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Tate tame symbol and the joint torsion of commuting operators

Jens Kaad and Ryszard Nest

We investigate determinants of Koszul complexes of holomorphic functions of a commuting tuple of bounded operators acting on a Hilbert space. Our main result shows that the analytic joint torsion, which compares two such determinants, can be computed by a local formula which involves a tame symbol of the involved holomorphic functions. As an application we are able to extend the classical tame symbol of meromorphic functions on a Riemann surface to the more involved setting of transversal functions on a complex analytic curve. This follows by spelling out our main result in the case of Toeplitz operators acting on the Hardy space over the polydisc.

1. Introduction

The main theme of this paper, which is a continuation of the work begun in [Kaad and Nest 2015; 2019], is the study of the algebraic structure of a Hilbert space $\mathscr{H}$ as a module over an algebra of holomorphic functions of a finite family of bounded commuting operators on $\mathscr{H}$.

The simplest example is the case of the Toeplitz operator T_z acting on the Hardy space over the open unit disc $H^2(\mathbb{D})$. Here z denotes the complex coordinate in $\mathbb{C}$. The holomorphic functional calculus of bounded operators furnishes the

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Hilbert space $H^2(\mathbb{D})$ with the structure of a module over the ring $\mathcal{O}(\overline{\mathbb{D}})$ of germs of functions holomorphic in a neighborhood of the closed unit disc, and analysis of Toeplitz operators is closely related to the algebraic properties of this module structure.

The prototype of results of the type we are interested in is the following:

Theorem 1.1 (Fritz Noether). *Given a holomorphic function f on a neighborhood of $\overline{\mathbb{D}}$ and invertible on its boundary $\partial\mathbb{D}$, the Toeplitz operator T_f is Fredholm and its index is given by* (*minus*) *the winding number of f*,

$$\operatorname{Ind}(T_f) = -\frac{1}{2\pi i}\int_{\mathbb{T}} f^{-1}\,df = -\sum_{|\lambda|<1} \operatorname{res}_\lambda(f^{-1}\,df). \tag{1.2}$$

A way of interpreting this result is to notice that the left-hand side of this equality is an *analytic object*, the Euler characteristic of the cochain complex

$$0 \to H^2(\mathbb{D}) \xrightarrow{T_f} H^2(\mathbb{D}) \to 0,$$

while the right-hand side has an algebraic K-theory interpretation. Indeed, let $\mathscr{K}$ denote the field of fractions of $\mathcal{O}(\overline{\mathbb{D}})$. Then an element $f \in \mathcal{O}(\overline{\mathbb{D}})$ whose restriction to the unit circle is invertible determines an element

$$[f] \in K_1^{\mathrm{alg}}(\mathscr{K})$$

and, in this situation, the right-hand side of (1.2) becomes the residue of $-d\log f$. This residue should be thought of as the result of the composition of the regulator map

$$K_1^{\mathrm{alg}}(\mathscr{K}) \ni [f] \mapsto [\log f] \in \varinjlim_{S\subseteq\mathbb{D}} H^1_{\mathcal{D}}(\mathbb{D}\setminus S, \mathbb{Z}(1))$$

from algebraic K-theory of $\mathscr{K}$ to the *Deligne cohomology* $\varinjlim_{S\subseteq\mathbb{D}} H^*_{\mathcal{D}}(\mathbb{D}\setminus S)$ (where the direct limit is taken over finite subsets $S \subseteq \mathbb{D}$) with the residue map (in fact minus the exterior derivative followed by the residue; see, for example, [Brylinski 2008]).

While this particular case does not really justify the reference to algebraic K-theory, the corresponding computations in the case of K_2^{alg} become much more demanding. In their work on algebraic K-theory of the quotient of the algebra of bounded operators by the ideal of trace class operators, Carey and Pincus introduced an analytic object, the joint torsion $\mathrm{JT}(A, B)$ of a pair of commuting Fredholm operators A and B. In the particular case of a pair of Toeplitz operators T_f and T_g on the Hardy space $H^2(\mathbb{D})$ and with symbols $f, g \in \mathcal{O}(\overline{\mathbb{D}})$, they proved the following theorem (in fact, Carey and Pincus work more generally with symbols in $H^\infty(\mathbb{D})$).

Theorem 1.3 [Carey and Pincus 1999, Proposition 1]. *Suppose that both T_f and T_g are Fredholm. Then*

$$\mathrm{JT}(T_f, T_g) = \prod_{\lambda\in\mathbb{D}} (-1)^{m_\lambda(g)\cdot m_\lambda(f)} \lim_{z\to\lambda} \frac{g(z)^{m_\lambda(f)}}{f(z)^{m_\lambda(g)}} \in \mathbb{C}^*, \tag{1.4}$$

where $m_\lambda(f)$ is the multiplicity of a zero of f at λ.

The left-hand side of this equality is a certain *analytic* invariant of the pair of operators (T_f, T_g), while the right-hand side is again best understood via evaluating the regulator map, this time on the Steinberg symbol $[f, g] \in K_2^{\mathrm{alg}}(\mathscr{K})$. The regulator map provides a class

$$K_2^{\mathrm{alg}}(\mathscr{K}) \ni [f, g] \mapsto [\log f] \cup [\log g] \in \varinjlim_{S\subseteq\mathbb{D}} H^2_{\mathcal{D}}(\mathbb{D}\setminus S, \mathbb{Z}(2)),$$

where the element $[\log f]\cup[\log g]$ in the direct limit of Deligne cohomology groups $\varinjlim_{S\subseteq\mathbb{D}} H^2_{\mathcal{D}}(\mathbb{D}\setminus S, \mathbb{Z}(2))$ is determined by the cup product in Deligne cohomology. The right-hand side of (1.4) then agrees with the residue map applied to the cohomology class $[\log f] \cup [\log g]$; see [Bloch 1981; Esnault and Viehweg 1988]. The right-hand side of (1.4) is known as the *Tate tame symbol* of the class $[f, g] \in K_2^{\mathrm{alg}}(\mathscr{K})$; see [Tate 1968; Deligne 1991]. A related construction in the case of a compact Riemann surface was considered in [Gustafsson and Tkachev 2009].

The subject of this paper is the generalization of these results to the case of a commuting n-tuple $A = (A_1, \dots, A_n)$ of bounded operators on a Hilbert space $\mathscr{H}$. The basic idea is to replace the single Toeplitz operator T_z by the Koszul complex $K(A, \mathscr{H})$ of our arbitrary n-tuple; see Section 3A. The generalization of the closed unit disc in the Toeplitz case becomes the Taylor spectrum $\mathrm{Sp}(A) \subseteq \mathbb{C}^n$, the set of $\lambda \in \mathbb{C}^n$ for which $K(A-\lambda, \mathscr{H})$, the Koszul complex of $A - \lambda = (A_1 - \lambda_1, \dots, A_n - \lambda_n)$, is not contractible. The notions of essential (resp. Fredholm) spectrum correspond to the values of λ for which $K(A-\lambda, \mathscr{H})$ has infinite-dimensional (resp. nontrivial finite-dimensional) cohomology. In the case that λ belongs to the Fredholm spectrum of A, the index $\mathrm{Ind}(A-\lambda)$ stands for the Euler characteristic of the cohomology of the Koszul complex $K(A-\lambda, \mathscr{H})$.

By a theorem of Taylor [1970a], the standard holomorphic functional calculus extends to a multivariable holomorphic functional calculus of such an n-tuple A, i.e.,

> *$\mathscr{H}$ is a module over the ring $\mathcal{O}(\mathrm{Sp}(A))$ of germs of functions holomorphic in a neighborhood of $\mathrm{Sp}(A)$.*

Our goal is to study the algebraic structure of this module.

The analogue of the Fritz Noether index theorem was dealt with in [Kaad and Nest 2015; Eschmeier and Putinar 1996]. Since it is needed to formulate the results of this paper, we recall the statement.

Theorem 1.5. *Let $f_1, \dots, f_n \in \mathcal{O}(\mathrm{Sp}(A))$. The n-tuple*

$$f(A) = (f_1(A_1, \dots, A_n), \dots, f_n(A_1, \dots, A_n))$$

is Fredholm if and only if the set $Z(f) \subseteq \mathrm{Sp}(A)$ of common zeroes of the f_i has no intersection with the essential spectrum of A. If $f(A)$ is Fredholm, the following hold:

(1) *The set $Z(f)$ of common zeroes is finite.*

(2) *We have*

$$\mathrm{Ind}(f(A)) = \sum_{\lambda \in Z(f)} \mathrm{Ind}_\lambda(f), \qquad \mathrm{Ind}_\lambda(f) = m_\lambda(f) \cdot \mathrm{Ind}(A - \lambda). \tag{1.6}$$

Here $m_\lambda(f)$ are the multiplicities of the points $\lambda \in Z(f)$ given by the dimensions

$$m_\lambda(f) = \dim_{\mathbb{C}}(\mathcal{O}_\lambda / \langle f \rangle_\lambda),$$

where $\langle f \rangle_\lambda$ is the ideal generated by f in the stalk $\mathcal{O}_\lambda$ of convergent power series near λ.

The joint torsion, the generalization of the torsion invariant of Carey and Pincus, is defined in the following context. First a bit of notation. A complex $\mathcal{C}$ of Hilbert spaces is Fredholm if it has finite-dimensional cohomology and, if $\mathcal{C}$ is Fredholm, its determinant line is the one-dimensional vector space

$$|\mathcal{C}| = \Lambda^{\mathrm{top}} H^+(\mathcal{C}) \otimes (\Lambda^{\mathrm{top}} H^-(\mathcal{C}))^\dagger$$

(see Section 2B for more details).

Definition 1.7. Let $\mathcal{C}$ be a complex of Hilbert spaces and $f, g : \mathcal{C} \to \mathcal{C}$ be two commuting morphisms of complexes. Suppose that the mapping cones C_f and C_g of f and g are both Fredholm. Then the long exact cohomology sequences of the mapping cone of f acting on C_g and of g acting on C_f provide two trivializations of the determinant line of $C_{f,g}$ and the joint torsion $\mathrm{JT}(\mathcal{C}; f, g)$ is the quotient of these two trivializations. See [Kaad 2012, Section 3.3] and Section 2A3 for details.[1]

The notion of joint torsion for the $\mathcal{O}(\mathrm{Sp}(A))$-module $\mathscr{H}$ appears in the following situation. Let $h_1, \dots, h_{n-1}, f, g$ be holomorphic functions defined in a neighborhood of $\mathrm{Sp}(A)$ and suppose that the zero sets $Z(h) \cap Z(f)$ and $Z(h) \cap Z(g)$ do not intersect the essential spectrum of A. In this case the commuting tuples

[1]The choice of signs in this paper is not standard, but is dictated by Theorem 2.17.

$(h_1(A), \dots, h_{n-1}(A), f(A))$ and $(h_1(A), \dots, h_{n-1}(A), g(A))$ are Fredholm and the joint torsion $\mathrm{JT}(K(h, \mathscr{H}); f, g) \in \mathbb{C}^*$ is well-defined. The main result of this paper is as follows.

Theorem 1.8 (Theorem 5.9). *Joint torsion is multiplicative, i.e.,*

$$\mathrm{JT}(K(h, \mathscr{H}); f, g) = \prod_{\lambda \in Z(h) \cap (Z(f) \cup Z(g))} c_\lambda(h; f, g)^{\mathrm{Ind}(A-\lambda)},$$

where the local terms $c_\lambda(h; f, g) \in \mathbb{C}^*$ (which are given explicitly in Theorem 1.9) *only depend on the image of the functions* $f, g, h_i, i = 1, \dots, n-1$ *in the stalk* $\mathcal{O}_\lambda$.

The local terms in the product above are given by the following result. Notice that the quantities $m_\mu(h^k, g^k)$ (and $m_\nu(h^k, f^k)$) appearing in the statement are multiplicities of the points $\mu \in Z(h^k) \cap Z(g^k) = Z(h^k, g^k)$ that are common zeroes of the n-tuple of holomorphic functions $(h_1^k, \dots, h_{n-1}^k, g^k)$; see the statement of Theorem 1.5. In particular, we see that a combination of Theorem 1.8 and Theorem 1.9 recovers Theorem 1.3 by specializing to the case where $n = 1$ and $A := T_z : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is just given by the single Toeplitz operator with symbol the inclusion $z : S^1 \to \mathbb{C}$ and spectrum $\mathrm{Sp}(A) = \overline{\mathbb{D}}$.

Theorem 1.9 (Theorem 5.6). *Let* $U \subseteq \mathbb{C}^n$ *be open with compact closure* $\overline{U}$ *and consider* $h_1, \dots, h_{n-1}, f, g \in \mathcal{O}(\overline{U})$. *Suppose that* $\lambda \in U$ *satisfies*

$$Z(h) \cap (Z(f) \cup Z(g)) \subseteq \{\lambda\}.$$

Then the sequence of quotients

$$c_\lambda(h^k; f^k, g^k) = \frac{\prod_{\mu \in U \cap Z(h^k) \cap Z(g^k)} f^k(\mu)^{m_\mu(h^k, g^k)}}{\prod_{\nu \in U \cap Z(h^k) \cap Z(f^k)} g^k(\nu)^{m_\nu(h^k, f^k)}}$$

converges to $c_\lambda(h; f, g)$ *for any sequences* $\{h_i^k\}, \{f^k\}, \{g^k\}$ *in* $\mathcal{O}(\overline{U})$ *which converge uniformly to* h_i, f, g *and for which*

$$Z(h^k) \cap Z(f^k) \cap Z(g^k) = \varnothing \quad \textit{for all } k \in \mathbb{N}.$$

Remark 1.10. The assumptions of Theorem 1.8 can be weakened. In fact it is sufficient for the functions f and g to be defined and holomorphic in a neighborhood of $X = Z(h) \cap \mathrm{Sp}(A)$. Using the methods of [Müller 2002], Taylor holomorphic calculus can be extended to define operators $f(A)$ and $g(A)$ acting on the Koszul complex $K(h, \mathscr{H})$ and the conclusions of the theorem still hold.

Remark 1.11. One can also define the "local joint torsion" by localizing the Hilbert space at the prime ideal $\mathfrak{p}_\lambda$ of functions in $\mathcal{O}(\mathrm{Sp}(A))$ vanishing at λ. The resulting numbers are conjecturally the same as the local joint torsion $c_\lambda(h; f, g)^{\mathrm{Ind}(A-\lambda)}$ as defined above.

As an application of our main results we are able to extend the definition of the Tate tame symbol from Riemann surfaces to more general complex analytic curves. More precisely, we work with a complex analytic curve $(X, \mathcal{O}_X)$ and a fixed point $x \in X$ such that there exists a local model $\big(Z(h), \mathcal{O}/\langle h\rangle|_{Z(h)}\big)$ near x which is determined by a holomorphic map

$$h = (h_1, \dots, h_{n-1}) : U \to \mathbb{C}^{n-1},$$

where $U \subseteq \mathbb{C}^n$ is an open set. In other words, X is a complete intersection in a neighborhood of x. For any two holomorphic functions $f, g \in \mathcal{O}_X(V)$ which are transversal to the curve near $x \in X$, we may then define the Tate tame symbol

$$c_x(X; f, g) \in \mathbb{C}^*$$

by applying the local description of Theorem 1.9. This invariant satisfies the properties which define a symbol in arithmetic; see [Tate 1971].

Remark 1.12. The conditions on the specific type of local model near $x \in X$ can be removed by working with more general resolutions than the Koszul complex. In particular, we will be able to study tame symbols of general complex analytic curves. We plan to carry out the details in a future publication.

The structure of the paper is a follows.

Section 2 is devoted to basic definitions involving determinants and torsion of Fredholm complexes. The definition essential to this paper, that of joint torsion, is given in Section 2A3.

The notion underlying these constructions is a determinant functor on the triangulated category of Fredholm complexes but, since this more abstract context is not necessary to understand what follows, we have avoided this language.

Section 3 is devoted to the generalities involving Koszul complexes of commuting families of bounded operators. After recalling the basic definitions (Section 3A) we describe the localization procedure involved in the computation of local indices and state the local index theorem (see Section 3C).

Section 4 contains the basic technical computations involving the joint torsion. The main result of this section, Theorem 4.3, gives a formula for joint torsion in the case when the $(n+1)$-tuple (h, f, g) has no common zeroes in $\mathrm{Sp}(A)$. The proof is based on the observation that in this case the joint torsion is given by the quotient of two determinants, both of which can be computed explicitly.[2]

[2]The fact that the joint torsion of two Fredholm operators (A_1, A_2) whose Koszul complex is contractible can be given as a quotient of two determinants holds in fact in a more general context. It is sufficient to assume that A_1 and A_2 commute up to a trace class operator and that, moreover, one can construct an acyclic complex $\mathscr{H} \xrightarrow{d_1} \mathscr{H} \oplus \mathscr{H} \xrightarrow{d_2} \mathscr{H}$, where the boundary maps are trace class perturbations of $(A_1, A_2)^t$ (and $(A_2, -A_1)$); see [Migler 2014].

Section 5 contains the proofs of the main theorems listed above. This section relies heavily on the continuity properties of the joint torsion as investigated by the authors in [Kaad and Nest 2019].

Section 6 contains the application of our results to the setting of complex analytic curves.

2. Determinants, torsion and joint torsion

Throughout this section $\mathbb{F}$ is a fixed field of characteristic zero.

2A. *Determinants of vector spaces.*

2A1. *Picard category of graded lines.*

Definition 2.1. $\mathfrak{L}$ denotes the category of $\mathbb{Z}$-graded lines over $\mathbb{F}$. The objects of $\mathfrak{L}$ are thus pairs (V, n), where V is a one-dimensional vector space over $\mathbb{F}$ and $n \in \mathbb{Z}$. The set of morphisms $\mathrm{Mor}((V, n), (W, m))$ is the set of isomorphisms $V \to W$ when $n = m$ and empty when $n \neq m$.

The category $\mathfrak{L}$ becomes a *Picard category* when equipped with the bifunctor

$$\otimes : ((V, n), (W, m)) \mapsto (V \otimes W, n + m),$$

which satisfies the obvious associativity constraint and the commutativity constraint

$$\psi_{(V,n),(W,m)} : (V, n) \otimes (W, m) \to (W, m) \otimes (V, n), \quad \xi \otimes \eta \mapsto (-1)^{n \cdot m} \eta \otimes \xi.$$

Let $\dagger : \mathfrak{L} \to \mathfrak{L}$ denote the covariant functor:

$$\dagger(V, n) := (V^*, -n) \text{ on objects} \quad \text{and} \quad \dagger(\alpha) := (\alpha^{-1})^* \text{ on morphisms},$$

where the superscript $(\cdot)^*$ denotes the linear dual (resp. transpose) of a vector space (resp. linear transformation). Below we often use the notation $(V, n)^\dagger := \dagger(V, n)$ and $\alpha^\dagger := \dagger(\alpha)$.

Together with the natural isomorphisms

$$c_{(V,n),(W,m)} : (V, n)^\dagger \otimes (W, m)^\dagger \to ((V, n) \otimes (W, m))^\dagger,$$
$$(c_{(V,n),(W,m)}(\lambda \otimes \mu))(\xi \otimes \eta) := \lambda(\xi) \cdot \mu(\eta) \cdot (-1)^{n \cdot m},$$

the covariant functor $\dagger$ becomes a monoidal functor. Furthermore, for any graded line (V, n), the image $(V, n)^\dagger = (V^*, -n)$, together with the isomorphism

$$\varepsilon_{(V,n)} : V \otimes V^* \to \mathbb{F}, \quad \varepsilon_{(V,n)} : \xi \otimes \lambda \mapsto \lambda(\xi),$$

is a fixed right inverse. Here the ground field $(\mathbb{F}, 0)$ and the obvious isomorphisms $V \otimes \mathbb{F} \cong V \cong \mathbb{F} \otimes V$ play the role of a fixed unit.

2A2. *Graded vector spaces.*

Definition 2.2. $\mathfrak{V}$ denotes the abelian category of *finite*-dimensional vector spaces over $\mathbb{F}$ and $\mathfrak{V}_{\mathrm{iso}}$ denotes the subcategory of $\mathfrak{V}$ with the same objects as $\mathfrak{V}$ and where

$$\mathrm{Mor}_{\mathfrak{V}_{\mathrm{iso}}}(V, W) = \{A \in \mathrm{Mor}_{\mathfrak{V}}(V, W) \mid A \text{ is invertible}\}.$$

We let $|\cdot|$ denote the determinant functor given by

$$\mathfrak{V}_{\mathrm{iso}} \ni V \to (\Lambda^{\mathrm{top}} V, \dim V) \in \mathfrak{L}$$

on objects and by $f \to \Lambda^{\mathrm{top}}(f)$ on morphisms (invertible linear transformations), where $\Lambda(V)$ denotes the exterior algebra over V.

The basic property of the determinant functor is the following simple observation. Given a short exact sequence of finite-dimensional vector spaces

$$\Delta : 0 \to V \xrightarrow{\iota} W \xrightarrow{\pi} Z \to 0,$$

there exists an associated canonical isomorphism

$$|\Delta| : |W| \to |V| \otimes |Z| \tag{2.3}$$

given as follows: Let $v_1, \dots, v_{\dim V}$ be a linear basis of the image of V in W and $w_1, \dots, w_{\dim Z}$ its completion to a linear basis of W. Then

$$\begin{aligned} &|\Delta|(v_1 \wedge \cdots \wedge v_{\dim V} \wedge w_1 \wedge \cdots \wedge w_{\dim Z}) \\ &\qquad = (-1)^{\dim V \cdot \dim Z} (\iota^{-1}(v_1) \wedge \cdots \wedge \iota^{-1}(v_{\dim V})) \otimes (\pi(w_1) \wedge \cdots \wedge \pi(w_{\dim Z})). \end{aligned}$$

It is straightforward to check that $|\Delta|$ is independent of the choices made. Remark that the extra sign $(-1)^{\dim V \cdot \dim Z}$ is nonstandard.

Remark 2.4. Note that the above rules determine a determinant functor as defined for example in [Breuning 2011, Definition 2.3]; see also [Deligne 1987] or [Knudsen 2002, Definition 1.4].

This determinant functor extends to the category of $\mathbb{Z}/2\mathbb{Z}$-graded finite-dimensional vector spaces. First some notation.

Notation 2.5. (1) $\mathfrak{V}^{\mathbb{Z}/2\mathbb{Z}}$ denotes the category of finite dimensional $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces with objects $V = V^+ \oplus V^-$ and morphisms

$$\alpha = \begin{pmatrix} \alpha^+ & 0 \\ 0 & \alpha^- \end{pmatrix} : V \to W,$$

where $\alpha^\pm : V^\pm \to W^\pm$ are linear maps.

(2) The map $[1]:\mathfrak{V}^{\mathbb{Z}/2\mathbb{Z}}\to\mathfrak{V}^{\mathbb{Z}/2\mathbb{Z}}$ is the self-equivalence of $\mathfrak{V}^{\mathbb{Z}/2\mathbb{Z}}$ given by change of grading:

$$V[1]^{\pm}=V^{\mp}\text{ on objects}\quad\text{and}\quad\alpha[1]^{\pm}=\alpha^{\mp}\text{ on morphisms.}$$

(3) $\mathfrak{V}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{iso}}$ is the category with the same objects as $\mathfrak{V}^{\mathbb{Z}/2\mathbb{Z}}$ and morphisms given by

$$\alpha=\begin{pmatrix}\alpha^{+} & 0\\ 0 & \alpha^{-}\end{pmatrix}$$

with $\alpha^{\pm}$ *invertible*.

Definition 2.6. The functor $|\cdot|:\mathfrak{V}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{iso}}\to\mathfrak{L}$ is given by

$$|V|=|V^{+}|\otimes|V^{-}|^{\dagger}\text{ on objects}\quad\text{and}\quad|\alpha|=|\alpha^{+}|\otimes|\alpha^{-}|^{\dagger}\text{ on morphisms.}$$

2A3. *Torsion.* The analogue of the isomorphism (2.3) in the context of $\mathbb{Z}$ (or $\mathbb{Z}/2\mathbb{Z}$) graded vector spaces has the following form.

Let V be a finite-dimensional vector space. The *degree map* $\varepsilon:\Lambda(V)\to\mathbb{N}_0$ on the exterior algebra over V is defined on homogeneous elements by $v_1\wedge\cdots\wedge v_k\mapsto k$. Suppose that L is a one-dimensional vector space and $t\in L$ a nonzero vector. Then $t^*\in L^*$ denotes the unique vector such that $t^*(t)=1$.

Suppose that $V_i=V_i^{+}\oplus V_i^{-}$ (for $i=1,2$) and $V=V^{+}\oplus V^{-}$ are $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces and that we are given grading-preserving linear maps

$$f:V_1\to V_2,\qquad i:V_2\to V,\qquad p:V\to V_1[1]$$

such that the following six-term sequence of finite dimensional vector spaces is exact:

$$\mathcal{V}:\qquad\begin{array}{ccccc} V_1^{+} & \xrightarrow{f^{+}} & V_2^{+} & \xrightarrow{i^{+}} & V^{+}\\ {\scriptstyle p^{-}}\uparrow & & & & \downarrow{\scriptstyle p^{+}}\\ V^{-} & \xleftarrow[i^{-}]{} & V_2^{-} & \xleftarrow[f^{-}]{} & V_1^{-}\end{array}\tag{2.7}$$

For future use, let us introduce the following.

Notation 2.8. We write the six-term exact sequence $\mathcal{V}$ above as the triangle

$$\mathcal{V}:\qquad\begin{array}{ccc} V_1 & \xrightarrow{\;\;f\;\;} & V_2\\ & {\scriptstyle p[1]}\nwarrow\quad\swarrow{\scriptstyle i} & \\ & V & \end{array}$$

and refer to it as an *exact triangle* of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces.

Definition 2.9. Suppose that we are given a six-term exact sequence $\mathcal{V}$ of the form (2.7). Set

$$(V_1^+)_{(0)} := \operatorname{Ker}(f^+), \qquad (V_2^+)_{(0)} := \operatorname{Ker}(i^+), \qquad (V^+)_{(0)} := \operatorname{Ker}(p^+),$$
$$(V_1^-)_{(0)} := \operatorname{Ker}(f^-), \qquad (V_2^-)_{(0)} := \operatorname{Ker}(i^-), \qquad (V^-)_{(0)} := \operatorname{Ker}(p^-).$$

For $i = 1, 2$, choose subspaces $(V_i^\pm)_{(1)} \subset V_i^\pm$ (resp. $(V^\pm)_{(1)} \subset V^\pm$) complementary to $(V_i^\pm)_{(0)} \subset V_i^\pm$ (resp. $(V^\pm)_{(0)}$) and nonzero vectors

$$t_i^\pm \in |(V_i^\pm)_{(1)}|, \quad i = 1, 2 \quad (\text{resp. } t^\pm \in |(V^\pm)_{(1)}|)$$

The *torsion isomorphism* of $\mathcal{V}$ is the isomorphism

$$|\mathcal{V}| : |V_2| \to |V_1| \otimes |V|$$

defined by

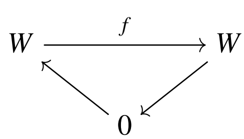
$$|\mathcal{V}|\big((f^+(t_1^+) \wedge t_2^+) \otimes (f^-(t_1^-) \wedge t_2^-)^*\big)$$
$$= (-1)^{\mu(\mathcal{V})}(p^-(t^-) \wedge t_1^+) \otimes (p^+(t^+) \wedge t_1^-)^* \otimes (i^+(t_2^+) \wedge t^+) \otimes (i^-(t_2^-) \wedge t^-)^*.$$

The sign exponent is defined by

$$\mu(\mathcal{V}) := (\varepsilon(t_2^+) + 1) \cdot (\varepsilon(t_1^-) + \varepsilon(t_1^+)) + \varepsilon(t_1^-) \cdot (\varepsilon(t^+) + \varepsilon(t^-))$$
$$+ \varepsilon(t^-) \cdot (\varepsilon(t_2^+) + \varepsilon(t_2^-)) + \varepsilon(t^+) \in \mathbb{N}_0.$$

It is a consequence of [Kaad 2012, Lemma 2.1.3] that the torsion isomorphism does not depend on the choices made. For future reference let us note the following simple fact:

Lemma 2.10. *Suppose that we are given a six-term exact sequence $\mathcal{V}$ of the form*

$\mathcal{V}:$ 

The torsion isomorphism of $\mathcal{V}$ is given by

$$|\mathcal{V}| = (-1)^{\dim(W^+) + \dim(W^-)} \cdot \frac{\det f^-}{\det f^+}.$$

Proof. This is a straightforward consequence of the definitions. □

2B. *Fredholm complexes.*

Definition 2.11. A *Fredholm complex* $\mathcal{X}$ is a finite cochain complex of (possibly infinite-dimensional) vector spaces

$$\mathcal{X} : \cdots \to X^k \xrightarrow{d^k} X^{k+1} \xrightarrow{d^{k+1}} X^{k+2} \to \cdots$$

such that the cohomology groups

$$H^k(\mathcal{X}) = \operatorname{Ker} d^k / \operatorname{Im} d^{k-1}$$

are finite-dimensional.

The *determinant* of the Fredholm complex $\mathcal{X}$ is the graded line

$$|\mathcal{X}| = |H^+(\mathcal{X})| \otimes |H^-(\mathcal{X})|^\dagger,$$

where $H^+(\mathcal{X}) := \bigoplus_{k\in\mathbb{Z}} H^{2k}(\mathcal{X})$ and $H^-(\mathcal{X}) := \bigoplus_{k\in\mathbb{Z}} H^{2k+1}(\mathcal{X})$. The *index* of $\mathcal{X}$ is the integer

$$\operatorname{Ind}(\mathcal{X}) = \dim(H^+(\mathcal{X})) - \dim(H^-(\mathcal{X})).$$

We let $\mathcal{X}[1]$ denote the *shift* of the Fredholm complex $\mathcal{X}$. $\mathcal{X}[1]$ is again a Fredholm complex with cochains $\mathcal{X}[1]^k := \mathcal{X}^{k+1}$ and with differentials

$$d[1]^k := -d^{k+1} : X^{k+1} \to X^{k+2}, \quad k \in \mathbb{Z}.$$

Definition 2.12. Let $\mathcal{X}$ and $\mathcal{Y}$ be finite cochain complexes and let $f : \mathcal{X} \to \mathcal{Y}$ be a cochain map. The *mapping cone* of f is the cochain complex C_f defined by $C_f^k := X^{k+1} \oplus Y^k$ and

$$d_f^k := \begin{pmatrix} -d_{\mathcal{X}}^{k+1} & 0 \\ f^{k+1} & d_{\mathcal{Y}}^k \end{pmatrix} : X^{k+1} \oplus Y^k \to X^{k+2} \oplus Y^{k+1}.$$

Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a cochain map of Fredholm complexes. The mapping cone is again a Fredholm complex and it fits into the *mapping cone triangle*

$$\Delta_f : \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{i} C_f \xrightarrow{p} \mathcal{X}[1],$$

where the cochain maps $i : \mathcal{Y} \to C_f$ and $p : C_f \to \mathcal{X}[1]$ are given by the inclusions $i^k := \binom{0}{1} : Y^k \to X^{k+1} \oplus Y^k$ and the projections $p^k := (\,1 \;\; 0\,) : X^{k+1} \oplus Y^k \to X^{k+1}$ for $k \in \mathbb{Z}$.

By passing to cohomology, the mapping cone triangle associated to f yields an exact triangle of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces:

$$H(\Delta_f) : \quad \begin{array}{ccccc} H^+(\mathcal{X}) & \xrightarrow{H^+(f)} & H^+(\mathcal{Y}) & \xrightarrow{H^+(i)} & H^+(C_f) \\ {\scriptstyle H^-(p)}\uparrow & & & & \downarrow{\scriptstyle H^+(p)} \\ H^-(C_f) & \xleftarrow[H^-(i)]{} & H^-(\mathcal{Y}) & \xleftarrow[H^-(f)]{} & H^-(\mathcal{X}) \end{array} \tag{2.13}$$

Definition 2.14. Let $f : \mathcal{X} \to \mathcal{Y}$ be a cochain map of Fredholm complexes. The *torsion isomorphism of* f is the torsion isomorphism of $H(\Delta_f)$,

$$|H(\Delta_f)| : |\mathcal{Y}| \to |\mathcal{X}| \otimes |C_f|$$

(compare with Definition 2.9).

2C. ***Joint torsion.*** Let $\mathcal{X}$ be a finite cochain complex and let $f : \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ be two commuting endomorphisms of $\mathcal{X}$. Remark that $\mathcal{X}$ is *not* assumed to be a Fredholm complex. Instead, we assume that the mapping cones C_f and C_g are Fredholm complexes. Since f and g commute we then obtain two cochain maps of Fredholm complexes:

$$\delta(g) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} : C_f \to C_f \quad \text{and} \quad \delta(f) = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} : C_g \to C_g.$$

Note that the two mapping cones $C_{\delta(f)}$ and $C_{\delta(g)}$ are in fact isomorphic: the isomorphism is given by the cochain map $\Phi : C_{\delta(g)} \to C_{\delta(f)}$ defined by

$$\Phi^k := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : X^{k+2} \oplus X^{k+1} \oplus X^{k+1} \oplus X^k \to X^{k+2} \oplus X^{k+1} \oplus X^{k+1} \oplus X^k \tag{2.15}$$

for all $k \in \mathbb{Z}$.

Definition 2.16. Let $\mathcal{X}$ be a finite cochain complex and let $f : \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ be two commuting morphisms of $\mathcal{X}$ such that both C_f and C_g are Fredholm complexes. The *joint torsion* of f and g is the nonzero number inducing the automorphism

$$\mathrm{JT}(\mathcal{X}; f, g) := (-1)^{\mathrm{Ind}(C_f)+\mathrm{Ind}(C_g)} \cdot |H(\Delta_{\delta(f)})|^{-1} \circ |\Phi| \circ |H(\Delta_{\delta(g)})| : \mathbb{F} \to \mathbb{F},$$

where we have used the canonical isomorphisms

$$|C_f| \otimes |C_f|^\dagger \cong |C_g| \otimes |C_g|^\dagger \cong \mathbb{F}$$

to identify the torsion isomorphisms of $\delta(g)$ and $\delta(f)$ with maps

$$|H(\Delta_{\delta(g)})| : \mathbb{F} \to |C_{\delta(g)}| \quad \text{and} \quad |H(\Delta_{\delta(f)})| : \mathbb{F} \to |C_{\delta(f)}|.$$

2C1. *Analyticity of joint torsion.* Let $\{X^k\}_{k\in\mathbb{Z}}$ be a fixed and finite family of *Hilbert spaces*; thus $X^k = \{0\}$ for all indices outside a finite subset of $\mathbb{Z}$.

Let us for future reference state the following result, which is a consequence of [Kaad and Nest 2019, Theorem 9.1].

Let $\mathscr{F}$ denote the set of triples $(\mathcal{X}; f, g)$, where

$$\mathcal{X} : \cdots X^k \xrightarrow{d^k} X^{k+1} \xrightarrow{d^{k+1}} X^{k+2} \to \cdots$$

is a cochain complex with each $d^k : X^k \to X^{k+1}$ a bounded operator and with $f, g : \mathcal{X} \to \mathcal{X}$ commuting cochain maps such that $f^k, g^k : X^k \to X^k$ are bounded operators and such that C_f and C_g are Fredholm. We can realize the elements of $\mathscr{F}$ as a set of bounded operators (d, f, g) on the $\mathbb{Z}$-graded Hilbert space $X = \bigoplus_{k\in\mathbb{Z}} X^k$ and hence endow $\mathscr{F}$ with the induced topology coming from the *operator norm*.

More precisely, $\mathscr{F}$ can be viewed as a subset of the C^*-algebra of bounded operators on $X \oplus X \oplus X$, $\mathscr{L}(X \oplus X \oplus X)$, by mapping a triple $(d, f, g) \in \mathscr{F}$ to the bounded operator given by $\iota(d, f, g) : (\xi_k, \eta_l, \zeta_m) \mapsto (d^k(\xi_k), f^l(\eta_l), g^m(\eta_m))$ for all $\xi_k \in X^k$, $\eta_l \in X^l$ and $\zeta_m \in X^m$. We emphasize that the individual Hilbert spaces X^k, $k \in \mathbb{Z}$, are fixed so that only the coboundary maps and cochain maps are allowed to vary. For an open subset $U \subseteq \mathbb{C}$ we then say that a map $\alpha : U \to \mathscr{F}$ is *analytic* when the associated map $\iota \circ \alpha : U \to \mathscr{L}(X \oplus X \oplus X)$ is analytic.

Theorem 2.17. *The map*

$$\mathscr{F} \ni (\mathcal{X}; f, g) \to \mathrm{JT}(\mathcal{X}; f, g) \in \mathbb{C}^*$$

is analytic. Thus, for any analytic map $\alpha : U \to \mathscr{F}$ defined on an open subset $U \subseteq \mathbb{C}$ we have that $\mathrm{JT} \circ \alpha : U \to \mathbb{C}^*$ is analytic.

The following variant of Theorem 2.17 also holds:

Theorem 2.18. *The map*

$$\mathscr{F} \ni (\mathcal{X}; f, g) \to \mathrm{JT}(\mathcal{X}; f, g) \in \mathbb{C}^*$$

is continuous.

Remark 2.19. We could have defined the joint torsion in the context of $\mathbb{Z}/2\mathbb{Z}$-graded Fredholm complexes instead of in the more restricted context of finite $\mathbb{Z}$-graded Fredholm complexes. The reason for staying with finite $\mathbb{Z}$-graded Fredholm complexes is that we do not have a proof of Theorem 2.17 for $\mathbb{Z}/2\mathbb{Z}$-graded Fredholm complexes.

3. Joint torsion of commuting operators

Throughout this section $A = (A_1, \dots, A_n) \in \mathscr{L}(\mathscr{H})^n$ denotes a commuting n-tuple of bounded operators on a Hilbert space $\mathscr{H}$. Thus, we have the relation $A_i A_j - A_j A_i = 0$ for all $i, j \in \{1, \dots, n\}$. Given $\lambda \in \mathbb{C}^n$, $A - \lambda$ denotes the n-tuple $(A_1 - \lambda_1, \dots, A_n - \lambda_n)$. We start by recalling some basic constructions and facts.

3A. ***The Koszul complex.*** Let $\Lambda(\mathbb{C}^n)$ denote the exterior algebra over $\mathbb{C}$ on n generators $e_1, \dots, e_n$.

For each subset $I = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ with $i_1 < \cdots < i_k$, let

$$e_I := e_{i_1} \wedge \cdots \wedge e_{i_k} \in \Lambda(\mathbb{C}^n),$$

where $\wedge : \Lambda(\mathbb{C}^n) \times \Lambda(\mathbb{C}^n) \to \Lambda(\mathbb{C}^n)$ denotes the wedge product.

The exterior algebra is then a $\mathbb{Z}$-graded algebra with respect to the decomposition

$$\Lambda(\mathbb{C}^n) = \bigoplus_{k \in \mathbb{Z}} \Lambda^k(\mathbb{C}^n),$$

where

$$\Lambda^k(\mathbb{C}^n) := \begin{cases} \{0\} & \text{for } k \notin \{0, \dots, n\}, \\ \operatorname{span}_{\mathbb{C}}\{e_I \mid I \subseteq \{1, \dots, n\}, |I| = k\} & \text{for } k \in \{0, \dots, n\}. \end{cases}$$

For each $j \in \{1, \dots, n\}$, the interior multiplication with the j-th generator is denoted by

$$\varepsilon_j^* : \Lambda(\mathbb{C}^n) \to \Lambda(\mathbb{C}^n),$$

$$\varepsilon_j^* : e_I \mapsto \begin{cases} 0 & \text{for } j \notin I, \\ (-1)^{m-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_m}} \wedge e_{i_{m+1}} \wedge \cdots \wedge e_{i_k} & \text{for } j = i_m. \end{cases}$$

This linear map has degree -1 with respect to the above $\mathbb{Z}$-grading.

Definition 3.1. By the *Koszul complex of the commuting n-tuple A* we understand the finite cochain complex of Hilbert spaces given by the following data:

(1) The Hilbert space $K^k(A, \mathscr{H}) := \mathscr{H} \otimes_{\mathbb{C}} \Lambda^{-k}(\mathbb{C}^n)$ for each $k \in \mathbb{Z}$.

(2) The differential of degree one,

$$d_A := \sum_{j=1}^n A_j \otimes \varepsilon_j^* : K^k(A, \mathscr{H}) \to K^{k+1}(A, \mathscr{H}).$$

We use the notation $K(A, \mathscr{H})$ for the Koszul complex and the notation $H^k(A, \mathscr{H})$, $k \in \mathbb{Z}$ for the cohomology groups of $K(A, \mathscr{H})$.

Remark 3.2. One may equally well define the Koszul complex of a commuting n-tuple $T = (T_1, T_2, \dots, T_n)$ of endomorphisms of a fixed vector space V, and we shall need this later on in the text; see Proposition 3.8.

Definition 3.3. We say that a commuting n-tuple A is *Fredholm* when the Koszul complex $K(A, \mathscr{H})$ is Fredholm. In this case, the *index* of A is the Euler characteristic (or Fredholm index) of the Fredholm complex $K(A, \mathscr{H})$,

$$\operatorname{Ind}(A) := \sum_{k \in \mathbb{Z}} (-1)^k \dim_{\mathbb{C}}(H^k(A, \mathscr{H})).$$

When A is Fredholm we have that $d_A^k : K^k(A, \mathscr{H}) \to K^{k+1}(A, \mathscr{H})$ has closed image for all $k \in \mathbb{Z}$. This is a consequence of [Curto 1981, Corollary 6.2].

The above Definition 3.1 is not quite the conventional definition of the Koszul complex. The reason for using interior instead of exterior multiplication on the exterior algebra stems from the fact that we want the mapping cone of a bounded operator $B : \mathscr{H} \to \mathscr{H}$ to be isomorphic to the Koszul complex $K(B, \mathscr{H})$ (without the dimension shift). Indeed, with the present convention, we have the following; see [Kaad and Nest 2015, Lemma 2.3].

Lemma 3.4. *Let $B : \mathscr{H} \to \mathscr{H}$ be a bounded operator such that $BA_j = A_j B$ for all $j \in \{1, \dots, n\}$. Then the mapping cone of the cochain map*

$$K^*(B) := B \otimes 1 : K^*(A, \mathscr{H}) \to K^*(A, \mathscr{H})$$

is cochain isomorphic to the Koszul complex $K^*((B, A), \mathscr{H})$.

Recall that the *Taylor spectrum* of A is the set

$$\mathrm{Sp}(A) := \{\lambda \in \mathbb{C}^n \mid H^*(A - \lambda, \mathscr{H}) \neq \{0\}\}.$$

The Taylor spectrum is a compact nonempty subset of $\mathbb{C}^n$ [Taylor 1970b, Theorem 3.1]. The essential Taylor spectrum of A is the set

$$\mathrm{Sp}_{\mathrm{ess}}(A) := \{\lambda \in \mathbb{C}^n \mid A - \lambda \text{ is not Fredholm}\}.$$

The unital commutative ring of germs of holomorphic functions on neighborhoods of $\mathrm{Sp}(A)$ is denoted by $\mathcal{O}(\mathrm{Sp}(A))$. The elements in $\mathcal{O}(\mathrm{Sp}(A))$ are thus equivalence classes of holomorphic functions $f : U \to \mathbb{C}$, where U is an open subset of $\mathbb{C}^n$ containing $\mathrm{Sp}(A)$.

The commuting tuple of bounded operators $A = (A_1, \dots, A_n)$ on the Hilbert space $\mathscr{H}$ provides us with a holomorphic functional calculus. To be more precise, the following holds; see [Taylor 1970a, Theorem 4.8].

Theorem 3.5. *Let $\mathcal{A} \subseteq \mathscr{L}(\mathscr{H})$ denote the smallest unital $\mathbb{C}$-algebra which contains the bounded operators $A_1, \dots, A_n$. There exists a unital homomorphism $\mathcal{O}(\mathrm{Sp}(A)) \to \mathcal{A}''$ (the double commutant of $\mathcal{A}$), $f \mapsto f(A)$ such that $z_i \mapsto A_i$. Furthermore, whenever $f = (f_1, \dots, f_m) : \mathrm{Sp}(A) \to \mathbb{C}^m$ is holomorphic (i.e., $f_k \in \mathcal{O}(\mathrm{Sp}(A))$, $k = 1, \dots, m$) the following identity holds:*

$$\mathrm{Sp}(f(A)) = f(\mathrm{Sp}(A)).$$

In particular, the above holomorphic functional calculus allows us to consider the Hilbert space $\mathscr{H}$ as a left module over $\mathcal{O}(\mathrm{Sp}(A))$.

Notation 3.6. (1) Given $f \in \mathcal{O}(\mathrm{Sp}(A))$, we denote the operator $f(A)$ on $\mathscr{H}$ by f. Given $g \in \mathcal{O}(\mathrm{Sp}(A))^m$, $K^*(g, \mathscr{H})$ denotes the Koszul complex of $g(A) = (g_1(A), \dots, g_m(A))$ and $H^*(g, \mathscr{H})$ denotes the cohomology of $K^*(g, \mathscr{H})$.

(2) For each $\lambda \in \mathrm{Sp}(A)$, $\mathscr{H}_\lambda$ denotes the localization of the module $\mathscr{H}$ with respect to the prime ideal $\mathfrak{p}_\lambda := \{f \in \mathcal{O}(\mathrm{Sp}(A)) \mid f(\lambda) = 0\} \subseteq \mathcal{O}(\mathrm{Sp}(A))$. More explicitly, the subset $S_\lambda := (\mathcal{O}(\mathrm{Sp}(A)) \setminus \mathfrak{p}_\lambda) \subseteq \mathcal{O}(\mathrm{Sp}(A))$ is a multiplicatively closed subset and we put $\mathscr{H}_\lambda := S_\lambda^{-1}\mathscr{H}$. For a vector $\xi \in \mathscr{H}$ and an element $s \in S_\lambda$ we let $\xi/s \in S_\lambda^{-1}\mathscr{H}$ denote the associated equivalence class in the localization. In particular, since the unit 1 lies in S_λ we obtain the map $\mathscr{H} \to S_\lambda^{-1}\mathscr{H}$ given by $\xi \mapsto \xi/1$. We emphasize that $\mathscr{H}_\lambda$ is *not* a Hilbert space but that $\mathscr{H}_\lambda$ remains a left module over $\mathcal{O}(\mathrm{Sp}(A))$ and a vector space over $\mathbb{C}$.

(3) Let $m \in \mathbb{N}$ and suppose that we are given $g = (g_1, \dots, g_m) \in \mathcal{O}(\mathrm{Sp}(A))^m$. We set

$$Z(g) := \{\lambda \in \mathrm{Sp}(A) \mid g_1(\lambda) = \cdots = g_m(\lambda) = 0\}.$$

Example 3.7 [Curto 1981]. Let $H^2(\mathbb{D}^n)$ be the Hardy-space over the polydisc $\mathbb{D}^n = \{z \in \mathbb{C}^n \mid |z_j| < 1 \text{ for all } j = 1, \dots, n\}$, and $A = (T_{z_1}, \dots, T_{z_n})$ be the n-tuple of multiplication operators by the coordinate functions $z_1, \dots, z_n$ on $\mathbb{C}^n$. Then

(1) $\mathrm{Sp}(A) = \overline{\mathbb{D}^n}$;

(2) for a function f holomorphic in a neighborhood of $\overline{\mathbb{D}^n}$, $f(A)$ coincides with the Toeplitz operator T_f of multiplication by f;

(3) $\mathrm{Sp}_{\mathrm{ess}}(A) = \partial\mathbb{D}^n$;

(4) for $\lambda \in \mathbb{D}^n$, $\mathrm{Ind}(A - \lambda) = 1$;

(5) for $\lambda \in \mathbb{D}^n$, $H^k(A - \lambda, H^2(\mathbb{D}^n)) = \{0\}$ for all $k \in \mathbb{Z} \setminus \{0\}$.

The next result is a consequence of [Kaad and Nest 2015, Proposition 4.5]. Notice that the cohomology groups involving the localizations $\mathscr{H}_\lambda$ are again Koszul cohomology groups but this time in the sense of Remark 3.2 (considering each $\mathscr{H}_\lambda$ as a vector space equipped with a commuting m-tuple of linear endomorphisms).

Proposition 3.8. *Suppose $g \in \mathcal{O}(\mathrm{Sp}(A))^m$ is such that $g(A) = (g_1(A), \dots, g_m(A))$ is Fredholm. Then $Z(g)$ is finite and the morphism of modules $\mathscr{H} \to \bigoplus_{\lambda \in Z(g)} \mathscr{H}_\lambda$, $\xi \mapsto \{\xi/1\}_{\lambda \in Z(g)}$, induces an isomorphism of cohomology groups,*

$$H^*(g, \mathscr{H}) \cong H^*\big(g, \textstyle\bigoplus_{\lambda \in Z(g)} \mathscr{H}_\lambda\big) \cong \textstyle\bigoplus_{\lambda \in Z(g)} H^*(g, \mathscr{H}_\lambda).$$

3B. *Joint torsion transition numbers.* Let $m \in \mathbb{N}$ and let $\langle m \rangle := \{1, \dots, m\}$. Fix an element $g = (g_1, \dots, g_m) \in \mathcal{O}(\mathrm{Sp}(A))^m$. For a subset $J = \{j_1, \dots, j_k\} \subseteq \langle m \rangle$ with $1 \le j_1 < \cdots < j_k \le m$, we define

$$g_J := (g_{j_1}, \dots, g_{j_k}) \in \mathcal{O}(\mathrm{Sp}(A))^k.$$

The following assumption remains in effect throughout this subsection:

> *Let $i, j \in \langle m \rangle$ and suppose that the commuting $(m-1)$-tuples $g_{\langle m \rangle \setminus \{i\}}(A)$ and $g_{\langle m \rangle \setminus \{j\}}(A)$ are Fredholm.*

It then follows by Lemma 3.4 that the mapping cones of the commuting cochain maps

$$K^*(g_i) \text{ and } K^*(g_j) : K^*(g_{\langle m \rangle \setminus \{i,j\}}, \mathscr{H}) \to K^*(g_{\langle m \rangle \setminus \{i,j\}}, \mathscr{H}) \tag{3.9}$$

are Fredholm complexes. In particular, the following definition makes sense:

Definition 3.10. The *joint torsion transition number* of $g(A)$ (in position i, j) is defined as the joint torsion of the cochain maps in (3.9). It is denoted by

$$\tau_{i,j}(g(A)) := \mathrm{JT}(K^*(g_{\langle m \rangle \setminus \{i,j\}}, \mathscr{H}); g_i, g_j) \in \mathbb{C}^*.$$

3B1. *The torsion line bundle.* As in the last subsection, let $g = (g_1, \dots, g_m)$ be a fixed element of $\mathcal{O}(\mathrm{Sp}(A))^m$. Recall that $\langle m \rangle := \{1, \dots, m\}$.

For each $i \in \{1, \dots, m\}$, define the open subset

$$U_i = \{\mu \in \mathbb{C}^m \mid (g - \mu)_{\langle m \rangle \setminus \{i\}}(A) \text{ is Fredholm}\}.$$

For each $i, j \in \{1, \dots, m\}$ and each $\mu \in U_i \cap U_j$, we then have the joint torsion transition number

$$\tau_{i,j}(g(A))(\mu) := \tau_{i,j}(g(A) - \mu) \in \mathbb{C}^*.$$

See Definition 3.10.

As a consequence of [Kaad 2012, Lemma 3.3.3] these functions satisfy the transition identities of a line bundle. Thus

$$\tau_{i,j}(g(A)) \cdot \tau_{j,k}(g(A)) = \tau_{i,k}(g(A)) \quad \text{and} \quad \tau_{i,j}(g(A)) = \tau_{j,i}(g(A))^{-1},$$

for all $i, j, k \in \{1, \dots, m\}$.

Furthermore, by Theorem 2.17, each $\tau_{i,j}(g(A)) : U_i \cap U_j \to \mathbb{C}^*$ is analytic. Hence the following makes sense:

Definition 3.11. The *torsion line bundle* of $g(A)$ is the analytic line bundle on $U := \bigcup_{i=1}^m U_i \subseteq \mathbb{C}^m$ with transition functions

$$\tau_{i,j}(g(A)) : U_i \cap U_j \to \mathbb{C}^*, \quad i, j = 1, \dots, m.$$

3C. ***Local indices.*** Let $\Omega := \mathrm{Sp}(A)^\circ$ denote the interior of the Taylor spectrum and let $\mathcal{O}$ denote the sheaf of analytic functions on Ω. Fix an element $g = (g_1, \dots, g_n)$ of $\mathcal{O}(\mathrm{Sp}(A))^n$. Notice that the number of holomorphic functions in g coincides with the number of operators in the n-tuple $A = (A_1, \dots, A_n)$.

> *Throughout this subsection we suppose that the commuting n-tuple $g(A) = (g_1(A), \dots, g_n(A))$ is Fredholm.*

This means precisely that the intersection $Z(g) \cap \mathrm{Sp}_{\mathrm{ess}}(A)$ is the trivial set. In particular, there is a well-defined index $\mathrm{Ind}(A - \lambda) \in \mathbb{Z}$ for each $\lambda \in Z(g)$.

Now, applying Proposition 3.8, it also follows from the Fredholmness of $g(A)$ that the set of common zeroes $Z(g) \subseteq \mathrm{Sp}(A)$ is finite. In particular, each element $\lambda \in \Omega \cap Z(g)$ is an isolated zero for the holomorphic map $g|_\Omega : \Omega \to \mathbb{C}^n$. This is equivalent to the finite dimensionality of the quotient vector space $\mathcal{O}_\lambda / \langle g \rangle_\lambda$, where $\langle g \rangle_\lambda \subseteq \mathcal{O}_\lambda$ is the ideal generated by the analytic functions $g_1, \dots, g_n : \Omega \to \mathbb{C}$ in the stalk $\mathcal{O}_\lambda$ of the sheaf $\mathcal{O}$ at λ. See [Grauert and Remmert 1984, Chapter 5, §1.2].

Definition 3.12. The *local degree* or (multiplicity) of g at $\lambda \in \Omega$ is the dimension $\dim_{\mathbb{C}}(\mathcal{O}_\lambda / \langle g \rangle_\lambda) \in \mathbb{N} \cup \{0\}$. The local degree is denoted by $m_\lambda(g)$.

Definition 3.13. The *local index* of $g(A)$ at $\lambda \in \mathrm{Sp}(A)$ is the Euler characteristic of the Koszul complex $K(g, \mathscr{H}_\lambda)$. The local index is denoted by $\mathrm{Ind}_\lambda(g(A)) \in \mathbb{Z}$. Thus

$$\mathrm{Ind}_\lambda(g(A)) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_{\mathbb{C}}(H^i(g, \mathscr{H}_\lambda)).$$

Note that the finite dimensionality of the cohomology groups $H^i(g, \mathscr{H}_\lambda)$, $i \in \mathbb{Z}$ is nonobvious. This is a consequence of Proposition 3.8 which also implies the identity

$$\dim_{\mathbb{C}} H^i(g(A), \mathscr{H}) = \sum_{\lambda \in Z(g)} \dim_{\mathbb{C}}(H^i(g, \mathscr{H}_\lambda)).$$

In particular we have that

$$\mathrm{Ind}(g(A)) = \sum_{\lambda \in Z(g)} \mathrm{Ind}_\lambda(g(A)).$$

The following "local" index theorem therefore yields the "global" index theorem of [Eschmeier and Putinar 1996, Theorem 10.3.13].

Theorem 3.14 (see [Kaad and Nest 2015, Theorem 8.5]). *Suppose that $g(A)$ is Fredholm and that $\lambda \in Z(g)$. The local index at λ is then given by*

$$\mathrm{Ind}_\lambda(g(A)) = m_\lambda(g) \cdot \mathrm{Ind}(A - \lambda).$$

Remark 3.15. By homotopy invariance of the Fredholm index, $\mathrm{Ind}(A - \lambda) = 0$ when $\lambda \in \partial(\mathrm{Sp}(A)) \cap (\mathbb{C}^n \setminus \mathrm{Sp}_{\mathrm{ess}}(A))$. The right-hand side of the equation in the above theorem should therefore be understood as 0 in this case, even though the local degree, $m_\lambda(g)$, is *not* defined.

4. Multiplicative Lefschetz numbers

Let $\mathcal{X}$ be a Fredholm complex over a fixed field $\mathbb{F}$ of characteristic 0, and let $f : \mathcal{X} \to \mathcal{X}$ be a cochain map.

> *Suppose that the induced maps $H^+(f) : H^+(\mathcal{X}) \to H^+(\mathcal{X})$ and $H^-(f) : H^-(\mathcal{X}) \to H^-(\mathcal{X})$ are invertible.*

Definition 4.1. The *multiplicative Lefschetz number* of $f : \mathcal{X} \to \mathcal{X}$ is the invertible element

$$M(\mathcal{X}; f) := \frac{\det(H^+(f))}{\det(H^-(f))} \in \mathbb{F}^*.$$

4A. ***Lefschetz numbers of Koszul complexes.*** Let $A = (A_1, \dots, A_n)$ be a commuting n-tuple of bounded operators on a Hilbert space $\mathscr{H}$, and let $h_1, \dots, h_{n-1}, f, g$ be in $\mathcal{O}(\mathrm{Sp}(A))$. We use the notation

$$(h, f) := (h_1, \dots, h_{n-1}, f) \in \mathcal{O}(\mathrm{Sp}(A))^n$$

(and similarly for (h, g)). Throughout this subsection we suppose the following:

The sets $Z(h,f)\cap \mathrm{Sp}_{\mathrm{ess}}(A)$ *and* $Z(h,g)\cap \mathrm{Sp}_{\mathrm{ess}}(A)$ *and* $Z(h,f,g)$ *are empty.*

It follows from this assumption that the Koszul complexes $K((h, f), \mathscr{H})$ and $K((h, g), \mathscr{H})$ are Fredholm. Furthermore, we have that the cohomology of the Koszul complex $K((h, f, g), \mathscr{H})$ is trivial.

Notice now that f and g induce cochain maps

$$\begin{aligned} f := K^*(f) : K((h, g), \mathscr{H}) \to K((h, g), \mathscr{H}), \\ g := K^*(g) : K((h, f), \mathscr{H}) \to K((h, f), \mathscr{H}) \end{aligned} \tag{4.2}$$

of Koszul complexes by means of the holomorphic functional calculus. As a consequence of the exactness of $K((h, f, g), \mathscr{H})$ we obtain that the induced maps

$$\begin{aligned} H^*(f) : H^*((h, g), \mathscr{H}) \to H^*((h, g), \mathscr{H}), \\ H^*(g) : H^*((h, f), \mathscr{H}) \to H^*((h, f), \mathscr{H}) \end{aligned}$$

are invertible.

The quantities involved in the next proposition are therefore well-defined.

Theorem 4.3. *The following identities hold*:

$$\mathrm{JT}(K(h, \mathscr{H});\, f, g) = \frac{M(K((h, g), \mathscr{H});\, f)}{M(K((h, f), \mathscr{H});\, g)} = \frac{\prod_{\lambda\in Z(h,g)} f(\lambda)^{m_\lambda(h,g)\cdot \mathrm{Ind}(A-\lambda)}}{\prod_{\mu\in Z(h,f)} g(\mu)^{m_\mu(h,f)\cdot \mathrm{Ind}(A-\mu)}}.$$

Proof. The first identity is an immediate consequence of the definition of the joint torsion and of Lemma 2.10; see also [Kaad 2012, Theorem 3.4.1].

To prove the second identity, it suffices to show that

$$M(K((h, g), \mathscr{H});\, f) = \prod_{\lambda\in Z(h)\cap Z(g)} f(\lambda)^{m_\lambda(h,g)\cdot \mathrm{Ind}(A-\lambda)}.$$

Let $i \in \{-n, \dots, 0\}$. Since $Z(h)\cap Z(g)\cap \mathrm{Sp}_{\mathrm{ess}}(A) = \varnothing$, the Koszul cohomology group $H^i((h, g), \mathscr{H})$ is finite-dimensional over $\mathbb{C}$. Let

$$H^i(A) := (H^i(A_1), \dots, H^i(A_n))$$

denote the commuting n-tuple of linear operators on $H^i((h, g), \mathscr{H})$ induced by $A = (A_1, \dots, A_n)$. For each $\lambda \in \mathbb{C}^n$, let $H^i((h, g), \mathscr{H})(\lambda) \subseteq H^i((h, g), \mathscr{H})$ denote the generalized eigenspace of the commuting n-tuple $H^i(A)$. Thus,

$$\begin{aligned} &H^i((h, g), \mathscr{H})(\lambda) \\ &\quad := \{\xi \in H^i((h, g), \mathscr{H}) \mid \exists m \in \mathbb{N} : \forall j \in \{1, \dots, n\}, H^i(A_j - \lambda_j)^m(\xi) = 0\big\} \end{aligned}$$

Recall now that the finite dimensionality of $H^i((h, g), \mathscr{H})$ implies that

$$\bigoplus_{\lambda\in Z(h,g)} H^i((h, g), \mathscr{H})(\lambda) \cong H^i((h, g), \mathscr{H})$$

and furthermore that each component $H^i((h,g),\mathscr{H})(\lambda) \subseteq H^i((h,g),\mathscr{H})$ admits a basis in which each of the restrictions

$$H^i(A_j)(\lambda) : H^i((h,g),\mathscr{H})(\lambda) \to H^i((h,g),\mathscr{H})(\lambda)$$

is upper triangular with only λ_j on the diagonal.

For each $\lambda \in Z(h) \cap Z(g)$, let

$$H^i(f(A))(\lambda) : H^i((h,g),\mathscr{H})(\lambda) \to H^i((h,g),\mathscr{H})(\lambda) \tag{4.4}$$

be the restriction of the isomorphism $H^i(f(A)) : H^i((h,g),\mathscr{H}) \to H^i((h,g),\mathscr{H})$. It then follows immediately from the above that

$$\det(H^i(f(A))) = \prod_{\lambda \in Z(h) \cap Z(g)} \det(H^i(f(A))(\lambda)). \tag{4.5}$$

The next lemma gives a computation of the determinant $\det(H^i(f(A))(\lambda))$ for each $\lambda \in Z(h) \cap Z(g)$.

Lemma 4.6. *Let* $\lambda \in Z(h) \cap Z(g)$. *The isomorphism* (4.4) *can be represented by an upper triangular matrix having* $f(\lambda) \in \mathbb{C}^*$ *as its only diagonal entry. In particular,*

$$\det(H^i(f(A))(\lambda)) = f(\lambda)^{\dim_{\mathbb{C}} H^i((h,g),\mathscr{H})(\lambda)}.$$

Proof. Suppose first that $f \in \mathcal{O}(\mathrm{Sp}(A))$ is the restriction of a polynomial p in the variables $z_1, \dots, z_n : \mathbb{C}^n \to \mathbb{C}$. In this case (4.4) is given by the polynomial $p(H^i(A)(\lambda))$, where each variable z_j has been replaced by the linear operator $H^i(A_j)(\lambda) : H^i((h,g),\mathscr{H})(\lambda) \to H^i((h,g),\mathscr{H})(\lambda)$. Choose a basis for $H^i((h,g),\mathscr{H})(\lambda)$ in which each of the operators $H^i(A_j)(\lambda)$ is represented by an upper triangular matrix having λ_j as the only diagonal entry. Represented in this basis $p(H^i(A)(\lambda))$ is an upper triangular matrix with $p(\lambda)$ as its only diagonal entry. This proves the claim of the lemma in this case.

To treat the general case, note that the action of $\mathcal{O}(\mathrm{Sp}(A))$ on $H^i((h,g),\mathscr{H})(\lambda)$ given by $k \mapsto H^i(k(A))(\lambda)$ factorizes through the local $\mathbb{C}$-algebra $\mathcal{O}_\lambda$ of convergent power series near λ by [Kaad and Nest 2015, Proposition 4.4]. Since the endomorphisms $H^i(A_j - \lambda_j)(\lambda) : H^i((h,g),\mathscr{H})(\lambda) \to H^i((h,g),\mathscr{H})(\lambda)$ are nilpotent for $j \in \{1,\dots,n\}$, this yields the existence of a polynomial $p \in \mathbb{C}[z_1,\dots,z_n]$ with $H^i(f(A))(\lambda) = H^i(p(A))(\lambda)$ and with $p(\lambda) = f(\lambda)$. To see this, remark that the nilpotency implies that the action of $\mathcal{O}_\lambda$ on $H^i(A_j - \lambda_j)(\lambda)$ factorizes through the quotient ring $\mathcal{O}_\lambda/(\mathfrak{m}_\lambda)^m\mathcal{O}_\lambda$ for some $m \in \mathbb{N}$, where $\mathfrak{m}_\lambda \subseteq \mathcal{O}_\lambda$ denotes the unique maximal ideal in the local $\mathbb{C}$-algebra $\mathcal{O}_\lambda$. But each element in $\mathcal{O}_\lambda/(\mathfrak{m}_\lambda)^m\mathcal{O}_\lambda$ can be represented by a polynomial.

Since it now has been established that $H^i(f(A))(\lambda) = H^i(p(A))(\lambda)$ for some polynomial with $p(\lambda) = f(\lambda)$, the first part of the proof yields the general result of the lemma. □

Let $\lambda \in Z(h) \cap Z(g)$ be fixed. To continue the proof of Theorem 4.3, remark that it follows by [Kaad and Nest 2015, Proposition 4.5], that the vector spaces $H^i((h, g), \mathscr{H})(\lambda)$ and $H^i((h, g), \mathscr{H}_\lambda)$ are isomorphic. Recall in this respect that $\mathscr{H}_\lambda$ denotes the localization of the module $\mathscr{H}$ over $\mathcal{O}(\mathrm{Sp}(A))$ at the prime ideal $\mathfrak{p}_\lambda := \{k \in \mathcal{O}(\mathrm{Sp}(A)) \mid k(\lambda) = 0\}$. In particular, it follows from Theorem 3.14 that

$$\begin{aligned} \dim_{\mathbb{C}} H^+((h, g), \mathscr{H})(\lambda) - \dim_{\mathbb{C}} H^-((h, g), \mathscr{H})(\lambda) \\ = \mathrm{Ind}_\lambda((h, g)(A)) = m_\lambda(h, g) \cdot \mathrm{Ind}(A - \lambda). \end{aligned} \tag{4.7}$$

The desired multiplicative Lefschetz number $M(K((h, g), \mathscr{H}); f) \in \mathbb{C}^*$ can now be computed as follows:

$$\begin{aligned} M(K((h, g), \mathscr{H}); f) & \\ &= \det(H^+(f)) \cdot \det(H^-(f))^{-1} \\ &= \prod_{\lambda \in Z(h) \cap Z(g)} \big(\det(H^+(f)(\lambda)) \cdot \det(H^-(f)(\lambda))^{-1}\big) \\ &= \prod_{\lambda \in Z(h) \cap Z(g)} \big(f(\lambda)^{\dim_{\mathbb{C}} H^+((h,g),\mathscr{H})(\lambda)} \cdot f(\lambda)^{-\dim_{\mathbb{C}} H^-((h,g),\mathscr{H})(\lambda)}\big) \\ &= \prod_{\lambda \in Z(h) \cap Z(g)} f(\lambda)^{\mathrm{Ind}_\lambda((h,g)(A))} = \prod_{\lambda \in Z(h) \cap Z(g)} f(\lambda)^{m_\lambda(h,g) \cdot \mathrm{Ind}(A - \lambda)}, \end{aligned}$$

where the second identity follows from (4.5), the third identity from Lemma 4.6, and the final two identities from (4.7). This proves the claim of Theorem 4.3. □

5. Localization of the joint torsion

Let $n \in \mathbb{N}$. For each open set $U \subseteq \mathbb{C}^n$ we let $\mathcal{O}(U)$ denote the unital $\mathbb{C}$-algebra of holomorphic functions on U with values in $\mathbb{C}$.

For each compact set $K \subseteq U$, we let $\mathscr{C}(K)$ denote the unital $\mathbb{C}$-algebra of continuous functions $f : K \to \mathbb{C}$ such that the restriction to the interior $f|_{K^\circ} : K^\circ \to \mathbb{C}$ is holomorphic. The unital $\mathbb{C}$-algebra $\mathscr{C}(K)$ becomes a Banach algebra when equipped with the supremum norm $\|\cdot\|_\infty : f \mapsto \sup_{z \in K} |f(z)|$. We let

$$r_K : \mathcal{O}(U) \to \mathscr{C}(K)$$

denote the restriction homomorphism.

Definition 5.1. Let $m \in \mathbb{N}$ and let $V \subseteq \mathbb{C}^m$. We say that a map $\alpha : V \to \mathcal{O}(U)$ is *holomorphic* when the composition

$$r_K \circ \alpha : V \to \mathscr{C}(K)$$

is holomorphic for each compact set $K \subseteq U$.

Let us fix a commuting n-tuple $A = (A_1, \dots, A_n)$ of bounded operators on the Hilbert space $\mathscr{H}$.

Proposition 5.2. *Let $U \supseteq \mathrm{Sp}(A)$ be an open set and $\alpha : V \to \mathcal{O}(U)$ be holomorphic. Then the map $V \to \mathscr{L}(\mathscr{H})$, $w \mapsto \alpha(w)(A)$ is holomorphic in operator norm.*

Proof. Choose a compact set $K \subseteq U$ such that $\mathrm{Sp}(A) \subseteq K^\circ$. By [Taylor 1970a, Theorem 4.3] the map $\mathscr{C}(K) \to \mathscr{L}(\mathscr{H})$, $f \mapsto f(A)$ is a bounded operator. Since the composition $r_K \circ \alpha : V \to \mathscr{C}(K)$ is holomorphic by definition, this proves the proposition. □

Let us write

$$\Omega := \mathrm{Sp}(A)^\circ$$

for the interior of the spectrum of A.

In the next theorem, we are using holomorphic maps $V \to \mathcal{O}(U)^{n+1}$ with certain properties, to approximate the tuple (h, f, g) and circumvent the existence of common zeroes for the n-tuples (h, f) and (h, g). In the very beginning of the proof of Theorem 5.3, we argue that this kind of approximation does in fact exist.

Theorem 5.3. *Let $U \supseteq \mathrm{Sp}(A)$ be an open set and let $h : U \to \mathbb{C}^{n-1}$ and $f, g : U \to \mathbb{C}$ be holomorphic maps. Suppose that $Z(h, f) \cap \mathrm{Sp}_{\mathrm{ess}}(A)$ and $Z(h, g) \cap \mathrm{Sp}_{\mathrm{ess}}(A)$ are empty. Then the joint torsion of $(K(h, \mathscr{H}); f, g)$ is given by*

$$\mathrm{JT}(K(h, \mathscr{H}); f, g) = \lim_{w \to 0} \left(\frac{\prod_{\lambda \in Z(h_w, g_w) \cap \Omega} f_w(\lambda)^{m_\lambda(h_w, g_w) \cdot \mathrm{Ind}(A-\lambda)}}{\prod_{\mu \in Z(h_w, f_w) \cap \Omega} g_w(\mu)^{m_\mu(h_w, f_w) \cdot \mathrm{Ind}(A-\mu)}} \right) \tag{5.4}$$

for any holomorphic map $V \to \mathcal{O}(U)^{n+1}$, $w \mapsto (h_w, f_w, g_w)$ satisfying

(1) $0 \in V$ *and* $(h_0, f_0, g_0) = (h, f, g)$;

(2) $Z(h_w, f_w, g_w) \cap \mathrm{Sp}(A) = \varnothing$ *for all* $w \in V \setminus \{0\}$;

(3) $Z(h_w, f_w) \cap \mathrm{Sp}_{\mathrm{ess}}(A)$ *and* $Z(h_w, g_w) \cap \mathrm{Sp}_{\mathrm{ess}}(A)$ *are empty for all* $w \in V$.

Proof. We first prove the existence of a holomorphic map $V \to \mathcal{O}(U)^{n+1}$ with the properties (1), (2), and (3).

Define the strictly positive numbers

$$\delta_0 := \inf\bigl\{|f(z)| \bigm| z \in Z(h) \cap \mathrm{Sp}_{\mathrm{ess}}(A)\bigr\},$$
$$\delta_1 := \inf\bigl\{|f(z)| \bigm| z \in Z(h, f) \cap \mathrm{Sp}(A) \text{ and } f(z) \neq 0\bigr\}.$$

Consider the open ball

$$\mathbb{B}_\delta(0) := \{w \in \mathbb{C} \mid |w| < \delta\}$$

of radius $\delta := \inf\{\delta_0, \delta_1\}$ and center $0 \in \mathbb{C}$. It can then be verified that the holomorphic map $\mathbb{B}_\delta(0) \to \mathcal{O}(U)^{n+1}$, $w \mapsto (h, f - w, g)$ has the properties (1), (2), and (3).

Let now $V \to \mathcal{O}(U)^{n+1}$ be any holomorphic map which satisfies (1), (2), and (3). By Theorem 2.17 and Proposition 5.2 we have that

$$\mathrm{JT}(K(h, \mathscr{H}); f, g) = \lim_{w \to 0} \mathrm{JT}(K(h_w, \mathscr{H}); f_w, g_w).$$

However, by Theorem 4.3 we may compute the joint torsion on the right-hand side,

$$\mathrm{JT}(K(h_w, \mathscr{H}); f_w, g_w) = \frac{\prod_{\lambda \in Z(h_w, g_w) \cap \Omega} f_w(\lambda)^{m_\lambda(h_w, g_w) \cdot \mathrm{Ind}(A - \lambda)}}{\prod_{\mu \in Z(h_w, f_w) \cap \Omega} g_w(\mu)^{m_\mu(h_w, f_w) \cdot \mathrm{Ind}(A - \mu)}}$$

for all $w \in V \setminus \{0\}$. This proves the theorem. □

Remark 5.5. It follows from the proof of Theorem 5.3 that $\mathrm{JT}(K(h, \mathscr{H}); f, g)$ can be computed more explicitly as the limit

$$\mathrm{JT}(K(h, \mathscr{H}); f, g) = \lim_{w \to 0} \left(\frac{\prod_{\lambda \in Z(h, g) \cap \Omega} (f(\lambda) - w)^{m_\lambda(h, g) \cdot \mathrm{Ind}(A - \lambda)}}{\prod_{\mu \in \Omega \cap Z(h) \cap f^{-1}(\{w\})} g(\mu)^{m_\mu(h, f - w) \cdot \mathrm{Ind}(A - \mu)}} \right),$$

where $w \in \mathbb{C}$ approaches zero in the Euclidean metric.

For each $\lambda \in \mathbb{C}^n$ and each $\varepsilon > 0$, we write

$$\mathbb{D}^n_\varepsilon(\lambda) := \left\{ z \in \mathbb{C}^n \;\middle|\; |z_j - \lambda_j| < \varepsilon \text{ for all } j \in \{1, \dots, n\} \right\}$$

for the open polydisc with radius $\varepsilon > 0$ and center λ.

Theorem 5.6. *Let $U \subseteq \mathbb{C}^n$ be open and let $h : U \to \mathbb{C}^{n-1}$ and $f, g : U \to \mathbb{C}$ be holomorphic. Suppose that $\lambda \in U$ is an isolated point in $Z(h, f) \cup Z(h, g)$. Then the limit*

$$c_\lambda(h; f, g) = \lim_{w \to 0} \left(\frac{\prod_{\mu \in Z(h_w, g_w) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda)} f_w(\mu)^{m_\mu(h_w, g_w)}}{\prod_{\nu \in Z(h_w, f_w) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda)} g_w(\nu)^{m_\nu(h_w, f_w)}} \right)$$

exists for any $\varepsilon > 0$ and any holomorphic map $V \to \mathcal{O}(\mathbb{D}^n_\varepsilon(\lambda))^{n+1}$, $w \mapsto (h_w, f_w, g_w)$ such that

(1) $\mathbb{D}^n_\varepsilon(\lambda) \subseteq U$ *and* $\mathbb{D}^n_\varepsilon(\lambda) \cap (Z(h, f) \cup Z(h, g)) = \{\lambda\}$;

(2) $0 \in V$ *and* $(h_0, f_0, g_0) = (h, f, g)$;

(3) $Z(h_w, f_w, g_w) \cap \overline{\mathbb{D}^n_{\varepsilon/2}(\lambda)} = \varnothing$ *for* $w \in V \setminus \{0\}$;

(4) $Z(h_w, f_w) \cap \partial \mathbb{D}^n_{\varepsilon/2}(\lambda)$ *and* $Z(h_w, g_w) \cap \partial \mathbb{D}^n_{\varepsilon/2}(\lambda)$ *are empty for all* $w \in V$.

Furthermore, $c_\lambda(h; f, g) \in \mathbb{C}^*$ only depends on the image of $(h, f, g) \in \mathcal{O}(U)^{n+1}$ in the stalk $\mathcal{O}^{n+1}_\lambda$ at $\lambda \in U$.

Proof. Consider the commuting n-tuple $A := (\varepsilon/2 T_{z_1}, \dots, \varepsilon/2 T_{z_n}) + \lambda$ of Toeplitz operators acting on the Hardy space over the polydisc $H^2(\mathbb{D}^n)$. It then follows that $\mathrm{Sp}(A) = \overline{\mathbb{D}^n_{\varepsilon/2}(\lambda)}$ and that $\mathrm{Sp}_{\mathrm{ess}}(A) = \partial \mathbb{D}^n_{\varepsilon/2}(\lambda)$. Furthermore, we have that $\mathrm{Ind}(A - \mu) = 1$ for all $\mu \in \mathbb{D}^n_{\varepsilon/2}(\lambda)$; see Example 3.7.

An application of Theorem 5.3 then yields that the limit $c_\lambda(h; f, g)$ exists and coincides with the joint torsion $\mathrm{JT}(K(h(A), H^2(\mathbb{D}^n)); f(A), g(A))$.

To see that $c_\lambda(h; f, g)$ only depends on the value of $(h, f, g) \in \mathcal{O}(U)$ in the stalk $\mathcal{O}_\lambda$, it suffices to check that it is independent of $\varepsilon > 0$ and of the holomorphic map $V \to \mathcal{O}(\mathbb{D}^n_\varepsilon(\lambda))^{n+1}$.

It follows immediately by Theorem 5.3 that $c_\lambda(h; f, g)$ is independent of the choice of holomorphic map $V \to \mathcal{O}(\mathbb{D}^n_\varepsilon(\lambda))^{n+1}$.

Let us thus choose an alternative $\varepsilon_0 > 0$ with $\varepsilon_0 < \varepsilon$. We may then find a holomorphic map $V_0 \to \mathcal{O}(\mathbb{D}^n_\varepsilon(\lambda))$, $w \mapsto (h_w, f_w, g_w)$ such that

$$(Z(h_w, f_w) \cup Z(h_w, g_w)) \cap \mathbb{D}^n_{\varepsilon_0/2}(\lambda) = (Z(h_w, f_w) \cup Z(h_w, g_w)) \cap \overline{\mathbb{D}^n_{\varepsilon/2}(\lambda)}$$

and such that (2) is satisfied as well. It is then clear that

$$\begin{aligned} c^\varepsilon_\lambda(h; f, g) &= \lim_{w\to 0} \left(\frac{\prod_{\mu \in Z(h_w, g_w) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda)} f_w(\mu)^{m_\mu(h_w, g_w)}}{\prod_{\nu \in Z(h_w, f_w) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda)} g_w(\mu)^{m_\nu(h_w, f_w)}} \right) \\ &= \lim_{w\to 0} \left(\frac{\prod_{\mu \in Z(h_w, g_w) \cap \mathbb{D}^n_{\varepsilon_0/2}(\lambda)} f_w(\mu)^{m_\mu(h_w, g_w)}}{\prod_{\nu \in Z(h_w, f_w) \cap \mathbb{D}^n_{\varepsilon_0/2}(\lambda)} g_w(\mu)^{m_\nu(h_w, f_w)}} \right) = c^{\varepsilon_0}_\lambda(h; f, g). \end{aligned}$$

This proves the theorem. □

Remark 5.7. The value $c_\lambda(h; f, g) \in \mathbb{C}^*$ may be computed more explicitly by the formula

$$\lim_{w\to 0} \left(\frac{(f(\lambda) - w)^{m_\lambda(h,g)}}{\prod_{\nu \in Z(h) \cap f^{-1}(\{w\}) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda)} g(\nu)^{m_\nu(h, f-w)}} \right),$$

where $w \in \mathbb{C}$ approaches zero in Euclidean metric on $\mathbb{C}$ and where $\varepsilon > 0$ is chosen such that

$$\mathbb{D}^n_\varepsilon(\lambda) \subseteq U \quad \text{and} \quad \mathbb{D}^n_\varepsilon(\lambda) \cap (Z(h, g) \cup Z(h, f)) = \{\lambda\}.$$

This is a consequence of the proof of Theorem 5.6 and Remark 5.5.

Remark 5.8. The quantity $c_\lambda(h; f, g) \in \mathbb{C}^*$ can be expressed as a limit of a sequence instead of as a limit point of a holomorphic function; see Theorem 1.9. To see this it suffices to apply Theorem 2.18 instead of Theorem 2.17 in the proof of Theorem 5.3 and Theorem 5.6.

The next theorem is the main result of this paper. It provides a local formula for the joint torsion.

Theorem 5.9. *Let $A = (A_1, \dots, A_n)$ be a commuting n-tuple on the Hilbert space $\mathscr{H}$ and let $h : \mathrm{Sp}(A) \to \mathbb{C}^{n-1}$ and $f, g : \mathrm{Sp}(A) \to \mathbb{C}$ be holomorphic. Suppose that*

$$Z(h, f) \cap \mathrm{Sp}_{\mathrm{ess}}(A) = \varnothing = Z(h, g) \cap \mathrm{Sp}_{\mathrm{ess}}(A).$$

Then

$$\mathrm{JT}(K(h,\mathscr{H});f,g)=\prod_{\lambda\in(Z(h,f)\cup Z(h,g))\cap\Omega} c_\lambda(h;f,g)^{\mathrm{Ind}(A-\lambda)},$$

where $\Omega:=\mathrm{Sp}(A)^\circ$.

Proof. Let $\tilde{h}:U\to\mathbb{C}^{n-1}$ and $\tilde{f},\tilde{g}:U\to\mathbb{C}$ be holomorphic representatives for $h\in\mathcal{O}(\mathrm{Sp}(A))^{n-1}$ and $f,g\in\mathcal{O}(\mathrm{Sp}(A))$ on an open set $U\supseteq\mathrm{Sp}(A)$.

Choose an $\varepsilon>0$ such that

$$\mathbb{D}^n_\varepsilon(\lambda)\subseteq U,\qquad \mathbb{D}^n_\varepsilon(\lambda)\cap(Z(h,f)\cup Z(h,g))=\{\lambda\},\qquad \mathbb{D}^n_\varepsilon(\lambda)\cap\mathrm{Sp}_{\mathrm{ess}}(A)=\varnothing$$

for all $\lambda\in Z(h,f)\cup Z(h,g)$. We may furthermore arrange that

$$\mathbb{D}^n_\varepsilon(\lambda)\subseteq\Omega$$

for all $\lambda\in(Z(h,f)\cup Z(h,g))\cap\Omega$. Finally, we may assume that

$$\mathbb{D}^n_\varepsilon(\lambda)\cap\mathbb{D}^n_\varepsilon(\mu)=\varnothing$$

whenever $\lambda\neq\mu$ and $\lambda,\mu\in Z(h,f)\cup Z(h,g)$.

Choose a $\delta>0$ such that

$$\delta\le\inf\bigl\{|f(z)|\bigm| z\in Z(h,g)\text{ and }f(z)\neq 0\bigr\},$$
$$\delta\le\inf\Bigl\{|f(z)|\Bigm| z\in\bigcap_{\lambda\in Z(h,g)\cup Z(h,f)}(\mathbb{C}^n\setminus\mathbb{D}^n_{\varepsilon/2}(\lambda))\cap Z(h)\Bigr\}.$$

The map $\mathbb{B}_\delta(0)\to\mathcal{O}(U)^{n+1}$, $w\mapsto(\tilde{h},\tilde{f}-w,\tilde{g})$ is then holomorphic and it satisfies conditions (1), (2), and (3) of Theorem 5.3. Furthermore, we have that

$$(Z(h,f-w)\cup Z(h,g))\cap\Biggl(\bigcup_{\lambda\in Z(h,g)\cup Z(h,f)}\mathbb{D}^n_{\varepsilon/2}(\lambda)\Biggr)=Z(h,f-w)\cup Z(h,g)$$

for all $w\in\mathbb{B}_\delta(0)$.

An application of Theorem 5.3 now yields that

$$\mathrm{JT}(K(h,\mathscr{H});f,g)=\lim_{w\to0}\Biggl(\prod_{\lambda\in Z(h,g)\cup Z(h,f)}\frac{f_w(\lambda)^{m_\lambda(h,g)\cdot\mathrm{Ind}(A-\lambda)}}{\prod_{\nu\in\mathbb{D}^n_{\varepsilon/2}(\lambda)\cap Z(h,f_w)\cap\Omega}g(\nu)^{m_\nu(h,f_w)\cdot\mathrm{Ind}(A-\nu)}}\Biggr),$$

where $f_w:=f-w$ for all $w\in\mathbb{B}_\delta(0)$.

Since the Fredholm index is a homotopy invariant and $\mathbb{D}^n_{\varepsilon/2}(\lambda)\cap\mathrm{Sp}_{\mathrm{ess}}(A)=\varnothing$ for all $\lambda\in Z(h,g)\cup Z(h,f)$, we obtain that

$$\mathrm{JT}(K(h,\mathscr{H});f,g)=\lim_{w\to0}\Biggl(\prod_{\lambda\in(Z(h,g)\cup Z(h,f))\cap\Omega}\frac{f_w(\lambda)^{m_\lambda(h,g)\cdot\mathrm{Ind}(A-\lambda)}}{\prod_{\nu\in\mathbb{D}^n_{\varepsilon/2}(\lambda)\cap Z(h,f_w)}g(\nu)^{m_\nu(h,f_w)\cdot\mathrm{Ind}(A-\lambda)}}\Biggr).$$

However, from Theorem 5.6 we know that the limit

$$\lim_{w\to 0} \frac{f_w(\lambda)^{m_\lambda(h,g)}}{\prod_{\nu\in\mathbb{D}^n_{\varepsilon/2}(\lambda)\cap Z(h,f_w)} g(\nu)^{m_\nu(h,f_w)}}$$

exists and agrees with $c_\lambda(h; f, g)$ for each $\lambda \in (Z(h, g)\cup Z(h, f))\cap\Omega$. This proves the present theorem. □

6. Application: tame symbols of complex analytic curves

6A. *Preliminaries on complex analytic spaces.* Consider an open set $U \subseteq \mathbb{C}^n$ together with holomorphic functions $h_1, \dots, h_m : U \to \mathbb{C}$. Define the zero-set

$$Z(h) := \{z \in U \mid h_1(z) = \cdots = h_m(z) = 0\}.$$

Let $\mathcal{O}_U$ denote the sheaf of holomorphic functions on U. For each $z \in Z(h)$, let

$$\langle h\rangle_z \subseteq \mathcal{O}_z$$

denote the ideal generated by $h_1, \dots, h_m : U \to \mathbb{C}$ in the stalk $\mathcal{O}_z$ of $\mathcal{O}_U$ at the point $z \in Z(h)$.

The *complex model space* associated to $h_1, \dots, h_m : U \to \mathbb{C}$ is the pair

$$\big(Z(h), \mathcal{O}/\langle h\rangle|_{Z(h)}\big)$$

consisting of the Hausdorff space $Z(h)$ and the restriction of the quotient sheaf $\mathcal{O}/\langle h\rangle$ to $Z(h)$. Thus, for an open set $V \subseteq \mathbb{C}^n$, a local section $s \in (\mathcal{O}/\langle h\rangle)(V\cap Z(h))$ is a collection

$$\{s_z\}_{z\in V\cap Z(h)}, \quad s_z \in \mathcal{O}_z/\langle h\rangle_z$$

such that for each $z_0 \in V \cap Z(h)$ there exists an open set $W \subseteq \mathbb{C}^n$ with $z_0 \in W$ and a holomorphic map $t : W \to \mathbb{C}$ with

$$s_z = [t_z] \quad \text{for all } z \in W \cap Z(h),$$

where $[t_z]$ denotes image of $t \in \mathcal{O}(W)$ under the map

$$\mathcal{O}(W) \to \mathcal{O}_z \to \mathcal{O}_z/\langle h\rangle_z.$$

We recall the following definition from [Grauert and Remmert 1984].

Definition 6.1. A *complex analytic space* is a pair $(X, \mathcal{O}_X)$, where X is a Hausdorff space and $\mathcal{O}_X$ is a sheaf of local $\mathbb{C}$-algebras on X, such that for each $x \in X$ there exist an open neighborhood $V \subseteq X$ and a complex model space $\big(Z(h), \mathcal{O}/\langle h\rangle|_{Z(h)}\big)$ together with an isomorphism

$$(\phi, \bar{\phi}) : \big(Z(h), \mathcal{O}/\langle h\rangle|_{Z(h)}\big) \to (V, \mathcal{O}_X|_V)$$

of sheaves of local $\mathbb{C}$-algebras. Thus, $\phi : Z(h) \to V$ is an isomorphism of topological spaces and $\bar{\phi}(W) : \mathcal{O}_X(W) \to (\mathcal{O}/\langle h\rangle)(\phi^{-1}(W))$ is an isomorphism of $\mathbb{C}$-algebras for each open set $W \subseteq V$.

We refer to $\big(Z(h), \mathcal{O}/\langle h\rangle|_{Z(h)}\big)$ as a *local model* for $(X, \mathcal{O}_X)$ near the point $x \in X$.

6B. ***The Tate tame symbol.*** Let $(X, \mathcal{O}_X)$ be a complex analytic space and let us fix a point $x \in X$.

Suppose that $\dim_x(X) = 1$.

Thus, there exists an open neighborhood $V \subseteq X$ of x and a section $f \in \mathcal{O}_X(V)$ such that $f(x) = 0$ and such that the quotient $\mathcal{O}_{X,x}/\langle f_x\rangle$ is a finite-dimensional vector space over $\mathbb{C}$, where $\langle f_x\rangle$ is the ideal generated by f in the stalk $\mathcal{O}_{X,x}$.

The following assumption also remains valid throughout this subsection.

Assumption 6.2. Suppose that X can be represented as a complete intersection in a neighborhood of x. Thus there exists a local model $\big(Z(h), \mathcal{O}/\langle h\rangle|_{Z(h)}\big)$ for $(X, \mathcal{O}_X)$ near $x \in X$, where

$$h = (h_1, \dots, h_{n-1}) : U \to \mathbb{C}^{n-1}$$

is holomorphic and $U \subseteq \mathbb{C}^n$ is open.

We are now ready to define the Tate tame symbol at the point $x \in X$.

Definition 6.3. Take an open neighborhood $V \subseteq X$ of $x \in X$ and let $f, g \in \mathcal{O}_X(V)$ with

$$\mathcal{O}_{X,x}/\langle f_x\rangle \quad \text{and} \quad \mathcal{O}_{X,x}/\langle g_x\rangle$$

of finite dimension over $\mathbb{C}$. Let $\big(Z(h), \mathcal{O}/\langle h\rangle|_{Z(h)}\big)$ be a local model for $(X, \mathcal{O}_X)$ near $x \in X$ as in Assumption 6.2 with associated isomorphism

$$(\phi, \bar{\phi}) : \big(Z(h), \mathcal{O}/\langle h\rangle|_{Z(h)}\big) \to (V, \mathcal{O}_X|_V).$$

The *Tate tame symbol* of $f, g \in \mathcal{O}_X(V)$ at $x \in X$ is defined by

$$c_x(X; f, g) := c_{\phi^{-1}(x)}(h|_\Omega; \tilde{f}, \tilde{g}),$$

where $\Omega \subseteq U$ is an open neighborhood of $\phi^{-1}(x) \in Z(h)$ and $\tilde{f}, \tilde{g} : \Omega \to \mathbb{C}$ are holomorphic functions such that the identities

$$[\tilde{f}_\mu] = \bar{\phi}(V)(f)_\mu \quad \text{and} \quad [\tilde{g}_\mu] = \bar{\phi}(V)(g)_\mu$$

hold in $\mathcal{O}_\mu/\langle h\rangle_\mu$ for all $\mu \in \Omega \cap Z(h)$.

We need to show that the Tate tame symbol is independent of the various choices. This is part of the next proposition, which also gives a more concrete expression for the tame symbol. Let us introduce the notation

$$m_x(f) := \dim_{\mathbb{C}} \mathcal{O}_{X,x}/\langle f_x\rangle$$

for any local section $f \in \mathcal{O}_X(V)$. We remark that $m_x(f) = 0$ whenever $f(x) \neq 0$.

Proposition 6.4. *Take an open neighborhood $V \subseteq X$ of $x \in X$ and let $f, g \in \mathcal{O}_X(V)$ with*

$$\mathcal{O}_{X,x}/\langle f_x \rangle \quad \text{and} \quad \mathcal{O}_{X,x}/\langle g_x \rangle$$

finite-dimensional. Then

$$c_x(X; f, g) = \lim_{w \to 0} \frac{(f(x) - w)^{m_x(g)}}{\prod_{y \in \Theta \cap f^{-1}(\{w\})} g(y)^{m_y(f-w)}},$$

where $w \in \mathbb{C}$ approaches 0 in the Euclidean metric on $\mathbb{C}$, and where $\Theta \subseteq V$ is any open neighborhood of x such that

(1) $\overline{\Theta} \subseteq V$ *and* $\overline{\Theta}$ *is compact*;

(2) $\overline{\Theta} \cap (Z(f) \cup Z(g)) \subseteq \{x\}$.

In particular, we have that $c_x(X; f, g)$ is well-defined.

Proof. Let us choose a local model $\big(Z(h), \mathcal{O}/\langle h \rangle|_{Z(h)}\big)$ for $(X, \mathcal{O}_X)$ near the point $x \in X$, and let

$$(\phi, \overline{\phi}) : \big(Z(h), \mathcal{O}/\langle h \rangle|_{Z(h)}\big) \to (V, \mathcal{O}_X|_V)$$

denote the associated isomorphism.

Put $\lambda := \phi^{-1}(x)$ and choose lifts

$$\tilde{f}, \tilde{g} : \Omega \to \mathbb{C}$$

of the sections $\overline{\phi}(V)(f), \overline{\phi}(V)(g) \in (\mathcal{O}/\langle h \rangle)(Z(h))$ near $\lambda \in \Omega$. We remark that

$$\tilde{f}(\mu) = f(\phi(\mu)) \quad \text{and} \quad \tilde{g}(\mu) = g(\phi(\mu))$$

for all $\mu \in \Omega \cap Z(h)$.

Furthermore, we notice that $(\phi, \overline{\phi})$ induces isomorphisms

$$\mathcal{O}_{X,\phi(\mu)}/\langle f_{\phi(\mu)} - w \rangle \cong \mathcal{O}_\mu/\langle h, \tilde{f} - w \rangle_\mu,$$
$$\mathcal{O}_{X,\phi(\mu)}/\langle g_{\phi(\mu)} \rangle \cong \mathcal{O}_\mu/\langle h, \tilde{g} \rangle_\mu$$

for all $\mu \in \Omega \cap Z(h)$ and all $w \in \mathbb{C}$. In particular, we have the identities

$$m_\mu(h, \tilde{f} - w) = m_{\phi(\mu)}(f - w) \quad \text{and} \quad m_\lambda(h, \tilde{g}) = m_x(g)$$

for all $\mu \in \Omega \cap Z(h)$ and all $w \in \mathbb{C}$.

Let us now choose an $\varepsilon > 0$ such that

$$(Z(h, \tilde{f}) \cup Z(h, \tilde{g})) \cap \mathbb{D}^n_\varepsilon(\lambda) \subseteq \{\lambda\} \quad \text{and} \quad \mathbb{D}_\varepsilon(\lambda) \subseteq \Omega.$$

Thus, by Remark 5.7 and the above observations, we obtain that

$$\begin{aligned} c_\lambda(h|_\Omega; \tilde{f}, \tilde{g}) &= \lim_{w \to 0} \frac{(\tilde{f}(\lambda) - w)^{m_\lambda(h, \tilde{g})}}{\prod_{\mu \in Z(h) \cap \tilde{f}^{-1}(\{w\}) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda)} \tilde{g}(\mu)^{m_\mu(h, \tilde{f} - w)}} \\ &= \lim_{w \to 0} \frac{(f(x) - w)^{m_x(g)}}{\prod_{\mu \in Z(h) \cap \tilde{f}^{-1}(\{w\}) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda)} g(\phi(\mu))^{m_{\phi(\mu)}(f - w)}} \\ &= \lim_{w \to 0} \frac{(f(x) - w)^{m_x(g)}}{\prod_{y \in \phi(Z(h) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda)) \cap f^{-1}(\{w\})} g(y)^{m_y(f - w)}}. \qquad (6.5) \end{aligned}$$

This proves the statement of the proposition for the open neighborhood

$$\Theta := \phi(Z(h) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda)) \subseteq V$$

of $x \in X$.

To prove the general statement, we now let $\Theta \subseteq V$ be an arbitrary open neighborhood of $x \in X$ such that the conditions (1) and (2) are satisfied. Since the limit in (6.5) is independent of the choice of $\varepsilon > 0$, we may assume that

$$\phi(Z(h) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda)) \subseteq \Theta.$$

It then suffices to find a $\delta > 0$ such that

$$\Theta \cap f^{-1}(\{w\}) \subseteq \phi(Z(h) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda)) \cap f^{-1}(\{w\})$$

for all $w \in \mathbb{B}_\delta(0)$. But this property is satisfied with

$$\delta := \inf\big\{|f(y)| \;\big|\; y \in \overline{\Theta} \setminus \phi(Z(h) \cap \mathbb{D}^n_{\varepsilon/2}(\lambda))\big\}. \qquad \square$$

Proposition 6.6. *Let $V \subseteq X$ be an open neighborhood of $x \in X$ and let $f_j \in \mathcal{O}_X(V)$ for $j = 1, 2, 3$ and $t \in \mathcal{O}_X(V)$ be local sections over V such that*

$$\mathcal{O}_{X,x}/\langle (f_j)_x \rangle, \quad \mathcal{O}_{X,x}/\langle t_x \rangle, \quad \text{and} \quad \mathcal{O}_{X,x}/\langle 1 - t_x \rangle$$

are finite-dimensional vector spaces over $\mathbb{C}$. Then the Tate tame symbol satisfies the properties

(1) $c_x(X; f_1, f_2) = c_x(X; f_2, f_1)^{-1}$;

(2) $c_x(X; f_1, f_2 f_3) = c_x(X; f_1, f_2) \cdot c_x(X; f_1, f_3)$;

(3) $c_x(X; t, 1 - t) = 1$.

Proof. This is an easy consequence of the definition of the Tate tame symbol; see Definition 6.3 and Theorem 5.6. □

For completeness, let us compute the resulting formula in the case when $x \in X$ above is a regular point.

Proposition 6.7. *Let X be a Riemann surface and let f, g be holomorphic functions defined in a neighborhood of $x \in X$. Then*

$$c_x(X; f, g) = (-1)^{m_x(f)m_x(g)} \lim_{w \to x} \frac{f(w)^{m_x(g)}}{g(w)^{m_x(f)}}. \tag{6.8}$$

Proof. Since the computation is local, we can just as well assume that $x = 0 \in \mathbb{C}$ and f and g are two functions holomorphic in a neighborhood of 0. Hence they can be written in the form

$$f(z) = z^{m_0(f)}\phi(z) \quad \text{and} \quad g(z) = z^{m_0(g)}\psi(z),$$

where both $\phi(0) \neq 0$ and $\psi(0) \neq 0$. Then, since the formula (6.8) has the properties listed in the proposition above, the computation reduces to checking that

$$c_0(\mathbb{C}; z, z) = -1, \quad c_0(\mathbb{C}; \phi, z) = \phi(0), \quad \text{and} \quad c_0(\mathbb{C}; \phi, \psi) = 1.$$

But this is obvious by Remark 5.7. □

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Witt and cohomological invariants of Witt classes

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We describe all Witt invariants and mod 2 cohomological invariants of the functor I^n as combinations of fundamental invariants; this is related to the study of operations on mod 2 Milnor K-theory. We also study behaviour of these invariants with respect to products, restrictions, similitudes and ramification.

Introduction

Building on classical constructions such as the discriminant and the Hasse–Witt invariant, cohomological invariants have become a standard tool in the study of quadratic forms. Cohomological invariants of quadratic forms are also related to cohomological invariants of algebraic groups, for split groups of orthogonal type.

In [Garibaldi et al. 2003], Serre introduces cohomological invariants over a field, and completely describes (away from characteristic 2) the invariants of Quad_n (nondegenerated n-dimensional quadratic forms) and $\mathrm{Quad}_{n,\delta}$ (those with prescribed determinant δ), and in particular this settles the case of invariants of split orthogonal and special orthogonal groups. In contrast, the case of split spin groups, corresponding to invariants of $\mathrm{Quad}_n \cap I^3$ (meaning that the Witt classes of the forms must be in I^3), is very much open, and has only been treated for small n (see for instance [Garibaldi 2009]) or for invariants of small degree (the case of degree 3 has been essentially solved by Merkurjev [2016]), one problem being that we do not have any satisfying parametrization of $\mathrm{Quad}_n \cap I^3$.

On the other hand, if we move from isometry classes to Witt classes, following the resolution of Milnor's conjecture by Voevodsky, we have at hand good descriptions of I^n (see for instance [Elman et al. 2008]), and at least one important cohomological invariant of I^n, $e_n : I^n(K) \to H^n(K, \mu_2)$. The goal of this article is to describe all mod 2 cohomological invariants of I^n, and study some of their basic properties.

Our starting point is a construction of Rost [1999], who defines a certain natural operation $P_n : I^n(K) \to I^{2n}(K)$ which behaves like a divided square in the sense that $P_n\left(\sum \varphi_i\right) = \sum_{i<j} \varphi_i \cdot \varphi_j$ if φ_i are n-fold Pfister forms. After composing with

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e_{2n} this gives a cohomological invariant of I^n of degree $2n$. We generalize this to operations $\bar{\pi}_n^d : I^n \to I^{dn}$ for all $d \in \mathbb{N}$ and thus cohomological invariants of degree dn. Since our constructions involve both Witt invariants and cohomological invariants, in order to avoid repeating very similar proofs in both settings, we choose to adopt a unified point of view and treat both cases simultaneously, using A to denote either the Witt ring or mod 2 cohomology.

We define two sets of generators for invariants, f_n^d (the invariants mentioned above, see Proposition 2.2) and g_n^d (Definition 4.4), each being useful depending on the situation. The invariants g_n^d have the important property that only a finite number of them are nonzero on a fixed form (Proposition 4.7), which allows us to take infinite combinations, and we show that any invariant of I^n is equal to such a combination (Theorem 4.9). They are also better behaved with respect to similitudes (Proposition 7.6). On the other hand, the f_n^d are preferable for handling products (Proposition 5.2 and Corollary 5.6) and restriction to I^{n+1} (Corollaries 6.3 and 6.4). We also study behaviour with respect to residues from discrete valuations (Proposition 8.1), and establish links with Serre's description of invariants of isometry classes (Proposition 9.5).

Our invariants may be related to other various constructions on Milnor K-theory and Galois cohomology, notably by Vial [2009]. The invariants defined here may be seen as lifting of Vial's to the level of I^n. See Section 10 for more details.

Finally, we adapt an idea of Rost [1999] (see also [Garibaldi 2009]) to study invariants of Witt classes in I^n that are divisible by an r-fold Pfister form, giving a complete description for $r = 1$ (Theorem 11.4).

Notations and some preliminaries

In all that follows, k is a fixed field of characteristic different from 2, and K denotes any field extension of k. The set of natural integers is denoted by $\mathbb{N}$, and the positive integers by $\mathbb{N}^*$; if $x \in \mathbb{R}$, $\lfloor x \rfloor \in \mathbb{Z}$ denotes its floor, and $\lceil x \rceil$ its ceiling. We extend the binomial coefficient $\binom{a}{b}$ for arbitrary $a, b \in \mathbb{Z}$ in the only way that still satisfies Pascal's triangle.

For all facts on quadratic forms, the reader is referred to [Elman et al. 2008]. All the quadratic forms we consider are assumed to be nondegenerated. The Grothendieck–Witt ring $\mathrm{GW}(K)$ has a fundamental ideal $\hat{I}(K)$, defined as the kernel of the dimension map $\mathrm{GW}(K) \to \mathbb{Z}$. We denote by $[q] \in W(K)$ the Witt class of an element $q \in \mathrm{GW}(K)$, and this ring morphism $\mathrm{GW}(K) \to W(K)$ induces an isomorphism between $\hat{I}(K)$ and the fundamental ideal $I(K) \subset W(K)$. If $x \in I(K)$, we write $\hat{x} \in \hat{I}(K)$ for its (unique) antecedent. If $n \in \mathbb{N}$ and $q \in W(K)$, then $nq = q + \cdots + q$ is not to be confused with $\langle n \rangle q$, which is pointwise multiplication by the scalar $n \in K^*$.

If $a \in K^*$, then we write $\langle\langle a\rangle\rangle = \langle 1, -a\rangle \in I(K)$, and if $a_1, \dots, a_n \in K^*$, then $\langle\langle a_1, \dots, a_n\rangle\rangle = \langle\langle a_1\rangle\rangle \cdots \langle\langle a_n\rangle\rangle \in I^n(K)$. Those elements are (the Witt classes of) the n-fold Pfister forms, and we use $\mathrm{Pf}_n(K) \subset I^n(K)$ for the set of such elements. We also write $\langle|a_1, \dots, a_n|\rangle$ for the antecedent of $\langle\langle a_1, \dots, a_n\rangle\rangle$ in $\hat{I}^n(K)$; we call such elements n-fold Grothendieck–Pfister elements, and we write $\widehat{\mathrm{Pf}}_n(K) \subset \hat{I}^n(K)$ for their set. For instance, $\langle|a|\rangle = \langle 1\rangle - \langle a\rangle$, so $\langle|1|\rangle = 0$. Notice that if $q \in W(K)$, then $2q = \langle\langle -1\rangle\rangle q$, and in particular if -1 is a square in K then $2q = 0$ in $W(K)$. Also, if $\varphi \in \mathrm{Pf}_n(K)$, then $\varphi^2 = 2^n\varphi$, since $\langle\langle a, a\rangle\rangle = \langle\langle -1, a\rangle\rangle = 2\langle\langle a\rangle\rangle$. This relation is also true if $\varphi \in \widehat{\mathrm{Pf}}_n(K)$.

By a filtered group A we mean that there are subgroups $A^{\geqslant d}$ for all $d \in \mathbb{Z}$, such that $A^{\geqslant d+1} \subset A^{\geqslant d}$. We say the filtration is *positive* if $A^{\geqslant d} = A$ for all $d \leqslant 0$, and that it is *separated* if $\bigcap_d A^{\geqslant d} = 0$. If A is a ring, it is a filtered ring if

$$A^{\geqslant d} \cdot A^{\geqslant d'} \subset A^{\geqslant d+d'},$$

and M is a filtered A-module if it is a filtered group such that $A^{\geqslant d} \cdot M^{\geqslant d'} \subset M^{\geqslant d+d'}$. For any $n \in \mathbb{Z}$, we denote by $M[n]$ the filtered module such that $(M[n])^{\geqslant d} = M^{\geqslant d+n}$. A morphism of filtered modules $f : M \to N$ is a module morphism such that $f(M^{\geqslant d}) \subset N^{\geqslant d}$.

Let $\mathrm{Fields}_{/k}$ be the category of field extensions of k. If we are given functors $T : \mathrm{Fields}_{/k} \to \mathrm{Sets}$ and $A : \mathrm{Fields}_{/k} \to \mathrm{Ab}$ (the category of abelian groups), then an invariant of T with values in A (over k) is a natural transformation from T to A. The set of such invariants is naturally an abelian group, denoted $\mathrm{Inv}(T, A)$. If T takes values in *pointed* sets, then we can define *normalized* invariants as the ones that send the distinguished element to 0. This subgroup is denoted $\mathrm{Inv}_0(T, A)$, and we have $\mathrm{Inv}(T, A) = A(k) \oplus \mathrm{Inv}_0(T, A)$.

Since we want to unify proofs for Witt and cohomological invariants, we use $A(K)$ for either $W(K)$ or $H^*(K, \mu_2)$, writing $A = W$ or $A = H$ if we want to distinguish cases. For $d \in \mathbb{N}$, we set $A^{\geqslant d}(K) = I^d(K)$ if $A = W$, and $A^{\geqslant d}(K) = \bigoplus_{i \geqslant d} H^i(K, \mu_2)$ if $A = H$. Then $A(K)$ is a filtered $A(k)$-algebra, and the filtration is separated and positive. Note that according to the resolution of Minor's conjecture by Voevodsky et al., the graded ring associated to $A(K)$ is in both cases the mod 2 cohomology ring $H^*(K, \mu_2)$.

For any $n \in \mathbb{N}^*$, we write $M(n) = \mathrm{Inv}(I^n, A)$, and $M^{\geqslant d}(n) = \mathrm{Inv}(I^n, A^{\geqslant d})$ for all $d \in \mathbb{N}$. Similarly, the subgroups of normalized invariants are denoted $M_0(n)$ and $M_0^{\geqslant d}(n)$. Then $M(n)$ is a filtered $A(k)$-algebra, and $M_0(n)$ is a submodule.

We list here the formal properties of A on which the article relies. We have a group morphism $f_n : I^n(K) \to A^{\geqslant n}(K)$ (either the identity if $A = W$, or the morphism e_n given by the Milnor conjecture if $A = H$) and we write

$$\{a_1, \dots, a_n\} = f_n(\langle\langle a_1, \dots, a_n\rangle\rangle)$$

(so it is either a Pfister form or a Galois symbol, depending on A). Note that

$$f_n(x)\cdot f_m(y)=f_{n+m}(xy). \tag{0.1}$$

We set $\delta=\delta(A)=1$ if $A=W$, and $\delta=0$ if $A=H$. Then we have

$$\forall a,b\in K^*,\ \{ab\}=\{a\}+\{b\}-\delta\{a,b\} \tag{0.2}$$

and

$$\delta\{-1\}=2\in A(K). \tag{0.3}$$

We also freely use the following lemmas:

Lemma 0.4. *If $x\in A(K)$ is such that for any extension L/K and any $\varphi\in \mathrm{Pf}_n(L)$ we have $f_n(\varphi)\cdot x\in A^{\geqslant d+n}(L)$, then $x\in A^{\geqslant d}(K)$. In particular, for any $n\in\mathbb{N}^*$, if $f_n(\varphi)\cdot x=0$ for all $\varphi\in\mathrm{Pf}_n(L)$, then $x=0$.*

Lemma 0.5. $\mathrm{Inv}(\mathrm{Pf}_n, A)=A(K)\oplus A(K)\cdot f_n$, *where we consider invariants defined over K.*

The first lemma can be proved by specialization, taking φ to be a generic Pfister form; the second corresponds to two theorems of Serre [Garibaldi et al. 2003, Theorem 18.1, Example 27.17].

1. Some pre-λ-ring structures

We refer to [Yau 2010] for the basic theory of λ-rings. If R is a commutative ring, a pre-λ-ring structure on R is the data of applications $\lambda^d : R\to R$ for all $d\in\mathbb{N}$ such that for all $x, y\in R$,

(i) $\lambda^0(x)=1$;

(ii) $\lambda^1(x)=x$;

(iii) $\forall d\in\mathbb{N},\ \lambda^d(x+y)=\sum_{k=0}^{d}\lambda^k(x)\lambda^{d-k}(y)$.

Example 1.1. The example we are interested in is $R=\mathrm{GW}(K)$. The λ^d are the exterior powers of bilinear forms, as defined in [Bourbaki 1970], and it is shown in [McGarraghy 2002] that they define a λ-ring structure on GW(K) (which is a pre-λ-ring structure with additional conditions).

We define $\Lambda(R)=1+tR[[t]]$, the subset of formal power series with coefficients in R that have a constant coefficient equal to 1. It is a group for the multiplication of formal series. If we set $\lambda_t(x)=\sum_{d\in\mathbb{N}}\lambda^d(x)t^d\in R[[t]]$, we see that a pre-λ-ring structure on R is equivalent to the data of a group morphism $\lambda_t:(R,+)\to(\Lambda(R),\cdot)$ such that for all $x\in R$ the degree 1 coefficient of $\lambda_t(x)$ is x. We switch freely between those two descriptions.

Example 1.2. For the canonical λ-ring structure on GW(K), we have $\lambda_t(\langle a\rangle)=1+\langle a\rangle t$ for all $a\in K^*$.

Recall that for any formal series $f, g \in R[[t]]$ such that the constant coefficient of f is zero, we can define the composition $g \circ f \in R[[t]]$. If furthermore the degree 1 coefficient of f is invertible in R, then f has an inverse for the composition, which we denote $f^{\circ -1}$.

Lemma 1.3. *Let R be a commutative ring. If $\lambda_t : R \to \Lambda(R)$ defines a pre-λ-ring structure on R, then for any $f \in t + t^2 R[[t]]$, the map*

$$\lambda_{f(t)} : R \to \Lambda(R), \quad x \mapsto \lambda_t(x) \circ f = \sum_{d \in \mathbb{N}} \lambda^d(x) f(t)^d$$

also defines a pre-λ-ring structure.

Proof. We have for any $x, y \in R$

$$\lambda_t(x+y) \circ f = (\lambda_t(x)\lambda_t(y)) \circ f = (\lambda_t(x) \circ f) \cdot (\lambda_t(y) \circ f).$$

Furthermore, since the degree 1 term of $f(t)$ is t, the degree 1 coefficient of $\lambda_t(x) \circ f$ is the same as that of $\lambda_t(x)$, which is x. □

We want to define for each $n \in \mathbb{N}^*$ a pre-λ-ring structure on $\mathrm{GW}(K)$ that vanishes for $d \geqslant 2$ on n-fold Grothendieck–Pfister elements. Our starting point is the following fundamental observation.

Lemma 1.4. *Let $a \in K^*$. For any $d \geqslant 1$, we have $\lambda^d(\langle\langle a \rangle\rangle) = \langle\langle a \rangle\rangle$. Therefore,*

$$\lambda_t(\langle\langle a \rangle\rangle) = 1 + \langle\langle a \rangle\rangle \theta(t),$$

where $\theta(t) = \sum_{d \geqslant 1} t^d = t/(1-t)$.

Proof. We have

$$\lambda_t(1 - \langle a \rangle) = \frac{\lambda_t(1)}{\lambda_t(\langle a \rangle)} = \frac{1+t}{1 + \langle a \rangle t} = 1 + \sum_{d \geqslant 1} (1 - \langle a \rangle) t^d,$$

using $\langle a \rangle^2 = 1$. □

We then define some formal series: for any $n \in \mathbb{N}^*$, $x_n(t) \in \mathbb{Z}[[t]]$ is defined recursively by

$$x_1(t) = \theta(t) = \frac{t}{1-t}, \quad x_{n+1} = x_n + 2^{n-1} x_n^2,$$

and $h_n(t) \in \mathbb{Q}[[t]]$ by

$$h_n = x_n^{\circ -1}.$$

Lemma 1.5. *For any $n \in \mathbb{N}^*$, we have $h_n(t) \in \mathbb{Z}[[t]]$. Furthermore, if a_n and b_n are the even part and odd part of x_n, respectively, then*

$$\begin{cases} a_{n+1} = 2^n b_n^2 = 2a_n + 2^n a_n^2, \\ b_{n+1} = b_n + 2^n a_n b_n. \end{cases}$$

Proof. Note first that $h_1(t) = t/(1+t) \in \mathbb{Z}[\![t]\!]$. Let $p_n(t) = t + 2^{n-1}t^2 \in \mathbb{Z}[\![t]\!]$; then by definition $x_{n+1} = p_n \circ x_n$, so $h_{n+1} = h_n \circ p_n^{\circ -1}$. Now a simple computation yields $p_n^{\circ -1} = tC(-2^{n-1}t)$, where $C(t) = (1-\sqrt{1-4t})/2t$ is the generating function of the Catalan numbers (this is essentially equivalent to the well-known functional equation for $C(t)$); in particular, $p_n^{\circ -1}$ has integer coefficients, so $h_n(t) \in \mathbb{Z}[\![t]\!]$.

Separating even and odd parts, the recursive definition of x_n yields

$$\begin{cases} a_{n+1} = a_n + 2^{n-1}a_n^2 + 2^{n-1}b_n^2, \\ b_{n+1} = b_n + 2^n a_n b_n. \end{cases}$$

So we need to show that for any $n \in \mathbb{N}^*$, $a_n + 2^{n-1}a_n^2 = 2^{n-1}b_n^2$. If $n = 1$, this is a direct computation, using $a_1(t) = t^2/(1-t^2)$ and $b_1(t) = t/(1-t^2)$.

Now suppose the formula holds until $n \in \mathbb{N}^*$. Then

$$a_{n+1} + 2^n a_{n+1}^2 = 2^n b_n^2 + 2^n(2^n b_n^2)^2 = 2^n b_n^2(1+2^{2n}b_n^2),$$
$$2^n b_{n+1}^2 = 2^n b_n^2(1+2^n a_n)^2 = 2^n b_n^2(1+2^{n+1}a_n + 2^{2n}a_n^2) = 2^n b_n^2(1+2^{2n}b_n^2),$$

which shows the expected formula. □

We can now use those formal series to define our pre-λ-ring structures.

Theorem 1.6. *For any $n \in \mathbb{N}^*$, the map $(\pi_n)_t = \lambda_{h_n(t)}$ defines a pre-λ-ring structure on $\mathrm{GW}(K)$ such that $\pi_n^d(\varphi) = 0$ for any $\varphi \in \widehat{\mathrm{Pf}}_n(K)$ and any $d \geqslant 2$.*

Proof. According to Lemma 1.3, $(\pi_n)_t$ does define a pre-λ-ring structure on $\mathrm{GW}(K)$. We show the statement about Grothendieck–Pfister elements by induction on n. For $n = 1$, the statement is equivalent to Lemma 1.4 since for any $\varphi \in \widehat{\mathrm{Pf}}_1(K)$, $\lambda_t(\varphi) = 1 + \varphi x_1(t)$ and $h_1 = x_1^{\circ -1}$.

Suppose the statement holds until $n \in \mathbb{N}^*$. Let $\varphi \in \widehat{\mathrm{Pf}}_{n+1}(K)$, and write $\varphi = \langle\!\langle a \rangle\!\rangle \psi$ with $a \in K^*$ and $\psi \in \widehat{\mathrm{Pf}}_n(K)$. We then need to show $\lambda_{h_{n+1}(t)}(\varphi) = 1 + \varphi t$, which is equivalent to

$$\lambda_t(\langle\!\langle a \rangle\!\rangle \psi) = 1 + \langle\!\langle a \rangle\!\rangle \psi x_{n+1}(t).$$

Note that for any $x \in \hat{I}(K)$, we have $-\langle a \rangle x = \langle -a \rangle x$, which implies that $\lambda^d(-\langle a \rangle x) = (-1)^d \langle a^d \rangle \lambda^d(x)$ for any $d \in \mathbb{N}$, and thus $\lambda_t(-\langle a \rangle x) = \lambda_{-\langle a \rangle t}(x)$. Therefore, we have in $\mathrm{GW}[\![t]\!]$

$$\begin{aligned} \lambda_t(\psi - \langle a \rangle \psi) &= \lambda_t(\psi)\lambda_{-\langle a \rangle t}(\psi) \\ &= (1 + \psi x_n(t))(1 + \psi x_n(-\langle a \rangle t)) \\ &= 1 + \psi\big(x_n(t) + x_n(-\langle a \rangle t) + 2^n x_n(t) x_n(-\langle a \rangle t)\big). \end{aligned}$$

Thus we can conclude if we show that

$$x_n(t) + x_n(-\langle a \rangle t) + 2^n x_n(t) x_n(-\langle a \rangle t) = (1 - \langle a \rangle) x_{n+1}(t).$$

If we decompose in even and odd parts, this amounts to

$$\begin{cases} a_n(t)+a_n(t)+2^n(a_n(t)^2-\langle a\rangle b_n(t)^2)=(1-\langle a\rangle)a_{n+1}(t), \\ b_n(t)-\langle a\rangle b_n(t)+2^n(b_n(t)a_n(t)-\langle a\rangle a_n(t)b_n(t))=(1-\langle a\rangle)b_{n+1}(t), \end{cases}$$

which are consequences of Lemma 1.5. □

Remark 1.7. Those are *not* λ-ring structures; for instance, $(\pi_1)_t(1)=1+h_1(t)=1+t-t^2+\cdots$, so $\pi_1^d(1)\neq 0$ for all $d\geqslant 2$.

Corollary 1.8. *Let* $n\in\mathbb{N}^*$, *and* $\varphi_1,\dots,\varphi_r\in\widehat{\mathrm{Pf}}_n(K)$. *Then*

$$\pi_n^d\left(\sum_{i=1}^r \varphi_i\right)=\sum_{1\leqslant i_1<\cdots<i_d\leqslant r}\varphi_{i_1}\cdots\varphi_{i_d}.$$

In particular, $\pi_n^d(\hat{I}^n(K))\subset \hat{I}^{nd}(K)$, *and* π_n^d *is zero on forms that are sums of* $d-1$ (*or less*) n*-fold Grothendieck–Pfister elements.*

Proof. The formula is proved by an easy induction, exactly similar to the proof of the formula for exterior powers of diagonal quadratic forms (or more generally λ-powers of a sum of elements of dimension 1 in any pre-λ-ring). If $x\in\hat{I}^n(K)$, then $x=x_1-x_2$, where the x_i are sums of elements of $\widehat{\mathrm{Pf}}_n(K)$, and $(\pi_n)_t(x)=(\pi_n)_t(x_1)\cdot((\pi_n)_t(x_2))^{-1}$. Now it is easy to see that since the degree d coefficient of $(\pi_n)_t(x_i)$ is in $\hat{I}^{nd}(K)$, then the same is true for $(\pi_n)_t(x)$. □

Note that the formula in Corollary 1.8 is not enough to completely describe π_n^d on $\hat{I}^n(K)$, even if we could show directly that it is well-defined (which is possible using the presentation of $I^n(K)$ given in [Elman et al. 2008, Theorem 42.4]), since not every element of $I^n(K)$ is a sum of Pfister forms.

The idea of similar "divided power" operations on related structures such as Milnor K-theory of Galois cohomology has been around for some time (see Section 10 for more details).

2. The fundamental invariants

We now use these various pre-λ-ring structures on $\mathrm{GW}(K)$ to define some invariants of I^n.

Definition 2.1. Let $n\in\mathbb{N}^*$ and $d\in\mathbb{N}$. Then we define

$$f_n^d: I^n(K)\xrightarrow{\sim}\hat{I}^n(K)\xrightarrow{\pi_n^d}\hat{I}^{nd}(K)\xrightarrow{\sim} I^{nd}(K)\xrightarrow{f_{nd}} A^{\geqslant nd}(K).$$

If $A=W$, then we sometimes write $f_n^d=\bar{\pi}_n^d$.
If $A=H$, then we sometimes write $f_n^d=u_{nd}^{(n)}$.

This is well-defined according to Corollary 1.8. The notation $u_{nd}^{(n)}$ may seem dissonant with the rest, but we chose to stick with the tradition to write the degree of cohomological invariants in the index, and the exponent serves to distinguish between, for instance, $u_6^{(2)} : I^2(K) \to H^6(K,\mu_2)$ and $u_6^{(3)} : I^3(K) \to H^6(K,\mu_2)$, which are completely different ($u_6^{(3)}$ is *not* the restriction of $u_6^{(2)}$ to I^3).

Proposition 2.2. *Let $n \in \mathbb{N}^*$. Then for any $d \in \mathbb{N}$, we have $f_n^d \in M^{\geqslant nd}(n)$, and $(f_n^d)_{d\in\mathbb{N}}$ is the only family of elements of $M(n)$ such that*

(i) $f_n^0 = 1$ *and* $f_n^1 = f_n$;

(ii) *for all* $q, q' \in I^n(K)$,

$$f_n^d(q+q') = \sum_{k=0}^{d} f_n^k(q)\cdot f_n^{d-k}(q');$$

(iii) *for all* $\varphi \in \mathrm{Pf}_n(K)$ *and* $d \geqslant 2$, $f_n^d(\varphi) = 0$.

Furthermore, for any $\varphi \in \mathrm{Pf}_n(K)$ and any $d \in \mathbb{N}^*$,

$$f_n^d(-\varphi) = (-1)^d\{-1\}^{n(d-1)} f_n(\varphi). \tag{2.3}$$

Proof. The fact that f_n^d is an invariant is clear by construction: the definition of π_n^d is made in terms of the exterior powers, which are of course compatible with field extensions, and the expression of the π_n^d in terms of the λ^d is given by a universal $h_n \in \mathbb{Z}[[t]]$.

The three properties are direct consequences of Theorem 1.6, after applying f_{nd} to the corresponding formulas for π_n^d (and using formula (0.1)).

The last formula on opposites of Pfister forms can be easily proved by induction using

$$0 = f_n^d(\varphi-\varphi) = f_n^d(-\varphi) + f_n^{d-1}(-\varphi) f_n(\varphi).$$

Uniqueness follows from property (ii) and the fact that Pfister forms additively generate $I^n(K)$, since the values of f_n^d are fixed on $\pm\varphi$ for any $\varphi \in \mathrm{Pf}_n(K)$. □

The following corollary is an immediate consequence of either Corollary 1.8 or Proposition 2.2.

Corollary 2.4. *Let $n \in \mathbb{N}^*$ and $\varphi_1, \dots, \varphi_r \in \mathrm{Pf}_n(K)$. Then*

$$f_n^d\left(\sum_{i=1}^r \varphi_i\right) = \sum_{1\leqslant i_1<\cdots<i_d\leqslant r} f_n(\varphi_{i_1})\cdots f_n(\varphi_{i_d}).$$

In particular, f_n^d is zero on forms that are sums of $d-1$ or less n-fold Pfister forms.

3. The shifting operator

Since $I^n(K)$ is additively generated by the n-fold Pfister forms, it is natural to study how the invariants behave under adding or subtracting a Pfister form.

Proposition 3.1. *Let $n \in \mathbb{N}^*$ and $\varepsilon = \pm 1$. There is a unique morphism of filtered $A(k)$-modules $\Phi_n^\varepsilon : M(n) \to M(n)[-n]$ such that*

$$\alpha(q + \varepsilon\varphi) = \alpha(q) + \varepsilon f_n(\varphi) \cdot \Phi_n^\varepsilon(\alpha)(q)$$

for all $\alpha \in M(n)$, $q \in I^n(K)$ and $\varphi \in \mathrm{Pf}_n(K)$.

Proof. Let $\alpha \in M(n)$ and $q \in I^n(K)$. For any extension L/K and any $\varphi \in \mathrm{Pf}_n(L)$, we set

$$\beta_q(\varphi) = \alpha(q + \varepsilon\varphi).$$

Then $\beta_q \in \mathrm{Inv}(\mathrm{Pf}_n, A)$, defined over K. According to Lemma 0.5, there are uniquely determined $x_q, y_q \in A(K)$ such that $\beta_q = x_q + y_q \cdot f_n$.

Taking $\varphi = 0$ we see that $x_q = \alpha(q)$, and we then set $\Phi_n^\varepsilon(\alpha)(q) = \varepsilon y_q$, which gives the expected formula, as well as the uniqueness of Φ_n^ε.

By definition, Φ_n^ε is clearly an $A(k)$-module morphism, and it is of degree $-n$ because if $\alpha \in M^{\geqslant d}(n)$, then for any $q \in I^n(K)$, $f_n(\varphi) \cdot \alpha^\varepsilon(q) \in A^{\geqslant d}(L)$ for all $\varphi \in \mathrm{Pf}_n(L)$ and any extension L/K. Thus $\alpha^\varepsilon(q) \in A^{\geqslant d-n}(K)$ by Lemma 0.4. □

We often write $\Phi^+ = \Phi_n^{+1}$ and $\Phi^- = \Phi_n^{-1}$, as there is in practice no confusion to what n is in the context. We also write $\alpha^+ = \Phi^+(\alpha)$ and $\alpha^- = \Phi^-(\alpha)$ for any $\alpha \in M(n)$. These two operators have natural links between each other:

Proposition 3.2. *Let $n \in \mathbb{N}^*$. The operators Φ_n^+ and Φ_n^- commute, and furthermore, for any $\alpha \in M(n)$ we have*

$$\alpha^+ - \alpha^- = \{-1\}^n \alpha^{+-} = \{-1\}^n \alpha^{-+}.$$

Proof. Let $q \in I^n(K)$ and $\varphi, \psi \in \mathrm{Pf}_n(L)$. We have

$$\begin{aligned}\alpha(q + \varphi - \psi) &= \alpha(q + \varphi) - f_n(\psi)\alpha^-(q + \varphi) \\ &= \alpha(q) + f_n(\varphi)\alpha^+(q) - f_n(\psi)\alpha^-(q) - f_n(\varphi) f_n(\psi)\alpha^{-+}(q),\end{aligned}$$

but also

$$\begin{aligned}\alpha(q + \varphi - \psi) &= \alpha(q - \psi) + f_n(\varphi)\alpha^+(q - \psi) \\ &= \alpha(q) - f_n(\psi)\alpha^-(q) + f_n(\varphi)\alpha^+(q) - f_n(\varphi) f_n(\psi)\alpha^{+-}(q).\end{aligned}$$

Thus $f_n(\varphi) f_n(\psi)\alpha^{-+}(q) = f_n(\varphi) f_n(\psi)\alpha^{+-}(q)$, and since this holds for any φ, ψ over any extension, by Lemma 0.4 we find $\alpha^{+-} = \alpha^{-+}$.

If we now take $\varphi = \psi$, the above formula gives

$$f_n(\varphi)\alpha^+(q) - f_n(\varphi)\alpha^-(q) = f_n(\varphi) f_n(\varphi)\alpha^{+-}(q),$$

which gives the result, using $f_n(\varphi) f_n(\varphi) = \{-1\}^n f_n(\varphi)$ and again Lemma 0.4. □

In view of this proposition, we may write $\alpha^{r+,s-} \in M(n)$ for any $\alpha \in M(n)$ and $r, s \in \mathbb{N}$, defined as applying r times Φ^+ to α, and s times Φ^-, in any order.

We also call $\Phi = \Phi^+$ the *shifting operator*, as justified by the following elementary result.

Proposition 3.3. *Let $n \in \mathbb{N}^*$. For any $d \in \mathbb{N}$, $\Phi(f_n^{d+1}) = f_n^d$ (and $\Phi(f_n^0) = 0$).*

Proof. We need to show $f_n^{d+1}(q + \varphi) = f_n^{d+1}(q) + f_n(\varphi) \cdot f_n^d(q)$, which is an immediate consequence of Proposition 2.2. □

The action of Φ^- on the f_n^d is more complicated, reflecting the fact that f_n^d behaves very nicely with respect to sums of Pfister forms, but quite poorly for differences of those.

Proposition 3.4. *Let $n, d \in \mathbb{N}^*$. Then*

$$(f_n^d)^- = \sum_{k=0}^{d-1} (-1)^{d-k-1} \{-1\}^{n(d-k-1)} f_n^k.$$

Proof. Let $q \in I^n(K)$ and $\varphi \in \mathrm{Pf}_n(K)$. Then

$$f_n^d(q-\varphi) = \sum_{k=0}^{d} f_n^k(q) f_n^{d-k}(-\varphi) = f_n^d(q) + \sum_{k=0}^{d-1} (-1)^{d-k} \{-1\}^{n(d-k-1)} f_n(\varphi) f_n^k(q)$$

using formula (2.3). □

Apart from its action on the f_n^d, the main property of Φ_n^ε is the following:

Proposition 3.5. *Let $n \in \mathbb{N}^*$ and $\varepsilon = \pm 1$. The morphism Φ_n^ε induces for any $d \in \mathbb{N}$ an exact sequence*

$$0 \to A(k)/A^{\geqslant d+n}(k) \to M(n)/M^{\geqslant d+n}(n) \xrightarrow{\Phi_n^\varepsilon} M(n)/M^{\geqslant d}(n).$$

In particular, the kernel of Φ_n^ε is the submodule of constant invariants in $M(n)$.

Proof. If $\alpha, \beta \in M(n)$ are congruent modulo $M^{\geqslant d+n}(n)$, then since $\Phi^\varepsilon(M^{\geqslant d+n}(n))$ is included in $M^{\geqslant d}(n)$, α^ε and β^ε are congruent modulo $M^{\geqslant d}(n)$.

Let $\alpha \in M(n)$ be such that $\alpha^\varepsilon \in M^{\geqslant d}(n)$. Then for any $q \in I^n(K)$ and any $\varphi \in \mathrm{Pf}_n(K)$, we have $\alpha(q + \varepsilon\varphi) \equiv \alpha(q)$ modulo $A^{\geqslant n+d}(K)$, and also by symmetry $\alpha(q - \varepsilon\varphi) \equiv \alpha(q)$. Since we can always write $q = q_1 - q_2$, where the q_i are sums of n-fold Pfister forms, then by simple induction on the lengths of the sums, $\alpha(q) \equiv \alpha(0)$ modulo $A^{\geqslant n+d}(K)$ (where $\alpha(0)$ is seen as a constant invariant).

Taking a large enough d, and since the filtration on $A(K)$ is separated, we see that $\alpha^\varepsilon = 0$ implies $\alpha = \alpha(0)$. □

Corollary 3.6. *Let $n \in \mathbb{N}^*$ and let $\varepsilon = \pm 1$. If $M'(n)$ is the submodule of $M(n)$ generated by the f_n^d for $d \in \mathbb{N}$, then Φ_n^ε induces an exact sequence of filtered $A(k)$-modules*

$$0 \to A(k) \to M'(n) \xrightarrow{\Phi_n^\varepsilon} M'(n)[-n] \to 0.$$

Proof. The only thing left to check is surjectivity, but this is easily implied by Propositions 3.3 for Φ^+ and 3.4 for Φ^-. □

Remark 3.7. All this implies that Φ may be seen as some kind of differential operator: if we know α^+ for some invariant α, we may "integrate" to find α, with a certain integration constant. Precisely, if $\alpha^+ = \sum a_d f_n^d$, then $\alpha = \alpha(0) + \sum a_d f_n^{d+1}$ (and we show in the next section that such a decomposition always holds). We use this method extensively to compute some invariants α by "induction on shifting".

4. Classification of invariants

The main goal of this article, and this section, is to show that any $\alpha \in M(n)$ can be expressed uniquely as a combination $\sum_d a_d f_n^d$. The next proposition gives the first step:

Proposition 4.1. *Let $n \in \mathbb{N}^*$ and $d \in \mathbb{N}$. The $A(k)/A^{\geqslant d}(k)$-module $M(n)/M^{\geqslant d}(n)$ is generated by the f_n^k with $nk < d$.*

Proof. We use induction on d. For $d = 0$, this is trivial since $M^{\geqslant 0}(n) = M(n)$. Suppose the property holds up to $d-1$, and let $\alpha \in M(n)$; we write $\bar{\alpha} \in M(n)/M^{\geqslant d}(n)$ for its residue class. By induction, $\Phi(\bar{\alpha}) = \sum a_k f_n^k$ with $nk < d - n$, so if we set $\beta = \alpha - \sum a_k f_n^{k+1}$ we get $\Phi(\bar{\beta}) = 0$. From there, β is congruent modulo $M^{\geqslant d}(n)$ to a constant invariant a_{-1}, hence $\bar{\alpha} = \sum a_{k-1} f_n^k$ with $nk < d$. □

The problem is that to express an invariant in terms of the f_n^d, it is in general necessary to use an infinite combination, as the following example illustrates.

Example 4.2. Consider the case $A = W$. Let $\alpha(q) = \langle \operatorname{disc}(q) \rangle$; it is a Witt invariant of I. Then $\alpha^+ = -\alpha$; indeed,

$$\langle \operatorname{disc}(q + \langle\langle a \rangle\rangle) \rangle = \langle \operatorname{disc}(q)a \rangle = \langle \operatorname{disc}(q) \rangle - \langle\langle a \rangle\rangle \langle \operatorname{disc}(q) \rangle.$$

Thus α cannot be written as a finite combination of the f_1^d (since the length of such a combination strictly decreases when applying Φ^+). On the other hand, we may write it (at least formally for now) as

$$\alpha = \sum_{d \in \mathbb{N}} (-1)^d f_1^d.$$

But such an infinite combination may not always be well-defined: since the f_n^d take values in $A^{\geqslant m}$ for increasing values of m, any $\sum_{d \in \mathbb{N}} a_d f_n^d$ is well-defined as an invariant with values in the completion of A with respect to its filtration, but usually not in A itself, as the next example shows.

Example 4.3. If k is formally real, then $\sum_d f_1^d$ sends $-\langle\langle -1 \rangle\rangle$ to $\sum_{d \in \mathbb{N}} (-1)^d \{-1\}^d$, which is not in $A(k)$ (but is in its completion).

It readily appears that the trouble is the bad behaviour of the f_n^d with respect to the opposites of Pfister forms. To get a satisfying description of $M(n)$, we introduce a new "basis", with better balance between sums and differences of Pfister forms, such that any infinite combination does take values in A.

Definition 4.4. Let $n \in \mathbb{N}^*$. For any $d \in \mathbb{N}$, we define $g_n^d \in M^{\geqslant nd}(n)$ by

- $g_n^0 = 1$;
- if $d \in \mathbb{N}^*$ is odd, $(g_n^d)^- = g_n^{d-1}$ and $g_n^d(0) = 0$;
- if $d \in \mathbb{N}^*$ is even, $(g_n^d)^+ = g_n^{d-1}$ and $g_n^d(0) = 0$.

If $A = W$ (resp. $A = H$), we sometimes write γ_n^d (resp. $\upsilon_{nd}^{(n)}$) for g_n^d.

Corollary 3.6 ensures that these are well-defined. This definition, which balances Φ^+ and Φ^-, gives a reasonable behaviour under both operators:

Proposition 4.5. *Let $n \in \mathbb{N}^*$ and $d \in \mathbb{N}$. Then*

$$(g_n^{d+2})^{+-} = (g_n^{d+2})^{-+} = g_n^d;$$

$$(g_n^{d+1})^+ = \begin{cases} g_n^d & \text{if } d \text{ is odd,} \\ g_n^d + \{-1\}^n g_n^{d-1} & \text{if } d \text{ is even;} \end{cases}$$

$$(g_n^{d+1})^- = \begin{cases} g_n^d & \text{if } d \text{ is even,} \\ g_n^d - \{-1\}^n g_n^{d-1} & \text{if } d \text{ is odd.} \end{cases}$$

Proof. If d is even, then $(g_n^{d+2})^- = g_n^{d+1}$ and $(g_n^{d+1})^+ = g_n^d$, and if d is odd, $(g_n^{d+2})^+ = g_n^{d+1}$ and $(g_n^{d+1})^- = g_n^d$. In any case the first formula is satisfied.

For the remaining two, we use $(g_n^{d+1})^+ - (g_n^{d+1})^- = \{-1\}^n g_n^{d-1}$ coming from Proposition 3.2. We may conclude, arguing according to the parity of d. □

We can now write the precise relation between f_n^d and g_n^d:

Proposition 4.6. *Let $n \in \mathbb{N}^*$. For any $d \in \mathbb{N}^*$,*

$$g_n^d = \sum_{k=\lfloor d/2 \rfloor + 1}^{d} \binom{\left\lfloor \frac{d-1}{2} \right\rfloor}{k - \left\lfloor \frac{d}{2} \right\rfloor - 1} \{-1\}^{n(d-k)} f_n^k,$$

$$f_n^d = \sum_{k=1}^{d} (-1)^{d-k} \binom{d - \left\lfloor \frac{k+1}{2} \right\rfloor - 1}{\left\lfloor \frac{k}{2} \right\rfloor - 1} \{-1\}^{n(d-k)} g_n^k.$$

In particular, $(f_n^i)_{i \leqslant d}$ and $(g_n^i)_{i \leqslant d}$ generate the same submodule of $M(n)$.

Proof. Denote by α_d the invariant defined by the right-hand side of the formula for g_n^d. If $d = 2m$, the formula becomes

$$\alpha_d = \sum_{k=m+1}^{2m} \binom{m-1}{k-m-1} \{-1\}^{n(2m-k)} f_n^k,$$

which gives

$$\alpha_d^+ = \sum_{k=m+1}^{2m} \binom{m-1}{k-m-1} \{-1\}^{n(2m-k)} f_n^{k-1},$$

and if $d = 2m+1$ then we get

$$\alpha_d = \sum_{k=m+1}^{2m+1} \binom{m}{k-m-1} \{-1\}^{n(2m+1-k)} f_n^k.$$

Hence

$$\alpha_d^+ = \sum_{k=m+1}^{2m+1} \binom{m}{k-m-1} \{-1\}^{2m+1-k} f_n^{k-1}.$$

We thus have to check that in both cases we find the correct induction formula for α_{d+1}^+ (coming from Proposition 4.5). If $d = 2m+1$, we have to show $\alpha_{2m+2}^+ = \alpha_{2m+1}$, which is immediate given the above formulas. If $d = 2m$, we have to show $\alpha_{2m+1}^+ = \alpha_{2m} + \{-1\}^n \alpha_{2m-1}$, so we need to compare

$$\sum_{k=m}^{2m} \binom{m}{k-m} \{-1\}^{n(2m-k)} f_n^k$$

and

$$\sum_{k=m+1}^{2m} \binom{m-1}{k-m-1} \{-1\}^{n(2m-k)} f_n^k + \sum_{k=m}^{2m-1} \binom{m-1}{k-m} \{-1\}^{n(2m-k)} f_n^k,$$

which are easily seen as being equal using Pascal's triangle.

The formula for f_n^d can be obtained either in a similar fashion, or by inverting the one for g_n^d. Let β_d be the invariant defined by the right-hand side. Then we show that $\beta_d^+ = \beta_{d-1}$, separating the sums according to the parity of k:

$$\begin{aligned}
\beta_d^+ = (-1)^d \sum_m \binom{d-m-1}{m-1} \{-1\}^{n(d-2m)} (g_n^{2m})^+ \\
+ (-1)^{d+1} \sum_m \binom{d-m-2}{m-1} \{-1\}^{n(d-2m-1)} (g_n^{2m+1})^+ \\
= (-1)^d \sum_m \binom{d-m-1}{m-1} \{-1\}^{n(d-2m)} g_n^{2m-1} \\
+ (-1)^{d+1} \sum_m \binom{d-m-2}{m-1} \{-1\}^{n(d-2m-1)} (g_n^{2m} + \{-1\}^n g_n^{2m-1})
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{d+1}\sum_m \binom{d-m-2}{m-1}\{-1\}^{n(d-2m-1)} g_n^{2m} \\
&\qquad + (-1)^d \sum_m \left(\binom{d-m-1}{m-1} - \binom{d-m-2}{m-1}\right)\{-1\}^{n(d-2m)} g_n^{2m-1} \\
&= (-1)^{d-1}\sum_m \binom{d-1-m-1}{m-1}\{-1\}^{n(d-1-2m)} g_n^{2m} \\
&\qquad + (-1)^{d-1+1}\sum_m \binom{d-m-1}{m-2}\{-1\}^{n(d-2m)} g_n^{2m-1},
\end{aligned}
$$

which does give α_{d-1}.

The last statement comes from the fact that the transition matrix from $(f_n^d)_d$ to $(g_n^d)_d$ is triangular unipotent. □

The next proposition gives an important consequence of the balance of g_n^d.

Proposition 4.7. *Let $n \in \mathbb{N}^*$, and let $q \in I^n(K)$ be such that $q = \sum_{i=1}^s \varphi_i - \sum_{i=1}^t \psi_i$, where $\varphi_i, \psi_i \in \mathrm{Pf}_n(K)$. Then for any $d > 2\max(s,t)$, we have $g_n^d(q) = 0$.*

Proof. We may add hyperbolic forms in either sum so that $s = t$. Then we prove the statement by induction on s: if $s = 0$ then $q = 0$, so for $d > 0$ we have indeed $g_n^d(q) = 0$ by construction.

If the result holds up to $s-1$ for some $s \in \mathbb{N}^*$, then write $q' = q - \varphi_s$ and $q'' = q' + \psi_s$. We get

$$
\begin{aligned}
g_n^d(q) &= g_n^d(q') + f_n(\varphi_s)(g_n^d)^+(q') \\
&= g_n^d(q'') - f_n(\psi_s)(g_n^d)^-(q'') + f_n(\varphi_s)(g_n^d)^+(q'') - f_n(\varphi_s) f_n(\psi_s)(g_n^d)^{+-}(q'').
\end{aligned}
$$

Now according to Proposition 4.5, $(g_n^d)^-$, $(g_n^d)^+$ and $(g_n^d)^{+-}$ may all be expressed as combinations of some g_n^k with $k \geqslant d-2$, so we may apply the induction hypothesis with q''. □

Corollary 4.8. *If $q \in I(K)$ is the Witt class of an r-dimensional form, then $g_1^d(q) = 0$ for any $d > r$.*

Proof. Writing $r = 2m$, if $q = \langle a_1, b_1, \ldots, a_m, b_m\rangle$, then $q = \sum_{i=1}^m \langle\langle -a_i\rangle\rangle - \langle\langle b_i\rangle\rangle$, which allows us to conclude using the previous proposition. □

We may now put it all together to prove the central theorem:

Theorem 4.9. *Let $n \in \mathbb{N}^*$, and let $N(n) = A(k)^{\mathbb{N}}$, which is a filtered $A(k)$-module for the filtration $N^{\geqslant m}(n) = \{(a_d)_{d\in\mathbb{N}} \mid a_d \in A^{\geqslant m-nd}\}$. The following applications are mutually inverse isomorphisms of filtered $A(k)$-modules:*

$$
\begin{aligned}
F &: N(n) \xrightarrow{\sim} M(n), \quad (a_d)_{d\in\mathbb{N}} \mapsto \textstyle\sum_{d\in\mathbb{N}} a_d g_n^d, \\
G &: M(n) \xrightarrow{\sim} N(n), \quad \alpha \mapsto (\alpha^{[d]}(0))_{d\in\mathbb{N}},
\end{aligned}
$$

where $\alpha^{[d]} = \alpha^{m+,m-}$ if $d = 2m$, and $\alpha^{[d]} = \alpha^{(m+1)+,m-}$ if $d = 2m+1$.

Proof. First, the application F is well-defined, since according to Proposition 4.7, for any fixed $q \in I^n(K)$ we have $g_n^d(q) = 0$ for large enough d. Then F and G are clearly module morphisms, and the fact that they respect the filtrations is just a reformulation of the fact that g_n^d takes values in $A^{\geqslant nd}$, and that Φ_n^ε has degree $-n$. Let $\alpha = \sum_d a_d g_n^d$. Using Proposition 3.5, we see that for any $r, s \in \mathbb{N}$, we can ignore the terms for d large enough when we compute $\alpha^{r+,s-}(0)$. Thus it is easy to see from Proposition 4.5 that $a_{2m} = \alpha^{m+,m-}(0)$ and $a_{2m+1} = \alpha^{(m+1)+,m-}(0)$, which shows that $G \circ F = \mathrm{Id}$.

We now prove that G is injective, which finishes the proof of the theorem. Let $\alpha \in \mathrm{Ker}(G)$, and let $d \in \mathbb{N}$. According to Proposition 4.1, and using the last statement of Proposition 4.6, we see that α is congruent to some combination $\sum_{nk<d} a_k g_n^k$ modulo $M^{\geqslant d}(n)$. Now the exact sequence in Proposition 3.5 shows that $a_k \equiv \alpha^{[k]}(0)$ modulo $A^{\geqslant d-nk}(k)$, so since $\alpha^{[k]}(0) = 0$, $a_k \in A^{\geqslant d-nk}(k)$. This in turn implies that $\sum_{nk\leqslant d} a_k g_n^k \in M^{\geqslant d}(n)$, and thus $\alpha \in M^{\geqslant d}(n)$. Since this is true for any $d \in \mathbb{N}$, we may conclude that $\alpha = 0$. □

Corollary 4.10. *Let $n \in \mathbb{N}^*$ and let $\varepsilon = \pm 1$. There is an exact sequence of filtered $A(k)$-modules*

$$0 \to A(k) \to M(n) \xrightarrow{\Phi_n^\varepsilon} M(n)[-n] \to 0.$$

Proof. Like for Corollary 3.6, the only thing left to prove after Proposition 3.5 is the surjectivity of Φ_n^ε, but it is an easy consequence of Theorem 4.9. □

Corollary 4.11. *Let $n \in \mathbb{N}^*$ and $\alpha \in M(n)$. There is a unique sequence $(a_d)_{d\in\mathbb{N}}$ with $a_d \in A(k)$ such that for any $q \in I^n(K)$, the infinite sum $\sum_{d\in\mathbb{N}} a_d f_n^d(q)$ exists in $A(K)$ and is equal to $\alpha(q)$. Furthermore, for all $d \in \mathbb{N}$, we have $a_d = \alpha^{d+}(0)$.*

Proof. If such a sequence exists, then using Proposition 3.5 we find that $\alpha^{i+}(0) \equiv a_i$ modulo $A^{\geqslant dn}(k)$ for all $i \leqslant d$, so for a fixed i we can make d go to infinity, and we find that indeed $a_d = \alpha^{d+}(0)$, which shows uniqueness.

For existence, write $\alpha = \sum_d b_d g_n^d$, and decompose each g_n^d in terms of the f_n^i using Proposition 4.6. Then we find a decomposition of α in terms of f_n^d which is valid pointwise, and the a_d we find are well-defined in $A(k)$ since each a_d is a combination of a finite number of b_i (using that f_n^i appears in the decomposition of g_n^d only if $d \leqslant 2i$). □

Remark 4.12. In particular, any invariant of I^n with values in $H^d(-, \mu_2)$ may be lifted to an invariant with values in I^d.

Remark 4.13. If k is not a formally real field, then for large enough d we have $\{-1\}^d = 0$, and thus according to formula (2.3), $f_n^d(-\varphi) = 0$ for any $\varphi \in \mathrm{Pf}_n(K)$. This implies that in this case, for any $q \in I^n(K)$ we have $f_n^d(q) = 0$ for large enough d (for the same reasons as in Corollary 2.4), and so we may use the f_n^d instead of the g_n^d in the theorem (with $G(\alpha) = (\alpha^{d+}(0))_d$). In the extreme case where -1 is a square in k, we actually even get $f_n^d = g_n^d$, as can be seen from Proposition 4.6. On the other hand, Example 4.3 shows that we cannot use the f_n^d if k is formally real. What happens in this case is that an arbitrary infinite combination of the f_n^d does correspond to a combination of the g_n^d (using Proposition 4.6), but with coefficients in the completion of A with respect to its filtration.

Remark 4.14. We may construct cohomological invariants α such that, even though the degree of $\alpha(q)$ is bounded for fixed q, it is unbounded when q varies (for instance, $\alpha = \sum_d g_n^d$). This reflects in some sense the "infinite" nature of I^n, and it is a behaviour that does not appear for invariants of algebraic groups. The submodule $M'(n)$ of uniformly bounded cohomological invariant is the submodule generated by the f_n^d (or by the g_n^d). We may write that $M(n) = \mathrm{Inv}(I^n, \varinjlim H^{\leqslant d}(-, \mu_2))$, while $M'(n) = \varinjlim \mathrm{Inv}(I^n, H^{\leqslant d}(-, \mu_2))$.

5. Algebra structure

Since $M(n)$ is not only an $A(k)$-module, but also an algebra, we wish to understand how the product can be expressed in terms of the basic elements f_n^d.

For this section, if $d, p, q \in \mathbb{N}$ are such that $p + q \leqslant d$, we set

$$C_{p,q}^d = \frac{d!}{p! \cdot q! \cdot (d-p-q)!}.$$

This is just a more compact notation for the usual multinomial

$$\binom{d}{p,\, q,\, d-p-q}.$$

Proposition 5.1. *Let $n \in \mathbb{N}^*$, and $\varepsilon = \pm 1$. Then for any $\alpha, \beta \in M(n)$,*

$$\Phi^\varepsilon(\alpha\beta) = \Phi^\varepsilon(\alpha)\beta + \alpha\Phi^\varepsilon(\beta) + \varepsilon\{-1\}^n \Phi^\varepsilon(\alpha)\Phi^\varepsilon(\beta).$$

Proof. Let $q \in I^n(K)$ and $\varphi \in \mathrm{Pf}_n(K)$. Then

$$\begin{aligned}(\alpha\beta)(q+\varepsilon\varphi) &= (\alpha(q)+\varepsilon f_n(\varphi)\alpha^\varepsilon(q))\cdot(\beta(q)+\varepsilon f_n(\varphi)\beta^\varepsilon(q)) \\ &= (\alpha\beta)(q)+\varepsilon f_n(\varphi)\big((\alpha^\varepsilon\beta)(q)+(\alpha\beta^\varepsilon)(q)+\varepsilon\{-1\}^n(\alpha^\varepsilon\beta^\varepsilon)(q)\big). \quad \square\end{aligned}$$

Proposition 5.2. *Let $n \in \mathbb{N}^*$ and $s, t \in \mathbb{N}$. Then*

$$f_n^s \cdot f_n^t = \sum_{d=\max(s,t)}^{s+t} C_{d-s,d-t}^d \{-1\}^{n(s+t-d)} f_n^d.$$

Proof. First note that both sides of the equality have the same value in 0 (which is 1 if $s = t = 0$ and 0 otherwise). So we just need to show that applying Φ to both sides of the equation gives the same expression.

Now Proposition 5.1 gives

$$\Phi(f_n^s \cdot f_n^t) = f_n^s \cdot f_n^{t-1} + f_n^{s-1} \cdot f_n^t + \{-1\}^n f_n^{s-1} \cdot f_n^{t-1}. \tag{5.3}$$

We proceed by induction, say on (s, t) with lexicographical order. First the result is clear if $s = 0$ or $t = 0$. Then by induction we can replace each term in (5.3) and rearrange them to find, for $\Phi(f_n^s \cdot f_n^t)$,

$$\binom{s}{t}\{-1\}^{nt} f_n^{s-1} + \binom{s+t}{t} f_n^{s+t-1} + \sum_{d=s}^{s+t-2} C^{d+1}_{d-s+1,d-t+1}\{-1\}^{n(s+t-d-1)} f_n^d, \tag{5.4}$$

where for the coefficient before f_n^{s-1} we use $\binom{s-1}{t} + \binom{s-1}{t-1} = \binom{s}{t}$, for that of f_n^{s+t-1} we use $\binom{s+t-1}{t} + \binom{s+t-1}{t-1} = \binom{s+t}{t}$, and for the other terms we use

$$C^d_{d-s+1,d-t} + C^d_{d-s,d-t+1} + C^d_{d-s+1,d-t+1} = C^{d+1}_{d-s+1,d-t+1}.$$

We can then compute that applying Φ to the right-hand side of the equality in the statement of the proposition yields exactly (5.4). □

Of course there is a corresponding formula for the products of the g_n^d, but it turns out that it is much more involved, and we do not address it here. This means that although we have a nice module isomorphism between $M(n)$ and $A(k)^{\mathbb{N}}$, transporting the algebra structure of $M(n)$ to $A(k)^{\mathbb{N}}$ is not as convenient. On the other hand, if we use the f_n^d we only have a module isomorphism between $M(n)$ and a submodule of $A(k)^{\mathbb{N}}$, which is hard to describe, but we can transport the product in a reasonably easy way.

There are several cases where the formula of Proposition 5.2 can be greatly simplified by studying the parity of the multinomials that appear. We introduce some notation: if $s, t \in \mathbb{N}$, we write $s \vee t$ (resp. $s \wedge t$) for the integer obtained by applying a bitwise *or* (resp. a bitwise *and*) to the binary representations of s and t. In particular, $s \vee t + s \wedge t = s + t$.

Lemma 5.5. *Let $d, s, t \in \mathbb{N}$ be such that $\max(s, t) \leqslant d \leqslant s + t$. Then $C^d_{d-s,d-t}$ is odd if and only if $d = s \vee t$.*

Proof. It is well-known that for any $a \in \mathbb{N}$, the 2-adic valuation of $a!$ is $a - f(a)$, where $f(a)$ is the number of 1's in the binary representation of a. Then

$$\begin{aligned} &v_2(C^d_{d-s,d-t}) \\ &\quad = (d-f(d))-(s+t-d-f(s+t-d))-(d-s-f(d-s))-(d-t-f(d-t)) \\ &\quad = f(s+t-d)+f(d-s)+f(d-t)-f(d). \end{aligned}$$

But it is easily seen that for any $a, b \in \mathbb{N}$, $f(a+b) \leqslant f(a) + f(b)$, with equality if and only if $a \wedge b = 0$. Thus $C^d_{d-s,d-t}$ is odd if and only if $s + t - d$, $d - s$ and $d - t$ have pairwise disjoint binary representations.

We claim this is equivalent to $d = s \vee t$. Indeed, if $d = s \vee t$ it is obvious, and if $d \neq s \vee t$, consider the weakest bit where d and $s \vee t$ differ; there are several possibilities for the bits of s, t and d in this slot: s has 1 and d has 0, t has 1 and d has 0, or s and t have 0 and d has 1. In all these cases, at least two numbers among $d - s$, $d - t$ and $s + t - d$ have a 1 in this slot, and their binary representations are thus not disjoint. □

Then we can state the following:

Corollary 5.6. *Let $n \in \mathbb{N}^*$ and $s, t \in \mathbb{N}$. If $A = H$, then*

$$u_{ns}^{(n)} \cup u_{nt}^{(n)} = (-1)^{n(s\wedge t)} \cup u_{n(s\vee t)}^{(n)}.$$

Proof. Since $H^*(k, \mu_2)$ is a ring of characteristic 2, by Lemma 5.5 the only potentially nonzero term in the formula of Proposition 5.2 is $\{-1\}^{s\wedge t} f_n^{s\vee t}$. □

Remark 5.7. This is very reminiscent of the formula for the product of Stiefel–Whitney classes, since $w_s \cup w_t = (-1)^{s\wedge t} \cup w_{s\vee t}$. When -1 is a square, this is easily explained by the fact that $u_d^{(1)}$ coincides with the Stiefel–Whitney map w_d (see Remark 9.10), but in general w_d is not well-defined on Witt classes so the formulas are really different phenomena.

Corollary 5.8. *Let $n \in \mathbb{N}^*$ and $s, t \in \mathbb{N}$. If -1 is a square in k, then $f_n^s \cdot f_n^t$ equals f_n^{s+t} if $s \wedge t = 0$, and 0 otherwise.*

Proof. Note that in this situation $A(k)$ is also a ring of characteristic 2, so the same reasoning as in Corollary 5.6 applies, but this time if $s \wedge t \neq 0$ the term is also 0. □

Remark 5.9. Consider the case $A = H$, and the submodule $M'(n) \subset M(n)$ generated by the $u_{nd}^{(n)}$, which is the subalgebra of cohomological invariants with uniformly bounded degree. Then from Corollary 5.6 we find a very simple algebra presentation of $M'(n)$: the (commuting) generators are $x_i = u_{n2^i}^{(n)}$, and the relations are given by $x_i^2 = \{-1\}^{n2^i} x_i$.

6. Restriction from I^n to I^{n+1}

For any $m, n \in \mathbb{N}^*$ with $m \geqslant n$, there is an obvious restriction morphism

$$\rho_{n,m} : M(n) \to M(m), \quad \alpha \mapsto \alpha_{|I^m}. \tag{6.1}$$

Given the definition of f_n^d, if we want to express $(f_n^d)_{|I^{n+1}}$ in terms of the f_{n+1}^k, it is natural to try to express π_n^d in terms of the π_{n+1}^k in $\mathrm{GW}(K)$.

Proposition 6.2. *Let $n \in \mathbb{N}^*$. For any $d \in \mathbb{N}^*$, we have*

$$\pi_n^d = \sum_{d/2 \leqslant k \leqslant d} \binom{k}{d-k} 2^{(d-k)(n-1)} \pi_{n+1}^k.$$

Proof. We define $p_n(t) = t + 2^{n-1}t^2 \in \mathbb{Z}[t]$. Then recall that $(\pi_n)_t = \lambda_{h_n(t)}$, where $h_n = x_n^{\circ -1}$, and x_n is defined recursively by $x_{n+1} = p_n \circ x_n$. Thus we have the formula $h_n = h_{n+1} \circ p_n$, and

$$(\pi_n)_t = (\pi_{n+1})_{p_n(t)}.$$

Therefore we find

$$\sum_d \pi_n^d \cdot t^d = \sum_k \pi_{n+1}^k (t + 2^{n-1}t^2)^k = \sum_k \sum_{k \leqslant d \leqslant 2k} \binom{k}{d-k} 2^{(d-k)(n-1)} \pi_{n+1}^k \cdot t^d,$$

which gives the result. □

Then we deduce the corresponding results for our invariants.

Corollary 6.3. *Let $n, d \in \mathbb{N}^*$. If $A = W$ then*

$$(\bar{\pi}_n^d)_{|I^{n+1}} = \sum_{d/2 \leqslant k \leqslant d} \binom{k}{d-k} \langle\langle -1 \rangle\rangle^{(d-k)(n-1)} \bar{\pi}_{n+1}^k.$$

Proof. This is an immediate consequence of the proposition, given that in $W(K)$ we have $\langle\langle -1 \rangle\rangle = 2$. □

Corollary 6.4. *Let $n, d \in \mathbb{N}^*$. If $A = H$ then*

$$(u_{nd}^{(n)})_{|I^{n+1}} = \begin{cases} (-1)^{m(n-1)} \cup u_{(n+1)m}^{(n+1)} & \text{if } d = 2m, \\ 0 & \text{if } d \text{ is odd.} \end{cases}$$

Proof. This is also a consequence of the proposition, but we have to notice that when we apply e_{nd} to the formula, the terms corresponding to $k > d/2$ vanish. Indeed, in this case $\langle\langle -1 \rangle\rangle^{(d-k)(n-1)} \pi_{n+1}^k$ sends $\hat{I}^{n+1}(K)$ to $\hat{I}^r(K)$ with

$$r = (d-k)(n-1) + k(n+1) = d(n-1) + 2k > nd.$$

Thus, composing with e_{nd} gives zero.

So only the term $k = d/2$ remains (and only when d is even). □

Remark 6.5. In particular, for cohomological invariants, and when $n = 1$, we get the simple formula $(u_{2d}^{(1)})_{|I^2} = u_{2d}^{(2)}$, which shows that any cohomological invariant of I^2 extends (not uniquely) to I. On the other hand, for $n \geqslant 3$ and $d \geqslant 1$, $u_{nd}^{(n)}$ never extends to I^{n-1}. This vastly generalizes the familiar facts that e_2 extends to I, but e_3 does not extend to I^2.

Remark 6.6. Suppose -1 is a square in k, and take $n \geqslant 2$. Then in the case of Witt invariants, $\bar{\pi}_n^d$ is independent of n, and in the case of cohomological invariants the restriction of any $\alpha \in M(n)$ to I^{n+1} is constant.

As an application of Corollary 6.4, we may improve a result of Kahn [2005]: he shows in the proof of Proposition 3.3 that if $H^r(K, \mu_2)$ has symbol length at most $l \in \mathbb{N}$, then any element of $H^{r(l+1)}(K, \mu_2)$ is a multiple of $(-1) \in H^1(K, \mu_2)$. We would like to thank Karim Becher for fruitful discussions about this application during a visit in Antwerp.

Proposition 6.7. *Let $r \in \mathbb{N}^*$, and assume that $H^r(K, \mu_2)$ has symbol length at most $l \in \mathbb{N}$. Then for any $d > l$, we have*

$$H^{rd}(K, \mu_2) \subset (-1)^{(r-1)\lceil (d-l)/2 \rceil} \cup H^*(K, \mu_2).$$

In particular, any element of $H^m(K, \mu_2)$ for $m \geqslant r(l+1)$ is a multiple of

$$(-1)^{r-1} \in H^{r-1}(K, \mu_2).$$

Proof. It is enough to prove the result for Galois symbols: let $\alpha \in H^{rd}(K, \mu_2)$ be a symbol, and write $\alpha = \alpha_1 \cup \cdots \cup \alpha_d$ with $\alpha_i \in H^r(K, \mu_2)$. Then we set $\varphi_i \in \mathrm{Pf}_r(K)$ such that $e_r(\varphi_i) = \alpha_i$, and $q = \sum_i \varphi_i \in I^r(K)$. According to Corollary 2.4, we have $\alpha = u_{rd}^{(r)}(q)$.

Now by hypothesis, $q = q' + x$, where $q' \in I^r(K)$ can be written as a sum of l or less r-fold Pfister forms, and $x \in I^{r+1}(K)$. We have

$$\alpha = u_{rd}^{(r)}(q' + x) = \sum_{k=0}^{rd} u_{rk}^{(r)}(q') \cup u_{r(d-k)}^{(r)}(x). \tag{6.8}$$

But Corollary 2.4 shows that $u_{rk}^{(r)}(q') = 0$ when $k > l$, and Corollary 6.4 shows that $u_{r(d-k)}^{(r)}(x)$ is a multiple of $(-1)^{(r-1)\lceil (d-k)/2 \rceil}$. It thus follows from (6.8) that α is a multiple of $(-1)^{(r-1)\lceil (d-l)/2 \rceil}$. □

7. Similitudes

In this section we study the behaviour of invariants with respect to similitudes.

Proposition 7.1. *There is a unique morphism of filtered $A(k)$-modules*

$$\Psi : \mathrm{Inv}(W, A) \to \mathrm{Inv}_0(W, A)[-1], \quad \alpha \mapsto \widetilde{\alpha}$$

such that

$$\alpha(\langle \lambda \rangle q) = \alpha(q) + \{\lambda\} \widetilde{\alpha}(q) \tag{7.2}$$

for any $\alpha \in \mathrm{Inv}(W, A)$, $q \in F(K)$ and $\lambda \in K^*$.

If F is a subfunctor of W such that $F(L)$ is stable under similitudes for any L/k, and $0 \in F(k)$, then Ψ restricts to a morphism $\mathrm{Inv}(F, A) \to \mathrm{Inv}_0(F, A)[-1]$. *In particular, for any $n \in \mathbb{N}^*$ we get a filtered morphism $M(n) \to M_0(n)[-1]$.*

Proof. Let $\alpha \in \mathrm{Inv}(F, A^{\geqslant d})$ for some $d \in \mathbb{N}$ and $q \in F(K)$. For any $\lambda \in L^*$, where L/K is any field extension, we set $\beta_q(\lambda) = \alpha(\langle \lambda \rangle q)$.

Then β_q is an invariant over K of square classes, with values in A. Now the functor of square classes is isomorphic to Pf_1, so we may apply Lemma 0.5: there are uniquely determined $x_q, y_q \in A(K)$ such that $\beta_q(\lambda) = x_q + \{\lambda\} \cdot y_q$ for all λ. Taking $\lambda = 1$ we see that $x_q = \alpha(q)$, and we set $\widetilde{\alpha}(q) = y_q$.

The uniqueness of y_q implies that $\widetilde{\alpha} \in \mathrm{Inv}(F, A)$. The fact that $\{\lambda\} \cdot y_q \in A^{\geqslant d}(L)$ for all $\lambda \in L^*$ shows according to Lemma 0.4 that $y_q \in A^{\geqslant d-1}(K)$, so as a filtered morphism Ψ has degree -1. It is clear that if $q = 0$, then $\alpha(\langle\lambda\rangle q) = \alpha(q) + \{\lambda\} \cdot 0$, so $\widetilde{\alpha}(0) = 0$, which means that $\widetilde{\alpha}$ is normalized. □

We first establish some basic properties of Ψ:

Proposition 7.3. *Let* $\alpha, \beta \in \mathrm{Inv}(W, A)$. *Then*

$$\Psi(\alpha\beta) = \Psi(\alpha)\beta + \alpha\Psi(\beta) + \{-1\}\Psi(\alpha)\Psi(\beta).$$

Proof. Let $q \in W(K)$ and $\lambda \in K^*$. Then

$$\begin{aligned}(\alpha\beta)(\langle\lambda\rangle q) &= (\alpha(q) + \{\lambda\}\widetilde{\alpha}(q))(\beta(q) + \{\lambda\}\widetilde{\beta}(q)) \\ &= (\alpha\beta)(q) + \{\lambda\}\big((\widetilde{\alpha}\beta)(q) + (\alpha\widetilde{\beta})(q) + \{-1\}(\widetilde{\alpha}\widetilde{\beta})(q)\big).\end{aligned}$$ □

Proposition 7.4. *We have* $\Psi^2 = -\delta(A)\Psi$.

Proof. For any extension L/K and any $\lambda, \mu \in L^*$,

$$\begin{aligned}\alpha(\langle\lambda\mu\rangle q) &= \alpha(\langle\lambda\rangle q) + \{\mu\}\widetilde{\alpha}(\langle\lambda\rangle q) \\ &= \alpha(q) + \{\lambda\}\widetilde{\alpha}(q) + \{\mu\}\widetilde{\alpha}(q) + \{\lambda, \mu\}\widetilde{\widetilde{\alpha}}(q) \\ &= \alpha(q) + \{\lambda\mu\}\widetilde{\alpha}(q) + \{\lambda, \mu\}(\delta\widetilde{\alpha}(q) + \widetilde{\widetilde{\alpha}}(q)),\end{aligned}$$

using formula (0.2) for the last equality. We also have

$$\alpha(\langle\lambda\mu\rangle q) = \alpha(q) + \{\lambda\mu\}\widetilde{\alpha}(q),$$

so $\{\lambda, \mu\}(\delta\widetilde{\alpha}(q) + \widetilde{\widetilde{\alpha}}(q)) = 0$. Since this holds for any λ, μ over any extension, we may conclude that $\widetilde{\widetilde{\alpha}}(q) = -\delta\widetilde{\alpha}(q)$. □

Remark 7.5. By definition, $\widetilde{\alpha} = 0$ if and only if $\alpha(\langle\lambda\rangle q) = \alpha(q)$, that is to say, α is *invariant under similitudes*. But the previous proposition suggests that in the case $A = W$, $\widetilde{\alpha} = -\alpha$ should also be an interesting property (notably, it is always satisfied by invariants of the form $\widetilde{\beta}$). And indeed, it is easily seen to be equivalent to $\alpha(\langle\lambda\rangle q) = \langle\lambda\rangle\alpha(q)$, in which case we say α is *compatible with similitudes*. Then the proposition shows that any α may be uniquely decomposed as a sum $\alpha = \beta + \gamma$ with β compatible with similitudes, and γ invariant under similitudes. Precisely, $\beta = -\widetilde{\alpha}$ and $\gamma = \alpha + \widetilde{\alpha}$.

From a less intrinsic point of view, if α is a finite combination of the f_n^d, then by definition of the f_n^d it can be seen as a composition

$$I^n(K) \xrightarrow{\sim} \hat{I}^n(K) \subset \mathrm{GW}(K) \xrightarrow{h} \mathrm{GW}(K) \to W(K),$$

where h is a combination of the λ^i. Then β corresponds to selecting only the odd i, while γ corresponds to the even terms. Thus it makes sense to call β the *odd part* of α, and γ its *even part*. This decomposition has no clear equivalent for cohomological invariants.

We now want to describe the action of Ψ on our basic invariants. It turns out that it is much easier to deal with the g_n^d than the f_n^d in this situation.

Proposition 7.6. *Let $n, d \in \mathbb{N}^*$. Then*

$$\widetilde{g_n^d} = \begin{cases} -\delta(A) g_n^d & \text{if } d \text{ is odd,} \\ \{-1\}^{n-1} g_n^{d-1} & \text{if } d \text{ is even.} \end{cases}$$

Proof. We prove the proposition by induction on d. If $d = 1$, the statement means that

$$f_n(\langle\lambda\rangle q) = f_n(q) - \delta\{\lambda\} f_n(q),$$

which is true whether $A = W$ or $A = H$.

Now suppose the proposition holds until $d-1$, for some $d \geqslant 2$. Since $\widetilde{g_n^d}$ is normalized, it is enough to compute $\widetilde{g_n^d}^+$. Let L/K be any extension, and take $q \in I^n(K)$, $\varphi \in \mathrm{Pf}_n(L)$ and $\lambda \in L^*$. Then

$$\begin{aligned} g_n^d(\langle\lambda\rangle(q+\varphi)) &= g_n^d(q+\varphi) + \{\lambda\}\widetilde{g_n^d}(q+\varphi) \\ &= g_n^d(q) + f_n(\varphi)(g_n^d)^+(q) + \{\lambda\}\widetilde{g_n^d}(q) + \{\lambda\} f_n(\varphi)\widetilde{g_n^d}^+(q), \end{aligned}$$

so if we consider generic λ and φ and take residues, we find exactly $\widetilde{g_n^d}^+(q)$.

On the other hand, if we write $\varphi = \langle\langle a\rangle\rangle\psi$, we can compute

$$g_n^d(\langle\lambda\rangle(q+\varphi)) = g_n^d(\langle\lambda\rangle q + \langle\langle\lambda a\rangle\rangle\psi - \langle\langle\lambda\rangle\rangle\psi)$$

using successively on each term Φ^+ relative to $\langle\langle\lambda a\rangle\rangle$, Φ^- relative to $\langle\langle\lambda\rangle\rangle$ and Ψ relative to $\langle\lambda\rangle$, to get an 8-term sum. Again considering generic λ, a and ψ, taking residues, and comparing to the previous computation, we find

$$\widetilde{g_n^d}^+ = -\delta(g_n^d)^+ - \widetilde{(g_n^d)^+} + \{-1\}^{n-1}(g_n^d)^{+-} + \{-1\}^n \widetilde{(g_n^d)^{+-}}, \tag{7.7}$$

using equations (0.2) and (0.3) several times.

If d is even, then $(g_n^d)^+ = g_n^{d-1}$ and $(g_n^d)^{+-} = g_n^{d-2}$, so by induction

$$\widetilde{(g_n^d)^+} = -\delta g_n^{d-1} \quad \text{and} \quad \widetilde{(g_n^d)^{+-}} = \{-1\}^{n-1} g_n^{d-3}.$$

Thus, from (7.7), we get

$$\widetilde{g_n^d}^+ = -\delta g_n^{d-1} + \delta g_n^{d-1} + \{-1\}^{n-1}(g_n^{d-2} + \{-1\}^n g_n^{d-3}) = \{-1\}^{n-1}(g_n^{d-1})^+,$$

which is the expected formula (we need to be a little careful with the case $d = 2$, but we can check that the reasoning still holds if we say that $g_n^{-1} = 0$).

Similarly, if d is odd, $(g_n^d)^+ = g_n^{d-1} + \{-1\}^n g_n^{d-2}$ and $(g_n^d)^{+-} = g_n^{d-2}$, so

$$\widetilde{(g_n^d)^+} = \{-1\}^{n-1} g_n^{d-2} - \delta\{-1\}^n g_n^{d-2} = -\{-1\}^{n-1} g_n^{d-2} \quad \text{and} \quad \widetilde{(g_n^d)^{+-}} = -\delta g_n^{d-2}.$$

Then from (7.7),

$$\widetilde{g_n^d}^+ = -\delta(g_n^d)^+ + \{-1\}^{n-1} g_n^{d-2} + \{-1\}^{n-1} g_n^{d-2} - \delta\{-1\}^n g_n^{d-2} = -\delta(g_n^d)^+$$

using (0.3), which gives the conclusion. □

Corollary 7.8. *The module* $\mathrm{Inv}(I^n/{\sim}, A)$ *of invariants of similarity classes of elements in* I^n *is given by the combinations* $\sum_{d\in\mathbb{N}} a_d g_n^d$ *with* $\{-1\}^{n-1} a_{2i+2} = \delta(A) a_{2i+1}$ *for all* $i \in \mathbb{N}$.

Proof. The module $\mathrm{Inv}(I^n/{\sim}, A)$ is naturally isomorphic to the kernel of Ψ, and if $\alpha = \sum_{d\in\mathbb{N}} a_d g_n^d$, we get

$$\widetilde{\alpha} = \sum_{i\in\mathbb{N}} (\{-1\}^{n-1} a_{2i+2} - \delta(A) a_{2i+1}) g_n^{2i+1},$$

which gives the result. □

The formula for $\widetilde{f_n^d}$ is not particularly enlightening (see Remark 7.10), but we may at least give the values of f_n^d on general Pfister forms (which amounts to computing the values of $\widetilde{f_n^d}$ on Pfister forms). This may be deduced from the previous proposition using Proposition 4.6, but we can give a direct proof.

Proposition 7.9. *Let* $n \in \mathbb{N}^*$ *and* $d \geqslant 2$. *Then for any* $\varphi \in \mathrm{Pf}_n(K)$ *and* $\lambda \in K^*$ *we have*

$$f_n^d(\langle\lambda\rangle\varphi) = (-1)^d \{-1\}^{n(d-1)-1} \{\lambda\} f_n(\varphi).$$

Proof. Write $\varphi = \langle\langle x\rangle\rangle\psi$. Then since $\langle\lambda\rangle\langle\langle x\rangle\rangle = \langle\langle \lambda x\rangle\rangle - \langle\langle\lambda\rangle\rangle$, using (2.3), we get

$$\begin{aligned}
f_n^d(\langle\lambda\rangle\varphi) &= f_n^d(\langle\langle\lambda x\rangle\rangle\psi - \langle\langle\lambda\rangle\rangle\psi) \\
&= f_n^d(-\langle\langle\lambda\rangle\rangle\psi) + \{\lambda x\} f_n^{d-1}(-\langle\langle\lambda\rangle\rangle\psi) \\
&= (-1)^d \{-1\}^{n(d-1)} \{\lambda\} f_{n-1}(\psi) \\
&\qquad + \{\lambda x\} f_{n-1}(\psi) (-1)^{n(d-1)} \{-1\}^{n(d-2)} \{\lambda\} f_{n-1}(\psi) \\
&= (-1)^d \{-1\}^{n(d-1)-1} (\{-1\}\{\lambda\} - \{\lambda x\}\{\lambda\}) f_{n-1}(\psi) \\
&= (-1)^d \{-1\}^{n(d-1)-1} \{\lambda\}\{x\} f_{n-1}(\psi).
\end{aligned}$$

□

Remark 7.10. We can give the general formula for $\widetilde{f_n^d}$ for the record, though we do not prove it:

$$\widetilde{f_n^d} = (-1)^d \sum_{k=1}^{d-1} \binom{d-1}{k-1} \{-1\}^{n(d-k)-1} f_n^k + \begin{cases} 0 & \text{if } d \text{ even,} \\ -\delta(A) f_n^d & \text{if } d \text{ odd.} \end{cases}$$

We can check that if we evaluate this on a Pfister form we retrieve Proposition 7.9, and as an even more special case, formula (2.3).

8. Ramification of invariants

In this short section we establish the behaviour of invariants with respect to residues of discrete valuations (which incidentally was one of the main initial motivations of this article). Let thus (K, v) be a valued field, where v is a rank 1 discrete k-valuation, with valuation ring $\mathcal{O}_K$ and residue field κ (in particular, κ is an extension of k, so it has characteristic not 2).

Recall from [Elman et al. 2008, Lemma 19.10] the so-called second residue map $\partial_\pi : W(K) \to W(\kappa)$, which depends on the choice of a uniformizing element $\pi \in K$. We say that $q \in W(K)$ is *unramified* if $\partial_\pi(q) = 0$, which is independent of the choice of π. Then q is unramified if and only if it has a diagonalization $\langle a_1, \dots, a_r\rangle$ with $a_i \in \mathcal{O}_K^*$.

Recall also from [Garibaldi et al. 2003, §7.9, p. 18] the canonical residue map $\partial : H^d(K, \mu_2) \to H^{d-1}(\kappa, \mu_2)$, which extends to $\partial : H^*(K, \mu_2) \to H^*(\kappa, \mu_2)$. We say that $x \in H^*(K, \mu_2)$ is unramified if $\partial(x) = 0$.

Moreover, from [Elman et al. 2008, Lemma 19.14], we have $\partial_\pi(I^d(K)) \subset I^{d-1}(\kappa)$, and using for instance [Elman et al. 2008, Proposition 101.8] we get for any $d \in \mathbb{N}^*$ a commutative diagram

$$\begin{array}{ccc} I^d(K) & \xrightarrow{\partial_\pi} & I^{d-1}(\kappa) \\ \Big\downarrow{\scriptstyle e_n} & & \Big\downarrow{\scriptstyle e_{d-1}} \\ H^d(K, \mu_2) & \xrightarrow{\partial} & H^{d-1}(\kappa, \mu_2). \end{array}$$

Proposition 8.1. *Let $n \in \mathbb{N}^*$ and $q \in I^n(K)$, where K is endowed with a rank 1 discrete k-valuation. If q is unramified, then $\alpha(q) \in A(K)$ is unramified for any $\alpha \in M(n)$.*

Proof. By hypothesis, $\hat{q} \in \hat{I}^n(K)$ comes from an element of $\mathrm{GW}(\mathcal{O}_K)$, so any $\lambda^i(\hat{q})$ also comes from $\mathrm{GW}(\mathcal{O}_K)$, and is unramified. Since π_n^d is a combination of the λ^i with integer coefficients, $\pi_n^d(\hat{q}) \in \hat{I}^{nd}(K)$ is unramified.

Now, tautologically if $A = W$, and applying the above commutative diagram if $A = H$, this implies that $f_n^d(q) \in A^{\geqslant nd}(K)$ is unramified.

Since any $\alpha \in M(n)$ is a combination of the f_n^d with coefficients in $A(k)$, and v is a k-valuation, we can conclude that $\alpha(q) \in A(K)$ is unramified. □

9. Invariants of Quad_{2r}

In [Garibaldi et al. 2003], Serre gives a complete description of $\mathrm{Inv}(\mathrm{Quad}_m, A)$: it is a free $A(k)$-module of rank $n + 1$, with basis $(\lambda^d)_{0\leqslant d\leqslant m}$ for $A = W$, and the Stiefel–Whitney classes $(w_d)_{0\leqslant d\leqslant m}$ for $A = H$ (see [Garibaldi et al. 2003, Theorem 27.16 and §17.1]). Clearly any invariant of I restricts to an invariant of Quad_m for any even m, and we want to express it in terms of the given basis.

For practical purposes it is more convenient to introduce a different basis for $\mathrm{Inv}(\mathrm{Quad}_m, W)$ which is the equivalent of the Stiefel–Whitney classes for Witt invariants. We use the notations and definitions from Section 1. Recall from [Elman et al. 2008, §5] that the total Stiefel–Whitney map $w_t : \mathrm{GW}(K) \to \Lambda(H^*(K, \mu_2))$ is the only group morphism such that $w_t(\langle a\rangle) = 1 + (a)t$ for all $a \in K^*$. We generalize this construction:

Proposition 9.1. *There is a unique group morphism*

$$h_t : \mathrm{GW}(K) \to \Lambda(A(K)), \quad x \mapsto h_t(x) = \sum_{d\in\mathbb{N}} h^d(x)t^d$$

such that $h_t(\langle a\rangle) = 1 + \{a\}t$ for all $a \in K^*$. The map h^d takes values in $A^{\geqslant d}(K)$. For any $m \in \mathbb{N}^*$, we write $h^d_m : \mathrm{Quad}_m(K) \to A(K)$ for the restriction of h^d to forms of dimension m. Then $h^d_m \in \mathrm{Inv}(\mathrm{Quad}_m, A^{\geqslant d})$.

If $A = H$, then h^d is the Stiefel–Whitney map w_d. If $A = W$, we write $P^d = h^d$ and $P^d_m = h^d_m$; then for any $q \in \mathrm{Quad}_m(K)$,

$$P^d_m(q) = \sum_{k=0}^{d} (-1)^k \binom{m-k}{d-k} \lambda^k(q). \tag{9.2}$$

In both cases, $(h^d_m)_{0\leqslant d\leqslant m}$ is a basis of the $A(k)$-module $\mathrm{Inv}(\mathrm{Quad}_m, A)$.

Proof. The uniqueness of h_t is obvious since $\mathrm{GW}(K)$ is generated by the $\langle a\rangle$ as an additive group. For $A = H$, the existence can either be deduced from the case $A = W$, or from the classical existence of Stiefel–Whitney maps. For $A = W$, we define P^d piecewise on quadratic forms, using (9.2) for P^d_m in each dimension m. We see immediately from the definition that $P^d_1(\langle a\rangle)$ is 1 if $d = 0$, $\langle\langle a\rangle\rangle$ if $d = 1$ and 0 if $d \geqslant 2$. The fact that this extends to a group morphism $\mathrm{GW}(K) \to \Lambda(\mathrm{GW}(K))$ can be deduced using the universal property of Grothendieck groups if we can show that for any $q \in \mathrm{Quad}_m(K)$, $q' \in \mathrm{Quad}_n(K)$, we have

$$P^d_{m+n}(q + q') = \sum_{k=0}^{d} P_m(q) P_n(q').$$

And indeed we find

$$\begin{aligned}\sum_{k=0}^{d} P_m(q) P_n(q') &= \sum_{k=0}^{d}\sum_{i=0}^{k}\sum_{j=0}^{d-k} (-1)^{i+j} \binom{m-i}{k-i}\binom{n-j}{d-k-j} \lambda^i(q)\lambda^j(q') \\ &= \sum_{l=0}^{d} (-1)^l \sum_{i+j=l} \left(\sum_{k=i}^{d-j} \binom{m-i}{k-i}\binom{n-j}{d-k-j} \right) \lambda^i(q)\lambda^j(q') \\ &= \sum_{l=0}^{d} (-1)^l \binom{m+n-l}{d-l} \sum_{i+j=l} \lambda^i(q)\lambda^j(q') = P_{m+n}(q+q').\end{aligned}$$

From the group property we easily see that

$$h_m^d(\langle a_1, \ldots, a_m\rangle) = \sum_{i_1<\cdots<i_d} \{a_{i_1}, \ldots, a_{i_d}\}, \tag{9.3}$$

so $h_m^d(q) \in I^d(K)$ if $q \in \mathrm{Quad}_m(K)$. The fact that h_m^d is an invariant is obvious given the definition with the λ-powers, or can be deduced from the uniqueness statement. Finally, the fact that the h_m^d for a basis of $\mathrm{Inv}(\mathrm{Quad}_m, A)$ is a consequence of Serre's result, directly for $A = H$, and observing for $A = W$ that the transition matrix from $(P_m^d)_{0\leqslant d\leqslant m}$ to $(\lambda^d)_{0\leqslant d\leqslant m}$ is triangular unipotent. □

Remark 9.4. Note that this does not define a pre-λ-ring structure on $\mathrm{GW}(K)$ since P^1 is not the identity (indeed, $P^1(\langle a\rangle) = \langle\langle a\rangle\rangle$).

Proposition 9.5. *Let $m = 2r \in \mathbb{N}^*$, $d \in \mathbb{N}$ and $q \in \mathrm{Quad}_m(K)$. Then*

$$f_1^d(q) = \sum_{i=0}^{d} (-1)^i \binom{r-i}{d-i} \{-1\}^{d-i} h_m^i(q),$$

$$g_1^d(q) = \sum_{i=0}^{d} (-1)^i \binom{r-i-1+\lfloor \frac{d+1}{2} \rfloor}{d-i} \{-1\}^{d-i} h_m^i(q).$$

Proof. We prove the statement concerning f_1^d; the case of g_1^d may be deduced by a lengthy but straightforward computation using Proposition 4.6, or can be directly proved by the same method.

Write α_m^d for the invariant of Quad_m defined by the right-hand side of the equation. It is clear by definition that $\alpha_m^0 = 1$ coincides with f_1^0 on Quad_m. We claim that it is enough to show that for any $d \in \mathbb{N}^*$,

$$\alpha_2^d(\langle\langle 1\rangle\rangle) = 0, \tag{9.6}$$

and for any $m = 2r \in \mathbb{N}^*$, any $q \in \mathrm{Quad}_m(K)$ and any $a \in K^*$,

$$\alpha_{m+2}^d(q + \langle\langle a\rangle\rangle) = \alpha_m^d(q) + \{a\}\alpha_m^{d-1}(q). \tag{9.7}$$

Indeed, taking $a = 1$ in (9.7) shows that $\alpha_m^d(q)$ depends only on the Witt class of $q \in \mathrm{Quad}_m(K)$, so it defines an invariant $\alpha^d \in M(1)$. Then (9.6) shows that α^d is normalized, and (9.7) shows that $(\alpha^d)^+ = \alpha^{d-1}$, so by an immediate induction, $\alpha^d = f_1^d$.

From (9.3) we easily see that $h_2^0(\langle\langle a\rangle\rangle) = 1$, $h_2^1(\langle\langle a\rangle\rangle) = \{-a\}$, and $h_2^i(\langle\langle a\rangle\rangle) = 0$ if $i \geqslant 2$. Thus $\alpha_2^d(\langle\langle 1\rangle\rangle) = \{-1\}^d - \{-1\}^{d-1}\cdot\{-1\} = 0$, which shows (9.6).

Furthermore, if $i \in \mathbb{N}$ and $q \in \mathrm{Quad}_m(K)$,

$$\begin{aligned} h_{m+2}^i(q + \langle\langle a\rangle\rangle) &= h_m^i(q) + \{-a\}h_m^{i-1}(q) \\ &= \big(h_m^i(q) + \{-1\}h_m^{i-1}(q)\big) - \{a\}h_m^{i-1}(q) \end{aligned} \tag{9.8}$$

(where by convention $h_m^{-1}=0$). Therefore,

$$\begin{aligned}\alpha_{m+2}(q+\langle\langle a\rangle\rangle) &= \sum_{i=0}^{d}(-1)^i\binom{r+1-i}{d-i}\{-1\}^{d-i}\big(h_m^i(q)+\{-1\}h_m^{i-1}(q)\big)\\ &\qquad -\{a\}\sum_{i=0}^{d}(-1)^i\binom{r+1-i}{d-i}\{-1\}^{d-i}h_m^{i-1}(q)\\ &= \sum_{i=0}^{d-1}\left((-1)^i\binom{r-i+1}{d-i}+(-1)^{i+1}\binom{r-i}{d-i-1}\right)\{-1\}^{d-i}h_m^i(q)\\ &\qquad +(-1)^d h_m^d(q)-\{a\}\sum_{i=0}^{d-1}(-1)^{i+1}\binom{r-i}{d-i-1}\{-1\}^{d-i-1}h_m^i(q)\\ &= \sum_{i=0}^{d}(-1)^i\binom{r-i}{d-i}\{-1\}^{d-i}h_m^i(q)\\ &\qquad +\{a\}\sum_{i=0}^{d-1}(-1)^i\binom{r-i}{d-1-i}\{-1\}^{d-1-i}h_m^i(q),\end{aligned}$$

which gives the expected formula. □

Remark 9.9. In particular, looking carefully at the binomial coefficients in the formula and remembering that $h_m^i=0$ if $i>m$, we retrieve the fact that g_1^d is zero if $d>m$ (recall Corollary 4.8). On the other hand, we see that f_1^d can be nonzero for arbitrarily high values of d, even for fixed m.

Remark 9.10. If -1 is a square in k, then $f_1^d=g_1^d=h_m^d$ on Quad_m for any even $m\in\mathbb{N}^*$.

Corollary 9.11. *For any even $m\in\mathbb{N}^*$, the restrictions of f_1^d (or g_1^d) for $0\leqslant d\leqslant m$ form an $A(k)$-basis of* $\mathrm{Inv}(\mathrm{Quad}_m, A)$. *In particular, any invariant of* Quad_m *can be extended to I.*

Remark 9.12. Serre also describes the cohomological invariants of $\mathrm{Quad}_{m,\delta}$, meaning of forms with prescribed determinant δ, and in particular this gives a description of invariants of $\mathrm{Quad}_m\cap I^2$. They are given by Stiefel–Whitney classes, plus one invariant that does not extend to Quad_m in general. Since any invariant of I^2 extends to I, this shows that there are invariants of $\mathrm{Quad}_m\cap I^2$ that do not extend to I^2.

There are also examples in the literature of some invariants of $\mathrm{Quad}_m\cap I^3$ that one can show, using the results in this article, do not extend to I^3 (for instance the invariant a_5 mentioned in Section 11).

Remark 9.13. Let us consider the cohomological invariants of $\mathrm{Quad}_m/\!\sim$ (the similarity classes of quadratic forms of dimension n). This is of course the same thing as an invariant of Quad_m which is constant on similarity classes, so according to Corollary 9.11 any such invariant is a unique combination of the $v_d^{(1)}$ with $0\leqslant d\leqslant m$.

Now Corollary 7.8 shows that such a combination is constant on similarity classes if and only if the only d that appear are odd. This is exactly the description that Rost gives in [1998, Lemma 2], where he proves that any invariant of $\mathrm{Quad}_m/\!\sim$ is a unique combination of invariants he calls v_{2i+1}, and a simple computation shows that $v_{2i+1} = v_{2i+1}^{(1)}$.

On the other hand, our present tools cannot a priori describe all cohomological invariants of similarity classes in $\mathrm{Quad}_m \cap I^2$, since not all invariants of $\mathrm{Quad}_m \cap I^2$ extend to I^2. What we can say from the previous remark and Corollary 7.8 is that those which do extend to I^2 can be uniquely written as $\sum_{d=0}^{r} a_d \cup v_{2d}^{(2)}$ with $(-1) \cup a_d = 0$ if $d > 0$ is even. However, Rost describes in [1998, Theorem 6] the invariants of similarity classes $\mathrm{Quad}_m \cap I^2$, and proves that they are combinations of invariants η_d. It turns out that $\eta_d = v_{2d}^{(2)}$, so this shows that even though some invariants of isometry classes in $\mathrm{Quad}_m \cap I^2$ do not extend to I^2, all invariants of *similarity classes* in $\mathrm{Quad}_m \cap I^2$ do extend to I^2 (and therefore to I), and Rost's description is exactly the same as ours.

10. Operations on mod 2 cohomology

In this section we are specifically interested in cohomological invariants. It was observed by Serre that one may define some sorts of divided squares on mod 2 cohomology:

$$\begin{aligned} H^n(K, \mu_2) &\to H^{2n}(K, \mu_2)/(-1)^{n-1} \cup H^{n+1}(K, \mu_2), \\ \sum_i \alpha_i &\mapsto \sum_{i<j} \alpha_i \cup \alpha_j. \end{aligned}$$

The quotient on the right-hand side is necessary for the map to be well-defined. Similarly, one may define higher divided powers:

$$\begin{aligned} H^n(K, \mu_2) &\to H^{dn}(K, \mu_2)/(-1)^{n-1} \cup H^{(d-1)n+1}(K, \mu_2), \\ \sum_i \alpha_i &\mapsto \sum_{i_1<\cdots<i_d} \alpha_{i_1} \cup \cdots \cup \alpha_{i_d}. \end{aligned}$$

On the other hand, Vial [2009] characterizes natural operations

$$H^n(K, \mu_2) \to H^*(K, \mu_2)$$

(his statement is formulated for mod 2 Milnor K-theory, which is equivalent according to the resolution of Milnor's conjecture). The precise statement, slightly reformulated, is the following (the original statement forgets to explicitly assume that operations must have uniformly bounded degree):

Proposition 10.1 [Vial 2009, Theorem 2]. *If $n \in \mathbb{N}^*$, the $H^*(k, \mu_2)$-module of operations $H^n(K, \mu_2) \to H^*(K, \mu_2)$ with uniformly bounded degree is*

$$H^*(k, \mu_2) \cdot 1 \oplus H^*(k, \mu_2) \cdot \mathrm{Id} \oplus \bigoplus_{d \in \mathbb{N}} \mathrm{Ker}(\tau_n) \cdot \theta_d,$$

where $\tau_n : H^*(k, \mu_2) \to H^*(k, \mu_2)$ *is defined by* $\tau_n(x) = (-1)^{n-1} \cup x$, *and if* $a \in \mathrm{Ker}(\tau_n)$, *then*

$$a \cdot \theta_d\Big(\sum_{1\leqslant i\leqslant r} x_i\Big) = a \cdot \sum_{i_1<\cdots<i_d} x_{i_1} \cup \cdots \cup x_{i_d},$$

where the x_i *are symbols.*

Note that the "divided power operation" θ_d is not defined on its own, but $a \cdot \theta_d$ is well-defined when $a \in \mathrm{Ker}(\tau_n)$. This is similar to how for Serre's operations it was necessary to consider some quotient on the right-hand side of the map; here one has to put some restriction on the left-hand side, in both cases to annihilate appropriate powers of the symbol $(-1) \in H^1(K, \mu_2)$. The remarkable phenomenon is that when we work on the level of I^n, we can lift those θ_d with no restriction: this is our $u_{nd}^{(n)}$.

Moreover, it is not too difficult to retrieve Vial's theorem using our results about invariants of I^n: operations on $H^n(K, \mu_2)$ are none other than invariants $\alpha \in M(n)$ (with $A = H$) such that

$$\alpha(q + \varphi) = \alpha(q) \quad \text{for all } q \in I^n(K), \varphi \in \mathrm{Pf}_{n+1}(K). \tag{10.2}$$

Consider the following easy lemma:

Lemma 10.3. *Let* $n \in \mathbb{N}^*$, *and let us restrict to* $A = H$. *For any* $\alpha \in M(n)$, *any* $q \in I^n(K)$ *and any* $\varphi \in \mathrm{Pf}_{n+1}(K)$, *we have*

$$\alpha(q + \varphi) = \alpha(q) + (-1)^{n-1} \cup e_{n+1}(\varphi) \cup \alpha^{++}(q).$$

Proof. Up to taking linear combinations, we may restrict to the case of $\alpha = u_{nd}^{(n)}$. Using Corollary 6.4, we see that $u_{nd}^{(n)}(\varphi)$ is 1 if $d = 0$, $(-1)^{n-1} \cup e_{n+1}(\varphi)$ if $d = 2$, and 0 otherwise. Then using the sum formula for $u_{nd}^{(n)}$ we find

$$u_{nd}^{(n)}(q + \varphi) = u_{nd}^{(n)}(q) + (-1)^{n-1} \cup e_{n+1}(\varphi) \cup u_{n(d-2)}^{(n)}(q). \qquad \square$$

Then $\alpha \in M(n)$ satisfies condition (10.2) if and only if $(-1)^{n-1} \cup \alpha^{++} = 0$, which precisely means that if we write $\alpha = \sum_d a_d \cup u_{nd}^{(n)}$ then, for $d \geqslant 2$, $a_d \in \mathrm{Ker}(\tau_n)$, and we indeed retrieve Vial's description.

11. Invariants of semifactorized forms

Garibaldi [2009, §20] defines a cohomological invariant on $\mathrm{Quad}_{12} \cap I^3$ in the following way: any such form can be written $q = \langle\langle c\rangle\rangle q'$, where $q' \in I^2(K)$, and we set $a_5(q) = e_5(\langle\langle c\rangle\rangle \bar{\pi}_2^2(q')) = (c) \cup u_4^{(2)}(q')$ (using our notation). Of course, the nontrivial ingredient is that $\langle\langle c\rangle\rangle \bar{\pi}_2^2(q')$ is actually independent of the decomposition of q.

This construction does not correspond to any of the tools we developed so far, since it does not give an invariant of I^3. However, it is easy to see that the construction works for any Witt class $q \in I^3(K)$ that factorizes as $q = \langle\langle c\rangle\rangle q'$. This leads us to the more general definition:

Definition 11.1. Let $n \in \mathbb{N}^*$ and $r \in \mathbb{N}$ such that $r \leqslant n$. We set

$$I^{n,r}(K) = \{\varphi \cdot q \mid \varphi \in \mathrm{Pf}_r(K),\ q \in I^{n-r}(K)\}.$$

We also define $M(n,r) = \mathrm{Inv}(I^{n,r}, A)$, and similarly $M_0(n,r)$, $M^{\geqslant d}(n,r)$ and $M_0^{\geqslant d}(n,r)$. In particular, $I^{n,0} = I^n$, so $M(n,0) = M(n)$ and so on.

Remark 11.2. A consequence of Milnor's conjecture proved in [Elman et al. 2008, Theorem 41.7] is that $I^{n,r}(K) = I^{r,r}(K) \cap I^n(K)$, so in particular

$$I^{n,r}(K) \cap I^{n+1}(K) = I^{n+1,r}(K).$$

Clearly, if $(m,s) \geqslant (n,r)$, then $I^{m,s}(K) \subset I^{n,r}(K)$, so we have a restriction morphism

$$\rho_{(n,r),(m,s)} : M(n,r) \to M(m,s), \quad \alpha \mapsto \alpha_{|I^{m,s}},$$

which is a morphism of filtered $A(k)$-algebras, and sends $M_0(n,r)$ to $M_0(m,s)$. In particular, when $r = s = 0$, we retrieve the restriction morphism $\rho_{n,m}$ defined in (6.1). We usually drop the indexes and simply write $\rho : M(n,r) \to M(m,s)$, since the indexes can be inferred from the source and target modules.

We can also define a morphism that goes in the other direction:

Proposition 11.3. *Let $n, r, t \in \mathbb{N}$ with $t \leqslant r < n$. There is a unique morphism of filtered $A(k)$-modules*

$$\Delta^t_{(n,r)} : M(n,r) \to M(n-t, r-t)[-t], \quad \alpha \mapsto \alpha^{(t)},$$

such that $\alpha^{(t)}(0) = \alpha(0)$, and if $\alpha \in M_0(n,r)$ then

$$\alpha(\varphi \cdot q) = f_t(\varphi) \cdot \alpha^{(t)}(q)$$

for any $\varphi \in \mathrm{Pf}_t(K)$ and $q \in I^{n-t,r-t}(K)$. Furthermore, $\Delta^t_{(n,r)}$ is injective.

Proof. Since $M(n,r) = A(k) \oplus M_0(n,r)$, this piecewise definition of $\Delta^t_{(n,r)}$ determines the whole function. Let $\alpha \in M_0^{\geqslant d}(n,r)$ and $q \in I^{n-t,r-t}(K)$. Then $\varphi \mapsto \alpha(\varphi \cdot q)$ defines an invariant of Pf_t over K with values in $A^{\geqslant d}$. Using Lemma 0.5, there are unique $x(q), y(q) \in A(K)$ such that

$$\alpha(\varphi \cdot q) = x(q) + f_t(\varphi) \cdot y(q)$$

and by uniqueness those are invariants of $I^{n-t,r-t}$, with $x = \alpha(0) = 0$. We then set $\alpha^{(t)} := y$. Furthermore, using Lemma 0.4, we see that $y(q) \in A^{\geqslant d-t}(K)$, so

$\alpha^{(t)} \in M_0^{\geqslant d-t}(n-t, r-t)$. The injectivity is clear since any element of $I^{n,r}(K)$ is of the form φq with φ and q as in the statement, and $\alpha(\varphi q)$ is determined by $\alpha^{(t)}$. □

We usually drop the indexes and simply write $\Delta^t : M(n, r) \to M(n-t, r-t)[-t]$. Using this notation, it is clear by definition that $\Delta^t \circ \Delta^{t'} = \Delta^{t+t'}$. The natural question is then as follows:

Question. What is the image of $\Delta^t : M(n+t, r+t) \to M(n, r)$?

This can be rephrased to ask for which $\beta \in M_0(n, r)$ is it true that for all $\varphi \in \mathrm{Pf}_t(K)$ and $q \in I^{n,r}(K)$, $f_t(\varphi)\beta(q)$ only depends on φq? With this point of view, the existence of the invariant a_5 given at the beginning of the section (which is [Garibaldi 2009, Corollary 20.7]) is exactly equivalent to the fact that $f_2^2 \in M(2, 0)$ is in the image of $\Delta^1 : M(3, 1) \to M(2, 0)$. The main result of the section is a generalization of this fact:

Theorem 11.4. *For any $n \in \mathbb{N}^*$, $\Delta^1 : M_0(n+1, 1) \to M_0(n)[-1]$ is an isomorphism of filtered $A(k)$-modules.*

Remark 11.5. This means that $\Delta^1 : M(n+1, 1) \to M(n)[-1]$ is a module isomorphism, but it is not a *filtered* module isomorphism, since it is the identity on the constant components, and while the identity is a bijective filtered morphism from $A(k)$ to $A(k)[-1]$, it is of course not a filtered isomorphism.

Before we prove Theorem 11.4, we construct a common generalization of ρ and Δ^t, which allows us to make simple statements about the general properties of both those morphisms. Most of that is not useful for the proof of the theorem, but has some independent interest.

Definition 11.6. Let $m, n \in \mathbb{N}^*$ and $r, s \in \mathbb{N}$ be such that $r < n$ and $s < m$. We say that a filtered $A(k)$-module morphism $M(n, r) \to M(m, s)[-t]$ is of type Ω^t if it is a composition of morphisms $\omega_i : M(n_i, r_i)[-a_i] \to M(n_{i+1}, r_{i+1})[-a_i - t_i]$ for $i = 0, \dots, d$, with $(n_0, r_0) = (n, r)$, $a_0 = 0$, $(n_{d+1}, r_{d+1}) = (m, s)$, $t = \sum_i t_i$, and ω_i is either ρ (so $t_i = 0$) or Δ^{t_i}.

In particular, we define ω of type Ω^1:

$$\omega : M(n, r) \xrightarrow{\rho} M(n+1, r+1) \xrightarrow{\Delta^1} M(n, r)[-1].$$

Remark 11.7. It is not hard to see that there is a morphism $M(n, r) \to M(m, s)[-t]$ of type Ω^t if and only if $t \geqslant n - m$ and $t \geqslant r - s$.

Proposition 11.8. *Let $m, n \in \mathbb{N}^*$ and $r, s \in \mathbb{N}$ be such that $r < n$ and $s < m$, and let $t \in \mathbb{N}$ be such that $t \geqslant n - m$ and $t \geqslant r - s$. Then there is exactly one morphism $M(n, r) \to M(m, s)[-t]$ of type Ω^t, and we call it simply Ω^t. The morphism $\Omega^t : M(n, r) \to M(n, r)[-t]$ is ω^t.*

In particular, let $t' \geqslant t$. Then the following diagram of filtered $A(k)$-modules commutes:

$$\begin{array}{ccc} & M(n,r) & \\ {}^{\Omega^t}\swarrow & & \searrow{}^{\Omega^{t'}} \\ M(m,s)[-t] & \xrightarrow{\omega^{t'-t}} & M(m,s)[-t'] \end{array}$$

Proof. The only thing to prove is that there is at most one morphism of type Ω^t. The fact that $\Omega^t = \omega^t$ then follows, since ω^t is of type Ω^t by definition, and the commutativity of the diagram comes from the fact that both compositions are of type $\Omega^{t'}$.

To show this uniqueness, it is enough to show that the following diagram commutes whenever it makes sense:

$$\begin{array}{ccc} M(n,r) & \xrightarrow{\rho} & M(m,s) \\ {\scriptstyle\Delta^t}\downarrow & & \downarrow{\scriptstyle\Delta^t} \\ M(n-t,r-t)[-t] & \xrightarrow{\rho} & M(m-t,s-t)[-t] \end{array}$$

Indeed, if we can prove this, then we can show by induction on the length of the composition that in the definition of a morphism of type Ω^t we can always assume that the first morphisms are all of the form ρ, and the remaining ones are all of the form Δ^{t_i}. But then the result is clear, since a composition of restriction morphisms is a restriction morphism, and $\Delta^t \circ \Delta^s = \Delta^{t+s}$ (with the only indices that make sense), so the morphism is entirely characterized by its source, its target and t.

We now show that the diagram commutes. Let $\alpha \in M(n,r)$. Since all morphisms are the identity on the constant components, we may assume $\alpha \in M_0(n,r)$. Let us write $\beta = (\alpha^{(t)})_{|I^{m-t,s-t}}$, and take $\varphi \in \mathrm{Pf}_t(K)$, $\psi \in \mathrm{Pf}_{s-t}(K)$ and $q \in I^{m-s}(K)$. We can set $\psi = \psi_1\psi_2$ with $\psi_1 \in \mathrm{Pf}_{r-t}(K)$ and $\psi_2 \in \mathrm{Pf}_{s-r}(K)$; then if $q' = \psi_2 q \in I^{m-r}(K)$, we have

$$\alpha(\varphi\psi q) = \alpha(\varphi\psi_1 q') = f_t(\varphi)\alpha^{(t)}(\psi_1 q') = f_t(\varphi)\beta(\psi q),$$

which shows that $\beta = (\alpha_{|I^{m,s}})^{(t)}$. □

Example 11.9. The morphism $\Omega^0 : M(n,r) \to M(m,s)$ exists when $(m,s) \geqslant (n,r)$, and it is the restriction morphism ρ. The morphism $\Omega^t : M(n,r) \to M(n-t,r-t)[-t]$ exists when $t \leqslant r$, and it is Δ^t.

Example 11.10. There is a morphism $\Omega^t : M(n) \to M(m)[-t]$ when $t \geqslant n-m$, and if $n = m$ it is ω^t, with $\omega : M(n) \to M(n)[-1]$.

We can now collect some basic properties of the morphisms Ω^t.

Proposition 11.11. *Let $n, m, r, s, t \in \mathbb{N}$ be as in Proposition 11.8. Then for any $\alpha, \beta \in M_0(n,r)$, we have*

$$\Omega^t(\alpha\beta) = \{-1\}^t \Omega^t(\alpha)\Omega^t(\beta).$$

Proof. Since the restriction morphisms obviously preserve the product of invariants, we may assume that $\Omega^t = \Delta^t$. Then for any $\varphi \in \mathrm{Pf}_t(K)$, $\psi \in \mathrm{Pf}_{r-t}(K)$ and $q \in I^{n-r}(K)$, we have

$$(\alpha\beta)(\varphi\psi q) = (f_t(\varphi)\alpha^{(t)}(\psi q))(f_t(\varphi)\beta^{(t)}(\psi q)) = \{-1\}^t f_t(\varphi)(\alpha^{(t)}\beta^{(t)})(\psi q),$$

hence the result. □

We may note from Proposition 7.1 that we have well-defined filtered morphisms

$$\Psi : M(n,r) \to M(n,r)[-1]$$

for any $n, r \in \mathbb{N}$ such that $r < n$.

Proposition 11.12. *Let $n, m, r, s, t \in \mathbb{N}$ be as in Proposition 11.8. Then the following diagram of filtered $A(k)$-modules commutes*:

$$\begin{array}{ccc} M(n,r) & \xrightarrow{\Omega^t} & M(m,s)[-t] \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ M(n,r)[-1] & \xrightarrow{\Omega^t} & M(m,s)[-t-1] \end{array}$$

Proof. The definition of Ψ makes it clear that it commutes with restriction morphisms, since it is defined on the whole $\mathrm{Inv}(W, A)$. Thus we may assume $\Omega^t = \Delta^t$. Let $\alpha \in M(n,r)$, $\varphi \in \mathrm{Pf}_t(K)$, $\psi \in \mathrm{Pf}_{r-t}(K)$, $q \in I^{n-r}(K)$ and $\lambda \in K^*$. Then

$$\alpha(\langle\lambda\rangle\varphi\psi q) = f_t(\varphi)\alpha^{(t)}(\langle\lambda\rangle\psi q) = f_t(\varphi)\alpha^{(t)}(\psi q) + f_t(\varphi)\{\lambda\}\widetilde{\alpha^{(t)}}(\psi q),$$

but also

$$\alpha(\langle\lambda\rangle\varphi\psi q) = \alpha(\varphi\psi q) + \{\lambda\}\widetilde{\alpha}(\varphi\psi q) = f_t(\varphi)\alpha^{(t)}(\psi q) + \{\lambda\} f_t(\varphi)\widetilde{\alpha}^{(t)}(\psi q),$$

which gives $\widetilde{\alpha^{(t)}} = \widetilde{\alpha}^{(t)}$. □

Since we saw in Corollary 6.3 that Φ^+ is far from commuting with the restriction morphisms, we cannot expect such a good compatibility with the morphisms Ω^t, but we still get the following:

Proposition 11.13. *Let $n \in \mathbb{N}^*$ and let $t \in \mathbb{N}$ be such that $t < n$. Then the following diagram of filtered $A(k)$-modules commutes for any $\varepsilon = \pm 1$*:

$$\begin{array}{ccc} M(n) & \xrightarrow{\Omega^t} & M(n-t)[-t] \\ {\scriptstyle\Phi^\varepsilon}\downarrow & & \downarrow{\scriptstyle\Phi^\varepsilon} \\ M(n)[-n] & \xrightarrow{\{-1\}^t\Omega^t} & M(n-t)[-n] \end{array}$$

Proof. The diagram obviously commutes for the constant components (since we find 0 in both cases), so we may consider $\alpha \in M_0(n)$. Let $\varphi \in \mathrm{Pf}_t(K)$, $\psi \in \mathrm{Pf}_{n-t}(K)$ and $q \in I^{n-t}(K)$. Then

$$\begin{aligned}\alpha(\varphi(q+\varepsilon\psi)) &= \alpha(\varphi q) + \varepsilon f_n(\varphi\psi)\alpha^{\varepsilon}(\varphi q)\\ &= f_t(\varphi)\alpha^{(t)}(q) + \varepsilon\{-1\}^t f_n(\varphi\psi)(\alpha^+)^{(t)}(q)\end{aligned}$$

as well as

$$\begin{aligned}\alpha(\varphi(q+\varepsilon\psi)) &= f_t(\varphi)\alpha^{(t)}(q+\varepsilon\psi)\\ &= f_r(\varphi)\alpha^{(r)}(q) + \varepsilon f_r(\varphi) f_{n-r}(\psi)(\alpha^{(r)})^+(q),\end{aligned}$$

which proves that $((\alpha_{|I^{n,t}})^{(t)})^+ = \{-1\}^t(\alpha^+)^{(t)}_{|I^{n,t}}$. □

Corollary 11.14. *Let $n, t \in \mathbb{N}$ be such that $t < n$. Then for any $d \in \mathbb{N}^*$, the morphism $\Omega^t : M(n) \to M(n-t)[-t]$ satisfies*

$$\Omega^t(f_n^d) = \{-1\}^{t(d-1)} f_{n-t}^d.$$

In particular, if $\varphi \in \mathrm{Pf}_t(K)$ and $q \in I^n(K)$ is a multiple of φ, then $f_n^d(q)$ is a multiple of $f_t(\varphi)$.

Proof. The formula follows from an induction on d, using Proposition 11.13. For the last statement, note that according to Remark 11.2, there is $q' \in I^{n-t}(K)$ such that $q = \varphi q'$. Then according to the formula,

$$f_n^d(q) = f_n^d(\varphi q') = \{-1\}^{t(d-1)} f_t(\varphi) f_{n-t}^d(q'). \qquad \square$$

We now turn to the proof of Theorem 11.4. We first need a preliminary lemma.

Lemma 11.15. *Let $a, b \in K^*$, and consider $q \in \hat{I}(K)$ of the form*

$$q = \sum_{i=1}^{r} \langle x_i\rangle\langle|c_i|\rangle,$$

where c_i is represented by $\langle\langle ab\rangle\rangle$. Then for any $k \in \mathbb{N}^*$,

$$\langle\langle a\rangle\rangle\lambda^k(q) = \langle\langle b\rangle\rangle\lambda^k(q).$$

In particular, for any $n, d \in \mathbb{N}^*$, $\langle\langle a\rangle\rangle\pi_n^d(q) = \langle\langle b\rangle\rangle\pi_n^d(q)$.

Proof. We have

$$\lambda^k(q) = \sum_{d_1+\cdots+d_r=k} \lambda^{d_1}(\langle x_1\rangle\langle|c_1|\rangle)\cdots\lambda^{d_r}(\langle x_r\rangle\langle|c_r|\rangle).$$

Now at least one of the d_i is nonzero, so we may conclude since

$$\langle\langle a\rangle\rangle\lambda^{d_i}(\langle x_i\rangle\langle|c_i|\rangle) = \langle x_i^d\rangle\langle\langle a\rangle\rangle\langle|c_i|\rangle = \langle x_i^d\rangle\langle\langle b\rangle\rangle\langle|c_i|\rangle = \langle\langle b\rangle\rangle\lambda^{d_i}(\langle x_i\rangle\langle|c_i|\rangle),$$

using Lemma 1.4 and the fact that if c is represented by $\langle\langle ab\rangle\rangle$ then $\langle\langle a, c\rangle\rangle = \langle\langle b, c\rangle\rangle$. The statement about π_n^d follows since by definition π_n^d is a combination of the λ^k with $1 \leqslant k \leqslant d$. □

We can finally prove the main result of the section:

Proof of Theorem 11.4. It suffices to show that f_n^d is in the image of Δ^1 for all $d \geqslant 1$, which amounts to saying that $\langle\langle a\rangle\rangle q \mapsto \{a\} f_n^d(q)$ is well-defined, in other words that if $q, q' \in I^n(K)$ and $a, b \in K^*$, then $\langle\langle a\rangle\rangle q = \langle\langle b\rangle\rangle q'$ implies $\{a\} f_n^d(q) = \{b\} f_n^d(q')$.

Assume first that $a = b$. Then according to [Elman et al. 2008, Corollary 6.23],

$$q - q' = \sum_{i \in J} \langle\langle c_i\rangle\rangle q_i,$$

where $q_i \in W(K)$ and c_i is represented by $\langle\langle a\rangle\rangle$. We may then reason by induction on $|J|$, and we are reduced to the case where $q' = q + \langle\langle c\rangle\rangle q_0$, with c represented by $\langle\langle a\rangle\rangle$. But according to Corollary 11.14, for any $k \in \mathbb{N}^*$, $f_n^k(\langle\langle c\rangle\rangle q_0)$ is divisible by $\{c\}$, so $\{a\} f_n^k(\langle\langle c\rangle\rangle q_0) = 0$. From there,

$$\{a\} f_n^d(q') = \{a\} \sum_{k=0}^{d} f_n^k(q) f_n^{d-k}(\langle\langle c\rangle\rangle q_0) = \{a\} f_n^d(q).$$

Suppose now that $a \neq b$. Then Hoffmann shows in [Garibaldi 2009, Corollary B.5] that we have

$$\langle\langle a\rangle\rangle q = \langle\langle a\rangle\rangle q_0 = \langle\langle b\rangle\rangle q_0 = \langle\langle b\rangle\rangle q',$$

where $q_0 = \sum_{i \in J} \langle x_i\rangle \langle\langle c_i\rangle\rangle \in I^n(K)$, and c_i is represented by $\langle\langle ab\rangle\rangle$. The previous discussion shows that $\{a\} f_n^d(q) = \{a\} f_n^d(q_0)$ and $\{b\} f_n^d(q) = \{b\} f_n^d(q_0)$, so it just remains to show that $\{a\} f_n^d(q_0) = \{b\} f_n^d(q_0)$ for any q_0 admitting a decomposition as above. This is a direct consequence of Lemma 11.15. □

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The Godbillon–Vey invariant and equivariant KK-theory

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We construct a groupoid equivariant Kasparov class for transversely oriented foliations in all codimensions. In codimension 1 we show that the Chern character of an associated semifinite spectral triple recovers the Connes–Moscovici cyclic cocycle for the Godbillon–Vey secondary characteristic class.

1. Introduction

In this paper we construct a semifinite spectral triple for codimension 1 foliations whose Chern character is a global, non-étale version of the cyclic cocycle, constructed by Connes and Moscovici [2005], representing the Godbillon–Vey class. The construction passes through groupoid equivariant Kasparov theory, and this initial part of the construction works in all codimensions.

Associated to any foliated manifold $(M, \mathcal{F})$ of codimension q is a canonical real rank q vector bundle $N = TM/T\mathcal{F}$ called the normal bundle. One of the foundational results of the theory of foliated manifolds is *Bott's vanishing theorem*, which states that the Pontrjagin classes $p^i(N)$ of the normal bundle N must vanish for all $i > 2q$ [Bott 1970]. This vanishing theorem guarantees the existence of new characteristic classes for M called *secondary characteristic classes*, which have been studied extensively [Bott 1972; Bott and Haefliger 1972; Kamber and Tondeur 1974]. It has been shown in particular that all such classes arise under the image of a characteristic map from the Gelfand–Fuks cohomology of the Lie algebra of formal vector fields [Gelfand and Fuks 1970] to the cohomology of M [Bott 1976; Bott and Haefliger 1972].

The most famous example of a secondary characteristic class is the Godbillon–Vey invariant, first discovered by Godbillon and Vey [1971], which arises in the context of transversely orientable foliations and can be constructed explicitly at the level of differential forms. More specifically, transverse orientability of a codimension q foliated manifold $(M, \mathcal{F})$ amounts to the existence of a nonvanishing section of the top degree line bundle $\Lambda^q N^*$ of the conormal bundle N^* over M. Any identification of N^* with a subbundle of T^*M, obtained say by equipping M

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with a Riemannian metric, identifies such a section with a nonvanishing differential form $\omega \in \Omega^q(M)$ such that

$$\omega(X_1 \wedge \cdots \wedge X_q) = 0 \tag{1.1}$$

whenever any one of the X_j is contained in the space $\Gamma(T\mathcal{F})$ of vector fields which are tangent to the foliation. Since the subbundle $T\mathcal{F} \subset TM$ is integrable, by the Frobenius theorem one is guaranteed the existence of a 1-form $\eta \in \Omega^1(M)$ for which

$$d\omega = \eta \wedge \omega.$$

The differential form $\eta \wedge (d\eta)^q$ is closed, and its class GV in de Rham cohomology is independent of the choices of ω and η. The Godbillon–Vey invariant has been shown to be closely related to measure theory and dynamics; see [Cantwell and Conlon 1984; Duminy 1982; Heitsch and Hurder 1984; Hurder 1986] for example.

Building on work of Winkelnkemper [1983] which associated to any foliated manifold $(M, \mathcal{F})$ its holonomy groupoid $\mathcal{G}$, Connes [1982] initiated the study of foliated manifolds as noncommutative geometries using the convolution algebra $C_c^\infty(\mathcal{G})$. While the convolution algebra $C_c^\infty(\mathcal{G})$ associated to the full holonomy groupoid is necessary when considering leafwise phenomena [Connes and Skandalis 1984], when considering *transverse geometry only* one can simplify matters in the following way. Choose a q-dimensional submanifold T of M which intersects each leaf of $\mathcal{F}$ at least once and which is everywhere transverse to $\mathcal{F}$ (such a T can be found by taking the disjoint union of local transversals in M defined by a covering of foliated charts). Then the restricted groupoid

$$(\mathcal{G})_T^T := \{u \in \mathcal{G} : s(u), r(u) \in T\}$$

inherits a differential topology from $\mathcal{G}$ under which it is an étale Lie groupoid [Crainic and Moerdijk 2001, Lemma 2]. Importantly, the groupoids $(\mathcal{G})_T^T$ and $\mathcal{G}$ are *Morita equivalent* [Crainic and Moerdijk 2001, Lemma 2]. They therefore have the same cyclic (co)homology [Crainic and Moerdijk 2001; 2004], and have Morita equivalent C^*-algebras [Muhly and Williams 2008] so are the same as far as K-theory is concerned also. In treating the transverse geometry of a foliation it has therefore become standard in the literature to use the étale groupoid $(\mathcal{G})_T^T$ in the place of the full holonomy groupoid $\mathcal{G}$ [Connes 1986; Connes and Moscovici 1998; 2001; 2005; Gorokhovsky 1999; 2002; Crainic and Moerdijk 2004; Moscovici and Rangipour 2007].

A reasonable model for any such étale groupoid $(\mathcal{G})_T^T$ is simply the action groupoid $V \rtimes \Gamma$, where V is an oriented manifold (a stand-in for the transversal T), and where Γ is a discrete group of orientation-preserving diffeomorphisms of V

(which is a stand-in for the action of the holonomy groupoid on T). It is in this setting that Connes [1986, Theorem 7.15] shows that all Gelfand–Fuks cohomology classes (hence all secondary characteristic classes) can be represented by cyclic cocycles on $C_c^\infty(V)\rtimes\Gamma$. Connes gives in particular an explicit formula for the cyclic cocycle defined by the Godbillon–Vey invariant when $V=S^1$. If dx denotes the standard volume form on S^1, then associated to any $g\in\Gamma$ is the $\mathbb{R}$-valued group cocycle

$$\ell(g):=\log\biggl(\frac{d(x\cdot g^{-1})}{dx}\biggr).$$

Connes shows that the formula

$$\phi_{\mathrm{GV}}(f^0,f^1,f^2) := \sum_{g_0g_1g_2=1_\Gamma}\int_{S^1} f^0(x)f^1(x\cdot g_0)f^2(x\cdot g_0g_1)(d\ell(g_1g_2)\ell(g_2)-\ell(g_1g_2)d\ell(g_2)) \tag{1.2}$$

defines a cyclic 2-cocycle on $C_c^\infty(V)\rtimes\Gamma$, and that the class of this 2-cocycle coincides with that defined by the Godbillon–Vey invariant.

More recently, Connes and Moscovici [1998] used a deep link with Hopf symmetry to construct a characteristic map sending Gelfand–Fuks cocycles to cyclic cocycles on the convolution algebra $C_c^\infty(F^+(V))\rtimes\Gamma$ of the groupoid $F^+(V)\rtimes\Gamma$ associated to the lift of Γ to the oriented frame bundle $F^+(V)$ for V. Connes and Moscovici [2005] show that the formula

$$\tilde{\phi}_{\mathrm{GV}}(a^0,a^1):=\sum_{g_0g_1=1_\Gamma}\int_{F^+(V)} a^0(y)(\delta_1a^1)(y\cdot g_0)\tilde{\omega}(y), \tag{1.3}$$

where δ_1 is a derivation of $C_c^\infty(F^+(V))\rtimes\Gamma$ related to $d\ell$ and where $\tilde{\omega}$ is a G-invariant transverse volume form on $F^+(V)$, defines a 1-cocycle on $C_c^\infty(F^+(V))\rtimes\Gamma$ that represents the Godbillon–Vey invariant. As will be shown in this paper, the derivation δ_1 can be realized in the *non-étale setting* of the *full holonomy groupoid* $\mathcal{G}$ of a foliated manifold, where it arises as a commutator between convolution along $\mathcal{G}$ with a dual Dirac operator on a Hilbert space of sections of an exterior algebra bundle. In noncommutative geometry, the Godbillon–Vey invariant has since been further explored in groupoid cohomology [Crainic and Moerdijk 2004], in cyclic cohomology [Gorokhovsky 1999; 2002], via its pairing with the indices of longitudinal Dirac operators [Moriyoshi and Natsume 1996], and in relation to manifolds with boundary [Moriyoshi and Piazza 2012].

Accompanying his introduction of the formula (1.2) for the cyclic cocycle ϕ_{GV}, Connes [1986, p. 4] remarks that the pairing of ϕ_{GV} with K-theory will not in general be integer-valued, which implies that ϕ_{GV} must not arise as the Chern character of a spectral triple on $C_c^\infty(V)\rtimes\Gamma$. Such constraints do not apply to *semifinite spectral triples*, whose pairings with K-theory need not lie in the integers [Connes and Cuntz 1988; Benameur and Fack 2006; Carey and Phillips 1998].

In this paper we will recover the analogue of (1.3) in the global setting of the full holonomy groupoid $\mathcal{G}$, from a semifinite spectral triple. Bearing in mind the close relationship between semifinite spectral triples and KK-theory [Kaad et al. 2012], this fact can be seen already in the étale case of an action groupoid of the form $V \rtimes \Gamma$ using the formalism of differential forms on jet bundles arising from Gelfand–Fuks cohomology [Connes and Moscovici 2005, Proposition 19]. An entirely novel nuance of our constructions, however, is that they are *global in nature*, applying immediately to foliated manifolds *without* needing to choose a complete transversal and pass through a Morita equivalence. This has the advantage of producing cocycles that are defined in terms of global geometric data for $(M, \mathcal{F})$, which previously has not been attempted.

We now outline the layout of the paper. Section 2 will discuss the background required on Clifford bundles, groupoid actions, semifinite spectral triples, and groupoid equivariant KK-theory. Section 3 will detail the constructions of the KK-classes required. The constructions of this section are very natural for foliations of arbitrary codimension, so will be carried out at this level of generality. Section 4 will consist of the proof of an index theorem in codimension 1 which states that the pairing with K-theory of the semifinite spectral triple obtained using the constructions of Section 3 coincides with the pairing coming from the Connes–Moscovici Godbillon–Vey cyclic cocycle. Finally in Section 5 we describe how, in codimension 1, our constructions can be viewed as a global geometric analogue of the jet bundle approach described by Connes [1986]. In particular this justifies our claim that the index formula we obtain really does represent the Godbillon–Vey invariant.

We remark that while the spectral triple itself can be easily constructed for foliations of arbitrary codimension, it is at this stage unclear whether the corresponding index pairing continues to compute the pairing of the higher codimension Godbillon–Vey invariant with K-theory. We leave this question to future work.

2. Background

Here we recall some basic facts about groupoid actions on spaces, Clifford algebras, semifinite spectral triples, groupoid actions on algebras, and the resulting equivariant Kasparov theory.

We will assume that the reader is familiar with locally compact groupoids and their associated convolution algebras [Connes 1982; Renault 1980]. All Hilbert spaces are assumed to be separable. For such a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the bounded operators on $\mathcal{H}$ and by $\mathcal{K}(\mathcal{H})$ the compact operators on $\mathcal{H}$. Inner products on Hilbert modules and Hilbert spaces are assumed to be conjugate-linear in the left variable and linear in the right.

If X, Y, and Z are sets with maps $f : Y \to X$ and $g : Z \to X$, we denote by $Y \times_{f,g} Z$ the fibred product $\{(y, z) \in Y \times Z : f(y) = g(z)\}$ of Y and Z.

2A. ***Clifford algebras.*** For our constructions we will need some facts regarding Clifford algebras and their representations on exterior algebra bundles. First, if $(V, \langle \cdot, \cdot \rangle)$ is a real inner product space with nondegenerate inner product, we denote by $\mathbb{C}\mathrm{liff}(V)$ the *complex Clifford algebra* of V, which is the complexification of the real Clifford algebra $\mathrm{Cliff}(V, \langle \cdot, \cdot \rangle)$.

There exists a linear isomorphism $\psi_V : \Lambda^* V \to \mathrm{Cliff}(V, \langle \cdot, \cdot \rangle)$ between the exterior algebra and the Clifford algebra of V defined with respect to any orthonormal basis $\{e_1, \dots, e_{\mathrm{rank}(V)}\}$ by

$$\psi_V(e_{i_1} \wedge \cdots \wedge e_{i_r}) := e_{i_1} \cdots e_{i_r}$$

for any multi-index $(i_1, \dots, i_r)$ with $r \le \mathrm{rank}(V)$. The isomorphism ψ_V determines the structure of a Clifford bimodule on $\Lambda^*(V)$, with left action given by

$$c_L(a)w := \psi_V^{-1}(a \cdot \psi_V(w))$$

and right action given by

$$c_R(a)w := \psi_V^{-1}(\psi_V(w) \cdot a)$$

for $a \in \mathrm{Cliff}(V)$ and $w \in \Lambda^*(V)$. We have the following important lemma describing how these representations behave with respect to orthogonal maps.

Lemma 2.1. *Let V and W be finite dimensional inner product spaces and let $\psi_V : \Lambda^* V \to \mathrm{Cliff}(V)$ and $\psi_W : \Lambda^* W \to \mathrm{Cliff}(W)$ be the corresponding linear isomorphisms. Then if $A : V \to W$ is an orthogonal transformation with induced algebra isomorphisms $A_\Lambda : \Lambda^* V \to \Lambda^* W$ and $A_{\mathrm{Cliff}} : \mathrm{Cliff}(V) \to \mathrm{Cliff}(W)$, we have*

$$A_{\mathrm{Cliff}} \circ \psi_V = \psi_W \circ A_\Lambda.$$

Proof. Regard V as a subspace of $\Lambda^* V$ in the usual way, let $\iota : V \to \mathrm{Cliff}(V)$ denote the inclusion map, and consider the map $j := (\psi_W \circ A_\Lambda)|_V : V \to \mathrm{Cliff}(W)$. Since A is orthogonal, we have $j(v)^2 = \|v\|^2 1_{\mathrm{Cliff}(W)}$ and so by the universal property of the Clifford algebra, there is a unique algebra isomorphism $\phi : \mathrm{Cliff}(V) \to \mathrm{Cliff}(W)$ such that $\phi \circ \iota = j$. Given any vector $v \in V$ we see that

$$j(v) = A_{\mathrm{Cliff}} \circ \iota(v)$$

so that $\phi = A_{\mathrm{Cliff}}$. Given an orthonormal basis $\{e_1, \dots, e_{\dim(V)}\}$ for V, and a multi-index $(i_1, \dots, i_k)$ we calculate

$$\begin{aligned} A_{\mathrm{Cliff}} \circ \psi_V(e_{i_1} \wedge \cdots \wedge e_{i_k}) &= A_{\mathrm{Cliff}}(\iota(e_{i_1}) \cdots \iota(e_{i_k})) \\ &= A_{\mathrm{Cliff}}(\iota(e_{i_1})) \cdots A_{\mathrm{Cliff}}(\iota(e_{i_k})) \\ &= \psi_W(A_\Lambda(e_{i_1})) \wedge \cdots \wedge \psi_W(A_\Lambda(e_{i_k})) \\ &= \psi_W \circ A_\Lambda(e_{i_1} \wedge \cdots \wedge e_{i_k}), \end{aligned}$$

where the first line is due to the equality $\psi_V|_V = \iota$, and the second is since A_{Cliff} is an algebra homomorphism. By linearity we obtain the required identity. □

By abuse of notation, we have a linear isomorphism $\psi_V : \Lambda^*(V) \otimes \mathbb{C} \to \mathbb{C}\mathrm{liff}(V)$, which gives, by the same formulae as in the real case, commuting actions c_L and c_R of $\mathbb{C}\mathrm{liff}(V)$ on $\Lambda^*(V) \otimes \mathbb{C}$. Any orthogonal map $A : V \to W$ of inner product spaces has the property that the induced maps $A_{\mathbb{C}\mathrm{liff}} : \mathbb{C}\mathrm{liff}(V) \to \mathbb{C}\mathrm{liff}(W)$ and $A_{\Lambda_{\mathbb{C}}} : \Lambda^*(V) \otimes \mathbb{C} \to \Lambda^*(W) \otimes \mathbb{C}$ satisfy $A_{\mathbb{C}\mathrm{liff}} \circ \psi_V = \psi_W \circ A_{\Lambda_{\mathbb{C}}}$.

If Y is a manifold and $E \to Y$ is a Euclidean vector bundle, we obtain a corresponding Clifford algebra bundle $\mathrm{Cliff}(E)$ and exterior bundle $\Lambda^*(E)$, as well as corresponding complexifications $\mathbb{C}\mathrm{liff}(E) = \mathrm{Cliff}(E) \otimes \mathbb{C}$ and $\Lambda^*(E) \otimes \mathbb{C}$. Operating pointwise, we have an isomorphism $\psi_E : \Lambda^*(E) \otimes \mathbb{C} \to \mathbb{C}\mathrm{liff}(E)$ of vector spaces giving $\Lambda^*(E) \otimes \mathbb{C}$ the structure of a $\mathbb{C}\mathrm{liff}(E)$-bimodule, with left and right actions denoted, again by abuse of notation, by c_L and c_R, respectively. We will denote by $\mathbb{C}\ell(E)$ the continuous sections vanishing at infinity of the bundle $\mathbb{C}\mathrm{liff}(E)$ over Y. This $\mathbb{C}\ell(E)$ is a C^*-algebra and is $\mathbb{Z}_2$-graded by even and odd elements.

2B. ***$\mathcal{G}$-spaces and $\mathcal{G}$-bundles.*** Let $\mathcal{G}$ be a groupoid, with unit space X and range and source maps $r : \mathcal{G} \to X$ and $s : \mathcal{G} \to X$, respectively. We say that $\mathcal{G}$ *acts on* (*the left of*) *a set* Y or that Y *is a $\mathcal{G}$-space* if there exists a map $a : Y \to X$ called the *anchor map* and a map $m : \mathcal{G} \times_{s,a} Y \to Y$, denoted $m(u, y) := u \cdot y$, such that

(1) $a(u \cdot y) = r(u)$ for all $(u, y) \in \mathcal{G} \times_{s,a} Y$,

(2) $(uv) \cdot y = u \cdot (v \cdot y)$ for all $(v, y) \in \mathcal{G} \times_{s,a} Y$ and $(u, v) \in \mathcal{G}^{(2)}$,

(3) $a(y) \cdot y = y$ for all $y \in Y$.

If $\mathcal{G}$ and Y are topological or smooth spaces, we require the maps a and m to be continuous or smooth, respectively. The simplest example of a $\mathcal{G}$-space is the unit space X of $\mathcal{G}$.

If $\mathcal{G}$ acts on Y, we denote by $Y \rtimes \mathcal{G}$ the space $Y \times_{a,r} \mathcal{G}$, regarded as a groupoid whose unit space is Y, with range and source maps $r(y, u) := y$ and $s(y, u) := u^{-1} \cdot y$, respectively, and with multiplication defined by

$$(y, u) \cdot (u^{-1} \cdot y, v) := (y, uv)$$

for all $(y, u) \in Y \times_{a,r} \mathcal{G}$ and $(u, v) \in \mathcal{G}^{(2)}$. If $\mathcal{G}$ and Y are topological or smooth spaces, the groupoid $Y \rtimes \mathcal{G}$ is equipped with a topological or smooth structure from its containment as a subspace of the topological or smooth space $Y \times \mathcal{G}$, respectively. While for left $\mathcal{G}$-spaces it is more natural to consider the analogous (and isomorphic) groupoid $\mathcal{G} \ltimes Y$ obtained from the set $\mathcal{G} \times_{s,a} Y$, it will be easier for our purposes to use $Y \rtimes \mathcal{G}$ because, as we will see, our convention in using $\mathcal{G}$-equivariant Kasparov theory consists of forming pullbacks using the range map rather than the source.

We say that a vector bundle $\pi : E \to X$ is *$\mathcal{G}$-equivariant* if E is a $\mathcal{G}$-space, with $\mathcal{G}$-action conventionally denoted $(u, e) \mapsto u_* e$ and with anchor map π, and if for each $u \in \mathcal{G}$ the map $(u, e) \mapsto u_* e$ defined on $E_{s(u)} := \pi^{-1}\{s(u)\}$ is a vector space isomorphism $E_{s(u)} \to E_{r(u)}$. More generally, if $\pi : E \to Y$ is a vector bundle over a $\mathcal{G}$-space Y, we say that E is *$\mathcal{G}$-equivariant* if it is $Y \rtimes \mathcal{G}$-equivariant as a bundle over Y, in which case we will often denote the map $(Y \rtimes \mathcal{G}) \times_{s,\pi} E \to E$, $((y, u), e) \mapsto (y, u)_* e$, by simply $(u, e) \mapsto u_* e$. If $\pi : E \to X$ admits a Euclidean or Hermitian structure, we say that E is a *$\mathcal{G}$-equivariant Euclidean or Hermitian bundle* if for all $(y, u) \in Y \rtimes \mathcal{G}$ the linear isomorphism $E_{u^{-1} \cdot y} \to E_y$ defined by $(u, e) \mapsto u_* e$ is orthogonal or unitary, respectively.

If $\pi : E \to Y$ is a $\mathcal{G}$-equivariant vector bundle over Y, then by functoriality $\Lambda^*(E) \otimes \mathbb{C}$ is also an equivariant bundle over Y, with action of $u \in \mathcal{G}$ denoted by $u_* : \Lambda^*(E|_{Y_{s(u)}}) \otimes \mathbb{C} \to \Lambda^*(E|_{Y_{r(u)}}) \otimes \mathbb{C}$. If moreover E is an equivariant Euclidean bundle, then by functoriality $\mathbb{C}\mathrm{liff}(E)$ is also an equivariant bundle, with action of $u \in \mathcal{G}$ denoted by $u_\diamond : \mathbb{C}\mathrm{liff}(E|_{Y_{s(u)}}) \to \mathbb{C}\mathrm{liff}(E|_{Y_{r(u)}})$. In this case, by Lemma 2.1 we have

$$u_*(c_L(a)e) = c_L(u_\diamond a)(u_* e), \tag{2.2}$$

$$u_*(c_R(a)e) = c_R(u_\diamond a)(u_* e) \tag{2.3}$$

for all $u \in \mathcal{G}$, $a \in \mathbb{C}\mathrm{liff}(E|_{Y_{s(u)}})$, and $e \in \Lambda^*(E|_{Y_{s(u)}})$.

When $(M, \mathcal{F})$ is a foliated manifold with holonomy groupoid $\mathcal{G}$, the normal bundle $N = TM/T\mathcal{F} \to M$ is a $\mathcal{G}$-equivariant bundle. As this fact is fundamental for our constructions, let us briefly review why it is the case. We choose a countable covering of M by foliated charts $\phi_i : U_i \cong T_i \times P_i$, where $T_i \subset \mathbb{R}^q$ and $P_i \subset \mathbb{R}^p$ are open balls, with change-of-chart maps $\varphi_{j,i} := \phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ of the form

$$\varphi_{j,i}(t, p) = (h_{j,i}(t), \tilde{\varphi}_{j,i}(t, p)),$$

such that the $h_{i,j}$ are *compatible* in the sense that they satisfy

$$h_{i,k} = h_{i,j} \circ h_{j,k}$$

whenever $U_i \cap U_j \cap U_k \neq \varnothing$. That such a covering can be chosen can be regarded as the definition of the foliation $\mathcal{F}$ on M [Candel and Conlon 2000, Chapter 1.2]. We say that a path $\gamma : [0, 1] \to M$ is *leafwise* if its image is entirely contained in a leaf L of M, and we refer to its endpoints $\gamma(0)$ and $\gamma(1)$ as its *source* and *range*, denoted $s(\gamma)$ and $r(\gamma)$, respectively. Any leafwise path γ whose image is contained in a union $U_0 \cup U_1$ of charts such that $U_0 \cap U_1 \neq \varnothing$, and with $s(\gamma) \in U_0$ and $r(\gamma) \in U_1$, determines a local diffeomorphism $h_\gamma := h_{1,0}$ on a small neighborhood of $T_0 \subset \mathbb{R}^q$. More generally, if the image of a leafwise path γ is covered by a chain of charts $\{U_0, \dots, U_k\}$ such that for each $0 \leq j < k$ we have $U_j \cap U_{j+1} \neq \varnothing$, on a sufficiently small neighborhood of T_0 we may define a local diffeomorphism

$$h_\gamma := h_{k,k-1} \circ h_{k-1,k-2} \circ \cdots \circ h_{1,0}$$

mapping onto a small neighborhood of T_k. Because of the compatibility of the $h_{i,j}$, the germ of h_γ at $s(\gamma)$ does not depend on the chain of charts chosen in its definition. By definition, the holonomy groupoid $\mathcal{G}$ consists of equivalence classes of leafwise paths γ for which $\gamma_1 \sim \gamma_2$ if and only if γ_1 and γ_2 have the same source and range and the germ at $s(\gamma_1) = s(\gamma_2)$ of h_{γ_1} is equal to that of h_{γ_2}.

In the coordinates defined by a chart U_j, the fibres of N identify with tangent vectors to the transversal neighborhood T_j, and via this identification it follows that for any leafwise path γ in M, the derivative of h_γ furnishes a linear isomorphism

$$dh_\gamma : N_{s(\gamma)} \to N_{r(\gamma)}.$$

It can be seen from the definition of h_γ that $dh_{\gamma_1} \circ dh_{\gamma_2} = dh_{\gamma_1\gamma_2}$ whenever the range of γ_2 is equal to the source of γ_1, where $\gamma_1\gamma_2$ is the path obtained by concatenating γ_1 and γ_2. Since local diffeomorphisms with the same germ at a point have the same derivative at that point, to any $u \in \mathcal{G}$ corresponds a well-defined linear isomorphism $u_* := h_\gamma : N_{s(u)} \to N_{s(u)}$ for any path γ that represents u. Since $dh_{\gamma_1\gamma_2} = dh_{\gamma_1} \circ dh_{\gamma_2}$, we have $(uv)_* = u_* \circ v_*$ for all $(u, v) \in \mathcal{G}^{(2)}$, and so N is indeed a $\mathcal{G}$-equivariant bundle over M.

We remark that in general the normal bundle N of a foliated manifold $(M, \mathcal{F})$ will not admit the structure of a $\mathcal{G}$-equivariant Euclidean bundle. Indeed, the existence of a $\mathcal{G}$-equivariant Euclidean structure for N implies the existence of a $\mathcal{G}$-invariant transverse volume form ω for $(M, \mathcal{F})$, and hence implies the existence of a faithful normal semifinite trace on the von Neumann algebra of $(M, \mathcal{F})$ defined by restricting functions in the weakly dense algebra $C_c(\mathcal{G})$ to M, and then integrating with respect to ω. If the Godbillon–Vey invariant of $(M, \mathcal{F})$ is nonzero, however, then by results of Hurder and Katok [1984, Theorem 2] and, in codimension 1, Connes [1986, Theorem 7.14], the von Neumann algebra of $(M, \mathcal{F})$ contains a type III factor and so admits no nonzero semifinite normal traces. Examples of foliated manifolds with nonzero Godbillon–Vey invariant are known to be plentiful [Thurston 1972].

2C. *Equivariant KK-theory for locally Hausdorff groupoids.* Equivariant KK-theory for Hausdorff topological groupoids was first developed by Le Gall [1994]. Since foliated manifolds generally have only locally Hausdorff holonomy groupoids, Le Gall's treatment requires extension for applications to foliation theory. Androulidakis and Skandalis [2019] have developed an equivariant KK-theory for the holonomy groupoids arising from singular foliations, whose topologies are generally even worse than the locally Hausdorff topologies on the holonomy groupoids of regular foliations, and which include all regular foliation groupoids as a subclass.

This section will summarize the required results and definitions of Androulidakis and Skandalis in the setting of locally Hausdorff Lie groupoids, as well as giving the unbounded picture in parallel with work of Pierrot [2006a]. See also [Muhly and Williams 2008; Tu 2004] for useful perspectives on non-Hausdorff groupoid actions which have further informed the exposition.

Let $\mathcal{G}$ be a locally Hausdorff Lie groupoid with locally compact Hausdorff unit space X, and let $\{U_i\}_{i\in I}$ be a countable cover of $\mathcal{G}$ by Hausdorff open sets. For each $i \in I$ we let $r_i := r|_{U_i}$ and $s_i := s|_{U_i}$ be the restrictions of range and source, respectively, to the set U_i.

Definition 2.4. A $C_0(X)$-*algebra* is a C^*-algebra A together with a homomorphism $\theta : C_0(X) \to \mathcal{M}(A)$ into the multiplier algebra of A such that $\theta(C_0(X))A = A$. For $a \in A$ and $f \in C_0(X)$, we will often denote $\theta(f)a$ by $f \cdot a$.

For $x \in X$, the *fibre over* x is the algebra $A_x := A/I_xA$, where I_x is the kernel of the evaluation functional $C_0(X) \ni f \mapsto f(x)$ on $C_0(X)$.

If A and B are $C_0(X)$-algebras, a homomorphism $\phi : A \to B$ is said to be a $C_0(X)$-*homomorphism* if $\phi(f \cdot a) = f \cdot \phi(a)$ for all $f \in C_0(X)$ and $a \in A$. Such a homomorphism induces a family $\phi_x : A_x \to B_x$ of homomorphisms between the fibres.

The simplest nontrivial example of a $C_0(X)$-algebra is $C_0(Y)$, where Y is a locally compact Hausdorff space equipped with a continuous map $p : Y \to X$. The $C_0(X)$-structure of $C_0(Y)$ is given by $\theta(f)g(y) := f(p(y))g(y)$ for all $f \in C_0(X)$ and $g \in C_0(Y)$, and the fibre over $x \in X$ is $C_0(Y)_x = C_0(Y_x)$, where $Y_x := p^{-1}\{x\}$.

Definition 2.5. Let A be a $C_0(X)$-algebra, and let $p : Y \to X$ be a continuous map of locally compact Hausdorff spaces. Then the *pullback* of A by p is the $C_0(Y)$-algebra $p^*A := C_0(Y) \otimes_{p,C_0(X)} A$, where we take the balanced tensor product by regarding the $C_0(X)$-algebras $C_0(Y)$ and A as $C_0(X)$-modules. If there is no ambiguity about the map p, it will often be omitted from the notation, so that $p^*A = C_0(Y) \otimes_{C_0(X)} A$.

It is easy to check that if A is a $C_0(X)$-algebra and $p : Y \to X$ is a continuous map of locally compact Hausdorff spaces, the fibre over $y \in Y$ of p^*A is $A_{p(y)}$. Equipped with the notion of pullbacks, we can define what is meant by a $\mathcal{G}$-algebra.

Definition 2.6. Let A be a $C_0(X)$-algebra. A *$\mathcal{G}$-action* on A is a family $\alpha = \{\alpha^i : s_i^*A \to r_i^*A\}_{i\in I}$ of grading-preserving $C_0(U_i)$-isomorphisms, such that $\alpha^i|_{s|^*_{U_i\cap U_j}A} = \alpha^j|_{s|^*_{U_i\cap U_j}A}$ for all $i, j \in I$, and such that the induced homomorphisms $\alpha_u : A_{s(u)} \to A_{r(u)}$ satisfy $\alpha_{uv} = \alpha_u \circ \alpha_v$. If A admits a $\mathcal{G}$-action α, we call (A, α) a *$\mathcal{G}$-algebra.*

The simplest nontrivial example of a $\mathcal{G}$-algebra is $C_0(Y)$, where Y is a $\mathcal{G}$-space with anchor map $p : Y \to X$, and where $C_0(Y)$ is equipped with the $\mathcal{G}$-action

$$\alpha_u(f)(y) := f(u^{-1} \cdot y)$$

for all $u \in \mathcal{G}$ and $f \in C_0(Y_{r(u)})$.

Now suppose that E is a Hilbert module over a $\mathcal{G}$-algebra A. For $x \in X$, we can consider the fibre $E_x := E \otimes_A A_x$, which is a Hilbert A_x-module, and if $p : Y \to X$ is a continuous map of locally compact Hausdorff spaces, we can consider the pullback $p^*E := E \otimes_A p^*A$, which is a Hilbert p^*A-module. If T is an A-linear operator on E, we let $p^*T := T \otimes 1_{p^*A}$ be its pullback to a p^*A-linear operator on p^*E.

Definition 2.7. Let (A, α) be a $\mathcal{G}$-algebra, and let E be a $\mathbb{Z}_2$-graded Hilbert A-module. A *$\mathcal{G}$-action* on E consists of a family $W = \{W^i : s_i^*E \to r_i^*E\}_{i\in I}$ of grading-preserving isometric Banach space isomorphisms, such that $W^i|_{s|^*_{U_i\cap U_j}E} = W^j|_{s|^*_{U_i\cap U_j}E}$ for all $i, j \in I$, and such that the induced isomorphisms $W_u : E_{s(u)} \to E_{r(u)}$ on the fibres satisfy $W_{uv} = W_u \circ W_v$, $\langle W_u\rho_1, W_u\rho_2\rangle_{r(u)} = \alpha_u(\langle\rho_1, \rho_2\rangle_{s(u)})$, and $W_u(\rho \cdot a) = W_u(\rho) \cdot \alpha_u(a)$ for all $(u, v) \in \mathcal{G}^{(2)}$, $a \in A_{s(u)}$, and $\rho, \rho_1, \rho_2 \in E_{s(u)}$. If E admits a $\mathcal{G}$-action W, we call (E, W) a *$\mathcal{G}$-Hilbert A-module.*

If $V \to Y$ is a $\mathcal{G}$-equivariant Hermitian vector bundle over a $\mathcal{G}$-space Y, then the continuous sections vanishing at infinity $\Gamma_0(Y; V)$ of V over Y is a $\mathcal{G}$-Hilbert $C_0(Y)$-module, with pointwise inner product and right action by $C_0(Y)$, and with $\mathcal{G}$-action defined by

$$(W_u\rho)(y) := u_*\rho(u^{-1} \cdot y) \tag{2.8}$$

for all $\rho \in \Gamma_0(Y_{r(u)}; V|_{Y_{r(u)}})$. All $\mathcal{G}$-Hilbert module constructions in this paper will arise from some variant of the action (2.8).

Definition 2.9. If B is a $\mathcal{G}$-algebra, and $\pi : A \to \mathcal{L}(E)$ is a representation of a $\mathcal{G}$-algebra (A, α) on a $\mathcal{G}$-Hilbert B-module (E, W), we say that π is *equivariant* if for all $i \in I$ we have

$$\mathrm{Ad}_{W^i}(\pi_i^s(a)) = \pi_i^r(\alpha^i(a))$$

for all $a \in A$. Here $\pi_i^s := 1_{C_b(U_i)} \otimes \pi$ and $\pi_i^r := 1_{C_b(U_i)} \otimes \pi$ are the induced homomorphisms $s_i^* A = C_0(U_i) \otimes_{s, C_0(X)} A \to \mathcal{L}(s_i^* E)$ and $r_i^* A = C_0(U_i) \otimes_{r, C_0(X)} A \to \mathcal{L}(r_i^* E)$, respectively.

The definition of the equivariant KK-groups now follows in the usual way.

Definition 2.10. Let (A, α) and (B, β) be $\mathcal{G}$-C^*-algebras. A *$\mathcal{G}$-equivariant Kasparov A-B-module* is a triple $(A, {}_\pi E_B, F)$, where (E, W) is a $\mathcal{G}$-equivariant Hilbert B-module carrying an equivariant representation $\pi : A \to \mathcal{L}(E)$, and where $F \in \mathcal{L}(E)$ is homogeneous of degree 1 such that for all $a \in A$ one has

(1) $\pi(a)(F - F^*) \in \mathcal{K}(E)$,

(2) $\pi(a)(F^2 - 1) \in \mathcal{K}(E)$,

(3) $[F, \pi(a)] \in \mathcal{K}(E)$,

and such that for all $i \in I$

(4) $\pi_i^r(r_i^*(a))(r_i^* F - W^i \circ s_i^* F \circ (W^i)^{-1}) \in r_i^* \mathcal{K}(E)$.

We say that two $\mathcal{G}$-equivariant Kasparov A-B-modules $(A, {}_\pi E_B, F)$, $(A, {}_{\pi'} E'_B, F')$ are *unitarily equivalent* if there exists a $\mathcal{G}$-equivariant unitary $V : E \to E'$ of degree 0 such that $VFV^* = F'$ and $V\pi(a)V^* = \pi'(a)$ for all $a \in A$. We denote by $\mathbb{E}^{\mathcal{G}}(A, B)$ the set of all unitary equivalence classes of $\mathcal{G}$-equivariant Kasparov A-B-modules.

A *homotopy* in $\mathbb{E}^{\mathcal{G}}(A, B)$ is an element of $\mathbb{E}^{\mathcal{G}}(A, B[0, 1])$, and we denote by $KK^{\mathcal{G}}(A, B)$ the set of homotopy equivalence classes in $\mathbb{E}^{\mathcal{G}}(A, B)$.

The direct sum of $\mathcal{G}$-equivariant Kasparov A-B-modules makes $KK^{\mathcal{G}}(A, B)$ into an abelian group.

We also need *unbounded representatives* of equivariant KK-classes. The definition for such representatives is the natural extension of that due to Pierrot [2006a] to the locally Hausdorff case. We remark here that if $\mathcal{A}$ is a dense $*$-subalgebra of a $C_0(X)$-algebra A, then we will assume that $C_0(X) \cdot \mathcal{A} \subset \mathcal{A}$, which will be true in our examples. We will denote by $\mathcal{A}_x := \mathcal{A}/I_x\mathcal{A}$ the fibre over $x \in X$, where as before I_x is the kernel of the evaluation functional $f \mapsto f(x)$ on $C_0(X)$.

Definition 2.11. Let A and B be $\mathcal{G}$-algebras. An *unbounded $\mathcal{G}$-equivariant Kasparov A-B-module* is a triple $(\mathcal{A}, {}_\pi E, D)$, where (E, W) is a $\mathcal{G}$-Hilbert B-module carrying an equivariant representation π of A in $\mathcal{L}(E)$, and D is a densely defined, odd, unbounded, self-adjoint, and regular operator on E commuting with the right action of B, and where $\mathcal{A}$ is a dense $*$-subalgebra of A preserved by the action of $\mathcal{G}$ such that for all $a \in \mathcal{A}$ one has

(1) $\pi(a)\,\mathrm{dom}(D) \subset \mathrm{dom}(D)$,

(2) $[D, \pi(a)]$ extends to an element of $\mathcal{L}(E)$,

(3) $\pi(a)(1+D^2)^{-1/2} \in \mathcal{K}(E)$,

and such that for all $i \in I$, $a \in \mathcal{A}$, and $f \in C_c(U_i)$ one has

(4) $f \cdot \pi_i^r(r_i^*(a)) \cdot (r_i^* D - W^i \circ s_i^* D \circ (W^i)^{-1})$ extends to an element of $\mathcal{L}(r_i^* E)$,

(5) $\mathrm{dom}((r_i^* D) f) = W^i \,\mathrm{dom}((s_i^* D) f)$.

That all unbounded equivariant Kasparov modules define classes in $KK^{\mathcal{G}}$ is an easy consequence of the corresponding result by Pierrot for Hausdorff groupoids.

Proposition 2.12. *Let A and B be $\mathcal{G}$-algebras, and let $(\mathcal{A}, {}_\pi E, D)$ be an unbounded $\mathcal{G}$-equivariant Kasparov A-B-module. Then $(A, {}_\pi E, D(1+D^2)^{-1/2})$ is a $\mathcal{G}$-equivariant Kasparov A-B-module.*

Proof. That $(A, {}_\pi E, D(1+D^2)^{-1/2})$ satisfies the first three requirements of Definition 2.10 is a consequence of the corresponding result in the nonequivariant case [Baaj and Julg 1983]. That the fourth requirement is met is a consequence of restricting the corresponding result of Pierrot [2006a, Théorème 6] to each of the Hausdorff open subsets U_i of $\mathcal{G}$. □

We now come to the descent map in equivariant KK-theory, for which we need to discuss groupoid crossed products. We will assume for this that $\mathcal{G}$ comes equipped with a bundle $\Omega^{1/2} \to \mathcal{G}$ of leafwise half-densities, as in [Connes 1994, Chapter 2.8]. Regard a $C_0(X)$-algebra A as the continuous sections vanishing at infinity $\Gamma_0(X; \mathfrak{A})$ of the upper-semicontinuous bundle $\mathfrak{A} \to X$ of C^*-algebras whose fibre over $x \in X$ is A_x [Le Gall 1994; Muhly and Williams 2008]. Thus, a $\mathcal{G}$-algebra (A, α) can be regarded as the continuous sections vanishing at infinity of the $\mathcal{G}$-space $\mathfrak{A}$ over X, where $\alpha_u : A_{s(u)} \to A_{r(u)}$ determines the action of $\mathcal{G}$ on the bundle $\mathfrak{A}$.

Define $\Gamma_c(\mathcal{G}; r^*\mathfrak{A} \otimes \Omega^{1/2})$ to be the space of finite linear combinations of sections of the bundle $r^*\mathfrak{A} \otimes \Omega^{1/2} \to \mathcal{G}$ which have compact support and are continuous in one of the U_i. The space $\Gamma_c(\mathcal{G}; r^*\mathfrak{A} \otimes \Omega^{1/2})$ is a $*$-algebra equipped with the convolution product

$$(f * g)_u := \int_{v \in \mathcal{G}^{r(u)}} f_v \alpha_v(g_{v^{-1}u}) \quad \text{and with involution} \quad (f^*)_u := \alpha_u((f_{u^{-1}})^*).$$

The appropriate completion of $\Gamma_c(\mathcal{G}; r^*\mathfrak{A} \otimes \Omega^{1/2})$ to a reduced C^*-algebra $A \rtimes_r \mathcal{G}$ has been given in [Khoshkam and Skandalis 2004, §3.7].

In a similar manner, if A is a $\mathcal{G}$-algebra we can regard any $\mathcal{G}$-Hilbert A-module E as the continuous sections vanishing at infinity of an upper-semicontinuous bundle $\mathfrak{E} \to X$ whose fibre over $x \in X$ is E_x. We define $\Gamma_c(\mathcal{G}; r^*\mathfrak{E} \otimes \Omega^{1/2})$ to be the space of finite linear combinations of sections of the bundle $r^*\mathfrak{E} \otimes \Omega^{1/2} \to \mathcal{G}$ that have compact support and are continuous in one of the U_i. The formulae

$$\langle \rho^1, \rho^2 \rangle_u^{\mathcal{G}} := \int_{v \in \mathcal{G}^{r(u)}} \alpha_v \langle \rho^1_{v^{-1}}, \rho^2_{v^{-1}u} \rangle \quad \text{and} \quad (\rho \cdot f)_u := \int_{v \in \mathcal{G}^{r(u)}} \rho_v \alpha_v (f_{v^{-1}u})$$

defined for $\rho^1, \rho^2, \rho \in \Gamma_c(\mathcal{G}; r^*\mathfrak{E} \otimes \Omega^{1/2})$ and $f \in \Gamma_c(\mathcal{G}; r^*\mathfrak{A} \otimes \Omega^{1/2})$ determine an $A \rtimes_r \mathcal{G}$-valued inner product and right action, respectively, on $\Gamma_c(\mathcal{G}; r^*\mathfrak{E} \otimes \Omega^{1/2})$, and we may complete in the norm arising from $\langle \cdot , \cdot \rangle^{\mathcal{G}}$ to obtain a Hilbert $A \rtimes_r \mathcal{G}$-module which we denote by $E \rtimes_r \mathcal{G}$. If T is an A-linear operator on E, we denote by $\mathfrak{Dom}(T)$ the bundle over X whose fibre over $x \in X$ is $\mathrm{dom}(T) \otimes_A A_x$. As in [Pierrot 2006a, Définition 2, Proposition 3] we define $r^*(T)$ on $\Gamma_c(\mathcal{G}; r^* \mathfrak{Dom}(T) \otimes \Omega^{1/2})$ by

$$(r^*(T)\rho)_u := T_{r(u)} \rho_u.$$

If $T \in \mathcal{L}(E)$, one can use the norm of T to bound that of $r^*(T)$, and then one can use T^* to show that $r^*(T) \in \mathcal{L}(E \rtimes_r \mathcal{G})$.

Lemma 2.13. *For any densely defined A-linear operator $T : \mathrm{dom}(T) \to E$, we have $r^*(T^*) \subset r^*(T)^*$. Moreover, $\overline{r^*(T^*)} = r^*(T)^*$.*

Proof. Fix $\xi \in \mathrm{dom}(r^*(T^*)) = \Gamma_c(\mathcal{G}; r^* \mathfrak{Dom}(T^*) \otimes \Omega^{1/2})$, and assume without loss of generality that ξ has compact support in some Hausdorff open subset U_i of $\mathcal{G}$. For each $u \in \mathcal{G}$, use the fact that $\xi_u \in \mathfrak{Dom}(T^*)_{r(u)} \otimes \Omega_u^{1/2}$ to define a section η of $r^*\mathfrak{E} \otimes \Omega^{1/2} \to \mathcal{G}$ by

$$\eta_u := T^*_{r(u)} \xi_u.$$

Because ξ is continuous with compact support in U_i, so too is η; therefore, $\eta \in \Gamma_c(\mathcal{G}, r^*\mathfrak{E} \otimes \Omega^{1/2})$. For any $\rho \in \mathrm{dom}(r^*(T)) = \Gamma_c(\mathcal{G}; r^* \mathfrak{Dom}(T) \otimes \Omega^{1/2})$ we can then calculate

$$\begin{aligned} \langle \xi, r^*(T)\rho \rangle_u^{\mathcal{G}} &= \int_{v \in \mathcal{G}^{r(u)}} \alpha_v (\langle \xi_{v^{-1}}, T_{s(v)} \rho_{v^{-1}u} \rangle) \\ &= \int_{v \in \mathcal{G}^{r(u)}} \alpha_v (\langle T^*_{s(v)} \xi_{v^{-1}}, \rho_{v^{-1}u} \rangle) = \langle \eta, \rho \rangle_u^{\mathcal{G}} \end{aligned}$$

for all $u \in \mathcal{G}$, so that $\xi \in \mathrm{dom}(r^*(T)^*)$. The above calculation also shows that $r^*(T)^*\xi = \eta = r^*(T^*)\xi$, so that we indeed have $r^*(T^*) \subset r^*(T)^*$.

Fix $\xi \in \mathrm{dom}(r^*(T)^*)$. Then we will show that $\xi \in \overline{r^*(T^*)}$. Let $\{\xi^n\}_{n \in \mathbb{N}} \subset \Gamma_c(\mathcal{G}; r^* \mathfrak{Dom}(T^*) \otimes \Omega^{1/2})$ be a sequence converging in $E \rtimes_r \mathcal{G}$ to ξ. Then the sequence $\{\langle \xi^n, r^*(T)\rho \rangle^{\mathcal{G}}\}_{n \in \mathbb{N}}$ of elements of $\Gamma_c(\mathcal{G}; r^*\mathfrak{A} \otimes \Omega^{1/2})$ defined for $u \in \mathcal{G}$ by

$$\langle \xi^n, r^*(T)\rho \rangle_u^{\mathcal{G}} = \int_{v \in \mathcal{G}^{r(u)}} \alpha_v (\langle \xi^n_{v^{-1}}, T_{s(v)} \rho_{v^{-1}u} \rangle) = \int_{v \in \mathcal{G}^{r(u)}} \alpha_v (\langle T^*_{s(v)} \xi^n_{v^{-1}}, \rho_{v^{-1}u} \rangle) \tag{2.14}$$

converges in $A \rtimes_r \mathcal{G}$ for all $\rho \in \Gamma_c(\mathcal{G}; r^* \mathfrak{dom}(T) \otimes \Omega^{1/2})$. For each $v \in \mathcal{G}^{r(u)}$ one can on the right-hand side of (2.14) take bump functions ρ with support of decreasing radius about $v^{-1}u$ to show that we have convergence of $\{(r^*(T^*)\xi^n)_{v^{-1}} = T^*_{s(v)}\xi^n_{v^{-1}}\}_{n\in\mathbb{N}}$ to an element of $E_{s(v)}$, and doing this for all $v \in \mathcal{G}^{r(u)}$ and all $u \in \mathcal{G}$ shows that in fact $\{r^*(T^*)\xi^n\}_{n\in\mathbb{N}}$ converges in $E \rtimes_r \mathcal{G}$, implying that $\xi^n \to \xi$ in the graph norm on $\mathrm{dom}(r^*(T^*))$ as claimed. □

Finally, we observe that if A and B are $\mathcal{G}$-algebras, and if (E, W) is a $\mathcal{G}$-Hilbert B-module with an equivariant representation $\pi : A \to \mathcal{L}(E)$, then the formula

$$((\pi \rtimes_r \mathcal{G})(f)\rho)_u := \int_{v\in\mathcal{G}^{r(u)}} \pi(f_v) W_v(\rho_{v^{-1}u})$$

defined for $f \in \Gamma_c(\mathcal{G}; r^*\mathfrak{A} \otimes \Omega^{1/2})$ and $\rho \in \Gamma_c(\mathcal{G}; r^*\mathfrak{E} \otimes \Omega^{1/2})$ determines a representation $\pi \rtimes_r \mathcal{G} : A \rtimes_r \mathcal{G} \to \mathcal{L}(E \rtimes_r \mathcal{G})$.

Proposition 2.15. *Let A and B be $\mathcal{G}$-algebras and $(\mathcal{A}, {}_\pi E, D)$ a $\mathcal{G}$-equivariant unbounded Kasparov A-B-module. Let $\tilde{\mathcal{A}}$ denote the bundle of $*$-algebras over X whose fibre over $x \in X$ is $\mathcal{A}_x$. Then*

$$(\Gamma_c(\mathcal{G}; r^*\tilde{\mathcal{A}} \otimes \Omega^{1/2}), {}_{\pi\rtimes_r\mathcal{G}}E \rtimes_r \mathcal{G}, r^*(D))$$

is an unbounded Kasparov $A \rtimes_r \mathcal{G}$-$B \rtimes_r \mathcal{G}$-module.

Proof. Since D is odd for the grading of E, $r^*(D)$ is odd for the induced grading of $E \rtimes_r \mathcal{G}$. Symmetry of D gives symmetry of $r^*(D)$, so without loss of generality we may assume that $r^*(D)$ is closed. Self-adjointness of $r^*(D)$ is then a consequence of the self-adjointness of D together with Lemma 2.13.

Regularity of $r^*(D)$ is a consequence of the regularity of D. Indeed, for any $\rho \in \Gamma_c(\mathcal{G}; r^* \mathfrak{dom}(D) \otimes \Omega^{1/2})$ we have

$$((1 + r^*(D)^2)\rho)_u = (1_{r(u)} + D^2_{r(u)})\rho_u.$$

Hence, the range of the operator $(1+r^*(D)^2)$ when restricted to $\Gamma_c(\mathcal{G}; r^* \mathfrak{dom}(D) \otimes \Omega^{1/2})$ is $\Gamma_c(\mathcal{G}; r^* \mathfrak{range}(1 + D^2) \otimes \Omega^{1/2})$, where $\mathfrak{range}(1 + D^2)$ denotes the bundle over X whose fibre over $x \in X$ is $\mathrm{range}(1 + D^2) \otimes_A A_x$, which by regularity of D is dense in $E_x = E \otimes_A A_x$. Thus, the range of $(1 + r^*(D)^2)$ contains the dense subspace $\Gamma_c(\mathcal{G}; r^* \mathfrak{range}(1 + D^2) \otimes \Omega^{1/2})$ of $E \rtimes_r \mathcal{G}$, and it follows that $r^*(D)$ is regular.

Regarding commutators, a simple calculation tells us that for any $u \in \mathcal{G}$, the vector $([r^*(D), (\pi \rtimes_r \mathcal{G})(f)]\rho)_u$ is equal to

$$\int_{v\in\mathcal{G}^{r(u)}} ([D_{r(u)}, \pi(f_v)] + \pi(f_v)(D_{r(v)} - W_v \circ D_{s(v)} \circ W_{v^{-1}}))(W_v\rho_{v^{-1}u})$$

for all $\rho \in \Gamma_c(\mathcal{G}; r^*\mathfrak{Dom}(T) \otimes \Omega^{1/2})$, so properties (2) and (4) in Definition 2.11 imply that the operator $[r^*(D), (\pi \rtimes_r \mathcal{G})(f)]$ extends to an element of $\mathcal{L}(E \rtimes_r \mathcal{G})$, with adjoint $[r^*(D), (\pi \rtimes_r \mathcal{G})(f^*)]$.

The only thing remaining to check is compactness of $(\pi \rtimes_r \mathcal{G})(f)(1+r^*(D)^2)^{-1/2}$ for $f \in \Gamma_c(\mathcal{G}; r^*\tilde{\mathcal{A}} \otimes \Omega^{1/2})$. For any $\rho \in \Gamma_c(\mathcal{G}; r^*\mathfrak{E} \otimes \Omega^{1/2})$ the definition of $r^*(D)$ gives

$$\begin{aligned}((1+r^*(D)^2)^{-1/2}(\pi \rtimes_r \mathcal{G})(f^*)\rho)_u &= (1+D_{r(u)}^2)^{-1/2}\int_{v\in\mathcal{G}^{r(u)}} \pi((f)_v^*)W_v(\rho_{v^{-1}u}) \\ &= \int_{v\in\mathcal{G}^{r(u)}} (1+D_{r(v)}^2)^{-1/2}\pi((f)_v^*)W_v(\rho_{v^{-1}u}),\end{aligned}$$

and since $(1+D_{r(v)}^2)^{-1/2}\pi((f)_v^*) \in \mathcal{K}(E)_{r(v)}$ for all $v \in \mathcal{G}^{r(u)}$ by property (3) in Definition 2.11, it follows that $(1+r^*(D)^2)^{-1/2}(\pi \rtimes_r \mathcal{G})(f^*)$ is an element of $\Gamma_c(\mathcal{G}; r^*\mathcal{K}(E) \otimes \Omega^{1/2})$. A similar argument to the one used in [Kasparov 1988, p. 172] then tells us that $(1+r^*(D))^{-1/2}(\pi \rtimes_r \mathcal{G})(f^*)$ can be approximated by finite-rank operators on $E \rtimes_r \mathcal{G}$ so is an element of $\mathcal{K}(E \rtimes_r \mathcal{G})$, and hence so too is its adjoint $(\pi \rtimes_r \mathcal{G})(f)(1+r^*(D)^2)^{-1/2}$. □

Let us remark finally that if Y is a locally compact Hausdorff $\mathcal{G}$-space, with corresponding bundle $C_0(\mathfrak{Y}) \to X$ whose fibre over $x \in X$ is $C_0(Y_x)$, then we have an inclusion $\Gamma_c(Y \rtimes \mathcal{G}; \Omega^{1/2}) \ni f \mapsto \tilde{f} \in \Gamma_c(\mathcal{G}; r^*C_0(\mathfrak{Y}) \otimes \Omega^{1/2})$ defined by

$$\tilde{f}_u(y) := f(y, u).$$

For ease of notation we will usually just refer to $\tilde{f}$ as f. By density of $C_c(Y_x)$ in $C_0(Y_x)$ for each $x \in X$, the subalgebra $\Gamma_c(Y \rtimes \mathcal{G}; \Omega^{1/2})$ is dense in $C_0(Y) \rtimes_r \mathcal{G}$. We will use this fact in the construction of our Godbillon–Vey spectral triple.

2D. ***Semifinite spectral triples.*** One of the defining features of a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is that the operators $a(1+\mathcal{D}^2)^{-1/2}$ are contained in the compact operators $\mathcal{K}(\mathcal{H})$ for all $a \in \mathcal{A}$. These compact operators come equipped with a trace Tr, which is used to measure the rank of projections that appear in the definition of the index, and subsequent index formulae [Connes and Moscovici 1995; Higson 2004].

Semifinite spectral triples are a generalization of spectral triples for which the rank of projections is measured by a different trace. More precisely we require a faithful normal semifinite trace τ on a semifinite von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$. We denote by $\mathcal{K}_\tau(\mathcal{N})$ the norm-closed ideal in $\mathcal{N}$ generated by projections of finite τ-trace, and refer to $\mathcal{K}_\tau(\mathcal{N})$ as the ideal of *τ-compact operators* [Fack and Kosaki 1986].

Definition 2.16. Let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra, regarded as an algebra of operators on a Hilbert space $\mathcal{H}$. A *semifinite spectral triple relative to* $(\mathcal{N}, \tau)$ is a triple $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$ consisting of a $*$-algebra $\mathcal{A}$ represented in $\mathcal{N}$ by

$\pi : \mathcal{A} \to \mathcal{N} \subset \mathcal{B}(\mathcal{H})$, and a densely defined, unbounded, self-adjoint operator $\mathcal{D}$ affiliated to $\mathcal{N}$ such that

(1) $\pi(a)\,\mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$ so that $[\mathcal{D}, \pi(a)]$ is densely defined, and moreover extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$,

(2) $\pi(a)(1+\mathcal{D}^2)^{-1/2} \in \mathcal{K}_\tau(\mathcal{N})$ for all $a \in \mathcal{A}$.

We say that $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$ is *even* if $\mathcal{A}$ is even and $\mathcal{D}$ is odd for some $\mathbb{Z}_2$-grading on $\mathcal{H}$, and otherwise we call $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$ *odd*.

Connes' original notion of spectral triple defines a subclass of semifinite spectral triples, for which $(\mathcal{N}, \tau) = (\mathcal{B}(\mathcal{H}), \mathrm{Tr})$. Just as the bounded transform of a spectral triple $(\mathcal{A}, {}_\pi\mathcal{H}, \mathcal{D})$ defines a Fredholm module (over the C^*-completion A of $\mathcal{A}$), and hence a class in $KK_*(A, \mathbb{C})$, semifinite spectral triples have a close relationship with KK-theory.

To see this, we first suppose B is a C^*-algebra, X_B is a Hilbert B-module with inner product $\langle \cdot, \cdot \rangle_B$, and τ is a faithful norm lower-semicontinuous semifinite trace on B. We can form the GNS space $L^2(B, \tau)$, or $L^2(X, \tau)$ with inner product $(x \mid y) = \tau(\langle x, y\rangle_B)$. These two Hilbert spaces are related by $X \otimes_B L^2(B, \tau) \cong L^2(X, \tau)$.

Then by results in [Laca and Neshveyev 2004], we obtain a faithful normal semifinite trace Tr_τ, called the *dual trace*, on the weak closure $\mathcal{N} = \mathrm{End}_B(X)'' \subset \mathcal{B}(L^2(X_B, \tau))$ of the adjointable B-linear operators on X_B. The functional Tr_τ satisfies

$$\mathrm{Tr}_\tau(\Theta_{\xi,\eta}) := \tau(\langle \eta, \xi\rangle_B).$$

Proposition 2.17. *Let $(\mathcal{A}, {}_\pi X_B, \mathcal{D})$ be an even or odd unbounded Kasparov A-B-module, and suppose that τ is a faithful norm lower semicontinuous semifinite trace on B. Let $(\mathcal{N}, \mathrm{Tr}_\tau)$ be the semifinite von Neumann algebra obtained from X_B and τ as above. Then (with a slight abuse of notation)*

$$(\mathcal{A}, {}_{\pi\hat{\otimes}1} X_B \hat{\otimes}_B L^2(B, \tau), \mathcal{D} \hat{\otimes} 1) = (\mathcal{A}, {}_\pi L^2(X_B, \tau), \mathcal{D})$$

is an even or odd semifinite spectral triple relative to $(\mathcal{N}, \mathrm{Tr}_\tau)$, respectively.

Proof. Clearly $\mathcal{A} \subset \mathcal{N}$, and the commutant of $\mathcal{N}$ is just B''. Since $\mathcal{D}$ is B-linear, every unitary in B'' preserves the domain of $\mathcal{D} \hat{\otimes} 1$, whence $\mathcal{D} \hat{\otimes} 1$ is affiliated to $\mathcal{N}$. That $[\mathcal{D} \hat{\otimes} 1, \pi(a) \hat{\otimes} 1]$ is bounded for all $a \in \mathcal{A}$ is a consequence of the corresponding fact for the Kasparov module $(\mathcal{A}, {}_\pi X_B, \mathcal{D})$, and that $(\pi(a)\hat{\otimes}1)(1+\mathcal{D}\hat{\otimes}1^2)^{-1/2}$ is τ-compact is true since the algebra $\mathcal{K}(X_B)$ is contained in $\mathcal{K}_\tau(\mathcal{N})$ by construction. □

In fact, a converse to Proposition 2.17 is also true: namely, every semifinite spectral triple can be factorized into a KK-class and a trace [Kaad et al. 2012]. Although we will not need this converse result, it provides a useful way of thinking about semifinite spectral triples.

One of the most useful features of (nice) spectral triples is that their pairing with K-theory can be computed using the local index formula [Connes and Moscovici 1995]. The same is true for (nice) semifinite spectral triples. There are now numerous results generalizing the Connes–Moscovici local index formula for spectral triples to semifinite spectral triples [Benameur and Fack 2006; Carey et al. 2004; 2006a; 2006b; 2008; 2012; 2014].

3. Construction of the Kasparov modules

In this section, $(M, \mathcal{F})$ will denote a transversely orientable foliated manifold of codimension q, with holonomy groupoid $\mathcal{G}$ and normal bundle $N = TM/T\mathcal{F} \to M$. The normal bundle is a $\mathcal{G}$-equivariant vector bundle, as explained at the end of Section 2B, and for $u \in \mathcal{G}$ we let $u_* : N_{s(u)} \to N_{r(u)}$ be the corresponding map $n \mapsto u_* n$. We assume $\mathcal{G}$ to be equipped with a countable cover $\mathcal{U} := \{U_i\}_{i \in I}$ by Hausdorff open subsets. We do not assume K-orientability at any point, working with exterior algebra bundles instead of spinor bundles.

The first of the two constructions, the Connes fibration, will not be featured in the index theorem in the final section. The Kasparov module of the Connes fibration provides a Thom-type isomorphism which does not conceptually affect our final index formulae. We include the Connes fibration for the sake of completeness, and to show that the whole construction does indeed factor through groupoid equivariant KK-theory.

3A. ***The Connes fibration.*** We begin this section with a revision of a construction due to Connes [1986]. Connes starts with an oriented manifold M of dimension n with an action of a discrete group Γ of orientation-preserving diffeomorphisms. Such a setting provides an étale model of the transverse geometry of a transversely oriented foliation.

Connes shows that if $W \to M$ denotes the "bundle of Euclidean metrics" for the tangent bundle TM over M, then one can construct a dual Dirac class in $KK^{\Gamma}_{n(n+1)/2}(C_0(M), C_0(W))$. The manifold W has the advantage that the pullback of TM to W admits a Γ-invariant Euclidean metric, even though one need not exist on M in general. We show that Connes' construction can be carried out directly in the groupoid equivariant setting, as it may be useful for future work in constructing the Godbillon–Vey invariant as a semifinite spectral triple in arbitrary codimension.

We let $\pi_F : F^+(N) \to M$ be the principal $GL^+(q, \mathbb{R})$-bundle of positively oriented frames for the vector bundle $N \to M$, whose fibre $(F^+(N))_x$ over $x \in M$ consists of positively oriented linear isomorphisms $\phi : \mathbb{R}^q \to N_x$. Then $F^+(N)$ is a $\mathcal{G}$-space with anchor map $\pi_F : F^+(N) \to M$ and action defined by

$$u \cdot \phi := u_* \circ \phi : \mathbb{R}^q \to N_{r(u)} \tag{3.1}$$

for $\phi : \mathbb{R}^q \to N_{s(u)}$ in $F^+(N)_{s(u)}$. Observe that this action of $\mathcal{G}$ commutes with the right action of $GL^+(q, \mathbb{R})$ on the principal $GL^+(q, \mathbb{R})$-bundle $F^+(N) \to M$.

The vertical subbundle $\ker(d\pi_F) = VF^+(N) \to F^+N$ of $TF^+(N)$ admits a trivialization $VF^+(N) \to F^+(N) \times \mathfrak{gl}(q, \mathbb{R})$, where $\mathfrak{gl}(q, \mathbb{R}) = M_q(\mathbb{R})$ is the Lie algebra of $GL^+(q, \mathbb{R})$ consisting of all $q \times q$ real matrices. The trivialization is given by the formula

$$F^+(N) \times \mathfrak{gl}(q, \mathbb{R}) \ni (\phi, v) \mapsto v_\phi := \frac{d}{dt}(\phi \cdot \exp(tv))\Big|_{t=0} \in VF^+(N).$$

For $u \in \mathcal{G}$, the differential $u_* : VF^+(N)_{s(u)} \to VF^+(N)_{r(u)}$ of $u\cdot : F^+(N)_{s(u)} \to F^+(N)_{r(u)}$ in the fibres defines on $VF^+(N)$ the structure of a $\mathcal{G}$-equivariant vector bundle. Since the left action of $\mathcal{G}$ commutes with the right action of $GL^+(q, \mathbb{R})$, one has

$$u_* v_\phi = \frac{d}{dt}(u \cdot (\phi \cdot \exp(tv)))\Big|_{t=0} = \frac{d}{dt}((u \cdot \phi) \cdot \exp(tv))\Big|_{t=0} = v_{u \cdot \phi} \tag{3.2}$$

for all $\phi \in (F^+(N))_{s(u)}$, and so with respect to the trivialization $F^+(N) \times \mathfrak{gl}(q, \mathbb{R})$ of $VF^+(N)$ we have

$$u_*(\phi, v) = (u \cdot \phi, v) \tag{3.3}$$

for all $\phi \in F^+(N)$ and $v \in \mathfrak{gl}(q, \mathbb{R})$.

Consider now the quotient $Q := F^+(N)/SO(q, \mathbb{R})$ of $F^+(N)$ by the right action of $SO(q, \mathbb{R})$. The projection $\pi_F : F^+(N) \to M$ descends to a projection $\pi_Q : Q \to M$, which defines a fibre bundle with typical fibre $S_q^+ := GL^+(q, \mathbb{R})/SO(q, \mathbb{R})$, the space of positive definite, symmetric $q \times q$ matrices. Moreover, since the action of $\mathcal{G}$ on $F^+(N)$ commutes with the right action of $SO(q, \mathbb{R})$, it follows that Q is a $\mathcal{G}$-space with anchor map $\pi_Q : Q \to M$, and with action of $u \in \mathcal{G}$ given by

$$u \cdot [\phi] := [u \cdot \phi] = [u_* \circ \phi] \tag{3.4}$$

for all $[\phi] \in Q_{s(u)}$. Following [Benameur and Heitsch 2018; Zhang 2017], we refer to $\pi_Q : Q \to M$ as the Connes fibration.

Definition 3.5. The fibre bundle $\pi_Q : Q \to M$ is a $\mathcal{G}$-space called the *Connes fibration* for the normal bundle N.

Let us consider the geometry of the fibres of $Q \to M$. Since $SO(q, \mathbb{R})$ is compact, the pair $(GL^+(q, \mathbb{R}), SO(q, \mathbb{R}))$ is a Riemannian symmetric pair and hence the space S_q^+ can be equipped with a $GL^+(q, \mathbb{R})$-invariant metric under which it is, by [Helgason 1962, Proposition 3.4], a globally symmetric Riemannian space. The Riemannian space S_q^+ is moreover of noncompact type, so by [Helgason 1962, Theorem 3.1] has everywhere nonpositive sectional curvature. We can find a locally finite open cover $\mathcal{U}$ of M by sets U for which the vertical bundle $VQ|_U \cong U \times TS_q$, and then choosing a partition of unity subordinate to $\mathcal{U}$ allows us to equip the bundle $VQ \to Q$ with a Euclidean structure. We will assume from here on that $VQ \to Q$ is equipped with a Euclidean structure in this way.

Proposition 3.6. *The bundle $VQ \to Q$ is a $\mathcal{G}$-equivariant Euclidean bundle over the $\mathcal{G}$-space Q. Consequently $\mathbb{C}\mathrm{liff}(VQ)$ and $\mathbb{C}\mathrm{liff}(V^*Q)$ are $\mathcal{G}$-equivariant bundles.*

Proof. Fix $u \in \mathcal{G}$ and suppose that U_s and U_r are open sets in M containing $s(u)$ and $r(u)$, respectively, such that we have local trivializations $N|_{U_s} \cong U \times \mathbb{R}^q$ and $N|_{U_r} \cong U \times \mathbb{R}^q$, with respect to which the holonomy action $u_* : N_{s(u)} \to N_{r(u)}$ is the action on $\mathbb{R}^q$ of an element $\tilde{u} \in GL^+(q, \mathbb{R})$.

We obtain corresponding local trivializations $F^+(N)|_{U_s} \cong U \times GL^+(q, \mathbb{R})$ and $F^+(N)|_{U_r} \cong U \times GL^+(q, \mathbb{R})$ of the local frame bundles over U_s and U_r, in which the holonomy action $u\cdot : F^+(N)_{s(u)} \to F^+(N)_{r(u)}$ is left multiplication on $GL^+(q, \mathbb{R})$ by $\tilde{u}$, and taking the quotient by $SO(q, \mathbb{R})$ we get local trivializations $Q|_{U_s} \cong U \times S_q^+$ and $Q|_{U_r} \cong U \times S_q^+$ in which $u\cdot : Q_{s(u)} \to Q_{r(u)}$ is the isometry of $S_q^+ = GL^+(q, \mathbb{R})/SO(q, \mathbb{R})$ defined by left multiplication by $\tilde{u} \in GL^+(q, \mathbb{R})$. Thus, $\mathcal{G}$ acts by orientation-preserving isometries between the fibres of Q, inducing an action by special orthogonal transformations on the Euclidean bundle $VQ \to Q$ of vectors tangent to the fibres of $Q \to M$, hence making $VQ \to Q$ a $\mathcal{G}$-equivariant Euclidean bundle over the $\mathcal{G}$-space Q. The final statement follows from functoriality of Clifford algebras with respect to orthogonal maps. □

That the fibres have nonpositive sectional curvature allows us to define a dual Dirac class for Q over M in a similar manner to Connes [1986]. First, let $\mathbb{C}\ell(V^*Q)$ be equipped with the $\mathcal{G}$-structure arising from the action of $\mathcal{G}$ on the equivariant bundle $\mathbb{C}\mathrm{liff}(V^*Q)$ over the $\mathcal{G}$-space Q, denoted for $u \in \mathcal{G}$ by $u_\diamond : \mathbb{C}\mathrm{liff}(V^*_{[\phi]}Q) \to \mathbb{C}\mathrm{liff}(V^*_{u\cdot[\phi]}Q)$ for all $[\phi] \in Q_{s(u)}$. That is, we define for any $u \in \mathcal{G}$ an isomorphism $\alpha^1_u : \mathbb{C}\ell(V^*Q|_{Q_{s(u)}}) \to \mathbb{C}\ell(V^*Q|_{Q_{s(u)}})$ by

$$\alpha^1_u(a)([\phi]) := u_\diamond a(u^{-1} \cdot [\phi]) \tag{3.7}$$

for all $[\phi] \in Q_{r(u)}$. Also let

$$E^1 := \Lambda^*(V^*Q) \otimes \mathbb{C}$$

be the complexified exterior algebra bundle of the bundle of vertical covectors V^*Q over Q. Here we equip V^*Q with the Euclidean structure coming from its dual VQ, which determines a Hermitian structure on $V^*Q \otimes \mathbb{C}$ and hence on E^1. Observe that

$$X_{E^1} := \Gamma_0(Q; E^1)$$

is a Hilbert $\mathbb{C}\ell(V^*Q)$-module under the inner product

$$\langle \rho^1, \rho^2 \rangle_{\mathbb{C}\ell(V^*Q)}([\phi]) := \psi_{V^*Q}(\rho^1([\phi]))\psi_{V^*Q}(\rho^2([\phi]))$$

and right action

$$(\rho \cdot a)([\phi]) := c_R(a([\phi]))\rho([\phi]),$$

where c_R is the right action of $\mathrm{Cliff}(V^*Q)$ on the Clifford bimodule E^1.

The isometric action of $\mathscr{G}$ on the Euclidean bundle VQ over Q gives rise to a unitary action of $\mathscr{G}$ on E^1, denoted for each $u \in \mathscr{G}$ by $u_* : E^1_{[\phi]} \to E^1_{u\cdot[\phi]}$ for all $[\phi] \in Q_{s(u)}$, and hence determines an isomorphism $W^1_u : \Gamma_0(Q_{s(u)}; E^1|_{Q_{s(u)}}) \to \Gamma_0(Q_{r(u)}; E^1|_{Q_{r(u)}})$ of Banach spaces given by the formula

$$(W^1_u\rho)([\phi]) := u_*\rho(u^{-1} \cdot [\phi])$$

for all $[\phi] \in Q_{r(u)}$. A routine calculation using Lemma 2.1 shows that

$$\langle W^1_u\rho^1, W^1_u\rho^2\rangle_{\mathbb{C}\ell(V^*Q)} = \alpha^1_u(\langle \rho^1, \rho^2\rangle_{\mathbb{C}\ell(V^*Q)}),$$

so (X_{E^1}, W^1) is a $\mathscr{G}$-equivariant Hilbert $\mathbb{C}\ell(V^*Q)$-module.

Choose now a Euclidean metric for N. Such a choice is determined by a section $\sigma : M \to Q$ of $\pi_Q : Q \to M$. For $[\phi_1], [\phi_2]$ in the same fibre Q_x, denote by $h([\phi_1], [\phi_2])$ the geodesic distance between $[\phi_1]$ and $[\phi_2]$ in the fibre, and then for any $[\phi_0] \in Q$ let $h^{[\phi_0]} : Q \to \mathbb{R}$ be the function

$$h^{[\phi_0]}([\phi]) := h([\phi_0], [\phi]).$$

In particular, for $x \in M$ and $[\phi] \in Q_x$, $h^{\sigma(x)}([\phi])$ gives the distance in the fibre between $[\phi]$ and the section σ. Consider now the vertical 1-form

$$Z_{[\phi]} := h^{\sigma(\pi_Q([\phi]))}([\phi])dh^{\sigma(\pi_Q([\phi]))}_{[\phi]},$$

where d denotes the exterior derivative in the fibre. Define an operator B_1 on the dense submodule $X^c_{E^1} := \Gamma_c(Q; E^1)$ of X_{E^1} by the formula

$$(B_1\rho)([\phi]) := c_L(Z_{[\phi]})\rho([\phi]),$$

where c_L is the left representation of $\mathrm{Cliff}(V^*Q)$ on the Clifford bimodule E^1. Since c_L and c_R commute, B_1 commutes with the right action of $\mathbb{C}\ell(V^*Q)$. Finally, we let m be the representation of $C_0(M)$ on X_{E^1} by multiplication, that is,

$$(m(f)\rho)([\phi]) := f(\pi_Q([\phi]))\rho([\phi])$$

for all $f \in C_0(M)$ and $\rho \in X_{E^1}$. Equivariance of the map π_Q tells us that m is an equivariant representation.

Proposition 3.8. *The triple $(C_0(M), {}_mX_{E^1}, B_1)$ is an unbounded $\mathscr{G}$-equivariant Kasparov $C_0(M)$-$\mathbb{C}\ell(V^*Q)$-module and hence defines a class*

$$[B_1] \in KK^{\mathscr{G}}(C_0(M), \mathbb{C}\ell(V^*Q)).$$

Proof. The first thing we need to prove is that B_1 is self-adjoint and regular. Observe first that B_1 is clearly symmetric. For each $[\phi] \in Q$, the localization $(X_{E^1})_{[\phi]}$ of X_{E^1} in the sense of [Pierrot 2006b; Kaad and Lesch 2012] is just the finite-dimensional Hilbert space

$$\mathcal{H}_{[\phi]} := \Lambda^*(V^*_{[\phi]}Q) \otimes \mathbb{C}$$

with the inner product coming from the Hermitian structure on $\Lambda^*(V^*_{[\phi]}Q) \otimes \mathbb{C}$, and the action of the localized operator $(B_1)_{[\phi]}$ on $\mathcal{H}_{[\phi]}$ is

$$(B_1)_{[\phi]}\eta := c_L(Z_{[\phi]})\eta.$$

Since $(B_1)_{[\phi]}$ is then self-adjoint on $\mathcal{H}_{[\phi]}$, it follows from [Pierrot 2006b, Théorème 1.18] that B_1 is self-adjoint and regular.

That $m(f)(1 + B_1^2)^{-1/2}$ is a compact operator for all $f \in C_0(M)$ follows from the definition of Clifford multiplication. Indeed, one has $c_L(Z_{[\phi]})^2 = \|Z_{[\phi]}\|^2 = h^{\sigma(\pi_Q([\phi]))}([\phi])^2$ since $dh^{\sigma(\pi_Q([\phi]))}_{[\phi]}$ has norm 1 for all $[\phi]$ as the dual of the tangent to the unique unit-speed geodesic joining $\sigma(\pi_Q([\phi]))$ to $[\phi]$, and so for any $f \in C_0(M)$, one simply has

$$(m(f)(1 + B_1^2)^{-1/2}\rho)([\phi]) = \frac{f(\pi_Q([\phi]))}{(1 + h^{\sigma(\pi_Q([\phi]))}([\phi])^2)^{1/2}}\rho([\phi]).$$

Since f vanishes at infinity on the base M of $Q \to M$, and because $[\phi] \mapsto (1 + h^{\sigma(\pi_Q([\phi]))}([\phi])^2)^{-1/2}$ vanishes at infinity on the fibres of $Q \to M$, the function $[\phi] \mapsto f(\pi_Q([\phi]))(1 + h^{\sigma(\pi_Q([\phi]))}([\phi])^2)^{-1/2}$ is an element of $C_0(Q)$, so that $m(f)(1 + B_1^2)^{-1/2}$ is indeed a compact operator on the $\mathbb{C}\ell(V^*Q)$-module X_{E^1}.

Concerning commutators, it is clear that B_1 commutes with the representation m of $C_0(M)$. Thus, it only remains to prove that B_1 is appropriately equivariant. The idea of this is essentially the unbounded version of analogous results by Connes [1986, Lemma 5.3] and Kasparov [1988, §5.3], but the details are somewhat technical so we give them here. Fix $u \in \mathcal{G}$ and $\rho \in \Gamma_c(Q_{r(u)}; E^1|_{Q_{r(u)}})$. We calculate

$$\begin{aligned}(B_1 - W^1_u B_1 W^1_{u^{-1}})\rho([\phi]) &= c_L(Z_{[\phi]})\rho([\phi]) - u_*(B_1 W^1_{u^{-1}}\rho)(u^{-1}\cdot[\phi]) \\ &= c_L(Z_{[\phi]})\rho([\phi]) - u_*(c_L(Z_{u^{-1}\cdot[\phi]})(W^1_{u^{-1}}\rho)(u^{-1}\cdot[\phi])) \\ &= c_L(Z_{[\phi]})\rho([\phi]) - u_*(c_L(Z_{u^{-1}\cdot[\phi]})(u_*^{-1}\rho([\phi]))) \\ &= c_L(Z_{[\phi]} - u_* Z_{u^{-1}\cdot[\phi]})\rho([\phi])\end{aligned}$$

where on the third line we have used the identity (2.2). Thus, we must calculate a bound for the norm of the covector $Z_{[\phi]} - u_* Z_{u^{-1}\cdot[\phi]}$.

Denote $\sigma_r := \sigma(r(u))$ and $\sigma_s := \sigma(s(u))$. With this notation, we have

$$Z_{[\phi]} - u_* Z_{u^{-1}\cdot[\phi]} = h^{\sigma_r}([\phi])dh^{\sigma_r}_{[\phi]} - u_* h^{\sigma_s}(u^{-1}\cdot[\phi])dh^{\sigma_s}_{u^{-1}\cdot[\phi]}.$$

For any vector $\gamma \in V_{[\phi]}Q$ we have

$$(u_* dh^{\sigma_s}_{u^{-1}\cdot[\phi]})(\gamma) = dh^{\sigma_s}_{u^{-1}\cdot[\phi]}(u_*^{-1}\gamma) = d(h^{\sigma_s} \circ u^{-1})_{[\phi]}(\gamma),$$

giving $u_* dh^{\sigma_s}_{u^{-1}\cdot[\phi]} = d(h^{\sigma_s} \circ u^{-1})_{[\phi]}$, and since the action of $\mathcal{G}$ is isometric on the fibres we get

$$(h^{\sigma_s} \circ u^{-1})([\phi]) = h(\sigma_s, u^{-1} \cdot [\phi]) = h(u \cdot \sigma_s, [\phi]) = h^{u\cdot\sigma_s}([\phi]).$$

Thus,

$$u_* dh^{\sigma_s}_{u^{-1}\cdot[\phi]} = dh^{u\cdot\sigma_s}_{[\phi]}.$$

We then see that

$$\begin{aligned} h^{\sigma_r}([\phi]) dh^{\sigma_r}_{[\phi]} - u_* h^{\sigma_s}(u^{-1}\cdot[\phi]) dh^{\sigma_s}_{u^{-1}\cdot[\phi]} &= h^{\sigma_r}([\phi]) dh^{\sigma_r}_{[\phi]} - h^{u\cdot\sigma_s}([\phi]) dh^{u\cdot\sigma_s}_{[\phi]} \\ &= \tfrac{1}{2} d((h^{\sigma_r})^2 - (h^{u\cdot\sigma_s})^2)_{[\phi]} \\ &= \tfrac{1}{2} d((h^{\sigma_r} - h^{u\cdot\sigma_s})(h^{\sigma_r} + h^{u\cdot\sigma_s}))_{[\phi]}. \end{aligned}$$

By the argument [Kasparov 1988, Lemma 5.3], we have

$$\| dh^{\sigma_r}_{[\phi]} - dh^{u\cdot\sigma_s}_{[\phi]} \| \leq 2h(\sigma_r, u\cdot\sigma_s)(h^{\sigma_r}([\phi]) + h^{u\cdot\sigma_s}([\phi]))^{-1},$$

which we use to estimate

$$\begin{aligned} &\| h^{\sigma_r}([\phi]) dh^{\sigma_r}_{[\phi]} - u_* h^{\sigma_s}(u^{-1}\cdot[\phi]) dh^{\sigma_s}_{u^{-1}\cdot[\phi]} \|^2 \\ &\quad \leq \tfrac{1}{4} \| (dh^{\sigma_r}_{[\phi]} - dh^{u\cdot\sigma_s}_{[\phi]})(h^{\sigma_r}([\phi]) + h^{u\cdot\sigma_s}([\phi])) \|^2 \\ &\qquad\qquad + \tfrac{1}{4} \| (h^{\sigma_r}([\phi]) - h^{u\cdot\sigma_s}([\phi]))(dh^{\sigma_r}_{[\phi]} + dh^{u\cdot\sigma_s}_{[\phi]}) \|^2 \\ &\quad \leq h(\sigma_r, u\cdot\sigma_s)^2 + (h(\sigma_r, [\phi]) - h(u\cdot\sigma_s, [\phi]))^2 \\ &\quad = h(\sigma_r, u\cdot\sigma_s)^2 + h(\sigma_r, [\phi])^2 + h(u\cdot\sigma_s, [\phi])^2 - 2h(\sigma_r, [\phi]) h(u\cdot\sigma_s, [\phi]) \\ &\quad \leq 2h(\sigma_r, u\cdot\sigma_s)^2, \end{aligned}$$

where the last line is a consequence of the cosine inequality for spaces of nonpositive sectional curvature [Helgason 1962, Corollary 13.2].

Thus, for all $[\phi] \in Q_{r(u)}$, we have that $\|Z_{[\phi]} - u_* Z_{u^{-1}\cdot[\phi]}\|^2 \leq 2h(\sigma(r(u)), u\cdot\sigma(s(u)))^2$ independently of $[\phi] \in Q_{r(u)}$, implying that $B_1 - W^1_u B_1 W^1_{u^{-1}}$ extends to a bounded operator on $(X_{E^1})_{r(u)}$. Moreover, $u \mapsto h(\sigma(r(u)), u\cdot\sigma(s(u)))$ is continuous and hence bounded on compact Hausdorff sets, so for any element U_i of the cover $\mathcal{U} = \{U_i\}_{i\in I}$ of $\mathcal{G}$ by Hausdorff open subsets, and for any $\varphi \in C_c(U_i)$ and $f \in C_0(M)$, we have that

$$\varphi \cdot m^r_i(r^*_i(f)) \cdot (r^*_i B_1 - (W^1)^i \circ s^*_i B_1 \circ ((W^1)^i)^{-1}) \in \mathcal{L}(r^*_i X_{E^1}).$$

It follows that $(C_0(M), {}_m X_{E^1}, B_1)$ is an unbounded equivariant Kasparov $C_0(M)$-$\mathbb{C}\ell(V^*Q)$-module. □

3B. ***The foliation of the Connes fibration.*** Before we can construct a second Kasparov module and the semifinite spectral triple associated to it, we need a closer study of the geometry and groupoid representation theory at our disposal.

Definition 3.9. Assume M to be equipped with a Riemannian metric g, with Levi-Civita connection ∇^{LC}. Then N identifies with the subbundle $N = T\mathcal{F}^{\perp}$ of TM, and we use the notation $X = X_{\mathcal{F}} + X_N$ for the corresponding decomposition of vector fields into their normal and leafwise components. Define a connection $\nabla^{\flat}$ on N by the formula

$$\nabla^{\flat}_X(Y) = [X_{\mathcal{F}}, Y]_N + \nabla^{\mathrm{LC}}_{X_N}(Y)_N, \quad X \in \Gamma^{\infty}(M; TM),\ Y \in \Gamma^{\infty}(M; N).$$

We refer to $\nabla^{\flat}$ as a *torsion-free Bott connection.*

The terminology "torsion-free" in Definition 3.9 refers to the fact that for any such connection $\nabla^{\flat}$ the associated torsion tensor

$$T_{\nabla^{\flat}}(X, Y) := \nabla^{\flat}_X(Y_N) - \nabla^{\flat}_Y(X_N) - [X, Y]_N$$

vanishes for all $X, Y \in \Gamma^{\infty}(M; TM)$. This fact follows from an easy calculation using the corresponding property of the Levi-Civita connection. Bott connections more generally are characterized by the formula $\nabla^{\flat}_{X_{\mathcal{F}}}(Y) = [X_{\mathcal{F}}, Y]_N$ for all smooth sections Y of N and $X_{\mathcal{F}}$ of $T\mathcal{F}$, and are the key ingredient for the Chern–Weil proof of Bott's vanishing theorem [1970]. For us, the use of the Levi-Civita connection in Definition 3.9 serves a purpose that will become apparent in Section 5.

Let us now come back to the frame bundle $\pi_F : F^+(N) \to M$. The total space of this bundle carries a foliation $\mathcal{F}_F$ by the orbits of the action of $\mathcal{G}$ on $F^+(N)$ defined in (3.1), whose normal we denote by $N_F := TF^+(N)/T\mathcal{F}_F$. The foliation $\mathcal{F}_F$ is everywhere transverse to the fibres of $F^+(N)$, and its tangent bundle $T\mathcal{F}_F$ projects fibrewise-isomorphically onto $T\mathcal{F}$. It is readily verified using a calculation in foliated coordinates that the connection form $\alpha^{\flat} \in \Omega^1(F^+(N); \mathfrak{gl}(q, \mathbb{R}))$ associated to any Bott connection $\nabla^{\flat}$ on N contains $T\mathcal{F}_F$ in its kernel $HF^+(N) := \ker(\alpha^{\flat})$. Consequently we find that the normal bundle to the foliation $\mathcal{F}_F$ admits a decomposition

$$N_F = VF^+(N) \oplus (HF^+(N)/T\mathcal{F}_F). \tag{3.10}$$

The normal bundle N_F is again a $\mathcal{G}$-equivariant bundle, and with respect to the splitting (3.10) we write

$$u_* = \begin{pmatrix} \tilde{a}(u) & \tilde{c}(u) \\ 0 & \tilde{d}(u) \end{pmatrix}$$

for the action of $u \in \mathcal{G}$ on N_F. Note that the zero appearing in the bottom-left corner is a consequence of the fact that by (3.1), $\mathcal{G}$ acts via diffeomorphisms between the fibres $GL^+(q,\mathbb{R})$ of $F^+(N)\to M$, and so preserves the bundle $VF^+(N)\to M$ of vectors tangent to the fibres.

Now we are not so interested in the frame bundle $F^+(N)$ as the Connes fibration Q. Since the action of $\mathcal{G}$ on $F^+(N)$ commutes with the right action of $SO(q,\mathbb{R})$, however, we find that we also obtain a foliation on the total space of $\pi_Q : Q\to M$.

To be more specific, let $k : F^+(N)\to Q$ be the quotient map. Then $T\mathcal{F}_Q := dk(T\mathcal{F}_F)$ is an integrable subbundle of TQ, which determines a foliation $\mathcal{F}_Q$ of Q. Since $\pi_Q\circ k=\pi_F$, we see that $d\pi_Q$ maps $T\mathcal{F}_Q$ isomorphically onto $T\mathcal{F}$ making $\pi_Q : Q\to M$ a foliated bundle. The normal bundle N_Q of $\mathcal{F}_Q$ also admits a splitting

$$N_Q = VQ\oplus(HQ/T\mathcal{F}_Q),$$

where HQ is the isomorphic image under dk of the horizontal subbundle $HF^+(N)\subset TF^+(N)$. For convenience, we will denote $HQ/T\mathcal{F}_Q$ simply by H. Thus,

$$N_Q = VQ\oplus H.$$

Now, $d\pi_Q$ maps the fibres of HQ isomorphically onto those of TM, and maps the fibres of $T\mathcal{F}_Q$ isomorphically onto those of $T\mathcal{F}$. It follows that $d\pi_Q$ induces an isomorphism of the fibres of $H = HQ/T\mathcal{F}_Q$ onto those of $N = TM/T\mathcal{F}$. We can then equip H with a Euclidean metric in the following way [Connes 1986, §5].

Proposition 3.11. *For $h_1, h_2\in H_{[\phi]}$ and with $\cdot$ denoting the Euclidean inner product in $\mathbb{R}^q$, the formula*

$$m^H_{[\phi]}(h_1,h_2) := \phi^{-1}(d\pi_Q(h_1))\cdot\phi^{-1}(d\pi_Q(h_2))$$

determines a well-defined Euclidean metric on the bundle $H\to Q$.

Proof. Suppose we were to choose a different representation $\phi' = \phi\circ A$ of $[\phi]$, where A is some matrix in $SO(q,\mathbb{R})$. Then by the invariance of the Euclidean inner product under special orthogonal transformations we have

$$\begin{aligned}(\phi')^{-1}(d\pi_Q(h_1))\cdot(\phi')^{-1}(d\pi_Q(h_2)) &= (A^{-1}\phi^{-1}(d\pi_Q(h_1)))\cdot(A^{-1}\phi^{-1}(d\pi_Q(h_2)))\\ &= \phi^{-1}(d\pi_Q(h_1))\cdot\phi^{-1}(d\pi_Q(h_2)),\end{aligned}$$

giving well-definedness. That we have defined a metric follows from the linearity of the maps ϕ and $d\pi_Q$, and the fact that the Euclidean inner product is a metric on $\mathbb{R}^q$. □

Remarkably, holonomy translations are orthogonal with respect to this Euclidean structure of H.

Proposition 3.12. *The normal bundle $N_Q\to Q$ of the foliation $\mathcal{F}_Q$ of Q is a $\mathcal{G}$-equivariant vector bundle over the $\mathcal{G}$-space Q. Moreover, with respect to the* *splitting $N_Q = VQ \oplus H$, for $u \in \mathcal{G}$ and $[\phi] \in Q_{s(u)}$ the holonomy action $u_* : (N_Q)_{[\phi]} \to (N_Q)_{u\cdot[\phi]}$ has the form*

$$u_* = \begin{pmatrix} a(u) & c(u) \\ 0 & d(u) \end{pmatrix}, \tag{3.13}$$

with $a(u) : V_{[\phi]}Q \to V_{u\cdot[\phi]}Q$ and $d(u) : H_{[\phi]} \to H_{u\cdot[\phi]}$ orthogonal and orientation-preserving.

Proof. The holonomy groupoid for the foliation $\mathcal{F}_Q$ of Q is precisely the groupoid $Q \rtimes \mathcal{G}$, under which the normal bundle $N_Q \to Q$ is therefore equivariant. Thus, $N_Q \to Q$ is a $\mathcal{G}$-equivariant vector bundle over the $\mathcal{G}$-space Q.

Proposition 3.6 tells us that $a(u) : V_{[\phi]}Q \to V_{u\cdot[\phi]}Q$ is orthogonal and orientation-preserving, and that the vertical bundle is preserved under holonomy translation, which accounts for the 0 appearing in the bottom-left corner of (3.13). Since $\pi_Q : Q \to M$ is the anchor map for the $\mathcal{G}$-space Q, it is $\mathcal{G}$-equivariant, implying that the identification $d\pi_Q$ of fibres of H with those of N is also $\mathcal{G}$-equivariant.

That $d(u) : H_{[\phi]} \to H_{u\cdot[\phi]}$ is orientation-preserving is then a consequence of the fact that $d(u)$ may be identified with the orientation-preserving action of u on the fibres of N. That $d(u)$ is orthogonal is a consequence of the following calculation for $h_1, h_2 \in H_{[\phi]}$:

$$\begin{aligned} m^H_{u\cdot[\phi]}&(d(u)h_1, d(u)h_2) \\ &= (u_* \circ \phi)^{-1}((d\pi_Q \circ d(u))(h_1)) \cdot (u_* \circ \phi)^{-1}((d\pi_Q \circ d(u))(h_1)) \\ &= (\phi^{-1} \circ u_*^{-1})((u_* \circ d\pi_Q)(h_1)) \cdot (\phi^{-1} \circ u_*^{-1})((u_* \circ d\pi_Q)(h_2)) \\ &= \phi^{-1}(d\pi_Q(h_1)) \cdot \phi^{-1}(d\pi_Q(h_2)) = m^H_{[\phi]}(h_1, h_2), \end{aligned}$$

where for the second equality, we have used the equivariance of the anchor map $d\pi_Q$ between H and N. □

The triangular shape of the matrix in Proposition 3.12 is what is referred to as an *almost isometric* or *triangular structure* by Connes [1986] and Connes and Moscovici [1995], respectively.

The map $c(u) : H_{[\phi]} \to V_{u\cdot[\phi]}Q$, for $u \in \mathcal{G}$ and $[\phi] \in Q_{s(u)}$, is where the interesting representation theory is encoded. Currently, however, the range of $c(u)$ is too high in dimension to be of much use, and these extra dimensions need to be "traced out". Observing that there is indeed a canonical trace $\mathrm{tr}_{F^+(N)} : VF^+(N) \to \mathbb{R}$ induced fibrewise by the usual matrix trace on $\mathfrak{gl}(q, \mathbb{R}) = M_q(\mathbb{R})$, we now check that we can apply this map to VQ also.

Lemma 3.14. *The map $\mathrm{tr}_{F^+(N)} : VF^+(N) \to \mathbb{R}$ descends to a well-defined map $\mathrm{tr}_Q : VQ \to \mathbb{R}$ for which $\mathrm{tr}_Q \circ a(u) = \mathrm{tr}_Q$ for all $u \in \mathcal{G}$.*

Proof. For $A \in GL^+(q, \mathbb{R})$, we denote by $R_A : F^+(N) \to F^+(N)$ the map $\phi \mapsto \phi \cdot A$. By definition, the action of $A \in SO(q, \mathbb{R})$ on $VF^+(N)$ is then given for $\phi \in F^+(N)$ and $v_\phi \in V_\phi F^+(N)$ by

$$v_\phi \cdot A := (dR_A)_\phi(v_\phi).$$

We compute

$$(dR_A)_\phi(v_\phi) = \frac{d}{dt}(\phi \cdot \exp(tv) \cdot A)\Big|_{t=0} = \frac{d}{dt}((\phi \cdot A) \cdot (A^{-1} \exp(tv)A))\Big|_{t=0} = (A^{-1} v A)_{\phi \cdot A},$$

from which we deduce that the action of $A \in SO(q, \mathbb{R})$ in the trivialization $VF^+(N) = F^+(N) \times \mathfrak{gl}(q, \mathbb{R})$ is given by

$$(\phi, v) \cdot A = (\phi \cdot A, A^{-1} v A)$$

for all $\phi \in F^+(N)$, $v \in \mathfrak{gl}(q, \mathbb{R})$. Now, $\mathrm{tr}_{F^+(N)} : F^+(N) \times \mathfrak{gl}(q, \mathbb{R}) \to \mathbb{R}$ is by definition

$$\mathrm{tr}_{F^+(N)}(\phi, v) := \mathrm{tr}(v),$$

with tr denoting the usual matrix trace on $q \times q$ matrices, and with the range $\mathbb{R}$ of $\mathrm{tr}_{F^+(N)}$ carrying the trivial action of $SO(q, \mathbb{R})$. Then since the matrix trace is invariant under conjugation, we see that $\mathrm{tr}_{F^+(N)}$ is equivariant,

$$\mathrm{tr}_{F^+(N)}((\phi, v) \cdot A) = \mathrm{tr}(A^{-1} v A) = \mathrm{tr}(v) = \mathrm{tr}_{F^+(N)}(\phi, v) \cdot A,$$

and so descends to a well-defined map $\mathrm{tr}_Q : VQ \to \mathbb{R}$.

For the second assertion, note that since u commutes with the quotient map $Q : F^+N \to Q$ and since u_* acts as the identity on the fibres of $VF^+(N) = F^+(N) \times \mathbb{R}^{q^2}$ by (3.3), we have

$$\mathrm{tr}_Q \circ a(u) \circ dQ = \mathrm{tr}_Q \circ dQ \circ \mathrm{id} = \mathrm{tr}_Q \circ dQ.$$

Since dQ is surjective, we conclude that

$$\mathrm{tr}_Q \circ a(u) = \mathrm{tr}_Q$$

as claimed. □

Remark 3.15. Note that what makes Lemma 3.14 possible is the fact that the map $v \mapsto \mathrm{tr}(v)$ on $\mathfrak{gl}(q, \mathbb{R})$ is invariant under conjugation by invertible matrices. Thus, in fact we could replace tr with any other invariant polynomial on $\mathfrak{gl}(q, \mathbb{R})$, paralleling the Chern–Weil construction of characteristic classes, and still obtain a well-defined (but no longer necessarily linear) map on the vertical tangent bundle of the Connes fibration. This observation is due to M. T. Benameur.

Let us put Lemma 3.14 to use in simplifying the groupoid representation theory. For $u \in \mathcal{G}$ and $[\phi] \in Q_{s(u)}$, define

$$\delta(u) := \mathrm{tr}_Q \circ c(u) : H_{[\phi]} \to \mathbb{R}.$$

This $\delta(u)$ is linear, and so can be regarded as an element of $H^*_{[\phi]}$. We also define

$$\theta(u) := d(u^{-1})^t : H^*_{[\phi]} \to H^*_{u\cdot[\phi]},$$

the action on the covector bundle for H coming from the transpose of $d(u^{-1}) : H_{u\cdot[\phi]} \to H_{[\phi]}$. We have the following "$ax+b$ group"-type transformation laws.

Lemma 3.16. *For all $u, v \in \mathcal{G}^{(2)}$, we have*

$$\theta(uv) = \theta(u)\theta(v), \quad \textit{and} \quad \delta(uv) = \delta(v) + \theta(v^{-1})\delta(u).$$

Proof. These identities follow from the triangular structure of the matrices (3.13) and Lemma 3.14. Specifically, since $\mathcal{G}$ acts on N_Q we have

$$\begin{pmatrix} a(uv) & c(uv) \\ 0 & d(uv) \end{pmatrix} = \begin{pmatrix} a(u) & c(u) \\ 0 & d(u) \end{pmatrix} \begin{pmatrix} a(v) & c(v) \\ 0 & d(v) \end{pmatrix} = \begin{pmatrix} a(u)a(v) & a(u)c(v)+c(u)d(v) \\ 0 & d(u)d(v) \end{pmatrix},$$

from which we immediately deduce that $d(uv) = d(u)d(v)$ and hence $\theta(uv) = \theta(u)\theta(v)$. We also calculate

$$\begin{aligned} \delta(uv) &= \mathrm{tr}_Q \circ c(uv) = \mathrm{tr}_Q \circ a(u) \circ c(v) + \mathrm{tr}_Q \circ c(u) \circ d(v) \\ &= \mathrm{tr}_Q \circ c(v) + \mathrm{tr}_Q \circ c(u) \circ d(v) = \delta(v) + \theta(v^{-1})\delta(u), \end{aligned}$$

using Lemma 3.14 for the third equality, giving the desired identities. □

3C. *The Vey Kasparov module.* We now go about constructing a second Kasparov module, referred to in this paper as the Vey Kasparov module since it appears to be analogous to the Vey homomorphism considered in previous work [Hurder 1986; Duminy 1982]. Our first job in constructing a second Kasparov module is to endow the total space H^* of the horizontal covector bundle $\pi_{H^*} : H^* \to Q$ with an action of $\mathcal{G}$ that encodes both θ and δ from Lemma 3.16.

Proposition 3.17. *For $u \in \mathcal{G}$ and $\eta \in H^*|_{Q_{s(u)}}$, the formula*

$$u \cdot \eta := \theta(u)\eta + \delta(u^{-1})$$

determines the structure of a $\mathcal{G}$-space on H^* with anchor map $\pi_Q \circ \pi_{H^*} : H^* \to M$.

Proof. It is clear that $(\pi_Q \circ \pi_{H^*})(u \cdot \eta) = r(u)$ for all $u \in \mathcal{G}$ and $\eta \in H^*|_{Q_{s(u)}}$, and since by Lemma 3.16 θ is the identity on units and δ is zero on units, we get $(\pi_Q \circ \pi_{H^*})(\eta) \cdot \eta = \eta$ for all η. Thus, it remains only to check that $(uv) \cdot \eta = u \cdot (v \cdot \eta)$ for all $(u, v) \in \mathcal{G}^{(2)}$ and $\eta \in H^*|_{Q_{s(v)}}$. For this we simply have

$$(uv) \cdot \eta = \theta(uv)\eta + \delta(v^{-1}u^{-1}) = \theta(u)(\theta(v)\eta + \delta(v^{-1})) + \delta(u^{-1}) = u \cdot (v \cdot \eta),$$

with the second equality being a consequence of Lemma 3.16. □

We can now construct another dual Dirac class in much the same way as we did for the Connes fibration. Consider the bundle $VH^* := \ker(d\pi_{H^*})$ of vertical tangent vectors over the horizontal covector bundle $\pi_{H^*} : H^* \to Q$, and denote by $\pi_H : H \to Q$ the projection for the horizontal bundle. Since the fibres of H^* are vector spaces, we have $V_\eta H^*_{[\phi]} \cong H^*_{[\phi]}$ for all $[\phi] \in Q$ and $\eta \in H^*_{[\phi]}$. Thus, the dual space $V^*_\eta H^*_{[\phi]}$ is a copy of $H_{[\phi]}$ and so we can write V^*H^* as the fibred product

$$V^*H^* \cong H^* \times_{\pi_{H^*},\pi_H} H,$$

regarded as a vector bundle over H^* by using the projection onto the first factor. Since H is a $\mathcal{G}$-equivariant Euclidean bundle over Q via the map d in Proposition 3.12, for all $u \in \mathcal{G}$, $\eta \in H^*|_{Q_{s(u)}}$, and $h \in H|_{Q_{s(u)}}$, the formula

$$u_*(\eta, h) := (u \cdot \eta, d(u)h) = (\theta(u)\eta + \delta(u^{-1}), d(u)h)$$

defines on V^*H^* the structure of a $\mathcal{G}$-equivariant Euclidean bundle over the $\mathcal{G}$-space H^*. Then by functoriality $\mathbb{C}\mathrm{liff}(V^*H^*)$ is a $\mathcal{G}$-equivariant bundle over H^*, and we denote the action of $u \in \mathcal{G}$ on $k \in \mathbb{C}\mathrm{liff}(V^*H^*|_{H^*_{[\phi]}})$ by $k \mapsto u_\diamond k$ for all $[\phi] \in Q_{s(u)}$. Using these facts together with Proposition 3.17, the following result is clear.

Proposition 3.18. *The formula*

$$\alpha^2_u(\zeta)(\eta) := u_\diamond\zeta(u^{-1} \cdot \eta) = u_\diamond\zeta(\theta(u^{-1})\eta + \delta(u)),$$

defined for $\zeta \in \mathbb{C}\ell(V^*H^*)$, $u \in \mathcal{G}$, and $\eta \in H^*_{[\phi]}$ with $[\phi] \in Q_{r(u)}$, determines the structure of a $\mathcal{G}$-algebra on $\mathbb{C}\ell(V^*H^*)$.

We now come to the definition of an appropriate Hilbert module. Let

$$E^2 := \Lambda^*(V^*H^*) \otimes \mathbb{C}$$

be the complexified exterior algebra bundle of V^*H^* over H^*, and define

$$X_{E^2} := \Gamma_0(H^*; E^2),$$

which is a Hilbert $\mathbb{C}\ell(V^*H^*)$-module whose structure as such is determined in the same way as for X_{E^1} using the identification of E^2 with $\mathbb{C}\mathrm{liff}(V^*H^*)$ as vector bundles.

By equivariance of V^*H^* over H^* and functoriality, for $u \in \mathcal{G}$, $[\phi] \in Q_{s(u)}$, and $\eta \in H^*_{[\phi]}$ we obtain a unitary holonomy transport map $u_* : E^2_\eta \to E^2_{u\cdot\eta}$ and an isomorphism $W^2_u : \Gamma_0(H^*_{[\phi]}; E^2|_{H^*_{[\phi]}}) \to \Gamma_0(H^*_{u\cdot[\phi]}; E^2|_{H^*_{u\cdot[\phi]}})$ of Banach spaces defined by

$$(W^2_u\zeta)(\eta) := u_*\zeta(u^{-1} \cdot \eta) = u_*\zeta(\theta(u^{-1})\eta + \delta(u)).$$

Using Lemma 2.1, we observe that

$$\begin{aligned}\langle W_u^2\zeta_1, W_u^2\zeta_2\rangle_{\mathbb{C}\ell(V^*H^*)_{r(u)}}(\eta) &= u_\diamond\langle\zeta_1(\theta(u^{-1})\eta+\delta(u)), \zeta_2(\theta(u^{-1})\eta+\delta(u))\rangle \\ &= \alpha_u^2(\langle\zeta_1, \zeta_2\rangle_{\mathbb{C}\ell(V^*H^*)_{s(u)}})(\eta)\end{aligned}$$

for all $u \in \mathcal{G}$, $[\phi] \in Q_{r(u)}$, and $\eta \in H^*_{[\phi]}$, so (X_{E^2}, W^2) is a $\mathcal{G}$-Hilbert $\mathbb{C}\ell(V^*H^*)$-module.

We define an unbounded operator B_2 on the dense submodule $X^c_{E^2} = \Gamma_c(H^*; E^2)$ of X_{E^2} by the formula

$$(B_2\zeta)(\eta) := c_L(\eta)\zeta(\eta),$$

where for $c_L(\eta)$ we regard $\eta \in H^*$ as a vertical covector in $V^*H^* = H^* \times_{\pi_{H^*},\pi_H} H$ using the Euclidean metric on H.

Finally, we take m^2 to be the representation of $C_0(Q)$ on X_{E^2} defined by

$$m^2(f)\zeta(\eta) := f(\pi_{H^*}(\eta))\zeta(\eta).$$

Using the fact that π_{H^*} is an equivariant map and that $\pi_{H^*}(\eta+\eta') = \pi_{H^*}(\eta) = [\phi]$ for all $[\phi] \in Q$ and $\eta, \eta' \in H^*_{[\phi]}$, a routine calculation shows that m^2 is an equivariant representation.

Proposition 3.19. *The triple $(C_0(Q), {}_{m^2}X_{E^2}, B_2)$ is an unbounded $\mathcal{G}$-equivariant Kasparov $C_0(Q)$-$\mathbb{C}\ell(V^*H^*)$-module, defining a class*

$$[B_2] \in KK^{\mathcal{G}}(C_0(Q), \mathbb{C}\ell(V^*H^*)).$$

Proof. The proof is essentially the same as the proof of Proposition 3.8. The only part that must be changed is checking the equivariance condition. For any $u \in \mathcal{G}$, $[\phi] \in Q_{r(u)}$, and $\eta \in H^*_{[\phi]}$, we have

$$\begin{aligned}(W_u^2 B_2 W_{u^{-1}}^2)\zeta(\eta) &= u_*(B_2 W_{u^{-1}}^2\zeta)(\theta(u^{-1})\eta+\delta(u)) \\ &= u_*(c_L(\theta(u^{-1})\eta+\delta(u))(W_{u^{-1}}^2\zeta)(\theta(u^{-1})\eta+\delta(u))) \\ &= u_*(c_L(\theta(u^{-1})\eta+\delta(u))(u_*^{-1}\zeta(\theta(u)(\theta(u^{-1})\eta+\delta(u))+\delta(u^{-1})))) \\ &= u_*(c_L(\theta(u^{-1})\eta+\delta(u))(u_*^{-1}\zeta(\eta))) \\ &= c_L(\eta-\delta(u^{-1}))\zeta(\eta)\end{aligned}$$

where the last line follows from the identity $\theta(u)\delta(u) = -\delta(u^{-1})$ arising from Lemma 3.16, together with the identity (2.2). We then have

$$B_2 - W_u^2 B_2 W_{u^{-1}}^2 = c_L(\delta(u^{-1})),$$

which defines a bounded operator on $(X_{E^2})_{r(u)}$. The rest of the proof is then the same as in Proposition 3.8. □

4. The index theorem

4A. ***Some simplifications in codimension 1.*** There are important simplifications in the codimension 1 case. Observe that for a codimension 1, transversely orientable foliation $\mathcal{F}$ of M, the conormal bundle $N^* \to M$ is trivialized by a choice of orientation, which is given by a choice of a transverse volume form ω. Such a choice determines a dual section ω^* of $N \to M$ and hence a map $t : N \to \mathbb{R}$ defined by the equality $n = t(n)\omega^*$ for $n \in N$. Thus,

$$N = M \times \mathbb{R}.$$

The action of $u \in \mathcal{G}$ on N will then be denoted by

$$u_*(s(u), n) := (r(u), \Delta(u)n), \tag{4.1}$$

with $\Delta : \mathcal{G} \to \mathbb{R}^*_+$ a multiplicative homomorphism. Observe that under the correspondence $\omega \mapsto \omega^*$, this $\Delta(u)$ is precisely the Radon–Nikodym derivative of the transverse volume form ω with respect to the holonomy translation u. The principal $\mathbb{R}^*_+$-bundle $F^+(N)$ of positively oriented frames for N, which coincides with the Connes fibration Q since $SO(1, \mathbb{R}) = 1$, is then also trivial under the map $\phi \mapsto (\pi_Q(\phi), t \circ \phi)$:

$$Q = M \times \mathbb{R}^*_+.$$

The action of u on the fibres of Q, defined by (3.1) since $q = 1$, is induced by the same homomorphism $\Delta(u)$:

$$u \cdot (s(u), b) := (r(u), \Delta(u)b).$$

We will assume for ease of calculation that

$$Q = M \times \mathbb{R}$$

using the logarithm map on the fibres, so that the action of a groupoid element $u \in \mathcal{G}$ on Q is now given by

$$u \cdot (s(u), c) = (r(u), c + \log \Delta(u)).$$

The horizontal and vertical bundles are both trivial line bundles, so

$$N_Q = VQ \oplus H = Q \times (\mathbb{R} \oplus \mathbb{R}).$$

Here we regard the horizontal bundle $H = Q \times \mathbb{R}$ as a Euclidean bundle with metric m arising from Q defined as in Proposition 3.11 by

$$m^H_{(x,c)}(h_1, h_2) := (e^{-c}h_1) \cdot (e^{-c}h_2) = e^{-2c}h_1h_2.$$

We use the metric m^H to identify H with its dual H^* by mapping $h \in H$ to the functional $m^H(h, \cdot)$. More precisely, we identify $h \in H_{(x,c)} = \mathbb{R}$ with $\eta_h := e^{-2c}h \in H^*_{(x,c)}$. We then find that the resulting metric on H^* is

$$m^{H^*}_{(x,c)}(\eta_h, \eta_{h'}) := m^H_{(x,c)}(h, h') = e^{-2c}hh' = e^{2c}\eta_h\eta_{h'}.$$

Under this identification, the map $\theta(u) : H^*_{(s(u),c)} \to H^*_{(r(u),c+\log\Delta(u))}$ is precisely $\eta \mapsto \Delta(u^{-1})\eta$.

With no need to trace over the vertical fibres in the codimension 1 case, we can then write the triangular structure of a holonomy transformation $u \in \mathcal{G}$ as

$$u_* = \begin{pmatrix} 1 & \delta(u) \\ 0 & \Delta(u) \end{pmatrix}.$$

This action of u_* on $VQ \oplus H \subset TQ$ is the differential of the action of u on Q. It follows then that $\delta(u)$ is the derivative with respect to the transverse coordinate in M of the map $c \mapsto c + \log\Delta(u)$ on the fibres of Q. Since the normal bundle N over M has been trivialized, we can write this derivative as the scalar $\delta(u) = \partial \log\Delta(u)$, with ∂ denoting the derivative with respect to the transverse coordinate. Thus,

$$u_* = \begin{pmatrix} 1 & \partial\log\Delta(u) \\ 0 & \Delta(u) \end{pmatrix}.$$

Let us now consider the Kasparov module $[B_2]$. The right-hand algebra in this case is $\mathbb{C}\ell(V^*H^*)$, and since for each $(x, c, \eta) \in H^*$ we can identify vertical tangent vectors in $V_{(x,c,\eta)}H^*$ with vectors in $H^*_{(x,c)}$, it follows that we can identify vertical covectors in $V^*_{(x,c,\eta)}H^*$ with linear functionals $H^*_{(x,c)} \to \mathbb{R}$. Observe then that there is a nonvanishing section κ of $V^*H^* \to H^*$ defined by

$$\kappa(x, c, \eta) := e^c\eta \quad \text{for } (x, c, \eta) \in H^*.$$

One has

$$\kappa(r(u), c + \log\Delta(u), \Delta(u^{-1})\eta) = e^{c+\log\Delta(u)}\Delta(u^{-1})\eta = e^c\eta = \kappa(s(u), c, \eta),$$

so κ is invariant under the action of $\mathcal{G}$ and therefore defines a trivialization $V^*H^* \cong H^* \times \mathbb{R}$ for which the action of $\mathcal{G}$ is given by

$$u_*(s(u), c, \eta, s) = (r(u), c + \log\Delta(u), \Delta(u^{-1})\eta, s) \quad \text{for } c \in Q,\ s \in \mathbb{R},\ \eta \in H^*_{(s(u),c)}.$$

It follows that we can take $\mathbb{C}\ell(V^*H^*)$ to be $C_0(H^*) \otimes \mathrm{Cliff}(\mathbb{R})$, where $\mathcal{G}$ acts trivially on $\mathrm{Cliff}(\mathbb{R})$. That is, for all $f \otimes e \in C_0(H^*) \otimes \mathrm{Cliff}(\mathbb{R})$ we have

$$\alpha^2_u(f \otimes e)(r(u), c, \eta) = f(s(u), c - \log\Delta(u), \Delta(u)\eta + \partial\log\Delta(u)) \otimes e, \qquad \eta \in H^*_{(r(u),c)}.$$

We therefore define an action α of $\mathcal{G}$ on $C_0(H^*)$ by

$$\alpha_u(f)(r(u), c, \eta) := f(s(u), c - \log \Delta(u), \Delta(u)\eta + \partial \log \Delta(u))$$

for $f \in C_0(H^*)$, so that $\alpha^2_u(f \otimes e) = \alpha_u(f) \otimes e$ for all $u \in \mathcal{G}$ and $e \in \mathbb{C}\mathrm{liff}(\mathbb{R})$.

The same remarks carry over to the exterior bundle $\Lambda^* V^* H^*$, so that $\Gamma_0(H^*; \Lambda^*(V^* H^*) \otimes \mathbb{C})$ is just $C_0(H^*) \otimes \mathbb{C}\mathrm{liff}(\mathbb{R})$, on which the representation W^2 of $\mathcal{G}$ is defined by the same formula as α^2:

$$W^2_u(\rho \otimes e)(r(u), c, \eta) = \rho(s(u), c - \log \Delta(u), \Delta(u)\eta + \partial \log \Delta(u)) \otimes e$$

for all $\rho \otimes e \in C_0(H^*) \otimes \mathbb{C}\mathrm{liff}(\mathbb{R})$. We thus define an action W of $\mathcal{G}$ on the Hilbert $C_0(H^*)$-module $C_0(H^*)$ by

$$W_u(\rho)(r(u), c, \eta) := \rho(s(u), c - \log \Delta(u), \Delta(u)\eta + \partial \log \Delta(u))$$

for all $\rho \in C_0(H^*)$, and we see that $W^2_u(\rho \otimes e) = W_u(\rho) \otimes e$ for all $u \in \mathcal{G}$ and $e \in \mathbb{C}\mathrm{liff}(\mathbb{R})$.

Finally, the operator B_2 acts on $C_0(H^*) \otimes \mathbb{C}\mathrm{liff}(\mathbb{R})$ by

$$(B_2 \rho \otimes e)(x, c, \eta) := e^c \eta \rho(x, c, \eta) \otimes c_L(e_1)e, \quad e \in \mathbb{C}\mathrm{liff}(\mathbb{R}), \ \eta \in H^*_{(x,c)},$$

where c_L is the left Clifford multiplication and e_1 is a fixed element of $\mathbb{C}\mathrm{liff}(\mathbb{R})$ with square 1. We can now proceed with the construction of a spectral triple from this data and the proof of the index theorem relating the spectral triple to the Godbillon–Vey invariant.

4B. ***The spectral triple.*** Applying the descent map to the equivariant Kasparov module $(C_0(Q), {}_{m^2}X_{E^2}, B_2)$ of Proposition 3.19 in codimension 1 gives us by Proposition 2.15 a Kasparov module

$$(\Gamma_c(Q \rtimes \mathcal{G}, \Omega^{1/2}), X_{E^2} \rtimes_r \mathcal{G}, r^* B_2) \tag{4.2}$$

which defines a class in $KK(C_0(Q) \rtimes_r \mathcal{G}, \mathbb{C}\ell(V^* H^*) \rtimes_r \mathcal{G})$. By the remarks of the previous section, we actually have

$$\mathbb{C}\ell(V^* H^*) \rtimes_r \mathcal{G} = (C_0(H^*) \otimes \mathbb{C}\mathrm{liff}(\mathbb{R})) \rtimes_r \mathcal{G} = (C_0(H^*) \rtimes_r \mathcal{G}) \otimes \mathbb{C}\mathrm{liff}(\mathbb{R})$$

since $\mathcal{G}$ acts trivially on $\mathbb{C}\mathrm{liff}(\mathbb{R})$. Thus, the module (4.2) can be replaced [Connes 1994, Proposition 13, Appendix A, Chapter 4] by the odd Kasparov $C_0(Q) \rtimes_r \mathcal{G}$-$C_0(H^*) \rtimes_r \mathcal{G}$-module

$$(\Gamma_c(Q \rtimes \mathcal{G}, \Omega^{1/2}), C_0(H^*) \rtimes_r \mathcal{G}, \mathcal{B}) \tag{4.3}$$

where we define $\mathcal{B}$ on $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2}) \subset C_0(H^*) \rtimes_r \mathcal{G}$ by

$$(\mathcal{B}\rho)_u(x, c, \eta) := (\mathcal{B}_{r(u)} \rho_u)(x, c, \eta) := e^c \eta \rho_u(x, c, \eta), \quad \eta \in \mathbb{R}.$$

Here we are using density of $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2})$ in $C_0(H^*) \rtimes_r \mathcal{G}$ and density of $\Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ in $C_0(Q) \rtimes_r \mathcal{G}$ as in the final paragraph of Section 2C.

The $\mathcal{G}$-invariant transverse volume forms of interest on Q and H^*, respectively, are

$$d\nu_Q = e^{-c}\omega \wedge dc, \qquad d\nu_{H^*} = \omega \wedge dc \wedge d\eta, \tag{4.4}$$

and we denote by τ_Q and τ_{H^*} the corresponding traces on $\Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ and $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2})$ defined by integration against $d\nu_Q$ and $d\nu_{H^*}$.

Putting the trace τ_{H^*} together with the odd Kasparov module (4.3), by Proposition 2.17 we obtain an odd semifinite spectral triple

$$(\mathcal{A}, \mathcal{H}, \mathcal{B})$$

relative to $(\mathcal{N}, \tau)$ where:

(1) $\mathcal{A} = \Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ acts by convolution operators on

(2) $\mathcal{H}$, the Hilbert space completion of $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2})$ in the inner product

$$(\rho_1 \mid \rho_2) = \tau_{H^*}(\rho_1^* * \rho_2),$$

(3) $\mathcal{B}$ is regarded as an operator on $\mathcal{H}$ with domain $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2})$,

(4) $\mathcal{N}$ is the weak closure of $\Gamma_c(H^* \rtimes \mathcal{G}; \Omega^{1/2})$ in the bounded operators on $\mathcal{H}$,

(5) τ is the normal extension of τ_{H^*} to $\mathcal{N}$.

We now apply the semifinite local index formula to $(\mathcal{A}, \mathcal{H}, \mathcal{B})$ to prove the codimension 1 Godbillon–Vey index theorem.

4C. *The index theorem.* We will apply the residue cocycle of [Carey et al. 2014, Definition 3.2] to prove the following theorem.

Theorem 4.5. *Let $(M, \mathcal{F})$ be a foliated manifold of codimension* 1. *The Chern character of the semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{B})$ given in Section 4B is, up to a factor of $(2\pi i)^{1/2}$, the global, non-étale analogue of Godbillon–Vey cyclic cocycle of Connes and Moscovici* [2005, Proposition 19].

To apply the local index formula of [Carey et al. 2014], we need to check the summability and smoothness of the spectral triple.

Lemma 4.6. *The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{B})$ is smoothly summable of spectral dimension $p = 1$ and has isolated spectral dimension.*

Proof. We first check finite summability. For $s \in \mathbb{R}$, $a \in \Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$, and $\rho \in \mathcal{H}$, we calculate

$$\begin{aligned}(a(1+\mathcal{B}^2)^{-s/2}\rho)_u(x,c,\eta) &= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(W_v(1+\mathcal{B}^2_{s(v)})^{-s/2}\rho_{v^{-1}u})(x,c,\eta)\\ &= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(1+e^{2(c-\log\Delta(v))}(\Delta(v)\eta+\partial\log\Delta(v))^2)^{-s/2}(W_v\rho_{v^{-1}u})(x,c,\eta)\\ &= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(1+e^{2c}\Delta(v^{-1})^2(\Delta(v)\eta+\partial\log\Delta(v))^2)^{-s/2}(W_v\rho_{v^{-1}u})(x,c,\eta)\\ &= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(1+e^{2c}(\eta-\partial\log\Delta(v^{-1}))^2)^{-s/2}(W_v\rho_{v^{-1}u})(x,c,\eta),\end{aligned}$$

where on the last line we have used Lemma 3.16 in simplifying $\Delta(v^{-1})\partial\log\Delta(v) = -\partial\log\Delta(v^{-1})$. So $a(1+\mathcal{B}^2)^{-s/2}$ is the half-density on $H^* \rtimes \mathcal{G}$ defined by

$$((x,c,\eta),u) \mapsto a_u(x,c)(1+e^{2c}(\eta-\partial\log\Delta(u^{-1}))^2)^{-s/2},$$

compactly supported in the u and (x,c) variables. Thus,

$$\begin{aligned}\tau_{H^*}(a(1+\mathcal{B}^2)^{-s/2}) &= \int_{M\times\mathbb{R}\times\mathbb{R}} a(x,c)(1+e^{2c}\eta^2)^{-s/2}\omega\wedge dc\wedge d\eta\\ &= \int_Q a(x,c)\,d\nu_Q\int_{\mathbb{R}}(1+t^2)^{-s/2}\,dt,\end{aligned}$$

where we've made the substitution $t = e^c\eta$. It is then clear that $\tau_{H^*}(a(1+\mathcal{B}^2)^{-s/2})$ is finite for all $s > 1$. For smoothness, we fix $a \in \Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ and calculate

$$\begin{aligned}([\mathcal{B}^2,a]\rho)_u(x,c,\eta) &= e^{2c}\eta^2\int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(W_v\rho_{v^{-1}u})(x,c,\eta)\\ &\qquad -\int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)(W_v\mathcal{B}^2_{s(v)}\rho_{v^{-1}u})(x,c,\eta)\\ &= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^{2c}(\eta^2-\Delta(v^{-1})^2(\Delta(v)\eta+\partial\log\Delta(v))^2)(W_v\rho_{v^{-1}u})(x,c,\eta)\\ &= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^{2c}(2\eta\partial\log\Delta(v^{-1})-(\partial\log\Delta(v^{-1}))^2)(W_v\rho_{v^{-1}u})(x,c,\eta)\end{aligned}$$

so that $[\mathcal{B}^2, a]$ is convolution by the half-density on $H^* \rtimes \mathcal{G}$ defined by

$$((x,c,\eta),u) \mapsto a_u(x,c)e^{2c}(2\eta\partial\log\Delta(u^{-1})-(\partial\log\Delta(u^{-1}))^2).$$

We also calculate

$$
\begin{aligned}
([\mathcal{B}^2,[\mathcal{B},a]]\rho)_u(x,c,\eta) &= e^{2c}\eta^2([\mathcal{B},a]\rho)_u(x,c,\eta)-([\mathcal{B},a]\mathcal{B}^2\rho)_u(x,c,\eta)\\
&= e^{2c}\eta^2\int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^c\partial\log\Delta(v^{-1})(W_v\rho_{v^{-1}u})(x,c,\eta)\\
&\qquad -\int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^c\partial\log\Delta(v^{-1})(W_v\mathcal{B}^2_{s(v)}\rho_{v^{-1}u})(x,c,\eta)\\
&= e^{2c}\eta^2\int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^c\partial\log\Delta(v^{-1})(W_v\rho_{v^{-1}u})(x,c,\eta)\\
&\qquad -\int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^{3c}\partial\log\Delta(v^{-1})\Delta(v^{-1})^2(\Delta(v)\eta+\partial\log\Delta(v))^2\\
&\qquad\qquad\times(W_v\rho_{v^{-1}u})(x,c,\eta)\\
&= \int_{v\in\mathcal{G}^{r(u)}} a_v(x,c)e^{3c}(2\eta\partial\log\Delta(v^{-1})-(\partial\log\Delta(v^{-1}))^2)\\
&\qquad\qquad\times\partial\log\Delta(v^{-1})(W_v\rho_{v^{-1}u})(x,c,\eta),
\end{aligned}
$$

so that $[\mathcal{B}^2,[\mathcal{B},a]]$ is the half-density on $H^*\rtimes\mathcal{G}$ defined by

$$((x,c,\eta),u)\mapsto a_u(x,c)e^{3c}(2\eta\partial\log\Delta(u^{-1})-(\partial\log\Delta(u^{-1}))^2)\partial\log\Delta(u^{-1}).$$

More generally, setting $T^{(0)}:=T$ and then inductively defining $T^{(k)}:=[\mathcal{B}^2,T^{(k-1)}]$, we see that $[\mathcal{B},a]^{(k)}$ is the half-density on $H^*\rtimes\mathcal{G}$ defined by

$$((x,c,\eta),u)\mapsto a_u(x,c)e^{(2k+1)c}(2\eta\partial\log\Delta(u^{-1})-(\partial\log\Delta(u^{-1}))^2)^k\partial\log\Delta(u^{-1}).$$

Now these computations show that for $a\in\mathcal{A}$ and $k\in\mathbb{N}$, the operators $a^{(k)}$ and $[\mathcal{B},a]^{(k)}$ are half-densities on $H^*\rtimes\mathcal{G}$, with compact support in the $((x,c),u)\in Q\rtimes\mathcal{G}$ variables equal to that of a, and growing like η^k in the fibre variable $\eta\in H^*_{(x,c)}$ for all $(x,c)\in Q$. Hence, both $a^{(k)}(1+\mathcal{B}^2)^{-k/2}$ and $[\mathcal{B},a]^{(k)}(1+\mathcal{B}^2)^{-k/2}$ are bounded with compact support in the $Q\rtimes\mathcal{G}$ directions. Hence, for all $a\in\mathcal{A}$ the operator

$$(1+\mathcal{B}^2)^{-k/2-s/4}(a^{(k)})^*a^{(k)}(1+\mathcal{B}^2)^{-k/2-s/4}$$

is trace class whenever the real part of s is greater than 1, and similarly with a replaced by $[\mathcal{B},a]$. Thus, $\mathcal{A}\cup[\mathcal{B},\mathcal{A}]\subset B_2^\infty(\mathcal{B},1)$ in the notation of [Carey et al. 2014]. Thus, $\mathcal{A}^2$, the span of products from $\mathcal{A}$, satisfies $\mathcal{A}^2\cup[\mathcal{B},\mathcal{A}^2]\subset B_1^\infty(\mathcal{B},1)$, showing that the semifinite spectral triple over $\mathcal{A}^2$ is smoothly summable.

The last step to establish smooth summability is to observe that $\mathcal{A}$ has a (left) approximate unit for the inductive limit topology by [Muhly and Williams 2008, Proposition 6.8]. This ensures that any compactly supported section in $\mathcal{A}=\Gamma_c(Q\rtimes\mathcal{G};\Omega^{1/2})$ can be approximated by products while preserving summability.

Finally the computations also show that $(\mathcal{A},\mathcal{H},\mathcal{B})$ has isolated spectral dimension, as in [Carey et al. 2014, Definition 3.1], since for all multi-indices k of length

$m \geq 0$ we have proved that

$$\tau_{H^*}(a_0[\mathcal{B}, a_1]^{(k_1)} \cdots [\mathcal{B}, a_m]^{(k_m)}(1+\mathcal{B}^2)^{-|k|-m/2-s})$$

has a meromorphic continuation in a neighborhood of $s = 0$. □

Finally we can prove Theorem 4.5.

Proof of Theorem 4.5. Since the spectral dimension $p = 1$ and since the parity of the spectral triple is 1, the only nonzero term in the residue cocycle is ϕ_1 as defined in [Carey et al. 2014, Definition 3.2]. For any $a \in \Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ we have

$$\begin{aligned}([\mathcal{B}, a]\rho)_u(x, c, \eta) &= \mathcal{B}_{r(u)} \int_{v \in \mathcal{G}^{r(u)}} a_v(x, c)(W_v \rho_{v^{-1}u})(x, c, \eta) \\ &\qquad - \int_{v \in \mathcal{G}^{r(u)}} a_v(x, c)(W_v \mathcal{B}_{s(v)} \rho_{v^{-1}u})(x, c, \eta) \\ &= \int_{v \in \mathcal{G}^{r(u)}} a_v(x, c)(\mathcal{B}_{r(v)} - W_v \mathcal{B}_{s(v)} W_{v^{-1}})(W_v \rho_{v^{-1}u})(x, c, \eta) \\ &= \int_{v \in \mathcal{G}^{r(u)}} a_v(x, c) e^c \partial \log \Delta(v^{-1})(W_v \rho_{v^{-1}u})(x, c, \eta) \\ &= (\delta_1(a)\rho)_u(x, c, \eta),\end{aligned}$$

where δ_1 is the derivation of $\Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$ defined by

$$\delta_1(a)_u(x, c) := e^c \partial \log \Delta(u^{-1}) a_u(x, c).$$

The derivation δ_1 is the non-étale analogue of that given in [Connes and Moscovici 2005, p. 39]. Thus, for $a_0, a_1 \in \Gamma_c(Q \rtimes \mathcal{G}; \Omega^{1/2})$, we calculate

$$\begin{aligned}\phi_1(a_0, a_1) &= 2(2\pi i)^{1/2} \operatorname{res}_{z=0} \tau_{H^*}(a_0[\mathcal{B}, a_1](1+\mathcal{B}^2)^{-1/2-z}) \\ &= 2(2\pi i)^{1/2} \tau_Q(a_0 \delta_1(a_1)) \operatorname{res}_{z=0} \int_{\mathbb{R}} (1+t^2)^{-1/2-z}\, dh \\ &= 2(2\pi i)^{1/2} \tau_Q(a_0 \delta_1(a_1)) \operatorname{res}_{z=0} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(z)}{2\Gamma\left(\frac{1}{2}+z\right)} \\ &= (2\pi i)^{1/2} \tau_Q(a_0 \delta_1(a_1)).\end{aligned}$$

This is, up to the factor $(2\pi i)^{1/2}$, the non-étale analogue of the Godbillon–Vey cyclic cocycle from [Connes and Moscovici 2005, Proposition 19]. □

5. Relation with Connes' approach

We will outline in this section how our construction can, in codimension 1, be reconciled with Connes' approach to realize Gelfand–Fuks cocycles as cyclic cocycles for a convolution algebra. In doing so we will be able to justify why our spectral triple represents the Godbillon–Vey invariant. Let us first briefly recall Connes' approach [1986, Theorem 7.15].

5A. *Connes' approach.* Connes considers a discrete group Γ of orientation preserving diffeomorphisms of an oriented manifold V of dimension n. Associated to V and any $k \in \mathbb{N} \cup \{0\}$ is its *positively oriented k-th order jet bundle* $J_k^+(V)$, whose fibre over $x \in V$ consists of equivalence classes of local diffeomorphisms $\varphi : U \to V$, where U is an open neighborhood of 0 in $\mathbb{R}^n$, for which $\varphi(0) = x$. Two such diffeomorphisms φ and ψ are said to have the same *k-jet at* 0, denoted $j_0^k(\varphi) = j_0^k(\psi)$, if in any local coordinate system about x all partial derivatives of φ and ψ of order less than or equal to k coincide at 0. All the bundles $J_k^+(V)$, for $k \geq 1$, carry canonical right actions of $GL^+(n, \mathbb{R})$, and we write $\underline{J}_k^+(V) := J_k^+(V)/SO(n, \mathbb{R})$. The $\underline{J}_k^+(V)$ have contractible fibres [Bott 1976, p. 132].

It is then well-known [Bott 1976, (3.2)] that associated to any ($SO(n, \mathbb{R})$-relative) Gelfand–Fuks cocycle ϖ is $k \in \mathbb{N}$ and a canonical, diffeomorphism-invariant differential form, also denoted ϖ, on the quotient $\underline{J}_k^+(V)$. Connes uses the exterior derivative on V to manufacture ϖ into a cyclic cocycle ϖ_c for the algebra $C_c^\infty(\underline{J}_k^+(V) \rtimes \Gamma)$. Letting W denote the bundle of metrics over V, Connes invokes [Kasparov 1995] to deduce the existence of a class $j_{1,k} \in KK^\Gamma(C_0(W), C_0(\underline{J}_k^+(V)))$, whose descent $j_{1,k}^\Gamma \in KK(C_0(W \rtimes \Gamma), C_0(\underline{J}_k^+(V) \rtimes \Gamma))$ implements an isomorphism on K-theory via the Kasparov product. Letting $j_{0,1}^\Gamma \in KK(C_0(V \rtimes \Gamma), C_0(W \rtimes \Gamma))$ denote the bundle of metrics Kasparov module constructed in [Connes 1986, §5], Connes obtains a linear map

$$K_*(C_0(V) \rtimes \Gamma) \ni a \mapsto \varpi_c(a \otimes_{C_0(V) \rtimes \Gamma} j_{0,1}^\Gamma \otimes_{C_0(W) \rtimes \Gamma} j_{1,k}^\Gamma) \in \mathbb{R}$$

defined by ϖ. In the case when $V = S^1$, one has $J_2^+(S^1) = S^1 \times \mathbb{R}_+^* \times \mathbb{R}$, and if dx is the standard volume form on S^1, then the Godbillon–Vey invariant is represented by the invariant differential form [Connes 1986, Lemma 7.7]

$$\varpi = \frac{1}{y^3} dx \wedge dy \wedge dy_1, \quad (x, y, y_1) \in S^1 \times \mathbb{R}_+^* \times \mathbb{R}. \tag{5.1}$$

The associated cyclic cocycle ϖ_c is the trace on $C_c^\infty(J_2^+(S^1)) \rtimes \Gamma$ obtained by integration with respect to ϖ, and an involved calculation [Connes 1986, Theorem 7.3] shows that the linear map thus obtained on $K_0(C_0(V) \rtimes \Gamma)$ is the cyclic cocycle given by (1.2). Alternatively, one can obtain by essentially the same method a map

$$K_1(C_0(W) \rtimes \Gamma) \ni a \mapsto \varpi_c(a \otimes_{C_0(W) \rtimes \Gamma} j_{1,2}^\Gamma) \in \mathbb{R} \tag{5.2}$$

on the K-theory of $C_0(W) \rtimes \Gamma$. In the next subsection we will indicate how the index pairing induced by the spectral triple in Theorem 4.5 can be thought of as a non-étale version of the map in (5.2).

5B. ***The case of a general foliation.*** In the setting of a transversely orientable foliated manifold $(M, \mathcal{F})$ of codimension q, one has access to the *transverse jet bundles* $J_k^+(\mathcal{F})$ [Morita 2001, pp. 113–114]. The fibre $J_k^+(\mathcal{F})_x$ over $x \in M$ now consists of the k-jets $j_0^k(\varphi)$ of orientation-preserving local diffeomorphisms φ sending an open neighborhood of $0 \in \mathbb{R}^q$ to a local *transversal* about $x = \varphi(0)$. Note in particular that $J_1^+(\mathcal{F})$ is the same thing as the oriented transverse frame bundle $F^+(N)$.

We have natural projections $\pi_k : J_k^+(\mathcal{F}) \to M$ and $\pi_{k+1,k} : J_{k+1}^+(\mathcal{F}) \to J_k^+(\mathcal{F})$ defined respectively by forgetting all partial derivatives, and by forgetting all partial derivatives of order $k+1$. Moreover the holonomy groupoid $\mathcal{G}$ acts naturally on each $J_k^+(\mathcal{F})$, and its orbits define a foliation $\mathcal{F}_k$ of $J_k^+(\mathcal{F})$. There is a canonical right action of $GL^+(q, \mathbb{R})$ on all of the $J_k^+(\mathcal{F})$, $k \geq 1$, which commutes with the action of $\mathcal{G}$.

Proposition 5.3. *Let $S^2(\mathbb{R}^q)$ denote the symmetric polynomials of degree 2 over the vector space $\mathbb{R}^q$. Then the bundle $J_2^+(\mathcal{F})$ is an affine bundle over $J_1^+(\mathcal{F})$, modeled on the vector bundle $\pi_1^*(N) \otimes S^2(\mathbb{R}^q)$. Moreover, if $(M, \mathcal{F})$ is of codimension 1, a choice of Bott connection $\nabla^\flat$ on N determines an affine isomorphism of $J_2^+(\mathcal{F})$ with the total space of the bundle $H^* = (\ker(\alpha^\flat / T\mathcal{F}_1))^*$ over $J_1^+(\mathcal{F})$.*

Proof. That $J_2^+(\mathcal{F})$ is an affine bundle over $J_1^+(\mathcal{F})$ modeled on $\pi_1^*(N) \otimes S^2(\mathbb{R}^q)$ can be seen using local coordinates. Indeed, if $U \subset \mathbb{R}^q$ is an open neighborhood of 0 and T_β is a local transversal in M, then the 2-jet at 0 of any diffeomorphism $\varphi : U \to T$ is distinguished from its 1-jet at 0 by the partial derivatives

$$\frac{\partial^2 \varphi^i}{\partial y^j \partial y^k}\bigg|_0, \quad i, j, k = 1, \dots, q,$$

which are elements of $N_x \otimes S^2(\mathbb{R}^q)$ (here we have identified $\mathbb{R}^q$ with its dual in the natural way). The chain rule implies that whenever $c_{\alpha\beta} : T_\beta \to T_\alpha$ is the diffeomorphism defined by a transverse change of coordinates, then the corresponding coordinate change in the fibre $J_2^+(\mathcal{F})_{j_0^1(\varphi)}$ is given by the affine transformation

$$\left(\frac{\partial^2 \varphi^i}{\partial y^j \partial y^k}\bigg|_0\right)_{i,j,k=1,\dots,q} \mapsto \left(\frac{\partial c^i_{\alpha\beta}}{\partial y^l}\bigg|_{\varphi(0)} \frac{\partial^2 \varphi^l}{\partial y^j \partial y^k}\bigg|_0 + \frac{\partial^2 c^i_{\alpha\beta}}{\partial y^j \partial y^k}\bigg|_{\varphi(0)}\right)_{i,j,k=1,\dots,q},$$

where we have used the Einstein summation convention.

Suppose now that ∇ is a torsion-free connection on N. Then on any sufficiently small transversal $\mathcal{Y}_x$ about $x \in M$, ∇ restricts to a torsion-free affine connection and is therefore associated with an exponential map $\exp_x^\nabla$ which sends an open neighborhood of $0 \in N_x = T_x\mathcal{Y}_x$ diffeomorphically onto $\mathcal{Y}_x$. We therefore obtain a section $\sigma_\nabla : J_1^+(\mathcal{F}) \to J_2^+(\mathcal{F})$ defined by

$$\sigma_\nabla(j_0^1(\varphi)) := j_0^2(\exp_x^\nabla \circ j_0^1(\varphi)), \tag{5.4}$$

where we consider $j_0^1(\varphi)$ as a frame $\mathbb{R}^q \to N_x$, which by the chain rule is equivariant under the right action of $GL^+(q,\mathbb{R})$. Therefore, ∇ determines an affine isomorphism of $J_2^+(\mathcal{F})$ with the vector bundle $\pi_1^*(N) \otimes S^2(\mathbb{R}^q)$ on which it is modeled. In particular, if $(M,\mathcal{F})$ is codimension 1, ∇ determines an affine isomorphism of $J_2^+(\mathcal{F})$ with $\pi_1^*(N)$. Therefore, if $\nabla = \nabla^\flat$ is a Bott connection, we attain the stated isomorphism of $J_2^+(\mathcal{F})$ with $H \cong \pi_1^*(N)$, and hence with $H^* \cong H$ via the tautological Euclidean structure of Proposition 3.11. □

Now any Gelfand–Fuks class ϖ admits a canonical representative in the $\mathcal{G}$-invariant forms on $\underline{J}_k^+(\mathcal{F})$ for sufficiently large k [Morita 2001, pp. 117–119]. Let us briefly describe this canonical representative in the case of the Godbillon–Vey invariant of a codimension-1 foliation. Let $\varphi_t : U \to \mathcal{Y}$ be a 1-parameter family of diffeomorphisms of some open neighborhood U of $0 \in \mathbb{R}$ onto a local transversal $\mathcal{Y}$ in M, determining tangent vectors

$$X_k := \frac{d}{dt}\bigg|_{t=0} j_0^{k+1}(\varphi_t)$$

in $J_{k+1}^+(\mathcal{F})$, for all $k \in \mathbb{N} \cup \{0\}$. Let $\varphi := \varphi_0$, and define a 1-parameter family of coordinates $x_t := \varphi^{-1} \circ \varphi_t$. Then we define the *k-th tautological 1-form* ω^k on $J_{k+1}^+(\mathcal{F})$ by the formula

$$\omega^k(X_k) = \frac{d}{dt}\bigg|_{t=0}\left(\frac{d^k x_t}{dy^k}\bigg|_{y=0}\right) \tag{5.5}$$

for all $k \in \mathbb{N} \cup \{0\}$. Such tautological forms were introduced (in the nonfoliated case) by Kobayashi [1961]. Notice that if $u \in \mathcal{G}$ is represented by a diffeomorphism $h : \mathcal{Y} \to \mathcal{Y}'$ of local transversals, then

$$(h^*\omega^k)(X_k) = \omega^k\left(\frac{d}{dt}\bigg|_{t=0} j_0^{k+1}(h \circ \varphi_t)\right) = \frac{d}{dt}\bigg|_{t=0}\left(\frac{d^k(\varphi^{-1} \circ h^{-1} \circ h \circ \varphi_t)}{dy^k}\right) = \omega^k(X_k),$$

so the ω^k are $\mathcal{G}$-invariant. Pulling back via the projections, let us assume that ω^0, ω^1, and ω^2 are all defined on $J_3^+(\mathcal{F})$. Then, using the $\mathcal{G}$ invariance of these forms together with [Morita 2001, Proposition 3.23], it can be shown that the ω^i satisfy the structure equations

$$d\omega^0 = -\omega^1 \wedge \omega^0, \qquad d\omega^1 = -\omega^2 \wedge \omega^0. \tag{5.6}$$

Now by (5.5) the form ω^0 on $J_1^+(\mathcal{F})$ is, by definition, simply the solder form on the transverse frame bundle. Therefore, the first of the equations (5.6) says that ω^1 on $J_2^+(\mathcal{F})$ behaves like a torsion-free connection form: indeed, the pullback $\sigma_\nabla^*\omega^1$ of ω^1 by the section $\sigma_\nabla : J_1^+(\mathcal{F}) \to J_2^+(\mathcal{F})$ defined by any torsion-free connection ∇ on N is precisely the connection form corresponding to ∇. Therefore, the second of the equations (5.6) is a formula for the "curvature" of the "connection form" ω^1.

In particular, the Godbillon–Vey form on $J_3^+(\mathcal{F})$ is simply the (negative of) the "connection" wedged with its "curvature" [Bott 1972, Lemma 10.9]:

$$\mathrm{gv} = -\omega^1 \wedge d\omega^1 = \omega^0 \wedge \omega^1 \wedge \omega^2. \tag{5.7}$$

It is by no means obvious that the form in (5.7) is related to the form $d\nu_{H^*}$ on H^* considered in (4.4). The next result tells us that in fact these forms are the same, and therefore justifies our claim that the cocycle obtained as the index formula from Theorem 4.5 truly does represent the Godbillon–Vey invariant.

Theorem 5.8. *Let $(M, \mathcal{F})$ be a transversely oriented foliation of codimension* 1, *with transverse volume form $\omega \in \Omega^1(M)$. Suppose moreover we are gifted with a torsion-free Bott connection on N, giving an identification $J_2^+(\mathcal{F}) \cong H^*$ as in Proposition 5.3. Then the form* gv *of* (5.7) *on $J_3^+(\mathcal{F})$ descends to a form on $J_2^+(\mathcal{F}) \cong H^*$, which, in the trivialization $H^* = M \times \mathbb{R} \times \mathbb{R}$ determined by ω as in Section 4A, coincides with the form $d\nu_{H^*} = \omega \wedge dc \wedge d\eta$ of* (4.4).

Proof. Associated to the transverse volume form ω is a nonvanishing normal vector field $Z \in \Gamma^\infty(M; N)$ characterized by the equation $\omega(Z) \equiv 1$. Fix $x \in M$ and let $\mathcal{Y}_x$ be a local transversal through x. Then the torsion-free Bott connection $\nabla^\flat$ on N restricts to an affine connection on $\mathcal{Y}_x$, and so determines an exponential map $\exp^{\nabla^\flat} : U \to \mathcal{Y}_x$ which is a local diffeomorphism defined on an open neighborhood U of $0 \in T_x\mathcal{Y}_x$. Rescaling ω if necessary, we can always assume that $Z_x \in U$ and we obtain a coordinate $u_0 : \mathcal{Y}_x \to \mathbb{R}$ defined by the equation

$$u_0(x')Z_x = (\exp^{\nabla^\flat})^{-1}(x'), \quad x' \in \mathcal{Y}_x.$$

Now fix a local diffeomorphism φ from an open neighborhood of $0 \in \mathbb{R}$ to $\mathcal{Y}_x$. The coordinate u_0 on $\mathcal{Y}_x$ identifies φ with a local diffeomorphism $\tilde{\varphi} := u_0 \circ \varphi$ of $\mathbb{R}$, so the 3-jet $j_0^3(\varphi)$ is determined by the polynomial

$$\tilde{\varphi}(0) + \frac{d\tilde{\varphi}}{dy}\bigg|_0 y + \frac{d^2\tilde{\varphi}}{dy^2}\bigg|_0 y^2 + \frac{d^3\tilde{\varphi}}{dy^3}\bigg|_0 y^3$$

where we use y to denote the standard coordinate in $\mathbb{R}$. We thus define coordinates $u_i(j_0^3(\varphi)) := \frac{d^i\tilde{\varphi}}{dy^i}\big|_0$ for $i = 1, 2, 3$ for $j_0^3(\varphi) \in J_3^+(\mathcal{Y}_x)$. Suppose now that φ_t is a 1-parameter family of local diffeomorphisms from an open neighborhood of $0 \in \mathbb{R}$ to $\mathcal{Y}_x$, with $\varphi_0 = \varphi$. The coordinate representation $\tilde{\varphi}_t := u_0 \circ \varphi_t$ of the family φ_t determines a curve

$$j_0^3(\tilde{\varphi}_t) = \left(\tilde{\varphi}_t(0), \frac{d\tilde{\varphi}_t}{dx}\bigg|_0, \frac{d^2\tilde{\varphi}_t}{dx^2}\bigg|_0 \frac{d^3\tilde{\varphi}_t}{dx^3}\bigg|_0\right)$$

in $J_3^+(\mathbb{R})$; hence, we can write the tangent vector $X = \frac{d}{dt}\big|_0 j_0^3(\varphi_t)$ on $J_3^+(\mathcal{Y}_x)$ determined by the curve $j_0^3(\varphi_t)$ in the form

$$X = \frac{d\tilde{\varphi}_t}{dt}\bigg|_0 \frac{\partial}{\partial u^0} + \frac{d}{dt}\bigg|_0 \bigg(\frac{d\tilde{\varphi}_t}{dy}\bigg|_0\bigg)\frac{\partial}{\partial u^1} + \frac{d}{dt}\bigg|_0 \bigg(\frac{d^2\tilde{\varphi}_t}{dy^2}\bigg|_0\bigg)\frac{\partial}{\partial u^2} + \frac{d}{dt}\bigg|_0 \bigg(\frac{d^3\tilde{\varphi}_t}{dy^3}\bigg|_0\bigg)\frac{\partial}{\partial u^3}.$$

Setting $h_t := \varphi^{-1}\circ\varphi_t$ we have $\tilde{\varphi}_t = \tilde{\varphi}\circ h_t$. Then using the chain rule together with the fact that $h_0 = \mathrm{id}_{\mathbb{R}}$, we compute

$$\begin{aligned}
\frac{d\tilde{\varphi}_t}{dt}\bigg|_0 &= \frac{d}{dt}\bigg|_0(\tilde{\varphi}\circ h_t) = u_1\frac{dh_t}{dt}\bigg|_0,\\
\frac{d}{dt}\bigg|_0\bigg(\frac{d\tilde{\varphi}_t}{dy}\bigg|_0\bigg) &= \frac{d}{dt}\bigg|_0\bigg(\frac{d(\tilde{\varphi}\circ h_t)}{dy}\bigg|_0\bigg) = u_2\frac{dh_t}{dt}\bigg|_0 + u_1\frac{d}{dt}\bigg|_0\bigg(\frac{dh_t}{dy}\bigg|_0\bigg),\\
\frac{d}{dt}\bigg|_0\bigg(\frac{d^2\tilde{\varphi}_t}{dy^2}\bigg|_0\bigg) &= \frac{d}{dt}\bigg|_0\bigg(\frac{d^2(\tilde{\varphi}\circ h_t)}{dy^2}\bigg|_0\bigg)\\
&= u_3\frac{dh_t}{dt}\bigg|_0 + 2u_2\frac{d}{dt}\bigg|_0\bigg(\frac{dh_t}{dy}\bigg|_0\bigg) + u_1\frac{d}{dt}\bigg|_0\bigg(\frac{d^2h_t}{dy^2}\bigg|_0\bigg).
\end{aligned}$$

Therefore, by (5.5) we find that[1]

$$du_0 = u_1\omega^0, \qquad du_1 = u_2\omega^0 + u_1\omega^1, \qquad du_2 = u_3\omega^0 + 2u_2\omega^1 + u_1\omega^2, \tag{5.9}$$

and we deduce that

$$\omega^0\wedge\omega^1\wedge\omega^2 = \frac{1}{u_1^3}du_0\wedge du_1\wedge du_2, \tag{5.10}$$

which is a well-defined form on $J_2^+(\mathcal{Y}_x)$.

Now we come to transporting the form of (5.10) on $J_2^+(\mathcal{Y}_x)\subset J_2^+(\mathcal{F})$ to the total space of the bundle H^* over $J_1^+(\mathcal{F})$ as in Proposition 5.3. The transverse volume form $\omega\in\Omega^1(M)$ determines a trivialization

$$J_1^+(\mathcal{F})\ni\phi_x\mapsto(x,\omega_x(\phi_x(1))) =: (x,t)\in M\times\mathbb{R}_+^*,$$

where we think of $\phi_x = d\varphi_0$ as a frame $\mathbb{R}\to N_x$. The transverse vector field $Z\in\Gamma^\infty(M;N)$ corresponding to ω determines a trivialization

$$N\ni hZ_x\mapsto(x,h)\in M\times\mathbb{R}, \quad x\in M,$$

[1] The formulae in (5.9) that we compute here differ slightly from the analogous equations of Kobayashi [1961, §4] and Connes and Moscovici [2005, p. 45], for whom the summands containing ω^0 in the second and third equations have factors of 2 and 3, respectively. The reader can easily verify using elementary calculus that our own computations do *not* give rise to these factors. In the absence of any explicit computations provided by Kobayashi and Connes–Moscovici, it is difficult to determine why these additional factors appear in their equations. In any case, these additional factors have no impact on the coordinate expression we obtain for the Godbillon–Vey differential form.

of N and therefore a corresponding trivialization $H \cong J_1^+(\mathcal{F}) \times \mathbb{R} \cong M \times \mathbb{R}_+^* \times \mathbb{R}$ of $H \cong \pi_1^* N$. Unlike the coordinates u_i used for the transversal $\mathcal{Y}_x$ in the first part of the proof, the trivialization $H \cong M \times \mathbb{R}_+^* \times \mathbb{R}$ is global, and we must show that on $\mathcal{Y}_x$, we have equalities $du_0 = \omega$, $u_1 = t$, and $u_2 = h$.

Now $du_0(Z) \equiv 1$ by definition of the coordinate u_0 on $\mathcal{Y}_x$, so $du_0 = \omega$. For u_1 we see that

$$u_1 = \frac{d(u_0 \circ \varphi)}{dy} = du_0 \circ d\varphi(1) = \omega_x(d\varphi(1)) = t$$

by definition of the variable t. Finally, in the trivial bundle $J_2^+(\mathbb{R}) = \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}$ that is the image of $J_2^+(\mathcal{Y}_x)$ under the coordinates (u_0, u_1, u_2), the u_2 variable identifies with the tangent variable for $\mathbb{R}$ in the manner of Proposition 5.3. Viewed as coordinates on $J_2^+(\mathcal{Y}_x)$ and $T\mathcal{Y}_x$, respectively, we then have $u_2 = h$ and therefore

$$\mathrm{gv} = \frac{1}{u_1^3}\, du_0 \wedge du_1 \wedge du_2 = -\frac{1}{t^3}\, \omega \wedge dt \wedge dh.$$

Finally, as in Section 4A, we make the substitutions $t = e^c$ giving $dt = e^c dc$, and $h = e^{2c}\eta$ giving $dh = e^{2c} d\eta$. Thus, on H^*, we find that

$$\mathrm{gv} = \omega \wedge dc \wedge d\eta = d\nu_{H^*}$$

as claimed. □

6. Concluding remarks

It is tempting to view the higher-codimension version of the codimension-1 Kasparov module and spectral triple as analogous data representing the Godbillon–Vey invariant in higher codimension. Sadly, despite the naturality of the constructions presented here, it is far from clear that such an interpretation is warranted. Without an identification of the Chern character of these spectral triples with the Godbillon–Vey class, they must remain an interesting construction.

One final remark on the constructions presented here: they all pass to real algebras and real KK-theory. All our constructions are Real [Kasparov 1980] for the obvious variations of complex conjugation, in part because of our systematic use of the exterior algebra rather than the spinor bundle. This means that we can at all stages retain contact with homology of manifolds with real coefficients.

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The extension problem for graph C^*-algebras

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We give a complete K-theoretical description of when an essential extension of two simple graph C^*-algebras is again a graph C^*-algebra.

1. Introduction

Whenever a class $\mathcal{C}$ of C^*-algebras is closed under passing to ideals and quotients, one may ask whether or not in any extension

$$\mathfrak{e} : 0 \to \mathfrak{I} \xrightarrow{\iota} \mathfrak{A} \xrightarrow{\pi} \mathfrak{A}/\mathfrak{I} \to 0 \tag{1.1}$$

with $\mathfrak{I}, \mathfrak{A}/\mathfrak{I} \in \mathcal{C}$, it is automatic that $\mathfrak{A} \in \mathcal{C}$. The class of AF algebras [Effros 1981, Chapter 9], the class of Type I C^*-algebras [Pedersen 1979, 6.2.6], the class of nuclear C^*-algebras [Brown and Ozawa 2008, Exercise 3.8.1], and the class of purely infinite C^*-algebras [Kirchberg and Rørdam 2000, Theorem 4.19] all have this property, but there are also many important classes when such a permanence result fails in obvious ways. In this case, one may usefully ask instead whether there is a K-theoretical description of when one may conclude from membership of $\mathcal{C}$ at the extremes to membership of $\mathcal{C}$ in the middle. In most, if not all, of the instances when such results are known, the K-theoretical data used comes from the six term exact sequence

$$\begin{array}{ccccc} K_0(\mathfrak{I}) & \xrightarrow{\iota_0} & K_0(\mathfrak{A}) & \xrightarrow{\pi_0} & K_0(\mathfrak{A}/\mathfrak{I}) \\ {\scriptstyle \partial_1}\uparrow & & & & \downarrow{\scriptstyle \partial_0} \\ K_1(\mathfrak{A}/\mathfrak{I}) & \xleftarrow[\pi_1]{} & K_1(\mathfrak{A}) & \xleftarrow[\iota_1]{} & K_1(\mathfrak{I}) \end{array} \tag{1.2}$$

as summarized in Table 1.

In Theorem 4.1 of this paper, we provide a result of this nature for the class of graph C^*-algebras under the assumption that $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$ are simple, and $\mathfrak{I}$ is not a complemented ideal of $\mathfrak{A}$. Reflecting complications arising from the fact that simple graph C^*-algebras may be AF as well as purely infinite, the K-theoretical

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Class	Obstruction	Reference
Real rank zero	$\partial_0 = 0$	[Brown and Pedersen 1991]
Stable rank one	$\partial_1 = 0$	[Lin and Rørdam 1995]
Stably finite	$\operatorname{Im} \partial_1 \cap K_0(\mathfrak{I})_+ = 0$	[Spielberg 1988]
AT algebras of real rank zero	$\partial_* = 0$	[Lin and Rørdam 1995]
Cuntz–Krieger algebras of real rank zero	$\partial_0 = 0$	[Bentmann 2019]

Table 1. Sample extension results.

Case	$\mathfrak{I}$	$\mathfrak{A}/\mathfrak{I}$
$[\mathbf{11}]$	AF	AF
$[\mathbf{1}\infty]$	AF	Kirchberg
$[\infty\mathbf{1}]$	Kirchberg	AF
$[\infty\infty]$	Kirchberg	Kirchberg

Case	$\mathfrak{A}$	$\mathfrak{A}/\mathfrak{I}$
$[-]_0$	Nonunital	Nonunital
$[-]_1$	Nonunital	Unital
$[-]_2$	Unital	Unital

Table 2. Notation for the 12 cases we consider.

obstructions (which we show by example are all necessary) are rather more complicated than in the cases previously known, with the possible exception of [Dadarlat and Loring 1994] (which we have not listed in Table 1 because the class to which it applies is too complicated to describe in the table). We establish our results by a two-step procedure of invoking classification results by K-theory paired with range results, thereby proving that $\mathfrak{A}$ is a graph C^*-algebra by first constructing a graph E whose C^*-algebra has the necessary K-theoretic data, and then applying classification results to conclude $\mathfrak{A} \cong C^*(E)$.

In the case where $\mathfrak{I}$ is a complemented ideal, $\mathfrak{A}$ is always a graph C^*-algebra since this class is closed under direct sums. Note also that strictly speaking, the class of graph C^*-algebras is not closed under passing to ideals and quotients, as only so-called gauge-invariant ideals respect the structure. Hence we work only with graph C^*-algebras with finitely many ideals, where all ideals are gauge-invariant. Importantly for our approach, a graph C^*-algebra with finitely many ideals automatically has real rank zero.

Providing such a characterization has been the ambition of some of the authors for almost a decade, and the path to obtaining a complete result has been unusually indirect. To describe the contribution of the paper at hand, we introduce notation for the 12 different cases we have to address in proving the result. As already mentioned, the ideal and quotient are either AF or purely infinite by the dichotomy of simple graph C^*-algebras, and we denote the four possible cases as indicated

$[-]_0$		
[**11**]	[Elliott 1976]	[Eilers et al. 2014a]
[**1**∞]	[Eilers et al. 2009]	[Eilers et al. 2016a]
[∞**1**]	[Eilers et al. 2009]	[Eilers et al. 2016a]
[∞∞]	[Rørdam 1997]	[Eilers et al. 2016a]

$[-]_1$		
[**11**]	[Elliott 1976]	[Eilers et al. 2014a]
[**1**∞]	[Gabe and Ruiz 2020]	Theorem 3.8
[∞**1**]	[Eilers et al. 2014b]	[Eilers et al. 2016a]
[∞∞]	[Eilers et al. 2014b]	[Eilers et al. 2016a]

$[-]_2$		
[**11**]	[Elliott 1976]	[Eilers et al. 2014a]
[**1**∞]	[Eilers et al. 2013]	[Eilers et al. 2016a]
[∞**1**]	[Eilers et al. 2013]	[Eilers et al. 2016a]
[∞∞]	[Eilers and Restorff 2006]	[Eilers et al. 2016a]

Table 3. References for the classification result (listed first) and range result (listed second) in each of our 12 cases.

in the left part of Table 2. By our assumptions, $\mathfrak{I}$ does not have a unit. Further, when $\mathfrak{A}$ has a unit, so does $\mathfrak{A}/\mathfrak{I}$, and hence the number of units among the three C^*-algebras $\mathfrak{A}$, $\mathfrak{I}$, and $\mathfrak{A}/\mathfrak{I}$ uniquely determines which of these three C^*-algebras has a unit. We denote the cases by $[-]_*$ with the symbol $* \in \{0, 1, 2\}$ indicating the number of units among the three C^*-algebras, as indicated in the right part of Table 2.

Back in 2011, the four senior authors thought to have completed the case where $\mathfrak{I}$ is stable (this is automatic by [Eilers and Tomforde 2010] in all but the $[\mathbf{11}]_*$ cases) and in fact Eilers announced this at the Kyoto conference "C^*-Algebras and Applications" at RIMS. We then went on to pursue the remaining $[\mathbf{11}]_*$ case, and had succeeded in solving that by 2013 in [Eilers et al. 2014a]. However, before publishing a proof of our combined result, [Gabe 2016] exposed that a claim from [Elliott and Kucerovsky 2001] used in the paper [Eilers et al. 2014b] was false (see [Eilers et al. 2016b] for a detailed corrigendum), thereby rendering incomplete the proof in a single case, namely the one denoted $[\mathbf{1}\infty]_1$. The example Gabe provided to show that classification by the standard invariant fails was, however, not a counterexample to our permanence statement, so we tried at several occasions to remedy this situation. Finding a way took us several years: Indeed, the missing piece of the puzzle was provided recently in [Gabe and Ruiz 2020], to the effect of providing a complete classification result applicable to the case left open. Joining forces with Gabe, we were finally able to finish the extension result by providing the necessary range result in the paper at hand. In Table 3 we list, as pairs, references to the classification results appropriate range results for each of our 12 cases.

We note that among the results listed in Table 1, the one by Bentmann is closely and subtly related to the result presented here, and it is worth discussing how our work is related to [Bentmann 2019]. Indeed, it was an important open question whether a unital extension of Cuntz–Krieger algebras is itself in the Cuntz–Krieger class (it is necessary to stabilize the ideal for such a statement to be nontrivial), but since Restorff's classification result for Cuntz–Krieger algebras [Restorff 2006] applies only when the middle C^*-algebra is known to be Cuntz–Krieger, we were only able to argue in such a way in the few cases when an appropriate external classification result was known; cf. [Arklint 2013]. For instance, the necessary result for extensions of simple Cuntz–Krieger algebras was provided in [Eilers and Restorff 2006].

Bentmann was able to circumvent this issue by elaborating on an idea from [Cuntz 1981] to make contact to a deep result by Kirchberg [2000], providing a complete solution of the extension problem in the Cuntz–Krieger case, and leading us to the following conjecture:

Conjecture 1.3. *Let $C^*(E_1)$ and $C^*(E_3)$ be unital graph C^*-algebras with finitely many ideals, and consider the unital extension*

$$0 \to C^*(E_1) \otimes \mathbb{K} \to \mathfrak{X} \to C^*(E_3) \to 0.$$

Then $\mathfrak{X}$ is a graph C^*-algebra if and only if $\partial_0 : K_0(C^*(E_3)) \to K_1(C^*(E_1))$ vanishes.

The forward implication is known to hold, and Bentmann's result establishes the converse when both E_1 and E_3 are finite graphs with no sinks and no sources. By a slight refinement of our main result, we are able to remove both the finiteness and the "no sinks and no sources" hypotheses in the very special case when both $C^*(E_1)$ and $C^*(E_3)$ are simple (see Corollary 4.2). At this stage we do not have a workable conjecture for the general extension problem for graph C^*-algebras with finitely many ideals.

The paper is organized as follows: We summarize notation and provide a few preliminary lemmas in Section 2. In Section 3 we solve the case hitherto left open. In Section 4 we state our extension result and succinctly explain how existing results may be combined to establish the extension result in the remaining 11 cases mentioned above.

2. Preliminaries

We follow the notation and definition for graph C^*-algebras in [Fowler et al. 2000]. In particular, our arrows are drawn in the direction for which sinks and infinite emitters are singular.

Definition 2.1. Let $\mathfrak{A}$ be a C^*-algebra. An element $a \in \mathfrak{A}$ is *norm-full* if a is not contained in any closed two-sided ideal of $\mathfrak{A}$. An extension

$$\mathfrak{e} : 0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$$

is *full* if for every nonzero $x \in \mathfrak{A}/\mathfrak{I}$, $\tau_{\mathfrak{e}}(x)$ is norm-full in $\mathcal{Q}(\mathfrak{I})$, where $\mathcal{M}(\mathfrak{I})$ is the multiplier algebra of $\mathfrak{I}$, $\mathcal{Q}(\mathfrak{I}) = \mathcal{M}(\mathfrak{I})/\mathfrak{I}$, and $\tau_{\mathfrak{e}} : \mathfrak{A}/\mathfrak{I} \to \mathcal{Q}(\mathfrak{I})$ is the Busby invariant associated to $\mathfrak{e}$.

Lemma 2.2. *Let $\mathfrak{A}$ be a C^*-algebra with exactly one proper nontrivial ideal $\mathfrak{I}$ satisfying the following four conditions*:

(1) $\mathfrak{I}$ *is AF*,

(2) $\mathfrak{A}/\mathfrak{I}$ *is purely infinite*,

(3) $\mathfrak{A}$ *is separable*, *and*

(4) $K_0(\mathfrak{A})_+ = K_0(\mathfrak{A})$.

Then $\mathfrak{I}$ is stable, *and $\mathfrak{e}$ is a full extension.*

Proof. Suppose (1)–(4) hold. Since the embedding from $\mathfrak{A}$ to $\mathfrak{A} \otimes \mathbb{K}$ induces an order isomorphism from $K_0(\mathfrak{A})$ to $K_0(\mathfrak{A} \otimes \mathbb{K})$, we have $K_0(\mathfrak{A} \otimes \mathbb{K})_+ = K_0(\mathfrak{A} \otimes \mathbb{K})$. Note that $\mathfrak{A} \otimes \mathbb{K}$ has exactly one proper nontrivial ideal $\mathfrak{I} \otimes \mathbb{K}$ (which is an AF algebra) and $\mathfrak{A} \otimes \mathbb{K}/(\mathfrak{I} \otimes \mathbb{K}) \cong (\mathfrak{A}/\mathfrak{I}) \otimes \mathbb{K}$ is a purely infinite simple C^*-algebra. Also, since ∂_0 must vanish, $\mathfrak{A}$ has real rank zero by [Brown and Pedersen 1991, Theorem 3.19], and by [Brown and Pedersen 1991, Proposition 3.15], so does $\mathfrak{A} \otimes \mathbb{K}$. Therefore, $\mathfrak{A} \otimes \mathbb{K}$ has stable weak cancellation by [Eilers et al. 2014b, Lemma 3.15], and by [Eilers et al. 2014b, Corollary 3.22],

$$\mathfrak{e}^s : 0 \to \mathfrak{I} \otimes \mathbb{K} \to \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{A}/\mathfrak{I} \otimes \mathbb{K} \to 0$$

is a full extension.

Let $\{e_n\}$ be an approximate identity for $\mathfrak{A}$ consisting of projections, and let $\{e_{i,j}\}$ be a system of matrix units for $\mathbb{K}$. Without loss of generality, we may assume that e_n is not an element of $\mathfrak{I}$ for all n. Since

$$\mathfrak{I} \cong \mathfrak{I} \otimes e_{1,1} = \overline{\bigcup_n (e_n \otimes e_{1,1})(\mathfrak{I} \otimes \mathbb{K})(e_n \otimes e_{1,1})}$$

with $(e_n \otimes e_{1,1})(\mathfrak{I} \otimes \mathbb{K})(e_n \otimes e_{1,1}) \subseteq (e_{n+1} \otimes e_{1,1})(\mathfrak{I} \otimes \mathbb{K})(e_{n+1} \otimes e_{1,1})$, by [Hjelmborg and Rørdam 1998, Corollary 4.1], $\mathfrak{I}$ is stable if $(e_n \otimes e_{1,1})(\mathfrak{I} \otimes \mathbb{K})(e_n \otimes e_{1,1})$ is stable for all n.

Let $n \in \mathbb{N}$. Let $\tau_{\mathfrak{e}}^{s} : (\mathfrak{A} \otimes \mathbb{K})/(\mathfrak{I} \otimes \mathbb{K}) \to \mathcal{Q}(\mathfrak{I} \otimes \mathbb{K})$ be the Busby map and $\sigma_{\mathfrak{e}}^{s} : \mathfrak{A} \otimes \mathbb{K} \to \mathcal{M}(\mathfrak{I} \otimes \mathbb{K})$ the injective $*$-homomorphism such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I}\otimes\mathbb{K} & \longrightarrow & \mathfrak{A}\otimes\mathbb{K} & \xrightarrow{\pi} & (\mathfrak{A}\otimes\mathbb{K})/(\mathfrak{I}\otimes\mathbb{K}) & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\scriptstyle \sigma_{\mathfrak{e}}^{s}} & & \Big\downarrow{\scriptstyle \tau_{\mathfrak{e}}^{s}} & & \\ 0 & \longrightarrow & \mathfrak{I}\otimes\mathbb{K} & \longrightarrow & \mathcal{M}(\mathfrak{I}\otimes\mathbb{K}) & \longrightarrow & \mathcal{Q}(\mathfrak{I}\otimes\mathbb{K}) & \longrightarrow & 0 \end{array}$$

commutes. By the commutativity of the diagram,

$$(e_n \otimes e_{1,1})(\mathfrak{I} \otimes \mathbb{K})(e_n \otimes e_{1,1}) = \sigma_{\mathfrak{e}}^{s}((e_n \otimes e_{1,1}))(\mathfrak{I} \otimes \mathbb{K})\sigma_{\mathfrak{e}}((e_n \otimes e_{1,1})).$$

Since $\mathfrak{e}^{s}$ is a full extension and $e_n \otimes e_{1,1}$ is not an element of $\mathfrak{I} \otimes \mathbb{K}$, $\tau_{\mathfrak{e}}^{s}(\pi(e_n \otimes e_{1,1}))$ is a full projection in $\mathcal{Q}(\mathfrak{I} \otimes \mathbb{K})$. By [Eilers et al. 2014b, Lemma 3.3] and the commutativity of the diagram, $\sigma_{\mathfrak{e}}^{s}((e_n \otimes e_{1,1}))$ is a norm-full projection in $\mathcal{M}(\mathfrak{I} \otimes \mathbb{K})$. Since $\mathfrak{I} \otimes \mathbb{K}$ is an AF algebra, every norm-full projection in $\mathcal{M}(\mathfrak{I} \otimes \mathbb{K})$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(\mathfrak{I} \otimes \mathbb{K})}$. Consequently,

$$(e_n \otimes e_{1,1})(\mathfrak{I} \otimes \mathbb{K})(e_n \otimes e_{1,1}) = \sigma_{\mathfrak{e}}^{s}((e_n \otimes e_{1,1}))(\mathfrak{I} \otimes \mathbb{K})\sigma_{\mathfrak{e}}((e_n \otimes e_{1,1})) \cong \mathfrak{I} \otimes \mathbb{K},$$

which implies that $(e_n \otimes e_{1,1})(\mathfrak{I} \otimes \mathbb{K})(e_n \otimes e_{1,1})$ is a stable C^*-algebra.

By [Eilers et al. 2014b, Theorem 3.22], $\mathfrak{e}$ is a full extension. □

Definition 2.3. If $\mathfrak{A}$ is a C^*-algebra, the *scale* of $K_0(\mathfrak{A})$ is the subset

$$\Sigma\mathfrak{A} = \{x \in K_0(\mathfrak{A}) : x = [p] \text{ for some projection } p \text{ in } \mathfrak{A}\}.$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras, a homomorphism $\alpha : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ is *contractive* if $\alpha(\Sigma\mathfrak{A}) \subseteq \Sigma B$. We also define an invertible homomorphism $\alpha : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ to be *scale-preserving* if both α and α^{-1} are contractive, and in the event that $\mathfrak{A}$ and $\mathfrak{B}$ are both unital we also require $\alpha([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$. (If either of $\mathfrak{A}$ or $\mathfrak{B}$ is nonunital, this last condition imposes no requirement.)

Some comments on this definition are in order: When $\mathfrak{A}$ is unital and AF, the element $[1_{\mathfrak{A}}]$ is the maximal element of $\Sigma\mathfrak{A}$ and in fact,

$$\Sigma\mathfrak{A} = \{x \in K_0(\mathfrak{A}) : 0 \leq x \leq [1_{\mathfrak{A}}]\}.$$

Consequently, if $\mathfrak{A}$ and $\mathfrak{B}$ are both unital AF algebras, an order-preserving bijection $\alpha : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ is scale-preserving if and only if α and α^{-1} are contractive. (Thus, the condition $\alpha([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$ in the definition of scale-preserving is superfluous in the AF case.) On the other hand, if A is purely infinite and simple, then $\Sigma\mathfrak{A} = K_0(\mathfrak{A})$. Consequently, if $\mathfrak{A}$ and $\mathfrak{B}$ are both purely infinite and simple, any homomorphism between their K_0-groups is automatically contractive, so that a bijection $\alpha : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ is scale-preserving if and only if $\alpha([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$ in the event that $\mathfrak{A}$ and $\mathfrak{B}$ are unital, and always so in the event at least one of $\mathfrak{A}$ and $\mathfrak{B}$ is nonunital. (Thus, the condition that α and α^{-1} be contractive in the definition of scale-preserving is superfluous in the purely infinite case.)

When $\mathfrak{A}$ is AF, the scale $\Sigma\mathfrak{A}$ is a subset of $K_0(\mathfrak{A})$ with the following properties:

- generating $(\forall x \in K_0(\mathfrak{A})_+,\ \exists y_1, \dots, y_n \in \Sigma\mathfrak{A} : x = y_1 + \cdots + y_n)$;
- hereditary $(\forall x \in K_0(\mathfrak{A})_+,\ \forall y \in \Sigma\mathfrak{A} : x \leq y \Rightarrow x \in \Sigma\mathfrak{A})$; and
- upward directed $(\forall y_1, y_2 \in \Sigma\mathfrak{A},\ \exists z \in \Sigma\mathfrak{A} : y_1 \leq z \text{ and } y_2 \leq z)$.

We recall that by the result of Effros, Handelman and Shen [Effros et al. 1980], the range of ordered K_0-groups associated to (separable) AF algebras are exactly the (countable) *dimension groups*: ordered groups with the Riesz interpolation property that are also unperforated. As explained in [Effros 1981] it is possible to amend this result by earlier work of Elliott [1976] to see that any dimension group (G, G_+) with a subset $\Sigma \subseteq G_+$ that is generating, hereditary, and upward directed arises as the scaled K_0-group of an AF algebra.

Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{I}$ be an ideal of $\mathfrak{A}$. Let $K_{\mathsf{six}}(\mathfrak{A}, \mathfrak{I})$ denote the six-term exact sequence (1.2) in K-theory induced by the extension (1.1). $K^+_{\mathsf{six}}(\mathfrak{A}, \mathfrak{I})$ denotes the same sequence as in (1.2), where the three K_0-groups are considered as preordered groups. We let $K^{+,\Sigma}_{\mathsf{six}}(\mathfrak{A}, \mathfrak{I})$ denote the same sequence as in (1.2) but $K_0(\mathfrak{A})$ is considered as a preordered group, and $K_0(\mathfrak{A}/\mathfrak{I})$ and $K_0(\mathfrak{I})$ are considered as scaled preordered groups.

For a C^*-algebra $\mathfrak{B}$ with ideal $\mathfrak{J}$, a homomorphism

$$(\alpha_\bullet, \beta_\bullet) : K_{\mathsf{six}}(\mathfrak{A}, \mathfrak{I}) \to K_{\mathsf{six}}(\mathfrak{B}, \mathfrak{J})$$

consists of six group homomorphisms $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, making the following diagram commute:

$$
\begin{array}{ccccccc}
K_0(\mathfrak{I}) & & \xrightarrow{\iota_0} & K_0(\mathfrak{A}) & \xrightarrow{\pi_0} & & K_0(\mathfrak{A}/\mathfrak{I}) \\
& \searrow^{\alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \swarrow^{\alpha_3} & \\
& & K_0(\mathfrak{J}) \xrightarrow{\iota_0} & K_0(\mathfrak{B}) & \xrightarrow{\pi_0} K_0(\mathfrak{B}/\mathfrak{J}) & & \\
\uparrow{\scriptstyle \partial_1} & & \uparrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_0} & & \downarrow{\scriptstyle \partial_0} \\
& & K_0(\mathfrak{B}/\mathfrak{J}) \xleftarrow{\pi_1} & K_1(\mathfrak{B}) & \xleftarrow{\iota_1} K_1(\mathfrak{J}) & & \\
& \nearrow^{\beta_3} & & \uparrow{\scriptstyle \beta_2} & & \nwarrow^{\beta_1} & \\
K_1(\mathfrak{A}/\mathfrak{I}) & & \xleftarrow{\pi_1} & K_1(\mathfrak{A}) & \xleftarrow{\iota_1} & & K_1(\mathfrak{I})
\end{array}
\tag{2.4}
$$

An isomorphism $(\alpha_\bullet, \beta_\bullet) : K_{\mathsf{six}}(\mathfrak{A}, \mathfrak{I}) \xrightarrow{\cong} K_{\mathsf{six}}(\mathfrak{B}, \mathfrak{J})$ is defined in the obvious way.

Homomorphisms and isomorphisms from $K^+_{\mathsf{six}}(\mathfrak{A}, \mathfrak{I})$ to $K^+_{\mathsf{six}}(\mathfrak{B}, \mathfrak{J})$ and from $K^{+,\Sigma}_{\mathsf{six}}(\mathfrak{A}, \mathfrak{I})$ to $K^{+,\Sigma}_{\mathsf{six}}(\mathfrak{B}, \mathfrak{J})$ are defined in a similar way with the requirement that the additional structure is preserved. We recall that a scale-preserving isomorphism must preserve the classes of the units also in the purely infinite unital case, when this would not follow from the map only be required to send scales to scales.

3. The outstanding $[\infty 1]_1$ case

In this section we prove the K-theoretical existence result necessary to resolve the extension problem in the case where $\mathfrak{I}$ is AF, $\mathfrak{A}/\mathfrak{I}$ is unital and purely infinite, and $\mathfrak{A}$ is nonunital. To do so, we carefully describe the invariant used in [Gabe and Ruiz 2020] and elaborate on methods from [Eilers et al. 2016a] to make contact to a result by Katsura, Sims, and Tomforde [Katsura et al. 2009] which establishes that certain AF algebras are always graph C^*-algebras.

Given a short exact sequence of C^*-algebras

$$\mathfrak{e} : 0 \to \mathfrak{I} \xrightarrow{\iota} \mathfrak{A} \xrightarrow{\pi} \mathfrak{A}/\mathfrak{I} \to 0$$

such that the quotient $\mathfrak{A}/\mathfrak{I}$ has a unit $1_{\mathfrak{A}/\mathfrak{I}}$ and this unit lifts to a projection in $\mathfrak{A}$, we define $\widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\mathfrak{A}, \mathfrak{I})$ to be

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_0^{\Sigma}(\mathfrak{I}) & \xrightarrow{\iota_0} & K_0^{\Sigma}(\mathfrak{D}) & \xrightarrow{\pi_0} & K_0^{\Sigma}(\mathbb{C}1_{\mathfrak{A}/\mathfrak{I}}) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mathrm{id}_0} & & \big\downarrow{\scriptstyle \iota_0} & & \big\downarrow{\scriptstyle \iota_0} & & \\
 & & K_0(\mathfrak{I}) & \xrightarrow{\iota_0} & K_0(\mathfrak{A}) & \xrightarrow{\pi_0} & K_0^{\Sigma}(\mathfrak{A}/\mathfrak{I}) & & \\
 & & {\scriptstyle \partial_1}\big\uparrow & & & & \big\downarrow{\scriptstyle \partial_0} & & \\
 & & K_1(\mathfrak{A}/\mathfrak{I}) & \xleftarrow[\pi_1]{} & K_1(\mathfrak{A}) & \xleftarrow[\iota_1]{} & K_1(\mathfrak{I}) & &
\end{array} \tag{3.1}$$

where $\mathfrak{D} := \pi^{-1}(\mathbb{C}1_{\mathfrak{A}/\mathfrak{I}}) \subseteq \mathfrak{A}$. The top row is exact since $K_1(\mathbb{C}1_{\mathfrak{A}/\mathfrak{I}})$ vanishes and since $1_{\mathfrak{A}/\mathfrak{I}}$ lifts to a projection in $\mathfrak{A}$. The latter observation further shows that there is a splitting map for the top row. The notation K_0^{Σ} for K_0 helps to distinguish the two groups in the top left corner, and to remind us that at the positions of these four groups, we will impose conditions of contractiveness and scale-preservation.

Lemma 3.2. *If $\mathfrak{e}$ is a full extension, then so is*

$$\tilde{\mathfrak{e}} : 0 \to \mathfrak{I} \xrightarrow{\iota} \mathfrak{D} \xrightarrow{\pi} \mathbb{C}1_{\mathfrak{A}/\mathfrak{I}} \to 0.$$

Proof. This follows from the fact that $\tau_{\mathfrak{e}}(1_{\mathfrak{A}/\mathfrak{I}}) = \tau_{\tilde{\mathfrak{e}}}(1)$. □

An isomorphism $(\tilde{\alpha}_\bullet, \alpha_\bullet, \beta_\bullet) : \widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\mathfrak{A}, \mathfrak{I}) \xrightarrow{\cong} \widetilde{K}^{+,\Sigma}_{\mathrm{six}}(\mathfrak{B}, \mathfrak{J})$ consists of nine group homomorphisms such that $(\alpha_\bullet, \beta_\bullet)$ induces an isomorphism from $K^{+,\Sigma}_{\mathrm{six}}(\mathfrak{A}, \mathfrak{I})$ to $K^{+,\Sigma}_{\mathrm{six}}(\mathfrak{B}, \mathfrak{J})$ as defined in the previous section, and so that the $\tilde{\alpha}_\bullet$ are all scale-preserving. All maps must commute with the morphisms in (3.1).

We will use the following result:

Theorem 3.3 [Gabe and Ruiz 2020, Theorem B]. *With notation and assumptions as above, assume that $\mathfrak{A}$ is a C^*-algebra with exactly one proper nontrivial ideal $\mathfrak{I}$ so that the following hold:*

(i) $\mathfrak{I}$ *is stable and AF*,

(ii) $\mathfrak{A}/\mathfrak{I}$ *is a unital Kirchberg algebra in the UCT class*, *and*

(iii) $\mathfrak{e}$ *is a full extension.*

Then $\widetilde{K}_{\mathrm{six}}^{+,\Sigma}(\mathfrak{A},\mathfrak{I})$ *is a complete invariant for* $\mathfrak{A}$ *in the sense that when* $\mathfrak{A}'$ *is another such* C^**-algebra with exactly one proper nontrivial ideal* $\mathfrak{I}'$ *satisfying* (i)–(iii), *then*

$$\widetilde{K}_{\mathrm{six}}^{+,\Sigma}(\mathfrak{A},\mathfrak{I}) \cong \widetilde{K}_{\mathrm{six}}^{+,\Sigma}(\mathfrak{A}',\mathfrak{I}') \iff \mathfrak{A} \cong \mathfrak{A}'.$$

The class of invariants $\widetilde{\mathcal{E}}$ we consider may be abstractly characterized as consisting of a cyclic six-term exact sequence

$$\begin{array}{ccccc} G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 \\ {\scriptstyle\delta_1}\uparrow & & & & \downarrow{\scriptstyle\delta_0} \\ F_3 & \xleftarrow{\gamma'} & F_2 & \xleftarrow{\epsilon'} & F_1 \end{array} \tag{3.4}$$

of abelian groups, and a short exact sequence

$$0 \to H_1 \xrightarrow{\tilde{\epsilon}} H_2 \xrightarrow{\tilde{\gamma}} H_3 \to 0 \tag{3.5}$$

of scaled ordered groups together with three group homomorphisms $\eta_i : H_i \to G_i$ such that the diagram

$$\begin{array}{ccccc} H_1 & \xrightarrow{\tilde{\epsilon}} & H_2 & \xrightarrow{\tilde{\gamma}} & H_3 \\ \downarrow{\scriptstyle\eta_1} & & \downarrow{\scriptstyle\eta_2} & & \downarrow{\scriptstyle\eta_3} \\ G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 \end{array} \tag{3.6}$$

commutes. We summarize the above information into the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1 & \xrightarrow{\tilde{\epsilon}} & H_2 & \xrightarrow{\tilde{\gamma}} & H_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\eta_1} & & \downarrow{\scriptstyle\eta_2} & & \downarrow{\scriptstyle\eta_3} & & \\ & & G_1 & \xrightarrow{\epsilon} & G_2 & \xrightarrow{\gamma} & G_3 & & \\ & & {\scriptstyle\delta_1}\uparrow & & & & \downarrow{\scriptstyle\delta_0} & & \\ & & F_3 & \xleftarrow{\gamma'} & F_2 & \xleftarrow{\epsilon'} & F_1 & & \end{array} \tag{3.7}$$

where we require throughout that $\tilde{\epsilon}$ and $\tilde{\gamma}$ are positive and contractive homomorphisms, and that $\eta_1, \eta_2, \eta_3, \epsilon, \gamma$ are positive homomorphisms.

Theorem 3.8. *Suppose* $\widetilde{\mathcal{E}}$ *is an invariant so that the following nine conditions hold*:

(i) H_1 *is a simple dimension group with* $\Sigma H_1 = H_1^+$,

(ii) $H_2^+ = \tilde{\epsilon}(H_1^+) \cup \tilde{\gamma}^{-1}(\{1,2,\ldots\})$ *and* $\Sigma H_2 \cap \tilde{\gamma}^{-1}(\{1\}) \neq \varnothing$,

(iii) ΣH_2 *is generating, hereditary, and upward directed, and does not have a largest element;*

(iv) $(H_3, (H_3)_+, \Sigma H_3) \cong (\mathbb{Z}, \mathbb{N} \cup \{0\}, \{0, 1\})$,

(v) η_1 *is an isomorphism of ordered groups,*

(vi) $(G_2)_+ = G_2$,

(vii) G_3 *is finitely generated and* $(G_3)_+ = G_3$,

(viii) $F_1 = 0$, *and*

(ix) F_3 *is a free group with* $\operatorname{rank} F_3 \leq \operatorname{rank} G_3$.

Then there is a graph C^**-algebra* $C^*(E)$ *with exactly one proper nontrivial ideal* $\mathfrak{I}$ *so that* $C^*(E)/\mathfrak{I}$ *is unital and* $\widetilde{K}_{\mathrm{six}}^{+,\Sigma}(C^*(E), \mathfrak{I}) \cong \widetilde{\mathcal{E}}$.

To prove Theorem 3.8 we need the following auxiliary results. The first result, which is a generalization of [Eilers et al. 2016a, Proposition 4.8], will allow us to glue together two graphs representing H_1, G_1, F_1 and H_3, G_3, F_3, respectively. Note that in [Eilers et al. 2016a, Proposition 4.8], $n_1 < \infty$ is assumed, and $x \in \mathbb{Z}^{n_1}$ in the proposition below is given by **1**. The version given below also corrects a number of regrettable typos in the statement of [Eilers et al. 2016a, Proposition 4.8].

We recall the setting from [Eilers et al. 2016a, Proposition 4.3]. As in (3.4) above, we let $\mathcal{E}$ denote an exact sequence of abelian groups with F_1, F_2, and F_3 free and suppose that there exist column-finite matrices $A \in M_{n_1,n_1'}(\mathbb{Z})$ and $B \in M_{n_3,n_3'}(\mathbb{Z})$ for some $n_1, n_1', n_3, n_3' \in \{0, 1, 2, \ldots, \infty\}$ with isomorphisms

$$\alpha_1 : \operatorname{coker} A \to G_1, \qquad \beta_1 : \ker A \to F_1,$$
$$\alpha_3 : \operatorname{coker} B \to G_3, \qquad \beta_3 : \ker B \to F_3.$$

We prove in [Eilers et al. 2016a, Proposition 4.3] that there exist a column-finite matrix $Y \in M_{n_1,n_3'}(\mathbb{Z})$ and isomorphisms

$$\alpha_2 : \operatorname{coker}\begin{pmatrix} A & Y \\ 0 & B \end{pmatrix} \to G_2, \qquad \beta_2 : \ker\begin{pmatrix} A & Y \\ 0 & B \end{pmatrix} \to F_2 \tag{3.9}$$

such that α_i and β_i for $i = 1, 2, 3$ give an isomorphism (see (2.4)) from the exact sequence

$$\begin{array}{ccccc} \operatorname{coker} A & \xrightarrow[I]{} & \operatorname{coker}\begin{pmatrix} A & Y \\ 0 & B \end{pmatrix} & \xrightarrow[P]{} & \operatorname{coker} B \\ {\scriptstyle [Y]}\uparrow & & & & \downarrow{\scriptstyle 0} \\ \ker B & \xleftarrow{P'} & \ker\begin{pmatrix} A & Y \\ 0 & B \end{pmatrix} & \xleftarrow{I'} & \ker A \end{array} \tag{3.10}$$

to $\mathcal{E}$, where I, I' and P, P' are induced by the obvious inclusions or projections.

In the proposition below, the ordering on matrices is given by entrywise ordering, i.e., for matrices A, B, we write $A \geq B$ if $A_{i,j} \geq B_{i,j}$ for all (i, j).

Proposition 3.11. *In the situation described above assume that* $n_3 < \infty$ *and* $g_2 \in G_2$ *is given with* $\alpha_3([\mathbf{1}]) = \gamma(g_2)$. *Choose* $x \in \mathbb{Z}^{n_1}$ *arbitrary and define* $y \in \mathbb{Z}^{n_1+n_3}$ *by*

$y = \left(\begin{smallmatrix} x \\ \mathbf{1} \end{smallmatrix}\right)$. *Suppose that B satisfies the condition that for some $1 \le i, j \le n_3$ we have*

$$B_{ik} > B_{jk}, \qquad 1 \le k \le n_3'.$$

Then for a given $Z \in \mathsf{M}_{n_1,n_3'}(\mathbb{Z})$, the matrix $Y \in \mathsf{M}_{n_1,n_3'}(\mathbb{Z})$ along with α_2, β_2 inducing the isomorphism may be chosen with the additional properties that $Y \ge Z$ and $\alpha_2([y]) = g_2$.

Proof. Take Y, α_2, and β_2 as in (3.9)–(3.10). We are going to define new Y' and α_2' (β_2 is unchanged) which satisfy (3.9)–(3.10) as well as $Y' \ge Z$ and $\alpha_2'([y]) = g_2$. Set

$$g_2' = \alpha_2([y]) - g_2,$$

and observe that $\gamma(g_2') = 0$ because

$$\gamma(\alpha_2([y])) = \alpha_3(P([y])) = \alpha_3([\mathbf{1}]) = \gamma(g_2).$$

Hence there exists $z \in \mathbb{Z}^{n_1}$ such that $\epsilon(\alpha_1([z])) = g_2'$. Choose $Q' \in \mathsf{M}_{n_1,n_3}(\mathbb{Z})$ such that $z = Q'\mathbf{1}$, which is possible because $n_3 \ge 1$. Let $Q'' \in \mathsf{M}_{n_1,n_3}(\mathbb{Z})$ be

$$(Q'')_{k,\ell} = \begin{cases} 1 & \text{if } \ell = i, \\ -1 & \text{if } \ell = j, \\ 0 & \text{otherwise}, \end{cases}$$

and note that each row of $Q''B$ is identically

$$\left(B_{i,1} - B_{j,1} \;\; B_{i,2} - B_{j,2} \;\; \cdots \;\; B_{i,n_3'} - B_{j,n_3'}\right),$$

which is strictly positive by assumption on B. Find an integer $c > 0$ so that we have

$$cQ''B \ge Z - Y - Q'B$$

and set $Q = cQ'' + Q'$. Since $Q''\mathbf{1} = 0$, we have $Q\mathbf{1} = z$. Set $Y' = Y + QB$. Since

$$\left(\begin{smallmatrix} I & -Q \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & Y \\ 0 & B \end{smallmatrix}\right)$$

we can define a map

$$\alpha_2' : \operatorname{coker}\left(\begin{smallmatrix} A & Y' \\ 0 & B \end{smallmatrix}\right) \to G_2$$

by

$$\alpha_2'([y]) = \alpha_2\left(\left[\left(\begin{smallmatrix} I & -Q \\ 0 & I \end{smallmatrix}\right)y\right]\right).$$

Now it is straightforward to see that Y', α_2' and β_2 satisfy (3.9)–(3.10). It is also easy to see $Y' \ge Z$, and we have $\alpha_2'([y]) = g_2$ because

$$\begin{aligned} \alpha_2'([y]) &= \alpha_2\left(\left[\left(\begin{smallmatrix} I & -Q \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ \mathbf{1} \end{smallmatrix}\right)\right]\right) = \alpha_2\left(\left[\left(\begin{smallmatrix} x \\ \mathbf{1} \end{smallmatrix}\right)\right]\right) - \alpha_2\left(\left[\left(\begin{smallmatrix} Q\mathbf{1} \\ \mathbf{0} \end{smallmatrix}\right)\right]\right) = \alpha_2([y]) - \alpha_2\left(\left[\left(\begin{smallmatrix} z \\ \mathbf{1} \end{smallmatrix}\right)\right]\right) \\ &= \alpha_2([y]) - \epsilon(\alpha_1([z])) = \alpha_2([y]) - g_2' = g_2. \end{aligned}$$

□

The second result will allow us to realize ΣH_2 by finding an AF graph C^*-algebra with the scale adjusted to our needs.

Lemma 3.12. *Let H_i satisfy* (i)–(iv) *of Theorem 3.8 and fix* $h_2 \in \Sigma H_2 \cap \tilde{\gamma}^{-1}(\{1\})$. *Then*

$$\Sigma := \{h \in H_1^+ : h_2 + \tilde{\epsilon}(h) \in \Sigma H_2\}$$

is a scale of H_1 that is generating, hereditary, and upward directed, and does not have a largest element, and consequently there is a graph E_1 with

$$\big(K_0(C^*(E_1)), K_0(C^*(E_1))_+, \Sigma C^*(E_1)\big) \cong (H_1, (H_1)_+, \Sigma).$$

Proof. It is obvious that Σ is hereditary. For the remaining claims we note that whenever $h_2 + \tilde{\epsilon}(h) \leq k \in \Sigma$, $\tilde{\gamma}(h_2) = 1 = \tilde{\gamma}(k)$ and hence $k - h_2 = \tilde{\epsilon}(h') \geq 0$ for some h'. We have

$$k = h_2 + \tilde{\epsilon}(h') \geq h_2 + \tilde{\epsilon}(h),$$

so we conclude that $\tilde{\epsilon}(h' - h) \geq 0$ in H_2. By (ii) this implies that $h' \geq h \geq 0$ in H_1, and $h' \in \Sigma$.

Since there is no largest element in ΣH_2, we may arrange that $h < h' \in \Sigma$ as above, and hence there is no largest element in Σ. In particular we have that $h' > 0$, and since H_1 is a simple dimension group, any positive element is dominated by nh' for some n. Using Riesz decomposition, this shows that any positive element is a finite sum of scale elements, and hence H_1 is generated by elements from Σ.

To see that Σ is upward directed, fix $h, \bar{h}$ in Σ and take $k \geq h_2 + \tilde{\epsilon}(h), h_2 + \tilde{\epsilon}(\bar{h})$. As above we may choose h' with $k = h_2 + \tilde{\epsilon}(h')$ and conclude that $h, \bar{h} \leq h'$.

We conclude that $(H_1, (H_1)_+, \Sigma)$ is a scaled dimension group with no largest scale element, and thus by [Katsura et al. 2009] we may choose E_1 as desired. □

Proof of Theorem 3.8. We choose h_2 and define $\Sigma \subseteq H_1$ as in Lemma 3.12 so that there exists a graph $E_1 = (E_1^0, E_1^1, r_{E_1}, s_{E_1})$ with $C^*(E_1)$ realizing $(H_1, (H_1)_+, \Sigma)$ as scaled ordered groups.

Let $g_2 := \eta_2(h_2) \in G_2$ and $g_3 := \gamma(g_2) \in G_3$. By [Eilers et al. 2016a, Proposition 3.8], there is a graph $E_3 = (E_3^0, E_3^1, r_{E_3}, s_{E_3})$ with finitely many vertices such that

(1) every vertex in E_3 is the base point of at least two loops,

(2) E_3 is transitive (so that, in particular, $C^*(E_3)$ is simple and purely infinite),

(3) $(K_0(C^*(E_3)), [1_{C^*(E_3)}]_0) \cong (G_3, g_3)$ and $K_1(C^*(E_3)) \cong F_3$, and

(4) there exist two vertices $v, w \in E_3^0$ such that $(R_{E_3} - I)(w, v') < (R_{E_3} - I)(v, v')$ for all $v' \in (E_3^0)_{\mathrm{reg}}$.

Arguing as in [Eilers et al. 2016a, Proposition 5.5], but applying Proposition 3.11 in place of [Eilers et al. 2016a, Proposition 4.7 and 4.8], we construct a graph

$$E_2 = (E_2^0, E_2^1, r_{E_2}, s_{E_2})$$

such that $K_{\mathsf{six}}^{+}(C^*(E_2), \mathfrak{I})$ is isomorphic to the 6-term part $\mathcal{E}$ of the invariant $\widetilde{\mathcal{E}}$, and with the further property that the isomorphism $\alpha_2 : K_0(C^*(E_2)) \to G_2$ sends $[\mathsf{p}_{E_3}]$ to $g_2 \in G_2$, where

$$\mathsf{p}_{E_3} = \sum_{v \in E_3^0} p_v.$$

As in [Eilers et al. 2016a], $E_2^0 = E_1^0 \sqcup E_3^0$, and E_2^1 contains all edges in $E_1^1 \sqcup E_3^1$ as well as a number of additional edges chosen carefully to obtain the relevant K-theoretical data.

We note that there is a natural surjection $\pi : C^*(E_2) \to C^*(E_3)$ whose kernel is isomorphic to $\mathfrak{I}$. Thus we may identify $C^*(E_3)$ with the quotient $C^*(E_2)/\mathfrak{I}$, having π as the quotient map. We see that the projection $\mathsf{p}_{E_3} \in C^*(E_2)$ is a lift of $1_{C^*(E_3)}$, and note that $\mathfrak{D} := \pi^{-1}(\mathbb{C}1_{C^*(E_3)}) \subseteq C^*(E_2)$ coincides with $\mathfrak{I} + \mathbb{C}\mathsf{p}_{E_3}$.

We see that $C^*(E_1)$ is naturally a subalgebra of $C^*(E_2)$, which is a full and hereditary subalgebra of $\mathfrak{I}$, and that

$$\mathfrak{D} \cap \{\mathsf{p}_{E_3}\}^{\perp} = C^*(E_1)$$

under this identification. We claim that

$$\Sigma\mathfrak{D} = \{x \in K_0(\mathfrak{D})_+ : x \le [\mathsf{p}_{E_3}] + [q] \text{ for some projection } q \in C^*(E_1)\}. \quad (3.13)$$

Since $\mathfrak{D} \cap \{\mathsf{p}_{E_3}\}^{\perp} = C^*(E_1)$ and since $\Sigma\mathfrak{D}$ is hereditary,

$$\{x \in K_0(\mathfrak{D})_+ : x \le [\mathsf{p}_{E_3}] + [q] \text{ for some projection } q \in C^*(E_1)\} \subseteq \Sigma\mathfrak{D}.$$

We now show the other set containment. Let p be a projection in $\mathfrak{D}$. Since $\Sigma\mathfrak{D}$ is upward directed, there exists a projection r in $\mathfrak{D}$ such that $[p], [\mathsf{p}_{E_3}] \le [r]$. Since the unitization of $\mathfrak{D}$, which we denote $\mathfrak{D}^{\dagger}$, has stable rank one, there exists a unitary $u \in \mathfrak{D}^{\dagger}$ such that $\mathsf{p}_{E_3} \le uru^*$. Hence, $q = uru^* - \mathsf{p}_{E_3} \in \mathfrak{D} \cap \{\mathsf{p}_{E_3}\}^{\perp} = C^*(E_1)$. Consequently,

$$[p] \le [r] = [\mathsf{p}_{E_3}] + [r] - [\mathsf{p}_{E_3}] = [\mathsf{p}_{E_3}] + [uru^*] - [\mathsf{p}_{E_3}] = [\mathsf{p}_{E_3}] + [q].$$

This proves the claim.

Next, we construct group homomorphisms $\tilde{\alpha}_\bullet$ for $\bullet = 1, 2, 3$ so that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0^{\Sigma}(\mathfrak{I}) & \xrightarrow{\iota_0} & K_0^{\Sigma}(\mathfrak{D}) & \xrightarrow{\pi_0} & K_0^{\Sigma}(\mathbb{C}1_{C^*(E_3)}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{\alpha}_1} & & \downarrow{\scriptstyle \tilde{\alpha}_2} & & \downarrow{\scriptstyle \tilde{\alpha}_3} & & \\ 0 & \longrightarrow & H_1 & \xrightarrow{\tilde{\epsilon}} & H_2 & \xrightarrow{\tilde{\gamma}} & H_3 & \longrightarrow & 0 \end{array} \quad (3.14)$$

commutes. We also need to show that the maps $\tilde{\alpha}_\bullet$ intertwine $\eta_\bullet$ and are isomorphisms of scaled ordered groups.

We set $\tilde{\alpha}_1 = \eta_1^{-1} \circ \alpha_1 \circ \mathrm{id}_0$ and note that $\tilde{\alpha}_1$ is an order isomorphism since it is a composition of three order isomorphisms. Since $\mathfrak{I}$ is stable by [Eilers and Tomforde 2010], the scale of $K_0^{\Sigma}(\mathfrak{I})$ coincides with its positive cone. Since the same is assumed for H_1, we conclude that $\tilde{\alpha}_1$ is an isomorphism of scaled ordered groups. We also define $\tilde{\alpha}_3$ by $\tilde{\alpha}_3([1_{C^*(E_3)}]) = \tilde{\gamma}(h_2)$, and note that both $\tilde{\alpha}_1$ and $\tilde{\alpha}_3$ intertwine the appropriate $\eta_\bullet$ by construction.

Next, we define $\tilde{\alpha}_2$ as the unique group homomorphism satisfying $\tilde{\alpha}_2 \circ \iota_0 = \tilde{\epsilon} \circ \tilde{\alpha}_1$ and $\tilde{\alpha}_2([\mathsf{p}_{E_3}]) = h_2$. The map $\tilde{\alpha}_2$ is automatically an order isomorphism as a consequence of the commutativity in (3.14), the fact that $\tilde{\alpha}_1$ and $\tilde{\alpha}_3$ are order isomorphisms, and the way in which the order structures on H_2 and $K_0^{\Sigma}(\mathfrak{D})$ depend on the order structures at the extremes of the diagram. Also, $\tilde{\alpha}_2$ intertwines η_2 on the image of ι_0 since

$$\eta_2 \circ \tilde{\alpha}_2 \circ \iota_0 = \eta_2 \circ \tilde{\epsilon} \circ \tilde{\alpha}_1 = \epsilon \circ \eta_1 \circ \eta_1^{-1} \circ \alpha_1 \circ \mathrm{id}_0 = \alpha_2 \circ \iota_0$$

and on $[\mathsf{p}_{E_3}]$ by construction. Thus all that remains is to prove that $\tilde{\alpha}_2(\Sigma\mathfrak{D}) = \Sigma H_2$.

When $x = [p] \in \Sigma\mathfrak{D}$, $[p] \le [q] + [\mathsf{p}_{E_3}]$ for some projection q in $C^*(E_1)$ by (3.13). Therefore,

$$0 \le \tilde{\alpha}_2([p]) \le \tilde{\epsilon}(\tilde{\alpha}_1([q])) + h_2 \in \Sigma H_2,$$

because $\tilde{\alpha}_1([q]) \in \Sigma$ by construction. Since ΣH_2 is hereditary, $\tilde{\alpha}_2([p]) \in \Sigma H_2$.

For fixed $h \in \Sigma H_2$, take $k \in \Sigma H_2$ so that $h, h_2 \le k$. As in the proof of Lemma 3.12, we have $k = h_2 + \tilde{\epsilon}(h')$ with $h' \in \Sigma$, proving that $k = \tilde{\alpha}_2(\Sigma\mathfrak{D})$. Since $\tilde{\alpha}_2(\Sigma\mathfrak{D}))$ is an order-isomorphic image of a hereditary set, it is also hereditary, and we conclude that $h \in \tilde{\alpha}_2(\Sigma\mathfrak{D})$. □

4. The main result

We are now ready to state and prove the main result of the paper. Recall that the (torsion-free) rank of an abelian group G is the dimension of the $\mathbb{Q}$-vector space $G \otimes \mathbb{Q}$.

Theorem 4.1. *Let $\mathfrak{A}$ be a C^*-algebra with exactly one proper nontrivial ideal $\mathfrak{I}$ so that $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$ are graph C^*-algebras. Then $\mathfrak{A}$ is a graph C^*-algebra if and only if the following three conditions hold*:

(1) *The exponential map $\partial_0 : K_0(\mathfrak{A}/\mathfrak{I}) \to K_1(\mathfrak{I})$ is zero.*

(2) *If $K_0(\mathfrak{A}/\mathfrak{I})_+ = K_0(\mathfrak{A}/\mathfrak{I})$, then $K_0(\mathfrak{A})_+ = K_0(\mathfrak{A})$.*

(3) *If $\mathfrak{A}$ is a unital C^*-algebra, then*
   - (a) *$K_0(\mathfrak{I})$ is finitely generated*,
   - (b) $\mathrm{rank}(K_1(\mathfrak{I})) \le \mathrm{rank}(K_0(\mathfrak{I}))$, *and*
   - (c) *$K_0(\mathfrak{I})_+ \neq K_0(\mathfrak{I})$ implies that $K_0(\mathfrak{I}) \cong \mathbb{Z}$.*

Proof. Step 1: Necessity.

Suppose $\mathfrak{A}$ is a graph C^*-algebra, i.e., $\mathfrak{A} \cong C^*(G)$ for some graph G. Since $\mathfrak{A}$ has finitely many ideals, G satisfies Condition (K) (see the proof of [Eilers and Tomforde 2010, Lemma 3.1]). Hence, by [Hong and Szymański 2003, Theorem 2.6] (also see [Jeong 2004, Theorem 3.5]), $C^*(G)$ has real rank zero and hence, $\mathfrak{A}$ has real rank zero. By [Brown and Pedersen 1991, Theorem 3.19], (1) must then hold.

Suppose $K_0(\mathfrak{A}/\mathfrak{I})_+ = K_0(\mathfrak{A}/\mathfrak{I})$, so that we have that $\mathfrak{A}/\mathfrak{I}$ is a purely infinite simple C^*-algebra by the dichotomy of simple graph C^*-algebras. By [Eilers and Tomforde 2010, Proposition 6.4], $\mathfrak{I}$ is stable. We further prove that $\mathfrak{e}$ is full. Indeed, since $\mathfrak{A}/\mathfrak{I}$ is simple, it is enough to show that for some $a \in \mathfrak{A}/\mathfrak{I}$, $\tau_{\mathfrak{e}}(a)$ is full in $\mathcal{Q}(\mathfrak{I})$. By [Eilers and Tomforde 2010, Proposition 3.10], there is a projection $p \in \mathfrak{A}$ such that $p\mathfrak{I}p$ is stable. By [Brown 1988, Theorem 4.23], $p \sim 1_{M(\mathfrak{I})}$. Thus, $\tau_{\mathfrak{e}}(\pi(p)) \sim 1_{\mathcal{Q}(\mathfrak{I})}$, and hence $\tau_{\mathfrak{e}}(\pi(p))$ is full in $\mathcal{Q}(\mathfrak{I})$. By [Eilers et al. 2014b, Proposition 4.2], $K_0(\mathfrak{A})_+ = K_0(\mathfrak{A})$.

Finally, suppose further that $\mathfrak{A}$ is a unital C^*-algebra. Then $K_*(\mathfrak{A})$ is the cokernel and kernel, respectively, of a map from $\mathbb{Z}^{m_1+m_2}$ to $\mathbb{Z}^{n_1+n_2}$ given by a block triangular matrix in which the $m_1 \times n_1$-block specifies $K_*(\mathfrak{I})$ and the $m_2 \times n_2$-block specifies $K_*(\mathfrak{A}/\mathfrak{I})$. We have $m_i \le n_i < \infty$, so all these groups are finitely generated, and rank $K_1(\mathfrak{I}) \le$ rank $K_0(\mathfrak{I})$, rank $K_1(\mathfrak{A}) \le$ rank $K_0(\mathfrak{A})$ and rank $K_1(\mathfrak{A}/\mathfrak{I}) \le$ rank $K_0(\mathfrak{A}/\mathfrak{I})$, establishing (3)(a) and (3)(b) in particular.

For (3)(c), suppose $K_0(\mathfrak{I})_+ \neq K_0(\mathfrak{I})$. Then $\mathfrak{I}$ is an AF algebra. Let H be a nontrivial saturated, hereditary subset of G^0 corresponding to $\mathfrak{I}$. Since $\mathfrak{A}$ is unital, G^0 is a finite set, and hence so is H. Note that $\mathfrak{I} \otimes \mathbb{K} \cong C^*(E_H) \otimes \mathbb{K}$ with $E_H = (H, r_G^{-1}(H), r_G|_{r_G^{-1}(H)}, s_G|_{r_G^{-1}(H)})$. Since $E_H^0 = H$ is a finite set and $C^*(E_H)$ is a simple AF algebra, $C^*(E_H) \otimes \mathbb{K} \cong \mathbb{K}$ by [Raeburn 2005, Proposition 1.18]. Therefore, $\mathfrak{I} \otimes \mathbb{K} \cong \mathbb{K}$ and hence $K_0(\mathfrak{I}) \cong \mathbb{Z}$.

We now establish sufficiency. Thus from here onwards, we assume that (1)–(3) hold.

Step 2: The $[\mathbf{11}]_*$ case.

To complete the result in the $[\mathbf{11}]_*$ cases, we recall that by [Eilers et al. 2014a, Theorem 5.9], it suffices to prove that every unital quotient of $\mathfrak{A}$ is a Type I C^*-algebra. This is vacuously true in the $[\mathbf{11}]_0$ subcase.

In the $[\mathbf{11}]_1$ subcase we note that since $\mathfrak{A}/\mathfrak{I}$ is a unital, simple graph AF algebra, by [Raeburn 2005, Proposition 1.18] again, $\mathfrak{A}/\mathfrak{I} \cong \mathsf{M}_k$ which is a Type I C^*-algebra.

In the $[\mathbf{11}]_2$ subcase, we first note that by (3)(c), $K_0(\mathfrak{I}) = \mathbb{Z}$ since $K_0(\mathfrak{I})_+ \neq K_0(\mathfrak{I})$. Since (up to isomorphism) the only simple, nonunital AF algebra with $K_0(\mathfrak{I}) \cong \mathbb{Z}$ is $\mathbb{K}$, we have that $\mathfrak{I} \cong \mathbb{K}$. Further, as above we have $\mathfrak{A}/\mathfrak{I} \cong \mathsf{M}_k$. Since $\mathbb{K}$ and M_k are Type I C^*-algebras, by permanence of this class (see [Pedersen 1979, 6.2.6]), $\mathfrak{A}$ is a Type I C^*-algebra.

Step 3: Stability and fullness.

We first note that in all remaining cases, $\mathfrak{I}$ is stable. This follows from [Zhang 1992, Theorem 1.2(i)] whenever $\mathfrak{I}$ is purely infinite, so we may assume that $\mathfrak{I}$ is stably finite and appeal to Lemma 2.2. We further see that in all remaining cases, $\mathfrak{e}$ is full. This is again easy to see whenever $\mathfrak{I}$ is purely infinite since $M(\mathfrak{I})/\mathfrak{I}$ is simple by [Lin 1989, Theorem 2.3], and follows from Lemma 2.2 in all other cases.

Step 4: The $[\infty\infty]_*$ and $[\infty\mathbf{1}]_*$ cases.

Consider first the unital cases $[\infty\infty]_2$ and $[\infty\mathbf{1}]_2$. We know that $\mathfrak{A}/\mathfrak{I}$ is a unital graph C^*-algebra, so as in Step 1 we conclude that $K_0(\mathfrak{A}/\mathfrak{I})$ is finitely generated and that $\operatorname{rank}(K_1(\mathfrak{A}/\mathfrak{I})) \leq \operatorname{rank}(K_0(\mathfrak{A}/\mathfrak{I}))$. This shows that [Eilers et al. 2016a, Theorem 6.4] applies, so there exists a graph E with finitely many vertices such that $C^*(E)$ has exactly one proper nontrivial ideal $\mathfrak{I}_1$ and there exists an isomorphism $(\alpha_\bullet, \beta_\bullet) : K_{\mathsf{six}}(\mathfrak{A}, \mathfrak{I}) \xrightarrow{\cong} K_{\mathsf{six}}(C^*(E), \mathfrak{I}_1)$ such that all $\alpha_\bullet$ are positive isomorphisms and $\alpha_2([1_\mathfrak{A}]) = [1_{C^*(E)}]$. Appealing to [Eilers and Restorff 2006, Corollary 12] (see [Restorff and Ruiz 2007, Theorem 2.4]) in the $[\infty\infty]_2$ case and to [Eilers et al. 2013, Corollary 4.16] in the $[\infty\mathbf{1}]_2$ case, $\mathfrak{A} \cong C^*(E)$.

In the four remaining cases, we first replace, if necessary, the graph presentation of $\mathfrak{I}$ by a left adhesive graph [Eilers et al. 2016a, Lemma 5.4(ℓ1)]. By [Eilers et al. 2016a, Proposition 5.5] we then obtain a graph E such that $C^*(E)$ has exactly one proper nontrivial ideal $\mathfrak{I}_1$ and such that $(\alpha_\bullet, \beta_\bullet) : K_{\mathsf{six}}(\mathfrak{A}, \mathfrak{I}) \xrightarrow{\cong} K_{\mathsf{six}}(C^*(E), \mathfrak{I}_1)$ can be chosen with all $\alpha_\bullet$ positive isomorphisms and α_3 scale-preserving (in particular, preserving the class of the unit of the quotient in the $[\infty\infty]_1$ and $[\infty\mathbf{1}]_1$ cases). Appealing to an appropriate classification result we get that $\mathfrak{A} \cong C^*(E)$: [Rørdam 1997] for $[\infty\infty]_0$, [Restorff and Ruiz 2007, Theorem 2.2] for $[\infty\infty]_1$, [Eilers et al. 2009, Theorem 2.3], and [Eilers et al. 2016b, Proposition 2] for $[\infty\mathbf{1}]_1$ and $[\infty\mathbf{1}]_0$.

Step 5: The $[\mathbf{1}\infty]_0$ and $[\mathbf{1}\infty]_2$ cases.

In the unital case $[\mathbf{1}\infty]_2$, we note as in Step 4 that $K_0(\mathfrak{A}/\mathfrak{I})$ is finitely generated and that $\operatorname{rank}(K_1(\mathfrak{A}/\mathfrak{I})) \leq \operatorname{rank}(K_0(\mathfrak{A}/\mathfrak{I}))$. By (3)(c), $K_0(\mathfrak{I}) \cong \mathbb{Z}$. Since (up to isomorphism) the only nonunital simple AF algebra with K_0-group isomorphic to $\mathbb{Z}$ is $\mathbb{K}$, we have that $\mathfrak{I} \cong \mathbb{K}$. By [Eilers et al. 2016a, Theorem 6.4], there exists a graph E with finitely many vertices such that $C^*(E)$ has exactly one proper nontrivial ideal $\mathfrak{I}_1$ and there exists an isomorphism $(\alpha_\bullet, \beta_\bullet) : K_{\mathsf{six}}(\mathfrak{A}, \mathfrak{I}) \xrightarrow{\cong} K_{\mathsf{six}}(C^*(E), \mathfrak{I}_1)$ such that $\alpha_\bullet$ are positive isomorphisms and $\alpha_2([1_\mathfrak{A}]) = [1_{C^*(E)}]$. Hence, by [Eilers et al. 2013, Corollary 4.20], $\mathfrak{A} \cong C^*(E)$.

In the $[\mathbf{1}\infty]_0$ case we argue as follows. By replacing, if necessary, the graph presentation of $\mathfrak{A}/\mathfrak{I}$ by a right adhesive graph [Eilers et al. 2016a, Lemma 5.4(r1)], we may apply [Eilers et al. 2016a, Proposition 5.5] to create a graph $\mathfrak{A}$ such that $C^*(E)$ has exactly one proper nontrivial ideal $\mathfrak{I}_1$ and there exists an isomorphism

$(\alpha_\bullet, \beta_\bullet) : K_{\mathsf{six}}(\mathfrak{A}, \mathfrak{I}) \xrightarrow{\cong} K_{\mathsf{six}}(C^*(E), \mathfrak{I}_1)$ such that all $\alpha_\bullet$ are positive isomorphisms. By [Eilers et al. 2009, Theorem 2.3] and [Eilers et al. 2016b, Proposition 2], $\mathfrak{A} \cong C^*(E)$.

Step 6: The $[\infty\mathbf{1}]_1$ case.

Because of Theorem 3.3, we only need to check that the invariant in (3.1) satisfies the conditions in Theorem 3.8. We see that (vi) holds by condition (2), and we get (vii)–(ix) because we are in the $[\infty\mathbf{1}]_1$ case.

Condition (i) follows from the fact that $\mathfrak{I}$ is a simple, stable AF algebra. The first half of (ii) — that the extension is lexicographic in the sense of [Handelman 1982] — follows from Lemma 3.2 and [Eilers et al. 2014b, Corollary 3.22] as a direct consequence of the fullness of $\mathfrak{e}$ established in Step 3. The second half follows by noting that any lift of $1_{\mathfrak{A}/\mathfrak{I}}$ lies in the intersection. Condition (iii) follows because $\mathfrak{D}$ is a nonunital AF algebra, and (iv) and (v) follow by construction. □

Corollary 4.2. *Let $C^*(E_1)$ and $C^*(E_3)$ be unital and simple graph C^*-algebras and consider the unital extension*

$$0 \to C^*(E_1) \otimes \mathbb{K} \to \mathfrak{X} \to C^*(E_3) \to 0.$$

The following are equivalent:

(a) *$\mathfrak{X}$ is a graph C^*-algebra*;

(b) *$\mathfrak{X}$ has real rank zero*;

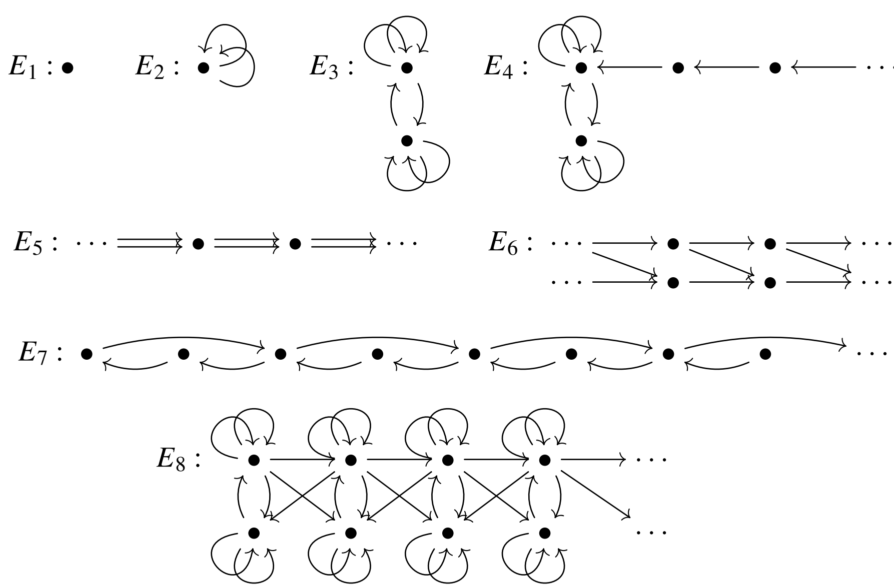
(c) *$\partial_0 : K_0(C^*(E_3)) \to K_1(C^*(E_1))$ vanishes.*

Proof. As noted in Step 1 of the proof of Theorem 4.1, we know that $C^*(E_1)$ and $C^*(E_3)$ have real rank zero from the outset. Hence, by [Brown and Pedersen 1991, Theorem 3.19], (b) and (c) are equivalent. Since $C^*(E_3)$ is simple, the Busby invariant of the above extension is either the zero map or injective. In the latter case, we have that every nonzero ideal of $\mathfrak{X}$ has a nontrivial intersection with $C^*(E_1)\otimes\mathbb{K}$. Thus, as $C^*(E_3)$ and $C^*(E_1)$ are assumed to be simple, either $\mathfrak{X}$ is isomorphic to $C^*(E_1) \otimes \mathbb{K} \oplus C^*(E_3)$ (the Busby map is the zero map) or $C^*(E_1) \otimes \mathbb{K}$ is the only proper nontrivial ideal of $\mathfrak{X}$ (the Busby map is injective). We conclude that $C^*(E_1) \otimes \mathbb{K}$ is the only proper nontrivial ideal of $\mathfrak{X}$ since $\mathfrak{X}$ is unital and $C^*(E_1)\otimes\mathbb{K}\oplus C^*(E_3)$ is nonunital. Consequently we can apply Theorem 4.1 since $C^*(E_1) \otimes \mathbb{K}$ is itself a graph C^*-algebra. When (a) holds, by Theorem 4.1 we get (c).

In the other direction, we need to establish (2) and (3) of Theorem 4.1 separately under the assumption of (c). Here, (3) follows from the fact that $C^*(E_1)$ is unital and that any unital graph C^*-algebra satisfies (a)–(c) of (3) in Theorem 4.1, so it remains to establish (2). There is nothing to check in the $[\mathbf{11}]_2$ and $[\infty\mathbf{1}]_2$ cases. For the cases, $[\infty\infty]_2$ and $[\mathbf{1}\infty]_2$, by [Eilers et al. 2014b, Corollary 3.22] it is enough to show that the extension is full. In both cases, $\mathcal{Q}(C^*(E_1) \otimes \mathbb{K})$ is simple since $C^*(E_1) \otimes \mathbb{K}$ is either a purely infinite simple C^*-algebra or isomorphic to $\mathbb{K}$. We conclude that the extension is full in the $[\infty\infty]_2$ and $[\mathbf{1}\infty]_2$ cases since the Busby invariant of the extension is injective. □

Example 4.3. The graphs given in Figure 1 can be used to demonstrate necessity of all the individual conditions in Theorem 4.1 in the sense of producing an extension $\mathfrak{e}$ satisfying all conditions of Theorem 4.1 but one, and where $\mathfrak{A}$ is not a graph C^*-algebra. We have that $C^*(E_1) = \mathbb{C}$, $C^*(E_2) = \mathcal{O}_2$, $C^*(E_5) = \mathsf{M}_{2^\infty} \otimes \mathbb{K}$, that $C^*(E_4)$, $C^*(E_7)$ and $C^*(E_8)$ are the stable UCT Kirchberg algebras with K-groups $\mathbb{Z} \oplus \mathbb{Z}$, $0 \oplus \mathbb{Z}$ and $\mathbb{Z}^\infty \oplus 0$, respectively, and that $C^*(E_3)$ is the unital UCT Kirchberg algebra with K-groups $\mathbb{Z} \oplus \mathbb{Z}$ and vanishing class of the identity. Finally, $C^*(E_6)$ is the stable AF algebra with dimension group $\mathbb{Z} + \varphi\mathbb{Z}$, where φ is the golden mean.

In all cases but the one with condition (2) this amounts to arranging the K-theory in an obvious way clashing with the given condition, finding first an appropriate KK_*-element and realizing it as explained in [Rørdam 1997]. The example to show necessity of (2) is more involved and was explained in [Eilers et al. 2014b, Example 4.1].



Condition not met:	(1)	(2)	(3)(a)	(3)(b)	(3)(c)
$\mathfrak{I}$	$C^*(E_4)$	$C^*(E_5)$	$C^*(E_8)$	$C^*(E_7)$	$C^*(E_6)$
$\mathfrak{A}/\mathfrak{I}$	$C^*(E_3)$	$C^*(E_2)$	$C^*(E_1)$	$C^*(E_1)$	$C^*(E_1)$

Figure 1. Examples showing necessity of the conditions in Theorem 4.1.

Remark 4.4. Note that although $C^*(E) \oplus C^*(F)$ is isomorphic to the graph C^*-algebra $C^*(E \sqcup F)$, condition (2) of Theorem 4.1 is not automatically met in this case. It is possible to obtain a result which works for any situation when $\mathfrak{A}$ has an ideal $\mathfrak{I}$ so that $\mathfrak{I}$ and $\mathfrak{A}/\mathfrak{I}$ are both simple graph C^*-algebras, but since this is somewhat convoluted, we refrain from doing so here.

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On line bundles in derived algebraic geometry

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We give the first example of a derived scheme X and a line bundle $\mathscr{L}$ on the truncation tX so that $\mathscr{L}$ does not extend to the original derived scheme X. In other words the pullback map $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$, and hence also the pullback map $K^0(X) \to K^0(tX)$, is not surjective. The derived schemes we construct have the further property that while their truncations are projective hypersurfaces, they fail to have any nontrivial line bundles, and therefore they are not quasiprojective.

1. Introduction

Derived algebraic geometry and related methods have recently seen numerous applications in the study of classical schemes, usually by giving either new results about, or new constructions of, various cohomology theories. For example, Kerz et al. [2018] use derived algebraic geometry to prove pro cdh-descent for the algebraic K-theory of Noetherian schemes, and subsequently to settle a conjecture of Weibel on the vanishing of negative K-groups. A more recent example is given in [Annala 2019b], where the algebraic cobordism of general quasiprojective (derived) schemes over a field of characteristic 0 is carried out. Given the position of algebraic cobordism as the universal oriented cohomology theory, and the fact that the cycles in the construction of [loc. cit.] are certain maps from derived schemes, one has ample motivation to study the geometry of general derived k-schemes, and not just, say, specific examples of derived moduli.

It is a well known fact [Kerz et al. 2018] that for an affine derived scheme X, the induced pullback map $K^0(X) \to K^0(tX)$ is an isomorphism. However, due to the form that the descent spectral sequence takes, one would expect the K^0 of a derived scheme to be different from that of its truncation in general. The problem of finding concrete examples of such behavior may be approached via a more computable invariant — the Picard group of X — which is a summand of $K^0(X)$: we have a map $\mathrm{Pic}(X) \to K^0(X)$ that sends a line bundle to its K-theory class, and a one-sided inverse is induced by the perfect determinant map of Schürg, Toën, and Vezzosi [Schürg et al. 2015] as the determinant of a line bundle $\mathscr{L}$, regarded as

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a perfect complex, is again $\mathscr{L}$. We can therefore conclude that if the pullback map $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$ fails to be injective (surjective), then so does the map $K^0(X) \to K^0(tX)$.

It is not very hard to find examples of derived schemes X so that the map $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$ is not injective. Some kind of trivial derived enhancement will often have this property: for example, take the derived scheme whose underlying scheme is $\mathbb{P}^2$, and whose structure sheaf is given by the trivial square-zero extension $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)[-1]$. One can compute, either using the descent spectral sequence or the deformation sequences as in Section 2, that the Picard group of X is isomorphic to $\mathbb{Z} \oplus k$.

However, finding an example so that $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$ fails to be surjective is harder, as a trivial extension will not work anymore. It is also a much more interesting question, as many geometric properties of X are governed by the line bundles on X. Consider for example the question of whether or not a derived scheme X is quasiprojective. In [Annala 2019a] it was noted that a derived scheme X is quasiprojective if and only if it has an *ample line bundle*, i.e., a line bundle whose truncation is ample on tX. Hence, the question of whether or not X is quasiprojective can be divided into two parts:

(1) is the truncation tX quasiprojective and

(2) does an ample line bundle on tX extend to X;

and the second question is obviously related to the surjectivity of the map $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$.

The main purpose of this article is finding an example of a derived scheme X such that the pullback map $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$ is not surjective. However, in the examples we construct, $\mathrm{Pic}(X)$ is trivial while the truncation tX is a projective hypersurface, realizing the obstruction (2) to quasiprojectivity. The examples are constructed in Section 3, and, after reducing the problem to one in classical algebraic geometry, verifying the desired properties is an easy computation involving nothing else than just the basic graduate knowledge of algebraic geometry. However, justifying these computations takes a bit more work, and is done in Section 2.

Conventions. Throughout this article, we are going to work over a field k of characteristic 0. Derived schemes over k are ringed spaces (in the ∞-categorical sense), which are locally modeled by spectra of simplicial commutative k-algebras. It is known that under our characteristic assumption, one gets an equivalent theory by replacing simplicial commutative k-algebras by connective commutative k differential graded algebras or by $\mathbb{E}_\infty$-ring spectra over k. As everything in this paper should be assumed to be derived, we will often drop the word "derived" to not burden the exposition; on the contrary, if something is not derived, we will emphasize this by using descriptions such as *classical* or *truncated*. We will denote by $[n]$ the operation of n-fold suspension Σ^n, which in the dg-model corresponds to the homological shift upwards n times. Throughout the article, unless otherwise specified, X will be a derived scheme over k. All derived schemes are assumed to be separated. All rings are assumed to be commutative (in the homotopy theoretic sense), unless otherwise specified.

Given a derived scheme X, its *Picard group* $\mathrm{Pic}(X)$ is defined as the set of equivalence classes of line bundles (locally free sheaves of rank 1) on X, whose group operation is given by the tensor product [Lurie 2018, Definition 2.9.4.1]. A potential alternative definition for the Picard group is as equivalence classes of invertible elements (*invertible sheaves*) of the ∞-category $\mathrm{QCoh}(X)$ of quasicoherent sheaves on X. In [loc. cit.] this is called the *extended Picard group* and denoted by $\mathrm{Pic}^\dagger(X)$. By Corollary 2.9.5.7 therein the difference is not very large: if the underlying topological space of X is connected, then an invertible sheaf is equivalent to $\mathscr{L}[n]$ for some line bundle $\mathscr{L}$ and $n \in \mathbb{Z}$, and therefore $\mathrm{Pic}^\dagger(X)$ is a semidirect product of $\mathrm{Pic}(X)$ and $\mathbb{Z}$.

2. Derived deformation theory of line bundles

An important part of this paper is to have very precise control of the pullback morphism $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$ for certain types of derived enhancements $tX \hookrightarrow X$. Recall that the Picard group of a derived scheme X can be naturally identified with the first cohomology group $H^1(X; \mathcal{O}_X^\times)$ of the sheaf of units $\mathcal{O}_X^\times$ on X. The sheaf $\mathcal{O}_X^\times$ is defined by the formula

$$\mathcal{O}_X^\times(U) := (\mathcal{O}_X(U))^\times, \tag{2.1}$$

where the right-hand side denotes the space of units of the simplicial k-algebra $\mathcal{O}_X(U)$, i.e., the collection of the components of $\mathcal{O}_X(U)$ corresponding to the units of $\pi_0(\mathcal{O}_X(U))$. Multiplication of A induces on $A^\times$ the structure of a grouplike commutative simplicial monoid, i.e., a spectrum. As the functor $A \to A^\times$ is corepresented by the Laurent polynomials $k[t, t^{-1}]$, units commute with limits, and therefore the above rule truly yields a sheaf.

Suppose now that X is a classical scheme and $\mathcal{F}$ is a connective quasicoherent sheaf on X. Given a derived derivation $d : X \to \mathcal{F}[1]$, we can form the *square-zero extension* X_d of X as the derived enhancement of X whose structure sheaf is given by the top-left corner of the Cartesian diagram

$$\begin{array}{ccc} \mathcal{O}_{X_d} & \longrightarrow & \mathcal{O}_X \\ \downarrow & & \downarrow (1,d) \\ \mathcal{O}_X & \xrightarrow[(1,0)]{} & \mathcal{O}_X \oplus \mathcal{F}[1] \end{array} \tag{2.2}$$

where 0 is the zero derivation (see [Porta and Vezzosi 2015, Definition 1.1] or [Toën and Vezzosi 2008, Definition 1.2.1.7] for the local case). Note that (2.2) remains Cartesian if considered as a square of sheaves of spectra on the underlying topological space X_{top} of X (combine Propositions 2.1.0.3 and 2.2.4.1 of [Lurie 2018]). Moreover, under some assumptions on $\mathcal{F}$ (e.g., the cohomology of $\mathcal{F}$ is concentrated in a single degree), any derived extension $X \hookrightarrow \widetilde{X}$ so that the fiber of the natural map $\mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X$ can be identified with $\mathcal{F}$ (henceforth a *square-zero extension of X by* $\mathcal{F}$) arises in the above way from a derived derivation $d : \mathcal{O}_X \to \mathcal{F}[1]$ which is unique up to equivalence [Lurie 2017, Theorem 7.4.1.23] (see also [Porta and Vezzosi 2015, Theorem 3.1] for an easier version that is enough for the purposes of this paper).

Consider now the induced map $\mathcal{O}_{X_d}^{\times} \to \mathcal{O}_X^{\times}$. As units are stable under pullbacks, the square of the previous paragraph yields us a Cartesian square

$$\begin{array}{ccc} \mathcal{O}_{X_d}^{\times} & \longrightarrow & \mathcal{O}_X^{\times} \\ \downarrow & & \downarrow \\ \mathcal{O}_X^{\times} & \longrightarrow & (\mathcal{O}_X \oplus \mathcal{F}[1])^{\times} \end{array} \tag{2.3}$$

of sheaves of spectra. The lower horizontal map clearly has cofiber equivalent to $\mathcal{F}[1]$, and the induced map $(\mathcal{O}_X \oplus \mathcal{F}[1])^{\times} \to \mathcal{F}[1]$ is given on the level of simplicial sets by the degreewise formula

$$(a, m) \mapsto \frac{m}{a} \tag{2.4}$$

(as X is discrete, the units are invertible on the nose). We can therefore conclude that we have a distinguished triangle

$$\mathcal{F} \to \mathcal{O}_{X_d}^{\times} \to \mathcal{O}_X^{\times} \xrightarrow{\delta_d} \mathcal{F}[1] \tag{2.5}$$

inducing a long exact sequence of cohomology groups. The most important part of this exact sequence, at least for the purposes of this paper, is

$$H^1(X; \mathcal{F}) \to \mathrm{Pic}(X_d) \to \mathrm{Pic}(X) \xrightarrow{\delta_{d*}} H^2(X; \mathcal{F}) \tag{2.6}$$

where the middlemost morphism is the pullback morphism we are interested in. Therefore, if we can understand δ_d well enough, then we can understand the surjectivity of $\mathrm{Pic}(X_d) \to \mathrm{Pic}(X)$. But δ_d has an easy description, given by the following proposition.

Proposition 2.7. *Let X be a classical scheme and $\mathcal{F}$ a connective quasicoherent sheaf on X, and suppose we are given a derivation d on X taking values in $\mathcal{F}[1]$. Then the induced map $\delta_d : \mathcal{O}_X^{\times} \to \mathcal{F}[1]$ of sheaves of simplicial abelian groups as* *in* (2.5) *is equivalent to the* log derivation $d_{\log}$ *associated to the derivation* d; *i.e., it is defined by the degreewise formula*

$$a \mapsto \frac{d(a)}{a}. \tag{2.8}$$

Remark 2.9. Unless X is smooth, the above formula is deceptive in its simplicity, as the derived derivations of a k-algebra A taking values in a simplicial A-module M are given by degreewise k-derivations $d : \tilde{A} \to M$, where $\tilde{A} \to A$ is a cofibrant replacement [Porta and Vezzosi 2015, Definition 1.1]. Most of the complications go away if X is smooth over k as the cotangent complex is equivalent to the cotangent bundle $\Omega_{X/k}$: one merely takes a fibrant resolution (corresponding to an injective resolution via Dold–Kan) $\widetilde{\mathcal{F}[n]}$ of the quasicoherent sheaf $\mathcal{F}[n]$ and then a derived derivation is the same as a map $\Omega_{X/k} \to \widetilde{\mathcal{F}[n]}$, which in turn is the same as a degreewise derivation $\mathcal{O}_X \to \widetilde{\mathcal{F}[n]}$.

Proof. Taking horizontal cofibers of (2.3) we get an extended diagram

$$\begin{array}{ccccc} \mathcal{O}_{X_d}^{\times} & \longrightarrow & \mathcal{O}_X^{\times} & \xrightarrow{\delta_d} & \mathcal{F}[1] \\ \downarrow & & \downarrow{\scriptstyle (1,d)} & & \downarrow{\scriptstyle 1} \\ \mathcal{O}_X^{\times} & \longrightarrow & (\mathcal{O}_X \oplus \mathcal{F}[1])^{\times} & \longrightarrow & \mathcal{F}[1] \end{array} \tag{2.10}$$

where the bottom-right horizontal map was identified earlier in (2.4). As the lower composition of the rightmost square is clearly equivalent to $d_{\log}$, the claim follows from the commutativity of the diagram. □

Before drawing the final conclusion of this section, we need to make the following remark.

Remark 2.11. Suppose that $\mathcal{F}$ is a classical quasicoherent sheaf on a classical scheme X and $\mathcal{A}$ is a sheaf of abelian groups on the underlying topological space of X. If $\mathcal{U} = (U_i)_{i\in I}$ is an affine open cover of X, then any Čech cocycle $(\delta_{i_0\cdots i_n})$ of homomorphisms from $\mathcal{A}$ to $\check{C}^n_{\mathcal{U}}(\mathcal{F})$ represents a morphism $\delta : \mathcal{A} \to \mathcal{F}[n]$, and moreover the induced morphism $\check{H}^j_{\mathcal{U}}(\mathcal{A}) \to \check{H}^j_{\mathcal{U}}(\mathcal{F}) = H^{n+j}(\mathcal{F})$ can be described by the formula

$$(\delta a)_{i_0\cdots i_{n+j}} = (-1)^n \delta_{i_0\cdots i_n}(a_{i_n\cdots i_{n+j}}) \tag{2.12}$$

on cocycles.

Indeed, as $\check{C}^*_{\mathcal{U}}(\mathcal{A})$ is weakly equivalent to $\mathcal{A}$, one hopes that there would be only one map (up to homotopy) $\check{C}^*_{\mathcal{U}}(\mathcal{A}) \to \check{C}^{*+n}_{\mathcal{U}}(\mathcal{F})$ that extends the original map $\mathcal{A} \to \check{C}^n_{\mathcal{U}}(\mathcal{F})$. If this were true, then it would suffice to verify that the above formula for δ gives a well defined map of chain complexes $\check{C}^*_{\mathcal{U}}(\mathcal{A}) \to \check{C}^{*+n}_{\mathcal{U}}(\mathcal{F})$. This is indeed true, and the easy algebraic manipulation is left for the reader. Now the fact that the formula given for δ is the right one follows from the fact that the Čech complex maps (quasi-isomorphically) to an injective resolution of $\mathcal{F}[n]$, and that maps to injective resolutions preserve homotopy equivalences.

Proposition 2.13. *Suppose X is a smooth k-scheme and $\mathcal{F}$ is a classical quasi-coherent sheaf on X. Recalling that $\mathrm{Ext}_X^{n+1}(\Omega_{X/k}, \mathcal{F})$ is naturally equivalent to the set of equivalence classes of k-derivations $d : \mathcal{O}_X \to \mathcal{F}[n+1]$, the formula*

$$(d, \mathscr{L}) \mapsto \delta_{d*}(\mathscr{L}) \tag{2.14}$$

where δ_{d*} is defined as in (2.6) *determines a biadditive pairing*

$$\psi_{\mathcal{F},n} : \mathrm{Ext}_X^{n+1}(\Omega_X, \mathcal{F}) \times \mathrm{Pic}(X) \to H^{n+2}(X; \mathcal{F}) \tag{2.15}$$

that is k-linear in the first argument. Moreover, the pairing has the property that $\psi_{\mathcal{F},n}(d, \mathscr{L})$ vanishes if and only if $\mathscr{L}$ extends to a line bundle on the square-zero extension X_d.

Proof. The fact that $\psi_{\mathcal{F},n}$ detects whether or not a line bundle $\mathscr{L}$ extends to X_d follows directly from the definition, so we are left to show the biadditivity and the k-linearity with respect to the first argument. Proposition 2.7 identifies δ_d with the log derivation $d_{\log}$, and as every map $\Omega_{X/k} \to \mathcal{F}[n+1]$ in the derived category of X is represented by a Čech cocycle $\Omega_{X/k} \to \check{C}_{\mathcal{U}}^{n+1}(\mathcal{F})$, we see that the induced derivation d (and therefore also the induced log derivation $d_{\log}$) is represented by a Čech cocycle of derivations (log derivations).

We can therefore apply Remark 2.11 to our situation, and the formula (2.12) now translates into

$$\psi_{\mathcal{F},n}(d, \alpha)_{k_0,k_1,\dots,k_{n+2}} = d_{k_0,k_1,\dots,k_{n+1};\log}(\alpha_{k_{n+1},k_{n+2}}), \tag{2.16}$$

which is clearly k-linear for d. As log derivations are group homomorphisms, the additivity with respect to the other argument follows as well. □

3. The main results

In this section we are going to give the example. We are also changing our terminology for a more deformation theoretic one: if X is a smooth variety, then a deformation of X over $k \oplus k[i]$ is the same as a square-zero extension of X by the quasicoherent sheaf $\mathcal{O}_X[i]$. This follows for example from [Porta and Vezzosi 2015, Proposition 6.1] (stated and proved only for the $i = 1$ case, but which generalizes easily for $i > 1$ as well) or from a direct globalization of [Lurie 2017, Proposition 7.4.2.5], as equivalence classes of such deformations are identified with $H^{i+1}(X; T_{X/k}) \cong \mathrm{Ext}^{i+1}(\Omega_{X/k}, \mathcal{O}_X)$, $T_{X/k} \cong \Omega_{X/k}^{\vee}$ being the tangent bundle. Hence, the results of the previous section on deformation theory of line bundles can be immediately applied to deformations of X over the higher square-zero extensions $k \oplus k[i]$.

Let $X \hookrightarrow \mathbb{P}^n$ be a smooth hypersurface of degree $n+1$ defined as the vanishing locus of a homogeneous polynomial F. Without loss of generality we may assume that X does not contain the point $[1 : 0 : \cdots : 0]$ so that $\mathcal{U} = (U_i|_X)^n_{i\geq 1}$, where $(U_i)^n_{i\geq 0}$ is the standard open cover of $\mathbb{P}^n$, is an affine cover of X. Computing the Čech cohomology groups of the structure sheaf $\mathcal{O}_X$ and the tangent bundle $T_{X/k}$ associated to the above covering, one obtains the following two lemmas. The results are completely elementary, and can be worked out by nothing more than a few pages of diagram chasing. For completeness, however, we give short proofs.

Lemma 3.1. *Suppose $n \geq 3$. Then the cohomology group $H^{n-1}(X; \mathcal{O}_X)$ is isomorphic to k, and it is generated by the Čech cocycle*

$$\frac{\partial_0 F}{x_1 \cdots x_n}. \tag{3.2}$$

Moreover, $H^i(X; \mathcal{O}_X) \cong 0$ for $1 \leq i \leq n-2$.

Proof. All the claims other than that the given cocycle generates are standard. The last remaining claim follows from the fact that $\partial_0 F$ has a term cx_0^n, where $c \neq 0$, and $x_0^n/(x_1 \cdots x_n)$ is not a boundary (unlike all other possibilities). □

Lemma 3.3. *Suppose $n \geq 4$. Then the cohomology group $H^{n-2}(X; T_{X/k})$ is isomorphic to k, and it is generated by the Čech cocycle $d = (d_{12\cdots\hat{i}\cdots n})_{1\leq i\leq n}$, where*

$$d_{12\cdots\hat{i}\cdots n} = (-1)^i \frac{(\partial_0 F)\partial_i - (\partial_i F)\partial_0}{x_1 x_2 \cdots \hat{x_i} \cdots x_n}. \tag{3.4}$$

Remark 3.5. As is customary, we use the hat to denote an index or a term which is left out.

Proof. The fact that $H^{n-2}(X; T_{X/k}) \cong k$ follows immediately from Serre duality once we recall that by hard Lefschetz $H^1(X; \Omega^1_{X/k}) \cong H^1(\mathbb{P}^n; \Omega^1_{\mathbb{P}^n}) \cong k$. Moreover, d is a cocycle in derivations on X: clearly all the chosen derivations send F to 0, so they are derivations on X, and they do satisfy the cocycle condition

$$\begin{aligned}
\sum_{i=1}^n (-1)^i d_{12\cdots\hat{i}\cdots n} &= \sum_{i=1}^n \frac{(\partial_0 F)\partial_i - (\partial_i F)\partial_0}{x_1 x_2 \cdots \hat{x_i} \cdots x_n} \\
&= \sum_{i=1}^n \frac{x_i(\partial_0 F)\partial_i - x_i(\partial_i F)\partial_0}{x_1 x_2 \cdots x_n} \\
&= \frac{x_0(\partial_0 F)\partial_0 - x_0(\partial_0 F)\partial_0}{x_1 x_2 \cdots x_n} \\
&= 0
\end{aligned} \tag{3.6}$$

as the Euler form equals 0. Hence, d generates $H^{n-2}(X; T_{X/k})$ if it is nontrivial, but this in fact follows from the calculation (3.7) following the lemma. □

We are now ready to show that the line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)|_X$ does not extend to any nontrivial first-order deformation of X over $k \oplus k[n-3]$. Recall that the transition maps α_{ij} of $\mathcal{O}(-1)$ are defined as $\alpha_{ij} = x_j/x_i$.

We can now just apply the generating derivation d of Lemma 3.3 to $\mathcal{O}_{\mathbb{P}^n}(-1)|_X$ using the pairing of Proposition 2.13:

$$\begin{aligned}
\psi_{\mathcal{O}_X,n-3}(d,\alpha)_{12\cdots n} &= (d_{\log}\alpha)_{12\cdots n} \\
&\overset{(2.12)}{=} (-1)^{n-2} d_{12\cdots n-1;\log}(\alpha_{n-1,n}) \\
&= (-1)^{n-2}\frac{x_{n-1}}{x_n}(-1)^n \frac{(\partial_0 F)\partial_n - (\partial_n F)\partial_0}{x_1x_2\cdots x_{n-1}}\left(\frac{x_n}{x_{n-1}}\right) \\
&= \frac{x_{n-1}}{x_n}\frac{(\partial_0 F)x_{n-1}^{-1}}{x_1x_2\cdots x_{n-1}} \\
&= \frac{(\partial_0 F)}{x_1x_2\cdots x_n}
\end{aligned} \tag{3.7}$$

and the right-hand side is the generator of Lemma 3.1, and therefore known to be nonzero. We have proven the following:

Theorem 3.8. *Let $X \hookrightarrow \mathbb{P}^n$, $n \geq 4$, be a smooth projective hypersurface of degree $n+1$. Then the line bundle $\mathcal{O}(1)|_X$ does not extend to any nontrivial deformation X' of X over $k \oplus k[n-3]$.*

Proof. Indeed, as we noticed earlier, the obstruction class of a line bundle depends k-linearly on the deformation of X. We have shown that the obstruction is not 0 for the generating deformation, and therefore it will not be 0 for any nonzero multiple of it. □

Remark 3.9. Note that when $n = 3$, the derivation δ given in Lemma 3.3 is still a perfectly valid element of $H^1(X; T_X)$, and applying the log differential to $\mathcal{O}(1)|_X$ as above shows that $\mathcal{O}(1)|_X$ does not extend to the deformation associated to δ. This has a moduli theoretic interpretation. Indeed, it is a well known fact that the moduli space of *polarized K3 surfaces* (K3 surfaces equipped with an ample line bundle) is 19-dimensional. Therefore, the kernel, which can easily be checked to be 19-dimensional, of the map $H^1(X; T_X) \to H^2(X; \mathcal{O}_X)$ given by evaluating at $\mathcal{O}(1)|_X$ should be thought as the tangent space of the space of polarized K3 surfaces, sitting inside the tangent space of the moduli of K3 surfaces.

Assume again that $n \geq 4$. It is known that the Picard group of a smooth hypersurface $X \hookrightarrow \mathbb{P}^n$ is isomorphic to $\mathbb{Z}$ and generated by $\mathcal{O}(-1)|_X$. Hence:

Theorem 3.10. *Let $X \hookrightarrow \mathbb{P}^n$, $n \geq 4$, be a smooth projective hypersurface of degree $n+1$, and let X' be a nontrivial deformation of X over $k \oplus k[n-3]$. Then* $\mathrm{Pic}(X') \cong 0$ *and therefore X' fails to be quasiprojective.*

Proof. We have the deformation sequence

$$\cdots \to H^{n-2}(X; \mathcal{O}_X) \to \mathrm{Pic}(X') \to \mathrm{Pic}(X) \xrightarrow{d_{\log}} H^{n-1}(X; \mathcal{O}_X) \to \cdots . \quad (3.11)$$

The computation (3.7) proves $d_{\log}$ is injective: indeed $d_{\log}$ is a homomorphism, it obtains a nonzero value on the generator $\mathcal{O}(-1)|_X$, and its target $H^{n-1}(X; \mathcal{O}_X) \cong k$ (Lemma 3.1) has no torsion. The claim now follows from the fact that $H^{n-2}(X; \mathcal{O}_X)$ (second claim of Lemma 3.1). □

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On modules over motivic ring spectra

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We provide an axiomatic framework that characterizes the stable ∞-categories that are module categories over a motivic spectrum. This is done by invoking Lurie's ∞-categorical version of the Barr–Beck theorem. As an application, this gives an alternative approach to Röndigs and Østvær's theorem relating Voevodsky's motives with modules over motivic cohomology and to Garkusha's extension of Röndigs and Østvær's result to general correspondence categories, including the category of Milnor–Witt correspondences in the sense of Calmès and Fasel. We also extend these comparison results to regular Noetherian schemes over a field (after inverting the residue characteristic), following the methods of Cisinski and Déglise.

1. Introduction

Röndigs and Østvær [2006; 2008] employed the technology of motivic functors developed in [Dundas et al. 2003] to prove an important structural result regarding motivic cohomology, namely that there is an equivalence of model categories between motives and modules over motivic cohomology (at least over fields of characteristic 0). In particular, this implies that Voevodsky's triangulated categories of motives [2000] is equivalent to the homotopy category of modules over the motivic Eilenberg–Mac Lane spectrum. This result has been extended to bases which are regular schemes over a field in the work of Cisinski and Déglise [2015] on integral mixed motives in the equicharacteristic case. More recently, Röndigs and Østvær's result was extended to general categories of correspondences over a field by Garkusha [2019]. These theorems provide pleasant reinterpretations of Voevodsky's category of motives as modules over a highly structured ring spectrum. The analog in topology is the result that chain complexes over a ring R are equivalent (in an appropriate model categorical sense) to modules over the Eilenberg–Mac Lane spectrum $\mathrm{H}R$. This result was first obtained by Schwede and Shipley [2003] as part of the characterization of stable model categories.[1]

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[1] We remark that an ∞-categorical treatment of the Schwede–Shipley results can be found in [Lurie 2017, Theorem 7.1.2.1].

In the present paper, we aim to provide a general axiomatic approach to the above results. More precisely, by making use of Lurie's ∞-categorical version of the Barr–Beck theorem we derive a characterization of those stable ∞-categories that are equivalent to a module category over a motivic spectrum. These categories are instances of *motivic module categories* as defined in Definition 3.1. Examples include $\mathrm{DM}(k)$ in the sense of Voevodsky [Mazza et al. 2006] and $\widetilde{\mathrm{DM}}(k)$ in the sense of Déglise and Fasel [2017]. Our characterization then reads as follows:

Theorem 1.1 (see Theorem 5.2). *Let k be a field of exponential characteristic e, and suppose that $\mathscr{M}(k)$ is a motivic module category on k. Then there is an equivalence of presentably symmetric monoidal stable ∞-categories*

$$\mathscr{M}(k)\left[\tfrac{1}{e}\right] \simeq \mathrm{Mod}_{R_{\mathscr{M}}[\frac{1}{e}]}(\mathrm{SH}(k)),$$

where $R_{\mathscr{M}}$ is a motivic $\mathcal{E}_\infty$-ring spectrum in $\mathrm{SH}(k)$ corresponding to the monoidal unit in $\mathscr{M}(k)$. In particular, the associated triangulated categories are equivalent.

In fact, we formulate a parametrized version of motivic module categories and, under further hypotheses, we show that Theorem 1.1 extends to regular schemes over fields (see Theorem 5.5). The proof of the latter follows the approach of Cisinski and Déglise [2015], while the proof of Theorem 1.1 breaks down into three steps:

(1) Invoke the Barr–Beck–Lurie theorem to prove that a motivic module category $\mathscr{M}(k)$ on k is equivalent to the category of modules over some monad on $\mathrm{SH}(k)$.

(2) Produce a functor from modules over the monad to modules over a corresponding motivic spectrum (Lemma 3.6).

(3) Determine when this functor is an equivalence.

After proving Theorem 1.1 we proceed to give a way to engineer several examples of motivic module categories via the notion of *correspondence categories*, on which one can apply the usual constructions of motivic homotopy theory.

1A. ***Overview.*** Here is an outline of this paper:

- In Section 2 we collect some background material on the Barr–Beck–Lurie theorem, on compact rigid generation in motivic homotopy theory, and on premotivic categories.
- In Section 3 we provide an axiomatic framework characterizing the stable ∞-categories that are module categories over motivic spectra.
- In Section 4 we move on to discuss examples of categories satisfying the axioms of Section 3. The most prominent example is those arising from some sort of correspondences.

- Finally, in Section 5 we prove that the axioms of Section 3 are satisfied for the correspondence categories constructed in Section 4 in various situations.

1B. ***Conventions and notation.*** We will rely on the language of ∞-categories following Lurie's books [2009; 2017]. By a *base scheme* we mean a Noetherian scheme S of finite dimension. We denote by Sch the category of Noetherian schemes, and by Sm_S the category of smooth schemes of finite type over S. The symbol $\mathbb{T}$ will denote the Thom space of the trivial vector bundle of rank 1 over the base S. Thus, we have the standard motivic equivalences $\mathbb{T} \simeq \boldsymbol{A}^1/\boldsymbol{A}^1 \setminus 0 \simeq \mathbb{P}^1$. We set $\boldsymbol{S}^{p,q} := (S^1)^{\otimes(p-q)} \otimes \boldsymbol{G}_m^{\otimes q}$ and $\Sigma^{p,q} M := \boldsymbol{S}^{p,q} \otimes M$, suitably interpreted in the category of motivic spaces or spectra. We reserve the symbol $\mathbf{1}$ for the motivic sphere spectrum in $\mathrm{SH}(k)$ and write $\Sigma^{p,q}\mathbf{1}$ for the (p,q)-suspension of $\mathbf{1}$. If τ is a topology on Sm_S, we write $\mathrm{H}_\tau(S)$ and $\mathrm{SH}_\tau(S)$ for the unstable and the $\mathbb{T}$-stable motivic homotopy ∞-categories, respectively. If $\tau = \mathrm{Nis}$ we may drop the decoration.

2. Preliminaries

2A. ***The Barr–Beck–Lurie theorem.*** Let us start out by recalling the Barr–Beck–Lurie theorem characterizing modules over a monad, in the setting of ∞-categories. We use the terminology of [Gaitsgory and Rozenblyum 2017, §3.7].

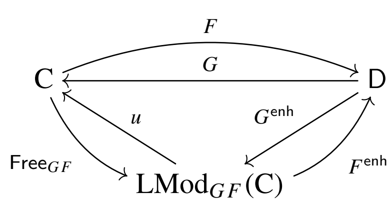
Let $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ be an adjunction. Then the endofunctor $GF : \mathsf{C} \to \mathsf{C}$ is a monad, and the functor $G : \mathsf{D} \to \mathsf{C}$ factors as

$$\mathsf{D} \xrightarrow{\mathrm{G}^{\mathsf{enh}}} \mathrm{LMod}_{GF}(\mathsf{C}) \xrightarrow{u} \mathsf{C},$$

where u is the forgetful functor. Moreover, the functor $\mathrm{G}^{\mathsf{enh}} : \mathsf{D} \to \mathrm{LMod}_{GF}(\mathsf{C})$ admits a left adjoint

$$\mathrm{F}^{\mathsf{enh}} : \mathrm{LMod}_{GF}(\mathsf{C}) \to \mathsf{D}.$$

2A1. The net result is that the adjunction $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ factors as



Here the functor $\mathsf{Free}_{GF} : \mathsf{C} \to \mathrm{LMod}_{GF}(C)$ is simply the left adjoint to the functor u appearing in the factorization of G above, and thus deserves to be called the "free GF-module" functor.

2A2. The Barr–Beck–Lurie theorem provides necessary and sufficient conditions for the functor $G^{\mathsf{enh}}\colon \mathsf{D} \to \mathrm{LMod}_{GF}(\mathrm{C})$ to be an equivalence. Before stating the theorem, recall first that a simplicial object $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathsf{D}$ is *split* if it extends to a split augmented object; in other words it extends to a functor $U\colon \Delta^{\mathrm{op}}_{-\infty} \to \mathsf{D}$. Here $\Delta_{-\infty}$ is the category whose objects are integers ≥ -1, and where $\mathrm{Hom}_{\Delta_{-\infty}}(n, m)$ consists of nondecreasing maps $n \cup \{-\infty\} \to m \cup \{-\infty\}$. Every split augmented simplicial diagram is a colimit diagram so that the map $\operatorname{colim} X_\bullet \to X_{-1}$ is an equivalence. If $G\colon \mathsf{D} \to \mathsf{C}$ is a functor, we say that a simplicial object $X_\bullet$ in D is *G-split* if $G \circ X_\bullet$ is split.

Theorem 2.1 (Barr–Beck–Lurie [Lurie 2017, Theorem 4.7.3.5]). *Let $G\colon \mathsf{D} \to \mathsf{C}$ be a functor of ∞-categories admitting a left adjoint $F\colon \mathsf{C} \to \mathsf{D}$. Then the following are equivalent*:

(1) *The functor* $\mathrm{G}^{\mathsf{enh}}$ *and* $\mathrm{F}^{\mathsf{enh}}$ *are mutually inverse equivalences.*

(2) *The functor* $\mathrm{G}^{\mathsf{enh}}$ *is conservative*, *and for any simplicial object $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathsf{D}$ which is G-split*, *$X_\bullet$ admits a colimit in D. Furthermore*, *any extension $\overline{X_\bullet}\colon (\Delta^{\mathrm{op}})^{\triangleright} \to \mathsf{D}$ is a colimit diagram if and only if $G \circ \overline{X_\bullet}$ is.*

Any adjunction (F, G) satisfying these equivalent conditions is called *monadic*.

2B. ***Compact and rigid objects in motivic homotopy theory.*** We now recall some facts about compact-rigid generation in motivic stable ∞-categories.

2B1. For now we work over an arbitrary base S. Denote by

(1) $\mathrm{SH}^{\omega}(S)$ the full subcategory of $\mathrm{SH}(S)$ spanned by the compact objects, and

(2) $\mathrm{SH}^{\mathsf{rig}}(S)$ the full subcategory of $\mathrm{SH}(S)$ spanned by the strongly dualizable objects.

The ∞-category $\mathrm{SH}(S)$ is generated under sifted colimits by $\Sigma^q_{\mathbb{T}}\Sigma^\infty_{\mathbb{T}} X_+$, where X is an affine smooth scheme over S and $q \in \boldsymbol{Z}$ [Khan 2016, Proposition 4.2.4]. Furthermore, each generator is a compact object in $\mathrm{SH}(S)$ since Nisnevich sheafification preserves filtered colimits (see, for example, [Hoyois 2017b, Proposition 6.4] where we set the group of equivariance to be trivial). Hence, the ∞-category $\mathrm{SH}^{\omega}(S)$ is generated under finite colimits and retracts by $\Sigma^q_{\mathbb{T}}\Sigma^\infty_{\mathbb{T}} X_+$, where $q \in \boldsymbol{Z}$ and X is affine. In particular the unit in $\mathrm{SH}(S)$ is compact and we have an inclusion

$$\mathrm{SH}^{\mathsf{rig}}(S) \subseteq \mathrm{SH}^{\omega}(S). \tag{2.2}$$

Over fields this inclusion is an equality — at least after an appropriate localization:

Lemma 2.3. *Let k be a field and suppose that ℓ is a prime which is coprime to the exponential characteristic e of k. Let $L_{(\ell)}\colon$ $\mathrm{SH}(k) \to \mathrm{SH}(k)$ be the localization* *endofunctor at ℓ. Then* (2.2) *induces equalities*

$$\mathrm{SH}^{\mathsf{rig}}(k)_{(\ell)} = \mathrm{SH}^{\omega}(k)_{(\ell)},$$
$$\mathrm{SH}^{\mathsf{rig}}(k)\left[\tfrac{1}{e}\right] = \mathrm{SH}^{\omega}(k)\left[\tfrac{1}{e}\right].$$

Proof. Since $\mathrm{SH}^{\omega}(k)$ is generated as a stable subcategory which is closed under retracts by $\Sigma^{\infty}_{\mathbb{T}} X_+$, where X is a smooth affine scheme, $\mathrm{SH}^{\omega}(k)_{(\ell)}$ is generated by the image of the same objects under $L_{(\ell)}$. Now, $\Sigma^{\infty}_{\mathbb{T}} X_+$ is dualizable whenever X is smooth and proper by [Riou 2005]; hence, it suffices to prove that $L_{(\ell)}(\Sigma^{\infty}_{\mathbb{T}} X_+)$ is a retract of some $L(\Sigma^{\infty}_{\mathbb{T}} Y_+)$, where Y is a smooth projective S-scheme. If k is perfect, then this is [Levine et al. 2019, Corollary B.2]. We note that this result is extended to the case of arbitrary fields in [Elmanto and Khan 2020, Theorem 3.2.1]. The result for e-inverted motivic spectra follows. □

Example 2.4. If S is a positive-dimensional base scheme, we should not expect (2.2) to be an equality in general even after localization [Cisinski and Déglise 2012, Corollary 3.2.7].

2B2. We adopt the following terminology.

Definition 2.5. Let k be a field and suppose that $L\colon \mathrm{SH}(k) \to \mathrm{SH}(k)$ is a localization endofunctor. We say that $L(\mathrm{SH}(k))$, or simply L, has *compact-rigid generation* if (2.2) is an equality after applying L.

Hence, Lemma 2.3 tells us that $\mathrm{SH}(k)_{(\ell)}$ and $\mathrm{SH}(k)\left[\tfrac{1}{e}\right]$ have compact-rigid generation.

2C. ***Premotivic categories and adjunctions.*** Lastly we recall Cisinski and Déglise's notion of a *premotivic category* [2019]. Suppose that $\mathscr{S}$ is a full subcategory of the category Sch of Noetherian schemes, and let $\mathscr{P}$ denote a class of admissible morphisms [Cisinski and Déglise 2019, §1]. In fact, the only example we care about is when $\mathscr{P}$ is the class of smooth morphisms. As in [Cisinski and Déglise 2019, §1] (see also [Cisinski and Déglise 2016, Appendix A] for a more succinct discussion), a functor

$$\mathscr{M}\colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$$

is called a *$\mathscr{P}$-premotivic category over $\mathscr{S}$* if for each morphism $f: T \to S$ in $\mathscr{S}$, the induced functor $f^*\colon \mathscr{M}(S) \to \mathscr{M}(T)$ admits a right adjoint f_*, and if f is admissible, it admits a left adjoint $f_\#$. The left adjoints are furthermore required to satisfy the *$\mathscr{P}$-base change formula*; i.e., the exchange morphism $\mathrm{Ex}^*_\#\colon q_\# g^* \to f^* p_\#$ is an equivalence whenever

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{p} & T \end{array}$$

is a Cartesian diagram in $\mathscr{S}$ such that p is a $\mathscr{P}$-morphism. See [Cisinski and Déglise 2019, §1.1.9] for details.

We refer the reader to the thesis of Khan [2016] for a detailed discussion of premotivic categories in the ∞-categorical setting. If the context is clear, we simply refer to $\mathscr{M}$ as a premotivic category. We may also speak of premotivic categories taking values in other (large) ∞-categories such as $\mathrm{Cat}_\infty^\otimes$, $\mathrm{Cat}_{\infty,\mathsf{stab}}$, or Pr^L.

2C1. We also have the appropriate notion of an adjunction between premotivic categories [Cisinski and Déglise 2019, Definition 1.4.6; 2016, Definition A.1.7]. Indeed, if $\mathscr{M}$ and $\mathscr{M}'$ are premotivic categories, then a *premotivic adjunction* is a transformation $\gamma^*\colon \mathscr{M} \to \mathscr{M}'$ such that

(1) for each $S \in \mathscr{S}$, the functor $\gamma_S^*\colon \mathscr{M}(S) \to \mathscr{M}'(S)$ admits a right adjoint γ_{S*} and

(2) for each morphism $f\colon T \to S \in \mathscr{S}$, the canonical transformation $f_\#\gamma_S^* \to \gamma_T^* f_\#$ is an equivalence.

Furthermore, we say that a premotivic adjunction γ^* is a *localization of premotivic categories* (or, simply, a *localization*) if for each $S \in \mathscr{S}$ the functor γ_{S*} is fully faithful, i.e., a localization in the sense of [Lurie 2009, Definition 5.2.7.2]. We say that a localization of premotivic categories is *smashing* if γ_{S*} preserves colimits. Suppose further that $\mathscr{M}$ takes values in $\mathrm{Cat}_\infty^\otimes$. In particular, the functors f^* are strongly symmetric monoidal. Then a localization L is *symmetric monoidal* if given any $S \in \mathscr{S}$ and any $\mathsf{E} \in \mathscr{M}(S)$ that is L-local, then for any $\mathsf{F} \in \mathscr{M}(S)$, $\mathsf{E}\otimes\mathsf{F}$ is L-local as well. This last condition implies that the symmetric monoidal structure on $\mathscr{M}(S)$ descends to one on the subcategory of L-local objects and that the localization functor is strongly symmetric monoidal [Lurie 2017, Proposition 2.2.1.9].

2C2. We recall two conditions on $\mathscr{M}$ which will be relevant to us later. In order to formulate them, we will now assume that $\mathscr{M}$ takes values in stable ∞-categories. Let $S \in \mathscr{S}$ be a scheme. Suppose that $i\colon Z \to S$ is a closed subscheme, and let $j\colon U \to S$ be its open complement.

Definition 2.6. Let $\mathscr{M}\colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Cat}_{\infty,\mathsf{stab}}$ be a premotivic category, and let $Z \xrightarrow{i} S \xleftarrow{j} U$ be as above. We say that $\mathscr{M}$ satisfies (Loc_i) if

$$\mathscr{M}(Z) \xrightarrow{i_*} \mathscr{M}(S) \xrightarrow{j^*} \mathscr{M}(U)$$

is a cofiber sequence of stable ∞-categories. We say that $\mathscr{M}$ satisfies (Loc) if (Loc_i) is satisfied for any closed immersion i.

Now let $c = (c_\alpha)_{\alpha\in I}$ be a collection of Cartesian sections of $\mathscr{M}$ (the only case we consider is $\{\Sigma^{p,q}\mathbf{1}\}_{p,q\in\mathbf{Z}}$). We denote by $\mathscr{M}_c(S) \subseteq \mathscr{M}(S)$ the smallest thick subcategory of $\mathscr{M}(S)$ which contains $f_\# f^* c_{\alpha,X}$ for any smooth morphism $f\colon T \to S$. Following [Cisinski and Déglise 2015, Definition 2.3], we call objects in $\mathscr{M}_c(S)$

c-constructible. We say that $\mathscr{M}$ is *c-generated* if for all $X \in \mathscr{S}$ the stable ∞-category $\mathscr{M}(S)$ is generated by $\mathscr{M}_c(S)$ under all small colimits.

Definition 2.7. Let $\mathscr{M}\colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Cat}_{\infty,\mathsf{stab}}$ be a premotivic category. Suppose that $\mathscr{A} \subseteq \mathscr{S}^{\Delta^1}$ is a collection of morphisms in $\mathscr{S}$. We say that $\mathscr{M}$ is *continuous with respect to* $\mathscr{A}$ if the following holds. Suppose that $X\colon I \to \mathscr{S}$ is a cofiltered diagram in $\mathscr{S}$ whose transition maps belongs to $\mathscr{A}$ and whose limit $X := \lim_{\alpha\in I} X_\alpha$ exists in $\mathscr{S}$. Then the canonical map

$$\mathscr{M}_c(X) \to \lim_{\alpha\in I} \mathscr{M}_c(X_\alpha)$$

is an equivalence.

3. Motivic module categories

In this section we formulate the notion of *motivic module categories* and relate it to categories of modules over a motivic $\mathcal{E}_\infty$-ring spectrum.

3A1. Let $\mathscr{S}$ be a full subcategory of Sch. By [Ayoub 2007; Cisinski and Déglise 2019] we then have a premotivic category $\mathrm{SH}|_{\mathscr{S}}\colon \mathscr{S} \to \mathrm{Pr}^{L,\otimes}_{\mathsf{stab}}$ whose value at $S \in \mathscr{S}$ is the motivic stable homotopy category $\mathrm{SH}(S)$ over S.

Definition 3.1. Let $\mathscr{S}$ be as above, and suppose that $L\colon \mathrm{SH}|_{\mathscr{S}} \to L(\mathrm{SH})|_{\mathscr{S}}$ is a localization which is symmetric monoidal in the sense of Section 2C1. We then define the following:

(1) Let $S \in \mathscr{S}$. An *L-local motivic module category on S* is a presentably symmetric monoidal stable ∞-category $\mathscr{M}(S)$ equipped with an adjunction

$$\gamma_S^* : L(\mathrm{SH}(S)) \rightleftarrows \mathscr{M}(S) : \gamma_{S*}$$

such that the left adjoint γ_S^* is symmetric monoidal, and the right adjoint γ_{S*} is conservative and preserves sifted colimits.

(2) An *L-local motivic module category over $\mathscr{S}$* (or, simply, a *motivic module category* if the context is clear) is a premotivic category

$$\mathscr{M}\colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Pr}^{L,\otimes}_{\mathsf{stab}}$$

valued in presentably symmetric monoidal stable ∞-categories, along with a premotivic adjunction

$$\gamma^*\colon L(\mathrm{SH})|_{\mathscr{S}} \to \mathscr{M}, \quad S \mapsto (\gamma_S^*\colon L(\mathrm{SH}(S)) \to \mathscr{M}(S)),$$

which evaluates to an L-local motivic module category $\mathscr{M}(S)$ on S for each $S \in \mathscr{S}$.

If L is the identity functor, then we simply say that $\mathscr{M}$ is a *motivic module category*. When the localization L is clear, we may denote a motivic module category by a pair $(\mathrm{SH}|_{\mathscr{S}}, \mathscr{M})$. Moreover, if the scheme S is implicitly understood, we may drop the S from the notation $(\gamma_S^*, \gamma_{S*})$.

In Section 4 we will give a way to construct motivic module categories using very general inputs.

Lemma 3.2. *Let $S \in \mathscr{S}$, and let $\mathbf{1}_S \in \mathrm{SH}(S)$ denote the motivic sphere spectrum over S. If $\mathscr{M}$ is an L-local motivic module category, then the spectrum $L\gamma_*\gamma^*(\mathbf{1}_S) \in \mathrm{SH}(S)$ is an $\mathcal{E}_\infty$-ring spectrum.*

Proof. As γ_* is lax symmetric monoidal, it follows that γ_* preserves $\mathcal{E}_\infty$-algebras. Since γ^* is symmetric monoidal, $\gamma^*(\mathbf{1}_S)$ is the unit object in $\mathscr{M}$ and is thus an $\mathcal{E}_\infty$-algebra. As L is symmetric monoidal, we conclude that $\gamma_*\gamma^*(\mathbf{1}_S)$ is an $\mathcal{E}_\infty$-ring spectrum. □

3A2. The Barr–Beck–Lurie theorem ensures that a motivic module category on S is always equivalent to modules over a monad, as the following lemma records. We will subsequently investigate when we can further enhance this equivalence to modules over the $\mathcal{E}_\infty$-ring spectrum $L\gamma_*\gamma^*(\mathbf{1}_S)$.

Lemma 3.3. *If $\mathscr{M}(S)$ is a motivic module category on S, then the induced adjunction*

$$\gamma^{*,\mathsf{enh}} : \mathrm{LMod}_{\gamma_*\gamma^*}(L(\mathrm{SH}(S))) \rightleftarrows \mathscr{M}(S) : \gamma_*^{\mathsf{enh}}$$

is an equivalence of ∞-categories.

Proof. By assumption, the conditions of Theorem 2.1 are satisfied. □

3B. ***Motivic module categories versus categories of modules.*** The following definition will be essential in relating a motivic module category to a category of modules over a motivic $\mathcal{E}_\infty$-ring spectrum.

Definition 3.4. Let $\mathscr{M}$ be an L-local motivic module category over $\mathscr{S}$ and let $S \in \mathscr{S}$. We say that the pair $(\mathrm{SH}|_{\mathscr{S}}, \mathscr{M})$ *admits the projection formula at S* if there is an equivalence

$$\gamma_*\gamma^*(\mathbf{1}_S) \otimes (\,\cdot\,) \xrightarrow{\simeq} \gamma_*\gamma^*$$

of endofunctors on $L(\mathrm{SH}(S))$. If $(\mathrm{SH}|_{\mathscr{S}}, \mathscr{M})$ admits the projection formula at any $S \in \mathscr{S}$, we say that $(\mathrm{SH}|_{\mathscr{S}}, \mathscr{M})$ *admits the projection formula*.

Theorem 3.5. *Let $\mathscr{M}$ be an L-local motivic module category over $\mathscr{S}$. Suppose that $S \in \mathscr{S}$ is a scheme such that $(\mathrm{SH}|_{\mathscr{S}}, \mathscr{M})$ admits the projection formula at S. Then there is an equivalence of presentably symmetric monoidal stable ∞-categories*

$$\mathscr{M}(S) \simeq \mathrm{Mod}_{L\gamma_*\gamma^*(\mathbf{1}_S)}(\mathrm{SH}(S)).$$

Consequently, if $(\mathrm{SH}|_{\mathscr{S}}, \mathscr{M})$ *admits the projection formula, then we have an equivalence of premotivic categories*

$$\mathscr{M} \simeq \mathrm{Mod}_{L\gamma_*\gamma^*(\mathbf{1})}(\mathrm{SH}(\,\cdot\,)).$$

3B1. In light of Lemma 3.3, we can prove Theorem 3.5 by means of relating modules over the monad $\gamma_*\gamma^*$ with modules over the motivic spectrum $\gamma_*\gamma^*(\mathbf{1}_S)$. Thus, given $S \in \mathscr{S}$ our task is to formulate a relationship between the two ∞-categories

$$\mathrm{LMod}_{\gamma_*\gamma^*}(\mathrm{SH}(S)) \quad \text{and} \quad \mathrm{LMod}_{\gamma_*\gamma^*(\mathbf{1}_S)\otimes(\,\cdot\,)}(\mathrm{SH}(S)).$$

To do so, it suffices to produce a map of monads

$$c\colon \gamma_*\gamma^*(\mathbf{1}_S) \otimes (\,\cdot\,) \to \gamma_*\gamma^*,$$

which will induce a functor

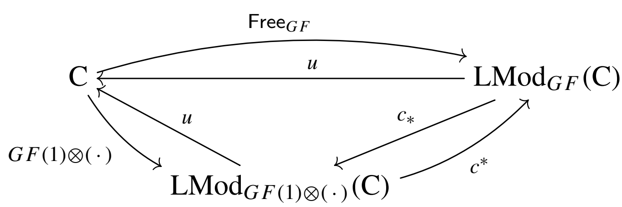
$$c^*\colon \mathrm{LMod}_{\gamma_*\gamma^*(\mathbf{1}_S)\otimes(\,\cdot\,)}(\mathrm{SH}(S)) \to \mathrm{LMod}_{\gamma_*\gamma^*}(\mathrm{SH}(S)).$$

For this, we appeal to a general lemma.

Lemma 3.6. *Let* C, D *be symmetric monoidal* ∞*-categories and suppose that we have an adjunction* $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ *such that* F *is symmetric monoidal* (*so that* G *is lax symmetric monoidal*). *Then there is a map of monads*

$$c\colon GF(1) \otimes (\,\cdot\,) \to GF, \tag{3.7}$$

which gives rise to a commutative diagram of adjunctions



Proof. Since F is monoidal and G is lax monoidal, the functor GF is lax monoidal. Hence, $GF(1)$ is an algebra object of C, and thus $GF(1) \otimes (\,\cdot\,)$ is indeed a monad. We construct the map of monads $c\colon GF(1) \otimes (\,\cdot\,) \to GF(\,\cdot\,)$ by letting c be the composite of the maps of monads

$$\begin{aligned} GF(1) \otimes (\,\cdot\,) &\simeq (GF(1) \otimes (\,\cdot\,)) \circ \mathrm{id} \\ &\xrightarrow{\mathrm{id}\circ\epsilon} (GF(1) \otimes (\,\cdot\,)) \circ GF(\,\cdot\,) \\ &\xrightarrow{\mu} G(F(1) \otimes F(\,\cdot\,)) \\ &\simeq GF. \end{aligned}$$

Here ϵ is the unit of the adjunction (F, G). The transformation ϵ is a map of monads via the triangle identities, and the map $\mathrm{id} \circ \epsilon$ is a map of monads since we are $\circ$-tensoring two maps of monads. The map μ is given by the lax monoidal structure of G; more precisely, we note that the endofunctor $G(A \otimes F(\,\cdot\,))$ is a monad for any algebra object A, and so $G(F(1) \otimes F(\,\cdot\,))$ is in particular a monad. We have a canonical equivalence of monads

$$(GF(1) \otimes (\,\cdot\,)) \circ GF(\,\cdot\,) \simeq GF(1) \otimes GF(\,\cdot\,).$$

The lax structure of G then provides a morphism of endofunctors

$$GF(1) \otimes GF(\,\cdot\,) \to G(F(1) \otimes F(\,\cdot\,)) \simeq GF(\,\cdot\,),$$

and the lax structure also verifies that this is a map of monads. This gives rise to a functor $c_* \colon \mathrm{LMod}_{GF}(\mathrm{C}) \to \mathrm{LMod}_{GF(1)\otimes(\,\cdot\,)}(\mathrm{C})$, which has a left adjoint by the adjoint functor theorem.

To obtain the desired factorizations, we note that we have the commutative diagram of forgetful functors

$$\begin{array}{ccc} \mathrm{C} & \xleftarrow{\quad u \quad} & \mathrm{LMod}_{GF}(\mathrm{C}) \\ & {\scriptstyle u}\nwarrow \qquad \swarrow{\scriptstyle c_*} & \\ & \mathrm{LMod}_{GF(1)\otimes(\,\cdot\,)}(\mathrm{C}) & \end{array}$$

Thus, the left adjoints also commute. □

3B2. We can now apply Lemma 3.6 to prove Theorem 3.5.

Proof of Theorem 3.5. We claim that the adjunction of Lemma 3.6,

$$c^* : \mathrm{LMod}_{\gamma_*\gamma^*(\mathbf{1}_S)}(\mathrm{SH}(S)) \rightleftarrows \mathrm{LMod}_{\gamma_*\gamma^*}(\mathrm{SH}(S)) : c_*,$$

is an equivalence. By the construction in the proof of Lemma 3.6, the above adjunction arises from a map of monads given by $c \colon \gamma_*\gamma^*(\mathbf{1}_S) \otimes (\,\cdot\,) \to \gamma_*\gamma^*$. Since $(\mathrm{SH}|_{\mathscr{S}}, \mathscr{M})$ satisfies the projection formula, we conclude that the adjunction (c^*, c_*) is an equivalence.

Now, note that Theorem 2.1 and Lemma 3.6 are phrased for $\mathcal{E}_1$-algebras and left modules. However, as $\gamma_*\gamma^*(\mathbf{1})$ is an $\mathcal{E}_\infty$-ring spectrum by Lemma 3.2, the ∞-categories of left and right $\gamma_*\gamma^*(\mathbf{1})$-modules are equivalent. We thus conclude that there is an natural equivalence

$$\mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1}_S)}(\mathrm{SH}(S)) \simeq \mathscr{M}(S)$$

of ∞-categories, which carries $\gamma_*\gamma^*(\mathbf{1}_S)$ to the unit object $\gamma^*(\mathbf{1}_S)$ of $\mathscr{M}(S)$. Finally, if $\mathscr{M}$ satisfies the projection formula at any $S \in \mathscr{S}$, then the naturality of the above equivalence furnishes the equivalence of premotivic categories $\mathscr{M} \simeq \mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})}(\mathrm{SH}(\cdot))$. □

Remark 3.8. In fact, the above reduction can be achieved using a more refined version of Lurie's Barr–Beck theorem [Lurie 2017, Proposition 4.8.5.8].

Remark 3.9. We were also informed by Niko Naumann that the above result is a consequence of [Mathew et al. 2017, Proposition 5.29].

In the following Sections 4 and 5 we will provide examples for which the hypotheses of Theorem 3.5 are satisfied.

4. Correspondence categories

The prime examples of motivic module categories are built from various notions of correspondences. In this section we will give an axiomatization of ∞-categories that behave like the category of framed correspondences as in [Elmanto et al. 2019a]; Suslin and Voevodsky's category of finite correspondences [Voevodsky et al. 2000; Mazza et al. 2006, Chapters 1 and 2]; Calmès and Fasel's finite Milnor–Witt correspondences [Calmès and Fasel 2017; Déglise and Fasel 2017]; Grothendieck–Witt correspondences [Druzhinin 2017]; and, more recently, the categories of correspondences studied in [Druzhinin and Kolderup 2019; Elmanto et al. 2020]. These examples will be discussed in Section 4C. To begin with, consider the discrete category Sch_{S+}, whose objects are S-schemes of the form $X_+ := X \amalg S$ and morphisms which preserve the base point. We consider the subcategory $\mathsf{Sm}_{S+} \subseteq \mathrm{Sch}_{S+}$ spanned by smooth S-schemes of finite type. We will heavily use the *nonabelian derived* ∞*-category* $\mathrm{P}_\Sigma(\mathrm{C})$ associated to an ∞-category C with finite products; more detailed treatments of this construction can be found in [Bachmann and Hoyois 2018, Chapter 1; Lurie 2009, §5.5.8].

Definition 4.1. A *correspondence category* (over a base scheme S) is a preadditive[2] ∞-category C equipped with a *graph functor*

$$\gamma_{\mathrm{C}} \colon \mathsf{Sm}_{S+} \to \mathrm{C} \tag{4.2}$$

satisfying the following conditions:

(1) The functor γ_{C} is essentially surjective and preserves finite coproducts,[3] so that we get an induced functor

$$\gamma_* \colon \mathrm{P}_\Sigma(\mathrm{C}) \to \mathrm{P}(\mathsf{Sm}_S), \quad \mathscr{F} \mapsto \mathscr{F} \circ \gamma_C.$$

[2] Recall that a preadditive ∞-category is one that is pointed, has finite products and coproducts, and is such that the map $X \amalg Y \to X \times Y$ is an equivalence for all $X, Y \in \mathrm{C}$.

[3] By requiring the functor γ_{C} to preserve finite coproducts we include also the empty coproduct, ensuring that γ_C preserves the base point of Sm_{S+}.

(2) The composite functor

$$\mathsf{Sm}_{S+} \to \mathrm{C} \to \mathrm{P}_\Sigma(\mathrm{C}) \xrightarrow{\gamma_*} \mathrm{P}_\Sigma(\mathsf{Sm}_{S+}) \tag{4.3}$$

has a right lax Sm_{S+}-linear structure. We abusively denote the composite (4.3) by $\gamma_C(\,\cdot\,)$; the context will always make it clear what is meant.

The ∞-category CorrCat of correspondence categories is defined as a full subcategory of the (large) ∞-category $\mathrm{PreAdd}_{\infty,\mathsf{Sm}_{S+}/}$ of small preadditive ∞-categories and functors which preserve finite coproducts equipped with a finite coproduct-preserving functor from Sm_{S+}.[4]

4A1. We begin with a couple of clarifying remarks and an example.

Remark 4.4. Informally, the Sm_{S+}-linear structure on $\gamma_\mathrm{C}(\,\cdot\,)$ encodes, for any $X, Y \in \mathsf{Sm}_S$, maps

$$X_+ \otimes \gamma_\mathrm{C}(Y_+) \to \gamma_\mathrm{C}(X_+ \otimes Y_+)$$

in $\mathrm{P}_\Sigma(\mathsf{Sm}_{S+}) \simeq \mathrm{P}_\Sigma(\mathsf{Sm}_S)_*$ which are subject to various compatibilities. For example, if $f\colon X_+ \to Z_+$ is a map in Sm_{S+}, then we have a 2-cell witnessing the commutativity of

$$\begin{array}{ccc} X_+ \otimes \gamma_C(Y_+) & \longrightarrow & \gamma_C(X_+ \otimes Y_+) \\ {\scriptstyle f\otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \gamma_C(f\otimes \mathrm{id})} \\ Z_+ \otimes \gamma_C(Y_+) & \longrightarrow & \gamma_C(Z_+ \otimes Y_+) \end{array}$$

Similarly, if $g\colon Y_+ \to Z_+$ is a map in Sm_{S+}, then we have a 2-cell witnessing the commutativity of

$$\begin{array}{ccc} X_+ \otimes \gamma_C(Y_+) & \longrightarrow & \gamma_C(X_+ \otimes Y_+) \\ {\scriptstyle \mathrm{id}\otimes g}\downarrow & & \downarrow{\scriptstyle \gamma_C(\mathrm{id}\otimes g)} \\ X_+ \otimes \gamma_C(Z_+) & \longrightarrow & \gamma_C(X_+ \otimes Z_+) \end{array}$$

These cells are required to satisfy an infinite list of coherences.

Remark 4.5. The Sm_{S+}-linearity assumption will be satisfied if C has a symmetric monoidal structure and the functor γ_C is symmetric monoidal. In more detail, we denote by $\mathrm{CorrCat}^\otimes$ the ∞-category of preadditive ∞-categories with a symmetric monoidal structure such that the graph functor $\gamma_\mathrm{C}\colon \mathsf{Sm}_{S+} \to \mathrm{C}$ is symmetric monoidal, is essentially surjective, and preserves finite coproducts. There is a forgetful functor $\mathrm{CorrCat}^\otimes \to \mathrm{CorrCat}$; the second part of Definition 4.1 is obtained from the strong symmetric monoidality of γ_C. This is the case in the examples

[4] More succinctly, CorrCat is the pullback of ∞-categories $\mathrm{PreAdd} \times_{\mathrm{Cat}_\infty^\amalg} \{\mathsf{Sm}_{S+}\}$, where $\mathrm{Cat}^\amalg$ denotes ∞-categories with finite coproducts and finite coproduct-preserving functors.

considered in this paper, but we include it as an axiom to clarify proofs of certain properties.

Example 4.6. Let Corr_S denote the discrete category whose objects are smooth S-schemes and morphisms are spans $X \leftarrow Y \rightarrow Z$. This is a preadditive category by [Bachmann and Hoyois 2018, Lemma C.3]. The graph functor witnesses Corr_S as a correspondence category.

4A2. We now provide some elementary properties of a correspondence category.

Proposition 4.7. *Let* C *be a preadditive* ∞*-category equipped with an essential surjection*

$$\gamma_{\mathrm{C}} \colon \mathsf{Sm}_S \to \mathrm{C}$$

which preserves coproducts, and let $\gamma_{\mathrm{C}*}$ *denote the induced functor*

$$\gamma_{\mathrm{C}*} \colon \mathrm{P}_\Sigma(\mathrm{C}) \to \mathrm{P}_\Sigma(\mathsf{Sm}_S), \quad \mathscr{F} \mapsto \mathscr{F} \circ \gamma_C.$$

Then the following properties hold:

(1) *The* ∞*-category* $\mathrm{P}_\Sigma(\mathrm{C})$ *is presentable and preadditive.*

(2) *The functor* γ_{C*} *preserves sifted colimits.*

(3) *The functor* γ_{C*} *is conservative.*

Proof. Presentability of $\mathrm{P}_\Sigma(\mathrm{C})$ is [Lurie 2009, Proposition 5.5.8.10(1)], while P_Σ applied to a preadditive ∞-category is again preadditive by [Gepner et al. 2015, Corollary 2.4]. The functor γ_{C*} preserves sifted colimits since sifted colimits are computed pointwise (a direct consequence of parts (4) and (5) of [Lurie 2009, Proposition 5.5.8.10]) while γ_{C*} is conservative since γ_{C} is essentially surjective. □

4A3. The composite of γ_{C} with Yoneda functor $\mathsf{Sm}_{S+} \xrightarrow{\gamma_{\mathrm{C}}} \mathrm{C} \xrightarrow{y} \mathrm{P}_\Sigma(\mathrm{C})$ has a canonical sifted colimit-preserving extension $\gamma_{\mathrm{C}}^* \colon \mathrm{P}_\Sigma(\mathsf{Sm}_{S+}) \to \mathrm{P}_\Sigma(C)$. It is easy to check that $\gamma_{\mathrm{C}*}$ is the right adjoint to γ_{C}^* and thus γ_{C}^* preserves all small colimits. As a result, we have an adjunction

$$\gamma_C^* : \mathrm{P}_\Sigma(\mathsf{Sm}_{S+}) \rightleftarrows \mathrm{P}_\Sigma(\mathrm{C}) : \gamma_{C*}. \tag{4.8}$$

It is also easy promote the Sm_{S+}-linear structure given by the second axiom of a correspondence category to a $\mathrm{P}_\Sigma(\mathsf{Sm}_{S+})$-linear structure so that the functor

$$\gamma_{C*} \circ \gamma_C^* \colon \mathrm{P}_\Sigma(\mathsf{Sm}_{S+}) \to \mathrm{P}_\Sigma(\mathsf{Sm}_{S+})$$

extends to a right lax $\mathrm{P}_\Sigma(\mathsf{Sm}_{S+})$-linear functor.

4A4. Now we would like to do motivic homotopy theory on C. Recall that if $X, Y \in \mathrm{P}_\Sigma(\mathsf{Sm}_{S+})$, then X is *$\boldsymbol{A}^1$-homotopy equivalent to Y* if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ and $\boldsymbol{A}^1$-homotopies $H\colon \boldsymbol{A}^1_+ \otimes X \to X$ and $H'\colon \boldsymbol{A}^1_+ \otimes Y \to Y$ from gf and fg to the respective identity morphisms. We note that any $\boldsymbol{A}^1$-homotopy equivalence is an $\mathrm{L}_{\boldsymbol{A}^1}$-equivalence [Morel and Voevodsky 1999, §2, Lemma 3.6].

Lemma 4.9. *The functor $\gamma_C\colon \mathrm{P}_\Sigma(\mathsf{Sm}_{S+}) \to \mathrm{P}_\Sigma(\mathsf{Sm}_{S+})$ preserves $\boldsymbol{A}^1$-homotopy equivalences.*

Proof. Suppose that we have a homotopy $H\colon \boldsymbol{A}^1_+ \otimes X_+ \to Y$ between maps $f, g\colon X \to Y$. We obtain, using the right lax-structure, a homotopy

$$\boldsymbol{A}^1_+ \otimes \gamma_C(X) \to \gamma_C(\boldsymbol{A}^1 \times X) \to \gamma_C(Y)$$

between $\gamma_C(f)$ and $\gamma_C(g)$. □

Lemma 4.10. *The functor $\gamma_C\colon \mathrm{P}_\Sigma(\mathsf{Sm}_{S+}) \to \mathrm{P}_\Sigma(\mathsf{Sm}_{S+})$ preserves $\mathrm{L}_{\boldsymbol{A}^1}$-equivalences.*

Proof. By definition the class of $\mathrm{L}_{\boldsymbol{A}^1}$-equivalences is the strong saturation, in the sense of [Lurie 2009, Proposition 5.5.4.5], of the maps in $\mathrm{P}_\Sigma(\mathsf{Sm}_{S+})$ by the (Yoneda image of) $\boldsymbol{A}^1$-projections $\pi_X\colon (\boldsymbol{A}^1 \times X)_+ \simeq \boldsymbol{A}^1_+ \otimes X_+ \to X_+$ for $X \in \mathsf{Sm}_S$. According to [Bachmann and Hoyois 2018, Lemma 2.10] the class of $\mathrm{L}_{\boldsymbol{A}^1}$-equivalences is then generated under 2-out-of-3 and sifted colimits by maps of the form $\pi_X \amalg \mathrm{id}_{Y_+}$ where $Y \in \mathsf{Sm}_S$.

Since π_X is an $\boldsymbol{A}^1$-homotopy equivalence, it follows from Lemma 4.9 that $\gamma_C(\pi_X)$ is an $\boldsymbol{A}^1$-homotopy equivalence. Since γ_C preserves coproducts by assumption, the same is true for the morphism

$$\gamma_C(\pi_X \amalg \mathrm{id}_{Y_+}) \simeq \gamma_C(\pi_X) \amalg \gamma_C(\mathrm{id}_{Y_+}).$$

The functor γ_C clearly preserves the 2-out-of-3-property. Lastly, the functor γ_C preserves sifted colimits by definition and sifted colimits are computed valuewise in $\mathrm{P}_\Sigma(\mathsf{Sm}_{S+})^{\Delta^1}$. Hence, we conclude that γ_C preserves $\mathrm{L}_{\boldsymbol{A}^1}$-equivalences. □

4A5. Now we take into account a topology that we might want to put on Sm_{S+}, namely, the topology of *coproduct decomposition*. This is a topology on Sm_{S+} defined by a cd-structure, denoted by $\amalg$, generated by squares

$$\begin{array}{ccc} S & \longrightarrow & U_+ \\ \downarrow & & \downarrow \\ V_+ & \longrightarrow & X_+ \end{array}$$

where U and V are clopen subschemes of X such that $U \amalg V = X$. Sheaves with respect to the topology generated by this cd-structure is precisely the nonabelian derived category on C. In other words we have

$$\mathrm{Shv}_{\amalg}(\mathsf{Sm}_{S+}) \simeq \mathrm{P}_{\Sigma}(\mathsf{Sm}_{S+})$$

by [Bachmann and Hoyois 2018, Lemma 2.4]. Hence, all topologies τ considered in this paper satisfy $\mathrm{Shv}_{\tau}(\mathsf{Sm}_{S+}) \subseteq \mathrm{P}_{\Sigma}(\mathsf{Sm}_{S+})$.

Definition 4.11. Let τ be a topology on Sm_S, and let C be a correspondence category with graph functor $\gamma_C \colon \mathsf{Sm}_{S+} \to \mathrm{C}$. Then C *is compatible with* τ if, for every τ-sieve $U \hookrightarrow X$ in Sm_S, the natural map

$$\gamma_{\mathrm{C}}(U_+) \to \gamma_{\mathrm{C}}(X_+)$$

is an L_τ-equivalence in $\mathrm{P}_{\Sigma}(\mathsf{Sm}_{S+})$.

Lemma 4.12. *Suppose that* C *is a correspondence category which is compatible with* τ. *Then the functor*

$$\gamma_C \colon \mathrm{P}_{\Sigma}(\mathsf{Sm}_{S+}) \to \mathrm{P}_{\Sigma}(\mathsf{Sm}_{S+})$$

preserves L_τ*-equivalences.*

Proof. By definition, the class of L_τ-equivalences is the strong saturation, in the sense of [Lurie 2009, Proposition 5.5.4.5], of the maps in $\mathrm{P}_{\Sigma}(\mathsf{Sm}_{S+})$ by the (Yoneda image of the) maps $i_+ \colon U_+ \hookrightarrow X_+$ where $X \in \mathsf{Sm}_S$ and i is a τ-sieve. According to [Bachmann and Hoyois 2018, Lemma 2.10], the class of L_τ-equivalences is then generated under 2-out-of-3 and sifted colimits by maps of the form $\pi_X \amalg \mathrm{id}_{Y_+}$ for $Y \in \mathsf{Sm}_S$. By the same reasoning as in Lemma 4.10 we need only check that $\gamma_{\mathrm{C}}(U_+) \to \gamma_{\mathrm{C}}(X_+)$ is an L_τ-equivalence, which is true by hypothesis. □

From now on, whenever we consider a correspondence category C, we make the following assumption on the topologies we discuss:

- The topology τ is at least as fine as the Nisnevich topology and is compatible in the sense of Definition 4.11.

4A6. If C is a correspondence category, then we can construct its unstable motivic homotopy ∞-category in the usual way, as we now do. We consider two full subcategories of $\mathrm{P}_{\Sigma}(\mathrm{C})$ spanned by objects $\mathscr{F}$ satisfying the following two axioms on homotopy invariance and τ-descent:

(Htpy): The presheaf $\mathscr{F} \circ \gamma_C \colon \mathsf{Sm}_S^{\mathrm{op}} \to \mathsf{Spc}$ is $\boldsymbol{A}^1$-invariant. We denote the ∞-category spanned by such $\mathscr{F}$'s by $\mathrm{P}_{\boldsymbol{A}^1}(\mathrm{C})$.

(τ-Desc): The presheaf $\mathscr{F} \circ \gamma_C \colon \mathsf{Sm}_S^{\mathrm{op}} \to \mathsf{Spc}$ is a τ-sheaf. We denote the ∞-category spanned by such $\mathscr{F}$'s by $\mathrm{Shv}_\tau(\mathrm{C})$.

Since $\mathrm{P}_\Sigma(\mathrm{C})$ is preadditive by Proposition 4.7, we have a canonical equivalence $\mathrm{CMon}(\mathrm{P}_\Sigma(\mathrm{C})) \simeq \mathrm{P}_\Sigma(\mathrm{C})$. The ∞-category of *unstable* C*-motives*, denoted by $\mathrm{H}_\tau(\mathrm{C})$, is then defined as $\mathrm{P}_{\mathbf{A}^1}(\mathrm{C}) \cap \mathrm{Shv}_\tau(\mathrm{C}) \subseteq \mathrm{P}_\Sigma(\mathrm{C})$. As usual we have localization functors $\mathrm{L}_\tau^{\mathrm{C}}: \mathrm{P}_\Sigma(\mathrm{C}) \to \mathrm{Shv}_\tau(\mathrm{C})$, $\mathrm{L}_{\mathbf{A}^1}^{\mathrm{C}}: \mathrm{P}_\Sigma(\mathrm{C}) \to \mathrm{P}_{\mathbf{A}^1}(\mathrm{C})$, and $\mathrm{L}_{\mathrm{mot},\tau}^{\mathrm{C}}: \mathrm{P}_\Sigma(\mathrm{C}) \to \mathrm{H}_\tau(\mathrm{C})$. From the construction of these localizations and the assumption on τ, the adjunction (4.8) descends to an adjunction

$$\gamma_C^*: \mathrm{H}_\tau(\mathsf{Sm}_{S+}) \simeq \mathrm{H}_\tau(S)_* \rightleftarrows \mathrm{H}_\tau(\mathrm{C}) : \gamma_{C*}. \tag{4.13}$$

Lemma 4.14. *The* ∞*-category* $\mathrm{H}_\tau(\mathrm{C})$ *is preadditive. Hence, we have a canonical equivalence* $\mathrm{CMon}(\mathrm{H}_\tau(\mathrm{C})^\times) \simeq \mathrm{H}_\tau(\mathrm{C})$.

Proof. The ∞-category $\mathrm{H}_\tau(\mathrm{C})$ is closed under finite products by checking that the conditions (Htpy) and (τ-Desc) are preserved under taking products which are computed pointwise. The statement follows since $\mathrm{P}_\Sigma(\mathrm{C})$ is preadditive by Proposition 4.7. □

Definition 4.15. The ∞-category of *effective* C*-motives* $\mathrm{H}_\tau(\mathrm{C})^{\mathrm{gp}}$ is defined to be the full subcategory of $\mathrm{H}_\tau(\mathrm{C})$ spanned by the grouplike objects, in the sense of [Gepner et al. 2015, Definition 1.2].

4A7. The next proposition captures the main property of categories of correspondences from the point of view of motivic homotopy theory.

Proposition 4.16. *Suppose that* C *is a correspondence category which is compatible with* τ*. Then the functor*

$$\gamma_{C*}: \mathrm{H}_\tau(\mathrm{C}) \to \mathrm{H}_\tau(S)_*$$

preserves sifted colimits and is conservative. Furthermore, $\mathrm{H}_\tau(\mathrm{C})$ *is canonically an* $\mathrm{H}(S)_*$*-module.*

Proof. For the first claim it suffices, after Proposition 4.7, to check that

$$\gamma_{\mathrm{C}*}: \mathrm{P}_\Sigma(\mathrm{C}) \to \mathrm{P}_\Sigma(\mathsf{Sm}_{S+}) \simeq \mathrm{P}_\Sigma(\mathsf{Sm}_S)_*$$

sends $\mathrm{L}_{\mathrm{mot},\tau}^{\mathrm{C}}$-equivalences to $\mathrm{L}_{\mathrm{mot},\tau}$-equivalences. This holds by Proposition 4.16 and Lemma 4.10. The assertion that $\mathrm{H}_\tau(\mathrm{C})$ is an $\mathrm{H}(S)_*$-module follows from the right lax structure of γ_{C*}. □

Remark 4.17. If τ is a topology finer than the Nisnevich topology, then the fully faithful functor $\mathrm{H}_\tau(S)_* \to \mathrm{H}(S)_*$ need not preserve colimits. Hence, the composite $\mathrm{H}_\tau(\mathrm{C}) \to \mathrm{H}_\tau(S)$ need not preserve colimits.

4A8. From the above point of view, we see that γ_{C*} is very close to preserving all colimits — we need only show that it preserves finite coproducts. The universal way to enforce this is to take commutative monoid objects on both sides with respect to Cartesian monoidal structures. We can do this for $\mathrm{H}_\tau(S)_*$ since it has finite products, and $\mathrm{CMon}(\mathrm{H}_\tau(\mathrm{C})^\times) \simeq \mathrm{H}_\tau(\mathrm{C})$ since it is preadditive [Gepner et al. 2015, Proposition 2.3]. We remark that the symmetric monoidal structure on $\mathrm{P}_\Sigma(\mathsf{Sm}_{S+})$ given by Day convolution is not Cartesian.[5]

To see this, consider the left adjoint to γ_{C*}, that is,

$$\gamma_{\mathrm{C}}^*\colon \mathrm{H}_\tau(S)_* \to \mathrm{H}_\tau(\mathrm{C}),$$

which preserves all small colimits. According to the universal property of CMon [Gepner et al. 2015, Corollary 4.9] we obtain an essentially unique functor $\gamma_{\mathrm{C}}^*\colon \mathrm{CMon}(\mathrm{H}_\tau(S)_*^\times) \to \mathrm{H}_\tau(\mathrm{C})$ since $\mathrm{H}_\tau(\mathrm{C})$ is preadditive by Proposition 4.7(1). This functor admits a right adjoint $\gamma_{C*}\colon \mathrm{H}_\tau(\mathrm{C}) \to \mathrm{CMon}(\mathrm{H}_\tau(S)_*^\times)$ which fits into a commutative diagram

$$\begin{array}{ccc} & & \mathrm{CMon}(\mathrm{H}_\tau(S)_*^\times) \\ & \overset{\gamma_{\mathrm{C}}^*}{\nearrow} & \downarrow \\ \mathrm{H}_\tau(\mathrm{C}) & \xrightarrow{\gamma_{C*}} & \mathrm{H}_\tau(S)_* \end{array} \tag{4.18}$$

That is, the functor γ_{C*} factors through the forgetful functor $\mathrm{CMon}(\mathrm{H}_\tau(S)_*^\times) \to \mathrm{H}_\tau(S)_*$.

Proposition 4.19. *Suppose that* C *is a correspondence category which is compatible with τ. Then the functor*

$$\gamma_{C*}\colon \mathrm{H}_\tau(\mathrm{C}) \to \mathrm{CMon}(\mathrm{H}_\tau(S)_*^\times)$$

preserves all small colimits and is conservative.

Proof. By the diagram (4.18), the functor γ_{C*} preserves sifted colimits because the horizontal arrow preserves sifted colimits by Proposition 4.16 and the vertical arrow preserves sifted colimits as a special case of [Gepner et al. 2015, Proposition B.4]. Since it is a right adjoint it preserves finite products, but since its domain and codomain are preadditive it preserves finite coproducts as well and we are done by [Bachmann and Hoyois 2018, Lemma 2.8]. The conservativity statement follows from Proposition 4.16 and the fact that the forgetful functor from commutative monoid objects is conservative. □

[5]On the other hand, the symmetric monoidal structure on $\mathrm{P}_\Sigma(\mathsf{Sm}_S)$ given by Day convolution is Cartesian, and the natural sifted-colimit-preserving functor $\mathrm{P}_\Sigma(\mathsf{Sm}_S) \to \mathrm{P}_\Sigma(\mathsf{Sm}_{S+})$ is symmetric monoidal.

4A9. $\mathbb{T}$-*stability.* We now introduce the notion of $\mathbb{T}$-stability along with the weaker notion of $\mathbb{T}$-prestability. This is inspired by the treatment of [Lurie 2018, Appendix C] on prestable ∞-categories.

Definition 4.20. Let C be an $\mathrm{H}(S)_*$-module in Cat_∞. Then C is $\mathbb{T}$-*prestable* if the endofunctor

$$\mathbb{T}\otimes(\cdot)\colon \mathrm{C}\to\mathrm{C} \tag{4.21}$$

is fully faithful. The ∞-category C is $\mathbb{T}$-*stable* if the endofunctor (4.21) is invertible.

4A10. From now on we assume that the correspondence category C is such that $\mathrm{H}_\tau(\mathrm{C})$ is $\mathbb{T}$-prestable.

Remark 4.22. The notion of a $\mathbb{T}$-stable ∞-category is a familiar one in motivic homotopy theory; indeed, the motivic stable homotopy category $\mathrm{SH}(S)$ is $\mathbb{T}$-stable. In fact, $\mathbb{T}$-prestability is a familiar concept as well: it is inspired by Voevodsky's *cancellation theorem* [2010], which asserts that $\mathrm{DM}^{\mathrm{eff}}(k;\mathbf{Z})$ is $\mathbb{T}$-prestable for any perfect field k. The analogous statement holds for Milnor–Witt motivic cohomology as proved in [Fasel and Østvær 2017]. For the ∞-category of framed motivic spaces, cancellation holds by [Elmanto et al. 2019a, Theorem 3.5.8], which in turn relies on the cancellation theorem of Ananyevskiy, Garkusha, and Panin [Ananyevskiy et al. 2016]. Moreover, for any base scheme S, the subcategory $\mathrm{SH}(S)^{\mathrm{eff}}\subseteq\mathrm{SH}(S)$ of effective motivic spectra is $\mathbb{T}$-prestable.

4A11. The thesis of Robalo [2015] provides a way to invert $\mathbb{T}$ for any $\mathrm{H}(S)_*$-module and obtain a symmetric monoidal stable ∞-category — in fact one that is a module over $\mathrm{SH}(S)$. We define the stable ∞-category of C-*motives* simply by

$$\mathrm{SH}_\tau(\mathrm{C}) := \mathrm{H}_\tau(\mathrm{C})[\mathbb{T}^{\otimes -1}],$$

with notation as in [Robalo 2015, Definition 2.6]. We then have the basic adjunction

$$\Sigma^\infty_{\mathbb{T},\mathrm{C}} : \mathrm{H}_\tau(\mathrm{C}) \rightleftarrows \mathrm{SH}_\tau(\mathrm{C}) : \Omega^\infty_{\mathbb{T},\mathrm{C}}.$$

The following summarizes the basic properties of $\mathrm{SH}_\tau(\mathrm{C})$:

Proposition 4.23. *If* C *is a correspondence category, then the following hold:*

(1) *The ∞-category* $\mathrm{SH}_\tau(\mathrm{C})$ *is a presentably symmetric monoidal stable ∞-category, and*

(2) *is generated under sifted colimits by objects of the form* $\{\mathbb{T}^{\otimes n}\otimes\Sigma^\infty_{\mathbb{T},\mathrm{C}}X\}_{n\in\mathbf{Z},X\in\mathrm{C}}$.

(3) *The ∞-category* $\mathrm{SH}_\tau(\mathrm{C})$ *is computed as the colimit in* $\mathrm{Mod}_{\mathrm{H}(\mathrm{Sm}_S)_*}(\mathrm{Pr}^L)$ *of*

$$\mathrm{H}_\tau(C)\xrightarrow{\mathbb{T}\otimes(\cdot)}\mathrm{H}_\tau(\mathrm{C})\xrightarrow{\mathbb{T}\otimes(\cdot)}\mathrm{H}_\tau(\mathrm{C})\xrightarrow{\mathbb{T}\otimes(\cdot)}\cdots. \tag{4.24}$$

(4) *The functor*

$$\gamma_{C*} \colon \mathrm{SH}_\tau(\mathrm{C}) \to \mathrm{SH}_\tau(\mathsf{Sm}_S)$$

is conservative and preserves colimits.

Proof. Stability follows from the standard equivalence $\mathbb{T} \simeq S^1 \otimes \boldsymbol{G}_m$ in $\mathrm{SH}(S)$, which remains true for modules over $\mathrm{SH}(S)$. The second assertion follows from the third via [Lurie 2009, Lemma 6.3.3.7] and the fact that $\mathrm{H}_\tau(\mathrm{C})$ is generated under sifted colimits by representables which are smooth affine by the argument of [Khan 2016, Proposition 2.2.9] (which works for any topology τ finer than Nis), while the third comes from [Robalo 2015, Corollary 2.22]. The last assertion follows from Proposition 4.19. □

4A12. The last part of Proposition 4.23 is the main point of our axiomatization: the adjunction $\mathrm{SH}_\tau(S) \rightleftarrows \mathrm{SH}_\tau(\mathrm{C})$ is monadic. In particular, if $\tau = \mathrm{Nis}$, then $\mathrm{SH}(S) \rightleftarrows \mathrm{SH}(\mathrm{C})$ is monadic.

4B. ***From categories of correspondences to motivic module categories.*** Suppose that we have a functor

$$\mathrm{C} \colon \mathscr{S}^{\mathrm{op}} \to \mathrm{CorrCat}^{\otimes}$$

which carries a morphism of schemes $f \colon T \to S$ to $f^* \colon \mathrm{C}_S \to \mathrm{C}_T$. By naturality of the preceding constructions[6] we obtain a functor

$$\mathrm{SH}_\tau \circ \mathrm{C} \colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Pr}^{L,\otimes}_{\mathsf{stab}}$$

equipped with a transformation $\mathrm{SH}|_{\mathscr{S}} \to \mathrm{SH}_\tau \circ \mathrm{C}$. We impose an additional assumption on C, inspired by [Cisinski and Déglise 2019, Lemma 9.3.7]:

- For each $p \colon T \to S$, a smooth morphism in $\mathscr{S}$, the functor p^* admits a left adjoint $p_\#$ such that the transformation $p_\# \gamma_{\mathrm{C}_T} \to \gamma_{\mathrm{C}_S} p_\#$ is an equivalence.

In this case, we say that C is *adequate*.

4B1. We employ the following additional notation: if $L \colon \mathrm{SH}(S) \to \mathrm{SH}(S)$ is a localization, denote by $L(\mathrm{SH}_\tau(\mathrm{C}_S))$ the subcategory of $\mathrm{SH}_\tau(\mathrm{C}_S)$ spanned by objects X such that $\gamma_{C*}X$ is L-local. Since γ_{C*} preserves limits, the inclusion $L(\mathrm{SH}_\tau(\mathrm{C}_S)) \hookrightarrow \mathrm{SH}_\tau(\mathrm{C}_S)$ is closed under limits and there is a localization functor (by the adjoint functor theorem)

$$L_{\mathrm{C}_S} \colon \mathrm{SH}_\tau(\mathrm{C}_S) \to L(\mathrm{SH}_\tau(\mathrm{C}_S))$$

[6] The most nontrivial of which is the universal property of $\mathbb{T}$-stabilization for which we can appeal to [Bachmann and Hoyois 2018, Lemma 4.1].

rendering the following diagram commutative (since their right adjoints commute):

$$\begin{array}{ccc} \mathrm{SH}(S) & \xrightarrow{\gamma^*_{\mathrm{C}_S}} & \mathrm{SH}_\tau(\mathrm{C}_S) \\ {\scriptstyle L}\big\downarrow & & \big\downarrow{\scriptstyle L_{\mathrm{C}_S}} \\ L(\mathrm{SH}(S)) & \xrightarrow{\gamma^*_{\mathrm{C}_S}} & L(\mathrm{SH}_\tau(\mathrm{C}_S)) \end{array}$$

Proposition 4.25. *If* $\mathrm{C}\colon \mathscr{S}^{\mathrm{op}} \to \mathrm{CorrCat}^{\otimes}$ *is adequate, then the following hold:*

(1) *We have premotivic adjunctions* $\mathrm{SH}|_{\mathscr{S}} \rightleftarrows \mathrm{SH}_\tau \circ C$.

(2) *If* L *is smashing and a symmetric monoidal localization of* $\mathrm{SH}|_{\mathscr{S}}$, *then we have a premotivic adjunction* $L(\mathrm{SH})|_{\mathscr{S}} \leftrightarrows L(\mathrm{SH} \circ \mathrm{C})$.

(3) *If* τ *is a topology such that for each* $S \in \mathscr{S}$, *the functor* $L(\mathrm{SH}_\tau(S)) \to L(\mathrm{SH}(S))$ *preserves sifted colimits, then the premotivic adjunction* $L(\mathrm{SH})|_{\mathscr{S}} \rightleftarrows L(\mathrm{SH}_\tau \circ \mathrm{C})$ *is a motivic module category (in particular, this holds when* $\tau = \mathrm{Nis}$).

Proof. The proof of (1) follows as in the case of Grothendieck abelian categories [Cisinski and Déglise 2019, Corollary 10.3.11] and Voevodsky's $\mathrm{C} = \mathsf{Corr}$ (in the sense of [Cisinski and Déglise 2019, §9]); we give only the main points. Since C is adequate, we get that the equivalence $p_\# \gamma_{\mathrm{C}_T} \to \gamma_{\mathrm{C}_S} p_\#$ persists on the level of $\mathbb{T}$-stabilizations. What we need to verify, just as in [Cisinski and Déglise 2019, Proposition 10.3.9], is that the transformation $L_\tau \gamma_{C*} \simeq \gamma_{C*} L_\tau$ is an equivalence on the unstable level, i.e., the "forgetful" functor $\mathrm{H}_\tau \circ C \to \mathrm{H}|_{\mathscr{S}}$ preserves τ-local objects. This is given by Lemma 4.12 under the standing assumption that C is compatible with τ. The next two statements are then immediate from the definition of motivic module categories and the last statement of Proposition 4.23. □

4C. *Examples.* We now discuss examples of the above constructions and results.

Example 4.26. Let $\mathscr{S} = \mathrm{Sch}_S$ and suppose that E is an $\mathcal{E}_\infty$-ring spectrum in $\mathrm{SH}(S)$. Then $\mathrm{Mod}_{\mathsf{E}} = (\mathsf{E} \otimes (\,\cdot\,)) \circ \mathrm{SH}$ furnishes the first examples of motivic module categories. We can also consider further localizations of the premotivic category $\mathrm{Mod}_{\mathsf{E}}$, such as in [Elmanto et al. 2019b] where $\mathscr{S} = \mathrm{Sch}_{\mathbb{Z}[\frac{1}{\ell}]}$ the localization functor is given by the composite of ℓ-completion and étale localization, and E is MGL; see [loc. cit.] for more details where results in this paper are used to describe the ∞-category of modules over étale cobordism.

Example 4.27. Consider a localization $L\colon \mathrm{SH}|_{\mathscr{S}} \to L(\mathrm{SH}|_{\mathscr{S}})$. If L is smashing, then $L(\mathrm{SH}|_{\mathscr{S}})$ is a motivic module category. Examples of these smashing localizations are given by *periodization of elements*; we refer the reader to [Hoyois 2017a, §3] for an extensive discussion in our context. For example, a theorem of Bachmann [2018] proves that periodizing the element ρ yields real étale localization. Consider $x\colon \Sigma^{p,q}\mathbf{1} \to \mathbf{1}$. Then the results of [Hoyois 2017a, §3] (or apply [Bachmann 2018, Lemma 15]) tell us that $\mathbf{1}[x^{-1}]$ is an $\mathcal{E}_\infty$-ring and the projection formula holds; hence, the category of x-periodic motivic spectra are modules over $\mathbf{1}[x^{-1}]$.

Example 4.28. The basic example of a category of correspondences is Voevodsky's category of finite correspondences Corr_S in the sense of [Mazza et al. 2006, Appendix 1A; Cisinski and Déglise 2019, §9], which is defined for any Noetherian scheme S [Cisinski and Déglise 2019, §9.1]. When S is essentially smooth over a perfect base field, the category of finite Milnor–Witt correspondences $\widetilde{\mathsf{Corr}}_S$ of Calmès and Fasel [2017] is defined and is also a category of correspondences. Over a perfect field (where both categories are defined), these categories are generalized by Garkusha's axioms in [2019]. When defined, these categories are adequate in the sense of Section 4B. All of these are examples of categories of correspondences, and thus gives rise motivic module categories.

Example 4.29. Let k be a perfect field. Given any $S \in \mathsf{Sm}_k$ and any good cohomology theory A on Sm_S in the sense of [Druzhinin and Kolderup 2019, §2], then [Druzhinin and Kolderup 2019, §3] defines an adequate category of correspondences Corr^A_S on Sm_S.

Example 4.30. The ∞-category of framed correspondences of [Elmanto et al. 2019a] is another example of a category of correspondences and is defined for any qcqs scheme S. The main theorem of [Hoyois 2018] asserts that the corresponding motivic module category is equivalent to $\mathrm{SH}(S)$, relying on the "recognition principle" of [Elmanto et al. 2019a].

Example 4.31. If $\mathsf{E} \in \mathrm{SH}(S)$ is a homotopy associative ring spectrum, [Elmanto et al. 2020] defines an hSpc-enriched category $\mathsf{hCorr}^{\mathsf{E}}_S$ of *finite* E*-correspondences*, which the authors expect to be the homotopy category of an ∞-category $\mathsf{Corr}^{\mathsf{E}}_S$ whenever E is an $\mathcal{A}_\infty$-ring. Setting $\mathrm{C} = \mathsf{Corr}^{\mathsf{E}}_S$, the ∞-category $\mathrm{SH}(\mathrm{C})$ in this paper corresponds to $\mathrm{DM}^{\mathsf{E}}(S)$ in [loc. cit]. We will return to this example in the next section.

5. Module categories over regular schemes

In this section we show that the hypotheses of Theorem 3.5 are satisfied for module categories over a field k, and more generally for module categories over regular k-schemes.

5A. ***The case of fields.*** We start by verifying that the projection formula holds at a field k. In this case, we can use the following computation to reduce to the case of compact-rigid generation.

Lemma 5.1. *Suppose we have an adjunction of symmetric monoidal ∞-categories*

$$F : \mathrm{C} \rightleftarrows \mathsf{D} : G$$

such that F is strongly symmetric monoidal. Let $1 \in \mathrm{C}$ denote the unit object of C. *If $\mathsf{E} \in \mathrm{C}$ is a strongly dualizable object, then the map $c\colon\ GF(1) \otimes \mathsf{E} \to GF(\mathsf{E})$ is an equivalence.*

Proof. This follows from a standard computation: let $\mathsf{E}' \in \mathrm{C}$ be arbitrary; then we have a string of equivalences

$$\begin{aligned}\mathrm{Maps}_{\mathsf{C}}(\mathsf{E}', GF(1) \otimes \mathsf{E}) &\simeq \mathrm{Maps}_{\mathsf{C}}(\mathsf{E}' \otimes \mathsf{E}^{\vee}, GF(1))\\ &\simeq \mathrm{Maps}_{\mathsf{D}}(F(\mathsf{E}' \otimes \mathsf{E}^{\vee}), F(1))\\ &\simeq \mathrm{Maps}_{\mathsf{D}}(F(\mathsf{E}') \otimes F(\mathsf{E})^{\vee}, F(1))\\ &\simeq \mathrm{Maps}_{\mathsf{D}}(F(\mathsf{E}'), F(\mathsf{E}))\\ &\simeq \mathrm{Maps}_{\mathsf{C}}(\mathsf{E}', GF(\mathsf{E})),\end{aligned}$$

which shows the claim. □

5A1. Thus, if $\mathrm{SH}(S)$ is generated by strongly dualizable objects, it follows that the projection formula holds:

Theorem 5.2. *Let k be a field. Suppose that ℓ is a prime which is coprime to the exponential characteristic e of k and let $\mathscr{M}$ be a motivic module category on k. Then we have the following equivalences of presentably symmetric monoidal stable ∞-categories*:

$$L_{(\ell)}(\mathscr{M}(k)) \simeq \mathrm{Mod}_{L_{(\ell)}\gamma_*\gamma^*(\mathbf{1}_S)}(\mathrm{SH}(k)),$$
$$\mathscr{M}(k)\left[\tfrac{1}{e}\right] \simeq \mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})[\frac{1}{e}]}(\mathrm{SH}(k)).$$

Proof. In light of Theorem 3.5 we need to verify the appropriate projection formulas. By assumption, the functor γ_* preserves sifted colimits and thus the functors $\gamma_*\gamma^*(\mathbf{1}_S) \otimes (\,\cdot\,)$ and $\gamma_*\gamma^*(\,\cdot\,)$ do as well. Now Lemma 5.1 tells us that the projection formula holds for strongly dualizable objects in $\mathrm{SH}(k)_{(\ell)}$. Thus, we will be done if we can prove that the inclusion of (2.2), $\mathrm{SH}^{\mathsf{rig}}(k)_{(\ell)} \subseteq \mathrm{SH}^{\omega}(k)_{(\ell)}$, is an equality. This amounts to showing that $\mathrm{SH}(k)_{(\ell)}$ is in fact generated by sifted colimits by strongly dualizable objects. But this follows by Lemma 2.3, which also verifies the theorem for the e-inverted case. □

5A2. We now obtain the following extension of [Röndigs and Østvær 2008, Theorem 1; Hoyois et al. 2017, Theorem 5.8; Garkusha 2019, Theorem 5.3; Bachmann and Fasel 2017, Lemma 5.3].

Corollary 5.3. *Let k be a field of exponential characteristic e and let $\gamma_{\mathrm{C}}\colon \mathsf{Sm}_k \to \mathrm{C}$ be a correspondence category over k. Then there is an equivalence of presentably* *symmetric monoidal stable ∞-categories*

$$\mathrm{SH}(\mathrm{C})\left[\tfrac{1}{e}\right] \simeq \mathrm{Mod}_{\gamma_{C*}\gamma_C^*(\mathbf{1})[\frac{1}{e}]}(\mathrm{SH}(k)).$$

5B. *The case $\mathscr{S} = \mathsf{Reg}_k$.* Following [Cisinski and Déglise 2015] we can extend the previous result to the category Reg_k of finite-dimensional Noetherian schemes that are regular over a field k, provided that we impose some additional assumptions on $\mathscr{M}$. For the rest of this section, we will therefore assume that $\mathscr{M}$ is a motivic module category which in addition satisfies the following property:

- The premotivic category $\mathscr{M}$ satisfies localization (Definition 2.6) and continuity (Definition 2.7).

Lemma 5.4. *Suppose that $f : T \to S$ is a morphism in Reg_k. In the following cases, the transformation*

$$f^*\gamma_* \to \gamma_* f^*$$

is an equivalence:

(1) *The scheme T is an inverse limit $\varprojlim_\alpha T_\alpha$ of S-schemes T_α such that the transition maps $T_\alpha \to T_\beta$ are dominant, affine, and smooth.*

(2) *The map f is a closed immersion and $S \cong \varprojlim_\alpha S_\alpha$, where each S_α is a smooth, separated k-scheme of finite type with flat affine transition maps.*

Proof. Under the continuity and localization assumption on $\mathscr{M}$, the proof in [Cisinski and Déglise 2015, Lemma 3.20] for the case of $\mathscr{M} = \mathrm{DM}(\,\cdot\,; R)$ applies verbatim. □

5B1. We now have the following extension of Theorem 5.2.

Theorem 5.5. *Let k be a field of exponential characteristic e, and let $\mathscr{M}$ be a motivic module category on Reg_k. Then the functor $\gamma^* \colon \mathrm{SH} \to \mathscr{M}$ induces a canonical equivalence*

$$\mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})[\frac{1}{e}]}(\mathrm{SH}(\,\cdot\,)) \xrightarrow{\simeq} \mathscr{M}\left[\tfrac{1}{e}\right]$$

of premotivic categories on Reg_k.

Proof. After Theorem 3.5, our goal is to verify that $(\mathrm{SH}|_{\mathsf{Reg}_k}, \mathscr{M})$ satisfies the projection formula. Suppose that $S \in \mathsf{Reg}_k$, and let $\mathsf{E} \in \mathrm{SH}(S)$. We claim that the map

$$\gamma_*\gamma^*(\mathbf{1}_S) \otimes \mathsf{E} \to \gamma_*\gamma^*(\mathsf{E}) \tag{5.6}$$

is an equivalence. To show this, we closely follow the logic of [Cisinski and Déglise 2015, Theorem 3.1].

First, assume that S is an essentially smooth scheme over a field. For each $x \in S$, we write S_x for the localization of S at x. Then the family of functors

$$\{\mathrm{SH}(S) \to \mathrm{SH}(S_x)\}$$

is conservative by [Cisinski and Déglise 2019, Proposition 4.3.9]. Hence, we are reduced to proving that the map (5.6) is an equivalence in the case S is furthermore *local*. In this case, let $i : x \hookrightarrow S_x$ be the closed point and write $j : U_x \to S_x$ for the open complement. By our assumption on S, U_x has dimension $< \dim S$. We consider the following commutative diagram, where the rows are cofiber sequences:

$$\begin{array}{ccccc} j_!(j^*\gamma_*\gamma^*(\mathbf{1}_S) \otimes j^*\mathsf{E}) & \longrightarrow & \gamma_*\gamma^*(\mathbf{1}_S) \otimes \mathsf{E} & \longrightarrow & i_*(i^*\gamma_*\gamma^*(\mathbf{1}_S) \otimes i^*\mathsf{E}) \\ \downarrow & & \downarrow & & \downarrow \\ j_!j^*\gamma_*\gamma^*(\mathsf{E}) & \longrightarrow & \gamma_*\gamma^*(\mathsf{E}) & \longrightarrow & i_*i^*\gamma_*\gamma^*(\mathsf{E}) \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle f_2} \\ j_!\gamma_*\gamma^*j^*\mathsf{E} & \longrightarrow & \gamma_*\gamma^*\mathsf{E} & \longrightarrow & i_*\gamma_*\gamma^*i^*\mathsf{E} \end{array} \tag{5.7}$$

Now:

- The left vertical composite is an equivalence because (1) j_* commutes with γ_* by definition of a morphism of premotivic categories, and (2) by the induction hypothesis.
- The right vertical composite is an equivalence using (1) Lemma 5.4(2) and (2) the case of fields, Theorem 5.2.

It therefore remains to show that f_1 and f_2 are equivalences:

- The map f_1 is an equivalence because j_* commutes with γ_*.
- That f_2 is an equivalence follows from Lemma 5.4(2).

Now, following the "*General case*" of [Cisinski and Déglise 2015], we explain how the bootstrap to regular k-schemes work. By continuity (appealing to [Cisinski and Déglise 2019, Proposition 4.3.9] again), we may again assume that S is a *Henselian local* regular k-scheme. As explained in [loc. cit.], there is a sequence of regular Noetherian k-schemes

$$T \xrightarrow{f} S' \xrightarrow{q} S$$

such that the following hold:

- The scheme S' has infinite residue field and the functor $q^* \colon \mathrm{SH}(S)\left[\frac{1}{e}\right] \to \mathrm{SH}(S')\left[\frac{1}{e}\right]$ is conservative.
- The scheme T is the ∞-*gonflement* of $\Gamma(S', \mathscr{O}_{S'})$ [Cisinski and Déglise 2015, Definition 3.21] and the functor $f^* \colon \mathrm{SH}(S')\left[\frac{1}{e}\right] \to \mathrm{SH}(T)\left[\frac{1}{e}\right]$ is conservative.

- Both f and q satisfy the hypotheses of Lemma 5.4.1, and thus f^* and q^* commute with γ_*.

Hence, to check that the map (5.6) is an equivalence it suffices to check that it is an equivalence after applying $(qf)^*$. Since T is, by construction, the spectrum of a filtered union of its smooth subalgebras, we invoke continuity of SH to conclude. □

5B2. Lastly, we provide the following class of examples of motivic module categories for which localization and continuity holds. We will make the following assumption:

- For a base scheme S and an $\mathcal{A}_\infty$-ring spectrum $\mathsf{E} \in \mathrm{SH}(S)$, there exists an ∞-category $\mathsf{Corr}^{\mathsf{E}}_S$ such that its homotopy category is the hSpc-enriched category $\mathrm{h}\mathsf{Corr}^{\mathsf{E}}_S$ of [Elmanto et al. 2020].

With this assumption in play, any motivic $\mathcal{A}_\infty$-ring spectrum E gives rise to the motivic module category DM^{E} as explained in Example 4.31 and [Elmanto et al. 2020]. While this makes the next results conditional, we will explain unconditional instances of these results in Example 5.12.

Proposition 5.8. *Let $\mathscr{S} \subseteq \mathrm{Sch}_S$. Then, for any $\mathcal{A}_\infty$-ring spectrum $\mathsf{E} \in \mathrm{SH}(S)$, the premotivic category $\mathrm{DM}^{\mathsf{E}} \colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ satisfies continuity for dominant affine morphisms.*

Proof. We first claim the analog of [Cisinski and Déglise 2019, Proposition 9.3.9] for E-correspondences. Let $(X_i)_{i \in I}$ be a cofiltered diagram of separated S-schemes of finite type with affine dominant transition morphisms. Let $X = \varprojlim_i X_i$, which is assumed to exist in Sch_S and is assumed to be Noetherian. Then we claim that for any separated S-scheme Y of finite type, the map

$$\operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathsf{Corr}^{\mathsf{E}}_S(X_i, Y) \to \mathsf{Corr}^{\mathsf{E}}_S(X, Y) \tag{5.9}$$

is an equivalence.

To do so, we use the dual of [Elmanto et al. 2019a, Lemma 4.1.26]. Denote by c_{X_i} and c_X the filtered posets of reduced subschemes of $X_i \times_S Y$ and $X \times_S Y$ which are finite and universally open over X_i and X, respectively. Furthermore, we denote by $\mathrm{Sub}(c_X)$ the poset of full subposets of c_X. We then have a functor $K \colon I \to \mathrm{Sub}(c_X)$ given by $i \mapsto K_i := c_{X_i}$, where c_{X_i} is regarded as a full subposet in the obvious way. By continuity of SH, the functor $\mathsf{E}^{\mathrm{BM}}(\,\cdot\,/X) \colon c_X \to \mathsf{Spc}$ of Borel–Moore E-homology spaces [Elmanto et al. 2020, §2] restricts to a functor $\mathsf{E}^{\mathrm{BM}}(\,\cdot\,/X_i) \colon c_{X_i} \to \mathsf{Spc}$. Hence, the map (5.9) is, by [Elmanto et al. 2020, Definition 4.1.1], equal to the map

$$\operatorname*{colim}_{I^{\mathrm{op}}} \operatorname*{colim}_{c_{X_i}} \mathsf{E}^{\mathrm{BM}}(Z_i/X_i) \to \operatorname*{colim}_{Z \in c_X} \mathsf{E}^{\mathrm{BM}}(Z/X),$$

which we claim is an equivalence. The hypotheses of [Elmanto et al. 2019a, Lemma 4.1.26] follow easily (under the hypotheses that the transition maps are affine and dominant) by [Cisinski and Déglise 2019, Propositions 8.3.6 and 8.3.9]. Hence, the desired claim follows. The rest of the proof follows as in the case of DM from [Cisinski and Déglise 2019, Theorem 11.1.24]. □

Proposition 5.10. *Let k be a field and let $\mathsf{E} \in \mathrm{SH}(k)$ be an $\mathcal{A}_\infty$-ring spectrum. Then the premotivic category $\mathrm{DM}^{\mathsf{E}} : \mathscr{S}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ satisfies (Loc_i) whenever i is a closed immersion of regular schemes.*

Proof. Since DM^{E} is constructed from Nisnevich local objects, it is Nisnevich separated. By [Cisinski and Déglise 2019, Proposition 6.3.14], it has the weak localization property; i.e., it has (Loc_i) for any closed immersion with smooth retractions. Arguing as in [Cisinski and Déglise 2019, Corollary 6.3.15], it has the localization property with respect to any closed immersion between smooth schemes. The rest of the argument then follows as in [Cisinski and Déglise 2015, Proposition 3.12], which uses the continuity results established in Proposition 5.8 as above. □

5B3. From this we conclude:

Corollary 5.11. *Let k be a field of exponential characteristic e and let $\mathsf{E} \in \mathrm{SH}(k)$ be an $\mathcal{A}_\infty$-ring spectrum. Then we have a canonical equivalence*

$$\mathrm{DM}^{\mathsf{E}}\left[\tfrac{1}{e}\right] \simeq \mathrm{Mod}_{\gamma_*\gamma^*(\mathbf{1})[\frac{1}{e}]}(\mathrm{SH}(\,\cdot\,))$$

of premotivic categories on Reg_k.

Example 5.12. As explained in [Elmanto et al. 2020, §4.1.10], the hypothetical ∞-category $\mathsf{Corr}^{\mathsf{E}}_S$ is equivalent to $\mathsf{hCorr}^{\mathsf{E}}_S$ whenever S is essentially smooth over a perfect field k and E is pulled back from the heart of the effective homotopy t-structure $\mathrm{SH}(k)^{\mathrm{eff},\heartsuit}$ over k. Hence, Corollary 5.11 holds unconditionally whenever E is pulled back from the prime subfield of k and lies in the heart of the effective homotopy t-structure there.

Examples of such spectra include the motivic cohomology spectrum $\mathrm{H}\mathbf{Z}$ and its Milnor–Witt counterpart $\mathrm{H}\widetilde{\mathbf{Z}}$. Furthermore, in [Elmanto et al. 2020, Propositions 4.3.6 and 4.3.19] it is proved that $\mathrm{DM}^{\mathrm{H}\mathbf{Z}}(S) \simeq \mathrm{DM}(S)$ and $\mathrm{DM}^{\mathrm{H}\widetilde{\mathbf{Z}}}(S) \simeq \widetilde{\mathrm{DM}}(S)$ whenever S is essentially smooth over a Dedekind domain or essentially smooth over a perfect field, respectively [Elmanto et al. 2020, Propositions 4.3.8 and 4.3.19]. By the continuity result of Proposition 5.8 we can enhance the comparison results for DM to regular schemes over fields. While $\widetilde{\mathrm{DM}}(S)$ is not defined outside of smooth schemes over perfect fields, Corollary 5.11 promotes the comparison results between $\widetilde{\mathrm{DM}}$ and modules over $\mathrm{H}\widetilde{\mathbf{Z}}$ of [Garkusha 2019; Bachmann and Fasel 2017] at least to smooth schemes over fields. We contend, however, that $\mathrm{DM}^{\mathrm{H}\widetilde{\mathbf{Z}}}(S)$ is a decent definition for $\widetilde{\mathrm{DM}}(S)$ in general.

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The Omega spectrum for mod 2 *KO*-theory

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The 8-periodic theory that comes from the *KO*-theory of the mod 2 Moore space is the same as the real first Morava *K*-theory obtained from the homotopy fixed points of the $\mathbb{Z}/(2)$ action on the first Morava *K*-theory. The first Morava *K*-theory, $K(1)$, is just mod 2 *KU*-theory. We compute the homology Hopf algebras for the spaces in this Omega spectrum.

1. Introduction

We have stable maps $2 : S^0 \to S^0$ and $\eta : S^1 \to S^0$ and we get a stable diagram

$$\begin{array}{ccccc}
S^1 & \xrightarrow{2} & S^1 & \longrightarrow & \Sigma^1 M \\
\downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} \\
S^0 & \xrightarrow{2} & S^0 & \longrightarrow & M \\
\downarrow & & \downarrow & & \downarrow \\
N & \xrightarrow{2} & N & \longrightarrow & NM
\end{array}$$

with M the mod 2 Moore space and N and NM the appropriate cofibers.

If we smash this diagram with connective *K*-theory, *bo*, and then only look at the low dimensional spaces in the Omega spectrum where we get periodicity, we get the diagram of fibrations

$$\begin{array}{ccccccc}
\underline{KO}_{i+1} & \xrightarrow{2} & \underline{KO}_{i+1} & \xrightarrow{\rho} & \underline{KR(1)}_{i+1} & \xrightarrow{\delta} & \underline{KO}_{i+2} \\
\downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} \\
\underline{KO}_{i} & \xrightarrow{2} & \underline{KO}_{i} & \xrightarrow{\rho} & \underline{KR(1)}_{i} & \xrightarrow{\delta} & \underline{KO}_{i+1} \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\
\underline{KU}_{i} & \xrightarrow{2} & \underline{KU}_{i} & \xrightarrow{\rho} & \underline{K(1)}_{i} & \xrightarrow{\delta} & \underline{KU}_{i+1} \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\
\underline{KO}_{i+2} & \xrightarrow{2} & \underline{KO}_{i+2} & \xrightarrow{\rho} & \underline{KR(1)}_{i+2} & \xrightarrow{\delta} & \underline{KO}_{i+3}
\end{array} \tag{1.1}$$

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The $\underline{KU}_i$ are the usual 2-periodic spaces for complex K-theory and the $\underline{KO}_i$ the 8-periodic spaces for real K-theory. The $\underline{K(1)}_i$ are 2-periodic and they are just the mod 2 KU-theory, or the first Morava K-theory. The spaces of interest are the $\underline{KR(1)}_i$, which are simultaneously the real version of the first Morava K-theory (see [Hu and Kriz 2001, Theorem 3.32]) and the mod 2 KO-theory.

Our interest is in computing the Hopf algebra $H_*(\underline{KR(1)}_i)$. We work with $\mathbb{Z}/(2)$ coefficients in homology. Our notation is that P is a polynomial algebra, E is an exterior algebra, $TP_4(x)$ is $P(x)/(x^4)$. The Frobenius F is just the map that takes x to x^2. The Verschiebung V is the dual of the Frobenius and gives us the coproduct structure on our Hopf algebras. Our notation is such that the subscript of an element denotes the degree it resides in. Keep in mind that the k used as a subscript for the tensor product in the main theorem is not a field, but an index. This is always the case throughout the paper when the tensor symbol has a subscript.

Our main theorem is easy to state.

Theorem 1.2. *The homology of the connected component of $\underline{KR(1)}_i$ is as follows. If the Verschiebung isn't described, it is zero. The index k runs over all $k > 0$.*

$i=0$	$E(x_k)\otimes_k P(y_{4k+2})$	$V(x_{2k})=x_k$
$i=1$	$P(x_{2k+1})\otimes_k P(y_{4k+2})$	$V(y_{4k+2})=x_{2k+1}$
$i=2$	$P(x_{8k+2})\otimes_k P(y_{4k+3})$	
$i=3$	$E(x_{8k+3})\otimes_k P(y_{8k+4})$	
$i=4$	$E(x_{4k})\otimes_k E(y_{8k+5})$	$V(x_{8k})=x_{4k}$
$i=5$	$E(x_{4k+1})\otimes_k E(y_{2k})$	$V(y_{4k})=y_{2k},\ V(y_{8k+2})=x_{4k+1}$
$i=6$	$\otimes_k TP_4(x_k)$	$V(x_{2k})=x_k$
$i=7$	$E(x_{2k})\otimes_k P(y_{2k+1})$	$V(x_{4k})=x_{2k}$

Remark 1.3. We began this research trying to give meaningful names to all of the algebra generators. Eventually, it became clear that it was easier to compute just using the degrees of the generators. We do know good names for all of the generators of $H_*(\underline{KO}_*)$, $H_*(\underline{KU}_*)$, and $H_*(\underline{K(1)}_*)$, and we are able to relate our poorly named generators to generators we are more familiar with, thus solving the naming problem after the fact. In order to be explicit about these results, we have to write down the known homologies first. We put that off until the next section. We can give the nonexplicit answer here.

Theorem 1.4. *The maps of the connected components $\underline{KO}_i \xrightarrow{\rho} \underline{KR(1)}_i \xrightarrow{\delta} \underline{KO}_{i+1}$ give rise to maps on homology*

$$H_*(\underline{KO}_i) \xrightarrow{\rho_*} H_*(\underline{KR(1)}_i) \xrightarrow{\delta_*} H_*(\underline{KO}_{i+1})$$

that are exact in the category of Hopf algebras at the middle term. For $i = 1, 2, 5$ and 6, this is a short exact sequence of Hopf algebras. In the case $i = 0$ we have a *long exact sequence*

$$H_*(\underline{KO}_1) \xrightarrow{\eta_*} H_*(\underline{KO}_0) \xrightarrow{2_*} H_*(\underline{KO}_0) \xrightarrow{\rho_*} H_*(\underline{KR(1)}_0) \xrightarrow{\delta_*} H_*(\underline{KO}_1) \xrightarrow{\rho_*} H_*(\underline{KU}_1).$$

In the above diagram there are 20 distinct spaces as i varies, KU and $K(1)$ are 2-periodic, and KO and $KR(1)$ are 8-periodic. We know the homology of 12 of them. It is the other 8 associated with $KR(1)$ that we are interested in. Counting the suspension maps, there are 98 maps to evaluate, 48 of them involving the $KR(1)$ spaces. For each map, there is a spectral sequence, and 56 of them involve the $KR(1)$ spaces. It is not necessary to know all of them to get our main results, but it is often helpful. Because I want to have access to this information, it has been written up as a supplement to this paper. Once you know the homology of all the spaces and also know the maps, it is fairly easy to figure out how all the spectral sequences behave. Also, for my personal benefit to have a reference, the long exact sequences of homotopy groups have been put in the appendix as well. In this paper we state, compute, and use, only what we need, but we assume results not involving the $\underline{KR(1)}_i$.

The spaces $\underline{KR(1)}_i$ have been around for a long time. When I tried to find a reference for the homotopy groups, the experts informed me that they were known in the 1960s to Mahowald and that there wasn't a reference because everyone already knew them. What might be new is that the $\underline{KR(1)}_i$ are also the real first Morava $K(1)$-theory. This comes from the work of Hu and Kriz [2001], where they compute the homotopy of all of the real Morava $K(n)$, $KR(n)$. This project got started because I thought $KR(2)$ would be interesting but that I should quickly take a look at $KR(1)$ first. From the point of view of personal satisfaction, the homology, $H_*(\underline{KR(1)}_6)$, was both the most difficult to compute and the most interesting. In the beginning, motivation was easy. I was hoping to find something interesting. After the fact, it isn't clear how to motivate. However, a quick look at the appendix might make this paper look elegant.

In Section 2 we give the homology of the spaces that are known already as well as state the details of Theorem 1.4. In Section 3 we state the spectral sequences we use and discuss how Hopf algebras help us with our computations. After that, each section is just the computation of some $H_*(\underline{KR(1)}_i)$. They are somewhat in order except that to do $H_*(\underline{KR(1)}_6)$, we need to have $H_*(\underline{KR(1)}_7)$ first, which is computed from $H_*(\underline{KR(1)}_0)$.

2. Connecting to known results

Our preferred generators for $H_*(\underline{KU}_*)$ and $H_*(\underline{KO}_*)$ come from Hopf rings. They are given elegant descriptions in [Cowen Morton and Strickland 2002]. In [Kitchloo and Wilson 2007, Section 25], there is an alternative Hopf ring description for

$H_*(\underline{KO}_*)$ and one can read off that for $H_*(\underline{KU}_*)$ from [Ravenel and Wilson 1977]. We do not write down these descriptions in this paper. It is enough to know they have nice Hopf ring names. In the case of $H_*(\underline{KR(1)}_*)$, we do not get Hopf ring names because $KR(1)$ is not a ring spectrum.

We give the descriptions of the homologies we need in this paper.

Theorem 2.1. *The homology of the connected component of $\underline{KU}_i$ is as follows. If the Verschiebung isn't described, it is zero. The index k runs over all $k > 0$.*

$$\begin{array}{lll} i=0 & \bigotimes_k P(x_{2k}) & V(x_{4k}) = x_{2k} \\ i=1 & \bigotimes_k E(x_{2k+1}) & \end{array}$$

Theorem 2.2. *The homology of the connected component of $\underline{KO}_i$ is as follows. If the Verschiebung isn't described, it is zero. The index k runs over all $k > 0$.*

$$\begin{array}{lll} i=0 & \bigotimes_k P(x_k) & V(x_{2k}) = x_k \\ i=1 & \bigotimes_k P(x_{2k+1}) & \\ i=2 & \bigotimes_k P(x_{4k+2}) & \\ i=3 & \bigotimes_k E(x_{4k+3}) & \\ i=4 & \bigotimes_k P(x_{4k}) & V(x_{8k}) = x_{4k} \\ i=5 & \bigotimes_k E(x_{4k+1}) & \\ i=6 & \bigotimes_k E(x_{2k}) & V(x_{4k}) = x_{2k} \\ i=7 & \bigotimes_k E(x_k) & V(x_{2k}) = x_k \end{array}$$

Theorem 2.3. *The homology of the connected component of $\underline{K(1)}_i$ is as follows. If the Verschiebung isn't described, it is zero. The index k runs over all $k > 0$.*

$$\begin{array}{lll} i=0 & TP_4(x_{4k+3}) \otimes_k E(y_{4k}) \otimes_k E(z_{8k+2}) & \begin{array}{c} V(y_{8k}) = y_{4k},\ V(y_{16k+4}) = z_{8k+2}, \\ V(y_{16k+12}) = (x_{4k+3})^2 \end{array} \\ i=1 & E(x_{4k+1}) \otimes_k P(y_{4k+2}) & V(y_{8k+2}) = x_{4k+1} \end{array}$$

In the paper this is from, [Wilson 1984], we computed $H_*(\underline{K(n)}_*)$ for all n and all primes. Slight adjustments had to be made all along the way for $p = 2$, and it seems that they weren't all made.

In the paper, we write

$$H_*(\underline{K(1)}_0) \simeq E(x_{4k+3}) \otimes_k E(x_{2k}),$$

but we missed the extension $x_{4k+3}^2 = x_{8k+6}$. So, what is in the paper is an associated graded version. When the spectral sequence there is used to compute $H_*(\underline{K(1)}_1)$, deep down in the gruesome depths of the paper there is a d_1, so the resulting answer is correct. Explicitly, what it shows in that paper is that we need (in the notation of the paper) $(e_1 a_{(0)})^2 = b_{(0)} b_{(1)}$. The rest follows from Hopf ring considerations as our generators there all have nice Hopf ring names. Something similar happens for $K(n)$ in that paper, but again, only for $p = 2$.

We can now use these results to connect to our new results.

Theorem 2.4. *The exactness at the middle term of*

$$H_*(\underline{KO}_i) \to H_*(\underline{KR(1)}_i) \to H_*(\underline{KO}_{i+1})$$

of Theorem 1.4 is given explicitly as follows, where, if not described, the element maps to zero. The index j runs over all $j > 0$.

$$i = 0 \qquad \bigotimes_j P(y_j) \xrightarrow{y_j \mapsto z_j} E(z_j) \otimes_j P(zz_{4j+2}) \xrightarrow{zz_{4j+2} \mapsto (w_{2j+1})^2} \bigotimes_j P(w_{2j+1})$$

$$i = 1 \qquad \bigotimes_j P(y_{2j+1}) \xrightarrow{y_{2j+1} \mapsto z_{2j+1}} P(z_{2j+1}) \otimes_j P(zz_{4j+2}) \xrightarrow{zz_{4j+2} \mapsto w_{4j+2}} \bigotimes_j P(w_{4j+2})$$

$$i = 2 \qquad \bigotimes_j P(y_{4j+2}) \xrightarrow[y_{8j+6} \mapsto (zz_{4j+3})^2]{y_{8j+2} \mapsto z_{8j+2}} P(z_{8j+2}) \otimes_j P(zz_{4j+3}) \xrightarrow{zz_{4j+3} \mapsto w_{4j+3}} \bigotimes_j E(w_{4j+3})$$

$$i = 3 \qquad \bigotimes_j E(y_{4j+3}) \xrightarrow{y_{8j+3} \mapsto z_{8j+3}} E(z_{8j+3}) \otimes_j P(zz_{8j+4}) \xrightarrow{zz_{8j+4} \mapsto w_{8j+4}} \bigotimes_j P(w_{4j})$$

$$i = 4 \qquad \bigotimes_j P(y_{4j}) \xrightarrow{y_{4j} \mapsto z_{4j}} E(z_{4j}) \otimes_j E(zz_{8j+5}) \xrightarrow{zz_{8j+5} \mapsto w_{8j+5}} \bigotimes_j E(w_{4j+1})$$

$$i = 5 \qquad \bigotimes_j E(y_{4j+1}) \xrightarrow{y_{4j+1} \mapsto z_{4j+1}} E(z_{4j+1}) \otimes_j E(zz_{2j}) \xrightarrow{zz_{2j} \mapsto w_{2j}} \bigotimes_j E(w_{2j})$$

$$i = 6 \qquad \bigotimes_j E(y_{2j}) \xrightarrow{y_{2j} \mapsto (z_j)^2} \bigotimes_j TP_4(z_j) \xrightarrow{z_j \mapsto w_j} \bigotimes_j E(w_j)$$

$$i = 7 \qquad \bigotimes_j E(y_j) \xrightarrow{y_{2j} \mapsto z_{2j}} E(z_{2j}) \otimes_j P(zz_{2j+1}) \xrightarrow{zz_{2j+1} \mapsto w_{2j+1}} \bigotimes_j P(w_j)$$

Remark 2.5. The long exact sequence for $i = 0$ of Theorem 1.4 consists of the above maps spliced together with well-understood maps that we will see throughout the paper.

3. Hopf algebras, fibrations, and spectral sequences

We need two spectral sequences. The homology version we use computes the homology of a base space from the homologies of the fiber and the total space. It is in [Moore 1961, Theorems 2.2 and 3.1]. I think of it as the bar spectral sequence, but it should perhaps be called the Moore spectral sequence. Unfortunately, Moore doesn't indulge appropriately with Hopf algebras as he clearly could have. Rothenberg and Steenrod [1965] really bring in the Hopf algebras, but neglected to do the more general case where the total space isn't contractible. Everyone seems to think they can do it by just slightly extending Rothenberg and Steenrod's proof, except those who think it is already in their paper. The cohomology version computes the cohomology of the fiber from the cohomologies of the base space and the total space. This seems to originate with Eilenberg and Moore [1966]. However, my favorite reference here is [Smith 1970] because this is where I learned to compute with Hopf algebras in these spectral sequences.

We state the two spectral sequences for the record and then discuss the use of Hopf algebras in their computations.

Proposition 3.1. *Let $F \to E \to B$ be a fibration of infinite loop spaces and maps.*

(1) *There is a first quadrant homology spectral sequence of Hopf algebras*

$$E^2_{*,*} = \mathrm{Tor}^{H_*(F)}_{*,*}(H_*(E), \mathbb{Z}/(2)) \Rightarrow H_*(B)$$

with $d_r : E_{u,v} \to E_{u-r,v+r-1}$.

(2) *There is a second quadrant cohomology spectral sequence of Hopf algebras*

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{H^*(B)}(H^*(E), \mathbb{Z}/(2)) \Rightarrow H^*(F)$$

with $d_r : E^{u,v} \to E^{u+r,v-r+1}$

Discussion of Hopf algebras,* Tor*, and differentials. Combining the above spectral sequences with Hopf algebras makes for a powerful tool. We only discuss the homology version but everything carries over to the cohomology version. The general reference for Hopf algebras is [Milnor and Moore 1965], but my computational reference is [Smith 1970].

We work with mod 2 homology throughout. The Borel structure theorem (see [Milnor and Moore 1965]) for our graded Hopf algebras over $\mathbb{Z}/(2)$ is that they are the tensor products of algebras of the form $P(x_i)$ (polynomial), $E(x_i)$ (exterior), and $TP_{2^j}(x_i) = P(x_i)/(x_i^{2^j})$ (truncated polynomial). (Recall our notation is that x_i is of degree i.) Sub-Hopf algebras of polynomial algebras must also be polynomial. In our Hopf algebras, we have $2_* = FV = VF$, where F is the Frobenius (i.e., $x \mapsto x^2$) and V is the Verschiebung (i.e., the dual of the Frobenius on cohomology). The Hopf algebra $\Gamma[x_i]$ is dual to $P(y_i)$ with y_i primitive. As such, it is $\mathbb{Z}/(2)$-free on elements $\gamma_k(x_i)$ in degree ki. As an algebra, it is an exterior algebra on the generators $\gamma_{2^j}(x_i)$ of degree $2^j i$. We have $V(\gamma_{2^{j+1}}(x_i)) = \gamma_{2^j}(x_i)$.

There are a number of situations that arise frequently in our computations. For example, we might find that we have an associated graded object that is $\bigotimes_i E(x_i)$, but we know that when the extensions are solved it must be polynomial. This becomes $\bigotimes_i P(x_{2i+1})$ for degree reasons. Similarly $\bigotimes_i E(x_{2i})$ and $\bigotimes_i E(x_{4i})$, if they are really polynomial algebras, become $\bigotimes_i P(x_{4i+2})$ and $\bigotimes_i P(x_{8i+4})$. If we have $\bigotimes_i \Gamma[x_i]$ as an associated graded object for what we know is polynomial, we get $\bigotimes_i P(x_i)$.

On the other hand, if there are no extension problems, as algebras, we have that $\bigotimes_i \Gamma[x_{2i+1}]$ is just $\bigotimes_i E(y_i)$, and $\bigotimes_i \Gamma[x_{4i+2}]$ is just $\bigotimes_i E(y_{2i})$.

If we have the differential Hopf algebra $E(x_i) \otimes P(y_{i+1})$ with $d^1(x_i) = y_{i+1}$, we know that the homology in positive degrees is zero. We are often confronted with the dual of this situation, where we have $E(x_i) \otimes \Gamma[y_{i+1}]$ with $d_1(y_{i+1}) = x_i$. Again, our homology here is zero in positive degrees. It is not always that simple though. It often happens that we have $E(x_{2i+1}) \otimes \Gamma[y_{i+1}]$ and have $d_2(\gamma_2(y_{i+1})) = x_{2i+1}$. This leaves $E(y_{i+1})$ as its homology. When this happens, we abuse notation and write

$$E(x_{2i+1}) \otimes \Gamma[y_{i+1}] \simeq E(x_{2i+1}) \otimes E(y_{i+1}) \otimes \Gamma[y_{2i+2}]$$

so we can see the differential and results more clearly. This is just the associated graded object we get from the short exact sequence of Hopf algebras

$$E(y_{i+1}) \to \Gamma[y_{i+1}] \to \Gamma[y_{2i+2}],$$

where we have written $\gamma_2(y_{i+1}) = y_{2i+2}$. Similarly, worse happens and we need

$$E(x_{4i+3}) \otimes \Gamma[y_{i+1}] \simeq E(x_{4i+3}) \otimes E(y_{i+1}) \otimes E(y_{2i+2}) \otimes \Gamma[y_{4i+4}],$$

where we have a differential taking y_{4i+4} to x_{4i+3} leaving only $E(y_{i+1}) \otimes E(y_{2i+2})$ but with $V(y_{2i+2}) = y_{i+1}$.

To deal with our spectral sequences, we must be able to evaluate Tor. The simple case of $\mathrm{Tor}_{0,*}$ is the Hopf algebra cokernel of the map $H_*(F) \to H_*(E)$. There are no differentials on this zero filtration and what remains after differentials hit it is a sub-Hopf algebra of $H_*(B)$, i.e., the image of $H_*(E) \to H_*(B)$. In general, for $\mathrm{Tor}_{i,j}$, this is our i-th filtration and an element has total degree $i + j$.

We have a few facts to accumulate.

(1) If A is the Hopf algebra kernel of the map $H_*(F) \to H_*(E)$, then the higher filtrations are given by $\mathrm{Tor}^A(\mathbb{Z}/(2), \mathbb{Z}/(2))$.

(2) Tor commutes with tensor products.

(3) $\mathrm{Tor}^{E(x_i)}(\mathbb{Z}/(2), \mathbb{Z}/(2)) = \Gamma[y_{i+1}]$ with y_{i+1} in bidegree $(1, i)$ and $\gamma_{2^j}(y_{i+1})$ in bidegree 2^j times this.

(4) $\mathrm{Tor}^{P(x_i)}(\mathbb{Z}/(2), \mathbb{Z}/(2)) = E(y_{i+1})$ with y_{i+1} in bidegree $(1, i)$.

(5) $\mathrm{Tor}^{TP_{2^k}(x_i)}(\mathbb{Z}/(2), \mathbb{Z}/(2)) = E(y_{i+1}) \otimes \Gamma[z_{2^k i+2}]$ with y_{i+1} in bidegree $(1, i)$ and $z_{2^k i+2}$ in bidegree $(2, 2^k i)$.

(6) Elements in filtrations zero and one are permanent cycles.

If the kernel A is trivial, the spectral sequence collapses and the cokernel is $H_*(B)$, giving us a short exact sequence $H_*(F) \to H_*(E) \to H_*(B)$.

Since the kernel A is a Hopf algebra, Borel's theorem applies and the above allows us to compute Tor completely. Differentials must start on the second or higher filtration and they must take generators to primitives. The primitives all live in filtrations 0, 1, or 2 and the primitives in filtrations 1 and 2 are all generators. All generators in filtrations 2 or higher are of even degree. Thus the targets of differentials must be odd degree elements in filtrations 0 or 1. A fact that we often use is that *any even degree element in filtrations* 0 *or* 1 *must survive*.

There is one more special case we need to discuss. If we have a short exact sequence $E(y_{2i}) \to TP_4(x_i) \to E(x_i)$ that takes y_{2i} to $(x_i)^2$, we can compute Tor of $TP_4(x_i)$ as above and get $E(z_{i+1}) \otimes \Gamma[w_{4i+2}]$. If we didn't know there was the square $x_i^2 = y_{2i}$ in the middle term, but thought the middle term might be $E(y_{2i}) \otimes E(x_i)$, then Tor would be $\Gamma[z_{i+1}] \otimes \Gamma[u_{2i+1}]$. If we had a reason to know that this was not correct, then $d_1(\gamma_2(z_{i+1})) = u_{2i+1}$ would leave us with the correct answer.

4. $H_*(\underline{KR(1)_0})$

We begin with the spectral sequence for

$$\underline{KO}_0 \xrightarrow{2} \underline{KO}_0 \to \underline{KR(1)}_0.$$

Computing 2_* is easy: we have

$$2_*(x_{2i}) = FV(x_{2i}) = F(x_i) = (x_i)^2 \quad \text{and} \quad 2_*(x_{2i+1}) = 0.$$

We can read off the cokernel as $\bigotimes_i E(x_i)$ and the kernel as $\bigotimes_i P(x_{2i+1})$. Computing Tor on the kernel, we get $\bigotimes_j E(y_{2j})$. Since all of these generators are in filtrations zero and one, the spectral sequence collapses. What we know at this stage is that we have

$$\bigotimes_i E(x_i) \subset H_*(\underline{KR(1)}_0), \qquad V(x_{2i}) = x_i$$

with quotient having an associated graded object $\bigotimes_j E(y_{2j})$.

We move now to a different spectral sequence, the one for

$$\underline{KO}_0 \to \underline{KR(1)}_0 \to \underline{KO}_1.$$

We have computed the image of $H_*(\underline{KO}_0) \to H_*(\underline{KR(1)}_0)$. It is just $\bigotimes_i E(x_i)$. The cokernel is the object with associated graded object $\bigotimes_i E(y_{2i})$ above. That is our zero filtration for this spectral sequence. The generators are all in even degrees and so must survive. This is all of the zeroth filtration and the zero filtration must be a sub-Hopf algebra of $H_*(\underline{KO}_1)$ which is polynomial, so the cokernel must be polynomial, and for degree reasons, this must be $\bigotimes_j P(y_{4j+2})$. This splits as algebras and coalgebras and so completes our computation.

5. $H_*(\underline{KR(1)}_1)$

We start with the spectral sequence for the fibration

$$\underline{KO}_1 \xrightarrow{2} \underline{KO}_1 \to \underline{KR(1)}_1.$$

The map 2_* is zero because all of the generators x_{2i+1} for $H_*(\underline{KO}_1)$ are primitive, so $V(x_{2i+1}) = 0$, giving $2_*(x_{2i+1}) = FV(x_{2i+1}) = 0$. The cokernel is $H_*(\underline{KO}_1) = \bigotimes_i P(x_{2i+1})$ and so is the kernel. We now know the zeroth filtration and taking Tor of the kernel, we get exterior generators y_{2i} in filtration 1. The spectral sequence collapses because all the generators are in filtrations 0 and 1. We still have extension problems though. Again, we move to the next spectral sequence for

$$\underline{KO}_1 \to \underline{KR(1)}_1 \to \underline{KO}_2.$$

We have computed the image of $H_*(\underline{KO}_1) \to H_*(\underline{KR(1)}_1)$. It is just $\bigotimes_i P(x_{2i+1})$. There is no kernel, so the spectral sequence collapses and is just the cokernel in the zeroth filtration. This becomes a short exact sequence of Hopf algebras

$$H_*(\underline{KO}_1) \to H_*(\underline{KR(1)}_1) \to H_*(\underline{KO}_2).$$

But this is just

$$\bigotimes_i P(x_{2i+1}) \to H_*(\underline{KR(1)}_1) \to \bigotimes_i P(y_{4i+2})$$

and so splits as algebras, giving us most of our answer. There is an extension problem to solve to get $V(y_{4i+2}) = x_{2i+1}$.

For that we use the spectral sequence for

$$\underline{KR(1)}_0 \to * \to \underline{KR(1)}_1.$$

Computing Tor of $H_*(\underline{KR(1)}_0)$ we get

$$\Gamma[w_k] \otimes_k E(ww_{4k+3}).$$

We should note that we have to use the $\mathbb{Z}/(2)$ in degree zero for $H_0(\underline{KR(1)}_0)$ to get the w_1 above.

This is way too big. Remember, we know the answer as algebras here. To get this down to size, we must take the first possible differential, i.e., we must have $d_3(\gamma_4(w_k)) \neq 0$. This element is degree $4k$ (in the fourth filtration) so the differential hits an element in the first filtration in degree $4k-1$. There are two possibilities, but it must hit one of them, and we don't need to know which just yet. All that is left after these differentials is an exterior algebra $\bigotimes_i E(z_i)$ with generators in filtration 1 and an exterior algebra $\bigotimes_i E(\gamma_2(w_i))$ with generators in filtration 2. This is precisely the size of the known answer so these differentials must indeed happen.

We know that the answer is polynomial, so the Frobenius must be injective. The Frobenius cannot raise filtration so the injective Frobenius on the first filtration gives us $\bigotimes_i P(z_{2i+1})$, forcing (to get the correct answer) the Frobenius to inject on the second filtration to get $\bigotimes_i P(w_{4i+2})$. The only ambiguity in the first filtration is about which elements in degrees $4i+3$ have survived. We know that the element in degree $2i+1$ in the first filtration must square to the element in degree $4i+2$, and this is unambiguously $x_{4i+2} = V(\gamma_2(x_{4i+2}))$. But we know that we must have $\gamma_2(x_{4i+2}) = (\gamma_2(x_{2i+1}))^2$ because of the injectivity of F. But now we have just computed VF on $\gamma_2(x_{2i+1})$ and found it nonzero. Consequently, $VF = FV$ must also be nonzero so that V is nonzero. We get our result that V of every generator of $P(y_{4i+2})$ is a generator of $P(x_{2i+1})$ as desired.

6. $H_*(\underline{KR(1)}_2)$

We start with the spectral sequence for

$$\underline{KO}_2 \xrightarrow{2} \underline{KO}_2 \to \underline{KR(1)}_2.$$

The map 2_* is zero because all of the generators x_{4i+2} for $H_*(\underline{KO}_2)$ are primitive, so $V(x_{4i+2}) = 0$, giving $2_*(x_{4i+2}) = FV(x_{4i+2}) = 0$. The cokernel is $H_*(\underline{KO}_2) = \bigotimes_i P(x_{4i+2})$ and so is the kernel. We now know the zeroth filtration and taking Tor of the kernel, we get $\bigotimes_i E(y_{4i+3})$. The spectral sequence collapses because all the generators are in filtrations 0 and 1. We still have extension problems though.

What we have from the spectral sequence is the short exact sequence

$$\bigotimes_i P(x_{4i+2}) \to H_*(\underline{KR(1)}_2) \to \bigotimes_i E(y_{4i+3}).$$

There is an extension problem we need to solve, namely $(y_{4i+3})^2$ from filtration 1 is x_{8i+6} in filtration 0. Once this is done, we would have the algebra structure.

To solve this problem we look at the spectral sequence for

$$\underline{KR(1)}_2 \xrightarrow{\eta} \underline{KR(1)}_1 \to \underline{K(1)}_1.$$

We have maps

$$H_*(\underline{KR(1)}_2) \to P(z_{2i+1}) \otimes_i P(zz_{4i+2}) \to E(w_{4i+1}) \otimes_i P(ww_{4i+2}).$$

Our calculation so far shows that $H_*(\underline{KR(1)}_2)$ is generated by primitives. We know from our computation of $H_*(\underline{KR(1)}_1)$ that $V(zz_{4i+2}) = z_{2i+1}$, so all the primitives in $H_*(\underline{KR(1)}_1)$ are in $\bigotimes_i P(z_{2i+1})$. Since primitives map to primitives, we see that $\bigotimes_i P(zz_{4i+2})$ is in the cokernel. It is even degree and in filtration zero so is a subalgebra of $H_*(\underline{K(1)}_1)$, so it must be our $\bigotimes_i P(ww_{4i+2})$ in our known answer. This accounts for all of the even degree generators and squares in $H_*(\underline{K(1)}_1)$.

If the element y_{4i+3} from $H_*(\underline{KR(1)}_2)$ is in the kernel, then Tor gives rise to an element in filtration 1 of degree $4i+4$. This element would have to survive, but we have all of the even degree generators and squares we need, so y_{4i+3} maps to z_{4i+3} because it is the only primitive in that degree. However, z_{4i+3} is a polynomial generator, so y_{4i+3} must also be a polynomial generator, solving our extension problem.

We can go one step further. If x_{8i+2} doesn't map to $(z_{4i+1})^2$, this last element would be even degree in the cokernel where we don't need any more even degree elements, so it does map accordingly. We get a rare short exact sequence:

$$H_*(\underline{KR(1)}_2) \to H_*(\underline{KR(1)}_1) \to H_*(\underline{K(1)}_1).$$

7. $H_*(\underline{KR(1)}_3)$

We start with the spectral sequence for

$$\underline{KR(1)}_2 \to * \to \underline{KR(1)}_3.$$

Since $H_*(\underline{KR(1)}_2) \simeq P(x_{8i+2}) \otimes_i P(x_{4i+3})$, computing Tor is easy: it is just

$$E(x_{8i+3}) \otimes_i E(x_{4i})$$

and since the generators are all in filtration 1, it collapses. All we have left are extension problems.

Next we use the spectral sequence for

$$\underline{KO}_3 \xrightarrow{2} \underline{KO}_3 \to \underline{KR(1)}_3.$$

The homology $H_*(\underline{KO}_3)$ is generated by primitives, so 2_* is zero. We get that the cokernel is $\bigotimes_i E(x_{4i+3})$ and so is the kernel. The E^2 term of the spectral sequence is

$$E(x_{4i+3}) \otimes_i \Gamma[y_{4i}].$$

This is much too big compared with our first spectral sequence. The only way to cut it down to the right size is with

$$d_2(\gamma_2(y_{4i})) = x_{8i-1}.$$

This leaves $E(x_{8i+3}) \otimes_i E(y_{4i})$ as with the first one, but now we know that the $E(x_{8i+3})$ is the image of $H_*(\underline{KO}_3)$ in $H_*(\underline{KR(1)}_3)$ and the cokernel has an associated graded object of $\bigotimes_i E(y_{4i})$.

We can move on to the spectral sequence for

$$\underline{KO}_3 \to \underline{KR(1)}_3 \to \underline{KO}_4.$$

We just computed the cokernel. It is even degree in filtration zero and all of the elements must survive. Since this cokernel is a subalgebra of the polynomial algebra $H_*(\underline{KO}_4)$, this solves all of our extension problems, giving $(y_{4i})^2 = y_{8i}$. So we have the expected polynomial algebra $\bigotimes_i P(y_{8i+4})$, completing our computation.

8. $H_*(\underline{KR(1)}_4)$

We use the spectral sequence coming from

$$\underline{KO}_4 \xrightarrow{2} \underline{KO}_4 \to \underline{KR(1)}_4.$$

As in the $\underline{KR(1)}_0$ case, $H_*(\underline{KO}_4)$ is bipolynomial. The cokernel of 2_* is just $\bigotimes_i E(x_{4i})$ and the kernel is $P(x_{8i+4})$. We take Tor of this to get exterior generators y_{8i+5} in filtration 1. Since all our generators are in filtrations zero and one, the spectral sequence collapses. For purely degree reasons, there can be no extension problems given that we know $\bigotimes_i E(x_{4i})$ is a subalgebra.

9. $H_*(\underline{KR(1)}_5)$

We use the spectral sequence for the fibration

$$\underline{KR(1)}_4 \to * \to \underline{KR(1)}_5.$$

Since $H_*(\underline{KR(1)}_4) \simeq E(x_{4k}) \otimes_k E(y_{8k+5})$, Tor is

$$\Gamma[z_{4k+1}] \otimes_k \Gamma[w_{8k+6}].$$

The only possible sources for differentials are in (total) degrees divisible by 4, but the only odd degree primitives are in bidegree $(1, 4k)$, total degree $4k+1$, so there can be no differentials (lowering total degree by 1). Furthermore, there are no algebra extension problems. In filtration one there are only elements z_{4k+1} and w_{8k+6}, so there is nothing for them to square to. In filtration two, the elements are in degrees $8k+2$ and $16k+12$ and again there are no elements in filtrations one or two to square to. Continue inductively on filtration. The degrees never work out to have extensions. This spectral sequence gives a complete description of V as well.

10. $H_*(\underline{KR(1)}_7)$

Note that we have skipped $H_*(\underline{KR(1)}_6)$. It is the hardest to compute and all our previous techniques failed us. We need $H_*(\underline{KR(1)}_7)$ to solve the problems with $H_*(\underline{KR(1)}_6)$.

We use the cohomology spectral sequence for the fibration

$$\underline{KR(1)}_7 \to \underline{KO}_0 \xrightarrow{2} \underline{KO}_0.$$

The homology of $\underline{KO}_0$ is bipolynomial with $H_*(\underline{KO}_0) \simeq \bigotimes_i P(x_i)$ and $V(x_{2i}) = x_i$. So we get that $H^*(\underline{KO}_0)$ is the same. Evaluating 2^* gives $2^*(x_{2i}) = FV(x_{2i}) = F(x_i) = (x_i)^2$. The cokernel is $\bigotimes_i E(x_i)$ with V as before. Since $V(x_{2i+1}) = 0$, the kernel is $\bigotimes_i P(x_{2i+1})$. Tor of the kernel is $\bigotimes_i E(w_{2i})$ with generators in the first filtration. Since all of the generators are in the first 2 filtrations, the spectral sequence collapses. Since we know the V on filtration zero ($\bigotimes_i E(x_i)$), we can dualize and we get that the homology has $\bigotimes_i P(y_{2i+1})$ in it. There is the $\bigotimes_i E(ww_{2i})$ (dual to $\bigotimes_i E(w_{2i})$) as well, but it could have extension problems we need to solve.

To show that the $\bigotimes_i E(ww_{2i})$ really is an exterior algebra, we take a quick look at the homology spectral sequence for

$$\underline{KO}_7 \xrightarrow{2} \underline{KO}_7 \to \underline{KR(1)}_7.$$

The first map is zero because $H_*(\underline{KO}_7) \simeq \bigotimes_i E(z_i)$ and F is zero, so the cokernel contains $\bigotimes_i E(z_{2i})$ in the zero filtration and this subalgebra must survive. We now have the desired exterior subalgebra.

11. $H_*(\underline{KR(1)}_6)$

This is both the hardest to compute and the most interesting. We start with our usual fibration

$$\underline{KO}_6 \xrightarrow{2} \underline{KO}_6 \to \underline{KR(1)}_6.$$

$H_*(\underline{KO}_6) \simeq \bigotimes_k E(x_{2k})$, so $2_* = VF = 0$ because F is zero. That means all of $H_*(\underline{KO}_6)$ is the cokernel and it all survives because it is even degree. Using our second spectral sequence for

$$\underline{KO}_6 \to \underline{KR(1)}_6 \to \underline{KO}_7,$$

we know that the first map injects so there is no kernel. The spectral sequence collapses with $H_*(\underline{KO}_7)$, the cokernel of the map. This gives the short exact sequence

$$\bigotimes_i E(x_{2i}) \to H_*(\underline{KR(1)}_6) \to \bigotimes_i E(y_i).$$

The goal here is to solve the extension problem $(y_i)^2 = x_{2i}$. We do already know that $V(x_{4i}) = x_{2i}$ on the first part and $V(y_{2i}) = y_i$ on the second part.

We use the spectral sequence

$$\underline{KR(1)}_6 \to * \to \underline{KR(1)}_7$$

to prove our result. Observe that Tor of $E(x_{2i}) \otimes_i E(y_i)$ is

$$\Gamma[w_{2i+1}] \otimes_i \Gamma[ww_i] = E(w_{2i+1}) \otimes_i \Gamma[w_{4i+2}] \otimes_i E(ww_i) \otimes_i \Gamma[ww_{2i}]$$

and that Tor of $\bigotimes_i TP_4(y_i)$ is

$$\Gamma[w_{4i+2}] \otimes_i E(ww_i).$$

If the extension exists, there is a $d_1(ww_{2i}) = w_{2i-1}$. We don't need d_1 for our result though. Note that no matter what the extension is, in Tor we have

$$\Gamma[w_{4i+2}] \otimes_i E(ww_i).$$

We rewrite this just a bit as

$$\Gamma[w_{4i+2}] \otimes_i E(ww_{2i}) \otimes_i E(ww_{2i+1}).$$

Note that this is precisely the correct size for our known result of $H_*(\underline{KR(1)}_7)$. That doesn't prove our result yet though. We do know that any even degree element in filtration 1 or 2 must survive, and so we know we have $E(w_{4i+2}) \otimes_i E(ww_{2i})$ already no matter what extensions there are. One of w_{4i+2} or ww_{4i+2} must be exterior and the other must square. The only thing to square to is ww_{8i+4}. Because we know the answer and all these elements must survive, this must be part of the polynomial part of the answer, so we must have $(ww_{4i})^2 = ww_{8i}$. It doesn't really matter which of the elements is exterior. What we know from this is that we have all the elements we need in degrees $4i+2$ that are generators, primitives, or squares.

If we had a case where $(y_{2i})^2 = 0$ in $H_*(\underline{KR(1)}_6)$, then from the above discussion, we would have Tor giving us a $\Gamma[z_{2i+1}] = E(z_{2i+1}) \otimes \Gamma[z_{4i+2}]$. We would not have the $\Gamma[z_{4i+2}]$ unless this happens. The z_{4i+2} is in filtration 2 so must survive, but we already have enough elements in this degree, so this cannot happen.

We now know that $(y_{2i})^2 = x_{4i}$ always. We have

$$x_{2i} = V(x_{4i}) = VF(y_{2i}) = FV(y_{2i}) = F(y_i) = (y_i)^2.$$

This solves the extension problem for all y_i with i odd.

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