THE OMEGA SPECTRUM FOR MOD 2 KO-THEORY: APPENDIX

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ABSTRACT. In the main paper we compute the homology Hopf algebras for the 8 spaces in the Omega spectrum for mod 2 KO-theory, which is the same as the first real Morava K-theory. There are a lot of maps into and out of these spaces and the spaces for KO-theory, KU-theory and the first Morava K-theory. For every one of these 98 maps (counting suspensions) there is a spectral sequence. It wasn't necessary to know all of these maps and spectral sequences for the main paper, but it often helped. So, for my own purposes, this appendix describes all 98 maps and spectral sequences. 48 of these maps involve our new spaces and 56 of the spectral sequences do. In addition, the maps on homotopy are all written down, again, just so I have them somewhere.

1. HOMOLOGIES

To make this appendix somewhat more self contained, we write down all of the homologies we need. In the main paper we write $\otimes_i P(x_i)$ to be clear that we are taking the polynomial algebra on generators x_i for all i. The tensor products clutter up the notation. That kind of precision isn't necessary in this appendix, so here, when we write $P(x_i)$, we mean $\otimes_i P(x_i)$. The tensor product is understood. This simplifies the notation significantly.

Theorem 1.1. *The homology of the connected component of $\underline{KR(1)}_i$ is as follows. If the Vershiebung isn't described, it is zero.*

$i=0$	$E(x_k)\otimes P(y_{4k+2})$	$V(x_{2k})=x_k$	
$i=1$	$P(x_{2k+1})\otimes P(y_{4k+2})$	$V(y_{4k+2})=x_{2k+1}$	
$i=2$	$P(x_{8k+2})\otimes P(y_{4k+3})$		
$i=3$	$E(x_{8k+3})\otimes P(y_{8k+4})$		
$i=4$	$E(x_{4k})\otimes E(y_{8k+5})$	$V(x_{8k})=x_{4k}$	
$i=5$	$E(x_{4k+1})\otimes E(y_{2k})$	$V(y_{4k})=y_{2k}$	$V(y_{8k+2})=x_{4k+1}$
$i=6$	$TP_4(x_k)$	$V(x_{2k})=x_k$	
$i=7$	$E(x_{2k})\otimes P(y_{2k+1})$	$V(x_{4k})=x_{2k}$	

Theorem 1.2. *The homology of the connected component of $\underline{KU}_i$ is as follows.*

$i=0$	$P(x_{2i})$	$V(x_{4i})=x_{2i}$
$i=1$	$E(x_{2i+1})$	

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Theorem 1.3. *The homology of the connected component of* $\underline{KO}_{\,i}$ *is as follows.*

$$\begin{array}{lll} i=0 & P(x_i) & V(x_{2i}) = x_i \\ i=1 & P(x_{2i+1}) & \\ i=2 & P(x_{4i+2}) & \\ i=3 & E(x_{4i+3}) & \\ i=4 & P(x_{4i}) & V(x_{8i}) = x_{4i} \\ i=5 & E(x_{4i+1}) & \\ i=6 & E(x_{2i}) & V(x_{4i}) = x_{2i} \\ i=7 & E(x_i) & V(x_{2i}) = x_i \end{array}$$

Theorem 1.4. *The homology of the connected component of* $\underline{K(1)}_{\,i}$ *is as follows.*

$$i=0 \qquad TP_4(x_{4k+3}) \otimes E(y_{4k}) \otimes E(z_{8k+2})$$

$$V(y_{8k}) = y_{4k} \qquad V(y_{16k+4}) = z_{8k+2} \qquad V(y_{16k+12}) = (x_{4k+3})^2$$

$$i=1 \qquad E(x_{4k+1}) \otimes P(y_{4k+2}) \qquad V(y_{8k+2}) = x_{4k+1}$$

2. Introduction and ground rules

We will not reproduce computations done in the main paper, [Wil19], hereafter referred to as the "main paper." Previous to the main paper, the maps, homologies, and spectral sequences of the spaces $\underline{KU}_{\,i}$, $\underline{KO}_{\,i}$, and $\underline{K(1)}_{\,i}$ were all known. We will not recompute these but only describe them. Many of the maps and spectral sequences associated with $\underline{KR(1)}_{\,i}$ are not computed in the main paper. However, relying on the main paper, we do know $H_*(\underline{KR(1)}_{\,i})$ as well as the homologies for all of the previously known spaces. Some of the maps and spectral sequences have already been computed in the main paper, but not, by any means, all. When we have a new map or spectral sequence we will do more than just describe it, but we will give details of the proof of what is new. For every spectral sequence we study here, we know the answer, which is often quite helpful. In fact, often the argument for a differential or the solution to an extension problem is "because we know the answer." Rather than keep repeating this phrase, we will just assert differentials and extension problem solutions if they must come about "because we know the answer."

Because there are 98 spectral sequences, we developed self-explanatory notation for them so we can refer to them if necessary. The general form is a sequence of fibrations

$$\underline{X}_{\,i} \longrightarrow \underline{Y}_{\,i} \longrightarrow \underline{Z}_{\,i} \longrightarrow \underline{X}_{\,i+1} \longrightarrow \underline{Y}_{\,i+1} \longrightarrow \underline{Z}_{\,i+1} \longrightarrow \underline{X}_{\,i+2} \longrightarrow \cdots$$

First we compute the map $H_*(\underline{X}_{\,i}) \longrightarrow H_*(\underline{Y}_{\,i})$ and from that we compute the spectral sequence for $H_*(\underline{Z}_{\,i})$. With the exception of one spectral sequence out of 98, this gives the map $H_*(\underline{Y}_{\,i}) \longrightarrow H_*(\underline{Z}_{\,i})$. When we move to the next spectral sequence, i.e. for $\underline{Y}_{\,i} \longrightarrow \underline{Z}_{\,i} \longrightarrow \underline{X}_{\,i+1}$, we already know the first map and we can repeat the story and move on to the next. Knowing the first map of each spectral sequence and the answer makes most of them quite easy to deal with.

Of the 4 such sequences we describe, only one requires some work to start with the first map, namely $H_*(\underline{KR(1)}_{\,1}) \xrightarrow{\eta} H_*(\underline{KR(1)}_{\,0})$. The only second map that doesn't come out of the spectral sequence is the map for $H_*(\underline{K(1)}_5) \to H_*(\underline{KR(1)}_7)$ from the spectral sequence **RKR557**. The problem is solved in the very next spectral sequence, **KRR576**.

We have also labeled the infrequent short exact sequences so the curious can find them easily.

3. $H_*(\underline{KO}_i) => H_*(\underline{KO}_{i+1})$ **OOi(i+1)**

We use the bar spectral sequence for

$$\underline{KO}_i \longrightarrow * \longrightarrow \underline{KO}_{i+1}$$

i=0, OO01

$$H_*(\underline{KO}_0) = P(x_i) => H_*(\underline{KO}_1) = P(y_{2i+1})$$

Tor of $P(x_i)$ is $E(y_i)$ with y_i in filtration 1. Solving all extension problems, $(y_i)^2 = y_{2i}$, gives $P(y_{2i+1})$.

i=1, OO12

$$H_*(\underline{KO}_1) = P(x_{2i+1}) => H_*(\underline{KO}_2) = P(y_{4i+2})$$

Tor of $P(x_{2i+1})$ is $E(y_{2i})$ with y_{2i} in filtration 1. Solving all extension problems, $(y_{2i})^2 = y_{4i}$, gives $P(y_{4i+2})$.

i=2, OO23

$$H_*(\underline{KO}_2) = P(x_{4i+2}) => H_*(\underline{KO}_3) = E(y_{4i+3})$$

Tor of $P(x_{4i+2})$ is $E(y_{4i+3})$ with y_{4i+3} in filtration 1. This gives $E(y_{4i+3})$.

i=3, OO34

$$H_*(\underline{KO}_3) = E(x_{4i+3}) => H_*(\underline{KO}_4) = P(y_{4i})$$

Tor of $E(x_{4i+3})$ is $\Gamma[y_{4i}]$ with y_{4i} in filtration 1. We have $(y_{4i})^2 = y_{8i}$ and corresponding formulas on the γs. This gives $P(y_{4i})$.

i=4, OO45

$$H_*(\underline{KO}_4) = P(x_{4i}) => H_*(\underline{KO}_5) = E(y_{4i+1})$$

Tor of $P(x_{4i})$ is $E(y_{4i+1})$ with y_{4i+1} in filtration 1. This gives $E(y_{4i+1})$.

i=5, OO56

$$H_*(\underline{KO}_5) = E(x_{4i+1}) => H_*(\underline{KO}_6) = E(y_{2i})$$

Tor of $E(x_{4i+1})$ is $\Gamma[y_{4i+2}] = E(y_{2i})$.

i=6, OO67

$$H_*(\underline{KO}_6) = E(x_{2i}) => H_*(\underline{KO}_7) = E(y_i)$$

Tor of $E(x_{2i})$ is $\Gamma[y_{2i+1}] = E(y_i)$.

i=7, OO70

$$H_*(\underline{KO}_7) = E(x_i) => H_*(\underline{KO}_0) = P(y_i)$$

Tor of $E(x_i)$ is $\Gamma[y_i]$ with y_i in filtration 1. We have $(y_i)^2 = y_{2i}$ and corresponding formulas on the γ generators. This gives $P(y_i)$.

$$4.\ H_*(\underline{KU}_{\,i}) => H_*(\underline{KU}_{\,i+1}) \quad \textbf{UUi(i+1)}$$

We use the bar spectral sequence for

$$\underline{KU}_{\,i} \longrightarrow * \longrightarrow \underline{KU}_{\,i+1}$$

i=0, UU01

$$H_*(\underline{KU}_{\,0}) = P(x_{2i}) => H_*(\underline{KU}_{\,1}) = E(y_{2i+1})$$

Tor of $P(x_{2i})$ is $E(y_{2i+1})$ with y_{2i+1} in filtration 1. This gives $E(y_{2i+1})$.

i=1, UU10

$$H_*(\underline{KU}_{\,1}) = E(x_{2i+1}) => H_*(\underline{KU}_{\,0}) = P(y_{2i})$$

Tor of $E(x_{2i+1})$ is $\Gamma[y_{2i}]$ with y_{2i} in filtration 1. Solving all extension problems, $(y_{2i})^2 = y_{4i}$ and similar formulas on the γ generators, gives $P(y_{2i})$.

$$5.\ H_*(\underline{K(1)}_{\,i}) => H_*(\underline{K(1)}_{\,i+1}) \quad \textbf{KKi(i+1)}$$

We use the bar spectral sequence for

$$\underline{K(1)}_{\,i} \longrightarrow * \longrightarrow \underline{K(1)}_{\,i+1}$$

i=0, KK01

$$H_*(\underline{K(1)}_{\,0}) = TP_4(x_{4i+3}) \otimes E(y_{4i}) \otimes E(z_{8i+2}) => H_*(\underline{K(1)}_{\,1}) = E(x_{4i+1}) \otimes P(y_{4i+2})$$

Tor of

$$TP_4(x_{4i+3}) \otimes E(y_{4i}) \otimes E(z_{8i+2})$$

is

$$E(x_{4i}) \otimes \Gamma[x_{16i+14}] \otimes \Gamma[y_{4i+1}] \otimes \Gamma[z_{8i+3}]$$

with x_{4i}, y_{4i+1}, and z_{8i+3} in filtration 1 and x_{16i+14} in filtration 2. We rewrite $\Gamma[y_{4i+1}]$ as $E(y_{4i+1}) \otimes \Gamma[y_{8i+2}]$ with the y_{8i+2} in filtration 2. We rewrite $\Gamma[z_{8i+3}]$ as $E(z_{8i+3}) \otimes \Gamma[z_{16i+6}]$ with the y_{16i+6} in filtration 2. Our Tor is now

$$E(x_{4i}) \otimes E(x_{4i+1}) \otimes E(x_{8i+3}) \otimes \Gamma[z_{4i+2}]$$

where the exterior generators are in filtration 1 and the Γ generator is in filtration 2. Now we rewrite $\Gamma[z_{4i+2}]$ as $E(z_{4i+2}) \otimes \Gamma[z_{8i+4}]$ with the Γ generator in filtration 4. We now have a d_3,

$$d_3(z_{8i+4}) = x_{8i+3}.$$

We are left with

$$E(x_{4i}) \otimes E(x_{4i+1}) \otimes E(z_{4i+2})$$

Solve the extension problems to get the known result $H_*(\underline{K(1)}_{\,1}) = E(x_{4i+1}) \otimes P(y_{4i+2})$. Those solutions are $(x_{4i})^2 = x_{8i}$ and $(z_{4i+2})^2 = x_{8i+4}$.

i=1, KK10

$$H_*(\underline{K(1)}_{\,1}) = E(x_{4i+1}) \otimes P(y_{4i+2}) => H_*(\underline{K(1)}_{\,0}) = TP_4(x_{4i+3}) \otimes E(y_{4i}) \otimes E(z_{8i+2})$$

Tor is

$$\Gamma[x_{4i+2}] \otimes E(y_{4i+3})$$

Rewrite $\Gamma[x_{4i+2}]$ as

$$\Gamma[x_{8i+2}] \otimes E(x_{8i+6}) \otimes \Gamma[x_{16i+12}]$$

There is one extension problem, $(y_{4i+3})^2 = x_{8i+6}$. After this we have

$$TP_4(y_{4i+3}) \otimes \Gamma[x_{8i+2}] \otimes \Gamma[x_{16i+12}]$$

Continuing to rewrite, this is

$$TP_4(y_{4i+3}) \otimes E(x_{8i+2}) \otimes \Gamma[x_{8i+4}]$$

and the $\Gamma[x_{8i+4}]$ is $E(x_{4i})$.

6. The sequence $\underline{KO}_1 \xrightarrow{\eta} \underline{KO}_0 \to \underline{KU}_0 \to \underline{KO}_2 \xrightarrow{\eta} \underline{KO}_1 \to \cdots$

- $\underline{KO}_1 \xrightarrow{\eta} \underline{KO}_0 \to \underline{KU}_0$ **OOU100**

$$P(x_{2i+1}) \xrightarrow{x_{2i+1} \to y_{2i+1}} P(y_i) \xrightarrow[y_{2i+1}\to 0]{y_{2i}\to z_{2i}} P(z_{2i})$$

The spectral sequence is just a short exact sequence

$$\textbf{S.E.S.} \qquad H_*(\underline{KO}_1) \xrightarrow{\eta_*} H_*(\underline{KO}_0) \longrightarrow H_*(\underline{KU}_0).$$

- $\underline{KO}_0 \to \underline{KU}_0 \to \underline{KO}_2$ **OUO002**

$$P(x_i) \xrightarrow[x_{2i+1}\to 0]{x_{2i}\to y_{2i}} P(y_{2i}) \xrightarrow{zero} P(z_{4i+2})$$

There is nothing in the cokernel. The kernel is $P(x_{2i+1})$ so Tor is $E(w_{2i})$. Solving the extension problems, $(w_{2i})^2 = w_{4i}$, gives $P(z_{4i+2})$.

- $\underline{KU}_0 \to \underline{KO}_2 \xrightarrow{\eta} \underline{KO}_1$ **UOO021**

$$P(x_{2i}) \xrightarrow{zero} P(y_{4i+2}) \xrightarrow{y_{4i+2}\to (z_{2i+1})^2} P(z_{2i+1})$$

The cokernel is $P(y_{4i+2})$ in filtration 0 and the kernel is $P(x_{2i})$. Tor on this is $E(w_{2i+1})$ with generators in filtration 1. We get $(w_{2i+1})^2 = y_{4i+2}$.

- $\underline{KO}_2 \xrightarrow{\eta} \underline{KO}_1 \to \underline{KU}_1$ **OOU211**

$$P(x_{4i+2}) \xrightarrow{x_{4i+2}\to (y_{2i+1})^2} P(y_{2i+1}) \xrightarrow[(y_{2i+1})^2\to 0]{y_{2i+1}\to z_{2i+1}} E(z_{2i+1})$$

We get a short exact sequence

$$\textbf{S.E.S.} \qquad H_*(\underline{KO}_2) \longrightarrow H_*(\underline{KO}_1) \longrightarrow H_*(\underline{KU}_1).$$

- $\underline{KO}_1 \to \underline{KU}_1 \to \underline{KO}_3$ **OUO113**

$$P(x_{2i+1}) \xrightarrow[(x_{2i+1})^2\to 0]{x_{2i+1}\to y_{2i+1}} E(y_{2i+1}) \xrightarrow{zero} E(z_{4i+3})$$

The cokernel is zero and the kernel is $P((x_{2i+1})^2)$. Tor of this is $E(w_{4i+3})$, our answer.

- $\underline{KU}_1 \to \underline{KO}_3 \xrightarrow{\eta} \underline{KO}_2$ **UOO132**

$$E(x_{2i+1}) \xrightarrow{zero} E(y_{4i+3}) \xrightarrow{zero} P(z_{4i+2})$$

The cokernel in filtration zero is $E(y_{4i+3})$ and the kernel is $E(x_{2i+1})$. Tor of this is $\Gamma[w_{2i}]$. There is a differential

$$d_2(\gamma(w_{2i})) = y_{4i-1}$$

leaving only $E(w_{2i})$. Solving the extension problems, $(w_{2i})^2 = w_{4i}$, gives $P(z_{4i+2})$.

- $\underline{KO}_3 \xrightarrow{\eta} \underline{KO}_2 \to \underline{KU}_2$ **OOU322**

$$E(x_{4i+3}) \xrightarrow{zero} P(y_{4i+2}) \xrightarrow{y_{4i+2} \to z_{4i+2}} P(z_{2i})$$

The cokernel in filtration zero is $P(z_{4i+2})$. The kernel is $E(x_{4i+3})$. Tor of this is $\Gamma[w_{4i}]$ starting in filtration 1. There are no differentials and everything in $\Gamma[w_{4i}]$ squares non-trivially to something in the same filtration, i.e. $(w_{4i})^2 = w_{8i}$ and all of the corresponding γ's do the same, giving $P(z_{4i})$.

- $\underline{KO}_2 \to \underline{KU}_2 \to \underline{KO}_4$ **OUO224**

$$P(x_{4i+2}) \xrightarrow{x_{4i+2} \to y_{4i+2}} P(y_{2i}) \xrightarrow[y_{4i+2} \to 0]{y_{4i} \to z_{4i}} P(z_{4i})$$

We get a short exact sequence

$$\textbf{S.E.S.} \qquad H_*(\underline{KO}_2) \longrightarrow H_*(\underline{KU}_2) \longrightarrow H_*(\underline{KO}_4).$$

- $\underline{KU}_2 \to \underline{KO}_4 \xrightarrow{\eta} \underline{KO}_3$ **UOO243**

$$P(x_{2i}) \xrightarrow[x_{4i+2} \to 0]{x_{4i} \to y_{4i}} P(y_{4i}) \xrightarrow{zero} E(z_{4i+3})$$

There is no cokernel. The kernel is $P(x_{4i+2})$. Tor on this is $E(w_{4i+3})$ and we are done.

- $\underline{KO}_4 \xrightarrow{\eta} \underline{KO}_3 \to \underline{KU}_3$ **OOU433**

$$P(x_{4i}) \xrightarrow{zero} E(y_{4i+3}) \xrightarrow{y_{4i+3} \to z_{4i+3}} E(z_{2i+1})$$

The cokernel is $E(y_{4i+3})$ and the kernel is $P(x_{4i})$. Tor on this is $E(y_{4i+1})$ and we have our answer.

- $\underline{KO}_3 \to \underline{KU}_3 \to \underline{KO}_5$ **OUO335**

$$E(x_{4i+3}) \xrightarrow{x_{4i+3} \to y_{4i+3}} E(y_{2i+1}) \xrightarrow[y_{4i+3} \to 0]{y_{4i+1} \to z_{4i+1}} E(z_{4i+1})$$

This gives a short exact sequence

$$\textbf{S.E.S.} \qquad H_*(\underline{KO}_3) \longrightarrow H_*(\underline{KU}_3) \longrightarrow H_*(\underline{KO}_5).$$

- $\underline{KU}_3 \to \underline{KO}_5 \xrightarrow{\eta} \underline{KO}_4$ **UOO354**

$$E(x_{2i+1}) \xrightarrow[x_{4i+3} \to 0]{x_{4i+1} \to y_{4i+1}} E(y_{4i+1}) \xrightarrow{zero} P(z_{4i})$$

There is no cokernel and the kernel is $E(x_{4i+3})$. Tor of this is $\Gamma[w_{4i}]$. Solving extensions, $(w_{4i})^2 = w_{8i}$ and corresponding formulas on the γ generators, gives $P(z_{4i})$.

- $\underline{KO}_5 \xrightarrow{\eta} \underline{KO}_4 \to \underline{KU}_4$ **OOU544**

$$E(x_{4i+1}) \xrightarrow{zero} P(y_{4i}) \xrightarrow{y_{4i}\to(z_{2i})^2} P(z_{2i})$$

The cokernel is $P(y_{4i})$ and the kernel $E(x_{4i+1})$. Tor of this is $\Gamma[w_{4i+2}] = E(w_{2i})$. We have $(w_{2i})^2 = y_{4i}$.

- $\underline{KO}_4 \to \underline{KU}_4 \to \underline{KO}_6$ **OUO446**

$$P(x_{4i}) \xrightarrow{x_{4i}\to(y_{2i})^2} P(y_{2i}) \xrightarrow[(y_{2i})^2\to 0]{y_{2i}\to z_{2i}} E(z_{2i})$$

We get a short exact sequence

$$\textbf{S.E.S.} \qquad H_*(\underline{KO}_4) \longrightarrow H_*(\underline{KU}_4) \longrightarrow H_*(\underline{KO}_6).$$

- $\underline{KU}_4 \to \underline{KO}_6 \xrightarrow{\eta} \underline{KO}_5$ **UOO465**

$$P(x_{2i}) \xrightarrow[(x_{2i})^2\to 0]{x_{2i}\to y_{2i}} E(y_{2i}) \xrightarrow{zero} E(z_{4i+1})$$

There is no cokernel. The kernel is $P((x_{2i})^2)$. The Tor of this is $E(w_{4i+1})$, our answer.

- $\underline{KO}_6 \xrightarrow{\eta} \underline{KO}_5 \to \underline{KU}_5$ **OOU655**

$$E(x_{2i}) \xrightarrow{zero} E(y_{4i+1}) \xrightarrow{zero} E(z_{2i+1})$$

The cokernel is $E(y_{4i+1})$ and the kernel is $E(x_{2i})$. Tor of this is $\Gamma[w_{2i+1}]$. We must have a differential

$$d_2(\gamma_2(w_{2i+1})) = y_{4i+1}$$

All that is left is the $E(w_{2i+1})$, our answer.

- $\underline{KO}_5 \to \underline{KU}_5 \to \underline{KO}_7$ **OUO557**

$$E(x_{4i+1}) \xrightarrow{zero} E(y_{2i+1}) \xrightarrow{y_{2i+1}\to z_{2i+1}} E(z_i)$$

The cokernel is $E(y_{2i+1})$ and the kernel is $E(x_{4i+1})$. Tor of this is $\Gamma[w_{4i+2}] = E(w_{2i})$.

- $\underline{KU}_5 \to \underline{KO}_7 \xrightarrow{\eta} \underline{KO}_6$ **UOU576**

$$E(x_{2i+1}) \xrightarrow{x_{2i+1}\to y_{2i+1}} E(y_i) \xrightarrow[y_{2i+1}\to 0]{y_{2i}\to z_{2i}} E(z_{2i})$$

We get a short exact sequence

$$\textbf{S.E.S.} \qquad H_*(\underline{KU}_5) \longrightarrow H_*(\underline{KO}_7) \longrightarrow H_*(\underline{KO}_6).$$

- $\underline{KO}_7 \xrightarrow{\eta} \underline{KO}_6 \to \underline{KU}_6$ **OOU766**

$$E(x_i) \xrightarrow[x_{2i+1}\to 0]{x_{2i}\to y_{2i}} E(y_{2i}) \xrightarrow{zero} P(z_{2i})$$

There is no cokernel. The kernel is $E(x_{2i+1})$. Tor of this is $\Gamma[w_{2i}]$. Solving the extension problems we get our answer, $P(z_{2i})$.

- $\underline{KO}_6 \to \underline{KU}_6 \to \underline{KO}_0$ **OUO660**

$$E(x_{2i}) \xrightarrow{zero} P(y_{2i}) \xrightarrow{y_{2i}\to(z_i)^2} P(z_i)$$

The cokernel is $P(y_{2i})$ and the kernel is $E(x_{2i})$. Tor of this is $\Gamma[w_{2i+1}] = E(w_i)$. The extension problem is solved by $(w_i)^2 = y_{2i}$.

- $\underline{KU}_6 \to \underline{KO}_0 \xrightarrow{\eta} \underline{KO}_7$ **UOO607**

$$P(x_{2i}) \xrightarrow{x_{2i}\to(y_i)^2} P(y_i) \xrightarrow[(y_i)^2\to 0]{y_i\to z_i} E(z_i)$$

We get a short exact sequence

$$\textbf{S.E.S.} \qquad H_*(\underline{KU}_6) \longrightarrow H_*(\underline{KO}_0) \to H_*(\underline{KO}_7).$$

- $\underline{KO}_0 \xrightarrow{\eta} \underline{KO}_7 \to \underline{KU}_7$ **OOU077**

$$P(x_i) \xrightarrow[(x_i)^2\to 0]{x_i\to y_i} E(y_i) \xrightarrow{zero} E(z_{2i+1})$$

The cokernel is zero and the kernel is $P((x_i)^2)$. Tor of this is $E(w_{2i+1})$.

- $\underline{KO}_7 \to \underline{KU}_7 \to \underline{KO}_1$ **OUO771**

$$E(x_i) \xrightarrow{zero} E(y_{2i+1}) \xrightarrow{zero} P(z_{2i+1})$$

The cokernel is $E(y_{2i+1})$ and the kernel is $E(x_i)$. Tor of this is $\Gamma[w_i]$. There is a differential, $d_2(\gamma_2(w_i)) = y_{2i-1}$ leaving $E(w_i)$. Solving extensions, $(w_i)^2 = w_{2i}$, we get our $P(w_{2i+1})$.

- $\underline{KU}_7 \to \underline{KO}_1 \xrightarrow{\eta} \underline{KO}_0$ **UOO710**

$$E(x_{2i+1}) \xrightarrow{zero} P(y_{2i+1}) \xrightarrow{y_{2i+1}\to z_{2i+1}} P(z_i)$$

The cokernel is $P(z_{2i+1})$ and the kernel is $E(x_{2i+1})$. Tor of this is $\Gamma[w_{2i}]$. To get our answer we must have $(w_{2i})^2 = w_{4i}$ in filtration 1 and $FV = VF$ will give us the squares in all the higher filtrations ending with $P(z_{2i})$.

7. The sequence $\underline{KU}_0 \xrightarrow{2} \underline{KU}_0 \to \underline{K(1)}_0 \to \underline{KU}_1 \xrightarrow{2} \underline{KU}_1 \to \cdots$

- $\underline{KU}_0 \xrightarrow{2} \underline{KU}_0 \to \underline{K(1)}_0$ **UUK000**

$$P(x_{2i}) \xrightarrow[x_{4i+2}\to 0]{x_{4i}\to(y_{2i})^2} P(y_{2i}) \xrightarrow[y_{8i+6}\to(zz_{4i+3})^2]{y_{4i}\to z_{4i},\quad y_{8i+2}\to z_{8i+2}} TP_4(zz_{4i+3}) \otimes E(z_{4i}) \otimes E(z_{8i+2})$$

The cokernel is $E(y_{2i})$ and the kernel is $P(x_{4i+2})$. Tor of this is $E(w_{4i+3})$. We have only the one extension problem, $(w_{4i+3})^2 = x_{8i+6}$.

- $\underline{KU}_0 \to \underline{K(1)}_0 \to \underline{KU}_1$ **UKU001**

$$P(x_{2i}) \xrightarrow[x_{8i+6}\to(yy_{4i+3})^2]{x_{4i}\to y_{4i},\quad x_{8i+2}\to y_{8i+2}}$$

$$TP_4(yy_{4i+3}) \otimes E(y_{4i}) \otimes E(y_{8i+2}) \xrightarrow[y_{8i+2}\to 0,\quad (yy_{4i+3})^2\to 0]{y_{4i}\to 0,\quad yy_{4i+3}\to z_{4i+3}} E(z_{2i+1})$$

The cokernel is $E(yy_{4i+3})$ and the kernel is $P((x_{2i})^2)$. Tor of this is $E(w_{4i+1})$.

- $\underline{K(1)}_0 \to \underline{KU}_1 \xrightarrow{2} \underline{KU}_1$ **KUU011**

$$TP_4(xx_{4i+3}) \otimes E(x_{4i}) \otimes E(x_{8i+2})$$

$$\xrightarrow[x_{8i+2}\to 0,\quad (xx_{4i+3})^2\to 0]{x_{4i}\to 0,\quad xx_{4i+3}\to y_{4i+3}} E(y_{2i+1}) \xrightarrow{zero} E(z_{2i+1})$$

The cokernel is $E(y_{4i+1})$ and the kernel $E(x_{2i})$. Tor of the kernel is $\Gamma[w_{2i+1}] \simeq E(w_{2i+1}) \otimes \Gamma[w_{4i+2}]$. We get a differential

$$d_2(ww_{4i+2}) = y_{4i+1}$$

All that is left is $E(w_{2i+1})$.

- $\underline{KU}_1 \xrightarrow{2} \underline{KU}_1 \to \underline{K(1)}_1$ **UUK111**

$$E(x_{2i+1}) \xrightarrow{zero} E(y_{2i+1}) \xrightarrow[y_{4i+3}\to 0]{y_{4i+1}\to z_{4i+1}} E(z_{4i+1}) \otimes P(zz_{4i+2}).$$

The cokernel is $E(y_{2i+1})$ and the kernel is $E(x_{2i+1})$. Tor of this is $\Gamma[w_{2i}]$. We have a differential

$$d_2(\gamma_2(w_{2i})) = y_{4i-1}$$

We are left with $E(y_{4i+1})$ in filtration zero, and $E(w_{2i})$ with generators in filtration 1. This last all have squares, $(w_{2i})^2 = w_{4i}$, giving $P(w_{4i+2})$.

- $\underline{KU}_1 \to \underline{K(1)}_1 \to \underline{KU}_0$ **UKU110**

$$E(x_{2i+1}) \xrightarrow[x_{4i+3}\to 0]{x_{4i+1}\to y_{4i+1}} E(y_{4i+1}) \otimes P(yy_{4i+2}) \xrightarrow[y_{4i+1}\to 0]{yy_{4i+2}\to z_{4i+2}} P(z_{2i})$$

The cokernel is $P(yy_{4i+2})$ and the kernel is $E(x_{4i+3})$. Tor of this is $\Gamma[w_{4i}]$. Squaring everything in Γ gives $P(w_{4i})$.

- $\underline{K(1)}_1 \to \underline{KU}_0 \xrightarrow{2} \underline{KU}_0$ **KUU100**

$$E(x_{4i+1}) \otimes P(xx_{4i+2}) \xrightarrow[x_{4i+1}\to 0]{xx_{4i+2}\to y_{4i+2}} P(y_{2i}) \xrightarrow[y_{4i+2}\to 0]{y_{4i}\to (z_{2i})^2} P(z_{2i})$$

The cokernel is $P(y_{4i})$ and the kernel is $E(x_{4i+1})$. Tor of this is $\Gamma[w_{4i+2}] = E(w_{2i})$. We have $(w_{2i})^2 = y_{4i}$ giving our answer.

8. THE SEQUENCE $\underline{KO}_0 \xrightarrow{2} \underline{KO}_0 \to \underline{KR(1)}_0 \to \underline{KO}_1 \xrightarrow{2} \underline{KO}_1 \to \cdots$

- $\underline{KO}_0 \xrightarrow{2} \underline{KO}_0 \to \underline{KR(1)}_0$ **OOR000**

$$P(x_i) \xrightarrow[x_{2i+1}\to 0]{x_{2i}\to (y_i)^2} P(y_i) \xrightarrow[(y_i)^2\to 0]{y_i\to z_i} E(z_i)\otimes P(zz_{4i+2})$$

The cokernel is $E(y_i)$ and the kernel is $P(x_{2i+1})$. Tor of this is $E(w_{2i})$. We have $(w_{2i})^2 = w_{4i}$ giving $P(w_{4i+2})$.

- $\underline{KO}_0 \to \underline{KR(1)}_0 \to \underline{KO}_1$ **ORO001**

$$P(x_i) \xrightarrow[(x_i)^2\to 0]{x_i\to y_i} E(y_i)\otimes P(yy_{4i+2}) \xrightarrow[y_i\to 0]{yy_{4i+2}\to (z_{2i+1})^2} P(z_{2i+1})$$

The cokernel is $P(yy_{4i+2})$ The kernel is $P((x_i)^2)$. Tor of this is $E(w_{2i+1})$. We have $(w_{2i+1})^2 = yy_{4i+2}$.

- $\underline{KR(1)}_0 \to \underline{KO}_1 \xrightarrow{2} \underline{KO}_1$ **ROO011**

$$E(x_i)\otimes P(xx_{4i+2}) \xrightarrow[x_i\to 0]{xx_{4i+2}\to (y_{2i+1})^2} P(y_{2i+1}) \xrightarrow{zero} P(z_{2i+1})$$

The cokernel is $E(y_{2i+1})$ and the kernel is $E(x_i)$. Tor of this is $\Gamma[w_i]$. We have differentials

$$d_2(\gamma_2(w_i)) = y_{2i-1}.$$

We are left with $E(w_i)$. We have $(w_i)^2 = w_{2i}$ so we get $P(w_{2i+1})$.

- $\underline{KO}_1 \xrightarrow{2} \underline{KO}_1 \to \underline{KR(1)}_1$ **OOR111**

$$P(x_{2i+1}) \xrightarrow{zero} P(y_{2i+1}) \xrightarrow{y_{2i+1}\to z_{2i+1}} P(z_{2i+1})\otimes P(zz_{4i+2})$$

The cokernel is $P(y_{2i+1})$ and the kernel is $P(x_{2i+1})$. Tor of this is $E(w_{2i})$ and We have $(w_{2i})^2 = w_{4i}$ so we get $P(w_{4i+2})$.

- $\underline{KO}_1 \to \underline{KR(1)}_1 \to \underline{KO}_2$ **ORO112**

$$P(x_{2i+1}) \xrightarrow{x_{2i+1}\to y_{2i+1}} P(y_{2i+1})\otimes P(yy_{4i+2}) \xrightarrow[y_{2i+1}\to 0]{yy_{4i+2}\to z_{4i+2}} P(z_{4i+2})$$

There is no kernel so this is a short exact sequence

$$\textbf{S.E.S.} \qquad H_*(\underline{KO}_1) \longrightarrow H_*(\underline{KR(1)}_1) \longrightarrow H_*(\underline{KO}_2).$$

- $\underline{KR(1)}_1 \to \underline{KO}_2 \xrightarrow{2} \underline{KO}_2$ **ROO122**

$$P(x_{2i+1})\otimes P(xx_{4i+2}) \xrightarrow[x_{2i+1}\to 0]{xx_{4i+2}\to y_{4i+2}} P(y_{4i+2}) \xrightarrow{zero} P(z_{4i+2})$$

The cokernel is zero and the kernel is $P(x_{2i+1})$. Tor of this is $E(w_{2i})$. We have $(w_{2i})^2 = w_{4i}$ so we get $P(w_{4i+2})$.

- $\underline{KO}_2 \xrightarrow{2} \underline{KO}_2 \to \underline{KR(1)}_2$ **OOR222**

$$P(x_{4i+2}) \xrightarrow{zero} P(y_{4i+2}) \xrightarrow[y_{8i+6}\to(zz_{4i+3})^2]{y_{8i+2}\to z_{8i+2}} P(z_{8k+2}) \otimes P(zz_{4i+3})$$

The cokernel is $P(y_{4i+2})$ and the kernel is $P(x_{4i+2})$. Tor of the kernel is $E(w_{4i+3})$ and we have $(w_{4i+3})^2 = x_{8i+6}$.

- $\underline{KO}_2 \to \underline{KR(1)}_2 \to \underline{KO}_3$ **ORO223**

$$P(x_{4i+2}) \xrightarrow[x_{8i+6}\to(yy_{4i+3})^2]{x_{8i+2}\to y_{8i+2}} P(y_{8k+2}) \otimes P(yy_{4i+3}) \xrightarrow[y_{8i+2}\to 0, \quad (yy_{4i+3})^2\to 0]{yy_{4i+3}\to z_{4i+3}} E(z_{4i+3})$$

The cokernel is $E(yy_{4i+3})$ and there is no kernel. We get a short exact sequence.

S.E.S. $H_*(\underline{KO}_2) \longrightarrow H_*(\underline{KR(1)}_2) \longrightarrow H_*(\underline{KO}_3)$.

- $\underline{KR(1)}_2 \to \underline{KO}_3 \xrightarrow{2} \underline{KO}_3$ **ROO233**

$$P(x_{8k+2}) \otimes P(xx_{4i+3}) \xrightarrow[x_{8i+2}\to 0, \quad (xx_{4i+3})^2\to 0]{xx_{4i+3}\to y_{4i+3}} E(y_{4i+3}) \xrightarrow{zero} E(z_{4i+3})$$

There is no cokernel. The kernel is $P(x_{8i+2}) \otimes P((xx_{4i+3})^2)$. Tor of this is $E(w_{8i+3}) \otimes E(ww_{8i+7})$.

- $\underline{KO}_3 \xrightarrow{2} \underline{KO}_3 \to \underline{KR(1)}_3$ **OOR333**

$$E(x_{4i+3}) \xrightarrow{zero} E(y_{4i+3}) \xrightarrow[y_{8i+7}\to 0]{y_{8i+3}\to z_{8i+3}} E(z_{8i+3}) \otimes P(zz_{8i+4})$$

The cokernel is $E(y_{4i+3})$ and the kernel is $E(x_{4i+3})$. Tor of this is $\Gamma[w_{4i}]$. We must have a differential

$$d_2(\gamma_2(w_{4i})) = y_{8i-1}$$

This leaves $E(y_{8i+3}) \otimes E(w_{4i})$. We have $(w_{4i})^2 = w_{8i}$ giving $P(zz_{8i+4})$.

- $\underline{KO}_3 \to \underline{KR(1)}_3 \to \underline{KO}_4$ **ORO334**

$$E(x_{4i+3}) \xrightarrow[x_{8i+7}\to 0]{x_{8i+3}\to y_{8i+3}} E(y_{8i+3}) \otimes P(yy_{8i+4}) \xrightarrow[y_{8i+3}\to 0]{yy_{8i+4}\to z_{8i+4}} P(z_{4i})$$

The cokernel is $P(yy_{8i+4})$ and the kernel is $E(x_{8i+7})$. Tor of this is $\Gamma[w_{8i}]$. We have $(w_{8i})^2 = w_{16i}$ and with $FV = VF$, this Γ becomes $P(z_{8i})$.

- $\underline{KR(1)}_3 \to \underline{KO}_4 \xrightarrow{2} \underline{KO}_4$ **ROO344**

$$E(x_{8i+3}) \otimes P(xx_{8i+4}) \xrightarrow[x_{8i+3}\to 0]{xx_{8i+4}\to y_{8i+4}} P(y_{4i}) \xrightarrow[y_{8i+4}\to 0]{y_{8i}\to(z_{4i})^2} P(z_{4i})$$

The cokernel is $P(y_{8i})$ and the kernel is $E(x_{8i+3})$. Tor of this is $\Gamma[w_{8i+4}] = E(w_{4i})$. We have $(w_{4i})^2 = y_{8i}$ to get $P(z_{4i})$.

- $\underline{KO}_4 \xrightarrow{2} \underline{KO}_4 \to \underline{KR(1)}_4$ **OOR444**

$$P(x_{4i}) \xrightarrow[x_{8i+4}\to 0]{x_{8i}\to(y_{4i})^2} P(y_{4i}) \xrightarrow[(y_{4i})^2\to 0]{y_{4i}\to z_{4i}} E(z_{4i}) \otimes E(zz_{8i+5})$$

The cokernel is $E(y_{4i})$. The kernel is $P(x_{8i+4})$. Tor of this is $E(w_{8i+5})$.

- $\underline{KO}_4 \to \underline{KR(1)}_4 \to \underline{KO}_5$ **ORO445**

$$P(x_{4i}) \xrightarrow[(x_{4i})^2 \to 0]{x_{4i}\to y_{4i}} E(y_{4i}) \otimes E(yy_{8i+5}) \xrightarrow[y_{4i}\to 0]{yy_{8i+5}\to z_{8i+5}} E(z_{4i+1})$$

The cokernel is $E(yy_{8i+5})$. The kernel is $P((x_{4i})^2)$. Tor of this is $E(w_{8i+1})$.

- $\underline{KR(1)}_4 \to \underline{KO}_5 \xrightarrow{2} \underline{KO}_5$ **ROO455**

$$E(x_{4i}) \otimes E(xx_{8i+5}) \xrightarrow[x_{4i}\to 0]{xx_{8i+5}\to y_{8i+5}} E(y_{4i+1}) \xrightarrow{zero} E(z_{4i+1})$$

The cokernel is $E(y_{8i+1})$ and the kernel is $E(x_{4i})$. Tor of this is $\Gamma[w_{4i+1}]$ with

$$d_2(\gamma_2(w_{4i+1})) = y_{8i+1}$$

What is left is $E(w_{4i+1})$.

- $\underline{KO}_5 \xrightarrow{2} \underline{KO}_5 \to \underline{KR(1)}_5$ **OOR555**

$$E(x_{4i+1}) \xrightarrow{zero} E(y_{4i+1}) \xrightarrow{y_{4i+1}\to z_{4i+1}} E(z_{4i+1}) \otimes E(zz_{2i})$$

The cokernel is $E(y_{4i+1})$ and the kernel is $E(x_{4i+1})$. Tor of this is $\Gamma[w_{4i+2}] = E(w_{2i})$.

- $\underline{KO}_5 \to \underline{KR(1)}_5 \to \underline{KO}_6$ **ORO556**

$$E(x_{4i+1}) \xrightarrow{x_{4i+1}\to y_{4i+1}} E(y_{4i+1}) \otimes E(yy_{2i}) \xrightarrow[y_{4i+1}\to 0]{yy_{2i}\to z_{2i}} E(z_{2i})$$

The cokernel is $E(yy_{2i})$ and there is no kernel. We get a short exact sequence

$$\textbf{S.E.S.} \qquad H_*(\underline{KO}_5) \longrightarrow H_*(\underline{KR(1)}_5) \longrightarrow H_*(\underline{KO}_6).$$

- $\underline{KR(1)}_5 \to \underline{KO}_6 \xrightarrow{2} \underline{KO}_6$ **ROO566**

$$E(x_{4i+1}) \otimes E(xx_{2i}) \xrightarrow[x_{4i+1}\to 0]{xx_{2i}\to y_{2i}} E(y_{2i}) \xrightarrow{zero} E(z_{2i})$$

There is no cokernel. The kernel is $E(x_{4i+1})$. Tor of this is $\Gamma[w_{4i+2}] = E(w_{2i})$.

- $\underline{KO}_6 \xrightarrow{2} \underline{KO}_6 \to \underline{KR(1)}_6$ **OOR666**

$$E(x_{2i}) \xrightarrow{zero} E(y_{2i}) \xrightarrow{y_{2i}\to (z_i)^2} TP_4(z_i)$$

The cokernel is $E(y_{2i})$ and the kernel is $E(x_{2i})$. Tor of this is $\Gamma[w_{2i+1}] = E(w_i)$. We have $(w_i)^2 = y_{2i}$.

- $\underline{KO}_6 \to \underline{KR(1)}_6 \to \underline{KO}_7$ **ORO667**

$$E(x_{2i}) \xrightarrow{x_{2i}\to (y_i)^2} TP_4(y_i) \xrightarrow{y_i\to z_i} E(z_i)$$

The cokernel is $E(y_i)$ and there is no kernel. We get a short exact sequence

$$\textbf{S.E.S.} \qquad H_*(\underline{KO}_6) \longrightarrow H_*(\underline{KR(1)}_6) \longrightarrow H_*(\underline{KO}_7).$$

- $\underline{KR(1)}_6 \to \underline{KO}_7 \xrightarrow{2} \underline{KO}_7$ **ROO677**

$$TP_4(x_i) \xrightarrow{x_i \to y_i} E(y_i) \xrightarrow{zero} E(z_i)$$

The cokernel is zero and the kernel is $E((x_i)^2)$. Tor of this is $\Gamma[w_{2i+1}] = E(w_i)$.

- $\underline{KO}_7 \xrightarrow{2} \underline{KO}_7 \to \underline{KR(1)}_7$ **OOR777**

$$E(x_i) \xrightarrow{zero} E(y_i) \xrightarrow[y_{2i+1}\to 0]{y_{2i}\to z_{2i}} E(z_{2i}) \otimes P(zz_{2i+1})$$

The cokernel is $E(y_i)$ and the kernel is $E(x_i)$. Tor is $\Gamma[w_i]$. We need

$$d_2(\gamma_2(w_i)) = y_{2i-1}.$$

We are left with $E(y_{2i}) \otimes E(w_i)$. We must have $(w_i)^2 = w_{2i}$ to get our $P(zz_{2i+1})$.

- $\underline{KO}_7 \to \underline{KR(1)}_7 \to \underline{KO}_0$ **ORO770**

$$E(x_i) \xrightarrow[x_{2i+1}\to 0]{x_{2i}\to y_{2i}} E(y_{2i}) \otimes P(yy_{2i+1}) \xrightarrow[y_{2i}\to 0]{yy_{2i+1}\to z_{2i+1}} P(z_i)$$

The cokernel is $P(yy_{2i+1})$ and the kernel is $E(x_{2i+1})$. Tor of this is $\Gamma[w_{2i}]$. We have $(w_{2i})^2 = w_{4i}$, which, along with $FV = VF$ gives $P(z_{2i})$.

- $\underline{KR(1)}_7 \to \underline{KO}_0 \xrightarrow{2} \underline{KO}_0$ **ROO700**

$$E(x_{2i}) \otimes P(xx_{2i+1}) \xrightarrow[x_{2i}\to 0]{xx_{2i+1}\to y_{2i+1}} P(y_i) \xrightarrow[y_{2i+1}\to 0]{y_{2i}\to (z_i)^2} P(z_i)$$

The cokernel is $P(y_{2i})$ and the kernel is $E(x_{2i})$. Tor of this is $\Gamma[w_{2i+1}] = E(w_i)$. We have $(w_i)^2 = y_{2i}$ to get $P(w_i)$.

9. $H_*(\underline{KR(1)}_i) => H_*(\underline{KR(1)}_{i+1})$ **RRi(i+1)**

We use the bar spectral sequence for

$$\underline{KR(1)}_i \longrightarrow * \longrightarrow \underline{KR(1)}_{i+1}$$

i=0, RR01

$$H_*(\underline{KR(1)}_0) = E(x_i) \otimes P(xx_{4i+2}) \longrightarrow * \longrightarrow H_*(\underline{KR(1)}_1) = P(y_{2i+1}) \otimes P(yy_{4i+2})$$

Tor is $\Gamma[w_i] \otimes E(ww_{4i+3})$. We have a differential

$$d_3(\gamma_4(w_i)) = ww_{4i-1}$$

This leaves $E(w_i)$ with generators in filtration 1 and $E(\gamma_2(w_i))$ with generators in filtration 2. We have $(w_i)^2 = w_{2i}$ giving $P(y_{2i+1})$ and $(\gamma_2(w_i))^2 = \gamma_2(w_{2i})$ giving $P(yy_{4i+2})$.

i=1, RR12

$$H_*(\underline{KR(1)}_1) = P(x_{2i+1}) \otimes P(xx_{4i+2}) \longrightarrow * \longrightarrow H_*(\underline{KR(1)}_2) = P(y_{8i+2}) \otimes P(yy_{4i+3})$$

Tor is $E(w_{2i}) \otimes E(ww_{4i+3})$. We have $(w_{2i})^2 = w_{4i}$ and $(ww_{4i+3})^2 = w_{8i+6}$.

i=2, RR23

$$H_*(\underline{KR(1)}_{\,2}) = P(x_{8i+2}) \otimes P(xx_{4i+3}) \longrightarrow * \longrightarrow H_*(\underline{KR(1)}_{\,3}) = E(y_{8i+3}) \otimes P(yy_{8i+4})$$

Tor is $E(w_{8i+3}) \otimes E(ww_{4i})$. We have $(ww_{4i})^2 = ww_{8i}$ giving $P(yy_{8i+4})$.

i=3, RR34

$$H_*(\underline{KR(1)}_{\,3}) = E(x_{8i+3}) \otimes P(xx_{8i+4}) \longrightarrow * \longrightarrow H_*(\underline{KR(1)}_{\,4}) = E(y_{4i}) \otimes E(yy_{8i+5})$$

Tor is $\Gamma[w_{8i+4}] \otimes E(ww_{8i+5})$ with $\Gamma[w_{8i+4}] = E(w_{4i})$.

i=4, RR45

$$H_*(\underline{KR(1)}_{\,4}) = E(x_{4i}) \otimes E(xx_{8i+5}) \longrightarrow * \longrightarrow H_*(\underline{KR(1)}_{\,5}) = E(y_{4i+1}) \otimes E(yy_{2i})$$

Tor is $\Gamma[w_{4i+1}] \otimes \Gamma[ww_{8i+6}]$. Rewrite this as $E(w_{4i+1}) \otimes \Gamma[ww_{4i+2}]$ and then again as our answer.

i=5, RR56

$$H_*(\underline{KR(1)}_{\,5}) = E(x_{4i+1}) \otimes E(yy_{2i}) \longrightarrow * \longrightarrow H_*(\underline{KR(1)}_{\,6}) = TP_4(y_i)$$

Tor is $\Gamma[w_{4i+2}] \otimes \Gamma[ww_{2i+1}]$. Rewritten, this is $E(w_{2i})$ and $E(ww_i)$. We have $(ww_i)^2 = w_{2i}$.

i=6, RR67

$$H_*(\underline{KR(1)}_{\,6}) = TP_4(x_i) \longrightarrow * \longrightarrow H_*(\underline{KR(1)}_{\,7}) = E(y_{2i}) \otimes P(yy_{2i+1})$$

Tor is $E(w_i) \otimes \Gamma[ww_{4i+2}]$. We have $(w_i)^2 = w_{2i}$ giving $P(yy_{2i+1})$ and $\Gamma[ww_{4i+2}]$ is just $E(y_{2i})$.

i=7, RR70

$$H_*(\underline{KR(1)}_{\,7}) = E(x_{2i}) \otimes P(xx_{2i+1}) \longrightarrow * \longrightarrow H_*(\underline{KR(1)}_{\,0}) = E(y_i) \otimes P(yy_{4i+2})$$

Tor is $\Gamma[w_{2i+1}] \otimes E(ww_{2i})$. We have $\Gamma[w_{2i+1}]$ is $E(y_i)$ and after $(ww_{2i})^2 = w_{4i}$, we get $P(yy_{4i+2})$.

10. THE SEQUENCE $\underline{KR(1)}_{\,1} \xrightarrow{\eta} \underline{KR(1)}_{\,0} \to \underline{K(1)}_{\,0} \to \underline{KR(1)}_{\,2} \xrightarrow{\eta} \underline{KR(1)}_{\,1} \to \cdots$

- $\underline{KR(1)}_{\,1} \xrightarrow{\eta} \underline{KR(1)}_{\,0} \to \underline{K(1)}_{\,0}$ **RRK100**

$$P(x_{2i+1}) \otimes P(xx_{4i+2}) \xrightarrow[x_{2i+1}\to y_{2i+1}, \quad (x_{2i+1})^2\to 0]{xx_{4i+2}\to y_{4i+2}+yy_{4i+2}} E(y_i) \otimes P(yy_{4i+2})$$

$$\xrightarrow[y_{2i+1}\to 0, \quad y_{4i+2}+yy_{4i+2}\to 0]{y_{8i+6}\to (z_{4i+3})^2, \quad y_{8i+2}\to zzz_{8i+2}, \quad y_{4i}\to zz_{4i}} TP_4(z_{4i+3}) \otimes E(zz_{4i}) \otimes E(zzz_{8i+2})$$

We are in new territory now because we don't know the first map. To compute it, we use

$$\begin{array}{ccc} \underline{KO}_1 & \longrightarrow & \underline{KO}_0 \\ \downarrow & & \downarrow \\ \underline{KR(1)}_1 & \longrightarrow & \underline{KR(1)}_0 \\ \downarrow & & \downarrow \\ \underline{KO}_2 & \longrightarrow & \underline{KO}_1 \end{array}$$

We know all of the maps in homology except the horizontal one in the middle. We know the top horizontal map from **OOU100** and the bottom horizontal map from **OOU211**. The left vertical maps are from **ORO112** and the right from **ORO001**. Algebraically, we have

$$\begin{array}{ccc} P(x_{2i+1}) & \xrightarrow{x_{2i+1}\to y_{2i+1}} & P(y_i) \\ \downarrow{\scriptstyle x_{2i+1}\to x_{2i+1}} & & \downarrow{\scriptstyle y_i\to y_i} \\ P(x_{2i+1})\otimes P(xx_{4i+2}) & \longrightarrow & E(y_i)\otimes P(yy_{4i+2}) \\ \downarrow{\scriptstyle xx_{4i+2}\to yy_{4i+2}} & & \downarrow{\scriptstyle yy_{4i+2}\to(yy_{2i+1})^2} \\ P(yy_{4i+2}) & \xrightarrow{yy_{4i+2}\to(yy_{2i+1})^2} & P(yy_{2i+1}) \end{array}$$

A diagram chase gives the first map as listed above including the unusual $xx_{4i+2}\to y_{4i+2}+yy_{4i+2}$. The extra term, y_{4i+2}, comes about because $V(xx_{4i+2})=x_{2i+1}$, so the image of xx_{4i+2} must have V of it be the image of x_{2i+1} and we know how V behaves on $H_*(\underline{KR(1)}_0)$. This odd map is, in some sense, dual to the problem we have in **KRR576**.

The cokernel is $E(y_{2i})$ and the kernel is $P((x_{2i+1})^2)$. Tor of this is $E(ww_{4i+3})$. We have $(ww_{4i+3})^2=y_{8i+6}$.

- $\underline{KR(1)}_0\to\underline{K(1)}_0\to\underline{KR(1)}_2$ **RKR002**

$$E(x_i)\otimes P(xx_{4i+2}) \xrightarrow[x_{2i+1}\to 0,\quad x_{4i+2}+xx_{4i+2}\to 0]{x_{8i+6}\to(y_{4i+3})^2,\quad x_{8i+2}\to yyy_{8i+2},\quad x_{4i}\to yy_{4i}}$$

$$TP_4(y_{4i+3})\otimes E(yy_{4i})\otimes E(yyy_{8i+2}) \xrightarrow{zero} P(z_{8i+2})\otimes P(zz_{4i+3})$$

The cokernel is $E(y_{4i+3})$ and the kernel is $E(x_{2i+1})\otimes P(x_{4i+2}+xx_{4i+2})$. Tor of this is $\Gamma[w_{2i}]\otimes E(ww_{4i+3})$. We need

$$d_2(\gamma_2(w_{2i}))=y_{4i-1}$$

This leaves us with $E(w_{2i})\otimes E(ww_{4i+3})$. We have $(w_{2i})^2=w_{4i}$ and $(ww_{4i+3})^2=w_{8i+6}$ to get our answer.

- $\underline{K(1)}_0\to\underline{KR(1)}_2\xrightarrow{\eta}\underline{KR(1)}_1$ **KRR021**

$$TP_4(x_{4i+3})\otimes E(xx_{4i})\otimes E(xxx_{8i+2}) \xrightarrow{zero}$$

$$P(y_{8i+2})\otimes P(yy_{4i+3}) \xrightarrow[y_{8i+2}\to(z_{4i+1})^2]{yy_{4i+3}\to z_{4i+3}} P(z_{2i+1})\otimes P(zz_{4i+2})$$

The cokernel is $P(y_{8i+2}) \otimes P(yy_{4i+3})$ and we know the kernel. Tor of the kernel, when you combine all 3 of the terms, is $\Gamma[wwww_{4i+2}]$ starting in filtration 2, with exterior generators in filtration 1 given by w_{4i}, ww_{4i+1}, and www_{8i+3}. We know from the main paper that the yy_{4i+3} inject to $H_*(\underline{KR(1)}_1)$ so can't be hit by a differential. So, we have

$$d_3(\gamma_2(wwww_{4i+2})) = www_{8i+3}$$

This leaves us with an exterior algebra with generators in filtration 1, $E(w_{4i}) \otimes E(ww_{4i+1})$ and an exterior algebra with generators in filtration 2, $E(wwww_{4i+2})$. We have $(w_{4i})^2 = w_{8i}$, $(wwww_{4i+2})^2 = w_{8i+4}$, and $(ww_{4i+1})^2 = y_{8i+2}$.

- $\underline{KR(1)}_2 \xrightarrow{\eta} \underline{KR(1)}_1 \to \underline{K(1)}_1$ **RRK211**

$$P(x_{8i+2}) \otimes P(xx_{4i+3}) \xrightarrow[x_{8i+2}\to(y_{4i+1})^2]{xx_{4i+3}\to y_{4i+3}} P(y_{2i+1}) \otimes P(yy_{4i+2})$$

$$\xrightarrow[y_{4i+3}\to 0,\quad (y_{4i+1})^2\to 0]{yy_{4i+2}\to zz_{4i+2},\quad y_{4i+1}\to z_{4i+1}} E(z_{4i+1}) \otimes P(zz_{4i+2})$$

The cokernel is $E(y_{4i+1}) \otimes P(yy_{4i+2})$ and there is no kernel. We get a rare short exact sequence.

$$\textbf{S.E.S.} \qquad H_*(\underline{KR(1)}_2) \longrightarrow H_*(\underline{KR(1)}_1) \longrightarrow H_*(\underline{K(1)}_1).$$

- $\underline{KR(1)}_1 \to \underline{K(1)}_1 \to \underline{KR(1)}_3$ **RKR113**

$$P(x_{2i+1}) \otimes P(xx_{4i+2}) \xrightarrow[x_{4i+3}\to 0,\quad (x_{4i+1})^2\to 0]{xx_{4i+2}\to yy_{4i+2},\quad x_{4i+1}\to y_{4i+1}}$$

$$E(y_{4i+1}) \otimes P(yy_{4i+2}) \xrightarrow{zero} E(z_{8i+3}) \otimes P(zz_{8i+4})$$

There is no cokernel and the kernel is $P(x_{4i+3}) \otimes P((x_{4i+1})^2)$. Tor is $E(w_{4i}) \otimes E(ww_{8i+3})$. We have $(w_{4i})^2 = w_{8i}$ giving the $P(zz_{8i+4})$.

- $\underline{K(1)}_1 \to \underline{KR(1)}_3 \xrightarrow{\eta} \underline{KR(1)}_2$ **KRR132**

$$E(x_{4i+1}) \otimes P(xx_{4i+2}) \xrightarrow{zero} E(y_{8i+3}) \otimes P(yy_{8i+4})$$

$$\xrightarrow[y_{8i+3}\to 0]{yy_{16i+4}\to(z_{8i+2})^2,\quad yy_{16i+12}\to(zz_{4i+3})^4} P(z_{8i+2}) \otimes P(zz_{4i+3})$$

The cokernel is $E(y_{8i+3}) \otimes P(yy_{8i+4})$ and the kernel $E(x_{4i+1}) \otimes P(xx_{4i+2})$. Tor is $\Gamma[w_{4i+2}] \otimes E(ww_{4i+3})$. We have

$$d_2(\gamma_2(w_{4i+2})) = y_{8i+3},$$

$(w_{4i+2})^2 = yy_{8i+4}$, and $(ww_{4i+3})^2 = w_{8i+6}$.

- $\underline{KR(1)}_3 \xrightarrow{\eta} \underline{KR(1)}_2 \to \underline{K(1)}_2$ **RRK322**

$$E(x_{8i+3}) \otimes P(xx_{8i+4}) \xrightarrow[x_{8i+3}\to 0]{xx_{16i+4}\to(y_{8i+2})^2,\quad xx_{16i+12}\to(yy_{4i+3})^4} P(y_{8i+2}) \otimes P(yy_{4i+3})$$

$$\xrightarrow[(yy_{4i+3})^4\to 0,\quad (y_{8i+2})^2\to 0]{y_{8i+2}\to zzz_{8i+2},\quad yy_{4i+3}\to z_{4i+3}} TP_4(z_{4i+3}) \otimes E(zz_{4i}) \otimes E(zzz_{8i+2})$$

The cokernel is $E(y_{8i+2}) \otimes TP_4(yy_{4i+3})$ and the kernel is $E(x_{8i+3})$. Tor of this is $\Gamma[w_{8i+4}] = E(w_{4i})$.

- $\underline{KR(1)}_2 \to \underline{K(1)}_2 \to \underline{KR(1)}_4$ **RKR224**

$$P(x_{8i+2}) \otimes P(xx_{4i+3}) \xrightarrow[(xx_{4i+3})^4 \to 0, \quad (x_{8i+2})^2 \to 0]{x_{8i+2} \to yyy_{8i+2}, \quad xx_{4i+3} \to y_{4i+3}}$$

$$TP_4(y_{4i+3}) \otimes E(yy_{4i}) \otimes E(yyy_{8i+2}) \xrightarrow[y_{4i+3} \to 0, \quad yyy_{8i+2} \to 0]{yy_{4i} \to z_{4i}} E(z_{4i}) \otimes E(zz_{8i+5})$$

The cokernel is $E(yy_{4i})$ and the kernel is $P((x_{8i+2})^2) \otimes P((xx_{4i+3})^4)$. Tor of this is $E(w_{16i+5}) \otimes E(ww_{16i+13})$.

- $\underline{K(1)}_2 \to \underline{KR(1)}_4 \xrightarrow{\eta} \underline{KR(1)}_3$ **KRR243**

$$TP_4(x_{4i+3}) \otimes E(xx_{4i}) \otimes E(xxx_{8i+2}) \xrightarrow[x_{4i+3} \to 0, \quad xxx_{8i+2} \to 0]{xx_{4i} \to y_{4i}}$$

$$E(y_{4i}) \otimes E(yy_{8i+5}) \xrightarrow{zero} E(z_{8i+3}) \otimes P(zz_{8i+4})$$

The cokernel is $E(yy_{8i+5})$ and the kernel is $TP_4(x_{4i+3}) \otimes E(xxx_{8i+2})$. Tor is

$$E(w_{4i}) \otimes \Gamma[ww_{16i+14}] \otimes \Gamma[www_{8i+3}]$$

There is no $E(yy_{8i+5})$ so it must be hit by a differential. Rewrite Tor as

$$E(w_{4i}) \otimes \Gamma[ww_{8i+6}] \otimes E(www_{8i+3})$$

where the ww_{8i+6} is in filtration 2. The differential is now obvious

$$d_2(ww_{8i+6}) = yy_{8i+5}$$

This leaves

$$E(w_{4i}) \otimes E(www_{8i+3})$$

We must have $(w_{4i})^2 = w_{8i}$ giving us our $P(w_{8i+4})$.

- $\underline{KR(1)}_4 \xrightarrow{\eta} \underline{KR(1)}_3 \to \underline{K(1)}_3$ **RRK433**

$$E(x_{4i}) \otimes E(xx_{8i+5}) \xrightarrow{zero} E(y_{8i+3}) \otimes P(yy_{8i+4})$$

$$\xrightarrow[y_{8i+3} \to 0]{yy_{8i+4} \to (zz_{4i+2})^2} E(z_{4i+1}) \otimes P(zz_{4i+2})$$

The cokernel is $E(y_{8i+3}) \otimes P(yy_{8i+4})$ and the kernel is $E(x_{4i}) \otimes E(xx_{8i+5})$. Tor is $\Gamma[w_{4i+1}] \otimes \Gamma[ww_{8i+6}]$. Rewriting the spectral sequence we have

$$E(y_{8i+3}) \otimes P(yy_{8i+4}) \otimes E(w_{4i+1}) \otimes E(w_{8i+2}) \otimes$$

$$\Gamma[w_{16i+4}] \otimes E(ww_{8i+6}) \otimes \Gamma[ww_{16i+12})$$

where the generating terms are in filtrations 0, 0, 1, 2, 4, 1, 2, respectively. To kill y_{8i+3} we need two differentials

$$d_2(ww_{16i+12}) = y_{16i+11} \quad \text{and} \quad d_4(ww_{16i+4}) = y_{16i+3}$$

After this, what is left is

$$P(yy_{8i+4}) \otimes E(w_{4i+1}) \otimes E(w_{8i+2}) \otimes E(ww_{8i+6})$$

Combining the last two terms we get $E(zz_{4i+2})$ with extension $(zz_{4i+2})^2 = yy_{8i+4}$ to get our answer.

- $\underline{KR(1)}_3 \to \underline{K(1)}_3 \to \underline{KR(1)}_5$ **RKR335**

$$E(x_{8i+3}) \otimes P(xx_{8i+4}) \xrightarrow[x_{8i+3}\to 0]{xx_{8i+4}\to(yy_{4i+2})^2} E(y_{4i+1}) \otimes P(yy_{4i+2})$$
$$\xrightarrow[(yy_{4i+2})^2\to 0]{y_{4i+1}\to z_{4i+1},\quad yy_{4i+2}\to zz_{4i+2}} E(z_{4i+1}) \otimes E(zz_{2i})$$

The cokernel is $E(y_{4i+1}) \otimes E(yy_{4i+2})$ and the kernel is $E(x_{8i+3})$. Tor is $\Gamma[w_{8i+4}] = E(w_{4i})$.

- $\underline{K(1)}_3 \to \underline{KR(1)}_5 \xrightarrow{\eta} \underline{KR(1)}_4$ **KRR354**

$$E(x_{4i+1}) \otimes P(xx_{4i+2}) \xrightarrow[(xx_{4i+2})^2\to 0]{x_{4i+1}\to y_{4i+1},\quad xx_{4i+2}\to yy_{4i+2}}$$
$$E(y_{4i+1}) \otimes E(yy_{2i}) \xrightarrow[y_{4i+1}\to 0,\quad yy_{4i+2}\to 0]{yy_{4i}\to z_{4i}} E(z_{4i}) \otimes E(zz_{8i+5})$$

The cokernel is $E(yy_{4i})$ and the kernel is $P((xx_{4i+2})^2)$. Tor is $E(w_{8i+5})$.

- $\underline{KR(1)}_5 \xrightarrow{\eta} \underline{KR(1)}_4 \to \underline{K(1)}_4$ **RRK544**

$$E(x_{4i+1}) \otimes E(xx_{2i}) \xrightarrow[x_{4i+1}\to 0,\quad xx_{4i+2}\to 0]{xx_{4i}\to y_{4i}}$$
$$E(y_{4i}) \otimes E(yy_{8i+5}) \xrightarrow{zero} TP_4(z_{4i+3}) \otimes E(zz_{4i}) \otimes E(zzz_{8i+2})$$

The cokernel is $E(yy_{8i+5})$ and the kernel is $E(x_{4i+1}) \otimes E(xx_{4i+2})$. Tor is $\Gamma[w_{4i+2}] \otimes \Gamma[ww_{4i+3}]$. There is no yy_{8i+5} in our answer and the only differential that could hit it is

$$d_2(\gamma_2(ww_{4i+3})) = yy_{8i+5}$$

This leaves us with $\Gamma[w_{4i+2}]\otimes E(ww_{4i+3}) = E(w_{2i})\otimes E(ww_{4i+3})$. We need $(ww_{4i+3})^2 = w_{8i+6}$.

- $\underline{KR(1)}_4 \to \underline{K(1)}_4 \to \underline{KR(1)}_6$ **RKR446**

$$E(x_{4i}) \otimes E(xx_{8i+5}) \xrightarrow{zero} TP_4(y_{4i+3}) \otimes E(yy_{4i}) \otimes E(yyy_{8i+2})$$
$$\xrightarrow[yy_{4i}\to (z_{2i}^2),\quad yyy_{8i+2}\to (z_{4i+1})^2]{y_{4i+3}\to z_{4i+3}} TP_4(z_i)$$

The cokernel is $TP_4(y_{4i+3})\otimes E(yy_{4i})\otimes E(yyy_{8i+2})$ and the kernel is $E(x_{4i})\otimes E(xx_{8i+5})$. Tor is $\Gamma[w_{4i+1}]\otimes\Gamma[ww_{8i+6}]$. We have $(w_{4i+1})^2 = yyy_{8i+2}$ leaving, in Tor, $\Gamma[www_{4i+2}] = E(www_{2i})$. We have $(www_{2i})^2 = yy_{4i}$.

- $\underline{K(1)}_4 \to \underline{KR(1)}_6 \xrightarrow{\eta} \underline{KR(1)}_5$ **KRR465**

$$TP_4(x_{4i+3}) \otimes E(xx_{4i}) \otimes E(xxx_{8i+2}) \xrightarrow[xx_{4i}\to (y_{2i})^2,\quad xxx_{8i+2}\to (y_{4i+1})^2]{x_{4i+3}\to y_{4i+3}}$$
$$TP_4(y_i) \xrightarrow[y_{4i+3}\to 0,\quad (y_i)^2\to 0]{y_{2i}\to zz_{2i},\quad y_{4i+1}\to z_{4i+1}} E(z_{4i+1}) \otimes E(zz_{2i})$$

The cokernel is $E(y_{2i})\otimes E(y_{4i+1})$ and there is no kernel, giving us another rare short exact sequence

$$\textbf{S.E.S.}\qquad H_*(\underline{K(1)}_4) \longrightarrow H_*(\underline{KR(1)}_6) \longrightarrow H_*(\underline{KR(1)}_5).$$

- $\underline{KR(1)}_6 \xrightarrow{\eta} \underline{KR(1)}_5 \rightarrow \underline{K(1)}_5$ **RRK655**

$$TP_4(x_i) \xrightarrow[x_{4i+3}\to 0,\,(x_i)^2\to 0]{x_{2i}\to yy_{2i},\, x_{4i+1}\to y_{4i+1}} E(y_{4i+1}) \otimes E(yy_{2i}) \xrightarrow{zero} E(z_{4i+1}) \otimes P(zz_{4i+2})$$

There is no cokernel. The kernel is

$$TP_4(x_{4i+3}) \otimes E((x_{2i})^2) \otimes E((x_{4i+1})^2)$$

Tor, in filtration 1, is

$$E(w_{4i}) \otimes E(ww_{4i+1}) \otimes E(www_{8i+3})$$

and in filtration 2, combining all of the Γ, we get $\Gamma[wwww_{4i+2}]$. (This takes some manipulation.) The $E(ww_{4i+1})$ must stay and the $E(www_{8i+3})$ must go. To make it go, we have a

$$d_3(\gamma_2(wwww_{4i+2})) = www_{8i+3}$$

All we have left now is

$$E(w_{4i}) \otimes E(ww_{4i+1}) \otimes E(wwww_{4i+2})$$

We have $(w_{4i})^2 = w_{8i}$ and $(wwww_{4i+2})^2 = w_{8i+4}$.

- $\underline{KR(1)}_5 \rightarrow \underline{K(1)}_5 \rightarrow \underline{KR(1)}_7$ **RKR557**

$$E(x_{4i+1}) \otimes E(xx_{2i}) \xrightarrow{zero} E(y_{4i+1}) \otimes P(yy_{4i+2})$$

$$\xrightarrow[y_{4i+1}\to 0]{yy_{4i+2}\to z_{4i+2}+(zz_{2i+1})^2} E(z_{2i}) \otimes P(zz_{2i+1})$$

We have cokernel $E(y_{4i+1}) \otimes P(yy_{4i+2})$ and kernel $E(x_{4i+1}) \otimes E(xx_{2i})$. Tor is $\Gamma[w_{4i+2}] \otimes \Gamma[ww_{2i+1}]$. We have

$$d_2(\gamma_2(ww_{2i+1})) = y_{4i+1}$$

and $(ww_{2i+1})^2 = yy_{4i+2}$. The second map is unusual and doesn't come from this. We'll pick it up in the next spectral sequence, **KRR576**.

Remark. So little interesting happens that I should point out when something does. The last spectral sequence did not completely describe the second map. The differential is correct, and we get the correct answer from the stated extension. However, what really happens is and $(ww_{2i+1})^2 = yy_{4i+2} + w_{4i+2}$. This doesn't change our answer, but if you look at the cokernel of the second map, we get a $TP_4(ww_{2i+1})$ this way. We can't see that in the last spectral sequence, but we'll see it in the next.

- $\underline{K(1)}_5 \rightarrow \underline{KR(1)}_7 \xrightarrow{\eta} \underline{KR(1)}_6$ **KRR576**

$$E(x_{4i+1}) \otimes P(xx_{4i+2}) \xrightarrow[x_{4i+1}\to 0]{xx_{4i+2}\to y_{4i+2}+(yy_{2i+1})^2} E(y_{2i}) \otimes P(yy_{2i+1})$$

$$\xrightarrow[y_{4i+2}+(yy_{2i+1})^2\to 0]{y_{2i}\to (z_i)^2,\quad yy_{2i+1}\to z_{2i+1}} TP_4(z_i)$$

We have a new problem here, and that is that the previous spectral sequence, **RKR557**, didn't pick up the details of the second map there, i.e. the first map of this spectral sequence. We didn't need it there, but do here. All we really know from the previous spectral sequence is that the $P(xx_{4i+2})$ injects, but there are two ways to do that. (1) $xx_{4i+2} \to (yy_{2i+1})^2$ or (2) $xx_{4i+2} \to y_{4i+2} + (yy_{2i+1})^2$.

If we try the first, our cokernel is $E(y_{2i}) \otimes E(yy_{2i+1})$ and the kernel is $E(x_{4i+1})$. Taking Tor of this gives $\Gamma[w_{4i+2}]$. There can be no $E(yy_{2i+1})$ as a subalgebra of $H_*(\underline{KR(1)}_6)$. The only possibilities for differentials are on $\gamma_{2^j}(w_{4i+2})$ (with $j > 0$). These elements are all in degrees divisible by 4 so can never hit a yy_{4i+1}. Version (1) cannot be correct, so try (2). The cokernel now is $TP_4(yy_{2i+1}) \otimes E(y_{4i})$ and the kernel is still $E(x_{4i+1})$ giving Tor as $\Gamma[w_{4i+2}] = w_{2i}$. We have $(w_{2i})^2 = y_{4i}$.

This computes the first map here and the second map in the previous spectral sequence, **RKR557**.

- $\underline{KR(1)}_7 \xrightarrow{\eta} \underline{KR(1)}_6 \to \underline{K(1)}_6$ **RRK766**

$$E(x_{2i}) \otimes P(xx_{2i+1}) \xrightarrow[x_{4i+2}+(xx_{2i+1})^2\to 0]{x_{2i}\to(y_i)^2,\quad xx_{2i+1}\to y_{2i+1}} TP_4(y_i)$$

$$\xrightarrow[y_{2i+1}\to 0,\quad (y_i)^2\to 0,\quad y_{4i}\to zz_{4i}]{y_{8i+2}\to zzz_{8i+2},\quad y_{8i+6}\to(x_{4i+3})^2} TP_4(z_{4i+3}) \otimes E(zz_{4i}) \otimes E(zzz_{8i+2})$$

The cokernel is $E(y_{2i})$ and the kernel is $P(x_{4i+2} + (xx_{2i+1})^2)$. Tor of this is $E(w_{4i+3})$. We have $(w_{4i+3})^2 = y_{8i+6}$.

- $\underline{KR(1)}_6 \to \underline{K(1)}_6 \to \underline{KR(1)}_0$ **RKR660**

$$TP_4(x_i) \xrightarrow[x_{2i+1}\to 0,\quad (x_{2i})^2\to 0,\quad x_{4i}\to yy_{4i}]{x_{8i+2}\to yyy_{8i+2},\quad x_{8i+6}\to(x_{4i+3})^2} TP_4(y_{4i+3}) \otimes E(yy_{4i}) \otimes E(yyy_{8i+2})$$

$$\xrightarrow[(y_{4i+3})^2\to 0,\quad yyy_{8i+2}\to 0]{y_{4i+3}\to z_{4i+3},\quad yy_{4i}\to 0} E(z_i) \otimes P(zz_{4i+2})$$

The cokernel is $E(y_{4i+3})$ and the kernel is $TP_4(x_{2i+1}) \otimes E((x_{2i})^2)$. Tor of this is $E(w_{2i}) \otimes \Gamma[ww_{8i+6}] \otimes \Gamma[www_{4i+1}]$. We can rewrite this as algebras to be $E(w_{2i}) \otimes E(www_{4i+1}) \otimes \Gamma[ww_{4i+2}]$, which is $E(w_{2i}) \otimes E(www_{4i+1}) \otimes E(ww_{2i})$. We have $(w_{2i})^2 = w_{4i}$. This conclusion is a bit tricky. The ww_{2i} are all in higher filtrations and for degree reasons, they could only square to w_{4i}. However, if that were the case, we would not have enough even degree exterior generators.

- $\underline{K(1)}_6 \to \underline{KR(1)}_0 \xrightarrow{\eta} \underline{KR(1)}_7$ **KRR607**

$$TP_4(x_{4i+3}) \otimes E(xx_{4i}) \otimes E(xxx_{8i+2}) \xrightarrow[(x_{4i+3})^2\to 0,\quad xxx_{8i+2}\to 0]{x_{4i+3}\to y_{4i+3},\quad xx_{4i}\to 0}$$

$$E(y_i) \otimes P(yy_{4i+2}) \xrightarrow[y_{2i}\to z_{2i},\quad y_{2i+1}\to 0]{yy_{4i+2}\to(zz_{2i+1})^2} E(z_{2i}) \otimes P(zz_{2i+1})$$

The cokernel is $E(y_{4i+1}) \otimes E(y_{2i}) \otimes P(yy_{4i+2})$. The kernel is $E(xx_{4i}) \otimes E(xxx_{8i+2}) \otimes E((x_{4i+3})^2)$. Tor of this is $\Gamma[w_{2i+1}]$. We need

$$d_2(\gamma_2(w_{2i+1})) = y_{4i+1}$$

To finish things off we have $(w_{2i+1})^2 = yy_{4i+2}$.

- $\underline{KR(1)}_0 \xrightarrow{\eta} \underline{KR(1)}_7 \to \underline{K(1)}_7$ **RRK077**

$$E(x_i) \otimes P(xx_{4i+2}) \xrightarrow[x_{2i}\to y_{2i}, \quad x_{2i+1}\to 0]{xx_{4i+2}\to (yy_{2i+1})^2} E(y_{2i}) \otimes P(yy_{2i+1})$$

$$\xrightarrow[y_{2i}\to 0, \quad yy_{4i+3}\to 0, \quad (yy_{4i+1})^2\to 0]{yy_{4i+1}\to z_{4i+1}} E(z_{4i+1}) \otimes P(zz_{4i+2})$$

The cokernel is $E(yy_{2i+1})$ and the kernel is $E(x_{2i+1})$. Tor of this is $\Gamma[w_{2i}]$. We have

$$d_2(\gamma_2(w_{2i})) = yy_{4i-1}$$

This leaves $E(x_{4i+1}) \otimes E(w_{2i})$. We have $(w_{2i})^2 = w_{4i}$ to get our answer.

- $\underline{KR(1)}_7 \to \underline{K(1)}_7 \to \underline{KR(1)}_1$ **RKR771**

$$E(x_{2i}) \otimes P(xx_{2i+1}) \xrightarrow[x_{2i}\to 0, \quad xx_{4i+3}\to 0, \quad (xx_{4i+1})^2\to 0]{xx_{4i+1}\to y_{4i+1}}$$

$$E(y_{4i+1}) \otimes P(yy_{4i+2}) \xrightarrow[y_{4i+1}\to 0]{yy_{4i+2}\to (z_{2i+1})^2} P(z_{2i+1}) \otimes P(zz_{4i+2})$$

The cokernel is $P(yy_{4i+2})$. The kernel is $E(x_{2i}) \otimes P(xx_{4i+3}) \otimes P((xx_{4i+1})^2)$. Tor of this is $\Gamma[w_{2i+1}] \otimes E(ww_{4i}) \otimes E(www_{8i+3})$. The way to untangle this mess is to have

$$d_3(\gamma_4(w_{2i+1})) = www_{8i+3}$$

leaving $P(yy_{4i+2})$ in filtration zero, $E(w_{2i+1}) \otimes E(ww_{4i})$ with generators in filtration 1, and $E(w_{4i+2})$ with generators in filtration 2.

The only way this can work out is

$$(w_{2i+1})^2 = yy_{4i+2} \qquad (ww_{4i+2})^2 = ww_{8i+4} \qquad (ww_{4i})^2 = ww_{8i}$$

- $\underline{K(1)}_7 \to \underline{KR(1)}_1 \xrightarrow{\eta} \underline{KR(1)}_0$ **KRR710**

$$E(x_{4i+1}) \otimes P(xx_{4i+2}) \xrightarrow[x_{4i+1}\to 0]{xx_{4i+2}\to (y_{2i+1})^2} P(y_{2i+1}) \otimes P(yy_{4i+2})$$

$$\xrightarrow[(y_{2i+1})^2\to 0]{y_{2i+1}\to z_{2i+1}, \quad yy_{4i+2}\to zz_{4i+2}} E(z_i) \otimes P(zz_{4i+2})$$

The cokernel is $E(y_{2i+1}) \otimes P(yy_{4i+2})$ and the kernel is $E(x_{4i+1})$. Tor of this is $\Gamma[w_{4i+2}] = E(w_{2i})$.

11. Appendix to the appendix: Homotopy long exact sequences

Just for my purposes, I want to write down all the homotopy exact sequences associated with the fibrations from the main paper once and for all.

i	$\pi_i(\underline{KU}_0) \xrightarrow{2}$	$\pi_i(\underline{KU}_0) \to$	$\pi_i(\underline{K(1)}_0)$
1	0	0	0
0	$\mathbb{Z} \xrightarrow{2}$	$\mathbb{Z} \longrightarrow$	$\mathbb{Z}/(2)$

i	$\pi_i(\underline{KO}_1)$	$\xrightarrow{\eta}$	$\pi_i(\underline{KO}_0)$	$\longrightarrow$	$\pi_i(\underline{KU}_0)$
7	0		0		0
6	0		0		$\mathbb{Z}$
5	$\mathbb{Z}$		0		0
4	0		$\mathbb{Z}$	$\xrightarrow{2}$	$\mathbb{Z}$
3	$\mathbb{Z}/(2)$		0		0
2	$\mathbb{Z}/(2)$	$\xrightarrow{=}$	$\mathbb{Z}/(2)$		$\mathbb{Z}$
1	$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}/(2)$		0
0	0		$\mathbb{Z}$	$\xrightarrow{=}$	$\mathbb{Z}$

i	$\pi_i(\underline{KO}_0)$	$\xrightarrow{2}$	$\pi_i(\underline{KO}_0)$	$\longrightarrow$	$\pi_i(\underline{KR(1)}_0)$
7	0		0		0
6	0		0		0
5	0		0		0
4	$\mathbb{Z}$	$\xrightarrow{2}$	$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}/(2)$
3	0		0		$\mathbb{Z}/(2)$
2	$\mathbb{Z}/(2)$		$\mathbb{Z}/(2)$	$\xrightarrow{2}$	$\mathbb{Z}/(4)$
1	$\mathbb{Z}/(2)$		$\mathbb{Z}/(2)$	$\xrightarrow{=}$	$\mathbb{Z}/(2)$
0	$\mathbb{Z}$	$\xrightarrow{2}$	$\mathbb{Z}$	$\longrightarrow$	$\mathbb{Z}/(2)$

i	$\pi_i(\underline{KR(1)}_1)$	$\xrightarrow{\eta}$	$\pi_i(\underline{KR(1)}_0)$	$\longrightarrow$	$\pi_i(\underline{K(1)}_0)$
7	0		0		0
6	0		0		$\mathbb{Z}/(2)$
5	$\mathbb{Z}/(2)$		0		0
4	$\mathbb{Z}/(2)$	$\xrightarrow{=}$	$\mathbb{Z}/(2)$		$\mathbb{Z}/(2)$
3	$\mathbb{Z}/(4)$	$\longrightarrow$	$\mathbb{Z}/(2)$		
2	$\mathbb{Z}/(2)$	$\xrightarrow{2}$	$\mathbb{Z}/(4)$	$\longrightarrow$	$\mathbb{Z}/(2)$
1	$\mathbb{Z}/(2)$	$\xrightarrow{=}$	$\mathbb{Z}/(2)$		0
0	0		$\mathbb{Z}/(2)$	$\xrightarrow{=}$	$\mathbb{Z}/(2)$

References

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