorname{ch}_n^{\geq 0}$ in rational homotopy). Nonetheless, it is useful on account of the fact that the coefficients of the codomain of ch_n are computed. Furthermore, this map is highly structured:

Proposition 2.4. *The higher Chern character* $\operatorname{ch}_n : \mathfrak{A}_n \to H\mathbb{Q}[\beta_n^{\pm 1}]$ *is a map of E_∞ ring spectra. Further, the composite $\mathfrak{S}_n \to \mathfrak{A}_n \to H\mathbb{Q}[\beta_n^{\pm 1}]$ is an equivalence, and splits $H\mathbb{Q}[\beta_n^{\pm 1}]$ off of $\mathfrak{A}_n$.*

Proof. The work of [Blumberg et al. 2014] ensures that the Dennis trace is an E_∞ map. The original map[3] $\operatorname{ch}_n^{\geq 0} : \mathfrak{a}_n \to (\Sigma^\infty K(\mathbb{Z}, n+1)_+)_\mathbb{Q}$ in the definition of ch_n may then be seen to be E_∞: it is a composite of iterates of tr, rationalizations, an equivalence induced by a map of ring spectra, and the map ev, which is easily seen to be E_∞. Finally ch_n is also E_∞, being obtained from this by localization at β_n.

To obtain the second statement, it suffices to note that $\iota \in \pi_{n+1}\mathfrak{A}_n$ is carried to β_n, which was shown in the proof of Theorem 1.10. □

Consequently ch_n induces a map of spectra $\operatorname{ch}_n : \operatorname{gl}_1(\mathfrak{A}_n) \to \operatorname{gl}_1(H\mathbb{Q}[\beta_n^{\pm 1}])$. Further, since the composite

$$
\Sigma^\infty K(\mathbb{Z}, n+1)_+ \xrightarrow{t_n} \mathfrak{a}_n \xrightarrow{\operatorname{ch}_n^{\geq 0}} (\Sigma^\infty K(\mathbb{Z}, n+1)_+)_\mathbb{Q}
$$

---

[3]This need not agree with the connective cover of the map ch_n!

is the rationalization map, it follows that ch_n carries the twisting t_n of $\mathfrak{A}_n$ by H^{n+2} to the standard[4] twisting of periodic cohomology by H^{n+2}.

Corollary 2.5. *For each $H \in H^{n+2}(X)$, the higher Chern character ch_n is a natural transformation of twisted cohomology theories,*

$$\mathrm{ch}_n : \mathfrak{A}_n^*(X; H) \to H\mathbb{Q}^*(X; H)[\beta_n^{\pm 1}],$$

which splits the target off of the source.

2C. $\mathfrak{A}$-*orientations and Umkehr maps*. In order to construct the T-duality isomorphism in twisted $\mathfrak{A}_n$-theory via a Fourier–Mukai push-pull formula, we will need to have an Umkehr map in $\mathfrak{A}_n$-theory. Here we set out a framework for constructing and analyzing the Umkehr map associated to a sphere bundle. We will work with an arbitrary ring spectrum R equipped with a periodic twisting of n-gerbes $\phi : \mathfrak{G}_n \to R$, but we have $R = \mathfrak{A}_n$ in mind throughout.

Let $\pi : E \to X$ be a smooth fiber bundle with fiber the q-sphere S^q and $V \to E$ denote the vertical tangent bundle of the fiber bundle π. Consider the unit map $\eta : S \to R$ from the sphere spectrum, and the J-homomorphism $J : O \to \mathrm{GL}_1(S)$ to the space of units of S, with BJ its delooping. Choose a map $V : E \to BO$ that classifies the vertical tangent bundle and let

$$\tau_E : E \xrightarrow{V} BO \xrightarrow{BJ} B\,\mathrm{GL}_1(S) \xrightarrow{B\,\mathrm{GL}_1(\eta)} B\,\mathrm{GL}_1(R)$$

be the twist of R-theory that it determines. The structure of an R-orientation of V, i.e., a Thom class in R-theory for the Thom spectrum E^V, is equivalent to the datum of the homotopy class of a nullhomotopy of the map τ_E.

Definition 2.6. We call τ_E *the orientation twist of E*, since it is the obstruction to the R-orientability of the vertical tangent bundle V.

Associated to the sphere bundle $\pi : E \to X$ is a Pontrjagin–Thom collapse map $\pi^! : \Sigma^\infty X_+ \to E^{-V}$. Applying R-cohomology, we get a twisted Umkehr map $\pi_! : R^*(E^{-V}) \to R^*(X)$. Since the Thom spectrum associated to the twist $-\tau_E$ is the R-module spectrum $R \wedge E^{-V}$, we may rewrite the twisted Umkehr map as

$$\pi_! : R^*(E; \tau_E) \to R^*(X).$$

More generally, if $\omega : X \to B\,\mathrm{GL}_1(R)$ is a twist of R over X, then there is a product twist $\tau_E \otimes \pi^*\omega$ induced by the action of $B\,\mathrm{GL}_1 S$ on $B\,\mathrm{GL}_1 R$ coming from the

---

[4]Traditionally when considering twisted periodic cohomology, the period of the theory is 2, but of course higher degrees also arise [Sati 2010]. In this setting, the period is somewhat longer, since β_n has dimension $n + 1$. Since n is odd, the period is still even, so the two are naturally comparable.

S-algebra structure of R. The Pontrjagin–Thom collapse map induces a twisted Umkehr map of the form

$$\pi_! : R^*(E; \tau_E \otimes \pi^*\omega) \to R^*(X; \omega).$$

In the presence of an orientation, we can recover the usual Umkehr map along π.

Proposition 2.7. *A trivialization of* τ_E*, or equivalently an* R*-orientation of* V*, determines an* (*untwisted*) *Umkehr map*

$$\pi_! : R^*(E) \cong R^{*-q}(E; \tau_E) \to R^{*-q}(X),$$

defined as the composite of the resulting Thom isomorphism in R*-theory for* E^{-V} *with the twisted Umkehr map.*

Remark 2.8. If R admits a periodic twisting by n-gerbes for $n = 1$, then V is R-orientable if and only if it admits a $\mathrm{spin}_{\mathbb{C}}$ structure. In the case of odd $n > 1$, the spectrum R is an $H\mathbb{Q}$-algebra (Proposition 1.16) and the virtual tangent bundle V is R-orientable if and only if it is an orientable vector bundle in the usual sense.

The twisted Umkehr map $\pi_!$ is natural in the variable X, meaning that if

$$
\begin{array}{ccc}
E' & \xrightarrow{\ \tilde{f}\ } & E \\
{\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\
X' & \xrightarrow{\ f\ } & X
\end{array}
$$

is a pullback diagram of bundles, then there is a commutative diagram

$$
\begin{array}{ccc}
R^*(E'; \tau_{E'} \otimes \pi'^* f^*\omega) & \xrightarrow{\ \pi'_!\ } & R^*(X'; f^*\omega) \\
{\scriptstyle \tilde{f}^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
R^*(E; \tau_E \otimes \pi^*\omega) & \xrightarrow{\ \pi_!\ } & R^*(X; \omega)
\end{array}
\qquad (2.9)
$$

relating the Umkehr maps along π and π' with the contravariant functoriality of the twisted cohomology theory $R^*(\,\cdot\,; \omega)$. Notice that we use the canonical equivalence of twists

$$\tilde{f}^*(\tau_E \otimes \pi^*\omega) \simeq \tilde{f}^*\tau_E \otimes \tilde{f}^*\pi^*\omega \simeq \tau_{E'} \otimes \pi'^* f^*\omega$$

to identify the target of $\tilde{f}^*$ with the source of $\pi_!$.

We will also need the following lemma on Mayer–Vietoris sequences.

Lemma 2.10. *The Umkehr map* $\pi_!$ *is natural for the boundary operator* δ *in Mayer–Vietoris sequences for twisted* R*-theory; i.e., if* $X = U \cup V$ *is a decomposition of* X

into open subsets, and we write $E_{U \cap V}$ for the restriction of the bundle E to the open subset $U \cap V$, then the following diagram commutes:

$$
\begin{array}{ccc}
R^*(E_{U \cap V}; \tau_{E_{U \cap V}} \otimes \pi^* \omega) & \xrightarrow{\pi|_{U \cap V!}} & R^*(U \cap V; \omega) \\
\downarrow{\delta} & & \downarrow{\delta} \\
R^{*+1}(E; \tau_E \otimes \pi^* \omega) & \xrightarrow{\pi_!} & R^{*+1}(X; \omega)
\end{array}
$$

Proof. We first construct the Mayer–Vietoris boundary operator δ in R-theory twisted by ω. We write $X^{-\omega}$ and $U^{-\omega}$, for the R-module Thom spectra associated to the inverse of the twist $\omega \colon X \to B\,\mathrm{GL}_1(R)$ and to the restriction of ω to U, where the latter is a slight abuse of notation. The functoriality of the Thom spectrum functor gives the commutative square of spectra on the left, which we extend to a morphism of cofiber sequences

$$
\begin{array}{ccccccc}
(U \cap V)^{-\omega} & \longrightarrow & U^{-\omega} & \xrightarrow{c_1} & C_1 & \xrightarrow{\partial} & \Sigma(U \cap V)^{-\omega} \\
\downarrow & & \downarrow & & \vdots\,\phi & & \downarrow \\
V^{-\omega} & \longrightarrow & X^{-\omega} & \xrightarrow{c_2} & C_2 & \xrightarrow{\partial} & \Sigma V^{-\omega}
\end{array}
$$

The induced morphism ϕ on cofibers is an equivalence by the excision property of the parametrized cohomology theory associated to the parametrized R-module spectrum over X determined by ω [May and Sigurdsson 2006, §20]. Another point of view is that the Thom spectrum functor $(\cdot)^{-\omega}$ preserves homotopy pushouts [Ando et al. 2014a, Corollary 3.13][5], such as $X = U \cup_{U \cap V} V$, and thus the induced map on cofibers is an equivalence. The Mayer–Vietoris sequence boundary operator δ is defined to be the composite $\delta = c_2^* \circ (\phi^{-1})^* \circ \partial^*$ of the induced maps in R-cohomology, where ϕ^{-1} is a homotopy inverse to the equivalence ϕ.

There is a similar diagram defining the Mayer–Vietoris boundary operator associated to the decomposition $E = E_U \cup_{E_{U \cap V}} E_V$ and the naturality of the Pontrjagin–Thom construction constructs a commutative square of cofiber sequences relating the displayed diagram to the one for E. This gives the desired relation between δ and the twisted Umkehr map $\pi_!$. $\square$

3. T-duality for sphere bundles in theories twisted by n-gerbes

In this section, we define T-duality for sphere bundles and prove the T-duality isomorphism for cohomology theories admitting periodic twisting by n-gerbes. The

---

[5]The cited work uses an ∞-categorical model for the Thom spectrum functor and so the relevant result is expressed using the language of colimits in ∞-categories. It is a consequence of their work and the identification of homotopy colimits with ∞-colimits that any model for the Thom spectrum functor preserves homotopy colimits.

material closely follows that of [Bunke and Schick 2005] in the setting of T-duality for circle bundles. We will also prove that when $n > 1$, any cohomology theory for which such an isomorphism holds must be rational.

3A. *Formulation of T-duality for sphere bundles.* Let X be a topological space, and $\pi : E \to X$ and $\hat{\pi} : \widehat{E} \to X$ be S^q-bundles over X. For simplicity, we work with smooth fiber bundles, but the arguments can be adapted to the case of topological fiber bundles. Note that we do not require the sphere bundles to be orientable. In the analysis of classifying spaces in Section 4, we will work with a chosen structure group for our bundles, but for now that choice is not relevant.

Define $E *_X \widehat{E}$ to be the *fiberwise join* of E and $\widehat{E}$ over X. Note that this is a bundle with fiber the join $S^q * S^q = S^{2q+1}$, and that there are natural fiberwise embeddings

$$i : E \hookrightarrow E *_X \widehat{E} \quad \text{and} \quad \hat{i} : \widehat{E} \hookrightarrow E *_X \widehat{E}$$

given by inclusion of each factor in the join.

Definition 3.1. A *bundle Thom class*[6] for an S^m-bundle $p : Y \to X$ is an element $\mathrm{Th} \in H^m(Y; \mathbb{Z})$ with the property that its restriction to each fiber is a generator of $H^m(S^m; \mathbb{Z})$. Equivalently, $p_!(\mathrm{Th}) = \pm 1 \in H^0(X; \mathbb{Z})$.

We let $n = 2q - 1$, which is evidently odd, and consider representatives

$$H : E \to K(\mathbb{Z}, n+2) \quad \text{and} \quad \widehat{H} : \widehat{E} \to K(\mathbb{Z}, n+2)$$

for cohomology classes

$$[H] \in H^{n+2}(E; \mathbb{Z}) \quad \text{and} \quad [\widehat{H}] \in H^{n+2}(\widehat{E}; \mathbb{Z}).$$

Definition 3.2. We say that the pairs (E, H) and $(\widehat{E}, \widehat{H})$ are *T-dual* if there exists a bundle Thom class $\mathrm{Th} \in H^{n+2}(E *_X \widehat{E})$ with $i^* \mathrm{Th} = [H]$ and $\hat{i}^* \mathrm{Th} = [\widehat{H}]$.

Consider the *correspondence space* $E \times_X \widehat{E}$, which is an $S^q \times S^q$-bundle over X:

---

[6]This is simply called a *Thom class* in [Bunke and Schick 2005]; this usage conflicts with the usual notion of a Thom class as a generator θ for the cohomology of a Thom space. The two notions are closely related, since θ is carried to Th under the collapse map from a sectioned sphere bundle to its Thom space.

There is a tautological homotopy $h : I \times (E \times_X \widehat{E}) \to E *_X \widehat{E}$ from $i \circ p$ to $\hat{i} \circ \hat{p}$ that is given by the formula $h_t(e, \hat{e}) = (t, e, \hat{e})$ and recognizes the fiberwise join $E *_X \widehat{E}$ as a quotient of $I \times (E \times_X \widehat{E})$.

Pulling the twisting classes back over p and $\hat{p}$ gives cohomology classes $p^*[H]$ and $\hat{p}^*[\widehat{H}]$, respectively, on the correspondence space. But since $i^* \text{Th} = [H]$ and $\hat{i}^* \text{Th} = [\widehat{H}]$, we have a homotopy between the maps representing $p^*[H]$ and $\hat{p}^*[\widehat{H}]$:

$$\Lambda = \text{Th} \circ h : H \circ p = \text{Th} \circ i \circ p \to \text{Th} \circ \hat{i} \circ \hat{p} = \widehat{H} \circ \hat{p}.$$

The T-duality transformation T is the composite

$$R^*(E; \tau_E \otimes H) \xrightarrow{\;p^*\;} R^*(E \times_X \widehat{E}; p^*\tau_E \otimes p^*H)$$
$$\cong \Big\downarrow {\scriptstyle \Lambda}$$
$$R^*(E \times_X \widehat{E}; \tau_{E \times_X \widehat{E}} \otimes \hat{p}^*\widehat{H}) \xrightarrow{\;\hat{p}_!\;} R^*(\widehat{E}; \widehat{H})$$

where the middle isomorphism is induced by the homotopy Λ and the canonical isomorphism of the pullback along p of the vertical tangent bundle of $\pi : E \to X$ with the vertical tangent bundle of $\hat{p} : E \times_X \widehat{E} \to \widehat{E}$. We may now state our main theorem.

Theorem 3.3. *Let R be an A_∞ ring spectrum, and let $\phi : \mathfrak{S}_n \to R$ be a periodic twisting of R by n-gerbes. Suppose that X is homotopy equivalent to a finite CW complex, that (E, H) and $(\widehat{E}, \widehat{H})$ are T-dual, and write $\tau_E : X \to B\,\text{GL}_1(R)$ for the orientation twist of the vertical tangent bundle of E. Then the T-duality transformation*

$$T := \hat{p}_! \circ \Lambda \circ p^* : R^*(E; \tau_E \otimes H) \to R^*(\widehat{E}; \widehat{H})$$

is an isomorphism.

Recall that our main examples of such an R are $\mathfrak{S}_n$ itself (Definition 1.13), and $\mathfrak{A}_n$ (Definition 1.11).

3B. *Proof of Theorem 3.3.* We will establish Theorem 3.3 through a series of lemmata.

Lemma 3.4. *Theorem 3.3 holds when X is a point.*

Proof. In this case, the Atiyah–Hirzebruch spectral sequence for the R-cohomology of $E \times \widehat{E} = S^q \times S^q$ contains $R^*(E)$ and $R^*(\widehat{E})$ as direct summands on every page, and thus cannot have any nonzero differentials. This gives an isomorphism of rings

$$R^*(E \times \widehat{E}) \cong R^*[x, \hat{x}]/(x^2, \hat{x}^2),$$

where x and $\hat{x}$ are represented by singular cohomology classes of dimension q. Write $\beta = \phi_*(\iota) \in \pi_{n+1}(R)$ for the periodicity class; when $R = \mathfrak{A}_n$, $\beta = \beta_n$. We claim that Λ is given by multiplication by the degree-0 class $1 \pm \beta x \hat{x}$, where we are regarding β as a $-2q = -(n+1)$-dimensional cohomology class via the identification $R^*(\mathrm{pt}) = \pi_{-*}R$.

Granting that, the result follows: $\hat{p}_! \circ \Lambda \circ p^*(1) = \hat{p}_!(1 \pm \beta x \hat{x}) = \pm \beta \hat{x}$, and

$$\hat{p}_! \circ \Lambda \circ p^*(x) = \hat{p}_![(1 \pm \beta x \hat{x})x] = \hat{p}_!(x) = 1$$

since the Umkehr map $\hat{p}_!$ is given by division by the class x if possible, and is 0 otherwise.

To see the claim, we first recall that for a trivial cohomology class $\alpha = 0 \in H^m(Y; \mathbb{Z})$, represented by a map $\alpha : Y \to K(\mathbb{Z}, m)$, the set of homotopy classes of nullhomotopies of α is a torsor for

$$[Y, \Omega K(\mathbb{Z}, m)] = H^{m-1}(Y; \mathbb{Z}) = H^m(\Sigma Y, \mathbb{Z}),$$

by concatenation of homotopies. In particular, it is in bijection with this group.

We note that since $H^{n+2}(E) = 0 = H^{n+2}(\widehat{E})$, we have $H = \widehat{H} = 0$. So p^*H and $\hat{p}^*\widehat{H}$ are necessarily null. Further, since $H^{n+1}(E) = 0 = H^{n+1}(\widehat{E})$, there is a unique homotopy class of trivialization (nullhomotopy) of H and $\widehat{H}$, yielding preferred nullhomotopies of p^*H and $\hat{p}^*\widehat{H}$. Note that the nullhomotopies of the pulled-back classes are not unique, since the group $H^{n+1}(E \times \widehat{E}) = \mathbb{Z}$ affords many possibilities.

Indeed, the homotopy Λ of Theorem 3.3 is a nontrivial homotopy between the trivialized twists p^*H and $\hat{p}^*\widehat{H}$. We may therefore regard it as being represented through the action of an element of the group $H^{n+1}(E \times \widehat{E}) \cong H^{n+2}\Sigma(E \times \widehat{E})$ on the trivial twist. But by construction, Λ is given as a composite

$$\Sigma(E \times \widehat{E}) \to E * \widehat{E} = S^{n+2} \xrightarrow{\text{Th}} K(\mathbb{Z}, n+2).$$

Here the first map arises from the tautological homotopy $I \times E \times \widehat{E} \to E * \widehat{E}$; the factorization through $\Sigma(E \times \widehat{E})$ uses the fact that the inclusions of E and $\widehat{E}$ into $E * \widehat{E}$ are nullhomotopic. In particular, this map induces an isomorphism in H^{n+2}. By assumption, this composite is a generator of $H^{n+2}\Sigma(E \times \widehat{E})$. The map that Λ induces in R^* is therefore given by multiplication by the invertible class

The result follows, since the top-dimensional cohomology class of S^{n+1} pulls back to $x\hat{x}$ on $E \times \widehat{E}$. $\qquad\square$

We now establish the naturality properties of the T-duality transformation that we need to prove Theorem 3.3 by cellular induction.

Lemma 3.5. *Let $f : Y \to X$ be a continuous map, and define $F : f^*E \to E$ and $\widehat{F} : f^*\widehat{E} \to \widehat{E}$ to be induced maps from the pullbacks of E and $\widehat{E}$ along f. Then the T-duality map for the bundles E and $\widehat{E}$ on X pulls back over f; that is, the following diagram commutes*:

$$
\begin{array}{ccc}
R^*(E, \tau_E \otimes H) & \xrightarrow{\ \ T\ \ } & R^*(\widehat{E}, \widehat{H}) \\
{\scriptstyle F^*}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{F}^*} \\
R^*(f^*E, F^*(\tau_E \otimes H)) & \xrightarrow{\ \ T\ \ } & R^*(f^*\widehat{E}, \widehat{F}^*\widehat{H})
\end{array}
$$

Proof. The maps p^* and Λ commute with F^* and $\widehat{F}^*$ by the contravariant functoriality of twisted cohomology theories. The twisted Umkehr map $\hat{p}_!$ also commutes with F^* and $\widehat{F}^*$ by diagram (2.9). $\qquad\square$

Lemma 3.6. *If X is decomposed as the union $X = U \cup V$ of open subsets, we may restrict E and $\widehat{E}$ to these subsets, obtaining $E = E_U \cup E_V$ and $\widehat{E} = \widehat{E}_U \cup \widehat{E}_V$. Further, let δ and $\hat{\delta}$ be the boundary operators in the Mayer–Vietoris sequences for these decompositions of E and $\widehat{E}$, respectively. Then the T-duality transformation preserves these Mayer–Vietoris sequences; that is,*

$$
T \circ \delta = \hat{\delta} \circ T : R^*(E_{U \cap V}, (\tau_E \otimes H)|_{E_{U \cap V}}) \to R^{*+1}(\widehat{E}, \widehat{H}).
$$

Proof. We argue that each component of $T = \hat{p}_! \circ \Lambda \circ p^*$ commutes with the relevant boundary operator. The pullback map p^* is induced by a map of spectra over E and hence extends to a natural transformation of parametrized cohomology theories on spaces over E [May and Sigurdsson 2006, §20]. The data of the boundary map ∂ in the cofiber sequence associated to a pair of spaces over E, such as in a Mayer–Vietoris sequence, is preserved by such a natural transformation. Similarly, the map Λ is induced by a map of spectra over $E \times_X \widehat{E}$, so commutes with the boundary operators. The twisted Umkehr map $\hat{p}_!$ commutes with the boundary operators by Lemma 2.10. $\qquad\square$

The proof of Theorem 3.3 immediately follows by cellular induction, using the previous three results, Lemmas 3.4–3.6.

Remark 3.7. Note that if R' is an A_∞ ring spectrum and $\psi : R \to R'$ an A_∞ map, the composite $\psi \circ \phi$ is a periodic twisting of R' by n-gerbes, and so the results of this theorem also hold for R'. Further, it is evident from the proof that ψ throws the T-duality isomorphism for R onto that for R'.

Summarizing this result in the case that $R = \mathfrak{A}_n$, $R' = H\mathbb{Q}[\beta_n^{\pm 1}]$, and $\psi = \mathrm{ch}_n$, we have:

Corollary 3.8. *The higher Chern character* ch_n *throws the T-duality isomorphism for* $\mathfrak{A}_n$ *onto that of periodic rational cohomology, in the sense that the following diagram commutes*:

$$
\begin{array}{ccc}
\mathfrak{A}_n^*(E; \tau_E \otimes H) & \xrightarrow[\cong]{\;\; T_{\mathfrak{A}_n} \;\;} & \mathfrak{A}_n^*(\widehat{E}; \widehat{H}) \\
{\scriptstyle \mathrm{ch}_n} \downarrow & & \downarrow {\scriptstyle \mathrm{ch}_n} \\
H\mathbb{Q}^*(E; \tau_E \otimes H)[\beta_n^{\pm 1}] & \xrightarrow[T_{H\mathbb{Q}[\beta_n^{\pm 1}]}]{\cong} & H\mathbb{Q}^*(\widehat{E}; \widehat{H})[\beta_n^{\pm 1}]
\end{array}
$$

3C. A T-duality isomorphism implies rationality. Let R be an A_∞ ring spectrum which admits a twisting by $n + 2$-dimensional cohomology classes. That is, we take as given a map

$$
\tau : K(\mathbb{Z}, n + 2) \to B\,\mathrm{GL}_1(R),
$$

whence we may define twisted cohomology groups $R^*(X, H)$ for $H \in H^{n+2}(X, \mathbb{Z})$. Note we do *not* assume that this twisting is periodic in the sense of [Definition 1.15](). In other words, the map

$$
t : \Sigma_+^\infty K(\mathbb{Z}, n + 1) \to R
$$

of ring spectra adjoint to τ is not assumed to factor through the Arthan spectrum $\mathfrak{S}_n = \Sigma_+^\infty K(\mathbb{Z}, n + 1)[\iota^{-1}]$. In particular, we may not immediately conclude from [Proposition 1.16]() that R is a rational spectrum.

One may ask whether a T-duality isomorphism holds in this setting. That is, if (E, H) and $(\widehat{E}, \widehat{H})$ are T-dual S^q-bundles[7] over X in the sense of [Definition 3.2](), we may, as before, form the T-duality transformation T as the composite

$$
\begin{array}{ccc}
R^*(E; \tau_E \otimes H) & \xrightarrow{\;\; p^* \;\;} & R^*(E \times_X \widehat{E}; p^*\tau_E \otimes p^*H) \\
& & \cong \downarrow \Lambda \\
& R^*(E \times_X \widehat{E}; \tau_{E \times_X \widehat{E}} \otimes \hat{p}^*\widehat{H}) \xrightarrow{\;\; \hat{p}_! \;\;} & R^*(\widehat{E}; \widehat{H})
\end{array}
$$

and ask whether T is an isomorphism.

Theorem 3.9. *The following are equivalent*:

(1) *The T-duality transformation T is an isomorphism for all base spaces X of the homotopy type of a finite CW complex.*

(2) *The T-duality transformation T is an isomorphism for X a point.*

---

[7]We still require $n = 2q + 1$.

(3) *The twisting τ is periodic.*

In particular, any cohomology theory which admits a T-duality isomorphism for S^q-bundles with $q > 1$ must be a rational cohomology theory.

Proof. Clearly the first statement implies the second. Conversely, the proof of Theorem 3.3 showed that the general case reduced to the case X a point using naturality (Lemma 3.5) and the Mayer–Vietoris sequence (Lemma 3.6) — note that these results do not require the twisting to be periodic. We have shown that the T-duality isomorphism holds when X is a point under the assumption of a periodic twisting (Lemma 3.4), so the third statement implies the second. So we need only show that the second statement implies the third.

Let us define $\beta : S^{n+1} \to R$ as we have done in Section 1B, via the composite

$$S^{n+1} \xrightarrow{\iota} \Sigma^\infty K(\mathbb{Z}, n+1)_+ \xrightarrow{t} R$$

where t is adjoint to τ. We will show that β is an invertible element of $\pi_* R$ under the assumption that T-duality holds over a point. This then implies that the map of ring spectra t factors through the Arthan spectrum $\mathfrak{S}_n$, and in particular makes R an algebra spectrum over $\mathfrak{S}_n$.

As in the proof of Lemma 3.4, we may write $R^*(E \times \widehat{E}) = R^*[x, \hat{x}]/(x^2, \hat{x}^2)$ and compute the T-duality map to be the R_*-linear operation given by

$$1 \mapsto \pm\beta\hat{x} \quad \text{and} \quad x \mapsto 1.$$

This can only be an isomorphism if β is invertible.   □

4. Classifying spaces for T-dual pairs

We continue to insist that $n = 2q - 1$ be odd. In this section, we introduce spaces $R_n(G)$ and $P_n(G)$ that classify the objects considered in the previous section, at least in the case where the bundles E and $\widehat{E}$ are oriented. More precisely, let G be a topological group equipped with a homomorphism to $\mathrm{Homeo}^+(S^q)$, the group of orientation-preserving homeomorphisms of the q-sphere S^q.

We will construct natural isomorphisms that give the following identifications:

- $[X, R_n(G)]$ is the set of equivalence classes of pairs (E, H), where $E \to X$ is an S^q-bundle with structure group G, and H is a map representing a twisting class $[H] \in H^{n+2}(E)$.

- $[X, P_n(G)]$ is the set of equivalence classes of triples $(E, \widehat{E}, \mathrm{Th})$, where E and $\widehat{E}$ are S^q-bundles with structure group G, and $\mathrm{Th} \in H^{n+2}(E *_X \widehat{E})$ is a bundle Thom class.

There is a map $P_n(G) \to R_n(G)$ that induces the natural transformation which carries $(E, \widehat{E}, \mathrm{Th})$ to $(E, i^* \mathrm{Th})$ (where we make a slight abuse of notation and

write i^* Th for a choice of representing map). We will analyze the homotopy type of the spaces involved and show that this map is an equivalence precisely when $q = 1$.

Remark 4.1. We may regard $[X, P_n(G)]$ as the set of T-dual pairs over X. The failure of this map to be an equivalence for $q > 1$ shows that it is not the case that every pair (E, H) has a T-dual $(\widehat{E}, \widehat{H})$, and that when a T-dual exists, there is no guarantee that it is unique. However, we will encounter special cases towards the end of Section 4C where T-duals do exist and can be unique.

4A. *Euler classes and the classifying space $P_n(G)$.* Let $p : E \to X$ be a fiber bundle with fiber S^m. If E is the subsphere bundle $E = S(V)$ of an oriented real vector bundle V over X of rank $m + 1$, then there is an *Euler class*

$$e(E) = e(V) \in H^{m+1}(X)$$

which is the pullback of the (usual notion of the) Thom class of X^V under the zero-section.

More generally, if E is an oriented S^m-bundle, then the fiberwise unreduced suspension

$$\Sigma_X E := I \times E / \sim, \quad \text{where } (t, e) \sim (t, e') \text{ for } t = 0, 1 \text{ when } p(e) = p(e'),$$

is a topological S^{m+1}-bundle over X which inherits an orientation from E. There are two canonical sections $s_t : X \to \Sigma_X E$, where $t = 0, 1$, given by $s_t(x) = [t, e]$ for any e in the fiber over x. The associated Thom space

$$X^{E \oplus 1} := \Sigma_X E / s_1(X)$$

is obtained by collapsing the image of s_1 to a point. Since $\Sigma_X E$ is oriented, there is a Thom class $\theta_E \in H^{m+1}(X^{E \oplus 1})$ in integral cohomology with the property that the map

$$H^*(X) \to \widetilde{H}^{*+m+1}(X^{E \oplus 1}) \quad \text{given by} \quad \alpha \mapsto (\Sigma_X p)^*(\alpha) \cup \theta_E$$

is an isomorphism. We define the Euler class of E by the formula $e(E) = s_0^*(\theta_E)$. The following is a straightforward application of the Leray–Hirsch theorem and the Gysin sequence; in the case $m = 1$, see [Bunke and Schick 2005, §2.2; Bouwknegt et al. 2004, §3.1].

Proposition 4.2. *The following are equivalent*:

(1) *E admits a bundle Thom class* $\mathrm{Th} \in H^m(E)$.

(2) *$H^*(E)$ is a free $H^*(X)$-module on 1, Th.*

(3) *$e(E) = 0$.*

The set of homotopy classes of nullhomotopies of a representative cocycle for $e(E) \in H^{m+1}(X)$, if trivial, is a torsor under $H^m(X)$. The same is true of bundle Thom classes $\mathrm{Th} \in H^m(E)$: for any $\alpha \in H^m(X)$, $\mathrm{Th} + p^*(\alpha)$ is another such bundle Thom class. This structure is indeed a torsor (as it arises from a free and transitive action of $H^m(X)$) by virtue of point (2) in the above proposition. Thus we may in fact conclude:

> *the set of homotopy classes of trivializations of $e(E)$ is naturally in bijection with the set of bundle Thom classes $\mathrm{Th} \in H^m(E)$.*

The next result implies that our definition of T-duality may be described in terms of characteristic classes, as in the cases $q = 1, 3$ [Bouwknegt et al. 2004; Bunke and Schick 2005; Bouwknegt et al. 2015a].

Proposition 4.3. *Suppose that E and $\widehat{E}$ are orientable S^q-bundles over X, and that (E, H) and $(\widehat{E}, \widehat{H})$ are T-dual. Then the equations*

$$e(E) = \pm \widehat{\pi}_!\,[\widehat{H}] \quad and \quad e(\widehat{E}) = \pm \pi_!\,[H]$$

hold in $H^{q+1}(X)$.

Proof. Let $\Sigma_X E$ and $\Sigma_X \widehat{E}$ be the oriented S^{q+1}-bundles over X formed by taking the fiberwise unreduced suspension. Form the fiberwise smash product $\Sigma_X E \wedge_X \Sigma_X \widehat{E}$ with respect to the canonical sections s_1 on either side. This is an S^{2q+2}-bundle and that inherits an orientation from E and $\widehat{E}$. In other words, we equip the Thom space

$$X^{E \oplus \widehat{E} \oplus 2} := (\Sigma_X E \wedge_X \Sigma_X \widehat{E})/s_1(X)$$

with the Thom class $\theta_E \cdot \theta_{\widehat{E}} \in H^{2q+2}(X^{E \oplus \widehat{E} \oplus 2})$. Consequently, the associated Euler class satisfies

$$e(E *_X \widehat{E}) = e(E) \cdot e(\widehat{E}).$$

The S^{2q+1}-bundle $E *_X \widehat{E}$ also inherits an orientation from E and $\widehat{E}$; the Thom class on the associated Thom space agrees with $\theta_E \cdot \theta_{\widehat{E}}$ under the homeomorphism

$$X^{(E *_X \widehat{E}) \oplus 1} \cong X^{E \oplus \widehat{E} \oplus 2}.$$

Using these orientations to define Thom isomorphisms and their associated Umkehr maps, the inclusion $i : E \to E *_X \widehat{E}$ induces a commutative diagram relating the Gysin sequences for $\rho : E *_X \widehat{E} \to X$ and $\pi : E \to X$:

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H^{2q+1}(X) & \xrightarrow{\rho^*} & H^{2q+1}(E *_X \widehat{E}) & \xrightarrow{\rho_!} & H^0(X) & \xrightarrow{e(E *_X \widehat{E})} & H^{2q+2}(X) & \longrightarrow \cdots \\
& \Big\| & & \Big\downarrow{\scriptstyle i^*} & & \Big\downarrow{\scriptstyle e(\widehat{E})} & & \Big\| & \\
\cdots \longrightarrow & H^{2q+1}(X) & \xrightarrow[\pi^*]{} & H^{2q+1}(E) & \xrightarrow[\pi_!]{} & H^{q+1}(X) & \xrightarrow[e(E)]{} & H^{2q+2}(X) & \longrightarrow \cdots
\end{array}
$$

Now suppose that (E, H) and $(\widehat{E}, \widehat{H})$ are T-dual. In particular, we are given a bundle Thom class $\text{Th} \in H^{2q+1}(E *_X \widehat{E})$ satisfying $i^* \text{Th} = [H]$ and $\hat{i}^* \text{Th} = [\widehat{H}]$. We do not know if (the suspension of) Th is the pullback of our preferred Thom class $\theta_E \cdot \theta_{\widehat{E}}$, but we do know that $\rho_! \text{Th} = \pm 1$, and consequently that $\pi_![H] = \pm e(\widehat{E})$. The relation $\hat{\pi}_![\widehat{H}] = \pm e(E)$ follows from the analogous diagram involving the Gysin sequence for $\hat{\pi}$. $\square$

We now construct the space $P_n(G)$ which parametrizes triples $(E, \widehat{E}, \text{Th})$, where E and $\widehat{E}$ are S^q-bundles with structure group G, and Th is a bundle Thom class on $E *_X \widehat{E}$.

Definition 4.4. Let $P_n(G)$ be the homotopy fiber of the composite

$$BG \times BG \xrightarrow{B \, \text{join}} B \, \text{Homeo}^+(S^q * S^q) \xrightarrow{e} K(\mathbb{Z}, 2q + 2) = K(\mathbb{Z}, n + 3).$$

Here $\text{join} : G \times G \to \text{Homeo}^+(S^q * S^q)$ is the join of a pair of orientation-preserving homeomorphisms and e classifies the Euler class.

By construction, homotopy classes of maps $X \to P_n(G)$ are in bijection with the set of equivalence classes of the following data:

- a pair $(E, \widehat{E})$ of S^q-bundles over X with structure group G, and
- the homotopy class of a trivialization of the Euler class of the fiberwise join $E *_X \widehat{E}$.

Applying Proposition 4.2 (or rather, its subsequent refinement) in this setting yields:

Proposition 4.5. *There is a bijection between $[X, P_n(G)]$ and the set of equivalence classes of triples $(E, \widehat{E}, \text{Th})$, where $\text{Th} \in H^{n+2}(E \times_X \widehat{E})$ is a bundle Thom class.*

Remark 4.6. Up to equivalence, the data of a triple $(E, \widehat{E}, \text{Th})$ may be identified with the data of two T-dual pairs (E, H) and $(\widehat{E}, \widehat{H})$ via $(E, \widehat{E}, \text{Th}) \mapsto (E, i^* \text{Th}), (E, \hat{i}^* \text{Th})$. Thus, $P_n(G)$ is the classifying space for T-dual pairs.

4B. *The classifying space $R_n(G)$.* We now work with slightly more generality. Let F be a topological space and suppose that G is a topological group equipped with a continuous homomorphism $G \to \text{Homeo}(F)$. Consider pairs (E, H) consisting of a fiber bundle $\pi : E \to X$ with fiber F and structure group G, along with a map $H : E \to K(\mathbb{Z}, n + 2)$ representing a cohomology class $[H] \in H^{n+2}(E; \mathbb{Z})$ on the total space that is fiberwise trivial, i.e., for every $x \in X$, the restriction of $[H]$ to $H^{n+2}(E_x; \mathbb{Z})$ is zero. We will construct a classifying space $R_n(G)$ for such pairs (E, H). Our construction is a generalization of Bunke and Schick's classifying space for principal circle bundles equipped with a $U(1)$-gerbe [2005].

Let $\text{Map}_0(F, K(\mathbb{Z}, n+2))$ be the connected component of $\text{Map}(F, K(\mathbb{Z}, n+2))$ containing the basepoint. In other words, $\text{Map}_0(F, K(\mathbb{Z}, n+2))$ is the subspace of

nullhomotopic maps. The group G acts on $\mathrm{Map}(F, K(\mathbb{Z}, n+2))$ by $(g \cdot f)(x) = f(g^{-1}x)$.

Proposition 4.7. *Equivalence classes of pairs (E, H) are classified by the space*

$$R_n(G) := EG \times_G \mathrm{Map}_0(F, K(\mathbb{Z}, n+2)).$$

Proof. Starting with a group G, one can extend the G-Borel construction $EG \times_G (\cdot)$ to the two-sided bar construction $B(\cdot, G, \cdot)$, where the first and last entries are right and left G-spaces, respectively. Applying this to the space of functions from F to $K = K(\mathbb{Z}, n+2)$ gives $R = B(*, G, \mathrm{Map}_0(F, K))$. The total space of the universal pair over R is

$$E_{\mathrm{univ}} = B(F, G, \mathrm{Map}_0(F, K)) \to B(*, G, \mathrm{Map}_0(F, K)) = R$$

with projection to R induced by $F \to *$. Note the evaluation map $F \times \mathrm{Map}_0(F, K) \to K$ is G-invariant and hence descends to the homotopy quotient:

$$H_{\mathrm{univ}} : B(F, G, \mathrm{Map}_0(F, K)) \to K.$$

This map represents the universal cohomology class $[H_{\mathrm{univ}}] \in H^{n+2}(E_{\mathrm{univ}}; \mathbb{Z})$. The pair $(E_{\mathrm{univ}}, H_{\mathrm{univ}})$ is the universal pair.

Given a pair (E, H), we now construct a classifying map $X \to R$. Let $P \to X$ be the principal G-bundle associated to E. The usual contracting simplicial homotopy shows that the projection $B(G, G, P) \to P$ induced by the action of G on P is a homotopy equivalence. Now form the following diagram of principal G-bundles:

$$
\begin{array}{ccccc}
P & \xleftarrow{\simeq} & B(G, G, P) & \longrightarrow & B(G, G, \mathrm{Map}_0(F, K)) \\
\downarrow & & \downarrow & & \downarrow \\
X & \xleftarrow{\simeq}{\dashrightarrow} & B(*, G, P) & \longrightarrow & B(*, G, \mathrm{Map}_0(F, K)) = R
\end{array}
$$

The map on the lower left is an equivalence by the five lemma, so we may choose a wrong-way homotopy inverse as indicated by the dashed arrow. The maps pointed to the right are induced by the G-map $P \to \mathrm{Map}_0(F, K)$ adjoint to the map $H : P \times_G F = E \to K$. The composite along the bottom gives the classifying map for the pair (E, H).

To see that the fiber bundle $E \to X$ is the pullback of the fiber bundle $E_{\mathrm{univ}} \to R$ along this classifying map, apply $F \times_G (\cdot)$ to the total spaces of the principal G-bundles in the diagram. $\qquad\square$

Remark 4.8. More generally, if we replace $K(\mathbb{Z}, n+2)$ by any space K, then R classifies equivalence classes of pairs (E, H) consisting of a fiber bundle $E \to X$ with fiber F and structure group G and a map $H : E \to K$ that is fiberwise null-homotopic.

4C. *Comparing $R_n(G)$ and $P_n(G)$.* We now return to the usual setting of oriented S^q-bundles with structure group G and set $n = 2q - 1$. We will analyze the map $f : P_n(G) \to R_n(G)$ that represents the forgetful functor taking a T-dual pair (E, H) and $(\widehat{E}, \widehat{H})$ to the first item (E, H). Abbreviate $P = P_n(G)$ and $R = R_n(G)$ and note that by construction R fits into a fiber sequence

$$\text{Map}(S^q, K(\mathbb{Z}, n+2)) \to R \to BG.$$

Also note that, by the connectivity of the Eilenberg–MacLane space, $\text{Map}(S^q, K(\mathbb{Z}, n+2))$ is connected, so we are automatically restricting to nullhomotopic maps as in the construction of R.

The classifying space P also participates in a fiber sequence, which we extend to the left by one entry:

$$\cdots \to K(\mathbb{Z}, n+2) \to P \to BG \times BG \xrightarrow{e(\cdot * \cdot)} K(\mathbb{Z}, n+3).$$

The composite of the map from P with the projection $\text{pr}_i : BG \times BG \to BG$ to the first or second factor classifies a principal G-bundle with fiber S^q over P, which we call E and $\widehat{E}$, respectively. The fiberwise join $E * \widehat{E}$ carries a canonical bundle Thom class by [Proposition 4.5](). Pulling it back along the inclusion $i_1 : E \to E * \widehat{E}$ gives a class $[H] \in H^{n+2}(E)$. Choose a representing map $H : E \to K(\mathbb{Z}, n+2)$. By [Proposition 4.7](), the pair (E, H) over P is classified by a map $f : P \to R$ which fits into a map of fiber sequences

$$
\begin{array}{ccccc}
K(\mathbb{Z}, n+2) & \longrightarrow & P & \longrightarrow & BG \times BG \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \text{pr}_1} \\
\text{Map}(S^q, K(\mathbb{Z}, n+2)) & \longrightarrow & R & \longrightarrow & BG
\end{array}
\tag{4.9}
$$

Consider the new fibration $P \to BG$ given by passing from the upper middle to lower right of this diagram; this now has fiber $BG \times K(\mathbb{Z}, n+2)$. There is a morphism of fiber sequences over BG:

$$
\begin{array}{ccccc}
BG \times K(\mathbb{Z}, n+2) & \longrightarrow & P & \longrightarrow & BG \\
\downarrow & & \downarrow{\scriptstyle f} & & \| \\
\text{Map}(S^q, K(\mathbb{Z}, n+2)) & \longrightarrow & R & \longrightarrow & BG
\end{array}
\tag{4.10}
$$

To compare the homotopy types of P and R, we may then compare the homotopy type of their fibers. Define a function $\tilde{e} : BG \to \text{Map}(S^q, K(\mathbb{Z}, n+2))$ as the composite

$$BG \xrightarrow{e} K(\mathbb{Z}, q+1) \simeq \Omega^q K(\mathbb{Z}, n+2) \to \text{Map}(S^q, K(\mathbb{Z}, n+2)).$$

Here, e is the Euler class of an S^q bundle with structure group G, and the second map is the inclusion of the space of based maps into all maps. Note that the fiber $\mathrm{Map}(S^q, K(\mathbb{Z}, n+2))$ of the bottom fibration in (4.10) is an H-space, via multiplication of maps in the target.

Lemma 4.11. *In the comparison between the fibers over P and R:*

(1) *The induced map of fibers $K(\mathbb{Z}, n+2) \to \mathrm{Map}(S^q, K(\mathbb{Z}, n+2))$ in (4.9) is homotopic to the inclusion $x \mapsto \mathrm{const}_x$ of constant maps.*

(2) *The fiber map $BG \times K(\mathbb{Z}, n+2) \to \mathrm{Map}(S^q, K(\mathbb{Z}, n+2))$ in (4.10) is homotopic to the product of the maps $\tilde{e}$ and const.*

Proof. The fundamental class $\iota : S^{n+2} \to K(\mathbb{Z}, n+2)$ represents the bundle Thom class of the trivial sphere bundle over a point. Thus, the induced map of fibers in (4.9) is adjoint to the composite

$$S^q \times K(\mathbb{Z}, n+2) \xrightarrow{i_1 \times 1} S^{2q+1} \times K(\mathbb{Z}, n+2)$$
$$\xrightarrow{\iota \times 1} K(\mathbb{Z}, n+2) \times K(\mathbb{Z}, n+2) \xrightarrow{+} K(\mathbb{Z}, n+2).$$

This composite is homotopic to the projection to the second factor, which proves the first claim.

The second claim is a parametrized form of the first. The identity map of the space P classifies a triple $(E, \widehat{E}, \mathrm{Th})$ over P of S^q-bundles E and $\widehat{E}$, and a bundle Thom class $\mathrm{Th} \in H^{n+2}(E *_P \widehat{E})$. In these terms, f carries this to the map $f : P \to R$ representing the pair $(E, i_1^*(\mathrm{Th}))$, where $i_1 : E \to E *_P \widehat{E}$ is the natural inclusion.

Write $X_b \cong BG \times K(\mathbb{Z}, n+2)$ for the fiber over b, and J_b for the fiber inclusion $J_b : X_b \to P$ in the top row of (4.10). Then the composition $f \circ J_b$ represents the pair $(J_b^* E, i_1^* J_b^*(\mathrm{Th}))$ over X_b. However, $J_b^* E$ is trivial, since its classifying map factors through $\{b\}$. Therefore, the fiberwise join $J_b^*(E *_P \widehat{E}) = \Sigma_{X_b}^{q+1} \widehat{E}$ is the $(q+1)$-th fiberwise suspension of $\widehat{E}$. Then the pullback of the bundle Thom class $i_1^* J_b^*(\mathrm{Th})$ is the composite

$$S^q \times BG \times K(\mathbb{Z}, n+2) = S^q \times X_b \cong J_b^* E \xrightarrow{i_1} \Sigma_{X_b}^{q+1} \widehat{E} \xrightarrow{J_b} E *_P \widehat{E} \xrightarrow{\mathrm{Th}} K(\mathbb{Z}, n+2).$$

Now, the Euler class of the S^{n+2}-bundle $\Sigma_{X_b}^{q+1} \widehat{E}$ is precisely the restriction of the bundle Thom class $J_b^* \mathrm{Th}$ along the zero section. Restricting along i_1 gives an S^q-parametrized form of this fact: $i_1 \circ J_b \circ \mathrm{Th} : S^q \times X_b \to K(\mathbb{Z}, n+2)$ is the Euler class of $\Sigma_{X_b}^{q+1} \widehat{E}$ when restricted to any point in S^q. Via adjunction, this is precisely $\tilde{e}$ in the BG variable. □

Now the following result provides special cases when T-duals exist and are unique.

Corollary 4.12. *If the Euler class map $e : BG \to K(\mathbb{Z}, q+1)$ is m-connected, so too is the comparison map $f : P \to R$. Thus, over complexes of dimension less than m, there exists a unique T-dual $(\widehat{E}, \widehat{H})$ for any (E, H). In dimension m, such duals exist, but are not necessarily unique.*

Proof. The result then follows from Lemma 4.11 and the fact that for an H-space such as $K(\mathbb{Z}, n+2)$, the product of the constant maps and based maps gives an equivalence $K(\mathbb{Z}, n+2) \times \Omega^q K(\mathbb{Z}, n+2) \simeq \mathrm{Map}(S^q, K(\mathbb{Z}, n+2))$. □

Example 4.13 (principal circle bundles). When $n = q = 1$ and $G = U(1)$, the Euler class $e : BU(1) \to K(\mathbb{Z}, 2)$ is a weak homotopy equivalence; therefore, the map f is as well. This recovers the result of Bunke and Schick [2005] that in the case of circle bundles and $U(1)$-gerbes, T-dual pairs exist and are unique up to equivalence.

Example 4.14 (principal $SU(2)$-bundles). Suppose that $n = 5$, i.e., $q = 3$, and $G = SU(2)$, and consider pairs (E, H) consisting of a principal $SU(2)$-bundle $\pi : E \to X$ and a 5-gerbe $H : E \to K(\mathbb{Z}, 7)$ on E. The Euler class $e : B\,SU(2) \to K(\mathbb{Z}, 4)$ is 5-connected, and so the same holds for f. This recovers the result of Bouwknegt et al. [2015a] that when the base X has dim ≤ 4, the T-dual of (E, H) exists and is unique.

We can, furthermore, lift this statement to one of our newly constructed theories. Still when $n = 5$, the spectrum $\mathfrak{a}_5 = K^{(4)}(ku)$ is the fourth iterated algebraic K-theory of ku and $\mathfrak{A}_5 = K^{(4)}(ku)[\beta_5^{-1}]$ is the localization at the higher Bott element β_5 of degree 6. The map

$$\tau_5 : K(\mathbb{Z}, 7) \to B\,\mathrm{GL}_1\,K^{(4)}(ku)[\beta_5^{-1}] = B\,\mathrm{GL}_1\,\mathfrak{A}_5$$

defines a twisting of $\mathfrak{A}_5$ by 5-gerbes and we now consider the T-duality isomorphism. Since the total space of the universal principal $SU(2)$-bundle is contractible, the vertical tangent bundle of E is trivializable. Hence, the associated twist τ_E of $\mathfrak{A}_5$ is trivial and the T-duality isomorphism takes the form

$$T = \hat{p}_! \circ \Lambda \circ p^* : \mathfrak{A}_5^*(E, H) \to \mathfrak{A}_5^{*-3}(\widehat{E}, \widehat{H}).$$

By Corollary 3.8, the Chern character throws this T-duality isomorphism onto a T-duality isomorphism

$$T : H\mathbb{Q}^*(E; H)[\beta_5^{-1}] \xrightarrow{\cong} H\mathbb{Q}^{*-3}(\widehat{E}; \widehat{H})[\beta_5^{-1}]$$

in ordinary cohomology. This recovers the T-duality isomorphism for principal $SU(2)$-bundles studied by Bouwknegt et al. [2015a, Theorem 5.2].

Example 4.15 (nonprincipal $SU(2)$-bundles). We again suppose that $n = 5$, i.e., $q = 3$, and that $G = SO(4)$ acts on S^3 via the inclusion

$$SO(4) = \mathrm{Isom}^+(SU(2)) \to \mathrm{Homeo}^+(SU(2)) \cong \mathrm{Homeo}^+(S^3).$$

In this case, the map on fibers in the map of fiber sequences comparing P and R (4.10) is

$$\tilde{e} \cdot \mathrm{const} : B\,SO(4) \times K(\mathbb{Z}, 7) \to \mathrm{Map}(S^3, K(\mathbb{Z}, 7)) \simeq K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 7).$$

The effect on π_4 is the projection $+ : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$. This means that over S^4, each pair (E, H) of an S^3 bundle with structure group $SO(4)$ and a 5-gerbe H has an infinite rank-one lattice of possible T-duals $(\widehat{E}, \widehat{H})$. In particular, spherical T-duals are not unique in this case. This analysis recovers the construction of spherical T-duals given by Bouwknegt et al. [2015b] in the case of $M = S^4$. They construct an infinite family of T-duals over any compact, oriented, simply connected, 4-dimensional manifold M, but in this case the classification of all possible T-dual pairs is a more intricate problem. Given a T-dual pair (E, H) and $(\widehat{E}, \widehat{H})$, we have a T-duality isomorphism in $\mathfrak{A}_5$-cohomology and Bott-inverted rational cohomology as in the previous example.

Example 4.16 (nonorientable bundles). Baraglia [2014] extended topological T-duality, as an isomorphism in twisted K-theory, to the case of general circle bundles. Via a geometric approach, he proves existence and uniqueness of T-duals. This requires considering a twist using real line bundles, i.e., arising from the factor $K(\mathbb{Z}/2, 1) \subset B\,\mathrm{GL}_1(K)$, in addition to the usual determinantal factor $K(\mathbb{Z}, 3)$. Specifically: the map $O(2) \to \mathrm{Homeo}(S^1)$ is a homotopy equivalence (this is a theorem of Kneser [1926]), so a general S^1-bundle fails to be equivalent to a principal bundle only on the basis of orientability.

This was put in the homotopy-theoretic framework of Bunke and Schick by Mathai and Rosenberg [2014, Theorem 2.2], who constructed a classifying space for T-dual pairs in the not necessarily oriented setting. When we defined the classifying space $P_n(G)$, we needed our sphere bundles to be oriented to define the Euler class. We may extend Mathai and Rosenberg's construction of the classifying space for T-dual pairs to the case of higher dimensional sphere bundles by considering not necessarily oriented S^q-bundles E and $\widehat{E}$ with structure group G over a space X. Let $\widetilde{X}$ be the double cover of X which orients $E *_X \widehat{E}$, and Q the oriented S^{n+1}-bundle over $\widetilde{X}$ to which $E *_X \widehat{E}$ pulls back. One may show that triples $(E, \widehat{E}, \mathrm{Th})$, where $\mathrm{Th} \in H^{n+2}(Q)$, are classified by a space $\widetilde{P}_n(G)$, which is defined as the homotopy fiber of the composite

$$BG \times BG \xrightarrow{B\,\mathrm{join}} B\,\mathrm{Homeo}(S^q * S^q) \xrightarrow{(w_1, e)} E\mathbb{Z}/2 \times_{\mathbb{Z}/2} K(\mathbb{Z}, n+3).$$

Here $\mathbb{Z}/2$ acts on $K(\mathbb{Z}, n+3)$ by negating the fundamental class. The Borel construction fibers over $K(\mathbb{Z}/2, 1)$, with fiber $K(\mathbb{Z}, n+3)$; (w_1, e) is the data of the first Stiefel–Whitney class of the $S^q * S^q$ bundle $E *_X \widehat{E}$ (i.e., w_1 maps into $K(\mathbb{Z}/2, 1)$), and e is the Euler class of the oriented cover Q.

The construction of the classifying space $R_n(G)$ for pairs (E, H) is applicable in the case where E is a not necessarily oriented sphere bundle with structure group G. As in the oriented case, there is a comparison map $f : \widetilde{P}_n(G) \to R_n(G)$, and we may study its connectivity to determine the existence and uniqueness of T-dual pairs in this setting. In the case where $n = q = 1$ and $G = O(2)$, the map f is an equivalence, so we recover Baraglia's theorem on the existence and uniqueness of T-duality for nonprincipal circle bundles.

Our proof of the T-duality isomorphism applies to nonorientable bundles: looking back to the definition of what it means for (E, H) and $(\widehat{E}, \widehat{H})$ to be T-dual, we allow E and $\widehat{E}$ to be arbitrary smooth bundles with fiber S^q. This is enough to prove the T-duality isomorphism; indeed, the orientation twist τ_E precisely carries the correction term for nonorientable sphere bundles.

Lastly, we recall Smale's conjecture that the inclusion $O(q + 1) \to \mathrm{Diff}(S^q)$ is a homotopy equivalence; this has been verified by Smale [1959] for $q = 2$ and Hatcher [1983] for $q = 3$. These results allow us to extend to dimensions two and three the results described above for S^1-bundles, yielding a similar construction and analysis of the classifying space $\widetilde{P}_n(G)$ of pairs (E, H), where E is a not necessarily orientable S^q-bundle with structure group G, and $H \in H^{n+2}(E)$.

5. Higher categories, n-vector spaces, and iterated algebraic K-theory

In this section we return to the question raised in the introduction on the relationship between the iterated algebraic K-theory $\mathfrak{a}_n = K^{(n-1)}(ku)$, the Bott-inverted iterated algebraic K-theory $\mathfrak{A}_n = K^{(n-1)}(ku)[\beta_n^{-1}]$, and n-vector spaces. The discussion is entirely speculative and is independent of the results proved in the paper. Our aim is to sketch a conceptual framework that might be useful for analyzing the geometric content of T-duality for iterated algebraic K-theory. Making this material rigorous would, at a minimum, require a good theory of weighted colimits for enriched ∞-categories. Also, for the sake of clarity, we ignore the size issues that arise.

We start with the concept that one should be able to associate an object like a vector bundle to a principal $K(\mathbb{Z}, n)$-bundle. We call such an object a (rank-one) n-vector bundle. Often the structure group $K(\mathbb{Z}, n)$ is written $B^{n-1}U(1)$ to highlight the categorical level: the notion of an n-vector bundle is an $(n - 1)$-fold categorification of the notion of a vector bundle. The association of an n-vector bundle to a principal $K(\mathbb{Z}, n)$-bundle works similarly to the classical theory, via an action of the group $K(\mathbb{Z}, n)$ on some structure. We take the point of view that the

canonical object to act on is the linear $(n-1)$-category $(n-1)\mathcal{V}\text{ect}_{\mathbb{C}}$ of $(n-1)$-vector spaces. The iterative definition of n-vector space given below is arranged so that the $(n-1)$-category $(n-1)\mathcal{V}\text{ect}_{\mathbb{C}}$ is the basic example of a rank-one n-vector space.

For small values of n this is completely rigorous. When $n=1$ one has the canonical action of $K(\mathbb{Z},1) = U(1)$ on the complex vector space $\mathbb{C}$. When $n=2$ the group $K(\mathbb{Z},2) = BU(1)$ is equivalent to the Picard groupoid of the category of vector spaces, i.e., to the symmetric monoidal groupoid of 1-dimensional complex vector spaces under the tensor product. As such, $BU(1)$ acts on the category $\mathcal{V}\text{ect}_{\mathbb{C}}$ of finite-rank complex vector spaces via the tensor product. So to a $BU(1)$-principal bundle is canonically associated a bundle with fiber $\mathcal{V}\text{ect}_{\mathbb{C}}$. The 2-category $2\mathcal{V}\text{ect}_{\mathbb{C}}$ of 2-vector spaces is defined to be the 2-category of finite-rank $\mathcal{V}\text{ect}_{\mathbb{C}}$-modules in $\mathbb{C}$-linear categories, as in [Kapranov and Voevodsky 1994]. Since $\mathcal{V}\text{ect}_{\mathbb{C}}$ is a rank-one object in $2\mathcal{V}\text{ect}_{\mathbb{C}}$, the bundle associated to a principal $K(\mathbb{Z},2)$-bundle is a line 2-vector bundle.

The first few values of n have explicit applications [Kapranov and Voevodsky 1994; Sati et al. 2009; Freed et al. 2010]. Intuitively, the pattern continues to higher n, but the situation becomes successively more complicated and less clear as n grows.

We will sketch an approach to n-vector spaces using enriched higher categories. If $\mathcal{V}$ is a monoidal ∞-category, let us write $\mathcal{C}\text{at}_{\infty}^{\mathcal{V}}$ for the $(\infty,1)$-category of $\mathcal{V}$-enriched ∞-categories and $\mathcal{V}$-functors. Although our arguments are merely heuristics, much of what we say can be implemented in the model developed by Gepner and Haugseng [2015]. For example, $\mathcal{C}\text{at}_{\infty}^{\mathcal{S}\text{p}}$ denotes the ∞-category of ∞-categories enriched in the monoidal ∞-category $\mathcal{S}\text{p}$ of spectra. Since $\mathcal{S}\text{p}$ carries the additional structure of a symmetric monoidal ∞-category, the ∞-category $\mathcal{C}\text{at}_{\infty}^{\mathcal{S}\text{p}}$ is also symmetric monoidal. We can iterate this procedure further. Let

$$\mathcal{C}\text{at}_{(\infty,n)}^{\mathcal{S}\text{p}} = \mathcal{C}\text{at}_{\infty}^{\mathcal{C}\text{at}_{\infty}^{\cdots \mathcal{C}\text{at}_{\infty}^{\mathcal{S}\text{p}}}}$$

be the ∞-category of ∞-categories enriched in $\mathcal{C}\text{at}_{(\infty,n-1)}^{\mathcal{S}\text{p}}$, where $\mathcal{C}\text{at}_{(\infty,1)}^{\mathcal{S}\text{p}} = \mathcal{C}\text{at}_{\infty}^{\mathcal{S}\text{p}}$ [Gepner and Haugseng 2015, Remark 5.7.13]. We think of objects of $\mathcal{C}\text{at}_{(\infty,n)}^{\mathcal{S}\text{p}}$ as (∞,n)-categories where the collection of n-morphisms between a pair of $(n-1)$-morphisms forms a spectrum.

There is not currently a well developed theory of weighted limits and weighted colimits in enriched ∞-categories, but we proceed as if there were, using the corresponding notions from ordinary enriched category theory as a guide for our intuition. We write $\mathcal{S}\text{t}_{\infty}^{(n)}$ for the full subcategory of $\mathcal{C}\text{at}_{(\infty,n)}^{\mathcal{S}\text{p}}$ consisting of those objects admitting all $\mathcal{C}\text{at}_{(\infty,n-1)}^{\mathcal{S}\text{p}}$-weighted colimits. Assuming that this is a sensible notion, there is a localization adjunction

$$\mathcal{C}\text{at}_{(\infty,n)}^{\mathcal{S}\text{p}} \underset{\longleftarrow}{\overset{L^{(n)}}{\longrightarrow}} \mathcal{S}\text{t}_{\infty}^{(n)} \tag{5.1}$$

where the left adjoint freely adds all such colimits and the right adjoint is the inclusion. The case $n = 1$ was studied by Blumberg, Gepner, and Tabuada [Blumberg et al. 2013] and takes the form

$$\mathcal{C}at_\infty^{\mathcal{S}p} \xrightleftharpoons{L^{(1)}} \mathcal{S}t_\infty,$$

where $\mathcal{S}t_\infty = \mathcal{S}t_\infty^{(1)}$ is the full subcategory of $\mathcal{C}at_\infty^{\mathcal{S}p}$ consisting of those spectrally enriched ∞-categories admitting small $\mathcal{S}p$-weighted colimits, i.e., the cocomplete stable ∞-categories. This explains the notation $\mathcal{S}t_\infty^{(n)}$.

Suppose that R is an E_∞ ring spectrum. There is an object $\flat R$ of $\mathcal{C}at_\infty^{\mathcal{S}p}$ with a single object $\bullet_0$ whose endomorphism spectrum is $\mathrm{Hom}(\bullet_0, \bullet_0) = R$. The image of the object $\flat R$ under the localization functor $L^{(1)}$ is the presentable stable ∞-category

$$L^{(1)}\flat R = \mathrm{Mod}_R$$

of R-modules. The functor $L^{(1)}$ admits a symmetric monoidal structure and we write $(\mathrm{Mod}_R, \otimes)$ for the induced symmetric monoidal structure on the image of $\flat R$. For the purposes of defining K-theory, we restrict to the full subcategory $(\mathrm{Mod}_R, \otimes)^\circ$ of dualizable R-modules under the symmetric monoidal structure. It is here that we use the commutativity assumption. We could equivalently restrict to the subcategory of perfect R-modules, meaning the objects which span the thick subcategory generated by R after passage to the homotopy category. It is the dualizability notion that we will generalize below, so we concentrate on that here.

The $(\infty, 1)$-category $(\mathrm{Mod}_R, \otimes)^\circ$ is pointed and admits finite colimits, so we may use the ∞-categorical version of Waldhausen's K-theory construction [Blumberg et al. 2013; Barwick 2016] to define its algebraic K-theory:

$$K(R) = K(\mathrm{Mod}_R) := K((L^{(1)}\flat R, \otimes)^\circ).$$

Notice that "$K(R)$" and "$K(\mathrm{Mod}_R)$" are synonyms. Soon we will consider the K-theory of other module categories, but in that case we will never write the K-theory in terms of the underlying "ring", only in terms of the category used to construct the K-theory.

When $n = 2$, adjunction (5.1) takes the form

$$\mathcal{C}at_{(\infty,2)}^{\mathcal{S}p} = \mathcal{C}at_\infty^{\mathcal{C}at_\infty^{\mathcal{S}p}} \xrightleftharpoons{L^{(2)}} \mathcal{S}t_\infty^{(2)}$$

where the left adjoint $L^{(2)}$ freely adjoins all $\mathcal{C}at_\infty^{\mathcal{S}p}$-weighted colimits. There is an object $\flat^2 R$ of $\mathcal{C}at_\infty^{\mathcal{C}at_\infty^{\mathcal{S}p}}$ which has a single object $\bullet_0$ and a single 1-morphism $\bullet_1$ whose endomorphism spectrum is $\mathrm{Hom}(\bullet_1, \bullet_1) = R$. The image of $\flat^2 R$ under the

localization functor $L^{(2)}$ is the $(\infty, 2)$-category

$$L^{(2)}\mathfrak{b}^2 R = \mathrm{Mod}_{\mathrm{Mod}_R}$$

of Mod_R-module categories. An object of $\mathrm{Mod}_{\mathrm{Mod}_R}$ may be described as a presentable stable ∞-category $\mathcal{C}$ equipped with a suitable action $\mathrm{Mod}_R \otimes \mathcal{C} \to \mathcal{C}$ of Mod_R, defined with respect to the symmetric monoidal structure $\otimes$ on stable presentable ∞-categories. When considering dualizable Mod_R-modules, as we will do below, it is natural to restrict to the action of the subcategory Mod_R° of perfect R-modules. If $\mathcal{A}$ and $\mathcal{B}$ are Mod_R-module categories, the $(\infty, 1)$-category of morphisms from $\mathcal{A}$ to $\mathcal{B}$ is the category of exact R-linear functors

$$\mathrm{Mod}_{\mathrm{Mod}_R}(\mathcal{A}, \mathcal{B}) \simeq \mathrm{Fun}_R^{\mathrm{ex}}(\mathcal{A}, \mathcal{B}).$$

So far, the constructions we have made would still make sense if R were an E_2 ring spectrum, but as in the $n = 1$ case, we now restrict to the full subcategory spanned by the dualizable objects. The spectral $(\infty, 2)$-category $\mathrm{Mod}_{\mathrm{Mod}_R}$ inherits a symmetric monoidal structure by virtue of the symmetric monoidal structure on the functor $L^{(2)}$. We write $(L^{(2)}\mathfrak{b}^2 R, \otimes)^\circ \to L^{(2)}\mathfrak{b}^2 R$ for the inclusion of the subcategory spanned by the fully dualizable objects, as defined by Lurie [2009b, Claim 2.3.19]. This has the effect of discarding 1-morphisms that do not admit adjoints in the underlying homotopy bicategory and then discarding objects that are not dualizable in the underlying symmetric monoidal homotopy category. We define the K-theory of Mod_R-module categories to be the Waldhausen K-theory of the $(\infty, 1)$-category truncation of the $(\infty, 2)$-category of the fully dualizable objects of $\mathrm{Mod}_{\mathrm{Mod}_R}$

$$K(\mathrm{Mod}_{\mathrm{Mod}_R}) := K(\iota_1(L^{(2)}\mathfrak{b}^2 R, \otimes)^\circ).$$

In the case of symmetric monoidal bicategories with no higher morphisms, the analogous K-theory functor was constructed explicitly by Osorno [2012].

When $R = Hk$ is the Eilenberg–MacLane spectrum associated to a field k, the $(\infty, 2)$-category $\mathrm{Mod}_{\mathrm{Mod}_{Hk}}$ is equivalent to the $(\infty, 2)$-category of k-linear dg-categories. The fully dualizable objects of $\mathrm{Mod}_{\mathrm{Mod}_{Hk}}$ are precisely the smooth and proper dg-categories [Toën and Vaquié 2007; Antieau and Gepner 2014], and $\pi_0 K(\mathrm{Mod}_{\mathrm{Mod}_{Hk}})$ is isomorphic to the secondary K-theory $K_0^{\{2\}}(k)$ of the field k defined by Toën [2011; Tabuada 2016]. We also expect that $K(\mathrm{Mod}_{\mathrm{Mod}_{Hk}})$ is related to the K-theory of varieties studied by Campbell, Wolfson, and Zakharevich [Campbell 2019; Campbell et al. 2019].

Remark 5.2. A slight variation allows us to consider 2-vector spaces instead of dg-categories. We replace Mod_R with the category $\mathcal{V}\mathrm{ect}_{\mathbb{C}}$ of finite-dimensional complex vector spaces, considered as an $(\infty, 1)$-category enriched in spectra by delooping the usual enrichment in topological abelian groups. In this case, $\mathrm{Mod}_{\mathcal{V}\mathrm{ect}_{\mathbb{C}}}$

is the $(\infty, 2)$-category of $\mathcal{V}\text{ect}_{\mathbb{C}}$-module categories. In particular, the 2-category $2\mathcal{V}\text{ect}_{\mathbb{C}}$ of 2-vector spaces, as defined by Kapranov and Voevodsky [1994], embeds in the subcategory $(\text{Mod}_{\mathcal{V}\text{ect}_{\mathbb{C}}}, \otimes)^{\circ}$ of dualizable objects, and $K(\text{Mod}_{\mathcal{V}\text{ect}_{\mathbb{C}}})$ is equivalent to the bicategorical K-theory of 2-vector spaces $K(2\mathcal{V}\text{ect}_{\mathbb{C}})$ studied by Baas, Dundas, and Rognes [Baas et al. 2004; Osorno 2012].

Instead of applying $L^{(2)}$ to $\mathfrak{b}^2 R$, we could add colimits in a two-step process. Write $\mathcal{C}\text{at}_{\infty}^{\mathcal{C}\text{at}_{\infty}^{\mathcal{S}t}}$ for the ∞-category of ∞-categories enriched in stable ∞-categories. There is a localization adjunction

$$\mathcal{C}\text{at}_{\infty}^{\mathcal{C}\text{at}_{\infty}^{\mathcal{S}p}} \xrightleftharpoons{\mathfrak{b}L^{(1)}} \mathcal{C}\text{at}_{\infty}^{\mathcal{S}t_{\infty}}$$

where the left adjoint applies $L^{(1)}$ to the hom objects. In other words, we've only added colimits at the top level. In the case of $\mathfrak{b}^2 R$, we get the category

$$(\mathfrak{b}L^{(1)})(\mathfrak{b}^2 R) = \mathfrak{b}\text{Mod}_R$$

with a single 0-cell whose endomorphism object is the stable ∞-category of R-modules. Notice that composition in this category uses the symmetric monoidal structure $\otimes$ on Mod_R. Next, we apply $K(\cdot)$, taking care to first restrict to the dualizable objects. Formally, we are using the fact that $\mathfrak{b}R$ is a commutative monoid in $\mathcal{C}\text{at}_{\infty}^{\mathcal{S}p}$ and that the functors $L^{(1)}$ and K induce functors on categories enriched in commutative monoid objects. We have now associated to $\mathfrak{b}^2 R$ the spectral category

$$\mathfrak{b}K(R) = (\mathfrak{b}K)(\iota_1((\mathfrak{b}L^{(1)})\mathfrak{b}^2 R, \otimes)^{\circ})$$

with one object and $K(R)$ as its endomorphism spectrum. Here ι_1 denotes the truncation functor $\mathfrak{b}\iota_0$ taking an $(\infty, 2)$-category to the underlying $(\infty, 1)$-category that removes the noninvertible 2-morphisms. After applying the same procedure one categorical level down by adding all spectral colimits, restricting to dualizable objects and taking K-theory, we get the twice-iterated algebraic K-theory spectrum

$$K(K(R)) = K((L^{(1)}\mathfrak{b}K(R), \otimes)^{\circ}).$$

Both the K-theory of Mod_R-modules $K(\text{Mod}_{\text{Mod}_R})$ and the iterated K-theory $K(K(R))$ map to the spectral enhancement $K^{\{2\}}(R)$ of Toën's secondary K-theory defined by Hoyois, Scherotzke, and Sibilla [Hoyois et al. 2017], as we now explain. They consider a variant of the Blumberg–Gepner–Tabuada category of noncommutative motives [Blumberg et al. 2013] that is the recipient of the universal additive invariant of stable Mod_R°-module categories. More precisely, there is a functor

$$\mathcal{U} : \langle \text{Mod}_R^{\circ}\text{-modules in } \mathcal{C}\text{at}_{\infty}^{\text{perf}} \rangle \to \text{Mot}_R$$

from the ∞-category of small, stable, idempotent complete Mod_R°-module categories to the ∞-category of R-linear motives that preserves filtered colimits, preserves zero objects, sends split exact sequences to cofiber sequences, and is universal among functors with these three properties. The $(\infty, 1)$-category Mot_R and the functor $\mathcal{U}$ admit compatible symmetric monoidal structures, and we define the K-theory of Mot_R to be the Waldhausen K-theory of the full subcategory of Mot_R generated under finite colimits and retracts by the images of dualizable objects under $\mathcal{U}$:

$$K(\mathrm{Mot}_R) := K((\mathrm{Mot}_R, \otimes)^\circ).$$

K-theory becomes corepresentable after passing to the ∞-category Mot_R, meaning that if $\mathcal{A}$ and $\mathcal{B}$ are Mod_R°-module categories, and $\mathcal{A}$ is compact, then there is a natural equivalence of spectra

$$\mathrm{Mot}_R(\mathcal{U}(\mathcal{A}), \mathcal{U}(\mathcal{B})) \simeq K(\mathrm{Fun}_R^{\mathrm{ex}}(\mathcal{A}, \mathcal{B})).$$

When $\mathcal{A}$ and $\mathcal{B}$ are both fully dualizable as Mod_R-module categories, which in particular implies that they are compact [Hoyois et al. 2017, Proposition 4.19], then their associated motives are dualizable in Mot_R. Applying the natural transformation $\iota_0 \to K$ from the groupoid core functor to Waldhausen K-theory to the spectrally enriched $(\infty, 1)$-categories $\mathrm{Fun}_R^{\mathrm{ex}}(\mathcal{A}, \mathcal{B})$ gives a map

$$\iota_1 \mathrm{Mod}_{\mathrm{Mod}_R}(\mathcal{A}, \mathcal{B}) \simeq \iota_0 \mathrm{Fun}_R^{\mathrm{ex}}(\mathcal{A}, \mathcal{B}) \to K(\mathrm{Fun}_R^{\mathrm{ex}}(\mathcal{A}, \mathcal{B})) \simeq \mathrm{Mot}_R(\mathcal{U}(\mathcal{A}), \mathcal{U}(\mathcal{B}))$$

which assembles into a symmetric monoidal functor of $(\infty, 1)$-categories

$$\iota_1(\mathrm{Mod}_{\mathrm{Mod}_R})^\circ \to \mathrm{Mot}_R^\circ.$$

Applying Waldhausen K-theory gives a map of E_∞ ring spectra

$$K(\mathrm{Mod}_{\mathrm{Mod}_R}) \to K(\mathrm{Mot}_R). \tag{5.3}$$

On the other hand, the motive associated to the category Mod_R° of perfect R-modules is the unit of the symmetric monoidal category Mot_R of R-linear motives, and so the thick subcategory that it generates is equivalent to the category of modules over the endomorphism ring

$$\mathrm{Mot}_R(\mathcal{U}(\mathrm{Mod}_R^\circ), \mathcal{U}(\mathrm{Mod}_R^\circ)) \simeq K(\mathrm{Mod}_R^\circ) = K(R).$$

The inclusion of this thick subcategory induces a map of E_∞ ring spectra

$$K(K(R)) \to K(\mathrm{Mot}_R), \tag{5.4}$$

as observed by Hoyois et al. [2017, Remark 6.23].

Question 5.5. To what extent do the images of the maps (5.3) and (5.4) coincide in $K(\mathrm{Mot}_R)$?

When $R = Hk$ is the Eilenberg–MacLane spectrum associated to a field k, the work of Baas, Dundas, Richter, and Rognes [Baas et al. 2011] implies that $K(K(k))$ is equivalent to the K-theory of the full subcategory of $\mathrm{Mod}_{\mathrm{Mod}_{Hk}}$ spanned by the free 2-vector spaces of finite rank. Equivalently, these are the dg-categories with a single object and the free algebras k^n as the ring of morphisms. Similarly, if we replace Mod_R with $\mathcal{V}\mathrm{ect}_{\mathbb{C}}$, as in Remark 5.2, then $K(ku)$ is equivalent to the K-theory $K(2\mathcal{V}\mathrm{ect}_{\mathbb{C}}) \simeq K(\mathrm{Mod}_{\mathcal{V}\mathrm{ect}_{\mathbb{C}}})$ of 2-vector spaces.

We would like to ask a similar question about the iterated algebraic K-theory spectrum $K^{(n)}(R)$ for higher values of n. Let $\mathfrak{b}^n R \in \mathcal{C}\mathrm{at}^{\mathcal{S}\mathrm{p}}_{(\infty,n)}$ be the (∞, n)-category enriched in spectra with a single k-morphism for $0 \leq k \leq n-1$ and for which the endomorphism spectrum of the $(n-1)$-morphism is the ring R. Let

$$\mathrm{Mod}^{(n)}_R = L^{(n)}\mathfrak{b}^n R = \mathrm{Mod}_{\mathrm{Mod}_{\cdots\mathrm{Mod}_R}}$$

denote the image of $\mathfrak{b}^n R$ under the localization functor $L^{(n)}$. The fully dualizable part

$$\mathrm{Mod}^{\circ}_{\mathrm{Mod}_{\cdots\mathrm{Mod}_R}} = (\mathrm{Mod}^{(n)}_R, \otimes)^{\circ}$$

of the symmetric monoidal (∞, n)-category $\mathrm{Mod}^{(n)}_R$ is our higher categorical analog of the category of perfect R-modules, and in the case of $R = Hk$ we consider this to be a reasonable definition of the (∞, n)-category of n-chain complexes over k. Similarly, we define the (∞, n)-category of complex n-vector spaces to be the symmetric monoidal (∞, n)-category

$$n\mathcal{V}\mathrm{ect}_{\mathbb{C}} = (L^{(n)}\mathfrak{b}^{n-1}\mathcal{V}\mathrm{ect}_{\mathbb{C}}, \otimes)^{\circ} = \mathrm{Mod}^{\circ}_{\mathrm{Mod}_{\cdots\mathcal{V}\mathrm{ect}_{\mathbb{C}}}}.$$

Remark 5.6. We do not know how to construct the appropriate analog of the category of noncommutative R-local motives Mot_R that would accept maps from the spectra $K^{(n)}(R)$ and $K((\mathrm{Mod}^{(n)}_R, \otimes)^{\circ})$, and we do not know how to ask the analog of Question 5.5 when $n > 2$ (compare with [Hoyois et al. 2017, Remark 6.22]). Our results on T-duality for $\mathfrak{A}_n$-algebras suggest that a good understanding of the relationship between $K^{(n)}(R)$ and $K((\mathrm{Mod}^{(n)}_R, \otimes)^{\circ})$ would connect T-duality for S^q-bundles and the theory of n-vector spaces for $n = 2q + 1$.

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Hausdorffified algebraic K_1-groups and invariants for C^*-algebras with the ideal property

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Dedicated to the memory of Professor Ronald G. Douglas

A C^*-algebra A is said to have the ideal property if each closed two-sided ideal of A is generated as a closed two-sided ideal by the projections inside the ideal. C^*-algebras with the ideal property are a generalization and unification of real rank zero C^*-algebras and unital simple C^*-algebras. It was long expected that an invariant that we call $\mathrm{Inv}^0(A)$, consisting of the scaled ordered total K-group $(\underline{K}(A); \underline{K}(A)^+; \Sigma A)_\Lambda$ (used in the real rank zero case), along with the tracial state spaces $T(pAp)$ for each cut-down algebra pAp, as part of the Elliott invariant of pAp (for each $[p] \in \Sigma A$), with certain compatibility conditions, is the complete invariant for a certain well behaved class of C^*-algebras with the ideal property (e.g., AH algebras with no dimension growth). In this paper, we construct two nonisomorphic $A\mathbb{T}$ algebras A and B with the ideal property such that $\mathrm{Inv}^0(A) \cong \mathrm{Inv}^0(B)$, disproving this conjecture. The invariant to distinguish the two algebras is the collection of Hausdorffified algebraic K_1-groups $U(pAp)/\overline{DU(pAp)}$ (for each $[p] \in \Sigma A$), along with certain compatibility conditions. We will prove in a separate article that, after adding this new ingredient, the invariant becomes the complete invariant for AH algebras (of no dimension growth) with the ideal property.

1. Introduction

A C^*-algebra A is called an *AH* algebra [Blackadar 1993] if it is the inductive limit C^*-algebra of

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \longrightarrow \cdots \longrightarrow A_n \longrightarrow \cdots$$

with $A = \lim_{n\to\infty}\left(A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i})) P_{n,i}, \phi_{n,m}\right)$, where $X_{n,i}$ are compact metric spaces, t_n and $[n, i]$ are positive integers, and $P_{n,i} \in M_{[n,i]}(C(X_{n,i}))$ are projections. An *AH* algebra is called of no dimension growth, if one can choose the spaces $X_{n,i}$ such that $\sup_{n,i} \dim(X_{n,i}) < +\infty$. If all the spaces $X_{n,i}$ can be

chosen to be the single point space {pt}, then A is called an AF algebra. If all the spaces can be chosen to be the interval $[0, 1]$ or circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, then A is called an AI algebra or $A\mathbb{T}$ algebra, respectively.

G. Elliott [1993b] initiated the classification program by classifying all real rank zero $A\mathbb{T}$ algebras (without the condition of simplicity), and he conjectured that the scaled ordered K_*-group $(K_*(A); K_*(A)^+; \Sigma A)$, where $K_*(A) = K_0(A) \oplus K_1(A)$, is a complete invariant for separable nuclear C^*-algebras of real rank zero and stable rank one. Elliott [1993a] also successfully classified all unital simple AI algebras by the so called Elliott invariant

$$\text{Ell}(A) = (K_0(A); K_0(A)^+; \Sigma A, K_1(A); T(A); \rho_A),$$

where $T(A)$ is the space of all unital traces on A, and ρ_A is the nature map from $K_0(A)$ to AffT(A) (the ordered Banach space of all affine maps from $T(A)$ to $\mathbb{R}$).

Later, Gong [1998] constructed two nonisomorphic (not simple) real rank zero AH algebras (with 2-dimensional local spectra) A and B such that

$$(K_*(A); K_*(A)^+; \Sigma A) \cong (K_*(B); K_*(B)^+; \Sigma B),$$

which disproved the conjecture of Elliott for C^*-algebras of real rank zero and stable rank one. This result led to a sequence of research by Dadarlat and Loring [1996a; 1996b] and Eilers [1996] culminating with Dadarlat and Gong's [1997] complete classification of real rank zero AH algebras by scaled ordered total K-theory $(\underline{K}(A); \underline{K}(A)^+; \Sigma A)_\Lambda$, where $\underline{K}(A) = K_*(A) \oplus \bigoplus_{p=2}^{\infty} K_*(A, \mathbb{Z}/p\mathbb{Z})$ and Λ is the system of Bockstein operations; see also [Dadarlat 1995a; 1995b; Elliott and Gong 1996a; 1996b; Elliott et al. 1996; 1998; Gong 1997; 1998; Gong and Lin 2000; Lin 1996; 2001]. Elliott, Gong, and Li [Elliott et al. 2007] completely classified simple AH algebras of no dimension growth by Elliott invariant; see also [Elliott 1997; Elliott et al. 2005; 1997; Gong 2002; Li 1997; 1999; Lin 2007; Nielsen and Thomsen 1996; Thomsen 1994; 1997]. A natural generalization and unification of real rank zero C^*-algebras and unital simple C^*-algebras is the class of C^*-algebras with the ideal property: each closed two-sided ideal of the C^*-algebra is generated as a closed two-sided ideal by the projections inside the ideal. It was long expected that a combination of scaled ordered total K-theory (used in the classification of real rank zero C^*-algebras) and the Elliott invariant (used in the classification of simple C^*-algebras), including tracial state spaces $T(pAp)$ — part of the Elliott invariant of cut-down algebras $\{pAp\}_{[p] \in \Sigma A}$ with certain compatibility conditions, called $\text{Inv}^0(A)$ (see [Jiang 2011, 2.18]), is a complete invariant for certain well behaved C^*-algebras (e.g., AH algebras of no dimension growth or $\mathcal{Z}$-stable C^*-algebras, where $\mathcal{Z}$ is the Jiang–Su algebra of [Jiang and Su 1999]) with the ideal property; see [Stevens 1998; Pasnicu 2000; Ji and Jiang 2011; Jiang and Wang 2012; Jiang 2011].

The main purpose of this paper is to construct two unital $\mathcal{Z}$-stable $A\mathbb{T}$ algebras A and B with the ideal property such that $\text{Inv}^0(A) \cong \text{Inv}^0(B)$, but $A \not\cong B$. The invariant to distinguish these two C^*-algebras is the Hausdorffified algebraic K_1-groups $U(pAp)/\overline{DU(pAp)}$ of the cut-down algebra pAp (for each element $x \in \Sigma A$, we chose one projection $p \in A$ such that $[p] = x$) along with a certain compatibility condition, where $DU(A)$ is the group generated by commutators $\{uvu^*v^* : u, v \in U(A)\}$. In this paper, we introduce the invariant $\text{Inv}'(A)$ and its simplified version $\text{Inv}(A)$, by adding these new ingredients — the Hausdorffified algebraic K_1-groups of cut-down algebras along with certain compatibility conditions — to $\text{Inv}^0(A)$.

In [Gong et al. 2016], we will prove that $\text{Inv}(A)$ is a complete invariant for AH algebras (of no dimension growth) with the ideal property.

Note that for the above C^*-algebras A and B, we have that $Cu(A) \cong Cu(B)$ and $Cu(A \otimes C(S^1)) \cong Cu(B \otimes C(S^1))$. That is, the new invariant can not be detected by the Cuntz semigroup.

In Section 2, we define $\text{Inv}(A)$ and discuss its properties. These properties will be used in [Gong et al. 2016]. In Section 3, we present the construction of $A\mathbb{T}$ algebras A and B with the ideal property such that $\text{Inv}(A) \not\cong \text{Inv}(B)$ but $\text{Inv}^0(A) \cong \text{Inv}^0(B)$.

2. The invariant

In this section, we recall the definition of $\text{Inv}^0(A)$ from [Jiang 2011] (also see [Stevens 1998; Ji and Jiang 2011; Jiang and Wang 2012]), and then introduce the invariant $\text{Inv}(A)$. Furthermore, we discuss the properties of $\text{Inv}(A)$ in the context of AH algebras and $A\mathcal{HD}$ algebras (for the definition of $A\mathcal{HD}$ algebras, see 2.3 below), which will be used in [Gong et al. 2016].

2.1. In the notation for an inductive limit system $\lim(A_n, \phi_{n,m})$, we understand that

$$\phi_{n,m} = \phi_{m-1,m} \circ \phi_{m-2,m-1} \circ \cdots \circ \phi_{n,n+1},$$

where all $\phi_{n,m} : A_n \to A_m$ are homomorphisms.

We assume that, for any summand A_n^i in the direct sum $A_n = \bigoplus_{i=1}^{t_n} A_n^i$, necessarily $\phi_{n,n+1}(\mathbb{1}_{A_n^i}) \neq 0$, since otherwise, we could simply delete A_n^i from A_n without changing the limit algebra.

If $A_n = \bigoplus_i A_n^i$, $A_m = \bigoplus_j A_m^j$, we use $\phi_{n,m}^{i,j}$ to denote the partial map of $\phi_{n,m}$ from the i-th block A_n^i of A_n to the j-th block A_m^j of A_m. Also, we use $\phi_{n,m}^{-,j}$ to denote the partial map of $\phi_{n,m}$ from A_n to A_m^j. That is, $\phi_{n,m}^{-,j} = \bigoplus_i \phi_{n,m}^{i,j} = \pi_j \phi_{n,m}$, where $\pi_j : A_m \to A_m^j$ is the canonical projection. Sometimes, we also use $\phi_{n,m}^{i,-}$ to denote $\phi_{n,m}|_{A_n^i} : A_n^i \to A_m$.

2.2. As in [Elliott and Gong 1996b], let $T_{\mathrm{II},k}$ be the 2-dimensional connected simplicial complex with $H^1(T_{\mathrm{II},k}) = 0$ and $H^2(T_{\mathrm{II},k}) = \mathbb{Z}/k\mathbb{Z}$, and let I_k be the subalgebra of $M_k(C[0,1]) = C([0,1], M_k(\mathbb{C}))$ consisting of all functions f with the properties $f(0) \in \mathbb{C} \cdot \mathbb{1}_k$ and $f(1) \in \mathbb{C} \cdot \mathbb{1}_k$ (this algebra is called an Elliott dimension drop interval algebra). Denote by $\mathcal{HD}$ the class of algebras consisting of direct sums of the building blocks of the forms $M_l(I_k)$ and $PM_n(C(X))P$, with X being one of the spaces $\{\mathrm{pt}\}$, $[0,1]$, S^1, and $T_{\mathrm{II},k}$, and with $P \in M_n(C(X))$ being a projection. (In [Dadarlat and Gong 1997], this class is denoted by $SH(2)$, and in [Jiang 2011] by $\mathcal{B}$). We call a C^*-algebra an $A\mathcal{HD}$ algebra if it is an inductive limit of the algebras in $\mathcal{HD}$.

For each basic building block $A = PM_n(C(X))P$, where $X = \{\mathrm{pt}\}$, $[0,1]$, S^1, $T_{\mathrm{II},k}$, or $A = M_l(I_k)$, we have $K_0(A) = \mathbb{Z}$ or $\mathbb{Z}/k\mathbb{Z}$ (for the case $A = PM_n(C(T_{\mathrm{II},k}))P$). Hence there is a natural map $\mathrm{rank}: K_0(A) \to \mathbb{Z}$. This map also gives a map from $\{p \in (M_\infty(A)) : p \text{ is a projection}\}$ to $\mathbb{Z}_+$. For example, if $p \in A = PM_n(C(X))P$, then $\mathrm{rank}(p)$ is the rank of projection $p(x) \in P(x)M_n(\mathbb{C})P(x) \cong M_{\mathrm{rank}(P)}(\mathbb{C})$ for any $x \in X$; and if $p \in A = M_l(I_k)$, then $\mathrm{rank}(p)$ is the rank of projection $p(0) \in M_l(\mathbb{C})$. (Note that we regard $p(0)$ as in $M_l(\mathbb{C}) \cong \mathbb{1}_k \otimes M_l(\mathbb{C})$, not $M_{lk}(\mathbb{C})$.)

2.3. By $A\mathcal{HD}$ algebra, we mean the inductive limit of

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \longrightarrow \cdots \longrightarrow \cdots ,$$

where $A_n \in \mathcal{HD}$ for each n.

For an $A\mathcal{HD}$ inductive limit $A = \lim(A_n, \phi_{nm})$, we write $A_n = \bigoplus_{i=1}^{t_n} A_n^i$, where $A_n^i = P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$ or $A_n^i = M_{[n,i]}(I_{k_{n,i}})$. For convenience, even for a block $A_n^i = M_{[n,i]}(I_{k_{n,i}})$, we still use $X_{n,i}$ for $\mathrm{Sp}(A_n^i) = [0,1]$ — that is, A_n^i is regarded as a homogeneous algebra or a subhomogeneous algebra over $X_{n,i}$.

2.4. In [Gong et al. 2010; 2018], joint with Cornel Pasnicu, the authors proved the reduction theorem for AH algebras with the ideal property provided that the inductive limit systems have no dimension growth. That is, if A is an inductive limit of $A_n = \bigoplus A_n^i = \bigoplus P_{n,i}M_{[n,i]}C(X_{n,i})P_{n,i}$ with $\sup_{n,i}\dim(X_{n,i}) < +\infty$, and if we further assume that A has the ideal property, then A can be rewritten as an inductive limit of $B_n = \bigoplus B_n^j = \bigoplus Q_{n,j}M_{\{n,j\}}C(Y_{n,i})Q_{n,j}$, with $Y_{n,i}$ being one of $\{\mathrm{pt}\}$, $[0,1]$, S^1, $T_{\mathrm{II},k}$, $T_{\mathrm{III},k}$, S^2. In turn, Jiang [2017] proved (also see [Li 2006]) that the above inductive limit can be rewritten as the inductive limit of the direct sums of homogeneous algebras over $\{\mathrm{pt}\}$, $[0,1]$, S^1, $T_{\mathrm{II},k}$ and $M_l(I_k)$. Combining these two results, we know that all AH algebras of no dimension growth with the ideal property are $A\mathcal{HD}$ algebras. Let us point out that, as proved in [Dadarlat and Gong 1997], there are real rank zero $A\mathcal{HD}$ algebras which are not AH algebras.

2.5. Let A be a C^*-algebra. Then $K_0(A)^+ \subset K_0(A)$ is defined to be the semigroup of $K_0(A)$ generated by $[p] \in K_0(A)$, where $p \in M_\infty(A)$ are projections. For all

C^*-algebras considered in this paper — for example, $A \in \mathcal{HD}$, or A is an $A\mathcal{HD}$ algebra, or $A = B \otimes C(T_{\mathrm{II},k} \times S^1)$, where B is an $\mathcal{HD}$ or $A\mathcal{HD}$ algebra — we always have

$$K_0(A)^+ \cap (-K_0(A)^+) = \{0\} \quad \text{and} \quad K_0(A)^+ - K_0(A)^+ = K_0(A). \qquad (*)$$

Therefore $(K_0(A), K_0(A)^+)$ is an ordered group. Define $\Sigma A \subset K_0(A)^+$ to be

$$\Sigma A = \{[p] \in K_0(A)^+ : p \text{ is a projection in } A\}.$$

Then $(K_0(A), K_0(A)^+, \Sigma A)$ is a scaled ordered group. (Note that for purely infinite C^*-algebras or stable projectionless C^*-algebras, condition $(*)$ does not hold.)

2.6. Let $\underline{K}(A) = K_*(A) \oplus \left(\bigoplus_{k=2}^{+\infty} K_*(A, \mathbb{Z}/k\mathbb{Z}) \right)$ be as in [Dadarlat and Gong 1997]. Let $\wedge$ be the Bockstein operation on $\underline{K}(A)$ (see [Dadarlat and Gong 1997, 4.1]). It is well known that $K_*(A, Z \oplus \mathbb{Z}/k\mathbb{Z}) = K_0(A \otimes C(W_k \times S^1))$, where $W_k = T_{\mathrm{II},k}$.

As in [Dadarlat and Gong 1997], let $K_*(A, Z \oplus \mathbb{Z}/k\mathbb{Z})^+ = K_0(A \otimes C(W_k \times S^1)^+)$ and let $\underline{K}(A)^+$ be the semigroup generated by $\{K_*(A, \mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z})^+ : k = 2, 3, \dots \}$.

2.7. Let $\mathrm{Hom}_\wedge(\underline{K}(A), \underline{K}(B))$ be the set of homomorphisms between $\underline{K}(A)$ and $\underline{K}(B)$ compatible with the Bockstein operations $\wedge$. There is a surjective map (see [Dadarlat and Gong 1997])

$$\Gamma : KK(A, B) \to \mathrm{Hom}_\wedge(\underline{K}(A), \underline{K}(B)).$$

Following Rørdam [1995], we write $KL(A, B) \triangleq KK(A, B)/\mathrm{Pext}(K_*(A), K_{*+1}(B))$, where $\mathrm{Pext}(K_*(A), K_{*+1}(B))$ is identified with $\ker \Gamma$ by [Dadarlat and Loring 1996b]. The triple $(\underline{K}(A); \underline{K}(A)^+; \Sigma A)$ is part of our invariant. For two C^*-algebras A and B, by a "homomorphism"

$$\alpha : (\underline{K}(A); \underline{K}(A)^+; \Sigma A) \to (\underline{K}(B); \underline{K}(B)^+; \Sigma B)$$

we mean a system of maps

$$\alpha_k^i : K_i(A, \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z}), \quad i = 0, 1, \ k = 0, 2, 3, \dots$$

which are compatible with the Bockstein operations and $\alpha = \bigoplus_{k,i} \alpha_k^i$ satisfies $\alpha(\underline{K}(A)^+) \subset \underline{K}(B)^+$. And finally, $\alpha_0^0(\Sigma A) \subset \Sigma B$.

2.8. For a unital C^*-algebra A, let $T(A)$ denote the space of tracial states of A, i.e., $\tau \in T(A)$ if and only if τ is a positive linear map from A to $\mathbb{C}$ with $\tau(xy) = \tau(yx)$, and $\tau(\mathbb{1}) = 1$. Endow $T(A)$ with the weak-* topology, that is, for any net $\{\tau_\alpha\}_\alpha \subset T(A)$ and $\tau \in T(A)$, $\tau_\alpha \to \tau$ if and only if $\lim_\alpha \tau_\alpha(x) = \tau(x)$ for any $x \in A$. Then $T(A)$ is a compact Hausdorff space with convex structure, that is, if $\lambda \in [0, 1]$ and $\tau_1, \tau_2 \in T(A)$, then $\lambda \tau_1 + (1 - \lambda)\tau_2 \in T(A)$. $\mathrm{Aff}T(A)$ is the collection of all continuous affine maps from $T(A)$ to $\mathbb{R}$, which is a real Banach space with

$\|f\| = \sup_{\tau \in T(A)} |f(\tau)|$. Let $(\text{AffT}(A))_+$ be the subset of $\text{AffT}(A)$ consisting of all nonnegative affine functions. An element $\mathbb{1} \in \text{AffT}(A)$, defined by $\mathbb{1}(\tau) = 1$ for all $\tau \in T(A)$, is called the order unit (or scale) of $\text{AffT}(A)$. Note that any $f \in \text{AffT}(A)$ can be written as $f = f_+ - f_-$ with $f_1, f_2 \in \text{AffT}(A)_+$, $\|f_1\| \leq \|f\|$ and $\|f_2\| \leq \|f\|$. Therefore $(\text{AffT}(A), \text{AffT}(A)_+, \mathbb{1})$ forms a scaled ordered real Banach space. If $\phi : \text{AffT}(A) \to \text{AffT}(B)$ is a unital positive linear map, then ϕ is bounded and therefore continuous.

There is a natural homomorphism $\rho_A : K_0(A) \to \text{AffT}(A)$ defined by setting $\rho_A([p])(\tau) = \sum_{i=1}^n \tau(p_{ii})$ for $\tau \in T(A)$ and $[p] \in K_0(A)$ represented by the projection $p = (p_{ij}) \in M_n(A)$.

If $\phi : A \to B$ is a unital homomorphism, then ϕ induces a continuous affine map $T\phi : T(B) \to T(A)$, which, in turn, induces a unital positive linear map $\text{AffT}\,\phi : \text{AffT}(A) \to \text{AffT}(B)$.

If $\phi : A \to B$ is not unital, we still use $\text{AffT}\,\phi$ to denote the unital positive linear map

$$\text{AffT}\,\phi : \text{AffT}(A) \to \text{AffT}(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))$$

by regarding ϕ as the unital homomorphism from A to $\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A)$ — that is, for any $l \in \text{AffT}(A)$ represented by $x \in A_{s.a}$ as $l(t) = t(x)$ for any $t \in T(A)$, we define

$$((\text{AffT}\,\phi)(l))(\tau) = \tau(\phi(x)) \quad \text{for any } \tau \in T(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A)),$$

where $\phi(x)$ is regarded as an element in $\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A)$. In the above equation, if we regard $\phi(x)$ as element in B (rather than in $\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A)$), the homomorphism ϕ also induces a positive linear map, denoted by ϕ_T to avoid confusion, from $\text{AffT}(A)$ to $\text{AffT}(B)$ — that is, for the l as above,

$$((\phi_T)(l))(\tau) = \tau(\phi(x)) \quad \text{for any } \tau \in T(B),$$

where $\phi(x)$ is now regarded as an element in B. But this map does not preserve the unit $\mathbb{1}$. It has the property that $\phi_T(\mathbb{1}_{\text{AffT}(A)}) \leq \mathbb{1}_{\text{AffT}(B)}$.

In this paper, we often use the notation ϕ_T for the following situation: if $p_1 < p_2$ are two projections in A, and $\phi = \iota : p_1 A p_1 \to p_2 A p_2$ is the inclusion, then ι_T denotes the (not necessarily unital) map from $\text{AffT}(p_1 A p_1)$ to $\text{AffT}(p_2 A p_2)$ induced by ι.

2.9. If $\alpha : (\underline{K}(A); \underline{K}(A)^+; \Sigma A) \to (\underline{K}(B); \underline{K}(B)^+; \Sigma B)$ is a homomorphism as in 2.7, then for each projection $p \in A$, there is a projection $q \in B$ such that $\alpha([p]) = [q]$.

Since I_k has stable rank one and the spaces X involved in the definition of $\mathcal{HD}$ class (see $PM_n(C(X))P$ in 2.2) are of dimension at most two, we know that for all C^*-algebras A considered in this paper, $\mathcal{HD}$ class or $A\mathcal{HD}$ algebra, the following

statement is true: if $[p_1] = [p_2] \in K_0(A)$, then there is a unitary $u \in A$ such that $up_1u^* = p_2$. Therefore, $\mathrm{AffT}(pAp)$ and $\mathrm{AffT}(qBq)$ depend only on the classes $[p] \in K_0(A)$ and $[q] \in K_0(B)$, respectively. Furthermore, if $[p_1] = [p_2]$, then the identification of $\mathrm{AffT}(p_1Ap_1)$ and $\mathrm{AffT}(p_2Ap_2)$ via the unitary equivalence $up_1u^* = p_2$ is canonical — that is, it does not depend on the choice of unitary u. For classes $[p] \in \Sigma A (\subset K_0(A)^+ \subset K_0(A))$, we also take $\mathrm{AffT}(pAp)$ as part of our invariant. We consider a system of unital positive linear maps

$$\xi^{p,q} : \mathrm{AffT}(pAp) \to \mathrm{AffT}(qBq)$$

associated with all pairs of two classes $[p] \in \Sigma A$ and $[q] \in \Sigma B$, with $\alpha([p]) = [q]$. Such a system of maps is said to be compatible if for any $p_1 \le p_2$ with $\alpha([p_1]) = [q_1]$, $\alpha([p_2]) = [q_2]$, and $q_1 \le q_2$, the diagram

$$
\begin{array}{ccc}
\mathrm{AffT}(p_1Ap_1) & \xrightarrow{\ \xi^{p_1,q_1}\ } & \mathrm{AffT}(q_1Bq_1) \\
\iota_T \downarrow & & \downarrow \iota_T \\
\mathrm{AffT}(p_2Ap_2) & \xrightarrow{\ \xi^{p_2,q_2}\ } & \mathrm{AffT}(q_2Bq_2)
\end{array}
\tag{2.10}
$$

commutes, where the vertical maps are induced by the inclusions. (See [Ji and Jiang 2011] and [Stevens 1998].)

2.11. In this paper, we denote

$$(\underline{K}(A); \underline{K}(A)^+; \Sigma A; \{\mathrm{AffT}(pAp)\}_{[p]\in\Sigma A})$$

by $\mathrm{Inv}^0(A)$, where $\mathrm{AffT}(pAp)$ are scaled ordered Banach spaces as in 2.8. By a map between the invariants $\mathrm{Inv}^0(A)$ and $\mathrm{Inv}^0(B)$, we mean a map

$$\alpha : (\underline{K}(A); \underline{K}(A)^+; \Sigma A) \to (\underline{K}(B); \underline{K}(B)^+; \Sigma B)$$

as in 2.7, and for each pair $[p] \in \Sigma A$, $[q] \in \Sigma B$ with $\alpha[p] = [q]$, there is an associated unital positive linear map

$$\xi^{p,q} : \mathrm{AffT}(pAp) \to \mathrm{AffT}(qBq)$$

(which is automatically continuous, as pointed out in 2.8). These maps are compatible in the sense of 2.9 (that is, the diagram (2.10) is commutative for any pair of projections $p_1 \le p_2$).

2.12. Let $[p] \in \Sigma A$ be represented by $p \in A$. Let $\alpha([p]) = [q]$ for $q \in B$. Then α induces a map (still denoted by α) $\alpha : K_0(pAp) \to K_0(qBq)$. Note that the natural map $\rho := \rho_{pAp} : K_0(pAp) \to \mathrm{AffT}(pAp)$, defined in 2.8, satisfies $\rho(K_0(pAp)^+) \subseteq \mathrm{AffT}(pAp)_+$ and $\rho([p]) = \mathbb{1} \in \mathrm{AffT}(pAp)$. By [Ji and Jiang 2011, 1.20], the compatibility in 2.9 (diagram (2.10)) implies that the following diagram commutes:

$$
\begin{array}{ccc}
K_0(pAp) & \xrightarrow{\ \rho\ } & \mathrm{AffT}(pAp) \\
\alpha \downarrow & & \downarrow \xi^{p,q} \\
K_0(qBq) & \xrightarrow{\ \rho\ } & \mathrm{AffT}(qBq)
\end{array}
\tag{2.13}
$$

For $p = \mathbb{1}_A$, this compatibility (the commutativity of diagram (2.13)) is included as a part of the Elliott invariant for unital simple C^*-algebras. But this information is contained in our invariant $\mathrm{Inv}^0(A)$, as pointed out in [Ji and Jiang 2011].

2.14. Let A be a unital C^*-algebra, $B \in \mathcal{HD}$ and $\{p_i\}_{i=1}^n \subset B$ be mutually orthogonal projections with $\Sigma p_i = \mathbb{1}_B$. Write $B = \bigoplus_{j=1}^m B^j$ with B^j being either $PM_\bullet(C(X))P$ or $M_l(I_k)$, and for any $i = 1, 2, \ldots, n$ write $p_i = \bigoplus_{j=1}^m p_i^j$ with $p_i^j \in B^j$, for $j = 1, 2, \ldots, m$. Note that for all $\tau \in T(B^j)$,

$$
\tau(p_i^j) = \frac{\mathrm{rank}(p_i^j)}{\mathrm{rank}(\mathbb{1}_{B^j})}
$$

(see 2.2 for the definition of the rank function), which is independent of $\tau \in T(B^j)$.

Let $\xi_i = (\xi_i^1, \xi_i^2, \ldots, \xi_i^m) : \mathrm{AffT}(A) \to \mathrm{AffT}(p_i B p_i) = \bigoplus_{j=1}^m \mathrm{AffT}(p_i^j B^j p_i^j)$ be unital positive linear maps. Then we can define $\xi = (\xi^1, \xi^2, \ldots, \xi^m) : \mathrm{AffT}(A) \to \mathrm{AffT}(B) = \bigoplus_{j=1}^m \mathrm{AffT}(B^j)$ as below:

$$
\xi^j(f)(\tau) = \sum_{\{i : \tau(p_i^j) \neq 0\}} \tau(p_i^j) \xi_i^j(f)\left(\frac{\tau|_{p_i^j B^j p_i^j}}{\tau(p_i^j)}\right) \quad \text{for } f \in \mathrm{AffT}(A) \text{ and } \tau \in T(B^j).
$$

Note that $\tau|_{p_i^j B^j p_i^j}/\tau(p_i^j) \in T(p_i^j B^j p_i^j)$. So $\xi_i^j(f)$ can evaluate at $\tau|_{p_i^j B^j p_i^j}/\tau(p_i^j)$. Since the value of $\tau(p_i^j)$ is independent of $\tau \in T(B^j)$, it is straightforward to verify that $\xi^j \in \mathrm{AffT}(B^j)$. We denote such ξ by $\bigoplus \xi_i$. (For the case that B is general stably finite unital simple C^*-algebras with mutually orthogonal projections $\{p_i\}$ with sum $\mathbb{1}_B$, this kind of construction can be carried out by using of [Lin 2017, Lemma 6.4].)

If $\phi_i : A \to p_i B p_i$ are unital homomorphisms and $\phi = \bigoplus \phi_i : A \to B$, then

$$
(\mathrm{AffT}\,\phi)^j(f)(\tau) = \sum_{\{i : \tau(p_i^j) \neq 0\}} \tau(p_i^j)\, \mathrm{AffT}\,\phi_i^j(f)\left(\frac{\tau|_{p_i^j B^j p_i^j}}{\tau(p_i^j)}\right),
$$

where $\phi_i^j : A \to p_i^j B^j p_i^j$ is the j-th component of ϕ_i. That is, $\mathrm{AffT}\,\phi = \bigoplus \mathrm{AffT}\,\phi_i$. In particular, if $\|\mathrm{AffT}\,\phi_i(f) - \xi_i(f)\| < \varepsilon$ for all i, then

$$
\|\mathrm{AffT}\,\phi(f) - \xi(f)\| < \varepsilon.
$$

2.15. Now we introduce the new ingredient of our invariant, a simplified version of $U(pAp)/\overline{DU(pAp)}$ for any $[p] \in \Sigma A$, where $DU(pAp)$ is the commutator

subgroup of $U(pAp)$. Some notations and preliminary results are quoted from [Thomsen 1997; 1995; Nielsen and Thomsen 1996].

2.16. Let A be a unital C^*-algebra. Let $U(A)$ denote the group of unitaries of A and $U_0(A)$ the connected component of $\mathbb{1}_A$ in $U(A)$. Let $DU(A)$ and $DU_0(A)$ denote the commutator subgroups of $U(A)$ and $U_0(A)$, respectively. (Recall that the commutator subgroup of a group G is the subgroup generated by all elements of the form $aba^{-1}b^{-1}$, where $a, b \in G$.) One can introduce the following metric D_A on $U(A)/\overline{DU(A)}$ (see §3 of [Nielsen and Thomsen 1996]). For $u, v \in U(A)/\overline{DU(A)}$

$$D_A(u, v) = \inf\{\|uv^* - c\| : c \in \overline{DU(A)}\},$$

where, on the right-hand side of the equation, we use u, v to denote any elements in $U(A)$ which represent the elements $u, v \in U(A)/\overline{DU(A)}$.

Remark 2.17. Obviously, $D_A(u, v) \leq 2$. Also, if $u, v \in U(A)/\overline{DU(A)}$ define two different elements in $K_1(A)$, then $D_A(u, v) = 2$. (This fact follows from the fact that $\|u - v\| < 2$ implies $uv^* \in U_0(A)$.)

2.18. Let A be a unital C^*-algebra. Let $\mathrm{AffT}(A)$ and $\rho_A : K_0(A) \to \mathrm{AffT}(A)$ be as defined as in 2.8. For simplicity, we use $\rho K_0(A)$ to denote the set $\rho_A(K_0(A))$. The metric d_A on $\mathrm{AffT}(A)/\overline{\rho K_0(A)}$ is defined as follows (see §3 of [Nielsen and Thomsen 1996]).

Let d' denote the quotient metric on $\mathrm{AffT}(A)/\overline{\rho K_0(A)}$. That is, for f, g in $\mathrm{AffT}(A)/\overline{\rho K_0(A)}$, let

$$d'(f, g) = \inf\{\|f - g - h\| : h \in \overline{\rho K_0(A)}\}.$$

Define d_A by

$$d_A(f, g) = \begin{cases} 2 & \text{if } d'(f, g) \geq \frac{1}{2}, \\ |e^{2\pi i d'(f,g)} - 1| & \text{if } d'(f, g) < \frac{1}{2}. \end{cases}$$

Obviously, $d_A(f, g) \leq 2\pi d'(f, g)$.

2.19. For $A = PM_k(C(X))P$, let $SU(A)$ be the set of unitaries $u \in PM_k(C(X))P$ such that for each $x \in X$, $u(x) \in P(x)M_k(\mathbb{C})P(x) \cong M_{\mathrm{rank}(P)}(\mathbb{C})$ has determinant 1 (note that the determinant of $u(x)$ does not depend on the identification of $P(x)M_k(\mathbb{C})P(x) \cong M_{\mathrm{rank}(P)}(\mathbb{C})$). For $A = M_l(I_k)$, by $u \in SU(A)$ we mean that $u \in SU(M_{lk}(C[0, 1]))$, where we consider A to be a subalgebra of $M_{lk}(C[0, 1])$. For all basic building blocks $A \neq M_l(I_k)$, we have $SU(A) = \overline{DU(A)}$. But for $A = M_l(I_k)$, this is not true (see 2.20 and 2.21 below).

In [Elliott et al. 2007], the authors also defined $SU(A)$ for A a homogeneous algebra and a certain AH inductive limit C^*-algebra. This definition cannot be generalized to a more general class of C^*-algebras, but we define $\widetilde{SU(A)}$ for any unital C^* algebra A. Later, in our definition of $\mathrm{Inv}(A)$, we only make use of $\widetilde{SU(A)}$ (rather than $SU(A)$).

2.20. Let $A = I_k$. Then $K_1(A) = \mathbb{Z}/k\mathbb{Z}$, which is generated by $[u]$, where u is the unitary

$$u = \begin{pmatrix} e^{2\pi i t(k-1)/k} & & & \\ & e^{2\pi i(-t/k)} & & \\ & & \ddots & \\ & & & e^{2\pi i(-t/k)} \end{pmatrix} \in I_k.$$

(Note that $u(0) = \mathbb{1}_k$, $u(1) = e^{2\pi i(-1/k)} \cdot \mathbb{1}_k$.)

Note that the above u is in $SU(A)$, but not in $U_0(A)$, and therefore not in $DU(A)$.

2.21. By [Thomsen 1995] (or [Gong et al. 2015a]), $u \in M_l(I_k)$ is in $\overline{DU(A)}$ if and only if for any irreducible representation $\pi : M_l(I_k) \to B(H)$ ($\dim H < +\infty$), we have $\det(\pi(u)) = 1$. For the unitary u in 2.20, and irreducible representation π corresponding to 1, $\pi(u) = e^{2\pi i(-1/k)}$, whose determinant is $e^{2\pi i(-1/k)} \neq 1$. By [Thomsen 1997, 6.1] one knows that if $A = I_k$, then

$$U_0(A) \cap SU(A) = \{e^{2\pi i(j/k)} : j = 0, 1, \ldots, k-1\} \cdot \overline{DU(A)}.$$

If $A = M_l(I_k)$, then for any $j \in \mathbb{Z}$, $e^{2\pi i(j/l)} \cdot \mathbb{1}_A \in \overline{DU(A)}$. Consequently,

$$U_0(A) \cap SU(A) = \{e^{2\pi i(j/kl)} : j = 0, 1, \ldots, kl-1\} \cdot \overline{DU(A)}.$$

2.22. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then for any $A \in \mathcal{HD}$, $\mathbb{T} \cdot \overline{DU(A)} \subset U_0(A)$. From 2.19 and 2.21, we have either $SU(A) = \overline{DU(A)}$ or $U_0(A) \cap SU(A) \subset \mathbb{T} \cdot \overline{DU(A)}$.

Lemma 2.23. *Let* $A = PM_k(C(X))P \in \mathcal{HD}$. *For any* $u, v \in U(A)$, *if* $uv^* \in \mathbb{T} \cdot \overline{DU(A)}$ *(in particular if both* u, v *are in* $\mathbb{T} \cdot \overline{DU(A)}$*), then* $D_A(u, v) \leq 2\pi/\text{rank}(P)$.

Let $A = M_l(I_k)$. *For any* u, v, *if* $uv^* \in \mathbb{T} \cdot \overline{DU(A)}$, *then* $D_A(u, v) \leq 2\pi/l$.

Proof. There is $\omega \in \overline{DU(A)}$ such that $uv^* = \lambda\omega$ for some $\lambda \in \mathbb{T}$. Choose $\lambda_0 = e^{2\pi i j/\text{rank}(P)}$, $j \in \mathbb{N}$, such that $|\lambda - \lambda_0| < 2\pi/\text{rank}(P)$. Then $\lambda_0 \cdot P \in PM_k(C(X))P$ has determinant 1 everywhere and is in $\overline{DU(A)}$. And so does $\lambda_0\omega$. Also, we have $|uv^* - \lambda_0\omega| < 2\pi/\text{rank}(P)$.

The case $A = M_l(I_k)$ is similar. $\qquad\square$

2.24. Let $\text{path}(U(A))$ denote the set of piecewise smooth paths $\xi : [0, 1] \to U(A)$. Recall that the de la Harp–Skandalis determinant $\Delta : \text{path}(U(A)) \to \text{AffT}(A)$ is defined by

$$\Delta(\xi)(\tau) = \frac{1}{2\pi i} \int_0^1 \tau\left(\frac{d\xi}{dt} \cdot \xi^*\right) dt$$

(see [de la Harpe and Skandalis 1984]). It is proved there (see also [Thomsen 1995]) that Δ induces a map $\Delta^\circ : \pi_1(U_0(A)) \to \text{AffT}(A)$. For any two paths ξ_1, ξ_2 starting at $\xi_1(0) = \xi_2(0) = 1 \in A$ and ending at the same unitary $u = \xi_1(1) = \xi_2(1)$, we have that

$$\Delta(\xi_1) - \Delta(\xi_2) = \Delta(\xi_1 \cdot \xi_2^*) \subset \Delta^\circ(\pi_1(U_0(A))).$$

Consequently, Δ induces a map

$$\overline{\Delta} : U_0(A) \to \mathrm{AffT}(A)/\Delta^\circ(\pi_1(U_0(A)))$$

(see Section 3 of [Thomsen 1995]). Passing to matrix over A, we have a map $\overline{\Delta}_n : U_0(M_n(A)) \to \mathrm{AffT}(A)/\Delta_n^\circ(\pi_1(U_0(M_n(A))))$.

If $1 \le m < n$, then $\mathrm{path}(U(M_m(A)))$ (and $U_0(M_m(A))$) can be embedded into $\mathrm{path}(U(M_n(A)))$ (and $U_0(M_n(A))$) by sending $u(t)$ to $\mathrm{diag}(u(t), \mathbb{1}_{n-m})$. From the above definition, and the formula

$$\frac{d}{dt}(\mathrm{diag}(u(t), \mathbb{1}_{n-m})) = \mathrm{diag}\Big(\frac{d}{dt}(u(t)), 0_{n-m}\Big),$$

one gets

$$\overline{\Delta}_n|_{U_0(M_m(A))} = \overline{\Delta}_m.$$

Recall that the Bott isomorphism $b : K_0(A) \to K_1(SA)$ is given by the following: for any $x \in K_0(A)$ represented by a projection $p \in M_n(A)$, we have

$$b(x) = [e^{2\pi i t} p + (\mathbb{1}_n - p)] \in K_1(SA).$$

If $\xi(t) = e^{2\pi i t} p + (\mathbb{1}_n - p)$, then

$$(\Delta^\circ \xi)(\tau) = \frac{1}{2\pi i} \int_0^1 \tau\big((2\pi i e^{2\pi i t} p) \cdot (e^{-2\pi i t} p + (1 - p))\big) \, dt$$

$$= \frac{1}{2\pi i} \int_0^1 \tau(2\pi i p) \, dt = \tau(p).$$

Since the Bott map is an isomorphism, it follows that each loop in $\pi_1(U_0(A))$ is homotopic to a product of loops of the form $\xi(t)$. Consequently,

$$\Delta^\circ(\pi_1(U_0(M_n(A)))) \subset \rho_A K_0(A).$$

Hence $\overline{\Delta}_n$ can be regarded as a map

$$\overline{\Delta}_n : U_0(M_n(A)) \to \mathrm{AffT}(A)/\overline{\rho_A K_0(A)}.$$

Proposition 2.25. *For $A \in \mathcal{HD}$ or $A \in \mathcal{AHD}$, $\overline{DU_0(A)} = \overline{DU(A)}$.*

Proof. Let the determinant function

$$\overline{\Delta}_n : U_0(M_n(A)) \to \mathrm{AffT}(A)/\overline{\Delta_n^0(\pi_1 U_0(M_n(A)))}$$

be defined as in §3 of [Thomsen 1995] (see 2.24 above). As observed in [Nielsen and Thomsen 1996, top of p. 33], Lemma 3.1 of [Thomsen 1995] implies that $\overline{DU_0(A)} = U_0(A) \cap \overline{DU(A)}$. For the reader's convenience, we give a brief proof

of this fact. Namely, the equation

$$\begin{pmatrix} uvu^{-1}v^{-1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} v & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & v^{-1} \end{pmatrix} \begin{pmatrix} u^{-1} & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} v^{-1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & v \end{pmatrix}$$

implies that $\overline{DU(A)} \subset \overline{DU_0(M_3(A))}$. Therefore by Lemma 3.1 of [Thomsen 1995], $\overline{DU(A)} \subset \ker \bar{\Delta}_3$. If $x \in U_0(A) \cap \overline{DU(A)}$, then $\bar{\Delta}_1$ is defined at x. By calculation in 2.24, $\bar{\Delta}_3|_{U_0(A)} = \bar{\Delta}_1$. So we have $\bar{\Delta}_1(x) = 0$, and thus $x \in \overline{DU_0(A)} = \ker \bar{\Delta}_1$, by [Thomsen 1995, Lemma 3.1]. Note if $A \in \mathcal{HD}$ or $A\mathcal{HD}$, then $\overline{DU(A)} \subset U_0(A)$. □

(It is not known to the authors whether it is always true that $\overline{DU_0(A)} = \overline{DU(A)}$.)

2.26. There is a natural map $\alpha : \pi_1(U(A)) \to K_0(A)$, or more generally, for each $n \in \mathbb{N}$ a map $\alpha_n : \pi_1(U(M_n(A))) \to K_0(A)$. We need the following notation. For a unital C^*-algebra A, let $\mathcal{P}_n K_0(A)$ (see [Gong et al. 2015b]) be the subgroup of $K_0(A)$ generated by the formal difference of projections $p, q \in M_n(A)$ (instead of $M_\infty(A)$). Then

$$\mathcal{P}_n K_0(A) \subset \text{image}(\alpha_n).$$

In particular, if $\rho : K_0(A) \to \text{AffT}(A)$ satisfies $\rho(\mathcal{P}_n K_0(A)) = \rho K_0(A)$, then by Theorem 3.2 of [Thomsen 1995],

$$U_0(M_n(A))/\overline{DU_0(M_n(A))} \cong U_0(M_\infty(A))/\overline{DU_0(M_\infty(A))} \cong \text{AffT}(A)/\overline{\rho K_0(A)}.$$

Note that for all $A \in \mathcal{HD}$, we have $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$ (see below). Consequently,

$$U_0(A)/\overline{DU_0(A)} \cong \text{AffT}(A)/\overline{\rho K_0(A)}.$$

If A does not contain building blocks of the form $PM_n(C(T_{\text{II},k}))P$, then such A is the special case of [Thomsen 1997], and the above fact is observed in [Thomsen 1997] (for circle algebras in [Nielsen and Thomsen 1996] earlier) — in this special case, we ever have $\mathcal{P}_1 K_0(A) = K_0(A)$ (as used in [Nielsen and Thomsen 1996] and [Thomsen 1997] in the form of surjectivity of $\alpha : \pi_1(U(A)) \to K_0(A)$). For $A = PM_n(C(T_{\text{II},k}))P$, we do not have the surjectivity of $\alpha : \pi_1(U(A)) \to K_0(A)$ anymore. But $K_0(A) = \mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$ and image(α) $= \mathcal{P}_1 K_0(A)$ contains at least one element which corresponds to a rank one projection (any bundle over $T_{\text{II},k}$ has a subbundle of rank 1) — that is,

$$\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)(\subseteq \text{AffT}(A))$$

consisting of all constant functions from $T_{\text{II},k}$ to $(1/\text{rank}(P))\mathbb{Z}$.

As in [Nielsen and Thomsen 1996, Lemma 3.1; Thomsen 1997, Lemma 6.4], the map $\bar{\Delta} : U_0(A) \to \text{AffT}(A)/\overline{\rho_A(K_0(A))}$ (see 2.24) has $\ker \bar{\Delta} = \overline{DU(A)}$ and the following lemma holds.

Lemma 2.27. *Suppose that a unital C^*-algebra A satisfies $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$ and $\overline{DU_0(A)} = \overline{DU(A)}$ (see 2.26 and 2.25), and in particular, that $A \in \mathcal{HD}$ or $A \in \mathcal{AHD}$. Then the following hold:*

(1) *There is a split exact sequence*

$$0 \to \mathrm{AffT}(A)/\overline{\rho K_0(A)} \xrightarrow{\lambda_A} U(A)/\overline{DU(A)} \to K_1(A) \to 0.$$

(2) *λ_A is an isometry with respect to the metrics d_A and D_A.*

2.28. Recall from §3 of [Thomsen 1995], the de la Harpe–Skandalis determinant (see [de la Harpe and Skandalis 1984]) can be used to define

$$\overline{\Delta} : U_0(A)/\overline{DU(A)} \to \mathrm{AffT}(A)/\overline{\rho K_0(A)}.$$

With the condition of Lemma 2.27 above, this map is an isometry with respect to the metrics d_A and D_A. In fact, the inverse of this map is λ_A in Lemma 2.27.

It follows from the definition of $\overline{\Delta}$ [Thomsen 1995, §3] that

$$\overline{\Delta}(e^{2\pi i t p}) = t \cdot \rho([p]) \pmod{(\overline{\rho K_0(A)})}, \tag{2.29}$$

where $[p] \in K_0(A)$ is the element represented by projection $p \in A$.

It is convenient to introduce the extended commutator group $DU^+(A)$, which is generated by $DU(A) \subset U(A)$ and the set

$$\{e^{2\pi i t p} = e^{2\pi i t} p + (\mathbb{1} - p) \in U(A) : t \in \mathbb{R}, \, p \in A \text{ is a projection}\}.$$

Let $\widetilde{DU(A)}$ denote the closure of $DU^+(A)$. That is, $\widetilde{DU(A)} = \overline{DU^+(A)}$.

Let us use $\widetilde{\rho K_0(A)}$ to denote the real vector space spanned by $\rho K_0(A)$. That is,

$$\widetilde{\rho K_0(A)} := \overline{\{\Sigma \lambda_i \phi_i : \lambda_i \in \mathbb{R}, \, \phi_i \in \rho K_0(A)\}}.$$

Suppose that $\overline{\rho K_0(A)} = \overline{\rho(\mathcal{P}_1 K_0(A))}$. It follows from (2.29) that the image of $\widetilde{DU(A)}/\overline{DU(A)}$ under the map $\overline{\Delta}$ is exactly $\widetilde{\rho K_0(A)}/\overline{\rho K_0(A)}$. Therefore, λ_A takes $\widetilde{\rho K_0(A)}/\overline{\rho K_0(A)}$ to $\widetilde{DU(A)}/\overline{DU(A)}$. Hence $\overline{\Delta} : U_0(A)/\overline{DU(A)} \to \mathrm{AffT}(A)/\overline{\rho K_0(A)}$ also induces a quotient map (still denoted by $\overline{\Delta}$)

$$\overline{\Delta} : U_0(A)/\widetilde{DU(A)} \to \mathrm{AffT}(A)/\widetilde{\rho K_0(A)},$$

which is an isometry using the quotient metrics of d_A and D_A. The inverse of this quotient map $\overline{\Delta}$ gives rise to the isometry

$$\widetilde{\lambda}_A : \mathrm{AffT}(A)/\widetilde{\rho K_0(A)} \to U_0(A)/\widetilde{DU(A)} \hookrightarrow U(A)/\widetilde{DU(A)},$$

which is an isometry with respect to the quotient metrics $\widetilde{d_A}$ and $\overline{D_A}$ as described below.

For any $u, v \in U(A)/\widetilde{DU(A)}$,

$$\overline{D_A}(u, v) = \inf\{\|uv^* - c\| : c \in \widetilde{DU(A)}\}.$$

Let $\widetilde{d'}$ denote the quotient metric on $\mathrm{AffT}(A)/\widetilde{\rho K_0(A)}$ of $\mathrm{AffT}(A)$, that is,

$$\widetilde{d'}(f, g) = \inf\{\| f - g - h \| : h \in \widetilde{\rho K_0(A)}\} \quad \text{for all } f, g \in \mathrm{AffT}(A)/\widetilde{\rho K_0(A)}.$$

Define $\widetilde{d}_A$ by

$$\widetilde{d}_A(f, g) = \begin{cases} 2 & \text{if } \widetilde{d'}(f, g) \geq \frac{1}{2}, \\ |e^{2\pi i \widetilde{d'}(f,g)} - 1| & \text{if } \widetilde{d'}(f, g) < \frac{1}{2}. \end{cases}$$

The following result is a consequence of Lemma 2.27.

Lemma 2.30. *Suppose that a unital C^*-algebra A satisfies $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$ and $\overline{DU_0(A)} = \overline{DU(A)}$ (see 2.26 and 2.25), and in particular, that $A \in \mathcal{HD}$ or $A \in A\mathcal{HD}$. Then we have the following*:

(1) *There is a split exact sequence*

$$0 \to \mathrm{AffT}(A)/\widetilde{\rho K_0(A)} \xrightarrow{\tilde{\lambda}_A} U(A)/\widetilde{DU(A)} \xrightarrow{\pi_A} K_1(A) \to 0.$$

(2) *$\tilde{\lambda}_A$ is an isometry with respect to $\widetilde{d}_A$ and $\overline{D}_A$.*

Proof. As we mentioned in 2.28, the map λ_A in Lemma 2.27 takes $\widetilde{\rho K_0(A)}/\overline{\rho K_0(A)}$ to $\widetilde{DU(A)}/\overline{DU(A)}$. From the exact sequence in Lemma 2.27, passing to quotient, one gets the exact sequence in (1).

Note that $\widetilde{d}_A$ on $\mathrm{AffT}(A)/\widetilde{\rho K_0(A)}$ is the quotient metric induced by d_A on $\mathrm{AffT}(A)/\overline{\rho K_0(A)}$ and $\overline{D}_A$ on $U(A)/\widetilde{DU(A)}$ is the quotient metric induced by D_A on $U(A)/\overline{DU(A)}$. Hence $\tilde{\lambda}_A$ is an isometry, since so is λ_A. $\qquad\square$

2.31. Instead of $\widetilde{DU(A)}$, we need the group

$$\widetilde{SU(A)} := \overline{\{x \in U(A) : x^n \in \widetilde{DU(A)} \text{ for some } n \in \mathbb{Z}_+\backslash\{0\}\}}.$$

For $A \in \mathcal{HD}$, say $A = PM_l(C(X))P$ ($X = [0, 1]$, S^1 or $T_{\mathrm{II},k}$) or $A = M_l(I_k)$, $\widetilde{SU(A)}$ is the set of all unitaries $u \in P(M_l C(X))P$ or $u \in M_l(I_k)$ such that the determinant function

$$X \ni x \mapsto \det(u(x)) \quad \text{or} \quad (0, 1) \ni t \mapsto \det(u(t))$$

is a constant function. Comparing with the set $SU(A)$ in [Elliott et al. 2007] or 2.19 above (which only defines for $\mathcal{HD}$ blocks), where the function will be constant 1, here we allow the function to be an arbitrary constant in $\mathbb{T}$. Hence for a basic building block $A = PM_n(C(X))P \in \mathcal{HD}$ or $A = M_l(I_k)$,

$$\widetilde{SU(A)} = \mathbb{T} \cdot SU(A).$$

The notations $\widetilde{\rho K_0(A)}$, $\widetilde{DU(A)}$ and $\widetilde{SU(A)}$ reflect that they are constructed from $\rho K_0(A)$, $DU(A)$ and $SU(A)$, respectively. *To make the notation simpler, from now on we use $\widetilde{\rho K_0}(A)$ to denote $\widetilde{\rho K_0(A)} = \rho_A(\widetilde{K_0(A)})$, $\widetilde{DU}(A)$ to denote $\widetilde{DU(A)}$, and $\widetilde{SU}(A)$ to denote $\widetilde{SU(A)}$.*

Lemma 2.32. *Let α, $\beta : K_1(A) \to U(A)/\widetilde{DU}(A)$ be splittings of π_A in [Lemma 2.30](#). Then*

$$\alpha|_{\mathrm{tor}\, K_1(A)} = \beta|_{\mathrm{tor}\, K_1(A)}$$

and $\alpha(\mathrm{tor}\, K_1(A)) \subset \widetilde{SU}(A)/\widetilde{DU}(A)$. Furthermore, α identifies $\mathrm{tor}(K_1(A))$ with $\widetilde{SU}(A)/\widetilde{DU}(A)$.

Proof. For any $z \in \mathrm{tor}\, K_1(A)$, with $kz = 0$ for some integer $k > 0$, we have

$$\pi_A \alpha(z) = z = \pi_A \beta(z).$$

By the exactness of the sequence, there is an element $f \in \mathrm{AffT}(A)/\rho\widetilde{K}_0(A)$ such that

$$\alpha(z) - \beta(z) = \tilde{\lambda}_A(f).$$

Since $k\alpha(z) - k\beta(z) = \alpha(kz) - \beta(kz) = 0$, we have $\tilde{\lambda}_A(kf) = 0$. By the injectivity of $\tilde{\lambda}_A$, $kf = 0$. Note that $\mathrm{AffT}(A)/\rho\widetilde{K}_0(A)$ is an $\mathbb{R}$-vector space, $f = 0$. Furthermore, $k\alpha(z) = 0$ in $U(A)/\widetilde{DU}(A)$ implies that

$$\alpha(z) \in \widetilde{SU}(A)/\widetilde{DU}(A).$$

Thus we get $\alpha(\mathrm{tor}\, K_1(A)) \subset \widetilde{SU}(A)$. If $u \in \widetilde{SU}(A)/\widetilde{DU}(A)$ then $\alpha(\pi_A(u)) = u$. $\square$

2.33. Let $U_{\mathrm{tor}}(A)$ denote the set of unitaries $u \in A$ such that $[u] \in \mathrm{tor}\, K_1(A)$. For any C^*-algebra A we have $\widetilde{SU}(A) \subset U_{\mathrm{tor}}(A)$. If we further assume $\overline{DU_0(A)} = \overline{DU(A)}$, then

$$\widetilde{DU}(A) = U_0(A) \cap \widetilde{SU}(A) \quad \text{and} \quad U_{\mathrm{tor}}(A) = U_0(A) \cdot \widetilde{SU}(A).$$

We have $U_0(A)/\widetilde{DU}(A) \cong U_{\mathrm{tor}}(A)/\widetilde{SU}(A)$. The metric $\overline{D}_A$ on $U(A)/\widetilde{DU}(A)$ induces a metric $\widetilde{D}_A$ on $U(A)/\widetilde{SU}(A)$. And the above identification $U_0(A)/\widetilde{DU}(A)$ with $U_{\mathrm{tor}}(A)/\widetilde{SU}(A)$ is an isometry with respect to $\overline{D}_A$ and $\widetilde{D}_A$. Hence $\tilde{\lambda}_A$ in [2.28](#) can be regarded as a map (still denoted by $\tilde{\lambda}_A$):

$$\tilde{\lambda}_A : \mathrm{AffT}(A)/\rho\widetilde{K}_0(A) \to U_{\mathrm{tor}}(A)/\widetilde{SU(A)} \hookrightarrow U(A)/\widetilde{SU(A)}.$$

Similar to [Lemma 2.30](#), we have the following.

Lemma 2.34. *Suppose that a unital C^*-algebra A satisfies $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$ and $\overline{DU_0(A)} = \overline{DU(A)}$ (see [2.26](#) and [2.25](#)), and in particular, that $A \in \mathcal{HD}$ or $A \in \mathcal{AHD}$. Then the following hold:*

(1) *There is a split exact sequence*

$$0 \to \mathrm{AffT}(A)/\rho\widetilde{K}_0(A) \xrightarrow{\tilde{\lambda}_A} U(A)/\widetilde{SU}(A) \xrightarrow{\pi_A} K_1(A)/\mathrm{tor}\, K_1(A) \to 0.$$

(2) *$\tilde{\lambda}_A$ is an isometry with respect to the metrics $\tilde{d}_A$ and $\widetilde{D}_A$.*

2.35. For each pair of projections $p_1, p_2 \in A$ with $p_1 = u p_2 u^*$,

$$U(p_1 A p_1)/\widetilde{SU}(p_1 A p_1) \cong U(p_2 A p_2)/\widetilde{SU}(p_2 A p_2).$$

Also, since in any unital C^*-algebra A and unitaries $u, v \in U(A)$, v and uvu^* represent the same element in $U(A)/\widetilde{SU}(A)$, the above identification does not depend on the choice of u to implement $p_1 = u p_2 u^*$. That is, for any $[p] \in \Sigma A$, the group $U(pAp)/\widetilde{SU}(pAp)$ is well defined, which does not depend on choice of $p \in [p]$. We include this group (with metric) as part of our invariant. If $[p] \leq [q]$, then we can choose p, q such that $p \leq q$. In this case, there is a natural inclusion map $\iota : pAp \to qAq$, which induces

$$\iota_* : U(pAp)/\widetilde{SU}(pAp) \to U(qAq)/\widetilde{SU}(qAq),$$

where ι_* is defined by

$$\iota_*(u) = u \oplus (q - p) \in U(qAq) \quad \text{for all } u \in U(pAp).$$

A unital homomorphism $\phi : A \to B$ induces a contractive group homomorphism

$$\phi^\natural : U(A)/\widetilde{SU}(A) \to U(B)/\widetilde{SU}(B).$$

If ϕ is not unital, then the map

$$\phi^\natural : U(A)/\widetilde{SU}(A) \to U(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))/\widetilde{SU}(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))$$

is induced by the corresponding unital homomorphism. In this case, ϕ also induces the map $\iota_* \circ \phi^\natural : U(A)/\widetilde{SU}(A) \to U(B)/\widetilde{SU}(B)$, which is denoted by ϕ_* to avoid confusion. If ϕ is unital, then $\phi^\natural = \phi_*$. If ϕ is not unital, then $\phi^\natural$ and ϕ_* have different codomains. That is, $\phi^\natural$ has codomain $U(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))/\widetilde{SU}(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))$, but ϕ_* has codomain $U(B)/\widetilde{SU}(B)$. (See the last paragraph of 3.8 below for some further explanation with an example.)

Since $U(A)/\widetilde{SU}(A)$ is an abelian group, we call the unit $[\mathbb{1}] \in U(A)/\widetilde{SU}(A)$ the zero element. If $\phi : A \to B$ satisfies $\phi(U(A)) \subset \widetilde{SU}(\phi(\mathbb{1}_A)B\phi(\mathbb{1}_A))$, then $\phi^\natural = 0$. In particular, if the image of ϕ is of finite dimension, then $\phi^\natural = 0$.

2.36. In this paper and [Gong et al. 2016], we denote

$$(\underline{K}(A); \underline{K}(A)^+; \Sigma A; \{\text{AffT}(pAp)\}_{[p]\in\Sigma A}; \{U(pAp)/\widetilde{SU}(pAp)\}_{[p]\in\Sigma A})$$

by Inv(A). By a map from Inv(A) to Inv(B), we mean

$$\alpha : (\underline{K}(A); \underline{K}(A)^+; \Sigma A) \to (\underline{K}(B); \underline{K}(B)^+; \Sigma B)$$

as in 2.7, and for each pair $([p], [\bar{p}]) \in \Sigma A \times \Sigma B$ with $\alpha([p]) = [\bar{p}]$, there exist an associate unital positive (continuous) linear map

$$\xi^{p,\bar{p}} : \text{AffT}(pAp) \to \text{AffT}(\bar{p}B\bar{p})$$

and an associate contractive group homomorphism

$$\chi^{p,\bar{p}} : U(pAp)/\widetilde{SU}(pAp) \to U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p})$$

satisfying the following compatibility conditions. (Note that $\chi^{p,\bar{p}}$ is continuous, as it is a contractive group homomorphism from a metric group to another metric group.)

(a) If $p < q$, then the diagrams

$$
\begin{array}{ccc}
\text{AffT}(pAp) & \xrightarrow{\xi^{p,\bar{p}}} & \text{AffT}(\bar{p}B\bar{p}) \\
{\scriptstyle \iota_T}\downarrow & & \downarrow{\scriptstyle \iota_T} \\
\text{AffT}(qAq) & \xrightarrow{\xi^{q,\bar{q}}} & \text{AffT}(\bar{q}B\bar{q})
\end{array}
\tag{I}
$$

and

$$
\begin{array}{ccc}
U(pAp)/\widetilde{SU}(pAp) & \xrightarrow{\chi^{p,\bar{p}}} & U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p}) \\
{\scriptstyle \iota_*}\downarrow & & \downarrow{\scriptstyle \iota_*} \\
U(qAq)/\widetilde{SU}(qAq) & \xrightarrow{\chi^{q,\bar{q}}} & U(\bar{q}B\bar{q})/\widetilde{SU}(\bar{q}B\bar{q})
\end{array}
\tag{II}
$$

commutes, where the vertical maps are induced by inclusions.

(b) The diagram

$$
\begin{array}{ccc}
K_0(pAp) & \xrightarrow{\rho} & \text{AffT}(pAp) \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \xi^{p,\bar{p}}} \\
K_0(\bar{p}B\bar{p}) & \xrightarrow{\rho} & \text{AffT}(\bar{p}B\bar{p})
\end{array}
\tag{III}
$$

commutes, and therefore $\xi^{p,\bar{p}}$ induces a map (still denoted by $\xi^{p,\bar{p}}$)

$$\xi^{p,\bar{p}} : \text{AffT}(pAp)/\widetilde{\rho K_0}(pAp) \to \text{AffT}(\bar{p}B\bar{p})/\widetilde{\rho K_0}(\bar{p}B\bar{p}).$$

(The commutativity of (III) follows from the commutativity of (I), by [Ji and Jiang 2011, 1.20]. So this is not an extra requirement.)

(c) The diagrams

$$
\begin{array}{ccc}
\text{AffT}(pAp)/\widetilde{\rho K_0}(pAp) & \longrightarrow & U(pAp)/\widetilde{SU}(pAp) \\
{\scriptstyle \xi^{p,\bar{p}}}\downarrow & & \downarrow{\scriptstyle \chi^{p,\bar{p}}} \\
\text{AffT}(\bar{p}B\bar{p})/\widetilde{\rho K_0}(\bar{p}B\bar{p}) & \longrightarrow & U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p})
\end{array}
\tag{IV}
$$

and

$$
\begin{array}{ccc}
U(pAp)/\widetilde{SU}(pAp) & \longrightarrow & K_1(pAp)/\operatorname{tor} K_1(pAp) \\
{\scriptstyle \chi^{p,\bar{p}}}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\
U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p}) & \longrightarrow & K_1(\bar{p}B\bar{p})/\operatorname{tor} K_1(\bar{p}B\bar{p})
\end{array}
\tag{V}
$$

commute, where α_1 is induced by α.

We denote the map from $\mathrm{Inv}(A)$ to $\mathrm{Inv}(B)$ by

$$(\alpha, \xi, \chi) : \big(\underline{K}(A); \{\mathrm{AffT}(pAp)\}_{[p]\in\Sigma A}; \{U(pAp)/\widetilde{SU}(pAp)\}_{[p]\in\Sigma A}\big)$$
$$\to \big(\underline{K}(B); \{\mathrm{AffT}(\bar{p}B\bar{p})\}_{[\bar{p}]\in\Sigma B}; \{U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p})\}_{[\bar{p}]\in\Sigma B}\big).$$

Completely similar to [Nielsen and Thomsen 1996, Lemma 3.2] and [Thomsen 1997, Lemma 6.5], we have the following propositions.

Proposition 2.37. *Let unital C^*-algebras A, B satisfy $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$, $\rho(\mathcal{P}_1 K_0(B)) = \rho K_0(B)$ and $\overline{DU_0(A)} = \overline{DU(A)}$, $\overline{DU_0(B)} = \overline{DU(B)}$. In particular, let A, $B \in \mathcal{HD}$ or $A\mathcal{HD}$ be unital C^*-algebras. Assume that*

$$\psi_1 : K_1(A) \to K_1(B) \quad \text{and} \quad \psi_0 : \mathrm{AffT}(A)/\overline{\rho K_0(A)} \to \mathrm{AffT}(B)/\overline{\rho K_0(B)}$$

are group homomorphisms such that ψ_0 is a contraction with respect to d_A and d_B. Then there is a group homomorphism

$$\psi : U(A)/\overline{DU(A)} \to U(B)/\overline{DU(B)}$$

which is a contraction with respect to D_A and D_B such that the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{AffT}(A)/\overline{\rho K_0(A)} & \xrightarrow{\lambda_A} & U(A)/\overline{DU(A)} & \xrightarrow{\pi_A} & K_1(A) & \longrightarrow & 0 \\
& & \downarrow{\psi_0} & & \downarrow{\psi} & & \downarrow{\psi_1} & & \\
0 & \longrightarrow & \mathrm{AffT}(A)/\overline{\rho K_0(B)} & \xrightarrow{\lambda_B} & U(B)/\overline{DU(B)} & \xrightarrow{\pi_B} & K_1(B) & \longrightarrow & 0
\end{array}
$$

commutes. If ψ_0 is an isometric isomorphism and ψ_1 is an isomorphism, then ψ is an isometric isomorphism.

Proposition 2.38. *Let unital C^*-algebras A, B satisfy $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$, $\rho(\mathcal{P}_1 K_0(B)) = \rho K_0(B)$ and $\overline{DU_0(A)} = \overline{DU(A)}$, $\overline{DU_0(B)} = \overline{DU(B)}$. In particular, let A, $B \in \mathcal{HD}$ or $A\mathcal{HD}$ be unital C^*-algebras. Assume that*

$$\psi_1 : K_1(A) \to K_1(B) \quad \text{and} \quad \psi_0 : \mathrm{AffT}(A)/\rho\widetilde{K}_0(A) \to \mathrm{AffT}(B)/\rho\widetilde{K}_0(B)$$

are group homomorphisms such that ψ_0 is a contraction with respect to $\widetilde{d}_A$ and $\widetilde{d}_B$. Then there is a group homomorphism

$$\psi : U(A)/\widetilde{SU}(A) \to U(B)/\widetilde{SU}(B)$$

which is a contraction with respect to $\widetilde{D}_A$ and $\widetilde{D}_B$ such that the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{AffT}(A)/\rho\widetilde{K}_0(A) & \xrightarrow{\tilde{\lambda}_A} & U(A)/\widetilde{SU}(A) & \xrightarrow{\tilde{\pi}_A} & K_1(A)/\mathrm{tor}\, K_1(A) & \longrightarrow & 0 \\
& & \downarrow{\psi_0} & & \downarrow{\psi} & & \downarrow{\psi_1} & & \\
0 & \longrightarrow & \mathrm{AffT}(A)/\rho\widetilde{K}_0(B) & \xrightarrow{\tilde{\lambda}_B} & U(B)/\widetilde{SU}(B) & \xrightarrow{\tilde{\pi}_B} & K_1(B)/\mathrm{tor}\, K_1(B) & \longrightarrow & 0
\end{array}
$$

commutes. If ψ_0 is an isometric isomorphism and ψ_1 is an isomorphism, then ψ is an isometric isomorphism.

Remark 2.39. As in Proposition 2.38 (or Proposition 2.37), for each fixed pair $p \in A$, $\bar{p} \in B$ with $\alpha([p]) = [\bar{p}]$, if we have an isometric isomorphism between the quotients $\mathrm{AffT}(pAp)/\widetilde{\rho K_0}(pAp)$ and $\mathrm{AffT}(\bar{p}B\bar{p})/\widetilde{\rho K_0}(\bar{p}B\bar{p})$ (or between $\mathrm{AffT}(pAp)/\overline{\rho K_0(pAp)}$ and $\mathrm{AffT}(\bar{p}B\bar{p})/\overline{\rho K_0(\bar{p}B\bar{p})}$) and an isomorphism between $K_1(pAp)$ and $K_1(pBp)$, then we also have an isometric isomorphism between $U(pAp)/\widetilde{SU}(pAp)$ and $U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p})$ (or between $U(pAp)/\overline{DU(pAp)}$ and $U(\bar{p}B\bar{p})/\overline{DU(\bar{p}B\bar{p})}$) making both diagrams (IV) and (V) commute. This is the reason $U(A)/\overline{DU(A)}$ is not included in the Elliott invariant in the classification of simple C^*-algebras. For our setting, even though for each pair of projections $(p, \bar{p})$ with $\alpha([p]) = [\bar{p}]$, we can find an isometric isomorphism between $U(pAp)/\widetilde{SU}(pAp)$ and $U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p})$, provided that the other parts of invariants $\mathrm{Inv}^0(A)$ and $\mathrm{Inv}^0(B)$ are isomorphic, we still cannot make such a system of isometric isomorphisms compatible — that is, we cannot make the diagram (II) commute for $p < q$. We present two nonisomorphic C^*-algebras A and B in our class such that $\mathrm{Inv}^0(A) \cong \mathrm{Inv}^0(B)$ in the next section, where $\mathrm{Inv}^0(B)$ is defined in 2.11. Hence it is essential to include $\{U(pAp)/\widetilde{SU}(pAp)\}_{p \in \Sigma}$ with the compatibility as part of $\mathrm{Inv}(A)$.

2.40. Replacing $U(pAp)/\widetilde{SU}(pAp)$, one can also use $U(pAp)/\overline{DU(pAp)}$ as the part of the invariant. That is, one can define $\mathrm{Inv}'(A)$ as

$$\left(\underline{K}(A); \underline{K}(A)^+; \Sigma A; \{\mathrm{AffT}(pAp)\}_{[p] \in \Sigma A}; \{U(pAp)/\overline{DU(pAp)}\}_{[p] \in \Sigma A}\right),$$

with corresponding compatibility condition — one needs to change diagrams (IV) and (V) to the corresponding ones. It is not difficult to see that $\mathrm{Inv}'(A) \cong \mathrm{Inv}'(B)$ implies $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$. We choose the formulation of $\mathrm{Inv}(A)$, since it is much more convenient for the proof of the main theorem in [Gong et al. 2016] and it is formally a weaker requirement than the one to require the isomorphism between $\mathrm{Inv}'(A)$ and $\mathrm{Inv}'(B)$, and the theorem is formally stronger. (Let us point out that, in the construction of the example (and its proof) in Section 3 of this article, $\mathrm{Inv}'(A)$ is as convenient as $\mathrm{Inv}(A)$, and therefore if only for the sake of the example in Section 3 of this paper, it is not necessary to introduce $\widetilde{SU}(A)$.)

Furthermore, it is straightforward to check the following proposition:

Proposition 2.41. *Let unital C^*-algebras A, B satisfy $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$, $\rho(\mathcal{P}_1 K_0(B)) = \rho K_0(B)$ and $\overline{DU_0(A)} = \overline{DU(A)}$, $\overline{DU_0(B)} = \overline{DU(B)}$. In particular, let $A, B \in \mathcal{HD}$ or $A\mathcal{HD}$ be unital C^*-algebras. Suppose that $K_1(A) = \mathrm{tor}(K_1(A))$ and $K_1(B) = \mathrm{tor}(K_1(B))$. Then $\mathrm{Inv}^0(A) \cong \mathrm{Inv}^0(B)$ implies $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$.*

Proof. It follows from the fact that any isomorphism

$$\xi^{p,\bar{p}} : \text{AffT}(pAp)/\widetilde{\rho K_0}(pAp) \to \text{AffT}(\bar{p}B\bar{p})/\widetilde{\rho K_0}(\bar{p}B\bar{p})$$

induces a unique isomorphism

$$\chi^{p,\bar{p}} : U(pAp)/\widetilde{SU}(pAp) \to U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p}).$$

(Note that by the split exact sequence in Lemma 2.34, $\text{AffT}(pAp)/\widetilde{\rho K_0}(pAp) \cong U(pAp)/\widetilde{SU}(pAp)$.) □

The following calculations and notations will be used in [Gong et al. 2016].

2.42. In general, for $A = \bigoplus A^i$, we have

$$\widetilde{SU}(A) = \bigoplus_i \widetilde{SU}(A^i).$$

For $A = PM_l(C(X))P \in \mathcal{HD}$, we have $\widetilde{SU}(A) = \widetilde{DU}(A)$, and for $A = M_l(I_k)$, $\widetilde{SU}(A) = \widetilde{DU}(A) \oplus K_1(A)$. For both cases, $U(A)/\widetilde{SU}(A)$ can be identified with $C_1(X, S^1) := C(X, S^1)/\{\text{constant functions}\}$, or in the case $A = M_l(I_k)$, with $C_1([0, 1], S^1) = C([0, 1], S^1)/\{\text{constant functions}\}$.

Furthermore, $C_1(X, S^1)$ can be identified as the set of continuous functions from X to S^1 such that $f(x_0) = 1$ for a certain fixed base point $x_0 \in X$. For $X = [0, 1]$, we choose 0 to be the base point. For $X = S^1$, we choose $1 \in S^1$ to be the base point.

2.43. Let $A = \bigoplus_{i=1}^n A^i \in \mathcal{HD}$, $B = \bigoplus_{j=1}^m B^j \in \mathcal{HD}$. In this subsection we discuss some consequences of the compatibility of the maps between AffT spaces. Let

$$p = \bigoplus p^i < q = \bigoplus q^i \in A \quad \text{and} \quad \bar{p} = \bigoplus_{j=1}^m \bar{p}^j < \bar{q} = \bigoplus_{j=1}^m \bar{q}^j \in B$$

be projections satisfying $\alpha([p]) = [\bar{p}]$ and $\alpha([q]) = [\bar{q}]$. Suppose two unital positive linear maps $\xi_1 : \text{AffT}(pAp) \to \text{AffT}(\bar{p}B\bar{p})$ and $\xi_2 : \text{AffT}(qAq) \to \text{AffT}(\bar{q}B\bar{q})$ are compatible with α (see diagram (2.13)) and compatible with each other (see diagram (2.10)). Since the (not necessarily unital) maps $\text{AffT}(pAp) \to \text{AffT}(qAq)$ and $\text{AffT}(\bar{p}B\bar{p}) \to \text{AffT}(\bar{q}B\bar{q})$ induced by inclusions are injective, we know that the map ξ_1 is completely determined by ξ_2. Let

$$\xi_2^{i,j} : \text{AffT}(q^i A q^i) \to \text{AffT}(\overline{q^j} B^j \overline{q^j}) \quad \text{or} \quad \xi_1^{i,j} : \text{AffT}(p^i A p^i) \to \text{AffT}(\overline{p^j} B^j \overline{p^j}))$$

be the corresponding component of the map ξ_2 (or ξ_1). If $p^i \neq 0$ and $\bar{p}^j \neq 0$, then $\xi_1^{i,j}$ is given by the following formula: for any $f \in \text{AffT}(p^i A^i p^i) = C_{\mathbb{R}}(\text{Sp}(A^i))$ ($\cong \text{AffT}(q^i A q^i)$),

$$\xi_1^{i,j}(f) = \frac{\text{rank } \overline{q_j}}{\text{rank } \overline{p_j}} \cdot \frac{\text{rank } \alpha^{i,j}(p^i)}{\text{rank } \alpha^{i,j}(q^i)} \cdot \xi_2^{i,j}(f).$$

In particular, if $q = \mathbb{1}_A$ with $\bar{q} = \alpha_0[\mathbb{1}_A]$, and $\xi_2 = \xi : \operatorname{AffT}(A) \to \operatorname{Aff} \alpha_0[\mathbb{1}_A] B \alpha_0[\mathbb{1}_A]$ (note that since $\operatorname{AffT}(QBQ)$ only depends on the unitary equivalence class of Q, it is convenient to denote it as $\operatorname{AffT}([Q]B[Q])$), then we denote ξ_1 by $\xi|_{([p],\alpha[p])}$. Even for the general case, we can also write $\xi_1 = \xi_2|_{([p],\alpha[p])}$, when $p < q$ as above.

2.44. As in 2.43, let $A = \bigoplus_{i=1}^n A^i$, $B = \bigoplus_{j=1}^m B^j$ and $p < q \in A$, $\bar{p} < \bar{q} \in B$, with $\alpha_0[p] = [\bar{p}]$ and $\alpha_0[q] = [\bar{q}]$. If

$$\gamma_1 : U(pAp)/\widetilde{SU}(pAp) \to U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p})$$

is compatible with

$$\gamma_2 : U(qAq)/\widetilde{SU}(qAq) \to U(\bar{q}B\bar{q})/\widetilde{SU}(\bar{q}B\bar{q}),$$

then γ_1 is completely determined by γ_2 (since both maps

$$U(pAp)/\widetilde{SU}(pAp) \to U(qAq)/\widetilde{SU}(qAq),$$
$$U(\bar{p}B\bar{p})/\widetilde{SU}(\bar{p}B\bar{p}) \to U(\bar{q}B\bar{q})/\widetilde{SU}(\bar{q}B\bar{q})$$

are injective). Therefore we can denote γ_1 by $\gamma_2|_{([p],\alpha[p])}$.

2.45. Let us point out that, in 2.43 and 2.44, if $A \in A\mathcal{HD}$ and $B \in A\mathcal{HD}$, ξ_1 is not completely determined by ξ_2 and γ_1 is not completely determined by γ_2.

3. The counterexample

3.1. In this section, we present an example of $A\mathbb{T}$ algebras to prove that $\operatorname{Inv}'(A)$ or $\operatorname{Inv}(A)$ is not completely determined by $\operatorname{Inv}^0(A)$. That is, the Hausdorffified algebraic K_1-groups $\{U(pAp)/\overline{DU(pAp)}\}_{p \in \operatorname{proj}(A)}$ or $\{U(pAp)/\widetilde{SU}(pAp)\}_{p \in \operatorname{proj}(A)}$ with the corresponding compatibilities are indispensable as a part of the invariant for $\operatorname{Inv}'(A)$ or $\operatorname{Inv}(A)$. This is one of the essential differences between the simple C^*-algebras and the C^*-algebras with the ideal property. In fact, for all the unital C^*-algebras A satisfying a reasonable condition (e.g., $\rho(\mathcal{P}_1 K_0(A)) = \rho K_0(A)$ and $\overline{DU_0(A)} = \overline{DU(A)}$), we have

$$U(pAp)/\overline{DU(pAp)} \cong \operatorname{AffT}(pAp)/\overline{\rho K_0(pAp)} \oplus K_1(pAp),$$
$$U(pAp)/\widetilde{SU}(pAp) \cong \operatorname{AffT}(pAp)/\overline{\rho \widetilde{K}_0}(pAp) \oplus K_1(pAp)/\operatorname{tor} K_1(pAp),$$

i.e., the metric groups $U(pAp)/\overline{DU(pAp)}$ and $U(pAp)/\widetilde{SU}(pAp)$ themselves are completely determined by $\operatorname{AffT}(pAp)$ and $K_1(pAp)$, which are included in other parts of the invariants, i.e., they are determined by $\operatorname{Inv}^0(A)$, but the compatibilities make the difference. The point is that the above isomorphisms are not natural and therefore the isomorphisms corresponding to the cutting down algebras pAp and qAq ($p < q$) may not be chosen to be compatible.

As pointed out in 2.40, $\mathrm{Inv}'(A) \cong \mathrm{Inv}'(B)$ implies $\mathrm{Inv}(A) \cong \mathrm{Inv}(B)$. For the C^*-algebras A and B constructed in this paper, we only need to prove $\mathrm{Inv}^0(A) \cong \mathrm{Inv}^0(B)$ but $\mathrm{Inv}(A) \ncong \mathrm{Inv}(B)$. Consequently, $\mathrm{Inv}'(A) \ncong \mathrm{Inv}'(B)$.

3.2. Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11, \ldots, p_n$ be the first n prime numbers, and let $1 < k_1 < k_2 < k_3 < \cdots$ be a sequence of positive integers. Let

$$A_1 = B_1 = C(S^1),$$

$$A_2 = B_2 = M_{p_1^{k_1}}(C[0,1]) \oplus M_{p_1^{k_1}}(C(S^1)) = A_1^1 \oplus A_1^2 = B_1^1 \oplus B_1^2,$$

$$A_3 = B_3 = M_{p_1^{k_1} p_1^{k_2}}(C[0,1]) \oplus M_{p_1^{k_1} p_2^{k_2}}(C[0,1]) \oplus M_{p_1^{k_1} p_2^{k_2}}(C(S^1)),$$

$$A_4 = B_4 = M_{p_1^{k_1} p_2^{k_2} p_1^{k_3}}(C[0,1]) \oplus M_{p_1^{k_1} p_2^{k_2} p_2^{k_3}}(C[0,1])$$
$$\oplus M_{p_1^{k_1} p_2^{k_2} p_3^{k_3}}(C[0,1]) \oplus M_{p_1^{k_1} p_2^{k_2} p_3^{k_3}}(C(S^1)).$$

In general, let

$$A_n = B_n = \bigoplus_{i=1}^{n-1} M_{p_1^{k_1} p_2^{k_2} \cdots p_i^{k_i} p_i^{k_{i+1}} \cdots p_i^{k_{n-1}}}(C[0,1]) \oplus M_{p_1^{k_1} p_2^{k_2} \cdots p_{n-1}^{k_{n-1}}}(C(S^1))$$

$$= \bigoplus_{i=1}^{n-1} M_{\prod_{j=1}^{i} p_j^{k_j} \cdot \prod_{j=i+1}^{n-1} p_i^{k_j}}(C[0,1]) \oplus M_{\prod_{i=1}^{n-1} p_i^{k_i}}(C(S^1)).$$

For $1 \le i \le n-1$, let $[n, i] = \prod_{j=1}^{i} p_j^{k_j} \cdot \prod_{j=i+1}^{n-1} p_i^{k_j}$ and $[n, n] = [n, n-1]$. Then

$$A_n = B_n = \bigoplus_{i=1}^{n-1} M_{[n,i]}(C[0,1]) \oplus M_{[n,n]}(C(S^1)).$$

(Note that the last two blocks have the same size $[n, n] = [n, n-1]$.)

Note that $[n+1, i] = [n, i] \cdot p_i^{k_n}$ for all $i \in \{1, 2, \ldots, n-1\}$ and $[n+1, n+1] = [n+1, n] = [n, n] \cdot p_n^{k_n}$.

3.3. Let $\{t_n\}_{n=1}^{\infty}$ be a dense subset of $[0, 1]$ and $\{z_n\}_{n=1}^{\infty}$ be a dense subset of S^1. In this subsection, we define the connecting homomorphisms

$$\phi_{n,n+1} : A_n \to A_{n+1} \quad \text{and} \quad \psi_{n,n+1} : B_n \to B_{n+1}.$$

For $i \le n-1$, define $\phi_{n,n+1}^{i,i} = \psi_{n,n+1}^{i,i} : M_{[n,i]}(C[0,1]) \to M_{[n+1,i]}(C[0,1])$ $(= M_{[n,i] \cdot p_i^{k_n}}(C[0,1]))$ by

$$\phi_{n,n+1}^{i,i}(f)(t) = \psi_{n,n+1}^{i,i}(f)(t)$$
$$= \mathrm{diag}(\underbrace{f(t), f(t), \ldots, f(t)}_{p_i^{k_n}-1}, f(t_n)) \quad \text{for all } f \in M_{[n,i]}(C[0,1]).$$

Define $\phi_{n,n+1}^{n,n+1} = \psi_{n,n+1}^{n,n+1} : M_{[n,n]}(C(S^1)) \to M_{[n+1,n+1]}(C(S^1)) = M_{[n,n]\cdot p_n^{k_n}}(C(S^1))$
by

$$\phi_{n,n+1}^{n,n+1}(f)(z) = \psi_{n,n+1}^{n,n+1}(f)(z)$$
$$= \operatorname{diag}(f(z), \underbrace{f(z_n), f(z_n), \ldots, f(z_n)}_{p_n^{k_n}-1}) \quad \text{for all } f \in M_{[n,n]}(C(S^1)).$$

But $\phi_{n,n+1}^{n,n}$ and $\psi_{n,n+1}^{n,n}$ are defined differently — this is the only nonequal component of $\phi_{n,n+1}$ and $\psi_{n,n+1}$.

Let $l = p_n^{k_n} - 1$. Then

$$\phi_{n,n+1}^{n,n}(f)(t) = \operatorname{diag}\big(f(e^{2\pi it}), f(e^{-2\pi it}), f(e^{2\pi i/l}), \ldots, f(e^{2\pi i(l-1)/l})\big),$$
$$\psi_{n,n+1}^{n,n}(f)(t) = \operatorname{diag}\big(f(e^{2\pi il_n t}), f(e^{-2\pi i\cdot 0/l}), f(e^{2\pi i/l}), \ldots, f(e^{2\pi i(l-1)/l})\big)$$

for any $f \in M_{[n,n]}(C(S^1))$, where $l_n = 4^n \cdot [n+1,n] \in \mathbb{N}$.

Let all other parts $\phi_{n,n+1}^{i,j}$, $\psi_{n,n+1}^{i,j}$ of $\phi_{n,n+1}, \psi_{n,n+1}$ (except $i = j \le n$ or $i = n$, $j = n+1$, as defined above) be zero. Note that all $\phi_{n,n+1}^{i,j}$, $\psi_{n,n+1}^{i,j}$ are either injective or zero.

Let $A = \lim(A_n, \phi_{n,m})$, $B = \lim(B_n, \psi_{n,m})$. Then it follows from the density of the sets $\{t_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ that both A and B have the ideal property (see the characterization theorem for AH algebras with the ideal property [Pasnicu 2000]).

Proposition 3.4. *There is an isomorphism between* $\operatorname{Inv}^0(A)$ *and* $\operatorname{Inv}^0(B)$ *(see* 2.11*), that is, there is an isomorphism*

$$\alpha : (\underline{K}(A); \underline{K}(A)^+; \Sigma A) \to (\underline{K}(B); \underline{K}(B)^+; \Sigma B)$$

which is compatible with Bockstein operations, and for pairs (p,q) with $p \in \Sigma A$, $q \in \Sigma B$ and $\alpha([p]) = [q]$, there are associated unital positive linear maps

$$\xi^{p,q} : \operatorname{AffT}(pAp) \to \operatorname{AffT}(qBq)$$

which are compatible in the sense of 2.9 *(see diagram* (2.10)*).*

Proof. As $KK(\phi_{n,m}) = KK(\psi_{n,m})$ and $\phi_{n,m} \sim_h \psi_{n,m}$, the identity maps $\eta_n : A_n \to B_n$ induce a shape equivalence between $A = \lim(A_n, \phi_{n,m})$ and $B = \lim(B_n, \psi_{n,m})$, and therefore induce an isomorphism

$$\alpha : (\underline{K}(A); \underline{K}(A)^+; \Sigma A) \to (\underline{K}(B); \underline{K}(B)^+; \Sigma B).$$

Note that $\phi_{n,n+1}^{i,i} = \psi_{n,n+1}^{i,i}$ for $i \le n-1$, $\phi_{n,n+1}^{n,n+1} = \psi_{n,n+1}^{n,n+1}$, and

$$\left\| \operatorname{AffT} \phi_{n,n+1}^{n,n}(f) - \operatorname{AffT} \psi_{n,n+1}^{n,n}(f) \right\| \le \frac{2}{p_n^{k_n}} \|f\|$$

(see the definition of $\phi_{n,n+1}$ and $\psi_{n,n+1}$). Therefore,

$$\text{AffT}\,\eta_n : \text{AffT}(A_n) \to \text{AffT}(B_n) \quad \text{and} \quad \text{AffT}\,\eta_n^{-1} : \text{AffT}(B_n) \to \text{AffT}(A_n)$$

induce the approximately intertwining diagram

$$
\begin{array}{ccccccc}
\text{AffT}(A_1) & \longrightarrow & \text{AffT}(A_2) & \longrightarrow & \cdots & \longrightarrow & \text{AffT}(A) \\
\updownarrow & & \updownarrow & & & & \\
\text{AffT}(B_1) & \longrightarrow & \text{AffT}(B_2) & \longrightarrow & \cdots & \longrightarrow & \text{AffT}(B)
\end{array}
$$

in the sense of [Elliott 1993b]. Therefore, there is a unital positive isomorphism

$$\xi : \text{AffT}(A) \to \text{AffT}(B).$$

Also, for any projection $[P] \in K_0(A)$, there is a projection $P_n \in A_n = B_n$ (for n large enough) with $P_n^i = \text{diag}(1, \ldots, 1, 0, \ldots, 0) \in M_{[n,i]}(C(X_{n,i}))$, where $X_{n,i} = [0, 1]$ for $i \leq n - 1$, and $X_{n,n} = S^1$, such that $\phi_{n,\infty}([P_n]) = [P] \in K_0(A)$. Note that for any constant functions $f \in A_n^i = B_n^i$ (e.g., P_n^i above) and for any j, $\phi_{n,n+1}^{i,j}(f)$ and $\psi_{n,n+1}^{i,j}(f)$ are still constant functions, and $\phi_{n,n+1}^{i,j}(f) = \psi_{n,n+1}^{i,j}(f)$. That is, we have

$$\phi_{n,n+1}(P_n) = \psi_{n,n+1}(P_n) \quad (\text{denoted by } P_{n+1}),$$

$$\phi_{n,m}(P_n) = \psi_{n,m}(P_n) \quad (\text{denoted by } P_m).$$

Let $P_\infty = \phi_{n,\infty}(P_n)$ and $Q_\infty = \psi_{n,\infty}(P_n)$. Then the identity maps $\{\eta_m\}_{m>n}$ also induce the approximate intertwining diagram

$$
\begin{array}{ccccccc}
\text{AffT}(P_n A_n P_n) & \longrightarrow & \text{AffT}(P_{n+1} A_{n+1} P_{n+1}) & \longrightarrow & \cdots & \longrightarrow & \text{AffT}(P_\infty A P_\infty) \\
\updownarrow & & \updownarrow & & & & \\
\text{AffT}(P_n B_n P_n) & \longrightarrow & \text{AffT}(P_{n+1} B_{n+1} P_{n+1}) & \longrightarrow & \cdots & \longrightarrow & \text{AffT}(Q_\infty B Q_\infty)
\end{array}
$$

and hence induce a positive linear isomorphism

$$\xi^{[P],\alpha[P]} : \text{AffT}(P_\infty A P_\infty) \to \text{AffT}(Q_\infty B Q_\infty).$$

(Note that $[P_\infty] = [P]$ and $[Q_\infty] = \alpha[P]$ in $K_0(A)$ and $K_0(B)$, respectively.) Evidently those maps are compatible since, they are induced by the same sequence of homomorphisms $\{\eta_n\}$ and $\{\eta_n^{-1}\}$. $\qquad\qquad\square$

Definition 3.5 and Proposition 3.6 are inspired by [Elliott 1997].

Definition 3.5. Let $C = \lim(C_n, \phi_{n,m})$ be an $A\mathcal{H}\mathcal{D}$ inductive limit. We say the system $(C_n, \phi_{n,m})$ has the uniformly varied determinant if for any $C_n^i = M_{[n,i]}(C(S^1))$

(that is, C_n^i has spectrum S^1), C_{n+1}^j, and $f \in C_n^i$ defined by

$$f(z) = \begin{pmatrix} z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{[n,i] \times [n,i]} \quad \text{for all } z \in S^1,$$

we have either that $\det(\phi_{n,n+1}^{i,j}(f)(x))$ is constant for $x \in \mathrm{Sp}(C_{n+1}^j) \neq S^1$ or that $\det(\phi_{n,n+1}^{i,j}(f)(z)) = \lambda z^k$ ($\lambda \in \mathbb{C}$) for $z \in \mathrm{Sp}(C_{n+1}^j) = S^1$, where j satisfies $\phi_{n,n+1}^{i,j} \neq 0$ and the determinant is taken inside $\phi_{n,n+1}^{i,j}(\mathbb{1}_{C_n^i}) C_{n+1}^j \phi_{n,n+1}^{i,j}(\mathbb{1}_{C_n^i})$.

Proposition 3.6. *If the inductive limit system $C = (C_n, \phi_{n,m})$ has the uniformly varied determinant, then for any elements $[p] \in \sum C$, there are splitting maps*

$$K_1(pCp)/\operatorname{tor} K_1(pCp) \xrightarrow{S_{pCp}} U(pCp)/\widetilde{SU}(pCp)$$

of the exact sequences

$$0 \to \mathrm{AffT}(pCp)/\widetilde{\rho K_0}(pCp) \to U(pCp)/\widetilde{SU}(pCp)$$
$$\xrightarrow{\pi_{pCp}} K_1(pCp)/\operatorname{tor} K_1(pCp) \to 0$$

(that is, $\pi_{pCp} \circ S_{pCp} = \mathrm{id}$ on $K_1(pCp)/\operatorname{tor} K_1(pCp)$) such that the system of maps $\{S_{pCp}\}_{[p] \in \sum C}$ are compatible in the following sense: if $p < q$, then the diagram

$$\begin{array}{ccc} K_1(pCp)/\operatorname{tor} K_1(pCp) & \xrightarrow{S_{pCp}} & U(pCp)/\widetilde{SU}(pCp) \\ \downarrow & & \downarrow \\ K_1(qCq)/\operatorname{tor} K_1(qCq) & \xrightarrow{S_{qCq}} & U(qCq)/\widetilde{SU}(qCq) \end{array} \qquad (3.7)$$

commutes, where the vertical maps are induced by the inclusions $pCp \to qCq$.

Proof. Fix $p \in C$. Let $x \in K_1(pCp)/\operatorname{tor} K_1(pCp)$. There exist a C_n and $p_n \in C_n$ such that $[\phi_{n,\infty}(p_n)] = [p] \in K_0(C)$. Without lose of generality, we can assume $\phi_{n,\infty}(p_n) = p$. By increasing n if necessary, we can assume that there is an element $x_n \in K_1(p_n C_n p_n)/\operatorname{tor} K_1(p_n C_n p_n)$ such that

$$(\phi_{n,\infty})_*(x_n) = x \in K_1(pCp)/\operatorname{tor} K_1(pCp).$$

Write $p_n C_n p_n = D = \bigoplus D^i$. Let $I = \{i : \mathrm{Sp}(D^i) = S^1\}$. For $i \in I$, D^i can be identified with $M_{l_i}(C(S^1))$. Let $u_i \in D^i$ be defined by

$$u_i(z) = \begin{pmatrix} z & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{l_i \times l_i} \quad \text{for all } z \in S^1,$$

which represents the standard generator of $K_1(D^i)$. Then x_n can be represented by

$$u = \bigoplus_{i \in I} u_i^{k_i} \oplus \bigoplus_{j \notin I} \mathbb{1}_{D^j} \in \bigoplus_{i \in I} D^i \oplus \bigoplus_{j \notin I} D^j = D \subseteq p_n C_n p_n.$$

Define $S(x) = [\phi_{n,\infty}(u)] \in U(pCp)/\widetilde{SU}(pCp)$. Note that all unitaries with constant determinants are in $\widetilde{SU}$, and that the inductive system has the uniformly varied determinant. It is routine to verify that $S(x)$ is well defined and the system $\{S_{pCp}\}_{[p] \in \sum C}$ makes the diagram (3.7) commute. $\qquad\square$

3.8. Let $\mathcal{A}$ be a unital C^*-algebra. Then $\mathrm{AffT}(\mathcal{A})$ is a real Banach space with quotient space $\mathrm{AffT}(\mathcal{A})/\rho\widetilde{K}_0(\mathcal{A})$. Let us use $\|\cdot\|^\sim$ to denote the quotient norm. Note that $\tilde{\lambda}_{\mathcal{A}}$ identifies $U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ with $\mathrm{AffT}(\mathcal{A})/\rho\widetilde{K}_0(\mathcal{A})$. Thus, $U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ is regarded as a real Banach space, whose norm is also denoted by $\|\cdot\|^\sim$. In general, we have

$$U(\mathcal{A})/\widetilde{SU}(\mathcal{A}) \cong U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A}) \times K_1(\mathcal{A})/\mathrm{tor}\,K_1(\mathcal{A}),$$

but the identification is not canonical. Even though $U(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ is not a Banach space, it is an abelian group: for $[u], [v] \in U(\mathcal{A})/\widetilde{SU}(\mathcal{A})$, define $[u] - [v] = [uv^*]$.

The norm $\|\cdot\|^\sim$ is related to the metrics $\tilde{d}_{\mathcal{A}}$ (on $\mathrm{AffT}(\mathcal{A})/\rho\widetilde{K}_0(\mathcal{A})$; see 2.28) and $\widetilde{D}_{\mathcal{A}}$ (on $U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$; see 2.33) as below. Let $\varepsilon < 1$. For $f, g \in \mathrm{AffT}(\mathcal{A})/\rho\widetilde{K}_0(\mathcal{A})$,

$$\|f - g\|^\sim < \frac{\varepsilon}{2\pi} \quad \Longrightarrow \quad \tilde{d}_{\mathcal{A}}(f, g) < \varepsilon \quad \Longrightarrow \quad \|f - g\|^\sim < \frac{\varepsilon}{4}.$$

And for any $[u], [v] \in U(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ with $[u] - [v] = [uv^*] \in U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$,

$$\left\|[u] - [v]\right\|^\sim < \frac{\varepsilon}{2\pi} \quad \Longrightarrow \quad \widetilde{D}_{\mathcal{A}}([u], [v]) < \varepsilon \quad \Longrightarrow \quad \left\|[u] - [v]\right\|^\sim < \frac{\varepsilon}{4}.$$

For $\mathcal{A} = PM_l(C(X))P \in \mathcal{HD}$ or $\mathcal{A} = M_l(I_k)$ (in this case we also denote $[0, 1]$ by X), there are canonical identifications

$$U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A}) \cong \mathrm{AffT}(\mathcal{A})/\rho\widetilde{K}_0(\mathcal{A}) \cong C(X, \mathbb{R})/\{\text{constant functions}\}$$

(see 2.42). Choose a base point $x_0 \in X$. Let $C_{x_0}(X, \mathbb{R})$ be the set of functions $f \in C(X, \mathbb{R})$ with $f(x_0) = 0$. Then $C(X, \mathbb{R})/\{\text{constant functions}\} \cong C_{x_0}(X, \mathbb{R})$. For $[f] \in \mathrm{AffT}(\mathcal{A})/\rho\widetilde{K}_0(\mathcal{A})$ (or $[f] \in U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$) identified with a function $f \in C_{x_0}(X, \mathbb{R})$, we have

$$\|[f]\|^\sim = \frac{1}{2}\left(\max_{x \in X}(f(x)) - \min_{x \in X}(f(x))\right)$$

(rather than $\sup_{x \in X}\{|f(x)|\}$).

In the above case, if $p \in \mathcal{A}$ is a nonzero projection, then $U_{\mathrm{tor}}(p\mathcal{A}p)/\widetilde{SU}(p\mathcal{A}p) \cong \mathrm{AffT}(p\mathcal{A}p)/\rho\widetilde{K}_0(p\mathcal{A}p)$ is also identified with $C_{x_0}(X, \mathbb{R})$. Consider the inclusion map $\iota : p\mathcal{A}p \to \mathcal{A}$. Then the map ι_* as a map from $U_{\mathrm{tor}}(p\mathcal{A}p)/\widetilde{SU}(p\mathcal{A}p) \cong$

$\mathrm{AffT}(p\mathcal{A}p)/\widetilde{\rho K_0}(p\mathcal{A}p)$ to $U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ can be described as follows: if

$$u \in U_{\mathrm{tor}}(p\mathcal{A}p)/\widetilde{SU}(p\mathcal{A}p) \cong \mathrm{AffT}(p\mathcal{A}p)/\widetilde{\rho K_0}(p\mathcal{A}p)$$

is identified with $f \in C_{x_0}(X, \mathbb{R})$, then $\iota_*(u) \in U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$ is identified with

$$\frac{\mathrm{rank}(p)}{\mathrm{rank}(\mathbb{1}_A)} f.$$

But $\iota^\natural$ is the identity map from $U_{\mathrm{tor}}(p\mathcal{A}p)/\widetilde{SU}(p\mathcal{A}p) \cong \mathrm{AffT}(p\mathcal{A}p)/\widetilde{\rho K_0}(p\mathcal{A}p)$ to itself (not to $U_{\mathrm{tor}}(\mathcal{A})/\widetilde{SU}(\mathcal{A})$).

3.9. It is easy to see that $K_1(A) = K_1(B) = \mathbb{Z}$.

In the definition of $A_n = \bigoplus_{i=1}^n A_n^i$, only one block $A_n^n = M_{[n,n]}(C(S^1))$ has spectrum S^1, and only two partial maps $\phi_{n,n+1}^{n,j}$ for $j = n$, $j = n+1$ (of $\phi_{n,n+1}$ from A_n^n) are nonzero. Let $f \in A_n^n$ be defined as in Definition 3.5. Then $\det(\phi_{n,n+1}^{n,n+1}(f)(z)) = z$ and $\det(\phi_{n,n+1}^{n,n}(f)(t)) = e^{2\pi i t} e^{-2\pi i t} e^{2\pi i/l} e^{2\pi i(2/l)} \cdots e^{2\pi i(l-1)/l} = \pm 1$ (see 3.3). So the inductive limit system $(A_n, \phi_{n,m})$ has the uniformly varied determinant, and therefore the limit algebra A has compatible splitting maps $S_p : K_1(pAp) \to U(pAp)/\widetilde{SU}(pAp)$.

We prove that $B = \lim(B_n, \psi_{n,m})$ does not have such a compatible system of splitting maps $\{K_1(pBp) \to U(pBp)/\widetilde{SU}(pBp)\}_{[p] \in \sum B}$.

Before proving the above fact, let us describe the K_0-group of A and B. Let

$$G_1 = \left\{ \frac{m}{p_1^l} : m \in \mathbb{Z}, l \in \mathbb{Z}_+ \right\},$$

$$G_2 = \left\{ \frac{m}{p_1^{k_1} p_2^l} : m \in \mathbb{Z}, l \in \mathbb{Z}_+ \right\},$$

$$G_3 = \left\{ \frac{m}{p_1^{k_1} p_2^{k_2} p_3^l} : m \in \mathbb{Z}, l \in \mathbb{Z}_+ \right\},$$

$$\vdots$$

$$G_n = \left\{ \frac{m}{p_1^{k_1} p_2^{k_2} \cdots p_{n-1}^{k_{n-1}} p_n^l} : m \in \mathbb{Z}, l \in \mathbb{Z}_+ \right\},$$

$$G_\infty = \left\{ \frac{m}{p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}} : t \in \mathbb{Z}_+, m \in \mathbb{Z} \right\},$$

where $p_1 = 2, p_2 = 3, \ldots, p_i, \ldots$ and $k_1, k_2, \ldots, k_i \ldots$ are defined in 3.2. Then

$$K_0(A) = K_0(B)$$
$$= \left\{ (a_1, a_2, \ldots, a_n, \ldots) \in \prod_{n=1}^{\infty} G_n : \exists N \text{ such that } a_N = a_{N+1} = \cdots \in \mathbb{Q} \right\}$$
$$\triangleq \widetilde{G}.$$

Furthermore, their positive cones consist of the elements whose coordinates are nonnegative, and their order units are $[\mathbb{1}_A] = [\mathbb{1}_B] = (1, 1, \ldots, 1, \ldots) \in \prod_{n=1}^{\infty} G_n$. Let

$$\alpha_0 : (K_0(A), K_0(A)^+, [\mathbb{1}_A]) = \big(\widetilde{G}, \widetilde{G}^+, (1, 1, \ldots, 1, \ldots)\big)$$
$$\to (K_0(B), K_0(B)^+, [\mathbb{1}_B]) = \big(\widetilde{G}, \widetilde{G}^+, (1, 1, \ldots, 1, \ldots)\big)$$

be a scaled ordered isomorphism. Then $\alpha_0((1, 1, \ldots, 1, \ldots)) = (1, 1, \ldots, 1, \ldots)$. Note that an element $x \in \widetilde{G}$ is divisible by power p_1^n (for any n) of the first prime number $p_1 = 2$ if and only if $x = (t, 0, 0, \ldots, 0, \ldots) \in G_1 \subset \widetilde{G}$. Hence $\alpha_0((1, 0, 0, \ldots, 0, \ldots)) = (t, 0, 0, \ldots, 0, \ldots)$ for some $t \in G_1$ with $t > 0$. Hence

$$\alpha_0((0, 1, 1, \ldots, 1, \ldots)) = (1 - t, 1, 1, \ldots, 1, \ldots).$$

Since α_0 preserves the positive cone, we have $1 - t \geq 0$, which implies $t \leq 1$. On the other hand, $(\alpha_0)^{-1}$ takes $(1, 0, 0, \ldots, 0, \ldots)$ to $(1/t, 0, 0, \ldots, 0, \ldots)$. But $(\alpha_0)^{-1}$ also preserves the positive cone. Symmetrically, we get $t \geq 1$. That is, $\alpha_0((1, 0, 0, \ldots, 0, \ldots)) = (1, 0, 0, \ldots, 0, \ldots)$. Similarly, using the fact that G_k is the subgroup of all elements in $\widetilde{G}$ which can be divisible by any power of p_k — the k-th prime number, we can prove that

$$\alpha_0\big((\underbrace{0, \ldots, 0}_{k-1}, 1, 0, \ldots, 0, \ldots)\big) = \big(\underbrace{0, \ldots, 0}_{k-1}, 1, 0, \ldots, 0, \ldots\big) \in G_k \subset \widetilde{G}.$$

That is, α_0 is the identity on $\widetilde{G}$.

Note that $\mathrm{Sp}(A) = \mathrm{Sp}(B)$ is the one point compactification of $\{1, 2, 3 \ldots\}$ — or, in other words, $\{1, 2, 3 \ldots, \infty\}$. If we let I_n (or J_n) be the primitive ideal A (or B) corresponding to n (including $n = \infty$), then

$$K_0(A/I_n) = K_0(B/J_n) = G_n.$$

Note also that if $m' > m > n \in \mathbb{N}$, then $\phi_{m,m'}(A_m^n) \subset A_{m'}^n$ and $\psi_{m,m'}(B_m^n) \subset B_{m'}^n$. Hence $A/I_n = \lim_{n < m \to \infty}(A_m^n, \phi_{m,m'}|_{A_m^n})$ (resp. $B/J_n = \lim_{n < m \to \infty}(B_m^n, \psi_{m,m'}|_{B_m^n})$) are ideals of A (resp. B). But A/I_∞ (or B/J_∞) is not an ideal of A (or B).

Let $\alpha : (\underline{K}(A), \underline{K}(A)^+, \Sigma A) \to (\underline{K}(B), \underline{K}(B)^+, \Sigma B)$ be an isomorphism. By 3.9 the induced map α_0 on K_0 group is identity, when both $K_0(A)$ and $K_0(B)$ are identified with $\widetilde{G}$ as scaled ordered groups. That is, α_0 is the same as the α_0 induced by the shape equivalence in the proof of Proposition 3.4. In particular, if there is an isomorphism $\wedge : A \to B$, then for all $i \leq n-1$, $\wedge_*[(\phi_{n,\infty}(\mathbb{1}_{A_n^i}))] = [\psi_{n,\infty}(\mathbb{1}_{B_n^i})]$. This implies $\wedge(\phi_{n,\infty}(\mathbb{1}_{A_n^i})) = \psi_{n,\infty}(\mathbb{1}_{B_n^i})$, since $\psi_{n,\infty}(\mathbb{1}_{B_n^i}) = \mathbb{1}_{B/I_i}$, which is in the center of B (any element in the center of the C^*-algebra can only unitary equivalent to itself). Hence it is also true that $\wedge(\phi_{n,\infty}(\mathbb{1}_{A_n^i})) = \psi_{n,\infty}(\mathbb{1}_{B_n^i})$ for $i = n$.

3.10. Let $P_1 = \mathbb{1}_B = \psi_{1,\infty}(\mathbb{1}_{B_1})$ and $P_n = \psi_{n,\infty}(\mathbb{1}_{B_n^n})$ for $n > 1$. Then we have $P_1 > P_2 > \cdots > P_n > \cdots$. We prove that there are no splittings

$$K_1(P_n B P_n) \to U(P_n B P_n)/\widetilde{SU}(P_n B P_n)$$

which are compatible for all pairs of projections $P_n > P_m$ (see diagram (3.7)) in the next subsection. Before doing so, we need some preparations.

Set $Q_1 = P_1 - P_2$, $Q_2 = P_2 - P_3, \ldots, Q_n = P_n - P_{n+1}$. Then for each n, we have the inductive limit

$$Q_n B Q_n = \lim_{m \to \infty} (B_m^n, \psi_{m,m'}^{n,n})$$

(note that for $m > n$, $\psi_{m,m+1}^{n,j} = 0$ if $j \neq n$), which is the quotient algebra corresponding to the primitive ideal of $n \in \mathrm{Sp}(B) = \{1, 2, 3 \ldots, \infty\}$. Note that $Q_n B Q_n$ is a simple AI algebra. The inductive limit of the C^*-algebras

$$B_{n+1}^n \to B_{n+2}^n \to B_{n+3}^n \to \cdots \to Q_n B Q_n$$

induces the inductive limit of the ordered Banach spaces

$$\mathrm{AffT}(B_{n+1}^n) \xrightarrow{\xi_{n+1,n+2}} \mathrm{AffT}(B_{n+2}^n) \xrightarrow{\xi_{n+2,n+3}} \cdots \to \mathrm{AffT}(Q_n B Q_n),$$

whose connecting maps $\xi_{m,m+1} : C_{\mathbb{R}}([0, 1]) \to C_{\mathbb{R}}([0, 1])$ (for $m > n$) satisfy

$$\|\xi_{m,m+1}(f) - f\| \leq \frac{1}{p_n^{k_m}} \|f\| \quad \text{for all } f \in C_{\mathbb{R}}[0, 1], \ m > n.$$

Hence we have the following approximate intertwining diagram:

$$
\begin{array}{ccccccc}
C_{\mathbb{R}}[0, 1] & \xrightarrow{\xi_{n,n+1}} & C_{\mathbb{R}}[0, 1] & \xrightarrow{\xi_{n+1,n+2}} & C_{\mathbb{R}}[0, 1] & \longrightarrow \cdots \longrightarrow & \mathrm{AffT}(Q_n B Q_n) \\
\Updownarrow\Uparrow & & \Updownarrow\Uparrow & & \Updownarrow\Uparrow & & \\
C_{\mathbb{R}}[0, 1] & \xrightarrow{\ \mathrm{id}\ } & C_{\mathbb{R}}[0, 1] & \xrightarrow{\ \mathrm{id}\ } & C_{\mathbb{R}}[0, 1] & \longrightarrow \cdots \longrightarrow & C_{\mathbb{R}}[0, 1]
\end{array}
$$

Consequently, $\mathrm{AffT}(Q_n B Q_n) \cong C_{\mathbb{R}}[0, 1]$, and the maps

$$\xi_{m,\infty} : \mathrm{AffT}(B_m^n) = C_{\mathbb{R}}[0, 1] \to \mathrm{AffT}(Q_n B Q_n) \cong C_{\mathbb{R}}[0, 1]$$

(under the identification) satisfy

$$\|\xi_{m,\infty}(f) - f\| \leq \left(\frac{1}{p_n^{k_m}} + \frac{1}{p_n^{k_{m+1}}} + \cdots \right) \|f\| \leq \frac{1}{4} \|f\| \quad \text{for all } f \in C_{\mathbb{R}}[0, 1].$$

Therefore $\|\xi_{m,\infty}(f)\| \geq \frac{3}{4} \|f\|$.

Note that $\rho \widetilde{K}_0(Q_n B Q_n) = \mathbb{R} = \rho \widetilde{K}_0(B_m^n)$ consists of constant functions on $[0, 1]$. Take an element $h \in C_{\mathbb{R}}[0, 1] = \mathrm{AffT}(B_m^n)$. Considering $\xi_{m,\infty}(h)$ as an element of

$\mathrm{AffT}(Q_n B Q_n)/\rho \widetilde{K}_0(Q_n B Q_n)$, we have

$$\|\xi_{m,\infty}(h)\|^{\sim} \geq \frac{1}{2} \cdot \frac{3}{4}\Big(\max_{t \in [0,1]} h(t) - \min_{t \in [0,1]} h(t)\Big),$$

where $\|\cdot\|^{\sim}$ is defined in 3.8.

3.11. We now prove that no compatible splittings

$$S_n : K_1(P_n B P_n) \to U(P_n B P_n)/\widetilde{SU}(P_n B P_n)$$

exist. Suppose such splittings exist. Then consider the generator $x \in K_1(B) = \mathbb{Z}$.

Note that $x \in K_1(P_n B P_n) \cong K_1(B)$ for all P_n. Note also that the diagram

$$
\begin{array}{ccc}
K_1(P_{n+1} B P_{n+1}) & \xrightarrow{S_{n+1}} & U(P_{n+1} B P_{n+1})/\widetilde{SU}(P_{n+1} B P_{n+1}) \\
\mathrm{id} \downarrow & & \iota_* \downarrow \\
K_1(P_1 B P_1) & \xrightarrow{\quad S_1 \quad} & U(P_1 B P_1)/\widetilde{SU}(P_1 B P_1)
\end{array}
$$

commutes ($P_1 B P_1 = B$). The composition

$$U(P_{n+1} B P_{n+1})/\widetilde{SU}(P_{n+1} B P_{n+1}) \xrightarrow{\iota_*} U(P_1 B P_1)/\widetilde{SU}(P_1 B P_1)$$

$$\to \bigoplus_{i=1}^{n} U(Q_i B Q_i)/\widetilde{SU}(Q_i B Q_i)$$

is the zero map. (Note that $Q_i B Q_i$ is an ideal of B and is also the quotient B/J_i.) Consequently, we have

$$\pi_n^{\natural}(S_1(x)) = \pi_n^{\natural}(\iota_* S_{n+1}(x)) = 0, \tag{$*$}$$

where $\pi_n : B \to Q_n B Q_n$ is the quotient map. Let $S_1(x)$ be represented by a unitary $u \in U(B)$. Then there are an n (large enough) and $[u_n] \in U(B_n)/\widetilde{SU}(B_n)$, represented by unitary $u_n \in B_n$, such that

$$\psi_{n,\infty}^{\natural}([u_n]) - S_1(x) \in U_{\mathrm{tor}}(B_n)/\widetilde{SU}(B_n) \quad \text{and} \quad \|\psi_{n,\infty}^{\natural}([u_n]) - S_1(x)\|^{\sim} < \tfrac{1}{16}.$$

Note that

$$(\psi_{n,m})_* : K_1(B_n) \to K_1(B_m)$$

is the identify map from $\mathbb{Z}$ to $\mathbb{Z}$. Let $g \in M_{[n,n]}(C(S^1)) = B_n^n$ be defined by

$$g(z) = \begin{pmatrix} z & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}_{[n,n] \times [n,n]}$$

Then $[g^{-1}u_n] = 0$ in $K_1(B_n)$. By the exactness of the sequence

$$0 \to \mathrm{AffT}(B_n)/\rho \widetilde{K}_0(B_n) \to U(B_n)/\widetilde{SU}(B_n) \to K_1(B_1) \to 0,$$

there is an $h \in \bigoplus_{i=1}^{n} C_{\mathbb{R}}[0,1] \oplus C_{\mathbb{R}}(S^1) = \mathrm{AffT}(B_n)$ such that

$$[u_n] = [g] \cdot (e^{2\pi ih} \cdot \mathbb{1}_{B_n}) \in U(B_n)/\widetilde{SU}(B_n).$$

Let $\|h\| = M$. Choose $m > n$ such that $4^{m-1} > 8M + 8$.

Consider

$$\psi_{n,m}^{n,m-1} : B_n^n = M_{[n,n]}(C(S^1)) \to B_m^{m-1} = M_{[m,m-1]}(C([0,1])),$$

which is the composition

$$\psi_{m-1,m}^{m-1,m-1} \circ \psi_{n,m-1}^{n,m-1} : M_{[n,n]}(C(S^1)) \to M_{[m-1,m-1]}(C(S^1)) \to M_{[m,m-1]}(C([0,1])).$$

Let $g' = \psi_{n,m}^{n,m-1}(g)$. We know that

$$g'(t) = \psi_{n,m}^{n,m-1}(g)(t) = \begin{pmatrix} e^{2\pi il_{m-1}t} & & & & \\ & * & & & \\ & & * & & \\ & & & \ddots & \\ & & & & * \end{pmatrix}_{[m,m-1]\times[m,m-1]}$$

where the $*$'s represent constant functions on $[0,1]$, and therefore

$$g' = e^{2\pi ih'} \quad (\mathrm{mod}\ \widetilde{SU}(B_m^{m-1}))$$

with $h'(t) = \dfrac{l_{m-1}}{[m, m-1]} \cdot t \cdot \mathbb{1}_{[m,m-1]}$. When we identify $U(B_m^{m-1})/\widetilde{SU}(B_m^{m-1})$ with

$$\mathrm{AffT}(B_m^{m-1})/\rho \widetilde{K}_0(B_m^{m-1}) = C_{\mathbb{R}}[0,1]/\{\text{constants}\},$$

g' is identified with $\tilde{h} \in C_{\mathbb{R}}[0,1]$, where

$$\tilde{h}(t) = \frac{l_{m-1}}{[m, m-1]}t.$$

Since $\dfrac{l_{m-1}}{[m, m-1]} \geq 8M + 8$, we have

$$\|\tilde{h}\|^{\sim} = \frac{1}{2}\Big(\max_{t\in[0,1]} \tilde{h}(t) - \min_{t\in[0,1]} \tilde{h}(t)\Big) \geq 4M + 4$$

(see 3.8). On the other hand,

$$[u_n] = [g] + \tilde{\lambda}_{B_n}([h]) \in U(B_n)/\widetilde{SU}(B_n),$$

where $[h] \in \mathrm{AffT}(B_n)/\widetilde{\rho K_0}(B_n)$ is the element defined by h, and

$$\tilde{\lambda}_{B_n} : \mathrm{AffT}(B_n)/\widetilde{\rho K_0}(B_n) \to U(B_n)/\widetilde{SU}(B_n)$$

is the map defined in 2.33 (also see 2.28). Consequently,

$$\begin{aligned}
(\psi_{n,m}^{n,m-1})^{\natural}(u) &= \mathrm{AffT}\,\psi_{n,m}^{n,m-1}(h) + \tilde{h} \\
&\triangleq \tilde{\tilde{h}} \in \mathrm{AffT}(B_m^{m-1})/\widetilde{\rho K_0}(B_m^{m-1}) \cong U(B_m^{m-1})/\widetilde{SU}(B_m^{m-1})
\end{aligned}$$

with

$$\|\tilde{\tilde{h}}\|^{\sim} = \frac{1}{2}\Big(\max_{t\in[0,1]} \tilde{\tilde{h}}(t) - \min_{t\in[0,1]} \tilde{\tilde{h}}(t)\Big) \geq 4,$$

since $\|h\| \leq M$. Therefore,

$$\begin{aligned}
(\pi_{m-1} \circ \psi_{n,\infty})^{\natural}(u) &\in U(Q_{m-1}BQ_{m-1})/\widetilde{SU}(Q_{m-1}BQ_{m-1}) \\
&\cong \mathrm{AffT}(Q_{m-1}BQ_{m-1})/\widetilde{\rho K_0}(Q_{m-1}BQ_{m-1}),
\end{aligned}$$

satisfies

$$\begin{aligned}
&\|(\pi_{m-1} \circ \psi_{n,\infty})^{\natural}(u)\|^{\sim} \\
&= \frac{1}{2}\Big(\max_{t\in[0,1]} (\pi_{m-1} \circ \psi_{n,\infty})^{\natural}(u)(t) - \min_{t\in[0,1]} (\pi_{m-1} \circ \psi_{n,\infty})^{\natural}(u)(t)\Big) \geq \frac{3}{4}\cdot 4 = 3,
\end{aligned}$$

where $\pi_{m-1} : B \to Q_{m-1}BQ_{m-1}$ is the quotient map. On the other hand,

$$\pi_{m-1}^{\natural}(S_1(x)) = 0$$

as calculated in (∗). Recall that

$$\|(\psi_{n,\infty})^{\natural}(u) - S_1(x)\|^{\sim} < \frac{1}{16}.$$

We get

$$\|(\pi_{m-1} \circ \psi_{n,\infty})^{\natural}(u)\|^{\sim} < \frac{1}{16},$$

which is a contradiction. This contradiction proves that such a system of splittings does not exist. Hence $\mathrm{Inv}(A) \not\cong \mathrm{Inv}(B)$ and $A \not\cong B$.

3.12. One can easily verify that

$$\begin{aligned}
\mathrm{AffT}(A) &= \mathrm{AffT}(B) \\
&= \Big\{(f_1, f_2, \ldots, f_n, \ldots) \in \prod_{n=1}^{\infty} C_{\mathbb{R}}[0,1] : \exists\, r \in \mathbb{R} \text{ such that} \\
&\qquad\qquad\qquad\qquad f_n(x) \text{ converges to } r \text{ uniformly}\Big\},
\end{aligned}$$

$$\begin{aligned}
\overline{\rho K_0}(A)(&= \overline{\rho K_0}(B)) \\
&= \Big\{(r_1, r_2, \ldots, r_n, \ldots) \in \prod_{n=1}^{\infty} \mathbb{R} : \exists\, r \in \mathbb{R} \text{ such that } r_n \text{ converges to } r\Big\} \\
&\subset \mathrm{AffT}(A)(= \mathrm{AffT}(B)).
\end{aligned}$$

Since $\overline{\rho K_0(A)}(= \overline{\rho K_0(B)})$ is already a vector space, we have $\widetilde{\rho K_0}(A) = \overline{\rho K_0(A)}$ and $\widetilde{\rho K_0}(B) = \overline{\rho K_0(B)}$. Therefore,

$$U_{\text{tor}}(A)/\widetilde{SU}(A) \cong \text{AffT}(A)/\widetilde{\rho K_0}(A) = \text{AffT}(A)/\overline{\rho K_0(A)} \cong U_0(A)/\overline{DU(A)}.$$

On the other hand, $U_{\text{tor}}(A) = U_0(A)$. Hence $\widetilde{SU}(A) = \overline{DU(A)}$. Furthermore, the map $\lambda_A : \text{AffT}(A)/\overline{\rho K_0(A)} \to U(A)/\overline{DU(A)}$ can be identified with the map $\tilde{\lambda}_A : \text{AffT}(A)/\widetilde{\rho K_0}(A) \to U(A)/\widetilde{SU}(A)$. That is, $\text{Inv}'(A) = \text{Inv}(A)$. Similarly, $\text{Inv}(B) = \text{Inv}'(B)$.

3.13. A routine calculation (we omit the details) shows that for any finite subset $F \subset A_n$ and $\varepsilon > 0$, there is an $m > n$ and two finite dimensional unital sub-C^*-algebras $C, D \subset A_m$ with nonabelian central projection such that

$$\left\|[\phi_{n,m}(f), c]\right\| < \varepsilon\|c\| \quad \text{and} \quad \left\|[\psi_{n,m}(f), d]\right\| < \varepsilon\|d\| \quad \text{for all } f \in F, c \in C, d \in D.$$

Consequently, both C^*-algebras A and B are approximately divisible in the sense of [Blackadar et al. 1992, Definition 1.2]. By [Toms and Winter 2008, Theorem 2.3], both A and B are $\mathcal{Z}$-stable. That is, $A \otimes \mathcal{Z} \cong A$ and $B \otimes \mathcal{Z} \cong B$, where $\mathcal{Z}$ is the Jiang–Su algebra (see [Jiang and Su 1999]). Furthermore, by using [Tikuisis 2011] (see also [Coward et al. 2008]), one can prove that $Cu(A) \cong Cu(B)$ and $Cu(A \otimes C(S^1)) \cong Cu(B \otimes C(S^1))$.

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On refined metric and hermitian structures in arithmetic I: Galois–Gauss sums and weak ramification

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We use techniques of relative algebraic K-theory to develop a common refinement of the theories of metrized and hermitian Galois structures in arithmetic. As a first application of the general approach, we then use it to prove several new results, and to formulate several explicit new conjectures, concerning the detailed arithmetic properties of a natural class of wildly ramified Galois–Gauss sums.

1. Introduction

This article has essentially two main purposes. Firstly, we shall use techniques of relative algebraic K-theory to develop a natural, and very general, algebraic formalism that gives a common, and strong, refinement of the theory of "hermitian modules" and "hermitian classgroups" described by Fröhlich [1984] and of the

theory of "metrized modules and complexes" and "arithmetic classgroups" introduced by Chinburg, Pappas, and Taylor [Chinburg et al. 2002].

Secondly, as a first concrete application of this refined theory, we shall show that it gives considerable new insight on the detailed arithmetic properties of a natural class of wildly ramified Galois–Gauss sums.

To give a few more details we fix a finite group Γ and recall that a hermitian Γ-module is a pair comprising a finitely generated projective Γ-module together with a nondegenerate Γ-invariant pairing on this module. Fröhlich showed that such modules are naturally classified by a "discriminant" invariant that lies in the hermitian classgroup $\mathrm{HCl}(\Gamma)$ of Γ and is defined in terms of idelic-valued functions on the ring R_Γ of $\mathbb{Q}^c$-valued virtual characters of Γ.

This theory was developed with arithmetic applications in mind since for any tamely ramified Galois extension of number fields L/K with $\mathrm{Gal}(L/K) = \Gamma$ the ring of algebraic integers of L constitutes a hermitian Γ-module when endowed with its natural trace pairing. In this setting, Fröhlich conjectured, and Cassou-Noguès and Taylor [1983] subsequently proved, that the corresponding discriminant element uniquely characterizes the Artin root numbers of irreducible complex symplectic characters of Γ. The latter result is commonly regarded as the highlight of classical "Galois module theory", as had been developed in the 1970s and 1980s (for more details see [Fröhlich 1984])

To develop an analogous theory in the setting of arithmetic schemes admitting a tame action of Γ, Chinburg, Pappas, and Taylor subsequently defined a metrized Γ-module (respectively, complex of Γ-modules) to be a pair comprising a finitely generated projective Γ-module and a collection of suitable metrics on the isotypic components of the complexified module (respectively, a perfect complex of Γ-modules together with metrics on the isotypic components of the complexified cohomology modules). To classify such structures they defined the arithmetic classgroup $A(\Gamma)$ of Γ in terms of idelic-valued functions on R_Γ and showed each metrized Γ-module (respectively, complex) gives rise to an associated invariant in $A(\Gamma)$.

To describe a common refinement of the above algebraic theories we construct canonical homomorphisms $\Pi_\Gamma^{\mathrm{met}}$ and $\Pi_\Gamma^{\mathrm{herm}}$ from the relative algebraic K_0-group $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ of the ring inclusion $\mathbb{Z}[\Gamma] \to \mathbb{Q}^c[\Gamma]$ to the group $A(\Gamma)$ and to a natural extension of the group $\mathrm{HCl}(\Gamma)$, respectively.

We then show that $\Pi_\Gamma^{\mathrm{met}}$ and $\Pi_\Gamma^{\mathrm{herm}}$ send each of the natural generating elements of $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ and of the subgroup $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}[\Gamma])$ of $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ to the difference of the natural invariants of two metrized modules in $A(\Gamma)$ and of the discriminants of two hermitian modules in $\mathrm{HCl}(\Gamma)$, respectively.

To define the homomorphisms $\Pi_\Gamma^{\mathrm{met}}$ and $\Pi_\Gamma^{\mathrm{herm}}$ we rely on a description of the group $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ in terms of idelic-valued functions on R_Γ that is proved by Agboola and Burns [2006].

The strategy to apply this theory in arithmetic settings is then twofold. In any given setting, one first hopes to identify a canonical element of $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ that at least one of $\Pi_\Gamma^{\mathrm{met}}$ or $\Pi_\Gamma^{\mathrm{herm}}$ sends to arithmetic invariants that have been considered previously. Then one can hope to prove, or at least to formulate conjecturally, a precise relation in $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ that projects (under either $\Pi_\Gamma^{\mathrm{met}}$ or $\Pi_\Gamma^{\mathrm{herm}}$ or both) to recover pre-existing results, or conjectures, in A(Γ) and HCl(Γ).

In any case in which this can be achieved one can reasonably hope to obtain up to three significant outcomes.

Firstly, one will obtain strong refinements of earlier results in the literature since both of the homomorphisms $\Pi_\Gamma^{\mathrm{met}}$ and $\Pi_\Gamma^{\mathrm{herm}}$ have large kernels.

Secondly, one can hope to obtain an explanation of any parallel aspects of the nature of earlier results in A(Γ) and HCl(Γ).

Thirdly, and perhaps most importantly, since $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}[\Gamma])$ has a canonical direct sum decomposition as $\bigoplus_\ell K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$, where ℓ runs over all primes, theorems and conjectures in A(Γ) and HCl(Γ) that appeared to be intrinsically global in nature are replaced by problems that can admit natural local decompositions and hence become easier to study.

While there is, of course, no guarantee that this strategy can work in all natural settings, in this article we show that it works very well in the setting of hermitian and metrized modules that arise from fractional ideals of number fields and their links to classical Galois–Gauss sums.

In addition, in a subsequent article it will be shown that the same approach can also be used to refine the theory of Chinburg, Pappas, and Taylor related to connections between the Zariski cohomology complexes of sheaves of differentials on arithmetic schemes with a tame action of a finite group and the associated epsilon constants and, in particular, to explain the similarity between the results obtained in [Chinburg et al. 2002; 2003].

A little more precisely, in the present article we first use the above approach in the setting of tamely ramified extensions of number fields to quickly both refine and extend previous results of Burns and Chinburg [1996] related to the links between Galois–Gauss sums and the hermitian modules comprising fractional powers of the different of L/K endowed with the natural trace pairing.

In the main body of the article we then consider wildly ramified Galois–Gauss sums. While the arithmetic properties of such sums are still in general poorly understood, significant progress has been made by Erez and others (see, for example, [Erez and Taylor 1992]) in the case of Galois extensions L/K that are both of odd degree and "weakly ramified" in the sense of [Erez 1991].

We recall, in particular, that under these hypotheses there exists a unique fractional ideal $\mathscr{A}_{L/K}$ of L, the square of which is equal to the inverse of the different of L/K, and that the hermitian Galois structure of $\mathscr{A}_{L/K}$ has been shown in special

cases to be closely linked to the properties of Galois–Gauss sums twisted by second Adams operators.

Following the general strategy described above, we shall now show that for any such extension L/K, with $\mathrm{Gal}(L/K) = \Gamma$, there exists a canonical element $\mathfrak{a}_{L/K}$ of $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ that simultaneously controls both the hermitian and metrized structures that are naturally associated to $\mathscr{A}_{L/K}$.

We then prove that $\mathfrak{a}_{L/K}$ belongs to, and also has finite order in, the subgroup $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}[\Gamma])$ of $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ and furthermore that it behaves well functorially under change of extension. We also show that $\mathfrak{a}_{L/K}$ has a canonical decomposition as a sum of elements constructed from local fields and then use this decomposition to compute $\mathfrak{a}_{L/K}$ explicitly in several important cases.

By using these results we can then derive several unconditional results concerning the hermitian and metrized structures associated to $\mathscr{A}_{L/K}$ and thereby, for example, extend the main results of the celebrated article [Erez and Taylor 1992].

In the general case, these results also combine with extensive numerical computation to motivate us to formulate several new conjectures concerning the detailed arithmetic properties of the local Galois–Gauss sums that are attached to weakly ramified extensions.

In the first such conjecture (Conjecture 7.4) we predict a precise conjectural description of $\mathfrak{a}_{L/K}$ in terms of local "Galois–Jacobi" sums and the fundamental classes of local class field theory.

We show that this explicit conjecture is equivalent to a special case of the "local epsilon constant conjecture" formulated by Breuning [2004b] and hence provides the first concrete link between the theory of the square root of the inverse different and the general framework of Tamagawa number conjectures that originated with Bloch and Kato [1990].

At the same time, this link combines with the methods developed here to give a new, and effective, strategy for proving the epsilon constant conjecture formulated by Bley and Burns [2003] for certain new families of wildly ramified Galois extensions of number fields.

Then, in Conjecture 10.7, we predict that $\mathfrak{a}_{L/K}$ can also be directly computed in terms of a naturally defined "idelic twisted unramified characteristic" element. This simple (and, we feel, quite surprising) conjectural formula can be proved completely in certain important special cases and is also supported by extensive numerical computations.

Upon comparing the predictions made in Conjectures 7.4 and 10.7 one then derives a new, and explicit, conjectural formula for Galois–Jacobi sums in terms of local fundamental classes (for details see Remark 10.8).

This framework of new conjectures concerning the arithmetic properties of wildly ramified Galois–Gauss sums is surely worthy of further investigation.

However, to finish with an even more concrete example of the insight that comes from using techniques of relative algebraic K-theory we recall that Vinatier [2003] conjectures that $\mathscr{A}_{L/K}$ is a free Γ-module when $K = \mathbb{Q}$ and is able, by using the connection to twisted Galois–Gauss sums, to prove this conjecture if the decomposition groups in $\mathrm{Gal}(L/\mathbb{Q})$ of each wildly ramified prime are abelian [Vinatier 2001]. The conjecture is also known to hold if $L/\mathbb{Q}$ is tamely ramified by the work of Erez [1991]. However, aside from numerical verifications in a small (finite) number of cases [Vinatier 2002], there is still essentially nothing known about this conjecture in the nonabelian weakly and wildly ramified case.

By contrast, applying our approach in this setting now allows us to show easily that Vinatier's conjecture naturally decomposes into a family of corresponding conjectures concerning extensions of local fields. This observation leads directly to a general "finiteness result" for Vinatier's conjecture and hence renders the conjecture accessible to effective computation. In particular, in this way we are able to prove the conjecture for several new, and infinite, families of nonabelian wildly ramified Galois extensions.

Although we do not pursue it here, we believe it likely that the same local approach would also shed light on several of the explicit questions that were recently raised in the introduction to [Caputo and Vinatier 2016].

Finally, we would like to note that much of this work grew out of the King's College London PhD thesis of Hahn [2016].

Part I. The general approach and first examples

In this part of the article we shall first review some basic facts concerning relative algebraic K-theory and the theories of both arithmetic and hermitian classgroups. We then establish a new link between these theories that will play a key role in subsequent arithmetic applications.

Throughout the section we illustrate abstract definitions and results by means of arithmetic examples that are motivated by our later applications.

For any Galois extension of fields F/E we set $G(F/E) := \mathrm{Gal}(F/E)$. We write $\mathbb{Q}^c$ for the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and for any number field $E \subseteq \mathbb{Q}^c$ we also write Ω_E for the absolute Galois group $G(\mathbb{Q}^c/E)$.

For any finite group Γ we write $\widehat{\Gamma}$ for the set of irreducible $\mathbb{Q}^c$-valued characters of Γ. If ℓ denotes a rational prime, then we write $\widehat{\Gamma}_\ell$ for the set of irreducible $\mathbb{Q}^c_\ell$-valued characters.

2. Relative K-theory, metric structures, and hermitian structures

2A. *Relative algebraic K-theory.* We fix a finite group Γ and a Dedekind domain R of characteristic zero and write F for the field of fractions of R.

For any extension field E of F and any $R[\Gamma]$-module M we set $M_E := E \otimes_R M$ and for any homomorphism $\phi : M \to N$ of $R[\Gamma]$-modules we write $\phi_E : M_E \to N_E$ for the induced homomorphism of $E[\Gamma]$-modules.

2A1. We write $K_0(R[\Gamma], E[\Gamma])$ for the relative algebraic K_0-group that arises from the inclusion of rings $R[\Gamma] \subset E[\Gamma]$ and we use the description of this group in terms of explicit generators and relations that is given by Swan [1968, p. 215].

We recall in particular that in this description each element of $K_0(R[\Gamma], E[\Gamma])$ is represented by a triple $[P, \phi, Q]$ where P and Q are finitely generated projective left $R[\Gamma]$-modules and $\phi : P_E \to Q_E$ is an isomorphism of (left) $E[\Gamma]$-modules.

We write $\mathrm{Cl}(R[\Gamma])$ for the reduced projective classgroup of $R[\Gamma]$ (as discussed in [Curtis and Reiner 1987, §49]) and often use the fact that there exists a canonical exact commutative diagram

$$
\begin{array}{ccccccc}
K_1(R[\Gamma]) & \longrightarrow & K_1(E[\Gamma]) & \xrightarrow{\partial^1_{R,E,\Gamma}} & K_0(R[\Gamma], E[\Gamma]) & \xrightarrow{\partial^0_{R,E,\Gamma}} & \mathrm{Cl}(R[\Gamma]) \\
\| & & \iota \uparrow & & \iota' \uparrow & & \| \\
K_1(R[\Gamma]) & \longrightarrow & K_1(F[\Gamma]) & \xrightarrow{\partial^1_{R,F,\Gamma}} & K_0(R[\Gamma], F[\Gamma]) & \xrightarrow{\partial^0_{R,F,\Gamma}} & \mathrm{Cl}(R[\Gamma])
\end{array}
\tag{2.1}
$$

Here the map ι is induced by the inclusion $F[\Gamma] \subseteq E[\Gamma]$ and ι' sends each element $[P, \phi, Q]$ to $[P, E \otimes_F \phi, Q]$. These maps are injective and will usually be regarded as inclusions. The map $\partial^0_{R,E,\Gamma}$ sends each element $[P, \phi, Q]$ to $[P] - [Q]$. (For details of all the other homomorphisms that occur above see [Swan 1968, Theorem 15.5].)

We write $K_0 T(R[\Gamma])$ for the Grothendieck group of finite $R[\Gamma]$-modules that are of finite projective dimension and recall that there are natural isomorphisms of abelian groups

$$
K_0 T(R[\Gamma]) \cong K_0(R[\Gamma], F[\Gamma]) \cong \bigoplus_v K_0(R_v[\Gamma], F_v[\Gamma]).
\tag{2.2}
$$

We choose the normalization of the first isomorphism so that for any finite $R[\Gamma]$-module M of finite projective dimension, and any resolution of the form $0 \to P \xrightarrow{\theta} P' \to M \to 0$, where the modules P and P' are finitely generated and projective, the class of M in $K_0 T(R[\Gamma])$ is sent to $[P, \theta_F, P']$. In addition, the direct sum in (2.2) runs over all nonarchimedean places v of F and the second isomorphism is the diagonal map induced by the homomorphisms

$$
\pi_{\Gamma,v} : K_0(R[\Gamma], F[\Gamma]) \to K_0(R_v[\Gamma], F_v[\Gamma])
\tag{2.3}
$$

that sends each element $[X, \xi, Y]$ to $[X_v, \xi_v, Y_v]$, where we set $X_v := R_v \otimes_R X$ and $\xi_v := F_v \otimes_F \xi$.

We write $\zeta(A)$ for the center of a ring A. Then to compute in $K_1(E[\Gamma])$ one uses the "reduced norm" homomorphism

$$\mathrm{Nrd}_{E[\Gamma]} : K_1(E[\Gamma]) \to \zeta(E[\Gamma])^{\times}$$

which sends the class of each pair (V, ϕ), where V is a finitely generated free $E[\Gamma]$-module and ϕ is an automorphism of V (as $E[\Gamma]$-module), to the reduced norm of ϕ, considered as an element of the semisimple E-algebra $\mathrm{End}_{E[\Gamma]}(V)$. If $E \subseteq \mathbb{Q}^c$ is a number field and $|\Gamma|$ is odd, then $\mathrm{Nrd}_{E[\Gamma]}$ is bijective by the Hasse–Schilling–Maass norm theorem [Curtis and Reiner 1987, Theorem (45.3)]. The same is true for algebraically closed fields and p-adic fields. In particular we write

$$\delta_{\Gamma} : \zeta(\mathbb{Q}^c[\Gamma])^{\times} \to K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma]) \tag{2.4}$$

for the composite $\partial^1_{\mathbb{Z}, \mathbb{Q}^c, \Gamma} \circ (\mathrm{Nrd}_{\mathbb{Q}^c[\Gamma]})^{-1}$. For a rational prime ℓ we write

$$\delta_{\Gamma, \ell} : \zeta(\mathbb{Q}^c_\ell[\Gamma])^{\times} \to K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}^c_\ell[\Gamma])$$

for the composite $\partial^1_{\mathbb{Z}_\ell, \mathbb{Q}^c_\ell, \Gamma} \circ (\mathrm{Nrd}_{\mathbb{Q}^c_\ell[\Gamma]})^{-1}$.

2A2. In the sequel we make much use of the fact that $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ can be explicitly described in terms of idelic-valued functions on the characters of Γ.

To recall this description we write R_Γ for the free abelian group on $\widehat{\Gamma}$. Then the Galois group $\Omega_{\mathbb{Q}}$ acts on R_Γ via the rule $(\omega \circ \chi)(\gamma) = \omega(\chi(\gamma))$ for every $\omega \in \Omega_{\mathbb{Q}}$, $\chi \in \widehat{\Gamma}$, and $\gamma \in \Gamma$.

For each a in $\mathrm{GL}_n(\mathbb{Q}^c[\Gamma])$ we define an element $\mathrm{Det}(a)$ of $\mathrm{Hom}(R_\Gamma, \mathbb{Q}^{c\times})$ in the following way: if T is a representation over $\mathbb{Q}^c$ which has character ϕ, then $\mathrm{Det}(a)(\phi) := \det(T(a))$. This definition depends only on ϕ and not on the choice of representation T. Analogously, if w denotes a finite place of $\mathbb{Q}^c$, then each element a of $\mathrm{GL}_n(\mathbb{Q}^c_w[\Gamma])$ defines a homomorphism $\mathrm{Det}(a) : R_\Gamma \to (\mathbb{Q}^c_w)^{\times}$.

We write $J_f(\mathbb{Q}^c[\Gamma])$ for the group of finite ideles of $\mathbb{Q}^c[\Gamma]$ and view $\mathbb{Q}[\Gamma]^{\times}$ as a subgroup of $J_f(\mathbb{Q}^c[\Gamma])$ via the natural diagonal embedding. In particular, if a is any element of $\mathrm{GL}_n(J_f(\mathbb{Q}^c[\Gamma]))$, the above approach allows one to define an element $\mathrm{Det}(a)$ of $\mathrm{Hom}(R_\Gamma, J_f(\mathbb{Q}^c))$ which is easily seen to be $\Omega_{\mathbb{Q}}$-equivariant. We set

$$U_f(\mathbb{Z}[\Gamma]) := \prod_\ell \mathbb{Z}_\ell[\Gamma]^{\times} \subset J_f(\mathbb{Q}[\Gamma]),$$

with the product taken over all primes ℓ, and then define a homomorphism

$$\Delta^{\mathrm{rel}}_{\Gamma} : \mathrm{Det}(\mathbb{Q}[G]^{\times}) \to \frac{\mathrm{Hom}(R_\Gamma, J_f(\mathbb{Q}^c))^{\Omega_{\mathbb{Q}}}}{\mathrm{Det}(U_f(\mathbb{Z}[\Gamma]))} \times \mathrm{Det}(\mathbb{Q}^c[\Gamma]^{\times}), \quad \theta \mapsto ([\theta], \theta^{-1}),$$

$$\tag{2.5}$$

where $[\theta]$ denotes the class of θ modulo $\mathrm{Det}(U_f(\mathbb{Z}[\Gamma]))$. We recall that by the Hasse–Schilling–Maass norm theorem

$$\mathrm{Det}(\mathbb{Q}[G]^\times) = \mathrm{Hom}^+(R_\Gamma, \mathbb{Q}^{c\times})^{\Omega_{\mathbb{Q}}}$$

where the right-hand expression denotes Galois equivariant homomorphisms whose values on R_Γ^s, the group of virtual symplectic characters, are totally positive. In particular, if Γ has odd order, then $\mathrm{Det}(\mathbb{Q}[G]^\times) = \mathrm{Hom}(R_\Gamma, \mathbb{Q}^{c\times})^{\Omega_{\mathbb{Q}}}$.

It is shown in [Agboola and Burns 2006, Theorem 3.5] that there is a natural isomorphism of abelian groups

$$h_\Gamma^{\mathrm{rel}} : K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma]) \xrightarrow{\sim} \mathrm{Cok}(\Delta_\Gamma^{\mathrm{rel}}). \tag{2.6}$$

We shall often use the explicit description of this map given in the following result (taken from [Agboola and Burns 2006, Remark 3.8]).

In the sequel for any ordered set of d elements $\{e^j\}_{1 \leq j \leq d}$ we write $\underline{e^j}$ for the $d \times 1$ column vector with j-th entry e^j.

In addition, for any Γ-modules X and Y we write $\mathrm{Is}_{\mathbb{Q}[\Gamma]}(X_{\mathbb{Q}}, Y_{\mathbb{Q}})$ for the set of isomorphisms of $\mathbb{Q}[\Gamma]$-modules $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$.

Lemma 2.7. *Let $c = [X, \xi, Y]$ be an element of $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ with locally free $\mathbb{Z}[\Gamma]$-modules X and Y of rank d. Choose a $\mathbb{Q}[\Gamma]$-basis $\{y_0^j\}$ of $Y_{\mathbb{Q}}$ and, for each rational prime p, a $\mathbb{Z}_p[\Gamma]$-basis $\{y_p^j\}$ of Y_p and an $\mathbb{Z}_p[\Gamma]$-basis $\{x_p^j\}$ of X_p and define μ_p to be the element of $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ which satisfies $\underline{y_p^j} = \mu_p \cdot \underline{y_0^j}$. Fix θ in $\mathrm{Is}_{\mathbb{Q}[\Gamma]}(X_{\mathbb{Q}}, Y_{\mathbb{Q}})$, note $\{\theta^{-1}(y_0^j)\}$ is a $\mathbb{Q}[\Gamma]$-basis of $X_{\mathbb{Q}}$, and write λ_p for the matrix in $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ with $\underline{x_p^j} = \lambda_p \cdot \underline{\theta^{-1}(y_0^j)}$. Finally, write μ for the matrix in $\mathrm{GL}_d(\mathbb{Q}^c[\Gamma])$ that represents $\overline{\xi} \circ (\theta^{-1} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}^c})$ with respect to the $\mathbb{Q}^c[\Gamma]$-basis $\{y^j\}$ of $Y_{\mathbb{Q}^c}$.*

Then the element $h_\Gamma^{\mathrm{rel}}(c)$ is represented by the homomorphism pair

$$\left(\prod_p \mathrm{Det}(\lambda_p \cdot \mu_p^{-1}) \right) \times \mathrm{Det}(\mu) \in \mathrm{Hom}(R_\Gamma, J_f(\mathbb{Q}^c))^{\Omega_{\mathbb{Q}}} \times \mathrm{Det}(\mathbb{Q}^c[\Gamma]^\times).$$

2A3. We give a first example of elements of relative algebraic K-groups that naturally arise in arithmetic contexts.

To do this we fix a finite Galois extension of number fields L/K and set $G := G(L/K)$. Since $\mathbb{Q}^c \subset \mathbb{C}$ we identify the set $\Sigma(L)$ of field embeddings $L \to \mathbb{Q}^c$ with the set of embeddings $L \to \mathbb{C}$ and we write $H_L := \prod_{\Sigma(L)} \mathbb{Z}$.

Then the natural action of G on $\Sigma(L)$ endows H_L with the structure of a G-module (explicitly, if $\{w_\sigma : \sigma \in \Sigma(L)\}$ is the canonical $\mathbb{Z}$-basis of H_L, then $\gamma w_\sigma = w_{\sigma \circ \gamma^{-1}}$). This module is free of rank $[K : \mathbb{Q}]$ since, if one fixes an extension $\hat{\sigma}$ in $\Sigma(L)$ of each σ in $\Sigma(K)$, then the set $\{w_{\hat{\sigma}}\}_{\sigma \in \Sigma(K)}$ is a basis of H_L over $\mathbb{Z}[G]$.

In addition, the map

$$\kappa_L : \mathbb{Q}^c \otimes_{\mathbb{Q}} L \to \prod_{\Sigma(L)} \mathbb{Q}^c = \mathbb{Q}^c \otimes_{\mathbb{Z}} H_L$$

that sends each element $z \otimes \ell$ to $(\sigma(\ell)z)_{\sigma \in \Sigma(L)}$ is then an isomorphism of $\mathbb{Q}^c[G]$-modules.

As a result, any full projective $\mathbb{Z}[G]$-sublattice $\mathcal{L}$ of L gives rise to an associated element

$$[\mathcal{L}, \kappa_L, H_L]$$

of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$.

In the case that $\mathcal{L}$ is an $\mathbb{O}_K[G]$-module the recipe in Lemma 2.7 gives rise to a useful description of this element that we record in the next result.

In this result (and the sequel) we use the following notation. For each element b of L with $L = K[G] \cdot b$ and each character χ in $\widehat{G}$ that is represented by a homomorphism of the form $T_\chi : G \to \mathrm{GL}_{n_\chi}(\mathbb{Q}^c)$, one defines a resolvent element

$$(b \mid \chi) := \det\left(\sum_{g \in G} g(b) T_\chi(g^{-1})\right)$$

and then an associated "norm-resolvent" by setting

$$\mathcal{N}_{K/\mathbb{Q}}(b \mid \chi) := \prod_{\omega}(b \mid \chi^{\omega^{-1}})^{\omega},$$

where ω runs through a transversal of $\Omega_{\mathbb{Q}}$ modulo Ω_K.

For each finite place v of K we write K_v for the completion of K at v and note that $L_v := L \otimes_K K_v \simeq \prod_{w|v} L_w$ is a free $K_v[G]$-module of rank one. Then, in the same way as above, for each element b_v in L_v such that $L_v = K_v[G] \cdot b_v$ we define an idelic-valued resolvent $(b_v \mid \chi)$ and an idelic-valued norm resolvent $\mathcal{N}_{K/\mathbb{Q}}(b_v \mid \chi)$ (for more details see [Burns and Chinburg 1996, §4.1]). For an $\mathbb{O}_K$-module $\mathcal{L}$ we also set $\mathcal{L}_v := \mathcal{L} \otimes_{\mathbb{O}_K} \mathbb{O}_{K_v}$.

Lemma 2.8. *Fix a $\mathbb{Z}$-basis $\{a_\sigma\}_{\sigma \in \Sigma(K)}$ of $\mathbb{O}_K$, an element b of L such that $L = K[G] \cdot b$, and, for each finite place v of K, an element b_v of L_v such that $\mathcal{L}_v = \mathbb{O}_{K_v}[G] \cdot b_v$.*

Then the element $h_G^{\mathrm{rel}}([\mathcal{L}, \kappa_L, H_L])$ is represented by the homomorphism pair $(\theta_1 \theta_2^{-1}, \theta_2 \theta_3)$ where for χ in $\widehat{G}$ one has

$$\theta_1(\chi) := \prod_v \mathcal{N}_{K/\mathbb{Q}}(b_v \mid \chi), \qquad \theta_2(\chi) := \mathcal{N}_{K/\mathbb{Q}}(b \mid \chi), \qquad \theta_3(\chi) := \delta_K^{\chi(1)}$$

with $\delta_K := \det(\tau(a_\sigma))_{\sigma, \tau \in \Sigma(K)}$.

Proof. Since H_L is a free G-module, in terms of the notation of Lemma 2.7 we can and will use the basis $\{y_0^j\} = \{y_p^j\} = \{w_{\hat\sigma}\}_{\sigma \in \Sigma(K)}$ so that μ_p is the identity matrix for every prime p.

We write $\theta_b : L \to H_{L,\mathbb{Q}}$ for the $\mathbb{Q}[G]$-linear isomorphism that sends each element $a_\sigma \cdot b$ to $w_{\hat\sigma}$.

For each prime p we set $\mathbb{O}_{K,p} := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{O}_K \simeq \prod_{v|p} \mathbb{O}_{K_v}$ and $\mathscr{L}_p := \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathscr{L} \simeq \prod_{v|p} \mathscr{L}_v$. We note that the element $b_p := (b_v)_{v|p}$ is a $\mathbb{O}_{K,p}[G]$-generator of $\mathscr{L}_p$ and that the homomorphism of $\mathbb{Z}_p[G]$-modules $\theta_{b_p} : \mathscr{L}_p \to H_{L,p}$ that sends each element $a_\sigma \cdot b_p$ to $w_{\hat\sigma}$ is bijective.

For the basis $\{x_p^j\}$ which occurs in the statement of Lemma 2.7, we choose $\{a_\sigma \cdot b_p\}_{\sigma \in \Sigma(K)}$ and then write λ_p for the matrix in $\mathrm{GL}_d(\mathbb{Q}_p[G])$ which satisfies $a_\sigma \cdot b_p = \lambda_p \cdot \theta_b^{-1}(w_{\hat\sigma})$. We note, in particular, that λ_p is the coordinate matrix of the $\mathbb{Q}_p[G]$-linear map $(\mathbb{Q}_p^c \otimes_{\mathbb{Q}} \theta_b) \circ (\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \theta_{b_p})^{-1}$ with respect to the basis $\{w_{\hat\sigma}\}$.

Then Lemma 2.7 implies that $h_G^{\mathrm{rel}}([\mathscr{L}, \kappa_L, H_L])$ is represented by the homomorphism pair

$$\left(\prod_p \mathrm{Det}(\lambda_p)\right) \times \mathrm{Det}(\mu)$$

where μ is the coordinate matrix in $\mathrm{GL}_d(\mathbb{Q}^c[G])$ of $\kappa_L \circ (\mathbb{Q}^c \otimes_{\mathbb{Q}} \theta_b)^{-1}$ with respect to the basis $\{w_{\hat\sigma}\}$.

In addition, by [Bley and Burns 2003, (16) and (17)], one knows that $\mathrm{Det}(\mu)(\chi) = \delta_K^{\chi(1)} \cdot \mathcal{N}_{K/\mathbb{Q}}(b \mid \chi)$ for each character χ.

Finally to compute each homomorphism $\mathrm{Det}(\lambda_p)$ we note that

$$(\mathbb{Q}_p^c \otimes_{\mathbb{Q}} \theta_b) \circ (\mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} \theta_{b_p})^{-1}$$
$$= ((\mathbb{Q}_p^c \otimes_{\mathbb{Q}} \theta_b) \circ (\mathbb{Q}_p^c \otimes_{\mathbb{Q}^c} \kappa_L)^{-1}) \circ ((\mathbb{Q}_p^c \otimes_{\mathbb{Q}^c} \kappa_L) \circ (\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \theta_{b_p})^{-1})$$

and write $\lambda_{p,2}$ for the coordinate matrix of $(\mathbb{Q}_p^c \otimes_{\mathbb{Q}^c} \kappa_L) \circ (\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \theta_{b_p})^{-1}$.

Then using similar computations to those used to derive [Bley and Burns 2003, (16) and (17)] one finds that for each character χ one has

$$\mathrm{Det}(\lambda_{p,2})(\chi) = \mathcal{N}_{K/\mathbb{Q}}(b_p \mid \chi) = \prod_{v|p} \mathcal{N}_{K/\mathbb{Q}}(b_v \mid \chi),$$

as required to complete the proof. $\square$

2B. *Hermitian modules and classgroups.*

In this section we recall some of the basic theory of hermitian modules and classgroups. For more details see [Fröhlich 1984, Chapter II]. Note, however, that in contrast to the convention used in [loc. cit.] we consider all modules as left modules.

Definition 2.9. A *hermitian form* on a Γ-module X is a nondegenerate bilinear map

$$h : X_{\mathbb{Q}} \times X_{\mathbb{Q}} \to \mathbb{Q}[\Gamma]$$

that is $\mathbb{Q}[\Gamma]$-linear in the first variable and satisfies $h(x, y) = h(y, x)^{\sharp}$ with $z \mapsto z^{\sharp}$ the $\mathbb{Q}$-linear anti-involution of $\mathbb{Q}[\Gamma]$ which inverts elements of Γ.

A *hermitian Γ-module* is a pair (X, h) comprising a finitely generated projective Γ-module X and a hermitian form h on X.

Example 2.10. For any number field K and any finite group Γ we extend the field-theoretic trace $\text{tr}_{K/\mathbb{Q}} : K \to \mathbb{Q}$ to a linear map $K[\Gamma] \to \mathbb{Q}[\Gamma]$ by applying it to the coefficients of each element of $K[\Gamma]$.

This action then gives rise to a hermitian form

$$t_{K[\Gamma]} : K[\Gamma] \times K[\Gamma] \to \mathbb{Q}[\Gamma]$$

by setting $t_{K[\Gamma]}(x, y) = \text{tr}_{K/\mathbb{Q}}(xy^{\sharp})$. In particular, since $\mathbb{O}_K$ is a free $\mathbb{Z}$-module the pair $(\mathbb{O}_K[\Gamma], t_{K[\Gamma]})$ is a hermitian Γ-module.

Example 2.11. For any finite Galois extension L/K of number fields, with $G = G(L/K)$, one obtains a hermitian form

$$t_{L/K} : L \times L \to \mathbb{Q}[G]$$

by setting $t_{L/K}(x, y) = \sum_{g \in G} \text{tr}_{L/\mathbb{Q}}(x \cdot g(y))g$. For each full projective G-sublattice $\mathscr{L}$ of L the pair $(\mathscr{L}, t_{L/K})$ is then a hermitian G-module.

Example 2.12. Let X_1 and X_2 be finitely generated projective Γ-modules and ξ an isomorphism of $\mathbb{Q}[\Gamma]$-modules $X_{2,\mathbb{Q}} \cong X_{1,\mathbb{Q}}$. For any hermitian form h on X_1 we define the "pullback of h through ξ" to be the hermitian form $\xi^*(h)$ on X_2 that satisfies

$$\xi^*(h)(x_2, y_2) = h(\xi(x_2), \xi(y_2))$$

for all $x_2, y_2 \in X_2$.

To classify general hermitian Γ-modules Fröhlich defined (see, for example, [Fröhlich 1984, Chapter II, (5.3)]) the "hermitian classgroup" $\text{HCl}(\Gamma)$ of Γ to be the cokernel of the homomorphism

$$\Delta_{\Gamma}^{\text{herm}} : \text{Det}(\mathbb{Q}[\Gamma]^{\times}) \to \frac{\text{Hom}(R_{\Gamma}, J_f(\mathbb{Q}^c))^{\Omega_{\mathbb{Q}}}}{\text{Det}(U_f(\mathbb{Z}[\Gamma]))} \times \text{Hom}(R_{\Gamma}^s, \mathbb{Q}^{c\times})^{\Omega_{\mathbb{Q}}}, \quad \theta \mapsto ([\theta]^{-1}, \theta^s)$$

$$(2.13)$$

where R_{Γ}^s denotes the subgroup of R_{Γ} generated by the set of irreducible symplectic characters of Γ and θ^s denotes the restriction of θ to R_{Γ}^s.

To each hermitian Γ-module (X, h) Fröhlich then associated a canonical "discriminant" element $\text{Disc}(X, h)$ in $\text{HCl}(\Gamma)$ that is defined explicitly as follows.

Definition 2.14. Let (X, h) be a hermitian Γ-module and write d for the rank of the free $\mathbb{Q}[\Gamma]$-module $X_\mathbb{Q}$. Choose a $\mathbb{Q}[\Gamma]$-basis $\{x_0^j\}$ of $X_\mathbb{Q}$ and, for each prime p, a $\mathbb{Z}_p[\Gamma]$-basis $\{x_p^j\}$ of X_p. Then there exists an element λ_p of $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ with $\underline{x_p^j} = \lambda_p \cdot \underline{x_0^j}$ and the "discriminant class" $\mathrm{Disc}(X, h)$ is the element of $\mathrm{HCl}(\Gamma)$ represented by the pair

$$\left(\prod_p \mathrm{Det}(\lambda_p), \mathrm{Pf}(h(x_0^i, x_0^j)) \right).$$

Here Pf is the "Pfaffian" function in $\mathrm{Hom}(\mathrm{R}_\Gamma^s, \mathbb{Q}^{c\times})$ defined in [Fröhlich 1984, Chapter II, Proposition 4.3].

We end this section with a new definition that will be useful in the sequel.

Definition 2.15. The "extended hermitian classgroup" $\mathrm{eHCl}(\Gamma)$ of Γ is defined to the cokernel of the homomorphism that is defined just as $\Delta_\Gamma^{\mathrm{herm}}$ except that the term $\mathrm{Hom}(\mathrm{R}_\Gamma^s, \mathbb{Q}^{c\times})^{\Omega_\mathbb{Q}}$ on the right-hand side of (2.13) is replaced by $\mathrm{Hom}(\mathrm{R}_\Gamma^s, \mathbb{Q}^{c\times})$. We regard $\mathrm{HCl}(\Gamma)$ as a subgroup of $\mathrm{eHCl}(\Gamma)$ in the obvious way.

2C. *Metrized modules and classgroups.* We quickly recall the definition of metrized modules and classgroups. For further details we refer the reader to [Chinburg et al. 2002, §2 and §3.1].

For each ϕ in $\widehat{\Gamma}$ we write W_ϕ for the Wedderburn component of $\mathbb{Q}^c[\Gamma]$ which corresponds to the contragredient character $\bar\phi$ of ϕ. Thus, W_ϕ has character $\phi(1)\bar\phi$.

For any $\mathbb{Q}^c[\Gamma]$-module X we then set

$$X_\phi := \bigwedge_{\mathbb{Q}^c}^{\mathrm{top}} (X \otimes_{\mathbb{Q}^c} W_\phi)^\Gamma,$$

where "$\bigwedge_{\mathbb{Q}^c}^{\mathrm{top}}$" denotes the highest exterior power over $\mathbb{Q}^c$ which is nonzero, and Γ acts diagonally on the tensor product. We recall from [Chinburg et al. 2002, Lemma 2.3] that $X_\phi \simeq \overline{W}_\phi X$.

Recall that $\mathbb{Q}^c$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We write $\sigma_\infty : \mathbb{Q}^c \to \mathbb{C}$ for the inclusion and $\bar z$ for the conjugate of a complex number z.

Definition 2.16. A *metrized* Γ-module is a pair $(X, \{\|\cdot\|_\phi\}_{\phi\in\widehat{\Gamma}})$ comprising a finitely generated projective Γ-module X and a set $\{\|\cdot\|_\phi\}_{\phi\in\widehat{\Gamma}}$ of metrics on the complex lines $\mathbb{C} \otimes_{\mathbb{Q}^c} X_\phi$ induced by positive definite hermitian forms μ_ϕ on the spaces $\mathbb{C} \otimes_{\mathbb{Q}^c} X_\phi$.

In this situation, we usually abbreviate $(X, \{\|\cdot\|_\phi\}_{\phi\in\widehat{\Gamma}})$ to $(X, \mu_\bullet)$ and note that for each ϕ in $\widehat{\Gamma}$ and each element x of $\mathbb{C} \otimes_{\mathbb{Q}^c} X_\phi$ one has $\|x\|_\phi^2 = \mu_\phi(x, x)$.

Example 2.17. An important special case occurs when μ_ϕ arises as the "highest exterior power" of a positive definite hermitian form $\tilde{\mu}_\phi$ on the space

$$(X \otimes_{\mathbb{Z}} W_\phi)^\Gamma \otimes_{\mathbb{Q}^c} \mathbb{C} = ((X \otimes_{\mathbb{Z}} \mathbb{C}) \otimes_{\mathbb{C}} (W_\phi \otimes_{\mathbb{Q}^c} \mathbb{C}))^\Gamma.$$

In this case, for any $\mathbb{C}$-basis $v_1, \ldots, v_d$ of this space one has

$$\|v_1 \wedge \cdots \wedge v_d\|_\phi^2 = \det((\tilde{\mu}_\phi(v_i, v_j)_{1 \le i, j \le d})).$$

Let Γ be a finite group. Then the standard Γ-equivariant positive definite hermitian form $\mu_{\mathbb{C}[\Gamma]}$ on $\mathbb{C}[\Gamma]$ is defined (for example, in [Chinburg et al. 2002, §2.1]) by setting

$$\mu_{\mathbb{C}[\Gamma]}\left(\sum_{g \in \Gamma} x_g g, \sum_{h \in \Gamma} y_h h\right) = \sum_{g \in \Gamma} x_g \overline{y_g}.$$

The associated $\mathbb{C}[\Gamma]$-valued hermitian form is the so-called "multiplication form"

$$\hat{\mu}_{\mathbb{C}[\Gamma]} : \mathbb{C}[\Gamma] \times \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$$

that sends each pair (x, y) to $x \cdot \bar{y}$, where we extend complex conjugation to an anti-involution on $\mathbb{C}[\Gamma]$ by setting

$$\overline{\sum_{\gamma \in \Gamma} a_\gamma \gamma} := \sum_{\gamma \in \Gamma} \overline{a_\gamma} \gamma^{-1}.$$

Example 2.18. In this example we use the hypotheses and notation of Section 2A3.

(i) We write μ_L for the (unique) Γ-equivariant positive definite hermitian form on $\mathbb{C} \otimes_{\mathbb{Z}} H_L$ that satisfies

$$\mu_L\left(\sum_{\sigma \in \Sigma(L)} x_\sigma w_\sigma, \sum_{\sigma \in \Sigma(L)} y_\sigma w_\sigma\right) = \sum_{\sigma \in \Sigma(L)} x_\sigma \overline{y_\sigma}.$$

For each $\phi \in \widehat{\Gamma}$ the form μ_L together with the restriction of $\mu_{\mathbb{C}[\Gamma]}$ on $\mathbb{C} \otimes_{\mathbb{Q}^c} W_\phi$ induces a positive definite hermitian form $\tilde{\mu}_{L,\phi}$ on the tensor product

$$((\mathbb{C} \otimes_{\mathbb{Z}} H_L) \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{Q}^c} W_\phi))^\Gamma = \mathbb{C} \otimes_{\mathbb{Q}^c} (H_L \otimes_{\mathbb{Z}} W_\phi)^\Gamma.$$

We then write $\mu_{L,\phi}$ for the positive definite hermitian form on

$$\mathbb{C} \otimes_{\mathbb{Q}^c} \overset{\text{top}}{\underset{\mathbb{Q}^c}{\bigwedge}} (H_L \otimes_{\mathbb{Z}} W_\phi)^\Gamma = \overset{\text{top}}{\underset{\mathbb{C}}{\bigwedge}} (\mathbb{C} \otimes_{\mathbb{Q}^c} (H_L \otimes_{\mathbb{Z}} W_\phi)^\Gamma)$$

that is obtained as the highest exterior power of $\tilde{\mu}_{L,\phi}$ (as per the discussion in Example 2.17). The induced metric

$$\mu_{L,\bullet} := \{\mu_{L,\phi}\}_{\phi \in \widehat{\Gamma}}$$

on H_L plays an important role in the sequel.

(ii) There is a Γ-equivariant positive definite hermitian form h_L on $\mathbb{C} \otimes_{\mathbb{Q}} L$ defined by

$$h_L(z_1 \otimes m, z_2 \otimes n) = z_1 \overline{z_2} \sum_{\sigma \in \Sigma(L)} \sigma(m) \overline{\sigma(n)}.$$

(This form is a scalar multiple of the "Hecke form" defined by Chinburg et al. [2002, §5.2].) For each ϕ in $\widehat{\Gamma}$ we write $h_{L,\phi}$ for the positive definite hermitian form on $(\mathbb{C} \otimes_{\mathbb{Q}} L)_\phi$ that is obtained as the highest exterior power of the form on $(L \otimes_{\mathbb{Q}} W_\phi)^G$ which is induced by h_L on $\mathbb{C} \otimes_{\mathbb{Q}} L$ and by the restriction of $\mu_{\mathbb{C}[\Gamma]}$ on $\mathbb{C} \otimes_{\mathbb{Q}^c} W_\phi$.

We set

$$h_{L,\bullet} := \{h_{L,\phi}\}_{\phi \in \widehat{\Gamma}}$$

and note that if $\mathscr{L}$ is any full projective $\mathbb{Z}[\Gamma]$-sublattice of L, then the pair $(\mathscr{L}, h_{L,\bullet})$ is naturally a metrized Γ-module.

Example 2.19. Let $E \subseteq \mathbb{Q}^c$ be a subfield and let X_1 and X_2 be finitely generated locally free $\mathbb{Z}[\Gamma]$-modules. Let ξ denote an isomorphism of $E[\Gamma]$-modules $X_{2,E} \cong X_{1,E}$. For each ϕ in $\widehat{\Gamma}$ we write

$$\xi_\phi : (X_2 \otimes_{\mathbb{Z}} \mathbb{Q}^c)_\phi \otimes_{\mathbb{Q}^c} \mathbb{C} \cong (X_1 \otimes_{\mathbb{Z}} \mathbb{Q}^c)_\phi \otimes_{\mathbb{Q}^c} \mathbb{C}$$

for the isomorphism of complex lines which is induced by ξ. If h is any metric on X_1, then we define the "pullback" of h under ξ to be the (unique) metric $\xi^*(h)$ on X_2 which satisfies

$$\xi^*(h)_\phi(z) = h_\phi(\xi_\phi(z))$$

for all $\phi \in \widehat{\Gamma}$ and $z \in (X_2 \otimes_{\mathbb{Z}} \mathbb{Q}^c)_\phi \otimes_{\mathbb{Q}^c} \mathbb{C}$.

In order to classify metrized Γ-modules Chinburg et al. [2002, §3.1 and §3.2] defined the *arithmetic classgroup* $A(\Gamma)$ of Γ to be the cokernel of the homomorphism

$$\Delta_\Gamma^{\mathrm{met}} : \mathrm{Det}(\mathbb{Q}[\Gamma]^\times) \to \frac{\mathrm{Hom}(R_\Gamma, J_f(\mathbb{Q}^c))^{\Omega_{\mathbb{Q}}}}{\mathrm{Det}(U_f(\mathbb{Z}[\Gamma]))} \times \mathrm{Hom}(R_\Gamma, \mathbb{R}_{>0}^\times), \quad \theta \mapsto ([\theta], |\theta|)$$

where we write $|\theta|$ for the homomorphism which sends each character ϕ in $\widehat{\Gamma}$ to $|\theta(\phi)|^{-1}$. Note that we adopt here the convention of [Agboola and Burns 2006, §4.2 and Remark 4.4], i.e., our $|\theta|$ is the inverse of the map $|\theta|$ used in [Chinburg et al. 2002].

To each metrized Γ-module (X, h) one can then associate a canonical "arithmetic class" $[X, h]$ in $A(\Gamma)$.

We next recall the explicit definition of this element from [Chinburg et al. 2002, §3.2] (see also [Agboola and Burns 2006, Remark 4.6]) and to do this we use the notation of Lemma 2.7.

Definition 2.20. Let $(X, \mu_\bullet)$ be a metrized Γ-module, with X locally free over $\mathbb{Z}[\Gamma]$ of rank d. Choose a $\mathbb{Q}[\Gamma]$-basis $\{x_0^j\}$ of $X_\mathbb{Q}$ and, for each prime p, a $\mathbb{Z}_p[\Gamma]$-basis $\{x_p^j\}$ of X_p. Then there exists an element λ_p of $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ such that $\underline{x_p^j} = \lambda_p \cdot \underline{x_0^j}$.

For each x in $X_\mathbb{Q}$ we set

$$r(x) := \sum_{\gamma \in \Gamma} \gamma(x) \otimes \gamma \in X \otimes_\mathbb{Z} \mathbb{Q}^c[\Gamma].$$

We note that for each w in W_ϕ one has $r(x)(1 \otimes w) \in (X \otimes_\mathbb{Z} W_\phi)^\Gamma$ where for each w in W_ϕ the action of $r(x)$ on $1 \otimes w$ is defined by $r(x)(1 \otimes w) := \sum_{\gamma \in \Gamma} \gamma(x) \otimes \gamma(w)$.

Let $\{w_{\phi,k}\}_{1 \le k \le \phi(1)^2}$ be a $\mathbb{Q}^c$-basis of W_ϕ that is orthonormal with respect to the restriction of $\mu_{\mathbb{C}[\Gamma]}$ to W_ϕ. Then the set $\{r(x_0^j)(1 \otimes w_{\phi,k})\}_{j,k}$ is a $\mathbb{Q}^c$-basis of $(X \otimes_\mathbb{Z} W_\phi)^\Gamma$ and so

$$\bigwedge_j \bigwedge_k r(x_0^j)(1 \otimes w_{\phi,k})$$

is a $\mathbb{Q}^c$-basis of $(X \otimes_\mathbb{Z} \mathbb{Q}^c)_\phi$.

We then define $[X, \mu_\bullet]$ to be the element of $A(\Gamma)$ that is represented by the homomorphism on R_Γ which sends each character $\phi \in \widehat{\Gamma}$ to

$$\prod_p \mathrm{Det}(\lambda_p)(\phi) \times \left\| \left(\bigwedge_j \bigwedge_k r(x_0^j)(1 \otimes w_{\phi,k}) \right) \otimes 1 \right\|_\phi^{1/\phi(1)} \in J_f(\mathbb{Q}^c) \times \mathbb{R}_{>0}^\times. \quad (2.21)$$

We note that it is straightforward to show that $[X, \mu]$ is independent of the precise choices of bases $\{x_0^j\}$, $\{x_p^j\}$, and $\{w_{\phi,k}\}$.

As a concrete example, we now apply the above recipe in the setting of Example 2.18(i). To do this we recall from Section 2A3 that $\{w_\sigma : \sigma \in \Sigma(L)\}$ denotes the canonical $\mathbb{Z}$-basis of the G-module $H_L = \prod_{\Sigma(L)} \mathbb{Z}$. Moreover, in Example 2.18(i) we have defined a metric $\mu_{L,\bullet}$ on H_L so that the pair $(H_L, \mu_{L,\bullet})$ gives rise to an element $[H_L, \mu_{L,\bullet}]$ of $A(G)$.

The following result will play an important role in a later argument.

Lemma 2.22. *The element $[H_L, \mu_{L,\bullet}]$ of $A(G)$ is represented by the pair $(1, \theta)$ where θ sends each character ϕ of $\widehat{G}$ to $|G|^{[K:\mathbb{Q}](\phi(1)/2)}$.*

Proof. If $X = H_L$ and $\mu_\bullet = \mu_{L,\bullet}$, then in the notation of Definition 2.20 we can take both $\{x_0^j\}$ and $\{x_p^j\}$ to be the basis $\{w_{\hat\sigma}\}_{\sigma \in \Sigma(K)}$ described in Section 2A3 and so $\lambda_p = 1$.

In addition, for a character ϕ in $\widehat{G}$, embeddings σ and τ in $\Sigma(K)$, and integers k and ℓ with $1 \le k, \ell \le \phi(1)^2$ one has

$$(\mu_L \otimes \mu_{\mathbb{C}[G]})(r(w_{\hat{\sigma}})(1 \otimes w_{\phi,k}), r(w_{\hat{\tau}})(1 \otimes w_{\phi,\ell}))$$

$$= (\mu_L \otimes \mu_{\mathbb{C}[G]})\left(\sum_{g \in G} g(w_{\hat{\sigma}}) \otimes g(w_{\phi,k}), \sum_{h \in G} h(w_{\hat{\tau}}) \otimes h(w_{\phi,\ell})\right)$$

$$= \sum_{g,h} \mu_L(g(w_{\hat{\sigma}}), h(w_{\hat{\tau}})) \cdot \mu_{\mathbb{C}[G]}(g(w_{\phi,k}), h(w_{\phi,\ell}))$$

$$= \sum_{g,h} \delta_{g,h}\delta_{\hat{\sigma},\hat{\tau}} \cdot \mu_{\mathbb{C}[G]}(g(w_{\phi,k}), h(w_{\phi,\ell}))$$

$$= \delta_{\hat{\sigma},\hat{\tau}}\delta_{k,\ell} \cdot |G|.$$

From the explicit description given in Example 2.17 it thus follows that the second component of the representative (2.21) is equal to the $\phi(1)$-th root of

$$\det((\delta_{\hat{\sigma},\hat{\tau}}\delta_{k,\ell} \cdot |G|)_{(\sigma,k),(\tau,\ell)})^{1/2} = |G|^{[K:\mathbb{Q}] \cdot \phi(1)^2/2},$$

as suffices to give the claimed result.   □

3. Canonical homomorphisms and the universal diagram

In this section we establish a direct link between relative algebraic K-theory and the theories of metrized and hermitian modules reviewed above. The existence of such a link will then play a key role in subsequent arithmetic results.

For any finite group Γ we abbreviate $\mathrm{Cl}(\mathbb{Z}[\Gamma])$ to $\mathrm{Cl}(\Gamma)$ and we recall that there is a natural isomorphism of abelian groups

$$h_{\Gamma}^{\mathrm{red}} : \mathrm{Cl}(\Gamma) \cong \mathrm{Cok}(\Delta_{\Gamma}^{\mathrm{red}}) \tag{3.1}$$

where $\Delta_{\Gamma}^{\mathrm{red}}$ denotes the homomorphism

$$\Delta_{\Gamma}^{\mathrm{red}} : \mathrm{Hom}(R_{\Gamma}, \mathbb{Q}^{c\times})^{\Omega_{\mathbb{Q}}} \to \frac{\mathrm{Hom}(R_{\Gamma}, J_f(\mathbb{Q}^c))^{\Omega_{\mathbb{Q}}}}{\mathrm{Det}(U_f(\mathbb{Z}[\Gamma]))}, \quad \theta \mapsto [\theta].$$

Remark 3.2. We normalize the isomorphism $h_{\Gamma}^{\mathrm{red}}$ as in [Fröhlich 1983, Remark 1, p. 21]. To be specific, if X is a finitely generated projective $\mathbb{Z}[\Gamma]$-module, then one can give an explicit representative of the class $h_{\Gamma}^{\mathrm{red}}([X])$ as follows. We choose a $\mathbb{Q}[\Gamma]$-basis $\{x_0^j\}$ of $X_{\mathbb{Q}}$ and, for each rational prime p, a $\mathbb{Z}_p[\Gamma]$-basis $\{x_p^j\}$ of X_p. Let λ_p be the matrix in $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ which satisfies $\underline{x_p^j} = \lambda_p \cdot \underline{x_0^j}$. Then $h_{\Gamma}^{\mathrm{red}}([X])$ is represented by the function $\left(\prod_p \mathrm{Det}(\lambda_p)\right)$.

In the next result we shall use the canonical homomorphisms (of abelian groups)

$$\partial_\Gamma^{1,1} : \mathrm{Cok}(\Delta_\Gamma^{\mathrm{rel}}) \to A(\Gamma), \qquad ([\theta_1], \theta_2) \mapsto ([\theta_1], |\theta_2|)$$
$$\partial_\Gamma^{2,1} : \mathrm{Cok}(\Delta_\Gamma^{\mathrm{rel}}) \to \mathrm{eHCl}(\Gamma), \qquad ([\theta_1], \theta_2) \mapsto ([\theta_1], \theta_2^s)$$
$$\partial_\Gamma^{1,2} : \quad A(\Gamma) \to \mathrm{Cok}(\Delta_\Gamma^{\mathrm{red}}), \quad ([\theta_1], \theta_2) \mapsto [\theta_1]$$
$$\partial_\Gamma^{2,2} : \ \mathrm{eHCl}(\Gamma) \to \mathrm{Cok}(\Delta_\Gamma^{\mathrm{red}}), \quad ([\theta_1], \theta_2) \mapsto [\theta_1].$$

We shall also use the composite homomorphisms (defined using the isomorphisms h_Γ^{rel} and h_Γ^{red})

$$\Pi_\Gamma^{\mathrm{met}} := \partial_\Gamma^{1,1} \circ h_\Gamma^{\mathrm{rel}} \qquad : K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma]) \to A(\Gamma),$$
$$\Pi_\Gamma^{\mathrm{herm}} := \partial_\Gamma^{2,1} \circ h_\Gamma^{\mathrm{rel}} \qquad : K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma]) \to \mathrm{eHCl}(\Gamma),$$
$$\partial_\Gamma^{\mathrm{met}} := (h_\Gamma^{\mathrm{red}})^{-1} \circ \partial_\Gamma^{1,2} : A(\Gamma) \qquad\qquad \to \mathrm{Cl}(\Gamma),$$
$$\partial_\Gamma^{\mathrm{herm}} := (h_\Gamma^{\mathrm{red}})^{-1} \circ \partial_\Gamma^{2,2} : \mathrm{eHCl}(\Gamma) \qquad\qquad \to \mathrm{Cl}(\Gamma).$$

For convenience we shall use the same notation $\partial_\Gamma^{\mathrm{herm}}$ to denote the restriction of $\partial_\Gamma^{\mathrm{herm}}$ to the subgroup $\mathrm{HCl}(\Gamma)$.

Theorem 3.3. (i) *The homomorphism $\Pi_\Gamma^{\mathrm{met}}$ sends each class $[X, \xi, Y]$ to*

$$[X, \xi^*(\mu)] - [Y, \mu]$$

for any choice of metric μ on Y.

(ii) *The homomorphism $\Pi_\Gamma^{\mathrm{herm}}$ sends each element $[X, \xi, Y]$ of the subgroup*

$$K_0(\mathbb{Z}[\Gamma], \mathbb{Q}[\Gamma])$$
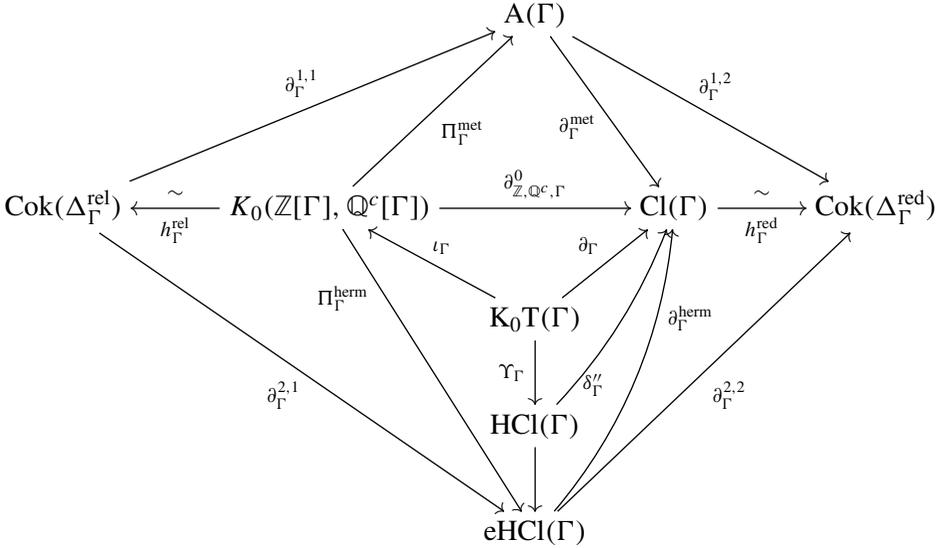
to

$$\mathrm{Disc}(X, \xi^*(h)) - \mathrm{Disc}(Y, h)$$

for any choice of hermitian form h on Y.

(iii) *The homomorphism $\partial_\Gamma^{\mathrm{met}}$ sends the class $[X, h]$ of a metrized module (X, h) to the class $[X]$.*

(iv) *The homomorphism $\partial_\Gamma^{\mathrm{herm}}$ sends the discriminant $\mathrm{Disc}(X, h)$ of a hermitian module (X, h) to the class $[X]$.*

(v) *The following diagram commutes*:

Here the unlabeled arrow is the natural inclusion $\mathrm{HCl}(\Gamma) \to \mathrm{eHCl}(\Gamma)$ *and the remaining homomorphisms that are not defined above are as follows.*

- ι_Γ *is the composition of the first isomorphism in* (2.2) *and the natural inclusion* $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}[\Gamma]) \to K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$,
- Υ_Γ *is the homomorphism defined in* [Fröhlich 1984, Chapter 2, §6],
- ∂_Γ *is the canonical map* (*as described in* [Fröhlich 1984, Chapter 1, (1.3)]),
- δ''_Γ *the homomorphism described in* [Fröhlich 1984, Chapter 2, (6.16)].

(*For further details of these maps see the argument below.*)

Proof. Claim (i) is proved by Agboola and Burns [2006, Theorem 4.11].

To prove claim (ii) we write d for the $\mathbb{Q}[\Gamma]$-rank of $X_\mathbb{Q} \cong Y_\mathbb{Q}$ and, just as in Definition 2.20, we fix a $\mathbb{Q}[\Gamma]$-basis $\{x_0^j\}_{1 \le j \le d}$ of $X_\mathbb{Q}$ and also, for each prime p, $\mathbb{Z}_p[\Gamma]$-bases $\{x_p^j\}_{1 \le j \le d}$ of X_p and $\{y_p^j\}_{1 \le j \le d}$ of Y_p.

We write λ_p and μ_p for the (unique) elements of $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ with $x_p^j = \lambda_p \cdot x_0^j$ and $y_p^j = \mu_p \cdot \xi(x_0^j)$, where in the last equality we use the fact that $\{\xi(x_0^j)\}_{1 \le j \le d}$ is a $\mathbb{Q}[\Gamma]$-basis of $Y_\mathbb{Q}$.

Then the explicit definition of h_Γ^{rel} as described in Lemma 2.7 ensures that $h_\Gamma^{\mathrm{rel}}([X, \xi, Y])$ is represented by the pair

$$\left(\prod_p \mathrm{Det}(\lambda_p) \cdot \mathrm{Det}(\mu_p)^{-1} \right) \times 1 \in \mathrm{Hom}(R_\Gamma, J_f(\mathbb{Q}^c))^{\Omega_F} \times \mathrm{Hom}(R_\Gamma, (\mathbb{Q}^c)^\times).$$

The assertion of claim (ii) thus follows because Definition 2.14 implies that for any hermitian form h on X the element $\mathrm{Disc}(X, \xi^*(h)) - \mathrm{Disc}(Y, h)$ is also

represented by

$$\left(\prod_p \mathrm{Det}(\lambda_p), \mathrm{Pf}(\xi^*(h)(x_0^i, x_0^j))\right) \times \left(\prod_p \mathrm{Det}(\mu_p)^{-1}, \mathrm{Pf}(h(\xi(x_0^i), \xi(x_0^j)))^{-1}\right)$$

$$= \left(\prod_p \mathrm{Det}(\lambda_p) \cdot \mathrm{Det}(\mu_p)^{-1}, \mathrm{Pf}(h(\xi(x_0^i), \xi(x_0^j))) \, \mathrm{Pf}(h(\xi(x_0^i), \xi(x_0^j)))^{-1}\right)$$

$$= \left(\prod_p \mathrm{Det}(\lambda_p) \cdot \mathrm{Det}(\mu_p)^{-1}, 1\right)$$

where the first equality follows immediately from the definition of the pullback $\xi^*(h)$.

Claims (iii) and (iv) are immediate consequences of the respective Hom descriptions of the groups $\mathrm{A}(\Gamma)$, $\mathrm{eHCl}(\Gamma)$, and $\mathrm{Cl}(\Gamma)$.

Turning to claim (v) we note at the outset that the upper and lower left- and right-hand-most triangles commute by definition of the maps involved and that the outer quadrilateral commutes since both of the composites $\partial_\Gamma^{1,2} \circ \partial_\Gamma^{1,1}$ and $\partial_\Gamma^{2,2} \circ \partial_\Gamma^{2,1}$ send each pair $([\theta_1], \theta_2)$ to the class of $[\theta_1]$.

We next note that the commutativity of the upper central triangle, namely the equality $\partial_{\mathbb{Z}, \mathbb{Q}^c, \Gamma}^0 = \partial_\Gamma^{\mathrm{met}} \circ \pi_\Gamma^{\mathrm{met}}$, will follow if we show that the composites $\partial_\Gamma^{1,2} \circ \partial_\Gamma^{1,1} \circ h_\Gamma^{\mathrm{rel}}$ and $h_\Gamma^{\mathrm{red}} \circ \partial_{\mathbb{Z}, \mathbb{Q}^c, \Gamma}^0$ coincide.

This is true because the explicit description of h_Γ^{rel} implies that $\partial_\Gamma^{1,2} \circ \partial_\Gamma^{1,1} \circ h_\Gamma^{\mathrm{rel}}$ sends each element $[X, \xi, Y]$ of $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ to the class represented by the homomorphism

$$\prod_p (\mathrm{Det}(\lambda_p) \cdot \mathrm{Det}(\mu_p)^{-1}) = \left(\prod_p \mathrm{Det}(\lambda_p)\right) \cdot \left(\prod_p \mathrm{Det}(\mu_p)\right)^{-1}$$

while

$$(h_\Gamma^{\mathrm{red}} \circ \partial_{\mathbb{Z}, \mathbb{Q}^c, \Gamma}^0)([X, \xi, Y]) = h_\Gamma^{\mathrm{red}}([X] - [Y]) = h_\Gamma^{\mathrm{red}}([X]) h_\Gamma^{\mathrm{red}}([Y])^{-1}$$

and Remark 3.2 implies that the classes $h_\Gamma^{\mathrm{red}}([X])$ and $h_\Gamma^{\mathrm{red}}([Y])$ are represented by the products $\prod_p \mathrm{Det}(\lambda_p)$ and $\prod_p \mathrm{Det}(\mu_p)$, respectively.

The above facts combine to directly imply commutativity of the lower central triangle, namely the equality $\partial_{\mathbb{Z}, \mathbb{Q}^c, \Gamma}^0 = \partial_\Gamma^{\mathrm{herm}} \circ \pi_\Gamma^{\mathrm{herm}}$, and so it only suffices to prove commutativity of the four triangles inside this triangle.

We shall now discuss these triangles clockwise, starting from the uppermost.

The commutativity of the first triangle follows directly from the fact that for any finite Γ-module M of finite projective dimension, and any resolution of the form

$$0 \to P \xrightarrow{\theta} P' \to M \to 0,$$

where P is finitely generated and locally free and P' is finitely generated and free, the class of M in $K_0 T(\mathbb{Z}[\Gamma])$ is sent by ι_Γ to $[P, \theta_{\mathbb{Q}}, P']$ and by ∂_Γ to $[P] - [P']$ ($= [P]$ as P' is free).

If for the above sequence we fix a $\mathbb{Z}[\Gamma]$-basis $\{x^i\}$ of P' and then for each prime p choose a matrix λ_p in $\mathrm{GL}_d(\mathbb{Q}_p[\Gamma])$ so that the components of the vector $\lambda_p \cdot \theta_{\mathbb{Q}}^{-1}(x^i)$ are a $\mathbb{Z}_p[\Gamma]$-basis of P_p, then the image of the class of M in $K_0 T(\mathbb{Z}[\Gamma])$ under Υ_Γ is represented by $\left(\prod_p \mathrm{Det}(\lambda_p), 1\right)$. This implies the commutativity of the second triangle since Remark 3.2 implies the class of $\partial_\Gamma(M) = [P]$ is represented by $\left(\prod_p \mathrm{Det}(\lambda_p)\right)$ while the definition of δ_Γ'' implies that it is induced by sending each pair $([\theta_1], \theta_2)$ to $(h_\Gamma^{\mathrm{red}})^{-1}([\theta_1])$.

The latter fact also directly implies commutativity of the third triangle and the fourth triangle commutes since, in terms of the above notation, the composite $h_\Gamma^{\mathrm{rel}} \circ \iota_\Gamma$ sends the class of M to the element represented by the pair $\left(\left(\prod_p \mathrm{Det}(\lambda_p)\right), 1\right)$. $\square$

In the next result we describe an explicit link between the elements in relative algebraic K-theory constructed in Section 2A3, the hermitian modules described in Example 2.11, and the metrized modules defined in Example 2.18.

This link explains the relevance of Theorem 3.3 to our later results.

Proposition 3.4. *Let L/K be a finite Galois extension of number fields with group G. Then for any full projective $\mathcal{O}_K[G]$-submodule $\mathcal{L}$ of L the following claims are valid.*

(i) *The image of $[\mathcal{L}, \kappa_L, H_L]$ under Π_G^{met} is equal to $[\mathcal{L}, h_{L,\bullet}] - [H_L, \mu_{L,\bullet}]$.*

(ii) *The image of $[\mathcal{L}, \kappa_L, H_L]$ under Π_G^{herm} is equal to $\mathrm{Disc}(\mathcal{L}, t_{L/K})$.*

Proof. The pullback with respect to κ_L of the metric $\mu_{L,\bullet}$ defined in Example 2.18 is equal to $h_{L,\bullet}$ [Agboola and Burns 2006, Example 4.10(i)]. This fact combines with Theorem 3.3(i) to directly imply the equality in claim (i).

To prove claim (ii) we use the representative $(\theta_1\theta_2^{-1}, \theta_2\theta_3)$ of $h_G^{\mathrm{rel}}([\mathcal{L}, \kappa_L, H_L])$ described in Lemma 2.8. We also recall that, with this notation, the general result of Fröhlich [1984, Corollary to Theorem 27] implies the element $\mathrm{Disc}(\mathcal{L}, t_{L/K}) - \mathrm{Disc}(\mathcal{O}_K[G], t_{K[G]})$ of $\mathrm{HCl}(G)$ is represented by $(\theta_1 \cdot \theta_2^{-1}, \theta_2^s)$, where the form $t_{K[G]}$ is as defined in Example 2.10.

Comparing these results one deduces that the element

$$\Pi_G^{\mathrm{herm}}([\mathcal{L}, \kappa_L, H_L]) - \mathrm{Disc}(\mathcal{L}, t_{L/K}) + \mathrm{Disc}(\mathcal{O}_K[G], t_{K[G]})$$

of $\mathrm{HCl}(G)$ is represented by the pair $(1, \theta_3^s)$.

To deduce claim (ii) from this it is thus enough to show that the pair $(1, \theta_3^s)$ also represents the element $\mathrm{Disc}(\mathcal{O}_K[G], t_{K[G]})$.

To check this we need only note that, in the terminology of [Fröhlich 1984, Chapter II, §5], the Pfaffian of the matrix $(t_{K[G]}(a_\sigma, a_\tau))_{\sigma,\tau \in \Sigma(K)}$ sends each character χ in R_G^s to $\delta_K^{\chi(1)} = \theta_3(\chi)$.

Then, by applying the recipe of Definition 2.14 with $\{x_0^j\} = \{x_p^j\} = \{a_\sigma\}_{\sigma \in \Sigma(K)}$ one finds that $\mathrm{Disc}(\mathbb{O}_K[G], t_{K[G]})$ is indeed represented by the pair $(1, \theta_3^s)$, as required. $\qquad\qquad\square$

Part II. Weak ramification and Galois–Gauss sums

In this part of the article we describe a first arithmetic application of the approach described in earlier sections by using Theorem 3.3 (and Proposition 3.4) to refine existing results concerning links between Galois–Gauss sums and certain metric and hermitian structures that arise naturally in arithmetic.

In this way, in Section 4 we refine the main results of Burns and Chinburg [1996] concerning relations between hermitian-metric structures involving fractional powers of the inverse different of a tamely ramified Galois extension of number fields and the associated Galois–Gauss sums (twisted by appropriate Adams operations).

In the remainder of the article we then focus on weakly ramified Galois extensions (of odd degree) and use Theorem 3.3 to refine key aspects of the extensive existing theory of the square root of the inverse different for such extensions.

4. Tamely ramified Galois–Gauss sums

4A. *Galois–Gauss sums, Adams operators, and Galois–Jacobi sums.* For the reader's convenience in this section we fix notation regarding various variants of Galois–Gauss sums that will play a role in the sequel.

To do this we fix an arbitrary finite Galois extension L/K of number fields in $\mathbb{Q}^c$ and set $G := G(L/K)$.

For each character χ in $\widehat{G}$ we obtain a primitive central idempotent of $\mathbb{Q}^c[G]$ by setting

$$e_\chi := \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) g^{-1}.$$

We use the fact that each element of $\zeta(\mathbb{Q}^c[G])$ can then be written uniquely in the form

$$x = \sum_{\chi \in \widehat{G}} e_\chi \cdot x_\chi \qquad\qquad (4.1)$$

with each x_χ in $\mathbb{Q}^c$.

For convenience we extend the assignment $x \mapsto x_\chi$ to arbitrary elements χ of R_G by multiplicativity.

4A1. We define the "equivariant global Galois–Gauss sum" for L/K by setting

$$\tau_{L/K} := \sum_{\chi \in \widehat{G}} e_\chi \cdot \tau(K, \chi) \in \zeta(\mathbb{Q}^c[G])$$

where each (global) Galois–Gauss sum $\tau(K, \chi)$ belongs to $\mathbb{Q}^c$ and is as defined, for example, by Fröhlich [1983, Chapter I, (5.22)].

We also define an "equivariant unramified characteristic" in $\zeta(\mathbb{Q}[G])$ by setting

$$y_{L/K} := \sum_{\chi \in \widehat{G}} e_\chi \cdot \prod_{v \mid d_L} y(K_v, \chi_v).$$

Here χ_v is the restriction of χ to the decomposition subgroup of some fixed place w of L above v and (following [Fröhlich 1983, Chapter IV, §1]) for any finite Galois extension of local fields F/E of group D and each ϕ in $\widehat{D}$ we set

$$y(E, \phi) := \begin{cases} 1 & \text{if } \phi|_I \neq 1, \\ -\phi(\sigma) & \text{if } \phi|_I = 1, \end{cases} \tag{4.2}$$

where I is the inertia subgroup of D and σ is a lift to D of the Frobenius element in D/I.

We then define the "modified equivariant (global) Galois–Gauss sum" for L/K by setting

$$\tau'_{L/K} := \tau_{L/K} \cdot y_{L/K}^{-1}.$$

Since we rely on certain results from [Bley and Burns 2003] we will also use the "absolute (global) Galois–Gauss sum for L/K" that is obtained by setting

$$\tau^\dagger_{L/K} := \sum_{\chi \in \widehat{G}} e_\chi \tau(\mathbb{Q}, \text{ind}_K^{\mathbb{Q}} \chi) \in \zeta(\mathbb{Q}^c[G])^\times.$$

In particular, it is useful to note that the inductivity property of Galois–Gauss sums combines with the fact $\tau(K, 1_K) = 1$ to imply

$$\tau^\dagger_{L/K} = \tau_K^G \cdot \tau_{L/K} \tag{4.3}$$

where τ_K^G is the invertible element of $\zeta(\mathbb{Q}^c[G])$ obtained by setting

$$\tau_K^G := \text{Nrd}_{\mathbb{Q}[G]}(\tau(\mathbb{Q}, \text{ind}_K^{\mathbb{Q}} \mathbf{1}_K))$$

so that $(\tau_K^G)_\chi = \tau(\mathbb{Q}, \text{ind}_K^{\widehat{\mathbb{Q}}} \mathbf{1}_K)^{\chi(1)}$ for all χ in $\widehat{G}$.

4A2. For each integer k that is coprime to $|G|$ we write ψ_k for the k-th Adams operator on R_G (for the relevant properties of which we refer to [Burns and Chinburg 1996, Lemma 3.1]).

We use this operator to construct endomorphisms of $\zeta(\mathbb{Q}^c[G])$ in the following way. For each pair of integers m and n we write $(m + n \cdot \psi_{k,*})(x)$ for the unique element of $\zeta(\mathbb{Q}^c[G])$ with $(m + n \cdot \psi_{k,*})(x)_\chi := (x_\chi)^m \cdot (x_{\psi_k(\chi)})^n$ for every χ in $\widehat{G}$.

We then define the "k-th Galois–Jacobi sum" for the extension L/K by setting

$$J_{k,L/K} := (\psi_{k,*} - k)(\tau_{L/K}).$$

In the sequel we shall often use the following key property of these sums.

Lemma 4.4. *For each integer k prime to $|G|$ one has $J_{k,L/K} \in \zeta(\mathbb{Q}[G])^{\times}$.*

Proof. An element x of $\zeta(\mathbb{Q}^c[G])$ belongs to $\zeta(\mathbb{Q}[G])$ if and only if one has $(x_\chi)^\omega = x_{\chi^\omega}$ for all $\chi \in \widehat{G}$ and all $\omega \in \Omega_{\mathbb{Q}}$.

To verify that the elements $J_{k,L/K}$ satisfy this criterion we recall how the absolute Galois group acts on Gauss sums. We let $\mathrm{Ver}_{K/\mathbb{Q}} : \Omega_{\mathbb{Q}}^{\mathrm{ab}} \to \Omega_K^{\mathrm{ab}}$ denote the transfer map and write $v_{K/\mathbb{Q}}$ for the cotransfer map from abelian characters of Ω_K to abelian characters of $\Omega_{\mathbb{Q}}$. Thus, for each $\chi \in \widehat{G}$ the map $v_{K/\mathbb{Q}} \det_\chi$ is an abelian character of $\Omega_{\mathbb{Q}}$. Then, by [Fröhlich 1983, Theorem 20B(ii)], one has $\tau(K, \chi^{\omega^{-1}})^\omega = \tau(K, \chi) \cdot (v_{K/\mathbb{Q}} \det_\chi)(\omega)$ for all $\chi \in \widehat{G}$ and all $\omega \in \Omega_{\mathbb{Q}}$.

Hence, it suffices to show that $((v_{K/\mathbb{Q}} \det_\chi)(\omega))^k = (v_{K/\mathbb{Q}} \det_{\psi_k(\chi)})(\omega)$ and this is true because $\det_{\psi_k(\chi)} = (\det_\chi)^k$ [Burns and Chinburg 1996, Lemma 3.1]. $\qquad \square$

With the results of [Bley and Burns 2003] in mind we finally note that if G has odd order, then an explicit comparison of the respective definitions shows that

$$\tau_K^G \cdot (\psi_{2,*} - 1)(\tau'_{L/K}) \cdot (\tau^\dagger_{L/K})^{-1} = J_{2,L/K} \cdot (\psi_{2,*} - 1)(y_{L/K}^{-1}). \qquad (4.5)$$

Remark 4.6. If F/E is a finite Galois extension of p-adic fields (for some p) with group D, then one can use the canonical local Gauss sum $\tau(E, \phi)$ (as discussed, for example, in [Fröhlich 1983, Chapter III, §2, Theorem 18 and Remark 1]) for each ϕ in $\widehat{D}$ to define natural analogs $\tau_{F/E}, y_{F/E}, \tau'_{F/E}, \tau^\dagger_{F/E}, \tau_E^D$, and $J_{k,F/E}$ in $\zeta(\mathbb{Q}^c[D])$ of the elements defined above. Then in the same way as above one can show that for each integer k that is coprime to $|D|$ the element $J_{k,F/E}$ belongs to $\zeta(\mathbb{Q}[D])^{\times}$ and can also prove the local analogs of the equalities (4.3) and (4.5)

$$\tau^\dagger_{F/E} = \tau_E^D \cdot \tau_{F/E} \qquad (4.7)$$

and

$$\tau_E^D \cdot (\psi_{2,*} - 1)(\tau'_{F/E}) \cdot (\tau^\dagger_{F/E})^{-1} = J_{2,F/E} \cdot (\psi_{2,*} - 1)(y_{F/E}^{-1}). \qquad (4.8)$$

4A3. In the next result we write $W_{L/K}$ for the so-called "Cassou-Noguès–Fröhlich root number class" in $\mathrm{Cl}(G)$.

We recall that this element plays a critical role in classical Galois module theory (as discussed by Fröhlich [1983; 1984]).

Lemma 4.9. *There exists a canonical element $W_{L/K}^{\mathrm{rel}}$ of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ that has all of the following properties.*

(i) *The image of $W_{L/K}^{\mathrm{rel}}$ under the connecting homomorphism $\partial^0_{\mathbb{Z},\mathbb{Q}^c,G}$ is $W_{L/K}$.*

(ii) *$W_{L/K}^{\mathrm{rel}}$ is trivial if the Artin root number of each symplectic character in $\widehat{G}$ is positive.*

(iii) *In all cases the element $2 \cdot W_{L/K}^{\mathrm{rel}}$ is trivial.*

Proof. The element $W_{L/K}$ is defined directly in terms of the Artin root numbers of symplectic characters in $\widehat{G}$ by means of the isomorphism h_G^{red} in (3.1).

One can use the isomorphism h_G^{rel} in (2.6) to define $W_{L/K}^{\mathrm{rel}}$ in a similarly explicit way. However, for later purposes, it is useful to adopt a different approach to the definition of $W_{L/K}^{\mathrm{rel}}$.

To do this we recall the element $\epsilon_{L/K}$ of $\zeta(\mathbb{R}[G])^{\times}$ that is defined in terms of epsilon constants in [Bley and Burns 2003, just after (9)].

Then, in view of the description of $\mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]})$ that is given by the Hasse–Schilling–Maass norm theorem, we can use the weak approximation theorem to choose an element λ of $\zeta(\mathbb{Q}[G])^{\times}$ with the property that $\lambda \cdot \epsilon_{L/K}$ belongs to $\mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]})$.

We then obtain an element of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ by setting

$$W_{L/K}^{\mathrm{rel}} := \delta_G(\lambda) - \sum_p \delta_{G,p}(\lambda)$$

where p runs over all primes and each $\delta_{G,p}(\lambda)$ is regarded as an element of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ by means of the composite inclusion

$$K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G]) \subset K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \subset K_0(\mathbb{Z}[G], \mathbb{Q}^c[G]).$$

This recipe is independent of the choice of λ since if λ' is any choice, then $\lambda^{-1}\lambda'$ belongs to $\mathrm{im}(\mathrm{Nrd}_{\mathbb{Q}[G]})$ and so one has

$$
\begin{aligned}
\delta_G(\lambda') - \delta_G(\lambda) &= \delta_G(\lambda^{-1}\lambda') \\
&= (\partial_{\mathbb{Z},\mathbb{Q},G}^1 \circ (\mathrm{Nrd}_{\mathbb{Q}[G]})^{-1})(\lambda^{-1}\lambda') \\
&= \sum_p (\partial_{\mathbb{Z}_p,\mathbb{Q}_p,G}^1 \circ (\mathrm{Nrd}_{\mathbb{Q}_p[G]})^{-1})(\lambda^{-1}\lambda') \\
&= \sum_p \delta_{G,p}(\lambda') - \sum_p \delta_{G,p}(\lambda).
\end{aligned}
$$

Given this definition of $W_{L/K}^{\mathrm{rel}}$, the property in claim (i) follows directly from the argument of [Bley and Burns 2003, Proposition 3.1].

In addition, claim (ii) is true because the given hypotheses imply that $\epsilon_{L/K}$ belongs to $\mathrm{im}(\mathrm{Nrd}_{\mathbb{R}[G]})$ so that one can compute $W_{L/K}^{\mathrm{rel}}$ by using the element $\lambda = 1$.

Finally, claim (iii) follows easily from the fact that the square of any element of $\zeta(\mathbb{Q}[G])^{\times}$ belongs to $\mathrm{im}(\mathrm{Nrd}_{\mathbb{Q}[G]})$. $\qquad\square$

4B. *Tame Galois–Gauss sums and fractional powers of the different.* We now assume the Galois extension L/K is tamely ramified and fix a natural number k that is both coprime to $|G|$ and so that the order of each inertia subgroup of G is congruent to 1 modulo k.

In any such case it follows immediately from Hilbert's formula for the different in terms of ramification invariants [Serre 1979, Chapter IV, Proposition 4] that there exists a unique fractional ideal $\mathfrak{D}_{L/K}^{-1/k}$ of $\mathcal{O}_L$ whose k-th power is equal to the inverse of the different $\mathfrak{D}_{L/K}$ of L/K and for any integer i we set $\mathfrak{D}_{L/K}^{-i/k} = (\mathfrak{D}_{L/K}^{-1/k})^i$.

Each ideal $\mathfrak{D}_{L/K}^{-i/k}$ is stable under the natural action of $\mathcal{O}_K[G]$ and, since L/K is assumed to be tamely ramified, the $\mathcal{O}_K[G]$-module $\mathfrak{D}_{L/K}^{-i/k}$ is known to be locally free [Ullom 1969].

In particular, since $\mathfrak{D}_{L/K}^{-i/k}$ is a full sublattice of L, the construction of Section 2A3 gives rise to a well defined element $[\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L]$ of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$.

We next write $\psi_k^\vee$ for the map which sends a function h on R_G to $h \circ \psi_k$ and recall that, since k is prime to $|G|$, Cassou-Noguès and Taylor [1985] have shown that the assignment

$$(\theta, \theta') \mapsto (\psi_k^\vee(\theta), \psi_k^\vee(\theta'))$$

induces (via the map (2.5) and isomorphism (2.6)) a well defined endomorphism Ψ_k of the group $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$.

We can now state the main result of this section. This result uses the invertible elements τ_K^G and $\tau_{L/K}'$ of $\zeta(\mathbb{Q}^c[G])$ that are defined in Section 4A as well as the relative Cassou-Noguès–Fröhlich root number class $W_{L/K}^{\mathrm{rel}}$ defined in Lemma 4.9.

Theorem 4.10. *Let L/K be a tamely ramified Galois extension of number fields with group G and k any natural number that is both coprime to $|G|$ and such that the order of each inertia subgroup of G is congruent to 1 modulo k. Then in $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ one has*

$$\sum_{i=0}^{k-1} [\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L] = \delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau_{L/K}')) + \Psi_k(W_{L/K}^{\mathrm{rel}}). \qquad (4.11)$$

Before proving this result we use it to derive certain explicit consequences concerning the metric and hermitian structures that arise in this setting.

In particular, the following result extends the results of Erez and Taylor [1992] on the hermitian modules $(\mathcal{O}_L, t_{L/K})$, corresponding to $k = 1$, and on $(\mathfrak{D}_{L/K}^{-1/2}, t_{L/K})$, corresponding to $k = 2$ and assuming G of odd order, to all integers k as in Theorem 4.10.

We recall the definition of the element δ_K from Lemma 2.8 and write d_K for the discriminant of $\mathcal{O}_K$.

In the sequel we will often use the fact that

$$\tau(\mathbb{Q}, \mathrm{ind}_K^{\mathbb{Q}} \mathbf{1}_K)^2 = d_K = \delta_K^2, \qquad (4.12)$$

as follows by combining [Neukirch 1999, Theorem (11.7)(iii)] together with [Fröhlich 1983, (5.23)].

Corollary 4.13. *Assume the notation and hypotheses of Theorem 4.10. Then both of the following claims are valid.*

(i) *In* A(G) *one has*

$$\sum_{i=0}^{k-1} [\mathfrak{D}_{L/K}^{-i/k}, h_{L,\bullet}] = \varepsilon_{L/K,k}^{\mathrm{met}} + \Pi_G^{\mathrm{met}}(\Psi_k(W_{L/K}^{\mathrm{rel}}))$$

where $h_{L,\bullet}$ is the metric defined in Example 2.18 and $\varepsilon_{L/K,k}^{\mathrm{met}}$ is represented by the pair $(1, |\theta_k|)$ with $\theta_k(\phi) = (|G|^{[K:\mathbb{Q}]}|d_K|)^{k(\phi(1)/2)} \cdot \tau(K, \psi_k(\phi))$ for all ϕ in R_G.

(ii) *In* HCl(G) *one has*

$$\sum_{i=0}^{k-1} \mathrm{Disc}(\mathfrak{D}_{L/K}^{-i/k}, t_{L/K}) = \varepsilon_{L/K,k}^{\mathrm{herm}} + \Pi_G^{\mathrm{herm}}(\Psi_k(W_{L/K}^{\mathrm{rel}}))$$

where the hermitian form $t_{L/K}$ is as defined in Example 2.11 and $\varepsilon_{L/K,k}^{\mathrm{herm}}$ is represented by the pair $(1, \tilde{\theta}_k)$ with $\tilde{\theta}_k(\phi) = d_K^{k(\phi(1)/2)} \cdot \tau(K, \psi_k(\phi))$ for all ϕ in R_G^s.

Proof. To prove claim (i) we note first Proposition 3.4(i) implies that for each i one has

$$\Pi_G^{\mathrm{met}}([\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L]) = [\mathfrak{D}_{L/K}^{-i/k}, h_{L,\bullet}] - [H_L, \mu_{L,\bullet}].$$

We next recall that for $\alpha = (\alpha_\chi)_{\chi \in \widehat{G}} \in \zeta(\mathbb{Q}^c[G])^\times$ the element $h_G^{\mathrm{rel}}(\delta_G(\alpha))$ is represented by the function $\chi \mapsto (1, \alpha_\chi)$.

This implies, in particular, that $h_G^{\mathrm{rel}}(\delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau_{L/K}')))$ is represented by the pair $(1, \theta_k')$ with $\theta_k'(\phi) := \tau(\mathbb{Q}, \mathrm{ind}_K^{\mathbb{Q}} \mathbf{1}_K)^{k\phi(1)} \cdot \tau'(K, \psi_k(\phi))$ for each ϕ in $\widehat{G}$.

Finally we recall that the element $[H_L, \mu_{L,\bullet}]$ has been explicitly computed in Lemma 2.22.

Putting these facts together with the equality in Theorem 4.10 one finds that the element

$$\sum_{i=0}^{k-1} [\mathfrak{D}_{L/K}^{-i/k}, h_{L,\bullet}] - \Pi_G^{\mathrm{met}}(\Psi_k(W_{L/K}^{\mathrm{rel}}))$$
$$= k \cdot [H_L, \mu_{L,\bullet}] + \Pi_G^{\mathrm{met}}\left(\sum_{i=0}^{k-1} [\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L] - \Psi_k(W_{L/K}^{\mathrm{rel}})\right)$$
$$= k \cdot [H_L, \mu_{L,\bullet}] + \Pi_G^{\mathrm{met}}(\delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau_{L/K}')))$$

of $A(G)$ is represented by the homomorphism pair $(1, |\theta_k|)$ where for each ϕ in $\widehat{G}$ one has

$$\theta_k(\phi) := |G|^{[K:\mathbb{Q}](k\phi(1)/2)} \cdot \tau(\mathbb{Q}, \mathrm{ind}_K^{\mathbb{Q}} \mathbf{1}_K)^{k\phi(1)} \cdot \tau'(K, \psi_k(\phi)).$$

But, taking account of both (4.12) and the fact that $y(K_v, \phi_v)$ is a root of unity for all ϕ in $\widehat{G}$, one finds that

$$|\theta_k|(\phi) = (|G|^{[K:\mathbb{Q}]}|d_K|)^{k(\phi(1)/2)} \cdot |\tau(K, \psi_k(\phi))|$$

and this proves claim (i).

It is enough to prove the equality of claim (ii) in $\mathrm{eHCl}(G)$ and to do this we note that the description in Proposition 3.4(ii) combines with Theorem 4.10 to imply that

$$\sum_{i=0}^{k-1} \mathrm{Disc}(\mathfrak{D}_{L/K}^{-i/k}, t_{L/K}) - \Pi_G^{\mathrm{herm}}(\Psi_k(W_{L/K}^{\mathrm{rel}}))$$
$$= \Pi_G^{\mathrm{herm}}\left(\sum_{i=0}^{k-1}[\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L] - \Psi_k(W_{L/K}^{\mathrm{rel}})\right)$$
$$= \Pi_G^{\mathrm{herm}}(\delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau_{L/K}'))).$$

In addition, by the definition of Π_G^{herm} one has the equality in $\mathrm{eHCl}(G)$

$$\Pi_G^{\mathrm{herm}}(\delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau_{L/K}'))) = (\partial_G^{2,1} \circ h_G^{\mathrm{rel}})(\delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau_{L/K}'))).$$

Hence, one deduces that the difference

$$\sum_{i=0}^{k-1} \mathrm{Disc}(\mathfrak{D}_{L/K}^{-i/k}, t_{L/K}) - \Pi_G^{\mathrm{herm}}(\Psi_k(W_{L/K}^{\mathrm{rel}}))$$

is represented by the pair $(1, (\theta_k')^s)$, where θ_k' is as defined in the proof of claim (i).

To deduce claim (ii) from this it is now enough to note that for ϕ in R_G^s one has

$$\theta_k'(\phi) = \tau(\mathbb{Q}, \mathrm{ind}_K^{\mathbb{Q}} \mathbf{1}_K)^{k\phi(1)} \cdot \tau'(K, \psi_k(\phi)) = d_K^{k\phi(1)/2} \cdot \tau(K, \psi_k(\phi)), \quad (4.14)$$

where, to derive the second equality, we have used (4.12) and the fact that for every ϕ in R_G^s the integer $\phi(1)$ is even and $y(K_v, \phi_v) = 1$ by [Fröhlich 1983, Theorem 29(i)]. $\qquad \square$

In the remainder of this section we shall prove Theorem 4.10 by combining results of Bley and Burns [2003] and Burns and Chinburg [1996].

To do this we fix a $K[G]$-generator b of L and a $\mathbb{Z}$-basis $\{a_\sigma\}_{\sigma \in \Sigma(K)}$ of $\mathbb{O}_K$. For each integer i with $0 \le i < k$ and each nonarchimedean place v of K we also fix an $\mathbb{O}_{K,v}[G]$-generator $b_{i,v}$ of $(\mathfrak{D}_{L/K}^{-i/k})_v$.

Then by Lemma 2.8 the element $h_G^{\mathrm{rel}}([\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L])$ is represented by the pair of homomorphisms $(\theta_{i,1} \cdot \theta_2^{-1}, \theta_2 \cdot \theta_3)$ where for each χ in R_G one has

$$\theta_{i,1}(\chi) := \prod_v \mathscr{N}_{K/\mathbb{Q}}(b_{i,v} \mid \chi), \quad \theta_2(\chi) := \mathscr{N}_{K/\mathbb{Q}}(b \mid \chi), \quad \theta_3(\chi) := \delta_K^{\chi(1)}. \quad (4.15)$$

With this notation, it is straightforward to check that

$$(\theta_2^{-k}, \theta_2^k) \equiv (\psi_k^\vee(\theta_2)^{-1}, \psi_k^\vee(\theta_2)) \pmod{\mathrm{im}(\Delta_G^{\mathrm{rel}})},$$

(see, for example, the end of the proof of [Burns and Chinburg 1996, Proposition 3.3]) and it is also clear $\psi_k^\vee(\theta_3) = \theta_3$.

In particular, if we denote the sum on the left-hand side of (4.11) by Σ_k, these observations combine with the above description of each element $h_G^{\mathrm{rel}}([\mathfrak{D}_{L/K}^{-i/k}, \kappa_L, H_L])$ and the congruence proved in Lemma 4.17 below to imply that $h_G^{\mathrm{rel}}(\Sigma_k)$ is represented by the pair

$$(\psi_k^\vee(\theta_{0,1} \cdot \theta_2^{-1}), \psi_k^\vee(\theta_2 \cdot \theta_3^k)) = (\psi_k^\vee(\theta_{0,1} \cdot \theta_2^{-1}), \psi_k^\vee(\theta_2 \cdot \theta_3)) \cdot (1, \theta_3^{k-1}).$$

It follows that, writing x_k for the element of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ for which $h_G^{\mathrm{rel}}(x_k)$ is represented by the pair $(1, \theta_3^{k-1})$, one has

$$\Sigma_k = \Psi_k([\mathbb{O}_L, \kappa_L, H_L]) + x_k.$$

We claim next that the results of [Bley and Burns 2003] imply that

$$[\mathbb{O}_L, \kappa_L, H_L] = \delta_G(\tau_K^G \cdot \tau'_{L/K}) + W_{L/K}^{\mathrm{rel}}. \qquad (4.16)$$

Before proving this equality we note that, if true, it would combine with the previous equality to imply that the element $h_G^{\mathrm{rel}}(\Sigma_k - \Psi_k(W_{L/K}^{\mathrm{rel}}))$ is represented by the homomorphism pair $(1, \theta_3^{k-1} \cdot \theta_3' \cdot \psi_k^\vee(\theta_4))$, where for each χ in R_G one has $\theta_3'(\chi) = \tau(\mathbb{Q}, \mathrm{ind}_K^\mathbb{Q} \mathbf{1}_K)^{\chi(1)}$ and $\theta_4(\chi) = (\tau'_{L/K})_\chi$.

On the other hand, from (4.12) one has $\tau(\mathbb{Q}, \mathrm{ind}_K^\mathbb{Q} \mathbf{1}_K) = \pm \delta_K$ so that

$$(1, \theta_3) \equiv (1, \theta_3') \pmod{\mathrm{im}(\Delta_G^{\mathrm{rel}})},$$

so $h_G^{\mathrm{rel}}(\delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau'_{L/K})))$ is also represented by the pair $(1, \theta_3^{k-1} \cdot \theta_3' \cdot \psi_k^\vee(\theta_4))$.

It would thus follow that $\Sigma_k - \Psi_k(W_{L/K}^{\mathrm{rel}}) = \delta_G((\tau_K^G)^k \cdot \psi_{k,*}(\tau'_{L/K}))$, as claimed.

To complete the proof of Theorem 4.10 it is therefore enough to prove (4.16). To do this we note that the notation $\mathscr{E}_{L/K}$ introduced in [Bley and Burns 2003, §3.1] denotes the element

$$\delta_G(\lambda \cdot \epsilon_{L/K}) - \sum_p \delta_{G,p}(\lambda) = \delta_G(\epsilon_{L/K}) + W_{L/K}^{\mathrm{rel}}$$

of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$, where λ in $\zeta(\mathbb{Q}[G])^\times$ is chosen as in the proof of Lemma 4.9. Note that here and in the sequel, to be able to apply the results of [Bley and Burns 2003] we are implicitly working in the group $K_0(\mathbb{Z}[G], \mathbb{C}[G])$, regarding both $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ and $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ as subgroups in the obvious way.

The definition of the element $\delta_{L/K}(\mathcal{O}_L)$ of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ given in [Bley and Burns 2003, §3.2] ensures that

$$
\begin{aligned}
\delta_G(\tau_{L/K}^\dagger) - [\mathcal{O}_L, \kappa_L, H_L] &= \delta_G(\epsilon_{L/K}) - \delta_{L/K}(\mathcal{O}_L) \\
&= \mathcal{E}_{L/K} - \delta_{L/K}(\mathcal{O}_L) - W_{L/K}^{\mathrm{rel}} \\
&= \delta_G(y_{L/K}) - W_{L/K}^{\mathrm{rel}}.
\end{aligned}
$$

The first equality here is a consequence of [Bley and Burns 2003, Remark 3.5] and the fact that the element $\tau_{L/K}$ and map ρ_L in [loc. cit.] correspond, in our notation, to $\tau_{L/K}^\dagger$ and κ_L. In addition, the third equality follows directly from [Bley and Burns 2003, Corollary 7.7].

To derive the required equality (4.16) from the last displayed formula, it is then enough to note that (4.3) implies $\tau_{L/K}^\dagger \cdot y_{L/K}^{-1}$ is equal to $\tau_K^G \cdot \tau_{L/K} \cdot y_{L/K}^{-1} = \tau_K^G \cdot \tau_{L/K}'$.

Lemma 4.17. *For the homomorphisms $\theta_{i,1}$ for $0 \le i < k$ that are defined in (4.15) one has*

$$
\prod_{i=0}^{k-1} \theta_{i,1} \equiv \psi_k^\vee(\theta_{0,1}) \pmod{\mathrm{Det}(U_f(\mathbb{Z}[G]))}.
$$

Proof. This is proved by a slight adaptation of the arguments in [Burns and Chinburg 1996] (and is implicitly used in the proof of [loc. cit., Corollary 2.2]). To be precise, we shall use the notation of [Burns and Chinburg 1996, §4.3.1] with our integer k corresponding the integer ℓ used in [loc. cit.].

Then the present hypotheses (on k) allow us to choose integer ℓ' to be $(1-e)/\ell$. In particular, if we set $N := 0$, then $N_\ell = 0$ and, for each i with $0 \le i < \ell$, also $N_i = -i\ell'(e-1) = -i\ell'e + N_i'$ with $N_i' := -i(e-1)/\ell$. Each element a_{N_i} can therefore be written as $c_i \cdot a_{N_i'}$ with c_i an element of B with $v_{\mathfrak{p}}(c_i) = -i\ell'$.

With this choice of ℓ' an explicit computation shows that the integer $M_{\mathfrak{p},\ell,\ell'}$ defined in [loc. cit., (2.4)] is equal to $\sum_{i=0}^{\ell-1} i\ell'$ and so one can take the element c chosen in [loc. cit., Corollary 4.5] to be the product $\prod_{i=0}^{\ell-1} c_i$. For this element there is for every χ in $\mathrm{Hom}(\Lambda, B^{c\times})$ an equality

$$
(ca_0 \mid \psi_\ell \chi) \prod_{i=0}^{\ell-1} (a_{N_i} \mid \chi)^{-1} = (a_0 \mid \psi_\ell \chi) \prod_{i=0}^{\ell-1} (a_{N_i'} \mid \chi)^{-1},
$$

and so [loc. cit., Corollary 4.5] asserts that the $\mathfrak{p}$-adic valuation of this element is zero.

It is now straightforward to derive the claimed congruence by combining this fact with the argument of [loc. cit., §5]. □

5. Weakly ramified Galois–Gauss sums and the relative element $\mathfrak{a}_{L/K}$

In the remainder of the article we study links between Galois–Gauss sums and hermitian and metric structures that arise in weakly ramified Galois extensions of odd degree. In this first section we define a canonical element in relative algebraic K-theory that is key to the theory we develop and then state some of the main results about this element that we establish in later sections.

At the outset we fix a finite odd-degree Galois extension of number fields L/K that is "weakly ramified" in the sense of Erez [1991] (that is, the second lower ramification subgroups in G of each place of L are trivial) and set $G := G(L/K)$.

Since L/K is of odd degree there exists a unique fractional $\mathbb{O}_L$-ideal $\mathscr{A}_{L/K}$ whose square is the inverse of the different $\mathfrak{D}_{L/K}$ (see the discussion at the beginning of Section 4B).

In addition, since L/K is weakly ramified, Erez [1991] showed that $\mathscr{A}_{L/K}$ is a locally free module with respect to the restriction of the natural action of $\mathbb{O}_K[G]$ on L.

We may therefore use the general construction of Section 2A3 to define a canonical element of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ by setting

$$\mathfrak{a}_{L/K} := [\mathscr{A}_{L/K}, \kappa_L, H_L] - \delta_G(\tau_K^G \cdot (\psi_{2,*} - 1)(\tau'_{L/K})) \tag{5.1}$$

where the Galois–Gauss sums τ_K^G and $\tau'_{L/K}$ are as defined in Section 4A.

Proposition 3.4 implies the projection of $[\mathscr{A}_{L/K}, \kappa_L, H_L]$ to each of the groups $A(G)$, $\mathrm{HCl}(G)$, and $\mathrm{Cl}(G)$ recovers arithmetic invariants related to $\mathscr{A}_{L/K}$ that have been studied in previous articles. By using this fact explicit information about the element $\mathfrak{a}_{L/K}$ can often constitute a strong refinement of pre-existing results or conjectures concerning the metric and hermitian structures that are associated to $\mathscr{A}_{L/K}$ and this observation motivates the systematic study of $\mathfrak{a}_{L/K}$ that we undertake in later sections.

In the next result (which will be proved in Section 8B) we collect some of the main results that we prove concerning $\mathfrak{a}_{L/K}$.

In the sequel we write $\mathscr{W}_{L/K}$ for the set of finite places v of K that ramify wildly in an extension L/K and $\mathscr{W}_{L/K}^{\mathbb{Q}}$ for the set of rational primes that lie below any place in $\mathscr{W}_{L/K}$.

We also let A_{tor} denote the torsion subgroup of an abelian group A.

Theorem 5.2. *Let L/K be a finite odd-degree weakly ramified Galois extension of number fields of group G. Then the following assertions are valid.*

(i) *The element $\mathfrak{a}_{L/K}$ belongs to the subgroup*

$$\bigoplus_{\ell \in \mathscr{W}_{L/K}^{\mathbb{Q}}} K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])_{\mathrm{tor}}$$

of $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$. In particular, if L/K is tamely ramified, then $\mathfrak{a}_{L/K} = 0$.

(ii) *In $A(G)$ one has*

$$[\mathscr{A}_{L/K}, h_{L,\bullet}] = \Pi_G^{\mathrm{met}}(\mathfrak{a}_{L/K}) + \varepsilon_{L/K}^{\mathrm{met}}$$

where the metric $h_{L,\bullet}$ is as defined in Example 2.18 and $\varepsilon_{L/K}^{\mathrm{met}}$ is represented by the pair $(1, \theta)$ with $\theta(\phi) = (|G|^{[K:\mathbb{Q}]}|\mathrm{d}_K|)^{\phi(1)/2} \cdot |\tau(K, \psi_2(\phi) - \phi)|$ for all ϕ in R_G.

(iii) *In $\mathrm{HCl}(G)$ one has*

$$\mathrm{Disc}(\mathscr{A}_{L/K}, t_{L/K}) = \Pi_G^{\mathrm{herm}}(\mathfrak{a}_{L/K}) + \varepsilon_{L/K}^{\mathrm{herm}}$$

where the hermitian form $t_{L/K}$ is as defined in Example 2.11 and $\varepsilon_{L/K}^{\mathrm{herm}}$ is represented by the pair $(1, \tilde{\theta})$ with $\tilde{\theta}(\phi) = \mathrm{d}_K^{\phi(1)/2} \cdot \tau(K, \psi_2(\phi) - \phi)$ for all ϕ in R_G^s.

(iv) *In $\mathrm{Cl}(G)$ one has $\partial_{\mathbb{Z}, \mathbb{Q}^c, G}^0(\mathfrak{a}_{L/K}) = [\mathscr{A}_{L/K}]$.*

Remark 5.3. In addition to the result of Theorem 5.2(i) it is also possible to explicitly compute $\mathfrak{a}_{L/K}$ for certain (weakly) wildly ramified extensions L/K (see, for example, Corollary 8.4 below). These results show, in particular, that $\mathfrak{a}_{L/K}$ does not in general vanish.

In Conjecture 7.4 below we shall offer a precise conjectural description of $\mathfrak{a}_{L/K}$ in terms of local (second) Galois–Jacobi sums and invariants related to fundamental classes arising in local class field theory. This description is related to certain "epsilon constant conjectures" that are already in the literature and hence to the general philosophy of Tamagawa number conjectures that originated with Bloch and Kato.

This connection gives a new perspective to the theory of the square root of the inverse different but does not itself help to compute $\mathfrak{a}_{L/K}$ explicitly in any degree of generality.

Nevertheless, our methods combine with extensive numerical experiments to suggest that, rather surprisingly, it might also be possible in general to describe $\mathfrak{a}_{L/K}$ very explicitly (see Section 10C). This possibility is definitely worthy of further investigation, not least because it could be used to obtain significant new evidence in the context of certain wildly ramified Galois extensions in support of the formalism of Tamagawa number conjectures.

In a different direction, Theorem 5.2 leads to effective "finiteness results" on the natural arithmetic invariants related to $\mathfrak{a}_{L/K}$ that arise as the extension L/K varies.

To give a simple example of such a result, for each number field K and finite abstract group Γ of odd order we write $\mathrm{WR}_K(\Gamma)$ for the set of fields L that are weakly ramified odd-degree Galois extensions of K and for which there exists an isomorphism of groups $\iota : G(L/K) \cong \Gamma$.

For each field $L \in \mathrm{WR}_K(\Gamma)$ we then write $\mathrm{Is}_L(\Gamma)$ for the set of group isomorphisms $\iota : G(L/K) \cong \Gamma$, and for each $\iota \in \mathrm{Is}_L(\Gamma)$ we consider the induced isomorphism of relative algebraic K-groups

$$\iota_* : K_0(\mathbb{Z}[G(L/K)], \mathbb{Q}^c[G(L/K)]) \cong K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma]).$$

We then define a subset of "realizable classes" in $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ by setting

$$R_K^{\mathrm{wr}}(\Gamma) := \{\iota_*(\mathfrak{a}_{L/K}) : L \in \mathrm{WR}_K(\Gamma), \ \iota \in \mathrm{Is}_L(\Gamma)\}.$$

Recalling that the group $K_0(\mathbb{Z}[G], \mathbb{Q}[G])_{\mathrm{tor}}$ is finite (see, for example, [Bley and Wilson 2009, Corollary 2.5]) the result of Theorem 5.2(i) leads directly to the following result.

Corollary 5.4. *The set $R_K^{\mathrm{wr}}(\Gamma)$ is finite.*

In Section 9 we explain how the set $R_K^{\mathrm{wr}}(\Gamma)$ can be computed effectively and then apply the general theory in the setting of an explicit conjecture of Vinatier [2003, §1, Conjecture] concerning the Galois structure of $\mathcal{A}_{L/K}$.

To end this section we prove an important preliminary result.

Proposition 5.5. *Let L/K be a finite odd-degree weakly ramified Galois extension of number fields of group G. Then $\mathfrak{a}_{L/K}$ belongs to the subgroup $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ of $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$.*

Proof. For x and y in $K_0(\mathbb{Z}[G], \mathbb{Q}^c[G])$ we write $x \equiv y$ if $x - y$ belongs to $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$.

Then $\mathfrak{a}_{L/K}$ is equal to

$$[\mathcal{A}_{L/K}, \kappa_L, H_L] - \delta_G(\tau_K^G \cdot (\psi_{2,*} - 1)(\tau'_{L/K}))$$
$$\equiv ([\mathcal{A}_{L/K}, \kappa_L, H_L] - \delta_G(\tau_{L/K}^{\dagger})) - \delta_G(J_{2,L/K})$$
$$\equiv [\mathcal{A}_{L/K}, \kappa_L, H_L] - \delta_G(\tau_{L/K}^{\dagger}),$$

where the first equivalence follows from (4.5) and the obvious containment

$$(\psi_{2,*} - 1)(y_{L/K}) \in \zeta(\mathbb{Q}[G])$$

and the second from Lemma 4.4 (with $k = 2$).

It thus suffices to note that the computations in [Bley and Burns 2003, pp. 555–556] (which rely heavily on a result of Fröhlich [1983, §9, (i)–(ii)]) show that $[\mathcal{A}_{L/K}, \kappa_L, H_L] \equiv \delta_G(\tau_{L/K}^{\dagger})$. $\square$

6. Functoriality properties of $\mathfrak{a}_{L/K}$

Following Proposition 5.5 we know each element $\mathfrak{a}_{L/K}$ belongs to $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$. In this section we prove the following result which establishes the basic functorial properties of these elements as the extension L/K varies.

Theorem 6.1. *Let L/K be a weakly ramified odd-degree Galois extension of number fields of group G, fix an intermediate field F of L/K, and set $J := G(L/F)$.*

(i) *The restriction map $\rho_J^G : K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \to K_0(\mathbb{Z}[J], \mathbb{Q}[J])$ sends $\mathfrak{a}_{L/K}$ to $\mathfrak{a}_{L/F}$.*

(ii) *Assume J is normal in G and write Γ for the quotient $G/J \cong G(F/K)$. Then the natural coinflation map $\pi_\Gamma^G : K_0(\mathbb{Z}[G], \mathbb{Q}[G]) \to K_0(\mathbb{Z}[\Gamma], \mathbb{Q}[\Gamma])$ sends $\mathfrak{a}_{L/K}$ to $\mathfrak{a}_{F/K}$.*

Proof. It is convenient to first prove claim (ii) in the statement of Theorem 6.1. To do this we use the commutative diagram

$$
\begin{array}{ccc}
\zeta(\mathbb{Q}^c[G])^\times & \xrightarrow{\delta_G} & K_0(\mathbb{Z}[G], \mathbb{Q}^c[G]) \\
\tilde{\pi}_\Gamma^G \downarrow & & \pi_\Gamma^G \downarrow \\
\zeta(\mathbb{Q}^c[\Gamma])^\times & \xrightarrow{\delta_\Gamma} & K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])
\end{array}
\tag{6.2}
$$

in which $\tilde{\pi}_\Gamma^G(z)_\phi = z_{\inf_\Gamma^G(\phi)}$ for all z in $\zeta(\mathbb{Q}^c[G])^\times$ and ϕ in $\widehat{\Gamma}$ (see, for example, [Bley and Burns 2003, p. 577]).

Then both equalities $\tilde{\pi}_\Gamma^G(\tau_K^G) = \tau_K^\Gamma$ and $\tilde{\pi}_\Gamma^G((\psi_{2,*}-1)(\tau_{L/K}')) = (\psi_{2,*}-1)(\tau_{F/K}')$ follow easily from the (well known) facts that Gauss sums and unramified characteristics are invariant under inflation and Adams operations commute with inflation.

Hence, the key point in proving claim (ii) is to prove $\pi_\Gamma^G([\mathscr{A}_{L/K}, \kappa_L, H_L]) = [\mathscr{A}_{F/K}, \kappa_F, H_F]$. To show this we write $\mathrm{tr}_{L/F}$ for the field-theoretic trace map $L \to F$. Since $\mathscr{A}_{L/K}$ is $\mathbb{Z}[G]$-projective it is also cohomologically trivial and so $\mathscr{A}_{L/K}^J = \mathrm{tr}_{L/F}(\mathscr{A}_{L/K}) = \mathscr{A}_{F/K}$, where the last equality follows, for example, from the explicit computations of Erez [1991, p. 246].

In addition, the natural identification of H_L^J with H_F induces a commutative diagram of $\mathbb{Q}^c[\Gamma]$-modules

$$
\begin{array}{ccc}
(\mathbb{Q}^c \otimes_{\mathbb{Q}} L)^J & \xrightarrow{\kappa_L^J} & (\mathbb{Q}^c \otimes_{\mathbb{Z}} H_L)^J \\
\| & & \| \\
\mathbb{Q}^c \otimes_{\mathbb{Q}} F & \xrightarrow{\kappa_F} & \mathbb{Q}^c \otimes_{\mathbb{Z}} H_F
\end{array}
$$

and, taken together, these facts imply that

$$
\pi_\Gamma^G([\mathscr{A}_{L/K}, \kappa_L, H_L]) = [\mathscr{A}_{L/K}^J, \kappa_L^J, H_L^J] = [\mathscr{A}_{F/K}, \kappa_F, H_F],
$$

as required to complete the proof of claim (ii) of Theorem 6.1.

To prove Theorem 6.1(i) we use the commutative diagram (see, for example, [Bley and Burns 2003, p. 575])

$$
\begin{array}{ccc}
\zeta(\mathbb{Q}^c[G])^\times & \xrightarrow{\ \delta_G\ } & K_0(\mathbb{Z}[G], \mathbb{Q}^c[G]) \\[2pt]
\tilde{\rho}_J^G \downarrow & & \rho_J^G \downarrow \\[2pt]
\zeta(\mathbb{Q}^c[J])^\times & \xrightarrow{\ \delta_J\ } & K_0(\mathbb{Z}[J], \mathbb{Q}^c[J])
\end{array}
\tag{6.3}
$$

Here, for each z in $\zeta(\mathbb{Q}^c[G])^\times$ and ϕ in $\hat{J}$, one has $\tilde{\rho}_J^G(z)_\phi = \prod_{\chi \in \widehat{G}} z_\chi^{\langle \chi, \mathrm{ind}_J^G(\phi) \rangle_G}$ where we write $\langle \cdot, \cdot \rangle_G$ for the natural pairing on R_G.

For each number field E we now set $\tau_E := \tau(\mathbb{Q}, \mathrm{ind}_E^{\mathbb{Q}} \mathbf{1}_E)$. We claim that

$$
\tilde{\rho}_J^G(\tau_K^G) = \mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_K^{[G:J]}).
\tag{6.4}
$$

In fact, for all $\phi \in \hat{J}$ one has

$$
\mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_K^{[G:J]})_\phi = \tau_K^{\phi(1)[G:J]} \quad \text{and} \quad \tilde{\rho}_J^G(\tau_K^G)_\phi = \prod_{\chi \in \widehat{G}} \tau_K^{\chi(1)\langle \chi, \mathrm{ind}_J^G \phi \rangle_G}
$$

and so the claimed equality is valid since $\sum_{\chi \in \widehat{G}} \chi(1)\langle \chi, \mathrm{ind}_J^G \phi \rangle_G = \phi(1)[G:J]$.

We next note that, since $|G|$ is odd, one has

$$
\mathrm{ind}_J^G(\psi_2(\phi)) = \psi_2(\mathrm{ind}_J^G(\phi))
$$

for all ϕ in $\widehat{G}$ (see, for example, [Erez 1991, Proposition-Definition 3.5]). Thus, given the commutativity of (6.3) and the (well known) inductivity in degree zero of both Galois–Gauss sums and nonramified characteristics one deduces that

$$
\rho_J^G(\delta_G((\psi_{2,*} - 1)(\tau_{L/K}'))) = \delta_J(\tilde{\rho}_J^G((\psi_{2,*} - 1)(\tau_{L/K}'))) = \delta_J((\psi_{2,*} - 1)(\tau_{L/F}')).
\tag{6.5}
$$

By combining (6.4) and (6.5) we obtain an equality

$$
\begin{aligned}
\rho_J^G(\delta_G(\tau_K^G \cdot (\psi_{2,*} - 1)(\tau_{L/K}'))) & \\
&= \delta_J(\mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_K^{[G:J]})) + \delta_J((\psi_{2,*} - 1)(\tau_{L/F}')) \\
&= \delta_J(\mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_F^{-1} \cdot \tau_K^{[G:J]})) + \delta_J(\tau_F^J \cdot (\psi_{2,*} - 1)(\tau_{L/F}')).
\end{aligned}
$$

To consider the corresponding behavior of the term $[\mathscr{A}_{L/K}, \kappa_L, H_L]$ under restriction the key point is that in the subgroup $K_0(\mathbb{Z}[J], \mathbb{Q}[J])$ of $K_0(\mathbb{Z}[J], \mathbb{Q}^c[J])$ there are equalities

$$
\begin{aligned}
\rho_J^G([\mathscr{A}_{L/K}, \kappa_L, H_L]) - [\mathscr{A}_{L/F}, \kappa_L, H_L] &= [\mathscr{A}_{L/K}, \kappa_L, H_L] - [\mathscr{A}_{L/F}, \kappa_L, H_L] \\
&= [\mathscr{A}_{L/K}, \mathrm{id}, \mathscr{A}_{L/F}] = [\mathscr{A}_{L/F}\mathscr{A}_{F/K}, \mathrm{id}, \mathscr{A}_{L/F}] = \delta_J(\mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_F^{-1} \cdot \tau_K^{[G:J]})).
\end{aligned}
$$

Here the first equality is obvious, the second follows from the defining relations of $K_0(\mathbb{Z}[J], \mathbb{Q}^c[J])$, the third from the (well known) multiplicativity property $\mathscr{A}_{L/K} = \mathscr{A}_{L/F}\mathscr{A}_{F/K}$, and the fourth from the result of Lemma 6.6 below.

Comparing the last two displayed equalities it follows directly that $\rho_J^G(\mathfrak{a}_{L/K}) = \mathfrak{a}_{L/F}$, as claimed. $\qquad\square$

Lemma 6.6. *With the subgroup J and field F as above,*

$$[\mathscr{A}_{L/F}\mathscr{A}_{F/K}, \mathrm{id}, \mathscr{A}_{L/F}] = \delta_J(\mathrm{Nrd}_{\mathbb{Q}[J]}(\tau_F^{-1} \cdot \tau_K^{[G:J]}))$$

in $K_0(\mathbb{Z}[J], \mathbb{Q}[J])$.

Proof. By Lemma 6.7 below it suffices to show that $N_{F/\mathbb{Q}}(\mathscr{A}_{F/K}) = \pm\tau_F^{-1} \cdot \tau_K^{[G:J]}$.

This equality is, in turn, a direct consequence of the fact that

$$N_{F/\mathbb{Q}}(\mathscr{A}_{F/K})^2 = N_{F/\mathbb{Q}}(\mathfrak{D}_{F/K})^{-1} = N_{F/\mathbb{Q}}(\mathfrak{D}_{F/\mathbb{Q}}^{-1}\mathfrak{D}_{K/\mathbb{Q}}) = d_{F/\mathbb{Q}}^{-1} \cdot N_{K/\mathbb{Q}}(\mathfrak{D}_{K/\mathbb{Q}})^{[F:K]}$$
$$= d_{F/\mathbb{Q}}^{-1} \cdot d_{K/\mathbb{Q}}^{[F:K]} = \tau_F^{-2} \cdot \tau_K^{2[G:J]},$$

where the last equality follows from (4.12). $\qquad\square$

Lemma 6.7. *Let E be a number field and G a finite group. Let N be a locally free $\mathbb{O}_E[G]$-module of rank one. Let $\mathfrak{a}$ denote a fractional $\mathbb{O}_E$-ideal. Then in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ one has*

$$[\mathfrak{a}N, \mathrm{id}, N] = \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(N_{E/\mathbb{Q}}(\mathfrak{a}))).$$

Proof. Recall that for each prime p and each $\mathbb{Z}$-module X we write X_p for the $\mathbb{Z}_p$-module $\mathbb{Z}_p \otimes_{\mathbb{Z}} X$.

In particular, there is an isomorphism $N_p \simeq (\mathbb{O}_{E,p}[G])^d$ of $\mathbb{O}_{E,p}[G]$-modules and hence in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p[G])$ an equality

$$[(\mathfrak{a}N)_p, \mathrm{id}, N_p] = [\mathfrak{a}_p[G]^d, \mathrm{id}, \mathbb{O}_{E,p}[G]^d] = d[\mathfrak{a}_p[G], \mathrm{id}, \mathbb{O}_{E,p}[G]].$$

It follows that $[\mathfrak{a}N, \mathrm{id}, N] = d[\mathfrak{a}[G], \mathrm{id}, \mathbb{O}_E[G]]$ in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$.

Now set $n := [E : \mathbb{Q}]$ and choose $\mathbb{Z}$-bases $\omega_1, \ldots, \omega_n$ for $\mathbb{O}_E$ and $\alpha_1, \ldots, \alpha_n$ for $\mathfrak{a}$. Then

$$\mathfrak{a}[G] = \bigoplus_{i=1}^n \mathbb{Z}[G]\alpha_i, \qquad \mathbb{O}_E[G] = \bigoplus_{i=1}^n \mathbb{Z}[G]\omega_i.$$

With respect to these bases the identity is represented by the matrix $B \in \mathrm{GL}_n(\mathbb{Q}) \subseteq \mathrm{GL}_n(\mathbb{Q}[G])$ defined by $B = (b_{ji})$ where $\alpha_i = \sum_{j=1}^n b_{ji}\omega_j$. Note that $|\det(B)| = N_{E/\mathbb{Q}}(\mathfrak{a})$.

By the defining relations in relative K-groups and the definitions of $\partial_{\mathbb{Z},\mathbb{Q},G}$ and δ_G we obtain

$$[\mathfrak{a}[G], \mathrm{id}, \mathbb{O}_E[G]] = [\mathbb{Z}[G]^n, B, \mathbb{Z}[G]^n] = \partial_{\mathbb{Z},\mathbb{Q},G}([\mathbb{Q}[G]^n, B]) = \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(B)).$$

Now $\mathrm{Nrd}_{\mathbb{Q}[G]}(B) = \sum_{\chi \in \hat{G}} x_\chi e_\chi$ with $x_\chi = \det(T_\chi(B)) = \det(B)^{\chi(1)}$, where T_χ is a representation with character χ. Hence, $\mathrm{Nrd}_{\mathbb{Q}[G]}(B) = \mathrm{Nrd}_{\mathbb{Q}[G]}(\det(B))$ and

$$[\mathfrak{a}[G], \mathrm{id}, \mathbb{O}_E[G]] = \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(\det(B)))$$
$$= \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(|\det(B)|)) = \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(N_{E/\mathbb{Q}}(\mathfrak{a}))),$$

where the second equality follows from $\delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(-1)) = 0$. $\qquad\square$

7. A canonical local decomposition of $\mathfrak{a}_{L/K}$

In this section we follow the approach of Breuning [2004b] to give a canonical decomposition of the term $\mathfrak{a}_{L/K}$ as a sum of terms which depend only upon the local extensions L_w/K_v for places v of K which ramify wildly (and weakly) in L/K.

7A. *The local relative element* $\mathfrak{a}_{F/E}$. We first define the canonical local terms that will occur in the decomposition of $\mathfrak{a}_{L/K}$.

To do this we fix a rational prime ℓ and an odd-degree weakly ramified Galois extension F/E of fields which are contained in $\mathbb{Q}_\ell^c$ and of finite degree over $\mathbb{Q}_\ell$ and we set $\Gamma := G(F/E)$.

We also fix an embedding of fields $j_\ell : \mathbb{Q}^c \to \mathbb{Q}_\ell^c$ and by abuse of notation also write $j_\ell : \zeta(\mathbb{Q}^c[\Gamma]) \to \zeta(\mathbb{Q}_\ell^c[\Gamma])$ for the induced ring embedding. We then write

$$j_{\ell,*} : K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma]) \to K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell^c[\Gamma])$$

for the homomorphism of abelian groups that sends each element $[P, \iota, Q]$ to $[P_\ell, \mathbb{Q}_\ell^c \otimes_{\mathbb{Q}^c, j_\ell} \iota, Q_\ell]$ and we note that $j_{\ell,*} \circ \delta_\Gamma = \delta_{\Gamma,\ell} \circ j_\ell$.

We write $\Sigma(F)$ for the set of embeddings $F \hookrightarrow \mathbb{Q}_\ell^c$ and

$$\kappa_F : \mathbb{Q}_\ell^c \otimes_{\mathbb{Q}_\ell} F \to \prod_{\Sigma(F)} \mathbb{Q}_\ell^c$$

for the isomorphism of $\mathbb{Q}_\ell^c[\Gamma]$-modules sending $x \otimes f$ to $(\sigma(f)x)_{\sigma \in \Sigma(F)}$ for $f \in F$ and $x \in \mathbb{Q}_\ell^c$.

We also write H_F for the submodule $\prod_{\Sigma(F)} \mathbb{Z}_\ell$ of $\prod_{\Sigma(F)} \mathbb{Q}_\ell^c$ and note that $[\mathscr{A}_{F/E}, \kappa_F, H_F]$ is then a well defined element of $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell^c[\Gamma])$.

We next write $U_{F/E}$ for the canonical "unramified" element of $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell^c[\Gamma])$ defined (for any Galois extension of local fields) by Breuning [2004b] and then define an element of $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell^c[\Gamma])$ by setting

$$\mathfrak{a}_{F/E} := [\mathscr{A}_{F/E}, \kappa_F, H_F] - \delta_{\Gamma,\ell}(j_\ell(\tau_E^\Gamma \cdot (\psi_{2,*} - 1)(\tau'_{F/E}))) - U_{F/E},$$

where the elements τ_E^Γ and $\tau'_{F/E}$ of $\zeta(\mathbb{Q}^c[\Gamma])^\times$ are constructed from local Galois–Gauss sums as in Remark 4.6.

The point of introducing the element $U_{F/E}$ is that it guarantees that $\mathfrak{a}_{F/E}$ is "rational" in the sense of the following proposition.

Proposition 7.1. *The element $\mathfrak{a}_{F/E}$ is independent of the choice of j_ℓ and belongs to $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$.*

Proof. The first assertion follows immediately from [Breuning 2004b, Lemma 2.2] and the containment

$$\tau_E^\Gamma \cdot (\psi_{2,*} - 1)(\tau'_{F/E}) \cdot (\tau_{F/E}^\dagger)^{-1} \in \zeta(\mathbb{Q}[\Gamma])^\times,$$

which itself follows directly from (4.8) and the local analog of Lemma 4.4 (see Remark 4.6).

The second claim follows by combining the same containment with the containment

$$[\mathscr{A}_{F/E}, \kappa_F, H_F] - \delta_{\Gamma,\ell}(j_\ell(\tau_{F/E}^\dagger)) - U_{F/E} \in K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$$

proved by Breuning's argument in [2004b, Proposition 3.4].   □

7B. $\mathfrak{a}_{F/E}$ and fundamental classes. In this section we reformulate the local epsilon constant conjecture formulated by Breuning in [2004b, Conjecture 3.2] in terms of the explicit element $\mathfrak{a}_{F/E}$.

To this end we recall that for any finite Galois extension of ℓ-adic fields F/E, of group Γ, Breuning's conjecture is an equality in $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$ of the form

$$T_{F/E} + C_{F/E} + U_{F/E} - M_{F/E} = 0. \tag{7.2}$$

Here, in addition to the element $U_{F/E}$ used in the previous section, the following elements also occur.

- $T_{F/E} := \delta_{\Gamma,\ell}(j_\ell(\tau_{F/E}^\dagger))$ is the equivariant local epsilon constant.
- $C_{F/E} = \mathscr{E}(\exp_\ell(\mathscr{L}))_\ell - [\mathscr{L}, \kappa_F, H_F]$, where $\mathscr{L}$ is any full projective $\mathbb{Z}_\ell[\Gamma]$-sublattice of $\mathbb{O}_F$ that is contained in a sufficiently large power of the maximal ideal $\mathfrak{p}_F$ of $\mathbb{O}_F$ to ensure the ℓ-adic exponential map $\exp_\ell$ converges on $\mathscr{L}$. For the precise definition of $\mathscr{E}(\exp_\ell(\mathscr{L}))_\ell$ we refer the reader to [Breuning 2004b, §2.4; Bley and Burns 2003, §3.2]. For the moment, we point out only that this element relies on local fundamental classes and is very difficult to compute explicitly in any degree of generality.
- $M_{F/E}$ is a simple and explicitly defined correction term [Breuning 2004b, §2.6].

To reinterpret (7.2) we assume F/E is weakly ramified. In this case the lattice $\mathscr{L}$ that occurs above can be taken to be $p^N \cdot \mathscr{A}_{F/E}$ for any sufficiently large integer N and the element

$$\mathscr{E}_{F/E} := \mathscr{E}(\exp_\ell(p^N \cdot \mathscr{A}_{F/E}))_\ell - \delta_{\Gamma,\ell}(\mathrm{Nrd}_{\mathbb{Q}_\ell[\Gamma]}(p^{N[E:\mathbb{Q}_\ell]}))$$

of $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$ is easily seen to be independent of the choice of N.

We next define an element of $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$ by setting

$$\mathfrak{c}_{F/E} := \delta_{\Gamma,\ell}((1 - \psi_{2,*})(y_{F/E})). \tag{7.3}$$

Then by combining Lemma 6.7 with (4.8) one finds that Breuning's conjectural equality (7.2) is equivalent to the following conjecture.

Conjecture 7.4. *Let F/E be a weakly ramified Galois extension of ℓ-adic fields with group Γ. Then in $K_0(\mathbb{Z}_\ell[\Gamma], \mathbb{Q}_\ell[\Gamma])$ one has*

$$\mathfrak{a}_{F/E} = \mathscr{E}_{F/E} - \delta_{\Gamma,\ell}(J_{2,F/E}) - \mathfrak{c}_{F/E} - M_{F/E},$$

where the second Galois–Jacobi sum $J_{2,F/E}$ of F/E is as discussed in Remark 4.6.

Remark 7.5. For later purposes we note that (4.2) implies that $(1 - \psi_{2,*})(y_{F/E}) = (1 - e_{\Gamma_0}) + \sigma^{-1}e_{\Gamma_0}$, with Γ_0 the inertia subgroup of Γ and σ an element of Γ that projects to the Frobenius in Γ/Γ_0, and hence that $\mathfrak{c}_{F/E} = \delta_{\Gamma,l}((1 - e_{\Gamma_0}) + \sigma^{-1}e_{\Gamma_0})$.

In particular, $\mathfrak{c}_{F/E}$ vanishes if F/E is tame (since then $(1 - e_{\Gamma_0}) + \sigma^{-1}e_{\Gamma_0} \in \mathbb{Z}_\ell[\Gamma]^\times$ and, in all cases, Γ/Γ_0 is abelian and so $\mathrm{Nrd}_{\mathbb{Q}_\ell[\Gamma]}((1 - e_{\Gamma_0}) + \sigma^{-1}e_{\Gamma_0}) = (1 - e_{\Gamma_0}) + \sigma^{-1}e_{\Gamma_0}$).

7C. *The decomposition result.* We can now state and prove the main result of this section. In this result we use, for each prime ℓ, each extension E of $\mathbb{Q}_\ell$, and each subgroup H of G, the natural induction map $\mathrm{i}_{H,E}^G : K_0(\mathbb{Z}_\ell[H], E[H]) \to K_0(\mathbb{Z}_\ell[G], E[G])$ on relative K-groups.

Theorem 7.6. *Let L/K be a weakly ramified odd-degree Galois extension of number fields of group G. Then in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ one has an equality*

$$\mathfrak{a}_{L/K} = \sum_\ell \sum_{v|\ell} \mathrm{i}_{G_w, \mathbb{Q}_\ell}^G(\mathfrak{a}_{L_w/K_v})$$

where the sum is over all primes ℓ and for each place v of K we fix a place w of L lying above v and identify the Galois group of L_w/K_v with the decomposition subgroup G_w of w in G.

Proof. Proposition 5.5 implies $\mathfrak{a}_{L/K}$ decomposes naturally as a sum $\sum_\ell \mathfrak{a}_{L/K,\ell}$ of ℓ-primary components and so it suffices to prove that for each ℓ there is in $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])$ an equality

$$\mathfrak{a}_{L/K,\ell} = \sum_{v|\ell} \mathrm{i}_{G_w, \mathbb{Q}_\ell}^G(\mathfrak{a}_{L_w/K_v}). \tag{7.7}$$

To do this we fix a prime ℓ and an embedding $j_\ell : \mathbb{Q}^c \to \mathbb{Q}_\ell^c$ and write $\mathbb{O}_\ell^t$ for the valuation ring of the maximal tamely ramified extension of $\mathbb{Q}_\ell$ in $\mathbb{Q}_\ell^c$.

We recall that Taylor's fixed point theorem for group determinants [1984, Chapter 8, Theorem 1.1] implies the following composite homomorphism is injective:

$$K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G]) \to K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell^c[G]) \xrightarrow{j_{\ell,*}^t} K_0(\mathbb{O}_\ell^t[G], \mathbb{Q}_\ell^c[G]) \qquad (7.8)$$

where the first arrow is the natural inclusion and $j_{\ell,*}^t$ sends $[X, \xi, Y]$ to $[\mathbb{O}_\ell^t \otimes_{\mathbb{Z}_\ell} X, \xi, \mathbb{O}_\ell^t \otimes_{\mathbb{Z}_\ell} Y]$. It is therefore enough to show that the equality (7.7) holds after applying $j_{\ell,*}^t$.

The key ingredients required to prove this fact are due to Breuning and are stated in Lemma 7.11 below.

In the sequel we abbreviate $i_{G_w, \mathbb{Q}_\ell}^G$ and $i_{G_w, \mathbb{Q}_\ell^c}^G$ to $i_{w,\ell}$ and $i_{w,\ell}^c$, respectively.

In particular, if for any finite Galois extension F/E of either local fields or number fields we set

$$\tau_{F/E,2} := \tau_E^{G(F/E)} \cdot (\psi_{2,*} - 1)(\tau_{F/E}'),$$

then Breuning's results as stated below combine with the explicit definitions of the terms $\mathfrak{a}_{L/K,\ell}$ and $\mathfrak{a}_{L_w/K_v}$ to imply that

$$\begin{aligned}
j_{\ell,*}^t &\left(\mathfrak{a}_{L/K,\ell} - \sum_{v|\ell} i_{w,\ell}(\mathfrak{a}_{L_w/K_v}) \right) \\
&= j_{\ell,*}^t(j_{\ell,*}([\mathscr{A}_{L/K,\ell}, \kappa_{L,\ell}, H_{L,\ell}])) - j_{\ell,*}^t(\delta_G(\tau_{L/K,2})) \\
&\qquad - \sum_{v|\ell} j_{\ell,*}^t(i_{w,\ell}^c([\mathscr{A}_{L_w/K_v}, \kappa_{L_w}, H_{L_w}]) \\
&\qquad - i_{w,\ell}^c(\delta_{G_w,\ell}(j_\ell(\tau_{L_w/K_v,2}))) - i_{w,\ell}^c(U_{L_w/K_v})) \\
&= -j_{\ell,*}^t(j_{\ell,*}(\delta_G(\tau_{L/K,2}))) + \sum_{v|\ell} j_{\ell,*}^t(i_{w,\ell}^c(\delta_{G_w,\ell}(j_\ell(\tau_{L_w/K_v,2})))) \\
&= -j_{\ell,*}^t(j_{\ell,*}(\delta_G(\tau_{L/K,2}(\tau_{L/K}^\dagger)^{-1}))) \\
&\qquad + \sum_{v|\ell} j_{\ell,*}^t(i_{w,\ell}^c(\delta_{G_w,\ell}(j_\ell(\tau_{L_w/K_v,2}(\tau_{L_w/K_v}^\dagger)^{-1})))) \\
&\qquad - j_{\ell,*}^t \left(\delta_{G,\ell} \left(\prod_{\substack{v|d_L \\ v \nmid \ell}} \tilde{i}_w(y_{L_w/K_v}) \right) \right). \qquad (7.9)
\end{aligned}$$

Here the first equality follows directly from the definitions and the second uses Lemma 7.11(i) and (ii). In addition, the third equality follows from Lemma 7.11(iii) below and uses the map

$$\tilde{i}_w : \zeta(\mathbb{Q}_\ell^c[G_w])^\times \to \zeta(\mathbb{Q}_\ell^c[G])^\times$$

that satisfies $\tilde{i}_w(x)_\chi = \prod_{\varphi \in \widehat{G}_w} x_\varphi^{\langle \mathrm{res}_{G_w}^G \chi, \varphi \rangle_{G_w}}$ for all x in $\zeta(\mathbb{Q}_\ell^c[G_w])^\times$ and χ in $\widehat{G}$.

Now, by (4.5), the first term in the expression (7.9) is equal to

$$-(j_{\ell,*}^t \circ \delta_{G,\ell} \circ j_\ell)(J_{2,L/K} \cdot (\psi_{2,*} - 1)(y_{L/K}^{-1})).$$

In the same way, equality (4.8) implies that the second term in (7.9) is

$$(j_{\ell,*}^t \circ i_{w,l}^c \circ \delta_{G_w,\ell} \circ j_\ell)\left(\prod_{v|\ell} J_{2,L_w/K_v} \cdot (\psi_{2,*} - 1)(y_{L_w/K_v}^{-1})\right).$$

These two expressions combine with the commutative diagram

$$
\begin{array}{ccc}
\zeta(\mathbb{Q}_\ell^c[G_w])^\times & \xrightarrow{\delta_{G_w,\ell}} & K_0(\mathbb{Z}_\ell[G_w], \mathbb{Q}_\ell^c[G_w]) \\
\tilde{i}_w \downarrow & & \downarrow i_{w,\ell}^c \\
\zeta(\mathbb{Q}_\ell^c[G])^\times & \xrightarrow{\delta_{G,\ell}} & K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell^c[G])
\end{array}
\tag{7.10}
$$

the fact that $j_\ell(J_{2,L/K}) = \prod_{v|d_L} \tilde{i}_w(j_\ell(J_{2,L_w/K_v}))$ by the decomposition of global Galois–Gauss sums as a product of local Galois–Gauss sums, and the explicit definition of $y_{L/K}$ to show that the sum in (7.9) is equal to the image under $j_{\ell,*}^t \circ \delta_{G,\ell}$ of

$$j_\ell(J_{2,L/K})^{-1} \cdot \prod_{v|\ell} \tilde{i}_w(j_\ell(J_{2,L_w/K_v})) \cdot \prod_{\substack{v|d_L \\ v\nmid\ell}} \tilde{i}_w(j_\ell((\psi_{2,*} - 2)(y_{L_w/K_v})))$$

$$= \prod_{\substack{v|d_L \\ v\nmid\ell}} \tilde{i}_w(j_\ell(J_{2,L_w/K_v}^{-1} \cdot (\psi_{2,*} - 2)(y_{L_w/K_v}))).$$

It is thus enough to note the image under $j_{\ell,*}^t \circ \delta_{G,\ell}$ of the latter element vanishes as a consequence of [Breuning 2004a, (9) and Lemma 5.3] and the second displayed equation on [loc. cit., p. 68]   □

Lemma 7.11.   (i) *For each prime ℓ one has*

$$j_{\ell,*}([\mathscr{A}_{L/K,\ell}, \kappa_{L,\ell}, H_{L,\ell}]) = \sum_{v|\ell} i_{G_w,\mathbb{Q}_\ell^c}^G([\mathscr{A}_{L_w/K_v}, \kappa_{L_w}, H_{L_w}]).$$

(ii) *For each $v \mid \ell$ the element $i_{G_w,\mathbb{Q}_\ell^c}^G(U_{L_w/K_v})$ belongs to $\ker(j_{\ell,*}^t)$.*

(iii) *One has*

$$j_{\ell,*}(\delta_G(\tau_{L/K}^\dagger)) - \sum_{v|\ell} i_{G_w,\mathbb{Q}_\ell}^G(\delta_{G_w,\ell}(j_\ell(\tau_{L_w/K_v}^\dagger)))$$

$$\equiv \delta_{G,\ell}\left(\prod_{\substack{v|d_L \\ v\nmid\ell}} \tilde{i}_w(y_{L_w/K_v})\right) \ (\mathrm{mod} \ \ker(j_{\ell,*}^t)).$$

Proof. To prove claim (i) one can just follow the proof of [Breuning 2004a, Lemma 5.4] verbatim, merely substituting $\mathscr{A}_{L/K}$ for the projective $\mathbb{Z}[G]$-sublattice $\mathscr{L}$ of $\mathbb{O}_L$ that is used in [loc. cit.].

The property stated in claim (ii) is part of the axiomatic characterization used by Breuning [2004a, Proposition 4.4] to define the elements U_{L_w/K_v}.

To prove claim (iii) we note that elements $\delta_G(\tau_{L/K}^{\dagger})$ and $\delta_{G_w,\ell}(\tau_{L_w/K_v}^{\dagger})$ are denoted by $\tau_{L/K}$ and T_{L_w/K_v}, respectively, in [Breuning 2004b] and that the claimed congruence is thus equivalent to the equality of [Breuning 2004a, (36)]. $\qquad\square$

8. Results in special cases

In this section we compute $\mathfrak{a}_{L/K}$ explicitly in some important special cases and also give a proof of Theorem 5.2.

8A. *Local results.* The following result uses the element $\mathfrak{c}_{F/E}$ defined in (7.3).

Theorem 8.1. *Let $E/\mathbb{Q}_\ell$ be a finite extension and F/E a weakly ramified Galois extension of odd degree with Galois group $\Gamma = G(F/E)$. Then $\mathfrak{a}_{F/E} = \mathfrak{c}_{F/E}$ if either F/E is tamely ramified or if $E/\mathbb{Q}_\ell$ is unramified and F/E is both abelian and has cyclic ramification subgroup.*

Proof. We fix an embedding $j_\ell : \mathbb{Q}^c \to \mathbb{Q}_\ell^c$ and use it to identify $\widehat{\Gamma}$ with the set of irreducible $\mathbb{Q}_\ell^c$-valued characters of Γ.

By Proposition 7.1 and Taylor's fixed point theorem it suffices to show that

$$j_{\ell,*}^t([\mathscr{A}_{F/E}, \kappa_F, H_F] - \delta_{\Gamma,\ell}(j_\ell(\tau_E^{\Gamma} \cdot (\psi_{2,*} - 1)(\tau_{F/E}'))) - U_{F/E} - \mathfrak{c}_{F/E} = 0 \quad (8.2)$$

with $j_{\ell,*}^t$ as in (7.8).

At the outset we note that $j_{\ell,*}^t(U_{F/E}) = 0$ [Breuning 2004a, Proposition 4.4] and that if θ is any element of F with $\mathscr{A}_{F/E} = \mathbb{O}_E[\Gamma] \cdot \theta$, then [Breuning 2004a, Lemma 4.16] implies

$$[\mathscr{A}_{F/E}, \kappa_F, H_F] = \delta_{\Gamma,\ell}\left(\sum_{\chi \in \widehat{\Gamma}} e_\chi \delta_E^{\chi(1)} \cdot \mathcal{N}_{E/\mathbb{Q}_\ell}(\theta \mid \chi)\right).$$

We now assume F/E is tamely ramified. In this case Remark 7.5 implies both that $\mathfrak{c}_{F/E}$ vanishes and $\delta_{\Gamma,\ell}((\psi_{2,*} - 1)\tau_{F/E}') = \delta_{\Gamma,\ell}((\psi_{2,*} - 1)\tau_{F/E})$ and so the element on the left-hand side of (8.2) is equal to the image under $j_{\ell,*}^t \circ \delta_{\Gamma,\ell}$ of $x_1 \cdot x_2$ where for each χ in $\widehat{\Gamma}$ one has (in terms of the notation in (4.1))

$$x_{1,\chi} := \frac{\delta_E^{\chi(1)}}{j_\ell(\tau(\mathbb{Q}_\ell, \mathrm{ind}_E^{\mathbb{Q}_\ell} 1_E)^{\chi(1)})} \quad \text{and} \quad x_{2,\chi} := \frac{\mathcal{N}_{E/\mathbb{Q}_\ell}(\theta \mid \chi)}{j_\ell(\tau(E, \psi_2(\chi) - \chi))}.$$

The equality (8.2) is therefore true in this case since both $(j^t_{\ell,*} \circ \delta_{\Gamma,\ell})(x_1) = 0$ (as a consequence of the obvious local analog of (4.12)) and $(j^t_{\ell,*} \circ \delta_{\Gamma,\ell})(x_2) = 0$, as indicated in the proof of [Erez 1991, Proposition 8.2].

In the remainder of the argument we assume that $E/\mathbb{Q}_\ell$ is unramified and F/E is both abelian and has cyclic ramification subgroup. The proof in this case will heavily rely on the computations of [Bley and Cobbe 2016] (which in turn rely on the work of Pickett and Vinatier [2013]) and so, for convenience, we switch to the notation introduced in [loc. cit., §3.1] (so that F, E, and $\Gamma = G(F/E)$ are now replaced by N, K, and G, respectively).

In particular, we define α_M as in [Bley and Cobbe 2016, just before Lemma 5.1.4], let $\theta_2 \in K'$ be such that $\mathbb{O}_K[G] \cdot \theta_2 = \mathbb{O}_{K'}$ and $T_{K'/K}(\theta_2) = 1$, and recall that the product $\theta = \alpha_M \cdot \theta_2$ satisfies $\mathscr{A}_{N/K} = \mathbb{O}_K[G] \cdot \theta$. (In this regard we observe that the assumption made in [Bley and Cobbe 2016] that $[K : \mathbb{Q}_p]$ and $[K' : K]$ are coprime is not needed for the results obtained in [loc. cit., §5].)

Each character $\psi \in \widehat{G}$ is of the form $\chi\phi$ with an unramified character ϕ of $G_{N/M}$ and χ a character of $G_{N/K'}$ and from [Bley and Cobbe 2016, Proposition 5.1.5] one has

$$\frac{\mathscr{N}_{K/\mathbb{Q}_\ell}(\theta \mid \chi\phi)}{\tau(K, \chi\phi)} = \begin{cases} \mathscr{N}_{K/\mathbb{Q}_\ell}(\theta_2 \mid \phi) & \text{if } \chi = \chi_0, \\ p^{-m} \cdot \chi(4) \cdot \mathscr{N}_{K/\mathbb{Q}_\ell}(\theta_2 \mid \phi) \cdot \phi(p^2) & \text{if } \chi \neq \chi_0, \end{cases}$$

where here and in the following we omit each occurrence of j_ℓ in our notation.

Now the proof of [Bley and Cobbe 2016, Proposition 5.2.1] shows $\tau(K, \chi\phi) = \tau(\mathbb{Q}_\ell, i_K^{\mathbb{Q}_\ell}(\chi\phi))$ and so (4.8) implies

$$\tau_K^G \cdot (\psi_{2,*} - 1)(\tau'_{N/K}) = \left(\sum_{\chi,\phi} e_{\chi\phi} \tau(K, \chi\phi) \right) \cdot J_{2,N/K} \cdot (\psi_{2,*} - 1)(y_{N/K}^{-1}).$$

It follows that for each χ and ϕ one has

$$\frac{\mathscr{N}_{K/\mathbb{Q}_\ell}(\theta \mid \chi\phi)}{(\tau_K^G \cdot (\psi_{2,*} - 1)(\tau'_{N/K}))_{\chi\phi}} = \frac{\mathscr{N}_{K/\mathbb{Q}_\ell}(\theta \mid \chi\phi)}{\tau(K, \chi\phi) \cdot \tau(K, \psi_2(\chi\phi) - 2\chi\phi) \cdot y(K, \chi\phi - \psi_2(\chi\phi))}$$

$$= \begin{cases} \mathscr{N}_{K/\mathbb{Q}_\ell}(\theta_2 \mid \phi) \dfrac{\tau(K, 2\chi\phi - \psi_2(\chi\phi))}{y(K, \chi\phi - \psi_2(\chi\phi))} & \text{if } \chi = \chi_0, \\ p^{-m} \cdot \chi(4) \cdot \phi(p^2) \cdot \mathscr{N}_{K/\mathbb{Q}_\ell}(\theta_2 \mid \phi) \dfrac{\tau(K, 2\chi\phi - \psi_2(\chi\phi))}{y(K, \chi\phi - \psi_2(\chi\phi))} & \text{if } \chi \neq \chi_0. \end{cases}$$

Furthermore, one has $\phi(p^2) = \phi((p^2, K'/K)) = \phi(\sigma^2)$ [Serre 1979, XIII, §4, Proposition 13] and so $\mathfrak{c}_{N/K}$ is equal to the element x_3 of $\mathbb{Q}_\ell^c[G]^\times$ that is characterized by the equalities for each χ and ϕ

$$x_{3,\chi\phi} = \begin{cases} y(K, \chi\phi - \psi_2(\chi\phi))^{-1} & \text{if } \chi = \chi_0, \\ \phi(p^2) y(K, \chi\phi - \psi_2(\chi\phi))^{-1} & \text{if } \chi \neq \chi_0. \end{cases}$$

Taken together, these facts imply that (8.2) is valid if $\ker(j^t_{\ell,*} \circ \delta_{G,\ell})$ contains the element x_4 of $\mathbb{Q}^c_\ell[G]^\times$ defined by

$$x_{4,\chi\phi} = \begin{cases} \mathcal{N}_{K/\mathbb{Q}_\ell}(\theta_2 \mid \phi) \cdot \tau(K, 2\chi\phi - \psi_2(\chi\phi)) & \text{if } \chi = \chi_0, \\ p^{-m} \cdot \chi(4) \cdot \mathcal{N}_{K/\mathbb{Q}_\ell}(\theta_2 \mid \phi) \cdot \tau(K, 2\chi\phi - \psi_2(\chi\phi)) & \text{if } \chi \neq \chi_0. \end{cases}$$

Now, as in the proof of [Bley and Cobbe 2016, Theorem 6.1, p. 1243], one can show that $\ker(j^t_{\ell,*} \circ \delta_{G,\ell})$ contains the element x'_4 of $\mathbb{Q}^c_\ell[G]^\times$ for which at all χ and ϕ one has

$$x'_{4,\chi\phi} = \begin{cases} \mathcal{N}_{K/\mathbb{Q}_\ell}(\theta_2 \mid \phi) & \text{if } \chi = \chi_0, \\ \chi(4) \cdot \mathcal{N}_{K/\mathbb{Q}_\ell}(\theta_2 \mid \phi) & \text{if } \chi \neq \chi_0. \end{cases}$$

In addition, [Bley and Cobbe 2016, Lemma 5.1.2] implies that for all χ and ϕ one has $\tau(K, 2\chi\phi - \psi_2(\chi\phi)) = \tau(K, 2\chi - \chi^2)$.

The required equality $(j^t_{\ell,*} \circ \delta_{G,\ell})(x_4) = 0$ is thus true if and only if

$$(j^t_{\ell,*} \circ \delta_{G,\ell})(x_5) = 0$$

with x_5 the element of $\mathbb{Q}^c_\ell[G]^\times$ for which at each χ and ϕ one has

$$x_{5,\chi\phi} = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ p^{-m}\tau(K, 2\chi - \chi^2) & \text{if } \chi \neq \chi_0. \end{cases}$$

But, by the last displayed formula in the proof of [Pickett and Vinatier 2013, Proposition 3.9], for each nontrivial character χ one has

$$\tau(K, \chi) = p^m \cdot \chi(c_\chi^{-1}) \cdot \psi_K(c_\chi^{-1}), \quad \tau(K, \chi^2) = p^m \cdot \chi^2((c_\chi/2)^{-1}) \cdot \psi_K((c_\chi/2)^{-1}),$$

with ψ_K the standard additive character and c_χ as described in [Pickett and Vinatier 2013, Proposition 3.9].

It follows that $\tau(K, 2\chi - \chi^2) = p^m \cdot \chi(4)^{-1}$ for nontrivial characters χ and hence that $x_{5,\chi\phi} = \chi(4)^{-1}$ for all χ and ϕ. Given this description, it is clear that $x_5 \in \ker(j^t_{\ell,*} \circ \delta_{G,\ell})$, as required to complete the proof of (8.2) in this case. $\qquad \square$

8B. *Global results.*

In this section we derive several consequences of Theorem 8.1, including a proof of Theorem 5.2.

8B1. We shall first give a proof of Theorem 5.2.

Following Proposition 5.5, for each prime ℓ we write $\mathfrak{a}_{L/K,\ell}$ for the image of $\mathfrak{a}_{L/K}$ in $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])$.

Then Theorem 7.6 combines with the vanishing of $\mathfrak{a}_{F/E}$ for each tamely ramified extension F/E of local fields (as proved in Theorem 8.1) to reduce the proof of Theorem 5.2(i) to showing that for each ℓ for which there is an ℓ-adic place v in $\mathcal{W}_{L/K}$ the element $\mathfrak{a}_{L/K,\ell}$ belongs to $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])_{\text{tor}}$.

In view of the explicit description of $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])_{\text{tor}}$ given in [Burns 2004, Theorem 4.1], it is thus enough to prove that for each such prime ℓ one has

$\pi_{H/J}^H(\rho_H^G(\mathfrak{a}_{L/K,\ell})) = 0$ for every cyclic subgroup H of G and every subgroup J of H with $|H/J|$ prime to ℓ.

Invoking the result of Theorem 6.1 it is thus enough to show that $\mathfrak{a}_{F/E,\ell}$ vanishes for all towers of number fields $K \subseteq E \subseteq F \subseteq L$ with L/E cyclic and the degree $[F : E]$ prime to ℓ. However, in any such case, all ℓ-adic places of E are tamely ramified in F/E and so Theorem 8.1 in conjunction with Theorem 7.6 (or (7.7)) implies $\mathfrak{a}_{F/E,\ell}$ vanishes, as required.

Claims (ii) and (iii) of Theorem 5.2 will follow from the same argument used to prove Corollary 4.13.

Finally we note that claim (iv) follows directly from the definition of $\mathfrak{a}_{L/K}$ and the facts that H_L is a free G-module and $\partial_{\mathbb{Z},\mathbb{Q},G} \circ \delta_G$ is the zero homomorphism.

This completes the proof of Theorem 5.2.

8B2. In order to describe a global consequence of Theorem 8.1 we define an "idelic twisted unramified characteristic" by setting

$$\mathfrak{c}_{L/K} := \sum_\ell \sum_{v|\ell} i_{G_w,\mathbb{Q}_\ell}^G(\mathfrak{c}_{L_w/K_v}). \tag{8.3}$$

If v is at most tamely ramified in L/K, then $\mathfrak{c}_{L_w/K_v}$ vanishes. This shows $\mathfrak{c}_{L/K}$ is a well defined element in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ and that

$$\mathfrak{c}_{L/K,\ell} = \begin{cases} 0 & \text{if } \ell \notin \mathcal{W}_{L/K}^{\mathbb{Q}}, \\ \sum_{v|\ell} i_{G_w,\mathbb{Q}_\ell}^G(\mathfrak{c}_{L_w/K_v}) & \text{if } \ell \in \mathcal{W}_{L/K}^{\mathbb{Q}}. \end{cases}$$

In particular, by combining Theorems 7.6 and 8.1 one obtains:

Corollary 8.4. *Let L/K be a weakly ramified odd-degree Galois extension of number fields. Then $\mathfrak{a}_{L/K} = \mathfrak{c}_{L/K}$ whenever all of the following conditions are satisfied at each v in $\mathcal{W}_{L/K}$.*

(i) *The decomposition subgroup of v is abelian.*

(ii) *The inertia subgroup of v is cyclic.*

(iii) *The extension $K_v/\mathbb{Q}_\ell$ is unramified, where $\ell = \ell(v)$ denotes the residue characteristic.*

Remark 8.5. Extensive numerical computations suggest that the equality $\mathfrak{a}_{L/K} = \mathfrak{c}_{L/K}$ proved in Corollary 8.4 may well be valid in all cases (see Section 10A3 for more details).

Corollary 8.4 immediately combines with Theorem 5.2(ii) and (iii) to give the following explicit consequence concerning the structures discussed in Examples 2.11 and 2.18.

Corollary 8.6. *Under the hypotheses of Corollary 8.4 one has*

$$[\mathcal{A}_{L/K}, h_{L,\bullet}] = \Pi_G^{\text{met}}(\mathfrak{c}_{L/K}) + \varepsilon_{L/K}^{\text{met}}, \qquad \text{Disc}(\mathcal{A}_{L/K}, t_{L/K}) = \Pi_G^{\text{herm}}(\mathfrak{c}_{L/K}) + \varepsilon_{L/K}^{\text{herm}}.$$

It is therefore of interest to know when the classes $\Pi_G^{\mathrm{met}}(\mathfrak{c}_{L/K})$ and $\Pi_G^{\mathrm{herm}}(\mathfrak{c}_{L/K})$ vanish and the next result shows that this is often the case.

Lemma 8.7. *The images of $\mathfrak{c}_{L/K}$ in each of the groups $\mathrm{Cl}(G)$, $A(G)$, and $\mathrm{HCl}(G)$ all vanish if for each $v \in \mathcal{W}_{L/K}$ one has either $I_w = G_w$ or I_w is of prime power order.*

Proof. We show that each of the individual terms in the definition of $\mathfrak{c}_{L/K}$ projects to zero. We fix v in $\mathcal{W}_{L/K}$ and set $\ell := \ell(v)$ and $\lambda_w := (1 - e_{I_w}) + \sigma_w^{-1} e_{I_w}$. If $G_w = I_w$, then $\lambda_w = 1$. In the other case I_w is necessarily of ℓ-power order. Hence, for any prime $p \neq \ell$ we have $\delta_{G_w}(\lambda_w) = 0$ in $K_0(\mathbb{Z}_p[G_w], \mathbb{Q}_p[G_w])$ since $\lambda_w \in \mathrm{Nrd}_{\mathbb{Q}_p[G_w]}(\mathbb{Z}_p[G_w]^\times)$.

Therefore, one has $\pi_{G,\ell}(i_{G_w}^G(\delta_{G_w}(\lambda_w))) = i_{G_w}^G(\delta_{G_w}(\lambda_w))$ in $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$, where $\pi_{G,\ell}$ is the homomorphism between relative K-groups defined in (2.3).

We next show that $\mathfrak{c}_v := i_{G_w}^G(\delta_{G_w}(\lambda_w))$ belongs to both $\ker(\partial_G^{1,1} \circ h_G^{\mathrm{rel}})$ and $\ker(\partial_G^{2,1} \circ h_G^{\mathrm{rel}})$.

To do this we recall first that for $\alpha = (\alpha_\chi)_{\chi \in \widehat{G}}$ in $\mathbb{Q}^c[G]^\times$ the element $h_G^{\mathrm{rel}}(\delta_G(\alpha))$ is represented by the function $\chi \mapsto (1, \alpha_\chi)$. Thus, the global analog of the commutative diagram (7.10) implies that $h_G^{\mathrm{rel}}(\mathfrak{c}_v)$ is represented by the pair $(1, \theta)$ with

$$\theta(\chi) = \prod_{\phi \in \widehat{G_w/I_w}} \phi(\sigma_w^{-1})^{\langle \mathrm{res}_{G_w}^G(\chi), \phi \rangle_{G_w}}.$$

The elements $\partial_G^{1,1}(h_G^{\mathrm{rel}}(\mathfrak{c}_v))$ and $\partial_G^{2,1}(h_G^{\mathrm{rel}}(\mathfrak{c}_v))$ are therefore represented by the pairs $(1, |\theta|)$ and $(1, \theta^s)$, respectively, and so it is enough to show that the maps $|\theta|$ and θ^s are both trivial.

Since $\theta(\chi)$ is a root of unity one has $|\theta|(\chi) = |\theta(\chi)| = 1$, and so $|\theta|$ is trivial.

In addition, the triviality of θ^s follows from the fact that if χ is a symplectic character of G, then both $\langle \mathrm{res}_{G_w}^G(\chi), \phi \rangle_{G_w} = \langle \mathrm{res}_{G_w}^G(\chi), \bar{\phi} \rangle_{G_w}$ and $\phi(\sigma_w)\bar{\phi}(\sigma_w) = 1$. □

Remark 8.8. In connection with Lemma 8.7 we note that if L_w/K_v is weakly ramified and abelian, then class field theory implies I_w is of prime-power order (as a consequence of [Serre 1979, Corollary 2, p. 70]). In fact, at this stage we know of no example in which the projection of $\mathfrak{c}_{L/K}$ to any of the groups $\mathrm{Cl}(G)$, $A(G)$, and $\mathrm{HCl}(G)$ does not vanish. It is, however, not difficult to show that the element $\mathfrak{c}_{L/K}$ itself does not always vanish. For example, if G is abelian, then $K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ identifies with the group of invertible $\mathbb{Z}[G]$-sublattices of $\mathbb{Q}[G]$. In particular, if $L/\mathbb{Q}$ is an abelian p-extension in which, for any p-adic place w of L, one has $I_w \subsetneq G_w = G$, then $(1 - e_{I_w}) + \sigma_w^{-1} e_{I_w}$ does not belong to $\mathbb{Z}[G]$ and so $\mathfrak{c}_{L/\mathbb{Q}} \neq 0$.

Remark 8.9. The element $\mathfrak{c}_{L/K}$ is in general different from and better behaved than the simpler variant $\delta_G((1 - \psi_{2,*})(y_{L/K}))$. In particular, while it is straightforward

to show that $\mathfrak{c}_{L/K}$ enjoys the same functorial properties under change of extension as those described in Theorem 6.1, the same isn't true of $\delta_G((1 - \psi_{2,*})(y_{L/K}))$.

9. Effective computations and Vinatier's conjecture

In this section we first refine Corollary 5.4 by explaining how to make an effective computation of the set of realizable classes $R_K^{\mathrm{wr}}(\Gamma)$.

We then apply this observation to consider a conjecture of Vinatier in the setting of two natural infinite families of extensions which will then be investigated numerically in Section 10.

In Section 9B2 we consider the family of extensions of smallest degree for which Vinatier's conjecture is not currently known to be valid and, while studying this case, we obtain evidence (described in Theorem 10.2) that $\mathfrak{a}_{L/K}$ may be controlled by the idelic twisted unramified characteristic $\mathfrak{c}_{L/K}$ in cases beyond those considered in Corollary 8.4.

Motivated by this last rather surprising observation, we consider in Section 9B3 a family of extensions of smallest possible degree for which the projection of $\mathfrak{c}_{L/K}$ to $\mathrm{Cl}(G(L/K))$ might not vanish, and hence that a close link between $\mathfrak{a}_{L/K}$ and $\mathfrak{c}_{L/K}$ need not be consistent with the validity of Vinatier's conjecture.

In all of the cases that we compute, however, we find both that Vinatier's conjecture is valid and the projection of $\mathfrak{c}_{L/K}$ to $\mathrm{Cl}(G(L/K))$ vanishes.

At the same time, our methods also give a proof of the central conjecture of [Bley and Burns 2003] for a new, and infinite, family of wildly ramified Galois extensions of number fields.

9A. *The general result.* Recall that for each number field K and finite abstract group Γ of odd order we write $\mathrm{WR}_K(\Gamma)$ for the set of fields L that are weakly ramified odd-degree Galois extensions of K and for which $G(L/K)$ is isomorphic to Γ.

Theorem 9.1. *Let K be a number field and Γ a finite abstract group whose order is both odd and coprime to the number of roots of unity in K.*

Then there exists a finite set $\mathrm{WR}_K^*(\Gamma)$ of Galois extensions E of K which have all of the following properties.

 (i) *There exists an injective homomorphism of groups $i_E : G(E/K) \to \Gamma$.*

 (ii) *There exists a unique place v of K that ramifies both wildly and weakly in E and for which there exists a unique place w of E above v.*

(iii) *All places of K other than v that divide $|\Gamma|$ are completely split in E/K.*

(iv) *For each L in* $\mathrm{WR}_K(\Gamma)$ *and every E in* $\mathrm{WR}_K^*(\Gamma)$ *there exists an integer* $n_{L,E} \in \{0, 1\}$ *so that in* $K_0(\mathbb{Z}[\Gamma], \mathbb{Q}^c[\Gamma])$ *one has*

$$i_{L,*}(\mathfrak{a}_{L/K}) = \sum_{E \in \mathrm{WR}_K^*(\Gamma)} n_{L,E} \cdot i^{\Gamma}_{\mathrm{im}(i_E)}(i_{E,*}(\mathfrak{a}_{E/K})).$$

Proof. We recall first that for each place v of K the set $R_v(K, \Gamma)$ of isomorphism classes of Galois extensions E/K_v for which $G(E/K_v)$ is isomorphic to a subgroup of Γ is finite.

We next fix a weakly ramified Galois extension L/K for which the group $G := G(L/K)$ is isomorphic to the given group Γ. We recall that Theorems 7.6 and 8.1 combine to imply that there is a finite sum decomposition

$$\mathfrak{a}_{L/K} = \sum_{\ell \in \mathscr{W}_{L/K}^{\mathbb{Q}}} \sum_{v \mid \ell} i^{G}_{G_w, \mathbb{Q}_\ell}(\mathfrak{a}_{L_w/K_v}). \tag{9.2}$$

For each place v in this sum the (weakly ramified) Galois extension L_w/K_v is isomorphic to one of the Galois extensions E/K_v in the finite set $R_v(K, \Gamma)$.

Further, since we are assuming $|\Gamma|$ is coprime to the number of roots of unity in K a result of Neukirch [1979, Corollary 2, p. 156] implies that there exists a finite Galois extension $\widetilde{E}/K$ with both of the following properties.

(P1) $\widetilde{E}$ has a unique place $\tilde{w}$ above v and the completion $\widetilde{E}_{\tilde{w}}/K_v$ is isomorphic to E/K_v (and hence to L_w/K_v).

(P2) If v' is any place of K which divides $|\Gamma|$, and $v' \neq v$, then v' is totally split in $\widetilde{E}/K$.

These conditions imply that the global extension $\widetilde{E}/K$ is weakly ramified and that the isomorphism of $\widetilde{E}_w/K_v$ with L_w/K_v induces a natural identification

$$G(\widetilde{E}/K) \cong G(L_w/K_v) \cong G_w. \tag{9.3}$$

In addition, since v is the only place of K that is not tamely ramified in $\widetilde{E}/K$ the results of Theorems 7.6 and 8.1(i) combine to imply

$$\mathfrak{a}_{\widetilde{E}/K} = \mathfrak{a}_{L_w/K_v}. \tag{9.4}$$

We now define $\mathrm{WR}_K^*(\Gamma)$ to be the finite set of extensions $\widetilde{E}/K$ that are obtained from the above construction as v runs over the places of K that divide $|\Gamma|$. We note that this set satisfies the claimed property (i) as a consequence of the isomorphisms (9.3), it satisfies properties (ii) and (iii) as a consequence of properties (P1) and (P2) above, and it satisfies property (iv) as a consequence of the equalities (9.2) and (9.4). □

Remark 9.5. The above argument also shows that $|\mathrm{WR}_K^*(\Gamma)| \leq \sum_{v||\Gamma|} \tilde{\nu}(K_v, \Gamma)$ where $\tilde{\nu}(K_v, \Gamma)$ denotes the number of nonisomorphic Galois extensions of K_v whose Galois group is isomorphic to a subgroup of Γ. In this context we recall that if Γ is a p-group, then $\tilde{\nu}(K_v, \Gamma)$ is explicitly computed by work of Shafarevitch [1947] and Yamagishi [1995]. We also recall that Pauli and Roblot [2001] have developed an algorithm for the computation of all extensions of a p-adic field of a given degree. One can therefore use the results of [Shafarevitch 1947; Yamagishi 1995] to design an algorithm to compute all p-extensions with a given p-group [Pauli and Roblot 2001, §10].

Remark 9.6. For any number k and any finite group Γ whose order is both odd and coprime to the number of roots of unity in k, write $\mathrm{WR}_k'(\Gamma)$ for the set of weakly ramified odd-degree Galois extensions L/K with $k \subseteq K$ and such that $G(L/K) \simeq \Gamma$ and $K_v = k_{v(k)}$ for each place v of K that ramifies wildly in L. Then a closer analysis of the proof of Theorem 9.1 shows that the stated result remains valid after one replaces each occurrence of K by k and then, in claim (iv), one replaces the terms L, $\mathrm{WR}_k(\Gamma)$, and $n_{L,E}$ by L/K, $\mathrm{WR}_k'(\Gamma)$, and $n_{L/K,E}$, respectively. This stronger version of Theorem 9.1 makes clear the advantage of the local nature of our computations.

9B. *Applications to Vinatier's conjecture.* Vinatier [2003, §1, Conjecture] has conjectured that for any weakly ramified odd-degree Galois extension L of $\mathbb{Q}$ the $G(L/\mathbb{Q})$-module $\mathscr{A}_{L/\mathbb{Q}}$ is free and we now apply our techniques to study this conjecture.

9B1. We first reformulate the conjecture in terms of the elements $\mathfrak{a}_{L/K}$ (global) and $\mathfrak{a}_{F/E}$ (local).

If F/E is a Galois extension of ℓ-adic fields, then we use the decomposition (2.2) to view $\mathfrak{a}_{F/E}$ as an element of $K_0(\mathbb{Z}[G(F/E)], \mathbb{Q}[G(F/E)])$.

Proposition 9.7. *The following are equivalent.*

(i) *For all odd-degree weakly ramified Galois extensions L/K of number fields the $G(L/K)$-module $\mathscr{A}_{L/K}$ is free.*

(ii) *For all odd-degree weakly ramified Galois extensions L/K of number fields the element $\mathfrak{a}_{L/K}$ projects to zero in $\mathrm{Cl}(G(L/K))$.*

(iii) *For all odd-degree weakly ramified Galois extensions F/E of local fields the element $\mathfrak{a}_{F/E}$ projects to zero in $\mathrm{Cl}(G(F/E))$.*

Proof. The equivalence of (i) and (ii) is Lemma 9.8 below and (ii) follows directly from (iii) and Theorem 7.6.

We finally assume (ii) and for a local extension F/E we choose a number field K and a place v of K such that K_v is isomorphic to E and $|G(F/E)|$ is coprime to

the number of roots of unity in K. (Since $G(F/E)$ is of odd order the existence of such a field K is easily implied by the main result of [Henniart 2001].)

Then by the construction in the proof of Theorem 9.1 we find a global extension $\widetilde{E}/K$ with the properties (P1) and (P2).

It follows that $\mathfrak{a}_{\widetilde{E}/K} = \mathfrak{a}_{F/E}$, and hence that $\mathfrak{a}_{F/E}$ projects to zero in $\mathrm{Cl}(G(F/E))$, as required to prove (iii). $\qquad\square$

Lemma 9.8. *Let L/K be an odd-degree weakly ramified Galois extension of number fields of group G. Then the G-module $\mathscr{A}_{L/K}$ is free if and only if the image of $\mathfrak{a}_{L/K}$ in $\mathrm{Cl}(G)$ vanishes.*

Proof. By Theorem 5.2(iv) one has $\partial_{\mathbb{Z},\mathbb{Q},G}(\mathfrak{a}_{L/K}) = [\mathscr{A}_{L/K}]$ in $\mathrm{Cl}(G)$. Given this, the equivalence of the stated conditions follows immediately from the fact that, as G has odd order, a finitely generated projective G-module is free if and only if its class in $\mathrm{Cl}(G)$ vanishes. $\qquad\square$

9B2. By [Vinatier 2001] Vinatier's conjecture is known to be true for extensions $L/\mathbb{Q}$ with the property that the decomposition group of each wildly ramified prime is abelian. The family of nonabelian Galois extensions of degree p^3, for some odd prime p, is thus the family of smallest possible degree for which Vinatier's conjecture is not known to be valid. Such extensions were considered (in special cases) by Vinatier [2002].

In the following result we study the number of corresponding local extensions of the base field $\mathbb{Q}_p$. This result (which will be proved at the end of this section) shows that the bounds on the number of such extensions that are discussed in Remark 9.5 can be improved if one imposes ramification conditions.

Proposition 9.9. *Let p be an odd prime. Then there exist exactly p (nonisomorphic) weakly ramified nonabelian Galois extensions of $\mathbb{Q}_p$ of degree p^3. Exactly one of these extensions has exponent p and the remaining $p - 1$ extensions have exponent p^2.*

As in the proof of Theorem 9.1, for each odd prime p and each weakly ramified nonabelian Galois extension F of $\mathbb{Q}_p$ of degree p^3 there exists a weakly ramified Galois extension $N/\mathbb{Q}$ of degree p^3 such that N has a unique p-adic place w and the corresponding completion $N_w/\mathbb{Q}_p$ is isomorphic to $F/\mathbb{Q}_p$. This fact motivates the following definitions.

For each odd prime p we fix a set $\mathscr{F}(p)$ of p weakly ramified Galois extensions $N/\mathbb{Q}$ of degree p^3 such that each field N has a unique p-adic place $w(N)$ and the corresponding completions $N_{w(N)}/\mathbb{Q}_p$ give the full set of local extensions that are described in Proposition 9.9.

For a finite set P of odd primes we define $\mathscr{L}(P)$ to be the set of Galois extensions of number fields L/K such that $\mathscr{W}_{L/K}^{\mathbb{Q}} \subseteq P$ and for each place v in $\mathscr{W}_{L/K}$ one has

both $K_v = \mathbb{Q}_{\ell(v)}$ and the order of the decomposition subgroup in $G(L/K)$ of any place of L above v divides $\ell(v)^3$.

Theorem 9.10. *For any finite set of odd primes P the following conditions are equivalent.*

(i) *For all L/K in $\mathcal{L}(P)$ the $G(L/K)$-module $\mathcal{A}_{L/K}$ is free.*

(ii) *For all $N/\mathbb{Q}$ in the finite set $\bigcup_{p \in P} \mathcal{F}(p)$ the $G(N/\mathbb{Q})$-module $\mathcal{A}_{N/\mathbb{Q}}$ is free.*

Proof. Obviously (i) implies (ii). For the reverse implication fix L/K in $\mathcal{L}(P)$. By Lemma 9.8 we have to show that the element $\mathfrak{a}_{L/K}$ projects to zero in $\mathrm{Cl}(G)$. By Theorem 7.6 together with Theorem 8.1(i) we have

$$\mathfrak{a}_{L/K} = \sum_{v \in \mathscr{W}_{L/K}} i^G_{G_w, \mathbb{Q}^c_{\ell(v)}}(\mathfrak{a}_{L_w/K_v}).$$

It is therefore enough to show that each of the terms $\mathfrak{a}_{L_w/K_v}$ projects to zero in $\mathrm{Cl}(\mathbb{Z}[G_w])$. By our assumptions $K_v = \mathbb{Q}_p$ for a prime $p \in P$ and $G(L_w/\mathbb{Q}_p)$ is a p-group of order at most p^3. If $|G(L_w/\mathbb{Q}_p)| \leq p^2$, then $L_w/\mathbb{Q}_p$ is abelian and $\mathfrak{a}_{L_w/\mathbb{Q}_p} = 0$ by the relevant case of Theorem 8.1. If $|G_{L_w/\mathbb{Q}_p}| = p^3$, then by the definition of $\mathcal{L}(P)$ the local extension $L_w/\mathbb{Q}_p$ is the localization of one of the extensions $N/\mathbb{Q}$ in $\mathcal{F}(p)$, so that we have $\mathfrak{a}_{N/\mathbb{Q}} = \mathfrak{a}_{L_w/\mathbb{Q}_p}$. The claim now follows from Lemma 9.8. $\qquad\square$

In the rest of this section we give the postponed proof of Proposition 9.9.

As a first step we recall that there are two isomorphism classes of nonabelian groups of order p^3, with respective presentations

$$\begin{cases} \langle a, b \mid a^{p^2} = 1 = b^p, \ b^{-1}ab = a^{1+p} \rangle, \\ \langle a, b, c \mid a^p = b^p = c^p = 1, \ ab = bac, \ ac = ca, \ bc = cb \rangle, \end{cases} \tag{9.11}$$

the first having exponent p^2 and the second exponent p (see, for example, [Hall 1959, §4.4]). In both cases the center $Z(G)$ of the group G has order p (being generated by a^p and c, respectively) and the quotient group $G/Z(G)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Any weakly ramified nonabelian Galois extension L of $\mathbb{Q}_p$ of degree p^3 must thus contain a subfield E that is Galois over $\mathbb{Q}_p$ and such that both $G(L/E)$ is central in $G(L/\mathbb{Q}_p)$ and $G(E/\mathbb{Q}_p)$ is isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Since $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^p$ has order p^2 local class field theory implies E is the compositum of the unique subextension E_1 of $\mathbb{Q}_p(\zeta_{p^2})$ of degree p over $\mathbb{Q}_p$ and of the unique unramified extension E_2 of $\mathbb{Q}_p$ of degree p (and hence is weakly ramified, as required). In the sequel we set $G := G(L/\mathbb{Q}_p)$, $H := G(L/E)$, $\Gamma := G(E/\mathbb{Q}_p)$, and $\Delta := G(E/E_1)$.

If L/E is a weakly ramified degree-p extension such that $L/\mathbb{Q}_p$ is Galois, then $L/\mathbb{Q}_p$ is weakly ramified. Indeed, $G_2 \cap H = H_2 = 1$ and hence $G_2 \simeq G_2 H/H$. By Herbrand's theorem we obtain $G_2 H/H = (G/H)_2$, which is trivial since $E/\mathbb{Q}_p$
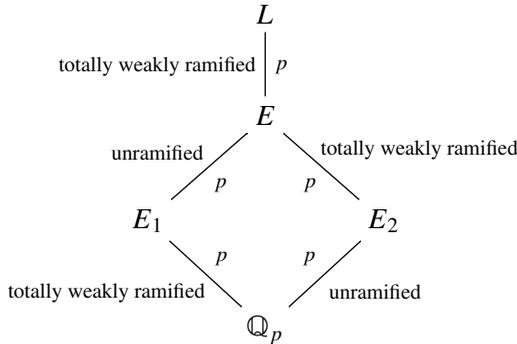
is weakly ramified. The required fields therefore correspond to weakly ramified degree-p extensions L of E which are Galois over $\mathbb{Q}_p$.

For each subfield F of E we write $\mathfrak{p}_F$ for the maximal ideal of the valuation ring $\mathbb{O}_F$ of F, $U_F^{(i)}$ for each natural number i for the group $1 + \mathfrak{p}_F^i$ of i-th principal units of F, and μ'_F for the maximal finite subgroup of $F^\times$ of order prime to p. If L/F is abelian we also write $\mathrm{rec}_{L/F}$ for the reciprocity map $F^\times \to G_{L/F}$.

If L/E is unramified, then the ramification degree of $L/\mathbb{Q}_p$ is p so that L contains both E_1 and the unramified extension of $\mathbb{Q}_p$ of degree p^2 and so L is abelian over $\mathbb{Q}_p$.

On the other hand, if L/E is ramified, then the inertia subgroup G_0 has order p^2. In addition, since $L/\mathbb{Q}_p$ is assumed to be weakly ramified, the group $G_0 = G_1$ identifies with G_1/G_2 and so is isomorphic to a subgroup of U_L^1/U_L^2 and therefore has exponent dividing p. It follows that G_0 is not cyclic and hence that $L/\mathbb{Q}_p$ is not abelian. We have therefore shown that $L/\mathbb{Q}_p$ is abelian if and only if L/E is unramified.

In summary, there is thus a field diagram of the following sort:

By an easy exercise one checks that L/E is weakly ramified if and only if the upper ramification subgroup H^2 vanishes. By local class field theory, the desired extensions L are therefore in bijective correspondence with subgroups N of $E^\times$ that are Γ-stable (as $L/\mathbb{Q}_p$ is Galois), contain $U_E^{(2)}$ [Serre 1979, Corollary 3, p. 228], contain $E^{\times p}$ (as $E^\times/N$ has exponent p), and contain $I_\Gamma(E^\times)$ (as Γ acts trivial on $E^\times/N \simeq Z(G)$), where I_Γ denotes the augmentation ideal of $\mathbb{Z}[\Gamma]$.

We note next that there are isomorphisms of abelian groups

$$(U_E^{(1)}/U_E^{(2)})_\Gamma \cong ((\mathfrak{p}_E/\mathfrak{p}_E^2)_\Delta)_{\Gamma/\Delta} \cong (\mathfrak{p}_{E_1}/\mathfrak{p}_{E_1}^2)_{\Gamma/\Delta} \cong (\mathbb{Z}/p\mathbb{Z})_{\Gamma/\Delta} = \mathbb{Z}/p\mathbb{Z} \quad (9.12)$$

where the first map is induced by the natural isomorphism $U_E^{(1)}/U_E^{(2)} \cong \mathfrak{p}_E/\mathfrak{p}_E^2$. The second isomorphism is induced by the field-theoretic trace Tr_{E/E_1}. Indeed, since E/E_1 is unramified, the induced map $(\mathfrak{p}_E/\mathfrak{p}_E^2)_\Delta \cong \mathfrak{p}_{E_1}/\mathfrak{p}_{E_1}^2$ is surjective with

kernel $\widehat{H}^{-1}(\Delta, \mathfrak{p}_E/\mathfrak{p}_E^2)$, which is trivial since $\mathfrak{p}_E^i$ is $\mathbb{Z}_p[\Delta]$-free for each nonnegative integer i. The third is induced by the fact that $\mathfrak{p}_{E_1}/\mathfrak{p}_{E_1}^2 \cong \mathbb{O}_{E_1}/\mathfrak{p}_{E_1}$ has order p (since $E_1/\mathbb{Q}_p$ is totally ramified).

To be explicit we fix a uniformizing parameter π of E_1 and recall that $E^\times = \langle \pi \rangle \times \mu'_E \times U_E^{(1)}$. Any $\gamma \in \Gamma$ can be written in the form $\gamma = \gamma_1 \gamma_2$ with $\gamma_1 \in G(E/E_1)$ and $\gamma_2 \in G(E/E_2)$. The wild inertia group Γ_1 is equal to $G(E/E_2)$ and hence we obtain $\pi^{\gamma-1} = \pi^{\gamma_2-1} \in U_{E_1}^{(1)} \subseteq U_E^{(1)}$. In addition, by (9.12) and the fact that Tr_{E/E_1} acts as multiplication by p on $\mathfrak{p}_{E_1}$, we see that $\pi^{\gamma-1}$ has trivial image in $(U_E^{(1)}/U_E^{(2)})_\Gamma$.

We set

$$T := \langle (E^\times)^p, U_E^{(2)}, I_\Gamma(E^\times) \rangle = \langle (E^\times)^p, U_E^{(2)}, I_\Gamma(U_E^{(1)}) \rangle$$

and note that the map

$$E^\times \to \langle \pi \rangle / \langle \pi^p \rangle \times (U_E^{(1)}/U_E^{(2)})_\Gamma,$$
$$\pi^a \epsilon y \mapsto (\pi^a \pmod{\langle \pi^p \rangle}, yU_E^{(2)} \pmod{I_\Gamma(U_E^{(1)}/U_E^{(2)}))}),$$

where $a \in \mathbb{Z}$, $\epsilon \in \mu'_E$, and $y \in U_E^{(1)}$, induces an isomorphism of the quotient group $Q := E^\times/T$ with the direct product $\langle \pi \rangle / \langle \pi^p \rangle \times (U_E^{(1)}/U_E^{(2)})_\Gamma \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

In particular, if we fix an element u of $U_E^{(1)}$ that generates $(U_E^{(1)}/U_E^{(2)})_\Gamma$, then the order-$p$ subgroups of Q correspond to the subgroups generated by the classes of the elements u and $\pi \cdot u^i$ for $i \in \{0, 1, 2, \ldots, p-1\}$.

In addition, since L/E is ramified if and only if N does not contain u, the quotients that we require correspond to the subgroups $Q_i := \langle \pi \cdot u^i \pmod{T} \rangle$ for $i \in \{0, 1, 2, \ldots, p-1\}$. The corresponding subgroups N_i of $E^\times$ are given by $N_i := \langle \pi u^i, T \rangle$ and we write L_i for the fields that correspond to N_i via local class field theory.

If $i \neq 0$, then Q_i does not contain the class of π so $G(L_i/E)$ is generated by $\mathrm{rec}_{L_i/E}(\pi) = \mathrm{rec}_{L_i/E_1}(N_{E/E_1}(\pi)) = \mathrm{rec}_{L_i/E_1}(\pi)^p$ and hence $G(L_i/E_1)$ is cyclic of order p^2 (and so $G(L_i/\mathbb{Q}_p)$ has exponent p^2).

Finally we claim that $G(L_0/\mathbb{Q}_p)$ has exponent p. To prove this it is enough, in view of the possible presentations (9.11), to show $G(L_0/\mathbb{Q}_p)$ contains two non-cyclic subgroups of order p^2. Hence, since its inertia subgroup $G(L_0/E_2)$ is one such subgroup (as $L_0/\mathbb{Q}_p$ is weakly ramified), it is enough to prove $G(L_0/E_1)$ also has exponent p.

To do this we note $G(L_0/E)$ is generated by $\mathrm{rec}_{L_0/E}(u) = \mathrm{rec}_{L_0/E_1}(N_{E/E_1}(u))$ and so $N_{E/E_1}(u)$ is an element of order p in $E_1^\times/N_{L_0/E_1}(L_0^\times)$. Since

$$N_{L_0/E_1}(L_0^\times) = N_{E/E_1}(N_0) = N_{E/E_1}(\langle \pi, T \rangle) \subseteq \langle \pi^p, U_{E_1} \rangle$$

we see that $\pi \notin N_{L_0/E_1}(L_0^\times)$ and $\pi^p \in N_{L_0/E_1}(L_0^\times)$. So it finally remains to show that π and $N_{E/E_1}(u)$ generate different subgroups of $E_1^\times/N_{L_0/E_1}(L_0^\times)$. But if $N_{E/E_1}(u)\pi^n$ were contained in $N_{L_0/E_1}(L_0^\times)$ for some integer n, then p would

divide n since E/E_1 is unramified of degree p. But this would then imply that $N_{E/E_1}(u)$ belongs to $N_{L_0/E_1}(L_0^\times)$, which is a contradiction.

This completes the proof of Proposition 9.9.

9B3. Following Lemma 8.7 and Remark 8.8, the weakly ramified Galois extensions $L/\mathbb{Q}$ of smallest degree for which the projection of $\mathfrak{c}_{L/\mathbb{Q}}$ to $\mathrm{Cl}(G(L/\mathbb{Q}))$ might not vanish are nonabelian and of degree $\ell^2 p$ for an odd prime p and an odd prime ℓ that divides $p - 1$. This motivates us to investigate such extensions numerically (in Section 10B) and the next result lays the groundwork for such investigations by determining a family of local extensions that satisfies the required conditions.

Proposition 9.13. *Let ℓ and p be odd primes with ℓ dividing $p - 1$. Then there exist exactly ℓ (nonisomorphic) weakly ramified nonabelian Galois extensions L of $\mathbb{Q}_p$ of degree $\ell^2 p$ with $G(E/\mathbb{Q}_p) \cong \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$, where $E := L^C$ and C is the unique Sylow-p-subgroup of $G(L/\mathbb{Q}_p)$.*

Proof. Let $L/\mathbb{Q}_p$ be an extension with the stated conditions and set $G := G(L/\mathbb{Q}_p)$.

As ℓ divides $p - 1$ the ℓ-th roots of unity are contained in $\mathbb{Q}_p$ and $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^\ell \simeq \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$, so that E is the maximal abelian extension of $\mathbb{Q}_p$ of exponent ℓ. Explicitly, $E = E_1 E_2$ where E_1 is the unramified extension of degree ℓ and $E_2 := \mathbb{Q}_p(\sqrt[\ell]{p})$. By local class field theory L corresponds to a subgroup X of $E^\times$ such that X is stable under the action of $\Gamma := G/C$, $|E^\times/X| = p$, and $U_E^{(2)} \subseteq X$ [Serre 1979, Corollary 3, p. 228].

Let H be a subgroup of Γ such that $|H| = \ell$. Since H is cyclic the extension L/E^H is abelian if and only if H acts trivially on $E^\times/X$. As a consequence $\Delta := G(E/E_1)$ acts nontrivially on $E^\times/X$ since otherwise $G(L/E_1) = G_0(L/\mathbb{Q}_p)$ would be abelian and by [Serre 1979, Corollary 2, p. 70] this contradicts $G_2(L/E_1) = 1$.

Since $p \nmid |\Gamma|$ the $\mathbb{F}_p[\Gamma]$-module $E^\times/X$ decomposes as $E^\times/X = \bigoplus_\phi e_\phi(E^\times/X)$ where ϕ runs over the $\mathbb{F}_p$-valued abelian characters of Γ and e_ϕ denotes the usual idempotent in $\mathbb{F}_p[\Gamma]$. In addition, since $|E^\times/X| = p$, exactly one of the components, for $\phi = \phi_0$ say, is nontrivial.

Since $H_0 := \ker(\phi_0)$ acts trivially on $e_{\phi_0}(E^\times/X)$ we deduce that $H_0 \neq \Delta$. Then, writing $T_H := \sum_{h \in H}$ for any subgroup H of Γ, one has $T_H(E^\times/X) = (E^\times/X)^H$ and so, since $(E^\times/X)^\Gamma \subseteq (E^\times/X)^\Delta = 0$, we deduce $H_0 \neq \Gamma$.

We claim that X contains $\langle \mu_E', \sqrt[\ell]{p}, U_E^{(2)}, I_{H_0}(U_E^{(1)}) \rangle$. To see this note $\mu_E' \subseteq X$ as $(\mu_E')^p = \mu_E'$. Since $T_\Delta(E^\times/X) = 0$ we obtain $T_\Delta(\sqrt[\ell]{p}) = N_{E_2/\mathbb{Q}_p}(\sqrt[\ell]{p}) = p = (\sqrt[\ell]{p})^\ell \in X$. As $\ell \neq p$ it follows that $\sqrt[\ell]{p} \in X$. Finally, as L/E^{H_0} is abelian, X must contain $I_{H_0}(E^\times)$, and hence also $I_{H_0}(U_E^{(1)})$, as required.

We will show below that for any subgroup H of Γ with $|H| = \ell$ and $H \neq \Delta$ the subgroup

$$X(H) := \langle \mu'_E, \sqrt[\ell]{p}, U_E^{(2)}, I_H(U_E^{(1)}) \rangle$$

is both stable under Γ and satisfies $|E^\times / X(H)| = p$.

This will show, in particular, that $X = X(H_0)$. Conversely, since each subgroup $X(H)$ corresponds by local class field theory (and [Serre 1979, p. 70, Corollary 2]) to a weakly ramified extension $L/\mathbb{Q}_p$ as in the proposition, we will also have proved that the extensions L in the proposition correspond uniquely to the subgroups H of Γ with $|H| = \ell$ and $H \neq \Delta$.

It thus remains to show that for each subgroup H as above the subgroup $X(H)$ is stable under Γ and such that $|E^\times / X(H)| = p$.

Since $\gamma(\sqrt[\ell]{p}) \equiv \sqrt[\ell]{p} \pmod{\mu'_E}$ for all $\gamma \in \Gamma$ it is immediate that $X(H)$ is Γ-stable. The extension E/E^H is unramified and therefore

$$(U_E^{(1)}/U_E^{(2)})^H \simeq (U_E^{(1)}/U_E^{(2)})_H \simeq (\mathfrak{p}_E/\mathfrak{p}_E^2)_H \simeq \mathfrak{p}_{E^H}/\mathfrak{p}_{E^H}^2 \simeq \mathbb{Z}/p\mathbb{Z},$$

where the first isomorphism holds since each $U_E^{(n)}$ is H-cohomologically trivial, the second is canonical, and the third is induced by the trace map tr_{E/E^H}. On the other hand, $(U_E^{(1)}/U_E^{(2)})_H = U_E^{(1)}/I_H(U_E^{(1)})U_E^{(2)}$ and so the decomposition $E^\times = \langle \sqrt[\ell]{p} \rangle \times \mu'_E \times U_E^{(1)}$ implies that the quotient $E^\times/X(H) \simeq U_E^{(1)}/I_H(U_E^{(1)})U_E^{(1)}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, as required. $\qquad\square$

Remark 9.14. Assume the situation of Proposition 9.13. Then the extension $E/\mathbb{Q}_p$ has ℓ subextensions $F_1, \ldots, F_\ell$ (corresponding to the subgroups H of Γ with $|H| = \ell$ and $H \neq \Delta$) that are ramified over $\mathbb{Q}_p$. For each such F_i there exists precisely one extension $L/\mathbb{Q}_p$ that satisfies the assumptions of Proposition 9.13 and is also such that L/F_i is abelian.

Remark 9.15. Our primary motivation for obtaining the explicit descriptions of wildly and weakly ramified nonabelian Galois extensions that are given above was to assist with attempts to make numerical investigations of the conjectures that we have discussed. However, such explicit descriptions are of course interesting in their own right. In this context we recall that Vinatier [2001, Corollary 2.2] has shown that for any positive multiple n of p there are exactly p nonisomorphic abelian wildly and weakly ramified extensions of $\mathbb{Q}_p$ of degree n and, moreover, that these extensions can be described explicitly.

10. Numerical examples

In this section we investigate numerically, and thereby prove, Vinatier's conjecture for two new, and infinite, families of nonabelian weakly ramified Galois extensions of $\mathbb{Q}$.

At the same time we shall also explicitly compute both sides of the equality in Conjecture 7.4 for all weakly ramified nonabelian Galois extensions of $\mathbb{Q}_3$ of degree 27, thereby verifying this conjecture, and hence also Breuning's local epsilon constant conjecture, in this case.

10A. *Extensions of degree* 27.

10A1. We first compute explicitly a set $\mathscr{F}(3)$ as in Section 9B2. To do this we have to find 3 weakly ramified Galois extensions L of $\mathbb{Q}$ of degree 27 with a unique 3-adic place w and such that $L_w/\mathbb{Q}_p$ runs over all extensions as in Proposition 9.9.

In the following p denotes 3. We shall also only consider Galois extensions $F/\mathbb{Q}$ that have a unique place w above p and so we write F_p in place of F_w.

We let E_1 be the subextension of $\mathbb{Q}(\zeta_{p^2})$ of degree p and E_2 an abelian extension of degree p such that p is inert in E_2. We set $E := E_1 E_2$ and let $\mathfrak{p}$ denote the unique prime ideal of $\mathbb{O}_E$ above p. We write Γ for the Galois group of $E/\mathbb{Q}$.

Set $Q_2 := \{\alpha \in (\mathbb{O}_E/\mathfrak{p}^2)^\times \mid \alpha \equiv 1 \pmod{\mathfrak{p}}\}$ and note that

$$(Q_2)_\Gamma \simeq (U_{E_p}^{(1)}/U_{E_p}^{(2)})_\Gamma \simeq \mathbb{Z}/p\mathbb{Z}_p$$

by (9.12). Let $u \in \mathbb{O}_E$ be such that the class of u generates $(Q_2)_\Gamma$ and let $\pi \in \mathbb{O}_E$ be a uniformizing element for $\mathfrak{p}$.

By algorithmic global class field theory we compute ray classgroups $\mathrm{cl}(q\mathfrak{p}^2)$ of conductor $q\mathfrak{p}^2$ for small positive integers q with $(q, p) = 1$ and search for subgroups $U \leq \mathrm{cl}(q\mathfrak{p}^2)$ of index p which are invariant under Γ and such that the corresponding extension L/E is ramified at $\mathfrak{p}$. Each such U corresponds to a Galois extension $L/\mathbb{Q}$ whose completion at p is one of the extensions of Proposition 9.9. As shown in the proof of Proposition 9.9 the local extensions $L_p/\mathbb{Q}_p$ are in one-to-one correspondence with the elements πu^b for $b \in \{0, 1, 2, \ldots, p-1\}$. More precisely, there is exactly one b such that $\mathrm{rec}_{L_p/\mathbb{Q}_p}(\pi u^b) = 1$. Thus, we have to find extensions $L/\mathbb{Q}$ such that the resulting integers b range from 0 to $p-1$. In order to compute $\mathrm{rec}_{L_p/\mathbb{Q}_p}(\pi u^b)$ we compute $\xi \in \mathbb{O}_E$ such that $\xi \equiv \pi \pmod{q}$ and $\xi \equiv u^{-b} \pmod{\mathfrak{p}^2}$. Then class field theory shows that $\mathrm{rec}_{L_p/\mathbb{Q}_p}(\pi u^b) = \mathrm{rec}_{L/\mathbb{Q}}(\xi\mathbb{O}_E)$, which can be computed globally.

This approach is implemented in MAGMA. For E_2 we used the cubic subextension of $\mathbb{Q}(\zeta_{19})$ and found a set of 3 global extensions $L/\mathbb{Q}$ by taking $q \in \{5, 19\}$. The results of these computations can be reproduced using the MAGMA implementation which can be downloaded from Bley's homepage.

10A2. Using results of [Bley and Boltje 2006] one can explicitly compute $\mathrm{Cl}(G)$ as an abstract group for each finite group G. In particular, for the two nonabelian groups of order 27 one finds in this way that $\mathrm{Cl}(G)$ is cyclic of order 9.

For each of the extensions $L/\mathbb{Q}$ computed in the last section we can use the algorithm described in [Bley and Wilson 2009, §5] to compute the logarithm of

$[\mathscr{A}_{L/\mathbb{Q}}]$ in $\mathrm{Cl}(G)$ with $G := G(L/\mathbb{Q})$. Since G is of odd order, $\mathscr{A}_{L/\mathbb{Q}}$ is a free G-module if and only if $[\mathscr{A}_{L/\mathbb{Q}}]$ is trivial.

In a little more detail, we first compute a normal basis element $\theta \in \mathbb{O}_L$ and the G-module $\mathscr{A}_\theta \subseteq \mathbb{Q}[G]$ such that $\mathscr{A}_\theta \theta = \mathscr{A}_{L/\mathbb{Q}}$. Then $\mathscr{A}_\theta \simeq \mathscr{A}_{L/\mathbb{Q}}$ and the element $[\mathscr{A}_\theta, \mathrm{id}, \mathbb{Z}[G]] \in K_0(\mathbb{Z}[G], \mathbb{Q}[G])$ projects to the class of $\mathscr{A}_{L/\mathbb{Q}}$ in $\mathrm{Cl}(G)$. The algorithm in [Bley and Wilson 2009] now solves the discrete logarithm problem for $[\mathscr{A}_{\theta,\ell}, \mathrm{id}, \mathbb{Z}_\ell[G]]$ in $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])$ for each of the primes ℓ dividing the generalized index $[\mathscr{A}_\theta : \mathbb{Z}[G]]$ and then uses the recipe described in [Bley and Wilson 2009, §5] to compute the logarithm of $[\mathscr{A}_{L/\mathbb{Q}}]$ in $\mathrm{Cl}(G)$.

However, for an arbitrary choice of θ the algorithm will in general fail because of efficiency problems since this set of primes ℓ is often too large and contains primes ℓ which are much too big. We therefore first compute a maximal order $\mathcal{M}$ in $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$ and an element $\delta \in \mathbb{Q}[G]$ such that $\mathcal{M}\mathscr{A}_\theta = \mathcal{M}\delta$. This is achieved by the method described in steps (1) to (5) of Algorithm 3.1 in [Bley and Johnston 2008]. We then set $\theta' := \delta(\theta)$ and start over again by computing $\mathscr{A}_{\theta'}$ such that $\mathscr{A}_{\theta'}\theta' = \mathscr{A}_{L/\mathbb{Q}}$. Then one has $\mathcal{M}\theta' = \mathcal{M}\mathscr{A}_\theta\theta = \mathcal{M}\mathscr{A}_{L/\mathbb{Q}} = \mathcal{M}\mathscr{A}_{\theta'}\theta'$.

Localizing at prime divisors ℓ of G we obtain $\mathbb{Z}_{(\ell)}[G]\theta' = \mathscr{A}_{\theta',(\ell)}\theta'$ and hence $\mathbb{Z}_{(\ell)}[G] = \mathscr{A}_{\theta',(\ell)}$. It follows that we only need to solve the discrete logarithm problem for $[\mathscr{A}_{\theta',\ell}, \mathrm{id}, \mathbb{Z}_\ell[G]]$ in $K_0(\mathbb{Z}_\ell[G], \mathbb{Q}_\ell[G])$ for primes ℓ dividing $|G|$.

The computations show that for each of the 3 extensions computed in the previous section the G-module $\mathscr{A}_{L/\mathbb{Q}}$ is free. As a consequence of these computations and Theorem 9.10 we obtain the following result.

Theorem 10.1. *For all extensions in $\mathscr{L}(3)$ the G-module $\mathscr{A}_{L/K}$ is free. In particular, Vinatier's conjecture holds for all nonabelian extensions $L/\mathbb{Q}$ of degree* 27.

10A3. We now show how to compute $\mathfrak{a}_{L/\mathbb{Q}}$ for the extension $L/\mathbb{Q}$ in $\mathscr{F}(3)$. By Theorems 7.6 and 8.1 we have $\mathfrak{a}_{L/\mathbb{Q}} = \mathfrak{a}_{L_p/\mathbb{Q}_p}$ and both $\mathfrak{a}_{L_p/\mathbb{Q}_p}$ and the right-hand side of the equality in Conjecture 7.4 can be computed by adapting the methods of [Bley and Debeerst 2013]. In the following we indicate where special care has to be taken to improve the performance of the general implementation used to obtain the results of [Bley and Debeerst 2013]

For the computation of $[\mathscr{A}_{L/\mathbb{Q}}, \kappa_L, H_L]$ we choose a normal basis element θ and write

$$[\mathscr{A}_{L/\mathbb{Q}}, \kappa_L, H_L] = [\mathbb{Z}[G] \cdot \theta, \kappa_L, H_L] + [p\mathscr{A}_{L/\mathbb{Q}}, \mathrm{id}, \mathbb{Z}[G] \cdot \theta] + \delta_G(\mathrm{Nrd}_{\mathbb{Q}[G]}(p)).$$

For computational reasons we proceed as in Section 10A2 and choose θ such that $\mathcal{M}\theta = \mathcal{M}\mathscr{A}_{L/\mathbb{Q}}$. The second and the third terms are straightforward to compute and the first term is given by norm resolvents (see, for example, [Bley and Debeerst 2013, (13)]).

For the computation of $\delta_{G,p}(j_p((\psi_{2,*} - 1)(\tau'_{L_p/\mathbb{Q}_p})))$ we first digress to describe the character theory of nonabelian groups of order p^3.

The center $Z = Z(G)$ of any such group G is equal to the commutator subgroup of G and the quotient G/Z is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ so that G has p^2 linear characters of order dividing p.

It is also easy to see that G has normal subgroups A that are isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and contain Z. We fix such a subgroup A and for each nontrivial character λ of Z we choose a character $\psi \in \hat{A}$ which restricts to give λ on Z. Then it can be shown that $\text{ind}_A^G(\psi)$ depends only on λ and does not depend on the choice of ψ. In addition, it is an irreducible character of G of degree p. Since $(p-1)p^2 + p^2 = p^3$ we have found all irreducible characters of G.

In particular, for $p = 3$ the characters of G comprise the trivial character, 8 linear characters of order 3, and 2 characters of degree 3.

Returning now to the computation of local Galois–Gauss sums we essentially proceed as described in [Bley and Breuning 2008, §2.5] but for reasons of efficiency must take care in the "Brauer induction step" of [loc. cit.].

The computation of $\tau(\mathbb{Q}_p, \chi)$ for abelian characters is clear. Let now $\chi = \text{ind}_A^G(\psi)$ be one of the characters of degree p. We set $M := L^A$ and $N := L^{\ker(\psi)}$ and use the equalities

$$\tau(\mathbb{Q}_p, \chi) = \tau(\mathbb{Q}_p, \text{ind}_A^G(\psi - 1_A)) \cdot \tau(\mathbb{Q}_p, \text{ind}_A^G(1_A))$$
$$= \tau(M_p, \psi - 1_A) \cdot \tau(\mathbb{Q}_p, \text{ind}_A^G(1_A))$$
$$= \tau(M_p, \psi) \cdot \prod_{\substack{\varphi \in \widehat{G} \\ \varphi|_A = 1_A}} \tau(\mathbb{Q}_p, \varphi).$$

The problematic part is the computation of $\tau(M_p, \psi)$. To explain why, we write $\mathfrak{f}(\psi)$ for the conductor of ψ and choose $c \in M_p$ such that $\mathfrak{f}(\psi)\mathfrak{D}_{M_p/\mathbb{Q}_p} = c\mathbb{O}_{M_p}$. Then, by the definition of local Gauss sums, one has

$$\tau(M_p, \psi) = \sum_x \psi(\text{rec}_{N_p/M_p}(x/c))\psi_{\text{add}}(x/c)$$

where ψ_{add} denotes the standard additive character and x runs over a set of representatives of $\mathbb{O}_{M_p}^\times$ modulo $U_{M_p}^{(2)}$. From an algorithmic point of view it is now important to choose the subgroup A such that $L^A/\mathbb{Q}$ is totally ramified (e.g., we may take $A = G(L/E_1)$) because then $\mathbb{O}_{M_p}^\times / U_{M_p}^{(2)}$ has order 6 as compared to order 702 if $M_p/\mathbb{Q}_p$ were the unique unramified extension of degree 3.

From the explicit description of the unramified characteristic in (4.2) it is now easy to compute $\tau'(\mathbb{Q}_p, \chi) = \tau(\mathbb{Q}_p, \chi)y(\mathbb{Q}_p, \chi)^{-1}$ for all $\chi \in \widehat{G}$ and based on this it is straightforward by the methods of [Bley and Wilson 2009] to compute the term $\delta_{G,p}(j_p((\psi_{2,*} - 1)(\tau'_{L_p/\mathbb{Q}_p})))).$

Our computations show that for all extensions $L/\mathbb{Q}$ in $\mathscr{F}(3)$ the element $\mathfrak{a}_{L_p/\mathbb{Q}_p}$ is equal to the twisted unramified characteristic $\mathfrak{c}_{L_p/\mathbb{Q}_p}$ defined in (7.3). Combining this with Theorems 7.6 and 8.1 we obtain the following result.

Theorem 10.2. *If L/K belongs to $\mathscr{L}(3)$, then $\mathfrak{a}_{L/K} = \mathfrak{c}_{L/K}$, where $\mathfrak{c}_{L/K}$ is as defined in (8.3).*

Remark 10.3. The equality $\mathfrak{a}_{L/K} = \mathfrak{c}_{L/K}$ in Theorem 10.2 combines with the results of Theorems 5.2(iv) and 10.1 to imply that the image of $\mathfrak{c}_{L/K}$ in $\mathrm{Cl}(G)$ vanishes. Under the stated conditions, this fact also follows directly from Lemma 8.7. Conversely, the results of Theorem 10.2, Theorem 5.2(iv), and Lemma 8.7 combine to give an alternative proof of Theorem 10.1.

Remark 10.4. By adapting the methods implemented for [Bley and Debeerst 2013] one can also compute the right-hand side of the equality in Conjecture 7.4 for all extensions $L/\mathbb{Q}$ in $\mathscr{F}(3)$. These computations show that

$$\mathscr{E}_{L_p/\mathbb{Q}_p} - J_{2,L_p/\mathbb{Q}_p} - \mathfrak{c}_{L_p/\mathbb{Q}_p} - M_{L_p/\mathbb{Q}_p} = \mathfrak{c}_{L_p/\mathbb{Q}_p},$$

and thereby verify Conjecture 7.4, and hence also Breuning's conjecture, for these extensions. Combining this fact with [Breuning 2004b, Theorem 4.1; Bley and Burns 2003, Corollary 7.6] one also finds that the central conjecture of [Bley and Burns 2003] is valid for all L/K in $\mathscr{L}(3)$ for which G_w has order 27 and exponent 3 for each wildly ramified place w of L.

10B. *Extensions of degree* 63.

10B1. Let ℓ and p be odd primes with ℓ dividing $p - 1$. We now sketch how to compute a set of Galois extensions $L/\mathbb{Q}$ of degree $\ell^2 p$ such that $L/\mathbb{Q}$ is at most tamely ramified outside p and the extensions $L_w/\mathbb{Q}_p$ cover the set of local extensions of Proposition 9.13 (where as usual w denotes the unique place of L above p).

We use a simple heuristic approach which is motivated by the proof of Proposition 9.13 and which works well for $\ell = 3$ and $p = 7$.

We fix a cyclic extension $E_1/\mathbb{Q}$ of degree ℓ such that p is inert and ℓ is unramified. Let E_2 denote the unique subextension of $\mathbb{Q}(\zeta_p)$ of degree ℓ. Let $E := E_1 E_2$ denote the compositum of E_1 and E_2 and let $F_1, \ldots, F_\ell$ be the subextensions of $E/\mathbb{Q}$ of degree ℓ which are ramified at p. Then the completions $F_{i,p}$ of the F_i at the unique prime above p range over the set of totally ramified cyclic extensions of $\mathbb{Q}_p$ of degree ℓ. By Remark 9.14 the extensions $N/\mathbb{Q}_p$ which are nonabelian and wildly and weakly ramified can be distinguished by the unique subfield $F_{i,p}$ such that $N/F_{i,p}$ is abelian.

Let now $\mathfrak{p}$ denote the unique prime of $\mathcal{O}_E$ above p. We then compute ray classgroups $\mathrm{cl}(q\mathfrak{p}^2)$ for small rational integers q with $(q, \ell p) = 1$ and search for

subgroups U of $\mathrm{cl}(q\mathfrak{p}^2)$ of index p which are invariant under $G(E/\mathbb{Q})$ and such that the corresponding extension L/E is both wildly and weakly ramified above $\mathfrak{p}$. Then $L_p/F_{i,p}$ is abelian, if and only if $G(E/F_i)$ acts trivially on the quotient $\mathrm{cl}(q\mathfrak{p}^2)/U$ (or equivalently, $I_{G(E/F_i)}\,\mathrm{cl}(q\mathfrak{p}^2) \subseteq U$). A search using MAGMA produces these extensions. The results can be reproduced with the MAGMA programs which can be downloaded from the Bley's homepage.

10B2. We now fix $\ell := 3$ and $p := 7$ and apply classgroup methods to verify Vinatier's conjecture for the three extensions $L/\mathbb{Q}$ described in the previous section. The principal approach is exactly the same as described in Section 10A2.

For the locally free classgroup of a nonabelian group G of order 63 such that $G/C \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ (where C denotes the Sylow-7-subgroup) one finds that $\mathrm{Cl}(G)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$.

Our computations show that for each of the three extensions L computed in the previous subsection Vinatier's conjecture is valid. Taken in conjunction with Theorem 7.6 and Lemma 9.8 this fact implies the following result.

Theorem 10.5. *Let L/K be a weakly ramified odd-degree Galois extension of number fields for which at each wildly ramified place v of K one has $K_v = \mathbb{Q}_7$, $|G_w| = 63$, and that G_w/C_w is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, where w denotes a fixed place of L above v, G_w its decomposition subgroup in $G(L/K)$, and C_w the Sylow-7-subgroup of G_w. Then $\mathscr{A}_{L/K}$ is a free $G(L/K)$-module.*

Remark 10.6. One can also use numerical methods to show that for each of the extensions L/K considered in Theorem 10.5 the projection of $\mathfrak{c}_{L/K}$ to $\mathrm{Cl}(G)$ vanishes.

10C. $\mathfrak{a}_{L/K}$ *and idelic twisted unramified characteristics.* An extension of the methods used in Section 10A3 also allowed us to numerically compute the element $\mathfrak{a}_{L_p/\mathbb{Q}_p}$ for one of the three extensions $L/\mathbb{Q}$ of degree 63 discussed in Section 10B1, so that $\ell = 3$ and $p = 7$. (For the other two extensions that occur in this setting, however, the necessary computations became too complex and did not finish in reasonable time.)

In particular, in this respect it is useful to note that groups of order $\ell^2 p$ are monomial, and hence that one can proceed as in Section 10A3 for the computation of the local Galois–Gauss sums.

These numerical computations showed that $\mathfrak{a}_{L_p/\mathbb{Q}_p} = \mathfrak{c}_{L_p/\mathbb{Q}_p}$. Taken in conjunction with Theorem 10.2, Corollary 8.4, and the observation in Remark 8.9, this fact motivates us to make the following remarkable conjecture.

Conjecture 10.7. *For any weakly ramified odd-degree Galois extension of number fields L/K one has $\mathfrak{a}_{L/K} = \mathfrak{c}_{L/K}$.*

Remark 10.8. Upon comparing Conjectures 7.4 and 10.7 one obtains, for each odd prime ℓ and each weakly ramified odd-degree Galois extension of ℓ-adic fields F/E, an explicit conjectural formula

$$\delta_{\Gamma,\ell}(J_{2,F/E}) = \mathscr{E}_{F/E} - 2c_{F/E} - M_{F/E}$$

that computes Galois–Jacobi sums in terms of fundamental classes and twisted unramified characters.

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Rigidity in equivariant algebraic *K*-theory

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If (R, I) is a henselian pair with an action of a finite group G and $n \geq 1$ is an integer coprime to $|G|$ such that $n \cdot |G| \in R^*$, then the reduction map of mod-n equivariant K-theory spectra

$$K^G(R)/n \xrightarrow{\simeq} K^G(R/I)/n$$

is an equivalence. We prove this by revisiting the recent proof of nonequivariant rigidity by Clausen, Mathew, and Morrow.

1. Introduction and statement of result

Rigidity is a fundamental feature of algebraic *K*-theory with finite coefficients which was established by Suslin [1983] for extensions of algebraically closed fields, and by Gabber and Gillet–Thomason [Gillet and Thomason 1984] for geometric henselian local rings. In [Gabber 1992], inspired by previous results of Suslin [1984] for henselian valuation rings of dimension one, Gabber proved a rigidity theorem for algebraic *K*-theory with finite coefficients for general henselian pairs:

Theorem 1.1 (Gabber). *If (R, I) is a henselian pair and $n \geq 1$ is an integer such that $n \in R^*$, then*

$$K(R)/n \xrightarrow{\simeq} K(R/I)/n$$

is an equivalence.

In all these results, the coefficients are assumed to be coprime to the characteristic. In [Clausen et al. 2018], the authors established the most comprehensive rigidity statement to date addressing the case of coefficients not necessarily coprime to the characteristic. To formulate it, we denote by K^{inv} the fiber of the cyclotomic trace $K \to \mathrm{TC}$. Then their result [Clausen et al. 2018, Theorem A] reads as follows:

Theorem 1.2 (Clausen, Mathew, Morrow). *If (R, I) is a henselian pair and $n \geq 1$ is an integer, then the reduction map*

$$K^{\mathrm{inv}}(R)/n \xrightarrow{\simeq} K^{\mathrm{inv}}(R/I)/n$$

is an equivalence.

The purpose of the present note is to generalize this result to an equivariant situation for an action of a finite, abstract group G.

Given a commutative ring R with an action of G, there is associated the twisted group ring $R \wr G$; see Section 2 for a reminder. Our main theorem is below:

Theorem 1.3. *If the finite group G acts on the henselian pair (R, I), $|G| \in R^*$, and $n \geq 1$ is an integer coprime to $|G|$, then the reduction map*

$$K^{\mathrm{inv}}(R \wr G)/n \xrightarrow{\simeq} K^{\mathrm{inv}}((R/I) \wr G)/n$$

is an equivalence.

The more traditional invariant in equivariant algebraic K-theory is the spectrum $K^G(R)$, defined to be the connective K-theory of the exact category of finitely generated projective R-modules together with a semilinear G-action. We deduce the next result about this.

Corollary 1.4. *Assume in the situation of Theorem 1.3 that, in addition, $n \in R^*$ holds. Then the reduction map*

$$K^G(R)/n \xrightarrow{\simeq} K^G(R/I)/n$$

is an equivalence.

Proof. Since $n \in R^*$, the TC-term in the definition of $K^{\mathrm{inv}}(R \wr G)$ vanishes mod n, i.e., $K^{\mathrm{inv}}(R \wr G)/n \simeq K(R \wr G)/n$. Since $|G| \in R^*$, a finitely generated projective left $R \wr G$-module is the same thing as a finitely generated projective R-module with a semilinear G-action; hence $K(R \wr G) \simeq K^G(R)$, and similarly with R replaced with R/I. $\square$

Remark 1.5.   • The appearance of $R \wr G$ might seem a bit spurious since all our results assume $|G| \in R^*$, which forces $K(R \wr G) \simeq K^G(R)$ (and similarly for TC). Since however several of the intermediate results work without assuming that $|G| \in R^*$, we decided to phrase things in terms of $R \wr G$.

   • The referee points out that Theorem 1.3 might lend credibility to the following expectation about rigidity: If (A, I) is a henselian pair, $n \geq 1$ an integer and B a finite A-algebra, then $K^{\mathrm{inv}}(B)/n \to K^{\mathrm{inv}}(B/IB)/n$ should be an equivalence.

Corollary 1.4 is a generalization of Theorem 1.1 for equivariant algebraic K-theory. Rigidity results for equivariant algebraic K-theory have been previously studied for henselian local rings with trivial group actions (but for more general algebraic groups) in [Yagunov and Østvær 2009; Krishna 2010] and in [Yagunov and Østvær 2009] and [Tabuada 2018] for extensions of algebraically closed fields and extensions of separably closed fields, respectively. In [Heller et al. 2018], Corollary 1.4 was proved in the geometric case, assuming that G is abelian and that k contains $|G|$-th roots of unity.

The proofs of our results are direct generalizations of those of [Clausen et al. 2018]. We made an effort to make this paper reasonably self-contained, which results in repeating some arguments from [Clausen et al. 2018].

We conclude the introduction with an overview of the sections. In Section 2, we establish the equivariant generalization of the key finiteness property, called pseudocoherence, isolated in [Clausen et al. 2018]. This allows us to generalize equivariant rigidity from certain nice geometric situations to general henselian pairs. In Section 3 we establish a sufficient supply of equivariant rigidity in nice situations (see Proposition 3.1) by combining the nonequivariant result with decomposition results of Vistoli and Tabuada–Van den Bergh. Section 4 collects further technical results. Finally, Section 5 assembles the pieces into a proof of Theorem 1.3.

2. G-projective pseudocoherence

The aim of this section is to establish the equivariant generalizations of the finiteness properties [Clausen et al. 2018, Propositions 4.21, 4.25] of algebraic K-theory and of topological cyclic homology with finite coefficients. Fix a finite group G throughout.

Let R be a commutative ring and $I \subset R$ an ideal. Recall that the pair (R, I) is called a henselian pair if for every $f(t) \in R[t]$, $\bar{a} \in R/I$, such that $\bar{a}$ is a simple root of $\bar{f}(t) \in (R/I)[t]$, there exists $a \in R$ such that $a \mapsto \bar{a}$ and $f(a) = 0$. By a result of Gabber [1992, Corollary 1], the property of (R, I) being a henselian pair depends only on the ideal I, regarded as a nonunital ring, and not on R. We now briefly recall the definition of nonunital henselian algebras. For a detailed discussion see [Clausen et al. 2018, Section 3].

For a commutative ring R, a nonunital R-algebra is an R-module I endowed with a multiplication $I \otimes_R I \to I$ which is associative and commutative. A nonunital R-algebra I is said to be henselian if for every $n \geq 0$ and every $g(t) \in I[t]$ of degree at most n, the polynomial $f(t) = t(1+t)^n + g(t)$ has a (necessarily unique) root in I. Let $\mathrm{Ring}_R^{\mathrm{nu,h}}$ denote the category of nonunital, henselian R-algebras.

Definition 2.1. We denote by $\mathrm{Ring}_R^{\mathrm{nu,h},G}$ the category of G-objects in $\mathrm{Ring}_R^{\mathrm{nu,h}}$.

To ease the notation, we abbreviate $\mathrm{Ring}^{\mathrm{nu,h},G} := \mathrm{Ring}_{\mathbb{Z}}^{\mathrm{nu,h},G}$.

It is observed in [Clausen et al. 2018, Remark 3.10] that the category $\mathrm{Ring}_R^{\mathrm{nu,h}}$ is bicomplete, and that the forgetful functor

$$R : \mathrm{Ring}_R^{\mathrm{nu,h}} \to \mathrm{Sets}$$

to sets is a conservative right adjoint which commutes with sifted colimits.[1]

---

[1]Equivalently, as the categories are discrete, it commutes with filtered colimits and split coequalizers.

Denoting by

$$F_R : \mathrm{Sets} \to \mathrm{Ring}_R^{\mathrm{nu,h}}$$

its left-adjoint, this is remarked to imply that the subcategory $(\mathrm{Ring}_R^{\mathrm{nu,h}})_\Sigma \subseteq \mathrm{Ring}_R^{\mathrm{nu,h}}$ of compact projective objects is the idempotent completion of the full subcategory spanned by the free objects

$$F_R(n) := F_R(\{1, \ldots, n\}) \quad (n \geq 0).$$

Moreover, $F_R(n)$ is identified in [Clausen et al. 2018, Example 3.9] as the ideal generated by the variables $X_1, \ldots, X_n$ in the R-algebra given by the henselization of $R[X_1, \ldots, X_n]$ along the ideal $(X_1, \ldots, X_n)$.

This generalizes to the equivariant setting as follows: The category $\mathrm{Ring}^{\mathrm{nu,h},G}$ is bicomplete and the forgetful functor

$$R' : \mathrm{Ring}_R^{\mathrm{nu,h},G} \to \mathrm{Ring}_R^{\mathrm{nu,h}}$$

is a conservative right-adjoint which commutes with all colimits. This is clear by thinking of $\mathrm{Ring}^{\mathrm{nu,h},G}$ as the category of presheaves on G with values in $\mathrm{Ring}^{\mathrm{nu,h}}$. Consequently, denoting the left-adjoint of R' by

$$F_R' : \mathrm{Ring}_R^{\mathrm{nu,h}} \to \mathrm{Ring}_R^{\mathrm{nu,h},G},$$

and letting $F_R'' := F_R' \circ F_R$, the subcategory $(\mathrm{Ring}_R^{\mathrm{nu,h},G})_\Sigma \subseteq \mathrm{Ring}_R^{\mathrm{nu,h},G}$ of compact projective objects is the idempotent completion of the full subcategory spanned by the free objects $F_R''(n) := F_R''(\{1, \ldots, n\})$ $(n \geq 0)$. These can be identified explicitly:

Proposition 2.2. *For every $n \geq 0$, $F_R''(n)$ is the ideal generated by the variables $X_{\sigma,i}$ ($\sigma \in G$, $1 \leq i \leq n$) in the R-algebra given by the henselization of the polynomial R-algebra $R[X_{\sigma,i} \mid \sigma \in G, 1 \leq i \leq n]$ along the ideal $(X_{\sigma,i})$, and G-action determined by $\sigma(x_{\tau,i}) = x_{\sigma\tau,i}$.*

Said a bit more invariantly, $F_R''(n)$ is the henselization along the origin of the affine R-space afforded by the direct sum of n copies of the regular representation of G over R.

Proof of Proposition 2.2. Since henselization is a left-adjoint, it suffices to see the analogous statement before henselization. Then using the equivalence between nonunital R-algebras and augmented R-algebras, the claim follows because the augmented R-algebra with G-action $R[X_{\sigma,i} \mid \sigma \in G, 1 \leq i \leq n]$ has the required mapping property. $\square$

For every $N \geq 1$, we denote by

$$[N] : F_R''(n) \to F_R''(n)$$

the "multiplication-by-N map", namely the unique map in $\mathrm{Ring}_R^{\mathrm{nu,h},G}$ which, under the identification of Proposition 2.2, maps every $X_{\sigma,i}$ to $N X_{\sigma,i}$.

Proposition 2.3. *For fixed $M \geq 1$ and $n \geq 0$, we have an isomorphism in $\mathrm{Ring}_{\mathbb{Z}[1/M]}^{\mathrm{nu,h},G}$*

$$\mathrm{colim}_{(N,M)=1} F''_{\mathbb{Z}[1/M]}(n) \simeq F''_{\mathbb{Q}}(n),$$

the (filtered) colimit being taken along the multiplication maps $[N]$ for all N coprime to M, partially ordered by divisibility.

Proof. This is proved exactly as in the special case $M = 1$, $G = \{e\}$, which is due to Gabber (see [Clausen et al. 2018, Corollary 3.20]). We leave the details to the reader. $\square$

Recall the twisted group ring (e.g., [Curtis and Reiner 1981, §28]): If R is a commutative ring with a (left) G-action, then the twisted group ring $R \wr G$ is the finite free R-module on the set $\{e_\sigma : \sigma \in G\}$ with multiplication determined by $(r e_\sigma)(r' e_\tau) = r\sigma(r') e_{\sigma\tau}$. This construction is functorial in R. It is rigged such that the datum of a left $R \wr G$-module is equivalent to the datum of an R-module together with a *semi*linear G-action. Observe that when the G-action on R is trivial, this construction gives the usual group ring, i.e., $R \wr G = R[G]$ in this case.

For an associative, unital ring A, we denote by $K(A)$ the connective K-theory spectrum of the category of finitely generated projective left A-modules; see [Quillen 1973]. Given any $I \in \mathrm{Ring}^{\mathrm{nu,h},G}$, we denote by $\mathbb{Z} \ltimes I$ the ring with G-action obtained from I by adjoining a unit (necessarily with trivial G-action). The augmentation $\mathbb{Z} \ltimes I \to \mathbb{Z}$ is G-equivariant and thus induces an augmentation

$$p : (\mathbb{Z} \ltimes I) \wr G \to \mathbb{Z}[G].$$

We need the following equivariant generalization of [Clausen et al. 2018, Lemma 4.20].

Proposition 2.4. *Given $I \in \mathrm{Ring}^{\mathrm{nu,h},G}$, the map $p^* : K_0((\mathbb{Z} \ltimes I) \wr G) \xrightarrow{\simeq} K_0(\mathbb{Z}[G])$ is an isomorphism.*

Proof. This is a special case of Proposition 4.6. $\square$

We denote by Sp the ∞-category of spectra. We recall from [Clausen et al. 2018, Definition 4.4] the notions of perfectness and pseudocoherence of spectrum-valued functors on a category relative to a subcategory: Given a small full subcategory $\mathcal{D}$ of a locally small category $\mathcal{C}$, a functor $F : \mathcal{C} \to \mathrm{Sp}$ is called $\mathcal{D}$-perfect if F belongs to the thick subcategory generated by the functors $\{\Sigma_+^\infty \mathrm{Hom}_{\mathcal{C}}(D, -) \mid D \in \mathcal{D}\}$ in the presentable, stable ∞-category $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp})$. A functor $F \in \mathrm{Fun}(\mathcal{C}, \mathrm{Sp})$ is said to be $\mathcal{D}$-pseudocoherent if for each $n \in \mathbb{Z}$, there exists a $\mathcal{D}$-perfect functor F_n and a map $F_n \to F$ such that $\tau_{\leq n} F_n(C) \to \tau_{\leq n} F(C)$ is an equivalence for all $C \in \mathcal{C}$. In the particular case when $\mathcal{D} = (\mathrm{Ring}^{\mathrm{nu,h}})_\Sigma \subseteq \mathcal{C} = \mathrm{Ring}^{\mathrm{nu,h}}$, F is called

projectively pseudocoherent; see [Clausen et al. 2018, Definition 4.12, (2)]. We pose the immediate equivariant generalization of this as a definition.

Definition 2.5. A functor $F : \mathrm{Ring}_R^{\mathrm{nu,h},G} \to \mathrm{Sp}$ is called G-projectively pseudocoherent (G-pscoh for short), if it is $(\mathrm{Ring}_R^{\mathrm{nu,h},G})_\Sigma$-pseudocoherent.

Our first aim then is to establish the following generalization of [Clausen et al. 2018, Proposition 4.21].

Proposition 2.6. *The functor* $\mathrm{Ring}^{\mathrm{nu,h},G} \to \mathrm{Sp}$, $I \mapsto K((\mathbb{Z} \ltimes I) \wr G)$ *is G-pscoh.*

Proof. Using the fiber sequence of functors

$$\tau_{\geq 1} K((\mathbb{Z} \ltimes (-)) \wr G) \to K((\mathbb{Z} \ltimes (-)) \wr G) \to \tau_{\leq 0} K((\mathbb{Z} \ltimes (-)) \wr G) = K_0((\mathbb{Z} \ltimes (-)) \wr G)$$

and the fact that G-pscoh functors form a thick subcategory [Clausen et al. 2018, Proposition 4.8, (1)], it suffices to see separately the G-projective pseudocoherence of $\tau_{\geq 1} K((\mathbb{Z} \ltimes (-)) \wr G)$ and of $K_0((\mathbb{Z} \ltimes (-)) \wr G)$.

For the latter, Proposition 2.4 yields an isomorphism $K_0((\mathbb{Z} \ltimes (-)) \wr G) \simeq K_0(\mathbb{Z}[G])$ to the constant functor with value the finitely generated abelian group $K_0(\mathbb{Z}[G])$; see [Kuku 2007, Theorem 2.2.1]. This settles the claim for this term.

To see that the other term is G-pscoh, we use the criterion [Clausen et al. 2018, Propositions 4.10 and 4.11] to reduce to seeing that the functor

$$H\mathbb{Z} \otimes \Sigma_+^\infty \Omega^\infty \tau_{\geq 1} K((\mathbb{Z} \ltimes (-)) \wr G)$$

is G-pscoh. It is well known (see [Weibel 2013, Chapter IV, §1]) that this functor is equivalent to $C_*(\mathrm{BGL}((\mathbb{Z} \ltimes (-)) \wr G); \mathbb{Z})$, the complex of integral chains on the classifying space of the infinite general linear group. We now use homology stability as given by [van der Kallen 1980, Theorem in Section 4.11] for the associative ring $A(-) := (\mathbb{Z} \ltimes (-)) \wr G$. To do so, we need to see that the stable range of $A(-)$ is bounded independently of the argument $- \in \mathrm{Ring}^{\mathrm{nu,h},G}$. Firstly, it is easy to see that dividing out a radical ideal does not change the stable range (see [Lam 1999, p. 32] and [Weibel 2013, Chapter I, Exercise 1.12(v)]), and at the beginning of the proof of Proposition 4.6 we will see that $(-) \wr G$ is a radical ideal in $A(-)$ with quotient ring $\mathbb{Z}[G]$. This already gives the independence of the stable range of $A(-)$ of the argument $(-)$, and since $\mathbb{Z}[G]$ is finite over its central subring $\mathbb{Z}$, this is bounded by (in fact, equal to) the stable range of $\mathbb{Z}$ (according to Bass's stable range theorem [1968, Chapter V, Theorem 3.5]). We conclude that for every $n \geq 1$ the obvious map on truncations

$$\tau_{\leq n} C_*(\mathrm{BGL}_{2n+1}((\mathbb{Z} \ltimes (-)) \wr G); \mathbb{Z}) \to \tau_{\leq n} C_*(\mathrm{BGL}((\mathbb{Z} \ltimes (-)) \wr G); \mathbb{Z})$$

is an equivalence. Renaming indices, this reduces us to seeing that for a fixed $n \geq 1$, the functor

$$C_*(\mathrm{BGL}_n((\mathbb{Z} \ltimes (-)) \wr G); \mathbb{Z})$$

is G-pscoh. There is a short exact sequence of groups

$$1 \to X(-) \to \mathrm{GL}_n((\mathbb{Z} \ltimes (-)) \wr G) \xrightarrow{\pi} \mathrm{GL}_n(\mathbb{Z}[G]) \to 1,$$

defining $X(-)$.[2] This gives an equivalence

$$C_*(\mathrm{BGL}_n((\mathbb{Z} \ltimes (-)) \wr G); \mathbb{Z}) \simeq (C_*(BX(-); \mathbb{Z}))_{h(\mathrm{GL}_n(\mathbb{Z}[G]))}.$$

To conclude the argument exactly as in the proof of [Clausen et al. 2018, Proposition 4.19], it remains to establish that, firstly, the functor $C_*(BX(-); \mathbb{Z})$ is G-pscoh and that, secondly, there is a finite index normal subgroup $N \subseteq \mathrm{GL}_n(\mathbb{Z}[G])$ such that its classifying space BN is equivalent to a finite CW-complex. The first claim follows as in [loc. cit.], because $X(I) \simeq I^{|G| \cdot n^2}$ (as sets), and the second claim follows from work of Borel and Serre, specifically [Serre 1971, Section 2.4, Théorème 4 and Section 1.5, Proposition 10], if we can show that $\mathrm{GL}_n(\mathbb{Z}[G])$ is an arithmetic subgroup of a suitable reductive group $\mathcal{G}$ over $\mathbb{Q}$. Indeed, one can take for $\mathcal{G}$ the group of units of the $\mathbb{Q}$-algebra $M_n(\mathbb{Q}[G])$: it is clear that

$$\mathrm{GL}_n(\mathbb{Z}[G]) \subseteq \mathcal{G}(\mathbb{Q}) = \mathrm{GL}_n(\mathbb{Q}[G])$$

is an arithmetic subgroup, and since $\mathbb{Q}[G] \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}[G]$ is a product of full matrix rings over $\mathbb{C}$, the group $\mathcal{G} \otimes_{\mathbb{Q}} \mathbb{C}$ is a finite product of $\mathrm{GL}_{i,\mathbb{C}}$ for various i, and hence is (connected and) reductive. $\square$

The following generalization of [Clausen et al. 2018, Proposition 4.25] is even more immediate.

Proposition 2.7. *For every prime p, the functor*

$$\mathrm{Ring}^{\mathrm{nu,h},G} \to \mathrm{Sp}, \quad I \mapsto \mathrm{TC}((\mathbb{Z} \ltimes I) \wr G)/p$$

is G-pscoh.

Proof. This is identical to [loc. cit.], and we leave the details to the reader. Recall at least that the core part of the argument, namely [Clausen et al. 2018, Proposition 2.19], is a result about $\mathrm{TC}(-)/p$ considered on the category of cyclotomic spectra, which applies equally well to the case at hand. $\square$

Recall that we write K^{inv} for the fiber of the cyclotomic trace $K \to \mathrm{TC}$. We introduce a relative term $K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G)$ to sit in a fiber sequence

$$K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G) \to K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G) \to K^{\mathrm{inv}}(\mathbb{Z} \wr G) = K^{\mathrm{inv}}(\mathbb{Z}[G]).$$

Combining Propositions 2.6 and 2.7 yields the following, which is the finiteness result to be used in the proof of Theorem 1.3.

---

[2]To see that π is onto, recall that the augmentation $(\mathbb{Z} \ltimes (-)) \wr G \to \mathbb{Z}[G]$ is *split* surjective.

Proposition 2.8. *For every prime* p, *the functor*

$$\mathrm{Ring}^{\mathrm{nu,h},G} \to \mathrm{Sp}, \quad I \mapsto K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G)/p$$

is G-*pscoh.*

3. A geometric special case

The purpose of this section is to establish a geometric special case of our main result, Theorem 1.3. This equivariant rigidity result will follow from its nonequivariant special case [Clausen et al. 2018, Theorem A] together with decomposition results of Vistoli and Tabuada–Van den Bergh [Tabuada and Van den Bergh 2018]. To formulate it, fix for the rest of this section a finite group G, a field k of characteristic not dividing $|G|$,[3] and a prime p not dividing $|G|$ (but possibly equal to the characteristic of k). Let X be an affine, smooth k-algebra with a G-action and assume given a rational point $x \in X(k)$ fixed by G. Then G acts canonically on the henselization $\mathcal{O}^h_{X,x}$ of the local ring $\mathcal{O}_{X,x}$, and the canonical map $\pi : \mathcal{O}^h_{X,x} \to k$ to the residue field is G-equivariant (for k endowed with the trivial G-action). Hence it induces a map on twisted group rings $\mathcal{O}^h_{X,x} \wr G \to k \wr G = k[G]$. The result then is the following.

Proposition 3.1. *In the above situation, the map*

$$K^{\mathrm{inv}}(\mathcal{O}^h_{X,x} \wr G)/p \xrightarrow{\simeq} K^{\mathrm{inv}}(k[G])/p \tag{3.2}$$

induced by π *is an equivalence.*

Proof. We start by setting the stage to apply [Tabuada and Van den Bergh 2018]. We let $E := \pi_*(K^{\mathrm{inv}}(-)/p)$, and observe that this is an additive invariant taking values in $\mathbb{Z}[1/|G|]$-modules and commuting with filtered colimits; for algebraic K-theory, this is classical and for $\mathrm{TC}(-)/p$ it follows from [Clausen et al. 2018, Theorem 2.7]. Now, [Tabuada and Van den Bergh 2018, Remark 1.3(ii) and (iii)] implies that

$$E([X/G]) \xrightarrow{\simeq} \left(\bigoplus_{\sigma \subseteq G \text{ cyclic}} \widetilde{E}(X^\sigma \times \mathrm{Spec}(k[\sigma])) \right)^G, \tag{3.3}$$

where $X^\sigma \subseteq X$ is the subscheme fixed by σ, and $\widetilde{E}$ refers to a certain functorially defined direct summand of E (depending on σ). Since we do not require knowledge of the exact shape of that summand, we do not review its definition here.

We observe that the G-fixed point $x \in X(k)$ determines a map

$$\bar{x} : [\mathrm{Spec}(k)/G] \to [X/G]$$

---

[3]By convention, this condition is satisfied if k is of characteristic zero.

such that $E(\bar{x})$ participates in a commutative diagram

$$
\begin{array}{ccc}
E([X/G]) & \xrightarrow{\simeq \ (3.3)} & \left(\bigoplus_{\sigma \subseteq G \text{ cyclic}} \widetilde{E}(X^\sigma \times \operatorname{Spec}(k[\sigma])) \right)^G \\
\downarrow{\scriptstyle E(\bar{x})} & & \downarrow{\scriptstyle \oplus_\sigma \widetilde{E}(x_\sigma \times \operatorname{id})} \\
E([\operatorname{Spec}(k)/G]) & \xrightarrow{\simeq} & \left(b \bigoplus_{\sigma \subseteq G \text{ cyclic}} \widetilde{E}(\operatorname{Spec}(k[\sigma])) \right)^G
\end{array}
\tag{3.4}
$$

where x_σ denotes the unique factorization of x through $X^\sigma \subseteq X$.

Next we want to pass to henselizations. To do this, we observe that everywhere in the above argument, one can replace $([X/G], x)$ with a pointed étale neighborhood (Y, y) such that $\kappa(x) \xrightarrow{\simeq} \kappa(y)$ is an isomorphism on residue fields. We obtain a commutative diagram generalizing (3.4):

$$
\begin{array}{ccc}
E([Y/G]) & \xrightarrow{\simeq} & \left(\bigoplus_{\sigma \subseteq G \text{ cyclic}} \widetilde{E}(Y^\sigma \times \operatorname{Spec}(k[\sigma])) \right)^G \\
\downarrow{\scriptstyle E(\bar{y})} & & \downarrow{\scriptstyle \oplus_\sigma \widetilde{E}(y_\sigma \times \operatorname{id})} \\
E([\operatorname{Spec}(k)/G]) & \xrightarrow{\simeq} & \left(\bigoplus_{\sigma \subseteq G \text{ cyclic}} \widetilde{E}(\operatorname{Spec}(k[\sigma])) \right)^G
\end{array}
\tag{3.5}
$$

Passing to the filtered colimit of all such (Y, y) and recalling that henselization commutes with the closed immersions $X^\sigma \subseteq X$ (and more generally with integral extensions [Stacks 2005–, tag 0DYE]), we obtain

$$
\begin{array}{ccc}
E([\operatorname{Spec}(\mathcal{O}^h_{X,x})/G]) & \xrightarrow{\simeq} & \left(\bigoplus_{\sigma \subseteq G \text{ cyclic}} \widetilde{E}(\operatorname{Spec}(\mathcal{O}^h_{X^\sigma,x}) \times \operatorname{Spec}(k[\sigma])) \right)^G \\
\downarrow{\scriptstyle E(\iota)} & & \downarrow{\scriptstyle \oplus_\sigma \widetilde{E}(\iota_\sigma \times \operatorname{id})} \\
E([\operatorname{Spec}(k)/G]) & \xrightarrow{\simeq} & \left(\bigoplus_{\sigma \subseteq G \text{ cyclic}} \widetilde{E}(\operatorname{Spec}(k[\sigma])) \right)^G
\end{array}
\tag{3.6}
$$

Here, $\iota_\sigma : \operatorname{Spec}(k) \hookrightarrow \operatorname{Spec}(\mathcal{O}^h_{X^\sigma,x})$ and $\iota : [\operatorname{Spec}(k)/G] \hookrightarrow [\operatorname{Spec}(\mathcal{O}^h_{X,x})/G]$ are (induced by) the canonical projection to the residue fields. Since each ι_σ is a closed immersion with henselian defining ideal, so is each $\iota_\sigma \times \operatorname{id}_{\operatorname{Spec}(k[\sigma])}$, and by [Clausen et al. 2018, Theorem A], every map $\widetilde{E}(\iota_\sigma \times \operatorname{id})$ is an isomorphism, and hence so is $E(\iota)$.

To equate $E(\iota)$ with $\pi_*((3.2))$, and thus to conclude the proof, it remains to recall that $E(-) = \pi_*(K^{\operatorname{inv}}(-)/p)$ and that since the order $|G|$ is invertible, a finitely generated projective module with a semilinear G-action is the same thing

as a finitely generated projective left module over the twisted group ring, so that we have an equivalence of ∞-categories of perfect modules

$$\mathrm{Perf}([\mathrm{Spec}(\mathcal{O}^h_{X,x})/G]) \simeq \mathrm{Perf}(\mathcal{O}^h_{X,x} \wr G),$$

and similarly with $\mathcal{O}^h_{X,x}$ replaced by k. $\hfill\square$

4. Nil-invariance, excision and exactness

Nil-invariance.

Proposition 4.1. *Let G be a finite group and $\pi : R \to R'$ a surjective homomorphism of commutative rings with a G-action such that $\ker(\pi)$ is nilpotent. Then $K^{\mathrm{inv}}(R \wr G) \xrightarrow{\simeq} K^{\mathrm{inv}}(R' \wr G)$ is an equivalence.*

Proof. This will follow from [Dundas et al. 2013, Chapter VII, Theorem 0.0.2] if we can show that the kernel of (the obviously surjective) ring homomorphism $\pi \wr G : R \wr G \to R' \wr G$ is nilpotent. However, an immediate computation shows that for every $n \geq 0$ we have

$$(\ker(\pi \wr G))^n \subseteq (\ker(\pi))^n \wr G. \hfill\square$$

Excision. Assume that

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle g} \\ R' & \longrightarrow & S' \end{array} \qquad (4.2)$$

is a Milnor square of commutative rings, i.e., a pull-back diagram of rings with g surjective; see [Bass 1968, Chapter IX, §5] for an early account and [Land and Tamme 2019] for a current development. If, in addition, a finite group G acts on (4.2), then the induced square of twisted group rings

$$\begin{array}{ccc} R \wr G & \longrightarrow & S \wr G \\ \downarrow & & \downarrow{\scriptstyle g \wr G} \\ R' \wr G & \longrightarrow & S' \wr G \end{array} \qquad (4.3)$$

is clearly still a Milnor square. Denoting by $\mathbb{K}$ nonconnective algebraic K-theory and by $\mathbb{K}^{\mathrm{inv}}$ the fiber of the cyclotomic trace $\mathbb{K} \to \mathrm{TC}$, we then deduce the following from [Land and Tamme 2019, Theorem 3.3].

Proposition 4.4. *In the above situation,*

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{inv}}(R \wr G) & \longrightarrow & \mathbb{K}^{\mathrm{inv}}(S \wr G) \\ \downarrow & & \downarrow \\ \mathbb{K}^{\mathrm{inv}}(R' \wr G) & \longrightarrow & \mathbb{K}^{\mathrm{inv}}(S' \wr G) \end{array} \qquad (4.5)$$

is a pull-back square.

To pass to connective K-theory here, we need the equivariant generalization of [Clausen et al. 2018, Corollary 4.34], namely Proposition 4.6 below.

For an associative, unital ring A, we denote by

$$\mathbb{P}\mathrm{roj}(A)$$

the set of isomorphism classes of finitely generated projective left A-modules.

We start by establishing the equivariant generalization of [Clausen et al. 2018, Lemma 4.20]:

Proposition 4.6. *Let (R, I) be a henselian pair and G a finite group acting on (R, I). Then the obvious homomorphism*

$$K_0(R \wr G) \xrightarrow{\simeq} K_0((R/I) \wr G)$$

is an isomorphism.

Proof. To see that the map is injective, according to [Bass 1968, Chapter IX, Proposition 1.3] it suffices to check that the kernel of the projection $R \wr G \to (R/I) \wr G$, namely

$$I \wr G := \left\{ \sum_{\sigma \in G} a(\sigma) e_\sigma \,\Big|\, a(\sigma) \in I \right\} \subseteq R \wr G$$

is contained in the radical of $R \wr G$. Otherwise, $I \wr G$ was not contained in some maximal left ideal $\mathfrak{n} \subseteq R \wr G$. Then the subset

$$I \wr G + \mathfrak{n} := \{ x + y \mid x \in I \wr G, \, y \in \mathfrak{n} \} \subseteq R \wr G$$

was a left-ideal properly containing $\mathfrak{n}$, and hence

$$I \wr G + \mathfrak{n} = R \wr G. \tag{4.7}$$

We consider $R = Re_e \subseteq R \wr G$ as a (noncentral!) subring. Then (4.7) holds as an equality of R-modules, and since $I \wr G = I(R \wr G)$ and $R \wr G$ is a finite (and free) R-module, Nakayama's lemma[4] implies that $\mathfrak{n} = R \wr G$, a contradiction which completes the proof of injectivity.

To see the surjectivity, we establish the stronger claim that the reduction map

$$\mathbb{P}\mathrm{roj}(R \wr G) \twoheadrightarrow \mathbb{P}\mathrm{roj}((R/I) \wr G) \tag{4.8}$$

is surjective. Write $\bar{R} := R/I$ and fix some $\bar{M} \in \mathbb{P}\mathrm{roj}(\bar{R} \wr G)$. We first descend everything to a situation of finite type over the integers. The ring with G-action $R = \bigcup_\alpha R_\alpha$ is the union of its finitely generated, G-stable subrings $R_\alpha \subseteq R$. Accordingly, we also have $\bar{R} = \bigcup_\alpha (R_\alpha / I_\alpha) =: \bigcup_\alpha \bar{R}_\alpha$ for $I_\alpha := R_\alpha \cap I$. Since then also $\bar{R} \wr G = \bigcup_\alpha (\bar{R}_\alpha \wr G)$, the given $\bar{M}$ descends to some $\bar{M}_\alpha \in \mathbb{P}\mathrm{roj}(\bar{R}_\alpha \wr G)$ for suitably large indices α.

---

[4]Recall that I is contained in the radical of R.

Write $(R_\alpha, I_\alpha)^h$ for the henselization of R_α along $I_\alpha \subseteq R_\alpha$, and note that G naturally acts on $(R_\alpha, I_\alpha)^h$, because henselization is functorial. It suffices to lift the given $\overline{M}_\alpha \in \mathbb{P}\mathrm{roj}(\overline{R}_\alpha \wr G)$ to some element of $\mathbb{P}\mathrm{roj}((R_\alpha, I_\alpha)^h \wr G)$, because the inclusion $R_\alpha \subseteq R$ factors through $(R_\alpha, I_\alpha)^h$ G-equivariantly.

We next claim an inclusion of (G-invariant) ideals for all sufficiently large $M \gg 0$, namely

$$I_\alpha^M \subseteq I_\alpha^G \cdot R_\alpha \subseteq I_\alpha \subseteq R_\alpha. \tag{4.9}$$

Indeed, the second inclusion is obvious, and the first one follows from the fact that I_α is finitely generated together with the below relation, valid for every $x \in I_\alpha$:

$$0 = \prod_{g \in G}(x - g(x)) =: x^{|G|} + \sum_{i=0}^{|G|-1} a_i x^i \quad \text{with } a_i \in I_\alpha \cap R_\alpha^G = I_\alpha^G,$$

which implies that $x^{|G|} \in I_\alpha^G \cdot R_\alpha$.

By (4.9), the kernel of the projection

$$R_\alpha / I_\alpha^G \cdot R_\alpha \simeq R_\alpha^G / I_\alpha^G \otimes_{R_\alpha^G} R_\alpha \twoheadrightarrow R_\alpha / I_\alpha = \overline{R}_\alpha$$

is nilpotent, and an easy calculation then shows that so is the kernel of the projection

$$(R_\alpha / I_\alpha^G \cdot R_\alpha) \wr G \twoheadrightarrow \overline{R}_\alpha \wr G$$

(see the proof of Proposition 4.1).

By [Bass 1968, Chapter III, Corollary 2.4 and Proposition 2.12] then, we can lift the given $\overline{M}_\alpha \in \mathbb{P}\mathrm{roj}(\overline{R}_\alpha \wr G)$ to some $\overline{M}'_\alpha \in \mathbb{P}\mathrm{roj}((R_\alpha / I_\alpha^G \cdot R_\alpha) \wr G)$.

As a final piece of preparation, we need to see what happens to the G-invariants under henselization. Since $R_\alpha^G \subseteq R_\alpha$ is integral, and (4.9) shows that

$$\sqrt{I_\alpha^G \cdot R_\alpha} = \sqrt{I_\alpha},$$

[Stacks 2005–, tag 0DYE] implies that the canonical map

$$(R_\alpha^G, I_\alpha^G)^h \otimes_{R_\alpha^G} R_\alpha \simeq (R_\alpha, I_\alpha)^h \tag{4.10}$$

is an isomorphism.

We are now in a position to lift the given $\overline{M}'_\alpha \in \mathbb{P}\mathrm{roj}((R_\alpha / I_\alpha^G \cdot R_\alpha) \wr G)$ using [Greco 1969, Theorem 4.1], as follows:[5]

As our henselian pair, we take $(R_\alpha^G, I_\alpha^G)^h$, and as our algebra, $A := (R_\alpha, I_\alpha)^h \wr G$. The algebra A is a finite $(R_\alpha^G)^h$-module, because it is clearly finite over R_α^h, and (4.10) shows that R_α^h is finite over $(R_\alpha^G)^h$, because $R_\alpha^G \subseteq R_\alpha$ is finite, being both integral and of finite type.

---

[5]The application of this theorem here is a bit involved because in general neither is $R \wr G$ an R-algebra in any obvious way (but only an R^G-algebra), nor is $R^G \subseteq R$ finite.

We then compute the reduction of $A = (R_\alpha, I_\alpha)^h \wr G$ to be

$$\bar{A} := A/I_\alpha^G \cdot A \simeq (R_\alpha/I_\alpha^G \cdot R_\alpha) \wr G.$$

Now, [Greco 1969, Theorem 4.1] shows that the given $\overline{M}'_\alpha \in \mathbb{P}\mathrm{roj}((R_\alpha/I_\alpha^G \cdot R_\alpha)\wr G) = \mathbb{P}\mathrm{roj}(\bar{A})$ lifts to some element of $\mathbb{P}\mathrm{roj}(A) = \mathbb{P}\mathrm{roj}((R_\alpha, I_\alpha)^h \wr G)$, as desired. $\qquad\square$

We can now start to work on the version of Proposition 4.4 for connective algebraic K-theory, at least for those diagrams (4.2) coming from suitable maps of henselian pairs. In the following, fix henselian pairs (R, I) and (S, J) with an action of the finite group G, and assume that $(R, I) \to (S, J)$ is a map of pairs which respects the G-action and maps I isomorphically to J. Then

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ S & \longrightarrow & S/J \end{array} \qquad (4.11)$$

is a diagram as in (4.2), i.e., a Milnor-square with a G-action.

We then define $K(R \wr G, I \wr G)$ by the fiber sequence

$$K(R \wr G, I \wr G) \to K(R \wr G) \to K((R/I) \wr G),$$

and analogously for $K(S \wr G, J \wr G)$ and with K replaced by $\mathbb{K}$. The map of pairs $(R, I) \to (S, J)$ induces a map

$$K(R \wr G, I \wr G) \to K(S \wr G, J \wr G),$$

and similarly for $\mathbb{K}$. Recall that there is a canonical transformation $K \to \mathbb{K}$.

Proposition 4.12. *In the above situation, the diagrams*

(i)
$$\begin{array}{ccc} K(R \wr G, I \wr G) & \longrightarrow & K(S \wr G, J \wr G) \\ \downarrow & & \downarrow \\ \mathbb{K}(R \wr G, I \wr G) & \longrightarrow & \mathbb{K}(S \wr G, J \wr G) \end{array} \qquad (4.13)$$

and

(ii)
$$\begin{array}{ccc} K^{\mathrm{inv}}(R \wr G, I \wr G) & \longrightarrow & K^{\mathrm{inv}}(S \wr G, J \wr G) \\ \downarrow & & \downarrow \\ \mathbb{K}^{\mathrm{inv}}(R \wr G, I \wr G) & \longrightarrow & \mathbb{K}^{\mathrm{inv}}(S \wr G, J \wr G) \end{array} \qquad (4.14)$$

are pull-back squares.

Proof. To prove part (i), let $F \to \mathbb{F}$ denote the map induced by (4.13) on horizontal fibers. The claim is that this map is an equivalence. Since $K \to \mathbb{K}$ induces an isomorphism on π_k for $k \geq 0$, we have $\pi_k(\mathbb{F}) \simeq \pi_k(F)$ for $k \geq 0$. The excision theorem of Milnor–Bass–Murthy [Bass 1968, Chapter XII, Theorem 8.3] applied

to the diagram obtained from (4.11) by passing to twisted group rings shows that $\pi_k(\mathbb{F}) = 0$ for all $k \leq -1$. It remains to see that $\pi_k(F) = 0$ in this range, too. Since $K_0(R \wr G) \xrightarrow{\simeq} K_0((R/I) \wr G)$ is an isomorphism (Proposition 4.6) and the maps

$$K_1(R \wr G) \to K_1((R/I) \wr G) \quad \text{and} \quad K_1(S \wr G) \to K_1((S/J) \wr G)$$

are surjections (this is true more generally for any surjective ring homomorphism with kernel contained in the radical [Bass 1968, Chapter IX, Proposition 1.3(1)]), we conclude that the fibers $K(R \wr G, I \wr G)$ and $K(S \wr G, J \wr G)$ are concentrated in degrees ≥ 1, and thus F is concentrated in degrees ≥ 0, as claimed.

Part (ii) follows from part (i) by passage to fibers over TC, because the canonical transformation $K \to$ TC factors as $K \to \mathbb{K} \to$ TC. $\qquad \square$

Corollary 4.15. *If* (R, I) *is a henselian pair with a G-action and $R \to S$ is a map of commutative rings with G-action mapping I isomorphically to an ideal $J \subseteq S$, then*

$$K^{\mathrm{inv}}(R \wr G, I \wr G) \xrightarrow{\simeq} K^{\mathrm{inv}}(S \wr G, J \wr G)$$

is an equivalence.

Proof. The diagram

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ S & \longrightarrow & S/J \end{array}$$

is a Milnor-square with G-action. Note that the pair (S, J) is also henselian by [Clausen et al. 2018, Lemma 3.18]. Therefore, an application of Proposition 4.12(ii) reduces our claim to the analogous statement with K^{inv} replaced with $\mathbb{K}^{\mathrm{inv}}$. This is then a special case of Proposition 4.4. $\qquad \square$

These results will be used to reduce rigidity of arbitrary pairs to rigidity of those pairs of the form $(\mathbb{Z} \ltimes I, I)$ already encountered in Section 2.

Corollary 4.16. *For a fixed finite group G, there is an equivalence of spectra, functorial in the henselian pair (R, I) with G-action*

$$K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G) \xrightarrow{\simeq} K^{\mathrm{inv}}(R \wr G, I \wr G).$$

Exactness. We call a sequence $I' \to I \to \bar{I}$ in $\mathrm{Ring}^{\mathrm{nu,h},G}$ *short exact* if it is so when considered nonequivariantly, i.e., in $\mathrm{Ring}^{\mathrm{nu,h}}$, i.e., if the underlying sequence of abelian groups is short exact; see [Clausen et al. 2018, Definition 3.4]. We consider the functor

$$F : \mathrm{Ring}^{\mathrm{nu,h},G} \to \mathrm{Sp}, \quad F(I) := K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G),$$

and claim that it is exact:

Proposition 4.17. *Given a short exact sequence $I' \to I \to \bar{I}$ in $\mathrm{Ring}^{\mathrm{nu,h},G}$, then $F(I') \to F(I) \to F(\bar{I})$ is a fiber sequence.*

Proof. We consider the following commutative diagram:

$$
\begin{array}{ccccc}
F(I') & \longrightarrow & F(I) & \longrightarrow & F(\bar{I}) \\
\downarrow & & \downarrow & & \downarrow \\
F(I') & \longrightarrow & K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G) & \longrightarrow & K^{\mathrm{inv}}((\mathbb{Z} \ltimes \bar{I}) \wr G) \\
& & \downarrow & & \downarrow \\
& & K^{\mathrm{inv}}(\mathbb{Z}[G]) & \xrightarrow{\quad = \quad} & K^{\mathrm{inv}}(\mathbb{Z}[G])
\end{array}
$$

The top row is the one we want to recognize as a fiber sequence. The two right columns are the fiber sequences defining $F(I)$ and $F(\bar{I})$. The indicated equality implies that the upper right square is a pull-back. Hence the top row is a fiber sequence if and only if so is the second row. We verify this by observing that using Corollary 4.15 for the obvious map of pairs with G-action $(\mathbb{Z} \ltimes I', I') \to (\mathbb{Z} \ltimes I, I')$ gives

$$
F(I') \overset{\mathrm{def}}{=} K^{\mathrm{inv}}((\mathbb{Z} \ltimes I') \wr G, I' \wr G) \overset{\simeq}{\to} K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I' \wr G)
$$
$$
\overset{\mathrm{def}}{=} \mathrm{fiber}\big(K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G) \to K^{\mathrm{inv}}(((\mathbb{Z} \ltimes I)/I') \wr G)\big).
$$

Since $(\mathbb{Z} \ltimes I)/I' \simeq \mathbb{Z} \ltimes \bar{I}$, this concludes the proof. $\qquad\square$

5. The proof of the main result

In this section, we give the proof of our main result, Theorem 1.3, which we restate for convenience.

Theorem 5.1. *If the finite group G acts on the henselian pair (R, I), $|G| \in R^*$, and $n \geq 1$ is an integer coprime to $|G|$, then the reduction map*

$$
K^{\mathrm{inv}}(R \wr G)/n \overset{\simeq}{\to} K^{\mathrm{inv}}((R/I) \wr G)/n
$$

is an equivalence.

Proof. We can assume that $n = p$ is a prime (not dividing $|G|$). Since

$$
K^{\mathrm{inv}}(R \wr G, I \wr G)/p \simeq K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G)/p
$$

(see Corollary 4.16), our claim is that the functor

$$
F : \mathrm{Ring}^{\mathrm{nu,h},G}_{\mathbb{Z}[1/|G|]} \to \mathrm{Sp}, \quad F(I) := K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G)/p
$$

is trivial. We start by collecting properties of F that were established previously. By Proposition 2.8,

$$
F \text{ is } G\text{-pscoh.} \tag{5.2}
$$

By Proposition 4.17,

$$F \text{ sends short exact sequences to fiber sequences} \qquad (5.3)$$

and by Proposition 4.1,

$$F \text{ vanishes on nilpotent arguments.} \qquad (5.4)$$

For every prime field Ω of characteristic not dividing $|G|$,[6] recall the compact projective generators $F''_\Omega(n) \in \text{Ring}_\Omega^{\text{nu,h},G}$ ($n \geq 0$) from Proposition 2.2. We deduce from Proposition 3.1 that $F(F''_\Omega(n)) = 0$ for all $n \geq 0$.[7] Since by (5.2), the restriction of F to $\text{Ring}_\Omega^{\text{nu,h},G}$ is G-pscoh, and in particular left Kan extended from its subcategory of compact projective objects (see [Clausen et al. 2018, Lemma 4.6]), which in turn is the idempotent completion of all the $F''_\Omega(n)$, we see that for every prime field Ω of characteristic not dividing $|G|$, we have

$$F(\text{Ring}_\Omega^{\text{nu,h},G}) = 0. \qquad (5.5)$$

We now boot-strap to see that for every $N \geq 1$, $I \in \text{Ring}_{\mathbb{Z}[1/|G|]}^{\text{nu,h},G}$:

$$\text{if } (N, |G|) = 1 \text{ and } NI = 0, \text{ then } F(I) = 0. \qquad (5.6)$$

Since F preserves finite products, we can assume that $N = q^r$ is a prime-power (with the prime q not dividing $|G|$) and then consideration of the short exact sequence $qI \to I \to I/qI$ together with (5.3), (5.4) and (5.5) (for $\Omega = \mathbb{F}_q$) proves (5.6).

Since F is bounded below, there is an integer $d \in \mathbb{Z}$ such that

$$\pi_k F = 0 \quad \text{for every } k < d. \qquad (5.7)$$

We will be done if we can show that the functor to abelian groups

$$F_0 := \pi_d F : \text{Ring}_{\mathbb{Z}[1/|G|]}^{\text{nu,h},G} \to \text{Ab}$$

vanishes, because d being arbitrary, this will imply that $F = 0$. To see this, we will establish that

there is some N coprime to $|G|$ such that for all $I \in \text{Ring}_{\mathbb{Z}[1/|G|]}^{\text{nu,h},G}$,

$$F_0(NI) \to F_0(I) \text{ is the zero map.} \qquad (5.8)$$

Given this, using (5.3) and (5.7), we obtain an exact sequence

$$F_0(NI) \xrightarrow{0} F_0(I) \to F_0(I/NI) \to 0,$$

where $F_0(I/NI) = 0$ by (5.6), and hence $F_0(I) = 0$.

---

[6]By convention, this is fulfilled for characteristic zero.

[7]This is the step which forces us to assume that p does not divide $|G|$, and that the characteristic of Ω does not divide $|G|$.

To prove (5.8), we recall (Proposition 2.3) that we have a relation between free objects

$$F_{\mathbb{Q}}''(n) = \mathrm{colim}_{(N,|G|)=1} F_{\mathbb{Z}[1/|G|]}''(n).$$

Since $F_0(F_{\mathbb{Q}}''(n)) = 0$ by (5.5) for $\Omega = \mathbb{Q}$ and F_0 commutes with filtered colimits, we deduce that for every $x \in F_0(F_{\mathbb{Z}[1/|G|]}''(n))$ there is some N coprime to $|G|$ (depending on x and n) such that $[N](x) = 0$. To deduce from this the more uniform statement (5.8) one uses that F_0 is finitely generated and takes the product of all N for the generators. Since the details of this step are literally the same as in the proof of [Clausen et al. 2018, Lemma 4.16], we omit them here.   □

The above proof establishes the following axiomatic rigidity result, which generalizes [Clausen et al. 2018, Proposition 4.15].

Proposition 5.9. *Assume G is a finite group and $F : \mathrm{Ring}_{\mathbb{Z}[1/|G|]}^{\mathrm{nu},\mathrm{h},G} \to \mathrm{Sp}$ is a G-pscoh functor such that*

(1) *for each prime field Ω of characteristic not dividing $|G|$ and $n \geq 0$, we have $F(F_{\Omega}''(n)) = 0$;*

(2) *F sends short exact sequences to fiber sequences;*

(3) *F vanishes on nilpotent arguments.*

Then $F = 0$.

Acknowledgement

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Functorality of the gamma filtration and computations for some twisted flag varieties

Eoin Mackall

We introduce techniques for uniformly studying the gamma filtration of projective homogeneous varieties. These techniques are utilized in some cases of inner-twisted flag varieties (of type A) to show that functorality known for the Chow rings of these varieties also extends to the associated graded rings for the gamma filtrations of the same varieties. As an application, we show that the associated graded groups for the gamma filtration of these varieties are torsion free in low homological degrees.

1. Introduction

In the past, the γ-filtration, along with the coniveau — or topological, or Chow — filtration, of the Grothendieck ring of a projective homogeneous variety X had been studied in order to gain information on the, often more elusive, Chow groups of X. For example, these filtrations facilitated the first calculation of torsion in the Chow groups of a projective quadric [Karpenko 1990] and the first computation of torsion in the Chow group of codimension 2 cycles of a Severi–Brauer variety [Karpenko 1995b]. In return, knowledge about the Chow groups of these varieties often led to information about these varieties, or related objects, including the construction of fields of u-invariant 9 [Izhboldin 2001] or showing generic central simple algebras of prime exponent were indecomposable [Karpenko 1995b].

More recently, the γ-filtration has been used to estimate the size of torsion in the Chow groups of higher codimension for Severi–Brauer varieties [Baek 2015] and to estimate torsion in the Chow groups of codimension 2 cycles and codimension 3 cycles for many other projective homogeneous varieties [Garibaldi and Zainoulline 2014]. The γ-filtration has also recently been shown to have connections to the theory of cohomological invariants [Merkurjev et al. 2015] due to the relations between the γ-filtration and the Chow group of codimension 2 cycles, and the Chow group of codimension 2 cycles for generic complete flag varieties and cohomological invariants of degree 3.

Very recently, Karpenko conjectured that the γ-filtration should completely compute the Chow ring for the class of generically split generic twisted flag varieties. More precisely, the Chow ring of such a variety X is generated by Chern classes [Karpenko 2018c]. This means that the γ-filtration and coniveau filtration for this X coincide, and Karpenko's conjecture is that the canonical epimorphism from the Chow ring of X to the associated graded ring for the coniveau filtration of X is an isomorphism. It's now known that this conjecture is false in general [Karpenko 2019] but it has been proved in a number of cases [Karpenko 2017b; 2018a; 2018b; Karpenko and Mackall 2019] and is still open in many more.

This paper is the result of studying the associated graded ring for the γ-filtration of an arbitrary Severi–Brauer variety (it is still open whether or not Karpenko's conjecture holds in this case; see [Karpenko and Mackall 2019] for partial results in this direction). We prove two main theorems in this regard: Theorem 4.11 and Theorem 5.1. The first of these theorems extends functorality that is known to hold for the Chow ring (and to the associated graded ring for the coniveau filtration) of a Severi–Brauer variety to functorality for the associated graded ring for the γ-filtration. The second is a direct computation of the group summands of the associated graded ring for the γ-filtration in low homological degrees; in particular, these summands are torsion free. Throughout this paper we also show how to extend these theorems to other (generically split) inner twisted flag varieties of type A.

It seems that, although all of our results are stated and worked out only in the case of Severi–Brauer and related varieties, the ideas contained here should extend to other classes of projective homogeneous varieties. For this reason we're going to spend some time explaining the aspects that should generalize to other settings.

The first aspect of this paper that should be explained is the use of τ-functorial replacements for a smooth variety X (Definition 4.1). Essentially, a τ-functorial replacement for X is a smooth variety Y that has the two properties that the associated graded rings, $\mathrm{gr}_\gamma K(X)$ and $\mathrm{gr}_\gamma K(Y)$, for the γ-filtrations of X and Y are isomorphic, and that the associated graded rings for the γ-filtration and coniveau filtration, $\mathrm{gr}_\gamma K(Y)$ and $\mathrm{gr}_\tau G(Y)$, of Y are canonically isomorphic. In particular, the ring $\mathrm{gr}_\gamma K(Y)$ computes the ring $\mathrm{gr}_\gamma K(X)$ and $\mathrm{gr}_\gamma K(Y)$ has all of the functorality of $\mathrm{gr}_\tau G(Y)$.

To the author's knowledge, the first time τ-functorial replacements appeared in the literature is in [Karpenko 1998], where they were used to compute the torsion subgroup of the Chow group of codimension 2 cycles of a Severi–Brauer variety in some generic cases. Here we use τ-functorial replacements to prove functorality results for the γ-filtration and its associated graded ring. As an example (see Corollary 4.12), we show that the associated graded ring for the γ-filtration of a Severi–Brauer variety X is a sum of copies of the associated graded ring for the γ-filtration of the minimal Severi–Brauer variety X' Brauer-equivalent to X.

Similarly, the general philosophy working with τ-functorial replacements should be that if one can obtain a decomposition of the motive of X depending only on some canonically associated subvarieties of, and projective bundles over, X, and if X has a τ-functorial replacement Y that also has this decomposition, then the ring $\mathrm{gr}_\gamma\, K(X)$ should decompose similarly. The reason for this is because the coniveau filtration already has most of the functorality one needs to make this claim, e.g., it has pushforwards. So, if Y is a τ-functorial replacement for X and if $f : Z \to Y$ is a proper morphism then it makes sense to talk about the pushforward $f_* : \mathrm{gr}_\gamma\, K(Z) \to \mathrm{gr}_\gamma\, K(Y)$ defined as the uniquely determined arrow making the following square commutative:

$$
\begin{array}{ccc}
\mathrm{gr}_\gamma\, K(Z) & \longrightarrow & \mathrm{gr}_\tau\, G(Z) \\
\Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\
\mathrm{gr}_\gamma\, K(Y) & =\!=\!= & \mathrm{gr}_\tau\, G(Y)
\end{array}
$$

And this should be sufficient to make the claim for $\mathrm{gr}_\gamma\, K(Y) = \mathrm{gr}_\gamma\, K(X)$.

The second aspect of this paper that should be explained is the method we use for calculating the γ-filtration of a Severi–Brauer variety. For any Severi–Brauer variety X associated to a p-primary indexed central simple algebra A, one can find a finite set of λ-ring generators for the Grothendieck ring $K(X)$. The finite set that we use is a collection of sheaves, or vector bundles, that comes from the data contained in the reduced behavior of A (Definition 5.2 and Lemma 5.3).

It follows almost immediately that any set of λ-ring generators for $K(X)$, with X a smooth variety, also determines generators for the γ-filtration of X: one can take as generators for γ^i those K-theoretic Chern classes of the λ-ring generating set (below we take the negatives of the duals of this set, since this is more convenient for computations). A possibly naïve, but still interesting, question would be whether this idea extends to other projective homogeneous varieties: is there a canonical set of sheaves, or vector bundles, associated to a discrete invariant of a projective homogeneous variety X that generates $K(X)$ as a λ-ring? As one could just take a basis for $K(X)$ for their λ-ring generating set, a better, but more difficult, question is: is there a canonical set S_X of sheaves, or vector bundles, associated to a discrete invariant of a projective homogeneous variety X such that S_X generates $K(X)$ as a λ-ring *and* S_X is minimal among such sets?

Now we give an overview of this paper. Sections 2 and 3 serve as background to Sections 4 and 5. In Section 2 we describe a nice presentation for the Grothendieck ring of a Severi–Brauer variety. Section 3 gives the definition of the γ and coniveau filtrations; we take the properties of these filtrations as known and refer to references when the reader needs them. Section 4 contains the main bulk of conceptual work. In this section we introduce τ-functorial replacements and prove that they

exist in a number of cases. Section 5 is, by contrast, mainly computational. We compute here the associated graded ring for the γ-filtration of a Severi–Brauer variety through entirely elementary means.

Notation and Conventions. We fix a field k throughout. All of our objects are defined over k unless stated otherwise.

If X is a variety considered over a field F, not necessarily equal to k, we write $\overline{X}$ for X over an algebraic closure of F.

If p is a prime, then v_p is the p-adic valuation.

2. Grothendieck groups of Severi–Brauer varieties

Throughout this section we fix a central simple algebra A of degree n and let

$$X = \mathrm{SB}(A) \subset \mathrm{Gr}(n, A)$$

be the Severi–Brauer variety of A of dimension $n - 1$ considered as a subvariety of the Grassmannian of n-planes in A. For any field F over k, the F-points of $\mathrm{SB}(A)$ are exactly the minimal right ideals of A_F. We write ζ_X for the tautological sheaf on X. By definition, ζ_X is the pullback of the universal subsheaf on $\mathrm{Gr}(n, A)$ so, for any k-algebra R and any R-point x of X corresponding to a right ideal $I \subset A \otimes_k R$, the sheaf $x^* \zeta_X$ can be canonically identified with I when considered as an R-module; in particular, ζ_X is a right module over the constant sheaf A.

By $K(X)$ we mean the Grothendieck ring of locally free sheaves on X. By $G(X)$ we mean the Grothendieck ring of coherent sheaves on X. The two groups are canonically isomorphic via the morphism sending the class of a locally free sheaf in $K(X)$ to the class of itself in $G(X)$. These groups have been computed in this case:

Theorem 2.1 [Quillen 1973, §8, Theorem 4.1]. *The homomorphism of K-groups*

$$\bigoplus_{i=0}^{\deg(A)-1} K(A^{\otimes i}) \to K(X)$$

sending the class of a left $A^{\otimes i}$-module M to $\zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M$ is an isomorphism.

In particular, $K(X)$ is a free $\mathbb{Z}$-module of rank $\deg(A)$ that is additively generated by the classes

$$\zeta_X(i) := \zeta_X^{\otimes i} \otimes_{A^{\otimes i}} M_i$$

as i varies between $0 \le i < \deg(A)$; here we denote by M_i a simple $A^{\otimes i}$-module. For any splitting field F of A the variety X_F is isomorphic with the projective space $\mathbb{P}_F^{n-1}$, and the extension of scalars map $K(X) \to K(X_F)$ is injective and identifies $K(X)$ as a subring of $K(X_F)$. More precisely, we have:

Theorem 2.2. *In the setting above, let ξ denote the class of $\mathcal{O}_{X_F}(-1)$ in $K(X_F)$. There is a ring isomorphism*

$$\mathbb{Z}[x]/(1-x)^n \xrightarrow{\sim} K(X_F)$$

sending x to ξ. Under this isomorphism $K(X)$ identifies with the subring of $\mathbb{Z}[x]/(1-x)^n$ generated by $\mathrm{ind}(A^{\otimes i})x^i$.

Proof. The isomorphism is well-known; see [Manin 1969]. Finally, we use that $\zeta_X \otimes_k F$ has class $\deg(A)\xi$ in $K(X_F)$ to get the remaining claim by computing the ranks of the $\zeta_X(i)$. $\qquad\qquad\square$

We also include here the following formulas. The first is just the binomial theorem (before and after a change of coordinates). The second applies the previous one.

Lemma 2.3. *In any commutative ring, there are equalities*

$$(1-x)^n = \sum_{i=0}^{n}(-1)^i \binom{n}{i}x^i \quad and \quad x^n = \sum_{i=0}^{n}(-1)^i \binom{n}{i}(1-x)^i, \qquad (2.4)$$

$$x^n - 1 = \sum_{i=1}^{n}(-1)^i \binom{n}{i}(1-x)^i, \qquad\qquad (2.5)$$

for all integers $n \geq i \geq 0$. $\qquad\qquad\square$

3. The gamma and coniveau filtrations

In this section we recall some results on the γ-filtration of $K(X)$ and on the coniveau (or topological or Chow) filtration on $G(X)$ for an arbitrary smooth variety X.

For the first, recall there are γ-operations defined on $K(X)$ as follows. The i-th-exterior power operation induces a well-defined map $\lambda^i : K(X) \to K(X)$ which is uniquely determined by sending the class of a locally free sheaf $\mathcal{F}$ to the class of $\bigwedge^i \mathcal{F}$. The i-th γ-operation $\gamma^i : K(X) \to K(X)$ is defined by sending an element x to the coefficient of t^i in the formal series

$$\gamma_t(x) = \sum_{j\geq 0} \lambda^j(x)\left(\frac{t}{1-t}\right)^j.$$

The γ-filtration on $K(X)$ is defined as $\gamma^0 = K(X)$, $\gamma^1 = \ker(\mathrm{rk})$, where $\mathrm{rk}:K(X)\to\mathbb{Z}$ is the map sending the class of a locally free sheaf $\mathcal{F}$ to its rank, and γ^i for $i \geq 2$ is generated by monomials $\gamma^{i_1}(x_1)\cdots\gamma^{i_r}(x_r)$ for any $r \geq 0$, $i_1 + \cdots + i_r \geq i$ and $x_1, \ldots, x_r$ elements of γ^1. We use the notation

$$\mathrm{gr}_\gamma^i K(X) := \gamma^{i/i+1} := \gamma^i/\gamma^{i+1} \quad and \quad \mathrm{gr}_\gamma K(X) := \bigoplus_{i\geq 0} \mathrm{gr}_\gamma^i K(X)$$

for the associated graded pieces of this filtration and for the associated graded ring of this filtration, respectively. When we need to be precise about which variety the γ-filtration is being considered for, we will specify by writing $\gamma^i(X)$ to mean the i-th piece of the γ-filtration for the variety X. For further properties of these operations we refer to the references [Manin 1969; SGA 6 1971].

For the second, recall the coniveau filtration on $G(X)$ is defined by setting τ^i, for any $i \geq 0$, to be the ideal

$$\tau^i := \sum_{x \in X^{(j)}} \ker(G(X) \to G(X \setminus \bar{x})),$$

where $j \geq i$, $X^{(j)}$ denotes the set of codimension j points of X, and the arrows are flat pullbacks with respect to the respective inclusions $X \setminus \bar{x} \subset X$ for varying points x. We use the notation

$$\mathrm{gr}_\tau^i G(X) := \tau^{i/i+1} := \tau^i/\tau^{i+1} \quad \text{and} \quad \mathrm{gr}_\tau G(X) := \bigoplus_{i \geq 0} \mathrm{gr}_\tau^i G(X)$$

for the associated graded pieces of this filtration and for the associated graded ring of this filtration, respectively. Sometimes when more precision is needed, we include the variety in our notation for the coniveau filtration, i.e., $\tau^i(X)$ for the i-th piece of the coniveau filtration of X.

The two filtrations are related:

Theorem 3.1. *Identify $K(X)$ with $G(X)$ under the canonical isomorphism. Then, for any $i \geq 0$ we have $\gamma^i \subset \tau^i$. Hence the isomorphism $K(X) \to G(X)$ induces a (graded) filtration-comparison morphism $\mathrm{gr}_\gamma K(X) \to \mathrm{gr}_\tau G(X)$. Moreover, if the filtration-comparison map is surjective, or injective, then the two filtrations are equal, i.e., $\gamma^i = \tau^i$ for all $i \geq 0$ (in particular, if either of these conditions hold then the filtration-comparison map is bijective).*

Proof. For the first claim, see [Manin 1969]. The second claim about surjectivity implying bijectivity originally appears in [Karpenko 1998] and is updated in [Karpenko and Merkurjev 2018, Proposition 3.3], where the claim about injectivity implying bijectivity also appears. $\square$

4. Reductions

The main purpose of this section is to provide a way to reduce computations of the associated graded ring for the γ-filtration of a Severi–Brauer variety X to the case $X = \mathrm{SB}(A)$ for a p-primary division algebra A. In this regard we utilize heavily the motivic techniques of Karpenko, e.g., [1995a, Corollary 1.3.2; 2017a, Lemma 3.5]. The reason we can use these results is due to the observation that for any Severi–Brauer variety X there is a Severi–Brauer variety Y so that the γ-filtrations of X

and Y are the same and, simultaneously, the γ-filtration and coniveau filtration for this Y are isomorphic as well. This allows us to prove results about X by first replacing it with a functorially nicer Y and then reducing to previously known results. This observation seems nice enough to name it.

Definition 4.1. Let X be an arbitrary smooth variety. We say that a smooth F-variety Y, with F being a field possibly different from k, is a τ-*functorial replacement of* X if the following conditions hold:

(1) there is an isomorphism of groups
$$\operatorname{coker}(K(X) \to K(\bar{X})) = \operatorname{coker}(K(Y) \to K(\bar{Y})),$$
   where the arrows are pullbacks along the projections,

(2) there is an isomorphism of graded rings $\operatorname{gr}_\gamma K(X) = \operatorname{gr}_\gamma K(Y)$,

(3) the filtration-comparison map $\operatorname{gr}_\gamma K(Y) \to \operatorname{gr}_\tau G(Y)$ is an isomorphism.

Remark 4.2. In the cases where we are concerned, condition (1) of Definition 4.1 will always imply condition (2). Most likely, condition (2) also implies condition (1) in these cases.

Note also that it's important to allow the variation of the field of definition of Y. Often when these τ-functorial replacements are known to exist, the field F is a much larger field than k.

We're going to rephrase Definition 4.1 so that, when X is a Severi–Brauer variety, a τ-functorial replacement can be constructed using only data that one can read off from the associated central simple algebra. To do this we introduce the following definition, which is a small generalization from one already in common use. From now on, we let A be an arbitrary central simple algebra and we set $X = \operatorname{SB}(A)$.

Definition 4.3. Suppose A has a decomposition $A = M_r(k) \otimes \left(\bigotimes_{p \text{ prime}} A_p\right)$ with each A_p a division algebra of p-primary power index. Then we define the *behavior* of A to be the sequence
$$\mathcal{B}eh(A) = \left(\operatorname{ind}(A), \operatorname{ind}(A^{\otimes 2}), \dots, \operatorname{ind}(A^{\otimes \exp(A)})\right).$$

We define the *p-behavior*, where p is a specified prime, to be the sequence
$$\mathcal{B}eh(p, A) = \left(\operatorname{ind}(A_p), \operatorname{ind}(A_p^{\otimes p}), \dots, \operatorname{ind}(A_p^{\otimes \exp(A_p)})\right).$$

Finally, we define the *reduced p-behavior* of A to be the sequence
$$r\mathcal{B}eh(p, A) = \left(v_p \operatorname{ind}(A_p), v_p \operatorname{ind}(A_p^{\otimes p}), \dots, v_p \operatorname{ind}(A_p^{\otimes \exp(A_p)})\right).$$

If A is a p-primary algebra then, in order to relieve some notational burden, we call the reduced p-behavior simply the reduced behavior of A, and write $r\mathcal{B}eh(A)$ instead of $r\mathcal{B}eh(p, A)$.

Remark 4.4. The reduced behavior is a strictly descending sequence ending in 0. Conversely, for every prime p and for every strictly descending sequence ending in 0 there is a p-primary algebra with reduced behavior the given sequence; see [Karpenko 1998, Lemma 3.10]. Note that it's possible to reconstruct the behavior of A from the p-behavior (or the reduced p-behavior) as p ranges over all primes.

An equivalent definition for a τ-functorial replacement Y of X, when Y is also a Severi–Brauer variety, then follows:

Lemma 4.5. *A Severi–Brauer variety $Y = \mathrm{SB}(B)$ associated to a central simple algebra B is a τ-functorial replacement for X if, and only if, the following conditions hold*:

(1) $\deg(A) = \deg(B)$,

(2) *for every prime p, the reduced p-behaviors $r\mathcal{B}eh(p, A) = r\mathcal{B}eh(p, B)$ are the same*,

(3) *the filtration-comparison map $\mathrm{gr}_\gamma K(Y) \to \mathrm{gr}_\tau G(Y)$ is an isomorphism.*

Proof. For the forward direction, it suffices to observe that conditions (1) and (2) of the lemma imply condition (1) of Definition 4.1 by Theorem 2.2. For condition (2) of Definition 4.1, this is observed in [Izhboldin and Karpenko 1999, Theorem 1.1 and Corollary 1.2].

For the reverse direction, we start by assuming $Y = \mathrm{SB}(B)$ is a τ-functorial replacement for X. Then

$$\deg(A) = \dim_\mathbb{Q}(\mathrm{gr}_\gamma K(X) \otimes \mathbb{Q}) = \dim_\mathbb{Q}(\mathrm{gr}_\gamma K(Y) \otimes \mathbb{Q}) = \deg(B)$$

proves condition (1) of the lemma statement. To see that condition (2) of the lemma statement holds, one can use the fact that tensoring the cokernel with $\mathbb{Z}_{(p)}$ gives a decomposition

$$\mathrm{coker}(K(X) \to K(\overline{X})) \otimes \mathbb{Z}_{(p)} = (\mathbb{Z}/p^{n_0}\mathbb{Z})^{\oplus r_0} \oplus \cdots \oplus (\mathbb{Z}/p^{n_{m-1}}\mathbb{Z})^{\oplus r_{m-1}}$$

for integers $n_0 > \cdots > n_{m-1} > 0$. Then $r\mathcal{B}eh(p, A) = (n_0, \ldots, n_{m-1}, 0)$ and, as the same is true for Y and B, we find $r\mathcal{B}eh(p, A) = r\mathcal{B}eh(p, B)$ for every prime p. $\square$

The remainder of this section is devoted to proving that, given an arbitrary Severi-Brauer variety like X, there exists a τ-functorial replacement Y of X such that Y is also a Severi-Brauer variety. Our starting point is that it's already known, from [Karpenko 1998, Theorem 3.7 and Lemma 3.10], that τ-functorial replacements exist for the Severi-Brauer varieties of p-primary division algebras for any prime p.

Lemma 4.6. *Fix a prime p and suppose A is a division algebra with $\mathrm{ind}(A) = p^n$, for some $n \geq 0$. Then there exists a τ-functorial replacement Y for $X = \mathrm{SB}(A)$ such that Y is also a Severi–Brauer variety.*

Proof. This is a restatement of [Karpenko 1998, Theorem 3.7 and Lemma 3.10]. We recall for later use how one constructs such a replacement. Let B be a division algebra over a field F with

$$\mathrm{ind}(B) = \exp(B) = \mathrm{ind}(A).$$

Let $r\mathcal{B}eh(A) = (n_0, \ldots, n_m)$ be the reduced behavior of A. Set $Z_i = \mathrm{SB}(p^{n_i}, B^{\otimes p^i})$ to be the generalized Severi–Brauer variety of right ideals of $B^{\otimes p^i}$ of dimension $r_i = \deg(B^{\otimes p^i}) p^{n_i}$ inside $\mathrm{Gr}(r_i, B^{\otimes p^i})$. Let $Z = Z_1 \times \cdots \times Z_m$. Then the τ-functorial replacement constructed in [Karpenko 1998] is exactly $Y = \mathrm{SB}(B_{F(Z)})$. □

To extend this lemma to arbitrary Severi–Brauer varieties takes some effort. We first show that, if a Severi–Brauer variety X has a τ-functorial replacement Y that is also a Severi–Brauer variety, then every Severi–Brauer variety X' Brauer equivalent to X also has a τ-functorial replacement Y' that is a Severi–Brauer variety. Together with Lemma 4.6, this proves that τ-functorial replacements exist for the Severi–Brauer variety of any central simple algebra that has p-primary index for some prime p. To extend this result to Severi–Brauer varieties of arbitrary central simple algebras A with no conditions on the index, one replaces the primary division algebra factors of A (in a particular way) and then takes a matrix ring over the tensor product of these replacements. That the Severi–Brauer variety of this algebra is a τ-functorial replacement of the Severi–Brauer variety of our original algebra A is proved in Theorem 4.11 below.

We start with a lemma:

Lemma 4.7. *Suppose A is an arbitrary central simple algebra, with $X = \mathrm{SB}(A)$. Let D_A be the underlying division algebra of A and set $X' = \mathrm{SB}(D_A)$. Then the following statements hold.*

(1) *Suppose there exists a Severi–Brauer variety $Y = \mathrm{SB}(B)$ that is a τ-functorial replacement for X; if D_B is the underlying division algebra of B, then $Y' = \mathrm{SB}(D_B)$ is a τ-functorial replacement for X'.*

(2) *Suppose there exists a Severi–Brauer variety $Y' = \mathrm{SB}(D_B)$ that is a τ-functorial replacement for X'; if $B = M_r(D_B)$ for some r with $\deg(A) = \deg(B)$, then $Y = \mathrm{SB}(B)$ is a τ-functorial replacement for X.*

Proof. In both statements (1) and (2), it's clear conditions (1) and (2) of Lemma 4.5 hold for the algebra associated to the Severi–Brauer variety that we are trying to check is a τ-functorial replacement. So, we only need to check condition (3).

Note that the projections $Y \times Y' \to Y'$ and $Y \times Y' \to Y$ are both projective bundles over their targets. Thus the following diagram commutes, where the vertical arrows are the filtration-comparison morphisms (or sums of these morphisms) and the horizontal equalities are from the projective bundle formulas for both $\mathrm{gr}_\gamma K$ and $\mathrm{gr}_\tau G$:

$$\bigoplus \mathrm{gr}_\tau \, G(Y) \Longrightarrow \mathrm{gr}_\tau \, G(Y \times Y') \Longrightarrow \bigoplus \mathrm{gr}_\tau \, G(Y')$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

$$\bigoplus \mathrm{gr}_\gamma \, K(Y) \Longrightarrow \mathrm{gr}_\gamma \, K(Y \times Y') \Longrightarrow \bigoplus \mathrm{gr}_\gamma \, K(Y')$$

It follows if the left, or the right, vertical arrow is a surjection then the middle vertical arrow is a surjection and therefore, by Theorem 3.1, an isomorphism. If the middle vertical arrow of this diagram is an isomorphism, then the outer two vertical arrows are isomorphisms as well. Hence the left vertical arrow is an isomorphism if and only if the right vertical arrow is an isomorphism, as claimed. $\qquad\square$

As a consequence of the above proof we get the following:

Proposition 4.8. *Suppose A has p-primary index. Then there exists a Severi–Brauer variety Y that is a τ-functorial replacement for $X = \mathrm{SB}(A)$.* $\qquad\square$

To extend Proposition 4.8 to an arbitrary central simple algebra (with no requirements on the index), we'll need the following description of the p-torsion in $\mathrm{gr}_\gamma \, K(X)$.

Lemma 4.9. *Fix a prime p. Let F be a finite field extension of k with degree $[F : k]$ not divisible by p. Then the pullback along the projection $K(X) \to K(X_F)$ induces an isomorphism*

$$\mathrm{gr}_\gamma \, K(X) \otimes \mathbb{Z}_{(p)} \to \mathrm{gr}_\gamma \, K(X_F) \otimes \mathbb{Z}_{(p)}.$$

Proof. Note that, by the projection formula, the pullback composed with the pushforward

$$K(X) \otimes \mathbb{Z}_{(p)} \to K(X_F) \otimes \mathbb{Z}_{(p)} \to K(X) \otimes \mathbb{Z}_{(p)}$$

is an isomorphism. As these are morphisms between free $\mathbb{Z}_{(p)}$-modules of the same rank, the composition in the other direction

$$K(X_F) \otimes \mathbb{Z}_{(p)} \to K(X) \otimes \mathbb{Z}_{(p)} \to K(X_F) \otimes \mathbb{Z}_{(p)}$$

is also an isomorphism. Thus the pullback itself $K(X) \otimes \mathbb{Z}_{(p)} \to K(X_F) \otimes \mathbb{Z}_{(p)}$ is an isomorphism.

Consider the following commuting ladder with exact rows and vertical arrows induced by the pullback along the projection $X_F \to X$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \gamma^{i+1}(X) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \gamma^i(X) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \gamma^{i/i+1}(X) \otimes \mathbb{Z}_{(p)} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \pi_{i+1}} & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_{i/i+1}} & & \\
0 & \longrightarrow & \gamma^{i+1}(X_F) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \gamma^i(X_F) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \gamma^{i/i+1}(X_F) \otimes \mathbb{Z}_{(p)} & \longrightarrow & 0
\end{array}
$$

The right vertical arrow is a surjection since $K(X) \otimes \mathbb{Z}_{(p)} \to K(X_F) \otimes \mathbb{Z}_{(p)}$ is a surjection (cf. [Mackall 2018, proof of Lemma 2.3]). By the snake lemma one

gets short exact sequences

$$0 \to \ker(\pi_{i/i+1}) \to \operatorname{coker}(\pi_{i+1}) \to \operatorname{coker}(\pi_i) \to 0.$$

Since for $j \geq \dim(X)$ one has $\operatorname{coker}(\pi_j) = 0$, the claim follows. $\qquad\square$

Remark 4.10. The above proof can be adapted to show that, for any morphism $X \to Y$ between smooth varieties X and Y, if the pullback $K(Y) \to K(X)$ is an isomorphism (resp. an isomorphism after tensoring with a flat ring R) then the pullback $\operatorname{gr}_\gamma K(Y) \to \operatorname{gr}_\gamma K(X)$ is an isomorphism (resp. an isomorphism after tensoring with a flat ring R); cf. [Karpenko and Merkurjev 2018, Lemma 3.6].

Finally, we've arrived at the main result of this section.

Theorem 4.11. *For an arbitrary central simple algebra A, there exists a Severi–Brauer variety Y that is a τ-functorial replacement for $X = \operatorname{SB}(A)$.*

Proof. Let $A = M_r(k) \otimes \left(\bigotimes_{p \text{ prime}} A_p \right)$ be a decomposition with each A_p a p-primary division algebra. We set $X_p = \operatorname{SB}(A_p)$ in the following.

Find a field F (e.g., $F = \mathbb{Q}$ works) such that, for each prime p, there exists division algebras B_p over F with $\operatorname{ind}(B_p) = \exp(B_p) = \operatorname{ind}(A_p)$. Fix any particular choice of prime p and consider the reduced p-behavior of A,

$$r\mathcal{B}eh(p, A) = (n_0, \ldots, n_m).$$

Set $Z_p^i = \operatorname{SB}(p^{n_i}, B_p^{\otimes p^i})$ as in Lemma 4.6. We set $Z_p = Z_p^1 \times \cdots \times Z_p^m$ to be the product of these varieties. In a similar fashion we construct varieties Z_q for all other primes $q \neq p$. Let

$$Z = \prod_{p \text{ prime}} Z_p \quad \text{and} \quad Z^p = \prod_{q \text{ prime}, q \neq p} Z_q$$

be the given products. Finally, set $B = M_r(F) \otimes \left(\bigotimes_{p \text{ prime}} B_p \right)$.

We claim $Y = \operatorname{SB}(B_{F(Z)})$ is a τ-functorial replacement for X. The proof proceeds in several steps. The first step we take is to show that

$$(B_p \otimes_F F(Z^p)) \otimes_{F(Z^p)} F(Z_{p, F(Z^p)}) = B_p \otimes_F F(Z) \qquad \text{(ts)}$$

is a τ-functorial replacement for A_p. But this is clear since, by index reduction [Merkurjev et al. 1996, equation (0.3)] one has

$$\operatorname{ind}(B_{p, F(Z^p)}) = \exp(B_{p, F(Z^p)}) = \operatorname{ind}(A_p)$$

and the left-hand side of the equation (ts) is the algebra constructed exactly as in Lemma 4.6.

The next step we take is to show that condition (3) of Lemma 4.5 is satisfied by $Y = \operatorname{SB}(B_{F(Z)})$. Since conditions (1) and (2) are clear for $B_{F(Z)}$ (applying again

index reduction in the same way as in Lemma 4.5), this will complete the proof of the theorem. To do this, we let L_p be a finite field extension of $F(Z)$ that splits B_q for all $q \neq p$ and such that $[L_p : F(Z)]$ is not divisible by p. Then the square

$$
\begin{array}{ccc}
\mathrm{gr}_\gamma K(Y) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \mathrm{gr}_\gamma K(Y_{L_p}) \otimes \mathbb{Z}_{(p)} \\
\downarrow & & \downarrow \\
\mathrm{gr}_\tau G(Y) \otimes \mathbb{Z}_{(p)} & \longrightarrow & \mathrm{gr}_\tau G(Y_{L_p}) \otimes \mathbb{Z}_{(p)}
\end{array}
\tag{D}
$$

is commuting, where the vertical arrows are filtration-comparison maps and the horizontal arrows are pullbacks with respect to the projection $Y_{L_p} \to Y$. Since the top horizontal arrow of (D) is an isomorphism by Lemma 4.9 and the right vertical arrow is an isomorphism, by the construction of Y_{L_p} and by the proof of Proposition 4.8, it follows that the left vertical arrow of (D) is an injection. Repeating this argument for all primes p allows us to conclude that the morphism $\mathrm{gr}_\gamma K(Y) \to \mathrm{gr}_\tau G(Y)$ is an injection since it is after localizing at every maximal ideal of $\mathbb{Z}$. But, the filtration-comparison map has the nice property that injectivity implies surjectivity (see Theorem 3.1), which completes the proof. $\square$

As an application of the existence of τ-functorial replacements for an arbitrary Severi–Brauer variety, let us show one way to extend known motivic results on Severi–Brauer varieties to statements for the associated graded ring of the γ-filtration.

Corollary 4.12. *Suppose A is an arbitrary central simple algebra and let D_A be the underlying division algebra of A. Write $X = \mathrm{SB}(A)$ and $X' = \mathrm{SB}(D_A)$. Then there is an isomorphism*

$$
\bigoplus_{i=1}^{\deg(A)/\deg(D_A)} \mathrm{gr}_\gamma K(X') \to \mathrm{gr}_\gamma K(X).
$$

Proof. Let $Y = \mathrm{SB}(B)$ be a τ-functorial replacement for X. Let $Y' = \mathrm{SB}(D_B)$, where D_B is the underlying division algebra for B. Let $r = \deg(A)/\deg(D_A) = \deg(B)/\deg(D_B)$. Then there is a canonical chain of isomorphisms

$$
\bigoplus_{i=1}^{r} \mathrm{gr}_\gamma K(X') \xrightarrow{\sim} \bigoplus_{i=1}^{r} \mathrm{gr}_\gamma K(Y') \xrightarrow{\sim} \bigoplus_{i=1}^{r} \mathrm{gr}_\tau G(Y') \\
\xrightarrow{\sim} G(Y) \xleftarrow{\sim} \mathrm{gr}_\gamma K(Y) \xleftarrow{\sim} \mathrm{gr}_\gamma K(X),
$$

using the definition of τ-functorial replacements and [Karpenko 1995a, Corollary 1.3.2], that defines the isomorphism of the corollary. $\square$

We end this section by making the following observation which generalizes the existence of a τ-functorial replacement to some other generalized flag varieties.

Corollary 4.13. *Let A be a central simple algebra with $\mathrm{ind}(A) = n$. Let $V_{i_1,\ldots,i_r}(A)$ be the variety of flags of ideals in A of reduced dimensions $i_1, \ldots, i_r$. If we have $\gcd(i_1, \ldots, i_r, n) = 1$, then there exists a τ-functorial replacement for $V_{i_1,\ldots,i_r}(A)$. Moreover, these τ-functorial replacements can be constructed as twisted flag varieties of the same kind.*

Proof. Let $Y = V_{i_1,\ldots,i_r}(A)$ and $X = \mathrm{SB}(A)$. Let X' be a Severi–Brauer variety that is a τ-functorial replacement of X, using Theorem 4.11, and let B be the central simple algebra corresponding to X'. Let $Y' = V_{i_1,\ldots,i_r}(B)$. Note that one has $\mathcal{B}eh(A) = \mathcal{B}eh(B)$. So, by the results of [Panin 1994], one also has

$$\mathrm{coker}(K(Y) \to K(\overline{Y})) = \mathrm{coker}(K(Y') \to K(\overline{Y'}))$$

and $\mathrm{gr}_\gamma K(Y) = \mathrm{gr}_\gamma K(Y')$ for exactly the same reasons as when X is a Severi–Brauer variety. It remains to show the γ-filtration and coniveau filtration for Y' are equal.

To finish the proof, we're going to show $\mathrm{gr}_\tau G(Y')$ is generated by Chern classes. It follows from this that the canonical map $\mathrm{gr}_\gamma K(Y') \to \mathrm{gr}_\tau G(Y')$ is a surjection, and therefore also an injection by Theorem 3.1. By [Petrov et al. 2008, Corollary 3.4] the projection $X' \times Y' \to X'$ is a cellular fiber bundle over X'. It follows that $\bigoplus \mathrm{gr}_\tau G(X') = \mathrm{gr}_\tau G(X' \times Y')$ is generated by Chern classes. Again by [Petrov et al. 2008, Corollary 3.4] the projection $X' \times Y' \to Y'$ is a projective bundle, and it follows that $\mathrm{gr}_\tau G(Y') \subset \bigoplus \mathrm{gr}_\tau G(Y') = \mathrm{gr}_\tau G(X' \times Y')$ is also generated by Chern classes. $\qquad\square$

5. Describing the γ-filtration

The goal of this section is to prove our main result, Theorem 5.1, that computes some of the graded groups associated to the γ-filtration in low homological degree for a Severi–Brauer variety $X = \mathrm{SB}(A)$ associated to a central simple algebra A with p-primary index.

Theorem 5.1. *Let A be a central simple algebra with $\mathrm{ind}(A) = p^n$ and set $X = \mathrm{SB}(A)$. Then there are equalities*

$$\mathrm{gr}_\gamma^{p^n-i} K(X) = p^n(\xi - 1)^{p^n-i}\mathbb{Z}$$

for all $1 \leq i \leq p - 1$.

In the above we're identifying $K(X)$ with its image in $K(X_F)$ for some splitting field F of A and letting ξ be the class of $\mathcal{O}_{X_F}(-1)$ in $K(X_F)$ as in Theorem 2.2.

Using results of the previous section, this computation immediately generalizes to an arbitrary central simple algebra and to more general twisted flag varieties.

Our proof of the main theorem works in the following way. We first consider the filtration on $K(X)$ generated by K-theoretic Chern classes in $\zeta_X(1)$ and show that,

inside of $K(X_F)$, this filtration is especially simple. Specifically, in low degrees this filtration is spanned by polynomials $p^n(\xi - 1)^i$ for large i. Then we write out a general generator of the γ-filtration on X in the same degree, considered also inside of $K(X_F)$, and show that p^n divides the coefficient of this general element. It follows that the γ-filtration is actually spanned, in these degrees, by K-theoretic Chern classes in $\zeta_X(1)$, and this allows us to conclude. The proof itself is long but entirely elementary.

Before proving this theorem, however, we describe a particular generating set for the γ-filtration for a Severi–Brauer variety $X = \mathrm{SB}(A)$ when A is a central simple algebra with p-primary index. This generating set appears in the literature already [Karpenko 1998; Baek 2015], but the justification for why it exists is conceptually clearer using the arguments given here. We also take this chance to uniformize the notation that is used throughout the remainder of this text. Recall then the following definition.

Definition 5.2 [Karpenko and Mackall 2019, Definition A.1]. Let A be a central simple algebra with $\mathrm{ind}(A) = p^n$ and let $X = \mathrm{SB}(A)$. Let

$$S_X = \{i : v_p\, \mathrm{ind}(A^{\otimes p^i}) < v_p\, \mathrm{ind}(A^{\otimes p^{i-1}}) - 1\}$$

be the given set of natural numbers. We call the cardinality $\#S_X$ the *level of A* or the *level of X*.

In other words, the level of A is the number of places where the reduced behavior decreases by more than one from one position to the next. The relevance of the level is contained in the following lemma.

Lemma 5.3 [Karpenko and Mackall 2019, Lemma A.6]. *The ring $K(X)$ is generated, as a λ-ring, by the classes of the sheaves of the set $\{\zeta_X(p^i)\}_i$, where i is an index for the set $\{0\} \cup S_X$.*

In particular, the above lemma implies the following lemma about a small generating set for the γ-filtration on $K(X)$.

Lemma 5.4. *Let A be a central simple algebra with p-primary index for a prime p. Set $X = \mathrm{SB}(A)$ to be the associated Severi–Brauer variety. Then the i-th piece of the γ-filtration, $\gamma^i \subset K(X)$, is generated additively by products*

$$\gamma^{j_1}(x_1 - \mathrm{rk}(x_1)) \cdots \gamma^{j_r}(x_r - \mathrm{rk}(x_r)),$$

where $j_1 + \cdots + j_r \geq i$ and $x_1, \ldots, x_r$ are elements of $\{\zeta_X(p^i)\}_i$, where i indexes the set $S_X \cup \{0\}$.

Proof. Note that the images of these products generate the graded group γ^i/γ^{i+1} since these are the images of K-theoretic Chern classes of the negatives of the duals

of the sheaves $\zeta_X(p^i)$. The claim can then be obtained by descending induction, since for $i = \dim(X)$ one has $\gamma^{i/i+1} = \gamma^i$. $\qquad\square$

Using the description of Theorem 2.2, the products appearing in the statement of Lemma 5.4 can be computed like so.

Lemma 5.5. *Let A be a central simple algebra with p-primary index for some prime p. Assume A has reduced behavior $r\mathcal{B}eh(A) = (n_0, \ldots, n_m)$. Fix a splitting field F of A and identify $K(X)$ with its image in $K(X_F)$. Let ξ be the class of $\mathcal{O}_{X_F}(-1)$. Then*

$$\gamma^i(\zeta_X(p^j) - p^{n_j}) = \binom{p^{n_j}}{i}(\xi^{p^j} - 1)^i.$$

Proof. This is computed in [Karpenko 1998]. It's done by observing

$$\gamma_t(p^{n_j}\xi^{p^j} - p^{n_j}) = \gamma_t(p^{n_j}(\xi^{p^j} - 1)) = \gamma_t(\xi^{p^j} - 1)^{p^{n_j}} = (1 + (\xi^{p^j} - 1)t)^{p^{n_j}},$$

which gives the claim. $\qquad\square$

We're almost in position to prove Theorem 5.1. The last ingredient we need for the proof is contained in the next definition and the following lemma.

Definition 5.6. Let $X = \mathrm{SB}(A)$ be the Severi–Brauer variety of a central simple algebra A with $\mathrm{ind}(A) = p^n$ for some prime p. Let $\eta^i(X)$ be the ideal of $K(X)$ generated by monomials

$$\gamma^{j_1}(\zeta_X(1) - p^n) \cdots \gamma^{j_r}(\zeta_X(1) - p^n)$$

with $j_1 + \cdots + j_r \geq i$. When it's clear from context, we simply write η^i for $\eta^i(X)$.

Lemma 5.7. *Let $X = \mathrm{SB}(A)$ be the Severi–Brauer variety of a central division algebra A with $\mathrm{ind}(A) = p^n$. Let F be a splitting field for X and make the identifications of Theorem 2.2. Then η^i defines a descending ring filtration on $K(X)$ and, for every $i \geq 0$, one has*

$$\eta^i = \bigoplus_{j \geq i} p^{n - v_p(j)}(\xi - 1)^j \mathbb{Z}.$$

Proof. The claim about being a filtration is clear. For the equality, we do this by showing both sides contain the other. The reverse direction

$$\eta^i \supset \bigoplus_{j \geq i} p^{n - v_p(j)}(\xi - 1)^j \mathbb{Z}$$

is clear since, for all $0 \leq i \leq p^n$, one has

$$\gamma^1(\zeta_X(1) - p^n) = p^n(\xi - 1) \quad \text{and} \quad \gamma^i(\zeta_X(1) - p^n) = \binom{p^n}{i}(\xi - 1)^i$$

by Lemma 5.5, and

$$\gcd\left(p^{in}, \binom{p^n}{i}\right) = p^{n-v_p(i)}$$

by [Karpenko 1998, Lemma 3.5].

For the inclusion

$$\eta^i \subset \bigoplus_{j \geq i} p^{n-v_p(j)} (\xi - 1)^j \mathbb{Z},$$

we note that $p^{n-v_p(j)}$ divides $\binom{p^n}{j_1} \cdots \binom{p^n}{j_r}$ whenever $j_1 + \cdots + j_r = j \leq p^n$. Indeed, let $v = \min_s\{v_p(j_s)\}$ and suppose, without loss of generality, that $v = v_p(j_1)$. Then

$$v_p\left(\binom{p^n}{j_1} \cdots \binom{p^n}{j_r}\right) = n - v + \sum_{s=2}^{r}(n - v_p(j_s)) \geq n - v_p(j).$$

This inequality, with the definition of η^i and Lemma 5.5, gives the result. □

Proof of Theorem 5.1. It suffices by Corollary 4.12 to assume A is a division algebra. Our proof works by showing p^n divides the coefficient of every element of $\gamma^{p^n-p+1} \supset \gamma^{p^n-i}$ when each of these elements is written as a polynomial in $(1-\xi)$. Note since there are inclusions

$$\gamma^{p^n-p+1}(X) \subset \tau^{p^n-p+1}(X) \subset \tau^{p^n-p+1}(X_F) = (1-\xi)^{p^n-p+1} K(X_F),$$

we can write every element y of γ^{p^n-p+1} as a sum

$$y = \sum_{j=p^n-p+1}^{p^n-1} a_j(1-\xi)^j$$

for some integers a_j. After we show p^n divides each of these a_j, it follows that we have inclusions

$$\eta^{p^n-p+1} \subset \gamma^{p^n-p+1} \subset \eta^{p^n-p+1},$$

and this will end the proof.

Suppose then

$$y = \gamma^{j_1}(x_1 - \mathrm{rk}(x_1)) \cdots \gamma^{j_r}(x_r - \mathrm{rk}(x_r))$$

is an arbitrary monomial generating γ^{p^n-p+1} like those described in Lemma 5.4. We work in two cases: each of $x_1, \ldots, x_k$ is equal to $\zeta_X(1)$ for some $1 \leq k \leq r$, or $\zeta_X(1)$ does not appear among the $x_1, \ldots, x_r$ at all.

Assuming we're in the former case, let us make one more reduction. We're trying to give a lower bound for the p-adic valuation of the coefficient in an expansion of y. We're also assuming each of $x_1, \ldots, x_k$ are equal to $\zeta_X(1)$ and, since

$$n - v_p(r) = v_p\left(\binom{p^n}{r}\right) \leq v_p\left(\binom{p^n}{j_1} \cdots \binom{p^n}{j_k}\right)$$

when $r = j_1 + \cdots + j_k$ (see the end of the proof of Lemma 5.7), we can therefore assume $k = 1$.

With these assumptions we can expand y as follows:

$$y = \binom{p^n}{j_1}(\xi - 1)^{j_1}\binom{p^{n-t_2}}{j_2}(\xi^{p^{s_2}} - 1)^{j_2} \cdots \binom{p^{n-t_r}}{j_r}(\xi^{p^{s_r}} - 1)^{j_r}$$

$$= \binom{p^n}{j_1}\binom{p^{n-t_2}}{j_2} \cdots \binom{p^{n-t_r}}{j_r}(\xi - 1)^{j_1}(\xi^{p^{s_2}} - 1)^{j_2} \cdots (\xi^{p^{s_r}} - 1)^{j_r},$$

where here we are writing $x_k = \zeta_X(p^{s_k})$ for some integers $s_2, \ldots, s_r \geq 1$ and $p^{n-t_k} = \mathrm{ind}(A^{\otimes p^{s_k}})$. Now by Lemma 2.3, equation (2.5), there is an expansion, for each integer k satisfying $2 \leq k \leq r$,

$$\xi^{p^{s_k}} - 1 = \sum_{i=1}^{p^{s_k}}(-1)^i\binom{p^{s_k}}{i}(1 - x)^i.$$

We set $x_{\mathrm{low}}(k) = \sum_{i=1}^{p-1}(-1)^i\binom{p^{s_k}}{i}(1-x)^i$ to be the sum containing the small degree summands of this latter sum and $x_{\mathrm{high}}(k) = \sum_{i=p}^{p^{s_k}}(-1)^i\binom{p^{s_k}}{i}(1-x)^i$ to be the sum containing the high degree summands. We still have an equality

$$\xi^{p^{s_k}} - 1 = x_{\mathrm{low}}(k) + x_{\mathrm{high}}(k)$$

for every $2 \leq k \leq r$ but it's useful to group the terms in this way since p divides each $x_{\mathrm{low}}(k)$ but one doesn't necessarily have that p divides any $x_{\mathrm{high}}(k)$.

Rewriting y in terms of the $x_{\mathrm{low}}(k)$ and $x_{\mathrm{high}}(k)$ gives

$$y = \binom{p^n}{j_1}\binom{p^{n-t_2}}{j_2} \cdots \binom{p^{n-t_r}}{j_r}(\xi-1)^{j_1}(x_{\mathrm{low}}(2)+x_{\mathrm{high}}(2))^{j_2} \cdots (x_{\mathrm{low}}(r)+x_{\mathrm{high}}(r))^{j_r}.$$

By applying the binomial theorem and expanding, we get

$$\prod_{k=2}^{r}(x_{\mathrm{low}}(k) + x_{\mathrm{high}}(k))^{j_k} = \prod_{k=2}^{r}\left(\sum_{l=0}^{j_k}\binom{j_k}{l}x_{\mathrm{low}}(k)^l x_{\mathrm{high}}(k)^{j_k-l}\right)$$

$$= \left(x_{\mathrm{low}}(2)^{j_k} \cdots x_{\mathrm{low}}(r)^{j_r} + \sum_{k=2}^{r}x_{\mathrm{high}}(k)q_k\right),$$

where q_k is a polynomial in the terms $x_{\mathrm{low}}(2), \ldots, x_{\mathrm{low}}(r), x_{\mathrm{high}}(2), \ldots, x_{\mathrm{high}}(r)$. If $x_{\mathrm{high}}(k) \neq 0$ then the lowest degree in $(1 - \xi)$ of $x_{\mathrm{high}}(k)$ is p, while the lowest degree of any $x_{\mathrm{low}}(k)$ is 1. In particular, the lowest degree in $(1 - \xi)$ of any $x_{\mathrm{high}}(k)q_k$ is $j_2 + \cdots + (j_k - 1 + p) + \cdots + j_r$. After multiplying by $(\xi - 1)^{j_1}$ it follows that

$$(\xi - 1)^{j_1}\prod_{k=2}^{r}(x_{\mathrm{low}}(k) + x_{\mathrm{high}}(k))^{j_k} = (\xi - 1)^{j_1}x_{\mathrm{low}}(2)^{j_k} \cdots x_{\mathrm{low}}(r)^{j_r}$$

because of
$$j_1 + j_2 + \cdots + (j_k - 1 + p) + \cdots + j_r \geq p^n - p + 1 - 1 + p = p^n$$

and [Theorem 2.2](#).

Thus we find

$$y = \binom{p^n}{j_1}\binom{p^{n-t_2}}{j_2} \cdots \binom{p^{n-t_r}}{j_r} (\xi - 1)^{j_1} x_{\text{low}}(2)^{j_2} \cdots x_{\text{low}}(r)^{j_r}$$

$$= \binom{p^n}{j_1}\binom{p^{n-t_2}}{j_2} \cdots \binom{p^{n-t_r}}{j_r} p^{j_2 + \cdots + j_r} (\xi - 1)^{j_1} \left(\frac{x_{\text{low}}(2)}{p}\right)^{j_2} \cdots \left(\frac{x_{\text{low}}(r)}{p}\right)^{j_r}$$

since each $x_{\text{low}}(k)$ is divisible by p.

Now set $\alpha = \binom{p^n}{j_1}\binom{p^{n-t_2}}{j_2} \cdots \binom{p^{n-t_r}}{j_r} p^{j_2 + \cdots + j_r}$. We have

$$v_p(\alpha) = n - v_p(j_1) + \sum_{k=2}^{r}(n - t_k - v_p(j_k)) + j_2 + \cdots + j_r$$

$$\geq n - v_p(j_1) + j_2 + \cdots + j_r.$$

We finish by showing $n - v_p(j_1) + j_2 + \cdots + j_r \geq n$ for all possible $j_1, \ldots, j_r$. Equivalently, assuming $j_1 + \cdots + j_r = p^n - i$ for some i with $0 < i < p$ we finish by showing
$$p^n - i \geq j_1 + v_p(j_1).$$

Assuming i is largest possible we can also show $p^n - p + 1 \geq j_1 + v_p(j_1)$. We can assume $v_p(j_1) > 0$, as otherwise p^n divides $\binom{p^n}{j_1}$. Hence we can assume $j_1 = a_1 p^{n-1} + \cdots + a_{n-r} p^r$ with $0 \leq a_1, \ldots, a_{n-r} < p$ and some minimal $r \geq 1$. This inequality becomes

$$p^n - p + 1 \geq a_1 p^{n-1} + \cdots + a_{n-r} p^r + r.$$

We make one last approximation and assume all $a_1, \ldots, a_{n-r}$ are equal to $(p-1)$, as this is the largest they can be. We're left checking

$$p^n - p + 1 \geq a_1 p^{n-1} + \cdots + a_{n-r} p^r + r = p^n - p^r + r.$$

Rearranging, we check
$$p^r - p \geq r - 1,$$

which is clear if $r = 1$ and is the same as

$$\frac{p^r - p}{r - 1} \geq 1$$

for $r > 1$. Using the mean value theorem, the left of this inequality equals $f'(c)$ for some c in the interval $[1, r]$ and $f(x) = p^x$. Since $f'(c) = \log(p)p^c \geq \log(p)p \geq \log(2)2 > 1$ we've completed this case.

We still need to check the second case, when $\zeta_X(1)$ is not a part of the γ-operations of our monomial. Following the same process as before, we're left to check the inequality $p^n - i \geq n$ for $0 < i < p$. But this is also readily checked to be true: we can assume we want to show $p^n - p + 1 \geq n$, and $p^n - p \geq n - 1$ is the same (ignoring the $n = 1$ case which is trivial) as

$$\frac{p^n - p}{n - 1} \geq 1,$$

which by the mean value theorem equals $f'(c)$ for some c in the interval $[1, n]$ and $f(x) = p^x$; for all such c we have $f'(c) = \log(p)p^c \geq \log(p)p \geq \log(2)2 > 1$. $\square$

We conclude with a series of corollaries that motivated this work.

Corollary 5.8. *Let B be a central simple algebra, and let $Y = \mathrm{SB}(B)$ be the Severi–Brauer variety of B. Suppose $\mathrm{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}$ is a prime factorization with primes $p_1 < \cdots < p_r$. Then for every pair of primes $p, q \in \{p_1, \ldots, p_r\}$ with $p \geq q$, and for all integers i satisfying $1 \leq i \leq q - 1$,*

$$\mathrm{gr}_\gamma^{d-i} K(Y) \otimes \mathbb{Z}_{(p)} = d(1 - \xi)^{d-i} \mathbb{Z}_{(p)},$$

where ξ is the class of $\mathcal{O}_{X_F}(-1)$ when identifying $K(X) \subset K(X_F)$ for a splitting field F of X.

Proof. Apply Lemma 4.9, Corollary 4.12, and Theorem 5.1. $\square$

Corollary 5.9. *Let B be a central simple algebra, and let $Y = \mathrm{SB}(B)$ be the Severi–Brauer variety of B. Suppose $\mathrm{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}$ is a prime factorization with primes $p_1 < \cdots < p_r$. Then for all integers i satisfying $1 \leq i \leq p_1 - 1$,*

$$\mathrm{gr}_\gamma^{d-i} K(Y) = d(1 - \xi)^{d-i},$$

where ξ is the class of $\mathcal{O}_{X_F}(-1)$ when identifying $K(X) \subset K(X_F)$ for a splitting field F of X.

Proof. This is true after localizing at every prime p by Corollary 5.8, so it's true in general. $\square$

Corollary 5.10. *Suppose B is an arbitrary central simple algebra and $X = \mathrm{SB}(B)$. Suppose $\mathrm{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}$ is a prime factorization with primes $p_1 < \cdots < p_r$. If $\mathrm{CH}(X)$ is generated by Chern classes and if the canonical epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau G(X)$, taking the class of an integral subvariety V to the class of its structure sheaf $[\mathcal{O}_V]$, is an isomorphism, then for every pair of primes $p, q \in \{p_1, \ldots, p_r\}$ with $p \geq q$,*

$$\mathrm{CH}_j(X) \otimes \mathbb{Z}_{(p)} = d\mathbb{Z}_{(p)} \quad \text{for all } j \leq q - 2.$$

Proof. In this setting, the rings $\mathrm{CH}(X)$ and $\mathrm{gr}_\gamma K(X)$ are isomorphic; see [Karpenko 2017c, Theorem 3.1]. $\square$

Corollary 5.11. *Suppose B is an arbitrary central simple algebra and $X = \mathrm{SB}(B)$. Suppose $\mathrm{ind}(B) = d = p_1^{n_1} \cdots p_r^{n_r}$ is a prime factorization with primes $p_1 < \cdots < p_r$. If $\mathrm{CH}(X)$ is generated by Chern classes and if the canonical epimorphism $\mathrm{CH}(X) \to \mathrm{gr}_\tau G(X)$ is an isomorphism, then*

$$\mathrm{CH}_j(X) = d\mathbb{Z} \quad \text{for all } j \leq p_1 - 2.$$

Proof. This is true after localizing at every prime p by Corollary 5.10, so it's true in general. $\qquad\square$

Remark 5.12. The conditions of Corollaries 5.10 and 5.11 hold, for example, when B is a central simple algebra corresponding to a generic Severi–Brauer variety; see [Karpenko 2017c]. These conditions also hold for a more general class of algebras [Karpenko and Mackall 2019]. In both of these cases, Corollaries 5.10 and 5.11 were already known to hold, so we've reproved and generalized this result.

These corollaries can also be extended to more general flag varieties by the following lemmas.

Lemma 5.13. *Let B be a central simple algebra with $\mathrm{ind}(B) = n$. Let $X = \mathrm{SB}(B)$ and let $Y = V_{i_1,\ldots,i_r}(B)$ be the variety of flags of ideals in B of reduced dimensions $i_1, \ldots, i_r$. If $\gcd(i_1, \ldots, i_r, n) = 1$, then the following statements hold:*

(1) $\mathrm{CH}^i(X)$ *is torsion free for all $i \leq j$ if, and only if, $\mathrm{CH}^i(Y)$ is torsion free for all $i \leq j$,*

(2) $\mathrm{CH}_i(X)$ *is torsion free for all $i \leq j$ if, and only if, $\mathrm{CH}_i(Y)$ is torsion free for all $i \leq j$.*

Replacing every CH with $\mathrm{gr}_\tau G$, the same statements hold.

Proof. In this case one has that $X \times Y \to Y$ is a projective bundle over Y and $X \times Y \to X$ is a cellular fibration over X by [Petrov et al. 2008, Corollary 3.4]. Therefore

$$\bigoplus \mathrm{CH}(X) = \mathrm{CH}(X \times Y) = \bigoplus \mathrm{CH}(Y)$$

and the claim follows by looking at torsion in the respective degrees. The same argument works replacing CH by $\mathrm{gr}_\tau G$. $\qquad\square$

Lemma 5.14. *Let B be a central simple algebra with $\mathrm{ind}(B) = n$. Let $X = \mathrm{SB}(B)$ and let $Y = V_{i_1,\ldots,i_r}(B)$ be the variety of flags of ideals in B of reduced dimensions $i_1, \ldots, i_r$. If $\gcd(i_1, \ldots, i_r, n) = 1$, then the following statements hold:*

(1) $\mathrm{gr}_\gamma^i K(X)$ *is torsion free for all $i \leq j$ if, and only if, $\mathrm{gr}_\gamma^i K(Y)$ is torsion free for all $i \leq j$,*

(2) $\mathrm{gr}_{\gamma,i} K(X)$ *is torsion free for all $i \leq j$ if, and only if, $\mathrm{gr}_{\gamma,i} K(Y)$ is torsion free for all $i \leq j$.*

Proof. First make a τ-functorial replacement of X with a Severi–Brauer variety X' associated to a central simple algebra C. Note that by the proof of Corollary 4.13 the variety $Y' = V_{i_1,\ldots,i_r}(C)$ is a τ-functorial replacement of Y. Since the associated graded ring for the γ-filtration doesn't change when making a τ-functorial replacement, the claim follows from Lemma 5.13 applied to $\mathrm{gr}_\gamma K(X') = \mathrm{gr}_\tau G(X')$ and $\mathrm{gr}_\gamma K(Y') = \mathrm{gr}_\tau G(Y')$. $\square$

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