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Coassembly is a homotopy limit map

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In memory of Bruce Williams

We prove a claim by Williams that the coassembly map is a homotopy limit map. As an application, we show that the homotopy limit map for the coarse version of equivariant A-theory agrees with the coassembly map for bivariant A-theory that appears in the statement of the topological Riemann–Roch theorem.

1. Introduction

In the celebrated paper [Dwyer et al. 2003], Dwyer, Weiss, and Williams give index-theoretic conditions that are necessary and sufficient for a perfect fibration $E \to B$ to be equivalent to a fiber bundle with fibers compact topological (or smooth) manifolds. Williams [2000] defines a bivariant version of A-theory for fibrations, which is contravariant in one variable and therefore comes with a coassembly map. He then reinterprets the condition from [Dwyer et al. 2003] as the condition that a certain class in bivariant A-theory (the Euler characteristic), after applying the coassembly map, lifts either along the assembly map or the inclusion of stable homotopy into $A(X)$.

In this paper, we show that coassembly maps in general agree with homotopy limit maps, the latter being more amenable to computations. In particular, this shows that the target of Williams' coassembly can be interpreted as a homotopy fixed point spectrum, which has an associated homotopy fixed point spectral sequence that computes its homotopy groups. Together with well known formulas

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for the assembly map, e.g., in [Malkiewich 2017, Definition 6.2], this means we get combinatorial formulas for each of the maps used in the statement of the bivariant topological and smooth Riemann–Roch theorems from [Williams 2000].

In general, the homotopy limit map is defined for any topological group G and G-space or G-spectrum X as the map from fixed points to homotopy fixed points,

$$X^G \to X^{hG}.$$

Atiyah proved that for KU with C_2-action induced by complex conjugation the homotopy limit map is an equivalence. In general, this is not the case, and the homotopy limit problem, beautifully described in [Thomason 1983], asks how close the homotopy limit map is to being an equivalence. Some of the classical examples of interest are Segal's conjecture where $X = \mathbb{S}_G$, the sphere spectrum for G finite, the Atiyah–Segal completion theorem, where $X = KU_G$, equivariant topological K-theory for G compact Lie, and Thomason's theorem, where $X = KE$, the algebraic K-theory of a finite Galois extension with Galois group action. In all of these cases, the homotopy limit map is shown to become an equivalence after suitable completion or inversion of an element in the homotopy groups of the fixed point spectrum. More recent solutions of homotopy limit problems appear in [Hu et al. 2011; Röndigs et al. 2018; Heard 2017], which study the homotopy limit problem for KGL, the motivic spectrum representing algebraic K-theory, with C_2-action.

On the other hand, the coassembly map considered in [Williams 2000] is defined for any reduced contravariant homotopy functor F, whose domain is the category of spaces over BG. It is a natural transformation $F \to F_{\%}$, one that universally approximates F by a functor that sends homotopy pushouts to homotopy pullbacks. It is formally dual to the assembly map of [Weiss and Williams 1995; Davis and Lück 1998], which by [Hambleton and Pedersen 2004; Davis and Lück 1998] coincides with the assembly map of the Farrell–Jones conjecture [1993]. A comprehensive recent survey on assembly maps is given in [Lück 2019]. The coassembly map is also a close analog of the linear approximation map of embedding calculus [Weiss 1999; Goodwillie and Weiss 1999]. Further applications of the coassembly map appear in [Cohen and Klein 2009; Raptis and Steimle 2014; Malkiewich 2017].

Our first result is a precise correspondence between these two constructions. We only consider topological groups G that are the realization of a simplicial group $G_\bullet$, and we focus on the case where F takes values in spectra, because the corresponding result for spaces is similar and a little easier. Without loss of generality, we assume that the homotopy functor F is enriched in simplicial sets, so that $F(EG)$ carries a continuous left action by G, and $F(BG)$ maps to its fixed points. We may then make $F(EG)$ into a G-spectrum whose fixed points are $F(BG)$. An analog of this result for the assembly map can be found in [Davis and Lück 1998, §5.2].

Theorem A (Theorem 3.6). *Let G be a group that is the realization of a simplicial group $G_\bullet$. The coassembly map on the terminal object $F(BG) \to F_{\%}(BG)$ is equivalent to the homotopy limit map of this G-spectrum, $F(BG) \to F(EG)^{hG}$.*

This is similar to a claim in [Williams 2000], when F is a contravariant form of algebraic K-theory and $G \simeq \Omega X$. Giving a precise proof amounts to showing that diagrams on a suitable category of contractible spaces over BG correspond to G-objects, plus a little more structure. Our version of the argument uses parametrized spectra to form a bridge between the two settings.

Our second result applies Theorem A to Williams' bivariant A-theory functor $\boldsymbol{A}(E \to B)$ to fibrations of the form $EG \times_G X \to BG$ where G is a finite group. This gives the homotopy limit map of the "coarse" equivariant A-theory G-spectrum from [Malkiewich and Merling 2019], equivalently the K-theory of group actions from [Barwick et al. 2020] applied to retractive spaces over X.

Theorem B (Theorem 4.2). *In the stable homotopy category, the homotopy limit map for $\boldsymbol{A}_G^{\text{coarse}}(X)$ is isomorphic to the coassembly map for bivariant A-theory*:

$$\begin{tikzcd} \boldsymbol{A}_G^{\text{coarse}}(X)^H \arrow[r] \arrow[d, "\Phi"', "\sim"] & \boldsymbol{A}_G^{\text{coarse}}(X)^{hH} \arrow[d, "\sim"] \\ \boldsymbol{A}(EG \times_H X \to BH) \arrow[r] & \boldsymbol{A}_{\%}(EG \times_H X \to BH) \end{tikzcd}$$

This is not quite a direct consequence of Theorem A because we have to show that the equivalence between the two theories preserves the G-actions and inclusions of fixed points, up to some coherent homotopies.

Remark. This provides one half of an argument that would significantly generalize the main theorem of [Malkiewich 2017]. The other half relies on a conjectural connection between assembly maps and the Adams isomorphism, which we do not pursue here.

Remark. This paper does not consider the homotopy limit problems for profinite groups, which involve a modified definition of homotopy fixed points that are associated to the continuous cohomology of the profinite group [Devinatz and Hopkins 2004]. Our homotopy limit map is the usual one from, e.g., [Bousfield and Kan 1972, Chapter XI, §3.5], and we only consider those topological groups that are geometric realizations of simplicial groups. The main example we have in mind is ΩX.

Conventions. Throughout, all of our topological spaces are compactly generated weak Hausdorff (CGWH) [Lewis 1978, Appendix A; Strickland 2009]. Unless otherwise noted, the term "spectra" can be interpreted to mean prespectra, symmetric spectra, or orthogonal spectra. See [Mandell et al. 2001] for more information about how to pass between these different models. The term "naïve G-spectrum" refers to a spectrum with an action by the group G, up to maps that are equivalences on all of the categorical fixed point spectra X^H subgroups $H \le G$. Equivalently, this can be viewed as a diagram of spectra on the orbit category $\mathbb{O}(G)^{\mathrm{op}}$. In fact, we will only be concerned with diagrams restricted to the trivial orbit G/G and the full orbit G/e, corresponding to the data of the G-fixed points of a naïve G-spectrum and its underlying spectrum with G-action.

2. Review of coassembly

Let B be an unbased space and let $\mathcal{U}_B$ denote the comma category of spaces over B. A commuting square in $\mathcal{U}_B$ is a homotopy pushout square if it is such when we forget the maps to B. A contravariant functor F from $\mathcal{U}_B$ to spectra is

- *reduced* if it sends $\varnothing \to B$ to a weakly contractible spectrum,
- a *homotopy functor* if it sends weak equivalences of spaces to stable equivalences of spectra, and
- *excisive* if it is a reduced homotopy functor that sends coproducts and homotopy pushout squares of spaces to products and homotopy pullback squares of spectra, respectively.

Note that this last condition can be stated in several equivalent ways, the simplest of which is that F takes all homotopy colimits to homotopy limits.

If F is a contravariant reduced homotopy functor from $\mathcal{U}_B$ to spectra, consider the comma category of excisive functors P with natural transformations $F \to P$. Define a weak equivalence of such functors to be a natural transformation $P \to P'$ (under F) that is a stable equivalence at every object. Inverting these equivalences gives the homotopy category of excisive functors under F.

Proposition 2.1 [Cohen and Klein 2009; Malkiewich 2017, Proposition 5.4; 2015, §7]. *The homotopy category of excisive functors under F has an initial object $F_\%$, in other words a universal approximation of F by an excisive functor. The natural transformation $F \to F_\%$ can be given by the formula*

$$F(X \to B) \to \operatorname*{holim}_{(\Delta^n \to X) \in \Delta_X^{\mathrm{op}}} F((\Delta^n \amalg B) \to B).$$

Here $\Delta_X = \Delta_{\mathrm{Sing}\, X}$ is the category of simplices in the simplicial set $\mathrm{Sing}\, X$. Concretely, it has an object for every continuous map $\Delta^n \to X$ and a morphism for every factorization $\Delta^p \to \Delta^q \to X$ where $\Delta^p \to \Delta^q$ is a composite of inclusions of a face. There is a natural "last vertex" operation that gives an equivalence $|\Delta_X| \xrightarrow{\sim} X$ [Goerss and Jardine 2009, Chapter III, §4; Malkiewich 2017, §5].

We could alternatively describe $F_\%(X \to B)$ as the spectrum of sections of a parametrized spectrum over X whose fiber over x is $F((x \amalg B) \to B)$. See [Weiss and Williams 1995; Williams 2000; Cohen and Klein 2009; Malkiewich 2015; 2017] for more details and other explicit constructions of the coassembly map.

3. Proof of Theorem A

The first step is to interpret both the homotopy limit map and the coassembly map as the unit of an adjunction.

Let $G_\bullet$ be a simplicial group with realization $G = |G_\bullet|$, and let BG be the topological bar construction of G. It will be convenient for us to let $\mathcal{U}_{BG}$ refer to the category of unbased spaces over BG that are homotopy equivalent to cell complexes, as opposed to all spaces over BG. Recall that $\Delta_{BG} \subseteq \mathcal{U}_{BG}$ is the subcategory of spaces over BG consisting only of the simplices $\Delta^p \to BG$ for varying $p \geq 0$ and the compositions of face maps. Note that a homotopy functor on this subcategory must send every map to a weak equivalence.

Proposition 3.1. *For reduced homotopy functors on spaces over BG, the coassembly map is the unit of the adjunction of homotopy categories*

$$\begin{tikzcd} \boxed{\begin{array}{c}\textit{reduced homotopy functors}\\ F:\mathcal{U}_{BG}^{\mathrm{op}}\to\mathcal{S}p\end{array}} \arrow[r, shift left=1.5, "\textit{restrict}"] \arrow[r, phantom, "\perp"] & \boxed{\begin{array}{c}\textit{homotopy functors}\\ F:\Delta_{BG}^{\mathrm{op}}\to\mathcal{S}p\end{array}} \arrow[l, shift left=1.5, "\operatorname*{holim}_{\Delta^p\to X} F(\Delta^p)"] \end{tikzcd}$$

Proof. We first examine the larger homotopy category of all functors. It is standard that the homotopy right Kan extension is the right adjoint of restriction. Furthermore, the canonical map of F into the extension of the restriction of F is the unit of this adjunction. By [Cohen and Klein 2009, §5] or [Malkiewich 2015, §7], this particular model for the homotopy right Kan extension sends homotopy functors to reduced homotopy functors, so the adjunction descends to these subcategories, with the same unit. $\square$

Let $\mathcal{B}G_\bullet$ be the simplicially enriched category with one object $[e]$ and morphism space $G_\bullet$. Note that $BG \cong |\mathcal{B}G_\bullet|$. Let $C(\mathcal{B}G_\bullet)$ be the "cone" category with one additional object $[G]$ and one additional nontrivial morphism $[G] \to [e]$. This is isomorphic to the full subcategory of the enriched orbit category $\mathcal{O}(G)^{\mathrm{op}}$ on the orbits G/e and G/G. Let $\iota : \mathcal{B}G_\bullet \to C(\mathcal{B}G_\bullet)$ be the inclusion.

Remark. If X is a G-space or naïve G-spectrum, then X^G and $X = X^{\{e\}}$ form a diagram over $C(\mathcal{B}G_\bullet)$. If X is a genuine orthogonal G-spectrum, the same is true for the genuine fixed points X^G, by taking a fibrant replacement, then passing to the underlying naïve G-spectrum.

Proposition 3.2. *For naïve G-spectra, the map $(-)^G \to (-)^{hG}$ is equivalent to the unit of the adjunction of homotopy categories*

$$\begin{tikzcd} \boxed{\begin{array}{c}\textit{enriched } C(\mathcal{B}G_\bullet)\\ \textit{diagrams of spectra}\end{array}} \arrow[r, shift left=1.5, "\iota^*"] \arrow[r, phantom, "\perp"] & \boxed{\begin{array}{c}\textit{enriched } \mathcal{B}G_\bullet\\ \textit{diagrams of spectra}\\ (\textit{i.e., spectra with } G\textit{-action})\end{array}} \arrow[l, shift left=1.5, "\begin{array}{c}\textit{enriched homotopy}\\ \textit{right Kan extension}\end{array}"] \end{tikzcd}$$

evaluated at $[G]$.

Proof. This is immediate from the local formula for an enriched homotopy right Kan extension [Riehl 2014, Example 7.6.6]. □

The next step is to relate the categories on the left-hand side of these adjunctions together. Morally, we take each homotopy functor F to the diagram on $C(\mathcal{B}G_\bullet)$ given by $F(BG)$ and $F(EG)$.

There are two problems to address here. The first problem is that this is not an equivalence of homotopy categories, but we can fix that by localizing the category of homotopy functors along the maps that are equivalences on BG and EG. The second problem is that G will not act on $F(EG)$ unless we make F simplicially enriched. We fix the second problem using the following result.

Lemma 3.3. *Every contravariant homotopy functor F to spaces or spectra can be replaced by a simplicially enriched functor, by a zig-zag of equivalences of functors*

$$F \xleftarrow{\sim} F' \xrightarrow{\sim} \widetilde{F}'$$

that is itself functorial in F.

Proof. This is by a variant of the trick used in [Waldhausen 1985] to replace functors by homotopy functors. It adapts from covariant to contravariant functors by replacing $\mathrm{Map}(\Delta^p, -)$ with $\Delta^p \times -$.

If F lands in orthogonal spectra, regard it as landing in prespectra or symmetric spectra, and replace the spectrum $F(X)$ at each level by $F'(X) = |\mathrm{Sing}\, F(X)|$. The effect of this is that each degeneracy map $\Delta^p \to \Delta^q$ induces a levelwise cofibration $F'(\Delta^q \times X) \to F'(\Delta^p \times X)$. Then pass back up to orthogonal spectra if desired, and replace $F'(X)$ again by the realization

$$\widetilde{F}'(X) = |n \mapsto F'(\Delta^n \times X)|.$$

This defines a functor that receives a map from F' by inclusion of simplicial level 0. The map is an equivalence on each spectrum level, because F' is a homotopy functor and the simplicial space defined above is good and therefore Reedy cofibrant [Lillig 1973]. We extend the functor structure on $\widetilde{F}'$ to a simplicial enrichment by taking each map $|Y_\bullet| \times X \to Z$ to the realization of the map that at level k is

$$Y_k \times F'(\Delta^k \times X) \to F'(\Delta^k \times Z),$$

obtained from the map of spaces

$$Y_k \times \Delta^k \times X \to \Delta^k \times Z$$

whose coordinates are the action $Y_k \times \Delta^k \times X \to Z$ and the projection to Δ^k. □

Proposition 3.4. *The forgetful functors in the following diagram are equivalences of homotopy categories. Here "enriched" means simplicially enriched*:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\textit{reduced homotopy functors}\\ F:\mathcal{U}_{BG}^{\mathrm{op}}\to \mathcal{S}p\ (\textit{localized})\end{array}} \\
\boxed{\begin{array}{c}\textit{enriched reduced homotopy functors}\\ F:\mathcal{U}_{BG}^{\mathrm{op}}\to \mathcal{S}p\ (\textit{localized})\end{array}} \arrow[u, "\sim"'] \arrow[d, "\sim"] \\
\boxed{\begin{array}{c}\textit{enriched reduced functors}\\ F:\mathcal{U}_{BG}^{\mathrm{op}}\to \mathcal{S}p\ (\textit{localized})\end{array}} \arrow[d, "\sim"] \\
\boxed{\begin{array}{c}\textit{enriched functors}\\ C(\mathcal{B}G_\bullet)\to \mathcal{S}p\end{array}}
\end{tikzcd}$$

Proof. The construction of Lemma 3.3 gives an inverse to the first equivalence. Note this is still well defined after localizing because the construction preserves the property of a map of functors $F \to F'$ being an equivalence on one particular space X. For the second pair of categories, by Whitehead's theorem any enriched functor is a homotopy functor on the cofibrant and fibrant objects. Hence we can invert the forgetful functor by composing each F with a fibrant replacement in $\mathcal{U}_{BG}$. To check this respects the localization, we note that when we turn an enriched functor into a homotopy functor, it will have equivalent values on EG and BG, because these two spaces are already fibrant. For the final pair of categories, the restriction functor has the enriched homotopy right Kan extension as its right adjoint, and this adjunction clearly descends to the localization. In fact, since $C(\mathcal{B}G_\bullet)$ is a full subcategory of $\mathcal{U}_{BG}^{\mathrm{op}}$, the counit is an equivalence, and therefore by the definition of our localization, the unit is also an equivalence; hence we get an equivalence of categories. □

Next we relate the categories on the right-hand side in Propositions 3.2 and 3.1 using parametrized spectra. To be definite, we will now assume that $\mathcal{S}p$ means orthogonal spectra. The category of parametrized orthogonal spectra is defined in [May and Sigurdsson 2006, Definition 11.2.3], and its homotopy category is obtained by inverting the π_*-isomorphisms from [May and Sigurdsson 2006, Definition 12.3.4].

The first part of the equivalence is as follows. Given a diagram F of orthogonal spectra over $\mathcal{C}$, at each spectrum level we can take its Bousfield–Kan homotopy colimit as a diagram of unbased spaces, giving a retractive space over $|\mathcal{C}|$. In total this gives a parametrized spectrum $\operatorname{hocolim}_{\mathcal{C}} F$ over $|\mathcal{C}|$ [Lind and Malkiewich 2018, §4]. See the diagram

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{homotopy functors}\\ F : \Delta^{\mathrm{op}}_{BG} \to \mathcal{S}p\end{array}} \arrow[d, "\operatorname{hocolim}_{\Delta^{\mathrm{op}}_{BG}} F(\Delta^p)", "\sim"'] \\
\boxed{\begin{array}{c}\text{parametrized spectra}\\ \text{over } |\Delta^{\mathrm{op}}_{BG}|\end{array}} \\
\boxed{\begin{array}{c}\text{enriched functors } \mathcal{B}G_\bullet \to \mathcal{S}p\\ \text{(spectra with left } G\text{-action)}\end{array}} \arrow[u, "l_!(QE\times_G -)"', "\sim"]
\end{tikzcd}$$

The second part of the equivalence is the Borel construction $EG \times_G -$, followed by pullback along the equivalence $|\Delta^{\mathrm{op}}_{BG}| \xrightarrow{\sim} BG$. Alternatively, we make the following construction. Let E be any weakly contractible space with a free right G-action, with a map $E/G \to |\Delta^{\mathrm{op}}_{BG}|$. Let QE be its cofibrant replacement as a free G-space, so that there is an equivalence $l : QE/G \xrightarrow{\sim} BG$. If X is a spectrum with G-action, take a cofibrant replacement if necessary so that its levels are well based, then take $QE \times_G X$, which is a parametrized spectrum over QE/G, and push it forward along l to $|\Delta^{\mathrm{op}}_{BG}|$. We will see in the next proposition that this is always equivalent to the Borel construction, but it is convenient to allow ourselves to pick a particular space E with this property, rather than having to use the pullback of EG to $|\Delta^{\mathrm{op}}_{BG}|$.

Proposition 3.5. *These are equivalences of homotopy categories, and the second is independent of the choice of E, up to isomorphism.*

Proof. For the first one, the homotopy category of homotopy functors on $\Delta^{\mathrm{op}}_{BG}$ is equivalent to the homotopy category of functors that are fibrant in the aggregate model structure of [Lind and Malkiewich 2018, Theorem 4.4]. Therefore, $\operatorname{hocolim}_{\Delta^{\mathrm{op}}_{BG}} F(\Delta^p)$ is naturally isomorphic as a map of homotopy categories to the left Quillen equivalence of [Lind and Malkiewich 2018, Theorem 4.5], and is thus an equivalence. On the other hand, for a G-space X the horizontal maps in the following square are equivalences:

$$\begin{tikzcd}
QE \times_G X \arrow[r, "\sim"] \arrow[d] & EG \times_G X \arrow[d] \\
|\Delta^{\mathrm{op}}_{BG}| \arrow[r, "\sim"] & BG
\end{tikzcd}$$

Hence the functor $QE \times_G -$ is equivalent to the Borel construction $EG \times_G -$ (which lands in spectra over BG) followed by the pullback from BG to $|\Delta^{\mathrm{op}}_{BG}|$. (Under the cofibrancy assumptions on X, the same is also true if we push $QE \times_G X$

forward along l.) This factorization into Borel-then-pullback also holds at the level of homotopy categories, since the Borel construction preserves all equivalences and outputs a fibration, on which the pullback preserves equivalences. Then the Borel construction is an equivalence by [Ando et al. 2018, Appendix B] or [Lind and Malkiewich 2018, Theorem 4.5], and the derived pullback is an equivalence by [May and Sigurdsson 2006, Proposition 12.6.7]. □

Now we may finish the proof of Theorem A.

Theorem 3.6. *For any reduced homotopy functor $F : \mathcal{U}_{BG}^{\mathrm{op}} \to \mathcal{S}p$, the coassembly map on BG is isomorphic in the homotopy category to the map $F(BG) \to F(EG)^{hG}$ induced by the functoriality of F.*

Proof. The adjunction from Proposition 3.1 descends to the localization we described above; hence we get the following diagram of adjunctions and equivalences of homotopy categories. It remains to check that the equivalences and left adjoints in this figure commute up to some natural isomorphism, so that the figure is an "equivalence of adjunctions":

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{reduced homotopy functors}\\ F : \mathcal{U}_{BG}^{\mathrm{op}} \to \mathcal{S}p \text{ (localized)}\end{array}} \arrow[rr, shift left, "\text{restrict}"] \arrow[rr, phantom, "\perp"] & & \boxed{\begin{array}{c}\text{homotopy functors}\\ F : \Delta_{BG}^{\mathrm{op}} \to \mathcal{S}p\end{array}} \arrow[ll, shift left, "X \mapsto \operatorname{holim}_{\Delta_X^{\mathrm{op}}} F(\Delta^p)"] \arrow[dd, "\sim"', "\operatorname{hocolim}_{\Delta_{BG}^{\mathrm{op}}} F(\Delta^p)"] \\
\boxed{\begin{array}{c}\text{enriched reduced}\\ \text{homotopy functors}\\ F : \mathcal{U}_{BG}^{\mathrm{op}} \to \mathcal{S}p \text{ (localized)}\end{array}} \arrow[u, "\sim"'] \arrow[d, "\sim"'] & & \\
\boxed{\begin{array}{c}\text{enriched reduced functors}\\ F : \mathcal{U}_{BG}^{\mathrm{op}} \to \mathcal{S}p \text{ (localized)}\end{array}} \arrow[d, "\sim"'] & & \boxed{\begin{array}{c}\text{parametrized spectra}\\ \text{over } |\Delta_{BG}^{\mathrm{op}}|\end{array}} \\
\boxed{\begin{array}{c}\text{enriched functors}\\ C(\mathcal{B}G_\bullet) \to \mathcal{S}p\end{array}} \arrow[rr, shift left, "\text{restrict}"] \arrow[rr, phantom, "\perp"] & & \boxed{\begin{array}{c}\text{enriched functors } \mathcal{B}G_\bullet \to \mathcal{S}p\\ \text{(spectra with left } G\text{-action)}\end{array}} \arrow[ll, shift left, "\begin{array}{c}\text{homotopy right}\\ \text{Kan extension}\end{array}"] \arrow[u, "\sim", "QE \times_G -"']
\end{tikzcd}
$$

To form this natural isomorphism, we assume that F is an enriched reduced homotopy functor on $\mathcal{U}_{BG}$. Composing with fibrant replacement, then reenriching by the equivalences in Proposition 3.4, we may assume that F sends equivalences of spaces to level equivalences of spectra. We may also compose with $|\mathrm{Sing} -|$ so that it is enriched in topological spaces. These manipulations are natural in F;

hence we can make these assumptions even if what we are after is an isomorphism that is natural in F.

We define

$$E = \operatorname*{hocolim}_{\Delta_{BG}^{\mathrm{op}}} \operatorname{Map}_{BG}(\Delta^p, EG)$$

with G acting on the right on EG. By Lemma 3.7 below, E is weakly contractible. Form the following diagram at each spectrum level, in which the second map along the top uses the enriched functoriality of F:

$$\begin{tikzcd}
QE \times F(EG) \arrow[r] \arrow[d] & \operatorname*{hocolim}_{\Delta^p \in \Delta_{BG}^{\mathrm{op}}} \operatorname{Map}_{BG}(\Delta^p, EG) \times F(EG) \arrow[r] \arrow[d] & \operatorname*{hocolim}_{\Delta^p \in \Delta_{BG}^{\mathrm{op}}} F(\Delta^p) \\
QE \times_G F(EG) \arrow[r] & \operatorname*{hocolim}_{\Delta^p \in \Delta_{BG}^{\mathrm{op}}} \operatorname{Map}_{BG}(\Delta^p, EG) \times_G F(EG) \arrow[ur, dashed] &
\end{tikzcd}$$

This map of spaces induces a map of parametrized spectra over $QE/G \to |\Delta_{BG}^{\mathrm{op}}|$, or a map from the pushforward of the first to the second over $|\Delta_{BG}^{\mathrm{op}}|$. To argue that the above map is an equivalence of parametrized spectra, it suffices to argue it is an equivalence at each spectrum level.

To check the composite along the bottom is an equivalence, it suffices to examine the induced map on their homotopy fibers over $|\Delta_{BG}^{\mathrm{op}}|$. In the target, by a variant of Quillen's theorem B [Meyer 1986; Grayson 1976], the map to $|\Delta_{BG}^{\mathrm{op}}|$ is a quasi-fibration, so the fiber $F(\Delta^p)$ is equivalent to the homotopy fiber. In the source, we pick a single G-orbit of QE and check that the inclusion of $G \times_G F(EG)$ into the homotopy fiber of $QE \times_G F(EG) \to QE/G$ is an equivalence, by replacing E by a space that is fibrant, then comparing to EG. Therefore the above map induces on homotopy fibers a map equivalent to $F(EG) \to F(\Delta^p)$, which is an equivalence because F is a homotopy functor. This proves that the left adjoints commute up to isomorphism. $\square$

Lemma 3.7. *The space* $E = \operatorname{hocolim}_{\Delta_{BG}^{\mathrm{op}}} \operatorname{Map}_{BG}(\Delta^p, EG)$ *is weakly contractible.*

Proof. We first rearrange the colimit using the string of weak equivalences

$$\begin{tikzcd}
\operatorname*{hocolim}_{\mathrm{Tw}(\Delta_{BG})^{\mathrm{op}}} \Delta^p \times_{BG} EG \arrow[r, "\sim"] & \operatorname*{hocolim}_{\Delta_{BG}} \Delta^p \times_{BG} EG \\
\operatorname*{hocolim}_{\mathrm{Tw}(\Delta_{BG})^{\mathrm{op}}} \Delta^p \times \operatorname{Map}_{BG}(\Delta^q, EG) \arrow[u, "\sim"'] \arrow[d, "\sim"] & \\
\operatorname*{hocolim}_{\mathrm{Tw}(\Delta_{BG})^{\mathrm{op}}} \operatorname{Map}_{BG}(\Delta^q, EG) \arrow[r, "\sim"] & \operatorname*{hocolim}_{\Delta_{BG}^{\mathrm{op}}} \operatorname{Map}_{BG}(\Delta^q, EG)
\end{tikzcd} \tag{3.8}$$

Here $\mathrm{Tw}(\Delta_{BG})^{\mathrm{op}}$ denotes (the opposite of) the twisted arrow category of Δ_{BG}. The objects are arrows in Δ_{BG}, and a morphism from $\Delta^p \to \Delta^q \to BG$ to $\Delta^{p'} \to \Delta^{q'} \to BG$ is a factorization

$$\begin{tikzcd} \Delta^p \arrow[r] \arrow[d] & \Delta^{p'} \arrow[d] \\ \Delta^q \arrow[d] & \Delta^{q'} \arrow[l] \arrow[d] \\ BG \arrow[r, equal] & BG \end{tikzcd}$$

In general, for a category $\mathscr{C}$, the twisted arrow category $\mathrm{Tw}(\mathscr{C})^{\mathrm{op}}$ is equipped with a "source" functor $s : \mathrm{Tw}(\mathscr{C})^{\mathrm{op}} \to C$ that remembers just the source of each arrow, and a "target" functor $t : \mathrm{Tw}(\mathscr{C})^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$ that remembers the target of the arrow.

It is straightforward to define the diagrams on the left-hand side of (3.8). The top horizontal map is the pullback of a diagram on Δ_{BG} along the source functor. Similarly, the horizontal diagram on the bottom is a pullback along the target functor. The bottom vertical arrow arises by collapsing Δ^p to a point and is thus a levelwise equivalence. The top vertical arrow arises from the levelwise maps

$$\Delta^p \times \mathrm{Map}_{BG}(\Delta^q, EG) \to \Delta^p \times_{BG} EG$$

defined by sending $(x, f) \mapsto (x, f(g(x)))$, where g is the given map $\Delta^p \to \Delta^q$. We check from the definition that this is indeed a map of $\mathrm{Tw}(\Delta_{BG})^{\mathrm{op}}$-diagrams. It is also an equivalence on each term, since restricting the Δ^p or Δ^q to a single point is an equivalence, and after this substitution we get a homeomorphism

$$\mathrm{Map}_{BG}(\{*\}, EG) \xrightarrow{\cong} \{*\} \times_{BG} EG.$$

The next step is to show that these four maps of colimits are weak equivalences. For the vertical maps, this follows because the two maps of diagrams are an equivalence on each term. For the horizontal arrows, this follows because the source and target functors are homotopy terminal. For the source functor, this means that for any object $j \in C$, the overcategory $(j \downarrow s)$ is contractible. To prove this, we note that the overcategory consists of pairs of arrows $j \to a \to b$ and morphisms of the form

$$\begin{tikzcd} j \arrow[r, equal] \arrow[d] & j \arrow[d] \\ a \arrow[r] \arrow[d] & c \arrow[d] \\ b & d \arrow[l] \end{tikzcd}$$

The inclusion of the subcategory of all arrows of the form $j = j \to b$ has a right adjoint, so that subcategory has an equivalent nerve. Furthermore, this subcategory has a terminal object $j = j = j$, so it is contractible. All together, this proves that s is homotopy terminal. A similar proof works for the target functor t.

We have now reduced to proving that $\mathrm{hocolim}_{\Delta_{BG}}(\Delta^p \times_{BG} EG)$ is weakly contractible. Since geometric realization commutes with finite limits, we get a homeomorphism

$$\operatorname*{hocolim}_{\Delta_{BG}}(\Delta^p \times_{BG} EG) \cong \big(\operatorname*{hocolim}_{\Delta_{BG}} \Delta^p\big) \times_{BG} EG.$$

Clearly $BG \times_{BG} EG \cong EG$ is contractible, so it is enough to prove that the map

$$\phi : \operatorname*{hocolim}_{\Delta_{BG}} \Delta^p \to BG,$$

which arises from all the individual maps $\Delta^p \to BG$, is an equivalence. There is an immediate equivalence

$$\operatorname*{hocolim}_{\Delta_{BG}} \Delta^p \xrightarrow{\sim} \operatorname*{hocolim}_{\Delta_{BG}} * \xrightarrow{\cong} |\Delta_{BG}| \xrightarrow{\sim} BG \tag{3.9}$$

but that is a different map. To show that ϕ is an equivalence, we extend it to a natural transformation of functors on unbased spaces

$$\operatorname*{hocolim}_{\Delta_X} \Delta^p \to X.$$

It is clearly an equivalence when X is empty or contractible. Furthermore, using (3.9), both sides are equivalent to the identity functor and are therefore excisive. A standard inductive argument then shows that ϕ is an equivalence on all spaces. This finishes the proof. $\square$

4. Review of coarse and bivariant A-theory

Let G be a finite group and X a G-space. Let $R(X)$ be the category of retractive spaces

$$X \xrightarrow{i} Y \xrightarrow{r} X, \quad ri = \mathrm{id},$$

with weak equivalences given by the weak homotopy equivalences and cofibrations given by maps that have the fiberwise homotopy extension property (FHEP). The category $R(X)$ has a G-action through exact functors induced by conjugation from the G-action on X [Malkiewich and Merling 2019, §3.1]. For taking K-theory, we restrict to the subcategory $R_{hf}(X) \subseteq R(X)$ of retractive spaces that are *homotopy finite*. These are the spaces that, in the homotopy category of retractive spaces, are a retract of a finite cell complex relative to X. We note the action respects this condition.

For each subgroup $H \leq G$, the homotopy fixed points are defined as

$$R_{hf}(X)^{hH} := \mathrm{Cat}(\mathscr{E}G, R_{hf}(X))^H,$$

where $\mathscr{E}G$ is the G-category with one object for each element of G and a unique morphism between any two objects, and $\mathrm{Cat}(\mathscr{E}G, R_{hf}(X))$ is the category of all functors and natural transformations, with G acting by conjugation [Malkiewich and Merling 2019, Definition 2.2].

The homotopy fixed point category $R_{hf}(X)^{hH}$ is equivalent to the Waldhausen category whose objects are H-spaces Y containing X as an H-equivariant retract, whose underlying space is homotopy finite [Malkiewich and Merling 2019, Proposition 3.1]. The morphisms are the H-equivariant maps of retractive spaces $Y \to Y'$. The cofibrations are the H-equivariant maps which are nonequivariantly cofibrations and the weak equivalences are the H-equivariant maps which are nonequivariantly weak equivalences.

We define $A_G^{\mathrm{coarse}}(X)$ to be the naïve G-spectrum obtained by applying $S_\bullet$ to the Waldhausen G-category $\mathrm{Cat}(\mathscr{E}G, R_{hf}(X))$. This is equivalent to the underlying naïve G-spectrum of a genuine Ω-G-spectrum [Malkiewich and Merling 2019, Theorem 2.21].

For a Hurewicz fibration $p : E \to B$, the bivariant A-theory $A(p)$ is defined to be the K-theory of the Waldhausen category of retractive spaces X over E, with the property that $X \to B$ is a fibration, and the map of fibers $E_b \to X_b$ is a retract up to homotopy of a relative finite complex. See [Williams 2000; Raptis and Steimle 2014].

In the present section we extend the following result of [Malkiewich and Merling 2019] to the coassembly map.

Proposition 4.1. *There is a natural equivalence of symmetric spectra*

$$A_G^{\mathrm{coarse}}(X)^H \simeq \boldsymbol{A}(EG \times_H X \to BH).$$

The equivalence is induced by the functor

$$\Phi : R_{hf}(X)^{hH} \to R_{hf}(EG \times_H X \xrightarrow{p} BH)$$

that applies $EG \times_H -$ to the retractive space (Y, i_Y, p_Y) over X, obtaining a retractive space over $EG \times_H X$:

$$EG \times_H X \xrightarrow{EG \times_H i_Y} EG \times_H Y \xrightarrow{EG \times_H p_Y} EG \times_H X.$$

To define the coassembly map, we observe that while bivariant A-theory is a functor of fibrations, it can be regarded as a contravariant functor on $\mathscr{U}_B$ in the following way. Fix a fibration $p : E \to B$. Then $\mathscr{U}_B$ is equivalent to the category whose objects are pullback squares

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r] & B \end{tikzcd}$$

and whose maps are commuting squares (necessarily pullback squares)

$$\begin{tikzcd} E'' \arrow[r] \arrow[d, "p''"'] & E' \arrow[d, "p'"] \\ B'' \arrow[r] & B' \end{tikzcd}$$

Along this equivalence, bivariant A-theory is a reduced homotopy functor from $\mathcal{U}_B^{\mathrm{op}}$ to spectra, so it has a coassembly map

$$c\alpha : \boldsymbol{A}(E' \xrightarrow{p'} B') \to \boldsymbol{A}_{\%}(E' \xrightarrow{p'} B').$$

We emphasize that the coassembly map depends on the choice of fibration $E \xrightarrow{p} B$ and map $B' \to B$. Different choices give rise to different coassembly maps.

Fix the fibration $EG \times_H X \to BH$ and the pullback square

$$\begin{tikzcd} EG \times_H X \arrow[r, "="] \arrow[d, "p"'] & EG \times_H X \arrow[d, "p"] \\ BH \arrow[r, "="] & BH \end{tikzcd}$$

and consider the resulting coassembly map. Our last remaining goal is to prove:

Theorem 4.2. *In the stable homotopy category, the map from fixed points to homotopy fixed points is isomorphic to the coassembly map for bivariant A-theory*:

$$\begin{tikzcd} \boldsymbol{A}_G^{\mathrm{coarse}}(X)^H \arrow[r] \arrow[d, "\sim"] & \boldsymbol{A}_G^{\mathrm{coarse}}(X)^{hH} \arrow[d, "\sim"] \\ \boldsymbol{A}(EG \times_H X \to BH) \arrow[r, "c\alpha"] & \boldsymbol{A}_{\%}(EG \times_H X \to BH) \end{tikzcd}$$

Furthermore the left-hand map in the above diagram can be taken to be the equivalence of Proposition 4.1.

5. Proof of Theorem B

Note that without loss of generality we may take $H = G$. Since G is finite, we may ignore issues of enrichment. By Theorem 3.6, the coassembly map for bivariant A-theory is equivalent to the homotopy limit map for the diagram on $C(\mathcal{B}G)$ given by bivariant A-theory on EG and BG. So it remains to compare the resulting diagram on $C(\mathcal{B}G)$ to the one defined by coarse A-theory.

Proposition 5.1. *The equivalence of Proposition 4.1 can be extended to an equivalence of diagrams of symmetric spectra over $C(\mathcal{B}G)$.*

We expect it is possible to compare these two as diagrams over $\mathbb{O}(G)^{\mathrm{op}}$, but this raises additional coherence issues, and is not necessary to prove Theorem 4.2.

Proof. We start by describing the $\mathbb{O}(G)^{\mathrm{op}}$-action on bivariant A-theory. To each map of G-sets $f : G/H \to G/K$ and G-space X we assign the pullback square

$$\begin{tikzcd} B(*, G, G \times_H X) \arrow[r] \arrow[d] & B(*, G, G \times_K X) \arrow[d] \\ B(*, G, G/H) \arrow[r, "EG \times_G f"] & B(*, G, G/K) \end{tikzcd}$$

The vertical maps collapse X to a point, and the top horizontal map

$$G \times_H X \to G \times_K X$$

sends (γ, x) to $(\gamma g^{-1}, gx)$, where g is any element such that $f(eH) = g^{-1}K$. Note that this formula is well defined because g is unique up to left multiplication by K. It is easy to check that these formulas give a functor from $\mathbb{O}(G)$ into the category of pullbacks of the fibration $EG \times_G X \to BG$, and therefore define the action of $\mathbb{O}(G)^{\mathrm{op}}$ on the bivariant A-theory spectra $\boldsymbol{A}(EG \times_H X \to EG/H)$. This action is strict by functoriality of bivariant A-theory [Raptis and Steimle 2014, Remark 3.5].

Now we restrict to $C(\mathcal{B}G)$, where we wish to prove that the functor Φ of Proposition 4.1 gives a map of $C(\mathcal{B}G)$ diagrams, in other words that the two squares below commute:

$$\begin{tikzcd} \boldsymbol{A}_G^{\mathrm{coarse}}(X)^G \arrow[r, "\Phi", "\sim"'] \arrow[d, "\text{include}"'] & \boldsymbol{A}(EG \times_G X \to EG/G) \arrow[d, "\text{include}"] \\ \boldsymbol{A}_G^{\mathrm{coarse}}(X)^{\{e\}} \arrow[r, "\Phi", "\sim"'] \arrow[d, "g\cdot"'] & \boldsymbol{A}(EG \times X \to EG) \arrow[d, "g\cdot"] \\ \boldsymbol{A}_G^{\mathrm{coarse}}(X)^{\{e\}} \arrow[r, "\Phi", "\sim"'] & \boldsymbol{A}(EG \times X \to EG) \end{tikzcd}$$

This turns out to be false, but only because the relevant functors of Waldhausen categories agree up to canonical isomorphism, rather than strictly. We therefore replace our two diagrams over $C(\mathcal{B}G)$ by equivalent ones on which the map Φ strictly commutes with the $C(\mathcal{B}G)$ action.

First we make the following reduction. We first show that in order to get a strictly commuting zig-zag of equivalences of $C(\mathcal{B}G)$-diagrams, it is enough to define a square of G-equivariant functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F_1"] \arrow[d, "I"'] & \mathscr{C}' \arrow[d, "I'"] \\ \mathscr{D} \arrow[r, "F_2"'] & \mathscr{D}' \end{tikzcd}$$

such that $\mathscr{C}$ and $\mathscr{C}'$ have trivial G-action, and such that the square commutes up to a G-fixed natural isomorphism η. Given such a square, we may replace $\mathscr{D}$ by the category $\mathscr{D}_I$ defined as follows:

- the objects $\mathscr{D}_I$ are $\operatorname{ob}\mathscr{C} \amalg \operatorname{ob}\mathscr{D}$, and
- the morphisms are given by $\mathscr{D}_I(d, d') = \mathscr{D}(d, d')$, $\mathscr{D}_I(d, c) = \mathscr{D}(d, Ic)$, and $\mathscr{D}_I(c, d) = \mathscr{D}(Ic, d)$ if c is an object of $\mathscr{C}$ and d, d' are objects of $\mathscr{D}$.

We define a new functor $\mathscr{D}_I \to \mathscr{D}'$ using F_2 on the full subcategory on $\operatorname{ob}\mathscr{D}$, $I' \circ F_1$ on the full subcategory on $\operatorname{ob}\mathscr{C}$, and on each morphism f between $c \in \operatorname{ob}\mathscr{C}$ and $d \in \operatorname{ob}\mathscr{D}$, the composite

$$I' \circ F_1(c) \overset{\cong}{\underset{\eta}{\longleftrightarrow}} F_2 \circ I(c) \underset{F_2(f)}{\longleftrightarrow} F_2(d).$$

It is easy to check this is indeed a functor and is G-equivariant. It is then straightforward to define the rest of the following diagram so that every functor is equivariant and every square of functors commutes strictly, giving a zig-zag of $C(\mathscr{B}G)$-diagrams of categories

$$\begin{tikzcd} \mathscr{C} \arrow[r, equal] \arrow[d, "I"'] & \mathscr{C} \arrow[r, "F_1"] \arrow[d] & \mathscr{C}' \arrow[d, "I'"] \\ \mathscr{D} & \mathscr{D}_I \arrow[l, "\sim"] \arrow[r] & \mathscr{D}' \end{tikzcd}$$

Note that if $\mathscr{C}$ and $\mathscr{D}$ are Waldhausen categories and all functors I, I', F_1, F_2 are exact, then the resulting diagram above is also a diagram of Waldhausen categories, where $\mathscr{D}_I$ has the Waldhausen structure inherited from computing maps in $\mathscr{D}$. With this reduction in hand, it is enough to make a square of functors of Waldhausen G-categories, in which the top row has trivial G-action, that commutes up to a G-fixed natural isomorphism. We will construct the square

$$\begin{tikzcd} \operatorname{Cat}(\mathscr{E}G, R_{hf}(X))^G \arrow[r, "\Phi"] \arrow[dd, "I"'] & R_{hf}(EG \times_G X \to BG) \arrow[d, "q^*"] \\ & R_{hf}(EG \times X \to EG) \arrow[d, "\mathrm{const}"] \\ \operatorname{Cat}(\mathscr{E}G, R_{hf}(X)) \arrow[r, "\widetilde{\Phi}"] & \operatorname{Cat}(\mathscr{E}G, R_{hf}(EG \times X \to EG)) \end{tikzcd}$$

The map Φ along the top is the one from Proposition 4.1 that applies $EG \times_H -$ to the retractive space (Y, i_Y, p_Y) over X, obtaining a retractive space over $EG \times_H X$.

The left-hand vertical map I includes the fixed points into the whole category, i.e., takes a retractive G-space (Y, i, p) to the G-tuple of retractive spaces $(Y, i \circ g^{-1}, g \circ p)$ with isomorphisms of retractive spaces

$$\phi_{g,h} : (Y, i \circ g^{-1}, g \circ p) \xrightarrow{h^{-1}g \cdot -} (Y, i \circ h^{-1}, h \circ p)$$

over the identity map of X. Along the right-hand edge, the first functor pulls back along the quotient map

$$q : EG \times X \to EG \times_G X.$$

The left action of $g \in G$ on the target is by pullback along the map

$$\rho_g : EG \times X \xrightarrow{- \cdot g \times g^{-1} \cdot -} EG \times X$$

and note that q^* lands in the G-fixed points because the composite function $q \circ \rho_g$ is equal to q. The second functor on the right-hand edge pulls back along the map of categories $\mathscr{E}G \to *$. To define the functor on the bottom, first form the functor

$$\begin{aligned} &\Phi : R_{hf}(X) \to R_{hf}(EG \times X \to EG), \\ &\Phi(Z, i, p) = EG \times (Z, i, p) = (EG \times Z, \mathrm{id} \times i, \mathrm{id} \times p). \end{aligned}$$

Then pick the isomorphisms

$$\begin{gathered} \theta_g : \Phi \circ g \to g \circ \Phi, \\ EG \times (Z, i \circ g^{-1}, g \circ p) \to \rho_g^*(EG \times (Z, i, p)) \end{gathered}$$

arising from the commuting diagram

$$\begin{tikzcd} EG \times X \arrow[r, "\cdot g, g^{-1} \cdot", "\rho_g"'] \arrow[d, "\mathrm{id}, i \circ g^{-1}"'] & EG \times X \arrow[d, "\mathrm{id}, i"'] \\ EG \times Z \arrow[r, "\cdot g, \mathrm{id}"] \arrow[d, "\mathrm{id}, g \circ p"'] & EG \times Z \arrow[d, "\mathrm{id}, p"'] \\ EG \times X \arrow[r, "\cdot g, g^{-1} \cdot", "\rho_g"'] & EG \times X \end{tikzcd}$$

We check the cocycle condition $g\theta_h \circ \theta_g = \theta_{gh}$, which reduces to the equality $(- \cdot g) \cdot h = - \cdot (gh)$ as self-maps of $EG \times Z$, and $\rho_h \circ \rho_g = \rho_{gh}$ as self-maps of $EG \times X$. Therefore by [Malkiewich and Merling 2019, Definition 2.5], the isomorphisms θ_g make Φ a pseudoequivariant functor. By [Malkiewich and Merling

2019, Proposition 2.10], after applying $\mathrm{Cat}(\mathscr{E}G, -)$ we get a strictly equivariant functor $\widetilde{\Phi}$.

The top route through our diagram of functors takes a retractive G-space Y over X to the functor $\mathscr{E}G \to R_{hf}(EG \times X \to EG)$ with values

$$g \mapsto q^*(EG \times_G (Y, i, p)), \quad (g \to h) \mapsto \mathrm{id}.$$

The bottom route produces the functor with values

$$g \mapsto \rho_g^*(EG \times (Y, i, p)).$$

To describe the maps, let us represent the space $\rho_g^*(EG \times (Y, i, p))$ by drawing the span along which we take the pullback to get it:

$$EG \times Y \xrightarrow{\mathrm{id}, p} EG \times X \xleftarrow[\rho_g]{\cdot g, g^{-1}\cdot} EG \times X \qquad\qquad \rho_g^*(EG \times (Y, i, p)).$$

Then our functor out of $\mathscr{E}G$ assigns the map $g \to h$ to the composite of the following isomorphisms:

$$\begin{tikzcd}
EG \times Y \arrow[r, "{\mathrm{id}, p}"] \arrow[d, "{\cdot g^{-1}, \mathrm{id}}"'] & EG \times X \arrow[d, "{\cdot g^{-1}, g\cdot}"] & EG \times X \arrow[l, "{\cdot g, g^{-1}\cdot}"', "\rho_g"] \arrow[d, equal] & \rho_g^*(EG \times (Y, i, p)) \arrow[d, "\theta_g^{-1}"] & \\
EG \times Y \arrow[r, "{\mathrm{id}, g \circ p}"] \arrow[d, "{\mathrm{id}, h^{-1}g\cdot}"'] & EG \times X \arrow[r, equal] \arrow[d, equal] & EG \times X \arrow[d, equal] & EG \times (Y, i \circ g^{-1}, g \circ p) \arrow[d, "{\mathrm{id}, (h^{-1}g\cdot)}"] & (*) \\
EG \times Y \arrow[r, "{\mathrm{id}, h \circ p}"] \arrow[d, "{\cdot h, \mathrm{id}}"'] & EG \times X \arrow[r, equal] \arrow[d, "{\cdot h, h^{-1}\cdot}"] & EG \times X \arrow[d, equal] & EG \times (Y, i \circ h^{-1}, h \circ p) \arrow[d, "\theta_h"] & \\
EG \times Y \arrow[r, "{\mathrm{id}, p}"] & EG \times X & EG \times X \arrow[l, "{\cdot h, h^{-1}\cdot}"', "\rho_h"] & \rho_h^*(EG \times (Y, i, p)) &
\end{tikzcd}$$

Now we will define a natural isomorphism η from the bottom route to the top route. Continuing to use this span notation, for each $g \in \mathscr{E}G$ we define an isomorphism η_g by the map of spans

$$\begin{tikzcd}
EG \times Y \arrow[r, "{\mathrm{id}, p}"] \arrow[d, "{\mathrm{id}, \mathrm{id}}"'] & EG \times X \arrow[d, "q"] & EG \times X \arrow[l, "\rho_g"'] \arrow[d, equal] & \rho_g^*(EG \times (Y, i, p)) \arrow[d, "\eta_g"] \\
EG \times_G Y \arrow[r, "{\mathrm{id}, p}"] & EG \times_G X & EG \times X \arrow[l, "q"'] & q^*(EG \times_G (Y, i, p))
\end{tikzcd}$$

This commutes with the maps $g \to h$ of $\mathscr{E}G$ because the composite of the three maps of spans from $(*)$ commutes with the map of spans just above. Naturality follows because each G-equivariant map $Y \to Y'$ induces maps on the source and target of η_g that commute with η_g for each g. Finally we check that η is a G-fixed natural transformation. The map $\gamma\eta_{\gamma^{-1}g} := \rho_\gamma^*\eta_{\gamma^{-1}g}$ comes from the map of spans

$$\begin{tikzcd}
EG\times Y \arrow[r, "\mathrm{id},p"] \arrow[d, "\mathrm{id},\mathrm{id}"'] & EG\times X \arrow[d, "q"] & EG\times X \arrow[l, "\rho_{\gamma^{-1}g}"] \arrow[d, equal] & EG\times X \arrow[l, "\rho_\gamma"] \arrow[d, equal] \arrow[ll, bend right, "\rho_g"'] \\
EG\times_G Y \arrow[r, "\mathrm{id},p"] & EG\times_G X & EG\times X \arrow[l, "q"'] & EG\times X \arrow[l, "\rho_\gamma"'] \arrow[ll, bend left, "q"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\rho_\gamma^*\rho_{\gamma^{-1}g}^*(EG\times(Y,i,p)) \arrow[d, "\rho_\gamma^*\eta_{\gamma^{-1}g}"] \\
\rho_\gamma^* q^*(EG\times_G(Y,i,p))
\end{tikzcd}$$

which is indeed the same map of spans that defines η_g. This finishes the construction of the square of equivariant functors that commutes up to equivariant isomorphism. In summary, using the reduction cited earlier in the proof, we have now constructed a strictly commuting zig-zag of $C(\mathcal{B}G)$-diagrams of Waldhausen categories

$$\begin{tikzcd}
\mathrm{Cat}(\mathcal{E}G, R_{hf}(X))^G \arrow[r] \arrow[d, equal] & \mathrm{Cat}(\mathcal{E}G, R_{hf}(X)) \\
\mathrm{Cat}(\mathcal{E}G, R_{hf}(X))^G \arrow[r] \arrow[d, "\Phi"'] & \mathrm{Cat}(\mathcal{E}G, R_{hf}(X))_I \arrow[u, "\sim"] \arrow[d] \\
R_{hf}(EG\times_G X\to BG) \arrow[r, "\mathrm{const}\circ q^*"'] \arrow[d, equal] & \mathrm{Cat}(\mathcal{E}G, R_{hf}(EG\times X\to EG)) \\
R_{hf}(EG\times_G X\to BG) \arrow[r, "q^*"'] & R_{hf}(EG\times X\to EG) \arrow[u, "\mathrm{const}", "\sim"']
\end{tikzcd}$$

Now we apply the K-theory functor to this diagram. By Proposition 4.1, the left map Φ induces an equivalence in K-theory. The right maps labeled $\sim$ are G-maps which are nonequivariant equivalences. It remains to show that the remaining vertical map gives an equivalence on K-theory. In general, for any pseudoequivariant functor $\Phi : \mathcal{C}\to\mathcal{D}$, we have a commutative diagram of nonequivariant categories

$$\begin{tikzcd}
\mathrm{Cat}(\mathcal{E}G, \mathcal{C}) \arrow[r, "\tilde{\Phi}"] \arrow[d, "\sim"'] & \mathrm{Cat}(\mathcal{E}G, \mathcal{D}) \arrow[d, "\sim"] \\
\mathcal{C} \arrow[r, "\Phi"'] & \mathcal{D}
\end{tikzcd}$$

where the vertical maps are nonequivariant equivalences. (Note that the diagram with those equivalences reversed doesn't commute.) Since Φ induces an equivalence on K-theory, so does $\widetilde{\Phi}$. Now use the factorization

$$\begin{tikzcd}
\mathrm{Cat}(\mathscr{E}G, R_{hf}(X)) \arrow[r, "\sim"] \arrow[rr, bend right, "\widetilde{\Phi}"'] & \mathrm{Cat}(\mathscr{E}G, R_{hf}(X))_I \arrow[r] & \mathrm{Cat}(\mathscr{E}G, R_{hf}(EG \times X \to EG))
\end{tikzcd}$$

to conclude that the remaining functor

$$\mathrm{Cat}(\mathscr{E}G, R_{hf}(X))_I \to \mathrm{Cat}(\mathscr{E}G, R_{hf}(EG \times X \to EG))$$

also gives an equivalence in K-theory. Thus we get a strictly commuting zig zag of equivalences of $C(\mathscr{B}G)$ diagrams in spectra. □

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References

[Ando et al. 2018] M. Ando, A. J. Blumberg, and D. Gepner, "Parametrized spectra, multiplicative Thom spectra and the twisted Umkehr map", *Geom. Topol.* **22**:7 (2018), 3761–3825. MR Zbl

[Barwick et al. 2020] C. Barwick, S. Glasman, and J. Shah, "Spectral Mackey functors and equivariant algebraic K-theory, II", *Tunis. J. Math.* **2**:1 (2020), 97–146. MR Zbl

[Bousfield and Kan 1972] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math. **304**, Springer, 1972. MR Zbl

[Cohen and Klein 2009] R. L. Cohen and J. R. Klein, "Umkehr maps", *Homology Homotopy Appl.* **11**:1 (2009), 17–33. MR Zbl

[Davis and Lück 1998] J. F. Davis and W. Lück, "Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory", *K-Theory* **15**:3 (1998), 201–252. MR Zbl

[Devinatz and Hopkins 2004] E. S. Devinatz and M. J. Hopkins, "Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups", *Topology* **43**:1 (2004), 1–47. MR Zbl

[Dwyer et al. 2003] W. Dwyer, M. Weiss, and B. Williams, "A parametrized index theorem for the algebraic K-theory Euler class", *Acta Math.* **190**:1 (2003), 1–104. MR Zbl

[Farrell and Jones 1993] F. T. Farrell and L. E. Jones, "Isomorphism conjectures in algebraic K-theory", *J. Amer. Math. Soc.* **6**:2 (1993), 249–297. MR Zbl

[Goerss and Jardine 2009] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Birkhäuser, Basel, 2009. MR Zbl

[Goodwillie and Weiss 1999] T. G. Goodwillie and M. Weiss, "Embeddings from the point of view of immersion theory, II", *Geom. Topol.* **3** (1999), 103–118. MR Zbl

[Grayson 1976] D. Grayson, "Higher algebraic K-theory, II: After Daniel Quillen", pp. 217–240 in *Algebraic K-theory* (Evanston, IL, 1976), edited by M. R. Stein, Lecture Notes in Math. **551**, Springer, 1976. MR Zbl

[Hambleton and Pedersen 2004] I. Hambleton and E. K. Pedersen, "Identifying assembly maps in K- and L-theory", *Math. Ann.* **328**:1–2 (2004), 27–57. MR Zbl

[Heard 2017] D. Heard, "The homotopy limit problem and the cellular Picard group of Hermitian K-theory", preprint, 2017. arXiv

[Hu et al. 2011] P. Hu, I. Kriz, and K. Ormsby, "The homotopy limit problem for Hermitian K-theory, equivariant motivic homotopy theory and motivic Real cobordism", *Adv. Math.* **228**:1 (2011), 434–480. MR Zbl

[Lewis 1978] L. G. J. Lewis, *The stable category and generalized Thom spectra*, Ph.D. thesis, University of Chicago, 1978, Available at https://search.proquest.com/docview/251798616. MR

[Lillig 1973] J. Lillig, "A union theorem for cofibrations", *Arch. Math.* (*Basel*) **24** (1973), 410–415. MR Zbl

[Lind and Malkiewich 2018] J. A. Lind and C. Malkiewich, "The Morita equivalence between parametrized spectra and module spectra", pp. 45–66 in *New directions in homotopy theory*, edited by N. Kitchloo et al., Contemp. Math. **707**, Amer. Math. Soc., Providence, RI, 2018. MR Zbl

[Lück 2019] W. Lück, "Assembly maps", preprint, 2019. arXiv

[Malkiewich 2015] C. Malkiewich, "A tower connecting gauge groups to string topology", *J. Topol.* **8**:2 (2015), 529–570. MR Zbl

[Malkiewich 2017] C. Malkiewich, "Coassembly and the K-theory of finite groups", *Adv. Math.* **307** (2017), 100–146. MR Zbl

[Malkiewich and Merling 2019] C. Malkiewich and M. Merling, "Equivariant A-theory", *Doc. Math.* **24** (2019), 815–855. MR Zbl

[Mandell et al. 2001] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, "Model categories of diagram spectra", *Proc. London Math. Soc.* (3) **82**:2 (2001), 441–512. MR Zbl

[May and Sigurdsson 2006] J. P. May and J. Sigurdsson, *Parametrized homotopy theory*, Mathematical Surveys and Monographs **132**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Meyer 1986] J.-P. Meyer, "Bar and cobar constructions, II", *J. Pure Appl. Algebra* **43**:2 (1986), 179–210. MR Zbl

[Raptis and Steimle 2014] G. Raptis and W. Steimle, "On the map of Bökstedt–Madsen from the cobordism category to A-theory", *Algebr. Geom. Topol.* **14**:1 (2014), 299–347. MR Zbl

[Riehl 2014] E. Riehl, *Categorical homotopy theory*, New Mathematical Monographs **24**, Cambridge Univ., 2014. MR Zbl

[Röndigs et al. 2018] O. Röndigs, M. Spitzweck, and P. A. Østvær, "The motivic Hopf map solves the homotopy limit problem for K-theory", *Doc. Math.* **23** (2018), 1405–1424. MR Zbl

[Strickland 2009] N. P. Strickland, "The category of CGWH spaces", course notes, 2009, Available at https://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf.

[Thomason 1983] R. W. Thomason, "The homotopy limit problem", pp. 407–419 in *Proceedings of the Northwestern Homotopy Theory Conference* (Evanston, IL, 1982), edited by H. R. Miller and S. B. Priddy, Contemp. Math. **19**, Amer. Math. Soc., Providence, R.I., 1983. MR Zbl

[Waldhausen 1985] F. Waldhausen, "Algebraic K-theory of spaces", pp. 318–419 in *Algebraic and geometric topology* (New Brunswick, NJ, 1983), edited by A. Ranicki et al., Lecture Notes in Math. **1126**, Springer, 1985. MR Zbl

[Weiss 1999] M. Weiss, "Embeddings from the point of view of immersion theory, I", *Geom. Topol.* **3** (1999), 67–101. MR Zbl

[Weiss and Williams 1995] M. Weiss and B. Williams, "Assembly", pp. 332–352 in *Novikov conjectures, index theorems and rigidity* (Oberwolfach, 1993), vol. 2, edited by S. C. Ferry et al., London Math. Soc. Lecture Note Ser. **227**, Cambridge Univ., 1995. MR Zbl

[Williams 2000] B. Williams, "Bivariant Riemann Roch theorems", pp. 377–393 in *Geometry and topology* (Aarhus, 1998), edited by K. Grove et al., Contemp. Math. **258**, Amer. Math. Soc., Providence, RI, 2000. MR Zbl

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Rational equivalence of cusps

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We prove that two cusps of the same dimension in the Baily–Borel compactification of some classical series of modular varieties are linearly dependent in the rational Chow group of the compactification. This gives a higher dimensional analogue of the Manin–Drinfeld theorem. As a consequence, we obtain a higher dimensional generalization of modular units as higher Chow cycles on the modular variety.

1. Introduction

The classical theorem of Manin [1972] and Drinfeld [1973] asserts that the difference of two cusps is torsion in the Picard group of the modular curve for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. This had stimulated the development of the theory of modular units and cuspidal class groups; see [Kubert and Lang 1981]. The original proof of Manin and Drinfeld used modular symbols and Hecke operators on the cohomology of the modular curve. Later, an interpretation in terms of the mixed Hodge structure of the modular curve minus the cusps was also found [Elkik 1990].

Our purpose in this paper is to prove a generalization of the Manin–Drinfeld theorem for cusps in the Baily–Borel compactification of some higher dimensional classical modular varieties. In higher dimensions, cusps are no longer divisors, but algebraic cycles of various codimension. We wish to clarify their contribution to the Chow group of the Baily–Borel compactification.

The modular varieties of our object of study are of the following three types:

(1) modular varieties of orthogonal type attached to rational quadratic forms of signature $(2, n)$, which have only 0-dimensional and 1-dimensional cusps;

(2) Siegel modular varieties attached to rational symplectic forms; and

(3) modular varieties of unitary type, including the Picard modular varieties, attached to Hermitian forms over imaginary quadratic fields.

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In Cartan's classification of irreducible Hermitian symmetric domains, these correspond to the domains $\mathcal{D}$ of type IV, III, and I, respectively. The Baily–Borel compactification [Baily and Borel 1966] of the modular variety $\Gamma\backslash\mathcal{D}$ for an arithmetic group Γ is obtained by adjoining rational boundary components to $\mathcal{D}$ and then taking the quotient by Γ. Below, by a *cusp* we mean the closure of the image of a rational boundary component in the Baily–Borel compactification.

Our main results are the following.

Theorem 1.1 (orthogonal case). *Let Λ be an integral quadratic lattice of signature $(2, n)$, Γ a congruence subgroup of the orthogonal group $\mathrm{O}^+(\Lambda)$, and X_Γ the Baily–Borel compactification of the modular variety defined by Γ. Let Z_1, Z_2 be two cusps of X_Γ of the same dimension, say $k \in \{0, 1\}$. Assume that $n \geq 3$ if $k = 1$. Then we have $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in the rational Chow group $\mathrm{CH}_k(X_\Gamma)_\mathbb{Q} = \mathrm{CH}_k(X_\Gamma) \otimes_\mathbb{Z} \mathbb{Q}$ of X_Γ.*

Theorem 1.2 (symplectic case). *Let Λ be an integral symplectic lattice, Γ a congruence subgroup of the symplectic group $\mathrm{Sp}(\Lambda)$, and X_Γ the Satake–Baily–Borel compactification of the Siegel modular variety defined by Γ. If Z_1, Z_2 are two cusps of X_Γ of the same dimension, say k, then $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_\mathbb{Q}$.*

Theorem 1.3 (unitary case). *Let K be an imaginary quadratic field, Λ a Hermitian lattice over $\mathcal{O}_K$, Γ a congruence subgroup of the unitary group $\mathrm{U}(\Lambda)$, and X_Γ the Baily–Borel compactification of the modular variety defined by Γ. If Z_1, Z_2 are two cusps of X_Γ of the same dimension, say k, then $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_\mathbb{Q}$.*

Note that the equality $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_\mathbb{Q}$ is the same as the equality $N_1[Z_1] = N_2[Z_2]$ in the integral Chow group $\mathrm{CH}_k(X_\Gamma)$ for some natural numbers N_1, N_2. When $k = 0$, we must have $N_1 = N_2$, so $[Z_1] - [Z_2]$ is torsion in $\mathrm{CH}_0(X_\Gamma)$.

In the symplectic case, when Λ has rank ≥ 4, every finite-index subgroup of $\mathrm{Sp}(\Lambda)$ is a congruence subgroup by [Mennicke 1965; Bass et al. 1964]. The case $\mathrm{rk}(\Lambda) = 2$ is just the case of modular curves.

The case $(n, k) = (2, 1)$ in the orthogonal case is indeed an exception. We have self products of modular curves as typical examples of X_Γ in $n = 2$, for which two transversal boundary curves are not homologically equivalent. On the other hand, we should note that some consideration in the case $n = 2$ is necessary for our proof for the case $n \geq 3$.

The proof of Theorems 1.1–1.3 is based on the same simple idea. We connect Z_1 and Z_2 by a chain of submodular varieties or their products, through the interior or the boundary, and use induction on the dimension of modular varieties. This eventually reduces the problem to the Manin–Drinfeld theorem for modular curves. The actual argument requires case-by-case construction depending on the combinatorics of rational boundary components. We need to argue the three cases separately, though the symplectic and the unitary cases are similar. Theorem 1.1 is proved in Section 2, Theorem 1.2 in Section 3, and Theorem 1.3 in Section 4.

In Section 5, as a consequence of these results, we associate an explicit nonzero element of the higher Chow group $\mathrm{CH}_k(\Gamma\backslash\mathcal{D}, 1)_{\mathbb{Q}}$ of the modular variety $\Gamma\backslash\mathcal{D}$ (before compactification) to each pair (Z_1, Z_2) of cusps of maximal dimension k. This gives a higher dimensional analogue of modular units from the viewpoint of algebraic cycles. If the span of all such higher Chow cycles on $\Gamma\backslash\mathcal{D}$ has dimension no less than the number of maximal cusps, we would then obtain a nontrivial subspace of $\mathrm{CH}_k(X_\Gamma, 1)_{\mathbb{Q}}$ for the Baily–Borel compactification X_Γ.

Throughout the paper $\Gamma(N)$ stands for the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N, and $X(N) = \Gamma(N)\backslash\mathbb{H}^*$ the (compactified) modular curve for $\Gamma(N)$. In Section 2 and Section 3, for a free $\mathbb{Z}$-module Λ of finite rank, we denote by $\Lambda^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ its dual $\mathbb{Z}$-module and define $\Lambda_F = \Lambda \otimes_{\mathbb{Z}} F$ for $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. For a $\mathbb{Q}$-vector space V we also write $V^\vee = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ and $V_F = V \otimes_{\mathbb{Q}} F$ when no confusion is likely to occur.

2. The orthogonal case

In this section we prove Theorem 1.1. We first recall orthogonal modular varieties; see [Scattone 1987; Looijenga 2016]. Let Λ be a free $\mathbb{Z}$-module of rank $2+n$ equipped with a nondegenerate symmetric bilinear form $(\,\cdot\,,\cdot\,) : \Lambda \times \Lambda \to \mathbb{Z}$ of signature $(2, n)$. Let

$$Q_\Lambda = \{[\mathbb{C}\omega] \in \mathbb{P}\Lambda_{\mathbb{C}} \mid (\omega, \omega) = 0\}$$

be the isotropic quadric in $\mathbb{P}\Lambda_{\mathbb{C}}$. The open set of Q_Λ defined by the condition $(\omega, \bar{\omega}) > 0$ consists of two connected components, and the Hermitian symmetric domain $\mathcal{D}_\Lambda$ attached to Λ is defined as one of them. This choice is equivalent to the choice of an orientation of a positive definite plane in $\Lambda_{\mathbb{R}}$.

Let $\mathrm{O}(\Lambda)$ be the orthogonal group of Λ, namely the group of isomorphisms $\Lambda \to \Lambda$ preserving the quadratic form. We write $\mathrm{O}^+(\Lambda)$ for the subgroup of $\mathrm{O}(\Lambda)$ preserving the component $\mathcal{D}_\Lambda$. For a natural number N let $\mathrm{O}^+(\Lambda, N) < \mathrm{O}^+(\Lambda)$ be the kernel of the reduction map $\mathrm{O}^+(\Lambda) \to \mathrm{GL}(\Lambda/N\Lambda)$. A subgroup Γ of $\mathrm{O}^+(\Lambda)$ is called a congruence subgroup if it contains $\mathrm{O}^+(\Lambda, N)$ for some level N. A typical example is the kernel of the reduction map $\mathrm{O}^+(\Lambda) \to \mathrm{GL}(\Lambda^\vee/\Lambda)$ for the discriminant group $\Lambda^\vee/\Lambda$.

There are two types of rational boundary components of $\mathcal{D}_\Lambda$: 0-dimensional and 1-dimensional components. The 0-dimensional components correspond to isotropic $\mathbb{Q}$-lines I in $\Lambda_{\mathbb{Q}}$: we take the point $p_I = [I_{\mathbb{C}}] \in Q_\Lambda$, which is in the closure of $\mathcal{D}_\Lambda$, for each such I. The 1-dimensional components correspond to isotropic $\mathbb{Q}$-planes J in $\Lambda_{\mathbb{Q}}$: we take the connected component of $\mathbb{P}J_{\mathbb{C}} - \mathbb{P}J_{\mathbb{R}} \simeq \mathbb{H} \sqcup \mathbb{H}$, say $\mathbb{H}_J$, that is in the closure of $\mathcal{D}_\Lambda$. The union

$$\mathcal{D}_\Lambda^* = \mathcal{D}_\Lambda \sqcup \bigsqcup_{\dim J=2} \mathbb{H}_J \sqcup \bigsqcup_{\dim I=1} p_I$$

is equipped with the Satake topology [Baily and Borel 1966; Borel and Ji 2005]. By [Baily and Borel 1966], the quotient space $X_\Gamma = \Gamma\backslash\mathcal{D}_\Lambda^*$ has the structure of a normal projective variety and contains $\Gamma\backslash\mathcal{D}_\Lambda$ as a Zariski open set.

In Section 2A we prove Theorem 1.1 for 0-dimensional cusps, and in Section 2B for 1-dimensional cusps. Throughout this section U stands for the rank 2 unimodular hyperbolic lattice with Gram matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. The symbol $\Lambda_1 \perp \Lambda_2$ stands for the orthogonal direct sum of two quadratic lattices (or spaces) Λ_1, Λ_2, while $\Lambda_1 \oplus \Lambda_2$ just stands for the direct sum of Λ_1, Λ_2 as $\mathbb{Z}$-module (or linear space) and does not necessarily mean that $(\Lambda_1, \Lambda_2) \equiv 0$.

2A. *0-dimensional cusps.* In this subsection we prove Theorem 1.1 for 0-dimensional cusps. Let $I_1 \neq I_2$ be two isotropic lines in $\Lambda_\mathbb{Q}$ and $p_1, p_2 \in X_\Gamma$ the corresponding 0-dimensional cusps. We consider separately the cases where $(I_1, I_2) \equiv 0$ or $(I_1, I_2) \not\equiv 0$. In the former case p_1 and p_2 are joined by a boundary curve, while in the latter case they are joined by a modular curve through the interior of X_Γ.

2A1. *The case* $(I_1, I_2) \equiv 0$. We first assume that $(I_1, I_2) \equiv 0$. The direct sum $J = I_1 \oplus I_2$ is an isotropic plane in $\Lambda_\mathbb{Q}$. Let $\mathbb{H}_J^* = \mathbb{H}_J \sqcup \bigsqcup_{I\subset J} p_I$ and $\Gamma_J \subset \mathrm{SL}(J)$ be the image of the stabilizer of J in Γ. We have a generically injective morphism $f : X_J \to X_\Gamma$ from the modular curve $X_J = \Gamma_J\backslash\mathbb{H}_J^*$ whose image is the 1-dimensional cusp associated to J.

Claim 2.1. *Γ_J is a congruence subgroup of* $\mathrm{SL}(J_\mathbb{Z})$, *where* $J_\mathbb{Z} = J \cap \Lambda$.

Proof. There exists a rank 2 isotropic sublattice $J'_\mathbb{Z}$ in $\Lambda_\mathbb{Q}$ such that $J'_\mathbb{Z} \simeq (J_\mathbb{Z})^\vee$ by the pairing. The lattice $\Lambda_1 = J_\mathbb{Z} \oplus J'_\mathbb{Z}$ is isometric to $U \perp U$. We set $\Lambda_2 = (\Lambda_1)^\perp \cap \Lambda$ and $\Lambda' = \Lambda_1 \perp \Lambda_2$. Recall that Γ contains $\mathrm{O}^+(\Lambda, N)$ for some level N. Since both Λ and Λ' are full lattices in $\Lambda_\mathbb{Q}$, we can find natural numbers N_1, N_2 such that

$$N_1\Lambda' \subset N\Lambda \subset \Lambda \subset N_2^{-1}\Lambda'.$$

If we set $N' = N_1 N_2$, this tells us that

$$\mathrm{O}^+(\Lambda', N') \subset \mathrm{O}^+(\Lambda, N) \subset \Gamma \tag{2.2}$$

inside $\mathrm{O}(\Lambda_\mathbb{Q}) = \mathrm{O}(\Lambda'_\mathbb{Q})$. Now we have the embedding

$$\mathrm{SL}_2(\mathbb{Z}) \simeq \mathrm{SL}(J_\mathbb{Z}) \hookrightarrow \mathrm{O}^+(\Lambda'), \qquad \gamma \mapsto (\gamma|_{J_\mathbb{Z}}) \oplus (\gamma^\vee|_{J'_\mathbb{Z}}) \oplus \mathrm{id}_{\Lambda_2},$$

whose image is contained in the stabilizer of J. Since this maps $\Gamma(N')$ into $\mathrm{O}^+(\Lambda', N') \subset \Gamma$, we see that Γ_J contains $\Gamma(N')$. □

Let q_1, q_2 be the cusps of X_J corresponding to I_1, I_2, respectively. By this claim we can apply the Manin–Drinfeld theorem to X_J. Therefore, $[q_1] = [q_2]$ in $\mathrm{CH}_0(X_J)_{\mathbb{Q}}$. Since $f(q_1) = p_1$ and $f(q_2) = p_2$, we obtain

$$[p_1] = f_*[q_1] = f_*[q_2] = [p_2]$$

in $\mathrm{CH}_0(X_\Gamma)_{\mathbb{Q}}$.

2A2. *The case* $(I_1, I_2) \not\equiv 0$. Next we assume that $(I_1, I_2) \not\equiv 0$. In this case $I_1 \oplus I_2$ is isometric to $U_{\mathbb{Q}}$. Its orthogonal complement $(I_1 \oplus I_2)^\perp$ has signature $(1, n-1)$. We choose a vector v of positive norm from $(I_1 \oplus I_2)^\perp$ and put $\Lambda'_{\mathbb{Q}} = I_1 \oplus I_2 \oplus \mathbb{Q}v$. Then $\Lambda'_{\mathbb{Q}}$ has signature $(2, 1)$. Let $\mathcal{D}_{\Lambda'}$ be the Hermitian symmetric domain attached to $\Lambda'_{\mathbb{Q}}$. We have the natural inclusion $\mathcal{D}^*_{\Lambda'} \subset \mathcal{D}^*_{\Lambda}$, which is compatible with the embedding of orthogonal groups

$$\iota : \mathrm{O}^+(\Lambda'_{\mathbb{Q}}) \hookrightarrow \mathrm{O}^+(\Lambda_{\mathbb{Q}}), \qquad \gamma \mapsto \gamma \oplus \mathrm{id}_{(\Lambda'_{\mathbb{Q}})^\perp}.$$

Claim 2.3. *There is a subgroup* $\Gamma' < \mathrm{O}^+(\Lambda'_{\mathbb{Q}})$ *such that* $\iota(\Gamma') \subset \Gamma$ *and* $X' = \Gamma' \backslash \mathcal{D}^*_{\Lambda'}$ *is naturally isomorphic to* $X(N)$ *for some level* N.

Proof. Let $\Lambda_1 = U \perp \langle 2 \rangle$. Then $\Lambda'_{\mathbb{Q}}$ is isometric to the scaling of $(\Lambda_1)_{\mathbb{Q}}$ by some positive rational number. This gives natural isomorphisms $\mathcal{D}^*_{\Lambda'} \simeq \mathcal{D}^*_{\Lambda_1}$ and $\mathrm{O}^+(\Lambda'_{\mathbb{Q}}) \simeq \mathrm{O}^+((\Lambda_1)_{\mathbb{Q}})$. The group $\mathrm{O}^+((\Lambda_1)_{\mathbb{Q}})$ is related to $\mathrm{SL}_2(\mathbb{Q})$ by the following well-known construction (cf. [Maclachlan and Reid 2003, §2.4]). Let $V \subset M_2(\mathbb{Q})$ be the space of 2×2 matrices with trace 0, equipped with the symmetric form $(A, B) = \mathrm{tr}(AB)$. Then $V \cap M_2(\mathbb{Z})$ is isometric to Λ_1. By conjugation $\mathrm{SL}_2(\mathbb{Q})$ acts on V. This defines a homomorphism

$$\varphi : \mathrm{SL}_2(\mathbb{Q}) \to \mathrm{O}^+(V) = \mathrm{O}^+((\Lambda_1)_{\mathbb{Q}})$$

with $\mathrm{Ker}(\varphi) = \{\pm I\}$. (We have $\mathrm{Im}(\varphi) = \mathrm{SO}^+(V)$, but we do not need this fact.) It is readily checked that $\varphi(\Gamma(N)) \subset \mathrm{O}^+(\Lambda_1, N)$ for every level N. Furthermore, φ is compatible with the Veronese isomorphism

$$\mathbb{H}^* \to \mathcal{D}^*_{\Lambda_1}, \qquad \tau \mapsto e + \tau v_0 - \tau^2 f,$$

where e, f are the standard basis of U and v_0 is a generator of $\langle 2 \rangle$. Now by the same argument as (2.2), there exists a level N such that the embedding ι maps $\mathrm{O}^+(\Lambda_1, N)$ into Γ. This proves our claim. $\square$

Let q_1, q_2 be the cusps of X' corresponding to the isotropic lines I_1, I_2 of $\Lambda'_{\mathbb{Q}}$. By this claim we have a finite morphism $f : X' \to X_\Gamma$ which sends q_1 to p_1 and q_2 to p_2. By the Manin–Drinfeld theorem for X' we have $[q_1] = [q_2]$ in $\mathrm{CH}_0(X')_{\mathbb{Q}}$. Applying f_*, we obtain $[p_1] = [p_2]$ in $\mathrm{CH}_0(X_\Gamma)_{\mathbb{Q}}$. This finishes the proof of Theorem 1.1 for 0-dimensional cusps.

Remark 2.4. If Λ has Witt index 2, $(I_1 \oplus I_2)^{\perp}$ contains an isotropic line, say I_3. Then we could also apply the result of Section 2A1 to I_1 vs. I_3 and to I_3 vs. I_2, thus obtaining $[p_1] = [p_2]$ via I_3. Together with the case of Section 2A1, this shows that when X_Γ contains at least one 1-dimensional cusp, then any two 0-dimensional cusps can be connected by a chain of 1-dimensional cusps of length ≤ 2, which provides their rational equivalence.

2B. 1-*dimensional cusps.* In this subsection we prove Theorem 1.1 for 1-dimensional cusps.

2B1. *Preliminaries in $n=2$.* Although the case $n=2$ is not included in Theorem 1.1 for 1-dimensional cusps, we need to study a specific example in $n=2$ as preliminaries for the proof for the case $n \geq 3$. We consider the lattice $2U = U \perp U$. Let e_1, f_1 be the standard basis of the first copy of U, and e_2, f_2 be that of the second U. Let $J_1' = \mathbb{Q}e_2 \oplus \mathbb{Q}e_1$ and $J_2' = \mathbb{Q}f_2 \oplus \mathbb{Q}f_1$, which are isotropic planes in $2U_{\mathbb{Q}}$. We take an arbitrary natural number N and consider the modular surface $S(N) = \mathrm{O}^+(2U, N)\backslash\mathcal{D}_{2U}^*$. Let C_1, C_2 be the boundary curves of $S(N)$ associated to J_1', J_2', respectively.

Lemma 2.5. *We have* $\mathbb{Q}[C_1] = \mathbb{Q}[C_2]$ *in* $\mathrm{CH}_1(S(N))_{\mathbb{Q}}$.

Proof. Recall that we have the Segre isomorphism

$$\mathbb{H} \times \mathbb{H} \to \mathcal{D}_{2U}, \qquad (\tau_1, \tau_2) \mapsto e_1 - \tau_1\tau_2 f_1 + \tau_1 e_2 + \tau_2 f_2. \tag{2.6}$$

This extends to $\mathbb{H}^* \times \mathbb{H}^* \to \mathcal{D}_{2U}^*$, and maps the boundary components $\mathbb{H} \times (\tau_2 = 0)$, $\mathbb{H} \times (\tau_2 = i\infty)$ of $\mathbb{H}^* \times \mathbb{H}^*$ to the boundary components $\mathbb{H}_{J_1'}$, $\mathbb{H}_{J_2'}$ of $\mathcal{D}_{2U}^*$, respectively.

Let $J_3' = \mathbb{Q}f_2 \oplus \mathbb{Q}e_1$ and $J_4' = \mathbb{Q}e_2 \oplus \mathbb{Q}f_1$. By the pairing we identify $J_2' \simeq (J_1')^\vee$ and $J_4' \simeq (J_3')^\vee$. Then we define an embedding

$$\mathrm{SL}_2(\mathbb{Q}) \times \mathrm{SL}_2(\mathbb{Q}) = \mathrm{SL}(J_1') \times \mathrm{SL}(J_3') \hookrightarrow \mathrm{O}^+(2U_{\mathbb{Q}})$$

by sending $\gamma_1 \in \mathrm{SL}(J_1')$ to $(\gamma_1|_{J_1'}) \oplus (\gamma_1^\vee|_{J_2'})$ and $\gamma_3 \in \mathrm{SL}(J_3')$ to $(\gamma_3|_{J_3'}) \oplus (\gamma_3^\vee|_{J_4'})$. This embedding of groups is compatible with the isomorphism (2.6) of domains, and it maps $\Gamma(N) \times \Gamma(N)$ into $\mathrm{O}^+(2U, N)$. We thus obtain a finite morphism $f : X(N) \times X(N) \to S(N)$ which maps the boundary curves

$$C_1' = X(N) \times (\tau_2 = 0), \quad C_2' = X(N) \times (\tau_2 = i\infty)$$

of $X(N) \times X(N)$ onto C_1, C_2, respectively. By the Manin–Drinfeld theorem for the second copy of $X(N)$, we have $[C_1'] = [C_2']$ in $\mathrm{CH}_1(X(N) \times X(N))_{\mathbb{Q}}$. Applying f_*, we obtain the assertion. $\square$

2B2. *The case* $J_1 \cap J_2 = \{0\}$. We go back to the proof of Theorem 1.1. Let Λ have signature $(2, n)$ with $n \geq 3$. Let $J_1 \neq J_2$ be two isotropic planes in $\Lambda_{\mathbb{Q}}$ and $Z_1, Z_2 \subset X_\Gamma$ the corresponding 1-dimensional cusps. We first consider the case where $J_1 \cap J_2 = \{0\}$. In this case the pairing between J_1 and J_2 is perfect because $J_i^\perp / J_i$ is negative definite. The direct sum $\Lambda'_{\mathbb{Q}} = J_1 \oplus J_2$ is isometric to $2U_{\mathbb{Q}}$. We can take an isometry $2U_{\mathbb{Q}} \to \Lambda'_{\mathbb{Q}}$, which maps J'_1, J'_2 to J_1, J_2, respectively. This gives an embedding of orthogonal groups

$$\mathrm{O}^+(2U_{\mathbb{Q}}) \simeq \mathrm{O}^+(\Lambda'_{\mathbb{Q}}) \hookrightarrow \mathrm{O}^+(\Lambda_{\mathbb{Q}}), \qquad \gamma \mapsto \gamma \oplus \mathrm{id}_{(\Lambda'_{\mathbb{Q}})^\perp}, \tag{2.7}$$

which is compatible with the embedding $\mathcal{D}_{2U} \simeq \mathcal{D}_{\Lambda'} \subset \mathcal{D}_\Lambda$ of domains. By the same argument as (2.2), we can find a level N such that the embedding (2.7) maps $\mathrm{O}^+(2U, N)$ into Γ. We thus obtain a finite morphism $f : S(N) \to X_\Gamma$ with $f(C_1) = Z_1$ and $f(C_2) = Z_2$. Sending the equality $\mathbb{Q}[C_1] = \mathbb{Q}[C_2]$ of Lemma 2.5 by f_*, we obtain $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_1(X_\Gamma)_{\mathbb{Q}}$.

2B3. *The case* $J_1 \cap J_2 \neq \{0\}$. We next consider the case where $J_1 \cap J_2 \neq \{0\}$. Let $I = J_1 \cap J_2$ and choose splittings $J_1 = I \oplus I_1$ and $J_2 = I \oplus I_2$. Since $(I_1, I_2) \not\equiv 0$, we have $I_1 \oplus I_2 \simeq U_{\mathbb{Q}}$. Let $\Lambda'_{\mathbb{Q}} = I_1 \oplus I_2$ and $\Lambda''_{\mathbb{Q}} = (\Lambda'_{\mathbb{Q}})^\perp$. Then $\Lambda''_{\mathbb{Q}}$ has signature $(1, n-1)$. Since $n - 1 \geq 2$ and $\Lambda''_{\mathbb{Q}}$ contains at least one isotropic line I, we find that $\Lambda''_{\mathbb{Q}}$ contains infinitely many isotropic lines. We can choose isotropic lines I_3, I_4 in $\Lambda''_{\mathbb{Q}}$ such that I, I_3, I_4 are linearly independent. Put $J_3 = I_4 \oplus I_2$ and $J_4 = I_3 \oplus I_1$. Then J_3, J_4 are isotropic of dimension 2 and we have

$$J_1 \cap J_3 = \{0\}, \quad J_3 \cap J_4 = \{0\}, \quad J_4 \cap J_2 = \{0\}.$$

If $Z_i \subset X_\Gamma$ is the 1-dimensional cusp associated to J_i, we can apply the result of Section 2B2 successively and obtain

$$\mathbb{Q}[Z_1] = \mathbb{Q}[Z_3] = \mathbb{Q}[Z_4] = \mathbb{Q}[Z_2]$$

in $\mathrm{CH}_1(X_\Gamma)_{\mathbb{Q}}$. This finishes the proof of Theorem 1.1 for 1-dimensional cusps.

3. The symplectic case

In this section we prove Theorem 1.2. We first recall Siegel modular varieties (see [Hulek et al. 1993; Looijenga 2016]). Let Λ be a free $\mathbb{Z}$-module of rank $2g$ equipped with a nondegenerate symplectic form $(\cdot,\cdot) : \Lambda \times \Lambda \to \mathbb{Z}$. Let $\mathrm{Sp}(\Lambda)$ be the symplectic group of Λ, namely the group of isomorphisms $\Lambda \to \Lambda$ preserving the symplectic form. For a natural number N we write $\mathrm{Sp}(\Lambda, N)$ for the kernel of the reduction map $\mathrm{Sp}(\Lambda) \to \mathrm{GL}(\Lambda/N\Lambda)$. A subgroup Γ of $\mathrm{Sp}(\Lambda)$ is called a congruence subgroup if it contains $\mathrm{Sp}(\Lambda, N)$ for some level N. When $g \geq 2$, every finite-index subgroup of $\mathrm{Sp}(\Lambda)$ is a congruence subgroup [Mennicke 1965; Bass et al. 1964].

Let

$$\mathrm{LG}_\Lambda = \{[V] \in G(g, \Lambda_{\mathbb{C}}) \mid (\,\cdot\,,\cdot\,)|_V \equiv 0\}$$

be the Lagrangian Grassmannian parametrizing g-dimensional (= maximal) isotropic $\mathbb{C}$-subspaces of $\Lambda_{\mathbb{C}}$. The Hermitian symmetric domain attached to Λ is defined as the open locus $\mathcal{D}_\Lambda \subset \mathrm{LG}_\Lambda$ of those $[V]$ such that the Hermitian form $i(\,\cdot\,,\bar{\cdot}\,)|_V$ on V is positive definite.

Rational boundary components of $\mathcal{D}_\Lambda$ correspond to isotropic $\mathbb{Q}$-subspaces I of $\Lambda_{\mathbb{Q}}$. To each such I we associate the locus $\mathcal{D}_I \subset \mathrm{LG}_\Lambda$ of those $[V]$ which contains I and for which $i(\,\cdot\,,\bar{\cdot}\,)|_V$ is positive semidefinite with kernel $I_{\mathbb{C}}$. If we consider the rational symplectic space $\Lambda'_{\mathbb{Q}} = I^\perp/I$, then $\mathcal{D}_I$ is canonically isomorphic to the Hermitian symmetric domain $\mathcal{D}_{\Lambda'}$ attached to $\Lambda'_{\mathbb{Q}}$ by mapping $[V] \in \mathcal{D}_I$ to $[V/I_{\mathbb{C}}] \in \mathcal{D}_{\Lambda'}$. The union

$$\mathcal{D}^*_\Lambda = \mathcal{D}_\Lambda \sqcup \bigsqcup_{I \subset \Lambda_{\mathbb{Q}}} \mathcal{D}_I$$

is equipped with the Satake topology [Baily and Borel 1966; Borel and Ji 2005; Hulek et al. 1993]. By [Baily and Borel 1966], the quotient space $X_\Gamma = \Gamma\backslash\mathcal{D}^*_\Lambda$ has the structure of a normal projective variety and contains $\Gamma\backslash\mathcal{D}_\Lambda$ as a Zariski open set.

Theorem 1.2 is proved by induction on g. The case $g = 1$ follows from the Manin–Drinfeld theorem. Let $g \geq 2$. Assume that the theorem is proved for every congruence subgroup of $\mathrm{Sp}(\Lambda')$ for every symplectic lattice Λ' of rank $< 2g$. We then prove the theorem for $\Gamma < \mathrm{Sp}(\Lambda)$ with Λ rank $2g$.

Let $I_1 \neq I_2$ be two isotropic $\mathbb{Q}$-subspaces of $\Lambda_{\mathbb{Q}}$ of the same dimension, say g', and $Z_1, Z_2 \subset X_\Gamma$ the corresponding cusps. If we write $g'' = g - g'$, then Z_i has dimension $k = g''(g''+1)/2$. We consider the following three cases separately:

(1) $I_1 \cap I_2 \neq \{0\}$;

(2) the pairing between I_1 and I_2 is perfect;

(3) $I_1 \cap I_2 = \{0\}$ but the pairing between I_1 and I_2 is not perfect.

The case (1) is studied in Section 3A, where Z_1 and Z_2 are joined by a modular variety in the boundary. The case (2) is studied in Section 3B, where Z_1 and Z_2 are joined by a product of two modular varieties (when $g' = 1$) or by a chain of boundary modular varieties (when $g' > 1$). The remaining case (3) is considered in Section 3C, where we combine the results of (1) and (2).

3A. ***The case $I_1 \cap I_2 \neq \{0\}$.*** Assume that $I_1 \cap I_2 \neq \{0\}$. Let $I = I_1 \cap I_2$. In this case $\mathcal{D}_{I_1}, \mathcal{D}_{I_2}$ are in the boundary of $\mathcal{D}_I$. We set $\Lambda'_{\mathbb{Q}} = I^\perp/I$, $I'_1 = I_1/I$, and $I'_2 = I_2/I$. Then I'_1, I'_2 are isotropic subspaces of $\Lambda'_{\mathbb{Q}}$. The isomorphism $\mathcal{D}_I \to \mathcal{D}_{\Lambda'}$ extends to $\mathcal{D}^*_I \to \mathcal{D}^*_{\Lambda'}$ and maps $\mathcal{D}_{I_i}$ to $\mathcal{D}_{I'_i}$. The stabilizer of I in Γ acts on $\Lambda'_{\mathbb{Q}}$ naturally. Let $\Gamma_I < \mathrm{Sp}(\Lambda'_{\mathbb{Q}})$ be its image in $\mathrm{Sp}(\Lambda'_{\mathbb{Q}})$. By a similar argument as Claim 2.1, Γ_I is a congruence subgroup of $\mathrm{Sp}(\Lambda')$ for some lattice $\Lambda' \subset \Lambda'_{\mathbb{Q}}$. If we put $X_I = \Gamma_I \backslash \mathcal{D}^*_{\Lambda'}$, we have a generically injective morphism $f : X_I \to X_\Gamma$ onto the I-cusp.

Let $Z'_1, Z'_2 \subset X_I$ be the cusps of X_I corresponding to $I'_1, I'_2 \subset \Lambda'_{\mathbb{Q}}$, respectively. By the induction hypothesis, we have $\mathbb{Q}[Z'_1] = \mathbb{Q}[Z'_2]$ in $\mathrm{CH}_k(X_I)_{\mathbb{Q}}$. Since $f(Z'_i) = Z_i$, applying f_* gives $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$.

3B. ***The case (I_1, I_2) perfect.*** Next we consider the case where the pairing between I_1 and I_2 is perfect. We distinguish the cases $g' > 1$ and $g' = 1$ (i.e., top dimensional cusps).

3B1. *The case $g' > 1$.* First let $g' > 1$. We can choose a proper subspace $J_1 \neq \{0\}$ of I_1. We put $J_2 = J_1^\perp \cap I_2$ and $I_3 = J_1 \oplus J_2$. Then I_3 is isotropic of dimension g'. By construction we have $I_1 \cap I_3 \neq \{0\}$ and $I_3 \cap I_2 \neq \{0\}$. Therefore we can apply the result of Section 3A to I_1 vs. I_3 and to I_3 vs. I_2. If Z_3 is the cusp of X_Γ associated to I_3, this gives $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_3] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$.

3B2. *The case $g' = 1$.* Next let $g' = 1$. We set $\Lambda'_{\mathbb{Q}} = I_1 \oplus I_2$, which is a nondegenerate symplectic space of dimension 2. Then $\Lambda''_{\mathbb{Q}} := (\Lambda'_{\mathbb{Q}})^\perp$ is also nondegenerate of dimension $2g-2$ and we have $\Lambda_{\mathbb{Q}} = \Lambda'_{\mathbb{Q}} \perp \Lambda''_{\mathbb{Q}}$. Let $\mathcal{D}_{\Lambda'}, \mathcal{D}_{\Lambda''}$ be the Hermitian symmetric domains attached to $\Lambda'_{\mathbb{Q}}, \Lambda''_{\mathbb{Q}}$, respectively. We have the embedding of domains

$$\mathcal{D}_{\Lambda'} \times \mathcal{D}_{\Lambda''} \hookrightarrow \mathcal{D}_\Lambda, \qquad (V', V'') \mapsto V' \oplus V''. \tag{3.1}$$

This is compatible with the embedding of groups

$$\mathrm{Sp}(\Lambda'_{\mathbb{Q}}) \times \mathrm{Sp}(\Lambda''_{\mathbb{Q}}) \hookrightarrow \mathrm{Sp}(\Lambda_{\mathbb{Q}}), \qquad (\gamma', \gamma'') \mapsto \gamma' \oplus \gamma''. \tag{3.2}$$

The isotropic lines I_1, I_2 in $\Lambda'_{\mathbb{Q}}$ correspond to the respective rational boundary points $[(I_1)_{\mathbb{C}}], [(I_2)_{\mathbb{C}}]$ of $\mathcal{D}_{\Lambda'} \simeq \mathbb{H}$. Then (3.1) extends to $\mathcal{D}^*_{\Lambda'} \times \mathcal{D}^*_{\Lambda''} \hookrightarrow \mathcal{D}^*_\Lambda$ and maps $[(I_i)_{\mathbb{C}}] \times \mathcal{D}_{\Lambda''}$ to $\mathcal{D}_{I_i}$.

We take some full lattices $\Lambda' \subset \Lambda'_{\mathbb{Q}}$ and $\Lambda'' \subset \Lambda''_{\mathbb{Q}}$. By the same argument as (2.2), we can find a level N such that (3.2) maps $\mathrm{Sp}(\Lambda', N) \times \mathrm{Sp}(\Lambda'', N)$ into Γ. If we put $X' = \mathrm{Sp}(\Lambda', N) \backslash \mathcal{D}^*_{\Lambda'}$ and $X'' = \mathrm{Sp}(\Lambda'', N) \backslash \mathcal{D}^*_{\Lambda''}$, we thus obtain a finite morphism $f : X' \times X'' \to X_\Gamma$. Let p_1, p_2 be the cusps of the modular curve X' corresponding to $I_1, I_2 \subset \Lambda'_{\mathbb{Q}}$, respectively. If we set

$$Z'_i = p_i \times X'' \subset X' \times X'',$$

the above consideration shows that $f(Z'_i) = Z_i$.

We have $[p_1] = [p_2]$ in $\mathrm{CH}_0(X')_{\mathbb{Q}}$ by the Manin–Drinfeld theorem. Taking the pullback by $X' \times X'' \to X'$, we obtain $[Z'_1] = [Z'_2]$ in $\mathrm{CH}_k(X' \times X'')_{\mathbb{Q}}$. Then, taking the pushforward by f, we obtain $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$.

3C. ***The remaining case.*** Finally we consider the remaining case, namely that $I_1 \cap I_2 = \{0\}$ but the pairing between I_1 and I_2 is not perfect. Let $J_1 \subset I_1$ and $J_2 \subset I_2$ be the kernels of the pairing between I_1 and I_2. We choose splittings $I_1 = J_1 \oplus K_1$ and $I_2 = J_2 \oplus K_2$. Then $\dim J_1 = \dim J_2$ and the pairing between K_1 and K_2 is perfect. (We may have $K_i = \{0\}$. This is the case, e.g., when $g' = 1$.) We set $\Lambda'_{\mathbb{Q}} = K_1 \oplus K_2$ and $\Lambda''_{\mathbb{Q}} = (\Lambda'_{\mathbb{Q}})^{\perp}$, which are nondegenerate subspaces of $\Lambda_{\mathbb{Q}}$ with $\Lambda_{\mathbb{Q}} = \Lambda'_{\mathbb{Q}} \perp \Lambda''_{\mathbb{Q}}$. By definition J_1 and J_2 are isotropic subspaces of $\Lambda''_{\mathbb{Q}}$ with $J_1 \cap J_2 = \{0\}$ and $(J_1, J_2) \equiv 0$. We can take another isotropic subspace J_0 of $\Lambda''_{\mathbb{Q}}$ of the same dimension as J_1, J_2 such that the pairings (J_0, J_1) and (J_0, J_2) are perfect. We set $I_3 = J_0 \oplus K_2$ and $I_4 = J_0 \oplus K_1$. Then I_3, I_4 are isotropic subspaces of $\Lambda_{\mathbb{Q}}$ of the same dimension as I_1, I_2. By construction the pairings (I_1, I_3) and (I_2, I_4) are perfect, and we have $I_3 \cap I_4 \neq \{0\}$. Then we can apply the result of Section 3B to I_1 vs. I_3 and to I_2 vs. I_4, and when $K_i \neq \{0\}$ the result of Section 3A to I_3 vs. I_4. (When $K_i = \{0\}$, so that $I_3 = I_4$, the latter process is skipped.) If Z_3, Z_4 are the cusps of X_Γ associated to I_3, I_4, respectively, this shows that

$$\mathbb{Q}[Z_1] = \mathbb{Q}[Z_3] = \mathbb{Q}[Z_4] = \mathbb{Q}[Z_2]$$

in $\mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$. This completes the proof of Theorem 1.2.

Remark 3.3. Summing up the argument in the case $g' > 1$, we see that if Z_1 and Z_2 are not top dimensional, we can obtain their rational equivalence through a chain of higher dimensional cusps of length ≤ 5.

4. The unitary case

In this section we prove Theorem 1.3. We first recall modular varieties of unitary type; see [Holzapfel 1998; Looijenga 2016]. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with $R = \mathcal{O}_K$ its ring of integers (or more generally an order in K). By a Hermitian lattice over R we mean a finitely generated torsion-free R-module Λ equipped with a nondegenerate Hermitian form $(\,\cdot\,,\,\cdot\,) : \Lambda \times \Lambda \to R$. We let $\Lambda_K = \Lambda \otimes_R K$ and $\Lambda_{\mathbb{C}} = \Lambda \otimes_R \mathbb{C}$, which are Hermitian spaces over K, $\mathbb{C}$, respectively, and in which Λ is naturally embedded. We may assume without loss of generality that the signature (p, q) of Λ satisfies $p \leq q$.

Let $\mathrm{U}(\Lambda)$ be the unitary group of Λ, namely the group of R-linear isomorphisms $\Lambda \to \Lambda$ preserving the Hermitian form. This is the same as K-linear isomorphisms $\Lambda_K \to \Lambda_K$ preserving the lattice Λ and the Hermitian form. We write $\mathrm{SU}(\Lambda)$ for the subgroup of $\mathrm{U}(\Lambda)$ of determinant 1. For a natural number N we write $\mathrm{U}(\Lambda, N)$ for the kernel of the reduction map $\mathrm{U}(\Lambda) \to \mathrm{GL}(\Lambda/N\Lambda)$. A subgroup Γ of $\mathrm{U}(\Lambda)$ is called a congruence subgroup if it contains $\mathrm{U}(\Lambda, N)$ for some level N.

Let $G_\Lambda = G(p, \Lambda_{\mathbb{C}})$ be the Grassmannian parametrizing p-dimensional $\mathbb{C}$-linear subspaces of $\Lambda_{\mathbb{C}}$. The Hermitian symmetric domain $\mathcal{D}_\Lambda$ attached to Λ is defined as the open locus

$$\mathcal{D}_\Lambda = \{[V] \in G_\Lambda \mid (\cdot\,,\cdot)|_V > 0\}$$

of subspaces V to which restriction of the Hermitian form is positive definite. When $p = 0$, this is one point; when $p = 1$, this is a ball in $\mathbb{P}\Lambda_{\mathbb{C}} \simeq \mathbb{P}^q$.

Rational boundary components of $\mathcal{D}_\Lambda$ correspond to isotropic K-subspaces I of Λ_K. For each such I we associate the locus $\mathcal{D}_I \subset G_\Lambda$ of those V which contain I and for which $(\cdot\,,\cdot)|_V$ is positive semidefinite with kernel $I_{\mathbb{C}}$. If we consider $\Lambda'_K = I^\perp/I$, this is a nondegenerate K-Hermitian space of signature $(p-r, q-r)$, where $r = \dim_K I$, and $\mathcal{D}_I$ is naturally isomorphic to the Hermitian symmetric domain $\mathcal{D}_{\Lambda'}$ attached to Λ'_K by sending $[V] \in \mathcal{D}_I$ to $[V/I_{\mathbb{C}}]$. The union

$$\mathcal{D}_\Lambda^* = \mathcal{D}_\Lambda \sqcup \bigsqcup_{I \subset \Lambda_K} \mathcal{D}_I$$

is equipped with the Satake topology [Baily and Borel 1966; Borel and Ji 2005]. By [Baily and Borel 1966], the quotient space $X_\Gamma = \Gamma\backslash\mathcal{D}_\Lambda^*$ has the structure of a normal projective variety and contains $\Gamma\backslash\mathcal{D}_\Lambda$ as a Zariski open set.

The proof of Theorem 1.3 proceeds by induction on q. The case $q = 1$ is the Manin–Drinfeld theorem; we explain this in Section 4A. The inductive argument is done in Section 4B. Since this is similar to the symplectic case, we will be brief in Section 4B.

4A. ***The case $q = 1$.*** Let $q = 1$. Then $r = p = q = 1$, so Λ_K is the (unique) K-Hermitian space of signature $(1, 1)$ containing an isotropic vector, and $\mathcal{D}_\Lambda$ is the unit disc in $\mathbb{P}\Lambda_{\mathbb{C}} \simeq \mathbb{P}^1$. The group $\mathrm{SU}(\Lambda_K)$ is naturally isomorphic to $\mathrm{SL}_2(\mathbb{Q})$, and $\Gamma \cap \mathrm{SU}(\Lambda)$ is mapped to a conjugate of a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ under this isomorphism. This is a classical fact, but since we could not find a suitable reference for the second assertion, we give below a self-contained account for the reader's convenience. Theorem 1.3 in the case $q = 1$ then follows from the Manin–Drinfeld theorem, because we have a natural finite morphism from $X_{\Gamma\cap\mathrm{SU}(\Lambda)}$ to X_Γ.

We embed $K = \mathbb{Q}(\sqrt{-D})$ into the matrix algebra $M_2(\mathbb{Q})$ by sending $\sqrt{-D}$ to $J_D = \left(\begin{smallmatrix} 0 & -D \\ 1 & 0 \end{smallmatrix}\right)$. Left multiplication by J_D makes $M_2(\mathbb{Q})$ a 2-dimensional K-linear space. We have a K-Hermitian form on $M_2(\mathbb{Q})$ defined by

$$(A, B) = \mathrm{tr}(AB^*) + \sqrt{-D}^{\,-1}\mathrm{tr}(J_D AB^*), \qquad A, B \in M_2(\mathbb{Q}),$$

where for $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we write $B^* = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. We denote $\Lambda_K = M_2(\mathbb{Q})$ when we want to stress this K-Hermitian structure. Then Λ_K has signature $(1, 1)$ and contains an isotropic vector, e.g., $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Right multiplication by $\mathrm{SL}_2(\mathbb{Q})$ on $M_2(\mathbb{Q})$ is K-linear and preserves this Hermitian form. This defines a homomorphism

$$\mathrm{SL}_2(\mathbb{Q}) \to \mathrm{SU}(\Lambda_K) \tag{4.1}$$

which in fact is an isomorphism; see, e.g., [Shimura 1964, §2].

Let $\Lambda \subset \Lambda_K$ be a full R-lattice. We shall show that for every level N the image of $\mathrm{SU}(\Lambda, N) = \mathrm{SU}(\Lambda_K) \cap \mathrm{U}(\Lambda, N)$ by (4.1) is conjugate to a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let

$$\mathcal{O} = \{X \in M_2(\mathbb{Q}) \mid \Lambda X \subset \Lambda\}.$$

This is an order in $M_2(\mathbb{Q})$; see [Maclachlan and Reid 2003, §2.2]. Then $\mathrm{SU}(\Lambda) = \mathcal{O}^1$, where for any subset $\mathcal{S}$ of $M_2(\mathbb{Q})$ we write $\mathcal{S}^1 = \mathcal{S} \cap \mathrm{SL}_2(\mathbb{Q})$. Take a maximal order $\mathcal{O}_{\max}$ of $M_2(\mathbb{Q})$ containing $\mathcal{O}$. Since $\mathcal{O}$ is of finite index in $\mathcal{O}_{\max}$, there exists a natural number N_0 such that $N_0\mathcal{O}_{\max} \subset \mathcal{O}$. Therefore,

$$I + NN_0\mathcal{O}_{\max} \subset I + N\mathcal{O} \subset \mathcal{O} \subset \mathcal{O}_{\max}.$$

Since $(I + N\mathcal{O})^1 \subset \mathrm{SU}(\Lambda, N)$, this implies that

$$(I + NN_0\mathcal{O}_{\max})^1 \subset \mathrm{SU}(\Lambda, N) \subset \mathrm{SU}(\Lambda) \subset \mathcal{O}^1_{\max}.$$

Since every maximal order of $M_2(\mathbb{Q})$ is conjugate to $M_2(\mathbb{Z})$, there exists $g \in \mathrm{GL}_2(\mathbb{Q})$ such that

$$\Gamma(NN_0) \subset \mathrm{Ad}_g(\mathrm{SU}(\Lambda, N)) \subset \mathrm{Ad}_g(\mathrm{SU}(\Lambda)) \subset \mathrm{SL}_2(\mathbb{Z}).$$

This proves our claim.

4B. ***Inductive step.*** Let $q \geq 2$. Suppose that Theorem 1.3 is proved for all Hermitian lattices of signature (p', q') with $p' \leq q' < q$. We then prove the theorem for Hermitian lattices of signature (p, q) with $p \leq q$. Since the argument is similar to the symplectic case, we will just indicate the outline. Let $I_1 \neq I_2$ be two isotropic K-subspaces of Λ_K of the same dimension, say r, and $Z_1, Z_2 \subset X_\Gamma$ the associated cusps. We make the following classification:

(1) $I_1 \cap I_2 \neq \{0\}$;

(2) the pairing between I_1 and I_2 is perfect;

(3) $I_1 \cap I_2 = \{0\}$ but the pairing between I_1 and I_2 is not perfect.

(1) This is similar to Section 3A. In this case Z_1 and Z_2 are joined by the cusp associated to $I_1 \cap I_2$, to which we can apply the induction hypothesis.

(2) The case $r = 1$ is similar to Section 3B2. If we set $\Lambda'_K = I_1 \oplus I_2$ and $\Lambda''_K = (\Lambda'_K)^\perp$, these are nondegenerate of signature $(1, 1)$ and $(p - 1, q - 1)$, respectively. Then Z_1 and Z_2 are joined by the embedding $\mathcal{D}_{\Lambda'} \times \mathcal{D}_{\Lambda''} \hookrightarrow \mathcal{D}_\Lambda$. We can apply the Manin–Drinfeld theorem to $\mathcal{D}_{\Lambda'}$.

The case $r > 1$ is similar to Section 3B1. We can interpolate Z_1 and Z_2 by a third cusp by taking a proper subspace $J_1 \neq \{0\}$ of I_1 and setting $I_3 = J_1 \oplus (J_1^\perp \cap I_2)$. Then we can apply the result of case (1) to I_1 vs. I_3 and to I_3 vs. I_2.

(3) This is similar to Section 3C. We take splittings $I_1 = J_1 \oplus K_1$ and $I_2 = J_2 \oplus K_2$ such that $(J_1, I_2) \equiv 0$, $(J_2, I_1) \equiv 0$ and (K_1, K_2) is perfect. We choose an isotropic subspace J_0 from $(K_1 \oplus K_2)^{\perp}$ with (J_1, J_0) and (J_2, J_0) perfect, and put $I_3 = J_0 \oplus K_2$ and $I_4 = J_0 \oplus K_1$. Then we apply case (2) to I_1 vs. I_3 and to I_4 vs. I_2, and case (1) to I_3 vs. I_4 when $K_i \neq \{0\}$. This proves Theorem 1.3.

Remark 4.2. As in the symplectic case, we see that when Z_1, Z_2 are not top dimensional, their rational equivalence can be obtained through a chain of higher dimensional cusps of length ≤ 5.

5. Modular units and higher Chow cycles

Let Γ, $\mathcal{D}_\Lambda$, and X_Γ be as in the previous sections. As a consequence of Theorems 1.1–1.3, we can associate to each pair of maximal cusps of X_Γ a nonzero higher Chow cycle of the modular variety $Y_\Gamma = \Gamma \backslash \mathcal{D}_\Lambda$. This gives a higher dimensional analogue of modular units [Kubert and Lang 1981] from the viewpoint of algebraic cycles.

Let $Z_1 \neq Z_2$ be two cusps of X_Γ of the same dimension, say k. By our result, we have $[Z_1] = \alpha[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$ for some $\alpha \neq 0 \in \mathbb{Q}$. On the other hand, we can also view Z_1, Z_2 as k-cycles on the boundary $\partial X_\Gamma = X_\Gamma - Y_\Gamma$, which is an equidimensional reduced closed subscheme of X_Γ.

Lemma 5.1. *When the cusps Z_1, Z_2 are not top dimensional, $[Z_1] = \alpha[Z_2]$ holds already in $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$.*

Proof. When Z_1, Z_2 are not top dimensional, the proofs of Theorems 1.1–1.3 and Remarks 2.4, 3.3, and 4.2 show that we can connect Z_1 and Z_2 by a chain of higher dimensional cusps. To be more precise, we have (congruence) modular varieties $X_1, \dots, X_N$, their cusps $Z_i^+, Z_i^- \subset X_i$ of dimension k, and a finite morphism $f_i : X_i \to X_\Gamma$ onto a cusp of X_Γ, such that $f_i(Z_i^-) = f_{i+1}(Z_{i+1}^+)$ for each i and $f_1(Z_1^+) = Z_1$, $f_N(Z_N^-) = Z_2$. By induction on dimension, we have $[Z_i^+] = \alpha_i[Z_i^-]$ in $\mathrm{CH}_k(X_i)_{\mathbb{Q}}$ for some $\alpha_i \in \mathbb{Q}$. Since f_i factors through

$$X_i \to \partial X_\Gamma \subset X_\Gamma,$$

we have

$$[f_i(Z_i^+)] = \alpha_i'[f_i(Z_i^-)]$$

in $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$ for some $\alpha_i' \in \mathbb{Q}$. It follows that

$$[Z_1] = \Big(\prod_i \alpha_i'\Big)[Z_2]$$

in $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$. □

Consider the localization exact sequence of higher Chow groups [Bloch 1986; 1994] for the Baily–Borel compactification

$$Y_\Gamma \overset{j}{\hookrightarrow} X_\Gamma \overset{i}{\hookleftarrow} \partial X_\Gamma.$$

The first few terms of this sequence are written as

$$\cdots \to \mathrm{CH}_k(X_\Gamma, 1)_{\mathbb{Q}} \xrightarrow{j^*} \mathrm{CH}_k(Y_\Gamma, 1)_{\mathbb{Q}} \xrightarrow{\delta} \mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}} \xrightarrow{i_*} \mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}} \to \cdots,$$

where δ is the connecting map. By Lemma 5.1, the $\mathbb{Q}$-linear subspace of $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$ generated by the k-dimensional cusps has dimension 1 if k is not the maximal dimension of cusps. On the other hand, when $k = \dim \partial X_\Gamma$, the k-dimensional (= maximal) cusps are irreducible components of ∂X_Γ, so $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$ is freely generated over $\mathbb{Q}$ by those cusps. Let t be the number of maximal cusps of X_Γ. Since the image of $i_* : \mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}} \to \mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$ has dimension 1 by Theorems 1.1–1.3, we find that

$$\dim \mathrm{Im}(\delta) = \dim \mathrm{Ker}(i_*) = t - 1.$$

Let us construct some explicit elements of $\mathrm{CH}_k(Y_\Gamma, 1)_{\mathbb{Q}}$ whose images by δ generate $\mathrm{Im}(\delta) = \mathrm{Ker}(i_*)$.

Let $Z_1 \neq Z_2$ be two maximal cusps of X_Γ, say of dimension $k = \dim \partial X_\Gamma$. As above, we have $i_*(Z_1 - \alpha Z_2) = 0$ in $\mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$ for some $\alpha \in \mathbb{Q}$. We construct an element of $\mathrm{CH}_k(Y_\Gamma, 1)_{\mathbb{Q}}$ whose image by δ is $Z_1 - \alpha Z_2$ in $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$. (Such an element must be nonzero because $Z_1 - \alpha Z_2$ is nonzero in $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$.) Recall from the proof of Theorems 1.1–1.3 that, in a basic case, we have a compactified modular curve $X' = X_{\Gamma'}$, its two cusps $p_1, p_2 \in X'$, a k-dimensional compactified modular variety $X'' = X_{\Gamma''}$, and a finite morphism $f : X' \times X'' \to X_\Gamma$ such that $f(p_i \times X'') = Z_i$. (In the orthogonal case X'' is one point when $k = 0$ and a modular curve when $k = 1$; in the symplectic case X'' is a Siegel modular variety of genus $g - 1$; in the unitary case X'' is associated to a unitary group of signature $(p - 1, q - 1)$.) The general case is a chain of such basic cases. For simplicity we assume that (Z_1, Z_2) is such a basic pair.

By the Manin–Drinfeld theorem for X', there exists a modular function F on X' such that $\mathrm{div}(F) = \beta(p_1 - p_2)$ for some natural number β. Let $Y' \subset X'$ and $Y'' \subset X''$ be the modular varieties before compactification. We can view F as an element of $\mathcal{O}^*(Y') = \mathrm{CH}_0(Y', 1)$. Then $\delta(F) = \beta(p_1 - p_2)$ for the connecting map $\delta : \mathrm{CH}_0(Y', 1) \to \mathrm{CH}_0(\partial X')$. Let $\pi : Y' \times Y'' \to Y'$ be the projection and, by abuse of notation, $f : Y' \times Y'' \to Y_\Gamma$ be the restriction of $f : X' \times X'' \to X_\Gamma$. We can pullback the higher Chow cycle F by the flat morphism π and then take its pushforward by the finite morphism f. The result, $f_*\pi^*F$, is an element of $\mathrm{CH}_k(Y_\Gamma, 1)$.

Proposition 5.2. *We have* $\mathbb{Q}\delta(f_*\pi^*F) = \mathbb{Q}(Z_1 - \alpha Z_2)$ *in* $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$.

Proof. We take a desingularization $\widetilde{X}'' \to X''$ of X'', and let $\widetilde{Y}'' \subset \widetilde{X}''$ be the inverse image of Y''. We have the commutative diagram

$$\begin{tikzcd}
\mathcal{O}^*(Y') \arrow[r, "\tilde{\pi}^*", "\simeq"'] \arrow[d, equal] & \mathcal{O}^*(Y' \times \widetilde{X}'') \arrow[r, "\tilde{j}^*"] \arrow[d, equal] & \mathcal{O}^*(Y' \times \widetilde{Y}'') \arrow[d, equal] & \\
\mathrm{CH}_0(Y', 1) \arrow[r, "\tilde{\pi}^*", "\simeq"'] \arrow[d, "\delta"'] & \mathrm{CH}_k(Y' \times \widetilde{X}'', 1) \arrow[r, "\tilde{j}^*"] \arrow[d, "\delta"'] & \mathrm{CH}_k(Y' \times \widetilde{Y}'', 1) \arrow[r, "\tilde{f}_*"] \arrow[d, "\delta"'] & \mathrm{CH}_k(Y_\Gamma, 1) \arrow[d, "\delta"'] \\
\mathrm{CH}_0(\partial X') \arrow[r, "\tilde{\pi}^*", "\simeq"'] & \mathrm{CH}_k(\partial X' \times \widetilde{X}'') \arrow[r, "\tilde{i}_*"] & \mathrm{CH}_k(\partial(X' \times \widetilde{X}'')) \arrow[r, "\tilde{f}_*"] & \mathrm{CH}_k(\partial X_\Gamma)
\end{tikzcd}$$

The various δ are the connecting maps of each localization sequence, $\tilde{\pi}: X' \times \widetilde{X}'' \to X'$ the projection, $\partial(X' \times \widetilde{X}'') = X' \times \widetilde{X}'' - Y' \times \widetilde{Y}''$, $\tilde{j} : Y' \times \widetilde{Y}'' \hookrightarrow Y' \times \widetilde{X}''$ the open immersion, $\tilde{i} : \partial X' \times \widetilde{X}'' \hookrightarrow \partial(X' \times \widetilde{X}'')$ the closed embedding, and $\tilde{f} : X' \times \widetilde{X}'' \to X_\Gamma$ the proper morphism induced from f. If we send $\mathbb{Q}F \subset \mathrm{CH}_0(Y', 1)_\mathbb{Q}$ through this diagram to $\mathrm{CH}_k(\partial X_\Gamma)_\mathbb{Q}$, the image is $\mathbb{Q}(Z_1 - \alpha Z_2)$. The assertion follows by noticing that $\tilde{f}_* \tilde{j}^* \tilde{\pi}^* = f_* \pi^*$. $\square$

In this way, as a "lift" from the modular unit F, we obtain an explicit nonzero element of $\mathrm{CH}_k(Y_\Gamma, 1)_\mathbb{Q}$ whose image by δ is $Z_1 - \alpha Z_2$. If we run (Z_1, Z_2) over all basic pairs of maximal cusps, we obtain a set of nonzero elements of $\mathrm{CH}_k(Y_\Gamma, 1)_\mathbb{Q}$ whose image by δ generate $\mathrm{Im}(\delta) = \mathrm{Ker}(i_*)$. In general, by this construction we could obtain more than $t - 1$ higher Chow cycles on Y_Γ. This is because

(1) the choice of $X' \times X'' \to X_\Gamma$ is not necessarily unique for the given pair (Z_1, Z_2), and

(2) the number of basic pairs could be larger than $t - 1$.

The point (1) amounts to the situation that two pairs (I_1, I_2), (I_1', I_2') of isotropic subspaces are not Γ-equivalent as pairs, although I_1 is Γ-equivalent to I_1' and I_2 is Γ-equivalent to I_2', respectively. A typical situation of (2) is that for three cusps Z_1, Z_2, Z_3, all pairs (Z_1, Z_2), (Z_2, Z_3), (Z_3, Z_1) are basic.

If the span $V \subset \mathrm{CH}_k(Y_\Gamma, 1)_\mathbb{Q}$ of all higher Chow cycles constructed in this way has dimension $\geq t$, the kernel of $\delta : V \to \mathrm{CH}_k(\partial X_\Gamma)_\mathbb{Q}$ would then give rise to a nontrivial subspace of $\mathrm{CH}_k(X_\Gamma, 1)_\mathbb{Q}$.

References

[Baily and Borel 1966] W. L. Baily, Jr. and A. Borel, "Compactification of arithmetic quotients of bounded symmetric domains", *Ann. of Math.* (2) **84** (1966), 442–528. MR Zbl

[Bass et al. 1964] H. Bass, M. Lazard, and J.-P. Serre, "Sous-groupes d'indice fini dans SL(n, $\mathbb{Z}$)", *Bull. Amer. Math. Soc.* **70** (1964), 385–392. MR Zbl

[Bloch 1986] S. Bloch, "Algebraic cycles and higher K-theory", *Adv. in Math.* **61**:3 (1986), 267–304. MR Zbl

[Bloch 1994] S. Bloch, "The moving lemma for higher Chow groups", *J. Algebraic Geom.* **3**:3 (1994), 537–568. MR Zbl

[Borel and Ji 2005] A. Borel and L. Ji, "Compactifications of symmetric and locally symmetric spaces", pp. 69–137 in *Lie theory*, edited by J.-P. Anker and B. Orsted, Progr. Math. **229**, Birkhäuser, Boston, 2005. MR Zbl

[Drinfeld 1973] V. G. Drinfeld, "Two theorems on modular curves", *Funkcional. Anal. i Priložen.* **7**:2 (1973), 83–84. In Russian; translated in *Funct. Anal. Appl.* **7** (1973), 155–156. MR Zbl

[Elkik 1990] R. Elkik, "Le théorème de Manin–Drinfeld", pp. 59–67 in *Séminaire sur les Pinceaux de Courbes Elliptiques (Paris, 1988)*, edited by L. Szpiro, Astérisque **183**, Soc. Math. France, Paris, 1990. MR Zbl

[Holzapfel 1998] R.-P. Holzapfel, *Ball and surface arithmetics*, Aspects of Math. **29**, Friedr. Vieweg & Sohn, Braunschweig, 1998. MR Zbl

[Hulek et al. 1993] K. Hulek, C. Kahn, and S. H. Weintraub, *Moduli spaces of abelian surfaces: compactification, degenerations, and theta functions*, de Gruyter Expositions in Math. **12**, Walter de Gruyter & Co., Berlin, 1993. MR Zbl

[Kubert and Lang 1981] D. S. Kubert and S. Lang, *Modular units*, Grundlehren der Math. Wissenschaften **244**, Springer, 1981. MR Zbl

[Looijenga 2016] E. Looijenga, "Moduli spaces and locally symmetric varieties", pp. 33–75 in *Development of moduli theory* (Kyoto, 2013), edited by O. Fujino et al., Adv. Stud. Pure Math. **69**, Math. Soc. Japan, Tokyo, 2016. MR Zbl

[Maclachlan and Reid 2003] C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Math. **219**, Springer, 2003. MR Zbl

[Manin 1972] J. I. Manin, "Parabolic points and zeta functions of modular curves", *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 19–66. In Russian; translated in *Math. USSR-Izv.* **6** (1972), 19–64. MR Zbl

[Mennicke 1965] J. Mennicke, "Zur Theorie der Siegelschen Modulgruppe", *Math. Ann.* **159** (1965), 115–129. MR Zbl

[Scattone 1987] F. Scattone, *On the compactification of moduli spaces for algebraic $K3$ surfaces*, Mem. Amer. Math. Soc. **374**, Amer. Math. Soc., Providence, RI, 1987. MR Zbl

[Shimura 1964] G. Shimura, "Arithmetic of unitary groups", *Ann. of Math.* (2) **79** (1964), 369–409. MR Zbl

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C_2-equivariant stable homotopy from real motivic stable homotopy

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We give a method for computing the C_2-equivariant homotopy groups of the Betti realization of a p-complete cellular motivic spectrum over $\mathbb{R}$ in terms of its motivic homotopy groups. More generally, we show that Betti realization presents the C_2-equivariant p-complete stable homotopy category as a localization of the p-complete cellular real motivic stable homotopy category.

1. Introduction

Let $\mathrm{SH}(K)$ denote the ∞-category of motivic spectra over a field K [Morel and Voevodsky 1999], whose equivalences are given by the stable $\mathbb{A}^1$-equivalences. This ∞-category has a bigraded family of spheres

$$S^{i,j} := S^{i-j} \wedge \mathbb{G}_m^j$$

of topological degree i and motivic weight j. These lead to bigraded homotopy groups

$$\pi_{i,j}^K X := [S^{i,j}, X]_K.$$

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A motivic spectrum is *cellular* if it is built from the spheres $S^{i,j}$ using cofiber sequences and filtered homotopy colimits. A map between cellular spectra is a stable $\mathbb{A}^1$-equivalence if and only if it is a $\pi^K_{*,*}$-isomorphism [Dugger and Isaksen 2005]. We shall let $\mathrm{SH}_{\mathrm{cell}}(K)$ denote the full subcategory of cellular spectra.

Complex and real Betti realization. If Z is a smooth scheme over $\mathbb{C}$, then its $\mathbb{C}$-points, or Betti realization

$$\mathrm{Be}(Z) := Z(\mathbb{C}),$$

form a topological space when endowed with the complex analytic topology. The resulting Betti realization functor

$$\mathrm{Be} : \mathrm{SH}(\mathbb{C}) \to \mathrm{Sp}$$

(where Sp denotes the ∞-category of spectra) is called *Betti realization* [Morel and Voevodsky 1999]. Since $\mathrm{Be}(S^{i,j}) = S^i$, Betti realization induces a map

$$\mathrm{Be} : \pi^{\mathbb{C}}_{i,j} X \to \pi_i \, \mathrm{Be}(X).$$

This map was well studied by Dugger and Isaksen [2010] (at the prime 2) and by Stahn [2016] (at odd primes). For a prime p, the p-complete motivic stable stems have an element

$$\tau \in \pi^{\mathbb{C}}_{0,-1}(S^{0,0})^{\wedge}_{p}.$$

The following result is a direct corollary of the results of Dugger and Isaksen and of Stahn (here, $\widehat{\mathrm{Be}}_p(-)$ denotes p-completed Betti realization).

Theorem 1.1 (see Theorem 8.18). *Let $X \in \mathrm{SH}(\mathbb{C})$ be p-complete and cellular. Then Betti realization induces an isomorphism of abelian groups*

$$\pi^{\mathbb{C}}_{i,j} X[\tau^{-1}] \xrightarrow{\cong} \pi_i \, \widehat{\mathrm{Be}}_p(X),$$

and thus an equivalence of ∞-categories

$$\widehat{\mathrm{Be}}_p : \mathrm{SH}_{\mathrm{cell}}(\mathbb{C})^{\wedge}_{p}[\tau^{-1}] \xrightarrow{\simeq} \mathrm{Sp}^{\wedge}_{p}.$$

In the real case, there is a real Betti realization functor

$$\mathrm{Be}_{\mathbb{R}} : \mathrm{SH}(\mathbb{R}) \to \mathrm{Sp}$$

which arises from associating to a smooth scheme Z over $\mathbb{R}$ its topological space of $\mathbb{R}$-points $Z(\mathbb{R})$, endowed with the real analytic topology. The inclusion

$$\rho : \{\pm 1\} \hookrightarrow \mathbb{G}_m$$

gives an element $\rho \in \pi^{\mathbb{R}}_{-1,-1} S^{0,0}$, which becomes an equivalence after real Betti realization. Bachmann [2018] proved the following:

Theorem 1.2 (see Theorem 8.10). *For all $X \in \mathrm{SH}(\mathbb{R})$, real Betti realization induces an isomorphism of abelian groups*

$$\mathrm{Be}_{\mathbb{R}} : \pi^{\mathbb{R}}_{i,j} X[\rho^{-1}] \xrightarrow{\cong} \pi_{i-j} \mathrm{Be}_{\mathbb{R}}(X),$$

and moreover[1] *an equivalence of ∞-categories*

$$\mathrm{SH}(\mathbb{R})[\rho^{-1}] \xrightarrow{\simeq} \mathrm{Sp}.$$

Statement of results. The results discussed above demonstrate that the homotopy groups of the complex and real Betti realizations of a cellular motivic spectrum can be obtained by localizing its motivic homotopy groups, and each of these Betti realization functors is a localization.

The purpose of this paper is to prove a similar result about the C_2-Betti realization functor

$$\mathrm{Be}^{C_2} : \mathrm{SH}(\mathbb{R}) \to \mathrm{Sp}^{C_2}.$$

Here, Sp^{C_2} denotes the ∞-category of genuine C_2-spectra. This functor arises from associating to a smooth scheme Z over $\mathbb{R}$ the C_2-topological space $Z(\mathbb{C})$, with the C_2-action given by complex conjugation.

For $Y \in \mathrm{Sp}^{C_2}$, the $RO(C_2)$-graded equivariant homotopy groups are bigraded by setting

$$\pi^{C_2}_{i,j} Y := [S^{(i-j)+j\sigma}, Y]^{C_2},$$

where σ is the sign representation. In G-equivariant homotopy theory, one takes the stable equivalences to be the π^H_*-isomorphisms, where π^H_* denotes the $\mathbb{Z}$-graded H-equivariant homotopy groups, and H ranges over the subgroups of G. However, in the case of $G = C_2$, a map in Sp^{C_2} is a stable equivalence if and only if it is a $\pi^{C_2}_{*,*}$-isomorphism (see the discussion following (6.1)). The C_2-equivariant homotopy groups of Y can be effectively analyzed from the homotopy pullback (isotropy separation square) [Greenlees and May 1995]

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y^{\Phi} \arrow[d] \\ Y^h \arrow[r] & Y^t \end{tikzcd} \tag{1.3}$$

where[2]

$$\begin{aligned} Y^h &:= F((EC_2)_+, Y) && \text{(homotopy completion)}, \\ Y^{\Phi} &:= Y \wedge \widetilde{EC_2} && \text{(geometric localization)}, \\ Y^t &:= (Y^h)^{\Phi} && \text{(equivariant Tate spectrum)}. \end{aligned}$$

[1] Bachmann's methods do not rely upon cellularity hypotheses.

[2] The terminology here comes from the fact that the fixed points Y^{hC_2}, $Y^{\Phi C_2}$, and Y^{tC_2} are the homotopy fixed points, geometric fixed points, and Tate spectrum of Y, respectively.

We let Sp^{hC_2} denote the full subcategory of Sp^{C_2} consisting of homotopically complete spectra, and let $\mathrm{Sp}^{\Phi C_2}$ denote the full subcategory consisting of geometrically local spectra. The C_2-geometric fixed points functor gives an equivalence of ∞-categories $\mathrm{Sp}^{\Phi C_2} \simeq \mathrm{Sp}$.

Bachmann's theorem (Theorem 1.2) effectively describes the homotopy theory of the geometric localization of C_2-Betti realization $\mathrm{Be}^{C_2}(-)^{\Phi}$. This is because

(1) for all $X \in \mathrm{SH}(\mathbb{R})$, we have

$$\mathrm{Be}_{\mathbb{R}}(X) = \mathrm{Be}^{C_2}(X)^{\Phi C_2},$$

(2) geometric localization is given by inverting $a := \mathrm{Be}^{C_2}(\rho) \in \pi^{C_2}_{-1,-1} S^{0,0}$:

$$Y^{\Phi} \simeq Y[a^{-1}]. \tag{1.4}$$

Thus, Bachmann's theorem (Theorem 1.2) can be restated in the following way.

Theorem 1.5 (see Theorem 8.24). *For all $X \in \mathrm{SH}(\mathbb{R})$, C_2-Betti realization induces an isomorphism*

$$\pi^{\mathbb{R}}_{*,*} X[\rho^{-1}] \xrightarrow{\cong} \pi^{C_2}_{*,*} \mathrm{Be}^{C_2}(X)^{\Phi},$$

and an equivalence

$$\mathrm{Be}^{C_2} : \mathrm{SH}(\mathbb{R})[\rho^{-1}] \xrightarrow{\simeq} \mathrm{Sp}^{\Phi C_2}.$$

We are thus left to describe the homotopy theory of the homotopy completion of the C_2-Betti realization.

We first note that a map

$$f : Y_1 \to Y_2$$

in Sp^{hC_2} is an equivalence if and only if the underlying map

$$f^e : Y_1^e \to Y_2^e$$

of spectra is a nonequivariant equivalence. We therefore first study $\mathrm{Be}^{C_2}(-)^e$. Consider the diagram of adjoint functors

$$\begin{tikzcd} \mathrm{SH}(\mathbb{R}) \arrow[r, shift left, "\mathrm{Be}^{C_2}"] \arrow[d, shift right, "\zeta^*"'] & \mathrm{Sp}^{C_2} \arrow[l, shift left, "\mathrm{Sing}^{C_2}"] \arrow[d, shift right, "\mathrm{Res}^{C_2}_e"'] \\ \mathrm{SH}(\mathbb{C}) \arrow[r, shift left, "\mathrm{Be}"] \arrow[u, shift right, "\zeta_*"'] & \mathrm{Sp} \arrow[l, shift left, "\mathrm{Sing}"] \arrow[u, shift right, "\mathrm{Ind}^{C_2}_e"'] \end{tikzcd} \tag{1.6}$$

where (ζ^*, ζ_*) are the base change functors associated to the morphism

$$\zeta : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{R})$$

and $(\mathrm{Res}_e^{C_2}, \mathrm{Ind}_e^{C_2})$ are the change of group functors associated to the inclusion

$$e \hookrightarrow C_2.$$

We will prove the following theorem, which has also been independently obtained by Isaksen, Kong, Wang, and Xu.

Theorem 1.7 (see Corollary 8.2). *Under the equivalence*

$$\zeta_* \zeta^* S^0 \simeq \Sigma^\infty_+ \mathrm{Spec}(\mathbb{C})$$

the adjunction (ζ^*, ζ_*) *induces an equivalence*

$$\mathrm{SH}(\mathbb{C}) \simeq \mathrm{Mod}_{\mathrm{SH}(\mathbb{R})}(\Sigma^\infty_+ \mathrm{Spec}(\mathbb{C})).$$

Let $C\rho \in \mathrm{SH}(\mathbb{R})$ denote the cofiber of $\rho \in \pi^{\mathbb{R}}_{-1,-1} S^{0,0}$. Since $\zeta^*(\rho)$ is null in $\pi^{\mathbb{C}}_{-1,-1} S^{0,0}$, there is a map

$$C(\rho) \to \Sigma^\infty_+ \mathrm{Spec}(\mathbb{C}) \tag{1.8}$$

which we show is a $\pi^{\mathbb{R}}_{*,*}$-isomorphism after p-completion (Proposition 8.3). The real motivic spectrum $\Sigma^\infty_+ \mathrm{Spec}(\mathbb{C})$ is not cellular (Remark 8.4), so $C\rho$ may be regarded as its p-complete cellular approximation. We deduce the following (which was also independently observed by Isaksen, Kong, Wang, and Xu):

Corollary 1.9 (see Corollary 8.6). *The adjunction* (ζ^*, ζ_*) *and equivalence* (1.8) *induces an equivalence*

$$\mathrm{SH}_{\mathrm{cell}}(\mathbb{C})^\wedge_p \simeq \mathrm{Mod}_{\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p}(C(\rho)).$$

In particular, for $X \in \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p$ *there is an isomorphism*

$$\pi^{\mathbb{C}}_{*,*}(\zeta^* X) \cong \pi^{\mathbb{R}}_{*,*}(X \wedge C\rho).$$

Combining Corollary 1.9 with Theorem 1.1, we deduce that for $X \in \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p$,

$$\pi_i(\widehat{\mathrm{Be}}_p^{C_2}(X)^e) \cong \pi^{\mathbb{R}}_{i,j}(X \wedge C\rho[\tau^{-1}]).$$

In particular, τ exists as a self map

$$\tau : \Sigma^{0,-1} C(\rho)^\wedge_p \to C(\rho)^\wedge_p.$$

Let $C(\rho^i)$ denote the cofiber of $\rho^i \in \pi^{\mathbb{R}}_{-i,-i} S^{0,0}$. We will prove:

Theorem 1.10 (see Theorem 7.10 and Proposition 7.11). *For each* $i \geq 1$, *there exists a* j *so that* $C(\rho^i)^\wedge_p$ *has a* τ^j*-self map*

$$\tau^j : \Sigma^{0,-j} C(\rho^i)^\wedge_p \to C(\rho^i)^\wedge_p.$$

Our proof of the existence of these τ-self maps at the prime 2 relies on first proving the existence of their C_2-Betti realizations, and then using a theorem of Dugger and Isaksen [2017b] to lift the self maps to the real motivic category. Because this approach involves some analysis of the C_2-equivariant stable stems, it may be of independent interest.

We shall let $C(\rho^i)^\wedge_p[\tau^{-1}]$ denote the telescope of this τ^j-self map. Define, for $X \in \mathrm{SH}(\mathbb{R})^\wedge_p$,[3]

$$X^\wedge_\rho[\tau^{-1}] := \operatorname*{holim}_i X \wedge C(\rho^i)[\tau^{-1}].$$

Our main theorem is the following:

Theorem 1.11 (see Corollary 8.21 and Theorem 8.26). *For $X \in \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p$, the p-completed C_2-Betti realization functor $\widehat{\mathrm{Be}}{}^{C_2}_p$ induces an isomorphism*

$$\pi^{\mathbb{R}}_{*,*} X^\wedge_\rho[\tau^{-1}] \xrightarrow{\cong} \pi^{C_2}_{*,*} \widehat{\mathrm{Be}}{}^{C_2}_p(X)^h$$

and the right adjoint

$$\mathrm{Cell}\,\mathrm{Sing}^{C_2} : (\mathrm{Sp}^{hC_2})^\wedge_p \to \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p$$

of p-complete, homotopy complete C_2-Betti realization is fully faithful.

Thus, Theorems 1.5 and 1.11 combine to express the $RO(C_2)$-graded equivariant homotopy groups of $\widehat{\mathrm{Be}}{}^{C_2}_p(X)^\Phi$ and $\widehat{\mathrm{Be}}{}^{C_2}_p(X)^h$ in terms of the real motivic homotopy groups of X. By the isotropy separation square (1.3), we just need to be able to compute $\pi^{C_2}_{*,*} \widehat{\mathrm{Be}}{}^{C_2}_p(X)^t$ (and the maps on homotopy groups) to recover $\pi^{C_2}_{*,*} \widehat{\mathrm{Be}}{}^{C_2}_p(X)$, but this is easily accomplished by combining Theorem 8.26 with (1.3) to deduce (for X cellular and p-complete) an isomorphism[4]

$$\pi^{C_2}_{*,*} \widehat{\mathrm{Be}}{}^{C_2}_p(X)^t \cong \pi^{\mathbb{R}}_{*,*} X^\wedge_\rho[\tau^{-1}][\rho^{-1}].$$

Finally, we will show that the isotropy separation square (1.3) implies that Theorems 1.5 and 1.11 combine to show that p-complete C_2-equivariant stable homotopy is a localization of real motivic cellular stable homotopy.

Theorem 1.12 (see Theorem 8.22). *The right adjoint to p-complete cellular C_2-Betti realization*

$$\mathrm{Cell}\,\mathrm{Sing}^{C_2} : (\mathrm{Sp}^{C_2})^\wedge_p \to \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p$$

is fully faithful.

We will apply our techniques to compute $\pi^{C_2}_{*,*} \widehat{\mathrm{Be}}{}^{C_2}_2 X$ from $\pi^{\mathbb{R}}_{*,*} X$, for X equal to

[3]For p odd, it turns out that independently of i, one can take $j = 2$ in Theorem 1.10 (see Proposition 7.11). Consequently, $X^\wedge_\rho$ has a τ^2-self map, and the spectrum $X^\wedge_\rho[\tau^{-1}]$ can be simply taken to be the telescope of this τ^2-self map on $X^\wedge_\rho$.

[4]For p odd, the situation is much simpler, as this Tate spectrum is contractible since $2 = |C_2|$ is invertible.

(1) $(H\mathbb{F}_2)_{\mathbb{R}}$, the real motivic mod 2 Eilenberg–MacLane spectrum, with

$$\widehat{\mathrm{Be}}_2^{C_2}(H\mathbb{F}_2)_{\mathbb{R}} \simeq H\underline{\mathbb{F}_2},$$

the C_2-equivariant Eilenberg–MacLane spectrum associated to the constant Mackey functor $\underline{\mathbb{F}_2}$,

(2) $(H\mathbb{Z}_2^\wedge)_{\mathbb{R}}$, the real motivic 2-adic Eilenberg–MacLane spectrum, with

$$\widehat{\mathrm{Be}}_2^{C_2}(H\mathbb{Z}_2^\wedge)_{\mathbb{R}} \simeq H\underline{\mathbb{Z}_2^\wedge},$$

the C_2-equivariant Eilenberg–MacLane spectrum associated to the constant Mackey functor $\underline{\mathbb{Z}_2^\wedge}$,

(3) $kgl_2^\wedge$, the 2-complete effective cover of the real motivic K-theory spectrum KGL, with

$$\widehat{\mathrm{Be}}_2^{C_2}\, kgl_2^\wedge \simeq kR_2^\wedge,$$

the 2-complete connective Real K-theory spectrum.

In the case of $(H\mathbb{F}_2)_{\mathbb{R}}$, the homotopy groups of the C_2-Betti realization differ from the motivic homotopy groups of the original spectrum through the addition of a notorious "negative cone" (see, e.g., [Dugger and Isaksen 2017b, Figure 1]). From the perspective of the mod 2 Adams spectral sequence, the presence of this "negative cone" makes the equivariant homotopy of the Betti realizations of the other examples similarly more complicated than the motivic homotopy of the original spectra. Our theory organically predicts the presence of the negative cone through a mechanism of local duality such as that studied in [Barthel et al. 2018], and thus gives a more direct route to these equivariant computations by starting with the simpler motivic analogs. This connection with local duality deserves further study.

Relationship to the work of Heller and Ormsby. Heller and Ormsby [2016; 2018] also study the relationship between real motivic and C_2-equivariant spectra (and their results extend to other real closed fields), but their analysis centers around the adjoint pair

$$c_{\mathbb{R}}^*\colon \mathrm{Sp}^{C_2} \rightleftarrows \mathrm{SH}(\mathbb{R}) : (c_{\mathbb{R}})_*$$

where $c_{\mathbb{R}}^*$ is the equivariant generalization of the constant functor (Definition 8.11).

Namely, Heller and Ormsby show that Sp^{C_2} is a *colocalization* of $\mathrm{SH}(\mathbb{R})$ by showing that $c_{\mathbb{R}}^*$ is fully faithful. Their results allow them to compute, for $X \in \mathrm{Sp}^{C_2}$, integer graded motivic homotopy groups of $c_{\mathbb{R}}^* X$ in terms of the integer graded equivariant homotopy groups of X.

Our results, by contrast, show that C_2-Betti realization exhibits $(\mathrm{Sp}^{C_2})_p^\wedge$ as a *localization* of $\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})_p^\wedge$, and this allows us to compute, for $X \in \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})_p^\wedge$, the equivariant $RO(C_2)$-graded homotopy groups of $\widehat{\mathrm{Be}}_p^{C_2}(X)$ in terms of the bigraded motivic homotopy groups of X. Nevertheless, we use the functor $c_{\mathbb{R}}^*$ to prove our localization theorem.

Organization of the paper. The first four sections of this paper are formal. In Section 2, we recall some facts concerning limits of presentable ∞-categories. In Section 3, we study both Bousfield localizations of symmetric monoidal ∞-categories and their relation to completion, and we discuss the interaction of these localizations with a monoidal Barr–Beck theorem of Mathew, Naumann, and Noel [Mathew et al. 2017]. In Section 4, we summarize some facts regarding cellularization in the ∞-categorical context, and the interaction of cellularization with localization and symmetric monoidal structures.

In Section 5, we recall the notion of a *recollement* of ∞-categories, which is a formalism for decomposing an ∞-category using two complementary localizations. We show that to prove an adjunction between two recollements is a localization, it suffices to check fully faithfulness on the constituents of the recollements.

In Section 6, we turn to the case of interest and recall some facts about motivic and equivariant homotopy theory that we will need later.

In Section 7, we show that James periodicity in the 2-primary equivariant stable stems results from the existence of *u-self maps* on $C(a^i)^\wedge_2$, where a is the Euler class of the sign representation. We then use an isomorphism theorem of Dugger and Isaksen [2017b] to lift these u-self maps to *τ-self maps* on $C(\rho^i)^\wedge_2$. For an odd prime p, we explain how the work of Stahn [2016] implies that every (p, ρ)-complete $\mathbb{R}$-motivic spectrum has a τ^2-self map.

Section 8 contains all of our main theorems, and their proofs, concerning the localizations induced by Betti realization.

Section 9 contains examples, where we take various real motivic spectra, and use our theory to compute the 2-primary $RO(C_2)$-graded C_2-equivariant homotopy groups of their Betti realizations from their 2-primary motivic homotopy groups. We also explain how to do these kinds of computations at an odd prime, where the story is much simpler.

2. Limits of presentable ∞-categories

We collect some necessary facts about limits in the ∞-category Pr^L of presentable ∞-categories.

Suppose $\mathscr{C}_\bullet : J \to \mathrm{Pr}^L$ is a diagram and let

$$\mathscr{X} = \textstyle\int \mathscr{C}_\bullet \to J$$

be the presentable fibration [Lurie 2009, Definition 5.5.3.2] classified by $\mathscr{C}_\bullet$. By [Lurie 2009, Proposition 5.5.3.13, Corollary 3.3.3.2], we have an equivalence

$$\mathscr{C} := \lim \mathscr{C}_\bullet \simeq \mathrm{Sect}(\mathscr{X}) := \mathrm{Fun}^{cocart}_{/J}(J, \mathscr{X})$$

between the limit $\mathscr{C}$ of $\mathscr{C}_\bullet$ and the ∞-category of cocartesian sections of $\mathscr{X}$. Let $\mathscr{D}$ be another presentable ∞-category and suppose that we have an extension

$$\overline{\mathscr{C}}_\bullet : J^{\triangleleft} \to \mathrm{Pr}^L$$

with the cone point sent to $\mathscr{D}$. Then we have an induced adjunction

$$F : \mathscr{D} \rightleftarrows \mathscr{C} : R.$$

Let

$$\overline{\mathscr{X}} \to J^{\triangleleft}$$

be the presentable fibration classified by $\overline{\mathscr{C}}_\bullet$. In terms of the description of C as $\mathrm{Sect}(\mathscr{X})$, we may describe F and R more explicitly as follows:

(1) The functor

$$F : \mathscr{D} \simeq \lim \overline{\mathscr{C}}_\bullet \to \mathscr{C} \simeq \lim \mathscr{C}_\bullet$$

is given by the contravariant functoriality of limits for the inclusion $J \subset J^{\triangleleft}$. Thus, under the equivalences $\mathrm{Sect}(\overline{\mathscr{X}}) \simeq \mathscr{D}$ and $\mathrm{Sect}(\mathscr{X}) \simeq \mathscr{C}$, the functor $F : \mathscr{D} \to \mathscr{C}$ corresponds to the functor

$$F : \mathrm{Sect}(\overline{\mathscr{X}}) \to \mathrm{Sect}(\mathscr{X})$$

given by restriction of cocartesian sections. In particular, an object $x \in \mathscr{D}$ corresponds to the cocartesian section

$$\overline{\sigma} : J^{\triangleleft} \to \overline{\mathscr{X}}$$

determined up to contractible choice by $\overline{\sigma}(v) = x$ for v the cone point, and then $F(x) = \overline{\sigma}|_J$.

(2) Let

$$p : \mathscr{X} \subset \overline{\mathscr{X}} \to \overline{\mathscr{X}}_v \simeq \mathscr{D}$$

be the cartesian pushforward to the fiber over the initial object $v \in J^{\triangleleft}$. Then for any object $\sigma \in \mathscr{C}$ viewed as a cocartesian section of $\mathscr{X}$ and $x \in \mathscr{D}$, we have the sequence of equivalences

$$\begin{aligned}\mathrm{Map}_{\mathscr{D}}(x, \lim p\sigma) &\simeq \lim \mathrm{Map}_{\mathscr{D}}(x, p\sigma(-)) \\ &\simeq \lim \mathrm{Map}_{\mathscr{C}_\bullet}(F_\bullet x, \sigma(-)) \\ &\simeq \mathrm{Map}_{\mathscr{C}}(Fx, \sigma),\end{aligned}$$

so there is an equivalence $R(\sigma) \simeq \lim p\sigma$.

3. Localization of symmetric monoidal ∞-categories with respect to a commutative algebra

Let $\mathscr{C}$, $\mathscr{D}$ be presentable stable symmetric monoidal ∞-categories, where we by default assume that the tensor product commutes with colimits separately in each variable.

Adjunctions and limits. We say that an adjunction

$$F : \mathscr{C} \rightleftarrows \mathscr{D} : R$$

is *monoidal* if F is (strong) symmetric monoidal, in which case R is lax monoidal.

Given a diagram of commutative algebras in $\mathscr{C}$

$$p : L \to \mathrm{CAlg}(\mathscr{C}),$$

we have a canonical monoidal adjunction in Pr^L

$$\phi\colon \mathrm{Mod}_{\mathscr{C}}(\lim_L p) \rightleftarrows \lim_L \mathrm{Mod}_{\mathscr{C}}(p(-)) : \psi. \tag{3.1}$$

Let $R = \lim_L p$. For $X \in \mathrm{Mod}_{\mathscr{C}}(R)$, the unit map $\eta : X \to \psi\phi X$ may be identified with the canonical map

$$X \to \lim_{i \in L} X \otimes_R p(i)$$

in view of the material in Section 2.

Moreover, for any functor $f : K \to L$, by functoriality of limits we have a commutative diagram in Pr^L

$$\begin{tikzcd}
\mathrm{Mod}_{\mathscr{C}}(\lim_L p) \arrow[r, "\phi"] \arrow[d] & \lim_L \mathrm{Mod}_{\mathscr{C}}(p(-)) \arrow[d] \\
\mathrm{Mod}_{\mathscr{C}}(\lim_K pf) \arrow[r, "\phi"] & \lim_K \mathrm{Mod}_{\mathscr{C}}(pf(-))
\end{tikzcd}$$

Bousfield localization. Recall [Lurie 2009, Definition 5.2.7.2] that a *localization* of an ∞-category $\mathscr{X}$ is an adjunction

$$L\colon \mathscr{X} \rightleftarrows \mathscr{X}_0 : R$$

where the right adjoint R is fully faithful. The left adjoint L is the localization functor.

When $\mathscr{X} = \mathscr{C}$ is our presentable stable symmetric monoidal ∞-category, we will be concerned with the special case of *Bousfield localization* with respect to an object $E \in \mathscr{C}$. We briefly recall this notion to fix terminology.

A map

$$X \to Y$$

in $\mathscr{C}$ is an *E-equivalence* if

$$E \otimes X \to E \otimes Y$$

is an equivalence. An object $X \in \mathscr{C}$ is *E-null* if

$$X \otimes E \simeq 0.$$

An object $X \in \mathscr{C}$ is *E-local* if for every E-equivalence

$$f : Y \to Z$$

the map

$$f^* : \mathrm{Hom}_{\mathscr{C}}(Z, X) \to \mathrm{Hom}_{\mathscr{C}}(Y, X)$$

is an equivalence, i.e., for every E-null object W,

$$\mathrm{Hom}_{\mathscr{C}}(W, X) \simeq 0.$$

Let $\mathscr{C}_E \subseteq \mathscr{C}$ denote the full subcategory consisting of the E-local objects. Then $\mathscr{C}_E$ is again a presentable stable ∞-category and we have the localization adjunction

$$L_E : \mathscr{C} \rightleftarrows \mathscr{C}_E : i_E.$$

With the tensor product on $\mathscr{C}_E$ defined by $L_E(- \otimes -)$, $\mathscr{C}_E$ is a symmetric monoidal ∞-category and $L_E \dashv i_E$ is a monoidal adjunction.

Example 3.2. Suppose $E = C(x)$ is the cofiber of a map

$$x : I \to 1$$

for $1 \in \mathscr{C}$ the unit. Then we also write $\mathscr{C}^{\wedge}_x$ for $\mathscr{C}_E$ and call this ∞-category the *x-completion* of $\mathscr{C}$.

Derived completion. If we further suppose that E is a dualizable E_∞-algebra $A \in \mathrm{CAlg}(\mathscr{C})$, then Bousfield localization can be computed as the A-completion. Specifically, we have the following:

(1) Let $\mathrm{C}^\bullet(A)$ be the Amitsur complex on A [Mathew et al. 2017, Construction 2.7]. By [Mathew et al. 2017, Proposition 2.21], for any $X \in \mathscr{C}$ we have an equivalence

$$L_A(X) \simeq \mathrm{Tot}(X \otimes \mathrm{C}^\bullet(A)) \simeq \lim_{n \in \Delta}(X \otimes A^{\otimes n+1}).$$

(2) By [Mathew et al. 2017, Theorem 2.30],[5] this equivalence of objects promotes to an equivalence of symmetric monoidal ∞-categories

$$\mathscr{C}_A \simeq \mathrm{Tot}\,\mathrm{Mod}_{\mathscr{C}}(\mathrm{C}^\bullet(A)) \simeq \lim_{n \in \Delta} \mathrm{Mod}_{\mathscr{C}}(A^{\otimes n+1}).$$

[5] Note that even though [Mathew et al. 2017, Hypotheses 2.26] are otherwise in effect in that section of the paper, the proof of [Mathew et al. 2017, Theorem 2.30] only uses that $A \in \mathrm{CAlg}(\mathscr{C})$ is dualizable.

Let I denote the fiber of the unit

$$I \xrightarrow{\iota} 1 \to A$$

and define

$$C(\iota^n) := \mathrm{cofib}(\iota^n : I^{\otimes n} \to 1).$$

Then there is an equivalence [Mathew et al. 2017, Proposition 2.14]

$$C(\iota^{n+1}) \simeq \mathrm{Tot}_n(\mathrm{C}^\bullet(A)).$$

Note that because the cosimplicial object $\mathrm{C}^\bullet(A)$ in $\mathscr{C}$ canonically lifts to a cosimplicial object in $\mathrm{CAlg}(\mathscr{C})$, the cofiber $C(\iota^n)$ obtains the structure of an E_∞-algebra as a limit and the maps

$$C(\iota^{n+1}) \to C(\iota^n)$$

are maps of E_∞-algebras.

The completion tower. For our dualizable E_∞-algebra A, we wish to reexpress the above descent description of $\mathscr{C}_A$ in terms of an inverse limit over the ∞-categories $\mathrm{Mod}_{\mathscr{C}}(C(\iota^n))$.

By (3.1), for all n we have canonical monoidal adjunctions

$$\phi_n : \mathrm{Mod}_{\mathscr{C}}(C(\iota^n)) \rightleftarrows \mathrm{Tot}_{n-1} \mathrm{Mod}_{\mathscr{C}}(\mathrm{C}^\bullet(A)) : \psi_n$$

where the left adjoints ϕ_n are compatible with restriction along $\Delta_{\leq n} \subset \Delta_{\leq m}$. Passage to the limit then yields the monoidal adjunction

$$\phi_\infty : \lim \mathrm{Mod}_{\mathscr{C}}(C(\iota^n)) \rightleftarrows \mathrm{Tot}\, \mathrm{Mod}_{\mathscr{C}}(\mathrm{C}^\bullet(A)) : \psi_\infty,$$

where $\phi_\infty\{X_n\} = \{\phi_n X_n\}$. By the universal property of the limit, and using that $C(\iota^n)$-modules are A-local, we also have the monoidal adjunctions

$$\phi : \mathscr{C}_A \rightleftarrows \lim \mathrm{Mod}_{\mathscr{C}}(C(\iota^n)) : \psi,$$

$$\phi' : \mathscr{C}_A \rightleftarrows \mathrm{Tot}\, \mathrm{Mod}_{\mathscr{C}}(\mathrm{C}^\bullet(A)) : \psi',$$

where the second adjunction is the adjoint equivalence of [Mathew et al. 2017, Theorem 2.30]. These adjunctions fit into a commutative diagram

$$\begin{tikzcd} \mathscr{C}_A \arrow[r, "\phi"'] \arrow[rr, bend left, "\phi'"] & \lim \mathrm{Mod}_{\mathscr{C}}(C(\iota^n)) \arrow[r, "\phi_\infty"'] & \mathrm{Tot}\, \mathrm{Mod}_{\mathscr{C}}(\mathrm{C}^\bullet(A)). \end{tikzcd}$$

Proposition 3.3. *Both $\phi \dashv \psi$ and $\phi_\infty \dashv \psi_\infty$ are adjoint equivalences of symmetric monoidal ∞-categories.*

Proof. It suffices to prove the first statement. We need to show that

(1) the unit $id \to \psi\phi$ is an equivalence,

(2) ψ is conservative.

For (1), given any A-local object X, the unit map

$$X \to \psi\phi X \simeq \lim X \otimes C(\iota^n) \simeq \operatorname{Tot} X \otimes \mathrm{C}^\bullet(A)$$

is already known to be an equivalence. For (2), we first note that ϕ_n is fully faithful, i.e., the unit map $id \to \psi_n\phi_n$ is an equivalence. Indeed, for *any* finite ∞-category K and functor $p : K \to \mathrm{CAlg}(\mathscr{C})$, if $R = \lim_K p$ and $X \in \mathrm{Mod}_C(R)$, then there is an equivalence

$$X \simeq \lim_K X \otimes_R p(-).$$

Now suppose that $\{X_n\}$ is a object in $\lim \mathrm{Mod}_{\mathscr{C}}(C(\iota^n))$ such that

$$\psi\{X_n\} = \lim X_n \simeq 0.$$

Note that since $\phi_\infty\{X_n\} = \{\phi_n X_n\}$, for the cosimplicial object $\phi_\bullet X_\bullet$ we have that $\mathrm{Tot}_n(\phi_\bullet X_\bullet) \simeq \psi_n\phi_n X_n \simeq X_n$, so

$$\psi'(\phi_\infty\{X_n\}) = \operatorname{Tot}\phi_\bullet X_\bullet \simeq \lim_n \mathrm{Tot}_n(\phi_\bullet X_\bullet) \simeq \lim_n X_n \simeq 0.$$

Therefore, because ψ' is an equivalence, $\phi_\infty\{X_n\} \simeq 0$. This means that for all n, $\phi_n X_n \simeq 0$, so $X_n \simeq \psi_n\phi_n X_n \simeq 0$ and $\{X_n\} \simeq 0$. $\square$

Remark 3.4. We have a commutative diagram of right adjoints

$$\begin{tikzcd}
\lim \mathrm{Mod}_{\mathscr{C}}(C(\iota^n)) \arrow[d] & \operatorname{Tot} \mathrm{Mod}_{\mathscr{C}}(\mathrm{C}^\bullet(A)) \arrow[l, "\psi_\infty"'] \arrow[d] \\
\mathrm{Fun}(\mathbb{Z}^{op}_{\geq 0}, \mathscr{C}) & \mathrm{Fun}(\Delta, \mathscr{C}) \arrow[l, "\mathrm{DK}"']
\end{tikzcd}$$

where DK is the functor that sends a cosimplicial object to its tower of partial totalizations. DK implements the equivalence of the ∞-categorical Dold–Kan correspondence [Lurie 2017, Theorem 1.2.4.1]. We may thus interpret Proposition 3.3 as a monoidal refinement of the Dold–Kan correspondence, with ϕ_∞ providing an explicit inverse.

We also record a useful corollary of the proof of Proposition 3.3. This result is a companion to the fact that

$$- \otimes A : \mathscr{C}_A \to \mathrm{Mod}_{\mathscr{C}}(A)$$

is conservative.

Lemma 3.5. *For every n, the base change functor*

$$\mathrm{Mod}_{\mathscr{C}}(C(\iota^n)) \to \mathrm{Mod}_{\mathscr{C}}(C(\iota)) = \mathrm{Mod}_{\mathscr{C}}(A)$$

is conservative.

Proof. We showed that the functor ϕ_n is fully faithful, and the restriction functor

$$\mathrm{Tot}_{n-1}\,\mathrm{Mod}_{\mathscr{C}}(\mathrm{C}^\bullet(A)) \to \mathrm{Mod}_{\mathscr{C}}(A)$$

is clearly conservative. □

The monoidal Barr–Beck theorem. Throughout, let

$$F : \mathscr{C} \rightleftarrows \mathscr{D} : R$$

be a monoidal adjunction between our presentable symmetric monoidal stable ∞-categories $\mathscr{C}$ and $\mathscr{D}$. We recall the *monoidal Barr–Beck theorem* of [Mathew et al. 2017], which will be a key technical device to many of the formal results of this paper.

Theorem 3.6 [Mathew et al. 2017, Theorem 5.29]. *Suppose that $F \dashv R$ satisfies the following conditions*:

(1) *R is conservative*,

(2) *R preserves colimits*,

(3) *(F, R) satisfies the projection formula*: *the natural map*

$$R(X) \otimes Y \to R(X \otimes F(Y))$$

is an equivalence for all $X \in \mathscr{D}$ and $Y \in \mathscr{C}$.

Then there is an equivalence

$$\mathscr{D} \simeq \mathrm{Mod}_{\mathscr{C}}(R(1_{\mathscr{D}}))$$

and $F \dashv R$ is equivalent to the free-forgetful adjunction.

We may descend Theorem 3.6 to subcategories of local objects.

Lemma 3.7. *Let $\mathscr{C}$ and $\mathscr{D}$ be presentable symmetric monoidal stable ∞-categories, let*

$$F : \mathscr{C} \rightleftarrows \mathscr{D} : R$$

be a monoidal adjunction, let $E \in \mathscr{C}$ be any object, and let $E' = F(E)$. Then the adjunction $F \dashv R$ induces a monoidal adjunction

$$F' : \mathscr{C}_E \rightleftarrows \mathscr{D}_{E'} : R'$$

between the ∞-categories of E-local and E'-local objects. Moreover:

(1) *If R is conservative, then R' is conservative.*

(2) *Suppose that (F, R) satisfies the projection formula. Then there is an equivalence*
$$L_E R \simeq R' L_{E'}$$
and (F', R') satisfies the projection formula. Moreover, if R in addition preserves colimits, then R' preserves colimits.

Therefore, we have
$$\mathrm{Mod}_{\mathscr{C}}(R(1_{\mathscr{D}}))_{E'} \simeq \mathrm{Mod}_{\mathscr{C}_E}(L_E R(1_{\mathscr{D}}))$$
and $F' \dashv R'$ is the free-forgetful adjunction.

Proof. Because the functor F is strong monoidal, F sends E-equivalences to E'-equivalences. Therefore, if X is E'-local, then $R(X)$ is E-local, so we may define
$$R' : \mathscr{D}_{E'} \to \mathscr{C}_E$$
to be the restriction of R. We may then define
$$F' : \mathscr{C}_E \to \mathscr{D}_{E'}$$
by $F' := L_{E'} F$ to obtain the induced monoidal adjunction
$$F' : \mathscr{C}_E \rightleftarrows \mathscr{D}_{E'} : R'.$$

For (1), if R is conservative, then because $i_E R' = R i_{E'}$, R' is conservative. For (2), if (F, R) satisfies the projection formula, then we have that for any E'-null object X,
$$R(X) \otimes E \simeq R(X \otimes E') \simeq 0,$$
so R sends E'-equivalences to E-equivalences. Therefore, we have $L_E R \simeq R' L_{E'}$. To see that (F', R') satisfies the projection formula, we use the sequence of equivalences
$$\begin{aligned} R'(X \otimes_{\mathscr{D}_{E'}} F'(Y)) &\simeq R'(L_{E'}(i_{E'} X \otimes_{\mathscr{D}} F(Y))) \\ &\simeq L_E R(i_{E'} X \otimes_{\mathscr{D}} F(Y)) \\ &\simeq L_E(R(i_{E'} X) \otimes_{\mathscr{C}} Y) \\ &\simeq R' X \otimes_{\mathscr{C}_E} Y. \end{aligned}$$
Now suppose that R preserves colimits. To see that R' preserves colimits, suppose being given a diagram $X_\bullet : J \to \mathscr{D}_{E'}$. Then we have
$$\operatorname{colim} X_\bullet \simeq L_{E'} \operatorname{colim} i_{E'} X_\bullet,$$

and we have the sequence of equivalences

$$\begin{aligned} R'(\operatorname{colim} X_\bullet) &\simeq R' L_{E'} \operatorname{colim} i_{E'} X_\bullet \\ &\simeq L_E R \operatorname{colim} i_{E'} X_\bullet \\ &\simeq \operatorname{colim} L_E R i_{E'} X_\bullet \\ &\simeq \operatorname{colim} R' L_{E'} i_{E'} X_\bullet \\ &\simeq \operatorname{colim} R' X_\bullet. \end{aligned}$$

Finally, the last statement is a consequence of Theorem 3.6. □

Example 3.8. In Lemma 3.7, let $E = C(x)$ for x as in Example 3.2. Then we see that

$$\mathrm{Mod}_{\mathscr{C}}(R(1_{\mathscr{D}}))^{\wedge}_x \simeq \mathrm{Mod}_{\mathscr{C}^{\wedge}_x}(R(1_{\mathscr{D}})^{\wedge}_x).$$

We also note a similar result when passing to module categories.

Lemma 3.9. *Let* $A \in \mathrm{CAlg}(\mathscr{C})$ *and* $A' = F(A)$, *and let*

$$F' : \mathrm{Mod}_{\mathscr{C}}(A) \rightleftarrows \mathrm{Mod}_{\mathscr{D}}(A') : R'$$

denote the induced monoidal adjunction. Then:

(1) *If* R *is conservative*, *then* R' *is conservative.*

(2) *If* R *preserves colimits*, *then* R' *preserves colimits.*

(3) *If* R *preserves colimits and* (F, R) *satisfies the projection formula*, *then* (F', R') *satisfies the projection formula.*

Proof. Because F' and R' are computed by F and R after forgetting the module structure, the first two results are clear. For the projection formula, under our assumptions the natural map

$$RM \otimes_A N \to R(M \otimes_{A'} FN)$$

is equivalent to the geometric realization of the map of simplicial diagrams

$$RM \otimes A^{\otimes \bullet} \otimes N \to R(M \otimes (A')^{\otimes \bullet} \otimes FN),$$

which is an equivalence in view of the projection formula for (F, R). □

Lifting localizations. For $A \in \mathrm{CAlg}(\mathscr{C})$ dualizable and

$$C(\iota^{n+1}) = \mathrm{Tot}_n \, \mathrm{C}^\bullet(A)$$

as before, let $A' = F(A)$ and $j = F(\iota)$, so that $F(C(\iota^n)) \simeq C(j^n)$. We have induced monoidal adjunctions

$$F_n\colon \mathrm{Mod}_{\mathscr{C}}(C(\iota^n)) \rightleftarrows \mathrm{Mod}_{\mathscr{D}}(C(j^n)) : R_n,$$

$$F_\infty\colon \mathscr{C}_A \rightleftarrows \mathscr{D}_{A'} : R_\infty.$$

We end this section with a result that allows us to lift the property of R_1 being fully faithful to R_n and R_∞.

Proposition 3.10. *Suppose that R preserves colimits, (F, R) satisfies the projection formula, and R_1 is fully faithful. Then R_n is fully faithful for all $1 \le n \le \infty$.*

Proof. First suppose $n < \infty$ and let $X \in \mathrm{Mod}_{\mathscr{D}}(C(j^n))$. We need to prove that the counit ϵ^n_X is an equivalence. Because the base change functor $- \otimes_{Cj^n} A'$ is conservative by Lemma 3.5, it suffices to show that $\epsilon^n_X \otimes_{Cj^n} A'$ is an equivalence. But by the projection formula for (F_n, R_n) established in Lemma 3.9, this map is equivalent to the counit $\epsilon^n_{(X\otimes_{Cj^n} A')}$. Because $X \otimes_{Cj^n} A'$ is an A'-module, $\epsilon^n_{(X\otimes_{Cj^n} A')}$ is lifted by the counit $\epsilon^1_{(X\otimes_{Cj^n} A')}$, which is an equivalence by assumption. The proof for the case $n = \infty$ is similar, where we instead use that

$$- \otimes A' : \mathscr{D}_{A'} \to \mathrm{Mod}_{\mathscr{D}}(A')$$

is conservative and the projection formula for (F_∞, R_∞) by Lemma 3.7. □

4. Cellularization

In this section, we collect a few technical lemmas that will be applied to study the ∞-category $\mathrm{SH}_{\mathrm{cell}}(S)$ of cellular motivic spectra. To begin with, we have the following variant of [Mathew et al. 2017, Proposition 2.27] (with the same conclusion), where we do not assume that E is an algebra object of $\mathscr{C}$.

Lemma 4.1. *Suppose $\mathscr{C}$ is a presentable symmetric monoidal stable ∞-category and E is a dualizable object in $\mathscr{C}$.*

(1) *For any object $X \in \mathscr{C}$, $E^\vee \otimes X$ is E-local. If $E^\vee \simeq E \otimes \kappa$, then $E \otimes X$ is also E-local.*

(2) *For any compact object $X \in \mathscr{C}$, $E^\vee \otimes X$ is compact in $\mathscr{C}_E$. If $E^\vee \simeq E \otimes \kappa$, then $E \otimes X$ is also compact in $\mathscr{C}_E$.*

(3) *If $\{X_i\}$ is a set of compact generators of $\mathscr{C}$, then $\{E^\vee \otimes X_i\}$ is a set of compact generators of $\mathscr{C}_E$. Therefore, if $\mathscr{C}$ is compactly generated, then $\mathscr{C}_E$ is compactly generated.*

Proof. (1) Let Z be an E-null object. Then

$$\mathrm{Hom}_{\mathscr{C}}(Z, E^\vee \otimes X) \simeq \mathrm{Hom}_{\mathscr{C}}(Z \otimes E, X) \simeq 0,$$

so $E^\vee \otimes X$ is E-local. If $E^\vee \simeq E \otimes \kappa$, then

$$\mathrm{Hom}_{\mathscr{C}}(Z, E \otimes X) \simeq \mathrm{Hom}_{\mathscr{C}}(Z \otimes E^\vee, X) \simeq \mathrm{Hom}_C(Z \otimes E \otimes \kappa, X) \simeq 0,$$

so $E \otimes X$ is E-local.

(2) Observe that the functor

$$\mathscr{C}_E \to \mathscr{C}$$

given by $Y \mapsto Y \otimes E$ preserves colimits [Mathew et al. 2017, Remark 2.20]. Let

$$Y_\bullet : J \to \mathscr{C}_E$$

be a functor and let us write $\operatorname{colim} Y_j$ for the colimit in $\mathscr{C}$ and $L_E(\operatorname{colim} Y_j)$ for the colimit in $\mathscr{C}_E$. Then we have

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(E^\vee \otimes X, L_E(\operatorname{colim} Y_j)) &\simeq \mathrm{Hom}_{\mathscr{C}}(X, E \otimes L_E(\operatorname{colim} Y_j))\\ &\simeq \mathrm{Hom}_{\mathscr{C}}(X, \operatorname{colim}(E \otimes Y_j))\\ &\simeq \operatorname{colim} \mathrm{Hom}_{\mathscr{C}}(X, E \otimes Y_j)\\ &\simeq \operatorname{colim} \mathrm{Hom}_{\mathscr{C}}(E^\vee \otimes X, Y_j),\end{aligned}$$

so $E^\vee \otimes X$ is compact. The second assertion is similar.

(3) This follows as in the proof of [Mathew et al. 2017, Proposition 2.27]. □

The following concerns the existence and basic properties of cellularization:

Lemma 4.2. *Let $\mathscr{C}$ be a compactly generated stable ∞-category, let $S = \{S_i : i \in \mathscr{I}\}$ be a small set of compact objects in $\mathscr{C}$, and let $\mathscr{C}'$ be the localizing subcategory generated by S (i.e., the smallest full stable subcategory containing S that is closed under colimits).*

(1) *$\mathscr{C}'$ is compactly generated and is a coreflective subcategory of $\mathscr{C}$ (i.e., the inclusion $j : \mathscr{C}' \subseteq \mathscr{C}$ admits a right adjoint). Moreover, if*

$$\mathrm{Cell} : \mathscr{C} \to \mathscr{C}'$$

denotes this right adjoint, then Cell *also preserves colimits.*

(2) *Suppose in addition that $\mathscr{C}$ is closed symmetric monoidal, the unit $1 \in \mathscr{C}$ is compact and in S, and for all $i, i' \in \mathscr{I}$, we have that $S_i \otimes S_{i'} \in S$. Then $\mathscr{C}' \subseteq \mathscr{C}$ is a symmetric monoidal subcategory.*

(3) *Suppose in addition to the assumptions of (2) that each S_i is dualizable. Then for all $X \in \mathscr{C}$ and $Y \in \mathscr{C}'$, the natural map $\theta : \mathrm{Cell}(X) \otimes Y \to \mathrm{Cell}(X \otimes Y)$ is an equivalence.*

Proof. For (1), $\mathscr{C}'$ is compactly generated by definition, so j admits a right adjoint by the adjoint functor theorem [Lurie 2009, Corollary 5.5.2.9]. Moreover, the set S furnishes a set of compact generators for $\mathscr{C}'$ that are sent to compact objects under j, so Cell preserves colimits. For (2), because the tensor product $\otimes$ commutes with colimits separately in each variable, our assumption ensures that if $X, Y \in \mathscr{C}'$, then $X \otimes Y \in \mathscr{C}'$. We may then invoke [Lurie 2017, Remark 2.2.1.2] to see that $\mathscr{C}' \subseteq \mathscr{C}$ is a symmetric monoidal subcategory. For (3), the assumption ensures that $\mathscr{C}'$ is generated by dualizable objects under colimits. Because Cell commutes with colimits, both the source and target of θ commute with colimits separately in each variable. We may thus suppose that Y is a dualizable object in $\mathscr{C}'$, with dual $Y^\vee$. Note that $Y^\vee$ is also the dual of Y in $\mathscr{C}$. For each generator S_i we have that

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(S_i, \mathrm{Cell}(X) \otimes Y) &\simeq \mathrm{Hom}_{\mathscr{C}}(S_i \otimes Y^\vee, \mathrm{Cell}(X)) \\ &\simeq \mathrm{Hom}_{\mathscr{C}}(S_i \otimes Y^\vee, X) \\ &\simeq \mathrm{Hom}_{\mathscr{C}}(S_i, X \otimes Y) \\ &\simeq \mathrm{Hom}_{\mathscr{C}}(S_i, \mathrm{Cell}(X \otimes Y)), \end{aligned}$$

so θ is an equivalence. □

The following two lemmas describe the interaction of cellularization with Bousfield localization and passage to module categories.

Lemma 4.3. *With the setup of Lemma* 4.2(2), *let E be a dualizable object in $\mathscr{C}'$. Then*:

(1) *If $X \in \mathscr{C}$ is $j(E)$-local, then* $\mathrm{Cell}(X) \in \mathscr{C}'$ *is E-local.*

(2) *For $X \in \mathscr{C}$, the natural map*

$$\mathrm{Cell}(X) \otimes E \to \mathrm{Cell}(X \otimes j(E))$$

is an equivalence. Hence, Cell *sends $j(E)$-equivalences to E-equivalences.*

(3) *The adjunction*

$$j : \mathscr{C}' \rightleftarrows \mathscr{C} : \mathrm{Cell}$$

induces a monoidal adjunction

$$j' : \mathscr{C}'_E \rightleftarrows \mathscr{C}_{j(E)} : \mathrm{Cell}'$$

such that $\mathrm{Cell}'(X) \simeq \mathrm{Cell}(X)$ *for $X \in \mathscr{C}_{j(E)}$, $j'(Y) \simeq L_{j(E)} j(Y)$ for $Y \in \mathscr{C}'_E$, and the functor j' is fully faithful.*

(4) *The functor* Cell$'$ *preserves colimits.*

(5) *Suppose in addition the condition of Lemma* 4.2(3). *Then for all* $X \in \mathscr{C}_{j(E)}$ *and* $Y \in \mathscr{C}'_E$, *we have the natural equivalence*

$$L_E(\mathrm{Cell}'(X) \otimes Y) \simeq \mathrm{Cell}'(L_E(X \otimes Y)).$$

Consequently, *the conclusion of Lemma* 4.5 *holds with* $j' \dashv \mathrm{Cell}'$ *in place of* $j \dashv \mathrm{Cell}$.

Proof. We consider each assertion in turn:

(1) If $Y \in \mathscr{C}'$ is E-null, then $j(Y) \in \mathscr{C}$ is $j(E)$-null since the inclusion $\mathscr{C}' \subseteq \mathscr{C}$ is strong monoidal. Then if $X \in \mathscr{C}$ is $j(E)$-local, we have for all $Y \in \mathscr{C}'$ E-null that $\mathrm{Hom}_{\mathscr{C}}(Y, \mathrm{Cell}\, X) \simeq \mathrm{Hom}_{\mathscr{C}}(Y, X) \simeq 0$, so $\mathrm{Cell}(X)$ is E-local.

(2) We write $j(E)$ as E for clarity. It suffices to observe that for all $i \in \mathscr{I}$,

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(S_i, \mathrm{Cell}(X) \otimes E) &\simeq \mathrm{Hom}_{\mathscr{C}}(S_i \otimes E^\vee, \mathrm{Cell}(X)) \\ &\simeq \mathrm{Hom}_{\mathscr{C}}(S_i \otimes E^\vee, X) \\ &\simeq \mathrm{Hom}_{\mathscr{C}}(S_i, X \otimes E) \\ &\simeq \mathrm{Hom}_{\mathscr{C}}(S_i, \mathrm{Cell}(X \otimes E)).\end{aligned}$$

(3) By (1), $\mathrm{Cell} : \mathscr{C} \to \mathscr{C}'$ restricts to a functor

$$\mathrm{Cell}' : \mathscr{C}_{j(E)} \to \mathscr{C}'_E.$$

Define

$$j' : \mathscr{C}'_E \to \mathscr{C}_{j(E)}$$

to be the composite

$$\mathscr{C}'_E \subseteq \mathscr{C} \xrightarrow{L_{j(E)}} \mathscr{C}_{j(E)}.$$

Then it is clear that $j' \dashv \mathrm{Cell}'$, the adjunction is monoidal with respect to the tensor products $L_E(- \otimes -)$ and $L_{j(E)}(- \otimes -)$ on $\mathscr{C}'_E$ and $\mathscr{C}_{j(E)}$, and the unit map

$$\eta : Y \to \mathrm{Cell}'\, j'Y$$

is equivalent to Cell of the unit map

$$\hat{\eta} : Y \to L_{j(E)}Y.$$

Because $\hat{\eta}$ is an $j(E)$-equivalence in $\mathscr{C}$, by (2) we see that $\mathrm{Cell}(\hat{\eta})$ is an equivalence.

(4) By Lemma 4.1, $\{S_i \otimes E^\vee : i \in \mathscr{I}\}$ are a set of compact generators for $\mathscr{C}'_E$, and are also compact and $j(E)$-local objects when regarded as being in $\mathscr{C}$. Therefore, the left adjoint j' sends compact generators to compact objects, which implies that the right adjoint Cell' preserves colimits.

(5) With our additional assumption, the $S_i \otimes E^\vee$ constitute a set of compact dualizable generators of $\mathscr{C}'_E$. The proof of Lemma 4.2(3) then applies to $j' \dashv \mathrm{Cell}'$. $\square$

Lemma 4.4. *With the setup of Lemma* 4.2(3), *let A be an E_∞-algebra in $\mathscr{C}$ and let $A' := \mathrm{Cell}(A)$ be the resulting E_∞-algebra in $\mathscr{C}'$. Then we have an induced adjunction*

$$j' : \mathrm{Mod}_{\mathscr{C}'}(A') \rightleftarrows \mathrm{Mod}_{\mathscr{C}}(A) : \mathrm{Cell}'$$

such that j' is fully faithful and identifies $\mathrm{Mod}_{\mathscr{C}'}(A')$ *with the localizing subcategory of* $\mathrm{Mod}_{\mathscr{C}}(A)$ *generated by $S_A := \{S_i \otimes A : i \in \mathscr{I}\}$.*

Proof. Note that $\mathrm{Mod}_{\mathscr{C}}(A)$ and $\mathrm{Mod}_{\mathscr{C}'}(A')$ are compactly generated stable symmetric monoidal ∞-categories, and the set $S_{A'} := \{S_i \otimes A'\}$ furnishes a set of compact dualizable generators for $\mathrm{Mod}_{\mathscr{C}'}(A')$. Because Cell is lax monoidal, it induces a functor $\mathrm{Cell}' : \mathrm{Mod}_{\mathscr{C}}(A) \to \mathrm{Mod}_{\mathscr{C}'}(A')$ such that the diagram of right adjoints

$$\begin{tikzcd} \mathscr{C}' & \mathscr{C} \arrow[l, "\mathrm{Cell}"'] \\ \mathrm{Mod}_{\mathscr{C}'}(A') \arrow[u, "U'"] & \mathrm{Mod}_{\mathscr{C}}(A) \arrow[l, "\mathrm{Cell}'"] \arrow[u, "U"'] \end{tikzcd}$$

commutes (where U and U' denote the forgetful functors). Since Cell preserves limits and U, U' create limits, Cell' also preserves limits and therefore admits a left adjoint j' such that the diagram of left adjoints

$$\begin{tikzcd} \mathscr{C}' \arrow[r, "j"] \arrow[d, "F'"] & \mathscr{C} \arrow[d, "F"] \\ \mathrm{Mod}_{\mathscr{C}'}(A') \arrow[r, "j'"] & \mathrm{Mod}_{\mathscr{C}}(A) \end{tikzcd}$$

commutes (where F and F' denote the free functors), so $j(S_{A'}) = S_A$. It remains to show that j' is fully faithful, i.e., that the unit map

$$\eta : M \to \mathrm{Cell}'\, j'M$$

is an equivalence for all $M \in \mathrm{Mod}_{\mathscr{C}'}(A')$. For this, note that Cell' preserves colimits since Cell preserves colimits by Lemma 4.2(2) and U, U' create colimits, so we may suppose that $M = S_i \otimes A'$. But then we have

$$\mathrm{Cell}'\, j'(S_i \otimes A') = \mathrm{Cell}'(S_i \otimes A) \simeq S_i \otimes A'$$

by Lemma 4.2(3), and it is easily checked that η implements this equivalence. □

Finally, we retain the projection formula after cellularization.

Lemma 4.5. *With the setup of Lemma* 4.2(3), *let $\mathscr{D}$ be a presentable symmetric monoidal stable ∞-category and let*

$$F : \mathscr{C} \rightleftarrows \mathscr{D} : R$$

be a monoidal adjunction such that R preserves colimits and (F, R) satisfies the projection formula. Then $\mathrm{Cell}\, R$ *preserves colimits and* $(Fj, \mathrm{Cell}\, R)$ *satisfies the projection formula.*

Proof. $\mathrm{Cell}\, R$ preserves colimits by Lemma 4.2(2). For the projection formula, we note that for all $X \in \mathcal{D}$ and $Y \in \mathcal{C}'$,

$$(\mathrm{Cell}\, R)(X) \otimes Y \simeq \mathrm{Cell}(RX \otimes Y) \simeq (\mathrm{Cell}\, R)(X \otimes LY),$$

where the first equivalence is by Lemma 4.2(3) and the second by our assumption on (L, R). $\square$

5. Recollements

Let $\mathcal{X}$ be an ∞-category which admits finite limits. Recall [Lurie 2017, §A.8; Barwick and Glasman 2016] that an ∞-category $\mathcal{X}$ is a *recollement* of two full subcategories $\mathcal{U}$ and $\mathcal{Z}$ if the inclusions j_*, i_* of these subcategories admit left adjoints j^*, i^*:

$$\mathcal{U} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathcal{X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{Z}$$

such that

(1) the subcategories $\mathcal{U}, \mathcal{Z} \subseteq \mathcal{X}$ are stable under equivalence,

(2) the left adjoints j^*, i^* are left exact,

(3) the functor $j^* i_*$ is equivalent to the constant functor at the terminal object,

(4) if f is a morphism of $\mathcal{X}$ such that $j^* f$ and $i^* f$ are equivalences, then f is an equivalence.

The following lemma shows that if $\mathcal{X}$ is a recollement of $\mathcal{U}$ and $\mathcal{Z}$, then to test whether a functor into $\mathcal{X}$ is a localization, it suffices to check this on $\mathcal{U}$ and $\mathcal{Z}$.

Lemma 5.1. *Let $\mathcal{C}$ and $\mathcal{X}$ be ∞-categories that admit finite limits and suppose that we have a recollement on $\mathcal{X}$*

$$\mathcal{U} \underset{j_*}{\overset{j^*}{\leftrightarrows}} \mathcal{X} \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{Z}$$

and an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{X} : R$ with F also left exact such that

(1) *the natural transformation $i^* F R j_* \Rightarrow i^* j_*$ induced by the counit of (F, R) is an equivalence*,

(2) *the functor $j^* F R i_*$ is equivalent to the constant functor at the terminal object*,

(3) *the two functors Rj_* and Ri_* are fully faithful.*

Then R is fully faithful.

Proof. We will show that for any $X \in \mathscr{X}$, the counit $\epsilon : FRX \to X$ is an equivalence. Because i^* and j^* are jointly conservative, it suffices to show that $i^*\epsilon$ and $j^*\epsilon$ are equivalences. Consider the pullback square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & i_*i^*X \arrow[d] \\ j_*j^*X \arrow[r] & i_*i^*j_*j^*X \end{tikzcd}$$

Applying i^*FR and using that Ri_* is fully faithful and $i^*FRj_* \simeq i^*j_*$, we obtain a pullback square

$$\begin{tikzcd} i^*FRX \arrow[r, "\simeq"] \arrow[d] & i^*FRi_*i^*X \simeq i^*X \arrow[d] \\ i^*FRj_*j^*X \simeq i^*j_*j^*X \arrow[r, "\simeq"] & i^*FRi_*i^*j_*j^*X \simeq i^*j_*j^*X \end{tikzcd}$$

from which it follows that $i^*\epsilon$ is an equivalence. Applying j^*FR and using that Rj_* is fully faithful and $j^*FRi_* \simeq 0$, we obtain a pullback square

$$\begin{tikzcd} j^*FRX \arrow[r] \arrow[d, "\simeq"] & j^*FRi_*i^*X \simeq 0 \arrow[d, "\simeq"] \\ j^*FRj_*j^*X \simeq j^*X \arrow[r] & j^*FRi_*i^*j_*j^*X \simeq 0 \end{tikzcd}$$

from which it follows that $j^*\epsilon$ is an equivalence. $\square$

6. Background on motivic and equivariant homotopy theory

The motivic stable homotopy category. Let S be a scheme and let $\mathrm{SH}(S)$ denote the symmetric monoidal ∞-category of motivic $\mathbb{P}^1$-spectra over S. Let $\mathrm{SH}_{\mathrm{cell}}(S)$ be the localizing subcategory of $\mathrm{SH}(S)$ generated by the motivic spheres $\{S^{p,q}\}$. A motivic spectrum E is *cellular* if it lies inside $\mathrm{SH}_{\mathrm{cell}}(S)$. Note that the full hypotheses of Lemma 4.2 apply.

We recall from [Elmanto and Kolderup 2020, §2.2] the following facts concerning compact and dualizable objects and generation in $\mathrm{SH}(S)$:

(1) For X an affine smooth scheme over S and $q \in \mathbb{Z}$, the motivic $\mathbb{P}^1$-spectrum $\Sigma^q X_+$ is compact; in particular, the bigraded motivic spheres $S^{p,q}$ are compact. Compactness of the unit then implies that every dualizable object in $\mathrm{SH}(S)$ is compact. Moreover, $\mathrm{SH}(S)$ is generated under sifted colimits by $\Sigma^q X_+$ and is thus compactly generated.

(2) If K is a field of characteristic 0, then every compact object in $\mathrm{SH}(K)$ is dualizable.

We collect a few facts concerning the functoriality of SH(−); see [Hoyois 2017] for a reference. Let $f : T \to S$ be a morphism of schemes. We always have a monoidal adjunction

$$f^* : \mathrm{SH}(S) \rightleftarrows \mathrm{SH}(T) : f_*.$$

The left adjoint $f_\sharp$ to f^* exists if f is smooth. If f is smooth and proper, we have the duality equivalence

$$f_* \simeq f_\sharp \Sigma^{-\Omega_f}.$$

In particular, if f is finite étale, then $f_* \simeq f_\sharp$ and the adjunction $f^* \dashv f_*$ is ambidextrous. On the other hand, if f is separated and of finite type, we have the adjunction

$$f_! : \mathrm{SH}(T) \rightleftarrows \mathrm{SH}(S) : f^!.$$

Moreover, $f_!$ coincides with f_* if f is proper. If f is finite étale, we have that $f^! \simeq f^*$. Finally, we have the projection formula

$$f_!(X \wedge f^*(Y)) \simeq f_!(X) \wedge Y.$$

Euler classes. Let

$$\rho = \rho_S : S^{-1,-1} \to S^{0,0}$$

be the map in SH(S) induced by the inclusion

$$S^{0,0} = \{\pm 1\} \hookrightarrow \mathbb{G}_m = S^{1,1}.$$

The equivariant analog is the element $a \in \pi^{C_2}_{-1,-1} S$ induced by the inclusion

$$S^0 \hookrightarrow S^\sigma.$$

The element a is the C_2-Betti realization of the element $\rho \in \pi^{\mathbb{R}}_{-1,-1}$, and also serves as the Euler class for the representation σ.

For $Y \in \mathrm{Sp}^{C_2}$, the cofiber sequence

$$\Sigma^\infty_+ C_2 \to S^0 \xrightarrow{\Sigma^\sigma a} S^\sigma \tag{6.1}$$

yields a long exact sequence

$$\cdots \to \pi^{C_2}_{i+1,1} Y \xrightarrow{a} \pi^{C_2}_i Y \to \pi^e_i Y \to \cdots .$$

It follows that a map of C_2-spectra is a stable equivalence if and only if it induces an isomorphism on the bigraded homotopy groups $\pi^{C_2}_{*,*}$, and that, in contrast to the $\mathbb{R}$-motivic case, every C_2-spectrum is stably equivalent to one built from representation spheres.

The cofiber sequence (6.1) results in an equivalence

$$Ca \simeq \Sigma^{1-\sigma} \Sigma^\infty_+ C_2. \tag{6.2}$$

More generally, the cofiber sequences

$$S(i\sigma)_+ \to S^0 \xrightarrow{\Sigma^{i\sigma} a^i} S^{i\sigma}$$

yield equivalences

$$Ca^i \simeq \Sigma^{1-i\sigma} S(i\sigma)_+ \tag{6.3}$$

and, taking Spanier–Whitehead duals, equivalences

$$Ca^i \simeq (S(i\sigma)_+)^\vee.$$

We therefore have, for any $Y \in \mathrm{Sp}^{C_2}$,

$$\begin{aligned} Y^h &= F((EC_2)_+, Y) \\ &\simeq \lim_i F(S(i\sigma)_+, Y) \\ &\simeq \lim_i Y \wedge C(a^i) \\ &\simeq Y^\wedge_a. \end{aligned} \tag{6.4}$$

Since we have

$$\begin{aligned} Y^\Phi &= Y \wedge \widetilde{EC}_2 \\ &\simeq \operatorname*{colim}_i Y \wedge S^{i\sigma} \\ &\simeq Y[a^{-1}] \end{aligned} \tag{6.5}$$

we deduce that the isotropy separation square (1.3) is equivalent to the a-arithmetic square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y[a^{-1}] \arrow[d] \\ Y^\wedge_a \arrow[r] & Y^\wedge_a[a^{-1}] \end{tikzcd}$$

Therefore, C_2-Betti realization takes the ρ-arithmetic square to the isotropy separation square.

η-completion and η-localization at odd primes. Let K be a perfect field. Bachmann [2018, Lemma 39] summarizes relations in $\pi^K_{*,*}S^{0,0}$ involving the Hopf map

$$\eta \in \pi^K_{1,1} S^{0,0}$$

and the element $\rho \in \pi^K_{-1,-1} S^{0,0}$, after 2 is inverted. Namely, the element[6]

$$\epsilon := -\eta\rho - 1$$

[6]Here we are following the convention that $\rho = [-1]$. Bachmann instead takes $\rho = -[-1]$, which results in the formula $\epsilon = \eta\rho - 1$ in his work.

is the interchange isomorphism

$$\epsilon : S^{1,1} \wedge S^{1,1} \to S^{1,1} \wedge S^{1,1}.$$

Therefore it satisfies $\epsilon^2 \simeq 1$, and hence for any $X \in \mathrm{SH}(K)[1/2]$, there is a corresponding decomposition into ± 1-eigenspaces

$$\pi^K_{*,*} X \cong \pi^K_{*,*} X^- \oplus \pi^K_{*,*} X^+. \tag{6.6}$$

Here $(-)^-$ is the $+1$ eigenspace, and $(-)^+$ is the -1 eigenspace. We have

$$\pi^K_{*,*} X^- = \pi^K_{*,*} X[\eta^{-1}] = \pi^K_{*,*} X[\rho^{-1}]$$

and on $\pi^K_{*,*} X^+$ multiplication by η and ρ^2 is zero.[7] We deduce:

Proposition 6.7. *For any $X \in \mathrm{SH}_{\mathrm{cell}}(K)[1/2]$, we have*

$$X[\eta^{-1}] \simeq X[\rho^{-1}]$$

and the homotopy groups of these spectra are $\pi^K_{*,*} X^-$, and we have

$$X^\wedge_\eta \simeq X^\wedge_\rho$$

and the homotopy groups of these spectra are $\pi^K_{*,*} X^+$.

Proof. From the discussion above we deduce that the maps

$$X[\eta^{-1}] \to X[\rho^{-1}][\eta^{-1}] \leftarrow X[\rho^{-1}],$$
$$X^\wedge_\eta \to X^\wedge_{\rho,\eta} \leftarrow X^\wedge_\eta$$

induce isomorphisms on bigraded homotopy groups, and hence are equivalences since the spectra are cellular. □

Finally we note that for $X \in \mathrm{SH}_{\mathrm{cell}}(K)[1/2]$, since $X^\wedge_\rho[\rho^{-1}] \simeq 0$, the ρ-arithmetic square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X[\rho^{-1}] \arrow[d] \\ X^\wedge_\rho \arrow[r] & X^\wedge_\rho[\rho^{-1}] \end{tikzcd}$$

yields a topological lift of the decomposition (6.6)

$$X \simeq X[\rho^{-1}] \vee X^\wedge_\rho. \tag{6.8}$$

[7] When $K = \mathbb{R}$, multiplication by ρ is zero on $\pi^{\mathbb{R}}_{*,*} X^+$. This follows from the presentation of the Milnor–Witt ring of $\mathbb{R}$ in the introduction of [Dugger and Isaksen 2017a].

On the other hand, for any $Y \in \mathrm{Sp}^{C_2}[1/2]$, the Tate spectrum Y^t is contractible, and the isotropy separation square reduces to a splitting

$$Y \simeq Y^{\Phi} \vee Y^h. \tag{6.9}$$

The discussion from the previous subsection implies that C_2-Betti realization carries the splitting (6.8) to (6.9).

Motivic and equivariant cohomology. Let $(H\mathbb{F}_p)_K$ denote the mod p motivic Eilenberg–MacLane spectrum over K. By [Voevodsky 2003; 2011; Stahn 2016],

$$\pi^K_{*,*}(H\mathbb{F}_p)_K = \begin{cases} \mathbb{F}_p[\tau], & K = \mathbb{C}, \\ \mathbb{F}_2[\tau, \rho], & K = \mathbb{R},\ p = 2, \\ \mathbb{F}_p[\tau^2], & K = \mathbb{R},\ p \text{ odd}. \end{cases}$$

Here, ρ is the Hurewicz image of $\rho_{\mathbb{R}}$ (and $\rho_{\mathbb{C}} \simeq 0$).

Eilenberg–MacLane spectra are stable under base change — in particular,

$$\zeta^*(H\mathbb{F}_p)_{\mathbb{R}} = (H\mathbb{F}_p)_{\mathbb{C}}$$

and the associated map

$$\pi^{\mathbb{R}}_{*,*}(H\mathbb{F}_p)_{\mathbb{R}} \to \pi^{\mathbb{C}}_{*,*}(H\mathbb{F}_p)_{\mathbb{C}}$$

is the quotient by the ideal generated by ρ if $p = 2$, and the evident inclusion if p is odd.

The C_2-Betti realization of the mod p motivic Eilenberg–MacLane spectrum is the C_2-equivariant Eilenberg–MacLane spectrum $H\underline{\mathbb{F}_p}$ associated to the constant Mackey functor $\underline{\mathbb{F}_p}$ [Heller and Ormsby 2016]:

$$\mathrm{Be}^{C_2}(H\mathbb{F}_p)_{\mathbb{R}} \simeq H\underline{\mathbb{F}_p}.$$

For $p = 2$ we have

$$\pi^{C_2}_{*,*} H\underline{\mathbb{F}_2} = \mathbb{F}_2[u, a] \oplus \frac{\mathbb{F}_2[u, a]}{(u^{\infty}, a^{\infty})}\{\theta\}$$

where a is the Hurewicz image of the element $a \in \pi^{C_2}_{-1,-1}$,

$$u = \mathrm{Be}^{C_2}(\tau) \in \pi^{C_2}_{0,-1} H\underline{\mathbb{F}_2},$$

and

$$\theta \in \pi^{C_2}_{0,2} H\underline{\mathbb{F}_2}.$$

For p odd we have

$$\pi^{C_2}_{*,*} H\underline{\mathbb{F}_p} = \mathbb{F}_p[u^{\pm 2}]$$

where

$$u^2 = \mathrm{Be}^{C_2}(\tau^2) \in \pi^{C_2}_{0,-2} H\underline{\mathbb{F}_p}.$$

7. τ-self maps

In this section we will construct τ^j-self maps on the spectra $C(\rho^i)^\wedge_p$. For $p = 2$, this will be accomplished in the first three subsections by first constructing the C_2-Betti realizations of the desired self maps, and then by using a theorem of Dugger and Isaksen [2017b] to lift these equivariant self maps to real motivic self maps. For p odd, we will observe in the last subsection that the work of Stahn [2016] implies that every ρ-complete spectrum has a τ^2-self map.

From now until the last subsection of this section, we implicitly assume everything is 2*-complete.*

$\mathbb{R}$-motivic and C_2-equivariant homotopy groups of spheres. For $j \in \mathbb{Z}$, let P_j^∞ denote the stunted projective spectrum given as the Thom spectrum

$$P_j^\infty := (\mathbb{R}P^\infty)^{j\xi}$$

where ξ is the canonical line bundle. The Segal conjecture for the group C_2 (Lin's theorem [1980]) implies the following:

Proposition 7.1. *There are isomorphisms*

$$\pi^{C_2}_{i,j} S^{0,0} \cong \pi_{i-j}([P_j^\infty]^\vee).$$

Proof. The Segal conjecture implies that for a finite C_2-spectrum Y, the map

$$Y \to Y^h = F((EC_2)_+, Y)$$

is a (2-adic) equivalence. Using the equivalence

$$P_j^\infty \simeq (S^{j\sigma})_{hC_2},$$

we have

$$\begin{aligned}
\pi^{C_2}_{i,j} &= [S^{i-j} \wedge S^{j\sigma}, S]^{C_2} \\
&\cong [S^{i-j} \wedge S^{j\sigma}, F((EC_2)_+, S)]^{C_2} \\
&\cong [S^{i-j}, F((EC_2)_+ \wedge S^{j\sigma}, S)]^{C_2} \\
&\cong [S^{i-j}, F((EC_2)_+ \wedge_{C_2} S^{j\sigma}, S)] \\
&= \pi_{i-j}([P_j^\infty]^\vee).
\end{aligned}$$

□

Applying $\pi^{C_2}_{*,*}$ to the norm cofiber sequence

$$(EC_2)_+ \to S^0 \to \widetilde{EC}_2 \tag{7.2}$$

gives the long exact sequence

$$\cdots \to \pi^s_{i-j+1} \to \lambda_{i,j} \to \pi^{C_2}_{i,j} \xrightarrow{\Phi^{C_2}} \pi^s_{j-i} \to \cdots$$

studied by Landweber[8] [1969]. Using the equivalences

$$\begin{aligned} [P_j^\infty]^\vee &\simeq \Sigma P_{-\infty}^{-j-1} && \text{[Bruner et al. 1986, Theorem V.2.14(iv)]}, \\ S^{-1} &\simeq P_{-\infty}^\infty && \text{[Lin 1980]}, \end{aligned}$$

there is an isomorphism of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \pi^s_{i-j+1} \arrow[r] \arrow[d, "\cong"] & \lambda_{i,j} \arrow[r, "\cong"] \arrow[d, "\cong"] & \pi^{C_2}_{i,j} \arrow[r, "\Phi^{C_2}"] \arrow[d, "\cong"] & \pi^s_{j-i} \arrow[r] \arrow[d, "\cong"] & \cdots \\ \cdots \arrow[r] & \pi_{i-j} P^\infty_{-\infty} \arrow[r] & \pi_{i-j} P^\infty_{-j} \arrow[r] & \pi_{i-j-1} P^{-j-1}_{-\infty} \arrow[r] & \pi_{i-j-1} P^\infty_{-\infty} \arrow[r] & \cdots \end{tikzcd} \tag{7.3}$$

where the bottom long exact sequence is the sequence obtained by applying π_* to the cofiber sequence

$$P_{-\infty}^{-j-1} \to P_{-\infty}^\infty \to P_{-j}^\infty.$$

By (6.5), the geometric fixed points map is the a-localization map

$$\begin{tikzcd} \pi^{C_2}_{i,j} S \arrow[r] \arrow[dr, "\Phi^{C_2}"'] & \pi^{C_2}_{i,j} S[a^{-1}] \arrow[d, "\cong"] \\ & \pi_{i-j} S \end{tikzcd}$$

Thus the groups $\pi^{C_2}_{*,*}$ consist of a-torsion, and a-towers, where the latter are in bijective correspondence with the nonequivariant stable stems. The generators of these a-towers correspond to the Mahowald invariants [Bruner and Greenlees 1995].

As explained in [Dugger and Isaksen 2017b], Landweber [1969] uses James periodicity to show that the a-torsion in $\pi^{C_2}_{i,j}$ is periodic in the j direction outside of a certain conic region.

Theorem 7.4 (Landweber). *Define*

$$\gamma(m) := \#\{k : 0 < k \leq m,\ k \equiv 0, 1, 2, 4 \bmod 8\}. \tag{7.5}$$

Outside of the region

$$j - 1 \leq i \leq 2j$$

there are isomorphisms

$$(\pi^{C_2}_{i,j})_{a\text{-tors}} \cong (\pi^{C_2}_{i,j+2^{\gamma(i-1)}})_{a\text{-tors}}.$$

[8] Here, we have indexed $\pi_{i,j}$ and $\lambda_{i,j}$ with respect to our bigrading convention, not Landweber's.

Proof. Outside of the region described, the map

$$\lambda_{i,j} \to (\pi^{C_2}_{i,j})_{a\text{-tors}}$$

is an isomorphism, and Landweber [1969, Theorem 2.4, Proposition 6.1] observed that James periodicity implies that there is an isomorphism

$$\lambda_{i,j} \cong \lambda_{i,j+2^{\gamma(i-1)}}. \qquad \square$$

Dugger and Isaksen [2017b] prove the following theorem.[9]

Theorem 7.6 (Dugger and Isaksen). *C_2-Betti realization induces an isomorphism*

$$\pi^{\mathbb{R}}_{i,j} S^{0,0} \to \pi^{C_2}_{i,j} S^{0,0}$$

for $i \geq 3j - 5$.

Figure 1 depicts the location of the a-torsion and the a-towers in $\pi^{C_2}_{*,*}$. The dashed line marks the region where Dugger and Isaksen proved these groups coincide with the groups $\pi^{\mathbb{R}}_{*,*}$ (Theorem 7.6). This cone in Theorem 7.4 is labeled the "nonperiodicity cone" in the figure. Outside of this cone, the map

$$\lambda_{i,j} \to (\pi^{C_2}_{i,j})_{a\text{-tors}}$$

is an isomorphism.

***u*-self maps.** Since the C_2-spectra $S^{1,0}$ and $S^{1,1}$ are nonequivariantly equivalent, the equivalence (6.2) results in a self-equivalence

$$u : \Sigma^{0,-1} Ca \to Ca.$$

We denote this map u, and shall refer to it as a *u-self map*, because it induces the multiplication by u map on the homology groups

$$(H\underline{\mathbb{F}_2})_{*,*}(Ca) \cong \mathbb{F}_2[u^{\pm}].$$

We invite the reader to think of a u-self map as analogous to the v_n-self maps of chromatic homotopy theory [Ravenel 1992]. For instance, the mod 2^i Moore spectrum admits a v_1^j-self map for certain values of j which depend on i. We have the following analog in the present situation.

Theorem 7.7. *The C_2-spectrum Ca^i admits a u-self map*

$$u_{2^{\gamma(i-1)}} : \Sigma^{0,-2^{\gamma(i-1)}} Ca^i \to Ca^i$$

and this map is an equivalence.

To prove Theorem 7.7 (and the forthcoming Theorem 7.10) we shall need the following lemma.

[9] Belmont, Guillou, and Isaksen [Belmont et al. 2020] have recently improved this isomorphism theorem to the region $i \geq 2j - 4$.

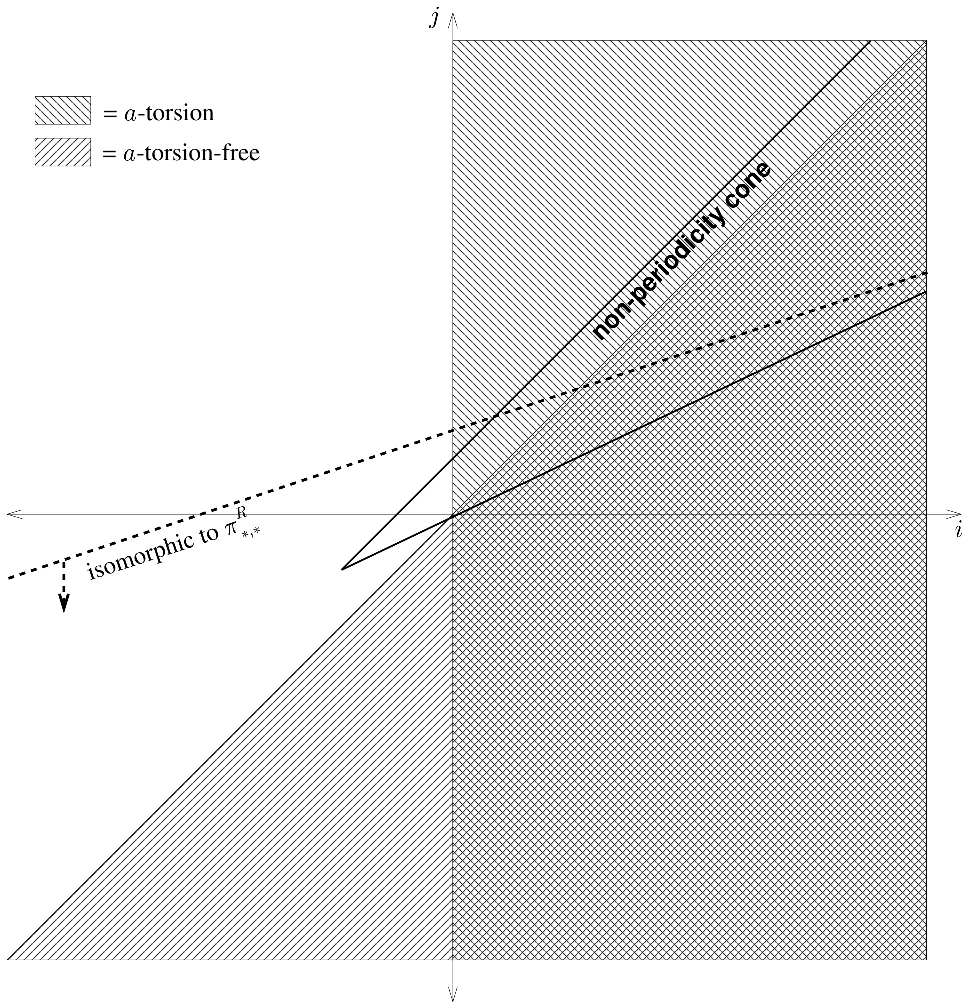

Figure 1. The structure of $\pi^{C_2}_{i,j} S^{0,0}$.

Lemma 7.8. *The spectra $C\rho^i \in \mathrm{SH}(\mathbb{R})$ and $Ca^i \in \mathrm{Sp}^{C_2}$ are E_∞-ring spectra.*

Proof. The case of $C\rho^i$ is explained in Remark 8.5. The case of Ca^i follows from the fact that C_2-Betti realization is monoidal. □

Proof of Theorem 7.7. Using the equivalence (6.3) and the Adams isomorphism,

$$\begin{aligned}\pi^{C_2}_{k,l} Ca^i &= [S^{k-l+l\sigma}, \Sigma^{1-i\sigma} S(i\sigma)_+]^{C_2} \\ &\cong [S^{k-l}, \Sigma S(i\sigma)_+ \wedge S^{(-l-i)\sigma}]^{C_2} \\ &\cong [S^{k-l}, \Sigma S(i\sigma)_+ \wedge_{C_2} S^{(-l-i)\sigma}] \\ &\cong \pi_{k-l} \Sigma P^{-l-1}_{-l-i}\end{aligned}$$

and a similar argument yields

$$(H\underline{\mathbb{F}_2})_{k,l}Ca^i \cong (H\mathbb{F}_2)_{k-l}\Sigma P_{-l-i}^{-l-1}. \tag{7.9}$$

It follows that

$$(H\underline{\mathbb{F}_2})_{*,*}Ca^i \cong \mathbb{F}_2[u^{\pm}, a]/(a^i)$$

where, under the isomorphism (7.9), the monomial $u^s a^t$ is the homology class coming from the $(s-1)$-cell of P_{s+t-i}^{s+t-1}.

By Lemma 7.8, to prove the theorem it suffices to prove that there is an element

$$u_{2^{\gamma(i-1)}} \in \pi^{C_2}_{0,-2^{\gamma(i-1)}}Ca^i$$

whose Hurewicz image is

$$u^{2^{\gamma(i-1)}} \in H^{C_2}_{0,-2^{\gamma(i-1)}}Ca^i.$$

Using the commutative diagram

$$\begin{tikzcd}
\pi^{C_2}_{0,-2^{\gamma(i-1)}}Ca^i \arrow[r, "\cong"] \arrow[d] & \pi_{2^{\gamma(i-1)}}\Sigma P^{2^{\gamma(i-1)}-1}_{2^{\gamma(i-1)}-i} \arrow[d] \\
H^{C_2}_{0,-2^{\gamma(i-1)}}Ca^i \arrow[r, "\cong"'] & H_{2^{\gamma(i-1)}}\Sigma P^{2^{\gamma(i-1)}-1}_{2^{\gamma(i-1)}-i}
\end{tikzcd}$$

relating equivariant and nonequivariant Hurewicz homomorphisms, the result follows from the fact [Bruner et al. 1986, Theorem V.2.14(v)] that $P^{2^{\gamma(i-1)}-1}_{2^{\gamma(i-1)}-i}$ is reducible.

The resulting self map $u_{2^{\gamma(i-1)}}$ induces multiplication by $u^{2^{\gamma(i-1)}}$ on homology, and therefore is a homology isomorphism, and hence is a equivalence. □

Note that we make no claims that these u-self maps have any uniqueness or compatibility properties.

τ-self maps.

Theorem 7.10. *The $\mathbb{R}$-motivic spectrum $C\rho^i$ admits a τ-self map*

$$\tau_{2^{\gamma(i-1)}} : \Sigma^{0,-2^{\gamma(i-1)}}C\rho^i \to C\rho^i.$$

Proof. By Lemma 7.8, it suffices to prove that there is an element

$$\tau_{2^{\gamma(i-1)}} \in \pi^{C_2}_{0,-2^{\gamma(i-1)}}C\rho^i$$

whose Hurewicz image is

$$\tau^{2^{\gamma(i-1)}} \in (H\underline{\mathbb{F}_2})_{0,-2^{\gamma(i-1)}}C\rho^i \cong \mathbb{F}_2[\tau, \rho]/(\rho^i).$$

By Theorem 7.6, there are isomorphisms in the map of long exact sequences

$$\begin{tikzcd}
\pi^{\mathbb{R}}_{i,i-2^{\gamma(i-1)}} \arrow[r, "\rho^i"'] \arrow[d, "\cong"'] & \pi^{\mathbb{R}}_{0,-2^{\gamma(i-1)}} \arrow[r] \arrow[d, "\cong"'] & \pi^{\mathbb{R}}_{0,-2^{\gamma(i-1)}} C\rho^i \arrow[r] \arrow[d] & \pi^{\mathbb{R}}_{i-1,i-2^{\gamma(i-1)}} \arrow[r, "\rho^i"'] \arrow[d, "\cong"'] & \pi^{\mathbb{R}}_{-1,-2^{\gamma(i-1)}} \arrow[d, "\cong"'] \\
\pi^{C_2}_{i,i-2^{\gamma(i-1)}} \arrow[r, "a^i"] & \pi^{C_2}_{0,-2^{\gamma(i-1)}} \arrow[r] & \pi^{C_2}_{0,-2^{\gamma(i-1)}} Ca^i \arrow[r] & \pi^{C_2}_{i-1,i-2^{\gamma(i-1)}} \arrow[r, "a^i"] & \cdots \pi^{C_2}_{-1,-2^{\gamma(i-1)}}
\end{tikzcd}$$

which, by the 5-lemma, allow us to deduce that there is an isomorphism

$$\pi^{\mathbb{R}}_{0,-2^{\gamma(i-1)}} C\rho^i \xrightarrow{\cong} \pi^{C_2}_{0,-2^{\gamma(i-1)}} Ca^i.$$

The desired element $\tau_{2^{\gamma(i-1)}}$ can be taken to be an element which corresponds, under this isomorphism, to the element $u_{2^{\gamma(i-1)}}$ of Theorem 7.7. $\square$

τ-self maps at an odd prime. *In this subsection, everything is implicitly p-complete for a fixed odd prime p.*

Consider the homotopy complete (p-complete) C_2-equivariant sphere S^h. We have

$$\begin{aligned}
\pi^{C_2}_{0,k} S^h &= [S^{k\sigma-k}, F((EC_2)_+, S)]^{C_2} \\
&\cong [(EC_2)_+ \wedge S^{k\sigma}, S^k]^{C_2} \\
&\cong [(EC_2)_+ \wedge_{C_2} S^{k\sigma}, S^k] \\
&\cong [P^\infty_k, S^k] \\
&\cong \begin{cases} \mathbb{Z}_p, & k \text{ even}, \\ 0, & k \text{ odd}, \end{cases}
\end{aligned}$$

where the last isomorphism comes from the fact that P^∞_k is p-adically contractible if k is odd, and inclusion of the bottom cell

$$S^k \hookrightarrow P^\infty_k$$

is a p-adic equivalence if k is even. Define u^2 to be a generator of $\pi^{C_2}_{0,-2} S^h$. Then the above calculation implies that

$$\pi^{C_2}_{0,*} S^h \cong \mathbb{Z}_p[u^{\pm 2}].$$

Thus the homotopy groups of any p-complete homotopy complete C_2-equivariant spectrum are u^2-periodic.

Proposition 7.11. *We have*

$$\pi^{\mathbb{R}}_{0,*} S^\wedge_\rho \cong \mathbb{Z}_p[\tau^2]$$

and every (p-complete) ρ-complete real motivic spectrum has a τ^2-self map. Moreover, we have

$$\widehat{\mathrm{Be}}^{C_2}_p(\tau^2) = u^2.$$

Proof. Let $BPGL$ be the odd primary real motivic Brown–Peterson spectrum constructed in [Stahn 2016]. By [Stahn 2016, Proposition 2.5], we have

$$\pi^{\mathbb{R}}_{*,*} BPGL = \mathbb{Z}_p[\tau^2, v_1, v_2, \dots]$$

with $|v_i| = (2p^i - 2, p^i - 1)$. Consider the associated (p-complete) real motivic Adams–Novikov spectral sequence[10]

$$\mathrm{Ext}_{BPGL_{*,*}BPGL}(BPGL_{*,*}, BPGL_{*,*}) \Rightarrow \pi^{\mathbb{R}}_{*,*} S^{\wedge}_{\eta}.$$

Stahn [2016] explains the odd primary analog of a recipe of Dugger and Isaksen [2010], which allows one to completely construct the motivic Adams–Novikov spectral sequence from the classical Adams–Novikov spectral sequence. In particular, using Proposition 6.7, we are able to deduce the first statement. The second statement follows by considering the composite (arising from the Hurewicz homomorphism, the map of [Hoyois 2015], and Betti realization)

$$\mathbb{Z}_p[\tau^2] = \pi^{\mathbb{R}}_{0,*} S^{\wedge}_{\rho} \to \pi^{\mathbb{R}}_{0,*} BPGL \to \pi^{\mathbb{R}}_{0,*} (H\mathbb{F}_p)_{\mathbb{R}} \to \pi^{C_2}_{0,*} H\underline{\mathbb{F}_p} = \mathbb{F}_p[u^{\pm 2}].$$

Theorem 4.18 of [Heller and Ormsby 2016] implies that C_2-Betti realization maps τ^2 to u^2. We deduce that

$$\widehat{\mathrm{Be}}^{C_2}_p(\tau^2) = \lambda u^2$$

with $\lambda \in \mathbb{Z}_p^{\times}$. Without loss of generality, we may choose the generator τ^2 so that $\lambda = 1$. □

8. The equivariant-motivic situation

The monoidal Barr–Beck theorem for étale base change. For a subgroup $H \leq G$, the restriction-induction adjunction

$$\mathrm{Res}^G_H : \mathrm{Sp}^G \rightleftarrows \mathrm{Sp}^H : \mathrm{Ind}^G_H$$

satisfies the hypotheses of Theorem 3.6 (cf. [Mathew et al. 2017, Theorem 5.32]).

Let ζ denote the map

$$\zeta : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$$

and consider the induced adjunction

$$\zeta^* : \mathrm{SH}(\mathbb{R}) \rightleftarrows \mathrm{SH}(\mathbb{C}) : \zeta_*.$$

[10] For convergence, the argument of [Dugger and Isaksen 2010, §8] shows that this spectral sequence converges to the $(H\mathbb{F}_p)_{\mathbb{R}}$-completion of the motivic sphere spectrum, which by [Hu et al. 2011], is the (p, η)-completion.

Note that since the adjunction $\zeta^* \dashv \zeta_*$ is monoidal, we have

$$\operatorname{Spec} \mathbb{C}_+ = \zeta_* 1 \in \operatorname{CAlg}(\operatorname{SH}(\mathbb{R})).$$

With the adjunction $\zeta^* \dashv \zeta_*$ being our main situation of interest, we now make the analogous observation in the motivic context.

Proposition 8.1. *If $f : T \to S$ is finite étale, then $f^* \dashv f_*$ satisfies the hypotheses of Theorem* 3.6, *and we have*

$$\operatorname{SH}(T) \simeq \operatorname{Mod}_{\operatorname{SH}(S)}(f_* 1).$$

Proof. In view of the properties of the base change functors outlined in Section 6, it only remains to show that f_* is conservative, so suppose $X \in \operatorname{SH}(T)$ such that $f_* X \simeq 0$. Consider the pullback square

$$\begin{array}{ccc} T \times_S T & \xrightarrow{g} & T \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle f} \\ T & \xrightarrow{f} & S \end{array}$$

Then $f^* f_* x \simeq g_* g^* x \simeq 0$. But $g_* g^* x$ is a finite coproduct of copies of x, using that f is finite étale. Hence, $x \simeq 0$. □

Corollary 8.2. $\operatorname{SH}(\mathbb{C}) \simeq \operatorname{Mod}_{\operatorname{SH}(\mathbb{R})}(\operatorname{Spec} \mathbb{C}_+)$.

The following is the key calculational observation behind this paper:

Proposition 8.3. *There is a noncanonical map*

$$C(\rho) \to \operatorname{Spec} \mathbb{C}_+$$

which becomes an equivalence after p-completion and cellularization.

Proof. Let

$$\xi : S^{0,0}_{\mathbb{R}} \to \operatorname{Spec} \mathbb{C}_+$$

be the unit map, which is adjoint to the identity in $\operatorname{SH}(\mathbb{C})$. By adjunction, we have

$$[S^{-1,-1}, \operatorname{Spec} \mathbb{C}_+]_{\mathbb{R}} \cong [S^{-1,-1}, S^{0,0}]_{\mathbb{C}}.$$

But since $\rho \simeq 0$ in $\operatorname{SH}(\mathbb{C})$, $\xi \circ \rho$ is null homotopic. Making a choice of null homotopy, we obtain a comparison map

$$\alpha : C(\rho) \to \operatorname{Spec} \mathbb{C}_+$$

that we wish to show is a p-complete cellular equivalence. Using the motivic Adams spectral sequence, it suffices to show that

$$\beta : (H\mathbb{F}_p)^{\mathbb{R}}_{*,*}(C(\rho)) \to (H\mathbb{F}_p)^{\mathbb{R}}_{*,*}(\operatorname{Spec} \mathbb{C}_+) \cong \pi^{\mathbb{C}}_{*,*}(H\mathbb{F}_p)_{\mathbb{C}}$$

is an isomorphism (for p odd, the motivic Adams spectral sequence only converges to the (p, η)-completion [Heller and Ormsby 2016], but both $C(\rho)$ and $\operatorname{Spec}\mathbb{C}_+$ are η-complete by Proposition 6.7).

For $p=2$, $\pi^{\mathbb{R}}_{*,*}$ of the map $(H\mathbb{F}_2)_{\mathbb{R}} \to \zeta_*(H\mathbb{F}_2)_{\mathbb{C}}$ is computed to be the surjection $\mathbb{F}_2[\tau,\rho] \to \mathbb{F}_2[\tau]$, which identifies β as the isomorphism $\mathbb{F}_2[\tau,\rho]/\rho \cong \mathbb{F}_2[\tau]$.

For p odd, $\pi^{\mathbb{R}}_{*,*}$ of the map $(H\mathbb{F}_p)_{\mathbb{R}} \to \zeta_*(H\mathbb{F}_p)_{\mathbb{C}}$ is computed [Stahn 2016, Proposition 1.1] to be the injection $\mathbb{F}_p[\tau^2] \to \mathbb{F}_p[\tau]$. Using the fact that ρ acts trivially, we deduce

$$(H\mathbb{F}_p)^{\mathbb{R}}_{*,*}C\rho \cong \mathbb{F}_p[\tau^2]\{1, \tau\}$$

and we conclude that β is an isomorphism. □

Remark 8.4. We claim that $\operatorname{Spec}\mathbb{C}_+$ is not cellular in $\mathrm{SH}(\mathbb{R})$. Indeed, upon applying ζ^*, the cofiber sequence

$$S^{-1,-1} \xrightarrow{\rho} S^{0,0} \to C(\rho)$$

becomes

$$S^{-1,-1} \xrightarrow{0} S^{0,0} \to \zeta^*(C(\rho))$$

and thus we have

$$\zeta^*(C\rho) \simeq S^{0,0} \vee S^{0,-1}.$$

But

$$\zeta^* \operatorname{Spec}\mathbb{C}_+ = \zeta^*\zeta_* 1 = S^{0,0} \vee S^{0,0}.$$

In effect, the presence of the motivic weight forbids $\operatorname{Spec}\mathbb{C}_+$ from being cellular.

Remark 8.5. Via Proposition 8.3 and Cell being lax monoidal, $C(\rho)^{\wedge}_p$ and therefore $C(\rho^n)^{\wedge}_p$ obtain the structure of E_∞-algebras in $\mathrm{SH}(\mathbb{R})^{\wedge}_p$.

Corollary 8.6. *There is an equivalence*

$$\mathrm{SH}_{\mathrm{cell}}(\mathbb{C})^{\wedge}_p \simeq \mathrm{Mod}_{\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^{\wedge}_p}(C(\rho))$$

and we have a diagram of commuting left adjoints

$$\begin{tikzcd}
\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^{\wedge}_p \arrow[r, hook] \arrow[d, "\zeta^*"] & \mathrm{SH}(\mathbb{R})^{\wedge}_p \arrow[d, "\zeta^*"] \\
\mathrm{SH}_{\mathrm{cell}}(\mathbb{C})^{\wedge}_p \arrow[r, hook] \arrow[d, "\simeq"] & \mathrm{SH}(\mathbb{C})^{\wedge}_p \arrow[d, "\simeq"] \\
\mathrm{Mod}_{\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^{\wedge}_p}(C(\rho)) \arrow[r, hook] & \mathrm{Mod}_{\mathrm{SH}(\mathbb{R})^{\wedge}_p}(\operatorname{Spec}\mathbb{C}_+)
\end{tikzcd}$$

where the horizontal right adjoints are given by the cellularization functor. In particular, for $X \in \mathrm{Sp}(\mathbb{R})^\wedge_p$, we have an induced isomorphism

$$\pi^{\mathbb{R}}_{*,*} X \wedge C(\rho) \cong \pi^{\mathbb{C}}_{*,*} \zeta^* X.$$

Proof. Combine Proposition 8.3, Proposition 8.1, Lemma 4.4 with $A = \mathrm{Spec}(\mathbb{C})_+$, and Lemma 4.3 for the p-completion. □

Warning 8.7. Cell is not strong monoidal, and indeed one may show that

$$\mathrm{Cell}(\mathrm{Spec}\,\mathbb{C}_+ \wedge \mathrm{Spec}\,\mathbb{C}_+) \not\simeq C(\rho)^{\wedge 2}.$$

Therefore, we don't have an induced adjunction between $\mathrm{Spec}\,\mathbb{C}_+$-local objects in $\mathrm{SH}(\mathbb{R})^\wedge_p$ and $C(\rho)$-local objects in $\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p$.

Betti realization. We next relate the motivic to the C_2-equivariant situation. We begin by recalling the Betti realization and constant functors, for which an ∞-categorical reference is [Bachmann and Hoyois 2018, §10.2, §11].

Definition 8.8. The complex Betti realization functor

$$\mathrm{Be} : \mathrm{SH}(\mathbb{C}) \to \mathrm{Sp}$$

is the unique colimit preserving functor that sends the complex motivic spectrum $\Sigma^\infty_+ X$ to $\Sigma^\infty_+ X(\mathbb{C})$ for X a smooth quasiprojective $\mathbb{C}$-variety, where $X(\mathbb{C})$ is endowed with the analytic topology. Likewise, the C_2-Betti realization functor

$$\mathrm{Be}^{C_2} : \mathrm{SH}(\mathbb{R}) \to \mathrm{Sp}^{C_2}$$

is the unique colimit preserving functor that sends the real motivic spectrum $\Sigma^\infty_+ X$ to $\Sigma^\infty_+ X(\mathbb{C})$ for X a smooth quasiprojective $\mathbb{R}$-variety, where $X(\mathbb{C})$ has C_2-action given by complex conjugation. We define p-complete Betti realization functors by

$$\widehat{\mathrm{Be}}_p(-) := \mathrm{Be}(-)^\wedge_p,$$
$$\widehat{\mathrm{Be}}^{C_2}_p(-) := \mathrm{Be}^{C_2}(-)^\wedge_p.$$

Both Be and Be^{C_2} are symmetric monoidal functors. Let Sing and Sing^{C_2} denote their respective right adjoints, so we have the following diagram of adjoint functors:

$$\begin{tikzcd}
\mathrm{SH}(\mathbb{R}) \arrow[r, shift left, "\mathrm{Be}^{C_2}"] \arrow[d, shift right, "\zeta^*"'] & \mathrm{Sp}^{C_2} \arrow[l, shift left, "\mathrm{Sing}^{C_2}"] \arrow[d, shift right, "\mathrm{Res}^{C_2}_e"'] \\
\mathrm{SH}(\mathbb{C}) \arrow[r, shift left, "\mathrm{Be}"] \arrow[u, shift right, "\zeta_*"'] & \mathrm{Sp} \arrow[l, shift left, "\mathrm{Sing}"] \arrow[u, shift right, "\mathrm{Ind}^{C_2}_e"']
\end{tikzcd} \tag{8.9}$$

We also have the real Betti realization functor

$$\mathrm{Be}_{\mathbb{R}} : \mathrm{SH}(\mathbb{R}) \to \mathrm{Sp}$$

that sends $\Sigma^\infty X_+$ to $\Sigma^\infty_+ X(\mathbb{R})$. By definition, $\Phi^{C_2} \mathrm{Be}^{C_2} \simeq \mathrm{Be}_{\mathbb{R}}$. Bachmann [2018, §10] has also identified his real-étale localization functor

$$\mathrm{SH}(\mathbb{R}) \to \mathrm{Sp}$$

with $\mathrm{Be}_{\mathbb{R}}$. If we let

$$i_* : \mathrm{Sp} \to \mathrm{Sp}^{C_2}$$

denote the right adjoint to geometric fixed points $(-)^{\Phi C_2}$, then it follows that $\mathrm{Sing}^{C_2} i_*$ is fully faithful.

Consider the ρ-inverted motivic sphere $S^{0,0}[\rho^{-1}]$ and the associated localization $\mathrm{SH}(S)[\rho^{-1}]$. The following main theorem of [Bachmann 2018] is essential.

Theorem 8.10 [Bachmann 2018]. *There is an equivalence of ∞-categories*

$$\mathrm{SH}(S)[\rho^{-1}] \simeq \mathrm{Sp}(\mathrm{Shv}(\mathrm{Sper}(S))),$$

where $\mathrm{Sper}(S)$ *is the real spectrum of* S [Bachmann 2018, §3]. *In particular*, *we have*

$$\mathrm{SH}(\mathbb{R})[\rho^{-1}] \simeq \mathrm{Sp}$$

and the following diagram commutes

$$\begin{tikzcd} \mathrm{SH}(\mathbb{R}) \arrow[rr, "\mathrm{Be}_{\mathbb{R}}"] \arrow[dr] & & \mathrm{Sp} \\ & \mathrm{SH}(\mathbb{R})[\rho^{-1}] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

Thus, *real Betti realization is localization with respect to* ρ.

We recall the definition of the constant functor, and Heller and Ormsby's equivariant generalization [2016].

Definition 8.11. The constant functor

$$c^*_{\mathbb{C}} : \mathrm{Sp} \to \mathrm{SH}(\mathbb{C})$$

is the unique colimit preserving functor that sends S^0 to $S^{0,0}$. The C_2-equivariant constant functor

$$c^*_{\mathbb{R}} : \mathrm{Sp}^{C_2} \to \mathrm{SH}(\mathbb{R})$$

is the unique colimit preserving functor that sends $S^0 = C_2/C_{2+}$ to $S^{0,0} = \mathrm{Spec}\,\mathbb{R}_+$ and $C_2/1_+$ to $\mathrm{Spec}\,\mathbb{C}_+$.

Lemma 8.12. *Betti realization splits the constant functor. In other words, we have equivalences*

$$\mathrm{Be} \circ c_{\mathbb{C}}^{*} \simeq id,$$
$$\mathrm{Be}^{C_2} \circ c_{\mathbb{R}}^{*} \simeq id.$$

Proof. The functors in question preserve colimits, so it suffices to observe that:

$$(\mathrm{Be}\, c_{\mathbb{C}}^{*})(S^0) = S^0,$$
$$(\mathrm{Be}^{C_2}\, c_{\mathbb{R}}^{*})(S^0) = S^0,$$
$$(\mathrm{Be}^{C_2}\, c_{\mathbb{C}}^{*})(C_2/1_+) = C_2/1_+. \qquad \square$$

Lemma 8.13. *The monoidal adjunctions*

$$\mathrm{Be}\colon \mathrm{SH}(\mathbb{C}) \rightleftarrows \mathrm{Sp} : \mathrm{Sing},$$
$$\mathrm{Be}^{C_2}\colon \mathrm{SH}(\mathbb{R}) \rightleftarrows \mathrm{Sp}^{C_2} : \mathrm{Sing}^{C_2}$$

satisfy the hypotheses of Theorem 3.6. *Therefore, we have*

$$\mathrm{Sp} \simeq \mathrm{Mod}_{\mathrm{SH}(\mathbb{C})}(\mathrm{Sing}\, S^0),$$
$$\mathrm{Sp}^{C_2} \simeq \mathrm{Mod}_{\mathrm{SH}(\mathbb{R})}(\mathrm{Sing}^{C_2}\, S^0).$$

Proof. We verify the second statement; the first will follow by a similar argument. Let us consider the hypotheses in turn:

(1) In view of Lemma 8.12, Sing^{C_2} is conservative as it is split by the right adjoint to the constant functor $c_{\mathbb{R}}^{*}$.

(2) Note that for X a smooth quasiprojective $\mathbb{R}$-variety, $X(\mathbb{R})$ and $X(\mathbb{C})$ have the homotopy types of finite CW-complexes; hence, $\mathrm{Be}^{C_2}(\Sigma_{+}^{\infty} X)$ is compact in Sp^{C_2}. Because the collection of motivic spectra $\{\Sigma_{+}^{\infty} X\}$ furnish a set of compact generators for $\mathrm{SH}(\mathbb{R})$, we deduce that Sing^{C_2} preserves colimits. To verify the projection formula

$$\mathrm{Sing}^{C_2}(A) \wedge B \simeq \mathrm{Sing}^{C_2}(A \wedge \mathrm{Be}^{C_2}\, B),$$

because both sides preserve colimits in the B variable, it suffices to check for $B = \Sigma_{+}^{\infty} X$. In this case, we need to show that for any $W \in \mathrm{SH}(\mathbb{R})$, the comparison map

$$[W, \mathrm{Sing}^{C_2}(A) \wedge B]_{\mathbb{R}} \to [W, \mathrm{Sing}^{C_2}(A \wedge \mathrm{Be}^{C_2}\, B)]_{\mathbb{R}}$$

is an isomorphism. Using that B is dualizable, under adjunction this is equivalent to

$$[\mathrm{Be}^{C_2}(W) \wedge \mathrm{Be}^{C_2}(B^{\vee}), A]^{C_2} \to [\mathrm{Be}^{C_2}(W), A \wedge \mathrm{Be}^{C_2}\, B]^{C_2}$$

where the conclusion follows because $\mathrm{Be}^{C_2} B$ is also dualizable with dual given by $\mathrm{Be}^{C_2}(B^\vee)$. $\square$

Using Lemma 3.7, we deduce the following p-complete variant.

Corollary 8.14. *For a prime* p, *we have*

$$\mathrm{Sp}_p^\wedge \simeq \mathrm{Mod}_{\mathrm{SH}(\mathbb{C})_p^\wedge}([\mathrm{Sing}\, S^0]_p^\wedge),$$
$$[\mathrm{Sp}^{C_2}]_p^\wedge \simeq \mathrm{Mod}_{\mathrm{SH}(\mathbb{R})_p^\wedge}([\mathrm{Sing}^{C_2}\, S^0]_p^\wedge).$$

We may also deduce the following cellular variant, which highlights an important difference between the $\mathbb{R}$- and $\mathbb{C}$-motivic settings.

Corollary 8.15. *The adjunction*

$$\mathrm{Be}\colon \mathrm{SH}_{\mathrm{cell}}(\mathbb{C}) \rightleftarrows \mathrm{Sp} : \mathrm{Cell}\,\mathrm{Sing}$$

satisfies the hypotheses of Theorem 3.6, *and therefore Betti realization gives an equivalence*

$$\mathrm{Mod}_{\mathrm{SH}_{\mathrm{cell}}(\mathbb{C})}(\mathrm{Cell}\,\mathrm{Sing}\, S^0) \simeq \mathrm{Sp}.$$

In particular, *we have an equivalence*

$$\mathrm{Mod}_{\mathrm{SH}_{\mathrm{cell}}(\mathbb{C})}(\mathrm{Cell}\,\mathrm{Sing}\, S^0) \simeq \mathrm{Mod}_{\mathrm{SH}(\mathbb{C})}(\mathrm{Sing}\, S^0).$$

In the real case, *the adjunction*

$$\mathrm{Be}^{C_2}\colon \mathrm{SH}_{\mathrm{cell}}(\mathbb{R}) \rightleftarrows \mathrm{Sp}^{C_2} : \mathrm{Cell}\,\mathrm{Sing}^{C_2}$$

satisfies these hypotheses after p*-completion*, *giving*

$$\mathrm{Mod}_{\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})_p^\wedge}([\mathrm{Cell}\,\mathrm{Sing}^{C_2}\, S^0]_p^\wedge) \simeq (\mathrm{Sp}^{C_2})_p^\wedge. \tag{8.16}$$

Proof. Lemma 4.5 implies every hypotheses of Theorem 3.6 holds for the cellular adjunctions except for the conservativity hypothesis. In the complex case, because $c_{\mathbb{C}}^*$ has essential image in $\mathrm{SH}_{\mathrm{cell}}(\mathbb{C})$, Cell Sing is conservative. However, in the real case,

$$c_{\mathbb{R}}^*(C_2/1_+) = \mathrm{Spec}\,\mathbb{C}_+$$

is not cellular. Nonetheless, because

$$(\mathrm{Cell}\,\mathrm{Spec}\,\mathbb{C}_+)_p^\wedge \simeq C(\rho)_p^\wedge \to (\mathrm{Spec}\,\mathbb{C}_+)_p^\wedge$$

is sent to an equivalence in $(\mathrm{Sp}^{C_2})_p^\wedge$, it follows that $\mathrm{Cell}\,\mathrm{Sing}^{C_2}$ is conservative after p-completion. $\square$

Remark 8.17. The observation (8.16) is not new — Ricka [2017, Theorem 2.4] proves this. However, Ricka's version does not have the p-completion. We believe the subtlety mentioned in the proof above may have been overlooked in his proof, however, and we do not know if (8.16) holds without the p-completion.

Betti realization as a localization. We will now show that in the p-complete setting, both Cell Sing and $\mathrm{Cell\,Sing}^{C_2}$ are fully faithful, implying $\widehat{\mathrm{Be}}_p$ and $\widehat{\mathrm{Be}}_p^{C_2}$ are localizations when restricted to p-complete cellular motivic spectra.

The complex case, summarized in the following theorem, was essentially proven by Dugger and Isaksen [2010] (in the case of $p = 2$) and Stahn [2016] (in the case of p odd).

Theorem 8.18. *The functor*

$$\mathrm{Cell\,Sing} : \mathrm{Sp}_p^\wedge \to \mathrm{SH}_{\mathrm{cell}}(\mathbb{C})_p^\wedge$$

is fully faithful with essential image consisting of those objects in $\mathrm{SH}_{\mathrm{cell}}(\mathbb{C})_p^\wedge$ *on which multiplication by* τ *is an equivalence. Therefore, given* $X \in \mathrm{SH}_{\mathrm{cell}}(\mathbb{C})_p^\wedge$, 2*-complete Betti realization induces an isomorphism*

$$\pi_{i,j}^{\mathbb{C}} X[\tau^{-1}] \xrightarrow{\cong} \pi_i \,\widehat{\mathrm{Be}}_p(X).$$

Proof. Because we already know that $\mathrm{Be} \dashv \mathrm{Cell\,Sing}$ satisfies the hypotheses of Theorem 3.6, it suffices to compute $(S^{0,0})_p^\wedge[\tau^{-1}] \simeq \mathrm{Cell\,Sing}(S^0)_p^\wedge$. But the natural map

$$(S^{0,0})_p^\wedge[\tau^{-1}] \to \mathrm{Sing}(S^0)_p^\wedge$$

is a cellular equivalence by the results of [Dugger and Isaksen 2010; Stahn 2016]. □

Our strategy will be to formally derive the real case from this, by lifting this localization up the ρ-completion tower, and combining with Bachmann's Theorem 8.10.

To this end, we consider the isotropy separation recollement on Sp^{C_2} given by

$$\mathrm{Sp}^{hC_2} \underset{j_*}{\overset{(-)^h}{\rightleftarrows}} \mathrm{Sp}^{C_2} \underset{i_*}{\overset{(-)^{\Phi C_2}}{\rightleftarrows}} \mathrm{Sp}.$$

Lemma 8.19. *We have equivalences of functors*

$$(\mathrm{Be}^{C_2}\,\mathrm{Sing}^{C_2}\, i_*(-))^h \simeq 0,$$
$$(\mathrm{Be}^{C_2}\,\mathrm{Cell\,Sing}^{C_2}\, i_*(-))^h \simeq 0.$$

Proof. Because $S^{0,0}[\rho^{-1}]$ is cellular, the essential image of

$$\mathrm{Sing}^{C_2}\, i_* : \mathrm{Sp} \to \mathrm{SH}(\mathbb{R})$$

is cellular as it is generated as a localizing subcategory by $S^{0,0}[\rho^{-1}]$. Therefore,

$$\mathrm{Cell\,Sing}^{C_2}\, i_* \simeq \mathrm{Sing}^{C_2}\, i_*,$$

so we may ignore cellularization in the proof. Because for $E \in \mathrm{Sp}^{C_2}$, $E^h \simeq 0$ if and only if $\mathrm{Res}_e^{C_2} E \simeq 0$, it suffices to show that

$$\mathrm{Res}_e^{C_2} \mathrm{Be}^{C_2} \mathrm{Sing}^{C_2} i_* \simeq 0.$$

Because $\mathrm{Res}_e^{C_2} \mathrm{Be}^{C_2} \simeq \mathrm{Be}\, \zeta^*$ for

$$\zeta : \mathrm{Spec}\, \mathbb{C} \to \mathrm{Spec}\, \mathbb{R},$$

this follows from the observation that

$$\zeta^* : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}(\mathbb{C})$$

vanishes on ρ-inverted objects. $\square$

Lemma 8.20. *The natural transformations*

$$(\mathrm{Be}^{C_2} \mathrm{Sing}^{C_2} j_*(-))^{\Phi C_2} \to (j_*(-))^{\Phi C_2},$$
$$(\mathrm{Be}^{C_2} \mathrm{Cell}\, \mathrm{Sing}^{C_2} j_*(-))^{\Phi C_2} \to (j_*(-))^{\Phi C_2}$$

induced by the counits of the adjunctions

$$\epsilon : \mathrm{Be}^{C_2} \mathrm{Sing}^{C_2} \to id,$$
$$\epsilon' : \mathrm{Cell}\, \mathrm{Be}^{C_2} \mathrm{Sing}^{C_2} \to id$$

are equivalences.

Proof. We first consider the noncellular assertion. Let $X \in \mathrm{Sp}^{hC_2}$ and $Y = j_* X$. Since i_* is fully faithful, it suffices to prove that

$$i_*([\mathrm{Be}^{C_2} \mathrm{Sing}^{C_2} Y]^{\Phi C_2}) = \widetilde{EC}_2 \wedge \mathrm{Be}^{C_2} \mathrm{Sing}^{C_2} Y \to i_*(Y^{\Phi C_2}) = \widetilde{EC}_2 \wedge Y$$

is an equivalence. For this, first note that because $\widetilde{EC}_2 = \mathrm{Be}^{C_2}(S^{0,0}[\rho^{-1}])$, using that Be^{C_2} is strong monoidal and the projection formula we have equivalences

$$\begin{aligned}\widetilde{EC}_2 \wedge \mathrm{Be}^{C_2} \mathrm{Sing}^{C_2} j_*(X) &\simeq \mathrm{Be}^{C_2}(\mathrm{Sing}^{C_2}(X) \wedge S^{0,0}[\rho^{-1}]) \\ &\simeq \mathrm{Be}^{C_2} \mathrm{Sing}^{C_2}(X \wedge \widetilde{EC}_2)\end{aligned}$$

under which $\widetilde{EC}_2 \wedge \epsilon_Y$ is identified with $\epsilon_{Y \wedge \widetilde{EC}_2}$. Next, by Lemma 8.19 and the fact that $\mathrm{Sing}^{C_2} i_*$ is fully faithful, for any $Z \in \mathrm{Sp}$ the fiber sequence of functors

$$(EC_2)_+ \wedge - \to id \to \widetilde{EC}_2 \wedge -$$

applied to $\mathrm{Be}^{C_2} \mathrm{Sing}^{C_2} i_* Z$ yields the equivalence

$$\mathrm{Be}^{C_2} \mathrm{Sing}^{C_2} i_* Z \to \widetilde{EC}_2 \wedge \mathrm{Be}^{C_2} \mathrm{Sing}^{C_2} i_* Z \simeq i_* Z.$$

In particular, the counit

$$\mathrm{Be}^{C_2} \mathrm{Sing}^{C_2}(X \wedge \widetilde{EC}_2) \to X \wedge \widetilde{EC}_2$$

is an equivalence.

Finally, the cellular assertion is proven in the same way, using now that $S^{0,0}[\rho^{-1}]$ is cellular and $\mathrm{Be}^{C_2} \dashv \mathrm{Cell\,Sing}^{C_2}$ is a monoidal adjunction that satisfies the projection formula by Lemma 4.5. $\square$

We have almost assembled all of the ingredients needed to prove Theorem 8.22. In view of Lemma 5.1, it only remains to prove the fully faithfulness of Sing^{C_2} on the Borel part of the recollement, which we turn to now.

Because we have $\mathrm{Be}^{C_2}(C(\rho)) = C(a)$ for the Euler class

$$a : S^{-\sigma} \to S^0$$

and $(\mathrm{Sp}^{C_2})^\wedge_a \simeq \mathrm{Sp}^{hC_2}$ (6.4), we obtain the induced adjunction

$$\widehat{\mathrm{Be}}_p^{hC_2} : \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_{p,\rho} \rightleftarrows (\mathrm{Sp}^{hC_2})^\wedge_p : \mathrm{Cell\,Sing}^{hC_2}$$

as in Lemma 3.7.

Corollary 8.21. *The functor* $\mathrm{Cell\,Sing}^{hC_2}$ *is fully faithful.*

Proof. Combine Theorem 8.18, Corollary 8.6, and Proposition 3.10. $\square$

We may now deduce the categorical half of our main theorem, which states that C_2-equivariant Betti realization, when restricted to p-complete cellular real motivic spectra, is a localization.

Theorem 8.22. $\mathrm{Cell\,Sing}^{C_2} : (\mathrm{Sp}^{C_2})^\wedge_p \to \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p$ *is fully faithful.*

Proof. The conditions of Lemma 5.1 apply in view of Lemma 8.19, Lemma 8.20, Bachmann's Theorem 8.10, and Corollary 8.21. $\square$

Computing Betti localization. In the complex case, Theorem 8.18 implies that Betti realization can be computed on p-complete cellular complex motivic spectra by inverting $\tau \in \pi^{\mathbb{C}}_{0,-1}(S^{0,0})^\wedge_p$.

We would like a similar result for the C_2-Betti realization of a p-complete cellular real motivic spectrum. In the real case, for $X \in \mathrm{SH}(\mathbb{R})$, the isotropy separation recollement implies that the homotopy type of the p-complete C_2-equivariant Betti realization can then be recovered by the pullback:

$$\begin{tikzcd} \widehat{\mathrm{Be}}_p^{C_2}(X) \arrow[r] \arrow[d] & \widehat{\mathrm{Be}}_p^{C_2}(X)^\Phi \arrow[d] \\ \widehat{\mathrm{Be}}_p^{C_2}(X)^h \arrow[r] & \widehat{\mathrm{Be}}_p^{C_2}(X)^t \end{tikzcd} \tag{8.23}$$

Therefore, it suffices to compute $\widehat{\mathrm{Be}}_p^{C_2}(X)^\Phi$, $\widehat{\mathrm{Be}}_p^{C_2}(X)^h$, $\widehat{\mathrm{Be}}_p^{C_2}(X)^t$, and the map

$$\widehat{\mathrm{Be}}_p^{C_2}(X)^\Phi \to \widehat{\mathrm{Be}}_p^{C_2}(X)^t.$$

For the geometric localization $\widehat{\mathrm{Be}}_p^{C_2}(X)^\Phi$, Bachmann's Theorem 8.10 has the following immediate consequence (which does not require p-completion or cellularization).

Theorem 8.24. *For $X \in \mathrm{SH}(\mathbb{R})$, equivariant Betti realization induces an isomorphism*

$$\pi^{\mathbb{R}}_{*,*} X[\rho^{-1}] \xrightarrow{\cong} \pi^{C_2}_{*,*} \mathrm{Be}^{C_2}(X)^\Phi.$$

Proof. Using Bachmann's Theorem 8.10, we have

$$\begin{aligned} \pi^{\mathbb{R}}_{i,j} X[\rho^{-1}] &\cong \pi_{i-j} \mathrm{Be}_{\mathbb{R}}(X) \\ &\cong \pi_{i-j} \mathrm{Be}^{C_2}(X)^{\Phi C_2} \\ &\cong \pi^{C_2}_{i,j} \mathrm{Be}^{C_2}(X)^\Phi. \end{aligned}$$ □

We will now show that if X is p-complete and cellular, the p-complete homotopy completion $\widehat{\mathrm{Be}}_p^{C_2}(X)^h$ can be computed by inverting τ on the ρ-completion tower. The Tate spectrum $\widehat{\mathrm{Be}}_p^{C_2}(X)^t$ may then computed by inverting ρ on the τ-inverted ρ-completion.

Let us now describe in detail how to invert τ on the ρ-completion tower. For every n, we have adjunctions

$$\widehat{\mathrm{Be}}_p^{C_2,n} : \mathrm{Mod}_{\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p}(C(\rho^n)) \rightleftarrows \mathrm{Mod}_{(\mathrm{Sp}^{C_2})^\wedge_p}(C(a^n)) : \mathrm{Sing}^{C_2,n}$$

where $\mathrm{Sing}^{C_2,n}$ is fully faithful by Theorem 8.18 and Proposition 3.10. The self map τ_N of $C(\rho^n)^\wedge_p$ constructed in Section 7 (where we take $\tau_N := \tau^2$ there in the case of p odd) allows us to explicitly compute the resulting localization functor in terms of τ_N-localization, as stated in the next lemma.

Lemma 8.25. *For $X \in \mathrm{Mod}_{\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p}(C(\rho^n))$, we have*

$$\mathrm{Sing}^{C_2,n} \widehat{\mathrm{Be}}_p^{C_2,n} X \simeq X[\tau_N^{-1}].$$

Thus, the image of the fully faithful right adjoint

$$\mathrm{Sing}^{C_2,n} : \mathrm{Mod}_{(\mathrm{Sp}^{C_2})^\wedge_p}(C(a^n)) \to \mathrm{Mod}_{\mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p}(C(\rho^n))$$

consists of those p-complete cellular $C(\rho^n)$-modules on which multiplication by τ_N is an equivalence.

Proof. For brevity, we implicitly assume everything is p-complete in this proof. We claim the self maps τ_N satisfy

(1) $\widehat{\mathrm{Be}}_p^{C_2,n}(\tau_N) = u_N$ is an self-equivalence of $C(a^n)$,

(2) $C(\rho^n)[\tau_N^{-1}] \wedge_{C(\rho^n)} C(\rho) \simeq C(\rho)[\tau^{-1}]$.

Statement (1) is proven in Theorem 7.7 (for $p = 2$) and Proposition 7.11 (for p odd). For statement (2), it suffices to show that the composite

$$\Sigma^{0,-N} C(\rho) \simeq \Sigma^{0,-N} C(\rho^n) \wedge_{C(\rho^n)} C(\rho) \xrightarrow{\tau_N \wedge 1} C(\rho^n) \wedge_{C(\rho^n)} C(\rho) \simeq C(\rho)$$

is equal to τ^N, up to multiplication by a unit. However, by Corollary 8.6, we have

$$\pi^{\mathbb{R}}_{0,*} C(\rho) \cong \pi^{\mathbb{C}}_{0,*} (S^{0,0})^\wedge_p \cong \mathbb{Z}_p[\tau].$$

In particular, the Hurewicz homomorphism

$$\pi^{\mathbb{R}}_{0,*} C(\rho) \to (H\mathbb{F}_p)^{\mathbb{R}}_{0,*} C\rho \cong \mathbb{F}_p[\tau]$$

is given by the obvious surjection, and the result follows from the fact that τ_N induces multiplication by τ^N on homology.

By (1), we have a comparison map

$$C(\rho^n)[\tau_N^{-1}] \to \mathrm{Sing}^{C_2,n}(C(a^n))$$

adjoint to the equivalence

$$C(a^n)[u_N^{-1}] \simeq C(a^n).$$

After base change to $C(\rho)$, this map is an equivalence by (2), hence is an equivalence as $- \wedge_{C(\rho^n)} C(\rho)$ is conservative. Because the adjunctions in question also satisfy the hypotheses of Theorem 3.6, we have that

$$\begin{aligned} \mathrm{Sing}^{C_2,n} \widehat{\mathrm{Be}}_p^{C_2,n} X &\simeq \mathrm{Sing}^{C_2,n}((\widehat{\mathrm{Be}}_p^{C_2,n} X) \wedge_{C(a^n)} C(a^n)) \\ &\simeq \mathrm{Sing}^{C_2,n}(C(a^n)) \wedge_{C(\rho^n)} X \\ &\simeq X[\tau_N^{-1}]. \end{aligned}$$ □

For p odd, every $X \in \mathrm{SH}(\mathbb{R})^\wedge_p$ has a τ^2-self map on its ρ-completion, and we can therefore form the telescope

$$X^\wedge_\rho[\tau^{-1}] := X^\wedge_\rho[\tau^{-2}].$$

For $p = 2$, because the periodicity of the elements τ_N increases as $n \to \infty$, we do not have an analogous construction. Nevertheless, given $X \in \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^\wedge_p$, the equivalences of Lemma 8.25 allow us to define maps

$$\begin{aligned} X \wedge C(\rho^n)[\tau_N^{-1}] &\simeq \mathrm{Sing}^{C_2,n} \widehat{\mathrm{Be}}_p^{C_2,n} X \wedge C(\rho^n) \\ &\to \mathrm{Sing}^{C_2,n-1} \widehat{\mathrm{Be}}_p^{C_2,n-1} X \wedge C(\rho^{n-1}) \\ &\simeq X \wedge C(\rho^{n-1})[\tau_{N'}^{-1}]. \end{aligned}$$

We may therefore define

$$X^{\wedge}_{\rho}[\tau^{-1}] := \lim_n X \wedge C(\rho^n)[\tau_N^{-1}].$$

We are now ready to deduce the computational half of our main theorem.

Theorem 8.26. *For $X \in \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^{\wedge}_p$, we have*

$$\operatorname{Cell}\operatorname{Sing}^{hC_2} \widehat{\mathrm{Be}}_p^{hC_2} X^{\wedge}_{\rho} \simeq X^{\wedge}_{\rho}[\tau^{-1}]$$

and C_2-Betti realization induces an isomorphism

$$\pi^{\mathbb{R}}_{*,*} X^{\wedge}_{\rho}[\tau^{-1}] \xrightarrow{\cong} \pi^{C_2}_{*,*} \widehat{\mathrm{Be}}_p^{C_2}(X)^h.$$

Proof. Since

$$\operatorname{Cell}\operatorname{Sing}^{hC_2} \widehat{\mathrm{Be}}_p^{hC_2} X^{\wedge}_{\rho} \simeq \lim_n \operatorname{Sing}^{C_2,n} \widehat{\mathrm{Be}}_p^{C_2,n} X \wedge C(\rho^n),$$

we deduce the first statement from Lemma 8.25. The second statement follows from the adjunction

$$\begin{aligned} \pi^{C_2}_{i,j}(\widehat{\mathrm{Be}}_p^{C_2}(X)^h) &= [\mathrm{Be}^{C_2} S^{i,j}, \widehat{\mathrm{Be}}_p^{C_2}(X)^h]^{C_2} \\ &\cong [S^{i,j}, \operatorname{Cell}\operatorname{Sing}^{C_2} \widehat{\mathrm{Be}}_p^{C_2}(X)^h]_{\mathbb{R}} \\ &\cong \pi^{\mathbb{R}}_{i,j} X^{\wedge}_{\rho}[\tau^{-1}]. \end{aligned}$$ □

9. Examples

We now demonstrate the effectiveness of our theory by computing the C_2-equivariant homotopy groups of the C_2-Betti realizations of some p-complete cellular real motivic spectra from their motivic homotopy groups.

For p odd, the computational implementation of our theory is straightforward. Given $X \in \mathrm{SH}_{\mathrm{cell}}(\mathbb{R})^{\wedge}_p$, we have (6.8)

$$X \simeq X[\rho^{-1}] \vee X^{\wedge}_{\rho}$$

and we have

$$\pi^{C_2}_{*,*} \widehat{\mathrm{Be}}_p^{C_2}(X) \simeq \pi^{\mathbb{R}}_{*,*} X[\rho^{-1}] \oplus \pi^{\mathbb{R}}_{*,*} X^{\wedge}_{\rho}[\tau^{-2}].$$

In the case of $p = 2$, the computations are more interesting, and we illustrate this with some examples. In each of these cases, the motivic homotopy groups are less complicated than the corresponding C_2-equivariant homotopy groups.[11]

[11] It is worth pointing out that in each of these examples the actual determination of these motivic homotopy groups is often the result of deep results in motivic homotopy theory, whereas the corresponding equivariant computations do not depend on similarly deep input.

We point out that the use of isotropy separation to organize the equivariant homotopy of the examples in this section is not new — see, for example, [Greenlees 2018].

mod 2 motivic cohomology. Let $(H\mathbb{F}_2)_{\mathbb{R}} \in \mathrm{SH}(\mathbb{R})$ denote the mod 2 real motivic Eilenberg–MacLane spectrum. Dugger and Isaksen [2005] proved that the motivic complex cobordism spectrum MGL is cellular. Work of Hopkins and Morel and of Hoyois [2015] implies that $(H\mathbb{Z})_{\mathbb{R}}$ (and hence $(H\mathbb{F}_2)_{\mathbb{R}}$) is a regular quotient of MGL, and is therefore cellular. Finally, Heller and Ormsby [2016, Theorem 4.17] prove that for any abelian group, the C_2-Betti realization of $(HA)_{\mathbb{R}}$ is $H\underline{A}$, the C_2-equivariant Eilenberg–MacLane spectrum associated to the constant Mackey functor $\underline{A}$, so we have

$$\mathrm{Be}^{C_2}(H\mathbb{F}_2)_{\mathbb{R}} \simeq H\underline{\mathbb{F}_2}.$$

We may therefore apply our theory to compute $\pi_*^{C_2} H\underline{\mathbb{F}_2}$.

Recall again that we have [Voevodsky 2003]

$$\pi_{*,*}^{\mathbb{R}}(H\mathbb{F}_2)_{\mathbb{R}} = \mathbb{F}_2[\tau, \rho].$$

Using Theorem 8.10, we have

$$\begin{aligned} \pi_{*,*}^{C_2} H\underline{\mathbb{F}_2}^{\Phi} &\cong \pi_{*,*}^{\mathbb{R}}(H\mathbb{F}_2)_{\mathbb{R}}[\rho^{-1}] \\ &= \mathbb{F}_2[\tau, \rho^{\pm}]. \end{aligned}$$

Using Theorem 8.26, we have

$$\begin{aligned} \pi_{*,*}^{C_2} H\underline{\mathbb{F}_2}^{h} &\cong \pi_{*,*}^{\mathbb{R}}(H\mathbb{F}_2)_{\mathbb{R}}[\tau^{-1}] \\ &= \mathbb{F}_2[\tau^{\pm}, \rho]. \end{aligned}$$

Because the Tate spectrum is the geometric localization of the homotopy completion, we may apply Theorem 8.10 to the above to deduce

$$\begin{aligned} \pi_{*,*}^{C_2} H\underline{\mathbb{F}_2}^{t} &\cong \pi_{*,*}^{\mathbb{R}}(H\mathbb{F}_2)_{\mathbb{R}}[\tau^{-1}][\rho^{-1}] \\ &= \mathbb{F}_2[\tau^{\pm}, \rho^{\pm}]. \end{aligned}$$

We may then use the Mayer–Vietoris sequence

$$\cdots \to \pi_{*+1,*}^{C_2} H\underline{\mathbb{F}_2}^{t} \xrightarrow{\partial} \pi_{*,*}^{C_2} H\underline{\mathbb{F}_2} \to \pi_{*,*}^{C_2} H\underline{\mathbb{F}_2}^{h} \oplus \pi_{*,*}^{C_2} H\underline{\mathbb{F}_2}^{\Phi} \to \cdots$$

associated to the isotropy separation square (8.23) to deduce

$$\pi_{*,*}^{C_2} H\underline{\mathbb{F}_2} = \mathbb{F}_2[\tau, \rho] \oplus \frac{\mathbb{F}_2[\tau, \rho]}{(\tau^{\infty}, \rho^{\infty})}\{\partial\rho^{-1}\tau^{-1}\}.$$

The calculation is displayed in Figure 2. The motivic homotopy $\pi_{*,*}^{\mathbb{R}}(H\mathbb{F}_2)_{\mathbb{R}}$ is displayed in the shaded region. In this figure, a dot represents a factor of $\mathbb{F}_2$,

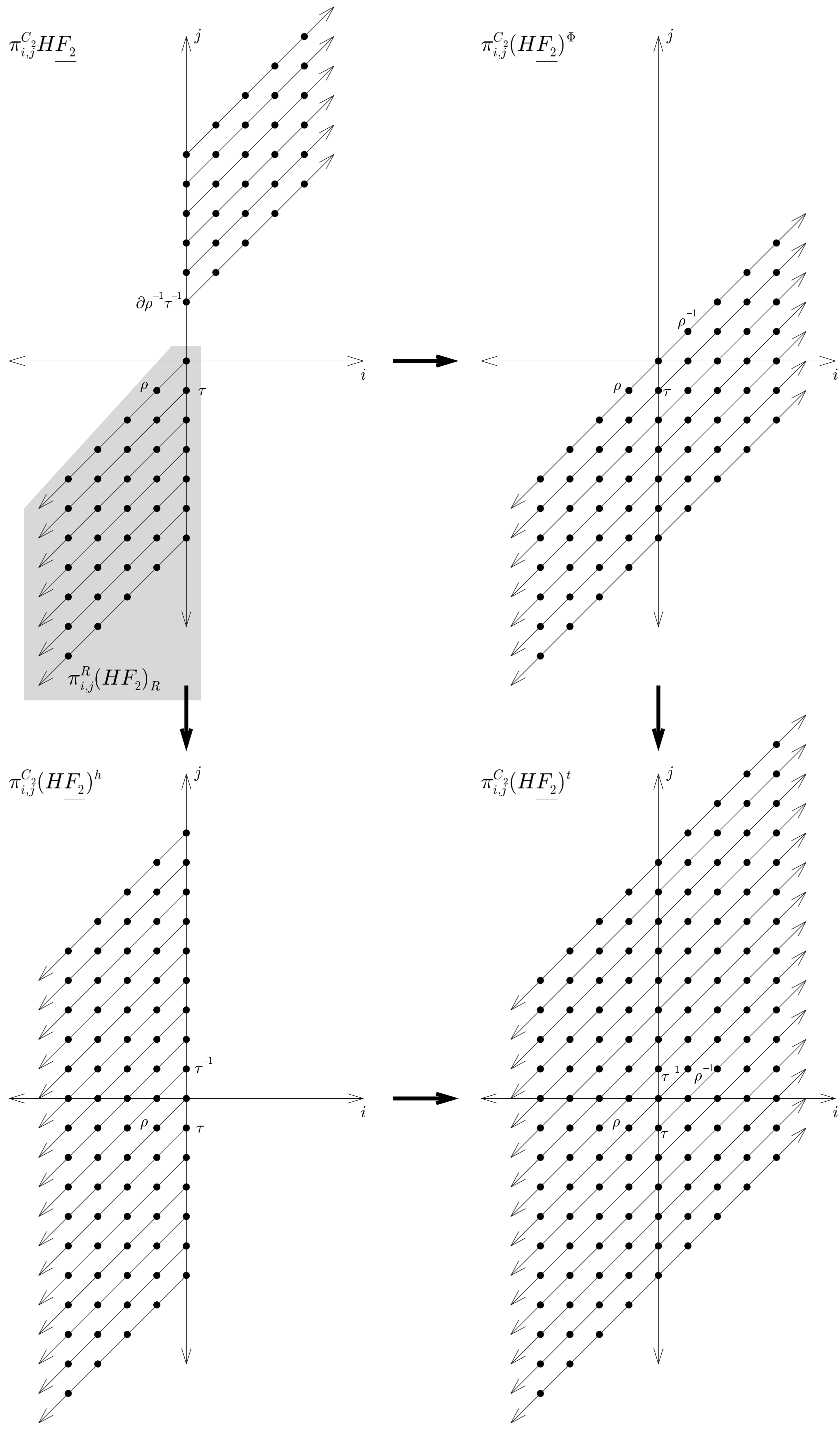

Figure 2. Computing $\pi^{C_2}_{*,*} H\underline{\mathbb{F}_2}$ from $\pi^{\mathbb{R}}_{*,*}(H\mathbb{F}_2)_{\mathbb{R}}$.

and a line represents multiplication by the element ρ. The other three quadrants are then obtained from this motivic homotopy by inverting τ, ρ, or both τ and ρ. The resulting equivariant homotopy, deduced from the Mayer–Vietoris sequence, is displayed in the upper-left-hand chart (the combination of the shaded and unshaded regions).

2-adic motivic cohomology. The discussion of the previous subsection also establishes that the 2-adic real motivic Eilenberg–MacLane spectrum $(H\mathbb{Z}_2)_{\mathbb{R}}$ is cellular (and it is clearly 2-complete). The coefficients of $(H\mathbb{Z}_2)_{\mathbb{R}}$ are given by (see, for example [Hill 2011][12])

$$\pi^{\mathbb{R}}_{*,*}(H\mathbb{Z}_2)_{\mathbb{R}} = \frac{\mathbb{Z}_2[\rho, \tau^2]}{(2\rho)}.$$

Again, [Heller and Ormsby 2016, Theorem 4.17] implies that

$$\mathrm{Be}^{C_2}(H\mathbb{Z}_2)_{\mathbb{R}} \simeq H\underline{\mathbb{Z}_2}.$$

We therefore deduce

$$\begin{aligned}
\pi^{C_2}_{*,*}H\underline{\mathbb{Z}_2}^{\Phi} &\cong \pi^{\mathbb{R}}_{*,*}(H\mathbb{Z}_2)_{\mathbb{R}}[\rho^{-1}] \\
&= \mathbb{F}_2[\tau^2, \rho^{\pm}], \\
\pi^{C_2}_{*,*}H\underline{\mathbb{Z}_2}^{h} &\cong \pi^{\mathbb{R}}_{*,*}(H\mathbb{Z}_2)_{\mathbb{R}}[\tau^{-2}] \\
&= \frac{\mathbb{Z}_2[\tau^{\pm 2}, \rho]}{(2\rho)}, \\
\pi^{C_2}_{*,*}H\underline{\mathbb{Z}_2}^{t} &\cong \pi^{\mathbb{R}}_{*,*}(H\mathbb{Z}_2)_{\mathbb{R}}[\tau^{-2}][\rho^{-1}] \\
&= \mathbb{F}_2[\tau^{\pm 2}, \rho^{\pm}].
\end{aligned}$$

We therefore deduce from the Mayer–Vietoris sequence

$$\pi^{C_2}_{*,*}H\underline{\mathbb{Z}_2} \cong \frac{\mathbb{Z}_2[\tau^2, \rho, 2\tau^{-2k}]}{(2\rho)} \oplus \frac{\mathbb{F}_2[\tau^2, \rho]}{(\tau^\infty, \rho^\infty)}\{\partial\rho^{-1}\tau^{-2}\}.$$

Note that there are implicitly defined relations in the above presentation, such as $\tau^2(2\tau^{-2k}) = 2\tau^{-2k+2}$ and $\rho(2\tau^{-2k}) = 0$.

The calculation is displayed in Figure 3. Everything is analogous to the notation of Figure 2, except that there are now boxes in addition to solid dots, which represent factors of $\mathbb{Z}_2$.

[12] Hill computes the homotopy of $BPGL\langle 0\rangle^{\wedge}_2$, which, by the work of Hopkins and Morel and of Hoyois [2015] is equivalent to $H\mathbb{Z}_2$.

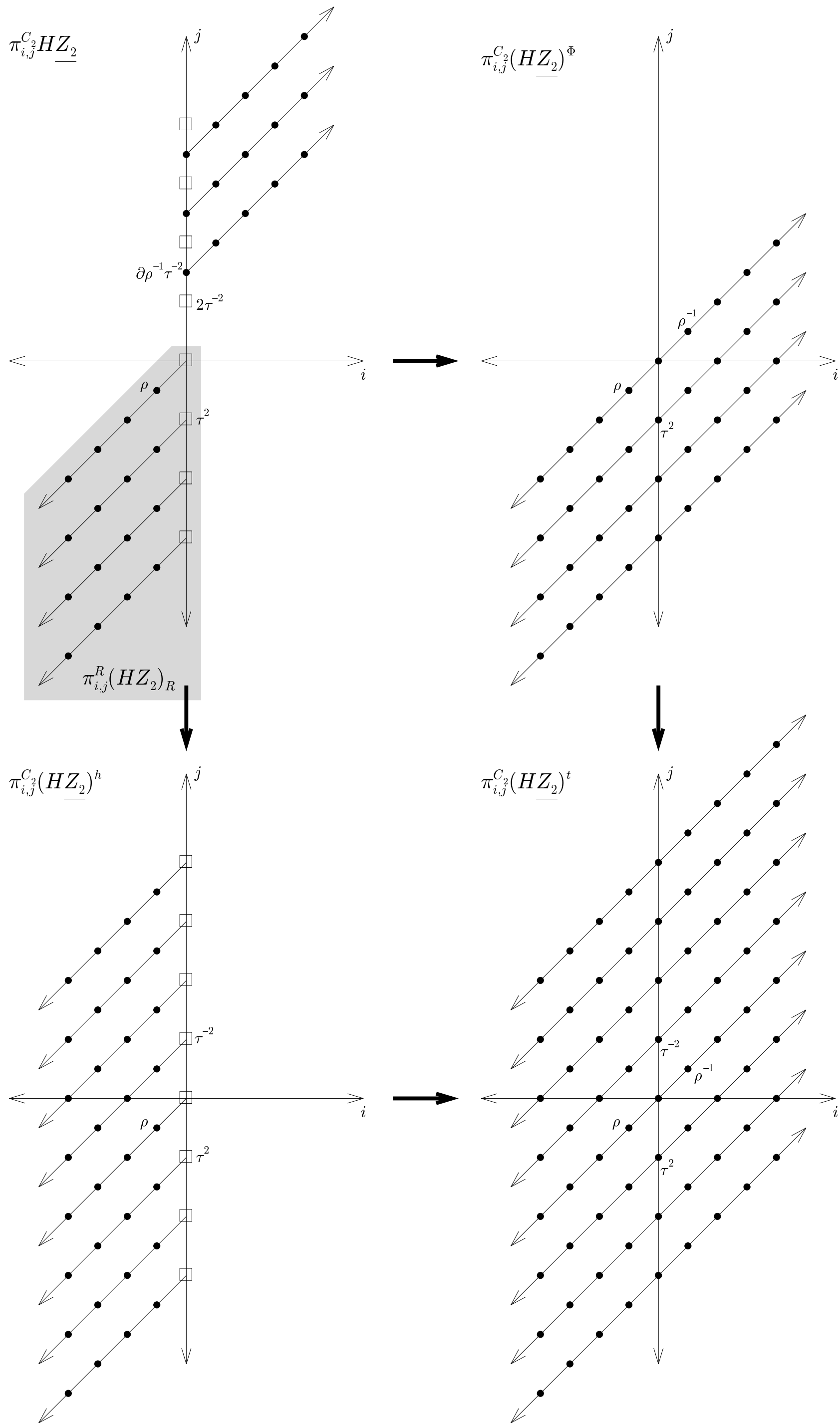

Figure 3. Computing $\pi^{C_2}_{*,*} H\underline{\mathbb{Z}_2}$ from $\pi^{\mathbb{R}}_{*,*}(H\mathbb{Z}_2)_{\mathbb{R}}$.

The effective cover of 2-adic algebraic K-theory. We now turn our attention to the spectrum kgl, the effective cover of KGL, the algebraic K-theory spectrum for the reals. Hill [2011] computes the 2-adic homotopy groups of this spectrum through the identification

$$kgl_2^\wedge \simeq BPGL\langle 1\rangle_2^\wedge.$$

In particular, $kgl_2^\wedge$ is cellular. We have

$$\pi^{\mathbb{R}}_{*,*} kgl_2^\wedge \cong \frac{\mathbb{Z}_2[\rho, 2\tau^2, \tau^4, v_1]}{(2\rho, v_1\rho^3)}$$

with

$$|v_1| = (2, 1).$$

Note that, just as in the previous subsection, our presentation has implicitly defined relations, such as $(2\tau^2)^2 = 4\tau^4$.

It is clear from the definition of KGL that we have

$$\mathrm{Be}^{C_2} KGL = KR$$

where KR is Atiyah's Real K-theory spectrum, and from [Heard 2019, Corollary 5.9] we deduce the connective analog

$$\mathrm{Be}^{C_2} kgl \simeq kR.$$

We deduce

$$\begin{aligned}
\pi^{C_2}_{*,*}(kR_2^\wedge)^\Phi &\cong \pi^{\mathbb{R}}_{*,*} kgl_2^\wedge[\rho^{-1}] \\
&= \mathbb{F}_2[\tau^4, \rho^\pm], \\
\pi^{C_2}_{*,*}(kR_2^\wedge)^h &\cong \pi^{\mathbb{R}}_{*,*} kgl_2^\wedge[\tau^{-4}] \\
&= \frac{\mathbb{Z}_2[\rho, 2\tau^2, \tau^{\pm 4}, v_1]}{(2\rho, v_1\rho^3)}, \\
\pi^{C_2}_{*,*}(kR_2^\wedge)^t &\cong \pi^{\mathbb{R}}_{*,*}(kgl_2^\wedge)[\tau^{-4}][\rho^{-1}] \\
&= \mathbb{F}_2[\tau^{\pm 4}, \rho^\pm].
\end{aligned}$$

We therefore deduce

$$\pi^{C_2}_{*,*} kR_2^\wedge \cong \frac{\mathbb{Z}_2[\rho, 2\tau^2, \tau^4, v_1, 2\tau^{-2k}, v_1\tau^{-4k}]}{(2\rho, v_1\rho^3)} \oplus \frac{\mathbb{F}_2[\tau^4, \rho]}{(\tau^\infty, \rho^\infty)}\{\partial\rho^{-1}\tau^{-4}\}.$$

The calculation is displayed in Figure 4. In this figure, dotted lines represent v_1-multiplication.

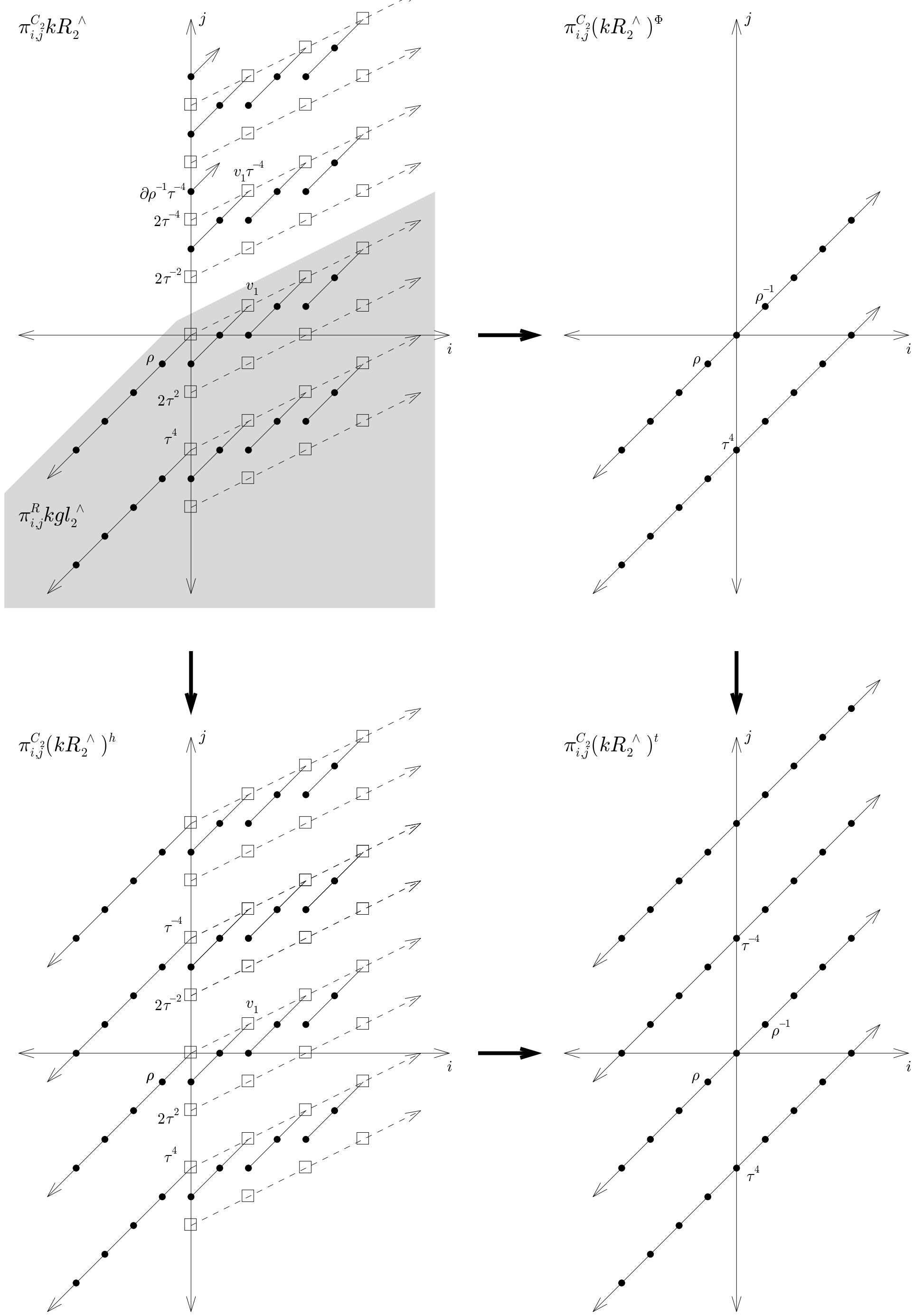

Figure 4. Computing $\pi^{C_2}_{*,*} kR_2^\wedge$ from $\pi^{\mathbb{R}}_{*,*}(kgl_2^\wedge)_{\mathbb{R}}$.

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References

[Bachmann 2018] T. Bachmann, "Motivic and real étale stable homotopy theory", *Compos. Math.* **154**:5 (2018), 883–917. MR Zbl

[Bachmann and Hoyois 2018] T. Bachmann and M. Hoyois, "Norms in motivic homotopy theory", preprint, 2018. arXiv

[Barthel et al. 2018] T. Barthel, D. Heard, and G. Valenzuela, "Local duality in algebra and topology", *Adv. Math.* **335** (2018), 563–663. MR Zbl

[Barwick and Glasman 2016] C. Barwick and S. Glasman, "A note on stable recollements", preprint, 2016. arXiv

[Belmont et al. 2020] E. Belmont, B. J. Guillou, and D. C. Isaksen, "C_2-equivariant and $\mathbb{R}$-motivic stable stems, II", preprint, 2020. arXiv

[Bruner and Greenlees 1995] R. Bruner and J. Greenlees, "The Bredon–Löffler conjecture", *Experiment. Math.* **4**:4 (1995), 289–297. MR Zbl

[Bruner et al. 1986] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ ring spectra and their applications*, Lecture Notes in Math. **1176**, Springer, 1986. MR Zbl

[Dugger and Isaksen 2005] D. Dugger and D. C. Isaksen, "Motivic cell structures", *Algebr. Geom. Topol.* **5** (2005), 615–652. MR Zbl

[Dugger and Isaksen 2010] D. Dugger and D. C. Isaksen, "The motivic Adams spectral sequence", *Geom. Topol.* **14**:2 (2010), 967–1014. MR Zbl

[Dugger and Isaksen 2017a] D. Dugger and D. C. Isaksen, "Low-dimensional Milnor–Witt stems over $\mathbb{R}$", *Ann. K-Theory* **2**:2 (2017), 175–210. MR Zbl

[Dugger and Isaksen 2017b] D. Dugger and D. C. Isaksen, "$\mathbb{Z}/2$-equivariant and $\mathbb{R}$-motivic stable stems", *Proc. Amer. Math. Soc.* **145**:8 (2017), 3617–3627. MR Zbl

[Elmanto and Kolderup 2020] E. Elmanto and H. Kolderup, "On modules over motivic ring spectra", *Ann. K-Theory* **5**:2 (2020), 327–355. MR

[Greenlees 2018] J. P. C. Greenlees, "Four approaches to cohomology theories with reality", pp. 139–156 in *An alpine bouquet of algebraic topology*, edited by C. Ausoni et al., Contemp. Math. **708**, Amer. Math. Soc., Providence, RI, 2018. MR Zbl

[Greenlees and May 1995] J. P. C. Greenlees and J. P. May, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. **543**, Amer. Math. Soc., Providence, RI, 1995. MR Zbl

[Heard 2019] D. Heard, "On equivariant and motivic slices", *Algebr. Geom. Topol.* **19**:7 (2019), 3641–3681. MR Zbl

[Heller and Ormsby 2016] J. Heller and K. Ormsby, "Galois equivariance and stable motivic homotopy theory", *Trans. Amer. Math. Soc.* **368**:11 (2016), 8047–8077. MR Zbl

[Heller and Ormsby 2018] J. Heller and K. Ormsby, "The stable Galois correspondence for real closed fields", pp. 1–9 in *New directions in homotopy theory*, edited by N. Kitchloo et al., Contemp. Math. **707**, Amer. Math. Soc., Providence, RI, 2018. MR Zbl

[Hill 2011] M. A. Hill, "Ext and the motivic Steenrod algebra over $\mathbb{R}$", *J. Pure Appl. Algebra* **215**:5 (2011), 715–727. MR Zbl

[Hoyois 2015] M. Hoyois, "From algebraic cobordism to motivic cohomology", *J. Reine Angew. Math.* **702** (2015), 173–226. MR Zbl

[Hoyois 2017] M. Hoyois, "The six operations in equivariant motivic homotopy theory", *Adv. Math.* **305** (2017), 197–279. MR Zbl

[Hu et al. 2011] P. Hu, I. Kriz, and K. Ormsby, "Convergence of the motivic Adams spectral sequence", *J. K-Theory* **7**:3 (2011), 573–596. MR Zbl

[Landweber 1969] P. S. Landweber, "On equivariant maps between spheres with involutions", *Ann. of Math.* (2) **89** (1969), 125–137. MR Zbl

[Lin 1980] W. H. Lin, "On conjectures of Mahowald, Segal and Sullivan", *Math. Proc. Cambridge Philos. Soc.* **87**:3 (1980), 449–458. MR Zbl

[Lurie 2009] J. Lurie, *Higher topos theory*, Annals of Math. Studies **170**, Princeton Univ., 2009. MR Zbl

[Lurie 2017] J. Lurie, "Higher algebra", preprint, 2017, Available at https://www.math.ias.edu/~lurie/papers/HA.pdf.

[Mathew et al. 2017] A. Mathew, N. Naumann, and J. Noel, "Nilpotence and descent in equivariant stable homotopy theory", *Adv. Math.* **305** (2017), 994–1084. MR Zbl

[Morel and Voevodsky 1999] F. Morel and V. Voevodsky, "$\mathbf{A}^1$-homotopy theory of schemes", *Inst. Hautes Études Sci. Publ. Math.* **90** (1999), 45–143. MR Zbl

[Ravenel 1992] D. C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, Annals of Math. Studies **128**, Princeton Univ., 1992. MR Zbl

[Ricka 2017] N. Ricka, "Motivic modular forms from equivariant stable homotopy theory", preprint, 2017. arXiv

[Stahn 2016] S.-T. Stahn, "The motivic Adams–Novikov spectral sequence at odd primes over $\mathbb{C}$ and $\mathbb{R}$", preprint, 2016. arXiv

[Voevodsky 2003] V. Voevodsky, "Motivic cohomology with $\mathbf{Z}/2$-coefficients", *Publ. Math. Inst. Hautes Études Sci.* **98** (2003), 59–104. MR Zbl

[Voevodsky 2011] V. Voevodsky, "On motivic cohomology with $\mathbf{Z}/l$-coefficients", *Ann. of Math.* (2) **174**:1 (2011), 401–438. MR Zbl

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Groups with Spanier–Whitehead duality

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Building on work by Kasparov, we study the notion of Spanier–Whitehead K-duality for a discrete group. It is defined as duality in the KK-category between two C^*-algebras which are naturally attached to the group, namely the reduced group C^*-algebra and the crossed product for the group action on the universal example for proper actions. We compare this notion to the Baum–Connes conjecture by constructing duality classes based on two methods: the standard "gamma element" technique, and the more recent approach via cycles with property gamma. As a result of our analysis, we prove Spanier–Whitehead duality for a large class of groups, including Bieberbach's space groups, groups acting on trees, and lattices in Lorentz groups.

Introduction

Alexander duality applies to the homology theory properties of the complement of a subspace inside a sphere in Euclidean space. More precisely, for a finite complex X contained in S^{n+1}, if $\widetilde{H}$ denotes reduced homology or cohomology with coefficients in a given abelian group, there is an isomorphism $\widetilde{H}_i(X) \cong \widetilde{H}^{n-i}(S^{n+1} \setminus X)$, induced by slant product with the pullback of the generator $\mu^*([S^n])$, via the duality map $\mu : X \times (S^{n+1} \setminus X) \to S^n$, $\mu(x, y) = (x - y)/\|x - y\|$.

Ed Spanier and J. H. C. Whitehead generalized this statement and adapted it to the context of stable homotopy theory. Their basic intuition was that sphere complements determine the homology, but not the homotopy type, in general. However, the stable homotopy type can be deduced and provides a first approximation to homotopy type [Spanier and Whitehead 1958]. Thus, the modern statement is phrased in terms of dual objects X, DX in the category of pointed spectra with the smash product as a monoidal structure, and by taking maps to an Eilenberg–Mac Lane spectrum one recovers Alexander duality formally.

The modern version of the duality implies Poincaré duality for compact manifolds and extends in a natural way to generalized cohomology theories such as K-theory. In this setting, a compact spin^c-manifold exhibits Poincaré duality in

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the sense that the K-homology class of the Dirac operator induces by cap product an isomorphism $K^*(M) \to K_{*+n}(M)$, where the shift is given by the dimension [Kasparov 1988].

More generally, the bivariant version of K-theory introduced by Kasparov, which we shall use extensively in this paper, showcases a close relationship to Alexander–Spanier duality; by this we mean that for X, Y finite complexes one has a chain of isomorphisms [Kaminker and Schochet 2019]

$$\mathrm{KK}_*(C(X), C(Y)) \cong \mathrm{KK}_*(\mathbb{C}, C(DX \wedge Y)) \cong K_*(C(DX \wedge Y)) \cong K^*(DX \wedge Y).$$

Having introduced C^*-algebras in this way, as they arise naturally in applications to topology, dynamics, and index theory, and are generally noncommutative, it is natural to seek for generalizations of Spanier–Whitehead duality in the framework of noncommutative geometry.

For a separable, nuclear C^*-algebra A represented on a Hilbert space, the commutant of its projection into the Calkin algebra has some of the properties reminiscent of a Spanier–Whitehead K-dual. This is the *Paschke dual* of A, and satisfies $K_*(P(A)) \cong K^*(A)$. However, in general $P(A)$ is neither separable nor nuclear, the Kasparov product is not defined, so that it seems desirable to explore different routes for the definition of a K-dual.

A. Connes [1994] introduced the appropriate formalism for this question, which shall be described shortly, and in [Connes 1996] he showed the first nontrivial example of a noncommutative Poincaré duality algebra, in the form of the irrational rotation algebra. H. Emerson [2003] proved the same result for the crossed product of a hyperbolic group acting on its Gromov boundary. Examples of pairs of algebras with Spanier–Whitehead duality were also given by Kaminker and Putnam [1997] in the case of Cuntz–Krieger algebras associated to M and its transpose, where M is a square $\{0, 1\}$-valued matrix. Their result is a special case of a more general one, in which the stable and unstable Ruelle algebras of a Smale space are shown to be in duality [Kaminker et al. 2017]. Duality in K-theory also appears in connection with string theory on noncommutative spacetimes [Brodzki et al. 2008; 2009].

In this paper, G is a discrete group which admits a G-compact model $\underline{E}G$ of the classifying space for proper actions [Baum et al. 1994]. We study the question of Spanier–Whitehead duality for the pair of C^*-algebras $C_r^*(G)$ and $C_0(\underline{E}G) \rtimes G$, where the latter is the crossed product for the group action on $\underline{E}G$.

This problem is tightly related to the Baum–Connes conjecture and in particular to the so-called Dirac dual-Dirac method. This goes back to the seminal work of Kasparov [1988, Sections 4 and 6] and is further explored in [Kasparov and Skandalis 1991, Section 6]. In a different direction, the relationship between the assembly map and Fourier–Mukai duality is discussed in [Block 2010].

The idea of an underlying noncommutative duality whenever Dirac and dual-Dirac classes are available is well-known to experts; see for example [Brodzki et al. 2008, Example 2.14; Echterhoff et al. 2008, Theorems 2.9 and 3.1]. In particular work of Emerson and Meyer [2010] shares many ideas with the present paper, while working in the context of equivariant KK-theory and groupoids. See page 472 and Remark 1.18 for more details.

Below are two main results of this paper. More details on statements and terminology are given in the sequel.

Theorem. *Suppose the γ-element exists. Then $C_0(\underline{E}G) \rtimes G$ is a Spanier–Whitehead K-dual of $C^*_r(G)$ (in a canonical way) if and only if G satisfies the strong Baum–Connes conjecture.*

Corollary. *For all a-T-menable groups G which admit a G-compact model of $\underline{E}G$, $C_0(\underline{E}G) \rtimes G$ is a Spanier–Whitehead K-dual of $C^*_r(G)$.*

Noncommutative Spanier–Whitehead duality. Let us see the main notions we will be working with. In what follows the C^*-tensor product is understood to be spatial.

Definition 0.1 (cf. [Brodzki et al. 2008, Section 2.7]). Let A, B be separable C^*-algebras. B is called a *weak Spanier–Whitehead K-dual* of A if there are elements

$$d \in \mathrm{KK}_i(A \otimes B, \mathbb{C}) \quad \text{and} \quad \delta \in \mathrm{KK}_{-i}(\mathbb{C}, A \otimes B)$$

such that the induced maps

$$\begin{aligned} d_j &: K_j(A) \to K^{j+i}(B), & d_j(x) &= x \,\widehat{\otimes}_A\, d, \\ \delta_j &: K^j(B) \to K_{j-i}(A), & \delta_j(x) &= \delta \,\widehat{\otimes}_B\, x \end{aligned}$$

are isomorphisms and inverses to each other.

Note that, unlike the case of topological spaces, in the noncommutative context the existence of d, given δ, is an additional requirement.

Some notation: $1_A \in \mathrm{KK}_0(A, A)$ stands for the ring unit, $\sigma : A \otimes B \cong B \otimes A$ denotes the flip isomorphism. Recall as well the homomorphism

$$\tau_B : \mathrm{KK}_*(A, A) \to \mathrm{KK}_*(A \otimes B, A \otimes B),$$

given on cycles as

$$(\phi, H, T) \mapsto (\phi \,\widehat{\otimes}\, 1, H \,\widehat{\otimes}\, B, T \,\widehat{\otimes}\, 1),$$

and equally defined via Kasparov product (over the complex numbers) by $\tau_B(x) = x \,\widehat{\otimes}\, 1_B = 1_B \,\widehat{\otimes}\, x$.

Lemma 0.2 [Emerson 2003, Lemma 9]. *In the setting of Definition 0.1, we have the identities*

$$(\delta_{j+i} \circ d_j)(y) = (-1)^{ij} y \,\widehat{\otimes}_A\, \Lambda_A \quad \textit{and} \quad (d_{j-i} \circ \delta_j)(y) = (-1)^{ij} \Lambda_B \,\widehat{\otimes}_B\, y,$$

where the elements $\Lambda_A \in \mathrm{KK}_0(A, A)$ *and* $\Lambda_B \in \mathrm{KK}_0(B, B)$ *are defined as*

$$\Lambda_A = \delta \mathbin{\widehat{\otimes}}_B d = (\delta \mathbin{\widehat{\otimes}} 1_A) \mathbin{\widehat{\otimes}}_{A\otimes B\otimes A} (1_A \mathbin{\widehat{\otimes}} \sigma^*(d)),$$
$$\Lambda_B = \delta \mathbin{\widehat{\otimes}}_A d = (\sigma_*(\delta) \mathbin{\widehat{\otimes}} 1_B) \mathbin{\widehat{\otimes}}_{B\otimes A\otimes B} (1_B \mathbin{\widehat{\otimes}} d).$$

Definition 0.3. Let A, B denote C^*-algebras in weak Spanier–Whitehead duality. With notation from Lemma 0.2, if we have $\Lambda_A = 1_A$ and $\Lambda_B = (-1)^i 1_B$, we say that A and B satisfy *Spanier–Whitehead K-duality*.

Note that this definition is symmetric, so that it can equivalently be phrased by saying that B is a Spanier–Whitehead K-dual of A, in alignment with the weak form introduced earlier.

Remark 0.4. In the tensor category $(\mathrm{KK}, \otimes)$, where objects are C^*-algebras and $\mathrm{Hom}(A, B) = \mathrm{KK}_0(A, B)$, the previous definition (for $i = j = 0$) can be reinterpreted as the statement that A is a dualizable object and B is its dual. In other words the triangle identity

$$\begin{tikzcd} & A \otimes B \otimes A \arrow[dr, "d\widehat{\otimes}1_A"] & \\ A \arrow[ur, "1_A\widehat{\otimes}\delta"] \arrow[rr, "1_A"] & & A \end{tikzcd}$$

(and its analogue swapping A and B) holds up to the unique isomorphisms coming from braiding and $A \otimes \mathbb{C} \cong A$.

The Spanier–Whitehead K-dual respects tensor products in the following sense: if the dual of A is B and the dual of A' is B', then the dual of $A \otimes B$ is KK-equivalent to $A' \otimes B'$, provided it exists; see [Kaminker and Schochet 2019].

Throughout this paper G denotes a countable discrete group admitting a G-compact model for its universal example for proper actions.

Definition 0.5. *G has (weak) Spanier–Whitehead K-duality* if $C_0(\underline{E}G) \rtimes G$ is a (weak) dual of $C_r^*(G)$.

Remark 0.6. It follows from [Anantharaman-Delaroche 2002, Proposition 2.2] that the action of G on $\underline{E}G$ is amenable. Then by [Anantharaman-Delaroche 2002, Theorem 5.3] the associated full and reduced crossed products are isomorphic. In particular, any covariant pair of representations for $C_0(\underline{E}G)$ and G gives rise to a representation of the reduced crossed product $C_0(\underline{E}G) \rtimes G$, namely the integrated form.

In short, the aim of this paper is identifying an element x belonging to the "representation ring" $\mathrm{KK}_0^G(\mathbb{C}, \mathbb{C})$, and constructing classes d and δ as above in such a way that $\Lambda_{C_r^*(G)}$ and $\Lambda_{C_0(\underline{E}G)\rtimes G}$ are both expressible in terms of x. Then the sought duality is reduced to studying the homotopy class of such an element.

Baum–Connes conjecture: the duality perspective. The Baum–Connes conjecture [Baum et al. 1994] states that the Baum–Connes assembly map

$$\mu^G : \mathrm{KK}_*^G(C_0(\underline{E}G), \mathbb{C}) \to \mathrm{KK}_*(\mathbb{C}, C_r^*(G)) \tag{0.7}$$

is an isomorphism of abelian groups. A generalization "with coefficients" can be introduced by inserting a G-algebra A in the right "slot" of the left-hand side of (0.7), and by considering the corresponding reduced crossed product in the target group:

$$\mu_A^G : \mathrm{KK}_*^G(C_0(\underline{E}G), A) \to \mathrm{KK}(\mathbb{C}, A \rtimes_r G). \tag{0.8}$$

Going back to the case with trivial coefficients (i.e., $A = \mathbb{C}$), since G is a discrete group, the (dual) Green–Julg isomorphism [Blackadar 1998; Kaad and Proietti 2018; Land 2015]

$$\mathrm{KK}_*^G(C_0(\underline{E}G), \mathbb{C}) \cong \mathrm{KK}_*(C_0(\underline{E}G) \rtimes_r G, \mathbb{C})$$

allows us to view the assembly map as a morphism

$$\mathrm{KK}_*(C_0(\underline{E}G) \rtimes G, \mathbb{C}) \to \mathrm{KK}_*(\mathbb{C}, C_r^*(G)). \tag{0.9}$$

We shall see that this map is given by Kasparov product with a certain element

$$\delta \in \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G)$$

(see Definition 1.1). Thus, the Baum–Connes conjecture for a discrete group G admitting a G-compact model $\underline{E}G$ is equivalent to the assertion that the element δ induces the isomorphism

$$\delta_* : K^*(C_0(\underline{E}G) \rtimes G) \xrightarrow{\cong} K_*(C_r^*(G)).$$

A priori, this isomorphism itself is not enough to conclude that G has weak Spanier–Whitehead K-duality. In this paper, under an assumption (see below), we identify an element

$$d \in \mathrm{KK}(C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C})$$

which induces a map

$$d_* : K_*(C_r^*(G)) \to K^*(C_0(\underline{E}G) \rtimes G)$$

which is the inverse of δ_* in favorable circumstances, namely if the Baum–Connes conjecture holds (it is a left inverse in general). Our assumption for constructing such an element d is the existence of the so-called gamma element, or alternatively the (γ)-element for G. Let us briefly review these notions.

The γ-element and the (γ)-element. The following notion of the gamma element originates in [Kasparov 1988].

Definition 0.10 [Tu 2000]. An element x in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ is called a *gamma element* for G if it satisfies the following:

(1) For any finite subgroup $F \subseteq G$, we have

$$\mathrm{Res}_G^F(x) = 1_{\mathbb{C}} \in \mathrm{KK}^F(\mathbb{C}, \mathbb{C}).$$

(2) For some separable, proper G-C^*-algebra P, we have

$$x = \beta \mathbin{\widehat{\otimes}}_P \alpha, \quad \text{where } \alpha \in \mathrm{KK}^G(P, \mathbb{C}),\ \beta \in \mathrm{KK}^G(\mathbb{C}, P).$$

A gamma element for G, if it exists, is a unique idempotent in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ which is characterized by the listed properties. Thus, we call it the gamma element for G and denote it by γ. The existence of the gamma element for G implies that the Baum–Connes assembly map is split-injective for all coefficients A [Tu 2000], and furthermore that the assembly map μ_A^G is surjective if and only if γ acts as the identity on $K_*(A \rtimes_r G)$ via ring homomorphisms

$$\mathrm{KK}^G(\mathbb{C}, \mathbb{C}) \to \mathrm{KK}^G(A, A) \to \mathrm{KK}(A \rtimes_r G, A \rtimes_r G) \to \mathrm{End}(K_*(A \rtimes_r G)). \quad (0.11)$$

The other composition $y = \alpha \mathbin{\widehat{\otimes}} \beta$ is an idempotent in $\mathrm{KK}(P, P)$ which may not be the identity on P in general. Upon replacing P with its "summand" $P_{\mathbb{C}} = yP$, which can be defined as a limit of $P \xrightarrow{y} P \xrightarrow{y} \cdots$ in the category KK^G [Neeman 2001, Proposition 1.6.8], we can arrange α (and β) above to be a weak-equivalence, meaning that $\mathrm{Res}_G^F(\alpha)$ is an isomorphism for any finite subgroup F of G. In this case, the element α in $\mathrm{KK}^G(P_{\mathbb{C}}, \mathbb{C})$ is called the *Dirac element* and can be characterized up to equivalence by the fact that α is a weak-equivalence from a "proper object" $P_{\mathbb{C}}$ to $\mathbb{C}$. Meyer and Nest [2006] showed that the Dirac element always exists for any group G but, in general, it is not known whether $P_{\mathbb{C}}$ can be taken to be a proper C^*-algebra. For most of the known examples, $P_{\mathbb{C}}$ can indeed be assumed to be proper, meaning that we may think $P = P_{\mathbb{C}}$. However, we emphasize that the algebra P appearing in the definition can be quite arbitrary, whereas $P_{\mathbb{C}}$ is a uniquely characterized object.

In [Nishikawa 2019], the first author introduced a notion called the (γ)-element, which can be thought of as an alternative description of the gamma element, bypassing the necessity of a proper algebra P for its definition.

Recall that we assume that G admits a G-compact model for $\underline{E}G$. We use $[-,-]$ to denote the commutator.

Definition 0.12 [Nishikawa 2019, Definition 2.2]. A cycle (H, T) representing an element $[H, T]$ in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ is said to have *property* (γ) if it satisfies the following:

(1) For any finite subgroup $F \subseteq G$, we have

$$\mathrm{Res}_G^F([H, T]) = 1_{\mathbb{C}} \in \mathrm{KK}^F(\mathbb{C}, \mathbb{C}).$$

(2) There is a nondegenerate G-equivariant representation of $C_0(\underline{E}G)$ on H such that

(2a) the function

$$g \mapsto [g \cdot f, T]$$

belongs to $C_0(G, K(H))$: it vanishes at infinity and is compact-operator-valued for any $f \in C_0(\underline{E}G)$;

(2b) for some cutoff function $c \in C_c(\underline{E}G)$ (i.e., c is nonnegative and satisfies $\sum_{g \in G} g(c)^2 = 1$), we have

$$T - \sum_{g \in G} (g \cdot c) T (g \cdot c) \in K(H).$$

An element x in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ is called a (γ)-element for G if it is represented by some cycle with property (γ).

A (γ)-element for G, if it exists, is a unique idempotent in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ which is characterized by the listed properties. Thus, we call it the (γ)-element for G. If there is a gamma element γ for G, there is a cycle with property (γ) representing γ. Thus the two notions, the γ-element and the (γ)-element for G, coincide when γ exists. The existence of the (γ)-element x for G implies that the Baum–Connes assembly map is split-injective for all coefficients A, and furthermore that the assembly map μ_A^G is surjective if and only if x acts as the identity on $K_*(A \rtimes_r G)$ via ring homomorphisms (0.11).

Given the existence of the (γ)-element, [Nishikawa 2019] introduced the so-called (γ)-morphism as a candidate for inverting the assembly map μ^G. This is given by Kasparov product with a certain element

$$\tilde{x} \in \mathrm{KK}^G(C_r^*(G) \otimes C_0(\underline{E}G), \mathbb{C}).$$

The Green–Julg isomorphism allows us to get the corresponding element

$$d \in \mathrm{KK}(C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C}).$$

Our proposed strategy aims at realizing weak Spanier–Whitehead duality through elements δ and d respectively corresponding to the assembly map and the (γ)-morphism, which seems to be a natural situation. Furthermore, as a result of Lemma 0.2, the surjectivity and injectivity of the assembly map are controlled by $\Lambda_{C_r^*(G)}$ and $\Lambda_{C_0(\underline{E}G) \rtimes G}$, respectively. This gives yet another interpretation of these two classes.

Equivariant Kasparov duality. In [Emerson and Meyer 2010] the authors study several duality isomorphisms between equivariant bivariant K-theory groups, generalizing Kasparov's first and second Poincaré duality isomorphisms. For many groupoids, both dualities apply to a universal proper G-space, which is the basis for the Dirac dual-Dirac method. In this setting they explain how to describe the Baum–Connes assembly map via localization of categories as in [Meyer and Nest 2006].

The main notion in [Emerson and Meyer 2010] is that of *equivariant Kasparov dual* for a G-space X. It involves an $X \rtimes G$-C^*-algebra P, an element $\alpha \in \mathrm{KK}^G(P, \mathbb{C})$, and an additional class $\Theta \in \mathrm{RKK}^G(X; \mathbb{C}, P)$ (see [Emerson and Meyer 2010, Definition 4.1] for more details). Recall that the category $\mathrm{RKK}^G(X)$ coincides with the range of the pullback functor $p_X^* : \mathrm{KK}^G \to \mathrm{KK}^{X \rtimes G}$ via the collapsing map $p : X \to *$.

The case $X = \underline{E}G$ is particularly relevant for our purposes. The class Θ may be thought as the "inverse" of α up to restriction to finite subgroups. More precisely, if a lifting $\beta \in \mathrm{KK}^G(\mathbb{C}, P)$ of Θ exists, then the axioms of equivariant Kasparov duality guarantee that $\beta \mathbin{\widehat{\otimes}_P} \alpha$ is the γ-element and $\alpha \mathbin{\widehat{\otimes}_{\mathbb{C}}} \beta = 1_P$. In particular, we have $P = P_{\mathbb{C}}$ and α is a weak equivalence, and hence a Dirac morphism.

Let Z denote the unit space of G and suppose the moment map from $\underline{E}G \to Z$ is proper. Then [Emerson and Meyer 2010, Theorem 5.7] establishes a connection to what we might call "equivariant" Spanier–Whitehead duality. We summarize it below for the convenience of the reader (see also Remark 1.18).

Theorem 0.13. *The triple (P, α, Θ) is a Kasparov dual for X if and only if $C_0(X)$ and P are dual objects in* KK^G (cf. Remark 0.4) *with duality unit and counit induced by Θ and α, respectively.*

Concerning the connection with the Baum–Connes assembly map, we have:

Theorem 0.14 [Emerson and Meyer 2010, Theorem 6.14]. *Suppose $\underline{E}G$ admits a local symmetric Kasparov dual. Then the assembly map μ_A^G is an isomorphism for all proper coefficient algebras A.*

Assuming $\underline{E}G$ to be G-compact, the proof of the previous theorem roughly goes as follows: the second Poincaré duality isomorphism [Emerson and Meyer 2010, Section 6] combined with the Green–Julg isomorphism for proper groupoids [Emerson and Meyer 2009, Theorem 4.2] translate the assembly map μ_A^G into the map $K_*((P \otimes A) \rtimes G) \to K_*(A \rtimes G)$ induced by α. Now it is easy to see from the definition of equivariant Kasparov dual that the element $\tau_A(\alpha) \in \mathrm{KK}^G(P \otimes A, A)$ is invertible when A is a proper C^*-algebra.

Main results. Let us summarize our main results. Recall that G is a countable discrete group with a G-compact model for $\underline{E}G$.

As we have explained in the previous sections, our main strategy for obtaining duality relies on (1) the γ-element, or (2) the (γ)-element. The choice of one over the other does not affect the expression for the unit of Spanier–Whitehead duality; nevertheless, the descriptions of the counit and the elements $\Lambda_{C_r^*(G)}$ and $\Lambda_{C_0(\underline{E}G)\rtimes G}$ depend on the method that we are employing. In practice, the latter elements will be expressible in terms of the γ-element in the first case, and in the terms of the (γ)-element in the second case.

Along this categorization, Theorem A and Corollary B below fall in the first scenario, while Theorem D is an instance of the second. Section 3 contains simple examples of possible applications of duality in K-theory.

Theorem A. *Suppose that the γ-element $\gamma \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ exists and let $P_{\mathbb{C}}$ be the source of the Dirac morphism $\alpha \in \mathrm{KK}^G(P_{\mathbb{C}}, \mathbb{C})$. Then the C^*-algebra $P_{\mathbb{C}} \rtimes G$ is Spanier–Whitehead K-dual to $C_0(\underline{E}G) \rtimes G$.*

A few more comments about this theorem. The source of the Dirac morphism (the "simplicial approximation" in [Meyer and Nest 2006]) can be obtained in a variety of ways: by appealing to the Brown representability theorem, by considering the left adjoint to the embedding functor of projective objects, or by constructing the appropriate homotopy colimit from a projective resolution of $\mathbb{C}$ (here, "projective" is to be understood in a relative sense, i.e., with respect to the homological ideal of weakly contractible objects). Even though $P_{\mathbb{C}}$ may not be a proper algebra in general, its reduced and maximal crossed products are KK-equivalent. This is because $P_{\mathbb{C}}$ belongs to the localizing subcategory of KK^G generated by proper algebras and the reduced and maximal crossed product functors are triangulated functors and commute with countable direct sums; see [Meyer and Nest 2006].

Theorem A provides a fourth characterization of $P_{\mathbb{C}}$, namely as the Spanier–Whitehead K-dual of the classifying space for proper actions. Note that even though our statement is only available after descent — that is, we can only get $P_{\mathbb{C}} \rtimes G$ and not $P_{\mathbb{C}}$ via duality — this is only a minor drawback in the case of discrete groups, for the left-hand side of (0.9) retains the full information of the "topological" K-theory group through the dual Green–Julg isomorphism

$$\mathrm{KK}^G(C_0(\underline{E}G), \mathbb{C}) \cong \mathrm{KK}(C_0(\underline{E}G) \rtimes G, \mathbb{C}).$$

In the situation where, at the KK-theory level, the simplicial approximation is equivalent to the data of G acting on the point, we can replace $P_{\mathbb{C}} \rtimes G$ with $C_r^*(G)$ and obtain Spanier–Whitehead duality for the group as in the next corollary. If the γ-element exists, we define the *strong* Baum–Connes conjecture to be the statement that $j_r^G(\gamma) = 1_{C_r^*(G)}$ in $\mathrm{KK}(C_r^*(G), C_r^*(G))$.

Corollary B. *Suppose the γ-element exists. Then G has Spanier–Whitehead duality if and only if it satisfies the strong Baum–Connes conjecture.*

In light of the result above, we can view the notion of Spanier–Whitehead K-duality for G as a homotopy-theoretic characterization of the strong Baum–Connes conjecture (cf. Remark 3.8).

The main application of the previous corollary is summarized in the result below.

Corollary C. *All a-T-menable groups which admit a G-compact model of $\underline{E}G$ have Spanier–Whitehead K-duality. Examples of a-T-menable groups are the following*:

- *all groups which act properly, affine-isometrically, and cocompactly on a finite-dimensional Euclidean space*,
- *all cocompact lattices of simple Lie groups* $\mathrm{SO}(n, 1)$ *or* $\mathrm{SU}(n, 1)$,
- *all groups which act cocompactly on a tree.*

Having such an explicit duality should be useful. For example, in principle, it allows us to compute the Lefschetz number of an automorphism of $C^*_r(G)$, or more generally of a morphism f in $\mathrm{KK}(C^*_r(G), C^*_r(G))$; see [Dell'Ambrogio et al. 2014; Emerson 2011].

If a cycle with property (γ) is found, then we can deduce the duality in complete analogy with the case of the γ-element (this is how the definition of property (γ) was designed). However, in this case we do not have information on the localization at the weakly contractible objects [Meyer and Nest 2010]. So we get the corresponding statement for Corollary B, but not for Theorem A.

Theorem D. *Suppose there is a (γ)-element $x \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ for G. If $j_r^G(x) = 1_{C^*_r(G)} \in \mathrm{KK}^G(C^*_r(G), C^*_r(G))$, then G has Spanier–Whitehead duality.*

1. General framework

Let G be a countable discrete group, and $\underline{E}G$ be a G-compact model of the universal proper G-space. Let A and B be C^*-algebras equipped with a G-action. If the G-action on B is trivial, we recall the dual Green–Julg isomorphism [Blackadar 1998; Kaad and Proietti 2018; Land 2015]

$$\mathrm{GJ} : \mathrm{KK}^G(A, B) \cong \mathrm{KK}(A \rtimes G, B).$$

Given $a \in A$, define $\delta^a_g \in C_c(G, A) \subseteq A \rtimes G$ to be the function

$$\delta^a_g(t) = \begin{cases} a & \text{if } t = g, \\ 0 & \text{if } t \neq g. \end{cases}$$

The *dual coaction* is defined as

$$\Delta : A \rtimes G \to C^*_r(G) \otimes A \rtimes G, \qquad \delta^a_g \mapsto g \otimes \delta^a_g.$$

Let $c \in C_c(\underline{E}G)$ be a cutoff function, and consider the associated projection $p_c \in C_c(G, C_0(\underline{E}G)) \subseteq C_0(\underline{E}G) \rtimes G$ defined by $p_c(g) = cg(c)$. This projection does not depend on c up to homotopy, hence we denote it p_G in the sequel.

Definition 1.1. We define the *canonical duality unit* to be the class

$$\delta = \delta_G = [\Delta(p_G)] \in \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G).$$

The notational dependence on G shall be dropped when clear from the context. In this paper, whenever we say that G has Spanier–Whitehead duality, we implicitly assume that the duality unit is given as above.

Let us recall the definition of Kasparov's descent homomorphism [1988], which plays an important role in this paper. It is denoted J^G below. Suppose (ϕ, H, T) is a Kasparov cycle defining an element of $\mathrm{KK}^G(A, B)$. The G-action on H is denoted $U : G \to \mathrm{End}_{\mathbb{C}}(H)$. The element $J^G([\phi, H, T]) \in \mathrm{KK}(A \rtimes G, B \rtimes G)$ is defined by the cycle $(\tilde{\phi}, H \rtimes G, \widetilde{T})$ given as follows.

The Hilbert C^*-module $H \rtimes G$ is the completion of $C_c(G, H)$ with respect to the norm induced by the $B \rtimes G$-valued inner product

$$\langle \xi | \zeta \rangle(t) = \sum_{g \in G} \beta_{g^{-1}}(\langle \xi(g) | \zeta(gt) \rangle),$$

where $\xi, \zeta \in C_c(G, H)$, $t \in G$, and β denotes the given G-action on B. The right action of $B \rtimes G$ is uniquely determined by the formula

$$(\xi \cdot f)(t) = \sum_{g \in G} \xi(g) \beta_g(f(g^{-1}t)),$$

where $\xi \in C_c(G, H)$, $f \in C_c(G, B)$, and $t \in G$. The representation of $A \rtimes G$ on $H \rtimes G$ is determined by

$$(\tilde{\phi}(f)(\xi))(t) = \sum_{g \in G} \phi(f(g))[U(g)(\xi(g^{-1}t))],$$

where $f \in C_c(G, A)$, $\xi \in C_c(G, H)$, and $t \in G$. Finally the operator $\widetilde{T}$ is defined by $(\widetilde{T}\xi)(t) = T(\xi(t))$ for $\xi \in C_c(G, H)$ and $t \in G$. By using reduced crossed products everywhere, we can similarly define a "reduced version" of the descent homomorphism, denoted J_r^G in the sequel.

Lemma 1.2 [Land 2015, Proposition 4.7]. *Kasparov's descent homomorphism can be factorized as follows*:

$$\begin{tikzcd}
\mathrm{KK}^G(A, \mathbb{C}) \arrow[r, "J^G"] \arrow[d, "\mathrm{GJ}"] & \mathrm{KK}(A \rtimes G, C^*(G)) \\
\mathrm{KK}(A \rtimes G, \mathbb{C}) \arrow[r, "\tau_{C^*(G)}"] & \mathrm{KK}(C^*(G) \otimes A \rtimes G, C^*(G)) \arrow[u, "\Delta^*"']
\end{tikzcd}$$

When the canonical map $A \rtimes G \to A \rtimes_r G$ is an isomorphism (e.g., if G acts properly on A), the version of the previous lemma with *reduced* crossed products also holds. See Remark 0.6.

Lemma 1.3 [Kaad and Proietti 2018, Section 2]. *Let A and B be G-C^*-algebras and suppose the G-action on B is trivial. Consider an element $x \in \mathrm{KK}^G(A, A)$. The following diagram commutes*:

$$\begin{array}{ccc} \mathrm{KK}^G(A, B) & \xrightarrow{\mathrm{GJ}} & \mathrm{KK}(A \rtimes G, B) \\ \big\downarrow{\scriptstyle x\widehat{\otimes}-} & & \big\downarrow{\scriptstyle j^G(x)\widehat{\otimes}-} \\ \mathrm{KK}^G(A, B) & \xrightarrow{\mathrm{GJ}} & \mathrm{KK}(A \rtimes G, B) \end{array}$$

It follows from Lemma 1.2 that we have the commutative diagram

$$\begin{array}{ccc} \mathrm{KK}^G(C_0(\underline{E}G), B) & \xrightarrow{\mu^G_B} & \mathrm{KK}(\mathbb{C}, B \otimes C^*_r(G)) \\ {\scriptstyle \mathrm{GJ}}\big\downarrow{\scriptstyle \cong} & & \big\downarrow{\scriptstyle =} \\ \mathrm{KK}(C_0(\underline{E}G) \rtimes G, B) & \xrightarrow{\delta\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}-} & \mathrm{KK}(\mathbb{C}, B \otimes C^*_r(G)) \end{array}$$

Since the definition of the duality counit requires additional information, and depends on the choice of "γ-like" element, the rest of this section gets split in two parts. The torsion-free case is treated in detail at the end of this section.

Argument based on the* (γ)*-element. Let (H, T) be a G-equivariant Kasparov cycle with property (γ). Let $x = [H, T]$ be the corresponding element in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$. Let

$$\tilde{x} = [H \otimes \ell^2(G), \rho \otimes \pi, (g(T))_{g\in G}] \in \mathrm{KK}^G(C^*_r(G) \otimes C_0(\underline{E}G), \mathbb{C}). \tag{1.4}$$

Here, $\pi : C_0(\underline{E}G) \to B(H)$ is the representation witnessing the conditions for property (γ) of (H, T), ρ stands for the right regular representation, and $C^*_r(G)$ has trivial G-action. By means of the Green–Julg isomorphism, we set

$$d = \mathrm{GJ}(\tilde{x}) \in \mathrm{KK}(C^*_r(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C}).$$

We set $\Lambda_{C^*_r(G)} = \delta \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)\rtimes G}\, d$ and $\Lambda_{C_0(\underline{E}G)\rtimes G} = \delta \mathbin{\widehat{\otimes}}_{C^*_r(G)}\, d$. We shall prove

(1) $\Lambda_{C^*_r(G)} = j^G_r(x)$ in $\mathrm{KK}(C^*_r(G), C^*_r(G))$;

(2) $\Lambda_{C_0(\underline{E}G)\rtimes G} = 1_{C_0(\underline{E}G)\rtimes G}$ in $\mathrm{KK}(C_0(\underline{E}G) \rtimes G, C_0(\underline{E}G) \rtimes G)$.

Proposition 1.5. *We have the equality $\Lambda_{C^*_r(G)} = j^G_r(x)$.*

Proof. We claim the Kasparov module

$$[p_G] \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)\rtimes G}\, j^G_r(\tilde{x}) \tag{1.6}$$

is equivalent to $j_r^G(x)$, i.e., there is an isomorphism of Hilbert C^*-modules intertwining the representations and the operators (up to a compact perturbation).

The class in (1.6) is represented by

$$(H \otimes \ell^2(G) \rtimes_r G, (\rho \otimes \pi \rtimes_r 1)(p_G \otimes -), (g(T))_{g\in G} \rtimes_r 1).$$

We have an isomorphism of $C_r^*(G)$-modules

$$H \rtimes_r G \cong (\rho \otimes \pi \rtimes_r 1)(p_G \otimes 1)(H \otimes \ell^2(G) \rtimes_r G) \tag{1.7}$$

given by the assignment

$$\xi \rtimes_r u_g \mapsto \sum_{h\in G} \pi(c)(h\cdot \xi) \otimes \delta_h \rtimes_r u_{hg},$$

where $\xi \in H$, $\delta_h \in \ell^2(G)$, and c is a cutoff function defining p_G. The inverse of the map above is given by the restriction of

$$(\xi)_{h\in G} \rtimes_r u_g \mapsto \sum_{h\in G} h^{-1}\cdot(\pi(c)\xi_h) \otimes \rtimes_r u_{h^{-1}g},$$

where $(\xi)_{h\in G} \in H \otimes \ell^2(G)$. Under the isomorphism in (1.7), the representation $(\rho\otimes\pi \rtimes_r 1)(p_G \otimes -)$ is identified with the left action of $C_r^*(G)$ on $H \rtimes_r G$, and the compressed operator

$$(\rho\otimes\pi\rtimes_r 1)(p_G\otimes 1)((g(T))_{g\in G}\rtimes_r 1)(\rho\otimes\pi\rtimes_r 1)(p_G\otimes 1)$$

is identified with $T' \rtimes_r 1$ on $H \rtimes_r G$, where we define

$$T' = \sum_{g\in G} (g\cdot c)T(g\cdot c).$$

Hence the claim follows by definition of property (γ).

By Lemma 1.2, we have

$$j_r^G(\tilde{x}) = \Delta \otimes_{C_0(\underline{E}G)\rtimes G} \mathrm{GJ}(\tilde{x}).$$

Thus, we have

$$\begin{aligned} j_r^G(x) &= [p_G]\,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, j_r^G(\tilde{x}) \\ &= [p_G]\,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, \Delta \otimes_{C_0(\underline{E}G)\rtimes G} \mathrm{GJ}(\tilde{x}) \\ &= \delta\,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d. \end{aligned}$$ □

Proposition 1.8. *We have the equality* $\Lambda_{C_0(\underline{E}G)\rtimes G} = 1_{C_0(\underline{E}G)\rtimes G}$.

In order to prove the proposition, a few preliminaries are in order. First we generalize the construction in (1.4) to include a coefficient algebra. This is easily done: simply replace $\ell^2(G)$ with the right Hilbert A-module $\ell^2(G, A)$ and define the right regular representation ρ_A^G of $A \rtimes_r G$ (equipped with trivial G-action)

$$a \mapsto (g(a))_{g\in G}, \qquad h \mapsto \rho_h : (a_g)_{g\in G} \mapsto (a_{gh})_{g\in G}$$

for $a \in A, h \in G$. Thus we get a class $\tilde{x}_A$ in $\mathrm{KK}^G(A \rtimes_r G \otimes C_0(\underline{E}G), A)$. We define a group homomorphism

$$\nu_A^G : \mathrm{KK}(\mathbb{C}, A \rtimes_r G) \to \mathrm{KK}^G(C_0(\underline{E}G), A)$$

as the one induced by the class $\tilde{x}_A$ via the index pairing

$$\mathrm{KK}(\mathbb{C}, A \rtimes_r G) \times \mathrm{KK}^G(A \rtimes_r G \otimes C_0(\underline{E}G), A) \to \mathrm{KK}^G(C_0(\underline{E}G), A).$$

This map is referred to as the (γ)-morphism in [Nishikawa 2019]. Note also that $\mathrm{GJ} \circ \nu_{\mathbb{C}}^G$ equals the map d_j from Definition 0.1 (choosing $B = C_0(\underline{E}G) \rtimes G$ as usual). The lemma below is about the naturality property of the assembly map and the (γ)-morphism.

Lemma 1.9. *The following diagrams commute for any $f \in \mathrm{KK}^G(A, B)$:*

$$\begin{tikzcd}
\mathrm{KK}^G(C_0(\underline{E}G), A) \arrow[r, "\mu_A^G"] \arrow[d, "-\widehat{\otimes} f"] & \mathrm{KK}(\mathbb{C}, A \rtimes_r G) \arrow[d, "-\widehat{\otimes} J_r^G(f)"] \\
\mathrm{KK}^G(C_0(\underline{E}G), B) \arrow[r, "\mu_A^G"] & \mathrm{KK}(\mathbb{C}, B \rtimes_r G)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{KK}(\mathbb{C}, A \rtimes_r G) \arrow[r, "\nu_A^G"] \arrow[d, "-\widehat{\otimes} J_r^G(f)"] & \mathrm{KK}^G(C_0(\underline{E}G), A) \arrow[d, "-\widehat{\otimes} f"] \\
\mathrm{KK}(\mathbb{C}, B \rtimes_r G) \arrow[r, "\nu_A^G"] & \mathrm{KK}^G(C_0(\underline{E}G), B)
\end{tikzcd}$$

Proof. The first diagram commutes by functoriality of descent and associativity of the Kasparov product. By results from [Meyer 2000] any morphism f in $\mathrm{KK}^G(A, B)$ can be written as a composition of $*$-homomorphisms and their inverses in KK. This means it suffices to check the commutativity of the second diagram with respect to $*$-homomorphisms $f : A \to B$. We omit this simple verification. $\square$

Proof of Proposition 1.8. Let $B = C_0(\underline{E}G) \rtimes G$ and regard it as a G-C^*-algebra with the trivial G-action. We have the following diagram:

$$\begin{tikzcd}
\mathrm{KK}^G(C_0(\underline{E}G), B) \arrow[r, "\mu_B^G"] \arrow[d, "\mathrm{GJ}"', "\cong"] & \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes B) \arrow[r, "\nu_B^G"] \arrow[d, "="] & \mathrm{KK}^G(C_0(\underline{E}G), B) \arrow[d, "\mathrm{GJ}"', "\cong"] \\
\mathrm{KK}(B, B) \arrow[r, "\delta \widehat{\otimes}_B -"] & \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes B) \arrow[r, "- \widehat{\otimes}_{C_r^*(G)} d"] & \mathrm{KK}(B, B)
\end{tikzcd}$$

If we prove that the composition on the top is the identity, then it follows that $\Lambda_{C_0(\underline{E}G)\rtimes G} = 1_{C_0(\underline{E}G)\rtimes G}$. Let $D_B : P_B \to B$ be a weak equivalence as in [Meyer and Nest 2006]. Because the diagram

$$\begin{tikzcd}
\mathrm{KK}^G(C_0(\underline{E}G), P_B) \arrow[r, "\mu^G_{P_B}"] \arrow[d, "D_{B*}"] & \mathrm{KK}(\mathbb{C}, P_B \rtimes G) \arrow[r, "\nu^G_{P_B}"] \arrow[d, "J^G_r(D_{B*})"] & \mathrm{KK}^G(C_0(\underline{E}G), P_B) \arrow[d, "D_{B*}"] \\
\mathrm{KK}^G(C_0(\underline{E}G), B) \arrow[r, "\mu^G_B"] & \mathrm{KK}(\mathbb{C}, B \rtimes_r G) \arrow[r, "\nu^G_B"] & \mathrm{KK}^G(C_0(\underline{E}G), B)
\end{tikzcd}$$

commutes, it suffices to show that $\nu^G_{P_B}$ is a left inverse of the assembly map $\mu^G_{P_B}$. Now, $\mu^G_{P_B}$ is invertible, hence it suffices to show that $\nu^G_{P_B}$ yields a right inverse. A minor generalization of the proof of Proposition 1.5 shows that $\mu^G_{P_B} \circ \nu^G_{P_B}$ coincides with the induced action of $x \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ on $K_*(P_B \rtimes G)$. Recall that x equals the identity when restricted to each finite subgroup $H \subseteq G$, and $P_B \rtimes G$ belongs to the localizing subcategory of KK generated by the $B \rtimes H$'s. Therefore the map $x \mathbin{\widehat{\otimes}} - : K_*(P_B \rtimes G) \to K_*(P_B \rtimes G)$ is the identity by [Meyer and Nest 2006, Theorem 9.3]. □

Remark 1.10. In parallel with Proposition 1.5, one can prove that

$$\Lambda_{C_0(\underline{E}G)\rtimes G} = J^G_r(x \mathbin{\widehat{\otimes}} 1_{C_0(\underline{E}G)}).$$

Again, we set $B = C_0(\underline{E}G) \rtimes G$ and first notice that $\nu^G_B \circ \mu^G_B = x \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)} -$. It is enough to show this when B is replaced by P_B, in which case we can invert the assembly map and write

$$\begin{aligned}
(J^G_r(x) \mathbin{\widehat{\otimes}} -) &= (J^G_r(x) \mathbin{\widehat{\otimes}} -) \circ \mu^G_B \circ (\mu^G_B)^{-1}, \\
\mu^G_B \circ \nu^G_B &= \mu^G_B \circ (x \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)} -) \circ (\mu^G_B)^{-1}, \\
\nu^G_B \circ \mu^G_B &= x \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)} -.
\end{aligned}$$

To complete the proof one must show that

$$\mathrm{GJ}(x \mathbin{\widehat{\otimes}}_{\mathbb{C}} \mathrm{GJ}^{-1}(1_B)) = J^G_r(x \mathbin{\widehat{\otimes}} 1_{C_0(\underline{E}G)}) \mathbin{\widehat{\otimes}}_B 1_B,$$

but this follows from Lemma 1.3.

We now come to the main result of this subsection.

Theorem 1.11. *Suppose there is a (γ)-element $x \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ for G. If*

$$J^G_r(x) = 1 \in \mathrm{KK}^G(C^*_r(G), C^*_r(G)),$$

then G has Spanier–Whitehead duality.

Argument based on the γ-element. Suppose there is a gamma element γ as in Definition 0.10. Following [Guentner et al. 2000, Chapter 15], we define a map s_A for any proper algebra A. This is the G-equivariant $*$-homomorphism

$$s_A : A \rtimes_r G \otimes C_0(\underline{E}G) \to K(A \otimes \ell^2(G)),$$

where $A \rtimes_r G$ is equipped with the trivial G-action, defined as the tensor product of the representation

$$C_0(\underline{E}G) \ni \phi \mapsto (\phi)_{g \in G} \in L(A \otimes \ell^2(G))$$

of $C_0(\underline{E}G)$ on $A \otimes \ell^2(G)$ and the right regular representation

$$A \ni a \mapsto (g(a))_{g \in G} \in L(A \otimes \ell^2(G)), \qquad G \ni g \mapsto 1 \otimes \rho_g$$

of $A \rtimes_r G$ on $A \otimes \ell^2(G)$, where ρ_g is the right translation by g. Here, the G-action on the Hilbert module $A \otimes \ell^2(G)$ is given by the tensor product of the action on A and the left-regular representation. The $*$-homomorphism s_A defines an element

$$s_A \in \mathrm{KK}^G(A \rtimes_r G \otimes C_0(\underline{E}G), A).$$

Proposition 1.12 (see [Guentner et al. 2000, Chapter 15]). *For any proper G-C^*-algebra A, the $*$-homomorphism s_A defines the inverse*

$$s_A : \mathrm{KK}(\mathbb{C}, A \rtimes_r G) \to \mathrm{KK}^G(C_0(\underline{E}G), A)$$

of the assembly map

$$\mu_A^G : \mathrm{KK}^G(C_0(\underline{E}G), A) \to \mathrm{KK}(\mathbb{C}, A \rtimes_r G).$$

Proof. The assembly map is an isomorphism for any proper algebra. Hence, we just show that the composition $\mu_A^G \circ s_A$ is the identity. Take a Kasparov cycle (E, F) for $\mathrm{KK}(\mathbb{C}, A \rtimes_r G)$ where E is a graded $A \rtimes_r G$-module and F is an odd, self-adjoint operator on E satisfying $1 - F^2 \equiv 0$ modulo compact operators.

By Kasparov's stabilization theorem, we can assume E is $A \otimes H \rtimes_r G$ for some graded Hilbert space H with the trivial G-action. The map s_A sends this cycle $(A \otimes H \rtimes_r G, F)$ to the G-equivariant cycle $(A \otimes H \otimes \ell^2(G), \pi, \rho(F))$ for $\mathrm{KK}^G(C_0(\underline{E}G), A)$, where π is a representation of $C_0(\underline{E}G)$ on $A \otimes H \otimes \ell^2(G)$ defined as follows: for ϕ in $C_0(\underline{E}G)$,

$$\pi(\phi)(a_g \otimes v_g \otimes \delta_g) = \phi a_g \otimes v_g \otimes \delta_g$$

and $\rho(F)$ is an operator in $L(A \otimes H \otimes \ell^2(G))$ determined by the map

$$\begin{aligned} L(A \otimes H \rtimes_r G) &= M(A \otimes K(H) \rtimes_r G) \\ &\xrightarrow{\rho} M(A \otimes K(H \otimes \ell^2(G))) = L(A \otimes H \otimes \ell^2(G)), \end{aligned}$$

which is a natural extension of the right regular representation ρ_A^G of $A \rtimes_r G$ on $A \otimes \ell^2(G)$ described before. Hence, the composition $\mu_A^G \circ s_A$ sends the cycle $(A \otimes H \rtimes_r G, F)$ to the cycle $(p_c(A \otimes H \otimes \ell^2(G) \rtimes_r G), p_c\rho(F) \rtimes_r 1 p_c)$, where we simply denote by p_c the image of a cutoff projection p_c in $C_0(\underline{E}G) \rtimes_G$ by the representation $\pi \rtimes_r 1$.

On the other hand, there is an isomorphism of right Hilbert $A \rtimes_r G$-modules

$$A \otimes H \rtimes_r G \to p_c(A \otimes H \otimes \ell^2(G) \rtimes_r G)$$

given by

$$\xi \rtimes_r u_g \mapsto \sum_{h \in G} c(h(\xi)) \otimes \delta_h \rtimes_r u_{hg} \quad \text{for } \xi \text{ in } A \otimes H.$$

The inverse map is given by

$$(\xi_h)_{h\in G} \rtimes_r u_g \mapsto \sum_{h \in G} h^{-1}(c\xi_h) \rtimes_r u_{h^{-1}g} \quad \text{for } (\xi_h)_{h\in G} \text{ in } A \otimes H \otimes \ell^2(G).$$

Under this isomorphism, the restriction $p_c\rho(F) \rtimes_r 1 p_c$ of $\rho(F) \rtimes_r 1$ on the subspace $p_c(A \otimes H \otimes \ell^2(G) \rtimes_r G)$ of $A \otimes H \otimes \ell^2(G) \rtimes_r G$ corresponds to the operator F on $A \otimes H \rtimes_r G$. In summary, the composition $\mu_A^G \circ s_A$ sends the cycle $(A \otimes H \rtimes_r G, F)$ to itself up to the isomorphism described above. □

For any separable G-C^*-algebra B, we have the commutative diagram

$$\begin{array}{ccc}
\mu_B^G : \mathrm{KK}^G(C_0(EG), B) & \longrightarrow & \mathrm{KK}(\mathbb{C}, B \rtimes_r G) \\
\Big\downarrow {\scriptstyle -\widehat{\otimes}_{\mathbb{C}}\beta} & & \Big\downarrow {\scriptstyle -\widehat{\otimes}_{B \rtimes_r G} J_r^G(\mathrm{id}_B \widehat{\otimes} \beta)} \\
\mu_{B\otimes P}^G : \mathrm{KK}^G(C_0(EG), B \otimes P) & \xrightarrow{\cong} & \mathrm{KK}(\mathbb{C}, (B \otimes P) \rtimes_r G) \\
\Big\downarrow {\scriptstyle -\widehat{\otimes}_P \alpha} & & \Big\downarrow {\scriptstyle -\widehat{\otimes}_{(B\otimes P) \rtimes_r G} J_r^G(\mathrm{id}_B \widehat{\otimes} \alpha)} \\
\mu_B^G : \mathrm{KK}^G(C_0(EG), B) & \longrightarrow & \mathrm{KK}(\mathbb{C}, B \rtimes_r G)
\end{array}$$

where the vertical composition on the left is the identity. With this observation and Proposition 1.12, we see that the element

$$(J_r^G(1_B \widehat{\otimes} \beta)) \widehat{\otimes}_{(B \widehat{\otimes} P)\rtimes_r G} s_{B\otimes P} \widehat{\otimes}_P \alpha \in \mathrm{KK}^G((B \rtimes_r G) \otimes C_0(\underline{E}G), B)$$

induces the left-inverse of the assembly map μ_B^G via Kasparov product. We remark that this is the standard technique for proving the split injectivity of the assembly map in the presence of a γ-element.

Now, we set d' to be the element in $\mathrm{KK}(C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C})$ which corresponds to

$$d'' = (J_r^G(\beta)) \widehat{\otimes}_{P \rtimes_r G} s_P \widehat{\otimes}_P \alpha \in \mathrm{KK}^G(C_r^*(G) \otimes C_0(\underline{E}G), \mathbb{C}).$$

Let
$$\delta = \delta_G \in \mathrm{KK}(\mathbb{C}, C_0(\underline{E}G) \rtimes G \otimes C_r^*(G))$$
as before. We set $\Lambda'_{C_r^*(G)} = \delta \,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d'$, $\Lambda'_{C_0(\underline{E}G)\rtimes G} = \delta \,\widehat{\otimes}_{C_r^*(G)}\, d'$.

Proposition 1.13. *We have*
$$\Lambda'_{C_r^*(G)} = j_r^G(\gamma) \in \mathrm{KK}(C_r^*(G), C_r^*(G))$$
and
$$\Lambda'_{C_0(\underline{E}G)\rtimes G} = 1_{C_0(\underline{E}G)\rtimes G} \in \mathrm{KK}(C_0(\underline{E}G) \rtimes G, C_0(\underline{E}G) \rtimes G).$$

Before giving a proof of Proposition 1.13, let us obtain our main results as its direct consequences:

Theorem 1.14. *If $j_r^G(\gamma) = 1_{C_r^*(G)}$, then G has Spanier–Whitehead duality.*

The previous result has a converse; see Theorem 3.3 for further details.

Theorem 1.15. *If $\mu_{\mathbb{C}}^G$ is an isomorphism, G has weak Spanier–Whitehead duality.*

Theorem 1.16. *In general, if $\gamma \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ exists, then $C_0(\underline{E}G) \rtimes G$ is a Spanier–Whitehead K-dual of $P_{\mathbb{C}} \rtimes G$.*

Proof. Note that $P_{\mathbb{C}} \rtimes G$ is a direct summand (in the category KK) of $C_r^*(G)$ corresponding to the idempotent $j_r^G(\gamma) \in \mathrm{KK}(C_r^*(G), C_r^*(G))$ (see [Neeman 2001, Proposition 1.6.8]). Namely, we have
$$i_{P_{\mathbb{C}}\rtimes G} \in \mathrm{KK}(P_{\mathbb{C}} \rtimes G, C_r^*(G)), \qquad q_{P_{\mathbb{C}}\rtimes G} \in \mathrm{KK}(C_r^*(G), P_{\mathbb{C}} \rtimes G),$$
so that $q_{P_{\mathbb{C}}\rtimes G} \circ i_{P_{\mathbb{C}}\rtimes G} = 1_{P_{\mathbb{C}}\rtimes G}$ and $i_{P_{\mathbb{C}}\rtimes G} \circ q_{P_{\mathbb{C}}\rtimes G} = j_r^G(\gamma)$. We set
$$\begin{aligned} d_{P_{\mathbb{C}}\rtimes G} &= i_{P_{\mathbb{C}}\rtimes G} \,\widehat{\otimes}_{C_r^*(G)}\, d' \in \mathrm{KK}(C_0(\underline{E}G) \rtimes G \otimes P_{\mathbb{C}} \rtimes G, \mathbb{C}), \\ \delta_{P_{\mathbb{C}}\rtimes G} &= \delta \,\widehat{\otimes}_{C_r^*(G)}\, q_{P_{\mathbb{C}}\rtimes G} \in \mathrm{KK}(\mathbb{C}, C_0(\underline{E}G) \rtimes G \otimes P_{\mathbb{C}} \rtimes G). \end{aligned}$$
Then we have
$$\delta_{P_{\mathbb{C}}\rtimes G} \,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d_{P_{\mathbb{C}}\rtimes G} = 1_{P_{\mathbb{C}}\rtimes G}, \qquad \delta_{P_{\mathbb{C}}\rtimes G} \,\widehat{\otimes}_{P_{\mathbb{C}}\rtimes G}\, d_{P_{\mathbb{C}}\rtimes G} = 1_{C_0(\underline{E}G)\rtimes G}.$$
This proves the statement. We only prove the first identity; the other one is proved similarly. For any C^*-algebra D, we have the following commutative diagram:
$$\begin{tikzcd} \mathrm{KK}(P_{\mathbb{C}} \rtimes G, D) \arrow[r] \arrow[d, "\delta_{P_{\mathbb{C}}\rtimes G}\widehat{\otimes}_{P_{\mathbb{C}}\rtimes G} -"'] & \mathrm{KK}(C_r^*(G), D) \arrow[d, "\delta\widehat{\otimes}_{C_r^*(G)} -"'] \\ \mathrm{KK}(\mathbb{C}, C_0(\underline{E}G) \rtimes G \otimes D) \arrow[r, "="] \arrow[d, "-\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d_{P_{\mathbb{C}}\rtimes G}"] & \mathrm{KK}(\mathbb{C}, C_0(\underline{E}G) \rtimes_r G \otimes D) \arrow[d, "-\widehat{\otimes}_{C_0(\underline{E}G)\rtimes_r G}\, d'"] \\ \mathrm{KK}(P_{\mathbb{C}} \rtimes G, D) \arrow[r] & \mathrm{KK}(C_r^*(G), D) \end{tikzcd}$$

Here, the top and the bottom horizontal arrows are induced by $i_{P_{\mathbb{C}}\rtimes G}$ and $q_{P_{\mathbb{C}}\rtimes G}$. The right vertical composition is induced by $j_r^G(\gamma)$. It follows that the left vertical composition is the identity. Taking $D = P_{\mathbb{C}} \rtimes G$, we get

$$\delta_{P_{\mathbb{C}}\rtimes G} \widehat{\otimes}_{C_0(\underline{E}G)\rtimes G} d_{P_{\mathbb{C}}\rtimes G} = 1_{P_{\mathbb{C}}\rtimes G}. \qquad \square$$

Proof of Proposition 1.13. We directly compute and prove

$$\delta \widehat{\otimes}_{C_0(\underline{E}G)\rtimes G} d' = j_r^G(\gamma) \in \mathrm{KK}(C_r^*(G), C_r^*(G)).$$

For simplicity, we prove this for the case when β is represented by a cycle (P, b) for b an essential unitary in $M(P)$, and α by a cycle (H, F), where P is represented on H nondegenerately and F is a G-equivariant essential unitary modulo P. Then d'' is represented by a cycle of the form

$$(H \otimes \ell^2(G), \rho \otimes \pi, N(g(b))_{g\in G} + M(g(F))_{g\in G}),$$

where the G-action on $H \otimes \ell^2(G)$ is the tensor product of the G-action on H and the left regular representation on $\ell^2(G)$, π is a representation of $C_0(\underline{E}G)$ on $H \otimes \ell^2(G)$ given by $\phi \mapsto (\phi)_{g\in G}$, and ρ is a representation of $C_r^*(G)$ on $H \otimes \ell^2(G)$ by the right regular representation $g \mapsto 1 \otimes \rho_g$. Here, M and N are given by the Kasparov technical theorem as usual [Higson 1987; Kasparov 1980; 1995]. If we compute $\delta \otimes_{C_0(\underline{E}G)\rtimes G} d'$, we get the cycle isomorphic to

$$(H \rtimes_r G, \pi_G, T_0 \rtimes_r 1) = j_r^G((H, T_0))$$

where (H, T_0) is a cycle for $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$, π_G is the natural left multiplication by $C_r^*(G)$, and $T_0 = N_0 b + M_0 F_0$. Here F_0 is the average of F: $F_0 = \int_G g(c) F g(c)\, d\mu_G$ and so are N_0 and M_0. The cycle (H, T_0) is (homotopic to) a Kasparov product of α and β. In other words, the element $[H, T_0]$ is the gamma element γ. It follows that

$$\delta \widehat{\otimes}_{C_0(\underline{E}G)\rtimes G} d' = j_r^G(\gamma).$$

Now we can prove

$$\delta \widehat{\otimes}_{C_r^*(G)} d' = 1_{C_0(\underline{E}G)\rtimes G} \in \mathrm{KK}(C_0(\underline{E}G) \rtimes G, C_0(\underline{E}G) \rtimes G)$$

using a simple trick. We have the following diagram for $B = C_0(\underline{E}G) \rtimes G$ with the trivial G action:

$$\begin{tikzcd}
\mathrm{KK}^G(C_0(\underline{E}G), B) \arrow[r, "\mu_B^G"] \arrow[d, "\cong"] & \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes B) \arrow[r, "(\mu_B^G)^{-1}"] \arrow[d, "="] & \mathrm{KK}^G(C_0(\underline{E}G), B) \arrow[d, "\cong"] \\
\mathrm{KK}(B, B) \arrow[r, "\delta\widehat{\otimes}_B -"] & \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes B) \arrow[r, "-\widehat{\otimes}_{C_r^*(G)} d"] & \mathrm{KK}(B, B)
\end{tikzcd}$$

Here, by $(\mu_B^G)^{-1}$ we simply mean the left inverse of μ_B^G. This shows $\delta \otimes_{C_r^*(G)} d'$ acts as the identity on $\mathrm{KK}(B, B)$, proving the claim. $\square$

Remark 1.17. The previous proof also shows that $d = d'$, as it is intuitive from the fact that the γ-element can be represented by a cycle satisfying property (γ) [Nishikawa 2019].

Remark 1.18. It is natural to use the duality class Θ from page 472 to prove Theorem 1.16. The argument is based on the following diagram, where we set $d' = \mathrm{GJ}(s_P \mathbin{\widehat{\otimes}_P} \alpha)$, and $\mu^G_{P\rtimes G,P}$ is a bivariant assembly map (see Section 3):

$$\begin{tikzcd}
\mathrm{KK}(\mathbb{C}, P\rtimes G\otimes C_0(\underline{E}G)\rtimes G) \arrow[r, "-\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G} d'"] & \mathrm{KK}(P\rtimes G, P\rtimes G) \\
\mathrm{KK}^G(C_0(\underline{E}G), P\otimes C_0(\underline{E}G)\rtimes G) \arrow[r, "-\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G} d'"] \arrow[u, "\mu^G_{P\otimes C_0(\underline{E}G)\rtimes G}", "\cong"'] \arrow[d, "p^*_{\underline{E}G}"', "\cong"] & \mathrm{KK}^G(C_0(\underline{E}G)\otimes P\rtimes G, P) \arrow[u, "\mu^G_{P\rtimes G,P}"'] \arrow[d, "\cong"', "p^*_{\underline{E}G}"] \\
\mathrm{RKK}^G(\underline{E}G; C_0(\underline{E}G), P\otimes C_0(\underline{E}G)\rtimes G) \arrow[r, "-\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G} d'"] & \mathrm{RKK}^G(\underline{E}G; C_0(\underline{E}G)\otimes P\rtimes G, P)
\end{tikzcd}$$

Set $e = \mathrm{GJ}^{-1}(1_{C_0(\underline{E}G)\rtimes G})$ and consider the element $\delta_0 = \Theta \mathbin{\widehat{\otimes}_{C_0(\underline{E}G)}} e$ in the bottom left group. Suppressing $p^*_{\underline{E}G}$ from the notation, we compute

$$\begin{aligned}\delta_0 \mathbin{\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}} d' &= \Theta \mathbin{\widehat{\otimes}_{C_0(\underline{E}G)}} e \mathbin{\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}} \mathrm{GJ}(s_P \mathbin{\widehat{\otimes}_P} \alpha) \\ &= (\Theta \mathbin{\widehat{\otimes}_P} \alpha) \mathbin{\widehat{\otimes}_{\mathbb{C}}} (e \mathbin{\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}} \mathrm{GJ}(s_P)) = s_P.\end{aligned}$$

Now it is routine to check that $\mu^G_{P\rtimes G,P}(s_P) = 1_{P\rtimes G}$. Hence, if we define

$$\delta_{P\rtimes G} \in \mathrm{KK}(\mathbb{C}, P \rtimes G \otimes C_0(\underline{E}G) \rtimes G)$$

by sending δ_0 through the left vertical isomorphism in the diagram above, we have

$$\delta_{P\rtimes G} \mathbin{\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}} d' = 1_{P\rtimes G}.$$

The other identity is similarly proved; we skip the details.

Note that this is an improvement over Theorem 1.16, because the existence of Θ is strictly weaker than having a gamma element. A similar argument shows that in general, if $P_{\mathbb{C}}$ is a (categorical) direct summand of some proper algebra, the conclusion of Theorem 1.16 holds, namely $C_0(\underline{E}G) \rtimes G$ is a Spanier–Whitehead K-dual of $P_{\mathbb{C}} \rtimes G$.

The torsion-free case. We treat the torsion-free case separately, partly because it is particularly simple (e.g., condition (1) of Definition 0.10 reduces to a statement in nonequivariant K-theory), and partly because it is among the first cases where the duality classes (i.e., unit and counit) have been identified (albeit in a slightly different language, cf. [Kasparov 1988, Theorems 6.6 and 6.7]).

We assume that G is a countable, discrete, torsion-free group. In this case, because proper actions are automatically free, the space $\underline{E}G$ is identified as the total space EG of the classifying space for principal G-bundles, and our assumption that G admits a G-compact model of $\underline{E}G$ translates into the assumption that G admits a compact model of BG. Denote by [MF] the class

$$[\mathrm{MF}] \in \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes C(BG))$$

associated to the module of sections of the Miščenko bundle. This is the Hermitian bundle of C^*-algebras obtained from the associated bundle construction

$$EG \times_G C_r^*(G) \to BG,$$

where G acts diagonally, acting on the reduced group C^*-algebra via the left regular representation [Miščenko and Fomenko 1979].

Proposition 1.19 ([Connes 1994]; for a proof see [Kaad and Proietti 2018]). *The Miščenko module* MF *is the finitely generated projective Hilbert C^*-module described as the completion of $C_c(EG)$ with respect to the norm induced by the $C_r^*(G) \mathbin{\widehat{\otimes}} C(BG)$-valued inner product*

$$\langle \xi | \zeta \rangle(t)(x) = \sum_{p(y)=x} \bar{\xi}(y)\zeta(y \cdot t), \tag{1.20}$$

where $\xi, \zeta \in C_c(EG)$, $t \in G$, $x \in BG$, and $p : EG \to BG$ is the quotient map. The right action of $C_r^*(G) \mathbin{\widehat{\otimes}} C(BG)$ on M is defined by

$$(\xi \cdot f)(y) = \sum_{g \in G} f(g)(p(y)) \cdot \xi(y \cdot g^{-1}), \tag{1.21}$$

where $\xi \in C_c(EG)$, $f \in C_c(G, C(BG))$, and $y \in EG$.

We have, for any separable C^*-algebra B with trivial G-action [Land 2015; Kaad and Proietti 2018],

$$\begin{tikzcd}
\mathrm{KK}(C(BG), B) \arrow[r, "{[\mathrm{MF}]\widehat{\otimes}_{C(BG)} -}"] \arrow[d, "\cong"] & \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes B) \arrow[d, "="] \\
\mathrm{KK}^G(C_0(EG), B) \arrow[r, "\mu_B^G"] & \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes B)
\end{tikzcd}$$

The vertical isomorphism above is implemented by the strong Morita equivalence between $C(BG)$ and $C_0(EG) \rtimes G$ [Rieffel 1976], whose associated KK-class is denoted $[Y^*]$ below (we use $[Y]$ for the opposite module).

If G admits a compact nonpositively curved manifold as a model for BG, then the element $\underline{d}$ was introduced by Kasparov [1988] as a "dual-Dirac" class

$$\underline{d} \in \mathrm{KK}(C_r^*(G) \mathbin{\widehat{\otimes}} C(BG), \mathbb{C}).$$

To be more consistent with the terminology of this paper, $\underline{d}$ should be called the duality counit induced by the γ-element (which exists in this situation). Kasparov went on to show that $\underline{d}$ defines a left inverse for the assembly map.

Hence we see that we are in a situation where Spanier–Whitehead duality comes into play very naturally, with the choice $\mathrm{MF} = \text{unit}$ and $\underline{d} = \text{counit}$. Note that, while the class $\underline{d}$ requires structural information on the group, the class of the Miščenko bundle relies on very little structure. This is in complete analogy with the canonical unit defined previously.

Proposition 1.22. *The class* MF *coincides with δ_G from Definition 1.1 up to* KK-*equivalence. More precisely, we have*

$$\delta_G = \mathrm{MF}\,\widehat{\otimes}_{C_r^*(G)\widehat{\otimes}C(BG)}\,\tau_{C_r^*(G)}([Y^*]).$$

Proof. Let us set $Z = \mathrm{GJ}^{-1}([Y]) \in \mathrm{KK}^G(C_0(EG), C(BG))$. It is shown in [Kaad and Proietti 2018] that Z is represented by a G-C^*-correspondence satisfying the isomorphism of Hilbert modules

$$\mathrm{MF} \cong i^*(Y^*)\,\widehat{\otimes}_{C_0(EG)\rtimes G}\,(Z \rtimes_r G)$$

(we are denoting by i the inclusion $\mathbb{C} \hookrightarrow C(BG)$ as constant functions). We want to prove

$$[p_G]\,\widehat{\otimes}_{C_0(EG)\rtimes G}\,[\Delta] = i^*([Y^*])\,\widehat{\otimes}_{C_0(EG)\rtimes G}\,J_r^G(Z)\,\widehat{\otimes}_{C_r^*(G)\widehat{\otimes}C(BG)}\,\tau_{C_r^*(G)}([Y^*]),$$

or equivalently, by Lemma 1.2,

$$\begin{aligned}[p_G]\,&\widehat{\otimes}_{C_0(EG)\rtimes G}\,[\Delta] \\ &= i^*([Y^*])\,\widehat{\otimes}_{C_0(EG)\rtimes G}\,\big([\Delta]\,\widehat{\otimes}\,\tau_{C_r^*(G)}(\mathrm{GJ}(Z))\big)\,\widehat{\otimes}_{C_r^*(G)\widehat{\otimes}C(BG)}\,\tau_{C_r^*(G)}([Y^*]).\end{aligned}$$

It is well-known that $[p_G] = i^*([Y^*])$ (see for example [Land 2015]), so that by associativity of the Kasparov product we have reduced the problem to showing

$$\tau_{C_r^*(G)}(\mathrm{GJ}(Z))\,\widehat{\otimes}\,\tau_{C_r^*(G)}([Y^*]) = \tau_{C_r^*(G)}(\mathrm{GJ}(Z)\,\widehat{\otimes}_{C(BG)}\,[Y^*]) = 1_{C_r^*(G)\widehat{\otimes}C_0(EG)\rtimes G}.$$

Now $\mathrm{GJ}(Z) = [Y]$ by construction, and hence the proof is complete. $\square$

Now suppose that G is a general torsion-free group, and that a (γ)-element $x = [H, T]$ exists. Inspired by Kasparov's construction, we define the class $\underline{d}$ in $\mathrm{KK}(C_r^*(G) \otimes C(BG), \mathbb{C})$ by setting

$$\underline{d} = [Y]\,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\,d.$$

The element $\underline{d}$ admits a simple description in terms of the cycle (H, T) with property (γ) as follows. The G-equivariant nondegenerate representation π of $C_0(\underline{E}G)$ on H extends to that of the multiplier algebra $C_b(\underline{E}G)$. Together with the representation π_G of G on H, it induces the representation $\pi_G \otimes \pi$ of $C_r^*(G) \otimes C(BG)$ on

H. Here, $C(BG)$ is naturally identified as the subalgebra $C_b(\underline{E}G)$ consisting of G-invariant functions. The representation π_G extends to the representation for $C_r^*(G)$ since π_G is weakly contained in the left regular representation. Indeed, π_G is contained in the (amplified) left regular representation as we have a G-equivariant embedding from H to $H \otimes \ell^2(G)$ given by

$$v \mapsto \sum_h \pi(h(c))v \otimes \delta_h.$$

Proposition 1.23. *The triple $(H, \pi_G \otimes \pi, T)$ defines a Kasparov cycle $[\pi_G \otimes \pi, H, T]$ for $\mathrm{KK}(C_r^*(G) \otimes C(BG), \mathbb{C})$. We have $[\pi_G \otimes \pi, H, T] = \underline{d}$.*

Proof. We need to show that for any G-invariant continuous function ϕ on $\underline{E}G$, the commutator $[T, \phi]$ is compact. By condition (2b) for property (γ), we just need to show that $[T', \phi]$ is compact, where $T' = \sum_{g \in G} g(c)Tg(c)$; c is a cutoff function on $\underline{E}G$. Take any compactly supported function χ on $\underline{E}G$ so that $c\chi = c$.

We have

$$[T', \phi] = \sum_{g \in G} g(c)[T, g(\chi\phi)]g(c) = \sum_{g \in G} g(c)T_g g(c),$$

where $T_g = [T, g(\chi\phi)]$ are compact operators whose norms vanish as g goes to infinity by condition (2a) for property (γ). It follows that $[T', \phi] = \sum_{g \in G} g(c)T_g g(c)$ is compact (see [Nishikawa 2019, Lemmas 2.5 and 2.6]).

We leave to the reader the straightforward check that the element $[H, \pi_G \otimes \pi, T]$ in $\mathrm{KK}(C_r^*(G) \otimes C(BG), \mathbb{C})$ corresponds to d in $\mathrm{KK}(C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C})$ by the Morita equivalence between $C(BG)$ and $C_0(EG) \rtimes G$. □

We set

$$\underline{\Delta}_{C_r^*(G)} = [\mathrm{MF}] \mathbin{\widehat{\otimes}}_{C(BG)} \underline{d}, \qquad \underline{\Delta}_{C(BG)} = [\mathrm{MF}] \mathbin{\widehat{\otimes}}_{C_r^*(G)} \underline{d}.$$

The following conclusions are immediate from the discussion above.

Theorem 1.24. *Let G be a torsion-free group and suppose that a (γ)-element $x \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ exists. We have*

$$\underline{\Delta}_{C_r^*(G)} = j_r^G(x), \qquad \underline{\Delta}_{C(BG)} = 1_{C(BG)}.$$

For example, this is the case when BG is a compact smooth manifold of nonpositive sectional curvature.

2. Examples

In this section we give a few examples and computations to put into context the abstract duality results that have been explained previously. We primarily treat the case of strong Spanier–Whitehead duality, and only briefly discuss the weak case, as it is mostly covered by other results in the literature (see, for example, [Brodzki et al. 2008, Examples 2.14 and 2.17]).

Groups with Spanier–Whitehead K-duality. Let G be a countable discrete group which satisfies (1) and either (2) or (3) of the following:

(1) G admits a G-compact model of $\underline{E}G$;

(2) G admits a γ-element γ such that $j_r^G(\gamma) = 1_{C_r^*(G)}$, or

(3) G admits a (γ)-element x such that $j_r^G(x) = 1_{C_r^*(G)}$.

We recall that the gamma element γ, if it exists, is represented by a cycle with property (γ). Therefore, condition (2) implies (3). Our previous argument shows that such a group G has Spanier–Whitehead K-duality. Thanks to the Higson–Kasparov theorem [2001], we obtain the following:

Theorem 2.1. *All a-T-menable groups which admit a G-compact model of $\underline{E}G$ have Spanier–Whitehead K-duality.*

Examples of such a-T-menable groups include the following:

- all groups which act properly, affine-isometrically, and cocompactly on a finite-dimensional Euclidean space,
- all cocompact lattices of simple Lie groups $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$,
- all groups which act cocompactly on a tree (or more generally on a CAT(0)-cube complex).

For any a-T-menable group G listed above, the gamma element γ can be represented by an explicit cycle with property (γ). Below, we describe an explicit cycle with property (γ) for these groups. As a consequence, we can obtain an explicit cycle d in $\mathrm{KK}(C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C})$ which, together with δ, induces the duality between $C_r^*(G)$ and $C_0(\underline{E}G) \rtimes G$.

To begin, we recall from [Kasparov 1988; Valette 2002] that the gamma element exists for any group G which acts properly and isometrically on a simply connected, complete Riemannian manifold M of nonpositive sectional curvature which is bounded from below. In this case, the gamma element for G is represented by an unbounded G-equivariant Kasparov cycle

$$(H_M, D_M),$$

where H_M is the Hilbert space $L^2(M, \Lambda^* T^*_{\mathbb{C}} M)$ of L^2-sections of the complexified exterior algebra bundles on M and where D_M is the self-adjoint operator

$$D_M = d_f + d_f^*$$

on M given by the Witten type perturbation

$$d_f = d + df \wedge$$

of the exterior derivative d; the function f is the squared distance $d_M^2(x_0, x)$ on M for some fixed point x_0 of M. Let

$$F_M = \frac{D_M}{(1+D_M^2)^{1/2}}$$

be the bounded transform of D_M. The element $[H_M, F_M]$ in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ is the gamma element for G. We now suppose furthermore that the action of G on M is cocompact. In this case, G admits a G-compact model of $\underline{E}G$, namely the manifold M.

Proposition 2.2. *The cycle (H_M, F_M) has property (γ).*

Proof. Since $[H_M, F_M]$ is the gamma element for G, it satisfies condition (1) of Definition 0.12. To show that condition (2) holds for $[H_M, F_M]$, we shall apply Theorem 6.1 of [Nishikawa 2019]. We use the natural nondegenerate representation of $C_0(M)$ on H_M by pointwise multiplication. We take the dense subalgebra B of $C_0(M)$ consisting of compactly supported smooth functions. Note that B contains a cutoff function of M. For any function h in B, we have

$$[D_M, g(h)] = [d + d^*, g(h)] = g(c(h)),$$

where $c(h)$ is the Clifford multiplication by the gradient of h which is bounded and compactly supported. We can now use [Nishikawa 2019, Theorem 6.1] to conclude that the bounded transform (H_M, F_M) satisfies condition (2) of property (γ). □

Corollary 2.3. *For all groups G which act properly, affine-isometrically, and cocompactly on a finite-dimensional Euclidean space $\mathbb{R}^n$, the G-equivariant cycle $(H_{\mathbb{R}^n}, F_{\mathbb{R}^n})$ has property (γ).*

Corollary 2.4. *For all cocompact closed subgroups G of a semisimple Lie group L, the G-equivariant cycle $(H_{L/K}, F_{L/K})$ has property (γ), where K is a maximal compact subgroup of L.*

Let us look at a few examples.

Poincaré–Langlands duality. In [Niblo et al. 2016] the authors examine the Baum–Connes correspondence for the (extended) affine Weyl group W_a associated to a compact connected semisimple Lie group G. This group can be realized as the group of affine isometries of the Lie algebra $\mathfrak{t}$ of a maximal torus $T \subseteq G$. The structure of W_a is that of a semidirect product $\Gamma \rtimes W$, where Γ is the lattice of translations in $\mathfrak{t}$, and W is the Weyl group of the root system of G.

Ultimately, it is shown that the Baum–Connes conjecture (which holds in this case) is equivalent to T-duality for the aforementioned torus T and the Pontryagin dual $\widehat{\Gamma}$ of the lattice Γ. From the viewpoint of Lie groups, $\widehat{\Gamma}$ equivariantly coincides with the maximal torus $T^\vee$ of the Langlands dual $G^\vee$ of G. In K-theory this is expressed by W-equivariant Spanier–Whitehead duality between the dual tori T and $T^\vee$, which is referred to as “Poincaré–Langlands duality” in [Niblo et al. 2016].

Propositions 1.19–1.23 and Theorem 1.24 can be equivalently applied to get these results, with $C(B\Gamma)$ playing the role of $C(T)$ and $C_r^*(\Gamma)$ playing the role of $C(T^\vee)$ through the Gelfand transform.

The (γ)-element, which belongs to $\mathrm{KK}^{W_a}(\mathbb{C}, \mathbb{C})$, in this case can be constructed as explained above with $M = \mathfrak{t}$ and distance function induced by a W-equivariant metric. Equivalently, the bounded transform of the Bott–Dirac operator

$$B_{\mathfrak{t}} = \sum_i (\mathrm{ext}(e_i) + \mathrm{int}(e_1))x_i + (\mathrm{ext}(e_i) - \mathrm{int}(e_i)) \frac{d}{dx_i}$$

yields a W-equivariant cycle with property (γ), provided that interior multiplication is defined through a W-equivariant metric. The cycle obtained this way is indeed isomorphic to the one obtained through the Witten type perturbation of the de Rham operator, and its KK-class coincides with the classical γ-element which is homotopic to the unit [Higson and Kasparov 2001].

In summary, we obtain equivariant duality classes $\underline{\delta}^W \in \mathrm{KK}^W(\mathbb{C}, C(\mathfrak{t}/\Gamma) \otimes C_r^*(\Gamma))$, derived from the Miščenko W-bundle associated to the principal Γ-bundle $\mathfrak{t} \to T$, and $\underline{d}^W \in \mathrm{KK}^W(C(\mathfrak{t}/\Gamma) \otimes C_r^*(\Gamma), \mathbb{C})$, derived from the (γ)-element described above. We can prove

$$\underline{\delta}^W \widehat{\otimes}_{C(T)} \underline{d}^W = j_r^\Gamma(\gamma),$$

where on the right-hand side we mean “partial” descent with respect to the normal subgroup $\Gamma \subseteq W_a$. As we know, $\gamma = 1_{\mathbb{C}}$ in $\mathrm{KK}^{\Gamma \times W}(\mathbb{C}, \mathbb{C})$, so that we get

$$\underline{\delta}^W \widehat{\otimes}_{C(T)} \underline{d}^W = 1, \qquad \underline{\delta}^W \widehat{\otimes}_{C(T^\vee)} \underline{d}^W = 1$$

in the equivariant groups $\mathrm{KK}^W(C(T^\vee), C(T^\vee))$, $\mathrm{KK}^W(C(T), C(T))$, respectively.

Lattices in $\mathrm{SO}(n, 1)$ **and** $\mathrm{SU}(n, 1)$**.** Let G be a cocompact lattice of a simple Lie group $L = \mathrm{SO}(n, 1)$, or $L = \mathrm{SU}(n, 1)$. Let K be a maximal compact subgroup of L. Corollary 2.4 shows that the G-equivariant cycle $(H_{L/K}, F_{L/K})$ has property (γ). The corresponding element $x = [H_{L/K}, F_{L/K}]$ is nothing but the gamma element γ for G, which is shown to be equal to 1_G [Higson and Kasparov 2001; Julg and Kasparov 1995].

Groups acting on trees. Let G be a countable discrete group which acts properly and cocompactly on a locally finite tree Y. The tree Y is the union of the sets Y^0, Y^1 of the vertices and edges of the tree. Without loss of generality, we assume a G-invariant typing on the tree. Namely, we assume a G-invariant decomposition $Y^0 = Y^0_0 \sqcup Y^0_1$ so that any two adjacent vertices have distinct types. This can be achieved by the barycentric subdivision of the tree. We take E as the geometric realization of the tree. This is a G-compact model of the universal proper G-space. We denote by d the edge path metric on E and hence on Y^0 such that each edge has length 1.

The ℓ^2 space $\ell^2(Y)$ is naturally a graded G-Hilbert space with the even and odd spaces being $\ell^2(Y^0)$, $\ell^2(Y^1)$, respectively. Let $H_{\mathbb{R}}$ be the graded Hilbert space $L^2(\mathbb{R}, \Lambda^*_{\mathbb{C}}(\mathbb{R}))$ as before, but now with the trivial G-action. We construct a Kasparov cycle with the property (γ) on the graded tensor product

$$H_Y = H_{\mathbb{R}} \mathbin{\widehat{\otimes}} \ell^2(Y).$$

Following [Kasparov and Skandalis 1991], we define a nondegenerate representation π of $C_0(E)$ on H_Y, which is diagonal with respect to Y. This is given by a family $(\pi_y)_{y\in Y}$ of representations of $C_0(E)$ on $H_{\mathbb{R}}$ indexed by y in Y. If y is a vertex, we define π_y by sending ϕ in $C_0(E)$ to the multiplication on $H_{\mathbb{R}}$ by the constant $\phi(y)$. If y is an edge with vertices y_0, y_1 of corresponding types, we identify y with the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ via the unique isometry sending y_j to $(-1)^j\frac{1}{2}$. We define π_y by sending ϕ in $C_0(E)$ to the multiplication on $H_{\mathbb{R}}$ by the restriction of ϕ to the edge y extended to the left and right constantly.

Now, like the operator D_M, we define an unbounded, odd, self-adjoint operator D_Y with compact resolvent of index 1, which is almost G-equivariant and has nice compatibility with functions in $C_0(E)$. The bounded transform F_Y of D_Y will give us a desired Kasparov cycle (H_Y, F_Y) with property (γ). For this, we fix a base point y_0 from Y^0. The following construction depends on the choice of y_0. We have the decomposition of H_Y

$$H_Y = H_{\mathbb{R}} \mathbin{\widehat{\otimes}} \mathbb{C}\delta_{y_0} \oplus \bigoplus_{y\in Y^0\setminus\{y_0\}} (H_{\mathbb{R}} \mathbin{\widehat{\otimes}} (\mathbb{C}\delta_y \oplus \mathbb{C}\delta_{e_y})),$$

where for each vertex $y \neq y_0$, e_y is the last edge appearing in the geodesic from y_0 to y and where the symbol δ_* denotes a delta function in $\ell^2(Y)$. Our operator D_Y is block-diagonal with respect to this decomposition. It is given by a family $(D_y)_{y\in Y^0}$ of an unbounded, odd, self-adjoint operators with compact resolvent.

For a vertex $y \in Y^0_j$ of type j, let $B_{\mathbb{R},y}$ be the Bott–Dirac operator on $H_{\mathbb{R}}$ with "origin shifted":

$$B_{\mathbb{R},y} = (\mathrm{ext}(e_1) + \mathrm{int}(e_1))(x - n_y) + (\mathrm{ext}(e_1) - \mathrm{int}(e_1))\,\frac{d}{dx},$$

where $n_y = (-1)^j\left(\frac{1}{2} + d(y, y_0)\right)$. For $y = y_0$, we simply set

$$D_{y_0} = B_{\mathbb{R},y_0} \mathbin{\widehat{\otimes}} 1 \quad \text{on } H_{\mathbb{R}} \mathbin{\widehat{\otimes}} \mathbb{C}\delta_{y_0}.$$

For $y \neq y_0$, we set

$$D_y = B_{\mathbb{R},y} \mathbin{\widehat{\otimes}} 1 + M_{\chi_y} \mathbin{\widehat{\otimes}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{on } H_{\mathbb{R}} \mathbin{\widehat{\otimes}} (\mathbb{C}\delta_y \oplus \mathbb{C}\delta_{e_y}),$$

where M_{χ_y} is the multiplication on $H_{\mathbb{R}}$ by the function χ_y on $\mathbb{R}$ defined as

$$\text{for } y \in Y_0^0, \quad \chi_y(x) = \begin{cases} 0, & x < \frac{1}{2}, \\ \left(x - \frac{1}{2}\right)^2, & \frac{1}{2} \le x < 1, \\ x - \frac{3}{4}, & 1 \le x < d(y, y_0), \\ -(x - n_y)^2 + d(y, y_0) - \frac{1}{2}, & d(y, y_0) \le x < n_y, \\ d(y, y_0) - \frac{1}{2}, & n_y \le x, \end{cases}$$

$$\text{for } y \in Y_1^0, \quad \chi_y(x) = \begin{cases} d(y, y_0) - \frac{1}{2}, & x < n_y, \\ -(x - n_y)^2 + d(y, y_0) - \frac{1}{2}, & n_y \le x < -d(y, y_0), \\ -x - \frac{3}{4}, & -d(y, y_0) \le y < -1, \\ \left(x + \frac{1}{2}\right)^2, & -1 \le x < -\frac{1}{2}, \\ 0, & -\frac{1}{2} \le x. \end{cases}$$

Note that for each $y \ne y_0$, D_y is a bounded perturbation of a self-adjoint operator $B_{\mathbb{R},y} \mathbin{\widehat{\otimes}} 1$ with compact resolvent of index 0, and hence so is D_y. All D_y are hence diagonalizable. Therefore, $D_Y = (D_y)_{y \in Y^0}$ is self-adjoint. In order to see that D_Y has compact resolvent, we compute

$$D_y^2 = B_{\mathbb{R},y}^2 \mathbin{\widehat{\otimes}} 1 + M_{\chi_y}^2 \mathbin{\widehat{\otimes}} 1 + \begin{pmatrix} 0 & -M_{\chi_y'} \\ M_{\chi_y'} & 0 \end{pmatrix} \mathbin{\widehat{\otimes}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where χ_y' is the derivative of χ_y. We see that D_y^2 has spectrum far away from 0 as y goes to infinity essentially because the derivatives χ_y' are uniformly bounded in y and because we have

$$(x - n_y)^2 + \chi_y^2 \ge 2\left(\tfrac{1}{2} d(y, y_0) - \tfrac{1}{8}\right)^2$$

everywhere. It follows that D_Y indeed has compact resolvent. Let F_Y be the bounded transform

$$F_Y = \frac{D_Y}{(1 + D_Y^2)^{1/2}}.$$

Proposition 2.5. *A pair (H_Y, F_Y) is a G-equivariant Kasparov cycle with property (γ).*

Proof. Almost G-equivariance follows from

$$D_Y - g(D_Y) = \text{bounded} \quad \text{for } g \in G,$$

which we leave to the reader. To see that $[H_Y, F_Y] = 1_F$ in $R(F)$ for any finite subgroup F of G, we note that the class $[H_Y, F_Y]$ does not depend on the choice of the base point y_0. Hence, we may assume that y_0 is a vertex fixed by the group F. In this case, it is not hard to see that F_Y is an odd, F-equivariant, self-adjoint operator whose graded index is the one-dimensional trivial representation of F spanned by $\xi_0 \mathbin{\widehat{\otimes}} \delta_{y_0}$ in $H_{\mathbb{R}} \mathbin{\widehat{\otimes}} \mathbb{C}\delta_{y_0}$, where $\xi_0 = e^{-x^2/2}$. This shows $[H_Y, F_Y] = 1_F$. To show that it has condition (2) of property (γ) with respect to the representation π of $C_0(E)$, we shall apply Theorem 6.1 of [Nishikawa 2019] for the dense subalgebra B of $C_0(E)$ consisting of compactly supported functions which are smooth inside each edge and constant near the vertices. Note that B contains a cutoff function of E. First, we can see that for each $y \neq y_0$, the operator $M_{\chi_y} \mathbin{\widehat{\otimes}} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ commutes with the representation π. This is due to the vanishing of χ_y for $y \in Y_0^0$ and $y \in Y_1^0$ over $x \leq \frac{1}{2}$ and $-\frac{1}{2} \leq x$, respectively. For ϕ in B, we compute the commutator $[D_Y, \pi(\phi)]$ as

$$
\begin{aligned}
[D_Y, \pi(\phi)] &= [B_{\mathbb{R},y_0} \mathbin{\widehat{\otimes}} 1, \pi(\phi)] + \sum_{y \in Y^0 \setminus \{y_0\}} \left[B_{\mathbb{R},y} \mathbin{\widehat{\otimes}} 1 + M_{\chi_y} \mathbin{\widehat{\otimes}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pi(\phi) \right] \\
&= [B_{\mathbb{R},y_0} \mathbin{\widehat{\otimes}} 1, \pi(\phi)] + \sum_{y \in Y^0 \setminus \{y_0\}} [B_{\mathbb{R},y} \mathbin{\widehat{\otimes}} 1, \pi(\phi)] \\
&= \sum_{y \in Y^0 \setminus \{y_0\}} \left[\begin{pmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}, \pi_{e_y}(\phi) \right] \mathbin{\widehat{\otimes}} 1 \\
&= \pi(\phi') \sum_{y \in Y^0 \setminus \{y_0\}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbin{\widehat{\otimes}} 1,
\end{aligned}
$$

where in the last two, each summand is an operator on $H_{\mathbb{R}} \mathbin{\widehat{\otimes}} \mathbb{C}\delta_{e_y}$ and where ϕ' is the derivative of ϕ. Note that each summation is a finite sum since ϕ is compactly supported. We can now use [Nishikawa 2019, Theorem 6.1] to conclude that the bounded transform (H_Y, F_Y) satisfies condition (2) of property (γ). □

Remark 2.6. The construction can be generalized to define a cycle with property (γ) for a group which acts properly and cocompactly on a Euclidean building in the sense of [Kasparov and Skandalis 1991]. In [Brodzki et al. 2019], a different construction is given which provides us a cycle with property (γ) for a group which acts properly and cocompactly on a finite-dimensional CAT(0) cube complex.

Groups with weak Spanier–Whitehead K-duality. Let G be a countable discrete group satisfying the conditions (1) and either (2)' or (3)' below:

(1) G admits a G-compact model of $\underline{E}G$;

(2)′ G admits a γ-element γ with $j_r^G(\gamma)$ acting as the identity on $K_*(C_r^*(G))$, or

(3)′ G admits a (γ)-element x with $j_r^G(x)$ acting as the identity on $K_*(C_r^*(G))$.

Our previous argument shows that such a group G has weak Spanier–Whitehead K-duality. For any word-hyperbolic group, the gamma element is shown to exist and the Baum–Connes conjecture has been verified [Lafforgue 2012; Kasparov and

Skandalis 2003; Mineyev and Yu 2002]. Moreover, any hyperbolic group is known to admit a G-compact model of $\underline{E}G$ [Meintrup and Schick 2002]. Hence, we have the following:

Theorem 2.7. *All word-hyperbolic groups G have weak Spanier–Whitehead K-duality.*

As an example of hyperbolic groups, we can take G to be a cocompact lattices of the simple Lie group $L = \mathrm{Sp}(n, 1)$. As before, the γ-element for G has an explicit representative $(H_{L/K}, F_{L/K})$ with property (γ). We remark that the gamma element $\gamma = [H_{L/K}, F_{L/K}]$ is well-known to be not homotopic to 1_G due to Kazhdan's property (T). Furthermore, Skandalis [1988] showed that $j_r^G(\gamma)$ is not equal to $1_{C_r^*(G)}$. More precisely, what he showed is that $C_r^*(G)$ is not K-nuclear, which in particular implies that it cannot be KK-equivalent to any nuclear C^*-algebra. The same remark that $j_r^G(\gamma) \neq 1_{C_r^*(G)}$ applies to any infinite hyperbolic property (T) group [Higson and Guentner 2004, Theorem 5.2]. In general, when the gamma element γ exists, the equality $j_r^G(\gamma) = 1_{C_r^*(G)}$ implies that $C_r^*(G)$ is KK-equivalent to $P_{\mathbb{C}} \rtimes G$, which satisfies the UCT [Meyer and Nest 2006, Proposition 9.5]; in particular it is K-nuclear. Therefore, if $C_r^*(G)$ is not K-nuclear, we have $j_r^G(\gamma) \neq 1_{C_r^*(G)}$.

3. Some applications

In this section we prove a few results by applying the theory of K-duality developed in the previous pages. Some of the material presented here has been previously treated in the literature via possibly different methods [Dadarlat 2009, Section 3; Emerson and Meyer 2010, Section 5; Kaminker et al. 2017, Section 4.4; Rosenberg and Schochet 1987, Section 7]. Nevertheless, we provide a brief account for completeness, to give a better idea of some applications of our main theorems.

We say a C^*-algebra A is KK-*compact* if the functor sending D to $\mathrm{KK}_*(A, D)$ commutes with filtered colimits. If A is a C^*-algebra with a Spanier–Whitehead K-dual B, then A is KK-compact because $\mathrm{KK}_*(A, D)$ is naturally isomorphic to $\mathrm{KK}_*(\mathbb{C}, D \otimes B)$ and the K-theory functor is continuous.

As explained after Theorem 6.6 of [Meyer and Nest 2006], a C^*-algebra satisfies the *universal coefficient theorem* (UCT) [Blackadar 1998, Section 23] if and only if it belongs to the localizing triangulated subcategory of the KK-category generated by the complex numbers (this category is denoted as $\langle * \rangle$ in [Meyer and Nest 2006]). As in [Dell'Ambrogio et al. 2011], let us denote this subcategory by $\mathcal{T}$. It is known that within this subcategory, an object is dualizable if and only if it is compact:

Proposition 3.1 [Dell'Ambrogio et al. 2011, Proposition 4.1]. *In the subcategory $\mathcal{T} \subseteq \mathrm{KK}$, the full triangulated subcategory $\mathcal{T}_c$ of compact objects coincides with the (closed) symmetric monoidal category $\mathcal{T}_d$ of dualizable objects. Furthermore, both* *these two subcategories are equal to the thick triangulated subcategory generated by the complex numbers.*

Corollary 3.2. *If G has Spanier–Whitehead duality then $C_r^*(G)$ satisfies the UCT.*

Proof. We know that $C_0(\underline{E}G) \rtimes G$ satisfies the UCT [Meyer and Nest 2006, Proposition 9.5]. By assumption, $C_0(\underline{E}G) \rtimes G$ has a Spanier–Whitehead K-dual $C_r^*(G)$. Thus, $C_0(\underline{E}G) \rtimes G$ is KK-compact. By Proposition 3.1, it is dualizable in $\mathcal{T}$. Namely, it has a Spanier–Whitehead K-dual, say A, which satisfies the UCT. On the other hand, it is fairly easy to see that a dual object is unique up to equivalence. Hence, $C_r^*(G)$ is KK-equivalent to A. The claim follows from this. □

The strong Baum–Connes conjecture was introduced in [Meyer and Nest 2006] as the assertion that the canonical Dirac morphism α in $\mathrm{KK}^G(P_{\mathbb{C}}, \mathbb{C})$ induces a KK-equivalence $j_r^G(\alpha)$ from $P_{\mathbb{C}} \rtimes G$ to $C_r^*(G)$. In the presence of the gamma element γ for G, this is equivalent to the assertion that $j_r^G(\gamma) = 1_{C_r^*(G)}$.

Theorem 3.3. *If G has Spanier–Whitehead duality then it satisfies the strong Baum–Connes conjecture. Moreover, if the γ-element exists and G satisfies the strong Baum–Connes conjecture, than G has Spanier–Whitehead duality.*

Proof. Suppose G has Spanier–Whitehead duality. Then we know that the Baum–Connes conjecture holds for G, and so the Dirac morphism α induces an isomorphism $j_r^G(\alpha)_*$ on K-groups from $P_{\mathbb{C}} \rtimes G$ to $C_r^*(G)$. Furthermore, both $P_{\mathbb{C}} \rtimes G$ and $C_r^*(G)$ satisfy the UCT by [Meyer and Nest 2006, Proposition 9.5] and by Corollary 3.2, respectively. It follows that $j_r^G(\alpha)$ is a KK-equivalence [Blackadar 1998, Theorem 23.10.1]. Conversely, if the strong Baum–Connes conjecture holds, we have $j_r^G(\gamma) = 1_{C_r^*(G)}$. Hence, G has Spanier–Whitehead duality by Theorem 1.14. □

As in [Blackadar 1998, Theorem 23.10.5], a C^*-algebra A satisfies the UCT if and only if it is KK-equivalent to a commutative C^*-algebra $C_0(X)$. Furthermore, this X can be taken to be a three-dimensional cell complex [Blackadar 1998, Corollary 23.10.3; Rosenberg and Schochet 1987, Proposition 7.4]. This is because the range of K-theory on such spaces exhausts all countable $\mathbb{Z}/(2)$-graded abelian groups. If $K_*(A)$ is finitely generated, then X can be chosen finite, and a Spanier–Whitehead K-dual exists for such spaces [Emerson and Meyer 2010, Proposition 5.9].

Lemma 3.4. *Suppose A has a Spanier–Whitehead K-dual and satisfies the UCT. Then it has finitely generated K-theory groups.*

Proof. As in the proofs of [Rosenberg and Schochet 1987, Proposition 7.4; Blackadar 1998, Corollary 23.10.3], let $C = C^0 \oplus C^1$ be a commutative C^*-algebra KK-equivalent to A, where C^0 is the mapping cone of a $*$-homomorphism on direct sums of $C_0(\mathbb{R})$, and C^1 is the suspension of such a mapping cone. It is easy to see that C is the inductive limit of subalgebras C_n, where C_n has finitely generated K-theory. Since $\mathrm{KK}_*(A, -)$ is continuous (since A is KK-compact), the equivalence $A \to C$ factors through C_n for some $n \in \mathbb{N}$. Then $K_*(A)$ is finitely generated because it is a quotient of $K_*(C_n)$, which enjoys this property. □

Proposition 3.5. *Suppose G satisfies the Baum–Connes conjecture and the γ-element exists. Then $C_r^*(G)$ has finitely generated K-theory groups.*

Proof. If $\gamma \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ exists, then $P_{\mathbb{C}} \rtimes G$ is dualizable by Theorem 1.16. It is known that $P_{\mathbb{C}} \rtimes G$ satisfies the UCT; see [Meyer and Nest 2006, Proposition 9.5]. Thus, $P_{\mathbb{C}} \rtimes G$ has finitely generated K-groups by Lemma 3.4. Recall that in the localization picture the assembly map appears as

$$K_*(P_{\mathbb{C}} \rtimes G) \to K_*(C_r^*(G)). \tag{3.6}$$

Therefore, if (3.6) is an isomorphism the right-hand side is finitely generated. □

Remark 3.7. More generally, $C_r^*(G)$ has finitely generated K-theory groups if G satisfies the Baum–Connes conjecture and the source $P_{\mathbb{C}}$ of the Dirac morphism is a (categorical) direct summand of a proper algebra. This is because by Remark 1.18, $P_{\mathbb{C}} \rtimes G$ has a Spanier–Whitehead K-dual.

Remark 3.8. By the results in [Dell'Ambrogio et al. 2011], there exists a functor $\mathbb{K}$ from the KK-category to the stable homotopy category satisfying $\pi_n(\mathbb{K}(A)) \cong K_n(A)$. This functor specializes to a full and faithful functor on the subcategory of dualizable objects satisfying the UCT, realizing C^*-algebras as perfect KU-modules (in particular, finite spectra). Hence, the previous results can also be obtained from the well-known fact that homotopy groups are finitely generated in this context.

Define the *n-th dimension-drop algebra* as

$$\mathbb{I}_n = \{ f \in C([0, 1], M_n(\mathbb{C})) \mid f(0) = 0, f(1) \in \mathbb{C}1_n \}.$$

We can use this to introduce the mod-n K-theory groups as follows:

$$K_*(B; \mathbb{Z}/(n)) = \mathrm{KK}_*(\mathbb{I}_n, B).$$

It is apparent from this definition that a Baum–Connes conjecture in mod-n K-theory for B would have to introduce coefficients on the left, and we can take this as motivation to find a satisfactory formulation for the full bivariant version of the Baum–Connes conjecture. The approach via localization immediately generalizes to this context, giving us a map

$$\mathrm{KK}_*(A, (P_{\mathbb{C}} \otimes B) \rtimes G) \to \mathrm{KK}_*(A, B \rtimes_r G) \tag{3.9}$$

defined as $y \mapsto y \otimes j_r^G(1_B \widehat{\otimes} \alpha)$, where $\alpha \in \mathrm{KK}(P_{\mathbb{C}}, \mathbb{C})$ is the Dirac morphism, for any (separable) C^*-algebra A and G-C^*-algebra B.

The original definition of the left-hand side (following [Baum et al. 2003] and [Uuye 2011]), what is called the "naive" topological K-group in [Uuye 2011], is given as

$$\varinjlim_{Y \subseteq \underline{E}G} \mathrm{KK}^G_*(C_0(Y, A), B),$$

where the limit ranges as usual over G-invariant G-compact subspaces of $\underline{E}G$. Unlike the simpler case of the conjecture, the definition making use of the naive topological group is *not* equivalent to the definition in (3.9). However, [Uuye 2011] shows that there are natural maps

$$\nu_Y : \mathrm{KK}^G_*(C_0(Y, A), B) \to \mathrm{KK}_*(A, (P_{\mathbb{C}} \otimes B) \rtimes G), \tag{3.10}$$

which make the obvious diagram commute. In addition, if A admits a Spanier–Whitehead K-dual, then (3.10) induces an isomorphism.

Theorem 3.11 [Uuye 2011]. *Suppose A has a Spanier–Whitehead K-dual. Then the comparison map induced by the ν_Y is an isomorphism.*

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References

[Anantharaman-Delaroche 2002] C. Anantharaman-Delaroche, "Amenability and exactness for dynamical systems and their C^*-algebras", *Trans. Amer. Math. Soc.* **354**:10 (2002), 4153–4178. MR Zbl

[Baum et al. 1994] P. Baum, A. Connes, and N. Higson, "Classifying space for proper actions and K-theory of group C^*-algebras", pp. 240–291 in *C^*-algebras: 1943–1993* (San Antonio, TX, 1993), edited by R. S. Doran, Contemp. Math. **167**, Amer. Math. Soc., Providence, RI, 1994. MR Zbl

[Baum et al. 2003] P. Baum, S. Millington, and R. Plymen, "Local-global principle for the Baum–Connes conjecture with coefficients", *K-Theory* **28**:1 (2003), 1–18. MR Zbl

[Blackadar 1998] B. Blackadar, *K-theory for operator algebras*, 2nd ed., Mathematical Sciences Research Institute Publications **5**, Cambridge Univ. Press, 1998. MR Zbl

[Block 2010] J. Block, "Duality and equivalence of module categories in noncommutative geometry", pp. 311–339 in *A celebration of the mathematical legacy of Raoul Bott*, edited by P. R. Kotiuga, CRM Proc. Lecture Notes **50**, Amer. Math. Soc., Providence, RI, 2010. MR Zbl

[Brodzki et al. 2008] J. Brodzki, V. Mathai, J. Rosenberg, and R. J. Szabo, "D-branes, RR-fields and duality on noncommutative manifolds", *Comm. Math. Phys.* **277**:3 (2008), 643–706. MR Zbl

[Brodzki et al. 2009] J. Brodzki, V. Mathai, J. Rosenberg, and R. J. Szabo, "Non-commutative correspondences, duality and D-branes in bivariant *K*-theory", *Adv. Theor. Math. Phys.* **13**:2 (2009), 497–552. MR Zbl

[Brodzki et al. 2019] J. Brodzki, E. Guentner, N. Higson, and S. Nishikawa, "On the Baum–Connes conjecture for groups acting on CAT(0)-cubical spaces", preprint, 2019. arXiv

[Connes 1994] A. Connes, *Noncommutative geometry*, Academic Press, San Diego, CA, 1994. MR Zbl

[Connes 1996] A. Connes, "Gravity coupled with matter and the foundation of non-commutative geometry", *Comm. Math. Phys.* **182**:1 (1996), 155–176. MR Zbl

[Dadarlat 2009] M. Dadarlat, "Continuous fields of C^*-algebras over finite dimensional spaces", *Adv. Math.* **222**:5 (2009), 1850–1881. MR Zbl

[Dell'Ambrogio et al. 2011] I. Dell'Ambrogio, H. Emerson, T. Kandelaki, and R. Meyer, "A functorial equivariant *K*-theory spectrum and an equivariant Lefschetz formula", preprint, 2011. arXiv

[Dell'Ambrogio et al. 2014] I. Dell'Ambrogio, H. Emerson, and R. Meyer, "An equivariant Lefschetz fixed-point formula for correspondences", *Doc. Math.* **19** (2014), 141–194. MR Zbl

[Echterhoff et al. 2008] S. Echterhoff, H. Emerson, and H. J. Kim, "*KK*-theoretic duality for proper twisted actions", *Math. Ann.* **340**:4 (2008), 839–873. MR Zbl

[Emerson 2003] H. Emerson, "Noncommutative Poincaré duality for boundary actions of hyperbolic groups", *J. Reine Angew. Math.* **564** (2003), 1–33. MR Zbl

[Emerson 2011] H. Emerson, "Lefschetz numbers for C^*-algebras", *Canad. Math. Bull.* **54**:1 (2011), 82–99. MR Zbl

[Emerson and Meyer 2009] H. Emerson and R. Meyer, "Equivariant representable *K*-theory", *J. Topol.* **2**:1 (2009), 123–156. MR Zbl

[Emerson and Meyer 2010] H. Emerson and R. Meyer, "Dualities in equivariant Kasparov theory", *New York J. Math.* **16** (2010), 245–313. MR Zbl

[Guentner et al. 2000] E. Guentner, N. Higson, and J. Trout, *Equivariant E-theory for C^*-algebras*, Mem. Amer. Math. Soc. **703**, Amer. Math. Soc., Providence, RI, 2000. MR Zbl

[Higson 1987] N. Higson, "On a technical theorem of Kasparov", *J. Funct. Anal.* **73**:1 (1987), 107–112. MR Zbl

[Higson and Guentner 2004] N. Higson and E. Guentner, "Group C^*-algebras and *K*-theory", pp. 137–251 in *Noncommutative geometry*, edited by S. Doplicher and R. Longo, Lecture Notes in Math. **1831**, Springer, 2004. MR Zbl

[Higson and Kasparov 2001] N. Higson and G. Kasparov, "*E*-theory and *KK*-theory for groups which act properly and isometrically on Hilbert space", *Invent. Math.* **144**:1 (2001), 23–74. MR Zbl

[Julg and Kasparov 1995] P. Julg and G. Kasparov, "Operator *K*-theory for the group SU(n, 1)", *J. Reine Angew. Math.* **463** (1995), 99–152. MR Zbl

[Kaad and Proietti 2018] J. Kaad and V. Proietti, "Index theory on the Miščenko bundle", preprint, 2018. arXiv

[Kaminker and Putnam 1997] J. Kaminker and I. Putnam, "K-theoretic duality of shifts of finite type", *Comm. Math. Phys.* **187**:3 (1997), 509–522. MR Zbl

[Kaminker and Schochet 2019] J. Kaminker and C. L. Schochet, "Spanier–Whitehead K-duality for C^*-algebras", *J. Topol. Anal.* **11**:1 (2019), 21–52. MR Zbl

[Kaminker et al. 2017] J. Kaminker, I. F. Putnam, and M. F. Whittaker, "K-theoretic duality for hyperbolic dynamical systems", *J. Reine Angew. Math.* **730** (2017), 263–299. MR Zbl

[Kasparov 1980] G. G. Kasparov, "The operator K-functor and extensions of C^*-algebras", *Izv. Akad. Nauk SSSR Ser. Mat.* **44**:3 (1980), 571–636. In Russian; translated in *Math. USSR-Izv.* **16**:3 (1981), 513–572. MR Zbl

[Kasparov 1988] G. G. Kasparov, "Equivariant KK-theory and the Novikov conjecture", *Invent. Math.* **91**:1 (1988), 147–201. MR Zbl

[Kasparov 1995] G. G. Kasparov, "K-theory, group C^*-algebras, and higher signatures (conspectus)", pp. 101–146 in *Novikov conjectures, index theorems and rigidity* (Oberwolfach, 1993), vol. 1, edited by S. C. Ferry et al., London Math. Soc. Lecture Note Ser. **226**, Cambridge Univ. Press, 1995. MR Zbl

[Kasparov and Skandalis 1991] G. G. Kasparov and G. Skandalis, "Groups acting on buildings, operator K-theory, and Novikov's conjecture", *K-Theory* **4**:4 (1991), 303–337. MR Zbl

[Kasparov and Skandalis 2003] G. G. Kasparov and G. Skandalis, "Groups acting properly on "bolic" spaces and the Novikov conjecture", *Ann. of Math.* (2) **158**:1 (2003), 165–206. MR Zbl

[Lafforgue 2012] V. Lafforgue, "La conjecture de Baum–Connes à coefficients pour les groupes hyperboliques", *J. Noncommut. Geom.* **6**:1 (2012), 1–197. MR Zbl

[Land 2015] M. Land, "The analytical assembly map and index theory", *J. Noncommut. Geom.* **9**:2 (2015), 603–619. MR Zbl

[Meintrup and Schick 2002] D. Meintrup and T. Schick, "A model for the universal space for proper actions of a hyperbolic group", *New York J. Math.* **8** (2002), 1–7. MR Zbl

[Meyer 2000] R. Meyer, "Equivariant Kasparov theory and generalized homomorphisms", *K-Theory* **21**:3 (2000), 201–228. MR Zbl

[Meyer and Nest 2006] R. Meyer and R. Nest, "The Baum–Connes conjecture via localisation of categories", *Topology* **45**:2 (2006), 209–259. MR Zbl

[Meyer and Nest 2010] R. Meyer and R. Nest, "Homological algebra in bivariant K-theory and other triangulated categories, I", pp. 236–289 in *Triangulated categories*, edited by T. Holm et al., London Math. Soc. Lecture Note Ser. **375**, Cambridge Univ. Press, 2010. MR Zbl

[Mineyev and Yu 2002] I. Mineyev and G. Yu, "The Baum–Connes conjecture for hyperbolic groups", *Invent. Math.* **149**:1 (2002), 97–122. MR Zbl

[Miščenko and Fomenko 1979] A. S. Miščenko and A. T. Fomenko, "The index of elliptic operators over C^*-algebras", *Izv. Akad. Nauk SSSR Ser. Mat.* **43**:4 (1979), 831–859, 967. In Russian; translated in *Math. USSR-Izv.* **15**:1 (1980), 87–112. MR Zbl

[Neeman 2001] A. Neeman, *Triangulated categories*, Annals of Math. Studies **148**, Princeton Univ. Press, 2001. MR Zbl

[Niblo et al. 2016] G. A. Niblo, R. Plymen, and N. Wright, "Extended Affine Weyl groups, the Baum–Connes correspondence and Langlands duality", preprint, 2016. arXiv

[Nishikawa 2019] S. Nishikawa, "Direct splitting method for the Baum–Connes conjecture", *J. Funct. Anal.* **277**:7 (2019), 2259–2287. MR Zbl

[Rieffel 1976] M. Rieffel, "Strong Morita equivalence of certain transformation group C^*-algebras", *Math. Ann.* **222**:1 (1976), 7–22. MR Zbl

[Rosenberg and Schochet 1987] J. Rosenberg and C. Schochet, "The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor", *Duke Math. J.* **55**:2 (1987), 431–474. MR Zbl

[Skandalis 1988] G. Skandalis, "Une notion de nucléarité en K-théorie (d'après J. Cuntz)", *K-Theory* **1**:6 (1988), 549–573. MR Zbl

[Spanier and Whitehead 1958] E. H. Spanier and J. H. C. Whitehead, "Duality in relative homotopy theory", *Ann. of Math.* (2) **67** (1958), 203–238. MR Zbl

[Tu 2000] J.-L. Tu, "The Baum–Connes conjecture for groupoids", pp. 227–242 in *C^*-algebras* (Münster, 1999), edited by J. Cuntz and S. Echterhoff, Springer, 2000. MR Zbl

[Uuye 2011] O. Uuye, "The Baum–Connes conjecture for KK-theory", *J. K-Theory* **8**:1 (2011), 3–29. MR Zbl

[Valette 2002] A. Valette, *Introduction to the Baum–Connes conjecture*, Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 2002. MR Zbl

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Homotopy equivalence in unbounded KK-theory

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We propose a new notion of unbounded KK-cycle, mildly generalizing unbounded Kasparov modules, for which the direct sum is well-defined. To a pair (A, B) of σ-unital C^*-algebras, we can then associate a semigroup $\overline{UKK}(A, B)$ of homotopy equivalence classes of unbounded cycles, and we prove that this semigroup is in fact an abelian group. In case A is separable, our group $\overline{UKK}(A, B)$ is isomorphic to Kasparov's KK-theory group $KK(A, B)$ via the bounded transform. We also discuss various notions of degenerate cycles, and we prove that the homotopy relation on unbounded cycles coincides with the relation generated by operator-homotopies and addition of degenerate cycles.

Introduction

Given two (σ-unital, $\mathbb{Z}_2$-graded) C^*-algebras A and B, Kasparov [1980] defined the abelian group $KK(A, B)$ as a set of homotopy equivalence classes of Kasparov A-B-modules, equipped with the direct sum. These groups simultaneously generalize K-theory (if $A = \mathbb{C}$) and K-homology (if $B = \mathbb{C}$).

It was shown by Baaj and Julg that every class in $KK(A, B)$ can also be represented by an *unbounded* Kasparov module. Many examples of elements in KK-theory which arise from geometric situations are most naturally described in the unbounded picture. The prototypical example is a first-order elliptic differential operator (e.g., the Dirac operator, signature operator, or de Rham operator) on a complete Riemannian manifold. The unbounded picture is also more suitable in the context of nonsmooth manifolds. Indeed, while on Lipschitz manifolds there is no pseudodifferential calculus, it makes perfect sense to consider first-order differential operators and thus to construct unbounded Kasparov modules on Lipschitz manifolds (see, e.g., [Teleman 1983; Hilsum 1985; 1989]). Furthermore, the Kasparov product is often easier to describe in the unbounded picture. In fact, under suitable assumptions, the Kasparov product of two unbounded Kasparov modules can be explicitly *constructed* [Mesland 2014; Kaad and Lesch 2013; Brain et al. 2016; Mesland and Rennie 2016]. These advantages of the unbounded picture of KK-theory motivate the following question.

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Question. *Can Kasparov's KK-groups equivalently be defined as the set of homotopy equivalence classes of* unbounded *Kasparov modules*?

A similar question is considered in [Kaad 2019], where it is shown that Kasparov's *KK*-groups can be obtained using the (a priori) weaker equivalence relation of *stable homotopy* of unbounded Kasparov modules. In the present paper we will provide a positive answer to the above question. Moreover, we will prove that the stable homotopy relation of [Kaad 2019] in fact coincides with ordinary homotopy equivalence.

The first problem one encounters when trying to answer the above question, is that the direct sum of unbounded Kasparov modules is not well-defined. To resolve this issue, we slightly weaken the standard definition of unbounded Kasparov modules in such a way that the set $\overline{\Psi}_1(A, B)$ of such *unbounded A-B-cycles* $(E, \mathscr{D})$ becomes closed under the direct sum operation. By considering the natural notion of homotopy equivalence on $\overline{\Psi}_1(A, B)$ (completely analogous to homotopies of bounded Kasparov modules), we thus obtain a semigroup $\overline{UKK}(A, B)$ given by the set of homotopy equivalence classes of $\overline{\Psi}_1(A, B)$. We will prove that $\overline{UKK}(A, B)$ is in fact a group.

To answer the aforementioned question, we must show that the group $\overline{UKK}(A, B)$ is isomorphic to Kasparov's *KK*-theory group $KK(A, B)$. The results of Baaj and Julg already show that the *bounded transform*

$$(E, \mathscr{D}) \mapsto (E, F_{\mathscr{D}} := \mathscr{D}(1+\mathscr{D}^2)^{-1/2})$$

induces a surjective homomorphism $\overline{UKK}(A, B) \to KK(A, B)$. This is proven by explicitly constructing an unbounded lift for any bounded Kasparov module.

The difficulty is to prove injectivity of the bounded transform. To be precise, given unbounded cycles $(E_0, \mathscr{D}_0)$ and $(E_1, \mathscr{D}_1)$ and a homotopy (E, F) between their bounded transforms, we can use the lifting results from Baaj and Julg to lift (E, F) to an unbounded homotopy $(E, \mathscr{S})$. However, it is in general not clear how the endpoints of $(E, \mathscr{S})$ are related to $(E_j, \mathscr{D}_j)$, and the main challenge is therefore to construct $(E, \mathscr{S})$ in such a way that its endpoints are in fact homotopic to $(E_j, \mathscr{D}_j)$.

For this purpose, we describe a general notion of *functional dampening*, which is the transformation $\mathscr{D} \mapsto f(\mathscr{D})$ for suitable "dampening functions" $f : \mathbb{R} \to \mathbb{R}$ which blow up towards infinity at a slow enough rate (such that $f(x)(1+x^2)^{-1/2}$ vanishes at infinity) and which are compatible with the Lipschitz structure obtained from $\mathscr{D}$. We prove that $(E, f(\mathscr{D}))$ is operator-homotopic to $(E, \mathscr{D})$ for any dampening function f, generalizing a result in [Kaad 2019].

With a careful adaptation of the lifting construction of [Baaj and Julg 1983; Kucerovsky 2000], using ideas from [Mesland and Rennie 2016], we then prove our first main result:

Theorem A. *If A is separable, then any homotopy (E, F) between $(E_0, F_{\mathcal{D}_0})$ and $(E_1, F_{\mathcal{D}_1})$ can be lifted to an unbounded Kasparov A-$C([0,1], B)$-module $(E, \mathcal{S})$ such that, for $j = 0, 1$, the endpoints $\mathrm{ev}_j(E, \mathcal{S})$ are unitarily equivalent to $(E_j, f_j(\mathcal{D}_j))$ for dampening functions $f_j : \mathbb{R} \to \mathbb{R}$.*

As mentioned above, functional dampening provides an operator-homotopy between $(E_j, \mathcal{D}_j)$ and $(E_j, f_j(\mathcal{D}_j))$, and thus we obtain a *positive answer* to the above question:

Theorem B. *If A is separable, then the bounded transform induces an isomorphism*

$$\overline{UKK}(A, B) \xrightarrow{\simeq} KK(A, B).$$

We continue to provide an alternative description of the homotopy equivalence relation at the unbounded level. In bounded *KK*-theory, it is well-known that the homotopy relation coincides with the relation obtained from unitary equivalences, operator-homotopies, and addition of degenerate modules. We will prove an analogous statement in unbounded *KK*-theory. We consider two notions of degenerate cycles, namely *spectrally degenerate* cycles (for which $\mathcal{D}$ is invertible and $\mathcal{D}|\mathcal{D}|^{-1}$ commutes with A) and *algebraically degenerate* cycles (for which A is represented trivially). We then consider the equivalence relation $\sim_{oh+d}$ obtained from unitary equivalences, operator-homotopies, and addition of algebraically and spectrally degenerate cycles. Our next main result then reads:

Theorem C. *Degenerate cycles are null-homotopic. Furthermore, if A is separable, then the homotopy equivalence relation $\sim_h$ on $\overline{\Psi}_1(A, B)$ coincides with the equivalence relation $\sim_{oh+d}$.*

We prove the first statement by explicitly constructing a homotopy between degenerate cycles and the zero cycle. The second statement is then obtained by combining [Kasparov 1980, §6, Theorem 1] with Theorem A.

Let us briefly compare our work with the existing literature on unbounded Kasparov modules. First, we note that, in the usual approach to unbounded *KK*-theory, it is necessary to make a fixed choice of a dense $*$-subalgebra $\mathcal{A} \subset A$, and to consider only those unbounded Kasparov A-B-modules $(E, \mathcal{D})$ for which $\mathcal{A} \subset \mathrm{Lip}(\mathcal{D})$, to ensure that the direct sum is well-defined. This means that any equivalence relation on unbounded Kasparov A-B-modules only applies to those unbounded Kasparov modules which are defined using the *same* choice of $\mathcal{A}$. Thus, it is impossible to compare unbounded Kasparov modules which are defined with respect to *different* choices of $\mathcal{A}$. One major advantage of our approach is that, instead of fixing a choice of $*$-subalgebra $\mathcal{A}$, we consider the slightly weaker notion of *unbounded cycles*, which only requires that $A \subset \overline{\mathrm{Lip}(\mathcal{D})}$. For such cycles the direct sum is well-defined in full generality. In particular, the notion of homotopy equivalence can then be used to compare *arbitrary* unbounded A-B-cycles. Nevertheless, we will show that Theorems A–C remain valid if we do fix a countably generated dense $*$-subalgebra $\mathscr{A} \subset A$, and replace $\overline{UKK}(A, B)$ by the semigroup $UKK(\mathscr{A}, B)$ given by homotopy equivalence classes of all those unbounded Kasparov modules $(\pi, E, \mathscr{D})$ for which $\pi(\mathscr{A}) \subset \mathrm{Lip}(\mathscr{D})$.

Other equivalence relations on unbounded Kasparov modules have already been considered in the literature, namely the bordism relation [Deeley et al. 2018] and the stable homotopy relation [Kaad 2019]. Both of these approaches rely on a fixed choice of a dense $*$-subalgebra $\mathscr{A} \subset A$. Let us discuss the relationships between homotopy equivalence, stable homotopy equivalence, and bordism. The paper [Deeley et al. 2018] studies a notion of bordism of unbounded Kasparov modules due to Hilsum [2010], and proves that there is a surjective homomorphism from the corresponding bordism group $\Omega(\mathscr{A}, B)$ to Kasparov's KK-group $KK(A, B)$. In particular, from Theorem B we obtain a surjective homomorphism to our $\overline{UKK}$-group, which means that the bordism relation is stronger than the homotopy relation. However, it remains an open question if these relations coincide or not. One technical tool appearing in [Deeley et al. 2018] is the notion of weakly degenerate module, which is shown to be null-bordant. As a spin-off from our study of Clifford symmetric modules, we give a direct proof in Lemma 4.15 that any weakly degenerate cycle is also null-homotopic (without assuming A to be separable).

After the appearance of [Deeley et al. 2018] as a preprint in 2015, there has been increased interest within the community regarding equivalence relations on unbounded Kasparov modules. Discussions between the authors and Kaad in November 2018 gave the problem new impetus. The subsequent paper [Kaad 2019] provides a first study of homotopies of unbounded Kasparov modules. The work in the present paper was initiated independently and the methods developed here are complementary to those in [Kaad 2019]. The main technical results, our Theorem A and [Kaad 2019, Proposition 6.2], are very distinct in spirit and lend themselves to different types of applications. Our proofs of Theorems A–C are independent of the results from [Kaad 2019]. Moreover, it should be noted that our Theorem B is stronger than the main result in [Kaad 2019] in the sense we now explain.

In [Kaad 2019], a countably generated dense $*$-subalgebra $\mathscr{A} \subset A$ is fixed and the notion of *stable homotopy* of unbounded Kasparov $\mathscr{A}$-B-modules is considered. Stable homotopy is a weakening of the homotopy equivalence relation obtained from homotopy equivalences and addition of "spectrally decomposable" modules. It is then proved that the resulting set of equivalence classes of unbounded Kasparov $\mathscr{A}$-B-modules forms an abelian group which (if A is separable) is isomorphic to Kasparov's KK-group. In particular, this group does not depend on the choice of the dense $*$-subalgebra $\mathscr{A} \subset A$ (up to isomorphism).

As described above, we avoid in the present paper the need to fix a countably generated dense $*$-subalgebra $\mathscr{A} \subset A$ in the definition of the unbounded KK-group. Even more importantly, thanks to our new approach towards lifting a homotopy in Theorem A (adapting the more refined lifting methods of [Kucerovsky 2000; Mesland and Rennie 2016]), we overcome the need to weaken the homotopy equivalence relation by addition of spectrally decomposable modules. Furthermore, we will also show that, in fact, adding spectrally decomposable modules does not weaken the homotopy equivalence relation after all. Indeed, any spectrally decomposable module is just a bounded perturbation of a spectrally degenerate module. Consequently, it follows from Theorem C that any spectrally decomposable cycle is null-homotopic, so that the relation of stable homotopy equivalence coincides with homotopy equivalence. We point out that, combined with the main results from [Kaad 2019], this provides a second and independent proof of Theorem B.

Finally, let us briefly summarize the layout of this paper. We start in Section 1 with our definition of unbounded cycles, and we show that the direct sum is well-defined. In Sections 1A and 1B we recall the lifting construction from [Baaj and Julg 1983], closely following the arguments of [Mesland and Rennie 2016; Kucerovsky 2000]. We collect some basic facts regarding regular self-adjoint operators in Section 1C.

In Section 2A we introduce the homotopy relation (as well as the special case of operator-homotopies), and construct the semigroup $\overline{UKK}(A, B)$. In Section 2B we show that the notion of functional dampening can be implemented via an operator-homotopy. In Section 2C we construct the lift of a homotopy and prove Theorem A (see Theorem 2.11). Combined with the operator-homotopy obtained from functional dampening, we then obtain Theorem B (see Theorem 2.12).

We introduce our notions of algebraically and spectrally degenerate cycles in Section 3, and we prove that degenerate cycles are null-homotopic (Lemma 3.2 and Proposition 3.7). In Section 3C we then show that any homotopy can be implemented as an operator-homotopy modulo addition of degenerate cycles (Theorem 3.10), which completes the proof of Theorem C.

We give a direct proof that $\overline{UKK}(A, B)$ is a group (and not just a semigroup) in Section 4. In the case where A is separable, this follows immediately from the isomorphism $\overline{UKK}(A, B) \simeq KK(A, B)$, but our direct proof works for any pair (A, B) of σ-unital C^*-algebras. The proof relies on the observation that the presence of certain symmetries induces homotopical triviality. After a brief discussion of Lipschitz regular cycles in Section 4A, we introduce the notion of spectrally symmetric cycles in Section 4B. These cycles are a mild generalization of the notion of spectrally decomposable modules introduced in [Kaad 2019]. We prove that any spectrally symmetric cycle is a bounded perturbation of a spectrally degenerate cycle, and therefore null-homotopic. In Section 4C we introduce the notion of Clifford symmetric cycles, which are elements of $\overline{\Psi}_1(A, B)$ which extend to $\overline{\Psi}_1(A \mathbin{\widehat{\otimes}} \mathbb{C}\mathrm{l}_1, B)$. We prove that every Clifford symmetric cycle is operator-homotopic to a spectrally symmetric cycle and therefore null-homotopic. The proof is easily generalized to show that in fact every weakly degenerate cycle is null-homotopic. We exploit such Clifford symmetries to prove in Section 4D that the semigroup $\overline{UKK}(A, B)$ is in fact a group.

Finally, the Appendix contains some basic facts regarding localizations of Hilbert C^*-modules and their dense submodules.

Notation and conventions. Let A and B denote σ-unital $\mathbb{Z}_2$-graded C^*-algebras. By an approximate unit for A we will always mean an even, positive, increasing, and contractive approximate unit for the C^*-algebra A. For elements $a, b \in A$ we denote by $[a, b]$ the graded commutator. If a and b are homogeneous, we denote by $\deg a, \deg b \in \mathbb{Z}_2$ their degree and $[a, b] := ab - (-1)^{\deg a \deg b} ba$. For general a, b we extend the graded commutator by linearity. Let E be a $\mathbb{Z}_2$-graded Hilbert C^*-module over B, or Hilbert B-module for short (for definitions and further details regarding Hilbert C^*-modules, we refer to the books [Lance 1995; Blackadar 1998]). Throughout this article, we will assume E is countably generated. We write $\mathrm{End}_B^*(E)$ for the adjointable operators on E, and $\mathrm{End}_B^0(E)$ for the compact operators on E. For any subset $W \subset \mathrm{End}_B^*(E)$, we write $\overline{W}$ for the closure of W with respect to the operator-norm of $\mathrm{End}_B^*(E)$.

1. Unbounded cycles

Kasparov [1980] defined the abelian group $KK(A, B)$ as a set of homotopy equivalence classes of Kasparov A-B-modules. We briefly recall the main definitions (more details can be found in, e.g., [Blackadar 1998, §17]).

A (bounded) *Kasparov A-B-module* is a triple (π, E, F) comprising a $\mathbb{Z}_2$-graded, countably generated, right Hilbert B-module E, a ($\mathbb{Z}_2$-graded) $*$-homomorphism $\pi : A \to \mathrm{End}_B^*(E)$, and an odd adjointable endomorphism $F \in \mathrm{End}_B^*(E)$ such that, for all $a \in A$,

$$\pi(a)(F - F^*), [F, \pi(a)], \pi(a)(F^2 - 1) \in \mathrm{End}_B^0(E).$$

Two Kasparov A-B-modules (π_0, E_0, F_0) and (π_1, E_1, F_1) are called *unitarily equivalent* (denoted with $\simeq$) if there exists an even unitary in $\mathrm{Hom}_B(E_0, E_1)$ intertwining the π_j and F_j (for $j = 0, 1$). A *homotopy* between (π_0, E_0, F_0) and (π_1, E_1, F_1) is given by a Kasparov A-$C([0, 1], B)$-module $(\widetilde{\pi}, \widetilde{E}, \widetilde{F})$ such that

$$\mathrm{ev}_j(\widetilde{\pi}, \widetilde{E}, \widetilde{F}) \simeq (\pi_j, E_j, F_j), \quad j = 0, 1.$$

A homotopy $(\widetilde{\pi}, \widetilde{E}, \widetilde{F})$ is called an *operator-homotopy* if there exists a Hilbert B-module E with a representation $\pi : A \to \mathrm{End}_B^*(E)$ such that $\widetilde{E}$ equals the

Hilbert $C([0,1], B)$-module $C([0,1], E)$ with the natural representation $\widetilde{\pi}$ of A on $C([0,1], E)$ induced from π, and if $\widetilde{F}$ is given by a *norm*-continuous family $\{F_t\}_{t\in[0,1]}$. A module (π, E, F) is called *degenerate* if for all $a \in A$ we have

$$\pi(a)(F - F^*) = [F, \pi(a)] = \pi(a)(F^2 - 1) = 0.$$

The KK-theory $KK(A, B)$ of A and B is defined as the set of homotopy equivalence classes of (bounded) Kasparov A-B-modules. Since homotopy equivalence respects direct sums, the direct sum of Kasparov A-B-modules induces a (commutative and associative) binary operation ("addition") on the elements of $KK(A, B)$ such that $KK(A, B)$ is in fact an abelian group [Kasparov 1980, §4, Theorem 1].

In this paper we will give a completely analogous description of KK-theory, based instead on unbounded Kasparov modules [Baaj and Julg 1983]. Recall that a closed densely defined symmetric operator $\mathscr{D} : \operatorname{Dom}\mathscr{D} \to E$ is self-adjoint and regular if the operators $\mathscr{D} \pm i : \operatorname{Dom}\mathscr{D} \to E$ have dense range. We refer to [Lance 1995, Chapters 9 and 10] for details on regular operators on Hilbert modules. For a self-adjoint regular operator $\mathscr{D} : \operatorname{Dom}\mathscr{D} \to E$, we write

$$\operatorname{Lip}(\mathscr{D}) := \{T \in \operatorname{End}^*_B(E) : T(\operatorname{Dom}\mathscr{D}) \subset \operatorname{Dom}\mathscr{D} \text{ and } [\mathscr{D}, T] \in \operatorname{End}^*_B(E)\}.$$

It is worth noting that, because $\mathscr{D}$ is densely defined, $\overline{\operatorname{Lip}(\mathscr{D})} \cap \operatorname{End}^0_B(E)$ is equal to $\operatorname{End}^0_B(E)$. However, in general $\overline{\operatorname{Lip}(\mathscr{D})}$ is not equal to $\operatorname{End}^*_B(E)$. We also introduce

$$\operatorname{Lip}^0(\mathscr{D}) := \{T \in \operatorname{Lip}(\mathscr{D}) : T(1+\mathscr{D}^2)^{-1/2},\, T^*(1+\mathscr{D}^2)^{-1/2} \in \operatorname{End}^0_B(E)\}.$$

We note that $\operatorname{Lip}^0(\mathscr{D})$ is a $*$-subalgebra of $\operatorname{End}^*_B(E)$. We introduce the following relaxation of the notion of unbounded Kasparov module.

Definition 1.1. An *unbounded A-B-cycle* $(\pi, E, \mathscr{D})$ consists of a $\mathbb{Z}_2$-graded, countably generated Hilbert B-module E, a $\mathbb{Z}_2$-graded $*$-homomorphism $\pi : A \to \operatorname{End}_B(E)$, and an odd regular self-adjoint operator $\mathscr{D}$ on E, such that

$$\pi(A) \subset \overline{\operatorname{Lip}^0(\mathscr{D})}.$$

The set of all unbounded A-B-cycles is denoted $\overline{\Psi}_1(A, B)$. We will often suppress the representation π in our notation and simply write $(E, \mathscr{D})$ instead of $(\pi, E, \mathscr{D})$.

Remark 1.2. (1) It follows immediately from the definition that $\pi(a)(1+\mathscr{D}^2)^{-1/2} \in \operatorname{End}^0_B(E)$ for any $a \in A$, i.e., $\mathscr{D}$ has "A-locally compact" resolvents.

(2) We point out that if $\pi(A) \subset \operatorname{End}^0_B(E)$ (i.e., A is represented as compact operators), then the condition $\pi(A) \subset \overline{\operatorname{Lip}^0(\mathscr{D})}$ is automatically satisfied, since $\operatorname{Lip}(\mathscr{D}) \cap \operatorname{End}^0_B(E) \subset \operatorname{Lip}^0(\mathscr{D})$ is always dense in $\operatorname{End}^0_B(E)$.

Remark 1.3. We use the term unbounded A-B-cycle since our definition is different from the usual definition of an unbounded Kasparov module, originally given in [Baaj and Julg 1983]. An unbounded A-B-cycle $(\pi, E, \mathscr{D})$ is an unbounded Kasparov module if there exists a dense $*$-subalgebra $\mathscr{A} \subset A$ such that $\pi(\mathscr{A}) \subset \operatorname{Lip}^0(\mathscr{D})$.

To avoid confusion we often refer to such cycles as *ordinary unbounded Kasparov modules.*

Our main reason for relaxing this definition is the following simple lemma.

Lemma 1.4. *The direct sum of unbounded A-B-cycles is well-defined, and therefore $\overline{\Psi}_1(A, B)$ is a semigroup.*

Proof. Given unbounded A-B-cycles $(\pi_i, E_i, \mathscr{D}_i)$, $i = 0, 1$, we have $\mathrm{Lip}^0(\mathscr{D}_0) \oplus \mathrm{Lip}^0(\mathscr{D}_1) \subset \mathrm{Lip}^0(\mathscr{D}_0 \oplus \mathscr{D}_1)$ and $\pi_i(A) \subset \overline{\mathrm{Lip}^0(\mathscr{D}_i)}$. It follows that

$$(\pi_0 \oplus \pi_1)(A) \subset \overline{\mathrm{Lip}^0(\mathscr{D}_0) \oplus \mathrm{Lip}^0(\mathscr{D}_1)} \subset \overline{\mathrm{Lip}^0(\mathscr{D}_0 \oplus \mathscr{D}_1)},$$

and therefore $(\pi_0 \oplus \pi_1, E_0 \oplus E_1, \mathscr{D}_0 \oplus \mathscr{D}_1)$ is also an unbounded A-B-cycle. □

Remark 1.5. Note that if there are dense $*$-subalgebras $\mathscr{A}_i \subset A$ such that $\pi_i(\mathscr{A}_i) \subset \mathrm{Lip}(\mathscr{D}_i)$, it may not be possible to find a dense $*$-subalgebra $\mathscr{A} \subset A$ such that

$$(\pi_0 \oplus \pi_1)(\mathscr{A}) \subset \mathrm{Lip}^0(\mathscr{D}_0 \oplus \mathscr{D}_1).$$

In fact, even if $E_0 = E_1$ and $\pi_0 = \pi_1 = \pi$, the intersection

$$\mathrm{Lip}(\mathscr{D}_0) \cap \mathrm{Lip}(\mathscr{D}_1) \cap \pi(A)$$

might not be dense in $\pi(A)$ (for an example, see for instance [Deeley et al. 2018, Appendix A]). Hence, the direct sum is not well-defined on *ordinary* unbounded Kasparov modules. The usual way around this problem is to *fix* a dense $*$-subalgebra $\mathscr{A} \subset A$, and to consider only those unbounded Kasparov modules $(\pi, E, \mathscr{D})$ for which $\pi(\mathscr{A}) \subset \mathrm{Lip}^0(\mathscr{D})$. With our relaxed condition $\pi(A) \subset \overline{\mathrm{Lip}^0(\mathscr{D})}$, we avoid the need to make such a choice for $\mathscr{A}$.

Lemma 1.6. *Let $(\pi, E, \mathscr{D})$ be an unbounded A-B-cycle, and suppose that A is separable. Then there exists a countable subset $W \subset \mathrm{Lip}^0(\mathscr{D})$ consisting of products of elements in $\mathrm{Lip}^0(\mathscr{D})$ (i.e., each $w \in W$ is of the form $w = T_1 T_2$ for $T_1, T_2 \in \mathrm{Lip}^0(\mathscr{D})$) such that $\pi(A) \subset \overline{W}$.*

Proof. Since A is separable, and since products are dense in any C^*-algebra, we may pick a countable dense subset of products $\{a_j b_j\}_{j\in\mathbb{N}} \subset A$. Since $\pi(A) \subset \overline{\mathrm{Lip}^0(\mathscr{D})}$, there exist sequences $\{v_{j,k}\}_{k\in\mathbb{N}}, \{w_{j,k}\}_{k\in\mathbb{N}} \subset \mathrm{Lip}^0(\mathscr{D})$ such that, for each j,

$$\lim_k \|a_j - v_{j,k}\| = \lim_k \|b_j - w_{j,k}\| = 0.$$

The statement then holds with $W := \{v_{j,k} w_{j,k}\}_{j,k\in\mathbb{N}}$. □

Baaj and Julg proved for any ordinary unbounded Kasparov module that the bounded transform $\mathscr{D} \mapsto F_{\mathscr{D}} := D(1 + \mathscr{D}^2)^{-1/2}$ yields a bounded Kasparov module and hence a KK-class. Before we continue, we need to show that this still holds for our relaxed definition of unbounded cycles.

Proposition 1.7 (cf. [Baaj and Julg 1983]). *If $(\pi, E, \mathscr{D})$ is an unbounded A-B-cycle (as in Definition 1.1), then the bounded transform $(\pi, E, F_{\mathscr{D}})$ is a bounded Kasparov module and hence defines an element in $KK(A, B)$.*

Proof. As remarked in [Blackadar 1998, Proposition 17.11.3], it suffices to show that $[F_{\mathscr{D}}, a]b$ is compact for any $a, b \in A$. By Definition 1.1, there is a sequence $T_n \in \mathrm{Lip}^0(\mathscr{D})$ such that $T_n \to a$ in norm, and then $[F_{\mathscr{D}}, T_n]b \to [F_{\mathscr{D}}, a]b$ in norm as well. It thus suffices to show that $[F, T]b \in \mathrm{End}_B^0(E)$ for $b \in A$ and $T \in \mathrm{Lip}^0(\mathscr{D})$. Compactness of $[F, T]b$ follows from the careful argument provided in the proof of [Carey and Phillips 1998, Proposition 2.4], after multiplication with b from the right. □

1A. *The algebras C_F and J_F.* Let E be a countably generated Hilbert B-module. The following result is well-known, and follows from the proof of [Blackadar 1998, Proposition 13.6.1] (which extends from $h \in \mathrm{End}_B^0(E)$ to arbitrary $h \in \mathrm{End}_B^*(E)$).

Lemma 1.8 (cf. [Blackadar 1998, Proposition 13.6.1]). *Let $h \in \mathrm{End}_B^*(E)$. Then hE is dense in E if and only if $h \cdot \mathrm{End}_B^0(E)$ is dense in $\mathrm{End}_B^0(E)$.*

For a bounded Kasparov A-B-module (E, F) with $F = F^*$ and $F^2 \leq 1$, we define

$$C_F := C^*(1 - F^2) + FC^*(1 - F^2), \qquad J_F := \mathrm{End}_B^0(E) + C_F.$$

The C^*-algebra J_F was introduced in [Mesland and Rennie 2016, Lemma 4.5], and plays an important role in the construction of the (unbounded) lift of a (bounded) Kasparov module.

Lemma 1.9. *The space C_F is a separable C^*-algebra, and $1 - F^2$ is a strictly positive element in C_F.*

Proof. It is explained in the proof of [Mesland and Rennie 2016, Lemma 4.5] that C_F is a separable C^*-algebra. By assumption, the spectrum $\mathrm{spec}(F)$ of F is contained in $[-1, 1]$, and by construction C_F can be identified with a $*$-subalgebra of $C_0(\mathrm{spec}(F) \setminus \{\pm 1\})$. Under this identification, the element $c = 1 - F^2$ corresponds to the function $x \mapsto 1 - x^2$. In particular, we have $c(t) \neq 0$ for each $t \in \mathrm{spec}(F) \setminus \{\pm 1\}$. Since C_F also separates points of $\mathrm{spec}(F) \setminus \{\pm 1\}$ (the elements $1 - F^2$ and $F(1 - F^2)$ suffice), the Stone–Weierstrass theorem implies that $C_F \simeq C_0(\mathrm{spec}(F) \setminus \{\pm 1\})$. Since c is a strictly positive function on $C_0(\mathrm{spec}(F) \setminus \{\pm 1\})$, it follows that $1 - F^2$ is a strictly positive element in C_F. □

Lemma 1.10. *The space J_F is a σ-unital C^*-algebra, and we have the inclusions*

$$AJ_F,\ J_F A,\ FJ_F,\ J_F F \subset J_F.$$

Furthermore, if $k \in \mathrm{End}_B^0(E)$ is a positive operator such that $k + (1 - F^2)$ has dense range in E, then $k + (1 - F^2)$ is strictly positive in J_F.

Proof. As E is countably generated, $\mathrm{End}^0_B(E)$ is a σ-unital C^*-algebra (see, e.g., [Lance 1995, Proposition 6.7]). Since $\mathrm{End}^0_B(E)$ is an ideal in $\mathrm{End}^*_B(E)$, it follows from [Kasparov 1980, §3, Lemma 2] that J_F is also a σ-unital C^*-algebra. The inclusions $FJ_F, J_FF \subset J_F$ are immediate, and the inclusions $AJ_F, J_FA \subset J_F$ follow because $a(1-F^2)$ and $[F,a]$ are compact for all $a \in A$.

Let $k \in \mathrm{End}^0_B(E)$ be a positive operator such that $h := k + (1-F^2)$ has dense range in E. Consider an element $l + c \in J_F$ where $l \in \mathrm{End}^0_B(E)$ and $c \in C_F$, and let $\varepsilon > 0$. Since $1-F^2$ is strictly positive in C_F by Lemma 1.9, there exists $b \in C_F$ such that $\|(1-F^2)b - c\| < \varepsilon$. Moreover, since $l - kb$ is compact, we know from Lemma 1.8 that there exists $a \in \mathrm{End}^0_B(E)$ such that $\|ha - (l-kb)\| < \varepsilon$. Hence,

$$\|h(a+b) - (l+c)\| \le \|ha - (l-kb)\| + \|(1-F^2)b - c\| < 2\varepsilon,$$

which proves that hJ_F is dense in J_F. □

1B. ***The lifting construction.*** Since our definition of unbounded cycle is more general than the usual definition of unbounded Kasparov module, it of course remains true that the bounded transform is surjective [Baaj and Julg 1983]. The way to prove this surjectivity is by showing that every bounded Kasparov module (E, F) can be lifted to an (ordinary) unbounded Kasparov module $(E, \mathscr{D})$ such that $F_{\mathscr{D}}$ is operator-homotopic to F. Because we will make essential use of the technical subtleties of this lifting procedure in the sequel, we present the proof here, closely following the arguments of [Mesland and Rennie 2016; Kucerovsky 2000]. Recall that all approximate units are assumed to be even, positive, increasing, and contractive for the C^*-algebra norm.

Lemma 1.11 [Mesland and Rennie 2016, proof of Theorem 1.25]. *Let C be a commutative separable C^*-algebra, $\{c_j\}_{j\in\mathbb{N}} \subset C$ a total subset, and $\{u_n\}_{n\in\mathbb{N}}$ a countable commutative approximate unit for C. If for some $0 < \varepsilon < 1$, $d_n := u_{n+1} - u_n$ satisfies*

$$\|d_n c_j\| \le \varepsilon^{2n} \quad \text{for all } j \le n,$$

then the series $l^{-1} := \sum \varepsilon^{-n} d_n$ defines an unbounded multiplier on C such that $l := (l^{-1})^{-1} \in C$ is strictly positive.

Proof. The series $l^{-1}c_j := \sum_n \varepsilon^{-n} d_n c_j$ is convergent for all j by our assumption that $\|d_n c_j\| \le \varepsilon^{2n}$ for all $n \ge j$, so l^{-1} is a densely defined unbounded multiplier. The partial sums $\sum_{n=0}^{k} \varepsilon^{-n} d_n$ are elements in the commutative C^*-algebra $C \simeq C_0(Y)$, where $Y = \operatorname{Spec} C$. Under this isomorphism, the approximate unit u_n is identified with a sequence of functions converging pointwise to 1. For fixed $t \in (0,1)$ set

$$Y_k := \{y \in Y : u_k(y) \ge t\},$$

which gives an increasing sequence of compact sets $Y_k \subset Y_{k+1}$ with $\bigcup_{k=0}^{\infty} Y_k = Y$. Let $y \in Y \setminus Y_k$ and $m \geq k$. We have the estimates

$$\begin{aligned}\sum_{n=0}^{\infty} \varepsilon^{-n} d_n(y) &\geq \sum_{n=k}^{n=m} \varepsilon^{-n} d_n(y) + \sum_{n=m+1}^{\infty} \varepsilon^{-n} d_n(y) \\ &\geq \varepsilon^{-k}(u_{m+1} - u_k)(y) + \sum_{n=m+1}^{\infty} \varepsilon^{-n} d_n(y) \\ &\geq \varepsilon^{-k}(u_{m+1}(y) - t) + \sum_{n=m+1}^{\infty} \varepsilon^{-n} d_n(y) \to \varepsilon^{-k}(1-t),\end{aligned}$$

as $m \to \infty$. This shows that l^{-1} is given by a function whose reciprocal is a strictly positive function in $C_0(Y)$, so this defines a strictly positive element $l \in C$. □

Proposition 1.12. *Let (E, F) be a bounded Kasparov A-B-module satisfying $F^* = F$ and $F^2 \leq 1$. Given a countable dense subset $\mathscr{A} \subset A$, there exists a positive operator $l \in J_F$ with dense range in E such that*

(1) *the (closure of the) operator $\mathfrak{D} := \frac{1}{2}(Fl^{-1} + l^{-1}F)$ makes $(E, \mathfrak{D})$ into an ordinary unbounded Kasparov A-B-module with $\mathscr{A} \subset \mathrm{Lip}^0(\mathfrak{D})$, and*

(2) *F and $F_{\mathfrak{D}}$ are operator-homotopic.*

Moreover, if $F^2 = 1$, we can ensure that l commutes with F and that $(1+\mathfrak{D}^2)^{-1/2}$ is compact.

Proof. Pick an even strictly positive element $h \in J_F$. Since we have (see Lemma 1.10)

$$AJ_F,\ J_F A,\ FJ_F,\ J_F F \subset J_F,$$

there exists by [Akemann and Pedersen 1977, Theorem 3.2] an approximate unit $u_n \in C^*(h)$ for J_F that is quasicentral for A and F. Let $\{a_i\}_{i\in\mathbb{N}}$ be an enumeration of $\mathscr{A}$, choose a countable dense subset $\{c_i\}_{i\in\mathbb{N}} \subset C^*(h)$, and fix a choice of $0 < \varepsilon < 1$. By selecting a suitable subsequence of u_n, we can furthermore achieve that, for each $n \in \mathbb{N}$, $d_n := u_{n+1} - u_n$ satisfies

(a) $\|d_n c_i\| \leq \varepsilon^{2n}$ for all $i \leq n$,

(b) $\|d_n(1-F^2)^{1/4}\| \leq \varepsilon^{2n}$,

(c) $\|d_n[F, a_i]\| \leq \varepsilon^{2n}$ for all $i \leq n$,

(d) $\|[d_n, a_i]\| \leq \varepsilon^{2n}$ for all $i \leq n$, and

(e) $\|[d_n, F]\| \leq \varepsilon^{2n}$.

Here properties (a)–(c) follow because u_n is an approximate unit for J_F (and c_i, $(1-F^2)^{1/4}$, and $[F, a_i]$ all lie in J_F), and properties (d)–(e) follow because u_n is quasicentral for A and F. By property (a) and Lemma 1.11 we obtain a strictly positive element $l \in C^*(h)$ such that $l^{-1} = \sum \varepsilon^{-n} d_n$. Since $lJ_F \supset lC^*(h)J_F$, $lC^*(h)$ is dense in $C^*(h)$, and $C^*(h)J_F$ is dense in J_F, it follows that lJ_F is dense in J_F and therefore l is strictly positive in J_F. In particular, l has dense range in E. From properties (b)–(e) it follows that $l \in C^*(h) \subset J_F$ satisfies [Mesland and Rennie 2016, Definition 4.6]. Then by [Mesland and Rennie 2016, Theorem 4.7; Kucerovsky 2000, Lemma 2.2] the (closure of the) operator

$$\mathcal{D} := \tfrac{1}{2}(Fl^{-1} + l^{-1}F)$$

is a densely defined and regular self-adjoint operator on E, and $(E, \mathcal{D})$ is an ordinary unbounded Kasparov A-B-module with $\mathcal{A} \subset \mathrm{Lip}^0(\mathcal{D})$. Furthermore, the proof of [Mesland and Rennie 2016, Theorem 4.7] (combined with [Blackadar 1998, Proposition 17.2.7]) shows that $F_{\mathcal{D}}$ is operator-homotopic to F.

For the final statement, suppose $F^2 = 1$, so that $J_F = \mathrm{End}^0_B(E)$. For any positive element $k \in J_F$ with dense range, we can consider $h := k + FkF \geq k$, which is also positive with dense range [Lance 1995, Corollary 10.2]. Then h is a strictly positive element in $\mathrm{End}^0_B(E)$ (see Lemma 1.8), and h commutes with F. We then proceed as above (conditions (b) and (e) now being redundant) to construct a compact operator $l \in C^*(h)$ which also commutes with F. Lastly, for $\mathcal{D} = Fl^{-1}$ we see $(1+\mathcal{D}^2)^{-1/2} = l(1+l^2)^{-1}$ is indeed compact. □

Proposition 1.12 immediately implies the surjectivity of the bounded transform:

Theorem 1.13 (cf. [Baaj and Julg 1983; Kucerovsky 2000; Blackadar 1998, Theorem 17.11.4]). *If A is separable, then the bounded transform gives a surjective map $\overline{\Psi}_1(A, B) \to KK(A, B)$.*

1C. *Regular self-adjoint operators.* Let $\mathcal{D}$ be a regular self-adjoint operator on a Hilbert B-module E. We recall from [Lance 1995, Theorem 10.9] that there exists a continuous functional calculus for $\mathcal{D}$, i.e., a $*$-homomorphism $f \mapsto f(\mathcal{D})$ from $C(\mathbb{R})$ to the regular operators on E, such that $\mathrm{id}(\mathcal{D}) = \mathcal{D}$ and $b(\mathcal{D}) = F_{\mathcal{D}} := \mathcal{D}(1+\mathcal{D}^2)^{-1/2}$ (where $b(x) = x(1+x^2)^{-1/2}$). In particular, if $f \in C(\mathbb{R})$ is real-valued, then $f(\mathcal{D})$ is regular self-adjoint.

If the operators $a(\mathcal{D} \pm i)^{-1}$ are compact for some $a \in \mathrm{End}^*_B(E)$, we note that also $ag(\mathcal{D})$ is compact for any $g \in C_0(\mathbb{R})$ (since the functions $x \mapsto (x \pm i)^{-1}$ generate $C_0(\mathbb{R})$). In particular, if $f \in C(\mathbb{R})$ is a real-valued function such that $\lim_{x\to\pm\infty} |f(x)| = \infty$, then $a(f(\mathcal{D}) \pm i)^{-1}$ and $a(1+f(\mathcal{D})^2)^{-1/2}$ are compact.

For completeness, we will show that the continuous functional calculus is compatible with $\mathbb{Z}_2$-gradings.

Lemma 1.14. *Let $\mathcal{D}$ be an odd regular self-adjoint operator on a $\mathbb{Z}_2$-graded Hilbert B-module E. If $f \in C(\mathbb{R})$ is an odd real-valued function, then the regular self-adjoint operator $f(\mathcal{D})$ is also odd.*

Proof. Let Γ denote the $\mathbb{Z}_2$-grading operator on E, and let us grade $C_0(\mathbb{R})$ by even and odd functions. As in the proof of [Higson and Roe 2000, Lemma 10.6.2], the identity

$$\Gamma(i \pm \mathscr{D})^{-1} = (i \mp \mathscr{D})^{-1}\Gamma$$

shows that Γ graded-commutes with $(i \pm \mathscr{D})^{-1}$ and hence with any element in $C_0(\mathbb{R})$. The linear subspace $\mathscr{E} := \{g(\mathscr{D})\psi : g \in C_c(\mathbb{R}),\ \psi \in E\}$ is a core for $f(\mathscr{D})$ [Lance 1995, Lemma 10.8]. Each $g \in C_c(\mathbb{R})$ is the sum of an even function $g_0 \in C_c(\mathbb{R})$ and an odd function $g_1 \in C_c(\mathbb{R})$. Then we have $\Gamma\mathscr{E} \subset \mathscr{E}$. Moreover, since $fg_0 \in C_c(\mathbb{R})$ is odd and $fg_1 \in C_c(\mathbb{R})$ is even, we find that

$$\Gamma f(\mathscr{D})g(\mathscr{D}) = -f(\mathscr{D})g_0(\mathscr{D})\Gamma + f(\mathscr{D})g_1(\mathscr{D})\Gamma = -f(\mathscr{D})\Gamma g(\mathscr{D}).$$

Thus, $[f(\mathscr{D}), \Gamma]_+ = 0$ on the core $\mathscr{E}$, and it follows that in fact Γ preserves $\operatorname{Dom} f(\mathscr{D})$ and $f(\mathscr{D})$ anticommutes with Γ. □

Lemma 1.15. *Let X be a locally compact Hausdorff space and $Y \subset X$ an open subset. Let $\{\mathscr{D}_y\}_{y \in Y}$ be a family of regular self-adjoint operators on a Hilbert B-module E, and assume there exists a dense submodule $\mathscr{E} \subset E$ which is a core for $\mathscr{D}_y$ for each $y \in Y$, such that for each $\psi \in \mathscr{E}$ the map $Y \to E$, $y \mapsto \mathscr{D}_y\psi$ is continuous. Then the operator $\widetilde{\mathscr{D}}$ on the Hilbert $C_0(X, B)$-module $C_0(Y, E)$ defined by*

$$\begin{aligned} \operatorname{Dom}\widetilde{\mathscr{D}} &:= \{\psi \in C_0(Y, E) : \psi(y) \in \operatorname{Dom}\mathscr{D}_y,\ \widetilde{\mathscr{D}}\psi \in C_0(Y, E)\}, \\ (\widetilde{\mathscr{D}}\psi)(y) &:= \mathscr{D}_y\psi(y) \end{aligned}$$

is regular and self-adjoint.

Proof. Consider the algebraic tensor product $\widetilde{\mathscr{E}} := C_c(Y) \otimes \mathscr{E}$. Since $y \mapsto \mathscr{D}_y\psi$ is continuous for each $\psi \in \mathscr{E}$, we note that $\widetilde{\mathscr{E}} \subset \operatorname{Dom}\widetilde{\mathscr{D}}$. In particular, since $\widetilde{\mathscr{E}}$ is dense in $C_0(Y, E)$, we know that $\widetilde{\mathscr{D}}$ is densely defined. Moreover, since $\mathscr{D}_y$ is closed on $\operatorname{Dom}\mathscr{D}_y$, it follows that also $\widetilde{\mathscr{D}}$ is closed on $\operatorname{Dom}\widetilde{\mathscr{D}}$. By assumption, the operators $\mathscr{D}_y \pm i : \mathscr{E} \to E$ have dense range in E for all $y \in Y$. Since $C_0(Y, E) \mathbin{\widehat{\otimes}}_{\mathrm{ev}_x} B = \{0\}$ for $x \notin Y$, it follows from Corollary A.3 that the operators $\widetilde{\mathscr{D}} \pm i : \widetilde{\mathscr{E}} \to C_0(Y, E)$ have dense range in $C_0(Y, E)$, and therefore $\widetilde{\mathscr{D}}$ is regular and self-adjoint. □

Remark 1.16. We will apply the above lemma to construct operator-homotopies over $X = [0, 1]$, and the two main cases of interest are $Y = X$ or $Y = (0, 1]$.

2. The unbounded homotopy relation

2A. ***The homotopy semigroup.*** For any $t \in [0, 1]$, we have the surjective $*$-homomorphism $\mathrm{ev}_t : C([0, 1], B) \to B$ given by $\mathrm{ev}_t(b) := b(t)$. Given an unbounded A-$C([0, 1], B)$-cycle $(\pi, E, \mathscr{D})$, we then define

$$\mathrm{ev}_t(\pi, E, \mathscr{D}) = (\pi_t, E_t, \mathscr{D}_t) := (\pi \mathbin{\widehat{\otimes}} 1, E \mathbin{\widehat{\otimes}}_{\mathrm{ev}_t} B, \mathscr{D} \mathbin{\widehat{\otimes}} 1).$$

Definition 2.1. Consider unbounded A-B-cycles $(\pi_0, E_0, \mathscr{D}_0)$ and $(\pi_1, E_1, \mathscr{D}_1)$. We introduce the following notions:

- *Unitary equivalence*. $(\pi_0, E_0, \mathscr{D}_0)$ and $(\pi_1, E_1, \mathscr{D}_1)$ are called *unitarily equivalent* (denoted $(\pi_0, E_0, \mathscr{D}_0) \simeq (\pi_1, E_1, \mathscr{D}_1)$) if there exists an even unitary $U : E_0 \to E_1$ such that $U\mathscr{D}_0 = \mathscr{D}_1 U$ and $U\pi_0(a) = \pi_1(a)U$ for all $a \in A$.
- *Homotopy*. A homotopy between $(\pi_0, E_0, \mathscr{D}_0)$ and $(\pi_1, E_1, \mathscr{D}_1)$ is given by an unbounded A-$C([0, 1], B)$-cycle $(\widetilde{\pi}, \widetilde{E}, \widetilde{\mathscr{D}})$ such that $\mathrm{ev}_j(\widetilde{\pi}, \widetilde{E}, \widetilde{\mathscr{D}}) \simeq (\pi_j, E_j, \mathscr{D}_j)$ for $j = 0, 1$.
- *Operator-homotopy*. A homotopy $(\widetilde{\pi}, \widetilde{E}, \widetilde{\mathscr{D}})$ is called an *operator-homotopy* if there exists a Hilbert B-module E with a representation $\pi : A \to \mathrm{End}^*_B(E)$ such that $\widetilde{E}$ equals the Hilbert $C([0, 1], B)$-module $C([0, 1], E)$ with the natural representation $\widetilde{\pi}$ of A on $C([0, 1], E)$ induced from π.

We denote by $\sim_{oh}$ the equivalence relation on $\overline{\Psi}_1(A, B)$ generated by operator-homotopies and unitary equivalences. The homotopy relation is denoted $\sim_h$.

Remark 2.2. If $(\pi, E, \mathscr{D})$ is an unbounded A-B-cycle such that $\pi(A) \subset \mathrm{End}^0_B(E)$ (i.e., A is represented as compact operators), then $(\pi, E, \mathscr{D})$ is operator-homotopic to $(\pi, E, 0)$, via the operator-homotopy given by $\mathscr{D}_t = t\mathscr{D}$ for $t \in [0, 1]$ (see also Remark 1.2(2)).

We note that it was shown in [Kaad 2019, Proposition 4.6] that the homotopy relation is an equivalence relation on unbounded Kasparov modules. We will show next that the proof extends to our more general notion of unbounded cycles from Definition 1.1, and for this purpose we recall some notation from [Kaad 2019, §4]. Consider two unbounded A-$C([0, 1], B)$-cycles $(\pi, E, \mathscr{D})$ and $(\pi', E', \mathscr{D}')$, and a unitary isomorphism $U : E \mathbin{\widehat{\otimes}}_{\mathrm{ev}_1} B \to E' \mathbin{\widehat{\otimes}}_{\mathrm{ev}_0} B$ satisfying

$$U(\pi(a) \mathbin{\widehat{\otimes}}_{\mathrm{ev}_1} 1)U^* = \pi'(a) \mathbin{\widehat{\otimes}}_{\mathrm{ev}_0} 1, \qquad U(\mathscr{D} \mathbin{\widehat{\otimes}}_{\mathrm{ev}_1} 1)U^* = \mathscr{D}' \mathbin{\widehat{\otimes}}_{\mathrm{ev}_0} 1,$$

for any $a \in A$. For $t \in [0, 1]$ we consider the localizations $E_t := E \mathbin{\widehat{\otimes}}_{\mathrm{ev}_t} B$, and for $e \in E$ we write $e_t := e \mathbin{\widehat{\otimes}}_{\mathrm{ev}_t} 1 \in E_t$ (as in the Appendix). We define the *concatenation*

$$E \times_U E' := \{(e, e') \in E \oplus E' : Ue_1 = e'_0\}.$$

The space $E \times_U E'$ is endowed with the right action of $C([0, 1], B)$ and the inner product described in [Kaad 2019, §4]. We note that $\pi \oplus \pi'$ and $\mathscr{D} \oplus \mathscr{D}'$ are well-defined on $E \times_U E'$, and that $\mathscr{D} \oplus \mathscr{D}'$ is a regular self-adjoint operator (see the proof of [Kaad 2019, Proposition 4.6]). For two linear subspaces $W \subset \mathrm{End}_{C([0,1],B)}(E)$ and $W' \subset \mathrm{End}_{C([0,1],B)}(E')$, we write

$$W \times_U W' := \{(w, w') \in W \oplus W' : U(w \mathbin{\widehat{\otimes}}_{\mathrm{ev}_1} 1)U^* = w' \mathbin{\widehat{\otimes}}_{\mathrm{ev}_0} 1\}.$$

We note that we have the inclusion $\mathrm{Lip}(\mathcal{D}) \times_U \mathrm{Lip}(\mathcal{D}') \subset \mathrm{Lip}(\mathcal{D} \oplus \mathcal{D}')$. In fact, using [Kaad 2019, Lemma 4.5], we obtain

$$\mathrm{Lip}^0(\mathcal{D}) \times_U \mathrm{Lip}^0(\mathcal{D}') \subset \mathrm{Lip}^0(\mathcal{D} \oplus \mathcal{D}').$$

Proposition 2.3 (cf. [Kaad 2019, Proposition 4.6]). *The homotopy relation on unbounded A-B-cycles is an equivalence relation.*

Proof. Reflexivity and symmetry are proven exactly as in [Kaad 2019, Proposition 4.6]. For transitivity, we need to show that the concatenation of two unbounded A-$C([0,1],B)$-cycles is again an unbounded A-$C([0,1],B)$-cycle.

We will first show that we may assume (without loss of generality) that any unbounded A-$C([0,1],B)$-cycle $(\pi, E, \mathcal{D})$ is "constant near the endpoints". We define

$$\widetilde{E} := C([0,1], E_0) \times_{\mathrm{Id}} E, \qquad \widetilde{\pi}(a) := \pi_0(a) \oplus \pi(a), \qquad \widetilde{\mathcal{D}} := \mathcal{D}_0 \oplus \mathcal{D}.$$

Here $\pi_0(a)$ and $\mathcal{D}_0$ denote the obvious extension to $C([0,1], E_0)$ of the operators $\pi(a) \mathbin{\widehat{\otimes}}_{\mathrm{ev}_0} 1$ and $\mathcal{D} \mathbin{\widehat{\otimes}}_{\mathrm{ev}_0} 1$ on E_0, respectively. Now consider $\varepsilon > 0$ and $a \in A$. Pick $S \in \mathrm{Lip}^0(\mathcal{D})$ such that $\|\pi(a) - S\| < \varepsilon$. Then we also have $\|\pi_0(a) - S_0\| < \varepsilon$ and therefore $\|\widetilde{\pi}(a) - S_0 \oplus S\| < \varepsilon$. This proves that we have the inclusions

$$\widetilde{\pi}(A) \subset \overline{\mathrm{Lip}^0(\mathcal{D}_0) \times_{\mathrm{Id}} \mathrm{Lip}^0(\mathcal{D})} \subset \overline{\mathrm{Lip}^0(\widetilde{\mathcal{D}})},$$

so $(\widetilde{\pi}, \widetilde{E}, \widetilde{\mathcal{D}})$ is an unbounded A-$C([0,1],B)$-cycle which is constant on $[0, \frac{1}{2}]$.

Now suppose we have two unbounded A-$C([0,1],B)$-cycles $(\pi, E, \mathcal{D})$ and $(\pi', E', \mathcal{D}')$, and a unitary isomorphism $U : E \mathbin{\widehat{\otimes}}_{\mathrm{ev}_1} B \to E' \mathbin{\widehat{\otimes}}_{\mathrm{ev}_0} B$ satisfying

$$U(\pi(a) \mathbin{\widehat{\otimes}}_{\mathrm{ev}_1} 1)U^* = \pi'(a) \mathbin{\widehat{\otimes}}_{\mathrm{ev}_0} 1, \qquad U(\mathcal{D} \mathbin{\widehat{\otimes}}_{\mathrm{ev}_1} 1)U^* = \mathcal{D}' \mathbin{\widehat{\otimes}}_{\mathrm{ev}_0} 1,$$

for any $a \in A$. As described above, we may assume (without loss of generality) that $(\pi', E', \mathcal{D}')$ is constant on $[0, \frac{1}{2}]$. We define

$$E'' := E \times_U E', \qquad \pi''(a) := \pi(a) \oplus \pi'(a), \qquad \mathcal{D}'' := \mathcal{D} \oplus \mathcal{D}'.$$

Now consider $\varepsilon > 0$ and $a \in A$. Pick $S \in \mathrm{Lip}^0(\mathcal{D})$ such that $\|\pi(a) - S\| < \varepsilon$. Then in particular we have

$$\|\pi'_0(a) - US_1U^*\| = \|\pi_1(a) - S_1\| < \varepsilon.$$

Pick a function $\chi \in C^\infty([0,1])$ such that $0 \le \chi \le 1$, $\chi(0) = 1$, and $\chi(t) = 0$ for all $\frac{1}{2} \le t \le 1$. Since E' is constant on $[0, \frac{1}{2}]$, we note that $\chi U S_1 U^*$ is a well-defined adjointable operator on E', which in fact lies in $\mathrm{Lip}^0(\mathcal{D}')$. If we also pick $R' \in \mathrm{Lip}^0(\mathcal{D}')$ such that $\|\pi'(a) - R'\| < \varepsilon$, then we obtain $T'' := S \oplus (\chi U S_1 U^* + (1-\chi)R') \in$

$\mathrm{Lip}^0(\mathcal{D}) \times_U \mathrm{Lip}^0(\mathcal{D}')$ and we have the estimate

$$\begin{aligned}\|\pi''(a) - T''\| &\leq \max\{\|\pi(a) - S\|, \|\pi'(a) - \chi U S_1 U^* + (1-\chi)R'\|\} \\ &\leq \max\Big\{\|\pi(a) - S\|, \sup_{t\in[0,1]} (\chi(t)\|\pi'_0(a) - U S_1 U^*\| + (1-\chi(t))\|\pi'_t(a) - R'\|)\Big\} \\ &< \varepsilon.\end{aligned}$$

This proves that we have the inclusions

$$\pi''(A) \subset \overline{\mathrm{Lip}^0(\mathcal{D}) \times_U \mathrm{Lip}^0(\mathcal{D}')} \subset \overline{\mathrm{Lip}^0(\mathcal{D}'')},$$

and we conclude that $(\pi'', E'', \mathcal{D}'')$ is again an unbounded A-$C([0, 1], B)$-cycle. □

Definition 2.4. We define $\overline{UKK}(A, B)$ as the set of homotopy equivalence classes of unbounded A-B-cycles.

We recall from Lemma 1.4 that the direct sum of two unbounded cycles is well-defined. Since the direct sum is also compatible with homotopies, we obtain a well-defined addition on $\overline{UKK}(A, B)$ induced by the direct sum. Moreover, this addition is associative and commutative (since homotopy equivalence is weaker than unitary equivalence). Hence, $\overline{UKK}(A, B)$ is an *abelian semigroup*, with the zero element given by the class of the zero cycle (0, 0).

2B. *Functional dampening.* The goal of this subsection is to show that, up to operator-homotopy, we can replace an unbounded cycle $(E, \mathcal{D})$ by $(E, f(\mathcal{D}))$ for suitable functions f which blow up towards infinity at a sublinear rate. One can think of $f(\mathcal{D})$ as a “dampened” version of $\mathcal{D}$, and we refer to the transformation $\mathcal{D} \mapsto f(\mathcal{D})$ as “functional dampening”. Our proof is partly inspired by the proof of [Kaad 2019, Proposition 5.1], where the special case $f(x) := x(1 + x^2)^{-r}$ (with $r \in (0, \frac{1}{2})$) is considered.

Definition 2.5. A *dampening function* is an odd continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x\to\infty} f(x) = \infty, \qquad \lim_{x\to\infty} f(x)(1 + x^2)^{-1/2} = 0.$$

Proposition 2.6. *Consider an unbounded A-B-cycle $(E, \mathcal{D})$ and a dampening function f. Assume that there exists a self-adjoint subset $W \subset \mathrm{Lip}^0(\mathcal{D}) \cap \mathrm{Lip}(f(\mathcal{D}))$ such that $\pi(A) \subset \overline{W}$. Then $(E, f(\mathcal{D}))$ is an unbounded A-B-cycle which is operator-homotopic to $(E, \mathcal{D})$.*

Proof. By Lemma 1.14, $f(\mathcal{D})$ is an odd regular self-adjoint operator on E. Since the function $x \mapsto (1 + f(x)^2)^{-1/2}$ lies in $C_0(\mathbb{R})$, we find that

$$\mathrm{Lip}^0(\mathcal{D}) \cap \mathrm{Lip}(f(\mathcal{D})) \subset \mathrm{Lip}^0(f(\mathcal{D})).$$

Hence $(E, f(\mathcal{D}))$ is indeed an unbounded A-B-cycle. To see that it is operator-homotopic to $(E, \mathcal{D})$, consider the functions $g(x) := (1 + x^2)^{-1/2}(1 + |f(x)|)$ and

$h(x) := xg(x)$. Then $g \in C_0(\mathbb{R})$ and since $f - h \in C_b(\mathbb{R})$, we see that $h(\mathcal{D})$ is a bounded perturbation of $f(\mathcal{D})$ (in particular, $(E, h(\mathcal{D}))$ is operator-homotopic to $(E, f(\mathcal{D}))$).

It remains to show that $(E, h(\mathcal{D}))$ is operator-homotopic to $(E, \mathcal{D})$. We consider the operator-homotopy given for $t \in [0, 1]$ by

$$\mathcal{D}_t := \mathcal{D} g_t(\mathcal{D}), \qquad g_t(x) := ((1-t)^{1/2} + g(x))^t.$$

We note that $g_0(x) = 1$ and $g_1(x) = g(x)$. Since $g(x)$ is bounded from below by a positive constant for $|x| < r$, we see that the map $[0, 1] \ni t \mapsto g_t(\,\cdot\,) \in C_b(\mathbb{R})$ is uniformly continuous on compact subsets of $\mathbb{R}$, and therefore $t \mapsto g_t(\mathcal{D})$ is strongly continuous (see, e.g., [Kaad and Lesch 2012, Lemma 7.2]). Consequently, $t \mapsto \mathcal{D}_t$ is strongly continuous on $\operatorname{Dom}\mathcal{D}$. Furthermore, for each $t \in [0, 1]$, $\operatorname{Dom}\mathcal{D}$ is a core for $\mathcal{D}_t$, so from Lemma 1.15 we obtain a regular self-adjoint operator $\widetilde{\mathcal{D}}$ on $C([0, 1], E)$.

Consider a self-adjoint element $w \in W$. Let us fix $0 < t < 1$ and write

$$Q_t(\mathcal{D}) := (1-t)^{1/2} + g(\mathcal{D}),$$

so that $g_t(\mathcal{D}) = Q_t(\mathcal{D})^t$. We note that $Q_t(\mathcal{D}) \in \operatorname{Lip}(\mathcal{D})$ and $[\mathcal{D}, Q_t(\mathcal{D})] = 0$, and we find that

$$\mathcal{D}[Q_t(\mathcal{D}), w] = \mathcal{D}[g(\mathcal{D}), w] = [h(\mathcal{D}), w] - [\mathcal{D}, w]g(\mathcal{D})$$

is bounded. Consider the integral formula (see the proof of [Pedersen 1979, Proposition 1.3.8])

$$Q_t(\mathcal{D})^t = \frac{\sin(\pi t)}{\pi} \int_0^\infty \lambda^{-t}(1 + \lambda Q_t(\mathcal{D}))^{-1} Q_t(\mathcal{D})\, d\lambda. \tag{2.7}$$

Since $Q_t(\mathcal{D})$ is bounded below by $(1-t)^{1/2}$, we know that $Q_t(\mathcal{D})$ is invertible, and that

$$\|(1 + \lambda Q_t(\mathcal{D}))^{-1}\| \le (1-t)^{-1/2}\lambda^{-1}. \tag{2.8}$$

In particular, $(1 + \lambda Q_t(\mathcal{D}))^{-1}$ is of order $\mathcal{O}(\lambda^{-1})$ as $\lambda \to \infty$. Using that $\operatorname{Dom}\mathcal{D}$ is a core for $Q_t(D)$ and $\mathcal{D}$ commutes with $Q_t(\mathcal{D})$, we then compute

$$\begin{aligned}[(1 + \lambda Q_t(\mathcal{D}))^{-1} Q_t(\mathcal{D}), w]\mathcal{D} &= (1 + \lambda Q_t(\mathcal{D}))^{-1}[Q_t(\mathcal{D}), w]\mathcal{D} \\ &\quad - \lambda(1 + \lambda Q_t(\mathcal{D}))^{-1}[Q_t(\mathcal{D}), w]\mathcal{D}(1 + \lambda Q_t(\mathcal{D}))^{-1} Q_t(\mathcal{D}),\end{aligned}$$

and we see that $\|[(1 + \lambda Q_t(\mathcal{D}))^{-1} Q_t(\mathcal{D}), w]\mathcal{D}\|$ is finite and of order $\mathcal{O}(\lambda^0)$ for $\lambda \to 0$, and of order $\mathcal{O}(\lambda^{-1})$ as $\lambda \to \infty$. By applying the above integral formula, we obtain that

$$S_t := [g_t(\mathcal{D}), w]\mathcal{D} = [Q_t(\mathcal{D})^t, w]\mathcal{D}$$
$$= \frac{\sin(\pi t)}{\pi} \int_0^\infty \lambda^{-t}[(1+\lambda Q_t(\mathcal{D}))^{-1} Q_t(\mathcal{D}), w]\mathcal{D}\, d\lambda$$

is a norm-convergent integral. It follows that S_t is a bounded operator. To show that S_t is in fact uniformly bounded in t, let us split the integral in two parts. First, since $\|(1+\lambda Q_t(\mathcal{D}))^{-1}\| \le 1$, we have

$$\left\| \frac{\sin(\pi t)}{\pi} \int_0^1 \lambda^{-t}[(1+\lambda Q_t(\mathcal{D}))^{-1} Q_t(\mathcal{D}), w]\mathcal{D}\, d\lambda \right\|$$
$$\le \frac{\sin(\pi t)}{\pi} \|[g(\mathcal{D}), w]\mathcal{D}\|(1+\|Q_t(\mathcal{D})\|) \int_0^1 \lambda^{-t}\, d\lambda$$
$$\le \frac{\sin(\pi t)}{\pi} \|[g(\mathcal{D}), w]\mathcal{D}\|(2+\|g(\mathcal{D})\|)(1-t)^{-1}.$$

Second, using (2.8) we estimate

$$\left\| \frac{\sin(\pi t)}{\pi} \int_1^\infty \lambda^{-t}[(1+\lambda Q_t(\mathcal{D}))^{-1} Q_t(\mathcal{D}), w]\mathcal{D}\, d\lambda \right\|$$
$$\le \frac{\sin(\pi t)}{\pi} \|[g(\mathcal{D}), w]\mathcal{D}\|((1-t)^{-1/2} + (1-t)^{-1}\|Q_t(\mathcal{D})\|) \int_1^\infty \lambda^{-t-1}\, d\lambda$$
$$\le \frac{\sin(\pi t)}{\pi} \|[g(\mathcal{D}), w]\mathcal{D}\|(2(1-t)^{-1/2} + (1-t)^{-1}\|g(\mathcal{D})\|)t^{-1}.$$

Using that $\sin(\pi t) = \mathcal{O}(t)$ as $t \to 0$ and $\sin(\pi t) = \mathcal{O}(1-t)$ as $t \to 1$, we see that both integrals are uniformly bounded in t. Thus, S_t is uniformly bounded. It then suffices to check strict continuity on the dense submodule $\operatorname{Dom}\mathcal{D}$. Since $g_t(\mathcal{D})$ is strongly continuous, we see that S_t is strongly continuous on $\operatorname{Dom}\mathcal{D}$. Furthermore, rewriting

$$\mathcal{D}[g_t(\mathcal{D}), w] = [\mathcal{D}g_t(\mathcal{D}), w] - [\mathcal{D}, w]g_t(\mathcal{D})$$
$$= [g_t(\mathcal{D}), w]\mathcal{D} + g_t(\mathcal{D})[\mathcal{D}, w] - [\mathcal{D}, w]g_t(\mathcal{D}),$$

we conclude that $S_t^* = -\mathcal{D}[g_t(\mathcal{D}), w]$ is also strongly continuous on $\operatorname{Dom}\mathcal{D}$. Thus, we have shown that the commutator

$$[\mathcal{D}_t, w] = [\mathcal{D}, w]g_t(\mathcal{D}) + \mathcal{D}[g_t(\mathcal{D}), w]$$

is uniformly bounded and strictly continuous, and therefore $[\widetilde{\mathcal{D}}, w]$ is bounded and adjointable on $C([0, 1], E)$.

Now consider the functions $R_t \in C_0(\mathbb{R})$ given by $R_t(x) := (i \pm x g_t(x))^{-1}$. We claim that $t \mapsto R_t$ is continuous with respect to the supremum-norm on $C_0(\mathbb{R})$. To prove this claim, first observe that $g_t(x) \ge g(x)^t \ge \min(1, g(x))$ for all $x \in \mathbb{R}$ and

$t \in [0,1]$. Hence, for each $\varepsilon > 0$ there exists $r \in (0,\infty)$ such that for all $t \in [0,1]$ we have $\sup_{|x|>r} |R_t(x)| \le \varepsilon$. Then for $t, s \in [0,1]$ we can estimate

$$\begin{aligned}\|R_t - R_s\| &\le 2\varepsilon + \sup_{|x|<r} \|R_t(x) - R_s(x)\| \le 2\varepsilon + \sup_{|x|<r} \|xg_t(x) - xg_s(x)\| \\ &\le 2\varepsilon + r \sup_{|x|<r} \|g_t(x) - g_s(x)\|.\end{aligned}$$

Since $g_t(x)$ is uniformly continuous for $|x| < r$, we see that $t \mapsto R_t$ is norm-continuous. Consequently, we conclude that $t \mapsto (i \pm \mathscr{D}_t)^{-1}$ is a norm-continuous map such that $w(i \pm \mathscr{D}_t)^{-1}$ is compact for each $w \in W$ and $t \in [0,1]$. Hence, $w(\widetilde{\mathscr{D}} \pm i)^{-1}$ is compact on $C([0,1], E)$. This completes the proof that $\mathscr{D}_t$ yields an operator-homotopy $(C([0,1], E), \widetilde{\mathscr{D}})$. □

Remark 2.9. A *higher order* unbounded Kasparov module is a pair $(E, \mathscr{D})$ such that there exist $0 < \varepsilon < 1$ and a dense $*$-subalgebra $\mathscr{A} \subset A$ for which the operators $[\mathscr{D}, a](1+\mathscr{D}^2)^{-(1-\varepsilon)/2}$ (for $a \in \mathscr{A}$) extend to bounded operators. The class of higher order Kasparov modules contains all *ordinary* unbounded Kasparov modules. In [Goffeng et al. 2019, Theorem 1.37] it was shown that the C^1-function

$$\operatorname{sgnlog}(x) := \operatorname{sgn}(x) \log(1 + |x|)$$

can be used to turn a higher order unbounded Kasparov module into an ordinary unbounded Kasparov module. In fact, the proof of [Goffeng et al. 2019, Theorem 1.37] shows that for any unbounded cycle $(E, \mathscr{D})$ (as in Definition 1.1) we have the inclusion $\operatorname{Lip}(\mathscr{D}) \subset \operatorname{Lip}(\operatorname{sgnlog}(\mathscr{D}))$. It then follows from Proposition 2.6 that any unbounded cycle $(E, \mathscr{D})$ is operator-homotopic to $(E, \operatorname{sgnlog}(\mathscr{D}))$.

Using the natural notion of homotopy for higher order modules, one can ask whether the transformation $(E, \mathscr{D}) \mapsto (E, \operatorname{sgnlog}(\mathscr{D}))$ can be implemented as an operator-homotopy within the class of higher order unbounded Kasparov modules, so that every higher order module would be operator-homotopic to an ordinary unbounded Kasparov module. It is not immediately clear if this is indeed the case.

2C. ***From bounded to unbounded homotopies.*** Recall the $*$-homomorphism $\mathrm{ev}_t : C([0,1], B) \to B$ given by $b \mapsto b(t)$. For a Hilbert $C([0,1], B)$-module E we write $E_t := E \mathbin{\widehat{\otimes}_{\mathrm{ev}_t}} B$ for the localization of E at $t \in [0,1]$. Moreover, for any $h \in \operatorname{End}^*_B(E)$, we consider the localization $h_t := h \mathbin{\widehat{\otimes}} 1$ on E_t. We describe some basic facts regarding these localizations in the Appendix.

Now consider two unbounded A-B-cycles $(E_0, \mathscr{D}_0)$ and $(E_1, \mathscr{D}_1)$, and assume that their bounded transforms are homotopic. Thus, there exists a homotopy (E, F) between $(E_0, F_{\mathscr{D}_0})$ and $(E_1, F_{\mathscr{D}_1})$, where E is a module over $C([0,1], B)$. For simplicity, let us assume that $\mathrm{ev}_j(E, F)$ is *equal* to $(E, F_{\mathscr{D}_j})$ (i.e., there is no unitary equivalence involved). We are ready to derive our main technical result.

Proposition 2.10. *Suppose A is separable, and B σ-unital. Consider two unbounded A-B-cycles $(E_0, \mathscr{D}_0)$ and $(E_1, \mathscr{D}_1)$, and let (E, F) be a homotopy between $(E_0, F_{\mathscr{D}_0})$ and $(E_1, F_{\mathscr{D}_1})$, satisfying $F = F^*$ and $F^2 \leq 1$. Let $W_j \subset \mathrm{Lip}^0(\mathscr{D}_j)$ be countable subsets consisting of products of elements in $\mathrm{Lip}^0(\mathscr{D}_j)$, such that $A \subset \overline{W_j}$ (for $j = 0, 1$). Then there exists a positive operator $l \in J_F \subset \mathrm{End}^*_{C([0,1],B)}(E)$ with dense range in E such that*

(1) *the (closure of the) operator $\mathscr{S} := \frac{1}{2}(Fl^{-1} + l^{-1}F)$ makes $(E, \mathscr{S})$ into an unbounded A-$C([0, 1], B)$-cycle, and*

(2) *writing $l_j := \mathrm{ev}_j(l)$ and $\mathscr{S}_j := \mathrm{ev}_j(\mathscr{S})$ (for $j = 0, 1$), we have*

$$l_j \in C^*((1+\mathscr{D}_j^2)^{-1}), \qquad \mathscr{S}_j = F_{\mathscr{D}_j} l_j^{-1}, \qquad W_j \subset \mathrm{Lip}(l_j^{-1}) \cap \mathrm{Lip}(\mathscr{S}_j),$$

and the operator $l_j^{-1}(1+\mathscr{D}_j^2)^{-1/4}$ extends to an adjointable endomorphism.

Proof. Note that (1) can be obtained by an application of Proposition 1.12. In order to achieve (2) simultaneously, we need to construct our lift more carefully. Consider again the σ-unital C^*-algebra $J_F = \mathrm{End}^0_{C([0,1],B)}(E) + C_F$. Let $k \in \mathrm{End}^0_{C([0,1],B)}(E)$ be an even strictly positive element and $\chi \in C([0, 1])$ be given by $\chi(t) := t(1-t)$. Then $\chi k \in \mathrm{End}^0_{C([0,1],B)}(E)$ (see Lemma A.1), and we define

$$h := \chi k + (1 - F^2) \in J_F.$$

Consider the localizations

$$h_t := \mathrm{ev}_t(h) = \chi(t)k_t + (1 - F_t^2).$$

For $t \in (0, 1)$ we have that $\chi(t) > 0$, and $\chi(t)k_t$ has dense range in E_t by Corollary A.3. Since $\chi(t)k_t \leq h_t$, h_t has dense range in E_t by [Lance 1995, Corollary 10.2]. For $t \in \{0, 1\}$, we have $h_t = (1 - F_t^2)^{1/2} = (1+\mathscr{D}_t^2)^{-1/2}$, which has dense range as well. Thus, applying Corollary A.3 again, we conclude that h has dense range in E. Moreover, from Lemma 1.10 it follows that h is a strictly positive element in J_F.

Let $\mathscr{A} := \{a_i\}_{i\in\mathbb{N}} \subset A$ be a countable dense subset of A, let $\{c_i\}_{i\in\mathbb{N}}$ be a countable dense subset of $C^*(h)$, and let $\{w_{j,i}\}_{i\in\mathbb{N}}$ be an enumeration of W_j. We have the inclusions AJ_F, J_FA, FJ_F, $J_FF \subset J_F$ (see Lemma 1.10). Since $\mathrm{ev}_j(F) = F_{\mathscr{D}_j}$ and $W_j \subset \mathrm{Lip}^0(\mathscr{D}_j)$, we have for all $w \in W_j$ that $w(1 - F^2_{\mathscr{D}_j}) = w(1+\mathscr{D}_j^2)^{-1} \in \mathrm{End}^0_B(E_j)$. Moreover, by assumption any $w \in W_j$ is of the form $w = T_1T_2$ for $T_1, T_2 \in \mathrm{Lip}^0(\mathscr{D}_j)$. Since $[F_{\mathscr{D}_j}, T_1]T_2$ is compact, as explained in the proof of Proposition 1.7, it follows that also $[F_{\mathscr{D}_j}, w] \in \mathrm{End}^0_B(E_j)$. It thus holds that

$$W_j J_{F_{\mathscr{D}_j}},\ J_{F_{\mathscr{D}_j}} W_j,\ F_{\mathscr{D}_j} J_{F_{\mathscr{D}_j}},\ J_{F_{\mathscr{D}_j}} F_{\mathscr{D}_j} \subset J_{F_{\mathscr{D}_j}}.$$

Furthermore, since $\mathrm{ev}_j : C([0, 1], B) \to B = \mathrm{End}^0_B(B)$ is a surjective $*$-homomorphism we have $\mathrm{End}^0_B(E_j) = \mathrm{End}^0_{C([0,1],B)}(E) \mathbin{\widehat{\otimes}}_{\mathrm{ev}_j} 1$ and hence $J_{F_{\mathscr{D}_j}} = J_F \mathbin{\widehat{\otimes}}_{\mathrm{ev}_j} 1$.

Therefore, any approximate unit $u_n \in J_F$ gives an approximate unit $\mathrm{ev}_j(u_n)$ for $J_{F_{\mathcal{D}_j}}$. The C^*-subalgebra $C^*(h) \subset J_F$ thus contains a commutative approximate unit u_n for J_F which is quasicentral for A and F, and such that for $j \in \{0, 1\}$, $\mathrm{ev}_j(u_n)$ is quasicentral for W_j [Akemann and Pedersen 1977, Theorem 3.2].

By fixing a choice of $0 < \varepsilon < 1$ and selecting a suitable subsequence of u_n, we can achieve that, for each $n \in \mathbb{N}$, $d_n := u_{n+1} - u_n$ satisfies properties (a)–(e) of the proof of Proposition 1.12 as well as

(c′) $\|\mathrm{ev}_j(d_n)[\mathrm{ev}_j(F), w_{j,i}]\| \leq \varepsilon^{2n}$ for all $i \leq n$ and for $j = 0, 1$, and

(d′) $\|[\mathrm{ev}_j(d_n), w_{j,i}]\| \leq \varepsilon^{2n}$ for all $i \leq n$ and for $j = 0, 1$.

As in Proposition 1.12, property (c′) follows because $\mathrm{ev}_j(u_n)$ is an approximate unit for $J_{F_{\mathcal{D}_j}}$ and (d′) follows because $\mathrm{ev}_j(u_n)$ is quasicentral for W_j. Thus, as in Proposition 1.12, we can construct a strictly positive element $l \in J_F$, such that the (closure of the) operator

$$\mathcal{S} := \tfrac{1}{2}(Fl^{-1} + l^{-1}F)$$

is a densely defined and regular self-adjoint operator on E, and $(E, \mathcal{S})$ is an unbounded Kasparov A-$C([0, 1], B)$-module for which we have $\mathcal{A} \subset \mathrm{Lip}^0(\mathcal{S})$. This proves (1).

For (2), we first note that $l_j \in C^*(h_j)$ and $h_j = (1 + \mathcal{D}_j^2)^{-1}$ for $j = 0, 1$. In particular, l_j commutes with $F_{\mathcal{D}_j}$ and $\mathcal{S}_j = F_{\mathcal{D}_j} l_j^{-1}$. Properties (c′) and (d′) ensure that $[\mathcal{S}_j, w]$ and $[l_j^{-1}, w]$ are bounded for all $w \in W_j$ $(j = 0, 1)$. Furthermore, from property (b) it follows that $l^{-1}(1 - F^2)^{1/4}$ is everywhere defined and bounded, and localizing in $j = 0, 1$ then shows that $l_j^{-1}(1 + \mathcal{D}_j^2)^{-1/4}$ is bounded. □

Theorem 2.11. *Suppose A is separable, and B σ-unital. Consider two unbounded A-B-cycles $(\pi_0, E_0, \mathcal{D}_0)$ and $(\pi_1, E_1, \mathcal{D}_1)$. Any homotopy (π, E, F) between $(\pi_0, E_0, F_{\mathcal{D}_0})$ and $(\pi_1, E_1, F_{\mathcal{D}_1})$ can be lifted to an unbounded A-$C([0, 1], B)$-cycle $(\pi, E, \mathcal{S})$ such that, for $j = 0, 1$,*

- *the endpoints $\mathrm{ev}_j(\pi, E, \mathcal{S})$ are unitarily equivalent to $(\pi_j, E_j, f_j(\mathcal{D}_j))$ for dampening functions $f_j : \mathbb{R} \to \mathbb{R}$, and*
- *there exist countable self-adjoint subsets $W_j \subset \mathrm{Lip}^0(\mathcal{D}_j) \cap \mathrm{Lip}(f_j(\mathcal{D}_j))$ such that $\pi_j(A) \subset \overline{W_j}$.*

Moreover, if (π, E, F) is an operator-homotopy, then $(\pi, E, \mathcal{S})$ is an operator-homotopy.

Proof. We may assume (without loss of generality) that $F = F^*$ and $F^2 \leq 1$ [Blackadar 1998, Proposition 17.4.3]. For $j = 0, 1$, we have unitary equivalences $U_j : \mathrm{ev}_j(E) \to E_j$ such that $\mathrm{ev}_j(F) = U_j^* F_{\mathcal{D}_j} U_j$. Then $\mathcal{D}_j$ on E_j is unitarily equivalent to $U_j^* \mathcal{D}_j U_j$ on $\mathrm{ev}_j(E)$. To simplify notation, we will from here on ignore this unitary equivalence and simply assume that $\mathrm{ev}_j(E, F)$ is *equal* to $(E_j, F_{\mathcal{D}_j})$.

We know by Lemma 1.6 that, for $j = 0, 1$, there exist countable self-adjoint subsets $W_j \subset \mathrm{Lip}^0(\mathcal{D}_j)$ consisting of products of elements in $\mathrm{Lip}^0(\mathcal{D}_j)$, such that $\pi_j(A) \subset \overline{W_j}$. From Proposition 2.10, we obtain an unbounded A-$C([0, 1], B)$-cycle $(E, \mathcal{S} := \frac{1}{2}(Fl^{-1} + l^{-1}F))$, which provides a homotopy between $(E_0, \mathcal{S}_0)$ and $(E_1, \mathcal{S}_1)$, where $\mathcal{S}_j := \mathrm{ev}_j(\mathcal{S})$. By property (2) of Proposition 2.10, we know that $l_j \in C^*((1+\mathcal{D}_j^2)^{-1})$, $\mathcal{S}_j = F_{\mathcal{D}_j} l_j^{-1}$, $W_j \subset \mathrm{Lip}(l_j^{-1}) \cap \mathrm{Lip}(\mathcal{S}_j)$, and $l_j^{-1}(1+\mathcal{D}_j^2)^{-1/4}$ is bounded. It follows that we can write $\mathcal{S}_j = f_j(\mathcal{D}_j)$ for some dampening function f_j, which proves the first statement. Furthermore, if we have in fact an *operator*-homotopy (E, F), then it is clear that the lift $(E, \mathcal{S})$ obtained from Proposition 2.10 is also an *operator*-homotopy. $\square$

2D. *The isomorphism with KK-theory.* Using the results from the previous sections, we can now prove that our semigroup $\overline{UKK}(A, B)$ is isomorphic to Kasparov's KK-group.

Theorem 2.12. *Suppose A is separable, and B σ-unital. The bounded transform induces a semigroup isomorphism $\overline{UKK}(A, B) \to KK(A, B)$, given by* $[(E, \mathcal{D})] \mapsto [(E, F_{\mathcal{D}})]$.

Proof. If there exists a homotopy $(E, \mathcal{D})$ between unbounded A-B-cycles $(E_0, \mathcal{D}_0)$ and $(E_1, \mathcal{D}_1)$, then $(E, F_{\mathcal{D}})$ provides a homotopy between $(E_0, F_{\mathcal{D}_0})$ and $(E_1, F_{\mathcal{D}_1})$. Moreover, the bounded transform is compatible with direct sums, so it induces a well-defined semigroup homomorphism. Furthermore, this homomorphism is surjective by Theorem 1.13, so it remains to prove that it is also injective.

Consider two unbounded A-B-cycles $(E_0, \mathcal{D}_0)$ and $(E_1, \mathcal{D}_1)$, with $[(E_0, F_{\mathcal{D}_0})] = [(E_1, F_{\mathcal{D}_1})]$. Then there exists a homotopy (E, F) between $(E_0, F_{\mathcal{D}_0})$ and $(E_1, F_{\mathcal{D}_1})$. From Theorem 2.11 we obtain an unbounded A-$C([0, 1], B)$-cycle $(E, \mathcal{S})$ such that, for $j = 0, 1$, the endpoints $\mathrm{ev}_j(E, \mathcal{S})$ are unitarily equivalent to $(E_j, f_j(\mathcal{D}_j))$ for dampening functions $f_j : \mathbb{R} \to \mathbb{R}$, and there exist self-adjoint subsets $W_j \subset \mathrm{Lip}^0(\mathcal{D}_j) \cap \mathrm{Lip}(f_j(\mathcal{D}_j))$ such that $\pi_j(A) \subset \overline{W_j}$. It then follows from Proposition 2.6 that $\mathcal{D}_j$ is operator-homotopic to $\mathcal{S}_j$. Thus, we have the composition of homotopies

$$\mathcal{D}_0 \sim_{oh} \mathcal{S}_0 \sim_h \mathcal{S}_1 \sim_{oh} \mathcal{D}_1,$$

which proves that $[(E_0, \mathcal{D}_0)] = [(E_1, \mathcal{D}_1)]$. $\square$

Remark 2.13. A priori, $\overline{UKK}(A, B)$ is a semigroup, and the isomorphism

$$\overline{UKK}(A, B) \to KK(A, B)$$

is an isomorphism of semigroups. Since $KK(A, B)$ is a group, it of course follows that $\overline{UKK}(A, B)$ is also a group. However, the isomorphism $\overline{UKK}(A, B) \to KK(A, B)$ requires the assumption that A is separable. In Theorem 4.16 we will give a direct proof that $\overline{UKK}(A, B)$ is a group, which avoids the bounded transform and therefore also works for nonseparable (σ-unital) C^*-algebras.

For any dense $*$-subalgebra $\mathscr{A} \subset A$, we define $\Psi_1(\mathscr{A}, B)$ as the set of those $(\pi, E, \mathscr{D}) \in \overline{\Psi}_1(A, B)$ for which $\pi(\mathscr{A}) \subset \mathrm{Lip}^0(\mathscr{D})$, and we define $UKK(\mathscr{A}, B)$ as the homotopy equivalence classes of elements in $\Psi_1(\mathscr{A}, B)$ (where it is understood that the homotopies are given by elements in $\Psi_1(\mathscr{A}, C([0,1], B))$). The natural inclusion $\Psi_1(\mathscr{A}, B) \hookrightarrow \overline{\Psi}_1(A, B)$ induces a well-defined semigroup homomorphism $UKK(\mathscr{A}, B) \to \overline{UKK}(A, B)$. We say that $\mathscr{A}$ is *countably generated* if $\mathscr{A}$ contains a countable subset that generates it as a $*$-algebra over $\mathbb{C}$. We emphasize that this does not involve taking closures of any kind. While, as we explained in Remark 1.5, it is *not necessary* to fix a countably generated dense $*$-subalgebra $\mathscr{A} \subset A$, we will show next that it is nevertheless *possible* to define unbounded *KK*-theory using any such fixed choice for $\mathscr{A} \subset A$.

Proposition 2.14. *Suppose A is separable, and B σ-unital. For any countably generated dense $*$-subalgebra $\mathscr{A} \subset A$, the map $UKK(\mathscr{A}, B) \to \overline{UKK}(A, B)$ is an isomorphism.*

Proof. We have the following commuting diagram:

$$\begin{tikzcd} UKK(\mathscr{A}, B) \arrow[rr] \arrow[dr] & & \overline{UKK}(A, B) \arrow[dl] \\ & KK(A, B) & \end{tikzcd}$$

We know from Theorem 2.12 that the map $\overline{UKK}(A, B) \to KK(A, B)$ is an isomorphism. Thus, we need to show that also $UKK(\mathscr{A}, B) \to KK(A, B)$ is an isomorphism. The assumption that A is separable ensures that the bounded transform $UKK(\mathscr{A}, B) \to KK(A, B)$ is surjective (see Theorem 1.13). Moreover, the proofs of Theorems 2.11 and 2.12 with the special choice $W_j = \pi_j(\mathscr{A})$ show that the bounded transform is also injective. $\square$

3. Degenerate cycles

In this section, we will consider two notions of degenerate cycles in unbounded *KK*-theory, namely “algebraically degenerate” and “spectrally degenerate” cycles. Our aim is to prove the following:

- any degenerate cycle is *null-homotopic*, i.e., homotopic to the zero cycle $(0, 0)$, and
- any homotopy can be implemented as an operator-homotopy modulo addition of degenerate cycles.

3A. *Algebraically degenerate cycles.*

Definition 3.1. An unbounded A-B-cycle $(\pi, E, \mathscr{D})$ is called *algebraically degenerate* if $\pi = 0$.

By considering the obvious homotopy $(C_0((0,1],E),\mathscr{D})$, we easily obtain:

Lemma 3.2. *An algebraically degenerate unbounded A-B-cycle $(E,\mathscr{D})$ is null-homotopic.*

As an application of the above lemma, we will show that two unbounded cycles $(\pi, E, \mathscr{D})$ and $(\pi, E, \mathscr{D}')$ are homotopic if the difference $\mathscr{D}-\mathscr{D}'$ is "*locally bounded*".

Proposition 3.3. *Let $(\pi, E, \mathscr{D})$ and $(\pi, E, \mathscr{D}')$ be unbounded A-B-cycles. Suppose there exists a subset $W \subset \mathrm{Lip}^0(\mathscr{D}) \cap \mathrm{Lip}^0(\mathscr{D}')$ with $\pi(A) \subset \overline{W}$ such that for each $w \in W$, the operator $(\mathscr{D}-\mathscr{D}')w$ extends to a bounded operator. Then $(\pi, E, \mathscr{D})$ and $(\pi, E, \mathscr{D}')$ are homotopic.*

Proof. Consider the unbounded A-$C([0,1],B)$-cycle $(\pi, C([0,1],E\oplus E), \mathscr{D}\oplus\mathscr{D}')$ with the representation given for $t \in [0,1]$ by $\pi_t(a) := (a \oplus a)P_t$ in terms of the norm-continuous family of projections

$$P_t := \begin{pmatrix} \cos^2(\frac{1}{2}\pi t) & \cos(\frac{1}{2}\pi t)\sin(\frac{1}{2}\pi t) \\ \cos(\frac{1}{2}\pi t)\sin(\frac{1}{2}\pi t) & \sin^2(\frac{1}{2}\pi t) \end{pmatrix}.$$

We note that $P_0 = 1 \oplus 0$ and $P_1 = 0 \oplus 1$. For homogeneous $w \in W$ we compute

$$[\mathscr{D}\oplus\mathscr{D}', (w\oplus w)P_t] = \begin{pmatrix} [\mathscr{D},w]\cos^2(\frac{1}{2}\pi t) & (\mathscr{D}w-(-1)^{\deg w}w\mathscr{D}')\cos(\frac{1}{2}\pi t)\sin(\frac{1}{2}\pi t) \\ (\mathscr{D}'w-(-1)^{\deg w}w\mathscr{D})\cos(\frac{1}{2}\pi t)\sin(\frac{1}{2}\pi t) & [\mathscr{D}',w]\sin^2(\frac{1}{2}\pi t) \end{pmatrix}.$$

We observe that $\mathscr{D}w-(-1)^{\deg w}w\mathscr{D}' = (\mathscr{D}-\mathscr{D}')w + [\mathscr{D}',w]$ is bounded, and similarly for $\mathscr{D}'w-(-1)^{\deg w}w\mathscr{D}$. Hence, $[\mathscr{D}\oplus\mathscr{D}', (w\oplus w)P_t]$ is uniformly bounded and norm-continuous in t, and we obtain $(w\oplus w)P_\bullet \subset \mathrm{Lip}(\mathscr{D}\oplus\mathscr{D}')$. Moreover, since the resolvents of $\mathscr{D}\oplus\mathscr{D}'$ are constant in t, we have in fact $(w\oplus w)P_\bullet \subset \mathrm{Lip}^0(\mathscr{D}\oplus\mathscr{D}')$. Thus, we have

$$\pi_\bullet(A) \subset \overline{\{(w\oplus w)P_\bullet : w \in W\}} \subset \overline{\mathrm{Lip}^0(\mathscr{D}\oplus\mathscr{D}')},$$

and we have a homotopy between $(\pi\oplus 0, E\oplus E, \mathscr{D}\oplus\mathscr{D}')$ and $(0\oplus\pi, E\oplus E, \mathscr{D}\oplus\mathscr{D}')$. Finally, since algebraically degenerate cycles are null-homotopic by Lemma 3.2, we note that $(\pi\oplus 0, E\oplus E, \mathscr{D}\oplus\mathscr{D}')$ is homotopic to $(\pi, E, \mathscr{D})$, and that $(0\oplus\pi, E\oplus E, \mathscr{D}\oplus\mathscr{D}')$ is homotopic to $(\pi, E, \mathscr{D}')$. □

Remark 3.4. The assumption that $(\mathscr{D}-\mathscr{D}')w$ is bounded for all $w \in W$ is interpreted as saying that $\mathscr{D}-\mathscr{D}'$ is *locally bounded*. In the above proposition, we have assumed that both $(E,\mathscr{D})$ and $(E,\mathscr{D}')$ are unbounded cycles. Under certain conditions, it suffices to assume only that $(E,\mathscr{D})$ is an unbounded cycle; using local boundedness of $\mathscr{D}-\mathscr{D}'$ one can then *prove* that $(E,\mathscr{D}')$ is also an unbounded cycle. We refer to [van den Dungen 2018] for further details.

3B. ***Spectrally degenerate cycles.*** We denote by $\mathrm{sgn} : \mathbb{R} \setminus \{0\} \to \{\pm 1\}$ the function $\mathrm{sgn}(x) := \frac{x}{|x|}$. We say that a regular self-adjoint operator $\mathcal{D} : \mathrm{Dom}\,\mathcal{D} \to E$ is *invertible* if there exists $\mathcal{D}^{-1} \in \mathrm{End}^*_B(E)$ that satisfies $\mathcal{D}\mathcal{D}^{-1} = \mathcal{D}^{-1}\mathcal{D} = 1$. It then follows that $\mathrm{Dom}\,\mathcal{D} = \mathrm{Ran}\,\mathcal{D}^{-1} = \mathrm{Ran}|\mathcal{D}|^{-1}$ and $\mathrm{Ran}\,\mathcal{D} = E$. Thus, if $\mathcal{D}$ is invertible, $\mathrm{sgn}(\mathcal{D})$ is well-defined and equal to $\mathcal{D}|\mathcal{D}|^{-1}$.

Definition 3.5. An unbounded A-B-cycle $(\pi, E, \mathcal{D})$ is called *spectrally degenerate* if $\mathcal{D}$ is invertible and there exists $W \subset \mathrm{Lip}^0(\mathcal{D})$ such that $\pi(A) \subset \overline{W}$ and $[\mathrm{sgn}(\mathcal{D}), w] = 0$ for all $w \in W$.

Lemma 3.6. *Let* $\mathcal{D} : \mathrm{Dom}\,\mathcal{D} \to E$ *be self-adjoint, regular, and invertible. If* $w \in \mathrm{End}^*_B(E)$ *is such that* $w : \mathrm{Dom}\,\mathcal{D} \to \mathrm{Dom}\,\mathcal{D}$ *and* $[\mathrm{sgn}(\mathcal{D}), w] = 0$, *then* $[\mathcal{D}, w]$ *is bounded if and only if* $[|\mathcal{D}|, w]$ *is bounded.*

Proof. This follows from the simple observation that $\mathrm{sgn}(\mathcal{D})$ is a self-adjoint unitary and $\mathcal{D} = \mathrm{sgn}(\mathcal{D})|D|$. We have

$$[\mathcal{D}, w] = \mathrm{sgn}(\mathcal{D})[|\mathcal{D}|, w], \qquad [|\mathcal{D}|, w] = \mathrm{sgn}(\mathcal{D})[\mathcal{D}, w],$$

whence $[\mathcal{D}, w]$ is bounded if and only if $[|\mathcal{D}|, w]$ is bounded. □

We have already seen in Lemma 3.2 that any algebraically degenerate cycle is null-homotopic. Here we shall prove that also any spectrally degenerate cycle $(E, \mathcal{D})$ is null-homotopic. The easiest way to prove this is by observing that the bounded transform $(E, F_{\mathcal{D}})$ is operator-homotopic to the degenerate cycle $(E, \mathrm{sgn}(\mathcal{D}))$ (which is null-homotopic), and then applying Theorem 2.11. However, we can only apply Theorem 2.11 if A is separable. But with only a bit more effort, we can in fact explicitly construct an *unbounded* homotopy between any spectrally degenerate cycle and the zero module.

Proposition 3.7. *Any spectrally degenerate unbounded* A-B*-cycle* $(E, \mathcal{D})$ *is null-homotopic.*

Proof. Consider for $t \in (0, 1]$ the family of regular self-adjoint operators

$$\mathcal{D}_t := t^{-1}\,\mathrm{sgn}(\mathcal{D})|\mathcal{D}|^t.$$

Since $t \mapsto |\mathcal{D}|^{t-1}$ is norm-continuous and $|\mathcal{D}|^t = |\mathcal{D}|^{t-1}|\mathcal{D}|$, we see that $|\mathcal{D}|^t$ is strongly continuous on $\mathrm{Dom}\,\mathcal{D}$. Since $\mathrm{Dom}\,\mathcal{D}$ is a core for $\mathcal{D}_t$ for each $t \in (0, 1]$, we obtain from Lemma 1.15 a regular self-adjoint operator $\widetilde{\mathcal{D}}$ on the Hilbert $C([0, 1], B)$-module $\widetilde{E} := C_0((0, 1], E)$. We claim that $(\widetilde{E}, \widetilde{\mathcal{D}})$ is an unbounded cycle, and therefore it provides a homotopy between $\mathrm{ev}_1(\widetilde{E}, \widetilde{\mathcal{D}}) = (E, \mathcal{D})$ and $\mathrm{ev}_0(\widetilde{E}, \widetilde{\mathcal{D}}) = (0, 0)$.

To prove the claim, choose $W \subset \mathrm{Lip}^0(\mathcal{D})$ such that $\pi(A) \subset \overline{W}$ and $[\mathrm{sgn}(\mathcal{D}), w] = 0$ for all $w \in W$. First consider the resolvents of $\mathcal{D}_t$. We compute

$$(\mathcal{D}_t \pm i)^{-1} = \mp it\,\mathrm{sgn}(\mathcal{D})|\mathcal{D}|^{-t}(t\,\mathrm{sgn}(\mathcal{D})|\mathcal{D}|^{-t} \mp i)^{-1}. \tag{3.8}$$

Since $\mathscr{D}$ is invertible, the operators $w|\mathscr{D}|^{-t}$ are compact for $0 < t \leq 1$ and for $w \in W$, and hence so are $w(\mathscr{D}_t \pm i)^{-1}$. Moreover, $t \mapsto |\mathscr{D}|^{-t}$ is norm-continuous on $(0, 1]$, and therefore $t \mapsto t \operatorname{sgn}(\mathscr{D})|\mathscr{D}|^{-t}$ is norm-continuous on $(0, 1]$. But then the composition with $x \mapsto x(x \pm i)^{-1}$ gives again a continuous function, and we see from (3.8) that $t \mapsto (\mathscr{D}_t \pm i)^{-1}$ is norm-continuous on $(0, 1]$. Furthermore, since $|\mathscr{D}|^{-t}$ is uniformly bounded and $t \operatorname{sgn}(\mathscr{D})|\mathscr{D}|^{-t}$ is self-adjoint, it also follows from (3.8) that

$$\lim_{t \searrow 0} \|(\mathscr{D}_t \pm i)^{-1}\| = \lim_{t \searrow 0} t \|\operatorname{sgn}(\mathscr{D})|\mathscr{D}|^{-t}(t \operatorname{sgn}(\mathscr{D})|\mathscr{D}|^{-t} \mp i)^{-1}\| = 0,$$

so we also obtain continuity at 0. Hence, $w(\widetilde{\mathscr{D}} \pm i)^{-1}$ is compact on $\widetilde{E}$.

Next, we consider the commutator $[\mathscr{D}_t, w] = t^{-1} \operatorname{sgn}(\mathscr{D})[|\mathscr{D}|^t, w]$ for some self-adjoint $w \in W$. We have seen above that $|\mathscr{D}|^t$ is strongly continuous on $\operatorname{Dom} \mathscr{D}$, and hence also $[\mathscr{D}_t, w]$ is strongly continuous on $\operatorname{Dom} \mathscr{D}$. To show that $[\mathscr{D}_t, w]$ is strongly continuous everywhere, it then suffices to show that $[\mathscr{D}_t, w]$ is uniformly bounded. For this purpose, we consider the operator inequality

$$\pm i[|\mathscr{D}|^{-1}, w] = \mp i|\mathscr{D}|^{-1}[|\mathscr{D}|, w]|\mathscr{D}|^{-1} \leq \|[|\mathscr{D}|, w]\||\mathscr{D}|^{-2},$$

where $[|\mathscr{D}|, w]$ is bounded by Lemma 3.6. Applying [Kucerovsky 2000, Proposition 2.11] to the function $f(x) := x^t$, we then find that

$$\begin{aligned}\pm i[|\mathscr{D}|^{-t}, w] = \pm i[f(|\mathscr{D}|^{-1}), w] &\leq f'(|\mathscr{D}|^{-1})\|[|\mathscr{D}|, w]\||\mathscr{D}|^{-2} \\ &= t|\mathscr{D}|^{1-t}\|[|\mathscr{D}|, w]\||\mathscr{D}|^{-2} = t\|[|\mathscr{D}|, w]\||\mathscr{D}|^{-1-t}.\end{aligned}$$

For any $\psi \in \operatorname{Dom} \mathscr{D}$, we therefore have

$$\begin{aligned}\langle \psi \mid \pm i[|\mathscr{D}|^t, w]\psi \rangle &= \langle \psi \mid \mp i|\mathscr{D}|^t[|\mathscr{D}|^{-t}, w]|\mathscr{D}|^t\psi \rangle = \langle |\mathscr{D}|^t\psi \mid \mp i[|\mathscr{D}|^{-t}, w]|\mathscr{D}|^t\psi \rangle \\ &\leq \langle |\mathscr{D}|^t\psi \mid t\|[|\mathscr{D}|, w]\||\mathscr{D}|^{-1}\psi \rangle = \langle \psi \mid t\|[|\mathscr{D}|, w]\||\mathscr{D}|^{t-1}\psi \rangle.\end{aligned}$$

Since both $\|[|\mathscr{D}|, w]\| \, |\mathscr{D}|^{t-1}$ and $[|\mathscr{D}|^t, w]$ are bounded for $t \in [0, 1]$ (for the latter, see for instance [Gracia-Bondía et al. 2001, Lemma 10.17]), we have the norm-inequality

$$\|[|\mathscr{D}|^t, w]\| = \|\pm i[|\mathscr{D}|^t, w]\| \leq t\|[|\mathscr{D}|, w]\| \, \||\mathscr{D}|^{t-1}\| \leq t\|[|\mathscr{D}|, w]\| \max\{1, \||\mathscr{D}|^{-1}\|\}.$$

We finally obtain

$$\|[\mathscr{D}_t, w]\| \leq t^{-1}\|\operatorname{sgn}(\mathscr{D})\| \, \|[|\mathscr{D}|^t, w]\| \leq \|[|\mathscr{D}|, w]\| \max\{1, \||\mathscr{D}|^{-1}\|\}.$$

Hence, $[\mathscr{D}_t, w]$ is uniformly bounded and strongly continuous as a function of $t \in (0, 1]$, and therefore the commutator $[\widetilde{\mathscr{D}}, w]$ is bounded on $\widetilde{E}$. Thus, we have shown that $W \subset \operatorname{Lip}^0(\widetilde{\mathscr{D}})$ and therefore $\widetilde{\pi}(A) \subset \overline{W} \subset \operatorname{Lip}^0(\widetilde{\mathscr{D}})$. ☐

3C. ***Operator-homotopies modulo degenerate cycles.*** In bounded *KK*-theory, it was shown by Kasparov that any homotopy can be implemented as an operator-homotopy modulo addition of degenerate modules [Kasparov 1980, §6, Theorem 1]. Using this result, we will prove that a similar statement holds in unbounded *KK*-theory.

Let $\sim_{oh+d}$ denote the equivalence relation on $\overline{\Psi}_1(A, B)$ given by operator-homotopies, unitary equivalences, and addition of spectrally degenerate and algebraically degenerate cycles. We already know from Lemma 3.2 and Proposition 3.7 that degenerate cycles are null-homotopic, so $\sim_{oh+d}$ is stronger than $\sim_h$. We will prove here that in fact these two relations coincide.

Lemma 3.9. *Suppose A is separable, and B σ-unital. Let (E, F) be a (bounded) Kasparov A-B-module, such that $F = F^*$, $F^2 = 1$, and $[F, a] = 0$ for all $a \in A$ (in particular, (E, F) is degenerate). Let $\mathcal{D} := Fl^{-1}$ be a lift of F, where l is a positive element in J_F with dense range in E obtained from Proposition 1.12. Then the unbounded A-B-cycle $(E, \mathcal{D})$ is spectrally degenerate.*

Proof. Since $F^2 = 1$ and $[F, l] = 0$, we have that $\mathcal{D}$ is invertible and that $\operatorname{sgn}(\mathcal{D}) = F$ (graded) commutes with the algebra A. Thus, $(E, \mathcal{D})$ is spectrally degenerate. □

Theorem 3.10. *Suppose A is separable, and B σ-unital. Then the homotopy equivalence relation $\sim_h$ on $\overline{\Psi}_1(A, B)$ coincides with the equivalence relation $\sim_{oh+d}$.*

Proof. We need to prove that the relation $\sim_{oh+d}$ is weaker than $\sim_h$. To this end let $(E_0, \mathcal{D}_0)$ and $(E_1, \mathcal{D}_1)$ be unbounded A-B-cycles which are homotopic. We then know that the bounded transforms $(\pi_0, E_0, F_{\mathcal{D}_0})$ and $(\pi_1, E_1, F_{\mathcal{D}_1})$ are also homotopic. By [Kasparov 1980, §6, Theorem 1], there exist degenerate bounded Kasparov modules (π'_0, E'_0, F'_0) and (π'_1, E'_1, F'_1) such that $(\pi_0 \oplus \pi'_0, E_0 \oplus E'_0, F_{\mathcal{D}_0} \oplus F'_0)$ is operator-homotopic to $(\pi_1 \oplus \pi'_1, E_1 \oplus E'_1, F_{\mathcal{D}_1} \oplus F'_1)$. Denote by $E_j^{\prime\mathrm{op}}$ the Hilbert B-module E'_j equipped with the opposite $\mathbb{Z}_2$-grading. By adding the algebraically degenerate module $(0, E_0^{\prime\mathrm{op}}, -F'_0) \oplus (0, E_1^{\prime\mathrm{op}}, -F'_1)$, we obtain the top line in the following diagram:

$$
\begin{tikzcd}
F_{\mathcal{D}_0} \oplus F'_0 \oplus -F'_0 \oplus -F'_1 \arrow[r, squiggly, no head, "oh"] \arrow[d, squiggly, no head, "oh"] & F_{\mathcal{D}_1} \oplus F'_1 \oplus -F'_0 \oplus -F'_1 \arrow[d, squiggly, no head, "oh"] \\
F_{\mathcal{D}_0} \oplus \widehat{F'_0} \oplus -F'_1 \arrow[d, squiggly, no head, "oh"] & F_{\mathcal{D}_1} \oplus \widehat{F'_1} \oplus -F'_0 \arrow[d, squiggly, no head, "oh"] \\
F_{\mathcal{D}_0} \oplus F_{\widehat{\mathcal{D}'_0}} \oplus -F_{\mathcal{D}'_1} \arrow[r, squiggly, no head, "oh"] & F_{\mathcal{D}_1} \oplus F_{\widehat{\mathcal{D}'_1}} \oplus -F_{\mathcal{D}'_0}
\end{tikzcd}
\tag{3.11}
$$

Since F'_j is degenerate, we know for all $a \in A$ that $[F'_j, \pi'_j(a)] = \pi'_j(a)(1-(F'_j)^2) = 0$. As in [Blackadar 1998, §17.6], $(\pi'_j \oplus 0, E'_j \oplus E'^{\mathrm{op}}_j, F'_j \oplus -F'_j)$ is operator-homotopic to the degenerate module

$$(\pi'_j \oplus 0, E'_j \oplus E'^{\mathrm{op}}_j, \widehat{F}'_j), \qquad \widehat{F}'_j := \begin{pmatrix} F'_j & (1-(F'_j)^2)^{1/2} \\ (1-(F'_j)^2)^{1/2} & -F'_j \end{pmatrix}.$$

This yields the vertical operator-homotopies between the first two lines in (3.11).

By construction, $(\widehat{F}'_j)^2 = 1$ and $[\widehat{F}'_j, \pi'_j(a)] = 0$. Hence, by Lemma 3.9 and Proposition 1.12 we can lift $\widehat{F}'_j$ to spectrally degenerate unbounded cycles $(\pi'_j \oplus 0, E'_j \oplus E'^{\mathrm{op}}_j, \widehat{\mathcal{D}}'_j)$, such that $\widehat{F}'_j \sim_{oh} F_{\widehat{\mathcal{D}}'_j}$. Moreover, using again Proposition 1.12, we can lift $-F'_j$ to algebraically degenerate unbounded cycles $(0, E'^{\mathrm{op}}_j, -\mathcal{D}'_j)$ such that $-F'_j \sim_{oh} -F_{\mathcal{D}'_j}$. This yields the vertical operator-homotopies between the last two lines in (3.11). Finally, by transitivity we obtain the horizontal operator-homotopy on the bottom line, and by Theorem 2.11 this operator-homotopy lifts to an unbounded operator-homotopy

$$\mathcal{D}_0 \oplus \widehat{\mathcal{D}}'_0 \oplus -\mathcal{D}'_1 \overset{oh}{\rightsquigarrow} \mathcal{D}_1 \oplus \widehat{\mathcal{D}}'_1 \oplus -\mathcal{D}'_0.$$

Thus, we have shown that $(E_0, \mathcal{D}_0) \sim_{oh+d} (E_1, \mathcal{D}_1)$. $\square$

4. Symmetries and the group structure

In this section we discuss various notions of symmetries for unbounded cycles. The presence of such symmetries induces homotopical triviality and can be used to give a direct proof of the fact that the semigroup $\overline{UKK}(A, B)$ is a group for any two σ-unital C^*-algebras.

4A. *Lipschitz regularity.* Let $0 < \alpha < 1$ and $f_\alpha \in C_0(\mathbb{R})$ be a function that behaves like x^α towards infinity. We will show here that we can use the functional dampening of Proposition 2.6 to replace any unbounded cycle $(E, \mathcal{D})$ by a *Lipschitz regular* cycle $(E, f_\alpha(\mathcal{D}))$.

Definition 4.1. An unbounded A-B-cycle $(\pi, E, \mathcal{D})$ is called *Lipschitz regular* if $\pi(A) \subset \mathrm{Lip}^0(\mathcal{D}) \cap \mathrm{Lip}(|\mathcal{D}|)$.

Remark 4.2. Since the map $x \mapsto |x| - (1+x^2)^{1/2}$ lies in $C_0(\mathbb{R})$, we have for $T \in \mathrm{End}^*_B(E)$ that $[|\mathcal{D}|, T]$ is bounded if and only if $[(1+\mathcal{D}^2)^{1/2}, T]$ is bounded, and therefore $\mathrm{Lip}(|\mathcal{D}|) = \mathrm{Lip}((1+\mathcal{D}^2)^{1/2})$.

The following result generalizes [Kaad 2019, Proposition 5.1], where the specific function $x \mapsto x(1+x^2)^{(\alpha-1)/2}$ was considered.

Proposition 4.3. *Let* $(E, \mathcal{D})$ *be an unbounded cycle*, $0 < \alpha < 1$, *and let* $f_\alpha : \mathbb{R} \to \mathbb{R}$ *be any odd continuous function such that* $\lim_{x\to\infty} f_\alpha(x) - x^\alpha$ *exists. Then*

$(E, f_\alpha(\mathcal{D}))$ defines a Lipschitz regular unbounded cycle that is operator homotopic to $(E, \mathcal{D})$.

Proof. We will show that $\mathrm{Lip}(\mathcal{D}) \subset \mathrm{Lip}(f_\alpha(\mathcal{D})) \cap \mathrm{Lip}(|f_\alpha(\mathcal{D})|)$, and the statement then follows from Proposition 2.6. Given two such functions f_α and g_α, both $f_\alpha - g_\alpha$ and $|f_\alpha| - |g_\alpha|$ lie in $C_b(\mathbb{R})$. Thus, $f_\alpha(\mathcal{D}) - g_\alpha(\mathcal{D})$ and $|f_\alpha(\mathcal{D})| - |g_\alpha(\mathcal{D})|$ are bounded operators, and we see that $\mathrm{Lip}(f_\alpha(\mathcal{D})) = \mathrm{Lip}(g_\alpha(\mathcal{D}))$ and $\mathrm{Lip}(|f_\alpha(\mathcal{D})|) = \mathrm{Lip}(|g_\alpha(\mathcal{D})|)$. Hence, it suffices to prove the statement for $f_\alpha(x) := x(1+x^2)^{(\alpha-1)/2}$. Using for $s \in (0, 1)$ the integral formula (which can be derived from (2.7) by replacing $Q_t(\mathcal{D})$ by $(1+\mathcal{D}^2)^{-1}$)

$$(1+\mathcal{D}^2)^{-s} = \frac{\sin(\pi s)}{\pi} \int_0^\infty \lambda^{-s}(1+\lambda+\mathcal{D}^2)^{-1}\, d\lambda,$$

it is shown in the proof of [Kaad 2019, Proposition 5.1] that $[(1+\mathcal{D}^2)^{(\alpha-1)/2}, T]\mathcal{D}$ extends to a bounded operator for each $T \in \mathrm{Lip}(\mathcal{D})$. Hence, $\mathrm{Lip}(\mathcal{D}) \subset \mathrm{Lip}(f_\alpha(\mathcal{D}))$.

To prove the Lipschitz regularity, we consider instead the function $g_\alpha(x) := \mathrm{sgn}(x)(1+x^2)^{\alpha/2}$. Using again the above integral formula, one can show similarly that

$$[|g_\alpha(\mathcal{D})|, T] = [(1+\mathcal{D}^2)^{\alpha/2}, T] = -(1+\mathcal{D}^2)^{\alpha/2}[(1+\mathcal{D}^2)^{-\alpha/2}, T](1+\mathcal{D}^2)^{\alpha/2}$$

is indeed bounded for each $T \in \mathrm{Lip}(\mathcal{D})$, and therefore $\mathrm{Lip}(\mathcal{D}) \subset \mathrm{Lip}(|g_\alpha(\mathcal{D})|)$. □

Remark 4.4. In addition to the two functions $x \mapsto x(1+x^2)^{(\alpha-1)/2}$ and $x \mapsto \mathrm{sgn}(x)(1+x^2)^{\alpha/2}$ considered in the proof of Proposition 4.3, another typical example of a function f_α as in Proposition 4.3 is the function $\mathrm{sgnmod}^\alpha : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto \mathrm{sgn}(x)|x|^\alpha$. Note that if $\mathcal{D}$ is invertible, then $\mathrm{sgnmod}^\alpha(\mathcal{D}) = \mathrm{sgn}(\mathcal{D})|\mathcal{D}|^\alpha = \mathcal{D}|\mathcal{D}|^{\alpha-1}$.

Remark 4.5. Recall from Remark 2.9 the function $\mathrm{sgnlog}(x) := \mathrm{sgn}(x)\log(1+|x|)$. In [Goffeng et al. 2019, Theorem 1.16], it is proved that the transformation $\mathcal{D} \mapsto \mathrm{sgnlog}(\mathcal{D})$ turns Lipschitz regular *twisted* unbounded Kasparov modules into ordinary unbounded Kasparov modules. Incorporating this "untwisting" procedure into the homotopy framework using Proposition 2.6 is of interest in the study of twisted local index formulae. This is beyond the scope of the present paper.

4B. *Spectral symmetries.*

Definition 4.6. An unbounded A-B-cycle $(E, \mathcal{D})$ is called

- *spectrally symmetric* if there exist an odd self-adjoint unitary S on E and a $W \subset \mathrm{Lip}^0(\mathcal{D})$ such that $\pi(A) \subset \overline{W}$, $[S, w] = 0$ for all $w \in W$, $S : \mathrm{Dom}\,\mathcal{D} \to \mathrm{Dom}\,\mathcal{D}$, $\mathcal{D}S - S\mathcal{D} = 0$, and $S\mathcal{D}$ is positive, and
- *spectrally decomposable* if there exists a spectral symmetry S such that both $(S \pm 1)\mathcal{D}$ are positive.

The definition of spectrally decomposable cycle is adapted from [Kaad 2019, Definition 4.1] (where it is phrased in terms of the projection $P = \frac{1}{2}(1+S)$). By definition, every spectrally decomposable cycle is also spectrally symmetric. Moreover, any spectrally degenerate cycle $(E, \mathcal{D})$ is clearly spectrally decomposable (hence spectrally symmetric) with spectral symmetry $\operatorname{sgn}(\mathcal{D})$.

Spectrally symmetric cycles are actually not much more general than spectrally degenerate cycles. Indeed, the following lemma shows that any spectral symmetry S more or less acts like $\operatorname{sgn}(\mathcal{D})$ (except that $\mathcal{D}$ may not be invertible, so there could be some freedom in how S acts on $\operatorname{Ker}\mathcal{D}$).

Lemma 4.7. *Let $(E, \mathcal{D})$ be an unbounded A-B-cycle with spectral symmetry S. Then $\mathcal{D} = S|\mathcal{D}|$, and $(E, \mathcal{D})$ is Lipschitz regular.*

Proof. On the $\mathbb{Z}_2$-graded module $E = E_+ \oplus E_-$ we can write

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix},$$

where $U : E_+ \to E_-$ is unitary. Since $\mathcal{D}S = S\mathcal{D}$, we see that $U\mathcal{D}_- = \mathcal{D}_+ U^*$. We then compute

$$\mathcal{D}^2 = \begin{pmatrix} \mathcal{D}_-\mathcal{D}_+ & 0 \\ 0 & \mathcal{D}_+\mathcal{D}_- \end{pmatrix} = \begin{pmatrix} U^*\mathcal{D}_+U^*\mathcal{D}_+ & 0 \\ 0 & U\mathcal{D}_-U\mathcal{D}_- \end{pmatrix}.$$

Since $S\mathcal{D}$ is positive, we know that $U^*\mathcal{D}_+$ and $U\mathcal{D}_-$ are positive, and we obtain

$$|\mathcal{D}| = \begin{pmatrix} U^*\mathcal{D}_+ & 0 \\ 0 & U\mathcal{D}_- \end{pmatrix} = S\mathcal{D}.$$

As in Lemma 3.6, it then follows that $\operatorname{Lip}(|\mathcal{D}|) = \operatorname{Lip}(\mathcal{D})$, so in particular $(E, \mathcal{D})$ is Lipschitz regular. ☐

Furthermore, the next proposition shows that any spectrally symmetric cycle is in fact just a bounded perturbation of a spectrally degenerate cycle.

Proposition 4.8. *Let $(E, \mathcal{D})$ be an unbounded A-B-cycle with spectral symmetry S. Then $(E, \mathcal{D} + S)$ is a spectrally degenerate unbounded A-B-cycle.*

Proof. Since S is bounded, self-adjoint, and odd, we know that $(E, \mathcal{D} + S)$ is again an unbounded A-B-cycle. Furthermore, since $(\mathcal{D} + S)^2 = \mathcal{D}^2 + 1 + 2S\mathcal{D}$ is positive and invertible, we know that also $\mathcal{D} + S$ is invertible. Moreover, noting that $(\mathcal{D}+S)^2 = (1+S\mathcal{D})^2$ and that $1+S\mathcal{D}$ is positive, we see that $|\mathcal{D}+S| = 1+S\mathcal{D}$. Hence, we find that

$$\operatorname{sgn}(\mathcal{D} + S) = (\mathcal{D} + S)|\mathcal{D} + S|^{-1} = S(S\mathcal{D} + 1)(1 + S\mathcal{D})^{-1} = S,$$

and we conclude that $(E, \mathcal{D} + S)$ is degenerate. ☐

In [Kaad 2019, Definition 4.8], the notion of spectrally decomposable module was used to define the equivalence relation of "stable homotopy" for unbounded Kasparov modules (i.e., homotopies modulo addition of spectrally decomposable modules). Here, we point out that in fact any spectrally symmetric cycle $(E, \mathcal{D})$ is null-homotopic. If A is separable, this follows from Theorem 2.11 by observing that, if S is a spectral symmetry of $(E, \mathcal{D})$, then the bounded transform $(E, F_{\mathcal{D}})$ is operator-homotopic to the degenerate cycle (E, S) (since $[F_{\mathcal{D}}, S] = 2SF_{\mathcal{D}}$ is positive [Blackadar 1998, Proposition 17.2.7]). In general, we simply combine Propositions 4.8 and 3.7 to obtain:

Corollary 4.9. *Any spectrally symmetric unbounded A-B-cycle is null-homotopic. Consequently, the relation of stable homotopy equivalence of* [Kaad 2019, Definition 4.8] *coincides with the relation $\sim_h$ of homotopy equivalence.*

In [Kaad 2019, Theorem 7.1] it was shown that, for any countable dense $*$-subalgebra $\mathcal{A} \subset A$, the stable homotopy equivalence classes of elements in $\Psi_1(\mathcal{A}, B)$ form a group which is isomorphic to $KK(A, B)$. In particular, this group is independent of the choice of $\mathcal{A}$. We emphasize here that Corollary 4.9, combined with [Kaad 2019, Theorem 7.1], then gives a second independent proof of the isomorphism $\overline{UKK}(A, B) \simeq KK(A, B)$ from Theorem 2.12.

As a further application of Corollary 4.9, the following proposition (adapted from the results of [Kaad 2019]) gives a criterion that ensures that two given unbounded cycles are homotopic.

Proposition 4.10 (cf. [Kaad 2019, Proposition 6.2]). *Let $(\pi, E, \mathcal{D})$ and $(\pi, E, \mathcal{D}')$ be unbounded A-B-cycles such that $\pi(A) \subset \overline{\mathrm{Lip}^0(\mathcal{D}) \cap \mathrm{Lip}^0(\mathcal{D}')}$. Suppose there exists an odd self-adjoint unitary $F : E \to E$ such that F commutes with both $\mathcal{D}$ and $\mathcal{D}'$, and such that we have the equalities $F\mathcal{D} = |\mathcal{D}|$ and $F\mathcal{D}' = |\mathcal{D}'|$. Then $(E, \mathcal{D})$ is homotopic to $(E, \mathcal{D}')$.*

Proof. Using Proposition 4.3, we may assume (without loss of generality) that $(E, \mathcal{D})$ and $(E, \mathcal{D}')$ are Lipschitz regular, and that $\pi(A) \subset \overline{W}$ for some

$$W \subset \mathrm{Lip}^0(\mathcal{D}) \cap \mathrm{Lip}(|\mathcal{D}|) \cap \mathrm{Lip}^0(\mathcal{D}') \cap \mathrm{Lip}(|\mathcal{D}'|).$$

We then note that the operator F satisfies the assumptions of [Kaad 2019, Proposition 6.2] (with the dense $*$-subalgebra $\mathcal{A} \subset A$ replaced by W), where we point out that the Lipschitz regularity of $\mathcal{D}$ ensures that

$$\mathcal{D}[F, w] = [\mathcal{D}F, w] - [\mathcal{D}, w]F = [|\mathcal{D}|, w] - [\mathcal{D}, w]F$$

is bounded for $w \in W$ (and similarly for $\mathcal{D}'$). Then we know from (the proof of) [Kaad 2019, Proposition 6.2] that $(E, \mathcal{D}) - (E, \mathcal{D}')$ is homotopic to a spectrally decomposable cycle. Using Corollary 4.9 we conclude that $(E, \mathcal{D}) - (E, \mathcal{D}')$ is null-homotopic, and therefore $(E, \mathcal{D})$ is homotopic to $(E, \mathcal{D}')$. □

Corollary 4.11. *Let $(\pi, E, \mathscr{D})$ and $(\pi, E, \mathscr{D}')$ be unbounded A-B-cycles. Suppose that $\mathscr{D}$ and $\mathscr{D}'$ are both invertible, and that* $\operatorname{sgn}(\mathscr{D}) = \operatorname{sgn}(\mathscr{D}')$. *Then $(E, \mathscr{D})$ is homotopic to $(E, \mathscr{D}')$.*

4C. *Clifford symmetries.*

Definition 4.12. An unbounded Kasparov A-B-cycle $(E, \mathscr{D})$ is called *Clifford symmetric* if there exists an odd self-adjoint unitary γ on E and a $W \subset \operatorname{Lip}^0(\mathscr{D})$ such that $\pi(A) \subset \overline{W}$, $[\gamma, w] = 0$ for all $w \in W$, $\gamma : \operatorname{Dom}\mathscr{D} \to \operatorname{Dom}\mathscr{D}$, and $\mathscr{D}\gamma = -\gamma\mathscr{D}$.

The idea here is that a Clifford symmetric A-B-cycle is in fact an $A \mathbin{\widehat{\otimes}} \mathbb{C}l_1$-$B$-cycle, and the image of the map $KK(A \mathbin{\widehat{\otimes}} \mathbb{C}l_1, B) \to KK(A, B)$ is zero. Indeed, one easily checks that the bounded transform $(E, F_{\mathscr{D}})$ of a Clifford symmetric unbounded cycle is operator-homotopic to the degenerate Kasparov module (E, γ). We prove here an analogous statement for unbounded cycles.

Lemma 4.13. *Let $(E, \mathscr{D})$ be an unbounded A-B-cycle with a Clifford symmetry γ and $0 < \alpha < 1$. Then $(E, \mathscr{D})$ is operator-homotopic to the spectrally symmetric unbounded cycle $(E, \gamma|\mathscr{D}|^{\alpha})$.*

Proof. Since γ commutes with $|\mathscr{D}|^{\alpha}$ and $(\gamma|\mathscr{D}|^{\alpha})^2 = |\mathscr{D}|^{2\alpha}$, we know that $\gamma|\mathscr{D}|^{\alpha}$ is regular and self-adjoint, and $T(1 + (\gamma|\mathscr{D}|^{\alpha})^2)^{-1/2}$ is compact for any $T \in \operatorname{Lip}^0(\mathscr{D})$. Moreover, since $\operatorname{Lip}(\gamma|\mathscr{D}|^{\alpha}) = \operatorname{Lip}(|\mathscr{D}|^{\alpha})$ contains $\operatorname{Lip}(\mathscr{D})$, we see that $\pi(A) \subset \operatorname{Lip}^0(\mathscr{D}) \subset \operatorname{Lip}^0(\gamma|\mathscr{D}|^{\alpha})$. Thus, $(E, \gamma|\mathscr{D}|^{\alpha})$ is indeed an unbounded cycle. We note that γ provides a spectral symmetry for $(E, \gamma|\mathscr{D}|^{\alpha})$. The operator-homotopy is obtained by composing the operator-homotopy between $\mathscr{D}$ and $\operatorname{sgnmod}^{\alpha}(\mathscr{D})$ (see Proposition 4.3 and Remark 4.4) with the operator-homotopy given for $t \in [0, 1]$ by

$$\mathscr{D}_t := \cos(\tfrac{1}{2}\pi t)\operatorname{sgnmod}^{\alpha}(\mathscr{D}) + \sin(\tfrac{1}{2}\pi t)\gamma|\mathscr{D}|^{\alpha}. \tag{4.14}$$

Note that γ anticommutes with $\operatorname{sgnmod}^{\alpha}(\mathscr{D})$ as the latter is given by an odd function of $\mathscr{D}$ (see Lemma 1.14). We then compute that $\mathscr{D}_t^2 = |\mathscr{D}|^{2\alpha}$, and thus $\operatorname{Lip}^0(\mathscr{D}) \subset \operatorname{Lip}^0(\mathscr{D}_t)$ for all $t \in [0, 1]$, so $\mathscr{D}_t$ is indeed an operator-homotopy. $\square$

As in [Deeley et al. 2018, Definition 3.1], we say that an unbounded cycle $(E, \mathscr{D})$ is *weakly degenerate* if $\mathscr{D}$ is given by a sum $\mathscr{D} = \mathscr{D}_0 + \mathscr{S}$, such that

- $\mathscr{D}_0$ and $\mathscr{S}$ are odd regular self-adjoint operators with $\operatorname{Dom}\mathscr{D} = \operatorname{Dom}\mathscr{D}_0 \cap \operatorname{Dom}\mathscr{S}$,
- $\mathscr{S}$ is invertible, $A \subset \operatorname{Lip}(\mathscr{S})$, and $\mathscr{S}a - a\mathscr{S} = 0$ for all $a \in A$, and
- there is a common core $\mathscr{E} \subset \operatorname{Dom}(\mathscr{S}\mathscr{D}_0) \cap \operatorname{Dom}(\mathscr{D}_0\mathscr{S})$ for $\mathscr{D}_0$ and $\mathscr{S}$ such that $\mathscr{D}_0\mathscr{S} + \mathscr{S}\mathscr{D}_0 = 0$ on $\mathscr{E}$.

Roughly speaking, this means that $\mathscr{S}$ is degenerate and $\mathscr{D}_0$ has Clifford symmetry $\gamma = \operatorname{sgn}(\mathscr{S})$. The proof of Lemma 4.13 can be adapted to weakly degenerate cycles.

Lemma 4.15. *Any weakly degenerate unbounded A-B-cycle $(E, \mathscr{D} = \mathscr{D}_0 + \mathscr{S})$ is operator-homotopic to spectrally symmetric unbounded A-B-cycle $(E, \operatorname{sgn}(\mathscr{S})|\mathscr{D}|^\alpha)$ for any $0 < \alpha < 1$. In particular, $(E, \mathscr{D})$ is null-homotopic.*

Proof. The proof is the same as for Lemma 4.13, but we need to show that (4.14) is again an operator-homotopy (with $\gamma = \operatorname{sgn}(\mathscr{S})$). We compute

$$\mathscr{D}_t^2 = |\mathscr{D}|^{2\alpha} + 2\sin(\tfrac{1}{2}\pi t)\cos(\tfrac{1}{2}\pi t)[\operatorname{sgnmod}^\alpha(\mathscr{D}), \gamma]|\mathscr{D}|^\alpha.$$

Since $\mathscr{S}$ is invertible, also $\mathscr{D}$ is invertible, and we find that

$$[\operatorname{sgnmod}^\alpha(\mathscr{D}), \gamma] = [\mathscr{D}, \gamma]|\mathscr{D}|^{\alpha-1} = 2|\mathscr{S}|\,|\mathscr{D}|^{\alpha-1}.$$

In particular, $[\operatorname{sgnmod}^\alpha(\mathscr{D}), \gamma]|\mathscr{D}|^\alpha$ is a positive operator and therefore $\mathscr{D}_t^2 \geq |\mathscr{D}|^{2\alpha}$ for all $t \in [0, 1]$. Hence, if $T(1+\mathscr{D}^2)^{-1/2}$ is compact for some $T \in \operatorname{End}_B(E)$, then also $T(1+|\mathscr{D}|^{2\alpha})^{-1/2}$ is compact, and therefore

$$T(1+\mathscr{D}_t^2)^{-1/2} = T(1+|\mathscr{D}|^{2\alpha})^{-1/2}(1+|\mathscr{D}|^{2\alpha})^{1/2}(1+\mathscr{D}_t^2)^{-1/2}$$

is compact. Thus, $\operatorname{Lip}^0(\mathscr{D}) \subset \operatorname{Lip}^0(\mathscr{D}_t)$ for all $t \in [0, 1]$, so $\mathscr{D}_t$ is indeed an operator-homotopy. Finally it follows from Corollary 4.9 that $(E, \gamma|\mathscr{D}|^\alpha)$ is null-homotopic. □

4D. *The unbounded KK-group.* As mentioned in Remark 2.13, the isomorphism $\overline{UKK}(A, B) \simeq KK(A, B)$ from Theorem 2.12 implies in particular that $\overline{UKK}(A, B)$ is a group. Here we give a direct proof of this fact, working only in the unbounded picture of KK-theory (hence avoiding the bounded transform entirely). In particular, the proof we give here (in contrast with Theorem 2.12) does not require the assumption that A is separable.

Given an unbounded A-B-cycle $(\pi, E, \mathscr{D})$, define its "inverse" as

$$-(\pi, E, \mathscr{D}) := (\pi^{\mathrm{op}}, E^{\mathrm{op}}, -\mathscr{D}),$$

where $E^{\mathrm{op}} = E$ with the opposite grading and the representation

$$\pi^{\mathrm{op}}(a) = (-1)^{\deg a}\pi(a)$$

for homogeneous elements $a \in A$.

Theorem 4.16. *For any σ-unital C^*-algebras A and B, the abelian semigroup $\overline{UKK}(A, B)$ is in fact a group. To be more precise, the inverse of $[(\pi, E, \mathscr{D})] \in \overline{UKK}(A, B)$ is given by $[-(\pi, E, \mathscr{D})]$.*

Proof. The sum $(\pi, E, \mathscr{D}) - (\pi, E, \mathscr{D})$ is given by the Clifford symmetric cycle

$$(\pi, E, \mathscr{D}) - (\pi, E, \mathscr{D}) = \left(\pi \oplus \pi^{\mathrm{op}}, E \oplus E^{\mathrm{op}}, \begin{pmatrix} \mathscr{D} & 0 \\ 0 & -\mathscr{D} \end{pmatrix}\right), \qquad \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where γ denotes the Clifford symmetry. From Lemma 4.13 we know that a Clifford symmetric cycle is operator-homotopic to a spectrally symmetric cycle. Furthermore, by Corollary 4.9, every spectrally symmetric cycle is null-homotopic. Thus, we have shown that $(\pi, E, \mathcal{D}) - (\pi, E, \mathcal{D})$ is null-homotopic, and therefore $[-(\pi, E, \mathcal{D})]$ is indeed the inverse of $[(\pi, E, \mathcal{D})]$. □

Appendix: On localizations of dense submodules

Let X be a locally compact Hausdorff space, B a C^*-algebra, and E a Hilbert $C_0(X, B)$-module. We will show in this appendix that a submodule of E is dense if and only if it is pointwise dense. One way to prove this could be by showing that E can be viewed as a continuous field of Banach spaces (where each Banach space is in fact a Hilbert B-module), and then applying the theory of continuous fields [Dixmier and Douady 1963] (for this approach, see for instance [Ebert 2018, Lemma 2.7, Corollary 2.8, and Proposition 2.21]). Here, we prefer instead to give our proof in the language of Hilbert C^*-modules.

For $x \in X$ we denote by $\mathrm{ev}_x : C_0(X, B) \to B$ the $*$-homomorphism $f \mapsto f(x)$. Let $\iota : B \to B^+$ be the embedding of B into its (minimal) unitization B^+. We define the *localization* $E_x := E \mathbin{\widehat{\otimes}}_{\mathrm{ev}_x} B^+$, and we note that there is a map $E \to E_x$ via $e \mapsto e_x := e \mathbin{\widehat{\otimes}} 1$. For a submodule $F \subset E$ we write

$$F_x := \{f_x \in E_x : f \in F\} \subset E_x,$$

for the image of F under the map $e \mapsto e_x$. We collect some basic facts regarding these localizations in the following lemma.

Lemma A.1. (1) *The Hilbert $C_0(X, B)$-module E is a central bimodule over $C_0(X)$, and the left $C_0(X)$ action is by adjointable operators.*

(2) *The map $E \to E_x$ given by $e \mapsto e_x := e \mathbin{\widehat{\otimes}} 1$ is surjective.*

(3) *We have a unitary isomorphism $E_x \simeq E \mathbin{\widehat{\otimes}}_{\mathrm{ev}_x} B$.*

(4) *We have the equality $\|e\|_E = \sup_{x\in X} \|e_x\|$, and the map $x \mapsto \|e_x\|$ lies in $C_0(X)$.*

Proof. For (1), see for instance [Kasparov 1988, Definition 1.5] and the discussion following it. For (2), it suffices to consider elements $e \mathbin{\widehat{\otimes}} b \in E_x$ with $e \in E$ and $b \in B$. Picking $f \in C_0(X)$ such that $f(x) = 1$ and defining $\tilde{b} \in C_0(X, B)$ by $\tilde{b}(y) := f(y)b$ for $y \in X$, we see that $e \mathbin{\widehat{\otimes}} b = e\tilde{b} \mathbin{\widehat{\otimes}} 1$, which proves (2). For (3), we note that the map $\mathrm{id} \mathbin{\widehat{\otimes}} \iota : E \mathbin{\widehat{\otimes}}_{\mathrm{ev}_x} B \to E \mathbin{\widehat{\otimes}}_{\mathrm{ev}_x} B^+$ is an isometry, so we only need to check that the range is dense. Using an approximate unit $u_n \in B$, we indeed find

$$\|e \mathbin{\widehat{\otimes}} 1 - e \mathbin{\widehat{\otimes}} u_n\|^2 = \|e \mathbin{\widehat{\otimes}} (1 - u_n)\|^2 = \|(1 - u_n)\, \mathrm{ev}_x(\langle e, e\rangle)(1 - u_n)\| \to 0.$$

The equality in (4) follows by direct calculation:

$$\|e\|_E^2 = \|\langle e, e\rangle\|_{C_0(X,B)} = \sup_{x\in X}\|\langle e, e\rangle(x)\|_B = \sup_{x\in X}\|\langle e \mathbin{\widehat{\otimes}} 1, e \mathbin{\widehat{\otimes}} 1\rangle_{E_x}\|_B = \sup_{x\in X}\|e_x\|^2.$$

Finally, for continuity of the norm, we use that $\|e_x\| = \|\langle e, e\rangle^{1/2}(x)\|$ and that the map $x \mapsto \langle e, e\rangle^{1/2}(x)$ is continuous. □

Proposition A.2. *If $F \subset E$ is a submodule, then F is dense in E if and only if for each $x \in X$, F_x is dense in E_x.*

Proof. We will freely use the facts from Lemma A.1. If F is dense in E, the equality $\|e\|_E = \sup_{x\in X}\|e_x\|$ shows that F_x is dense in E_x for each $x \in X$. Conversely, suppose F_x is dense in E_x for all $x \in X$. Fix $\varepsilon > 0$ and $\psi \in E$. For each $x \in X$, there exists $\phi \in F$ such that $\|\psi_x - \phi_x\| < \frac{\varepsilon}{2}$. By continuity of the norm, there exists a precompact open neighborhood U_x of x in X such that

$$\sup_{y\in U_x}\|\psi_y - \phi_y\| < \varepsilon.$$

There exists a compact subset $K \subset X$ such that $\sup_{x\in X\setminus K}\|\psi(x)\| < \varepsilon$. By compactness of K, we can choose finitely many points $\{x_i\}_{i=1}^N$ such that $K \subset \bigcup_{i=1}^N U_{x_i}$. Thus, on each $U_i := U_{x_i}$ there exists $\phi_i \in F$ such that $\sup_{y\in U_i}\|\psi_y - \phi_{i,y}\| < \varepsilon$. Let $U_0 := X \setminus K$, and let χ_i be a partition of unity subordinate to $\{U_i\}_{i=0}^N$. Let $\{u_n\}$ be an approximate unit for B, and choose n large enough such that $\|\phi_{i,y} - \phi_{i,y}u_n\| < \varepsilon$ for all $i = 1, \dots, N$ and $y \in U_i$. Let $\eta_i \in C_0(X, B)$ be given by $\eta_i(x) := \chi_i(x)u_n$. Then the element $\phi := \sum_{i=1}^N \phi_i\eta_i \in F$ is supported on $V := \bigcup_{i=1}^N U_i$, and we compute

$$\begin{aligned}
\|\psi - \phi\| &\le \sup_{x\in V}\|\psi_x - \phi_x\| + \sup_{x\in X\setminus V}\|\psi_x - \phi_x\| \\
&\le \sup_{x\in V\setminus K}\left\|\left(1 - \sum_{i=1}^N \chi_i(x)\right)\psi_x\right\| + \sup_{x\in V}\left\|\sum_{i=1}^N \chi_i(x)(\psi_x - \phi_{i,x})\right\| \\
&\qquad + \sup_{x\in V}\left\|\sum_{i=1}^N \chi_i(x)(\phi_{i,x} - \phi_{i,x}u_n)\right\| + \sup_{x\in X\setminus V}\|\psi_x\| \\
&\le 4\varepsilon.
\end{aligned}$$

It follows that F is dense in E. □

For any adjointable operator T on E, we write $T_x := \mathrm{ev}_x(T) := T \mathbin{\widehat{\otimes}} 1$ for the corresponding operator on $E_x = E \mathbin{\widehat{\otimes}}_{\mathrm{ev}_x} B^+$.

Corollary A.3. *Let E be a $C_0(X, B)$-module and $h \in \mathrm{End}_B^*(E)$. Then h has dense range in E if and only if for all $x \in X$, h_x has dense range in E_x.*

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References

[Akemann and Pedersen 1977] C. A. Akemann and G. K. Pedersen, “Ideal perturbations of elements in C^*-algebras”, *Math. Scand.* **41**:1 (1977), 117–139. MR Zbl

[Baaj and Julg 1983] S. Baaj and P. Julg, “Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens”, *C. R. Acad. Sci. Paris Sér. I Math.* **296**:21 (1983), 875–878. MR Zbl

[Blackadar 1998] B. Blackadar, *K-theory for operator algebras*, 2nd ed., Mathematical Sciences Research Institute Publications **5**, Cambridge University, 1998. MR Zbl

[Brain et al. 2016] S. Brain, B. Mesland, and W. D. van Suijlekom, “Gauge theory for spectral triples and the unbounded Kasparov product”, *J. Noncommut. Geom.* **10**:1 (2016), 135–206. MR Zbl

[Carey and Phillips 1998] A. Carey and J. Phillips, “Unbounded Fredholm modules and spectral flow”, *Canad. J. Math.* **50**:4 (1998), 673–718. MR Zbl

[Deeley et al. 2018] R. J. Deeley, M. Goffeng, and B. Mesland, “The bordism group of unbounded KK-cycles”, *J. Topol. Anal.* **10**:2 (2018), 355–400. MR Zbl

[Dixmier and Douady 1963] J. Dixmier and A. Douady, “Champs continus d’espaces hilbertiens et de C^*-algèbres”, *Bull. Soc. Math. France* **91** (1963), 227–284. MR Zbl

[van den Dungen 2018] K. van den Dungen, “Locally bounded perturbations and (odd) unbounded KK-theory”, *J. Noncommut. Geom.* **12**:4 (2018), 1445–1467. MR Zbl

[Ebert 2018] J. Ebert, “Elliptic regularity for Dirac operators on families of noncompact manifolds”, preprint, 2018. arXiv

[Goffeng et al. 2019] M. Goffeng, B. Mesland, and A. Rennie, “Untwisting twisted spectral triples”, *Internat. J. Math.* **30**:14 (2019), art. id. 1950076. MR Zbl

[Gracia-Bondía et al. 2001] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa, *Elements of noncommutative geometry*, Birkhäuser, Boston, 2001. MR Zbl

[Higson and Roe 2000] N. Higson and J. Roe, *Analytic K-homology*, Oxford University, 2000. MR Zbl

[Hilsum 1985] M. Hilsum, “Signature operator on Lipschitz manifolds and unbounded Kasparov bimodules”, pp. 254–288 in *Operator algebras and their connections with topology and ergodic theory* (Buşteni, Romania 1983), edited by H. Araki et al., Lecture Notes in Mathematics **1132**, Springer, 1985. MR Zbl

[Hilsum 1989] M. Hilsum, “Fonctorialité en K-théorie bivariante pour les variétés lipschitziennes”, *K-Theory* **3**:5 (1989), 401–440. MR Zbl

[Hilsum 2010] M. Hilsum, "Bordism invariance in KK-theory", *Math. Scand.* **107**:1 (2010), 73–89. MR Zbl

[Kaad 2019] J. Kaad, "On the unbounded picture of KK-theory", preprint, 2019. arXiv

[Kaad and Lesch 2012] J. Kaad and M. Lesch, "A local global principle for regular operators in Hilbert C^*-modules", *J. Funct. Anal.* **262**:10 (2012), 4540–4569. MR Zbl

[Kaad and Lesch 2013] J. Kaad and M. Lesch, "Spectral flow and the unbounded Kasparov product", *Adv. Math.* **248** (2013), 495–530. MR Zbl

[Kasparov 1980] G. G. Kasparov, "The operator K-functor and extensions of C^*-algebras", *Izv. Akad. Nauk SSSR Ser. Mat.* **44**:3 (1980), 571–636. In Russian; translated in *Math. USSR Izv.* **16**:3 (1981), 513–572. MR Zbl

[Kasparov 1988] G. G. Kasparov, "Equivariant KK-theory and the Novikov conjecture", *Invent. Math.* **91**:1 (1988), 147–201. MR Zbl

[Kucerovsky 2000] D. Kucerovsky, "A lifting theorem giving an isomorphism of KK-products in bounded and unbounded KK-theory", *J. Operator Theory* **44**:2 (2000), 255–275. MR Zbl

[Lance 1995] E. C. Lance, *Hilbert C^*-modules: a toolkit for operator algebraists*, London Mathematical Society Lecture Note Series **210**, Cambridge University, 1995. MR Zbl

[Mesland 2014] B. Mesland, "Unbounded bivariant K-theory and correspondences in noncommutative geometry", *J. Reine Angew. Math.* **691** (2014), 101–172. MR Zbl

[Mesland and Rennie 2016] B. Mesland and A. Rennie, "Nonunital spectral triples and metric completeness in unbounded KK-theory", *J. Funct. Anal.* **271**:9 (2016), 2460–2538. MR Zbl

[Pedersen 1979] G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs **14**, Academic, London, 1979. MR Zbl

[Teleman 1983] N. Teleman, "The index of signature operators on Lipschitz manifolds", *Inst. Hautes Études Sci. Publ. Math.* **58** (1983), 39–78. MR Zbl

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The p-completed cyclotomic trace in degree 2

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We prove that for a quasiregular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra R (in the sense of Bhatt–Morrow–Scholze), the cyclotomic trace map from the p-completed K-theory spectrum $K(R; \mathbb{Z}_p)$ of R to the topological cyclic homology $\mathrm{TC}(R; \mathbb{Z}_p)$ of R identifies on π_2 with a q-deformation of the logarithm.

1. Introduction

Fix a prime p. The aim of this paper is to concretely identify in degree 2, for a certain class of p-complete rings R, the p-completed cyclotomic trace

$$\mathrm{ctr} : K(R; \mathbb{Z}_p) \to \mathrm{TC}(R; \mathbb{Z}_p)$$

from the p-completed K-theory spectrum $K(R; \mathbb{Z}_p)$ of R to the topological cyclic homology $\mathrm{TC}(R; \mathbb{Z}_p)$ of R. Our main result is that on π_2 the p-completed cyclotomic trace is given by a q-logarithm

$$\log_q(x) := \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} \frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q},$$

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which is a q-deformation of the usual logarithm (where q is a parameter to be defined later). Before stating a precise version of the theorem, let us try to put it in context and to explain what the involved objects are.

K-theory and topological cyclic homology. We start with K-theory. For any commutative ring A, [Quillen 1973] defined the algebraic K-theory space $K(A)$ of A as a generalization of the Grothendieck group $K_0(A)$ of vector bundles on the scheme $\mathrm{Spec}(A)$. The (connective) K-theory spectrum $K(A)$ of a ring A is obtained by group completing[1] the $\mathbb{E}_\infty$-monoid of vector bundles on $\mathrm{Spec}(A)$ whose addition is given by the direct sum. In other words, for the full K-theory one mimics in a homotopy-theoretic context the definition of $K_0(A)$ with the set of isomorphism classes of vector bundles replaced by the groupoid of vector bundles. Algebraic K-theory behaves like a cohomology theory but has the nice feature — compared to other cohomology theories, like étale cohomology — that it only depends on the category of vector bundles on the ring (rather than on the ring itself) and thus enjoys strong functoriality properties, which makes it a powerful invariant attached to A.

Unfortunately, the calculation of the homotopy groups

$$K_i(A) := \pi_i(K(A)), \quad i \geq 1,$$

is in general rather intractable. There is for example a natural embedding

$$A^\times \to \pi_1(K(A)),$$

which is an isomorphism if A is local, but the higher K-groups are much more mysterious. One essential difficulty comes from the fact that K-theory, although it is a Zariski (and even Nisnevich) sheaf of spaces (see [Thomason and Trobaugh 1990]), does not satisfy étale descent. One could remedy this by étale sheafification, but one would lose the good properties of K-theory. This lead people to look for good *approximations* of K-theory, at least after profinite completions. By this, we mean invariants, still depending only on the category of vector bundles on the underlying ring, satisfying étale descent — and therefore, easier to compute — and close enough to (completed) K-theory, at least in some range.

The work of Thomason [1985] provides a good illustration of this principle. Thomason shows that the $K(1)$*-localization* of K-theory, with respect to a prime ℓ *invertible in* A, satisfies étale descent[2] and coincides with ℓ-adically completed (for short: ℓ-adic) K-theory in high degrees under some extra assumptions, later removed by [Rosenschon and Østvær 2006], building upon the work of Voevodsky

[1] See [Nikolaus 2017] for a discussion of homotopy-theoretic group completions and Quillen's $+$-construction.

[2] In fact, it even coincides with ℓ-adic étale K-theory on connective covers.

and Rost. When the prime p is not invertible in A, the situation is much more subtle. For instance, a theorem of Gabber [1992] shows that ℓ-adic K-theory is insensitive to replacing A by A/I if (A, I) forms a henselian pair; in particular, the computation of ℓ-adic K-theory of henselian rings (which form a basis of the Nisnevich topology) is reduced to the computation of the ℓ-adic K-theory of fields. This is not true anymore for p-adic K-theory. Nevertheless, the recent work of Clausen, Mathew and Morrow [Clausen et al. 2018], expresses this failure in terms of another noncommutative invariant attached to A, the *topological cyclic homology* of A, whose definition will be recalled below. Topological cyclic homology is related to K-theory via the *cyclotomic trace*

$$\mathrm{ctr} : K(A) \to \mathrm{TC}(A)$$

(see [Blumberg et al. 2013, Section 10.3; Bökstedt et al. 1993, Section 5]). Clausen, Mathew and Morrow prove, extending earlier work of Dundas, Goodwillie and McCarthy [Dundas et al. 2013] in the nilpotent case,[3] that the cyclotomic trace induces, for any ideal $I \subseteq A$ such that the pair (A, I) is henselian, an isomorphism

$$K(A, I)/n \cong \mathrm{TC}(A, I)/n$$

from the relative K-theory

$$K(A, I)/n := \mathrm{fib}(K(A)/n \to K(A/I)/n)$$

to the relative topological cyclic homology

$$\mathrm{TC}(A, I)/n := \mathrm{fib}(\mathrm{TC}(A)/n \to \mathrm{TC}(A/I)/n),$$

for *any* integer n. This has the consequence that p-completed TC provides a good approximation of p-adic K-theory, at least for rings henselian along (p): namely, it satisfies étale descent (because topological cyclic homology does) and coincides with p-adic K-theory in high degrees. Under additional hypotheses, one can even get better results: for instance, Clausen, Mathew and Morrow prove, among other things, that the cyclotomic trace induces an isomorphism

$$K(R; \mathbb{Z}_p) \cong \tau_{\geq 0}\mathrm{TC}(R; \mathbb{Z}_p)$$

for all rings R which are henselian along (p) and such that R/p is semiperfect (i.e., such that Frobenius is surjective); see [Clausen et al. 2018, Corollary 6.9].

Examples of such rings are the *quasiregular semiperfectoid rings* of [Bhatt et al. 2019]. A ring R is called quasiregular semiperfectoid if R is p-complete with

[3] This is not a generalization though, since the result of Dundas–Goodwillie–McCarthy applies also to noncommutative rings and is not restricted to finite coefficients.

bounded p^∞-torsion,[4] the p-completed cotangent complex $\widehat{L_{R/\mathbb{Z}_p}}$ has p-complete Tor-amplitude in $[-1,0]$ and there exists a surjective morphism $R' \to R$ with R' (integral) perfectoid. This class of rings is interesting as for R quasiregular semiperfectoid, the topological cyclic homology $\pi_*(\mathrm{TC}(R;\mathbb{Z}_p))$ can be computed in more concrete terms.

Let us recall the description of topological cyclic homology $\pi_*(\mathrm{TC}(R;\mathbb{Z}_p))$ from [Bhatt et al. 2019], which builds heavily on the foundational work of Nikolaus and Scholze [2018]. For this, we need some definitions. From now on, all spectra are assumed to be p-completed. One starts with the (p-completed) topological Hochschild homology spectrum $\mathrm{THH}(R;\mathbb{Z}_p)$ of R, which is equipped with a natural $\mathbb{T} = S^1$-action and a $\mathbb{T}$-equivariant map, the cyclotomic Frobenius,

$$\varphi_{\mathrm{cycl}} : \mathrm{THH}(R;\mathbb{Z}_p) \to \mathrm{THH}(R;\mathbb{Z}_p)^{tC_p}$$

to the Tate fixed points of the cyclic group $C_p \subseteq \mathbb{T}$. Then one takes the homotopy fixed points, the *negative topological cyclic homology*,

$$\mathrm{TC}^-(R;\mathbb{Z}_p) := \mathrm{THH}(R;\mathbb{Z}_p)^{h\mathbb{T}}$$

and the Tate fixed points, the *periodic topological cyclic homology*,

$$\mathrm{TP}(R;\mathbb{Z}_p) := \mathrm{THH}(R;\mathbb{Z}_p)^{t\mathbb{T}}.$$

From the cyclotomic Frobenius on $\mathrm{THH}(R;\mathbb{Z}_p)$ one derives a map[5]

$$\varphi_{\mathrm{cycl}}^{h\mathbb{T}} : \mathrm{TC}^-(R;\mathbb{Z}_p) \to \mathrm{TP}(R;\mathbb{Z}_p).$$

Then the topological cyclic homology $\mathrm{TC}(R;\mathbb{Z}_p)$ of R is defined via the fiber sequence

$$\mathrm{TC}(R;\mathbb{Z}_p) \to \mathrm{TC}^-(R;\mathbb{Z}_p) \xrightarrow{\mathrm{can}-\varphi_{\mathrm{cycl}}^{h\mathbb{T}}} \mathrm{TP}(R;\mathbb{Z}_p),$$

where $\mathrm{can} : \mathrm{TC}^-(R;\mathbb{Z}_p) \to \mathrm{TP}(R;\mathbb{Z}_p)$ is the canonical map from homotopy to Tate fixed points. The ring

$$\widehat{\Delta}_R := \pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p)) \cong \pi_0(\mathrm{TP}(R;\mathbb{Z}_p))$$

is p-complete, p-torsion free[6] and the cyclotomic Frobenius $\varphi_{\mathrm{cycl}}^{h\mathbb{T}}$ induces a Frobenius lift φ on $\widehat{\Delta}_R$; see [Bhatt and Scholze 2019, Theorem 11.10].

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[4]This means that there exists $N \geq 0$ such that $R[p^\infty] = R[p^N]$. This technical condition is useful when dealing with derived completions.

[5]Here one needs [Nikolaus and Scholze 2018, Lemma II.4.2], which implies $\mathrm{TP}(R;\mathbb{Z}_p) \cong (\mathrm{THH}(R;\mathbb{Z}_p)^{tC_p})^{h\mathbb{T}}$.

[6]Indeed, any element killed by p is killed by φ, as in the proof of [Bhatt and Scholze 2019, Lemma 2.28], and thus lies in all the steps of the Nygaard filtration.

Remark 1.1. The prismatic perspective of [Bhatt and Scholze 2019] gives an alternative description of $\widehat{\Delta}_R$: it is the completion with respect to the Nygaard filtration of the (derived) prismatic cohomology Δ_R of R. In particular, using the theory of δ-rings, one can give, when R is a p-complete with bounded p^∞-torsion quotient of a perfectoid ring by a regular sequence, a construction of $\widehat{\Delta}_R$ as the Nygaard completion of a concrete prismatic envelope; see [Bhatt and Scholze 2019, Proposition 3.12].

The choice of a morphism $R' \to R$ with R' perfectoid yields a distinguished element $\tilde{\xi}$ (up to a unit) of the ring $\widehat{\Delta}_R$. Using $\tilde{\xi}$ one defines the Nygaard filtration

$$\mathcal{N}^{\geq i}\widehat{\Delta}_R := \varphi^{-1}((\tilde{\xi}^i))$$

on $\widehat{\Delta}_R$. The graded rings $\pi_*(\mathrm{TC}^-(R;\mathbb{Z}_p))$ and $\pi_*(\mathrm{TP}(R;\mathbb{Z}_p))$ are then concentrated in even degrees and

$$\pi_{2i}(\mathrm{TC}^-(R;\mathbb{Z}_p)) \cong \mathcal{N}^{\geq i}\widehat{\Delta}_R, \qquad \pi_{2i}(\mathrm{TP}(R;\mathbb{Z}_p)) \cong \widehat{\Delta}_R$$

for $i \in \mathbb{Z}$; see [Bhatt and Scholze 2019, Theorem 11.10].[7] Moreover, on π_{2i} the cyclotomic Frobenius

$$\varphi^{h\mathbb{T}}_{\mathrm{cycl}} : \pi_{2i}(\mathrm{TC}^-(R;\mathbb{Z}_p)) \to \pi_{2i}(\mathrm{TP}(R;\mathbb{Z}_p))$$

identifies with the divided Frobenius $\varphi/\tilde{\xi}^i$. Thus, from the definition of $\mathrm{TC}(R;\mathbb{Z}_p)$ we obtain exact sequences

$$0 \to \pi_{2i}(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi=\tilde{\xi}^i} \to \mathcal{N}^{\geq i}\widehat{\Delta}_R \xrightarrow{1-(\varphi/\tilde{\xi}^i)} \widehat{\Delta}_R \to \pi_{2i-1}(\mathrm{TC}(R;\mathbb{Z}_p)) \to 0.$$

As mentioned in Remark 1.1, the ring $\widehat{\Delta}_R$ tends to be computable. For example, if R is perfectoid, then $\widehat{\Delta}_R \cong A_{\mathrm{inf}}(R)$ is Fontaine's construction applied to R and if $pR = 0$, then $\widehat{\Delta}_R$ is the Nygaard completion of the universal PD-thickening $A_{\mathrm{crys}}(R)$ of R. Thus, for quasiregular semiperfectoid rings the target of the cyclotomic trace is rather explicit.

Main results. The results of [Clausen et al. 2018] (together with those of [Bhatt et al. 2019]) therefore give a way of computing higher p-completed K-groups of quasiregular semiperfectoid rings. But there is at least one degree (except 0) where one can be more explicit, without using the cyclotomic trace map: namely, after p-completion of $K(R)$ there is a canonical morphism

$$T_p(R^\times) \to \pi_2(K(R;\mathbb{Z}_p))$$

[7] These identifications depend on the choice of a suitable generator $v \in \pi_{-2}(\mathrm{TC}^-(R;\mathbb{Z}_p))$. If R is an algebra over $\mathbb{Z}_p^{\mathrm{cycl}}$ we will clarify our choice in Section 6 carefully.

from the Tate module $T_p(R^\times)$ of the units of R, which is an isomorphism in many cases. The results explained in the previous paragraph show that the cyclotomic trace identifies $\pi_2(K(R;\mathbb{Z}_p))$ with

$$\pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi=\tilde{\xi}}.$$

What does the composite map

$$T_p(R^\times) \to \pi_2(K(R;\mathbb{Z}_p)) \xrightarrow{\mathrm{ctr}} \pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

look like? The main result of this paper, which we now state, provides a concrete description of it. Let R be a quasiregular semiperfectoid ring which admits a compatible system of morphisms $\mathbb{Z}[\zeta_{p^n}] \to R$ for $n \geq 0$. These morphisms give rise to the elements

$$\varepsilon = (1, \zeta_p, \dots) \in R^\flat = \varprojlim_{x \mapsto x^p} R, \quad q := [\varepsilon]_\theta \in \widehat{\Delta}_R \quad \text{and} \quad \tilde{\xi} := \frac{q^p - 1}{q - 1}.$$

Here

$$[-]_\theta : R^\flat \to \Delta_R$$

is the Teichmüller lift coming from the surjection $\theta : \Delta_R \to R$ (see the proof of Lemma 2.4).

Theorem 1.2 (cf. Theorem 6.7). *The composition*[8]

$$T_p(R^\times) \to \pi_2(K(R;\mathbb{Z}_p)) \xrightarrow{\mathrm{ctr}} \pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

is given by the q-logarithm

$$x \mapsto \log_q([x]_\theta) := \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} \frac{([x]_\theta - 1)([x]_\theta - q)\cdots([x]_\theta - q^{n-1})}{[n]_q}.$$

Here we embed

$$T_p(R^\times) \subseteq R^\flat, \quad (r_0 \in R^\times[p], r_1, \dots) \mapsto (1, r_0, r_1, \dots).$$

By

$$[n]_q := \frac{q^n - 1}{q - 1}$$

we denote the q-analog of $n \in \mathbb{Z}$.

Remark 1.3. A similar result can be found in Lemma 4.2.3 of [Geisser and Hesselholt 1999], but only before p-completion, on π_1 and in terms of TR_*, which is not enough to deduce Theorem 1.2 from their result.

[8] See Section 6 for a more precise description of the isomorphism $\pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$. We note that it depends on the choice of some compatible system $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots)$ of primitive p^n-th roots of unity.

As a consequence of [Clausen et al. 2018] and Theorem 1.2, one gets the following result.

Corollary 1.4. *Let R be a quasiregular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. The map*

$$\log_q([-]_\theta) : T_p(R^\times) \to \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

is a bijection.

This corollary is used in [Anschütz and Le Bras 2019], which studies a prismatic version of Dieudonné theory for p-divisible groups, and was our original motivation for proving Theorem 1.2.

Here is a short description of the proof of Theorem 1.2. By testing the universal case $R = \mathbb{Z}^{\mathrm{cycl}}\langle x^{1/p^\infty}\rangle/(x-1)$, one is reduced to the case where the pair $(p, \tilde{\xi})$ forms a regular sequence on $\widehat{\Delta}_R$, i.e., the prism $(\widehat{\Delta}_R, \tilde{\xi})$ is *transversal* (see Definition 3.2). In this situation, we prove that the reduction map

$$\widehat{\Delta}_R^{\varphi=\tilde{\xi}} \hookrightarrow \mathcal{N}^{\geq 1}\widehat{\Delta}_R/\mathcal{N}^{\geq 2}\widehat{\Delta}_R$$

is *injective* (Corollary 3.11). Thus it suffices to identify the composition

$$T_p(R^\times) \xrightarrow{\mathrm{ctr}} \widehat{\Delta}_R^{\varphi=\tilde{\xi}} \to \mathcal{N}^{\geq 1}\widehat{\Delta}_R/\mathcal{N}^{\geq 2}\widehat{\Delta}_R.$$

Using the results of [Bhatt et al. 2019] the quotient $\mathcal{N}^{\geq 1}\widehat{\Delta}_R/\mathcal{N}^{\geq 2}\widehat{\Delta}_R$ identifies with the p-completed Hochschild homology $\pi_2(\mathrm{HH}(R;\mathbb{Z}_p))$ (see Section 5) and thus the above composition identifies with the p-completed Dennis trace. A straightforward computation then identifies the p-completed Dennis trace (see Section 2), which allows us to conclude. We expect the results in Section 2 to be known, in some form, to the experts, but we did not find the results anywhere in the literature.

Let us end this introduction by a remark and a question. One could try to reverse the perspective from Corollary 1.4 and try to recover a (very) special case of the result of Clausen, Mathew and Morrow [Clausen et al. 2018] regarding the cyclotomic trace map using the concrete description furnished by Theorem 1.2. If R is of characteristic p, we have $q = 1$ and then the q-logarithm becomes the honest logarithm

$$\log([-]_\theta) : T_p(R^\times) \to A_{\mathrm{crys}}(R)^{\varphi=p}.$$

In [Scholze and Weinstein 2013], it is proven (using the exponential) that the map $\log([-])$ is an isomorphism, when R is the quotient of a perfect ring modulo a regular sequence. If R is the quotient of a perfectoid ring by a finite regular sequence and is p-torsion free, it is not difficult to deduce from Scholze and Weinstein's result that the map

$$\log_q([-]_\theta) : T_p(R^\times) \to \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

is a bijection when p is odd. Is there a way to prove it directly in general, for any p and any quasiregular semiperfectoid ring?

Plan of the paper. In Section 2 we concretely identify the p-completed Dennis trace on the Tate module of units (see Proposition 2.5) in the form we need it. In Section 3 we prove the crucial injectivity statement, namely Corollary 3.11, for transversal prisms. In Section 4 we make sense of the q-logarithm. Finally, in Section 6 we prove our main result, Theorem 1.2, and its consequence, Corollary 1.4.

2. The p-completed Dennis trace in degree 2

Fix some prime p and let $A = R/I$ be the quotient of a (p, I)-complete ring R. The aim of this section is to concretely describe in degree 2 the composition

$$T_p(A^\times) \to \pi_2(K(A; \mathbb{Z}_p)) \xrightarrow{\mathrm{Dtr}} \pi_2(\mathrm{HH}(A; \mathbb{Z}_p)) \to \pi_2(\mathrm{HH}(A/R; \mathbb{Z}_p)).$$

Here

$$K(A; \mathbb{Z}_p)$$

denotes the p-completed (connective) K-theory spectrum of A,

$$\mathrm{HH}(A; \mathbb{Z}_p) \quad \text{and} \quad \mathrm{HH}(A/R; \mathbb{Z}_p)$$

are the p-completed (derived) Hochschild homology of A as a $\mathbb{Z}$-algebra and R-algebra, respectively, and Dtr is the Dennis trace map. Before stating precisely our result, let us start by some reminders on the objects and the maps involved in the previous composition.

We first recall the construction of the first map $T_p(A^\times) \to \pi_2(K(A; \mathbb{Z}_p))$. Let

$$\mathrm{GL}(A) = \varinjlim_r \mathrm{GL}_r(A)$$

be the infinite general linear group over A. There is a canonical inclusion

$$A^\times = \mathrm{GL}_1(A) \to \mathrm{GL}(A)$$

of groups which on classifying spaces induces a map

$$B(A^\times) \to B(\mathrm{GL}(A)).$$

Composing with the morphism to Quillen's $+$-construction yields a canonical morphism

$$B(A^\times) \to B\mathrm{GL}(A) \to K(A) := B\mathrm{GL}(A)^+ \times K_0(A)$$

into the K-theory space $K(A)$ of A.[9] After p-completion of spaces we obtain a canonical morphism

$$\iota : B(A^\times)^\wedge_p \to K(A; \mathbb{Z}_p) := K(A)^\wedge_p.$$

[9] We use space as a synonym for Kan complex.

We recall (see [May and Ponto 2012, Theorem 10.3.2]) that the space $B(A^\times)^\wedge_p$ has two nontrivial homotopy groups which are given by

$$\pi_1(B(A^\times)^\wedge_p) \cong H^0(R\varprojlim_n(A^\times \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n)),$$
$$\pi_2(B(A^\times)^\wedge_p) \cong H^{-1}(R\varprojlim_n(A^\times \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n)) \cong T_p(A^\times).$$

In degree 2 we thus get a morphism

$$T_p(A^\times) = \pi_2(B(A^\times)^\wedge_p) \to \pi_2(K(A;\mathbb{Z}_p)),$$

which is the first constituent of the map

$$T_p(A^\times) \to \pi_2(K(A;\mathbb{Z}_p)) \xrightarrow{\mathrm{Dtr}} \pi_2(\mathrm{HH}(A;\mathbb{Z}_p)) \to \pi_2(\mathrm{HH}(A/R;\mathbb{Z}_p))$$

we want to describe.

Now we turn to the construction of Hochschild homology and the Dennis trace

$$\mathrm{Dtr} : K(A) \to \mathrm{HH}(A).$$

Let R be a (commutative) ring and A a (commutative) R-algebra. Let

$$\mathbb{T} := S^1 \cong B\mathbb{Z}$$

be the circle group. Then the Hochschild homology spectrum

$$\mathrm{HH}(A/R)$$

(simply denoted $\mathrm{HH}(A)$ when $R = \mathbb{Z}$) is the initial $\mathbb{T}$-equivariant[10] $E_\infty - R$-algebra with a nonequivariant map $A \to \mathrm{HH}(A/R)$ of $E_\infty - R$-algebras [Bhatt et al. 2019, Remark 2.4]. For a comparison with classical definitions, we refer to [Hoyois 2015].

The functor $A \mapsto \mathrm{HH}(A/R)$ extends to all simplicial R-algebras and as such is left Kan extended (as it commutes with sifted colimits) from the category of finitely generated polynomial R-algebras. By left Kan extending the (decreasing) Postnikov filtration $\tau_{\geq \bullet}\mathrm{HH}(A/R)$ on $\mathrm{HH}(A/R)$ for A a finitely generated polynomial R-algebra one obtains the $\mathbb{T}$-equivariant HKR-filtration

$$\mathrm{Fil}^n_{\mathrm{HKR}}\mathrm{HH}(A/R)$$

on $\mathrm{HH}(A/R)$ for A a general R-algebra. The ∞-category of $\mathbb{T}$-equivariant objects in the derived ∞-category $\mathcal{D}(R)$ of R is equivalent to the ∞-category of $R[\mathbb{T}]$-modules, where

$$R[\mathbb{T}] = R \otimes \Sigma^\infty_+ \mathbb{T}$$

[10] For an ∞-category $\mathcal{C}$ the category of $\mathbb{T}$-equivariant objects of $\mathcal{C}$ is by definition the ∞-category of functors $B\mathbb{T} \to \mathcal{C}$.

is the group algebra of $\mathbb{T}$ over R; see [Hoyois 2015, page 5]. Let

$$\gamma \in H_1(\mathbb{T}, R) \cong \mathrm{Hom}_{\mathcal{D}(R)}(R[1], R[\mathbb{T}])$$

be a generator.[11] The multiplication by γ induces a differential

$$d : \mathrm{HH}_i(A/R) \to \mathrm{HH}_{i+1}(A/R)$$

which makes $\mathrm{HH}_*(A/R)$ into a graded commutative dg-algebra over R all of whose elements of odd degree square to zero; see [Krause and Nikolaus 2017, Lemma 2.3]. By the universal property of the de Rham complex $\Omega^*_{A/R}$, the canonical morphism $A \to \mathrm{HH}_0(A/R)$ extends therefore to a morphism

$$\alpha_\gamma : \Omega^*_{A/R} \to \mathrm{HH}_*(A/R).$$

The Hochschild–Kostant–Rosenberg theorem affirms that α_γ is an isomorphism if $R \to A$ is smooth. By left Kan extension, one obtains for arbitrary $R \to A$ the natural description

$$\alpha_\gamma : \textstyle\bigwedge^i L_{A/R}[i] \cong \mathrm{gr}^i_{\mathrm{HKR}} \mathrm{HH}(A/R)$$

of the graded pieces of the HKR-filtration via exterior powers of the cotangent complex of A over R; see [Bhatt et al. 2019, Section 2.2].

In particular, we get after p-completion the following consequence in degree 2, which will be used to formulate our description of the Dennis trace below.

Lemma 2.1. *Let R be a ring and $I \subseteq R$ an ideal. Let $A = R/I$. Fix a generator γ of $H_1(\mathbb{T}, \mathbb{Z})$. There is a natural isomorphism*

$$\alpha_\gamma : (I/I^2)^\wedge_p \cong \pi_2(\mathrm{HH}(A/R; \mathbb{Z}_p)).$$

Here (and in the rest of the paper) we denote by $M^\wedge_p$ the *derived* p-adic completion of an abelian group M, i.e.,

$$M^\wedge_p := H^0(R \varprojlim_n M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n).$$

Proof. The first assertion follows from the HKR-filtration on $\mathrm{HH}(A/R; \mathbb{Z}_p)$ described above and the fact there is a canonical isomorphism

$$(I/I^2)^\wedge_p \cong H^{-1}((L_{A/R})^\wedge_p),$$

which is implied by [Stacks 2005–, Tag 08RA]. □

The Dennis trace can be defined abstractly [Blumberg et al. 2013, Section 10.2] as the composition of the unique natural transformation

$$K \to \mathrm{THH}$$

[11] We will mostly assume that γ is obtained by base change from some generator of $H_1(\mathbb{T}, \mathbb{Z})$.

of additive invariants of small stable ∞-categories from K-theory to topological Hochschild homology, which induces the identity on

$$\mathbb{Z} \cong \pi_0(K(\mathbb{S})) \to \pi_0(\mathrm{THH}(\mathbb{S})) \cong \mathbb{Z},$$

and the natural transformation (on rings) $\mathrm{THH} \to \mathrm{HH}$.

The only thing we need to use as an input regarding the Dennis trace is the following explicit description in degree 1. Recall from above that if A is a ring, each choice of a generator γ of $H_1(\mathbb{T}, \mathbb{Z})$ gives rise to an isomorphism

$$\alpha_\gamma : \pi_1(\mathrm{HH}(A/\mathbb{Z})) \cong \Omega^1_{A/\mathbb{Z}}$$

as $H^0(L_{A/\mathbb{Z}}) \cong \Omega^1_{A/\mathbb{Z}}$ for any A.

Lemma 2.2. *Let A be a commutative ring. There exists a unique bijection*

$$\delta_1 : \{\textit{generators of } H_1(\mathbb{T}, \mathbb{Z})\} \cong \{\pm 1\}$$

such that

$$A^\times \cong \pi_1(BA^\times) \xrightarrow{\mathrm{Dtr}} \pi_1(\mathrm{HH}(A)) \overset{\alpha_\gamma}{\cong} \Omega^1_{A/\mathbb{Z}}, \quad a \mapsto \delta_1(\gamma)\,\mathrm{dlog}(a)$$

for any generator $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$.

Proof. Let A be any commutative ring. The Hochschild homology $\mathrm{HH}(A)$ can be calculated as the geometric realization

$$\mathrm{HH}(A) := \varinjlim_{\Delta^{\mathrm{op}}} A^{\otimes^{\mathbb{L}}_{\mathbb{Z}} n+1}.$$

Note that this representation, which relies on the standard simplicial model of the circle $\Delta^1/\partial\Delta^1$, depends implicitly on the choice of a generator γ_0 of $H_1(\mathbb{T}, \mathbb{Z})$; see [Hoyois 2015, Theorem 2.3].[12] Replacing the derived tensor product by the non-derived one produces the classical, nonderived Hochschild homology $\mathrm{HH}^{\mathrm{usual}}(A)$ of A. As

$$\pi_1(\mathrm{HH}(A)) \cong \pi_1(\mathrm{HH}^{\mathrm{usual}}(A))$$

we may argue using $\mathrm{HH}^{\mathrm{usual}}$ instead of HH.

Using the above description of the classical Hochschild homology, the Dennis trace can be described more concretely; see [Bökstedt et al. 1993, Section 5; Loday 1992, Chapter 8.4]. It factors (on homotopy groups) through the integral group homology of $\mathrm{GL}(A)$, i.e., through $H_*(B\mathrm{GL}(A), \mathbb{Z})$, which is by definition (and the Dold–Kan correspondence) the homotopy of the space $\mathbb{Z}[B\mathrm{GL}(A)]$ obtained by taking the free simplicial abelian group on the simplicial $B\mathrm{GL}(A)$. As the $+$-construction

$$B\mathrm{GL}(A) \to B\mathrm{GL}(A)^+$$

[12] In this reference, γ_0 is called γ.

is an equivalence on integral homology [Weibel 2013, Chapter IV, Theorem 1.5], the morphism

$$\mathbb{Z}[B\mathrm{GL}(A)] \simeq \mathbb{Z}[B\mathrm{GL}(A)^+]$$

is an equivalence of simplicial abelian groups, and using the canonical inclusion

$$B\mathrm{GL}(A)^+ \to \mathbb{Z}[B\mathrm{GL}(A)^+]$$

we arrive at a canonical morphism

$$K(A) \to B\mathrm{GL}(A)^+ \to \mathbb{Z}[B\mathrm{GL}(A)^+] \simeq \mathbb{Z}[B\mathrm{GL}(A)].$$

We observe that for $r = 1$ the morphism $B\mathrm{GL}_1(A) \to B\mathrm{GL}_1(A)^+$ is an equivalence as $\mathrm{GL}_1(A) = A^\times$ is abelian. Thus there is a commutative diagram (up to homotopy)

$$\begin{tikzcd} B\mathrm{GL}_1(A) \arrow[r] \arrow[d] & \mathbb{Z}[B\mathrm{GL}_1(A)] \arrow[d] \\ K(A) \arrow[r] & \mathbb{Z}[B\mathrm{GL}(A)] \end{tikzcd}$$

with each morphism being the canonical one.

The Dennis trace factors as a composition

$$\mathrm{Dtr} : K(A) \to \mathbb{Z}[B\mathrm{GL}(A)] \xrightarrow{\mathrm{Dtr}'} \mathrm{HH}^{\mathrm{usual}}(A/\mathbb{Z}),$$

where by construction

$$\mathrm{Dtr}' : \mathbb{Z}[B\mathrm{GL}(A)] \to \mathrm{HH}^{\mathrm{usual}}(A)$$

is given as the colimit of compatible maps[13]

$$\mathrm{Dtr}'_r : \mathbb{Z}[B\mathrm{GL}_r(A)] \to \mathrm{HH}^{\mathrm{usual}}(A).$$

When $r = 1$, which is the only case relevant for us, the map Dtr'_1 is the linear extension of the map

$$BA^\times \to \mathrm{HH}^{\mathrm{usual}}(A)$$

which in simplicial degree n is given by

$$(a_1, \dots, a_n) \mapsto \frac{1}{a_1 \cdots a_n} \otimes a_1 \otimes \cdots \otimes a_n.$$

Fix a generator γ of $H_1(\mathbb{T}, \mathbb{Z})$. The choice of γ gives the HKR-isomorphism

$$\alpha_\gamma : \pi_1(\mathrm{HH}^{\mathrm{usual}}(A)) \cong \pi_1(\mathrm{HH}(A/\mathbb{Z})) \cong \Omega^1_{A/\mathbb{Z}}.$$

[13]Here compatible means up to some homotopy. To obtain strict compatibility one has to use the normalized Hochschild complex; see [Loday 1992, Section 8.4].

Using the above description of Hochschild homology as a geometric realization, the isomorphism α_γ is given by

$$\pi_1(\mathrm{HH}^{\mathrm{usual}}(A)) \cong \Omega^1_{A/\mathbb{Z}}, \quad a \otimes b \mapsto adb$$

with inverse $adb \mapsto a \otimes b$ if $\gamma = \gamma_0$, and by

$$\pi_1(\mathrm{HH}^{\mathrm{usual}}(A)) \cong \Omega^1_{A/\mathbb{Z}}, \quad a \otimes b \mapsto bda$$

with inverse $bda \mapsto a \otimes b$ if $\gamma = -\gamma_0$; this can be checked by analyzing compatibility with differentials and using [Hoyois 2015, Theorem 2.3]. In the first case, we set $\delta_1(\gamma) = 1$; in the second case, we set $\delta_1(\gamma) = -1$. Then on homotopy groups the map Dtr_1 is given by

$$A^\times \cong \pi_1(BA^\times) \to \pi_1(\mathrm{HH}(A)) \overset{\alpha_\gamma}{\cong} \Omega^1_{A/\mathbb{Z}}, \quad a \mapsto \delta_1(\gamma)\,\mathrm{dlog}(a) := \delta_1(\gamma)\,\frac{da}{a},$$

as claimed. □

Remark 2.3. Let A be a flat $\mathbb{Z}$-algebra. The description of $\mathrm{HH}(A) = \mathrm{HH}^{\mathrm{usual}}(A)$ as the geometric realization of the simplicial object

$$\mathrm{HH}(A/\mathbb{Z}) := \varinjlim_{\Delta^{\mathrm{op}}} A^{\otimes_{\mathbb{Z}}{}^{n+1}}$$

shows that $\mathrm{HH}(A; \mathbb{Z}_p)$ is computed by the complex

$$\cdots \to (A \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} A)^\wedge_p \to (A \otimes_{\mathbb{Z}} A)^\wedge_p \to A^\wedge_p.$$

One can then show that the p-completed Dennis trace $(BA^\times)^\wedge_p \to \mathrm{HH}(A; \mathbb{Z}_p)$ sends an element

$$(a_1, a_2, \ldots) \in T_p(A^\times) = \pi_2((BA^\times)^\wedge_p)$$

to the element represented, up to a sign, by the cycle

$$1 \otimes 1 \otimes 1 + \sum_{n=1}^{\infty} p^{n-1}\Big(\frac{1}{a_n^2} \otimes a_n \otimes a_n + \frac{1}{a_n^3} \otimes a_n^2 \otimes a_n + \cdots + \frac{1}{a_n^p} \otimes a_n^{p-1} \otimes a_n\Big).$$

We omit the proof, since we will not use this result.

We can now state and prove the main result of this section. Fix a generator γ of $H_1(\mathbb{T}, \mathbb{Z})$. We describe the image of some element $T_p(A^\times)$ under the composition

$$T_p(A^\times) \xrightarrow{\mathrm{Dtr}} \pi_2(\mathrm{HH}(A; \mathbb{Z}_p)) \to \pi_2(\mathrm{HH}(A/R; \mathbb{Z}_p)) \overset{\alpha_\gamma^{-1}}{\cong} (I/I^2)^\wedge_p,$$

using the notation of Lemma 2.1. Recall first the following standard lemma.

Lemma 2.4. *Let R be a ring, $I \subseteq R$ an ideal and assume that R is (p, I)-adically complete. Then the canonical map*

$$R^\flat := \varprojlim_{x \mapsto x^p} R \to A^\flat := \varprojlim_{x \mapsto x^p} A,$$

with $A = R/I$, is bijective.

Proof. It suffices to construct a well-defined, multiplicative map

$$[-] : A^\flat \to R$$

reducing to the first projection modulo I. Let

$$r := (r_0, r_1, \dots) \in A^\flat$$

be a p-power compatible system of elements in A with lifts $r_i' \in R$ of each r_i. Then the limit

$$\lim_{n \to \infty} (r_n')^{p^n}$$

exists and is independent of the lift. Thus

$$[r] := \lim_{n \to \infty} (r_n')^{p^n}$$

defines the desired map. □

The morphism

$$[-] : A^\flat \to R$$

is the Teichmüller lift for the surjection $\pi : R \to R/I$. If we want to make its dependence of the surjection clear, we write $[-]_\pi$. Let

$$T_p A^\times = \varprojlim_{x \mapsto x^p} A^\times[p^n]$$

be the Tate module of $A^\times$. Then we embed $T_p A^\times$ into $A^\flat$ as the sequences with first coordinate 1. For any $a \in A^\flat$ we define

$$[a] := r_0,$$

where $r = (r_0, r_1, \dots) \in R^\flat$ is the unique element reducing to a. If $a = (1, a_1, a_2, \dots)$ lies in $T_p A^\times$, then $[a] \in 1 + I$.

Proposition 2.5. *Fix a generator $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$. Let R be a ring and $I \subseteq R$ an ideal such that R is (p, I)-adically complete. Let $A = R/I$. Then the composition*

$$T_p(A^\times) \cong \pi_2((BA^\times)^\wedge_p) \xrightarrow{\mathrm{Dtr}} \pi_2(\mathrm{HH}(A/R; \mathbb{Z}_p)) \cong (I/I^2)^\wedge_p$$

is given by sending $a \in T_p(A^\times)$ to

$$\delta_1(\gamma)([a] - 1),$$

where $\delta_1(\gamma) \in \{\pm 1\}$ is the sign from Lemma 2.2.

Proof [14]. Fix $a \in T_p(A^\times)$. Then there exists, by (p, I)-adic completeness of R, a unique morphism $\mathbb{Z}[1/p] \to R^\times$ of abelian groups such that

$$1/p^n \mapsto [a^{1/p^n}].$$

By naturality, it therefore suffices to check that for

$$R := \mathbb{Z}[t^{1/p^\infty}] \cong \mathbb{Z}[\mathbb{Z}[1/p]] \quad \text{and} \quad A := R/(t-1) \cong \mathbb{Z}[\mathbb{Q}_p/\mathbb{Z}_p],$$

under the morphism

$$T_pA^\times \xrightarrow{\mathrm{Dtr}} \mathrm{HH}_2(A;\mathbb{Z}_p) \to \mathrm{HH}_2(A/R;\mathbb{Z}_p) \cong L_{A/R}[-1] \cong (t-1)/(t-1)^2$$

the element $(1, t^{1/p}, t^{1/p^2}, \ldots) \in A^\flat$ is mapped to the class of $\delta_1(\gamma)(t-1)$.

Observe first that the Hochschild homology

$$\mathrm{HH}_2(A)$$

vanishes. Indeed, it is easy to see that $L_{A/\mathbb{Z}}$ is concentrated in degree 0. Moreover, $\Omega^1_{A/\mathbb{Z}} \cong L_{A/\mathbb{Z}}$ is generated by one element. This implies that

$$\pi_0\big(\textstyle\bigwedge^n L_{A/\mathbb{Z}}\big) = 0$$

for $n \geq 2$ (see the proof of [Bhatt 2012, Corollary 3.13]). By the HKR-filtration, we get that $\mathrm{HH}_2(A) = 0$. Passing to p-completions we can conclude that

$$\mathrm{HH}_2(A;\mathbb{Z}_p) \cong T_p\mathrm{HH}_1(A) \overset{\alpha_\gamma}{\cong} T_p(\Omega^1_{A/\mathbb{Z}}),$$

where the last isomorphism is the HKR-isomorphism (for γ).

There is a commutative diagram

$$\begin{tikzcd}
\mathrm{HH}_2(A;\mathbb{Z}_p) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{HH}_2(A/R;\mathbb{Z}_p) \arrow[d, "\cong"] \\
T_p\Omega^1_{A/\mathbb{Z}} \cong \pi_1((L_{A/\mathbb{Z}})^\wedge_p) \arrow[r, "\cong"] & \pi_1((L_{A/R})^\wedge_p) \cong ((t-1)/(t-1)^2)^\wedge_p
\end{tikzcd}$$

Using Lemma 2.2, the element

$$(1, t^{1/p}, t^{1/p^2}, \ldots) \in T_pA^\times$$

is mapped to the element

$$\delta_1(\gamma)(0, d\log(t^{1/p}), d\log(t^{1/p^2}), \ldots) \in T_p(\Omega^1_{A/\mathbb{Z}}).$$

[14] The following argument is simpler than our original argument and was suggested by the referee. We thank her/him for allowing us to include it.

The effect of the bottom row can be calculated using the exact triangle

$$L_{R/\mathbb{Z}} \otimes^{\mathbb{L}}_R A \to L_{A/\mathbb{Z}} \xrightarrow{\beta} L_{A/R}$$

and applying p-completions. More precisely, rotating plus the isomorphisms

$$L_{R/\mathbb{Z}} \cong \Omega^1_{R/\mathbb{Z}}, \qquad L_{A/R} \cong (t-1)/(t-1)^2[1]$$

yield the exact triangle

$$(t-1)/(t-1)^2 \xrightarrow{d} \Omega^1_{R/\mathbb{Z}} \otimes_R A \to \Omega^1_{A/R} \to (t-1)/(t-1)^2[1],$$

where the first morphism is the differential. Now, applying (derived) p-completion to this exact triangle, the resulting connecting morphism

$$T_p(\Omega^1_{A/\mathbb{Z}}) \to (t-1)/(t-1)^2$$

sends $(0, d\log(t^{1/p}), d\log(t^{1/p^2}), \dots)$ to $t-1$ as $t-1 \equiv \dfrac{t-1}{t} \bmod (t-1)^2$ and

$$\frac{d(t-1)}{t} = d\log(t) = p^n d\log(t^{1/p^n})$$

for all $n \geq 0$.[15] Thus,

$$\beta((0, d\log(t^{1/p}), d\log(t^{1/p^2}), \dots) = t-1$$

as claimed. □

We recall the following lemma. For a perfect ring S we denote its ring of Witt vectors by $W(S)$.

Lemma 2.6. *Let S be a perfect ring and let A be a $W(S)$-algebra. Then the canonical morphism*

$$\mathrm{HH}(A; \mathbb{Z}_p) \to \mathrm{HH}(A/W(S); \mathbb{Z}_p)$$

is an equivalence.

Proof. By the HKR-filtration, it suffices to see that the canonical morphism

$$L_{A/\mathbb{Z}} \to L_{A/W(S)}$$

of cotangent complexes is a p-adic equivalence, i.e., an equivalence after $- \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p$. Computing the right-hand side by polynomial algebras over $W(S)$ we see that it suffices to consider the case that A is p-torsion free. Then by base change

$$L_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p \cong L_{(A/p)/\mathbb{F}_p} \quad \text{and} \quad L_{A/W(S)} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p \cong L_{(A/p)/S}$$

[15] If $0 \to M \to N \to Q \to 0$ is a short exact sequence of abelian groups, then the boundary map $T_p Q \to M^\wedge_p$ has the following description: Take $x := (q_i)_{i \geq 0} \in T_p Q$ and lift each q_i to some $n_i \in N$. Then $p^i n_i \in M$ and the limit $\varprojlim p^i n_i \in M^\wedge_p$ exists and is the image of x.

and the claim follows from the transitivity triangle

$$A/p \otimes^{\mathbb{L}}_S L_{S/\mathbb{F}_p} \to L_{(A/p)/\mathbb{F}_p} \to L_{(A/p)/S},$$

using the fact that S is perfect, which implies that the cotangent complex $L_{S/\mathbb{F}_p}$ of S over $\mathbb{F}_p$ vanishes. □

3. Transversal prisms

In this section we want to prove the crucial injectivity statement (Corollary 3.11) mentioned in the introduction. Let us recall the following definition from [Bhatt and Scholze 2019].

Definition 3.1. A *δ-ring* is a pair (A, δ), where A is a commutative ring, $\delta : A \to A$ a map of sets, with $\delta(0) = 0$, $\delta(1) = 0$, and

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p},$$
$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y),$$

for all $x, y \in A$.

A *prism* (A, I) is a δ-ring A with an ideal I defining a Cartier divisor on $\operatorname{Spec}(A)$, such that A is derived (p, I)-adically complete and $p \in (I, \varphi(I))$.

Here, the map

$$\varphi : A \to A, \quad x \mapsto \varphi(x) := x^p + p\delta(x)$$

denotes the lift of Frobenius induced from the δ-structure on A. We make the (usually harmless) assumption that $I = (\tilde{\xi})$ is generated by some distinguished element $\tilde{\xi} \in A$, i.e., $\tilde{\xi}$ is a nonzero divisor and $\delta(\tilde{\xi})$ is a unit.

Definition 3.2. We call a prism *transversal* if $(p, \tilde{\xi})$ is a regular sequence on A.

Let us fix a transversal prism (A, I). In particular, A is p-torsion free. Moreover, A is classically (p, I)-adically complete. Indeed, $(p, \tilde{\xi})$ being a regular sequence implies that

$$A \otimes^{\mathbb{L}}_{\mathbb{Z}[x,y]} \mathbb{Z}[x, y]/(x^n, y^n) \cong A/(p^n, \tilde{\xi}^n)$$

and therefore

$$A \cong R \varprojlim_n (A \otimes^{\mathbb{L}}_{\mathbb{Z}[x,y]} \mathbb{Z}[x, y]/(x^n, y^n)) \cong R \varprojlim_n (A/(p^n, \tilde{\xi}^n)) \cong \varprojlim_n A/(p^n, \tilde{\xi}^n),$$

using Mittag-Leffler for the last isomorphism.

We set

$$I_r := I\varphi(I) \cdots \varphi^{r-1}(I)$$

for $r \geq 1$ (where $\varphi^0(I) := I$). Then $I_r = (\tilde{\xi}_r)$ with

$$\tilde{\xi}_r := \tilde{\xi}\varphi(\tilde{\xi}) \cdots \varphi^{r-1}(\tilde{\xi}).$$

Lemma 3.3. *For all $r \geq 1$ the element*

$$\varphi^r(\tilde{\xi})$$

is a nonzero divisor and $(\varphi^r(\tilde{\xi}), p)$ is again a regular sequence. In particular, the elements $\tilde{\xi}_r$, $r \geq 1$, are nonzero divisors.

Proof. The regularity of the sequence $(p, \varphi^r(\tilde{\xi}))$, or equivalently of $(p, \tilde{\xi}^{p^r})$, follows from that of $(p, \tilde{\xi})$. The regularity of $(\varphi(\tilde{\xi}^{p^r}), p)$ follows from this and the fact that in any ring R with a regular sequence (r, s) such that R is r-adically complete the sequence (s, r) is again regular.[16] □

Lemma 3.4. *The ring A is complete for the topology induced by the ideals I_r, i.e.,*

$$A \cong \varprojlim_r A/I_r.$$

Proof. Each A/I_r is p-torsion free by Lemma 3.3. Therefore, both sides are p-complete and p-torsion free. Hence, it suffices to check the statement modulo p (note that by p-torsion freeness of each A/I_r, modding out p commutes with the inverse limit). But modulo p the topology defined by the ideals I_r is just the $\tilde{\xi}$-adic topology and A/p is $\tilde{\xi}$-adically complete. □

Lemma 3.5. *For $r \geq 1$ there is a congruence*

$$\varphi^r(\tilde{\xi}) \equiv pu \quad \textit{modulo } (\tilde{\xi})$$

with $u \in A^\times$ some unit.

Proof. For $r = 1$ this follows from

$$\varphi(\tilde{\xi}) = \tilde{\xi}^p + p\delta(\xi)$$

because by definition of distinguishedness the element $\delta(\xi) \in A^\times$ is a unit. For $r \geq 2$ we compute

$$\varphi^r(\tilde{\xi}) = \varphi^{r-1}(\tilde{\xi}^p + p\delta(\tilde{\xi})) = \varphi^{r-1}(\tilde{\xi})^p + p\varphi^{r-1}(\delta(\tilde{\xi})).$$

By induction we may write $\varphi^{r-1}(\tilde{\xi}) = pu + a\tilde{\xi}$ with $u \in A^\times$ some unit, and thus modulo $\tilde{\xi}$ we calculate

$$\varphi^r(\tilde{\xi}) \equiv (pu)^p + p\varphi(\delta(\tilde{\xi})) = p(\varphi(\delta(\tilde{\xi})) + p^{p-1}u^p)$$

with $\varphi(\delta(\tilde{\xi})) + p^{p-1}u^p \in A^\times$ some unit. □

Lemma 3.6. *For all $r \geq 1$ the sequences $(\varphi^r(\tilde{\xi}), \tilde{\xi})$ and $(\tilde{\xi}, \varphi^r(\tilde{\xi}))$ are again regular. Moreover, $I_r = \bigcap_{i=0}^{r-1} \varphi^i(I)$ for all $r \geq 1$.*

---

[16] Passing to the inverse limit of the injections $R/r^n \xrightarrow{s} R/r^n$ implies that $s \in R$ is a nonzero divisor. Thus, (r, s) is regular and s is regular, which implies that (s, r) is regular.

Proof. We can write $\varphi(\tilde{\xi}) = p\delta(\tilde{\xi}) + \tilde{\xi}^p$, where $\delta(\tilde{\xi}) \in A^\times$ is a unit. By Lemma 3.5 we get $\varphi^r(\tilde{\xi}) \equiv pu$ modulo $(\tilde{\xi})$ with $u \in A^\times$ a unit. As $(\tilde{\xi}, p)$ is a regular sequence we conclude (using [Stacks 2005–, Tag 07DW] and Lemma 3.3) that $(\varphi^r(\tilde{\xi}), \tilde{\xi})$ is a regular sequence. To prove the last statement we proceed by induction on r. First note the following general observation: If R is some ring and (f, g) a regular sequence in R, then $(f) \cap (g) = (fg)$. In fact, if $r = sg \in (f) \cap (g)$, then $sg \equiv 0$ modulo f, and hence $s \equiv 0$ modulo f as desired. Thus, it suffices to prove that $(\tilde{\xi}_r, \varphi^r(\tilde{\xi}))$ is a regular sequence for $r \geq 1$ (recall that $\tilde{\xi}_r = \tilde{\xi}\varphi(\tilde{\xi}) \cdots \varphi^{r-1}(\tilde{\xi})$). By induction, the morphism

$$A/(\tilde{\xi}_r) \to \prod_{i=0}^{r-1} A/(\varphi^i(\tilde{\xi}))$$

is injective. Hence, it suffices to show that for each $i = 0, \ldots, r-1$ the element $\varphi^r(\tilde{\xi})$ maps to a nonzero divisor in $A/(\varphi^i(\tilde{\xi}))$. But this follows from Lemma 3.5, which implies $\varphi^r(\tilde{\xi}) \equiv pu$ modulo $\varphi^i(\tilde{\xi})$ for some unit $u \in A^\times$. □

We can draw the following corollary.

Lemma 3.7. *Define* $\rho : A \to \prod_{r \geq 0} A/\varphi^r(I)$, $x \mapsto (x \bmod \varphi^r(I))$. *Then* ρ *is injective.*

Proof. This follows from Lemma 3.4 and Lemma 3.6, as the kernel of ρ is given by $\bigcap_{r=1}^\infty \varphi^r(I) = \bigcap_{r=1}^\infty I_r = 0$. □

We now define the Nygaard filtration of the prism (A, I) (see [Bhatt and Scholze 2019, Definition 11.1]).

Definition 3.8. Define

$$\mathcal{N}^{\geq n} A := \{x \in A \mid \varphi(x) \in I^n A\},$$

the n-th filtration step of the Nygaard filtration.

By definition, the Frobenius on A induces a morphism

$$\varphi : \mathcal{N}^{\geq 1} A \to I.$$

Note that we do not divide the Frobenius by $\tilde{\xi}$. Moreover, we define

$$\sigma : \prod_{i \geq 0} A/\varphi^i(I) \to \prod_{i \geq 0} A/\varphi^i(I), \quad (x_0, x_1, \ldots) \mapsto (0, \varphi(x_0), \varphi(x_1), \ldots).$$

Here we use the fact that if $a \equiv b \bmod \varphi^i(I)$, then $\varphi(a) \equiv \varphi(b) \bmod \varphi^{i+1}(I)$ to get that σ is well-defined. Then the diagram

$$\begin{array}{ccc} \mathcal{N}^{\geq 1}A & \xrightarrow{\rho} & \prod\limits_{i\geq 0} A/\varphi^i(I) \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\sigma} \\ I & \xrightarrow{\rho} & \prod\limits_{i\geq 0} A/\varphi^i(I) \end{array} \tag{3.9}$$

commutes, where ρ is the homomorphism from Lemma 3.7.

Lemma 3.10. *The reduction map*

$$A^{\varphi=\tilde{\xi}} \to A/I, \quad x \mapsto x \bmod (\tilde{\xi})$$

is injective.

Proof. Let $x \in A^{\varphi=\tilde{\xi}} \cap I$. We want to prove that $x = 0$. Clearly, $x \in \mathcal{N}^{\geq 1}A$. By Lemma 3.7 it suffices to prove that

$$x \equiv 0 \mod \varphi^i(I)$$

for all $i \geq 0$. Write

$$\rho(x) = (x_0, x_1, \ldots)$$

By the commutativity of the square (3.9) we get

$$\rho(\varphi(x)) = \sigma(\rho(x)) = (0, \varphi(x_0), \varphi(x_1), \ldots).$$

As $\varphi(x) = \tilde{\xi}x$ and therefore $\rho(\varphi(x)) = \tilde{\xi}\rho(x)$, we thus get

$$(\tilde{\xi}x_0, \tilde{\xi}x_1, \ldots) = (0, \varphi(x_0), \varphi(x_1), \ldots).$$

We assumed that $x \in I$, and thus $x_0 = 0 \in A/I$. Now we use that $\tilde{\xi}$ is a nonzero divisor modulo $\varphi^i(I)$ (see Lemma 3.6) for $i > 0$. Hence, if $x_i = 0$, then

$$0 = \varphi(x_i) = \tilde{\xi}x_{i+1} \in A/\varphi^{i+1}(I)$$

implies $x_{i+1} = 0$. Beginning with $x_0 = 0$, this shows that $x_i = 0$ for all $i \geq 0$, which implies our claim. □

The same proof shows that also the reduction map

$$A^{\varphi=\tilde{\xi}^n} \to A/I$$

is injective for $n \geq 1$.

The following corollary is crucially used in Theorem 6.7.

Corollary 3.11. *The reduction map*

$$A^{\varphi=\tilde{\xi}} \to \mathcal{N}^{\geq 1}A/\mathcal{N}^{\geq 2}A$$

is injective.

Proof. Let $x \in A^{\varphi=\tilde{\xi}} \cap \mathcal{N}^{\geq 2} A$. Then

$$\tilde{\xi} x = \varphi(x) = \tilde{\xi}^2 y$$

for some $y \in A$. As $\tilde{\xi}$ is a nonzero divisor in A we get $x \in I = (\tilde{\xi})$. But then $x = 0$ by Lemma 3.10. $\square$

Similarly, for each $n \geq 0$ the morphism

$$A^{\varphi=\tilde{\xi}^n} \to \mathcal{N}^{\geq i} A / \mathcal{N}^{\geq i+1} A \tag{3.12}$$

is injective. Let R be a quasiregular semiperfectoid ring (see [Bhatt et al. 2019, Definition 4.19]) which is p-torsion free. In this case,

$$A := \widehat{\Delta}_R$$

is transversal and (3.12) implies that for $i \geq 0$,

$$\pi_{2i}(\mathrm{TC}(R)) \to \pi_{2i}(\mathrm{THH}(R))$$

is injective; see [Bhatt et al. 2019, Theorem 1.12]. We ignore if there exists a direct topological proof, i.e., one which does not invoke prisms. Note that the p-torsion freeness is necessary. Indeed, by [Bhatt et al. 2019, Remark 7.20], $\pi_{2i}(\mathrm{TC}(R))$ is always p-torsion free.

4. The q-logarithm

In this section we recall the definition of the q-logarithm and prove some properties of it. Several statements in q-mathematics that we use are probably standard; see, e.g., [Scholze 2017] for more on q-mathematics. Recall that the q-analog of the integer $n \in \mathbb{Z}$ is defined to be

$$[n]_q := \frac{q^n - 1}{q - 1} \in \mathbb{Z}[q^{\pm 1}].$$

If $n \geq 1$, then we can rewrite

$$[n]_q = 1 + q + \cdots + q^{n-1}$$

and then the q-number actually lies in $\mathbb{Z}[q]$. For $n \geq 0$, we moreover get the relation

$$[-n]_q = \frac{q^{-n} - 1}{q - 1} = q^{-n} \frac{1 - q^n}{q - 1} = -q^{-n}[n]_q. \tag{4.1}$$

The q-numbers satisfy some basic relations, for example

$$[n+k]_q = q^k [n]_q + [k]_q \tag{4.2}$$

for $n, k \in \mathbb{Z}$, or

$$[m]_q = \frac{(q^n)^k - 1}{q^n - 1} \frac{q^n - 1}{q - 1} = \frac{(q^n)^k - 1}{q^n - 1} [n]_q \quad \text{if } n \mid m.$$

As further examples of q-analogs let us define the q-factorial for $n \geq 1$ as

$$[n]_q! := [1]_q \cdot [2]_q \cdots [n]_q \in \mathbb{Z}[q]$$

(with the convention that $[0]_q! := 1$) and, for $0 \leq k \leq n$, the q-binomial coefficient as

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Lemma 4.3. (1) *For $0 \leq k \leq n$, the q-binomial $\binom{n}{k}_q$ is in $\mathbb{Z}[q]$.*

(2) *For $1 \leq k \leq n$, the analog*

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

of Pascal's identity holds.

Proof. Part (1) follows from part (2) using induction and the easy case $\binom{n}{0}_q = 1$. Then part (2) can be proved as follows: Let $1 \leq k \leq n$; then

$$\begin{aligned} q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q &= \frac{[n-1]_q!}{[k-1]_q![n-1-k]_q!}\left(\frac{q^k}{[k]_q} + \frac{1}{[n-k]_q}\right) \\ &= \frac{[n-1]_q!}{[k-1]_q![n-1-k]_q!} \frac{q^k[n-k]_q + [k]_q}{[k]_q[n-k]_q} \\ &= \frac{[n-1]_q!}{[k-1]_q![n-1-k]_q!} \frac{[n]_q}{[k]_q[n-k]_q} \\ &= \binom{n}{k}_q \end{aligned}$$

using the addition rule (4.2). □

Let us define a generalized q-Pochhammer symbol by

$$(x, y; q)_n := (x+y)(x+yq)\cdots(x+yq^{n-1}) \in \mathbb{Z}[q^{\pm 1}, x, y]$$

for $n \geq 1$. Note that setting $x = 1$ and $y := -a$ recovers the known q-Pochhammer symbol

$$(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) = (1, -a; q)_n.$$

Moreover, we make the convention

$$(x, y; q)_0 := 1.$$

In the q-world the generalized q-Pochhammer symbol replaces the polynomial

$$(x+y)^n.$$

For example one can show (using Lemma 4.3) the q-binomial formula

$$(x, y; q)_n = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q x^{n-k} y^k. \tag{4.4}$$

Let us now come to q-derivations. We recall that the q-derivative $\nabla_q f$ of some polynomial $f \in \mathbb{Z}[q^{\pm 1}][x^{\pm 1}]$ is defined by

$$\nabla_q f(x) := \frac{f(qx) - f(x)}{qx - x} \in \mathbb{Z}[q^{\pm 1}][x^{\pm 1}].$$

Thus, for example, if $f(x) = x^n$, $n \in \mathbb{Z}$, then we can calculate

$$\nabla_q(x^n) = \frac{q^n x^n - qx}{qx - x} = \frac{q^n - 1}{q - 1} x^{n-1} = [n]_q x^{n-1}.$$

The q-derivative satisfies an analog of the Leibniz rule, namely

$$\nabla_q(f(x)g(x)) = \nabla_q(f(x))g(qx) + f(x)\nabla_q(g(x)).$$

Similarly to the classical rule

$$\nabla_x((x + y)^n) = n\nabla_x((x + y)^{n-1}),$$

we obtain the following relation for the generalized q-Pochhammer symbol.

Lemma 4.5. *Let $\nabla_q := \nabla_{q,x}$ denote the q-derivative with respect to x. Then the formula*

$$\nabla_q((x, y; q)_n) = [n]_q (x, y; q)_{n-1}$$

holds in $\mathbb{Z}[q^{\pm 1}][x^{\pm 1}, y^{\pm 1}]$.

Proof. We proceed by induction on n. Let $n = 1$. Then $(x, y; q)_n = x + y$ and

$$\nabla_q((x + y)) = 1.$$

Now let $n \geq 2$. We calculate using induction

$$\begin{aligned}
\nabla_q((x, y; q)_n) &= \nabla_q((x, y; q)_{n-1}(x + yq^{n-1})) \\
&= (x, y, q)_{n-1} \nabla_q(x + yq^{n-1}) + (qx + q^{n-1}y)\nabla_q((x, y; q)_{n-1}) \\
&= (x, y; q)_{n-1} \cdot 1 + q(x + q^{n-2}y)[n-1]_q (x, y; q)_{n-2} \\
&= (1 + q[n-1]_q)(x, y; q)_{n-1} \\
&= [n]_q (x, y; q)_{n-1},
\end{aligned}$$

where we used the q-Leibniz rule and (4.2). □

Similarly, as the polynomials

$$1, x-1, \frac{(x-1)^2}{2!}, \dots, \frac{(x-1)^n}{n!}, \dots$$

are useful for developing some function into a Taylor series around $x = 1$ (because the derivative of one polynomial is the previous one), the q-polynomials

$$1, (x, -1; q)_1, \frac{(x, -1; q)_2}{[2]_q!}, \dots, \frac{(x, -1; q)_n}{[n]_q!}, \dots$$

are useful for developing a q-polynomial into some "q-Taylor series" around $x = 1$. However, for this to make sense we have to pass to suitable completions and localize at $\{[n]_q\}_{n\geq 1}$. Let us be more precise about this. The $(q-1, x-1)$-completion $\mathbb{Z}[[q-1, x-1]]$ of $\mathbb{Z}[q, x]$ contains expressions of the form

$$\sum_{n=0}^{\infty} a_n (x, -1; q)_n$$

with $a_n \in \mathbb{Z}[[q-1]]$ because

$$(x, -1; q)_n = (x-1)(x-1+1-q)\cdots\left(x-1+(1-q)\frac{1-q^{n-1}}{1-q}\right) \in (q-1, x-1)^n.$$

Finally, the next calculations will take place in the ring[17]

$$\mathbb{Q}[[q-1, x-1]] \cong \mathbb{Z}[[q-1, x-1]][1/[n]_q \mid n \geq 1]^{\wedge}_{(q-1,x-1)}$$

because

$$\frac{(x, -1; q)_n}{[n]_q!} \in (q-1, x-1)_{\mathbb{Q}[[q-1,x-1]]}.$$

The ring $\mathbb{Q}[[q-1, x-1]]$ admits a surjection

$$\mathbb{Q}[[q-1, x-1]] \to \mathbb{Q}[[x-1]]$$

with kernel generated by $q-1$. Similarly, there is a morphism

$$\mathrm{ev}_1 : \mathbb{Q}[[q-1, x-1]] \to \mathbb{Q}[[q-1]]$$

with kernel generated by $x-1$. Finally, the q-derivative ∇_q extends to a q-derivation on $\mathbb{Q}[[q-1, x-1]]$ and it induces the usual derivative after modding out $q-1$. We denote by ∇_q^n the n-fold decomposition of ∇_q and by

$$f(x)_{|x=1} := \mathrm{ev}_1(f(x))$$

the evaluation at $x = 1$ of an element $f \in \mathbb{Q}[[q-1, x-1]]$.

[17] Note that inverting $[n]_q$ for $n \geq 0$ and then $(q-1)$-adically completing is the same as inverting n for $n \geq 0$ and then $(q-1)$-adically completing.

Lemma 4.6. *Take* $f(x) \in \mathbb{Q}[[q-1, x-1]]$. *If* $\nabla_q^n(f(x))_{|x=1} = 0$ *for all* $n \geq 0$, *then* $f(x) = 0$.

Proof. As ∇_q reduces to the usual derivative modulo $q-1$, we see that f must be divisible by $q-1$, i.e., we can write $f(x) = (q-1)g(x)$ with $g(x) \in \mathbb{Q}[[q-1, x-1]]$. But then $\nabla_q^n(g(x))_{x=1} = 0$ for all $n \geq 0$, and we can conclude as before that $q-1 \mid g(x)$, which in the end implies

$$f(x) \in \bigcap_{k=1}^{\infty} (q-1)^k = \{0\}$$

because $\mathbb{Q}[[q-1, x-1]]$ is $(q-1)$-adically separated. ☐

Now we can describe the q-Taylor expansion around $x = 1$ for elements in $\mathbb{Q}[[q-1, x-1]]$.

Proposition 4.7. *For any* $f(x) \in \mathbb{Q}[[q-1, x-1]]$ *there is the Taylor expansion*

$$f(x) = \sum_{n=0}^{\infty} \nabla_q^n(f(x))_{|x=1} \frac{(x, -1; q)_n}{[n]_q!}.$$

Proof. Because

$$\nabla_q\left(\frac{(x, -1; q)_n}{[n]_q!}\right) = \frac{(x, -1; q)_{n-1}}{[n-1]_q!}$$

we can directly calculate that both sides have equal higher derivatives at $x = 1$. Thus they agree by Lemma 4.6. ☐

Using this in Lemma 4.9 we can motivate the below formula for the q-logarithm.

Definition 4.8. We define the q-logarithm as

$$\log_q(x) := \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} \frac{(x, -1; q)_n}{[n]_q} \in \mathbb{Q}[[q-1, x-1]].$$

Note that $\log_q(x)$ is contained in a much smaller subring of $\mathbb{Q}[[q-1, x-1]]$: it suffices to adjoin the elements $(x, -1; q)_n/[n]_q$ for $n \geq 0$ to $\mathbb{Z}[q^{\pm 1}, x^{\pm 1}]$ and $(x-1)$-adically complete.

In the ring $\mathbb{Q}[[q-1, x-1]]$ the element x is invertible, as

$$\frac{1}{x} = \frac{1}{1-(1-x)} = 1 + (1-x) + (1-x)^2 + \cdots.$$

The q-derivative of the q-logarithm is $1/x$, like the usual logarithm.

Lemma 4.9. *The* q*-logarithm* $\log_q(x)$ *is the unique* $f(x) \in \mathbb{Q}[[q-1, x-1]]$ *satisfying* $f(1) = 0$ *and* $\nabla_q(f(x)) = 1/x$. *Moreover*,

$$\log_q(x) = \frac{q-1}{\log(q)} \log(x)$$

as elements in $\mathbb{Q}[[q-1, x-1]]$.

Proof. That $\log_q(x)$ has q-derivative $1/x$ can be checked using Proposition 4.7 after writing $1/x$ in its q-Taylor expansion. Moreover, $\log_q(1) = 0$. For the converse pick f as in the statement. By Proposition 4.7 we can write

$$f(x) = \sum_{n=0}^{\infty} \nabla_q^n(f(x))_{|x=1} \frac{(x, -1; q)_n}{[n]_q!},$$

and thus we have to determine

$$a_n := \nabla_q^n(f(x))_{|x=1}$$

for $n \geq 0$. By assumption we must have $a_0 = f(1) = 0$. Moreover, for $n \geq 1$,

$$a_n = \nabla_q^n(f(x))_{|x=1} = \nabla_q^{n-1}(x^{-1})_{|x=1} = [-n+1]_q \cdots [-1]_q.$$

Using $[-k]_q = -q^{-k}[k]_q$ for $k \in \mathbb{Z}$, the last expression simplifies to

$$[-n+1]_q \cdots [-1]_q = (-1)^{n-1} q^{-n(n-1)/2} [n-1]_q!.$$

Thus we can conclude

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} \frac{(x, -1; q)_n}{[n]_q} = \log_q(x).$$

For the last statement note that

$$f(x) := \frac{q-1}{\log(q)} \log(x)$$

exists in $\mathbb{Q}[\![q-1, x-1]\!]$ (because $n \in R_q^\times$ for all $n \geq 1$) and satisfies $f(1) = 0$. Moreover,

$$\nabla_q(f(x)) = \frac{f(qx) - f(x)}{qx - x} = \frac{q-1}{\log(q)} \frac{\log(q) + \log(x) - \log(x)}{(q-1)x} = \frac{1}{x},$$

which implies $f(x) = \log_q(x)$ by the proven uniqueness of the q-logarithm. $\square$

We now turn to prisms again. Define

$$\tilde{\xi} := [p]_q = 1 + q + \cdots + q^{p-1}$$

and

$$\tilde{\xi}_r := \tilde{\xi} \varphi(\tilde{\xi}) \cdots \varphi^{r-1}(\tilde{\xi})$$

for $r \geq 1$. Here, φ is the Frobenius lift on $\mathbb{Z}[q^{\pm 1}]$ satisfying $\varphi(q) = q^p$. Then $\tilde{\xi}$ is a distinguished element in the prism $\mathbb{Z}_p[\![q-1]\!]$. The $\tilde{\xi}_r$ are again q-numbers, namely

$$\tilde{\xi}_r = [p^r]_q.$$

Let us recall the following situation from crystalline cohomology. Assume that A is a p-complete ring with an ideal $J \subseteq A$ equipped with divided powers

$$\gamma_n : J \to J, \quad n \geq 1.$$

In this situation the logarithm

$$\log(x) := \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)!\gamma_n(x-1)$$

converges in A for every element $x \in 1 + J$. We now want to prove an analogous statement for the q-logarithm. Recall that for a prism (A, I) we defined in Definition 3.8 the Nygaard filtration

$$\mathcal{N}^{\geq n} A := \{x \in A \mid \varphi(x) \in I^n\}, \quad n \geq 0.$$

From now on, we assume that the prism (A, I) lives over $(\mathbb{Z}_p[\![q-1]\!], (\tilde{\xi}))$. The expression

$$\gamma_{n,q}(x-y) := \frac{(x-y)(x-qy)\cdots(x-q^{n-1}y)}{[n]_q!} \in \mathbb{Z}_p[\![q-1]\!][x, y][1/[m]_q \mid m \geq 0]$$

is called the n-th q-divided power of $x - y$; see [Pridham 2019, Remark 1.4].[18] We study the divisibility of

$$(x-y)(x-qy)\cdots(x-q^{n-1}y)$$

by

$$\tilde{\xi}, \varphi(\tilde{\xi}), \ldots.$$

The following statement is clear.

Lemma 4.10. *For $r \geq 1$ the polynomial (in q)*

$$\varphi^{r-1}(\tilde{\xi}) = \frac{q^{p^r} - 1}{q^{p^{r-1}} - 1}$$

is the minimal polynomial of a p^r-th root of unity ζ_{p^r}, i.e., the morphism

$$\mathbb{Z}[q]/(\varphi^{r-1}(\tilde{\xi})) \to \mathbb{Z}[\zeta_{p^r}], \quad q \mapsto \zeta_{p^r}$$

is injective.

Thus, reducing modulo $\varphi^{r-1}(\tilde{\xi})$ is the same as setting $q = \zeta_{p^r}$. Moreover, in $\mathbb{Z}[\zeta_{p^r}]$ there is the equality

$$z^{p^r} - 1 = \prod_{i=0}^{p^r-1} (z - \zeta_{p^r}^i).$$

[18] This terminology is, however, quite bad. The q-divided power depends on the pair (x, y) and not simply their difference $x - y$.

Setting $z = x/y$ one thus arrives at the congruence

$$x^{p^r} - y^{p^r} \equiv (x-y)(x-qy)\cdots(x-q^{p^r-1}y) \mod \varphi^{r-1}(\tilde{\xi}), \tag{4.11}$$

which will be useful.

Lemma 4.12. *Let $n \geq 1$, and for $r \geq 1$ write $n = a_r p^r + b_r$ with $a_r, b_r \geq 0$ and $b_r < p^r$. Then in $\mathbb{Z}_p[[q-1]]$,*

$$[n]_q! = u \prod_{r\geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{a_r}$$

for some unit $u \in \mathbb{Z}_p[[q-1]]^\times$.

Proof. We may prove the statement by induction on n. Thus let us assume that it is true for $m = n-1$, and for $r \geq 1$ write $m = c_r p^r + d_r$ with $c_r, d_r \geq 0$ and $d_r < p^r$. If n is prime to p, then $[n]_q$ is a unit in $\mathbb{Z}_p[[q-1]]$ and it suffices to see that the right-hand side is equal (up to some unit in $\mathbb{Z}_p[[q-1]]$) to

$$\prod_{r\geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{c_r}.$$

But n being prime to p implies that $b_r > 0$ for all $r \geq 1$. Thus $c_r = a_r$ and $d_r = b_r - 1$, which implies that both products are equal. Now assume that p divides n and write $n = p^s n'$ with n' prime to p. Moreover, write $m = n-1 = c_r p^r + d_r$ as above. Then we can conclude $a_r = c_r$ for $r > s$ while $c_r = a_r - 1$ for $1 \leq r \leq s$ (as $d_r = p^r - 1$ for such r). Altogether we therefore arrive at

$$[n]_q! = [n]_q[n-1]_q! = u'[n]_q \prod_{r\geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{c_r} = u'v \prod_{r\geq 1}^{\infty} \varphi^{r-1}(\tilde{\xi})^{a_r},$$

$u' \in \mathbb{Z}_p[[q-1]]^\times$, where we used the fact that

$$[n]_q = v[p^s]_q = v\varphi^{s-1}(\tilde{\xi})\cdots\tilde{\xi}$$

for some unit $v \in \mathbb{Z}_p[[q-1]]$. □

Proposition 4.13. *Let (A, I) be a prism over $(\mathbb{Z}_p[[q-1]], (\tilde{\xi}))$ and let $x, y \in A$ be elements of rank 1 such that $\varphi(x-y) = x^p - y^p \in \tilde{\xi}A$. Then for all $n \geq 1$ the ring A contains a q-divided power*

$$\gamma_{n,q}(x-y) = \frac{(x-y)(x-qy)\cdots(x-q^{n-1}y)}{[n]_q!}$$

of $x-y$.[19] *Moreover, $\gamma_{n,q}$ lies in fact in the n-th step $\mathcal{N}^{\geq n}A$ of the Nygaard filtration of A.*

[19]By this we mean that there exists an element, called $\gamma_{n,q}(x-y)$, such that $[n]_q!\gamma_{n,q}(x-y) = (x-y)(x-qy)\cdots(x-q^{n-1}y)$. The element $\gamma_{n,q}(x-y)$ need not be unique, but it is if A is

Proof. Replacing A, x, y by the universal case we may assume that A is flat over $\mathbb{Z}_p[\![q-1]\!]$. In particular, this implies that $\tilde{\xi}, \varphi(\tilde{\xi}), \ldots$ are pairwise regular sequences (see Lemma 3.6). Fix $n \geq 1$. For $r \geq 1$ we write n as

$$n = a_r p^r + b^r$$

with $a_r, b_r \geq 0$ and $0 \leq b^r < p^r$. We claim that for each $r \geq 0$

$$\varphi^{r-1}(\tilde{\xi})^{a_r}$$

divides

$$(x-y)(x-qy)\cdots(x-q^{n-1}y).$$

This implies the proposition, namely by Lemma 4.12 we have

$$[n]_q! = u \prod_{r\geq 1} \varphi^{r-1}(\tilde{\xi})^{a_r}$$

for some unit $u \in A^\times$ while furthermore the morphism

$$A/([n]_q!) \to \prod_{r\geq 1} A/(\varphi^{r-1}(\tilde{\xi}))^{a_r}$$

is injective by the proof of Lemma 3.6. Thus fix $r \geq 1$. To prove our claim we may replace n by $n - b_r$ as

$$(x-y)(x-qy)\cdots(x-q^{n-b_r-1}y)$$

divides

$$(x-y)(x-qy)\cdots(x-q^{n-1}y).$$

Thus let us assume that $n = a_r p^r$. We claim that each of the a_r-many elements

$$\begin{gathered}(x-y)(x-qy)\cdots(x-q^{p^r-1}y),\\ (x-q^{p^r}y)(x-q^{p^r+1}y)\cdots(x-q^{2p^r-1}y),\\ \vdots\\ (x-q^{(a_r-1)p^r}y)(x-q^{(a_r-1)p^r+1}y)\cdots(x-q^{a_r p^r-1}y)\end{gathered}$$

(note that their product is $(x-y)\cdots(x-q^{n-1}y)$) is divisible by $\varphi^{r-1}(\tilde{\xi})$. For this recall the congruence (4.11). Replacing in this congruence y by $q^{p^r}y, \ldots, q^{(a_r-1)p^r}y$ shows that each of the above a_r elements is congruent modulo $\varphi^{r-1}(\tilde{\xi})$ to an element of the form

$$x^{p^r} - q^k y^{p^r}$$

$[n]_q$-torsion free for any $n \geq 0$. Note that even in this torsion free case $\gamma_{n,q}(x-y)$ depends on the pair (x, y) and not merely on the difference $x - y$.

with $k \geq 0$ divisible by p^r. But we have

$$x^{p^r} - q^k y^{p^r} = (x^{p^r} - y^{p^r}) + y^{p^r}(1 - q^k)$$

and we claim that under our assumptions both summands are divisible by $\varphi^{r-1}(\tilde{\xi})$. For the first summand we use that x, y are of rank 1 to write

$$x^{p^r} - y^{p^r} = \varphi^{r-1}(x^p - y^p) = \varphi^{r-1}(\tilde{\xi})\varphi^{r-1}\Big(\frac{x^p - y^p}{\tilde{\xi}}\Big),$$

which makes sense as we assumed that

$$x^p - y^p \in \tilde{\xi} A.$$

For the second summand we note that

$$1 - q^k = \frac{1 - q^k}{1 - q^{p^r}} \varphi^{r-1}(\tilde{\xi})(1 - q^{p^{r-1}})$$

with all factors in $\mathbb{Z}_p[\![q-1]\!]$ as p^r divides k. It remains to prove that

$$\gamma_{n,q}(x - y) = \frac{(x - y)(x - qy) \cdots (x - q^{n-1}y)}{[n]_q!}$$

lies in $\mathcal{N}^{\geq n} A$. But

$$\varphi(\gamma_{n,q}) = \frac{(x^p - y^p)(x^p - q^p y^p) \cdots (x^p - q^{p(n-1)} y^p)}{\varphi([n]_q!)}$$

and as we saw above $\tilde{\xi}$ divides each of the n factors

$$(x^p - y^p), (x^p - q^p y^p), \ldots, (x^p - q^{p(n-1)} y^p).$$

But $\tilde{\xi}$ and $\varphi([n]_q!)$ form a regular sequence by Lemma 3.6, which implies that

$$(x^p - y^p)(x^p - q^p y^p) \cdots (x^p - q^{p(n-1)} y^p)$$

is divisible by $\tilde{\xi}^n \varphi([n]_q)$ as was to be proven. This finishes the proof of the proposition. □

As the proof shows there exists unique choice of a q-divided power

$$\gamma_{n,q}(x - y)$$

which is functorial in the triple (A, x, y) (with $x, y \in A$ satisfying the assumptions in Proposition 4.13). From now on we always assume that these q-divided powers are chosen. Moreover, we get the following lemma concerning the convergence of the q-logarithm.

Lemma 4.14. *Let (A, I) be a prism over $(\mathbb{Z}_p[[q-1]], (\tilde{\xi}))$. Then for every element $x \in 1+\mathcal{N}^{\geq 1}A$ of rank 1 the series*

$$\log_q(x) = \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} [n-1]_q! \gamma_{n,q}(x-1)$$

is well-defined and converges in A. Moreover, $\log_q(x) \in \mathcal{N}^{\geq 1}A$,

$$\log_q(x) \equiv x-1 \mod \mathcal{N}^{\geq 2}A$$

and

$$\log_q(xy) = \log_q(x) + \log_q(y)$$

for any $x, y \in 1+\mathcal{N}^{\geq 1}A$ of rank 1.

Proof. By our assumption on x we get $\varphi(x-1) \in \tilde{\xi}A$, and thus we may apply Proposition 4.13 to $x = x$ and $y = 1$. Thus the (canonical choice of) q-divided powers

$$\gamma_{n,q}(x-1) = \frac{(x-1)(x-q)\cdots(x-q^{n-1})}{[n]_q!}$$

in A are well-defined. Moreover, as

$$\log_q(x) = \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} [n-1]_q! \gamma_{n,q}(x-1)$$

and the elements $[n-1]_q!$ tend to zero in A for the (p, I)-adic topology, we can conclude that the series $\log_q(x)$ converges because A is $\tilde{\xi}$-adically complete. The claim concerning the Nygaard filtrations follows directly from $\gamma_{n,q}(x-1) \in \mathcal{N}^{\geq n}A$, which was proven in Proposition 4.13. That $\log_q$ is a homomorphism can be seen in the universal case in which A is flat over $\mathbb{Z}_p[[q-1]]$ (by [Bhatt and Scholze 2019, Proposition 3.13]). Then the formula $\log_q(xy) = \log_q(x) + \log_q(y)$ can be checked after base change to $\mathbb{Q}_p[[q-1]]$, where it follows from Lemma 4.9 as the usual logarithm is a homomorphism. □

5. Prismatic cohomology and topological cyclic homology

This section is devoted to the relation of the prismatic cohomology developed by Bhatt and Scholze [2019] with topological cyclic homology (as described by Bhatt, Morrow and Scholze [Bhatt et al. 2019]) following [Bhatt and Scholze 2019, Section 11.5].

Let R be a quasiregular semiperfectoid ring, and let S be any perfectoid ring with a map $S \to R$.

Proposition 5.1. *The category of prisms (A, I) with a map $R \to A/I$ admits an initial object $(\Delta_R^{\mathrm{init}}, I)$, which is a bounded prism. Moreover, Δ_R^{init} identifies with the derived prismatic cohomology $\Delta_{R/A_{\mathrm{inf}}(S)}$, for any choice of S as before.*

Proof. See [Bhatt and Scholze 2019, Propositions 7.2 and 7.10] or [Anschütz and Le Bras 2019, Proposition 3.4.2]. $\square$

In the following, we simply write $\Delta_R = \Delta_R^{\mathrm{init}} = \Delta_{R/A_{\mathrm{inf}}(S)}$.

Theorem 5.2. *Let R be a quasiregular semiperfectoid ring. There is a functorial* (*in R*) *δ-ring structure on $\widehat{\Delta}_R^{\mathrm{top}} := \pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p))$ refining the cyclotomic Frobenius. The induced map $\Delta_R = \Delta_R^{\mathrm{init}} \to \widehat{\Delta}_R^{\mathrm{top}}$ identifies $\widehat{\Delta}_R^{\mathrm{top}}$ with the completion with respect to the Nygaard filtration* (Definition 3.8) *of Δ_R, and is compatible with the Nygaard filtration on both sides.*

Proof. See [Bhatt and Scholze 2019, Theorem 11.10]. $\square$

The Nygaard filtration on $\widehat{\Delta}_R^{\mathrm{top}}$ is defined as the double-speed abutment filtration for the (degenerating) homotopy fixed point spectral sequence

$$E_2^{ij} := H^i(\mathbb{T}, \pi_{-j}(\mathrm{THH}(R;\mathbb{Z}_p))) \Rightarrow \pi_{-i-j}(\mathrm{TC}^-(R;\mathbb{Z}_p))$$

for the $\mathbb{T} = S^1$-action on $\mathrm{THH}(R;\mathbb{Z}_p)$. If $\eta \in H^2(\mathbb{T},\mathbb{Z})$ is a generator, then multiplication by any lift $v \in \pi_{-2}(\mathrm{TC}^-(R;\mathbb{Z}_p))$ of the image of η in $H^2(\mathbb{T}, \pi_0(\mathrm{THH}(R;\mathbb{Z}_p)))$ induces isomorphisms

$$\pi_{2i}(\mathrm{TC}^-(R;\mathbb{Z}_p))) \cong \mathcal{N}^{\geq i}\widehat{\Delta}_R^{\mathrm{top}}$$

for $i \in \mathbb{Z}$.

Remark 5.3. We will only use the fact that $\widehat{\Delta}_R$ is a prism in this paper (as we will apply the results of Section 3 to $\pi_0(\mathrm{TC}^-(R;\mathbb{Z}_p)))$ and that the topological Nygaard filtration, defined via the homotopy fixed point spectral sequence, agrees with the Nygaard filtration from Definition 3.8, but the way one proves this is by showing the stronger statement that $\widehat{\Delta}_R^{\mathrm{top}}$ is the Nygaard completion of Δ_R. We ignore whether there is a more direct way to produce the δ-structure on $\widehat{\Delta}_R$; see [Bhatt and Scholze 2019, Remark 1.14].

6. The p-completed cyclotomic trace in degree 2

Now we are ready to prove our main theorem on the identification of the p-completed cyclotomic trace. Recall that for any ring A the cyclotomic trace

$$\mathrm{ctr} : K(A) \to \mathrm{TC}(A)$$

from the algebraic K-theory of A to its topological cyclic homology is a natural morphism[20] refining the Dennis trace $\mathrm{Dtr} : K(A) \to \mathrm{HH}(A)$ introduced in Section 2; see [Blumberg et al. 2013, Section 10.3; Bökstedt et al. 1993, Section 5]. Let us carefully fix some notation. For the whole section we fix a generator $\gamma \in H_1(\mathbb{T},\mathbb{Z})$,

[20]When upgraded to a natural transformation of functors on small stable ∞-categories the cyclotomic trace is uniquely determined by these properties; see [Blumberg et al. 2013, Section 10.3].

but note that the formulas in Theorem 6.7 will be independent of this choice. Set $\mathbb{Z}_p^{\mathrm{cycl}}$ as the p-completion of $\mathbb{Z}_p[\mu_{p^\infty}]$ and choose some p-power compatible system of p-power roots of unity

$$\varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in (\mathbb{Z}_p^{\mathrm{cycl}})^\flat$$

with $\zeta_p \neq 1$. This choice determines several elements as we now discuss. Set

$$\begin{aligned} q &:= [\varepsilon]_\theta \in A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cycl}}) := W((\mathbb{Z}_p^{\mathrm{cycl}})^\flat) \cong \pi_0(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)), \\ \mu &:= q - 1, \\ \tilde{\xi} &:= [p]_q = \frac{q^p - 1}{q - 1} = 1 + q + \cdots + q^{p-1}, \end{aligned}$$

and

$$\xi := \varphi^{-1}(\tilde{\xi}).$$

Note that the ring $A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cycl}})$ is the $(p, q-1)$-adic completion of $\mathbb{Z}_p[q^{1/p^\infty}]$. We now construct elements

$$u \in \pi_2(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)), \qquad v \in \pi_{-2}(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$$

such that $uv = \xi \in \pi_0(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$.[21] The elements u, v will be uniquely determined by ε. Let

$$\mathrm{ctr} : T_p(\mathbb{Z}_p^{\mathrm{cycl}})^\times \to \pi_2(\mathrm{TC}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$$

be the cyclotomic trace in degree 2. We denote by the same symbol the composition

$$\mathrm{ctr} : T_p(\mathbb{Z}_p^{\mathrm{cycl}})^\times \to \pi_2(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$$

with the canonical morphism $\mathrm{TC}(-; \mathbb{Z}_p) \to \mathrm{TC}^-(-; \mathbb{Z}_p)$. Let

$$\mathrm{can} : \mathrm{TC}^-(-; \mathbb{Z}_p) \to \mathrm{TP}(-; \mathbb{Z}_p)$$

be the canonical morphism (from homotopy to Tate fixed points).

Lemma 6.1. *The element*

$$\mathrm{can}(\mathrm{ctr}(\varepsilon)) \in \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$$

is divisible by μ.

A similar statement (in terms of TF) is proven in [Hesselholt 2006, Proposition 2.4.2] (see also [Hesselholt 2018, Definition 4.1]) using the explicit description of the cyclotomic trace in degree 1 via TR from [Geisser and Hesselholt 1999, Lemma 4.2.3].

[21] We need a finer statement than [Bhatt et al. 2019, Propositions 6.2 and 6.3], which asserts the existence of some u, v as above with $uv = a\xi$ for some unspecified unit $a \in A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cycl}})^\times$.

Proof. Fix a generator
$$\sigma' \in \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)).$$
It suffices to show that $\mathrm{can} \circ \mathrm{ctr}(\varepsilon)$ maps to 0 under the composition
$$\pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)) \xrightarrow{\sigma'^{-1}} \pi_0(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)) \cong A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cycl}}) \to W(\mathbb{Z}_p^{\mathrm{cycl}})$$
because the kernel of $A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cycl}}) \to W(\mathbb{Z}_p^{\mathrm{cycl}})$ is generated by μ; see [Bhatt et al. 2018, Lemma 3.23]. It therefore suffices to prove the statement for $\mathcal{O}_C$ for $C/\mathbb{Q}_p^{\mathrm{cycl}}$ an algebraically closed, complete nonarchimedean extension. Over $\mathcal{O}_C$ we can (after changing σ') find
$$u' \in \pi_2(\mathrm{TC}^-(\mathcal{O}_C; \mathbb{Z}_p)), \qquad v' \in \pi_{-2}(\mathrm{TC}^-(\mathcal{O}_C; \mathbb{Z}_p))$$
such that
$$u'v' = \xi = \frac{\mu}{\varphi^{-1}(\mu)}, \qquad \mathrm{can}(v') = \sigma'^{-1}$$
and the cyclotomic Frobenius maps u' to σ'; see [Bhatt et al. 2019, Proposition 6.2., Proposition 6.3]. Then multiplication by v' induces an isomorphism
$$\pi_2(\mathrm{TC}(\mathcal{O}_C; \mathbb{Z}_p)) \cong A_{\mathrm{inf}}(\mathcal{O}_C)^{\varphi = \tilde{\xi}}.$$
By [Fargues and Fontaine 2018, Proposition 6.2.10]
$$(A_{\mathrm{inf}}(\mathcal{O}_C)[1/p])^{\varphi = \tilde{\xi}}$$
is 1-dimensional over $\mathbb{Q}_p$ and thus generated by μ (as $\mu \neq 0$ and $\varphi(\mu) = \tilde{\xi}\mu$). But μ is not divisible by p in $A_{\mathrm{inf}}(\mathcal{O}_C)$ as it maps to a unit in $W(C)$. This proves that $A_{\mathrm{inf}}(\mathcal{O}_C)^{\varphi = \tilde{\xi}} = \mathbb{Z}_p \mu$, which implies the claim. □

Let us define
$$\sigma := \frac{\mathrm{ctr}(\varepsilon)}{\mu} \in \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)) \quad \text{and} \quad u := \xi\sigma \in \pi_2(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}})).$$
More precisely, the element u is defined via $\mathrm{can}(u) = \xi\sigma$. Note that $\xi\sigma$ lies indeed in the image of
$$\mathrm{can} : \pi_2(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)) \to \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)),$$
due to the fact that the abutment filtration for the Tate fixed point spectral sequence on $\pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$ is the ξ-adic filtration.

Lemma 6.2. *The element u defined above lifts the class of*
$$\delta_1(\gamma)\xi \in \pi_2(\mathrm{THH}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)) \cong \pi_2(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)) \overset{\alpha_\gamma}{\cong} (\xi)/(\xi^2).$$

Proof. By definition

$$\mathrm{can}(u) = \frac{\xi}{\mu}\mathrm{ctr}(\varepsilon) \in \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)).$$

Now

$$\frac{\xi}{\mu} = \frac{1}{\varphi^{-1}(\mu)}$$

and $(\xi)/(\xi^2)$ is $\varphi^{-1}(\mu)$-torsion free as a module over $A_{\mathrm{inf}}(\mathbb{Z}_p^{\mathrm{cycl}})$, since

$$\theta(\varphi^{-1}(\mu)) = \zeta_p - 1 \neq 0 \in \mathbb{Z}_p^{\mathrm{cycl}}.$$

Moreover, the cyclotomic trace lifts the Dennis trace in Hochschild homology. Thus, by Proposition 2.5,

$$\alpha_\gamma(\mathrm{Dtr}(\varepsilon)) \equiv \delta_1(\gamma)([\varepsilon] - 1) \in (\xi)/(\xi^2)$$

and therefore

$$u \equiv \delta_1(\gamma)\frac{[\varepsilon]-1}{\varphi^{-1}(\mu)} = \delta_1(\gamma)\xi \in (\xi)/(\xi^2)$$

as desired. □

In particular, we see that the element

$$\sigma \in \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$$

is a generator. Set

$$v := \sigma^{-1} \in \pi_{-2}(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)) \overset{\mathrm{can}}{\cong} \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)).$$

Then

$$uv = \xi.$$

Recall that for any morphism of rings $R \to A$, the negative cyclic homology is defined to be

$$\mathrm{HC}^-(A/R) := \mathrm{HH}(A/R)^{h\mathbb{T}},$$

where $(-)^{h\mathbb{T}} := \varprojlim_{B\mathbb{T}}(-)$; see [Hoyois 2015] for a comparison with the classical definition in [Loday 1992, Definition 5.1.3]. The homotopy fixed point spectral sequence

$$H^i(B\mathbb{T}, \pi_{-j}(\mathrm{HH}(A/R)) \Rightarrow \pi_{-i-j}(\mathrm{HC}^-(A/R))$$

endows $\pi_*(\mathrm{HC}^-(A/R))$ with a (multiplicative) decreasing filtration, which we denote by

$$\mathcal{N}^{\geq\bullet}\mathrm{HC}^-(A/R).$$

Each generator $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ defines canonically a generator $\eta_\gamma \in H^2(B\mathbb{T}, \mathbb{Z})$. We abuse notation and denote by $\gamma \in H_1(\mathbb{T}, R)$ the image of $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$, and similarly for η_γ.

Proposition 6.3. *Let $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ be a generator and assume $A = R/(f)$ for some nonzero divisor $f \in R$. Then*

(1) $\mathrm{HH}_*(A/R)$ *is concentrated in even degrees and the homotopy fixed point spectral sequence*

$$H^i(B\mathbb{T}, \pi_{-j}(\mathrm{HH}(A/R)) \Rightarrow \pi_{-i-j}(\mathrm{HC}^-(A/R))$$

degenerates.

(2) *There exists a unique element $\delta_2 \in \{\pm 1\}$, independent of the choice of γ, such that the morphism*

$$(f)/(f)^2 \xrightarrow{\alpha_\gamma} \pi_2(\mathrm{HH}(A/R)) \xrightarrow{\tilde{\eta}_\gamma} \pi_0(\mathrm{HC}^-(A/R))/\mathcal{N}^{\geq 2}\mathrm{HC}^-(A/R)$$

sends the class of f to $\delta_2 f \cdot 1_{\pi_0(\mathrm{HC}^-(A/R))/\mathcal{N}^{\geq 2}\mathrm{HC}^-(A/R)}$. Here the first isomorphism is the one from Lemma 2.1. The second morphism is the multiplication by some lift $\tilde{\eta}_\gamma \in \pi_{-2}(\mathrm{HC}^-(A/R))$ of $\eta_\gamma \in H^2(B\mathbb{T}, \pi_0(\mathrm{HH}(A/R))$.[22]

Proof. The first claim follows from the HKR-filtration as the exterior powers

$$\bigwedge\nolimits^i L_{A/R}[i]$$

are concentrated in even degrees for all $i \geq 0$. For the second claim we can reduce by naturality to the universal case $R = \mathbb{Z}[x]$, $f = x$, in which case it is well-known that the elements

$$\tilde{\eta}_\gamma(\alpha_\gamma(f)), \quad f \cdot 1_{\pi_0(\mathrm{HC}^-(A/R))/\mathcal{N}^{\geq 2}\mathrm{HC}^-(A/R)}$$

are generators of the free A-module $\mathcal{N}^{\geq 1}\pi_0(\mathrm{HC}^-(A/R)/\mathcal{N}^{\geq 2}\pi_0(\mathrm{HC}^-(A/R)))$ of rank 1. This implies the existence of δ_2 as $A \cong \mathbb{Z}$. As the composition $\tilde{\eta}_\gamma \circ \alpha_\gamma$ is independent of the choice of $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ (because both α_γ and $\tilde{\eta}_\gamma$ are changed by a sign), the proof is finished. $\square$

Remark 6.4. We expect that $\delta_2 = 1$, but did not make the explicit computation, since we do not need it.

We need the following relation of v to η_γ.

Lemma 6.5. *Let $\underline{\eta_\gamma}, \underline{v} \in H^2(\mathbb{T}, \pi_0(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)))$ be the images of $\eta_\gamma \in H^2(\mathbb{T}, \mathbb{Z})$ and $v \in \pi_{-2}(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$ under the canonical morphisms*

$$H^2(\mathbb{T}, \mathbb{Z}) \to H^2(\mathbb{T}, \pi_0(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)))$$

and

$$\pi_{-2}(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p)) \to H^2(\mathbb{T}, \pi_0(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))),$$

respectively. Then $\underline{v} = \delta_2\delta_1(\gamma)\underline{\eta_\gamma}$.

[22] As we mod out by $\mathcal{N}^{\geq 2}$ and the spectral sequence degenerates, the second morphism does not depend on the choice of a lift.

Proof. By Lemma 6.2 we know that the image of u in

$$\pi_2(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)) \overset{\alpha_\gamma}{\cong} (\xi)/(\xi^2)$$

is

$$\alpha_\gamma(\delta_1(\gamma)\xi).$$

As $\underline{\eta_\gamma}, \underline{v} \in H^2(\mathbb{T}, \pi_0(\mathrm{HH}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)))$ there exists some unit $r \in \mathbb{Z}_p^{\mathrm{cycl}}$ such that $r\underline{\eta_\gamma} = \underline{v}$. We can calculate in $\pi_0(\mathrm{HC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))/\mathcal{N}^{\geq 2}\pi_0(\mathrm{HC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$ that

$$\xi = uv = v\alpha_\gamma(\delta_1(\gamma)\xi) = r\eta_\gamma\alpha_\gamma(\delta_1(\gamma)\xi) = r\delta_2\delta_1(\gamma)\xi$$

using Proposition 6.3. Thus, $r = \delta_2\delta_1(\gamma)$. □

One has the following (important) additional property (which, up to changing ξ by some unit, is implied by the conjunction of [Bhatt et al. 2019, Propositions 6.2 and 6.3]).

Lemma 6.6. *The cyclotomic Frobenius*

$$\varphi^{h\mathbb{T}} : \pi_2(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p)) \to \pi_2(\mathrm{TP}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$$

sends u to σ.

Proof. The cyclotomic Frobenius $\varphi^{h\mathbb{T}}$ is linear over the Frobenius on A_{inf}. Thus, noting $\xi/\mu = 1/\varphi^{-1}(\mu)$, we can calculate

$$\varphi^{h\mathbb{T}}(u) = \varphi\Big(\frac{\xi}{\mu}\Big)\varphi^{h\mathbb{T}}(\mathrm{ctr}(\varepsilon)) = \frac{1}{\mu}\varphi^{h\mathbb{T}}(\mathrm{ctr}(\varepsilon)).$$

But

$$\varphi^{h\mathbb{T}}(\mathrm{ctr}(\varepsilon)) = \mathrm{can}(\mathrm{ctr}(\varepsilon))$$

as the cyclotomic trace has image in $\pi_2(\mathrm{TC}(\mathbb{Z}_p^{\mathrm{cycl}};\mathbb{Z}_p))$. This implies that

$$\varphi^{h\mathbb{T}}(u) = \frac{\mathrm{ctr}(\varepsilon)}{\mu} = \sigma.$$ □

By Lemma 6.6 one can conclude that there is a commutative diagram

$$\begin{tikzcd}
\pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \arrow[r] \arrow[d, "\beta_\varepsilon"] & \pi_2(\mathrm{TC}^-(R;\mathbb{Z}_p)) \arrow[r, "\varphi^{h\mathbb{T}}-\mathrm{can}"] \arrow[d, "v"] & \pi_2(\mathrm{TP}(R;\mathbb{Z}_p)) \arrow[d, "\sigma^{-1}"] \\
\widehat{\Delta}_R^{\varphi=\tilde{\xi}} \arrow[r] & \mathcal{N}^{\geq 1}\widehat{\Delta}_R \arrow[r, "(\varphi/\tilde{\xi})-1"] & \widehat{\Delta}_R
\end{tikzcd}$$

whose vertical arrows are isomorphisms, for any quasiregular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra R. We remind the reader that the induced isomorphism

$$\beta_\varepsilon : \pi_2(\mathrm{TC}(R;\mathbb{Z}_p)) \cong \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

depends only on ε, not on γ.

For a quasiregular semiperfectoid ring R we denote the Teichmüller lift by

$$[-]_{\tilde{\theta}} : R^\flat = \varprojlim_{x \mapsto x^p} R \to \Delta_R.$$

More precisely, the canonical morphism $R \to \overline{\Delta}_R$ induces a morphism $\iota : R^\flat \to \overline{\Delta}_R^\flat$ and $[-]_{\tilde{\theta}}$ is the composition of ι with the Teichmüller lift for the surjection

$$\Delta_R \to \overline{\Delta}_R.$$

We set[23]

$$[-]_\theta := [(-)^{1/p}]_{\tilde{\theta}}.$$

We consider the p-adic Tate module

$$T_p R^\times = \varprojlim_{n \geq 0} R^\times[p^n]$$

of $R^\times$ as being embedded into $R^\flat$ as the elements with first coordinate equal to 1.

We are ready to state and prove our main theorem.

Theorem 6.7. *Let R be a quasiregular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. Then the composition*

$$T_p R^\times \to \pi_2(K(R; \mathbb{Z}_p)) \xrightarrow{\mathrm{ctr}} \pi_2(\mathrm{TC}(R; \mathbb{Z}_p)) \overset{\beta_\varepsilon}{\cong} \widehat{\Delta}_R^{\varphi = \tilde{\xi}}$$

is given by sending $x \in T_p(R^\times)$ to

$$\log_q([x]_\theta) = \sum_{n=1}^{\infty} (-1)^{n-1} q^{-n(n-1)/2} \frac{([x]_\theta - 1)([x]_\theta - q) \cdots ([x]_\theta - q^{n-1})}{[n]_q}.$$

Proof. Replacing R by the universal case $\mathbb{Z}_p^{\mathrm{cycl}}\langle x^{1/p^\infty}\rangle/(x-1)$, we may assume that R is p-torsion free and (thus) that $(\widehat{\Delta}_R, (\tilde{\xi}))$ is transversal (by Lemma 3.3 it suffices to see that (p, ξ) is a regular sequence, which follows as $\widehat{\Delta}_R/\xi \cong \widehat{L\Omega_{R/\mathbb{Z}_p^{\mathrm{cycl}}}}$, by [Bhatt et al. 2019, Theorem 7.2.(5)], is p-torsion free).

Let us define

$$\mathrm{ctr}_2 : T_p R^\times \to \pi_2(K(R; \mathbb{Z}_p)) \xrightarrow{\mathrm{ctr}} \pi_2(\mathrm{TC}(R; \mathbb{Z}_p)).$$

By Theorem 5.2 the canonical morphism

$$\iota : \Delta_R \to \pi_0(\mathrm{TC}^-(R; \mathbb{Z}_p))$$

is compatible with the Nygaard filtrations and identifies $\pi_0(\mathrm{TC}^-(R; \mathbb{Z}_p))$ with the Nygaard completion $\widehat{\Delta}_R$ of Δ_R. By Corollary 3.11 the morphism

$$\Delta_R^{\varphi = \tilde{\xi}} \hookrightarrow \mathcal{N}^{\geq 1}\Delta_R / \mathcal{N}^{\geq 2}\Delta_R \cong \mathcal{N}^{\geq 1}\widehat{\Delta}_R / \mathcal{N}^{\geq 2}\widehat{\Delta}_R$$

[23] This agrees with the definition of $[-]_\theta$ made in the introduction.

is injective. Hence it suffices to show that the morphisms $\log_q([-]_\theta)$ and $\beta_\varepsilon \circ \mathrm{ctr}$ agree modulo $\mathcal{N}^{\geq 2}\widehat{\Delta}_R$. Multiplication by the element $v \in \pi_{-2}(\mathrm{TC}^-(\mathbb{Z}_p^{\mathrm{cycl}}; \mathbb{Z}_p))$ constructed after Lemma 6.2 and the HKR-isomorphism (which depends on γ) induce an isomorphism

$$J/J^2 \overset{\alpha_\gamma}{\cong} \pi_2(\mathrm{THH}(R; \mathbb{Z}_p)) \overset{v}{\cong} \mathcal{N}^{\geq 1}\widehat{\Delta}_R/\mathcal{N}^{\geq 2}\widehat{\Delta}_R,$$

where J is the kernel of the surjection

$$\theta : W(R^\flat) \to R.$$

By Proposition 6.3 and Lemma 6.5 this isomorphism sends the class of $j \in J$ to

$$\delta_2^2\delta_1(\gamma) \cdot j \cdot 1_{\widehat{\Delta}_R/\mathcal{N}^{\geq 2}\widehat{\Delta}_R} = \delta_1(\gamma) \cdot j \cdot 1_{\widehat{\Delta}_R/\mathcal{N}^{\geq 2}\widehat{\Delta}_R}$$

for the canonical $W(R^\flat)$-algebra structure on

$$\begin{aligned}\widehat{\Delta}_R/\mathcal{N}^{\geq 2}\widehat{\Delta}_R &\cong \pi_0(\mathrm{TC}^-(R; \mathbb{Z}_p))/\mathcal{N}^{\geq 2}\pi_0(\mathrm{TC}^-(R; \mathbb{Z}_p)) \\ &\cong \pi_0\big(\mathrm{HC}^-(R/W(R^\flat))\big)/\mathcal{N}^{\geq 2}\pi_0\big(\mathrm{HC}^-(R/W(R^\flat))\big)\end{aligned}$$

(which lifts the morphism θ). Let $x \in T_pR^\times$. By Lemma 4.14

$$\log_q([x]_\theta) \equiv [x]_\theta - 1 \mod \mathcal{N}^{\geq 2}\widehat{\Delta}_R.$$

On the other hand, as the cyclotomic trace reduces to the Dennis trace Dtr, we can calculate, using Proposition 2.5 and Lemma 6.5,

$$\begin{aligned}\beta_\varepsilon(\mathrm{ctr}(x)) &\equiv v\,\mathrm{Dtr}(x) \\ &= v\delta_1(\gamma)([x]_\theta - 1) = \delta_1(\gamma)^2([x]_\theta - 1) \cdot 1_{\widehat{\Delta}_R/\mathcal{N}^{\geq 2}\widehat{\Delta}_R} \mod \mathcal{N}^{\geq 2}\widehat{\Delta}_R \\ &= ([x]_\theta - 1) \mod \mathcal{N}^{\geq 2}\widehat{\Delta}_R.\end{aligned}$$

Thus we can conclude

$$\log_q([x]_\theta) = \beta_\varepsilon \circ \mathrm{ctr}(x)$$

as desired. □

Corollary 6.8. *Let R be a quasiregular semiperfectoid $\mathbb{Z}_p^{\mathrm{cycl}}$-algebra. The map*

$$\log_q([-]_\theta) : T_p(R^\times) \to \widehat{\Delta}_R^{\varphi=\tilde{\xi}}$$

is a bijection.

Proof. Since both sides satisfy quasisyntomic descent,[24] one can assume, as in [Bhatt et al. 2019, Proposition 7.17], that R is w-local and $R^\times$ is divisible. In this

[24] For $T_p(-)^\times$ this follows from p-completely faithfully flat descent on p-complete rings with bounded p^∞-torsion [Anschütz and Le Bras 2019, Appendix]. For $\widehat{\Delta}_R^{\varphi=\tilde{\xi}}$ it is proven in [Bhatt et al. 2019].

case, the map

$$T_p(R^\times) \to \pi_2(K(R; \mathbb{Z}_p))$$

is a bijection. Moreover, [Clausen et al. 2018, Corollary 6.9] shows that

$$\mathrm{ctr} : \pi_2(K(R; \mathbb{Z}_p)) \to \pi_2(\mathrm{TC}(R; \mathbb{Z}_p))$$

is also bijective. As by Theorem 6.7 the composite of these two maps is the map $\log_q([-]_{\tilde{\theta}})$, this proves the corollary. □

Remark 6.9. As explained at the end of the introduction, one can give a direct and more elementary proof of Corollary 6.8 when R is the quotient of a perfect ring by a finite regular sequence [Scholze and Weinstein 2013] or when R is a p-torsion free quotient of a perfectoid ring by a finite regular sequence and p is odd. But we do not know how to prove it directly in general.

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References

[Anschütz and Le Bras 2019] J. Anschütz and A.-C. Le Bras, "Prismatic Dieudonné theory", preprint, 2019. arXiv

[Bhatt 2012] B. Bhatt, "p-adic derived de Rham cohomology", preprint, 2012. arXiv

[Bhatt and Scholze 2019] B. Bhatt and P. Scholze, "Prisms and prismatic cohomology", preprint, 2019. arXiv

[Bhatt et al. 2018] B. Bhatt, M. Morrow, and P. Scholze, "Integral p-adic Hodge theory", *Publ. Math. Inst. Hautes Études Sci.* **128** (2018), 219–397. MR Zbl

[Bhatt et al. 2019] B. Bhatt, M. Morrow, and P. Scholze, "Topological Hochschild homology and integral p-adic Hodge theory", *Publ. Math. Inst. Hautes Études Sci.* **129** (2019), 199–310. MR Zbl

[Blumberg et al. 2013] A. J. Blumberg, D. Gepner, and G. Tabuada, "A universal characterization of higher algebraic K-theory", *Geom. Topol.* **17**:2 (2013), 733–838. MR Zbl

[Bökstedt et al. 1993] M. Bökstedt, W. C. Hsiang, and I. Madsen, "The cyclotomic trace and algebraic K-theory of spaces", *Invent. Math.* **111**:3 (1993), 465–539. MR Zbl

[Clausen et al. 2018] D. Clausen, A. Mathew, and M. Morrow, "K-theory and topological cyclic homology of henselian pairs", preprint, 2018. arXiv

[Dundas et al. 2013] B. I. Dundas, T. G. Goodwillie, and R. McCarthy, *The local structure of algebraic K-theory*, Algebra and Applications **18**, Springer, 2013. MR Zbl

[Fargues and Fontaine 2018] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque **406**, Soc. Math. France, Paris, 2018. MR Zbl

[Gabber 1992] O. Gabber, "K-theory of Henselian local rings and Henselian pairs", pp. 59–70 in *Algebraic K-theory, commutative algebra, and algebraic geometry* (Santa Margherita Ligure, 1989), edited by R. K. Dennis et al., Contemp. Math. **126**, Amer. Math. Soc., Providence, RI, 1992. MR Zbl

[Geisser and Hesselholt 1999] T. Geisser and L. Hesselholt, "Topological cyclic homology of schemes", pp. 41–87 in *Algebraic K-theory* (Seattle, WA, 1997), edited by W. Raskind and C. Weibel, Proc. Sympos. Pure Math. **67**, Amer. Math. Soc., Providence, RI, 1999. MR Zbl

[Hesselholt 2006] L. Hesselholt, "On the topological cyclic homology of the algebraic closure of a local field", pp. 133–162 in *An alpine anthology of homotopy theory*, edited by D. Arlettaz and K. Hess, Contemp. Math. **399**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Hesselholt 2018] L. Hesselholt, "Topological Hochschild homology and the Hasse–Weil zeta function", pp. 157–180 in *An alpine bouquet of algebraic topology*, edited by C. Ausoni et al., Contemp. Math. **708**, Amer. Math. Soc., Providence, RI, 2018. MR Zbl

[Hoyois 2015] M. Hoyois, "The homotopy fixed points of the circle action on Hochschild homology", preprint, 2015. arXiv

[Krause and Nikolaus 2017] A. Krause and T. Nikolaus, "Lectures on topological Hochschild homology and cyclotomic spectra", lecture notes, 2017, available at https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/Lectures.pdf.

[Loday 1992] J.-L. Loday, *Cyclic homology*, Grundlehren der Math. Wissenschaften **301**, Springer, 1992. MR Zbl

[May and Ponto 2012] J. P. May and K. Ponto, *More concise algebraic topology: localization, completion, and model categories*, Univ. of Chicago Press, 2012. MR Zbl

[Nikolaus 2017] T. Nikolaus, "The group completion theorem via localizations of ring spectra", preprint, 2017, available at https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/Group_completion.pdf.

[Nikolaus and Scholze 2018] T. Nikolaus and P. Scholze, "On topological cyclic homology", *Acta Math.* **221**:2 (2018), 203–409. MR Zbl

[Pridham 2019] J. P. Pridham, "On q-de Rham cohomology via Λ-rings", *Math. Ann.* **375**:1-2 (2019), 425–452. MR Zbl

[Quillen 1973] D. Quillen, "Higher algebraic K-theory, I", pp. 85–147 in *Algebraic K-theory, I: Higher K-theories* (Seattle, 1972), edited by H. Bass, Lecture Notes in Math. **341**, Springer, 1973. MR Zbl

[Rosenschon and Østvær 2006] A. Rosenschon and P. A. Østvær, "Descent for K-theories", *J. Pure Appl. Algebra* **206**:1-2 (2006), 141–152. MR Zbl

[Scholze 2017] P. Scholze, "Canonical q-deformations in arithmetic geometry", *Ann. Fac. Sci. Toul. Math.* (6) **26**:5 (2017), 1163–1192. MR Zbl

[Scholze and Weinstein 2013] P. Scholze and J. Weinstein, "Moduli of p-divisible groups", *Camb. J. Math.* **1**:2 (2013), 145–237. MR Zbl

[Stacks 2005–] P. Belmans, A. J. de Jong, et al., "The Stacks project", electronic reference, 2005–, available at http://stacks.math.columbia.edu.

[Thomason 1985] R. W. Thomason, "Algebraic K-theory and étale cohomology", *Ann. Sci. École Norm. Sup.* (4) **18**:3 (1985), 437–552. MR Zbl

[Thomason and Trobaugh 1990] R. W. Thomason and T. Trobaugh, "Higher algebraic K-theory of schemes and of derived categories", pp. 247–435 in *The Grothendieck Festschrift*, vol. III, edited by P. Cartier et al., Progr. Math. **88**, Birkhäuser, Boston, 1990. MR Zbl

[Weibel 2013] C. A. Weibel, *The K-book: an introduction to algebraic K-theory*, Graduate Studies in Math. **145**, Amer. Math. Soc., Providence, RI, 2013. MR Zbl

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Nisnevich topology with modulus

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In the theory of motives à la Voevodsky, the Nisnevich topology on smooth schemes is used as an important building block. We introduce a Grothendieck topology on proper modulus pairs, which is used to construct a non-homotopy-invariant generalization of motives. We also prove that the topology satisfies similar properties to the Nisnevich topology.

1. Introduction

In the theory of motives à la Voevodsky [2000], the Nisnevich topology on the category of smooth schemes over a field k plays a fundamental role. In this paper, we introduce a Grothendieck topology on proper modulus pairs, which is used to construct a non-homotopy-invariant generalization of motives. We also prove that the topology satisfies similar properties to the Nisnevich topology.

A Nisnevich cover $f : Y \to X$ is an étale cover such that any point $x \in X$ admits a point $y \in Y$ with $f(y) = x$ and $k(y) = k(x)$. Therefore, the Nisnevich topology is finer than the Zariski topology and is coarser than the étale topology. Voevodsky defined the category of effective motives $\mathbf{DM}^{\mathrm{eff}}$ as the derived category of the abelian category of Nisnevich sheaves with transfers $\mathbf{NST}$, modulo $\mathbf{A}^1$-homotopy invariance:

$$\mathbf{DM}^{\mathrm{eff}} := \frac{\mathbf{D}(\mathbf{NST})}{(\mathbf{A}^1\text{-homotopy invariance})}. \tag{1.1.1}$$

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We briefly recall the definition of **NST**. Let **PST** be the category of additive abelian presheaves on the category of finite correspondences **Cor**. We have a natural functor **Sm** $\to$ **Cor**, where **Sm** denotes the category of smooth schemes over k. Then **NST** is defined to be the full subcategory of **PST** which consists of $F \in \mathbf{PST}$ such that the restriction $F|_{\mathbf{Sm}}$ is a Nisnevich sheaf on **Sm**.

The definition of **NST** is simple, but it is nontrivial that **NST** is an abelian category. It follows from the existence of a left adjoint to the inclusion functor **NST** $\to$ **PST**. A key ingredient of the proof of its existence is the following fact: for any Nisnevich cover $U \to X$, the Čech complex

$$\cdots \to \mathbb{Z}_{\mathrm{tr}}(U \times_X U) \to \mathbb{Z}_{\mathrm{tr}}(U) \to \mathbb{Z}_{\mathrm{tr}}(X) \to 0$$

is exact as a complex of Nisnevich sheaves, where $\mathbb{Z}_{\mathrm{tr}}(-) : \mathbf{Cor} \to \mathbf{PST}$ denotes the Yoneda embedding (see for example [Mazza et al. 2006, Proposition 6.12]). Moreover, the Nisnevich topology is *subcanonical*, i.e., every representable presheaf in **Sm** is a sheaf.

The category of motives $\mathbf{DM}^{\mathrm{eff}}$ has provided vast applications to the study of arithmetic geometry, but on the other hand, it has a fundamental constraint that it cannot capture *non-*$\mathbf{A}^1$*-homotopy-invariant phenomena*, e.g., wild ramification. Indeed, the arithmetic fundamental group $\pi_1(X)$, which captures the information of ramifications, is not $\mathbf{A}^1$-homotopy invariant.

An attempt to develop a theory of motives which captures non-$\mathbf{A}^1$-homotopy-invariant phenomena started in [Kahn et al. 2015]. The strategy is to extend Voevodsky's theory to *modulus pairs*. A *modulus pair* is a pair $M = (\overline{M}, M^\infty)$ of a scheme $\overline{M}$ and an effective Cartier divisor M^∞ on $\overline{M}$ such that the *interior* $M^{\mathrm{o}} := \overline{M} - M^\infty$ is smooth over k. We can define a reasonable notion of morphisms between modulus pairs, and we obtain a category of modulus pairs $\underline{\mathbf{M}}\mathbf{Sm}$. A modulus pair M is *proper* if $\overline{M}$ is proper over k, and we denote by **MSm** the full subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ consisting of proper modulus pairs (see Definition 2.1.1 for details).

These categories embed in categories of "modulus correspondences" $\underline{\mathbf{M}}\mathbf{Cor}$ and **MCor**, just as **Sm** embeds in **Cor** (see Definition 2.3.2). In [Kahn et al. 2015], categories of "modulus sheaves with transfers" $\underline{\mathbf{M}}\mathbf{NST}$ (relative to $\underline{\mathbf{M}}\mathbf{Cor}$) and **MNST** (relative to **MCor**) were introduced, in order to parallel the definition of (1.1.1). However, the proof that these categories are abelian was found to contain a gap. This gap was filled in [Kahn et al. 2019a] for $\underline{\mathbf{M}}\mathbf{NST}$, by showing that its objects are indeed the sheaves with transfers for a suitable Grothendieck topology on $\underline{\mathbf{M}}\mathbf{Sm}$.

In this paper, we construct a Grothendieck topology on **MSm** with nice properties. It will be shown in [Kahn et al. 2019b], using [Kahn and Miyazaki 2019], that the objects of **MNST** are the sheaves (with transfers) for this topology and that this category is abelian. Thus the present paper contains the tools to finish filling the gap of [Kahn et al. 2015]. Moreover, we prove an important exactness result.

Our guide is the following characterization of the Nisnevich topology on **Sm**: the Nisnevich topology is generated by coverings $U \sqcup V \to X$ associated with some commutative square S in **Sm** of the form

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

which satisfies the following properties:

(1) S is a cartesian square,

(2) the horizontal morphisms are open immersions,

(3) the vertical morphisms are étale, and

(4) the morphism $(V - W)_{\mathrm{red}} \to (X - U)_{\mathrm{red}}$ is an isomorphism.

Such squares are called *elementary Nisnevich squares*. Elementary Nisnevich squares form a *cd-structure* on **Sm** in the sense of [Voevodsky 2010]. A remarkable property of the Nisnevich cd-structure is the following fact: a presheaf of sets F on **Sm** is a Nisnevich sheaf if and only if $F(\varnothing) = \{*\}$ and for any elementary Nisnevich square as above, the square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(W) \end{array}$$

is cartesian. This equivalence holds for any cd-structure which is complete and regular; see [Voevodsky 2010, Definitions 2.3, 2.10, Corollary 2.17].

In [Kahn et al. 2019a], a cd-structure on $\underline{\mathbf{M}}\mathbf{Sm}$ is introduced. It is denoted $P_{\underline{\mathrm{MV}}}$, and satisfies properties similar to elementary Nisnevich squares. Its definition will be recalled in Section 4.1. For short, we call the topology on $\underline{\mathbf{M}}\mathbf{Sm}$ associated with $P_{\underline{\mathrm{MV}}}$ the $\underline{\mathrm{MV}}$-*topology.*

Our main result is the following.

Theorem. *The category of proper modulus pairs* **MSm** *admits a cd-structure* P_{MV} *such that the following assertions hold. For short, we call the topology associated with* P_{MV} *the* MV*-topology.*

(1) (Theorems 4.3.1, 4.4.1, 4.4.2) *The cd-structure* P_{MV} *is complete and regular. In particular, a presheaf of sets* F *on* **MSm** *is a sheaf for the* MV*-topology if*

and only if $F(\varnothing) = \{*\}$ *and for any square* $T \in P_{\mathrm{MV}}$ *of the form*

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & M \end{array}$$

the square

$$\begin{array}{ccc} F(M) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(W) \end{array}$$

is cartesian.

(2) (Theorem 4.5.1) *The* $\underline{\mathrm{MV}}$*-topology and the* MV*-topology are subcanonical.*

(3) (Corollary 5.2.7) *For any* $M \in \underline{\mathbf{M}}\mathbf{Sm}$, *consider the presheaf* $\mathbb{Z}_{\mathrm{tr}}(M)$ *on* $\underline{\mathbf{M}}\mathbf{Cor}$ *represented by* M, *which is a sheaf for the* $\underline{\mathrm{MV}}$*-topology by [Kahn et al. 2019a, Theorem* 2(2)*]. Then, for any square as above, the following complex of sheaves for the* $\underline{\mathrm{MV}}$*-topology is exact*:

$$0 \to \mathbb{Z}_{\mathrm{tr}}(W) \to \mathbb{Z}_{\mathrm{tr}}(U) \oplus \mathbb{Z}_{\mathrm{tr}}(V) \to \mathbb{Z}_{\mathrm{tr}}(M) \to 0.$$

The organization of the paper is as follows. In Section 2, we recall basic definitions and results on modulus pairs from [Kahn et al. 2019a]. In Section 3, we introduce "the off-diagonal functor", which is a key ingredient to define the cd-structure on the category of proper modulus pairs. In Section 4, we define the cd-structure on the category of proper modulus pairs, and prove that it satisfies completeness and regularity. Finally, in Section 5, we prove the exactness of the Mayer–Vietoris sequences associated with the distinguished squares with respect to the cd-structure.

Notation and convention. Throughout the paper, we fix a base field k. Let **Sm** be the category of separated smooth schemes of finite type over k, and let **Sch** be the category of separated schemes of finite type over k. For any scheme X and for any closed subscheme $F \subset X$, we denote by $\mathbf{Bl}_F(X)$ the blow-up of X along F.

2. Basics on modulus pairs

In this section, we introduce basic notions which we use throughout the paper.

2.1. ***Category of modulus pairs.*** We recall basic definitions on modulus pairs, introduced in [Kahn et al. 2019a]. We also introduce some new notation. In particular, the *canonical model of fiber product* is often useful (see Definition 2.2.2). Though our main interest in this paper is on *proper* modulus pairs, we introduce the general definition of modulus pairs for later use.

Definition 2.1.1. (1) A *modulus pair* is a pair $M = (\overline{M}, M^\infty)$ consisting of a scheme $\overline{M} \in \mathbf{Sch}$ (the *ambient space*) and an effective Cartier divisor M^∞ on $\overline{M}$ (the *modulus divisor*) such that the *interior* $M^o := \overline{M} \setminus |M^\infty|$ belongs to $\mathbf{Sm}$, where $|M^\infty|$ denotes the support of M^∞.

Note that M^o is a dense open subset of $\overline{M}$. Moreover, we can prove that $\overline{M}$ must be a reduced scheme by using the smoothness of M^o and the assumption that M^∞ is an effective Cartier divisor.

(2) A modulus pair M is called *proper* if the ambient space $\overline{M}$ is proper over k.

(3) An *admissible morphism* $f : M \to N$ of modulus pairs is a morphism between the interiors $f^o : M^o \to N^o$ in $\mathbf{Sm}$ which satisfies *the properness condition*:

– Let Γ be the graph of the rational map $\bar{f} : \overline{M} \dashrightarrow \overline{N}$ which is induced by f^o. Then the natural morphism $\Gamma \to \overline{M}$ is proper.

and *the modulus condition*:

– Let Γ^N be the normalization of Γ. Then we have the inequality

$$M^\infty|_{\Gamma^N} \geq N^\infty|_{\Gamma^N}$$

of effective Cartier divisors on Γ^N, where $M^\infty|_{\Gamma^N}$ and $N^\infty|_{\Gamma^N}$ denote the pullbacks of M^∞ and N^∞ along the natural morphisms $\Gamma^N \to \overline{M}$ and $\Gamma^N \to \overline{N}$. Note that the pullbacks are defined since the rational map $\bar{f}$ restricts to a morphism f^o, and since M^o is dense in $\overline{M}$.

If $f : M \to N$ and $g : N \to L$ are admissible morphisms, then the composite $g^o \circ f^o : M^o \to L^o$ defines an admissible morphism $M \to N$; see [Kahn et al. 2019a]. If N is proper, then the properness condition above is always satisfied.

(4) We let $\underline{\mathbf{M}}\mathbf{Sm}$ denote the category whose objects are modulus pairs and whose morphisms are admissible morphisms. The full subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ consisting of proper modulus pairs is denoted by $\mathbf{MSm}$.

(5) A morphism $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}$ is called *ambient* if $f^o : M^o \to N^o$ extends to a morphism $\overline{M} \to \overline{N}$ in $\mathbf{Sch}$. Such an extension is unique since $\overline{M}$ is reduced, M^o is dense in $\overline{M}$, and $\overline{N}$ is separated. We let $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ (resp. $\mathbf{MSm}^{\mathrm{fin}}$) denote the (nonfull) subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ (resp. $\mathbf{MSm}$) whose objects are modulus pairs (resp. proper modulus pairs) and whose morphisms are ambient morphisms.

(6) A morphism $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}$ is called *minimal* if f is ambient and satisfies $M^\infty = \bar{f}^* N^\infty$.

(7) We let $\underline{\Sigma}_{\mathrm{fin}}$ denote the subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ whose objects are the same as $\underline{\mathbf{M}}\mathbf{Sm}$ and whose morphisms are those morphisms $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ such that f is minimal, $\bar{f} : \overline{M} \to \overline{N}$ is proper, and $f^o : M^o \to N^o$ is an isomorphism

in $\mathbf{Sm}$. Then the canonical functor $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{Sm}$ induces an equivalence of categories $\Sigma_{\mathrm{fin}}^{-1} \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}} \xrightarrow{\simeq} \underline{\mathbf{M}}\mathbf{Sm}$ [Kahn et al. 2019a, Proposition 1.9.2].

(8) Let $\mathbf{Sq}$ be the product category $[0]\times[0]$, where $[0]=\{0\to 1\}$. For any category $\mathcal{C}$, we define $\mathcal{C}^{\mathbf{Sq}}$ to be the category of functors from $\mathbf{Sq}$ to $\mathcal{C}$. An object T of $\mathcal{C}^{\mathbf{Sq}}$ is given by a commutative diagram

$$\begin{array}{ccc} T(00) & \longrightarrow & T(01) \\ \downarrow & & \downarrow \\ T(10) & \longrightarrow & T(11) \end{array}$$

in $\mathcal{C}$, and a morphism $T_1 \to T_2$ in $\mathcal{C}^{\mathbf{Sq}}$ is given by a set of morphisms $T_1(ij) \to T_2(ij)$, $i, j = 0, 1$, which are compatible with all the edges of the squares.

(9) A morphism $T_1 \to T_2$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ is called *ambient* if for any $i, j = 0, 1$, the morphisms $T_1(ij) \to T_2(ij)$ in $\underline{\mathbf{M}}\mathbf{Sm}$ are ambient. A square $T \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ is called *ambient* if it is contained in $(\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}} \subset \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$.

The following lemma is often useful.

Lemma 2.1.2. *For any square $T \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$, there exists an ambient square T' which admits an ambient morphism $T' \to T$ which is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$.*

Proof. This is just a consequence of a repeated use of the graph trick [Kahn et al. 2019a, Lemma 1.3.6]. Or, the reader can consult the calculus of fractions in [Kahn et al. 2019a, Proposition 1.9.2]. The details are left to the reader. □

2.2. *Fiber products.* We discuss fiber products in $\underline{\mathbf{M}}\mathbf{Sm}$ and $\mathbf{MSm}$.

Lemma 2.2.1. *Let X be a scheme, and let D_1 and D_2 be effective Cartier divisors on X. Assume that the scheme-theoretic intersection* $\inf(D_1, D_2) := D_1 \times_X D_2$ *is also an effective Cartier divisor on X. Set $X^\infty := D_1 + D_2 - \inf(D_1, D_2)$.*

Then for any morphism $f : Y \to X$ in $\mathbf{Sch}$ *such that Y is normal and the image of any irreducible component of Y is not contained in $|X^\infty| = |D_1| \cup |D_2|$, we have*

$$f^* X^\infty = \sup(f^* D_1, f^* D_2),$$

where sup *is the supremum of Weil divisors on the normal scheme Y.*

Proof. Since $\inf(D_1, D_2) \times_X Y = \inf(f^*D_1, f^*D_2)$, we are reduced to the case $X = Y$. Moreover, an easy local computation shows that $D_1 - \inf(D_1, D_2)$ and $D_2 - \inf(D_1, D_2)$ do not intersect. The assertion immediately follows from this. See [Kahn et al. 2019a, Lemma 1.10.1, Definition 1.10.2, Remark 1.10.3] for more details. □

Definition 2.2.2. Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, and assume that the fiber product $P^{\mathrm{o}} := M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ exists in $\mathbf{Sm}$. We define a modulus pair P as follows. Let $\overline{P}_0$ be the scheme-theoretic closure of P^{o} in $\overline{M} \times_{\overline{N}} \overline{M}_2$, and let $\bar{p}_{0,i} : \overline{P}_0 \to \overline{M}_1 \times_{\overline{N}} \overline{M}_2 \xrightarrow{\mathrm{pr}_i} \overline{M}_i$ be the composite of the closed immersion followed by the i-th projection for $i = 1, 2$. Let

$$\overline{P} := \mathbf{Bl}_{(\bar{p}^*_{0,1} M_1^\infty) \times_{\overline{P}_0} (\bar{p}^*_{0,2} M_2^\infty)}(\overline{P}_0) \xrightarrow{\pi_P} \overline{P}_0$$

be the blow-up of $\overline{P}_0$ along the closed subscheme $(\bar{p}^*_{0,1} M_1^\infty) \times_{\overline{P}_0} (\bar{p}^*_{0,2} M_2^\infty)$. Set

$$P^\infty := \pi_P^* \bar{p}^*_{0,1} M_1^\infty + \pi_P^* \bar{p}^*_{0,2} M_2^\infty - E,$$

where $E := \pi_P^{-1}((\bar{p}^*_{0,1} M_1^\infty) \times_{\overline{P}_0} (\bar{p}^*_{0,2} M_2^\infty))$ denotes the exceptional divisor. Then we have $\overline{P} - |P^\infty| = P^{\mathrm{o}} \in \mathbf{Sm}$ by construction, and we obtain a modulus pair $P = (\overline{P}, P^\infty)$.

We call P the *canonical model of fiber product of* f_1 *and* f_2, and we often write

$$M_1 \times_N^{\mathrm{c}} M_2 := P.$$

By construction, we have a commutative diagram

$$\begin{tikzcd} M_1 \times_N^{\mathrm{c}} M_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & M_2 \arrow[d, "f_2"] \\ M_1 \arrow[r, "f_1"'] & N \end{tikzcd}$$

in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. Moreover, we have $(M_1 \times_N^{\mathrm{c}} M_2)^{\mathrm{o}} \cong M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$.

Theorem 2.2.3. *Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. Assume that the fiber product $M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ exists in* $\mathbf{Sm}$. *Then the canonical model of fiber product $M_1 \times_N^{\mathrm{c}} M_2$ represents the fiber product $M_1 \times_N M_2$ in* $\underline{\mathbf{M}}\mathbf{Sm}$. *Moreover, if M_1, M_2, N are proper, then $M_1 \times_N^{\mathrm{c}} M_2$ (hence $M_1 \times_N M_2$) is proper.*

Remark 2.2.4. $M_1 \times_N^{\mathrm{c}} M_2$ does not necessarily represent a fiber product in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, and it is not functorial in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. However, under some minimality conditions, they behave nicely in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.

Proof. We prove that $P := M_1 \times_N^{\mathrm{c}} M_2$ satisfies the universal property of fiber product in $\underline{\mathbf{M}}\mathbf{Sm}$. Let $g_1 : L \to M_1$ and $g_2 : L \to M_2$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}$ which coincide at N. Since $\underline{\mathbf{M}}\mathbf{Sm} \cong \underline{\Sigma}_{\mathrm{fin}}^{-1} \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, we can find morphisms $L_1 \to L$ in $\underline{\Sigma}_{\mathrm{fin}}$ such that the composite morphisms $L_1 \to L \to M_i$ are ambient for $i = 1, 2$, and such that $\overline{L}_1$ is normal. Since $L_1 \to L$ is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}$, we replace L with L_1 and assume that $\overline{L}$ is normal, and that g_1 and g_2 are ambient. Let $p_1 : P \to M_1$ and $p_2 : P \to M_2$ be the ambient morphisms as in Definition 2.2.2.

There exists a unique morphism $g^{\mathrm{o}} : L^{\mathrm{o}} \to P^{\mathrm{o}} = M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ in $\mathbf{Sm}$ which is compatible with g_1^{o}, g_2^{o}, p_1^{o}, and p_2^{o}. It suffices to prove that g^{o} defines a morphism

$L \to P$ in $\underline{\mathbf{M}}\mathbf{Sm}$. Let $\Gamma \subset \bar{L} \times \bar{P}$ be the closure of the graph of g^{o}, and let Γ^N be the normalization of Γ. Let $s : \Gamma^N \to \bar{L}$ and $t : \Gamma^N \to \bar{P}$ be the natural projections.

Then, for $i = 1, 2$, we obtain a commutative diagram

$$\begin{tikzcd} \Gamma^N \arrow[r, "t"] \arrow[d, "s"'] & \bar{P} \arrow[d, "\bar{p}_i"] \\ \bar{L} \arrow[r, "\bar{g}_i"'] \arrow[ur, dotted, "g^{\mathrm{o}}"] & \overline{M}_i \end{tikzcd}$$

where the commutativity follows from the fact that $\bar{p}_i t$ and $\bar{g}_i s$ coincide on the dense open subset $s^{-1}(L^{\mathrm{o}}) \subset \Gamma^N$.

By the construction of P and by Lemma 2.2.1, we have

$$t^* P^\infty = \sup(t^* \bar{p}_1^* M_1^\infty, t^* \bar{p}_2^* M_2^\infty) = \sup(s^* \bar{g}_1^* M_1^\infty, s^* \bar{g}_1^* M_2^\infty),$$

where the second equality follows from the commutativity of the above diagram. Since g_1 and g_2 are ambient and $\bar{L}$ is normal, we have $\bar{g}_i^* M_i^\infty \le L^\infty$. Therefore, we obtain

$$t^* P^\infty \le s^* L^\infty,$$

which shows that g^{o} defines a morphism $g : L \to P$. This proves the first assertion. The last assertion is obvious by construction. $\square$

Corollary 2.2.5. *Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be morphisms in* $\underline{\mathbf{M}}\mathbf{Sm}$. *Assume that the fiber product $M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ exists in* $\mathbf{Sm}$. *Then there exists a fiber product $M_1 \times_N M_2$ in* $\underline{\mathbf{M}}\mathbf{Sm}$. *Moreover, if M_1, M_2, and N are proper, then $M_1 \times_N M_2$ is proper.*

Proof. By [Kahn et al. 2019a, Lemma 1.3.6], for each $i = 1, 2$, there exists a morphism $M_i' \to M_i$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ which is invertible in $\underline{\mathbf{M}}\mathbf{Sm}$ and such that the composite $M_i' \to M_i \to N$ is ambient. Theorem 2.2.3 shows that the fiber product $M_1' \times_N M_2'$ exists in $\underline{\mathbf{M}}\mathbf{Sm}$. This also represents a fiber product $M_1 \times_N M_2$, proving the first assertion. The second assertion follows from the construction of the canonical model of fiber product. $\square$

Remark 2.2.6. The inclusion functor $\tau_s : \mathbf{MSm} \to \underline{\mathbf{M}}\mathbf{Sm}$ preserves fiber products by construction.

Given some minimality assumptions, we can say more about the canonical model of fiber product. We do not need this in this paper, but it will be used in the other papers, including [Kahn and Miyazaki 2019].

Proposition 2.2.7. (1) *Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be morphisms in* $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, *and assume that f_1 is minimal, $M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ is smooth over k and $\overline{M}_1 \times_{\overline{N}} M_2^\infty$ is an effective Cartier divisor on $\overline{M}_1 \times_{\overline{N}} \overline{M}_2$. Then we have*

$$M_1 \times_N^{\mathrm{c}} M_2 = (\overline{M}_1 \times_{\overline{N}} \overline{M}_2, \overline{M}_1 \times_{\overline{N}} M_2^\infty).$$

(2) *Consider the commutative diagram*

$$\begin{array}{ccccc} U_1 & \longrightarrow & V & \longleftarrow & U_2 \\ {\scriptstyle j_1}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j_2} \\ M_1 & \longrightarrow & N & \longleftarrow & M_2 \end{array}$$

in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, *such that* j_1 *and* j_2 *are minimal, and such that* $M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ *and* $U_1^{\mathrm{o}} \times_{V^{\mathrm{o}}} U_2^{\mathrm{o}}$ *are smooth over k. Then the morphism*

$$j_1 \times j_2 : U_1 \times_V^{\mathrm{c}} U_2 \to M_1 \times_N^{\mathrm{c}} M_2$$

in $\underline{\mathbf{M}}\mathbf{Sm}$, *induced by the universal property of fiber product, belongs to* $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ *and is minimal.*

(3) *In the situation of* (2), *if* $\bar{j}$, $\bar{j}_1$, $\bar{j}_2$ *are open immersions, if* $U_1 \to V$ *is minimal, and if* $\overline{U}_1 \times_{\overline{V}} \overline{U}_2$ *is normal, then*

$$\overline{j_1 \times j_2} : \overline{U}_1 \times_{\overline{V}} \overline{U}_2 = \overline{U_1 \times_V^{\mathrm{c}} U_2} \to \overline{M_1 \times_N^{\mathrm{c}} M_2}$$

is an open immersion, where the equality follows by (1).

Proof. (1): This follows from the construction of canonical model of fiber product; see also [Kahn et al. 2019a, Corollary 1.10.7].

(2): Let $\overline{P}$ be the closure of $M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ in $\overline{M}_1 \times_{\overline{N}} \overline{M}_2$, and $\overline{Q}$ the closure of $U_1^{\mathrm{o}} \times_{V^{\mathrm{o}}} U_2^{\mathrm{o}}$ in $\overline{U}_1 \times_{\overline{V}} \overline{U}_2$. Then the morphisms $\bar{j}_1$ and $\bar{j}_2$ induce a morphism

$$\bar{J} : \overline{Q} \to \overline{P}.$$

Then we obtain the commutative diagrams

$$\begin{array}{ccc} \overline{Q} & \xrightarrow{\bar{J}} & \overline{P} \\ {\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle p_i} \\ \overline{U}_i & \xrightarrow{\bar{j}_i} & \overline{M}_i \end{array}$$

in **Sch** for $i = 1, 2$, where p_i and q_i are the natural i-th projections. Set $F := p_1^* M_1^\infty \times_{\overline{P}} p_2^* M_2^\infty \subset \overline{P}$ and $G := q_1^* U_1^\infty \times_{\overline{Q}} p_2^* U_2^\infty \subset \overline{Q}$. Then the commutativity of the diagrams shows

$$\bar{J}^{-1} F := F \times_{\overline{P}} \overline{Q} = (q_i^* \bar{j}_1^* M_1^\infty) \times_{\overline{Q}} (q_i^* \bar{j}_2^* M_2^\infty) = q_i^* U_1^\infty \times_{\overline{Q}} q_i^* U_2^\infty = G,$$

where the equality in the second line follows from the minimality of j_1 and j_2. Let $\pi_P : \mathbf{Bl}_F(\overline{P}) \to \overline{P}$ and $\pi_Q : \mathbf{Bl}_G(\overline{Q}) \to \overline{Q}$ be the blow-ups. Then, by the universal property of blow-up, $\bar{J}$ lifts to a morphism

$$\bar{J}_1 : \overline{U_1 \times_V^{\mathrm{c}} U_2} = \mathbf{Bl}_G(\overline{Q}) \to \mathbf{Bl}_F(\overline{P}) = \overline{M_1 \times_N^{\mathrm{c}} M_2},$$

which makes the diagram

$$\begin{array}{ccc} \mathbf{Bl}_G(\bar{Q}) & \xrightarrow{\bar{J}_1} & \mathbf{Bl}_F(\bar{P}) \\ {\scriptstyle \pi_Q}\downarrow & & \downarrow{\scriptstyle \pi_P} \\ \bar{Q} & \xrightarrow{\bar{J}} & \bar{P} \end{array}$$

commute. Moreover, letting $F' := \pi_P^{-1}(F)$, $G' := \pi_Q^{-1}(G)$ be the exceptional divisors, the commutativity of the two diagrams as above shows

$$\begin{aligned} \bar{J}_1^*(M_1 \times_N^{\mathrm{c}} M_2)^\infty &= \bar{J}_1^*(\pi_P^* p_1^* M_1^\infty + \pi_P^* p_1^* M_1^\infty - F') \\ &= \pi_Q^* \bar{J}^* p_1^* M_1^\infty + p_Q^* \bar{J}^* \pi_2^* M_2^\infty - G' \\ &= \pi_Q^* q_1^* \bar{j}_1^* M_1^\infty + \pi_Q^* q_2^* \bar{j}_2^* M_2^\infty - G' \\ &= \pi_Q^* q_1^* U_1^\infty + \pi_Q^* q_2^* U_2^\infty - G' \\ &= (U_1 \times_V^{\mathrm{c}} U_2)^\infty, \end{aligned}$$

where the equality in the fourth line follows from the minimality of j_1 and j_2. Therefore, the morphism $\bar{J}_1$ defines a minimal morphism $U_1 \times_V^{\mathrm{c}} U_2 \to M_1 \times_N^{\mathrm{c}} M_2$, as desired.

(3): We take the notation as above. Then $\bar{U}_1 \times_{\bar{V}} \bar{U}_2$ is an open subset of $\bar{P}$. Since $\bar{J}^* F = G$, the minimality of $U_1 \to V$ shows $F \cap \bar{U}_1 \times_{\bar{V}} \bar{U}_2 = \bar{U}_1 \times_{\bar{V}} U_2^\infty$, where the right-hand side is an effective Cartier divisor on $\bar{U}_1 \times_{\bar{V}} \bar{U}_2$. Therefore, the blow-up π_P is an isomorphism over $\bar{U}_1 \times_{\bar{V}} \bar{U}_2$, and the open immersion $\bar{U}_1 \times_{\bar{V}} \bar{U}_2 \to \bar{P}$ uniquely lifts to an open immersion $\bar{U}_1 \times_{\bar{V}} \bar{U}_2 \to \mathbf{Bl}_F(\bar{P})$. □

2.3. ***A remark on elementary correspondences.*** In this subsection, we observe a relationship between cartesian squares and elementary correspondences. First we provide some definitions.

Definition 2.3.1. For any $M_1, M_2 \in \underline{\mathbf{M}}\mathbf{Sm}$, we define $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}$ to be the set of elementary finite correspondences $V : M_1^{\mathrm{o}} \to M_2^{\mathrm{o}}$ which satisfy the following *admissibility conditions*: let $\bar{V}$ be the closure of V in $\bar{M}_1 \times \bar{M}_2$, and let $\bar{V}^N \to \bar{V}$ be the normalization of $\bar{V}$. Let $\mathrm{pr}_i : \bar{V}^N \to \bar{M}_i$ be the i-th projections.

(1) pr_1 is proper.

(2) $\mathrm{pr}_1^* M_1^\infty \geq \mathrm{pr}_2^* M_2^\infty$.

Definition 2.3.2 [Kahn et al. 2019a, Definitions 1.1.1, 1.3.3]. A category $\underline{\mathbf{M}}\mathbf{Cor}$ is defined as follows: the objects are the same as $\underline{\mathbf{M}}\mathbf{Sm}$, and for $M, N \in \underline{\mathbf{M}}\mathbf{Cor}$, the set of morphisms is defined as the free abelian group generated on $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, N)$. Note that $\underline{\mathbf{M}}\mathbf{Cor}(M, N) \subset \mathbf{Cor}(M^{\mathrm{o}}, N^{\mathrm{o}})$ by definition. The composition is given by the composition of finite correspondences. Define $\mathbf{MCor}$ as the full subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ whose objects are proper modulus pairs.

Proposition 2.3.3. *For any modulus pair M, for any $f : N \to L$ in $\underline{\mathbf{M}}\mathbf{Sm}$, and for any $V \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, N)$, the image*

$$f_+(V) := (\mathrm{Id}_{M^{\mathrm{o}}} \times f^{\mathrm{o}})(V) \subset M^{\mathrm{o}} \times L^{\mathrm{o}}$$

is an irreducible closed subset, and we have $f_+(V) \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, L)$.

Thus, any modulus pair M is associated a covariant functor

$$\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, -) : \underline{\mathbf{M}}\mathbf{Sm} \to \mathbf{Set}.$$

Proof. By [Kahn et al. 2019a, Proposition 1.2.3], the composition of finite correspondences $W := \Gamma_{f^{\mathrm{o}}} \circ V$ belongs to $\underline{\mathbf{M}}\mathbf{Cor}(M, L)$, where $\Gamma_{f^{\mathrm{o}}}$ denotes the graph of $f^{\mathrm{o}} : M^{\mathrm{o}} \to N^{\mathrm{o}}$. By the definition of composition, we can verify that $|W| = f_+(V)$. This implies that $f_+(V)$ is a component of W. Therefore, we have $W \in \underline{\mathbf{M}}\mathbf{Cor}(M, L)$, as desired. □

Proposition 2.3.4. *Let T be a pull-back square in $\underline{\mathbf{M}}\mathbf{Sm}$ of the form*

$$\begin{array}{ccc} T(00) & \xrightarrow{v_T} & T(01) \\ {\scriptstyle q_T}\downarrow & & \downarrow{\scriptstyle p_T} \\ T(10) & \xrightarrow{u_T} & T(11) \end{array} \tag{2.3.5}$$

and let M be a modulus pair. Consider the associated commutative diagram of sets

$$\begin{array}{ccc} \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00)) & \xrightarrow{v_{T+}} & \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01)) \\ {\scriptstyle q_{T+}}\downarrow & & \downarrow{\scriptstyle p_{T+}} \\ \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(10)) & \xrightarrow{u_{T+}} & \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(11)) \end{array}$$

and set

$$\Pi := \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(10)) \times_{\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(11))} \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01)).$$

Then the induced map $\rho : \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00)) \to \Pi$ is surjective. Moreover, it is bijective if v_T^{o} is an immersion.

Remark 2.3.6. We can formulate another statement by replacing $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}$ with $\underline{\mathbf{M}}\mathbf{Cor}$ and $(-)_+$ with $(-)_*$, but it is false. Indeed, if α_1 and α_2 are distinct elementary correspondences which have the same image β under p_{T*}, then the image of the (nonelementary) finite correspondence $\alpha := \alpha_1 - \alpha_2$ is zero, which is trivially contained in the image of u_{T*}. But there is no reason why α is contained in the image of v_{T*}.

Proof. The latter statement is clear, since the composite $\mathrm{pr}_2 \circ \rho$ is equal to v_{T+}, which is injective if v_T^{o} is an immersion.

We prove the surjectivity of ρ. Consider any $\alpha_1 \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(10))$ and $\alpha_2 \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01))$, and assume $\beta := u_{T+}(\alpha_1) = p_{T+}(\alpha_2)$. Let ξ_i be the generic point of α_i for $i = 1, 2$.

We need to prove that there exists an element $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00))$ which maps to α_1 and α_2.

Let

$$\begin{aligned}\zeta \in (M^{\mathrm{o}} \times T(10)^{\mathrm{o}}) \times_{M^{\mathrm{o}} \times T^{\mathrm{o}}(11)} (M^{\mathrm{o}} \times T(01)^{\mathrm{o}}) &\cong M^{\mathrm{o}} \times T(10)^{\mathrm{o}} \times_{T^{\mathrm{o}}(11)} T(01)^{\mathrm{o}} \\ &\cong M^{\mathrm{o}} \times T(00)^{\mathrm{o}}\end{aligned}$$

be a point which lies over ξ_1 and ξ_2. Let $\gamma := \overline{\{\zeta\}}$ be the closure of ζ in $M^{\mathrm{o}} \times T(00)^{\mathrm{o}}$, endowed with the reduced scheme structure.

Claim 2.3.7. γ is an elementary correspondence from M^{o} to $T(00)^{\mathrm{o}}$.

Proof. We have to prove that γ is finite and surjective over a component of M^{o}. Since $\zeta = (\xi_1, \xi_2) \in \alpha_1 \times_{M^{\mathrm{o}}} \alpha_2$, the scheme γ is naturally a closed subscheme of $\alpha_1 \times_{M^{\mathrm{o}}} \alpha_2$. Moreover, since ζ maps to ξ_i via the projection $\mathrm{pr}_i : \alpha_1 \times_{M^{\mathrm{o}}} \alpha_2 \to \alpha_i$ for each $i = 1, 2$, we obtain dominant maps $\gamma \to \alpha_i$. These maps are finite (hence surjective) since each α_i is finite over M^{o}. Since the natural map $\gamma \to M^{\mathrm{o}}$ factors as $\gamma \to \alpha_1 \to M^{\mathrm{o}}$, and since α_1 is finite and surjective over a component, we obtain the claim. $\square$

Claim 2.3.8. $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00))$.

Proof. We make a preliminary reduction as follows: since the assertion depends only on the isomorphism class of T in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$, we may assume that T is ambient by Lemma 2.1.2. Moreover, since T is a pull-back diagram, we have $T(00) \cong T(10) \times^{c}_{T(11)} T(01)$, where the right-hand side is the canonical model of fiber product in Definition 2.2.2. Therefore, by replacing $T(00)$ with (the normalization of) $T(10) \times^{\mathrm{c}}_{T(11)} T(01)$ (this preserves the condition that T is ambient by the construction of canonical model), we may assume that $\bar{q}_T^* T(10)^\infty$ and $\bar{v}_T^* T^*(01)$ have a universal supremum in the sense of [Kahn et al. 2019a, Definition 1.10.2] and that $T(00)^\infty = \sup(\bar{q}_T^* T(10)^\infty, \bar{v}_T^* T(01)^\infty)$.

Let $\overline{\gamma}$ be the closure of γ in $\overline{M} \times \overline{T}(00)$. First we check that $\overline{\gamma}$ is proper over $\overline{M}$. Note that the natural map $\overline{\gamma} \to \overline{M}$ factors as $\overline{\gamma} \to \overline{\alpha}_1 \times_{\overline{M}} \overline{\alpha}_2 \to \overline{M}$. The first map is proper since the natural map $\overline{T}(00) \to \overline{T}(10) \times_{\overline{T}(11)} \overline{T}(01)$ is proper by construction of the canonical model of fiber product, and the latter map is proper since the $\overline{\alpha}_i$ are proper over $\overline{M}$ by assumption. This shows that $\overline{\gamma} \to \overline{M}$ is proper, as desired.

Next we check the modulus condition. Let $\overline{\gamma}^N$ be the normalization of $\overline{\gamma}$. Similarly, let $\overline{\alpha}_1$ be the closure of α_1 in $\overline{M} \times \overline{T}(10)$, $\overline{\alpha}_2$ the closure of α_2 in $\overline{M} \times \overline{T}(01)$, and $\overline{\alpha}_i^N$ the normalization of $\overline{\alpha}_i$. By assumption, we have $\alpha_1 \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(10))$

and $\alpha_2 \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01))$, which means

$$M^\infty|_{\overline{\alpha}_1^N} \geq T(10)^\infty|_{\overline{\alpha}_1^N} \quad \text{and} \quad M^\infty|_{\overline{\alpha}_2^N} \geq T(01)^\infty|_{\overline{\alpha}_2^N}.$$

Since $\gamma \to \alpha_i$ are dominant for $i = 1, 2$, we obtain morphisms $\overline{\gamma}^N \to \overline{\alpha}_i^N$ by the universal property of normalization. Therefore, the above inequalities imply

$$M^\infty|_{\overline{\gamma}^N} \geq \bar{q}_T^* T(10)^\infty|_{\overline{\gamma}^N} \quad \text{and} \quad M^\infty|_{\overline{\gamma}^N} \geq \bar{v}_T^* T(01)^\infty|_{\overline{\gamma}^N}.$$

Thus, since $\bar{q}_T^* T(10)^\infty$ and $\bar{v}_T^* T(01)^\infty$ have a universal supremum and $T(00)^\infty = \sup(\bar{q}_T^* T(10)^\infty, \bar{v}_T^* T(01)^\infty)$ by assumption, we obtain

$$\begin{aligned} M^\infty|_{\overline{\gamma}^N} &\geq \sup(\bar{q}_T^* T(10)^\infty|_{\overline{\gamma}^N}, \bar{v}_T^* T(01)^\infty|_{\overline{\gamma}^N}) \\ &= \sup(\bar{q}_T^* T(10)^\infty, \bar{v}_T^* T(01)^\infty)|_{\overline{\gamma}^N} \\ &= T(00)^\infty|_{\overline{\gamma}^N} \end{aligned}$$

by [Kahn et al. 2019a, Remark 1.10.3(3)]. This finishes the proof of the claim. □

By construction, we have $\alpha_1 = q_{T+}(\gamma)$ and $\alpha_2 = v_{T+}(\gamma)$. This finishes the proof of Proposition 2.3.4. □

3. Off-diagonal functor

We introduce the "off-diagonal" functor, which is a key notion used in the definition of the cd-structure on **MSm**.

Definition 3.1.1. Define $\underline{\mathbf{M}}\mathbf{Et}$ as a category such that

(1) objects are those morphisms $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}$ such that $f^{\mathrm{o}} : M^{\mathrm{o}} \to N^{\mathrm{o}}$ is étale, and

(2) morphisms of $f : M_1 \to N_1$ and $g : M_2 \to N_2$ are those pairs of morphisms $(s : M_1 \to M_2, t : N_1 \to N_2)$ which are compatible with f, g such that s^{o} and t^{o} are *open immersions*.

Define **MEt** as the full subcategory of $\underline{\mathbf{M}}\mathbf{Et}$ consisting of those $f : M \to N$ such that $M, N \in \mathbf{MSm}$.

Definition 3.1.2. For modulus pairs M and N, we define the *disjoint union of M and N* by

$$M \sqcup N := (\overline{M} \sqcup \overline{N}, M^\infty \sqcup N^\infty).$$

We have $(M \sqcup N)^{\mathrm{o}} = M^{\mathrm{o}} \sqcup N^{\mathrm{o}}$, and $M \sqcup N$ represents a coproduct of M and N in the category $\underline{\mathbf{M}}\mathbf{Sm}$.

Theorem 3.1.3. *There is a functor*

$$\mathrm{OD} : \underline{\mathbf{M}}\mathbf{Et} \to \underline{\mathbf{M}}\mathbf{Sm}$$

such that for any $f: M \to N$, *there exists a functorial decomposition*

$$M \times_N M \cong M \sqcup \mathrm{OD}(f).$$

Moreover, we have $\mathrm{OD}(f)^{\mathrm{o}} = M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}} \setminus \Delta(M^{\mathrm{o}})$, *where* $\Delta : M^{\mathrm{o}} \to M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}}$ *is the diagonal morphism. In particular, if* f^{o} *is an open immersion, then* $\mathrm{OD}(f)^{\mathrm{o}} = \varnothing$, *and hence* $\mathrm{OD}(f) = \varnothing$. *Moreover, the functor* OD *restricts to a functor*

$$\mathrm{OD} : \mathbf{MEt} \to \mathbf{MSm}.$$

We call these functors the off-diagonal functors.

Proof. First, we prove that for any $f: M \to N$ in $\underline{\mathbf{M}}\mathbf{Et}$, there exists a morphism $i : X \to M \times_N M$ such that the induced morphism

$$M \sqcup X \xrightarrow{\Delta \sqcup i} M \times_N M$$

is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}$. Take any object $f: M \to N$ in $\underline{\mathbf{M}}\mathbf{Et}$. Since f^{o} is étale and separated by the assumption, the diagonal morphism $\Delta : M^{\mathrm{o}} \to M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}}$ is an open and closed immersion. Therefore, we obtain a decomposition into two connected components:

$$M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}} = \Delta(M^{\mathrm{o}}) \sqcup (M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}} - \Delta(M^{\mathrm{o}})).$$

Let P denote the canonical model of fiber product $M \times_N^{\mathrm{c}} M$ as in Definition 2.2.2. Note that $P^{\mathrm{o}} = M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}}$.

Define a closed immersion $\bar{i}_\Delta : \bar{\Delta}(f) \to \bar{P}$ as the scheme-theoretic closure of the open immersion $\Delta(M^{\mathrm{o}}) \to P^{\mathrm{o}} \to \bar{P}$. Set

$$\Delta(f)^\infty := \bar{i}_\Delta^* P^\infty \quad \text{and} \quad \Delta(f) := (\bar{\Delta}(f), \Delta(f)^\infty).$$

Then $\bar{i}_\Delta$ induces a minimal morphism $i_\Delta : \Delta(f) \to P$, and we have $\Delta(f)^{\mathrm{o}} = \Delta(M^{\mathrm{o}})$.

Similarly, define a closed immersion $\bar{i}_{\mathrm{OD}} : \overline{\mathrm{OD}(f)} \to \bar{P}$ as the scheme-theoretic closure of the open immersion $M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}} - \Delta(M^{\mathrm{o}}) \to P^{\mathrm{o}} \to \bar{P}$. Set

$$\mathrm{OD}(f)^\infty := \bar{i}_{\mathrm{OD}}^* P^\infty \quad \text{and} \quad \mathrm{OD}(f) := (\overline{\mathrm{OD}(f)}, \mathrm{OD}(f)^\infty).$$

Then $\bar{i}_{\mathrm{OD}}$ induces a minimal morphism $i_{\mathrm{OD}} : \mathrm{OD}(f) \to P$. Moreover, we have $\mathrm{OD}(f)^{\mathrm{o}} = M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}} - \Delta(M^{\mathrm{o}})$.

The morphisms i_Δ and i_{OD} induce a minimal morphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$:

$$i_\Delta \sqcup i_{\mathrm{OD}} : \Delta(f) \sqcup \mathrm{OD}(f) \to P.$$

By (7) in Definition 2.1.1, this morphism is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}$ (not in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$) since $(i_\Delta \sqcup i_{\mathrm{OD}})^{\mathrm{o}} = i_\Delta^{\mathrm{o}} \sqcup i_{\mathrm{OD}}^{\mathrm{o}} : \Delta(f)^{\mathrm{o}} \sqcup \mathrm{OD}(f)^{\mathrm{o}} \to P^{\mathrm{o}} \cong M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}}$ is an isomorphism in $\mathbf{Sm}$, and since $\bar{i}_\Delta \sqcup \bar{i}_{\mathrm{OD}} : \overline{\Delta(f)} \sqcup \overline{\mathrm{OD}(f)} \to \bar{P}$ is proper by construction.

We claim $\Delta(f) \cong M$. Let $\Delta : M \to P$ ($\cong M \times_N M$) be the diagonal morphism. Then the composite $M \xrightarrow{\Delta} P \cong \Delta(f) \sqcup \mathrm{OD}(f)$ factors through $\Delta(f)$. The inverse morphism is given by $\Delta(f) \to P \xrightarrow{\mathrm{pr}_1} M$, where pr_1 denotes the first projection $P \cong M \times_N M \to M$.

Thus, for any $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Et}$, we have obtained a decomposition

$$M \times_N M \cong M \sqcup \mathrm{OD}(f).$$

Next we check the functoriality of $\mathrm{OD}(f)$. Let $(f_1 : M_1 \to N_1) \to (f_2 : M_2 \to N_2)$ be a morphism in $\underline{\mathbf{M}}\mathbf{Et}$, i.e., a commutative diagram

$$\begin{tikzcd} M_1 \arrow[r, "s"] \arrow[d, "f_1"'] & M_2 \arrow[d, "f_2"] \\ N_1 \arrow[r, "t"] & N_2 \end{tikzcd}$$

where f_1, f_2, s, and t are morphisms in $\underline{\mathbf{M}}\mathbf{Sm}$ such that f_1^{o} and f_2^{o} are étale and s^{o} and t^{o} are open immersions.

We claim that there exists a unique morphism $\mathrm{OD}(f_1) \to \mathrm{OD}(f_2)$ such that the diagram

$$\begin{tikzcd} M_1 \times_{N_1} M_1 \arrow[r] & M_2 \times_{N_2} M_2 \\ M_1 \sqcup \mathrm{OD}(f_1) \arrow[r] \arrow[u, "\cong"] & M_2 \sqcup \mathrm{OD}(f_2) \arrow[u, "\cong"'] \end{tikzcd}$$

commutes. The uniqueness is obvious by the commutativity of the above diagram. For the existence, we need to show that the composite

$$\mathrm{OD}(f_1) \to M_1 \times_{N_1} M_1 \to M_2 \times_{N_2} M_2 \cong M_2 \sqcup \mathrm{OD}(f_2)$$

factors through $\mathrm{OD}(f_2)$. To see this, it suffices to prove that the image of the morphism

$$M_1^{\mathrm{o}} \times_{N_1^{\mathrm{o}}} M_1^{\mathrm{o}} \setminus \Delta(M_1^{\mathrm{o}}) \to M_1^{\mathrm{o}} \times_{N_1^{\mathrm{o}}} M_1^{\mathrm{o}} \xrightarrow{s^{\mathrm{o}} \times s^{\mathrm{o}}} M_2^{\mathrm{o}} \times_{N_2^{\mathrm{o}}} M_2^{\mathrm{o}}$$

lands in $M_2^{\mathrm{o}} \times_{N_2^{\mathrm{o}}} M_2^{\mathrm{o}} \setminus \Delta(M_2^{\mathrm{o}})$, which easily follows from the injectivity of the open immersion s^{o}. This finishes the proof. □

The off-diagonal functor is compatible with base change.

Proposition 3.1.4. *Let $f : M \to N$ be an object of* $\underline{\mathbf{M}}\mathbf{Et}$, *and $N' \to N$ any morphism in* $\underline{\mathbf{M}}\mathbf{Sm}$. *Then the base change $g := f \times_N N'$ belongs to* $\underline{\mathbf{M}}\mathbf{Et}$, *and we have a natural isomorphism* $\mathrm{OD}(g) \cong \mathrm{OD}(f) \times_N N'$.

Proof. The first assertion holds since $g^{\mathrm{o}} = f^{\mathrm{o}} \times_{N^{\mathrm{o}}} N'^{\mathrm{o}}$ is étale as a base change of an étale morphism. We prove the second assertion. Note $(M \times_N M) \times_N N' \cong M' \times_{N'} M'$, where $M' := M \times_N N'$. Consider the following diagram in $\underline{\mathbf{M}}\mathbf{Sm}$:

$$\begin{tikzcd}
(M \times_N M) \times_N N' \arrow[d] & (M \sqcup \mathrm{OD}(f)) \times_N N' \arrow[l] & M' \sqcup (\mathrm{OD}(f) \times_N N') \arrow[l] \arrow[d, "h"] \\
M' \times_{N'} M' & & M' \sqcup \mathrm{OD}(g) \arrow[ll]
\end{tikzcd}$$

where all the arrows, except for h, are natural isomorphisms in $\underline{\mathbf{M}}\mathbf{Sm}$, and h is defined to be the composite. By diagram chase, h restricts to the identity map on M' and an isomorphism $\mathrm{OD}(f) \times_N N' \to \mathrm{OD}(g)$. □

4. The cd-structure

In this section, we introduce a cd-structure on **MSm**, and prove its fundamental properties.

4.1. $\underline{\mathrm{M}}\mathrm{V}$-*squares.* First, let us recall from [Kahn et al. 2019a] the cd-structure on $\underline{\mathbf{M}}\mathbf{Sm}$.

Definition 4.1.1. (1) An $\underline{\mathrm{M}}\mathrm{V}^{\mathrm{fin}}$-*square* is a square $S \in (\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}}$ such that the morphisms in S are minimal, and such that the resulting square

$$\begin{tikzcd}
\overline{S}(00) \arrow[r] \arrow[d] & \overline{S}(01) \arrow[d] \\
\overline{S}(10) \arrow[r] & \overline{S}(11)
\end{tikzcd}$$

is an elementary Nisnevich square (on **Sch**).

(2) An $\underline{\mathrm{M}}\mathrm{V}$-*square* is a square $S \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ which is isomorphic to the image of an $\underline{\mathrm{M}}\mathrm{V}^{\mathrm{fin}}$-square by the inclusion functor $(\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}} \to \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$.

Proposition 4.1.2 [Kahn et al. 2019a, Proposition 3.2.2]. *The* $\underline{\mathrm{M}}\mathrm{V}$*-squares form a complete and regular cd-structure* $P_{\underline{\mathrm{MV}}}$ *on* $\underline{\mathbf{M}}\mathbf{Sm}$. □

Definition 4.1.3. The topology on $\underline{\mathbf{M}}\mathbf{Sm}$ associated with the cd-structure $P_{\underline{\mathrm{MV}}}$ is called *the* $\underline{\mathrm{M}}\mathrm{V}$*-topology*.

In the following, we describe OD for $\underline{\mathrm{M}}\mathrm{V}^{\mathrm{fin}}$ and $\underline{\mathrm{M}}\mathrm{V}$-squares.

Lemma 4.1.4. *Let $f : U \to M$ be a minimal morphism such that $\bar{f} : \overline{U} \to \overline{M}$ is étale. Then we have*

$$\overline{\mathrm{OD}(f)} = \overline{U} \times_{\overline{M}} \overline{U} - \Delta(\overline{U}) \quad \textit{and} \quad \mathrm{OD}(f)^{\infty} = \pi^* M^{\infty} \cap \overline{\mathrm{OD}(f)},$$

where $\Delta : \overline{U} \to \overline{U} \times_{\overline{M}} \overline{U}$ is the diagonal, and $\pi : \overline{U} \times_{\overline{M}} \overline{U} \to \overline{M}$ is the natural morphism.

Proof. Since $U^{\mathrm{o}} \times_{M^{\mathrm{o}}} U^{\mathrm{o}} - \Delta(U^{\mathrm{o}})$ is dense in $\overline{U} \times_{\overline{M}} \overline{U} - \Delta(\overline{U})$ (as a complement of the divisor $U^{\infty} \times_{\overline{M}} \overline{U} \setminus \Delta(\overline{U})$), and since $U^{\infty} \times_{\overline{M}} \overline{U} = \overline{U} \times_{\overline{M}} U^{\infty} = \pi^* M^{\infty}$, the assertion follows from the construction of $\mathrm{OD}(f)$. □

Proposition 4.1.5. *Let S be an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square of the form*

$$\begin{array}{ccc} S(00) & \xrightarrow{v_S} & S(01) \\ {\scriptstyle q_S}\downarrow & & \downarrow{\scriptstyle p_S} \\ S(10) & \xrightarrow{u_S} & S(11) \end{array}$$

Then the morphism $\mathrm{OD}(q_S) \to \mathrm{OD}(p_S)$ *is an isomorphism in* $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.

Proof. Let S be an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square. Then, since $\overline{S}$ is an elementary Nisnevich square, we have a natural isomorphism

$$\overline{S}(00) \times_{\overline{S}(10)} \overline{S}(00) - \Delta_0(\overline{S}(00)) \xrightarrow{\sim} \overline{S}(01) \times_{\overline{S}(11)} \overline{S}(01) - \Delta_1(\overline{S}(01)),$$

where $\Delta_i : \overline{S}(0i) \to \overline{S}(0i) \times_{\overline{S}(1i)} \overline{S}(0i)$ is the diagonal for each $i = 0, 1$. Then, in view of Lemma 4.1.4, the minimality of u_S, p_S, q_S shows that the isomorphism as above induces an isomorphism $\mathrm{OD}(q_S) \to \mathrm{OD}(p_S)$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. □

Corollary 4.1.6. *Let S be an $\underline{\mathrm{MV}}$-square. The natural morphism* $\mathrm{OD}(q_S) \to \mathrm{OD}(p_S)$ *is an isomorphism in* $\underline{\mathbf{M}}\mathbf{Sm}$.

Proof. By definition of $\underline{\mathrm{MV}}$-square, there exists an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square S' which is isomorphic to S. Then, noting that there are natural isomorphisms $\mathrm{OD}(q_S) \cong \mathrm{OD}(q_{S'})$ and $\mathrm{OD}(p_S) \cong \mathrm{OD}(p_{S'})$ in $\underline{\mathbf{M}}\mathbf{Sm}$, the assertion follows from Proposition 4.1.5. □

4.2. MV-*squares.*

Definition 4.2.1. Let T be an object of $\mathbf{MSm}^{\mathbf{Sq}}$ of the form (2.3.5). Then T is called an MV-*square* if the following conditions hold:

(1) T is a pull-back square in $\mathbf{MSm}$.

(2) There exist an $\underline{\mathrm{MV}}$-square S such that $S(11) \in \mathbf{MSm}$ and a morphism $S \to T$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ such that the induced morphism $S^{\mathrm{o}} \to T^{\mathrm{o}}$ is an isomorphism in $\mathbf{Sm}^{\mathbf{Sq}}$ and $S(11) \to T(11)$ is an isomorphism in $\mathbf{MSm}$. In particular, T^{o} is an elementary Nisnevich square.

(3) $\mathrm{OD}(q_T) \to \mathrm{OD}(p_T)$ is an isomorphism in $\mathbf{MSm}$.

We let P_{MV} be the cd-structure on $\mathbf{MSm}$ consisting of MV-squares. The topology on $\mathbf{MSm}$ associated with the cd-structure P_{MV} is called *the* MV-*topology* for short.

Remark 4.2.2. (1) For any $T \in \mathbf{MSm}^{\mathbf{Sq}}$ with T^{o} an elementary Nisnevich square, the induced morphism $\mathrm{OD}(p_T)^{\mathrm{o}} \to \mathrm{OD}(q_T)^{\mathrm{o}}$ between interiors is an isomorphism in $\mathbf{Sm}$. This follows easily from the definition of elementary Nisnevich squares.

(2) If p_T^{o} and q_T^{o} are open immersions, then $\mathrm{OD}(q_T) = \mathrm{OD}(p_T) = \varnothing$. In particular, we have $\mathrm{OD}(q_T) \cong \mathrm{OD}(p_T)$.

Proposition 4.2.3. *Let T be a square in* $\mathbf{MSm}^{\mathbf{Sq}}$ *which satisfies condition (1), (2), or (3) of Definition 4.2.1. Then, for any morphism $M \to T(11)$ in* $\mathbf{MSm}$, *the base change square $T_M := T \times_{T(11)} M$ also satisfies (1), (2), or (3), respectively.*

Proof. Since base change of a pull-back diagram is a pull-back diagram, condition (1) is preserved by base change. Proposition 3.1.4 shows that (3) is preserved by the base change.

Finally, we prove that (2) is preserved by base change. Let $S \to T$ be a morphism as in (2), and let $M \to T(11)$ be any morphism in $\mathbf{MSm}$. Then we obtain a morphism $S_M \to T_M$, where $S_M := S \times_{S(11)} M$ and $T_M := T \times_{T(11)} M$. Since $S(11) \cong T(11)$, we obtain $S_M(11) \cong T_M(11)$. Moreover, S_M is an $\underline{\mathrm{MV}}$-square as the base change of an $\underline{\mathrm{MV}}$-square (see [Kahn et al. 2019a, Theorem 4.1.2]), and we have $S_M^{\mathrm{o}} \cong T_M^{\mathrm{o}}$. Therefore, the morphism $S_M \to T_M$ satisfies the requirement in (2). This finishes the proof. □

4.3. *Completeness.*

Theorem 4.3.1. *The cd-structure P_{MV} is complete.*

Proof. By [Voevodsky 2010, Lemma 2.5], it suffices to prove the following:

(1) Any morphism with values in $\varnothing = (\varnothing, \varnothing)$ is an isomorphism.

(2) For any $T \in P_{\mathrm{MV}}$ and any $M \to T(11)$ in $\mathbf{MSm}$, the square $T_M := T \times_{T(11)} M$, which is obtained by base change, belongs to P_{MV}.

But (1) is obvious, and (2) is a direct consequence of Proposition 4.2.3. □

4.4. *Regularity.*

Theorem 4.4.1. *The cd-structure P_{MV} is regular.*

Proof. By [Voevodsky 2010, Lemma 2.11], it suffices to prove that for any $T \in P_{\mathrm{MV}}$, the following assertions hold:

(1) T is a pull-back square in $\mathbf{MSm}$.

(2) $u_T : T(10) \to T(11)$ is a monomorphism.

(3) The fiber products $T(01) \times_{T(11)} T(01)$ and $T(00) \times_{T(10)} T(00)$ exist in $\mathbf{MSm}$, and the derived square

$$\begin{tikzcd} T(00) \arrow[r] \arrow[d, "\Delta_{q_T}"'] & T(01) \arrow[d, "\Delta_{p_T}"] \\ T(00) \times_{T(10)} T(00) \arrow[r] & T(01) \times_{T(11)} T(01) \end{tikzcd}$$

which we denote by $d(T)$, belongs to P_{MV}.

The definition of MV-squares gives (1), and (2) holds since $u_T^{\mathrm{o}} : T^{\mathrm{o}}(10) \to T^{\mathrm{o}}(11)$ is an open immersion. We prove (3) by checking the conditions in Definition 4.2.1 for $d(T)$.

Since $\Delta_{p_T}^{\mathrm{o}}$ and $\Delta_{q_T}^{\mathrm{o}}$ are open immersions, we have $\mathrm{OD}(\Delta_{q_T}) \cong \varnothing \cong \mathrm{OD}(\Delta_{p_T})$ by Theorem 3.1.3. Hence $d(T)$ satisfies (3) in Definition 4.2.1.

Note that $d(T)$ is isomorphic in $\mathbf{MSm}^{\mathbf{Sq}}$ to the diagram

$$\begin{array}{ccc} T(00) & \longrightarrow & T(01) \\ \downarrow & & \downarrow \\ T(00) \sqcup \mathrm{OD}(q_T) & \longrightarrow & T(01) \sqcup \mathrm{OD}(p_T) \end{array}$$

where the vertical maps are the canonical inclusions, and the horizontal maps are induced by v_T. It is easy to see that this diagram is a pull-back diagram, i.e., $d(T)$ satisfies (1) in Definition 4.2.1. Indeed, suppose that we are given a pair of morphisms $f : M \to T(01)$ and $g : M \to T(00) \sqcup \mathrm{OD}(q_T)$ which coincide at $T(01) \sqcup \mathrm{OD}(p_T)$. Then, one sees that $g^{\mathrm{o}} : M^{\mathrm{o}} \to T(00)^{\mathrm{o}} \sqcup \mathrm{OD}(q_T)^{\mathrm{o}}$ factors through $T(00)^{\mathrm{o}}$, which implies that g factors through $T(00)$.

We are reduced to checking Definition 4.2.1(2) for $d(T)$. Consider the following diagram in $\underline{\mathbf{M}}\mathbf{Sm}$:

$$\begin{array}{ccc} (T(00)^{\mathrm{o}}, \varnothing) & \longrightarrow & T(01) \\ \downarrow & & \downarrow \\ (T(00)^{\mathrm{o}}, \varnothing) \sqcup \mathrm{OD}(q_T) & \longrightarrow & T(01) \sqcup \mathrm{OD}(p_T) \end{array}$$

which we denote by $d(T)_0$, where the vertical maps are the canonical inclusions. Then $d(T)_0$ is an $\underline{\mathrm{M}}\mathrm{V}$-square since $\mathrm{OD}(q_T) \cong \mathrm{OD}(p_T)$, and there exists a natural morphism $d(T)_0 \to d(T)$. It induces an isomorphism $d(T)_0^{\mathrm{o}} \cong d(T)^{\mathrm{o}}$, and we have $d(T)_0(11) \cong d(T)(11)$. Therefore, $d(T)$ satisfies (2) in Definition 4.2.1. This finishes the proof. □

Theorem 4.4.2. *Let F be a presheaf with values in* **Sets** *on* **MSm**. *Then F is a sheaf with respect to the* MV*-topology if and only if $F(\varnothing) = 0$ and for any* $\underline{\mathrm{MV}}$*-square $T \in P_{\mathrm{MV}}$, the square*

$$\begin{array}{ccc} F(T(11)) & \longrightarrow & F(T(10)) \\ \downarrow & & \downarrow \\ F(T(01)) & \longrightarrow & F(T(00)) \end{array}$$

is cartesian.

Proof. This follows from [Voevodsky 2010, Corollary 2.17], Theorem 4.3.1, and Theorem 4.4.1. □

4.5. ***Subcanonicity.*** In this subsection, we prove the following result. Recall that a Grothendieck topology is *subcanonical* if every representable presheaf is a sheaf.

Theorem 4.5.1. *The* $\underline{\mathrm{MV}}$*-topology and the* MV*-topology are subcanonical.*

We need the following elementary observation.

Lemma 4.5.2. *Let P be a complete and regular cd-structure on a category $\mathcal{C}$. Then the topology associated with P is subcanonical if and only if every square $T \in P$ is cocartesian in $\mathcal{C}$.*

Proof. Let $\mathcal{Y}$ denote the Yoneda embedding of $\mathcal{C}$ into the category of presheaves on $\mathcal{C}$. All squares $T \in P$ are cocartesian in $\mathcal{C}$ if and only if for any $T \in P$ and for any $X \in \mathcal{C}$, the square

$$\begin{array}{ccc} \mathcal{Y}(X)(T(11)) & \xrightarrow{u_T^*} & \mathcal{Y}(X)(T(10)) \\ {\scriptstyle p_T^*}\downarrow & & \downarrow{\scriptstyle q_T^*} \\ \mathcal{Y}(X)(T(01)) & \xrightarrow{v_T^*} & \mathcal{Y}(X)(T(00)) \end{array} \tag{4.5.3}$$

is cartesian in $\mathcal{C}$. The latter condition is equivalent to that for any $X \in \mathcal{C}$, the representable presheaf $\mathcal{Y}(X)$ is a sheaf for the topology associated with P by [Voevodsky 2010, Corollary 2.17]. This finishes the proof. □

We also need the following results:

Lemma 4.5.4 [Krishna and Park 2012, Lemma 2.2]. *Let $f : X \to Y$ be a surjective morphism of normal integral schemes, and let D, D' be two Cartier divisors on Y. If $f^*D' \le f^*D$, then $D' \le D$.* □

Proposition 4.5.5. (1) *Any* $\underline{\mathrm{MV}}$*-square is cocartesian in* $\underline{\mathbf{M}}\mathbf{Sm}$.

(2) *Any* MV*-square is cocartesian in* $\underline{\mathbf{M}}\mathbf{Sm}$, *and hence in* $\mathbf{MSm}$.

Proof. (1): Let S be an $\underline{\mathrm{MV}}$-square. We may assume that S is an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square since cocartesianness is stable under isomorphisms. Let $S(10) \to M$ and $S(01) \to M$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}$ which coincide after restricting to $S(00)$. Since S^{o} is an elementary Nisnevich square, it is cocartesian in $\mathbf{Sm}$. Therefore, the morphisms $S(10)^{\mathrm{o}} \to M^{\mathrm{o}}$ and $S(01)^{\mathrm{o}} \to M^{\mathrm{o}}$ induce a unique morphism $h^{\mathrm{o}} : S(11)^{\mathrm{o}} \to M^{\mathrm{o}}$. It suffices to check that h^{o} induces a morphism $S(11) \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}$.

Let Γ be the graph of the rational map $\overline{S}(11) \dashrightarrow \overline{M}$, and let $\Gamma^N \to \Gamma$ be the normalization. For any $(ij) \in \mathbf{Sq}$, set

$$S_1(ij) := (\overline{S}(ij) \times_{\overline{S}(11)} \Gamma^N, S^\infty(ij) \times_{\overline{S}(11)} \Gamma^N).$$

The minimal morphisms $S_1(ij) \to S_1(kl)$ are induced by $S(ij) \to S(kl)$ for all $(ij) \to (kl)$ in $\mathbf{Sq}$, and they form an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square S_1. Moreover, $S_1(ij)$ are normal for all $(ij) \in Sq$, and the composites

$$\bar{h}_{ij} : \overline{S}_1(ij) \to \overline{S}(11) \dashrightarrow \overline{M}$$

are morphisms of schemes for all $(ij) \in \mathbf{Sq}$ by construction. Moreover, the morphisms $\overline{S}_1(ij) \to \overline{S}(ij)$ are proper (by the properness of Γ over $\overline{S}(11)$). Therefore, by the minimality of $S_1(ij) \to S(ij)$, the morphism $S_1 \to S$ is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$.

Claim 4.5.6. $S_1^\infty(11) \geq \bar{h}_{11}^* M^\infty$.

Proof. The admissibility of $S(10) \to M$ and $S(01) \to M$ implies that of $S_1(10) \to M$ and $S_1(01) \to M$. Since $\overline{S}_1(10)$ and $\overline{S}_1(01)$ are normal, we have $S_1(ij)^\infty \geq \bar{h}_{ij}^* M^\infty$ for $(ij) = (10), (01)$. Since $\overline{S}_1(10) \sqcup \overline{S}_1(01) \to \overline{S}_1(11)$ is a surjection between normal schemes and since $S_1(10) \to S_1(11)$ and $S_1(01) \to S_1(11)$ are minimal, Lemma 4.5.4 implies

$$S_1(11)^\infty \geq \bar{h}_{11}^* M^\infty. \qquad \square$$

By Claim 4.5.6, we have a morphism $S_1(11) \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. The composite $S(11) \xleftarrow{\sim} S_1(11) \to M$ gives the desired morphism. The uniqueness of the morphism follows from the fact that the elementary Nisnevich square S^{o} is cocartesian in $\mathbf{Sm}$. This finishes the proof of (1).

(2): Let T be an MV-square. Then condition (2) of Definition 4.2.1 shows that there are an $\underline{\mathrm{M}}\mathrm{V}$-square S and a morphism $S \to T$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ such that $S(11) \cong T(11)$. Let $f : T(10) \to M$ and $g : T(01) \to M$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}$ which coincide after restriction to $T(00)$. Then the composites

$$f_S : S(10) \to T(10) \to T(11) \quad \text{and} \quad g_S : S(01) \to T(01) \to T(11)$$

coincide after restriction to $S(00)$. Then f_S and g_S induce a unique morphism $h : T(11) \cong S(11) \to M$ since S is cocartesian in $\underline{\mathbf{M}}\mathbf{Sm}$ by (1). Since $S^{\mathrm{o}} \cong T^{\mathrm{o}}$, we have $h \circ u_T = f$ and $h \circ p_T = g$. This finishes the proof of Proposition 4.5.5. $\square$

Proof of Theorem 4.5.1. This follows from Lemma 4.5.2 and parts (1) and (2) of Proposition 4.5.5. $\square$

5. Mayer–Vietoris sequence

5.1. *Easy Mayer–Vietoris.*

Definition 5.1.1. For any sheaf F on a site $\mathcal{C}$, we denote by $\mathbb{Z}F$ the sheaf associated with the presheaf $\mathcal{C} \ni X \mapsto \mathbb{Z}(F(X))$, where for any set S, we denote by $\mathbb{Z}S$ the free abelian group generated on S.

For any $M \in \underline{\mathbf{M}}\mathbf{Sm}$ (or $\mathbf{MSm}$), we set $\mathbb{Z}(M) := \mathbb{Z}\mathcal{Y}(M)$, where $\mathcal{Y}(M)$ denotes the presheaf of sets represented by M.

Theorem 5.1.2. *Let T be an* MV*-square. Then the complex*

$$0 \to \mathbb{Z}(T(00)) \to \mathbb{Z}(T(10)) \oplus \mathbb{Z}(T(01)) \to \mathbb{Z}(T(11)) \to 0$$

of sheaves on **MSm** *is exact.*

Proof. This follows from [Voevodsky 2010, Lemma 2.18], Theorem 4.4.1, and Theorem 4.5.1. □

5.2. ***Mayer–Vietoris with transfers.***

Theorem 5.2.1. *Let $T \in \mathbf{MSm}^{\mathbf{Sq}}$. Assume that T^{o} is an elementary Nisnevich square, and that T satisfies (1) and (3) in Definition 4.2.1. Recall the notation*

$$\begin{array}{ccc} T(00) & \xrightarrow{v_T} & T(01) \\ {\scriptstyle q_T}\big\downarrow & & \big\downarrow{\scriptstyle p_T} \\ T(10) & \xrightarrow{u_T} & T(11) \end{array} \tag{5.2.2}$$

from Definition 4.2.1. Then for any $M \in \underline{\mathbf{M}}\mathbf{Sm}$, the complex

$$0 \to \mathbb{Z}_{\mathrm{tr}}(T(00))(M) \xrightarrow{(q_{T*}, v_{T*})} \mathbb{Z}_{\mathrm{tr}}(T(10))(M) \oplus \mathbb{Z}_{\mathrm{tr}}(T(01))(M) \xrightarrow{p_{T*} - u_{T*}} \mathbb{Z}_{\mathrm{tr}}(T(11))(M) \tag{5.2.3}$$

of abelian groups is exact.

Proof. The assertion is equivalent to requiring that the commutative square

$$\begin{array}{ccc} \underline{\mathbf{M}}\mathbf{Cor}(M, T(00)) & \xrightarrow{v_{T*}} & \underline{\mathbf{M}}\mathbf{Cor}(M, T(01)) \\ {\scriptstyle q_{T*}}\big\downarrow & & \big\downarrow{\scriptstyle p_{T*}} \\ \underline{\mathbf{M}}\mathbf{Cor}(M, T(10)) & \xrightarrow{u_{T*}} & \underline{\mathbf{M}}\mathbf{Cor}(M, T(11)) \end{array} \tag{5.2.4}$$

be cartesian. Note that the horizontal maps are injective.

The following lemma is key. Recall the notation from Proposition 2.3.3.

Lemma 5.2.5. *Let $\alpha_1, \alpha_2 \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01))$ be elementary correspondences with $\alpha_1 \neq \alpha_2$. Assume that $p_{T+}(\alpha_1) = p_{T+}(\alpha_2)$ holds in $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(11))$. Then α_1 and α_2 belong to the image of v_{T*}.*

Proof. Set $P := T(01) \times_{T(11)} T(01)$, and consider the commutative diagram

$$\begin{array}{ccc} \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, P) & \xrightarrow{\mathrm{pr}_{1+}} & \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01)) \\ \big\downarrow{\scriptstyle \mathrm{pr}_{2+}} & & \big\downarrow \\ \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01)) & \longrightarrow & \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(11)) \end{array}$$

in **Set**. By Proposition 2.3.4, there exists an element $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, P)$ such that $\mathrm{pr}_{1+}(\gamma) = \alpha_1$ and $\mathrm{pr}_{2+}(\gamma) = \alpha_2$.

We have a canonical identification

$$\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, P) \cong \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01)) \sqcup \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, \mathrm{OD}(p_T))$$

induced by $P \cong T(01) \sqcup \mathrm{OD}(p_T)$. Through this identification, we may regard $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, \mathrm{OD}(p_T))$ as a subset of $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, P)$.

Claim 5.2.6. $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, \mathrm{OD}(p_T))$.

Proof. Let ξ_1, ξ_2, and ζ be the generic points of α_1, α_2, and γ. Then ζ lies over ξ_1 and ξ_2. Since $\xi_1 \neq \xi_2$ by the assumption that $\alpha_1 \neq \alpha_2$, we have $\zeta \notin M^{\mathrm{o}} \times \Delta(T(01)^{\mathrm{o}})$, where $\Delta(T(01)^{\mathrm{o}})$ denotes the image of $\Delta : T(01)^{\mathrm{o}} \to T(01)^{\mathrm{o}} \times_{T(11)^{\mathrm{o}}} T(01)^{\mathrm{o}}$. This implies that $\zeta \in M^{\mathrm{o}} \times \mathrm{OD}(p_T)^{\mathrm{o}}$. Therefore, we have

$$\gamma \in \mathbf{Cor}(M^{\mathrm{o}}, \mathrm{OD}(p_T)^{\mathrm{o}}) \cap \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, P) = \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, \mathrm{OD}(p_T)). \qquad \square$$

By construction, we have $\alpha_i = \mathrm{pr}_i(\gamma) = |(\mathrm{pr}_i)_*(\gamma)|$, where

$$\mathrm{pr}_i : T(01)^{\mathrm{o}} \times_{T(11)^{\mathrm{o}}} T(01)^{\mathrm{o}} \to T(01)^{\mathrm{o}}, \quad i = 1, 2,$$

are the projections. Thus, in order to prove $\alpha_i \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(00))$ for $i = 1, 2$, it suffices to prove that $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(00) \times_{T(10)} T(00))$. Since by the above claim $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}(M, \mathrm{OD}(p_T))$, and since $\mathrm{OD}(q_T) \cong \mathrm{OD}(p_T)$ by condition (3) of Definition 4.2.1, we have $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}(M, \mathrm{OD}(q_T)) \subset \underline{\mathbf{M}}\mathbf{Cor}(M, T(00) \times_{T(10)} T(00))$. This finishes the proof of Lemma 5.2.5. $\square$

Now we are ready to prove that (5.2.4) is cartesian. Let $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(01))$ and assume $p_{T*}(\alpha) \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(10))$. Write $\alpha = \sum_{i \in I} m_i \alpha_i$, where I is a finite set, $m_i \in \mathbb{Z} - \{0\}$, and the α_i are elementary correspondences which are distinct from each other. Then we have $\alpha_i \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(01))$ for all $i \in I$. Set

$$J := \{i \in I \mid \exists j \in I - \{i\},\ |p_{T*}(\alpha_i)| = |p_{T*}(\alpha_j)|\}.$$

Then by Lemma 5.2.5, we have $\alpha_i \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(00))$ for all $i \in J$. Let $i \in I - J$, and set $\beta := |p_{T*}(\alpha_i)|$. Then the coefficient of β in $p_{T*}(\alpha)$ is nonzero, and therefore $\beta \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(10))$. By Proposition 2.3.4, there exists a unique element $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00))$ such that $v_{T+}(\gamma) = \alpha_i$ and $q_{T+}(\gamma) = \beta$. Since $T(00)^{\mathrm{o}} \to T(01)^{\mathrm{o}}$ is an open immersion, this implies $\alpha_i = \gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00))$. This finishes the proof of the exactness of (5.2.3). $\square$

Recall from [Kahn et al. 2019a, Theorem 2(2)] that for any $M \in \underline{\mathbf{M}}\mathbf{Sm}$, the presheaf $\mathbb{Z}_{\mathrm{tr}}(M)$ on $\underline{\mathbf{M}}\mathbf{Sm}$ is a sheaf for the $\underline{\mathrm{MV}}$-topology.

Corollary 5.2.7. *Let T be an* MV*-square. Then the complex*

$$0 \to \mathbb{Z}_{\mathrm{tr}}(T(00)) \xrightarrow{(q_{T*}, v_{T*})} \mathbb{Z}_{\mathrm{tr}}(T(10)) \oplus \mathbb{Z}_{\mathrm{tr}}(T(01)) \xrightarrow{p_{T*} - u_{T*}} \mathbb{Z}_{\mathrm{tr}}(T(11)) \to 0$$

of sheaves on $\underline{\mathbf{M}}\mathbf{Sm}$ for the $\underline{\mathrm{MV}}$-topology is exact.

Proof. By Theorem 5.2.1, it suffices to prove the surjectivity of the last maps of the complexes. Take a morphism $S \to T$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ as in (2) of Definition 4.2.1. Then the map

$$\mathbb{Z}_{\mathrm{tr}}(S(10)) \oplus \mathbb{Z}_{\mathrm{tr}}(S(01)) \to \mathbb{Z}_{\mathrm{tr}}(S(11)) = \mathbb{Z}_{\mathrm{tr}}(T(11))$$

is epi in $\underline{\mathbf{M}}\mathbf{NST}$ by [Kahn et al. 2019a, Theorem 4.5.7]. Since the map factors through

$$\mathbb{Z}_{\mathrm{tr}}(T(10)) \oplus \mathbb{Z}_{\mathrm{tr}}(T(01)),$$

we are done. □

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References

[Kahn and Miyazaki 2019] B. Kahn and H. Miyazaki, "Topologies on schemes and modulus pairs", preprint, 2019. To appear in *Nagoya Math J.* arXiv

[Kahn et al. 2015] B. Kahn, S. Saito, and T. Yamazaki, "Motives with modulus", preprint (withdrawn), 2015. arXiv

[Kahn et al. 2019a] B. Kahn, H. Miyazaki, S. Saito, and T. Yamazaki, "Motives with modulus, I: Modulus sheaves with transfers for non-proper modulus pairs", preprint, 2019. arXiv

[Kahn et al. 2019b] B. Kahn, H. Miyazaki, S. Saito, and T. Yamazaki, "Motives with modulus, II: Modulus sheaves with transfers for proper modulus pairs", preprint, 2019. arXiv

[Krishna and Park 2012] A. Krishna and J. Park, "Moving lemma for additive higher Chow groups", *Algebra Number Theory* **6**:2 (2012), 293–326. MR Zbl

[Mazza et al. 2006] C. Mazza, V. Voevodsky, and C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs **2**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Voevodsky 2000] V. Voevodsky, "Triangulated categories of motives over a field", pp. 188–238 in *Cycles, transfers, and motivic homology theories*, Ann. of Math. Stud. **143**, Princeton Univ. Press, 2000. MR Zbl

[Voevodsky 2010] V. Voevodsky, "Homotopy theory of simplicial sheaves in completely decomposable topologies", *J. Pure Appl. Algebra* **214**:8 (2010), 1384–1398. MR Zbl

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The Topological Period-Index Conjecture for spin^c 6-manifolds

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The Topological Period-Index Conjecture is a hypothesis which relates the period and index of elements of the cohomological Brauer group of a space. It was identified by Antieau and Williams as a topological analogue of the Period-Index Conjecture for function fields.

In this paper we show that the Topological Period-Index Conjecture holds and is in general sharp for spin^c 6-manifolds. We also show that it fails in general for 6-manifolds.

1. Introduction

This paper is about the *Topological Period-Index Problem* (TPIP), which was identified by Antieau and Williams [2014a; 2014b] as an important analogue of period-index problems in algebraic geometry. We give a brief introduction to the TPIP and refer the reader to [Antieau and Williams 2014a; 2014b] for more information.

Let X be a connected space with the homotopy type of a finite CW-complex. The cohomological Brauer group of X is defined to be the torsion subgroup of its third integral cohomology group:

$$\mathrm{Br}'(X) := TH^3(X).$$

Here and throughout integer coefficients are omitted. For $\alpha \in \mathrm{Br}'(X)$, the period of α is defined to be the order of α,

$$\mathrm{per}(\alpha) := \mathrm{ord}(\alpha).$$

Let $PU(n) := U(n)/U(1)$ be the n-dimensional projective unitary group, which is the quotient of the unitary group $U(n)$ by its centre. By a theorem of Serre [Grothendieck 1968, Corollaire 1.7], every class $\alpha \in TH^3(X)$ arises as the obstruction to lifting the structure group of some principal $PU(n)$-bundle $P \to X$ to the group $U(n)$. In this case one writes $\alpha = \delta(P)$ and defines the index of α by

$$\mathrm{ind}(\alpha) := \gcd(n : \alpha = \delta(P) \text{ for a } PU(n)\text{-bundle } P),$$

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so that the index defines the homotopy-invariant function

$$\mathrm{ind} : TH^3(X) \to \mathbb{Z}, \quad \alpha \mapsto \mathrm{ind}(\alpha).$$

From the definitions, one sees that $\mathrm{per}(\alpha) \mid \mathrm{ind}(\alpha)$ and by [Antieau and Williams 2014a, Theorem 3.1] the primes dividing $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ coincide. The TPIP is the problem of relating the index of a class α to its period and properties of X, like its dimension.

To investigate the TPIP, Antieau and Williams [2014b, Straw Man] formulated what is often called the *Topological Period-Index Conjecture* (TPIC) for X:

TPIC. *If X is homotopy equivalent to a CW-complex of dimension $2d$ and if α is an element of $\mathrm{Br}'(X)$, then*

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{d-1}.$$

Warning. The TPIC should be regarded as a *hypothesis* for investigating the TPIP and *not as a conjecture*, in the usual sense of the word.

Indeed, while the obstruction theory developed by Antieau and Williams [2014b, Theorem A] shows that the TPIC holds for any 4-dimensional complex, they also prove that the TPIC fails in general for 6-dimensional complexes, but at most by a factor of two.

Theorem 1.1 (cf. [Antieau and Williams 2014b, Theorems A and B]). *Let X be a 6-dimensional CW-complex, $\alpha \in \mathrm{Br}'(X)$ have period n, and set $\epsilon(n) := \mathrm{gcd}(n, 2)$. Then $\mathrm{ind}(\alpha) \mid \epsilon(n)n^2$.*

Moreover, if X is a 6-skeleton of the Eilenberg–Mac Lane space $K(\mathbb{Z}/2, 2)$ and we take the generator $\alpha \in H^3(X) = \mathbb{Z}/2$ (so that $\mathrm{per}(\alpha) = 2$), then $\mathrm{ind}(\alpha) = 8 > \mathrm{per}(\alpha)^2$.

An important motivation for Antieau and Williams in identifying the TPIC was the Algebraic Period-Index Conjecture (APIC) which was identified in the work of Colliot-Thélène [2002]. This is a statement in algebraic geometry concerning the Brauer group of certain algebras A. When $A = \mathbb{C}(V)$ is the function field of a smooth complex variety V, then the APIC for $\mathbb{C}(V)$ implies the TPIC for V. When the variety V has complex dimension $d = 1$, the APIC is trivially true, it was proven for $d = 2$ by de Jong [2004], and for $d \geq 3$ we have the *Antieau–Williams alternative*:

(A) either there exits a V violating the TPIC, in which case the APIC fails in general,

(B) or every V satisfies the TPIC (in which case we have identified an a priori new topological property of smooth complex varieties).

In this paper we show that for $d = 3$ the latter statement holds. This may be regarded as evidence for the APIC in complex dimension 3.

A smooth complex projective variety V is in particular a manifold: here and besides Remark 1.9, we use the word “manifold” to mean “closed smooth manifold”. Recall that a manifold M admits a spin^c structure if it is orientable and the second Stiefel–Whitney class of M has an integral lift. For example, every variety V as above admits a spin^c structure. More generally, it is well known that a 6-manifold admits a spin^c structure if and only if it admits an almost complex structure (as can be easily deduced from results in [Massey 1961]).

Theorem 1.2. *The Topological Period-Index Conjecture holds for spin^c 6-manifolds.*

As we explain in Section 2, Theorem 1.2 is an elementary consequence of results of Antieau and Williams [2014b] and:

Theorem 1.3. *Let N be a closed spin^c 6-manifold and let $x \in H^2(N; \mathbb{Z}/2)$. Then there exists a class $e_x \in H^2(N)$ such that*

$$\beta^{\mathbb{Z}/2}(x^2) = \beta^{\mathbb{Z}/2}(x)e_x \in H^5(N),$$

where $\beta^{\mathbb{Z}/2} : H^*(N; \mathbb{Z}/2) \to H^{*+1}(N)$ denotes the mod 2 Bockstein.

To discuss the TPIP further for 6-manifolds we recall that Teichner [1995] has already constructed orientable 6-manifolds N with $x \in H^2(N; \mathbb{Z}/2)$ such that $\beta^{\mathbb{Z}/2}(x^2) \neq 0$. The manifolds in Teichner’s examples are all the total-spaces of 2-sphere bundles over 4-manifolds, where the class x restricts to a generator of $H^2(S^2; \mathbb{Z}/2)$. We call pairs (N, x) coming from Teichner’s examples *Teichner pairs* (see Definition 5.3) and investigating their construction we prove:

Theorem 1.4. *For a Teichner pair (N, x), let $\alpha := \beta^{\mathbb{Z}/2}(x) \in TH^3(N)$.*

(1) *If the base 4-manifold of a Teichner pair (N, x) is orientable, then N is spin^c,* $\mathrm{per}(\alpha) = 2$, *and* $\mathrm{ind}(\alpha) = 4$.

(2) *There exist Teichner pairs (N, x) over nonorientable 4-manifolds where we have* $\mathrm{per}(\alpha) = 2$ *but* $\mathrm{ind}(\alpha) = 8$.

Summarising Theorems 1.2 and 1.4 we obtain the following result on the TPIP for 6-manifolds.

Theorem 1.5. *The TPIC fails in general for 6-manifolds but it holds and is in general sharp for spin^c 6-manifolds.*

Remark 1.6. One may view Theorem 1.3 as giving a cohomological obstruction to a closed 6-manifold admitting a spin^c structure. For instance, we do not currently know how to prove that the Teichner manifold N appearing in Theorem 1.4(2) (and Proposition 5.9) is not spin^c, except by invoking Theorem 1.3.

Remark 1.7. The nonvanishing of $\beta^{\mathbb{Z}/2}(x^2) \in H^5(N)$ is related to various nonrealisability phenomena, for which the examples in [Teichner 1995] are of minimal dimension. For example, $\beta^{\mathbb{Z}/2}(x^2)$ vanishes if $x \in H^2(N; \mathbb{Z}/2)$ can be realised as the second Stiefel–Whitney class $w_2(E)$ of some real vector bundle E over N, since $w_2(E)^2$ is the mod 2 reduction of the integral class $p_1(E)$, the first Pontrjagin class.

It is a classical result of Thom [1954] that $\beta^{\mathbb{Z}/2}(x^2)$ vanishes if the Poincaré dual of x in $H_4(N; \mathbb{Z}/2)$ is realised as the fundamental class of an embedded 4-manifold in N. More recently, in [Grant and Szűcs 2013] the second author and Szűcs showed that $\beta^{\mathbb{Z}/2}(x^2)$ vanishes if the Poincaré dual of x is realised by the fundamental class of an immersion of a 4-manifold in N and more precisely that the Poincaré dual of $\beta^{\mathbb{Z}/2}(x^2)$ is realised by the singular set of a generic smooth map realising the Poincaré dual of x. Notwithstanding Remarks 1.6 and 1.8, the geometric significance of the condition $\beta^{\mathbb{Z}/2}(x^2) \notin \beta^{\mathbb{Z}/2}(x)H^2(N)$ appearing in Section 2 remains somewhat mysterious.

Remark 1.8. The TPIP also arises in twisted K-theory, where classes $\alpha \in TH^3(X)$ define the twisting used to define the K-groups, $K^*_\alpha(X)$, of α-twisted vector bundles over X [Donovan and Karoubi 1970]. For $\alpha \in TH^3(X)$ and $i : * \to X$ the inclusion of a point, by [Antieau and Williams 2014a, Proposition 2.21], we have

$$i^*(K^0_\alpha(X)) = \mathrm{ind}(\alpha)K^0(*) = \mathrm{ind}(\alpha)\mathbb{Z}.$$

Hence $\mathrm{ind}(\alpha)$ is the index of the intersection $\bigcap_{i=1}^\infty \mathrm{Ker}(d_i) \subseteq H^0(X; K^0) \cong \mathbb{Z}$, where $d_i : H^0(X; K^0) \to H^i(X; K^{i-1})$ is the i-th differential in the twisted Atiyah–Hirzebruch spectra sequence computing $K^*_\alpha(X)$.

This perspective is behind the index formula [Antieau and Williams 2014b, Theorem A], which we use in Section 2, and also the recent work of Gu [2019] on the TPIP for 8-complexes. Gu shows that the 3-primary TPIP for 8-complexes involves controlling $\beta^{\mathbb{Z}/3}(x^3)/\beta^{\mathbb{Z}/3}(x)H^4(X)$ for classes $x \in H^2(X; \mathbb{Z}/3)$, just as the TPIP for 6-complexes involves controlling $\beta^{\mathbb{Z}/2}(x^2)/\beta^{\mathbb{Z}/2}(x)H^2(X)$ for classes $x \in H^2(X; \mathbb{Z}/2)$. We expect that the methods of this paper involving the integrality of Wu classes and the bilinear algebra of the subsection beginning on page 613 will generalise to combine with the work of Gu and prove the TPIC for odd-order Brauer classes over orientable 8-manifolds.

Remark 1.9. It is natural to wonder whether the singular spaces Z underlying singular complex 3-dimensional projective varieties satisfy the TPIC. In this direction, we note that the complement of the singular set in Z can often be compactified to give a spin^c manifold with boundary $(N, \partial N)$. The arguments of this paper can be generalised to prove that if $(N, \partial N)$ is a compact spin^c manifold with boundary where the first Chern class of N vanishes on ∂N and $TH_1(\partial N) \otimes \mathbb{Z}/2 = 0$, then the TPIC holds for quotients $N/\partial N$. As a consequence we believe that the TPIC holds for singular spaces underlying certain complex 3-dimensional varieties with isolated conical singularities.

Organisation. The rest of this paper is organised as follows. In Section 2 we prove Theorem 1.2 assuming Theorem 1.3. In Section 3 we establish some preliminary results about linking pairings and bilinear forms. In Section 4 we prove Theorem 1.3 and in Section 5 we discuss Teichner's examples and prove Theorem 1.4.

2. The Topological Period-Index Conjecture for spin^c 6-manifolds

In this section we prove that the Topological Period-Index Conjecture holds for spin^c 6-manifolds. This is an elementary consequence of Theorem 1.3 and results in [Antieau and Williams 2014b].

Let $\alpha \in \mathrm{Br}'(X) = TH^3(X)$ with $\mathrm{ord}(\alpha) = n$ and let

$$\beta^{\mathbb{Z}/n} : H^*(X;\mathbb{Z}/n) \to H^{*+1}(X)$$

be the mod n Bockstein, which lies in the exact sequence

$$H^*(X;\mathbb{Z}/n) \xrightarrow{\beta^{\mathbb{Z}/n}} H^{*+1}(X) \xrightarrow{\times n} H^{*+1}(X).$$

As $\mathrm{ord}(\alpha) = n$, we see that $\alpha = \beta^{\mathbb{Z}/n}(\xi)$ for some $\xi \in H^2(X;\mathbb{Z}/n)$. We consider the Pontrjagin square

$$P_2 : H^2(X;\mathbb{Z}/2m) \to H^4(X;\mathbb{Z}/4m)$$

and following Antieau and Williams define $\widetilde{Q}(\xi) \in H^5(X)/\alpha H^2(X)$ by the equation

$$\widetilde{Q}(\xi) := \begin{cases} [\beta^{\mathbb{Z}/n}(\xi^2)] & n \text{ is odd,} \\ [\beta^{\mathbb{Z}/2n}(P_2(\xi))] & n \text{ is even,} \end{cases}$$

where $[\gamma] \in H^5(X)/\alpha H^2(X)$ denotes the coset of $\gamma \in H^5(X)$. By [Antieau and Williams 2014b, Theorem A], the element $\widetilde{Q}(\xi)$ depends only on α and when X is a 6-dimensional CW-complex,

$$\mathrm{ind}(\alpha) = \mathrm{ord}(\widetilde{Q}(\xi))\,\mathrm{per}(\alpha).$$

Hence to verify the Topological Period-Index Conjecture in dimension 6, it suffices to show that $\mathrm{ord}(\widetilde{Q}(\xi)) \mid n$, i.e., $n\widetilde{Q}(\xi) = 0$. For this we consider the commutative diagram

$$\begin{tikzcd} H^2(X;\mathbb{Z}/2k) \arrow[r, "\beta^{\mathbb{Z}/2k}"] \arrow[d, "\rho_2"] & H^3(X) \arrow[d, "\times k"] \\ H^2(X;\mathbb{Z}/2) \arrow[r, "\beta^{\mathbb{Z}/2}"] & H^3(X) \end{tikzcd}$$

where ρ_2 denotes reduction modulo 2 and the diagram commutes as a consequence of the commutative diagram of coefficient short exact sequences

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\times 2k"] \arrow[d, "\times k"'] & \mathbb{Z} \arrow[r, "\rho_{2k}"] \arrow[d, "="] & \mathbb{Z}/2k \arrow[d, "\rho_2"] \\ \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "\rho_2"] & \mathbb{Z}/2 \end{tikzcd}$$

Hence for all $\xi \in H^2(X; \mathbb{Z}/2k)$ we have the equation

$$\beta^{\mathbb{Z}/2}(\rho_2(\xi)) = k\beta^{\mathbb{Z}/2k}(\xi). \tag{2.1}$$

Proof of the Topological Period-Index Conjecture for spin^c *6-manifolds.* Let (N, c_1) be a spin^c 6-manifold and $\alpha \in \mathrm{Br}'(N)$ have order n, and choose $\xi \in H^2(N; \mathbb{Z}/n)$ such that $\alpha = \beta^{\mathbb{Z}/n}(\xi)$. If n is odd, then $n\widetilde{Q}(\xi) = 0$ and so by [Antieau and Williams 2014b, Theorem A(3)] the Topological Period-Index Conjecture holds for α. If $n = 2m$, then set

$$x := \rho_2(\xi) \in H^2(N; \mathbb{Z}/2).$$

By Theorem 1.3, there is a $y \in H^2(N)$ such that $\beta^{\mathbb{Z}/2}(x^2) = \beta^{\mathbb{Z}/2}(x)y$. Applying (2.1) we obtain

$$\beta^{\mathbb{Z}/2}(x^2) = \beta^{\mathbb{Z}/2}(x)y = m\beta^{\mathbb{Z}/2m}(\xi)y = m\alpha y \in \alpha H^2(N) \subseteq H^5(N) \tag{2.2}$$

and so $[\beta^{\mathbb{Z}/2}(x^2)] = 0 \in H^5(N)/\alpha H^2(N)$. Applying (2.1) and (2.2) we obtain

$$\begin{aligned} 2m\widetilde{Q}(\xi) = 2m[\beta^{\mathbb{Z}/4m}(P_2(\xi))] &= [2m\beta^{\mathbb{Z}/4m}(P_2(\xi))] \\ &= [\beta^{\mathbb{Z}/2}(\rho_2(P_2(\xi)))] = [\beta^{\mathbb{Z}/2}(x^2)] = 0, \end{aligned}$$

where the second to last equality holds since P_2 satisfies $\rho_{2m}(P_2(\xi)) = \xi^2$, where ρ_{2m} denotes reduction to modulo $2m$. □

3. Linking pairings and bilinear forms

In this section we establish some elementary results used in the proof of Theorem 1.3.

Some properties of Bockstein homomorphisms. For a space X and a positive integer n recall that

$$\beta^{\mathbb{Z}/n} : H^*(X; \mathbb{Z}/n) \to H^{*+1}(X)$$

is the Bockstein associated to the coefficient sequence $\mathbb{Z} \to \mathbb{Z} \xrightarrow{\rho_n} \mathbb{Z}/n$.

Lemma 3.1. *Let $x \in H^*(X; \mathbb{Z}/n)$ and $y \in H^i(X)$, and consider $xy \in H^{*+i}(X; \mathbb{Z}/n)$. Then*

$$\beta^{\mathbb{Z}/n}(xy) = \beta^{\mathbb{Z}/n}(x)y.$$

Proof. Let $y \in C^i(X)$ be a cocycle representative for y and ρ_n denote reduction modulo n, and consider the commutative diagram below, in which the rows are short exact sequences of chain complexes:

$$\begin{tikzcd}
0 \arrow[r] & C^*(X) \arrow[r, "\times n"] \arrow[d, "\cup y"'] & C^*(X) \arrow[r, "\rho_n"] \arrow[d, "\cup y"'] & C^*(X;\mathbb{Z}/n) \arrow[r] \arrow[d, "\cup \rho_n(y)"'] & 0 \\
0 \arrow[r] & C^{*+i}(X) \arrow[r, "\times n"] & C^{*+i}(X) \arrow[r, "\rho_n"] & C^{*+i}(X;\mathbb{Z}/n) \arrow[r] & 0
\end{tikzcd}$$

Observe that the vertical arrows are chain maps, since the coboundary is a derivation and y is a cocycle. The result now follows from the naturality of connecting homomorphisms. (Compare to [Brown 1982, Chapter V, §3.3].) □

We also consider the Bockstein homomorphism

$$\beta^{\mathbb{Q}/\mathbb{Z}} : H^*(X;\mathbb{Q}/\mathbb{Z}) \to H^{*+1}(X),$$

which is associated to the coefficient sequence $\mathbb{Z} \to \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z}$. Let

$$\iota_n : \mathbb{Z}/n \to \mathbb{Q}/\mathbb{Z}$$

be the inclusion defined by sending $[1] \in \mathbb{Z}/n$ to $\left[\frac{1}{n}\right]$ and also write

$$\iota_n : H^*(X;\mathbb{Z}/n) \to H^*(X;\mathbb{Q}/\mathbb{Z})$$

for the map on homology induced by ι_n. The commutative diagram of coefficient sequences

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times n"] \arrow[d, "="] & \mathbb{Z} \arrow[r] \arrow[d, "\times\frac{1}{n}"] & \mathbb{Z}/n \arrow[d, "\iota_n"] \\
\mathbb{Z} \arrow[r] & \mathbb{Q} \arrow[r] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

gives rise to the equality

$$\beta^{\mathbb{Z}/n} = \beta^{\mathbb{Q}/\mathbb{Z}} \circ \iota_n : H^*(X;\mathbb{Z}/n) \to H^{*+1}(X). \tag{3.2}$$

The linking pairings of an oriented manifold. Let G and H be finite abelian groups. Recall that a bilinear pairing

$$\phi : G \times H \to \mathbb{Q}/\mathbb{Z}$$

is called *perfect* if $g = 0 \in G$ if and only if $\phi(g,h) = 0$ for all $h \in H$ and $h = 0 \in H$ if and only if $\phi(g,h) = 0$ for all $g \in G$.

Remark 3.3. A useful property of perfect pairings, which we leave for the reader to verify, is that $h_1 = h_2 \in H$ if and only if $\phi(g,h_1) = \phi(g,h_2)$ for all $g \in G$. An analogous statement holds for $g_1, g_2 \in G$.

Now let M be a closed, connected, oriented m-manifold with $[M] \in H_m(M)$ the fundamental class of M. For each $k = 2, \dots, m-2$, the *linking pairing* of M is the pairing

$$b_M : TH^{k+1}(M) \times TH^{m-k}(M) \to \mathbb{Q}/\mathbb{Z}, \quad (f, y) \mapsto b_M(f, y) := \langle \tilde{f} y, [M] \rangle,$$

where $\tilde{f} \in H^k(M; \mathbb{Q}/\mathbb{Z})$ is any class such that $\beta^{\mathbb{Q}/\mathbb{Z}}(\tilde{f}) = f$. The pairing b_M is well defined because if $\tilde{f}'$ is some other lift of f, then $\tilde{f} - \tilde{f}'$ itself lifts to $H^k(M; \mathbb{Q})$ and then $\langle \tilde{f} y, [M] \rangle - \langle \tilde{f}' y, [M] \rangle = \langle (\tilde{f} - \tilde{f}') y, [M] \rangle = 0$, since y is torsion.

Lemma 3.4. *The linking pairing $b_M : TH^{k+1}(M) \times TH^{m-k}(M) \to \mathbb{Q}/\mathbb{Z}$ is a perfect pairing such that for all $w \in H^k(M; \mathbb{Z}/n)$ and all $y \in TH^{m-k}(M)$*

$$b_M(\beta^{\mathbb{Z}/n}(w), y) = \iota_n(\langle wy, [M] \rangle).$$

Proof. That b_M is perfect is well known. The case $m = 2k+1$ is part of [Davis and Kirk 2001, Exercise 55]. The general case follows from results in [Seifert and Threlfall 1934]. Since we did not find a definitive reference in the literature, we give a proof below.

For a finite abelian group G, let $G^\wedge := \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ denote the *torsion dual* of G. A pairing $\phi : G \times H \to \mathbb{Q}/\mathbb{Z}$ of finite abelian groups induces adjoint homomorphisms $\hat{\phi} : H \to G^\wedge$, $h \mapsto [g \mapsto \phi(g, h)]$ and $\hat{\phi}' : G \to H^\wedge$, $g \mapsto [h \mapsto \phi(g, h)]$, and it is easily checked that ϕ is perfect if and only if either one of $\hat{\phi}$ or $\hat{\phi}'$ is an isomorphism.

Standard properties of cup and cap products give $\langle \tilde{f} y, [M] \rangle = \langle \tilde{f}, y \cap [M] \rangle$. Hence the adjoint homomorphism of b_M,

$$\hat{b}_M : TH^{m-k}(M) \to TH^{k+1}(M)^\wedge, \quad y \mapsto [f \mapsto b_M(f, y) = \langle \tilde{f}, y \cap [M] \rangle],$$

is equal to the composition $\hat{\phi}_M \circ PD$, where $PD : TH^{m-k}(M) \to TH_k(M)$ is the Poincaré duality isomorphism and $\hat{\phi}_M : TH_k(M) \to TH^{k+1}(M)^\wedge$ is an adjoint of the pairing

$$\phi_M : TH^{k+1}(M) \times TH_k(M) \to \mathbb{Q}/\mathbb{Z}, \quad (f, b) \mapsto \langle \tilde{f}, b \rangle,$$

for $\tilde{f} \in H^k(M; \mathbb{Q}/\mathbb{Z})$ a lift of f. Hence it suffices to prove that $\hat{\phi}_M$ is an isomorphism or equivalently that the other adjoint $\hat{\phi}'_M : TH^{k+1}(M) \to TH_k(M)^\wedge$ is an isomorphism. Since the finite groups $TH^{k+1}(M)$ and $TH_k(M)^\wedge$ have the same order by the universal coefficient theorem, it suffices to show that $\hat{\phi}'_M$ is injective.

Suppose that $\hat{\phi}'_M(f) = 0$ and let $\tilde{f} \in H^k(M; \mathbb{Q}/\mathbb{Z})$ be a lift of f. Then for all $b \in TH_k(M)$

$$\langle \tilde{f}, b \rangle = 0 \in \mathbb{Q}/\mathbb{Z}.$$

Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, another application of the universal coefficient theorem gives

$$H^k(M; \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(H_k(M), \mathbb{Q}/\mathbb{Z}) \cong TH_k(M)^\wedge \oplus \mathrm{Hom}(FH_k(M); \mathbb{Q}/\mathbb{Z}),$$

where $FH_k(M) := H_k(M)/TH_k(M)$. With respect to the above decomposition we have $\tilde{f} = (0, \bar{z})$ for some $\bar{z} \in \mathrm{Hom}(FH_k(M); \mathbb{Q}/\mathbb{Z})$. Now $\bar{z}$ can be lifted to $z \in H^k(M; \mathbb{Q})$ so that $\tilde{f} - \pi(z) = 0$ but then $f = \beta^{\mathbb{Q}/\mathbb{Z}}(\tilde{f}) = \beta^{\mathbb{Q}/\mathbb{Z}}(\tilde{f} - \pi(z)) = 0$ and so $\hat{\phi}'_M$ is injective.

The second statement follows directly from the definition of b_M and (3.2). □

Bilinear forms over $\mathbb{Z}/2$. In this subsection we establish a basic fact about symmetric bilinear forms over $\mathbb{Z}/2$. Let V be a finitely generated $(\mathbb{Z}/2)$-vector space and let

$$\lambda : V \times V \to \mathbb{Z}/2$$

be a symmetric bilinear form on V. If $V^* := \mathrm{Hom}(V, \mathbb{Z}/2)$ is the dual vector space to V, then the adjoint homomorphism of λ is the homomorphism

$$\hat{\lambda} : V \to V^*, \quad v \mapsto (w \mapsto \lambda(v, w)).$$

The form (λ, V) is *nonsingular* if $\hat{\lambda} : V \to V^*$ an isomorphism. Notice that the map

$$\gamma(\lambda) : V \to \mathbb{Z}/2, \quad v \mapsto \lambda(v, v)$$

is linear since

$$\begin{aligned}\lambda(v + w, v + w) &= \lambda(v, v) + \lambda(v, w) + \lambda(w, v) + \lambda(w, w)\\ &= \lambda(v, v) + 2\lambda(v, w) + \lambda(w, w) = \lambda(v, v) + \lambda(w, w).\end{aligned}$$

Thus $\gamma(\lambda) \in V^*$.

Lemma 3.5. *For all λ, $\gamma(\lambda) \in \mathrm{Im}(\hat{\lambda})$.*

Proof. For the orthogonal sum of bilinear forms, $\lambda_0 \oplus \lambda_1$, we have

$$\gamma(\lambda_0 \oplus \lambda_1) = \gamma(\lambda_0) \oplus \gamma(\lambda_1).$$

The lemma follows since every symmetric bilinear form over a finite field is isomorphic to the orthogonal sum of the zero form and a nonsingular form. □

Remark 3.6. Although we will not use this fact, it is worthwhile to note that Lemma 3.5 is equivalent to the following statement: let A be a symmetric matrix over $\mathbb{Z}/2$; then the diagonal of A lies in the column space of A.

Example 3.7. Let N be a closed, connected, oriented 6-manifold, and let $x \in H^2(N; \mathbb{Z}/2)$. We identify $H^6(N; \mathbb{Z}/2) = \mathbb{Z}/2$ and for the $(\mathbb{Z}/2)$-vector space

$$V := TH^2(N)/2TH^2(N)$$

we define the symmetric bilinear form

$$\lambda_x : V \times V \to \mathbb{Z}/2, \quad ([y], [z]) \mapsto yxz.$$

By Lemma 3.5, there is a vector $[d] \in V$ such that $\hat{\lambda}_x([d]) = \gamma(\lambda_x) \in V^*$. Hence for any $d_x \in [d] \subset TH^2(N)$ and all $y \in TH^2(N)$, we have

$$y^2 x = yxy = \lambda_x([y], [y]) = \lambda_x([y], [d_x]) = yxd_x.$$

4. The proof of Theorem 1.3

Let N be a closed, connected, oriented spin^c 6-manifold. To prove Theorem 1.3 it suffices to prove the following: for $x \in H^2(N; \mathbb{Z}/2)$ and all $y \in TH^2(N)$, there is a class $e_x \in H^2(N)$ such that

$$x^2 y = xe_x y \in H^6(N; \mathbb{Z}/2). \tag{4.1}$$

To see this we use the linking pairing of N, which is a perfect pairing by Lemma 3.4:

$$b_N : TH^5(N) \times TH^2(N) \to \mathbb{Q}/\mathbb{Z}.$$

From (4.1) and Lemmas 3.4 and 3.1, for all $y \in TH^2(N)$ we have

$$\begin{aligned} b_N(\beta^{\mathbb{Z}/2}(x^2), y) &= \iota_2(\langle x^2 y, [N]\rangle) = \iota_2(\langle xe_x y, [N]\rangle) \\ &= b_N(\beta^{\mathbb{Z}/2}(xe_x), y) = b_N(\beta^{\mathbb{Z}/2}(x)e_x, y). \end{aligned}$$

Thus $\beta^{\mathbb{Z}/2}(x^2) = \beta^{\mathbb{Z}/2}(x)e_x$, since b_N is perfect; see Remark 3.3.

To find e_x, we start with $v_2(N)$, the second Wu class of N. Since N is orientable, $v_2(N)$ coincides with $w_2(N)$, the second Stiefel–Whitney class of N. Since N is spin^c, the class $w_2(N)$ lifts to an integral class $c_1 \in H^2(N)$. In summary, we have

$$v_2(N) = w_2(N) = \rho_2(c_1) \in H^2(N; \mathbb{Z}/2). \tag{4.2}$$

By definition of the Wu class $v_2(N)$ we have

$$xyv_2(N) = Sq^2(xy) = x^2 y + xy^2, \tag{4.3}$$

where we have used the Cartan formula for $Sq^2(xy)$ and the fact that $Sq^1(\rho_2(y)) = 0$. By (4.2) we can replace $v_2(N)$ by c_1 in (4.3) and rearranging we obtain

$$x^2 y = xyc_1 + xy^2. \tag{4.4}$$

By Example 3.7, there is an element $d_x \in TH^2(N)$ such that $xy^2 = xyd_x$ and so

$$x^2y = xyc_1 + xyd_x = xye_x,$$

where $e_x := c_1 + d_x$. Hence we have found e_x as in (4.1), finishing the proof of Theorem 1.3.

5. Teichner's examples

In this section we recall a construction due to Teichner [1995] which produces closed smooth 6-manifolds N with classes $x \in H^2(N; \mathbb{Z}/2)$ such that $\beta^{\mathbb{Z}/2}(x^2) \neq 0$. The manifolds N are constructed as total spaces of sphere bundles of rank-3 vector bundles E over closed 4-manifolds. In the following, $\mathbb{Z}^{w_1(E)}$ denotes integral coefficients twisted by the first Stiefel–Whitney class of the bundle E.

Lemma 5.1 [Teichner 1995, Lemma 1]. *Let E be a 3-dimensional vector bundle over a path-connected space X, with sphere bundle $N = SE$.*

(i) *There exists a class $x \in H^2(N; \mathbb{Z}/2)$ which restricts to the generator in the cohomology $H^2(S^2; \mathbb{Z}/2)$ of the fibre if and only if $w_3(E) = 0$.*

(ii) *Assume that $w_2(E)$ is not the reduction of a class in $H^2(X; \mathbb{Z}^{w_1(E)})$. Then any class x as in* (i) *has $0 \neq \beta^{\mathbb{Z}/2}(x^2) \in H^5(N; \mathbb{Z})$.*

The next lemma guarantees the existence of such bundles with base $X = M$ a closed connected 4-manifold.

Lemma 5.2 [Teichner 1995, Lemma 2]. *Let M be a closed connected 4-manifold with fundamental group $\mathbb{Z}/4$. Then there exists a 3-dimensional bundle E over M with $w_3(E) = 0$, $w_1(E) = w_1(M)$, and $w_2(E)$ not the reduction of a class in $H^2(M; \mathbb{Z}^{w_1(E)})$.* □

Definition 5.3. The total space N of the sphere bundle of a bundle E satisfying the conditions of Lemma 5.2 is a closed connected 6-manifold, which by Lemma 5.1 supports a class $x \in H^2(N; \mathbb{Z}/2)$ satisfying $\beta^{\mathbb{Z}/2}(x^2) \neq 0$. We will call such a total space N a *Teichner manifold* and the pair (N, x) a *Teichner pair*.

Spin^c 6-manifolds N with $\beta^{\mathbb{Z}/2}(x^2) \neq 0 \in H^5(N)$. In this subsection we show that a Teichner manifold over an orientable base is spin^c.

Lemma 5.4. *Let N be a Teichner manifold over a closed connected 4-manifold M. Then*

(i) *N is orientable*,

(ii) *and if M is orientable*, *then N is spin^c.*

Proof. Let $\pi : N \to M$ be the bundle projection. Since the normal bundle of the sphere bundle in the total space of E is trivial, there are bundle isomorphisms

$$TN \oplus \mathbb{R} \cong TE|_N \cong \pi^*(TM) \oplus \pi^*(E).$$

Now part (i) follows from the equation

$$w_1(N) = \pi^* w_1(M) + \pi^* w_1(E) = 0.$$

For (ii), assume $w_1(M) = 0$ so that

$$w_2(N) = \pi^* w_2(M) + \pi^* w_1(M) \pi^* w_1(E) + \pi^* w_2(E) = \pi^* w_2(M) + \pi^* w_2(E).$$

Then

$$\beta^{\mathbb{Z}/2}(w_2(N)) = \pi^*(\beta^{\mathbb{Z}/2}(w_2(M))) + \pi^*(\beta^{\mathbb{Z}/2}(w_2(E))).$$

The first term vanishes since any orientable 4-manifold is spin^c; see [Morgan 1996] for example. The second term vanishes since $\beta^{\mathbb{Z}/2}(w_2(E)) \in H^3(M)$ is the Euler class of the orientable bundle E. □

The following proposition proves Theorem 1.4(i).

Proposition 5.5. *Let (N, x) be a Teichner pair over a closed, connected, orientable 4-manifold. Then N is spin^c and $\beta^{\mathbb{Z}/2}(x^2) \neq 0$, but $\beta^{\mathbb{Z}/2}(x^2) \in \beta^{\mathbb{Z}/2}(x) H^2(N)$.*

Furthermore, the element $\alpha = \beta^{\mathbb{Z}/2}(x) \in TH^3(N)$ has $\operatorname{per}(\alpha) = 2$ *and* $\operatorname{ind}(\alpha) = 4$.

Proof. The first statement is a consequence of Lemmas 5.4 and 5.1 and Theorem 1.3.

To prove the second statement, we recall that by [Antieau and Williams 2014b, Theorem A],

$$\operatorname{ind}(\alpha) = \operatorname{ord}(\widetilde{Q}(x)) \operatorname{per}(\alpha),$$

where $\widetilde{Q}(x) = [\beta^{\mathbb{Z}/4}(P_2(x))] \in H^5(N)/\alpha H^2(N)$. Note that by Theorem 1.3 and (2.1),

$$2\widetilde{Q}(x) = 2[\beta^{\mathbb{Z}/4}(P_2(x))] = [2\beta^{\mathbb{Z}/4}(P_2(x))] = [\beta^{\mathbb{Z}/2}(x^2)] = 0,$$

since N is spin^c. However $\widetilde{Q}(x) \neq 0$, since any element of $\alpha H^2(N)$ is 2-torsion, while

$$2\beta^{\mathbb{Z}/4}(P_2(x)) = \beta^{\mathbb{Z}/2}(x^2) \neq 0.$$

Hence $\operatorname{ord}(\widetilde{Q}(x)) = 2$ and we're done. □

6-manifolds violating the TPIC. In this subsection we give examples of Teichner pairs (N, x) over a nonorientable base which violate the Topological Period-Index Conjecture, i.e., $\beta^{\mathbb{Z}/2}(x^2) \notin \beta^{\mathbb{Z}/2}(x) H^2(N)$. We first prove an extension of [Teichner 1995, Lemma 2].

Lemma 5.6. *Let M be a closed connected 4-manifold with an element $a \in H_1(M)$ of order 4. Then there exists a 3-dimensional bundle E over M with $w_1(E) = w_1(M)$, $w_2(E)$ not coming from $H^2(M; \mathbb{Z}^{w_1(E)})$, and $w_3(E) = 0$.*

Proof. We use multiplicative notation for elements of $H_1(M) = \pi_1(M)_{\mathrm{ab}}$. The Poincaré dual of a^2 in $H^3(M; \mathbb{Z}^{w_1(M)})$ has order 2 and hence is the image of an element $z \in H^2(M; \mathbb{Z}/2)$ under the twisted Bockstein. As in Teichner's proof of [1995, Lemma 2], there are no obstructions to constructing a 3-bundle E with $(w_1(E), w_2(E)) = (w_1(M), z)$.

It remains to show that $w_3(E) = 0$. This follows from Theorem 2.3 of [Greenblatt 2006], which states that for any space X and twisting $w \in H^1(X; \mathbb{Z}/2)$, the composition of the twisted Bockstein $\beta^w : H^i(X; \mathbb{Z}/2) \to H^{i+1}(X; \mathbb{Z}^w)$ with reduction mod 2 is given by

$$\rho_2 \circ \beta^w(z) = Sq^1(z) + zw.$$

Hence we have

$$\begin{aligned} w_3(E) &= Sq^1(w_2(E)) + w_2(E)w_1(E) \\ &= \rho_2 \circ \beta^{w_1(M)}(w_2(E)) \\ &= 0, \end{aligned}$$

since $\beta^{w_1(M)}(w_2(E)) = \beta^{w_1(M)}(z)$ is even. □

In order to find an example with $\beta^{\mathbb{Z}/2}(x^2) \notin \beta^{\mathbb{Z}/2}(x)H^2(N)$ it turns out to be sufficient that there is an element $a \in H_1(M)$ of order 4 such that $0 \neq \tau_!(a^2) \in H_1(\widehat{M})$, where $\tau_! : H_1(M) \to H_1(\widehat{M})$ is the transfer associated to the orientation double cover $\tau : \widehat{M} \to M$.

To this end, we shall use a closed connected 4-manifold M with

$$\pi_1(M) = C_8 \rtimes C_2 = \langle a, b \mid b^{-1}ab = a^5, a^8, b^2 \rangle$$

and with $w_1(M) : \pi_1(M) \to C_2$ the projection onto the base of the semidirect product. Note that

$$H_1(M) = \langle a, b \mid a = a^5, a^8, b^2, [a, b] \rangle \cong C_4 \times C_2$$

has an element a of order 4. It is well known (see, e.g., [Ranicki 2002, Propostion 11.75]) that every homomorphism $w : \pi \to \mathbb{Z}/2$ from a finitely presented group π arises as $(\pi_1(X), w_1(X))$ for a 4-manifold X, so a 4-manifold M as above exists.

Lemma 5.7. *The transfer homomorphism $\tau_! : H_1(M) \to H_1(\widehat{M})$ does not map the element $a^2 \in H_1(M)$ to 0.*

Proof. Let $G = \pi_1(M)$ and let $H = \ker(w_1(M)) = C_8$, so that $[G : H] = 2$. The definition of the transfer in terms of coset representatives gives

$$\tau_! : G_{\mathrm{ab}} \to H_{\mathrm{ab}}, \quad g[G, G] \mapsto g^2[H, H].$$

Therefore $\tau_!(a^2) = a^4 \neq 0$ as claimed. □

Before continuing, we record the following lemma which will be useful in the proof of Proposition 5.9 below.

Lemma 5.8 [Dold 1980, Chapter VII, §8.10]. *Let $i : A \to X$ denote the inclusion of a CW-pair (X, A), and let $\delta : H^*(A) \to H^{*+1}(X, A)$ be the connecting homomorphism in the long exact cohomology sequence (with any coefficients). Then for all $x \in H^*(A)$ and $y \in H^*(X)$ we have*

$$\delta(xi^*(y)) = \delta(x)y. \qquad \square$$

The following proposition proves Theorem 1.4(ii).

Proposition 5.9. *Let (N, x) be a Teichner pair over a nonorientable 4-manifold M with $w_1(M) : \pi_1(M) \to \mathbb{Z}/2$ as above. Then $\beta^{\mathbb{Z}/2}(x^2) \notin \beta^{\mathbb{Z}/2}(x)H^2(N)$.*

Furthermore, the element $\alpha = \beta^{\mathbb{Z}/2}(x) \in TH^3(N)$ has $\mathrm{per}(\alpha) = 2$ and $\mathrm{ind}(\alpha) = 8$.

Proof. We first prove that $\beta^{\mathbb{Z}/2}(x^2) \notin \beta^{\mathbb{Z}/2}(x)H^2(N)$. Suppose towards a contradiction that $\beta^{\mathbb{Z}/2}(x^2) = \beta^{\mathbb{Z}/2}(x)Y$ for some $Y \in H^2(N)$. Let $i : N \hookrightarrow DE$ be the inclusion of the unit sphere bundle in the unit disc bundle of E. From the long exact sequence of the pair (DE, N), the twisted Thom isomorphism $H^3(DE, N) \cong H^0(M; \mathbb{Z}^w)$, and the fact that M is nonorientable, we see that $i^* : H^2(DE) \to H^2(N)$ is surjective. Hence $Y = i^*(y)$ for some $y \in H^2(DE) \cong H^2(M)$.

Let $t_E^w \in H^3(DE, N; \mathbb{Z}^w)$ be the twisted Thom class of E and $t_E \in H^3(DE, N; \mathbb{Z}/2)$ its mod 2 reduction. From the fact that x restricts to a generator in each fibre, it follows that $t_E = \delta(x)$, where $\delta : H^*(N; \mathbb{Z}/2) \to H^{*+1}(DE, N; \mathbb{Z}/2)$ is the connecting homomorphism (see the proof of Lemma 1 in [Teichner 1995]). Now we have

$$\delta(x^2) = \delta(Sq^2(x)) = Sq^2(\delta(x)) = Sq^2(t_E) = w_2(E)t_E$$

and since Bocksteins commute with connecting homomorphisms

$$\delta(\beta^{\mathbb{Z}/2}(x^2)) = \beta^{\mathbb{Z}/2}(\delta(x^2)) = \beta^{\mathbb{Z}/2}(w_2(E)t_E).$$

On the other hand, $\beta^{\mathbb{Z}/2}(x^2) = \beta^{\mathbb{Z}/2}(x)i^*(y)$ and so

$$\begin{aligned}
\delta(\beta^{\mathbb{Z}/2}(x^2)) &= \delta(\beta^{\mathbb{Z}/2}(x)i^*(y)) \\
&= \delta(\beta^{\mathbb{Z}/2}(xi^*(\rho_2(y)))) \\
&= \beta^{\mathbb{Z}/2}(\delta(xi^*(\rho_2(y)))) \\
&= \beta^{\mathbb{Z}/2}(\delta(x)\rho_2(y)) \\
&= \beta^{\mathbb{Z}/2}(t_E\rho_2(y)).
\end{aligned}$$

Here we have used Lemmas 3.1 and 5.8.

The above argument shows that $\beta^{\mathbb{Z}/2}(w_2(E)t_E) = \beta^{\mathbb{Z}/2}(t_E\rho_2(y))$, or equivalently $t_E(w_2(E) - \rho_2(y))$ is the reduction of an integral class. From the square

$$\begin{tikzcd} H^5(DE, N) \arrow[r, "\rho_2"] & H^5(DE, N; \mathbb{Z}/2) \\ H^2(M; \mathbb{Z}^w) \arrow[r, "\rho_2"] \arrow[u, "\cup t_E^w", "\cong"'] & H^2(M; \mathbb{Z}/2) \arrow[u, "\cup t_E", "\cong"'] \end{tikzcd}$$

which commutes since the Thom isomorphisms commute with reduction mod 2, we see that $w_2(E) - \rho_2(y)$ is the reduction of a twisted integral class, or equivalently

$$\beta^w(w_2(E)) = \beta^w(\rho_2(y)).$$

Next we lift this equation to the orientation cover, using the commutative square

$$\begin{tikzcd} H^2(\widehat{M}; \mathbb{Z}/2) \arrow[r, "\beta^{\mathbb{Z}/2}"] & H^3(\widehat{M}) \\ H^2(M; \mathbb{Z}/2) \arrow[r, "\beta^w"] \arrow[u, "\tau^*"] & H^3(M; \mathbb{Z}^w) \arrow[u, "\tau^*"] \end{tikzcd}$$

to conclude that

$$\tau^*\beta^w(w_2(E)) = \tau^*\beta^w(\rho_2(y)) = \beta^{\mathbb{Z}/2}(\tau^*(\rho_2(y))) = \beta^{\mathbb{Z}/2}\rho_2(\tau^*(y)) = 0.$$

However, Poincaré duality gives a commutative square

$$\begin{tikzcd} H^3(M; \mathbb{Z}^w) \arrow[r, "\tau^*"] \arrow[d, "\cap[M]_w", "\cong"'] & H^3(\widehat{M}) \arrow[d, "\cap[\widehat{M}]", "\cong"'] \\ H_1(M) \arrow[r, "\tau_!"] & H_1(\widehat{M}) \end{tikzcd}$$

Since the bundle E was chosen as in Lemma 5.6 so that $\beta^w(w_2(E)) \cap [M]_w = a^2$, and $\tau_!(a^2) \neq 0$ by Lemma 5.7, we see that $\tau^*\beta^w(w_2(E)) \neq 0$, a contradiction.

To prove the second statement, we have $\mathrm{per}(\alpha) = 2$ and since $\beta^{\mathbb{Z}/2}(x^2) \notin \beta^{\mathbb{Z}/2}(x)H^2(N)$,

$$2\widetilde{Q}(x) = [\beta^{\mathbb{Z}/4}(P_2(x))] = [2\beta^{\mathbb{Z}/4}(P_2(x))] = [\beta^{\mathbb{Z}/2}(x^2)] \neq 0.$$

Hence $\mathrm{ord}(\widetilde{Q}(x)) = 4$. As $\mathrm{ind}(\alpha) = \mathrm{ord}(\widetilde{Q}(x))\,\mathrm{per}(\alpha)$ by [Antieau and Williams 2014b, Theorem A], $\mathrm{ind}(\alpha) = 8$. $\square$

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References

[Antieau and Williams 2014a] B. Antieau and B. Williams, "The period-index problem for twisted topological K-theory", *Geom. Topol.* **18**:2 (2014), 1115–1148. MR Zbl

[Antieau and Williams 2014b] B. Antieau and B. Williams, "The topological period-index problem over 6-complexes", *J. Topol.* **7**:3 (2014), 617–640. MR Zbl

[Brown 1982] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics **87**, Springer, 1982. MR Zbl

[Colliot-Thélène 2002] J.-L. Colliot-Thélène, "Exposant et indice d'algèbres simples centrales non ramifiées", *Enseign. Math.* (2) **48**:1-2 (2002), 127–146. MR Zbl

[Davis and Kirk 2001] J. F. Davis and P. Kirk, *Lecture notes in algebraic topology*, Graduate Studies in Mathematics **35**, American Mathematical Society, Providence, RI, 2001. MR Zbl

[Dold 1980] A. Dold, *Lectures on algebraic topology*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **200**, Springer, 1980. MR Zbl

[Donovan and Karoubi 1970] P. Donovan and M. Karoubi, "Graded Brauer groups and K-theory with local coefficients", *Inst. Hautes Études Sci. Publ. Math.* **38** (1970), 5–25. MR Zbl

[Grant and Szűcs 2013] M. Grant and A. Szűcs, "On realizing homology classes by maps of restricted complexity", *Bull. Lond. Math. Soc.* **45**:2 (2013), 329–340. MR Zbl

[Greenblatt 2006] R. Greenblatt, "Homology with local coefficients and characteristic classes", *Homology Homotopy Appl.* **8**:2 (2006), 91–103. MR Zbl

[Grothendieck 1968] A. Grothendieck, "Le groupe de Brauer, I: Algèbres d'Azumaya et interprétations diverses", pp. 46–66 in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, North-Holland, Amsterdam, 1968. MR Zbl

[Gu 2019] X. Gu, "The topological period–index problem over 8-complexes, I", *J. Topol.* **12**:4 (2019), 1368–1395. Zbl

[de Jong 2004] A. J. de Jong, "The period-index problem for the Brauer group of an algebraic surface", *Duke Math. J.* **123**:1 (2004), 71–94. MR Zbl

[Massey 1961] W. S. Massey, "Obstructions to the existence of almost complex structures", *Bull. Amer. Math. Soc.* **67** (1961), 559–564. MR Zbl

[Morgan 1996] J. W. Morgan, *The Seiberg–Witten equations and applications to the topology of smooth four-manifolds*, Mathematical Notes **44**, Princeton University, 1996. MR Zbl

[Ranicki 2002] A. Ranicki, *Algebraic and geometric surgery*, Oxford University, 2002. MR Zbl

[Seifert and Threlfall 1934] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Teubner, Leipzig, 1934. Zbl

[Teichner 1995] P. Teichner, "6-dimensional manifolds without totally algebraic homology", *Proc. Amer. Math. Soc.* **123**:9 (1995), 2909–2914. MR Zbl

[Thom 1954] R. Thom, "Quelques propriétés globales des variétés différentiables", *Comment. Math. Helv.* **28** (1954), 17–86. MR Zbl

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Weibel's conjecture for twisted K-theory

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We prove Weibel's conjecture for twisted K-theory when twisting by a smooth proper connective dg-algebra. Our main contribution is showing we can kill a negative twisted K-theory class using a projective birational morphism (in the same twisted setting). We extend the vanishing result to relative twisted K-theory of a smooth affine morphism and describe counterexamples to some similar extensions.

1. Introduction

The so-called fundamental theorem for K_1 and K_0 states that for any ring R there is an exact sequence

$$0 \to K_1(R) \to K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t^{\pm}]) \to K_0(R) \to 0.$$

We see K_0 can be defined using K_1. There is an analogous exact sequence, truncated on the right, for K_0. Bass defines $K_{-1}(X)$ as the cokernel of the final morphism. He then iterates the construction to define a theory of negative K-groups [Bass 1968, §XII.7 and §XII.8].

Weibel's conjecture [1980] asks if $K_{-i}(R) = 0$ for $i > \dim R$ when R has finite Krull dimension. Kerz, Strunk, and Tamme [Kerz et al. 2018] have proven Weibel's conjecture for any Noetherian scheme of finite Krull dimension (see the introduction for a historical summary of progress) by establishing pro cdh-descent for algebraic K-theory. Land and Tamme [2019] have shown that a general class of localizing invariants satisfy pro cdh-descent. With this improvement, we extend Weibel's vanishing to some cases of twisted K-theory.

Theorem 1.1. *Let X be a Noetherian d-dimensional scheme and $\mathscr{A}$ a sheaf of smooth proper connective quasicoherent differential graded algebras over X; then $K_{-i}(\mathrm{Perf}(\mathscr{A}))$ vanishes for $i > d$.*

The original goal of this paper was to extend Weibel's conjecture to an Azumaya algebra over a scheme. To an Azumaya algebra $\mathscr{A}$ of rank r^2 on X we can associate a Severi–Brauer variety P of relative dimension $r - 1$ over X. Such a variety is

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étale-locally isomorphic over X to $\mathbb{P}_X^{r-1}$. Quillen [1973] generalizes the projective bundle formula to Severi–Brauer varieties showing (for $i \geq 0$)

$$K_i(P) \cong \bigoplus_{n=0}^{r-1} K_i(\mathscr{A}^{\otimes n}).$$

At the root of this computation is a semiorthogonal decomposition of $\mathrm{Perf}(P)$. Consequently, the computation lifts to the level of nonconnective K-theory spectra. Statements about the K-theory of Azumaya algebras can generally be extracted through this decomposition. In our case, the dimension of the Severi–Brauer variety jumps and so Weibel's conjecture (for our noncommutative dg-algebra) does not follow from the commutative setting.

We could remedy this by characterizing a class of morphisms to X, which should include Severi–Brauer varieties, and then show the relative K-theory vanishes under $-d-1$. In Remark 4.4, we show that smooth and proper morphisms (in fact, smooth and projective) are not sufficient. We warn the reader that we will use the overloaded words "smooth and proper" in both the scheme and dg-algebra settings.

For dg-algebras and dg-categories, properness and smoothness are module and algebraic finiteness conditions [Toën and Vaquié 2007, Definition 2.4]. Together, the two conditions characterize the dualizable objects in $\mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega})$, whose objects are ω-compactly generated R-linear stable presentable ∞-categories. More surprisingly, the invertible objects of $\mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega})$ are exactly the module categories over derived Azumaya algebras [Antieau and Gepner 2014, Theorem 3.15]. So Theorem 1.1 recovers the discrete Azumaya algebra case.

However, any connective derived Azumaya algebra is discrete. After base-changing to a field k, $\mathscr{A}_k \cong H_*\mathscr{A}_k$ is a connective graded k-algebra and $H_*\mathscr{A}_k \otimes_k (H_*\mathscr{A}_k)^{\mathrm{op}}$ is Morita equivalent to k. So $H_*\mathscr{A}_k$ is discrete. The scope of Theorem 1.1 is not wasted as smooth proper connective dg-algebras can be nondiscrete [Raedschelders and Stevenson 2019, §4.3].

The proof of Theorem 1.1 follows [Kerz 2018]. In Section 2, we define and study twisted K-theory. We kill a negative twisted K-theory class using a projective birational morphism in Section 3. Lastly, Section 4 holds the main theorem and we consider some extensions.

Conventions. We make very little use of the language of ∞-categories. For a commutative ring R, there is an equivalence of ∞-categories between the $\mathbb{E}_1$-ring spectra over HR and differential graded algebras over R localized at the quasi-isomorphisms [Lurie 2017, Proposition 7.1.4.6]. For a dg-algebra (or $\mathbb{E}_1$-ring) $\mathscr{A}$, we can consider the ∞-category $\mathrm{RMod}(\mathscr{A})$ of spectra which have a right $\mathscr{A}$-module structure. We will refer to this ∞-category as the derived category of $\mathscr{A}$ and denote it by $D(\mathscr{A})$. The subcategory $\mathrm{Perf}(\mathscr{A})$ consists of all compact objects of $\mathrm{RMod}(\mathscr{A})$, or the right $\mathscr{A}$-modules which corepresent a functor that commutes with filtered colimits. We shall refer to objects of $\mathrm{Perf}(\mathscr{A})$ as perfect complexes over $\mathscr{A}$.

We use $K(-)$ undecorated as nonconnective algebraic K-theory and consider it as a localizing invariant in the sense of Blumberg, Gepner, and Tabuada [Blumberg et al. 2013]. In particular, it is an ∞-functor from $\mathrm{Cat}_\infty^{\mathrm{perf}}$, the ∞-category of idempotent complete small stable infinity categories with exact functors, taking values in Sp, the ∞-category of spectra. For X a quasicompact quasiseparated scheme, $K(\mathrm{Perf}(X))$ is equivalent to the nonconnective K-theory spectrum of Thomason and Trobaugh [1990]. The ∞-category $\mathrm{Cat}_\infty^{\mathrm{perf}}$ has a symmetric monoidal structure which we will denote by $\widehat{\otimes}$. For R an $\mathbb{E}_\infty$-ring spectrum, $\mathrm{Perf}(R)$ is an $\mathbb{E}_\infty$ algebra in $\mathrm{Cat}_\infty^{\mathrm{perf}}$. We will restrict the domain of algebraic K-theory to $\mathrm{Mod}_{\mathrm{Perf}(R)}(\mathrm{Cat}_\infty^{\mathrm{perf}})$.

2. Twisted K-theory

In Grothendieck's original papers [1968a; 1968b; 1968c], he globalizes the notion of a central simple algebra over a field.

Definition 2.1. A locally free sheaf of $\mathscr{O}_X$-algebras $\mathscr{A}$ is a *sheaf of Azumaya algebras* if it is étale-locally isomorphic to $\mathscr{M}_n(\mathscr{O}_X)$ for some n.

An Azumaya algebra is then a PGL_n-torsor over the étale topos of X and so, by Giraud, isomorphism classes are in bijection with $H^1_{\text{ét}}(X, PGL_n)$. The central extension of sheaves of groups in the étale topology

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

leads to an exact sequence of nonabelian cohomology

$$\cdots \to H^1_{\text{ét}}(X, \mathbb{G}_m) \to H^1_{\text{ét}}(X, GL_n) \to H^1_{\text{ét}}(X, PGL_n) \to H^2_{\text{ét}}(X, \mathbb{G}_m).$$

For $d \mid n$ we have a morphism of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] & GL_n \arrow[r] & PGL_n \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[u, equal] & GL_d \arrow[r] \arrow[u] & PGL_d \arrow[r] \arrow[u] & 1
\end{tikzcd}$$

with the two right arrows given by block-summing the matrix along the diagonal n/d times. The Brauer group is the filtered colimit of cofibers

$$Br(X) := \varinjlim(\mathrm{cofib}(H^1_{\text{ét}}(X, GL_n) \to H^1_{\text{ét}}(X, PGL_n)))$$

along the partially ordered set of the natural numbers under division. This is the group of Azumaya algebras modulo Morita equivalence with group operation given by tensor product [Grothendieck 1968a]. We have an injection $Br(X) \hookrightarrow H^2_{\text{ét}}(X, \mathbb{G}_m)$ and when X is quasicompact this injection factors through the torsion subgroup. We will call $Br'(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tor}}$ the cohomological Brauer group. Grothendieck asked if the injection $Br(X) \hookrightarrow Br'(X)$ is an isomorphism.

This map is not generally surjective. Edidin, Hassett, Kresch, and Vistoli [Edidin et al. 2001] give a nonseparated counterexample by connecting the image of the Brauer group to quotient stacks. There are two ways to proceed in addressing the question. The first is to provide a class of schemes for when this holds. In [de Jong 2006], de Jong publishes a proof of O. Gabber that $Br(X) \cong Br'(X)$ when X is equipped with an ample line bundle. Along with reproving Gabber's result for affines, Lieblich [2004] shows that for a regular scheme with dimension less than or equal to 2 there are isomorphisms $Br(X) \cong Br'(X) \cong H^2_{\text{ét}}(X, \mathbb{G}_m)$.

The second perspective is to enlarge the class of objects considered. The Morita equivalence classes of $\mathbb{G}_m$-gerbes over the étale topos of a scheme X are in bijection with $H^2_{\text{ét}}(X, \mathbb{G}_m)$. Lieblich [2004] associates to any Azumaya algebra $\mathscr{A}$ a $\mathbb{G}_m$-gerbe of Morita-theoretic trivializations. Over an étale open $U \to X$, the gerbe gives a groupoid of Morita equivalences from $\mathscr{A}$ to $\mathscr{O}_X$. The gerbe of trivializations represents the boundary class $\delta([\mathscr{A}]) = \alpha \in H^2_{\text{ét}}(X, \mathbb{G}_m)$.

Any class $\alpha \in H^2_{\text{ét}}(X, \mathbb{G}_m)$ is realizable on a Čech cover. We can use this data to build a well defined category of sheaves of $\mathscr{O}_X$-modules which "glue up to α" [Căldăraru 2000, Chapter 1]. Let Mod^{α}_X denote the corresponding derived ∞-category and Perf^{α}_X the full subcategory of compact objects. $K(\text{Perf}^{\alpha}_X)$ is the classical definition of α-twisted algebraic K-theory. Determining when the cohomology class α is represented by an Azumaya algebra reduces to finding a twisted locally free sheaf with trivial determinant on a $\mathbb{G}_m$-gerbe associated to α [Lieblich 2004, §2.2.2]. The endomorphism algebra of the twisted locally free sheaf gives the Azumaya algebra and the twisted module represents the tilt $\text{Mod}^{\alpha}_X \simeq \text{Mod}_{\mathscr{A}}$.

Lieblich also compactifies the moduli of Azumaya algebras. This necessarily includes developing a definition of a derived Azumaya algebra.

Definition 2.2. A *derived Azumaya algebra* over a commutative ring R is a proper dg-algebra $\mathscr{A}$ such that the natural map of R-algebras

$$\mathscr{A} \otimes^{\mathbb{L}}_R \mathscr{A}^{\text{op}} \xrightarrow{\simeq} \mathbb{R}Hom_{D(R)}(\mathscr{A}, \mathscr{A})$$

is a quasi-isomorphism.

After Lieblich, Toën [2012] and (later) Antieau and Gepner [2014] consider the analogous problem posed by Grothendieck in the dg-algebra and $\mathbb{E}_\infty$-algebra settings, respectively. Antieau and Gepner construct an étale sheaf $\boldsymbol{Br}$ in the ∞-topos $\text{Shv}^{\text{ét}}_R$. For any étale sheaf X, we can now associate a Brauer space $\boldsymbol{Br}(X)$. For X a quasicompact quasiseparated scheme, they show $\pi_0(\boldsymbol{Br}(X)) \cong H^1_{\text{ét}}(X, \mathbb{Z}) \times H^2_{\text{ét}}(X, \mathbb{G}_m)$ and every such Brauer class is algebraic. Now for any (possibly nontorsion) $\alpha \in H^2_{\text{ét}}(X, \mathbb{G}_m)$ there is a derived Azumaya algebra $\mathscr{A}$ and an equivalence $\mathrm{Mod}_X^\alpha \simeq \mathrm{Mod}_{\mathscr{A}}$ of stable ∞-categories.

This reframes classical twisted K-theory as K-theory with coefficients in a particularly special dg-algebra in $D(X)$. For our purposes, we work with a generalized definition of twisted K-theory which allows "twisting" by any dg-algebra.

Definition 2.3. Let R be a commutative ring. For a dg-algebra $\mathscr{A}$ over R, we define the *$\mathscr{A}$-twisted K-theory* $K^{\mathscr{A}} : \mathrm{Mod}_{\mathrm{Perf}(R)}(\mathrm{Cat}_\infty^{\mathrm{perf}}) \to \mathrm{Sp}$ by

$$K^{\mathscr{A}}(\mathscr{C}) := K(\mathscr{C} \mathbin{\widehat{\otimes}}_{\mathrm{Perf}(R)} \mathrm{Perf}(\mathscr{A})).$$

When the dg-algebra "$\mathscr{A}$" is clear, we just write twisted K-theory. If our input to $K^{\mathscr{A}}$ is an R-algebra S, then

$$K^{\mathscr{A}}(S) = K(\mathrm{Perf}(S) \mathbin{\widehat{\otimes}}_{\mathrm{Perf}(R)} \mathrm{Perf}(\mathscr{A})) \simeq K(\mathrm{Perf}(S \otimes_R \mathscr{A})) \simeq K(S \otimes_R \mathscr{A}).$$

Our definition recovers the historical definition of twisted K-theory when $\mathscr{A}$ is a derived Azumaya algebra and we evaluate on the base ring R. The same definition works for a scheme X and $\mathscr{A} \in \mathrm{Alg}_{\mathbb{E}_1}(D_{\mathrm{qc}}(X))$. We will refer to such an $\mathscr{A}$ as *a sheaf of quasicoherent dg-algebras over X*. By [Blumberg et al. 2013, Theorem 9.36], twisted K-theory is a localizing invariant. When X is a quasicompact quasiseparated scheme, Clausen, Mathew, Naumann, and Noel [Clausen et al. 2020, Proposition A.15] establish Nisnevich descent when X is quasicompact quasiseparated.

Definition 2.4. A dg-algebra $\mathscr{A}$ over a ring R is *proper* if it is perfect as a complex over R and *smooth* if it is perfect over $\mathscr{A}^{\mathrm{op}} \otimes_R \mathscr{A}$.

The following is Lemma 2.8 of [Toën and Vaquié 2007] and is an essential property for our proof in Section 3.

Lemma 2.5. *Let $\mathscr{A}$ be a smooth proper dg-algebra over a ring R. Then a complex of $D(\mathscr{A})$ is perfect over $\mathscr{A}$ if and only if it is perfect as an object of $D(R)$.*

The previous definition and lemma both generalize to a sheaf of quasicoherent dg-algebras over a scheme as perfection is a local property. For the remainder of the section, we prove some basic properties of $\mathscr{A}$-twisted K-theory, often assuming $\mathscr{A}$ is connective. We will not use smooth and properness until the later sections.

Proposition 2.6. *Let $\mathscr{A}$, S be connective dg-algebras over R. Then the natural maps induce isomorphisms*

$$K_i^{\mathscr{A}}(S) \cong K_i^{\mathscr{A}}(\pi_0(S)) \cong K_i^{\pi_0(\mathscr{A})}(S) \cong K_i^{\pi_0(\mathscr{A})}(\pi_0(S))$$

for $i \leq 0$.

Proof. We have the following isomorphisms of discrete rings:

$$\pi_0(\mathscr{A} \otimes_R S) \cong \pi_0(\mathscr{A} \otimes_R \pi_0(S)) \cong \pi_0(\pi_0(\mathscr{A}) \otimes_R S) \cong \pi_0(\pi_0(\mathscr{A}) \otimes_R \pi_0(S)).$$

The lemma follows since $K_i(R) \cong K_i(\pi_0(R))$ for $i \le 0$ [Blumberg et al. 2013, Theorem 9.53]. $\square$

The previous proposition suggests we can work discretely and then transfer the results to the derived setting. This is true to some extent. However, taking π_0 of a connective dg-algebra does not preserve smoothness, which is a necessary property for our proof of Proposition 3.2. We will also need reduction invariance for low-dimensional K-groups.

Proposition 2.7. *Let R be a commutative ring and $\mathcal{A}$ a connective dg-algebra over R. Let S be a commutative ring under R, and let I be a nilpotent ideal of S. Then the induced morphism $K_i^{\mathcal{A}}(S) \xrightarrow{\cong} K_i^{\mathcal{A}}(S/I)$ is an isomorphism for $i \le 0$.*

Proof. By naturality of the fundamental exact sequence of twisted K-theory (see (†) and the surrounding discussion at the beginning of Section 3), we can restrict the proof to $K_0^{\mathcal{A}}$. By Proposition 2.6, we can assume $\mathcal{A}$ is a discrete algebra. Let $\varphi : S \twoheadrightarrow S/I$ be the surjection. After $- \otimes_R \mathcal{A}$ we have a surjection $(\ker \varphi) \otimes_R \mathcal{A} \twoheadrightarrow \ker(\varphi \otimes_R \mathcal{A})$. The nonunital ring $(\ker \varphi) \otimes_R \mathcal{A}$ is nilpotent. So $\ker(\varphi \otimes_R \mathcal{A})$ is nilpotent as well. The proposition follows from nil-invariance of K_0. $\square$

A Zariski descent spectral sequence argument gives us a global result.

Corollary 2.8. *Let X be a quasicompact quasiseparated scheme of finite Krull dimension d and $\mathcal{A}$ a sheaf of connective quasicoherent dg-algebras over X. The natural morphism $f : X_{\mathrm{red}} \to X$ induces isomorphisms*

$$K_{-i}^{f^*\mathcal{A}}(X_{\mathrm{red}}) \cong K_{-i}^{\mathcal{A}}(X)$$

for $i \ge d$.

Proof. We have descent spectral sequences

$$\begin{aligned} E_2^{p,q} &= H^p_{\mathrm{Zar}}(X, (\pi_q K^{\mathcal{A}})^\sim) &&\Rightarrow \pi_{q-p} K^{\mathcal{A}}(X), \\ E_2^{p,q} &= H^p_{\mathrm{Zar}}(X, f_*(\pi_q K^{f^*(\mathcal{A})})^\sim) &&\Rightarrow \pi_{q-p} K^{f^*\mathcal{A}}(X_{\mathrm{red}}) \end{aligned}$$

both with differential $d_2 = (2, 1)$. We let $F^\sim$ denote the Zariski sheafification of the presheaf F. The spectral sequences agree for $q \le 0$. By Corollary 3.27 of [Clausen and Mathew 2019], the spectral sequences vanishes for $p > d$. $\square$

In Theorem 4.3, we extend our main theorem across smooth affine morphisms. We will need reduction invariance in this setting.

Definition 2.9. For $f : S \to X$ a morphism of quasicompact quasiseparated schemes and $\mathcal{A}$ a sheaf of quasicoherent dg-algebras over X, the *relative $\mathcal{A}$-twisted K-theory of f* is

$$K^{\mathcal{A}}(f) := \mathrm{fib}(K^{\mathcal{A}}(X) \xrightarrow{f^*} K^{\mathcal{A}}(S)).$$

As defined, $K^{\mathcal{A}}(f)$ is a spectrum. There is an associated presheaf of spectra on the base scheme X given by $U \mapsto K^{\mathcal{A}}(f_{|U})$. This presheaf sits in a fiber sequence

$$K^{\mathcal{A}}(f) \to K^{\mathcal{A}} \to K_S^{\mathcal{A}}$$

where the presheaf $K_S^{\mathcal{A}}$ is also defined by pullback along f. Both presheaves $K^{\mathcal{A}}$ and $K_S^{\mathcal{A}}$ satisfy Nisnevich descent and so $K^{\mathcal{A}}(f)$ does as well.

Corollary 2.10. *Let $f : S \to X$ be an affine morphism of quasicompact quasiseparated schemes. Suppose X has Krull dimension d and let $\mathcal{A}$ be a sheaf of connective quasicoherent dg-algebras over X. Then the commutative diagram*

$$\begin{tikzcd} S_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d] & X_{\mathrm{red}} \arrow[d, "g"] \\ S \arrow[r, "f"] & X \end{tikzcd}$$

induces an isomorphism of relative twisted K-theory groups

$$K_{-i}^{g^*\mathcal{A}}(f_{\mathrm{red}}) \cong K_{-i}^{\mathcal{A}}(f)$$

for $i \geq d+1$.

Proof. We have two descent spectral sequences

$$\begin{aligned} E_2^{p,q} &= H^p_{\mathrm{Zar}}(X, (\pi_q K^{\mathcal{A}}(f))^{\sim}) && \Rightarrow \pi_{q-p} K^{\mathcal{A}}(f)(X), \\ E_2^{p,q} &= H^p_{\mathrm{Zar}}(X, g_*(\pi_q K^{g^*\mathcal{A}}(f_{\mathrm{red}}))^{\sim}) && \Rightarrow \pi_{q-p} K^{g^*\mathcal{A}}(f_{\mathrm{red}})(X_{\mathrm{red}}) \end{aligned}$$

with differential of degree $d = (2, 1)$ and $F^{\sim}$ the sheafification of the presheaf F. For an open affine $\operatorname{Spec} R \to X$ with pullback $\operatorname{Spec} A \to S$ we examine the morphism of long exact sequences when $q \leq 0$:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_q K^{\mathcal{A}}(R) \arrow[r] \arrow[d, "\cong"] & \pi_q K^{\mathcal{A}}(A) \arrow[r] \arrow[d, "\cong"] & \pi_{q-1} K^{\mathcal{A}}(f) \arrow[r] \arrow[d] & \pi_{q-1} K^{\mathcal{A}}(R) \arrow[r] \arrow[d, "\cong"] & \pi_{q-1} K^{\mathcal{A}}(A) \arrow[r] \arrow[d, "\cong"] & \cdots \\ \cdots \arrow[r] & \pi_q K^{\mathcal{A}}(R_{\mathrm{red}}) \arrow[r] & \pi_q K^{\mathcal{A}}(A_{\mathrm{red}}) \arrow[r] & \pi_{q-1} K^{\mathcal{A}}(f_{\mathrm{red}}) \arrow[r] & \pi_{q-1} K^{\mathcal{A}}(R_{\mathrm{red}}) \arrow[r] & \pi_{q-1} K^{\mathcal{A}}(A_{\mathrm{red}}) \arrow[r] & \cdots \end{tikzcd}$$

By the 5-lemma, this induces sheaf isomorphisms

$$g_*(\pi_q K^{g^*\mathcal{A}}(f_{\mathrm{red}}))^{\sim} \cong (\pi_q K^{\mathcal{A}}(f))^{\sim}$$

for $q < 0$ and, as in Corollary 2.8, cohomology vanishes for $p > d$. $\square$

We will need proexcision for abstract blow-up squares. Recall that an abstract blow-up square is a pullback square

$$\begin{tikzcd} D \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd} \tag{$*$}$$

with $Y \to X$ a closed immersion and $\widetilde{X} \to X$ a proper morphism which restricts to an isomorphism of open subschemes $\widetilde{X} \setminus D \to X \setminus Y$. The theorem is stated using the ∞-category of prospectra $\boldsymbol{Pro}(\mathrm{Sp})$, where an object is a small cofiltered diagram, $E : \Lambda \to \mathrm{Sp}$, valued in spectra. We write $\{E_n\}$ for the corresponding prospectrum. If the brackets and index are omitted, then the prospectrum is considered constant. After adjusting equivalence class representatives, we may assume the cofiltered diagram is fixed when working with a finite set of prospectra. Any morphism can then be represented by a natural transformation of diagrams (also known as a level map). We will need no knowledge of the ∞-category beyond the following definition.

Definition 2.11. A square of prospectra

$$\begin{tikzcd} \{E_n\} \arrow[r] \arrow[d] & \{F_n\} \arrow[d] \\ \{X_n\} \arrow[r] & \{Y_n\} \end{tikzcd}$$

is *procartesian* if and only if the induced map on the levelwise fiber prospectra is a weak equivalence [Land and Tamme 2019, Definition 2.27].

The following is Theorem A.8 of [Land and Tamme 2019]. The theorem holds much more generally for any k-connective localizing invariant [Land and Tamme 2019, Definition 2.5]. Twisted K-theory is 1-connective.

Theorem 2.12 [Land and Tamme 2019]. *Given an abstract blow-up square* $(*)$ *of schemes and a sheaf of dg-algebras* $\mathcal{A}$ *on* X, *then the square of prospectra*

$$\begin{tikzcd} K^{\mathcal{A}}(X) \arrow[r] \arrow[d] & K^{\mathcal{A}}(\widetilde{X}) \arrow[d] \\ \{K^{\mathcal{A}}(Y_n)\} \arrow[r] & \{K^{\mathcal{A}}(D_n)\} \end{tikzcd}$$

is procartesian (*where* Y_n *is the infinitesimal thickening of* Y).

The procartesian square of prospectra gives a long exact sequence of progroups

$$\cdots \to \{K^{\mathcal{A}}_{-i+1}(E_n)\} \to K^{\mathcal{A}}_{-i}(X) \to K^{\mathcal{A}}_{-i}(\widetilde{X}) \oplus \{K^{\mathcal{A}}_{-i}(Y_n)\} \to \{K^{\mathcal{A}}_{-i}(E_n)\} \to \cdots$$

which is the key to our induction argument.

3. Blowing up negative twisted K-theory classes

We turn to our main contribution of the existence of a projective birational morphism which kills a given negative twisted K-theory class (when twisting by a smooth proper connective dg-algebra). Let X be a quasicompact quasiseparated scheme and $\mathscr{A}$ a sheaf of quasicoherent dg-algebras on X. We first construct geometric cycles for negative twisted K-theory classes on X using a classical argument of Bass [1968, §XII.7] which works for a general additive invariant. We have an open cover

$$\begin{tikzcd} X[t^{\pm}] \arrow[r, "f"] \arrow[d, "g"] & X[t^{-}] \arrow[d, "j"] \\ X[t] \arrow[r, "k"] & \mathbb{P}^1_X \end{tikzcd}$$

Since twisted K-theory satisfies Zariski descent, there is an associated Mayer–Vietoris sequence of homotopy groups

$$\cdots \to K^{\mathscr{A}}_{-n}(\mathbb{P}^1_X) \xrightarrow{(j^*k^*)} K^{\mathscr{A}}_{-n}(X[t]) \oplus K^{\mathscr{A}}_{-n}(X[t^{-}]) \\ \xrightarrow{f^*-g^*} K^{\mathscr{A}}_{-n}(X[t^{\pm}]) \xrightarrow{\partial} K^{\mathscr{A}}_{-n-1}(\mathbb{P}^1_X) \to \cdots .$$

As an additive invariant, $K^{\mathscr{A}}(\mathbb{P}^1_X) \simeq K^{\mathscr{A}}(X) \oplus K^{\mathscr{A}}(X)$ splits as a $K^{\mathscr{A}}(X)$-module with generators

$$[\mathscr{O} \otimes_{\mathscr{O}_X} \mathscr{A}] = [\mathscr{A}] \quad \text{and} \quad [\mathscr{O}(1) \otimes_{\mathscr{O}_X} \mathscr{A}] = [\mathscr{A}(1)]$$

corresponding to the Beilinson semiorthogonal decomposition. Adjusting the generators to $[\mathscr{A}]$ and $[\mathscr{A}] - [\mathscr{A}(1)]$, we can identify the map (j^*, k^*) as it is a map of $K^{\mathscr{A}}(X)$-modules. The second generator vanishes under each restriction. This identifies the map as

$$K^{\mathscr{A}}(\mathbb{P}^1_X) \simeq K^{\mathscr{A}}(X)[\mathscr{A}] \oplus K^{\mathscr{A}}(X)([\mathscr{A}] - [\mathscr{A}(1)]) \xrightarrow{\Delta \oplus 0} K^{\mathscr{A}}(X[t]) \oplus K^{\mathscr{A}}(X[t^{-}])$$

with Δ the diagonal map corresponding to pulling back along the projections $X[t] \to X$ and $X[t^{-}] \to X$. As Δ is an embedding the long exact sequence splits as

$$0 \to K^{\mathscr{A}}_{-n}(X) \xrightarrow{\Delta} K^{\mathscr{A}}_{-n}(X[t]) \oplus K^{\mathscr{A}}_{-n}(X[t^{-}]) \xrightarrow{\pm} K^{\mathscr{A}}_{-n}(X[t^{\pm}]) \xrightarrow{\partial} K^{\mathscr{A}}_{-n-1}(X) \to 0. \quad (\dagger)$$

After iterating the complex

$$K^{\mathscr{A}}_{-n}(X[t]) \to K^{\mathscr{A}}_{-n}(X[t^{\pm}]) \twoheadrightarrow K^{\mathscr{A}}_{-n-1}(X),$$

we can piece together a complex

$$K^{\mathscr{A}}_{0}(\mathbb{A}^{n+1}_X) \to K^{\mathscr{A}}_{0}(\mathbb{G}^{n+1}_{m,X}) \twoheadrightarrow K^{\mathscr{A}}_{-n-1}(X).$$

Negative twisted K-theory classes have geometric representations as twisted perfect complexes on $\mathbb{G}^i_{m,X}$. There is even a sufficient geometric criterion implying a given representative is 0; it is the restriction of a twisted perfect complex on $\mathbb{A}^i_X$. Our proof of the main proposition of this section will use these representatives. We first need a lemma about extending finitely generated discrete modules in a twisted setting.

Lemma 3.1. *Let $j : U \to X$ be an open immersion of quasicompact quasiseparated schemes. Let $\mathscr{A}$ be a sheaf of proper connective quasicoherent dg-algebras on X and $j^*\mathscr{A}$ its restriction. Let $\mathscr{N}$ be a discrete $j^*\mathscr{A}$-module which is finitely generated as an $\mathscr{O}_U$-module. Then there exists a discrete $\mathscr{A}$-module $\mathscr{M}$, finitely generated over $\mathscr{O}_X$, such that $j^*\mathscr{M} \cong \mathscr{N}$.*

Proof. Note that $H_{\geq 1}(j^*\mathscr{A})$ necessarily acts trivially on $\mathscr{N}$. So the $j^*\mathscr{A}$-module structure on $\mathscr{N}$ comes from forgetting along the map $j^*\mathscr{A} \to H_0(j^*\mathscr{A})$ and the natural $H_0(j^*\mathscr{A})$-module structure. Under restriction,

$$j^* H_0(\mathscr{A}) \cong H_0(j^*\mathscr{A}).$$

We reduce to when $\mathscr{A}$ is a quasicoherent sheaf of discrete $\mathscr{O}_X$-algebras, finite over the structure sheaf. We have an isomorphism $\mathscr{N} \cong j^* j_* \mathscr{N}$. Write $j_*\mathscr{N}$ as a filtered colimit of its finitely generated $\mathscr{A}$-submodules $j_*\mathscr{N} \cong \varinjlim_\lambda \mathscr{M}_\lambda$. The pullback is exact, so we can write $\mathscr{N} \cong \varinjlim_\lambda j^*\mathscr{M}_\lambda$ as a filtered colimit of finitely generated submodules. As $\mathscr{N}$ is finitely generated itself, this isomorphism factors at some stage and $\mathscr{N} \cong j^*\mathscr{M}_\lambda$. □

Proposition 3.2. *Let X be a reduced scheme which is quasiprojective over a Noetherian affine scheme. Let $\mathscr{A}$ be a sheaf of smooth proper connective quasicoherent dg-algebras on X. Let $\gamma \in K^{\mathscr{A}}_{-i}(X)$ for $i > 0$. Then there is a projective birational morphism $\rho : \widetilde{X} \to X$ so that $\rho^*\gamma = 0 \in K^{\mathscr{A}}_{-i}(\widetilde{X})$.*

Proof. We fix a diagram of schemes over X

$$\begin{tikzcd} \mathbb{G}^i_{m,X} \arrow[rr, "j"] \arrow[dr, "\pi_1"'] & & \mathbb{A}^i_X \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

For any morphism $f : Y_1 \to Y_2$, we let $\tilde{f} : \mathbb{G}^i_{m,Y_1} \to \mathbb{G}^i_{m,Y_2}$ denote the pullback. Lift γ to a $K^{\mathscr{A}}_0(\mathbb{G}^i_{m,X})$-class $[P_\bullet]$, with $P_\bullet$ some $\pi_1^*\mathscr{A}$-twisted perfect complexes on $\mathbb{G}^i_{m,X}$.

The induction step. We induct on the range of homology of $P_\bullet$. As $\pi_1^*\mathscr{A}$ is a sheaf of proper quasicoherent dg-algebras, $P_\bullet$ is perfect on $\mathbb{G}^i_{m,X}$ by Lemma 2.5. Since $\mathbb{G}^i_{m,X}$ has an ample family of line bundles, we may choose $P_\bullet$ to be strict perfect without changing the quasi-isomorphism class. After some (de)suspension, we may assume $P_\bullet$ is connective as this only alters the K_0-class by ± 1. For the lowest nontrivial differential of $P_\bullet$, d_1, we utilize part (iv) of Lemma 6.5 of [Kerz et al. 2018] (with the morphism $\mathbb{G}^i_{m,X} \to X$) to construct a projective birational morphism $\rho : X_1 \to X$ so that $\operatorname{coker}(\tilde{\rho}^* d_1)$ $(= H_0(\tilde{\rho}^* P_\bullet))$ has tor-dimension ≤ 1 over X_1. Consider the distinguished triangle of $\tilde{\rho}^*\pi_1^*\mathcal{A}$-complexes on $\mathbb{G}^i_{m,X_1}$

$$F_\bullet \to \tilde{\rho}^* P_\bullet \to H_0(\tilde{\rho}^* P_\bullet) \cong \operatorname{coker} \tilde{\rho}^* d_1.$$

In Lemma 3.3 below, we cover the base induction step, when the homology is concentrated in a single degree. Using this, construct a projective birational morphism $\phi : X_2 \to X_1$ such that $L\tilde{\phi}^* H_0(\tilde{\rho}^* P_\bullet)$ is a perfect complex and is the restriction of a perfect complex from $\mathbb{A}^i_{X_2}$. By two out of three, $L\tilde{\phi}^* F_\bullet$ is perfect and $[\tilde{\phi}^*\tilde{\rho}^* P_\bullet] = [L\tilde{\phi}^* F_\bullet] + [L\tilde{\phi}^* H_0(\tilde{\rho}^* P_\bullet)]$ in $K_0^{\mathcal{A}}(\mathbb{G}^i_{m,X_2})$. We then repeat the entire induction step with $L\tilde{\phi}^* F_\bullet$.

We need the induction to terminate, which is the purpose of the first projective birational morphism of each step. Since $\operatorname{coker}(\tilde{\rho}^* d_1)$ has tor-dimension ≤ 1 over X_1, by [Kerz et al. 2018, Lemma 6.5], $L\tilde{\phi}^* \operatorname{coker}(\tilde{\rho}^* d_1) \cong \tilde{\phi}^* \operatorname{coker}(\tilde{\rho}^* d_1)$. This implies $L\tilde{\phi}^* F_\bullet$ will have no homology outside the original range of homology of $P_\bullet$. Since $\tilde{\phi}^* \operatorname{coker}(\tilde{\rho}^* d_1) \cong \operatorname{coker}(\tilde{\phi}^*\tilde{\rho}^* d_1)$, this guarantees $H_0(L\tilde{\phi}^* F_\bullet) = 0$, so the homology of $L\tilde{\phi}^* F_\bullet$ lies in a strictly smaller range than $\tilde{\phi}^*\tilde{\rho}^* P_\bullet$. Proposition 3.2 follows from the next lemma. $\square$

Lemma 3.3. *Let X be a reduced scheme which is quasiprojective over a Noetherian affine scheme. Let $\mathcal{A}$ be a sheaf of smooth proper connective quasicoherent dg-algebras on X. Let $\mathcal{N}$ be a discrete $\pi_1^*\mathcal{A}$-module which is coherent on $\mathbb{G}^i_{m,X}$. Then there exists a birational blow-up $\phi : \widetilde{X} \to X$ so that $\tilde{\phi}^*\mathcal{N}$ is perfect over $\tilde{\phi}^*\pi_1^*\mathcal{A}$ on $\mathbb{G}_{m,\widetilde{X}}$ and is the restriction of a perfect complex over the pullback of $\mathcal{A}$ to $\mathbb{A}^i_{\widetilde{X}}$.*

Proof. Using Lemma 3.1, extend $\mathcal{N}$ from $\mathbb{G}^i_{m,X}$ to a coherent $\pi_2^*\mathcal{A}$-module $\mathcal{M}$ on $\mathbb{A}^i_X$. Using the ample family, choose a resolution in $\mathcal{O}_{\mathbb{A}^i_X}$-modules of the form

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{M} \to 0$$

where $\mathcal{F}$ is a vector bundle and $\mathcal{K}$ is the coherent kernel. As X is reduced, $\mathcal{K}$ is flat over some dense open set U of X. By platification par éclatement [Raynaud and Gruson 1971, Theorem 5.2.2], there is a U-admissible blow-up $\phi : \widetilde{X} \to X$ so that the strict transform of $\mathcal{K}$ along the pullback morphism $p : \mathbb{A}^i_{\widetilde{X}} \to \mathbb{A}^i_X$ is flat over $\widetilde{X}$.

We now show the pullback $p^*\mathcal{M}$ is perfect as a $p^*\pi_2^*\mathcal{A}$-module. Let $j : \mathbb{A}^i_U \to \mathbb{A}^i_{\widetilde{X}}$ be the inclusion of the open set and Z the closed complement. For any sheaf of modules $\mathcal{G}$ on $\mathbb{A}^i_{\widetilde{X}}$, we let $\mathcal{G}_Z$ denote the subsheaf of sections supported on Z. We have a short exact sequence natural in $\mathcal{G}$

$$0 \to \mathcal{G}_Z \to \mathcal{G} \to j^{\mathrm{st}}\mathcal{G} \to 0.$$

We also obtain the following exact sequence of sheaves of abelian groups via pullback:

$$0 \to \mathcal{T}or_1^{p^{-1}\mathcal{O}_{\mathbb{A}^i_X}}(p^{-1}\mathcal{M}, \mathcal{O}_{\mathbb{A}^i_{\widetilde{X}}}) \to p^*\mathcal{K} \to p^*\mathcal{F} \to p^*\mathcal{M} \to 0.$$

To make our notation clearer, we set $\mathscr{T} = \mathscr{T}or_1^{p^{-1}\mathscr{O}_{\mathbb{A}^i_X}}(p^{-1}\mathcal{M}, \mathscr{O}_{\mathbb{A}^i_{\widetilde{X}}})$. We flesh both these exact sequences out into a (nonexact) commutative diagram of $p^{-1}\mathscr{O}_{\mathbb{A}^i_X}$-modules

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{T}_Z \arrow[r] \arrow[d] & \mathscr{T} \arrow[r] \arrow[d] & j^{\mathrm{st}}\mathscr{T} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (p^*\mathscr{K})_Z \arrow[r] \arrow[d] & p^*\mathscr{K} \arrow[r] \arrow[d] & j^{\mathrm{st}}p^*\mathscr{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (p^*\mathscr{F})_Z \arrow[r] \arrow[d] & p^*\mathscr{F} \arrow[r] \arrow[d] & j^{\mathrm{st}}p^*\mathscr{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (p^*\mathcal{M})_Z \arrow[r] \arrow[d] & p^*\mathcal{M} \arrow[r] \arrow[d] & j^{\mathrm{st}}p^*\mathcal{M} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We observe that every row and the middle column is exact. The first map in the left column is an injection and the last map in the right column is a surjection. Since $p^*\mathscr{F}$ is flat, we have $(p^*\mathscr{F})_Z = 0$. This induces a lifting of the injection

$$\begin{tikzcd}
\mathscr{T}_Z \arrow[r] \arrow[d] & \mathscr{T} \arrow[d] \\
(p^*\mathscr{K})_Z \arrow[r] \arrow[ur, dashed] & p^*\mathscr{K}
\end{tikzcd}$$

We finish the proof by showing $j^*\mathscr{T}or_1^{p^{-1}\mathscr{O}_{\mathbb{A}^i_X}}(p^{-1}\mathcal{M}, \mathscr{O}_{\mathbb{A}^i_{\widetilde{X}}}) = 0$. Since $j : \mathbb{A}^i_U \to \mathbb{A}^i_{\widetilde{X}}$ is flat, the sheaf is isomorphic to $\mathscr{T}or_1^{\mathbb{A}^i_U}(j^*p^{-1}\mathcal{M}, j^*\mathscr{O}_{\mathbb{A}^i_{\widetilde{X}}})$ and $j^*\mathscr{O}_{\mathbb{A}^i_{\widetilde{X}}} \cong \mathscr{O}_{\mathbb{A}^i_U}$. Our big diagram can be rewritten as

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{T}_Z \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathscr{T} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (p^*\mathscr{K})_Z \arrow[r] \arrow[d] & p^*\mathscr{K} \arrow[r] \arrow[d] & j^{\mathrm{st}}p^*\mathscr{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & p^*\mathscr{F} \arrow[r] \arrow[d] & j^{\mathrm{st}}p^*\mathscr{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (p^*\mathcal{M})_Z \arrow[r] \arrow[d] & p^*\mathcal{M} \arrow[r] \arrow[d] & j^{\mathrm{st}}p^*\mathcal{M} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and we can glue together to get a flat resolution of $p^*\mathcal{M}$ as an $\mathcal{O}_{\mathbb{A}^i_{\widetilde{X}}}$-module

$$0 \to j^{\mathrm{st}} p^*\mathcal{K} \to p^*\mathcal{F} \to p^*\mathcal{M} \to 0$$

implying globally finite Tor-amplitude. It remains to show the complex is pseudo-coherent. This follows since $\mathbb{A}^i_{\widetilde{X}}$ is Noetherian and $p^*\mathcal{M}$ is coherent. Since $p^*\pi_2^*\mathcal{A}$ is a sheaf of smooth quasicoherent dg-algebras over $\mathcal{O}_{\mathbb{A}^i_{\widetilde{X}}}$, the complex $p^*\mathcal{M}$ is perfect over $p^*\pi_2^*\mathcal{A}$ by Lemma 2.5. By commutativity, $p^*\mathcal{M}$ restricts to $\tilde{\phi}^*\mathcal{N}$ on $\mathbb{G}^i_{m,\widetilde{X}}$. This completes the proof of Proposition 3.2. □

We will need a relative version of Proposition 3.2.

Corollary 3.4. *Let $f : S \to X$ be a smooth quasiprojective morphism of Noetherian schemes with X reduced and quasiprojective over a Noetherian base ring. Let $\mathcal{A}$ be a sheaf of smooth proper connective quasicoherent dg-algebras over X and consider a negative twisted K-theory class $\gamma \in K_i^{\mathcal{A}}(S)$ for $i < 0$. Then there exists a projective birational morphism $\rho : \widetilde{X} \to X$ such that, under the pullback of the pullback morphism, $\rho_S^*\gamma = 0$.*

Proof. We will briefly check that we can run the induction argument in the proof of Proposition 3.2. The assumptions of this corollary are invariant under pullback along projective birational morphisms $\widetilde{X} \to X$. We need to ensure we can select projective birational morphisms to our base X. Lemma 6.5 of [Kerz et al. 2018] is stated in a relative setting. The proof also relies on platification par éclatement. This can still be applied in our relative setting as X is reduced [Kerz and Strunk 2017, Proposition 5]. □

4. Twisted Weibel's conjecture

We now prove Theorem 1.1 and an extension across a smooth affine morphism. We begin with the base induction step for both theorems. Kerz and Strunk [2017] use a sheaf cohomology result of Grothendieck along with a spectral sequence argument to show vanishing for a Zariski sheaf of spectra can be reduced to the setting of local ring.

Proposition 4.1. *Let R be a regular Noetherian ring of Krull dimension d over a local Artinian ring k. Let $\mathcal{A}$ be a smooth proper connective dg-algebra over R; then $K_i^{\mathcal{A}}(R) = 0$ for $i < 0$.*

Proof. By Corollary 2.10, we may assume k is a field. Proposition 5.4 of [Raedschelders and Stevenson 2019] shows that the t-structure on $D(\mathcal{A})$ restricts to a t-structure on $\mathrm{Perf}(\mathcal{A})$, which is observably bounded. The heart is the category of finitely generated modules over $H_0(\mathcal{A})$. As $H_0(\mathcal{A})$ is finite-dimensional over k, this is a Noetherian abelian category. By Theorem 1.2 of Antieau, Gepner, and Heller [Antieau et al. 2019], the negative K-theory vanishes. □

Theorem 1.1. *Let X be a Noetherian scheme of Krull dimension d and $\mathcal{A}$ a sheaf of smooth proper connective quasicoherent dg-algebras on X; then $K^{\mathcal{A}}_{-i}(X)$ vanishes for $i > d$.*

Proof. Proposition 4.1 covers the base case so assume $d > 0$. By the Kerz–Strunk spectral sequence argument and Corollary 2.8, we may assume X is a Noetherian reduced affine scheme.

Choose a negative $K^{\mathcal{A}}$-theory class $\gamma \in K^{\mathcal{A}}_{-i}(X)$ for $i \geq \dim X + 1$. Using Proposition 3.2, construct a projective birational morphism that kills γ and extend it to an abstract blow-up square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \widetilde{X} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

By [Land and Tamme 2019, Theorem A.8], there is a Mayer–Vietoris exact sequence of progroups

$$\cdots \to \{K^{\mathcal{A}}_{-i+1}(E_n)\} \to K^{\mathcal{A}}_{-i}(X) \to K^{\mathcal{A}}_{-i}(\widetilde{X}) \oplus \{K^{\mathcal{A}}_{-i}(Y_n)\} \to \{K^{\mathcal{A}}_{-i}(E_n)\} \to \cdots .$$

When $i \geq \dim X + 1$, by induction every nonconstant progroup vanishes and $K^{\mathcal{A}}_{-i}(X) \cong K^{\mathcal{A}}_{-i}(\widetilde{X})$ showing $\gamma = 0$. □

By [Antieau and Gepner 2014, Theorem 3.15], we recover Weibel's vanishing for discrete Azumaya algebras.

Corollary 4.2. *For X a Noetherian d-dimensional scheme and $\mathcal{A}$ a quasicoherent sheaf of discrete Azumaya algebras, then $K^{\mathcal{A}}_{-i}(X) = 0$ for $i > d$.*

The next result nearly covers the K-regularity portion of Weibel's conjecture, but we are missing the boundary case $K^{\mathcal{A}}_{-d}(X) \cong K^{\mathcal{A}}_{-d}(\mathbb{A}^n_X)$.

Theorem 4.3. *Let $f : S \to X$ be a smooth affine morphism of Noetherian schemes and $\mathcal{A}$ a sheaf of smooth proper connective quasicoherent dg-algebras on X. Then $K^{\mathcal{A}}_{-i}(f) = 0$ for $i > \dim X + 1$.*

Proof. The base case is covered by Proposition 4.1 and our reductions are analogous to those in the proof of Theorem 1.1. So assume X is a Noetherian reduced affine scheme of dimension d. Choose $\gamma \in K^{\mathcal{A}}_{-i}(S)$ with $i > d$. Using Corollary 3.4, construct a projective birational morphism $\rho : \widetilde{X} \to X$ that kills γ. We then build a morphism of abstract blow-up squares

$$\begin{tikzcd} D \arrow[rr] \arrow[dr] \arrow[dd] & & \tilde{S} \arrow[dr] \arrow[dd] & \\ & E \arrow[rr] \arrow[dd] & & \widetilde{X} \arrow[dd] \\ V \arrow[rr] \arrow[dr] & & S \arrow[dr] & \\ & Y \arrow[rr] & & X \end{tikzcd}$$

By Theorem 2.12, we again get a long exact sequence of progroups corresponding to the back square

$$\cdots \to \{K^{\mathscr{A}}_{-i+1}(D_n)\} \to K^{\mathscr{A}}_{-i}(S) \to K^{\mathscr{A}}_{-i}(\tilde{S}) \oplus \{K^{\mathscr{A}}_{-i}(V_n)\} \to \{K^{\mathscr{A}}_{-i}(D_n)\} \to \cdots .$$

When $i \geq \dim X + 1$, every nonconstant progroup vanishes by induction and we have an isomorphism $K^{\mathscr{A}}_{-i}(S) \cong K^{\mathscr{A}}_{-i}(\tilde{S})$ implying $\gamma = 0$. □

Remark 4.4. The conditions on the morphism in Corollary 3.4 are more general than those of Theorem 4.3. We might hope to generalize Theorem 4.3 to a smooth quasiprojective or smooth projective map of Noetherian schemes. Although the induction step is present, both base cases fail. Consider the descent spectral sequence

$$E_2^{p,q} := H^p(X, \widetilde{K}_q) \Rightarrow K_{q-p}(X) \quad \text{with } d_2 = (2, 1).$$

If $\dim X \leq 3$, then

$$E_3^{2,1} = E_\infty^{2,1} = \operatorname{coker}(H^0(X, \mathbb{Z}) \xrightarrow{d_2} H^2(X, \mathscr{O}_X^*))$$

contributes to $K_{-1}(X)$. The differential is zero as the edge morphism

$$K_0(X) \xrightarrow{\text{rank}} E_\infty^{0,0}$$

identifies $E_\infty^{0,0}$ with the rank component of K_0, implying $E_2^{0,0} = E_\infty^{0,0}$. We now construct a family of examples for schemes X with nontrivial $H^2(X, \mathscr{O}_X^*)$. Let X_{red} be quasiprojective smooth over a field k and form the cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X_{\text{red}} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(k[t]/(t^2)) & \longrightarrow & \operatorname{Spec} k \end{array}$$

The pullback X will be our counterexample. We have an isomorphism

$$\mathscr{O}_X^* \cong g_*(\mathscr{O}_{X_{\text{red}}}^*) \oplus g_*(\mathscr{O}_{X_{\text{red}}})$$

of sheaves of abelian groups on X with $g : X_{\text{red}} \to X$ the pullback of the reduction morphism $\operatorname{Spec} k \to \operatorname{Spec} k[t]/(t^2)$. Locally, $(R[t]/(t^2))^\times$ consists of all elements of the form $u + v \cdot t$ where $u \in R^\times$ and $v \in R$. Sheaf cohomology commutes with coproducts, so this turns into an isomorphism

$$\begin{aligned} H^2(X, \mathscr{O}_X^*) &\cong H^2(X, g_*(\mathscr{O}_{X_{\text{red}}}^*)) \oplus H^2(X, g_*(\mathscr{O}_{X_{\text{red}}})) \\ &\cong H^2(X_{\text{red}}, \mathscr{O}_{X_{\text{red}}}^*) \oplus H^2(X_{\text{red}}, \mathscr{O}_{X_{\text{red}}}). \end{aligned}$$

Now the problem reduces to finding a surface or 3-fold X_{red} with nontrivial degree-2 sheaf cohomology. Take a smooth quartic in $\mathbb{P}_k^3$ for a counterexample which is smooth and proper. Here is a counterexample which is smooth and quasiaffine. Let $(A, \mathfrak{m})$ be a 3-dimensional local ring which is smooth over a field k. Take $X = \operatorname{Spec} A \setminus \{\mathfrak{m}\}$ to be the punctured spectrum. Then $H^2(X, \mathscr{O}_X) \cong H^3_{\mathfrak{m}}(A)$, which is the injective hull of the residue field $A/\mathfrak{m}$.

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References

[Antieau and Gepner 2014] B. Antieau and D. Gepner, "Brauer groups and étale cohomology in derived algebraic geometry", *Geom. Topol.* **18**:2 (2014), 1149–1244. MR Zbl

[Antieau et al. 2019] B. Antieau, D. Gepner, and J. Heller, "K-theoretic obstructions to bounded t-structures", *Invent. Math.* **216**:1 (2019), 241–300. MR Zbl

[Bass 1968] H. Bass, *Algebraic K-theory*, W. A. Benjamin, New York, 1968. MR Zbl

[Blumberg et al. 2013] A. J. Blumberg, D. Gepner, and G. Tabuada, "A universal characterization of higher algebraic K-theory", *Geom. Topol.* **17**:2 (2013), 733–838. MR Zbl

[Căldăraru 2000] A. H. Căldăraru, *Derived categories of twisted sheaves on Calabi–Yau manifolds*, Ph.D. thesis, Cornell University, Ithaca, NY, 2000, https://search.proquest.com/docview/304592237. MR

[Clausen and Mathew 2019] D. Clausen and A. Mathew, "Hyperdescent and étale K-theory", preprint, 2019. arXiv

[Clausen et al. 2020] D. Clausen, A. Mathew, N. Naumann, and J. Noel, "Descent in algebraic K-theory and a conjecture of Ausoni–Rognes", *J. Eur. Math. Soc.* **22**:4 (2020), 1149–1200. MR Zbl

[Edidin et al. 2001] D. Edidin, B. Hassett, A. Kresch, and A. Vistoli, "Brauer groups and quotient stacks", *Amer. J. Math.* **123**:4 (2001), 761–777. MR Zbl

[Grothendieck 1968a] A. Grothendieck, "Le groupe de Brauer, I: Algèbres d'Azumaya et interprétations diverses", pp. 46–66 in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, North-Holland, Amsterdam, 1968. MR Zbl

[Grothendieck 1968b] A. Grothendieck, "Le groupe de Brauer, II: Théorie cohomologique", pp. 67–87 in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, North-Holland, Amsterdam, 1968. MR

[Grothendieck 1968c] A. Grothendieck, "Le groupe de Brauer, III: Exemples et compléments", pp. 88–188 in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, North-Holland, Amsterdam, 1968. MR Zbl

[de Jong 2006] A. J. de Jong, "A result of Gabber", preprint, 2006, https://www.math.columbia.edu/~dejong/papers/2-gabber.pdf.

[Kerz 2018] M. Kerz, "On negative algebraic K-groups", pp. 181–191 in *Proceedings of the International Congress of Mathematicians* (Rio de Janeiro, 2018), vol. II: Invited lectures, edited by B. Sirakov et al., World Scientific, Hackensack, NJ, 2018. MR

[Kerz and Strunk 2017] M. Kerz and F. Strunk, "On the vanishing of negative homotopy K-theory", *J. Pure Appl. Algebra* **221**:7 (2017), 1641–1644. MR Zbl

[Kerz et al. 2018] M. Kerz, F. Strunk, and G. Tamme, "Algebraic K-theory and descent for blow-ups", *Invent. Math.* **211**:2 (2018), 523–577. MR Zbl

[Land and Tamme 2019] M. Land and G. Tamme, "On the K-theory of pullbacks", *Ann. of Math.* (2) **190**:3 (2019), 877–930. MR Zbl

[Lieblich 2004] M. Lieblich, *Moduli of twisted sheaves and generalized Azumaya algebras*, Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, MA, 2004, https://search.proquest.com/docview/305088000. MR

[Lurie 2017] J. Lurie, "Higher algebra", preprint, 2017, https://www.math.ias.edu/~lurie/papers/HA.pdf.

[Quillen 1973] D. Quillen, "Higher algebraic K-theory, I", pp. 85–147 in *Algebraic K-theory* (Seattle, 1972), vol. I: Higher K-theories, edited by H. Bass, Lecture Notes in Math. **341**, Springer, 1973. MR Zbl

[Raedschelders and Stevenson 2019] T. Raedschelders and G. Stevenson, "The motive of a smooth proper connective DG-algebra is of unit type", preprint, 2019. arXiv

[Raynaud and Gruson 1971] M. Raynaud and L. Gruson, "Critères de platitude et de projectivité: techniques de 'platification' d'un module", *Invent. Math.* **13** (1971), 1–89. MR Zbl

[Thomason and Trobaugh 1990] R. W. Thomason and T. Trobaugh, "Higher algebraic K-theory of schemes and of derived categories", pp. 247–435 in *The Grothendieck Festschrift*, vol. III, edited by P. Cartier et al., Progr. Math. **88**, Birkhäuser, Boston, 1990. MR Zbl

[Toën 2012] B. Toën, "Derived Azumaya algebras and generators for twisted derived categories", *Invent. Math.* **189**:3 (2012), 581–652. MR Zbl

[Toën and Vaquié 2007] B. Toën and M. Vaquié, "Moduli of objects in dg-categories", *Ann. Sci. École Norm. Sup.* (4) **40**:3 (2007), 387–444. MR Zbl

[Weibel 1980] C. A. Weibel, "K-theory and analytic isomorphisms", *Invent. Math.* **61**:2 (1980), 177–197. MR Zbl

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Positive scalar curvature metrics via end-periodic manifolds

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We obtain two types of results on positive scalar curvature metrics for compact spin manifolds that are even-dimensional. The first type of result are obstructions to the existence of positive scalar curvature metrics on such manifolds, expressed in terms of end-periodic eta invariants that were defined by Mrowka, Ruberman and Saveliev (Mrowka et al. 2016). These results are the even-dimensional analogs of the results by Higson and Roe (2010). The second type of result studies the number of path components of the space of positive scalar curvature metrics modulo diffeomorphism for compact spin manifolds that are even-dimensional, whenever this space is nonempty. These extend and refine certain results in (Botvinnik and Gilkey 1995) and also (Mrowka et al. 2016). End-periodic analogs of K-homology and bordism theory are defined and are utilised to prove many of our results.

1. Introduction

Eta invariants were originally introduced by Atiyah, Patodi and Singer [Atiyah et al. 1975a; 1975b; 1976] as a correction term appearing in an index theorem for manifolds with odd-dimensional boundary. The eta invariant itself is a rather sensitive object, being defined in terms of the spectrum of a Dirac operator. However,

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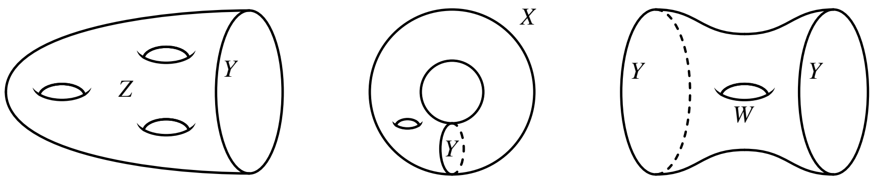

Figure 1. Pieces of an end-periodic manifold.

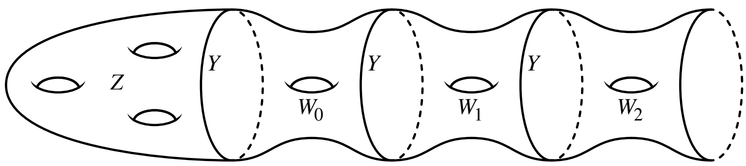

Figure 2. End-periodic manifold, where $Z_\infty = Z \bigcup_Y W_0 \bigcup_Y W_1 \bigcup_Y W_2 \cdots$.

when one considers the *relative* eta invariant (or *rho invariant*), defined by twisting the Dirac operator by a pair of flat vector bundles and subtracting the resulting eta invariants, many marvellous invariance properties emerge. For example, Atiyah, Patodi and Singer showed that the mod $\mathbb{Z}$ reduction of the relative eta invariant of the signature operator is in fact independent of the choice of Riemannian metric on the manifold. Key to the approach is their index theorem for even-dimensional manifolds with global boundary conditions, which they show is equivalent to studying manifolds with cylindrical ends and imposing (weighted) L^2 decay conditions.

The links between eta invariants and metrics of positive scalar curvature metrics have been studied using different approaches by Mathai [1992a; 1992b], Keswani [1999] and Weinberger [1988]. A conceptual proof of the approach by Keswani, was achieved by Higson and Roe [2010] using K-homology; see also [Deeley and Goffeng 2016; Benameur and Mathai 2013; 2014; 2015; Piazza and Schick 2007a; 2007b].

Our goal in this paper to use the results of Mrowka, Ruberman and Saveliev [Mrowka et al. 2016] instead of those by Atiyah, Patodi and Singer [Atiyah et al. 1975a]. Manifolds with cylindrical ends studied in [Atiyah et al. 1975a] are special cases of end-periodic manifolds studied in [Mrowka et al. 2016]. More precisely, let Z be a compact manifold with boundary Y and suppose that Y is a connected submanifold of a compact oriented manifold X that is Poincaré dual to a primitive cohomology class $\gamma \in H^1(X, \mathbb{Z})$. Let W be the fundamental segment obtained by cutting X open along Y (Figure 1).

If W_k are isometric copies of W, then we can attach $X_1 = \bigcup_{k\geq 0} W_k$ to the boundary component Y of Z, forming the *end-periodic manifold* Z_∞ (Figure 2). Often in the paper, we also deal with manifolds with more than one periodic end.

The motivations for considering such manifolds are from gauge theory; it was Taubes [1987] who originally developed the analysis of end-periodic elliptic operators on end-periodic manifolds, and successfully calculated the index of the end-periodic antiself dual operator in Yang–Mills theory.

We adapt the results by Higson and Roe [2010], using end-periodic K-homology, to obtain obstructions to the existence of positive scalar curvature metrics in terms of end-periodic eta invariants (see Section 3) that were defined by Mrowka, Ruberman and Saveliev [Mrowka et al. 2016] for even-dimensional manifolds, using the b-trace approach of Melrose [1993]. These obstructions are for the compact manifold X, and not the end-periodic manifold Z_∞; the end-periodic manifold is only a tool used to obtain the obstructions. This is established in Section 6. Roughly speaking, end-periodic K-homology is an analog of geometric K-homology, where the representatives have in addition, a choice of degree 1 cohomology class determining the codimension 1 submanifold. It is defined and studied in Section 2.

We also adapt the results by Botvinnik and Gilkey [1995], using end-periodic bordism, to obtain results on the number of components of the moduli space of Riemannian metrics of positive scalar curvature metrics in terms of end-periodic eta invariants. Such results have been obtained by Mrowka, Ruberman and Saveliev [Mrowka et al. 2016], and the introduction of end-periodic bordism provides a conceptualisation of their approach. Again, the information on path components is for the compact manifold X, and the end-periodic manifold is but a means to obtaining this information. End-periodic bordism is defined and studied in Section 4.

In Section 5 we define the end-periodic analogs of the structure groups of Higson and Roe, and study the end-periodic rho invariant on these groups.

Section 6 contains applications to positive scalar curvature, using the established end-periodic K-theory and end-periodic spin bordism of the previous sections.

In Section 7 we give a proof of the vanishing of the end-periodic rho invariant of the twisted Dirac operator with coefficients in a flat Hermitian vector bundle on a compact even dimensional Riemannian spin manifold X of positive scalar curvature using the representation variety of $\pi_1(X)$.

It seems to be a general theme that for any geometrically defined homology theory, there is an analogous theory tailored to the setting of end-periodic manifolds, and that this end-periodic theory is isomorphic to the original geometric theory in a natural way. These isomorphisms are built on the foundation of Poincaré duality.

2. End-periodic K-homology

2.1. ***Review of K-homology.*** We begin by reviewing the definition of K-homology of Baum and Douglas [1982], using the (M, S, f)-formulation introduced by Keswani [1999], and used by Higson and Roe [2010].

Definition 2.1. A *K-cycle* for a discrete group π is a triple (M, S, f), where M is a compact oriented odd-dimensional Riemannian manifold, S is a smooth Hermitian bundle over M with Clifford multiplication $c : TM \to \mathrm{End}(S)$, and $f : M \to B\pi$ is a continuous map to the classifying space of π.

Such a bundle S with the above data is called a *Dirac bundle*. We remark that M may be disconnected, and that its connected components are permitted to have different odd dimensions.

Definition 2.2. Two K-cycles (M, S, f) and (M', S', f') for $B\pi$ are said to be *isomorphic* if there is an orientation preserving diffeomorphism $\varphi : M \to M'$ covered by an isometric bundle isomorphism $\psi : S \to S'$ such that

$$\psi \circ c_M(v) = c_{M'}(\varphi_* v)$$

for all $v \in TM$, and such that $f' \circ \varphi = f$.

A *Dirac operator* for the cycle (M, S, f) is any first-order linear partial differential operator D acting on smooth sections of S whose principal symbol is the Clifford multiplication. That is to say, for any smooth function $\phi : M \to \mathbb{R}$ one has

$$[D, \phi] = c(\operatorname{grad} \phi) : \Gamma(S) \to \Gamma(S).$$

The K-homology group $K_1(B\pi)$ will consist of geometric K-cycles for π modulo an equivalence relation, which we will now describe.

Definition 2.3. A K-cycle (M, S, f) is a *boundary* if there exists a compact oriented even-dimensional manifold W with boundary $\partial W = M$ such that:

(a) W is isometric to the Riemannian product $(0, 1] \times M$ near the boundary.

(b) There is a $\mathbb{Z}_2$-graded Dirac bundle over W that is isomorphic to $S \oplus S$ in the collar with Clifford multiplication given by

$$c_W(v) = \begin{pmatrix} 0 & c_M(v) \\ c_M(v) & 0 \end{pmatrix}, \quad c_W(\partial_t) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

for $v \in TM$.

(c) The map $f : M \to B\pi$ extends to a continuous map $f : W \to B\pi$.

Remark 2.4. Our orientation convention for boundaries is the following: If W is an oriented manifold with boundary ∂W then the orientation on W at the boundary is given by the outward unit normal followed by the orientation of ∂W. The isometry in part (a) is required to be orientation preserving.

We define the *negative* of a K-cycle (M, S, f) to be $(-M, -S, f)$, where $-M$ is M with its orientation reversed, and $-S$ is S with the negative Clifford multiplication $c_{-S} = -c_S$. Two K-cycles (M, S, f) and (M', S', f') are *bordant* if the disjoint union $(M, S, f) \amalg (-M', -S', f')$ is a boundary, and we write $(M, S, f) \sim (M', S', f')$. This is the first of the relations defining K-homology; there are two more to define:

(1) *Direct sum/disjoint union*:

$$(M, S_1, f) \amalg (M, S_2, f) \sim (M, S_1 \oplus S_2, f).$$

(2) *Bundle modification*: Let (M, S, f) be a K-cycle. If P is a principal $SO(2k)$-bundle over M, we define

$$\widehat{M} = P \times_\rho S^{2k}.$$

Here ρ denotes the action of $SO(2k)$ on S^{2k} given by the standard embedding of $SO(2k)$ into $SO(2k+1)$. The metric on $\widehat{M}$ is any metric agreeing with that of M on horizontal tangent vectors and with that of S^{2k} on vertical tangent vectors. The map $\hat{f} : \widehat{M} \to B\pi$ is the composition of the projection $\widehat{M} \to M$ and $f : M \to B\pi$. Over S^{2k} is an $SO(2k)$-equivariant vector bundle $C\ell_\theta(S^{2k}) \subset C\ell(TS^{2k})$, defined as the $+1$ eigenspace of the *right* action by the oriented volume element θ on the Clifford bundle $C\ell(TS^{2k})$. The $SO(2k)$-equivariance of this bundle implies that it lifts to a well-defined bundle over $\widehat{M}$. We thus define the bundle

$$\widehat{S} = S \otimes C\ell_\theta(S^{2k})$$

over $\widehat{M}$. Clifford multiplication on $\widehat{S}$ is given by

$$c(v) = \begin{cases} c_M(v) \otimes \epsilon & \text{if } v \text{ is horizontal,} \\ I \otimes c_{S^{2k}}(v) & \text{if } v \text{ is vertical,} \end{cases}$$

where ϵ is the grading element of the Clifford bundle over S^{2k}. The K-cycle $(\widehat{M}, \widehat{S}, \hat{f})$ is called an *elementary bundle modification* of (M, S, f), and we write $(M, S, f) \sim (\widehat{M}, \widehat{S}, \hat{f})$. We remark also that individual bundle modifications are allowed to be made on connected components of M.

Remark 2.5. If D is a given Dirac operator for the cycle (M, S, f), then there is a preferred choice of Dirac operator for an elementary bundle modification $(\widehat{M}, \widehat{S}, \hat{f})$ of (M, S, f). If D_θ denotes the $SO(2k)$-equivariant Dirac operator acting on $C\ell(S^{2k})$, then the Dirac operator on $S \otimes C\ell_\theta(S^{2k})$ is

$$\widehat{D} = D \otimes \epsilon + I \otimes D_\theta,$$

where ϵ is the grading element of $C\ell_\theta(S^{2k})$.

Definition 2.6. The *K-homology group* $K_1(B\pi)$ is the abelian group of K-cycles modulo the equivalence relation generated by isomorphism of cycles, bordism, direct sum/disjoint union, and bundle modification. The addition of equivalence classes of K-cycles is given by disjoint union

$$(M, S, f) \amalg (M', S', f') = (M \amalg M', S \amalg S', f \amalg f').$$

One must of course check that this operation descends to a well-defined binary operation on K-homology which satisfies the group axioms. The details are straightforward.

Remark 2.7. There is another group $K_0(B\pi)$ defined in terms of even-dimensional cycles, which is well suited to the original Atiyah–Singer index theorem. We will not need it here.

2.2. *Definition of end-periodic K-homology.* With the definition of K-homology reviewed, we now adapt the definition to the setting of manifolds with periodic ends.

Definition 2.8. An *end-periodic K-cycle*, or simply a *K^{ep}-cycle* for a discrete group π is a quadruple (X, S, γ, f), where X is a compact oriented even-dimensional Riemannian manifold, $S = S^+ \oplus S^-$ is a $\mathbb{Z}_2$-graded Dirac bundle over X, $\gamma \in H^1(X, \mathbb{Z})$ is a cohomology class whose restriction to each connected component of X is primitive, and f is a continuous map $f : X \to B\pi$.

The $\mathbb{Z}_2$-graded structure of S includes a Clifford multiplication by tangent vectors to X which swaps the positive and negative subbundles. Again, the manifold X is allowed to be disconnected, with the connected components possibly having different even dimensions. Note that the definition of a K^{ep}-cycle imposes topological restrictions on X, namely each connected component of X must have nontrivial first cohomology in order for the class γ to be primitive on each component.

Definition 2.9. Two K^{ep}-cycles (X, S, γ, f) and (X', S', γ', f') are *isomorphic* if there exists an orientation preserving diffeomorphism $\varphi : X \to X'$ which is covered by a $\mathbb{Z}_2$-graded isometric bundle isomorphism $\psi : S \to S'$ such that

$$\psi \circ c_X(v) = c_{X'}(\varphi_* v)$$

for all $v \in TX$. The diffeomorphism φ must additionally satisfy $\varphi^*(\gamma') = \gamma$, and $f' \circ \varphi = f$.

We now define what it means for a K^{ep}-cycle (X, S, γ, f) to be a boundary. First, let $Y \subset X$ be a connected codimension-1 submanifold that is Poincaré dual to γ. The orientation of Y is such that for all closed forms α of codimension 1 (over each component of X),

$$\int_Y \iota^*(\alpha) = \int_X \gamma \wedge \alpha,$$

where $\iota : Y \to X$ is the inclusion and we abuse notation by writing γ for what is really a closed 1-form representing the cohomology class γ. In other words, the orientation of Y is such that the signs of the above two integrals always agree. Now, cut X open along Y to obtain a compact manifold W with boundary $\partial W = Y \amalg -Y$, with our boundary orientation conventions as in Remark 2.4. Glue infinitely many isometric copies W_k of W end to end along Y to obtain the complete oriented Riemannian manifold $X_1 = \bigcup_{k\geq 0} W_k$ with boundary $\partial X_1 = -Y$. There is a canonical map $X_1 \to X$ sending a point of W_k to its corresponding point in X. Pull back the Dirac bundle S on X via this map to get a $\mathbb{Z}_2$-graded Dirac bundle on X_1, also denoted S, and pull back the map f to get a map $f : X_1 \to B\pi$.

Definition 2.10. The K^{ep}-cycle (X, S, γ, f) is a *boundary* if there exists a compact oriented Riemannian manifold Z with boundary $\partial Z = Y$, which can be attached to X_1 along Y to form a complete oriented Riemannian manifold $Z_\infty = Z \cup_Y X_1$, such that the bundle S extends to a $\mathbb{Z}_2$-graded Dirac bundle on Z_∞ and the map f extends to a continuous map $f : Z_\infty \to B\pi$.

Remark 2.11. Being a boundary is clearly independent of the choice of Y; if Y' is another choice of submanifold Poincaré dual to γ we simply embed Y' somewhere in the periodic end of Z_∞, and take Z' to be the compact piece in Z_∞ bounded by Y'.

Definition 2.12. The manifold Z_∞ from Definition 2.10 is called an *end-periodic* manifold. It is convenient to say the end is *modelled* on (X, γ), or sometimes just X if γ is understood. Any object on Z_∞ whose restriction to the periodic end X_1 is the pullback of an object from X is called *end-periodic*. For example, the bundle S, the map f, and the metric on Z_∞ in the previous definition are all end-periodic.

Remark 2.13. We allow end-periodic manifolds to have multiple ends. This situation arises when the manifold X, on which the end of Z_∞ is modelled, is disconnected.

The *negative* of a K^{ep}-cycle (X, S, γ, f) is simply $(X, S, -\gamma, f)$. This is so that the disjoint union of a K^{ep}-cycle with its negative is a boundary — it is clear that the $\mathbb{Z}$-cover $\widetilde{X}$ of X corresponding to γ is an end-periodic manifold with end modelled on $(X \amalg X, \gamma \amalg -\gamma)$. The definitions of bordism and direct sum/disjoint union are exactly the same as before, with the class γ left unchanged. In the case of bundle modification, the class $\hat{\gamma}$ on $\widehat{X} = P \times_\rho S^{2k}$ is the pullback of γ by the projection $p : \widehat{X} \to X$, and we endow the tensor product bundle $S \otimes C\ell_\theta(S^{2k})$ with the standard tensor product grading of $\mathbb{Z}_2$-graded modules. There is also one more relation we define which relates the orientation on X to the one-form γ:

$$(X, S, -\gamma, f) \sim (-X, \Pi(S), \gamma, f)$$

where $-X$ is X with the reversed orientation and $\Pi(S)$ is S with its $\mathbb{Z}_2$-grading reversed. We call this relation *orientation/sign*, as it links the orientation on X to the sign of γ. The need for this relation will become apparent in (2) of the proof of Lemma 2.16.

Definition 2.14. The *end-periodic K-homology group*, $K_1^{\mathrm{ep}}(B\pi)$, is the abelian group consisting of K^{ep}-cycles up to the equivalence relation generated by isomorphism of K^{ep}-cycles, bordism, direct sum/disjoint union, bundle modification, and orientation/sign. Addition is given by disjoint union of cycles

$$(X, S, \gamma, f) \amalg (X', S', \gamma', f') = (X \amalg X', S \amalg S', \gamma \amalg \gamma', f \amalg f').$$

Remark 2.15. As for K-homology we could also define the group $K_0^{\mathrm{ep}}(B\pi)$ using odd-dimensional K^{ep}-cycles, although we will not pursue this here.

2.3. *The isomorphism.* We will now show that there is a natural isomorphism $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$.

First we describe the map $K_1(B\pi) \to K_1^{\mathrm{ep}}(B\pi)$. Let (M, S, f) be a K-cycle for $B\pi$. Define $X = S^1 \times M$ an even-dimensional manifold with the product orientation and Riemannian metric, the Dirac bundle $S \oplus S \to X$ with Clifford multiplication as in (b) of Definition 2.3, $\gamma = d\theta \in H^1(X, \mathbb{Z})$ the standard generator of the first cohomology of S^1, and $f : X \to B\pi$ the extension of $f : M \to B\pi$. We map the equivalence class of (M, S, f) in $K_1(B\pi)$ to the equivalence class of $(S^1 \times M, S \oplus S, d\theta, f)$ in $K_1^{\mathrm{ep}}(\pi)$.

Lemma 2.16. *The map sending a cycle (M, S, f) to the end-periodic cycle $(S^1 \times M, S \oplus S, d\theta, f)$ descends to a well-defined map of K-homologies.*

Proof. It must be checked that each of the relations defining $K_0(B\pi)$ are preserved by this map.

(1) **Boundaries**: Let (M, S, f) be a boundary. Then we have a compact manifold W with boundary $\partial W = M$ satisfying conditions (a) and (b) in Definition 2.3. To show that $(S^1 \times M, S \oplus S, d\theta, f)$ is a boundary, we attach W to the half-cover $X_1 = \mathbb{R}_{\geq 0} \times M$ to obtain a Riemannian manifold Z_∞. Over X_1 is the bundle $S \oplus S$, and over W is a bundle isomorphic to $S \oplus S$. We use the isomorphism to glue the bundles together and define $S \oplus S$ over Z_∞. The assumptions on the Clifford multiplication imply that it extends over this bundle. Since the map f on M extends to W, the map f on $S^1 \times M$ extends to Z_∞.

(2) **Negatives**: The negative of (M, S, f) is $(-M, -S, f)$, which maps to $(-S^1 \times M, -S \oplus -S, d\theta, f)$. The negative of $(-S^1 \times M, -S \oplus -S, d\theta, f)$ is

$$(-S^1 \times M, -S \oplus -S, -d\theta, f) \sim (S^1 \times M, \Pi(-S \oplus -S), d\theta, f)$$

by the orientation/sign relation. The only difference between this cycle and $(X, S \oplus S, d\theta, f)$ is that the Clifford multiplication is negative; Clifford multiplication by vectors tangent to M has become negative and reversing the $\mathbb{Z}_2$-grading has caused ∂_θ to act negatively. This cycle is isomorphic to

$$(S^1 \times M, S \oplus S, d\theta, f)$$

via the identity map $\varphi : M \to M$ and the isometric bundle isomorphism $\psi : -S \oplus -S \to S \oplus S$, $\psi(s \oplus t) = c(\omega)(s \oplus t)$, where ω is the oriented volume element of $S^1 \times M$. Hence negatives are preserved by the mapping.

(3) **Disjoint union**: Obvious.

(4) **Bordism**: Since negatives map to negatives, boundaries map to boundaries, and disjoint union is preserved, it follows that bordism is also preserved.

(5) **Direct sum/disjoint union**: Also obvious.

(6) **Bundle modification**: Let $(\widehat{M}, \widehat{S}, \hat{f})$ be an elementary bundle modification for (M, S, f) associated to the principal $SO(2k)$-bundle $P \to M$. We pullback P to a bundle over $X = S^1 \times M$, and use it to construct our bundle modification $(\widehat{X}, (S \oplus S)\hat{}, d\theta, f)$ of $(S^1 \times M, S \oplus S, d\theta, f)$. It is clear that $\widehat{X} = S^1 \times \widehat{M}$. Now $\widehat{S} = S \otimes C\ell_\theta(S^{2k})$, so

$$\widehat{S} \oplus \widehat{S} \cong (S \oplus S) \otimes C\ell_\theta(S^{2k}) = (S \oplus S)\hat{}.$$

It is straightforward yet tedious to verify that Clifford multiplication is preserved by this isomorphism. So the K^{ep}-cycle obtained via bundle modification then mapping, is isomorphic to the K^{ep}-cycle obtained by mapping then bundle modification. □

Now for the inverse map. Let (X, S, γ, f) be an end-periodic cycle. Choose a submanifold $Y \subset X$ Poincaré dual to γ, oriented as in the paragraph after Definition 2.9. We map the cycle (X, S, γ, f) to (Y, S^+, f), where S^+ and f are restricted to Y. If ω is an oriented volume form for Y then we let ∂_t be the unit normal to Y such that $\partial_t \wedge \omega$ is the orientation on X. The Clifford multiplication on S^+ is then defined to be

$$c_Y(v) = c_X(\partial_t) c_X(v)$$

for $v \in TY$. Note that this agrees with the conventions of (b) in Definition 2.3. One easily verifies that this indeed defines a Clifford multiplication on S^+.

Lemma 2.17. *The map sending an end-periodic cycle (X, S, γ, f) to the cycle (Y, S^+, f) described above, descends to a well-defined map of K-homologies.*

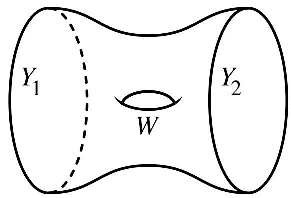

Figure 3. Compact bordism between Y_1 and Y_2.

Proof. We must not only check that the relations defining end-periodic K-homology are preserved, but that the class in K-homology obtained is independent of the choice of Y.

(1) **Boundaries**: Let (X, S, γ, f) be a boundary. Then there is a compact oriented manifold Z with boundary $\partial Z = Y$ over which the $\mathbb{Z}_2$-graded Dirac bundle S and map f extend. We modify the metric near the boundary of Z to make it a product. It follows that the cycle (Y, S^+, f) is a boundary.

(2) **Choice of** Y: Suppose Y_1 and Y_2 are submanifolds of X that are Poincaré dual to γ. We can take functions $f_1, f_2 : X \to S^1$ both having $1 \in S^1$ as a regular value and satisfying $f_i^{-1}(1) = Y_i$ for $i = 1, 2$. Since Y_1 and Y_2 are both Poincaré dual to γ, the functions f_i are homotopic. Let $\tilde{f}_i : \widetilde{X} \to \mathbb{R}$ be the lift of $f_i : X \to S^1$, where $\widetilde{X} \to X$ is the $\mathbb{Z}$-cover determined by γ. The preimage $\tilde{f}_i^{-1}(m)$ gives an embedding of Y_i in $\widetilde{X}$ for any $m \in \mathbb{Z}$. Choosing a tubular neighbourhood $Y_1 \times (-\epsilon, \epsilon) \subset \widetilde{X}$, we can homotopy $\tilde{f}_1$ into $\tilde{f}_2$ over the interval $(-\epsilon, \epsilon)$. Letting F be the resulting function on $\widetilde{X}$, we may take $F^{-1}[-m, m]$ for some large integer m to be a bordism W between Y_1 and Y_2. Since f_1 and f_2 are proper, so is F, and the resulting bordism is compact; see Figure 3.

We pull back the bundle S and the map f to W, and modify the metric near the boundary so that it is a product. The result is that $Y_1 \amalg -Y_2$ is a boundary.

(3) **Negatives**: Reversing the sign of γ changes the orientation of Y. Clifford multiplication on Y also becomes negative, since changing the orientation on Y reverses the unit normal to Y. Hence negatives of cycles map to negatives.

(4) **Disjoint union**: Obvious.

(5) **Bordism**: Since boundaries map to boundaries, negatives map to negatives, and disjoint union is preserved, it follows that bordism is also preserved.

(6) **Direct sum/disjoint union**: Obvious.

(7) **Orientation/sign**: From (3) in this proof, the K-cycle obtained from $(X, S, -\gamma, f)$ is the negative of the cycle (Y, S^+, f). Now consider the K-cycle obtained from $(-X, \Pi(S), \gamma, f)$. Reversing the orientation on X will also reverse it on Y. Instead

of S^+, we now take S^- with Clifford multiplication

$$c_{S^-}(v) = c(-\partial_t)c(v) = -c(\partial_t)c(v)$$

where $v \in TY$ and $-\partial_t$ is the unit normal to $-Y$. We now show $(-Y, S^+, f)$ and $(-Y, S^-, f)$ are isomorphic. Let ω be the oriented volume element of $+Y$ (or $-Y$, it does not matter) and define a map $\psi : S^+ \to S^-$ by $\psi(s) = c(\omega)s$. Then

$$\psi \circ c_{S^+}(v) = c_{S^-}(v) \circ \psi$$

and the cycles are therefore isomorphic.

(8) **Bundle modification**: Let $(\widehat{X}, \widehat{S}, \hat{\gamma}, \hat{f})$ be an elementary bundle modification for (X, S, γ, f), associated to the principal $SO(2k)$-bundle $P \to X$. We restrict this principal bundle to Y and consider the corresponding bundle modification $(\widehat{Y}, \widehat{S}^+, \hat{f})$ for (Y, S^+, f). It is clear that $\widehat{Y} \subset \widehat{X}$ is Poincaré dual to $\hat{\gamma}$. The bundle

$$\widehat{S} = S \otimes C\ell_\theta(S^{2k})$$

has even part

$$\widehat{S}^+ = (S^+ \otimes C\ell_\theta^+(S^{2k})) \oplus (S^- \otimes C\ell_\theta^-(S^{2k})),$$

while over $\widehat{Y}$ we have the bundle

$$\widehat{S^+} = S^+ \otimes C\ell_\theta(S^{2k}).$$

Identifying S^+ with S^- via the isomorphism $c(\partial_t)$, we see that $\widehat{S}^+ \cong \widehat{S^+}$. It is routine to check that the Clifford multiplications are preserved under this isomorphism. □

Theorem 2.18. *The above maps between K-homologies define an isomorphism of groups $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$.*

Proof. We must check that the above maps on K-homologies are inverse to each other. If we begin with a cycle (M, S, f), this maps to $(S^1 \times M, S \oplus S, d\theta, f)$. Mapping this again, we get (M, S, f) back, so this direction is easy. Now suppose we begin with a cycle (X, S, γ, f). This maps to (Y, S^+, f) which then maps to $(S^1 \times Y, S^+ \oplus S^+, d\theta, f)$. We will show this cycle is bordant to the original cycle (X, S, γ, f). Consider the half cover X_1 of X obtained using $-\gamma$. Near the boundary, this is diffeomorphic to a product $(-\delta, 0] \times Y$. The half cover of $S^1 \times Y$ obtained from $d\theta$ is $\mathbb{R}_{\geq 0} \times Y$. The two half covers clearly glue together to produce and end-periodic manifold with two ends. The Dirac bundles and maps to $B\pi$ extend over this manifold, and hence the two cycles are bordant. □

3. Relative eta/rho invariants

In this section, we use the end-periodic eta invariant of Mrowka, Ruberman and Saveliev [Mrowka et al. 2016] to define homomorphisms from the end-periodic K-homology group $K_1^{\mathrm{ep}}(B\pi)$ to $\mathbb{R}/\mathbb{Z}$. Any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$ will determine such a homomorphism, and we see that this homomorphism agrees with that constructed in Higson and Roe [2010] under the natural isomorphism $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$.

3.1. ***Rho invariant for K-homology.*** Let (M, S, f) be a K-cycle. Any Dirac operator for this cycle is a self-adjoint elliptic first-order operator on S, and so has a discrete spectrum of real eigenvalues. The *eta function* of this operator is defined to be the sum over the nonzero eigenvalues of D

$$\eta(s) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda)|\lambda|^{-s},$$

which converges absolutely for $\mathrm{Re}(s)$ sufficiently large. It is a theorem of Atiyah, Patodi and Singer (APS) that this function admits a meromorphic continuation to the complex plane, and that this continuation takes a finite value $\eta(0)$ at the origin. The *eta invariant* of the chosen Dirac operator D is by definition

$$\eta(D) = \tfrac{1}{2}(\eta(0) - h) \tag{3.1}$$

where $h = \dim\ker(D)$ is the multiplicity of the zero eigenvalue.

The eta invariant plays a central role in the Atiyah–Patodi–Singer index theorem, appearing as a correction term for the boundary. Suppose W is an even-dimensional manifold with boundary $\partial W = M$, equipped with a Dirac bundle satisfying the conditions of Definition 2.3. Further, suppose we have a Dirac operator $D(W)$ on W so that

$$D(W) = \begin{pmatrix} 0 & -\partial_t + D \\ \partial_t + D & 0 \end{pmatrix} \tag{3.2}$$

in a product neighbourhood of the boundary, where D is the Dirac operator on M. In this instance we say that $D(W)$ *bounds* D. Then the APS index theorem [Atiyah et al. 1975a] states

$$\mathrm{Ind}_{\mathrm{APS}}\, D^+(W) = \int_W \boldsymbol{I}(D^+(W)) - \eta(D). \tag{3.3}$$

The left-hand side is the index of $D^+(W)$ with respect to a certain global boundary condition — the projection onto the nonnegative eigenspace of D must vanish. The integrand $\boldsymbol{I}(D^+(W))$ is the constant term in the asymptotic expansion of the supertrace of the heat operator for $D^+(W)$, called the *index form* of the Dirac operator.

Remark 3.1. In (3.3), the eta invariant is as in (3.1), where the sign of the term $h = \dim \ker D$ is negative. This is contingent on the orientation of M being consistent with the boundary orientation inherited from W. If the orientations are not compatible, then the sign of h is reversed in (3.3).

The map f in the cycle (M, S, f) determines a principle π-bundle over M. Given a representation $\sigma_1 : \pi \to U(N)$, we can then form a flat vector bundle $E_1 \to M$ and twist the Dirac operator D on S to obtain a Dirac operator D_1 acting on sections of $S \otimes E_1$. Given a second representation $\sigma_2 : \pi \to U(N)$ we form another operator D_2 on $S \otimes E_2$ in the same way.

Definition 3.2. The *relative eta invariant*, or *rho invariant* associated to the two unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the K-cycle (M, S, f) for $B\pi$, and the choice of Dirac operator D for the K-cycle, is defined to be

$$\rho\,(\sigma_1, \sigma_2\,; M, S, f) = \eta(D_1) - \eta(D_2).$$

The eta invariant of an operator depends sensitively on the operator itself, whereas the relative eta invariant is much more robust. The following is a restatement of [Higson and Roe 2010, Theorem 6.1], and is the reason for our omission of D in the above notation for the rho invariant.

Theorem 3.3. *The* mod $\mathbb{Z}$ *reduction of the rho invariant $\rho(\sigma_1, \sigma_2\,; M, S, f)$ for representations $\sigma_1, \sigma_2 : \pi \to U(N)$, depends only on the equivalence class of (M, S, f) in $K_1(B\pi)$, and on σ_1, σ_2. There is therefore a well-defined group homomorphism*

$$\rho\,(\sigma_1, \sigma_2) : K_1(B\pi) \to \mathbb{R}/\mathbb{Z}.$$

The most complicated part of the proof is showing invariance under bundle modification. We will not repeat the full proof, however we will show invariance under bordism since the argument serves to motivate the end-periodic case.

Proof. Let (M, S, f) be a boundary — we will show that the rho invariant $\rho\,(\sigma_1, \sigma_2;$ $M, S, f)$ vanishes modulo $\mathbb{Z}$. Let W be as in Definition 2.3 and let $D(W)$ be a Dirac operator on W which bounds the Dirac operator D on M. Since the map f to $B\pi$ extends to W, we find twisted Dirac operators $D_1(W)$ and $D_2(W)$ on W bounding the twisted operators D_1 and D_2 on M. Applying the APS index theorem separately to these operators gives

$$\operatorname{Ind}_{\mathrm{APS}} D_i^+(W) = \int_W \boldsymbol{I}(D_i^+(W)) - \eta(D_i) \tag{3.4}$$

for $i = 1, 2$. Since $D_1(W)$ and $D_2(W)$ are both twists of the same Dirac operator $D(W)$ by flat bundles of dimension N, we have

$$\boldsymbol{I}(D_1^+(W)) = \boldsymbol{I}(D_2^+(W)) = N \cdot \boldsymbol{I}(D^+(W)).$$

Subtracting the two equations (3.4) from each other therefore yields

$$\rho(\sigma_1, \sigma_2\,; M, S, f) = \eta(D_1) - \eta(D_2) = \mathrm{Ind}_{\mathrm{APS}}\, D_2^+(W) - \mathrm{Ind}_{\mathrm{APS}}\, D_1^+(W),$$

which is an integer.

Now, consider the negative cycle $(-M, -S, f)$ for (M, S, f). If D is a Dirac operator for (M, S, f), then $-D$ is a Dirac operator for $(-M, -S, f)$. From the definition of the eta invariant (3.1) and from Remark 3.1, we see that $\eta(-D) = -\eta(D)$. Finally, the eta invariant is clearly additive under disjoint unions of cycles. It follows that if two cycles are bordant, then their eta invariants agree modulo integers. □

Higson and Roe [2010] used this map on K-homology to obtain obstructions to positive scalar curvature for odd-dimensional manifolds. Our isomorphism of K-homologies will allow us to transfer their results to the even-dimensional case.

3.2. ***Index theorem for end-periodic manifolds [Mrowka et al. 2016].*** Mrowka, Ruberman and Saveliev [2016] proved an index theorem for end-periodic Dirac operators on end-periodic manifolds, which generalises the Atiyah–Patodi–Singer index theorem. Rather than the eta invariant appearing as a correction term for the end, a new invariant called the *end-periodic eta invariant* appears, and this new invariant agrees with the eta invariant of Atiyah–Patodi–Singer in the case of a cylindrical end. In this section, we review the end-periodic index theorem of MRS, and give the necessary definitions and theorems required to define the end-periodic rho invariants. There is nothing new here, so the reader who is already familiar with the MRS index theorem may safely skip to Section 3.3

Let (X, S, γ, f) be a K^{ep}-cycle, and let $D(X)$ be a Dirac operator for the cycle. Let $\widetilde{X}$ be the $\mathbb{Z}$-cover associated to γ, and let $F : \widetilde{X} \to \mathbb{R}$ be the map which covers the classifying map $X \to S^1$ for the $\mathbb{Z}$-cover $\widetilde{X}$. Then F satisfies $F(x+1) = F(x)+1$, where $x+1$ denotes the image of $x \in \widetilde{X}$ under the fundamental covering translation. It follows that dF descends to a well-defined one-form on X, also denoted dF. Fixing a branch of the complex logarithm, define a family of operators

$$D_z(X) = D(X) - \ln(z)\, c(dF)$$

on X, where $c(dF)$ is Clifford multiplication by dF, and $z \in \mathbb{C}^*$. These are in fact the operators obtained by conjugating the Dirac operator on $\widetilde{X}$ with the *Fourier-Laplace transform* — see Section 2.2 of [Mrowka et al. 2016] for more details. The *spectral set* of this family of operators is defined to be the set of z for which $D_z(X)$ is not invertible. The spectral sets of the families $D_z^{\pm}(X)$ are defined similarly.

Henceforth, we will take Z_∞ to be an end-periodic manifold with end modelled on (X, γ). All objects on Z_∞ will be taken to be end-periodic, unless stated otherwise. Now, the Fredholm properties of the end-periodic operator $D^+(Z_\infty)$

are linked to the spectral set of the family $D_z^+(X)$. In fact, it follows from [Taubes 1987, Lemma 4.3], that $D^+(Z_\infty)$ is Fredholm if and only if the spectral set of the family $D_z^+(X)$ is disjoint from the unit circle $S^1 \subset \mathbb{C}$. Thus, a necessary (but not sufficient) condition for $D^+(Z_\infty)$ to be Fredholm is that $\operatorname{Ind} D^+(X) = 0$.

Definition 3.4 (Mrowka et al. 2016). Suppose that the spectral set of the family $D_z^+(X)$ is disjoint from the unit circle $S^1 \subset \mathbb{C}$. The *end-periodic eta invariant* for the Dirac operator $D^+(X)$ is then defined as

$$\eta^{\mathrm{ep}}(D^+(X)) = \frac{1}{\pi i}\int_0^\infty \oint_{|z|=1} \operatorname{Tr}\bigl(c(dF)\cdot D_z^+ \exp(-tD_z^- D_z^+)\bigr)\frac{dz}{z}\,dt,$$

where the Dirac operators in the integral are on X, and the contour integral over the unit circle is taken in the anticlockwise direction.

Remark 3.5. There is an equivalent definition of the eta invariant in terms of the von Neumann trace — see [Mrowka et al. 2016, Proposition 6.2], also [Atiyah 1976] for information on the von Neumann trace.

Suppose $X = S^1 \times Y$, where Y is a compact oriented odd dimensional manifold, and X is endowed with the product Riemannian metric. Assume the Dirac operator $D(X)$ on X takes the form of that in the RHS of Equation (3.2), with D being the Dirac operator on Y. Then it is shown in [Mrowka et al. 2016, §6.3] that for $dF = d\theta$,

$$\eta^{\mathrm{ep}}(D^+(X)) = \eta(D).$$

We now state the end-periodic index theorem of Mrowka, Ruberman and Saveliev, in the case when the end-periodic operator $D^+(Z_\infty)$ is Fredholm. Recall that for $D^+(Z_\infty)$ to be Fredholm, it is necessary that $\operatorname{Ind} D^+(X) = 0$. The Atiyah–Singer index theorem then implies that the index form $\boldsymbol{I}(D^+(X))$ is exact, so one can find a form ω on X satisfying $d\omega = \boldsymbol{I}(D^+(X))$.

Theorem 3.6 (MRS index theorem, [Mrowka et al. 2016, Theorem A]). *Suppose that the end-periodic operator $D^+(Z_\infty)$ is Fredholm, and choose a form ω on X such that $d\omega = \boldsymbol{I}(D^+(X))$. Then*

$$\operatorname{Ind} D^+(Z_\infty) = \int_Z \boldsymbol{I}(D^+(Z)) - \int_Y \omega + \int_X dF \wedge \omega - \tfrac{1}{2}\,\eta^{\mathrm{ep}}(X). \tag{3.5}$$

Remarks 3.7. The form ω is called the *transgression class* — see [Gilkey 1984b, p. 306] for more details. In the case that the metric is a product near Y, one can choose F so that the two integrals involving the transgression class cancel, leaving a formula similar to the original APS formula. The theorem reduces to the APS index theorem [Atiyah et al. 1975a] when Z_∞ only has cylindrical ends.

When $D^+(Z_\infty)$ is *not* Fredholm, Mrowka, Ruberman and Saveliev are still able to prove an index theorem under the assumptions that the spectrum of the family $D_z^+(X)$ is discrete, which in particular implies $\operatorname{Ind} D^+(X) = 0$. This is analogous to the case in the APS index theorem when the Dirac operator D on the boundary has a nonzero kernel, and the correction $h = \dim\ker D$ appears in the formula.

The key is to introduce the *weighted Sobolev spaces* on Z_∞ as follows. First recall that the Sobolev space $L_k^2(Z_\infty, S)$ for an integer $k \geq 0$, is defined as the completion of $C_0^\infty(Z_\infty, S)$ in the norm

$$\|u\|^2_{L_k^2(Z_\infty,S)} = \sum_{j\leq k} \int_{Z_\infty} |\nabla^j u|^2$$

for a fixed choice of end-periodic metric and compatible end-periodic Clifford connection on Z_∞. Now, restrict the upstairs covering map $F : \widetilde{X} \to \mathbb{R}$ to the half-cover $X_1 = \bigcup_{k\geq 0} W_k$, and choose an extension of this map to Z_∞, which we continue to denote F. Given a weight $\delta \in \mathbb{R}$ and an integer $k \geq 0$, we say that $u \in L^2_{k,\delta}(Z_\infty, S)$ if $e^{\delta F} u \in L^2_k(Z_\infty, S)$. Define the $L^2_{k,\delta}$-norm by

$$\|u\|_{L^2_{k,\delta}(Z_\infty,S)} = \|e^{\delta F} u\|_{L^2_k(Z_\infty,S)}.$$

It is easy to check that up to equivalence of norms, this is independent of the choice of extension of F to Z_∞, since the region over which we are choosing an extension is compact. The spaces $L^2_{k,\delta}(Z_\infty, S)$ are all complete in this norm, and the operator $D^+(Z_\infty)$ extends to a bounded operator

$$D^+(Z_\infty) : L^2_{k+1,\delta}(Z_\infty, S^+) \to L^2_{k,\delta}(Z_\infty, S^-) \tag{3.6}$$

for every k and δ. The following theorem of [Taubes 1987] classifies Fredholmness of the operator (3.6) in terms of the family $D_z^+(X) = D^+(X) - \ln(z)\, c(dF)$.

Lemma 3.8 (Taubes 1987, Lemma 4.3). *The operator*

$$D^+(Z_\infty) : L^2_{k+1,\delta}(Z_\infty, S^+) \to L^2_{k,\delta}(Z_\infty, S^-)$$

is Fredholm if and only if the operators $D_z^+(X)$ are invertible for all z on the circle $|z| = e^\delta$.

The usual L^2-case corresponds to the weighting $\delta = 0$, and hence we see by setting $z = 1$:

Corollary 3.9. *A necessary condition for the operator $D^+(Z_\infty)$ to be Fredholm is that* $\operatorname{Ind} D^+(X) = 0$.

The following result on the spectral set of the family is also due to Taubes, which suffices for our purposes.

Theorem 3.10 (Taubes 1987, Theorem 3.1). *Suppose that* $\operatorname{Ind} D^+(X) = 0$ *and that the map* $c(dF) : \ker D^+(X) \to \ker D^-(X)$ *is injective. Then the spectral set of the family* $D_z^+(X)$ *is a discrete subset of* $\mathbb{C}^*$*, and the operator* $D^+(Z_\infty)$ *is a Fredholm operator.*

It follows that the operator $D^+(Z_\infty)$ acting on the Sobolev spaces of weight δ is Fredholm for all but a closed discrete set of $\delta \in \mathbb{R}$.

Remark 3.11. There are two important instances where the hypothesis of Theorem 3.10 is satisfied:

(1) When $X = S^1 \times M$ with the product metric, and the Dirac operator on X taking the form of Equation (3.2). In this case $dF = d\theta$, and $c(d\theta)$ is as in part (b) of Definition 2.3. This example shows that every class in $K^{\mathrm{ep}}(B\pi)$ has a representative with discrete spectral set.

(2) When X is spin with positive scalar curvature and $D^+(X)$ is the spin Dirac operator on X (or more generally, $D^+(X)$ twisted by a flat bundle). In this case Lichnerowicz' vanishing theorem implies that $\ker D^+(X)$ and $\ker D^-(X)$ are trivial. In the applications to positive scalar curvature, we will always assume X to be spin, so that this assumption is satisfied.

[Mrowka et al. 2016, Theorem C] extends Theorem 3.6 to the non-Fredholm case that applies to operators such as the signature operator and is analogous to the extended L^2 case considered in [Atiyah et al. 1975a].

We allow for the case where the family has poles lying on the unit circle, in which case the operator $D^+(X)$ is not Fredholm. By discreteness of the spectral set, the family $D_z^+(X)$ has no poles for z sufficiently close to (but not lying on) the unit circle, and hence there is $\epsilon > 0$ such that for all $0 < \delta < \epsilon$ the operators $D_z^+(Z_\infty)$ acting on the δ-weighted Sobolev spaces are all Fredholm (see Lemma 3.8). The index does not change under small variations of δ in this region, and we denote it by $\operatorname{Ind}_{\mathrm{MRS}} D^+(Z_\infty)$. This is the regularised form of the index which appears in the full MRS index theorem.

There are two more quantities to define which appear in the full MRS index theorem. First of all, the end-periodic eta invariant in Definition 3.4 is no longer well defined if the family $D_z^+(X)$ has poles on the unit circle. Letting $\epsilon > 0$ be sufficiently small so that there are no poles in $e^{-\epsilon} < |z| < e^{\epsilon}$ except for those with $|z| = 1$, define

$$\eta_\epsilon^{\mathrm{ep}}(D^+(X)) = \frac{1}{\pi i} \int_0^\infty \oint_{|z|=e^\epsilon} \operatorname{Tr}\bigl(df \cdot D_z^+ \exp(-t(D_z^+)^* D_z^+)\bigr) \frac{dz}{z}\, dt, \tag{3.7}$$

where the integral is taken to be the constant term of its asymptotic expansion in powers of t. Define

$$\eta^{\mathrm{ep}}_{\pm}(D^+(X)) = \lim_{\epsilon\to 0\pm} \eta^{\mathrm{ep}}_{\epsilon}(D^+(X)),$$

and

$$\eta^{\mathrm{ep}}(D^+(X)) = \tfrac{1}{2}\,[\eta^{\mathrm{ep}}_{+}(D^+(X)) + \eta^{\mathrm{ep}}_{-}(D^+(X))]. \tag{3.8}$$

It is this incarnation of the eta invariant which will appear in the MRS index theorem. Since $(D_z^+)^* = D_z^-$ for $|z| = 1$ this definition of $\eta^{\mathrm{ep}}(X)$ agrees with Definition 3.4 when there are no poles on the unit circle.

The last term to define is the analog of $h = \dim\ker D$ appearing in the APS index theorem. The family $D_z^+(X)^{-1}$ is meromorphic, so if $z \in S^1$ is a pole then it has some finite order m. Define $d(z)$, as in [Mrowka et al. 2011, §6.3], to be the dimension of the vector space solutions $(\varphi_1, \dots, \varphi_m)$ to the system of equations

$$\begin{aligned} D_z^+(X)\varphi_1 &= c(dF)\varphi_2,\\ &\vdots\\ D_z^+(X)\varphi_{m-1} &= c(dF)\varphi_m,\\ D_z^+(X)\varphi_m &= 0. \end{aligned}$$

For z not in the spectral set of the family $D_z^+(X)$, we have $d(z) = 0$. The term h in the MRS index theorem is defined as the finite sum of integers

$$h = \sum_{|z|=1} d(z).$$

Remark 3.12. The integers $d(z)$ give a formula for the change in index when one varies the weight δ; if $\mathrm{Ind}_\delta\, D^+(Z_\infty)$ denotes the index of $D^+(Z_\infty)$ acting on the δ-weighted Sobolev spaces, then one has for $\delta < \delta'$ that

$$\mathrm{Ind}_\delta\, D^+(Z_\infty) - \mathrm{Ind}_{\delta'}\, D^+(Z_\infty) = \sum_{e^\delta < |z| < e^{\delta'}} d(z).$$

Theorem 3.13 (MRS index theorem [Mrowka et al. 2016, Theorem C]). *Suppose the spectral set of $D_z^+(X)$ is a discrete subset of $\mathbb{C}^*$, and let ω be a form on X such that $d\omega = \boldsymbol{I}(D^+(X))$. Then*

$$\mathrm{Ind}_{\mathrm{MRS}}\, D^+(Z_\infty) = \int_Z \boldsymbol{I}(D^+(Z)) - \int_Y \omega + \int_X dF \wedge \omega - \tfrac{1}{2}(h + \eta^{\mathrm{ep}}\,(D^+(X))).$$

3.3. ***End-periodic $\mathbb{R}/\mathbb{Z}$-index theorem.*** Let $\sigma_1, \sigma_2 : \pi \to U(N)$ be unitary representations of the discrete group π. Using the end-periodic eta invariant of MRS, we will define an end-periodic rho invariant $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2)$ analogous to the rho invariant in the APS case. This will determine a map from end-periodic K-homology to

$\mathbb{R}/\mathbb{Z}$, however we must be more careful about how we define the rho invariant due to the MRS index theorem not being applicable to all operators.

Definition 3.14. Let (X, S, γ, f) be a K^{ep}-cycle. Assume we can choose a covering function $F : \widetilde{X} \to \mathbb{R}$ so that the spectral sets of the families of the twisted operators $D_1^+(X)$ and $D_2^+(X)$ are discrete. Then we define the *end-periodic rho invariant* to be

$$\rho^{\text{ep}}(\sigma_1, \sigma_2; X, S, \gamma, f) = \tfrac{1}{2}[h_1 + \eta^{\text{ep}}(D_1^+(X)) - h_2 - \eta^{\text{ep}}(D_2^+(X))].$$

By [Mrowka et al. 2016, Lemma 8.2], this definition is independent of the choice of such function F, if it exists.

Theorem 3.15. *Whenever it is defined, the mod $\mathbb{Z}$ reduction of the end-periodic rho invariant $\rho^{\text{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f)$ associated to $\sigma_1, \sigma_2 : \pi \to U(N)$ depends only on the representations σ_1, σ_2 and the equivalence class of (X, S, γ, f) in $K_1^{\text{ep}}(B\pi)$. Moreover, every equivalence class has a representative with a well-defined rho invariant. Hence there is a well-defined group homomorphism*

$$\rho^{\text{ep}}(\sigma_1, \sigma_2) : K_1^{\text{ep}}(B\pi) \to \mathbb{R}/\mathbb{Z}.$$

Furthermore, the following diagram commutes:

$$\begin{tikzcd}
K_1^{\text{ep}}(B\pi) \arrow[rr, leftrightarrow, "\sim"] \arrow[dr, "\rho^{\text{ep}}(\sigma_1,\sigma_2)"'] & & K_1(B\pi) \arrow[dl, "\rho\,(\sigma_1,\sigma_2)"] \\
& \mathbb{R}/\mathbb{Z} &
\end{tikzcd}$$

Hence, even if the spectral set of $D^+(X)$ is not discrete, we can still define its $\mathbb{R}/\mathbb{Z}$ end-periodic rho invariant in a perfectly reasonable and consistent manner. This allows us to define the $\mathbb{R}/\mathbb{Z}$ invariant, for instance, in the case where $\operatorname{Ind} D^+(X) \neq 0$. For the applications to positive scalar curvature, the end-periodic rho invariant is well-defined and given by the usual formula (3.8), since in Remark 3.11 we have noted that the spectral sets of its twisted operators are discrete.

Proof. That every equivalence class in K^{ep}-homology has a representative with discrete spectral set follows from the proof of Theorem 3.3 — the cycle (X, S, γ, f) is bordant to the cycle $(S^1 \times Y, S^+ \oplus S^+, d\theta, f)$, which has discrete spectral set by part (1) of Remark 3.11.

As we shall see, it is only necessary to prove invariance of ρ^{ep} under bordism, and then Theorem 3.3 will imply invariance under the other relations defining K^{ep}-homology. First suppose that (X, S, γ, f) is a boundary with Dirac operator $D^+(X)$ such that the families associated to the twisted operators $D_1^+(X)$ and $D_2^+(X)$ have discrete spectral sets. We apply the MRS index theorem to each operator separately to get

$$\operatorname{Ind}_{\mathrm{MRS}} D_i^+(Z_\infty) = \int_Z \boldsymbol{I}(D_i^+(Z)) - \int_Y \omega_i + \int_X dF \wedge \omega_i - \tfrac{1}{2} h_i + \eta^{\mathrm{ep}}(D_i^+(X))$$

for $i = 1, 2$. Now, since we are twisting by flat vector bundles, both the index form and the transgression classes for the twisted operators are constant multiples of the index form and transgression class of the original operator. Hence when we subtract the two equations, the terms involving these vanish and we are left with

$$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2; X, S, \gamma, f) = \operatorname{Ind}_{\mathrm{MRS}} D_2^+(Z_\infty) - \operatorname{Ind}_{\mathrm{MRS}} D_1^+(Z_\infty),$$

which is an integer. The end-periodic rho invariant behaves additively under disjoint unions of cycles and changes sign when the negative of a cycle is taken. This proves bordism invariance mod $\mathbb{Z}$.

Now the K^{ep}-cycle (X, S, γ, f) with discrete spectral sets is bordant to $(S^1 \times Y, S^+ \oplus S^+, d\theta, f)$, where Y is Poincaré dual to γ. By [Mrowka et al. 2016, §6.3], the end-periodic rho invariant of $(S^1 \times Y, S^+ \oplus S^+, d\theta, f)$ is equal to the rho invariant of the K-cycle (Y, S^+, f). Hence

$$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2; X, S, \gamma, f) = \rho(\sigma_1, \sigma_2; Y, S^+, f) \mod \mathbb{Z}.$$

The isomorphism $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$ then immediately implies the theorem. □

4. End-periodic bordism groups

In this section, we recall the definition of the spin bordism groups, and introduce the analogous bordism groups in the end-periodic setting. As for K-homology, there are natural isomorphisms between the spin bordism groups and the end-periodic spin bordism groups. We also consider the PSC spin bordism groups described in Botvinnik and Gilkey [1995], and define the corresponding end-periodic PSC spin bordism groups. Throughout, we take $m \geq 5$ to be a positive odd integer.

4.1. ***Spin bordism and end-periodic spin bordism.*** We recall the definition of the spin bordism group $\Omega_m^{\mathrm{spin}}(B\pi)$ for a discrete group π.

Definition 4.1. An Ω_m^{spin}-*cycle* for $B\pi$ is a triple (M, σ, f), where M is a compact oriented Riemannian spin manifold of dimension m, σ is a choice of spin structure on M, and $f : M \to B\pi$ is a continuous map.

The *negative* of an Ω_m^{spin}-cycle (M, σ, f) is $(-M, \sigma, f)$, where $-M$ is M with the reversed orientation. An Ω_m^{spin}-cycle (M, σ, f) is a *boundary* if there exists a compact oriented Riemannian manifold W with boundary $\partial W = M$, a spin structure on W whose restriction to the boundary is the spin structure σ, and a continuous map $W \to B\pi$ extending the map f. Two Ω_m^{spin}-cycles (M, σ, f) and (M', σ', f') are *bordant* if $(M, \sigma, f) \amalg (-M', \sigma', f')$ is a boundary.

Definition 4.2. The m-dimensional spin bordism group $\Omega_m^{\mathrm{spin}}(B\pi)$ for $B\pi$, consists of Ω_m^{spin}-cycles for $B\pi$ modulo the equivalence relation of bordism. It is an abelian group with addition given by disjoint union of cycles.

The end-periodic spin bordism group $\Omega_m^{\mathrm{ep,spin}}(B\pi)$, is defined in an analogous way to the end-periodic K-homology group.

Definition 4.3. An $\Omega_m^{\mathrm{ep,spin}}$-cycle for $B\pi$ is a quadruple (X, σ, γ, f) where X is a compact oriented Riemannian spin manifold of dimension $m+1$, σ is a spin structure on X, γ is a cohomology class in $H^1(X, \mathbb{Z})$ that is primitive on each component of X, and $f : X \to B\pi$ is a continuous map.

The definition of a boundary is essentially the same as for end-periodic K-homology.

Definition 4.4. An $\Omega_m^{\mathrm{ep,spin}}$-cycle (X, σ, γ, f) is a *boundary* if there exists an end-periodic oriented Riemannian spin manifold Z_∞ with end modelled on (X, γ), such that the pulled back spin structure σ on the periodic end extends to Z_∞, as does the pulled back map f to $B\pi$.

The *negative* of a cycle (X, σ, γ, f) is $(X, \sigma, -\gamma, f)$. As before, we introduce the additional relation of *orientation/sign*:

$$(X, \sigma, -\gamma, f) \sim (-X, \sigma, \gamma, f).$$

Two $\Omega_m^{\mathrm{ep,spin}}$-cycles (X, γ, σ, f) and $(X', \gamma', \sigma', f')$ are *bordant* if $(X, \sigma, \gamma, f) \amalg (X, \sigma, -\gamma, f)$ is a boundary.

Definition 4.5. The m-dimensional *end-periodic spin bordism group* $\Omega_m^{\mathrm{ep,spin}}(B\pi)$ consists of $\Omega_m^{\mathrm{ep,spin}}$-cycles modulo the equivalence relation generated by bordism and orientation/sign, with addition given by disjoint union.

Analogous to the K-homology groups from Section 2, there is a canonical isomorphism between the spin bordism and end-periodic spin bordism groups which we will now describe.

The map $\Omega_m^{\mathrm{spin}}(B\pi) \to \Omega_m^{\mathrm{ep,spin}}(B\pi)$ takes a $\Omega_m^{\mathrm{spin}}(B\pi)$-cycle (M, σ, f) to $(S^1 \times M, 1 \times \sigma, d\theta, f)$, where $S^1 \times M$ has the product orientation and Riemannian metric, $1 \times \sigma$ is the product spin structure of the trivial spin structure 1 on S^1 with the spin structure σ on M, $d\theta$ is the standard generator of the first cohomology of S^1, and f is the obvious extension of $f : M \to B\pi$ to $S^1 \times M$.

Proposition 4.6. *The map which sends an $\Omega_m^{\mathrm{spin}}(B\pi)$-cycle (M, σ, f) to the*

$$\Omega_m^{\mathrm{ep,spin}}(B\pi)\text{-}cycle\ (S^1 \times M, 1 \times \sigma, d\theta, f)$$

is well-defined on spin bordism groups.

Proof. If (M, σ, f) and (M', σ', f') are bordant, with W bounding their disjoint union, then $\mathbb{R}_{\geq 0} \times M$ and $\mathbb{R}_{\leq 0} \times M'$ can be joined using W to form an end-periodic manifold Z_∞ with multiple ends. All structures extend to Z_∞ by assumption, hence the two $\Omega_m^{\mathrm{ep,spin}}(B\pi)$-cycles $(S^1 \times M, 1 \times \sigma, d\theta, f)$ and $(-S^1 \times M, 1 \times \sigma', -d\theta, f')$ are bordant. Using the orientation/sign relation, we see that $(S^1 \times M, 1 \times \sigma, d\theta, f)$ and $(S^1 \times M', 1 \times \sigma', d\theta, f')$ are equivalent. □

Now for the map $\Omega_m^{\mathrm{ep,spin}}(B\pi) \to \Omega_m^{\mathrm{spin}}(B\pi)$. Let (X, σ, γ, f) be an $\Omega_m^{\mathrm{ep,spin}}$-cycle for $B\pi$, and Y be a submanifold of X Poincaré dual to γ. We equip Y with the induced spin structure and orientation from γ. Explicitly, the orientation of Y is as in the paragraph after Definition 2.9, and the restricted spin structure is obtained first by cutting X open along Y to get a manifold W with boundary $\partial W = Y \amalg -Y$, and then taking the boundary spin structure on the positively oriented component Y of ∂W. This yields an Ω_m^{spin}-cycle (Y, σ, f), where σ and f are restricted to Y.

Proposition 4.7. *The map taking an $\Omega_m^{\mathrm{ep,spin}}(B\pi)$-cycle (X,σ,γ, f) to the $\Omega_m^{\mathrm{spin}}(B\pi)$-cycle (Y, σ, f) described above is well-defined on bordism groups.*

Proof. Independence of the choice of Y is proved as for the K-homology case, only with spin structures instead of Dirac bundles. It is clear that the orientation/sign relation is respected, since both $(X, \sigma, -\gamma, f)$ and $(-X, \sigma, \gamma, f)$ get sent to $(-Y, \sigma, f)$. If (X, σ, γ, f) and $(X', \sigma', \gamma', f')$ are bordant, then there is a compact manifold Z with boundary $\partial Z = Y \amalg -Y'$ such that the spin structures and maps extend over Z. But this shows that (Y, σ, f) and (Y', σ', f') are bordant. □

Theorem 4.8. *The above maps of bordism groups are inverse to each other, and so define a natural isomorphism of abelian groups $\Omega_m^{\mathrm{spin}}(B\pi) \cong \Omega_m^{\mathrm{ep,spin}}(B\pi)$.*

Proof. A cycle (M, σ, f) gets mapped to $(S^1 \times M, 1 \times \sigma, d\theta, f)$, which gets returned to $(M, 1 \times \sigma, f)$, where the latter two entries are restricted to M. It is straightforward to check that the product spin structure $1 \times \sigma$ restricted to M yields the original spin structure σ. Therefore we obtain our original cycle (M, σ, f) after mapping it to and from end-periodic bordism.

Now let (X, σ, γ, f) be an end-periodic cycle, with submanifold Y Poincaré dual to γ. This maps to a cycle (Y, σ, f), where the latter two structures are restricted from X, and this maps back to $(S^1 \times Y, 1 \times \sigma, d\theta, f)$. The same argument as in the proof of Definition 2.9 shows that this is bordant to (X, σ, γ, f). □

4.2. ***PSC spin bordism and end-periodic PSC spin bordism.*** Botvinnik and Gilkey [1995] use a variant of spin cobodism tailored to the setting of manifolds with positive scalar curvature, which we now recall.

Definition 4.9. A $\Omega_m^{\mathrm{spin},+}$-cycle is a quadruple (M, g, σ, f), where M is a compact oriented Riemannian spin manifold of dimension m with a metric g of positive scalar curvature, σ is a spin structure on M, and $f : M \to B\pi$ is a continuous map.

The negative of (M, g, σ, f) is $(-M, g, \sigma, f)$, as before. A cycle (M, g, σ, f) is called a *boundary* if there is a compact oriented Riemannian spin manifold W with boundary $\partial W = M$ so that the spin structure σ and map f extend to W. It is also required that W has a metric of positive scalar curvature that is a product metric $dt^2 + g$ in a neighbourhood of the boundary. Two cycles are *bordant* if the disjoint union of one with the negative of the other is a boundary.

Definition 4.10. The *PSC spin bordism group* $\Omega_m^{\mathrm{spin},+}(B\pi)$ for $B\pi$ consists of $\Omega_m^{\mathrm{spin},+}$-cycles modulo bordism, with addition given by disjoint union.

We now define the end-periodic PSC spin bordism group $\Omega_m^{\mathrm{ep,spin},+}(B\pi)$ for $B\pi$.

Definition 4.11. An $\Omega_m^{\mathrm{ep,spin},+}$*-cycle* is a quintuple $(X, g, \sigma, \gamma, f)$, where X is a compact oriented Riemannian spin manifold of dimension $m+1$ with a metric g of positive scalar curvature, σ is a choice of spin structure on X, γ is a cohomology class in $H^1(X, \mathbb{Z})$ whose restriction to each component of X is primitive, and $f : X \to B\pi$ is a continuous map. We further require that there is a submanifold Y of X that is Poincaré dual to γ, such that the induced metric on Y has positive scalar curvature, and the metric on X is a product metric $dt^2 + g_Y$ in a neighbourhood of Y.

Let $(X, g, \sigma, \gamma, f)$ be an $\Omega_m^{\mathrm{ep,spin},+}$-cycle and take $Y \subset X$ to be a submanifold with PSC that is Poincaré dual to γ, having the product metric in a tubular neighbourhood. As before we form $X_1 = \bigcup_{k\geq 0} W_k$, where the W_k are isometric copies of X cut open along Y. For $(X, g, \sigma, \gamma, f)$ to be a *boundary* means that there is a compact oriented Riemannian spin manifold Z of positive scalar curvature, whose metric is a product near the boundary, which can be attached to X_1 along Y to form a complete oriented Riemannian spin manifold of PSC $Z_\infty = Z \cup_Y X_1$, such that the pulled back spin structure σ and map f on X_1 extend over Z.

The *negative* of $(X, g, \sigma, \gamma, f)$ is $(X, g, \sigma, -\gamma, f)$, and we have the *orientation/sign* relation

$$(X, g, \sigma, -\gamma, f) \sim (-X, g, \sigma, \gamma, f).$$

Two $\Omega_m^{\mathrm{ep,spin},+}$-cycles are *bordant* if the disjoint union of one with the negative of the other is a boundary.

Definition 4.12. The m-dimensional *end-periodic PSC spin bordism group* $\Omega_m^{\mathrm{ep,spin},+}(B\pi)$ for $B\pi$ consists of $\Omega_m^{\mathrm{ep,spin},+}$-cycles modulo bordism and orientation/sign, with addition given by disjoint union.

Theorem 4.13. *There is a canonical isomorphism* $\Omega_m^{\mathrm{spin},+}(B\pi) \cong \Omega_m^{\mathrm{ep,spin},+}(B\pi)$.

The maps are exactly as for the spin bordism theories, only when mapping from $\Omega_m^{\mathrm{ep,spin},+}(B\pi)$ to $\Omega_m^{\mathrm{spin},+}(B\pi)$ the Poincaré dual submanifold Y must be taken to have PSC and a product metric in a tubular neighbourhood.

Proof. As before. □

4.3. ***Rho invariants.*** Given a triple (M, σ, f) and two unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, we define the rho invariant $\rho(\sigma_1, \sigma_2; M, \sigma, f)$ as before, using the spin Dirac operator for the cycle (M, S, f). We also define the end-periodic rho invariant for cycles (X, σ, γ, f) in an entirely analogous manner, using the end-periodic eta invariant of MRS instead. Of course, we must again be careful with the definition, allowing only the rho invariant for cycles whose twisted operators have discrete spectral sets to be defined in terms of the true end-periodic eta invariants — all others are defined by taking bordant cycles with discrete spectra. We remark also that in the case of positive scalar curvature, the h-terms appearing in the definition of the rho invariants vanish.

Theorem 4.14. *The rho invariant extends to a well-defined homomorphism*

$$\rho(\sigma_1, \sigma_2) : \Omega_m^{\mathrm{spin}}(B\pi) \to \mathbb{R}/\mathbb{Z},$$

as does the end-periodic rho invariant

$$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2) : \Omega_m^{\mathrm{ep,spin}}(B\pi) \to \mathbb{R}/\mathbb{Z}.$$

Furthermore, the following diagram commutes:

$$\begin{tikzcd} \Omega_m^{\mathrm{ep,spin}}(B\pi) \arrow[rr, leftrightarrow, "\sim"] \arrow[dr, "\rho^{\mathrm{ep}}(\sigma_1,\sigma_2)"'] & & \Omega_m^{\mathrm{spin}}(B\pi) \arrow[dl, "\rho(\sigma_1,\sigma_2)"] \\ & \mathbb{R}/\mathbb{Z} & \end{tikzcd}$$

Proof. Apply the APS and MRS index theorems respectively, and use the isomorphism of Theorem 4.8. □

Now for the positive scalar curvature case.

Theorem 4.15. *The rho invariant extends to a well-defined homomorphism*

$$\rho(\sigma_1, \sigma_2) : \Omega_m^{\mathrm{spin},+}(B\pi) \to \mathbb{R},$$

as does the end-periodic rho invariant

$$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2) : \Omega_m^{\mathrm{ep,spin},+}(B\pi) \to \mathbb{R}.$$

Furthermore, the following diagram commutes:

$$\begin{tikzcd} \Omega_m^{\mathrm{ep,spin},+}(B\pi) \arrow[rr, leftrightarrow, "\sim"] \arrow[dr, "\rho^{\mathrm{ep}}(\sigma_1,\sigma_2)"'] & & \Omega_m^{\mathrm{spin},+}(B\pi) \arrow[dl, "\rho(\sigma_1,\sigma_2)"] \\ & \mathbb{R} & \end{tikzcd}$$

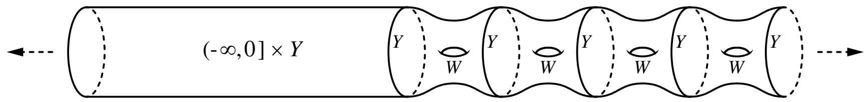

Figure 4. End-periodic manifold with two ends.

Remark 4.16. The end-periodic rho invariant appearing in the theorem is given on all representatives of equivalence classes as the genuine difference of the twisted eta invariants as in formula (3.8), due to Remark 3.11.

For the proof, we will need the following (see [Mrowka et al. 2016, Proposition 8.5 (ii)]).

Lemma 4.17. *If* $(X, g, \sigma, \gamma, f)$ *is an* $\Omega_m^{\mathrm{ep,spin},+}$*-cycle and* (Y, g, σ, f) *is the* $\Omega_m^{\mathrm{spin},+}$*-cycle it maps to, then*

$$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,;\, X, g, \sigma, \gamma, f) = \rho(\sigma_1, \sigma_2\,;\, Y, g, \sigma, f).$$

Proof. We join $\mathbb{R}_{\geq 0} \times Y$ to $X_1 = \bigcup_{k\geq 0} W_k$ together as in Figure 4 to form an end-periodic spin manifold Z_∞ with two ends. [Mrowka et al. 2016, Lemma 8.1] (which uses the results of Gromov and Lawson [1983]) gives that the spin Dirac operator $D^+(Z_\infty)$ is Fredholm and has zero index. The same holds for its twisted counterparts. Applying the MRS index theorem to the two twisted spin Dirac operators $D_1^+(Z_\infty)$ and $D_2^+(Z_\infty)$, and subtracting the equations as per usual then yields the result. □

Proof of Theorem 4.15. See [Botvinnik and Gilkey 1995, Theorem 1.1] for the proof that the map $\rho\,(\sigma_1, \sigma_2) : \Omega_m^{\mathrm{spin},+}(B\pi) \to \mathbb{R}$ is well-defined. Lemma 4.17 and the isomorphism of Theorem 4.13 then immediately imply the result. □

5. End-periodic structure group

Let $\sigma_1, \sigma_2 : \pi \to U(N)$ be unitary representations of the discrete group π. Recall the definition of the structure group $S_1(\sigma_1, \sigma_2)$ of Higson–Roe, starting from [Higson and Roe 2010, Definition 8.7].

Definition 5.1. An *odd* (σ_1, σ_2)*-cycle* is a quintuple (M, S, f, D, n) where (M, S, f) is an odd K-cycle for $B\pi$, D is a Dirac operator for (M, S, f), and $n \in \mathbb{Z}$.

A (σ_1, σ_2)-cycle (M, S, f, D, n) is a *boundary* if the K-cycle (M, S, f) is bounded by a manifold W (as in Definition 2.3) and there are Dirac operators $D_1(W)$ and $D_2(W)$ on W which bound the twisted Dirac operators D_1 and D_2 on M, such that

$$\mathrm{Ind}_{\mathrm{APS}}\, D_1^+(W) - \mathrm{Ind}_{\mathrm{APS}}\, D_2^+(W) = n.$$

Since we are no longer looking at rho invariants modulo integers or at spin Dirac operators, we will denote by $\rho(\sigma_1, \sigma_2\,; D, f)$ the rho invariant of Definition 3.2, indicating its possible dependence on the Dirac operator D.

Lemma 5.2 (Higson and Roe 2010, Lemma 8.10). *If a (σ_1, σ_2)-cycle (M, S, f, D, n) is a boundary, then $\rho(\sigma_1, \sigma_2; D, f) + n = 0$.*

Definition 5.3. The *relative eta invariant*, or *rho invariant* of the (σ_1, σ_2)-cycle (M, S, f, D, n) is $\rho(\sigma_1, \sigma_2\,; D, f) + n$.

The *disjoint union* of (σ_1, σ_2)-cycles is defined as,

$$(M, S, f, D, n) \amalg (M', S', f', D', n') = (M \amalg M', S \amalg S', f \amalg f', D \amalg D', n + n').$$

The *negative* of a (σ_1, σ_2)-cycle (M, S, f, D, n), is defined as,

$$-(M, S, f, D, n) = (M, -S, f, -D, h_1 - h_2 - n),$$

where $h_1 = \dim\ker(D_1)$ and $h_2 = \dim\ker(D_2)$. Two (σ_1, σ_2)-cycles are *bordant* if the disjoint union of one cycle with the negative of the other is a boundary.

The two remaining relations to define are:

- *Direct sum/disjoint union*:

$$(M, S \oplus S', f, D \oplus D', n) \sim (M \amalg M, S \amalg S', f \amalg f, D \amalg D', n).$$

- *Bundle Modification*: If $(\widehat{M}, \widehat{S}, \hat{f})$ is an elementary bundle modification of (M, S, f) with the Dirac operator $\widehat{D}$ from 2.5, then

$$(M, S, f, D, n) \sim (\hat{M}, \hat{S}, \hat{f}, \hat{D}, n).$$

Definition 5.4. The structure group $S(\sigma_1, \sigma_2)$, is the set of equivalence classes of (σ_1, σ_2)-cycles under the equivalence relation generated by bordism, direct sum/disjoint union, and bundle modification. It is an abelian group with addition is given by disjoint union.

In [Higson and Roe 2010, Proposition 8.14], it is proved that the relative eta invariant of a (σ_1, σ_2)-cycle depends only on the class that the cycle determines in $S(\sigma_1, \sigma_2)$. Hence there is a well-defined group homomorphism $\rho : S(\sigma_1, \sigma_2) \to \mathbb{R}$, defined by

$$\rho(M, S, f, D, n) = \rho(\sigma_1, \sigma_2\,; D, f) + n.$$

5.1. *End-periodic structure group.* We define in a parallel manner the end-periodic structure group $S_1^{\mathrm{ep}}(\sigma_1, \sigma_2)$.

Definition 5.5. An *odd* $(\sigma_1, \sigma_2)^{\mathrm{ep}}$*-cycle* is a sextuple (X, S, γ, f, D, n) where (X, S, γ, f) is a K^{ep}-cycle for $B\pi$, D is a Dirac operator for (X, S, γ, f), and $n \in \mathbb{Z}$. We additionally assume that the spectral set of the family $D_z^+(X)$ is discrete.

A $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycle (X, S, f, γ, D, n) is a *boundary* if the K^{ep}-cycle (X, S, γ, f) is a boundary (Definition 2.10), and moreover there is a Dirac operator $D(Z_\infty)$ on the manifold Z_∞ extending the Dirac operator D on $X_1 = \bigcup_{k\geq 0} W_k$ such that the difference of the MRS indices

$$\mathrm{Ind}_{\mathrm{MRS}}(D_1^+(Z_\infty)) - \mathrm{Ind}_{\mathrm{MRS}}(D_2^+(Z_\infty)) = n.$$

Here the $D_i^+(Z_\infty)$ are the twists of $D^+(Z_\infty)$ by the flat vector bundles determined by the extension of f to Z_∞ and by σ_1, σ_2. We can show the analog of Lemma 5.2

Lemma 5.6. *If a* $(\sigma_1, \sigma_2)^{\mathrm{ep}}$*-cycle* (X, S, γ, f, D, n) *is a boundary, then*

$$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,;\, D, f, \gamma) + n = 0.$$

We call the quantity $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,;\, D, f, \gamma) + n$ the *end-periodic rho invariant* of the $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycle (X, S, γ, f, D, n).

The *disjoint union* of $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycles is defined as

$$(X, S, f, \gamma, D, n) \amalg (X', S', \gamma', f', D', n') \\ = (X \amalg X', S \amalg S', \gamma \amalg \gamma', f \amalg f', D \amalg D', n + n').$$

The *negative* of a $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycle (X, S, γ, f, D, n), is

$$-(X, S, \gamma, f, D, n) = (X, S, -\gamma, f, D, h_1 - h_2 - n),$$

where h_1, h_2 are the integers occurring in the MRS index theorem associated to σ_1, σ_2. Two $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycles are *bordant* if the disjoint union of one with the negative of the other is a boundary. We also have:

- *Direct sum/disjoint union*:

$$(X, S \oplus S', \gamma + \gamma' f, D \oplus D', n) \sim (X \amalg M, S \amalg S', \gamma \amalg \gamma', f \amalg f, D \amalg D', n).$$

- *Bundle modification*: If $(\hat{X}, \hat{S}, \hat{\gamma}, \hat{f})$ is an elementary bundle modification of (X, S, γ, f) and $\hat{D}$ is the Dirac operator of Remark 2.5, then

$$(X, S, \gamma, f, D, n) \sim (\hat{X}, \hat{S}, \hat{\gamma}, \hat{f}, \hat{D}, n).$$

- *Orientation/sign*:

$$(X, S, -\gamma, f, D, n) \sim (-X, \Pi(S), \gamma, f, D, n).$$

Definition 5.7. The end-periodic structure group, denoted by $S_1^{\mathrm{ep}}(\sigma_1, \sigma_2)$, is the set of equivalence classes of $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycles under the equivalence relation generated by bordism, direct sum/disjoint union, bundle modification, and orientation/sign. It is an abelian group with unit and addition is given by disjoint union.

Define the group homomorphism $\rho^{\mathrm{ep}} : S_1^{\mathrm{ep}}(\sigma_1, \sigma_2) \to \mathbb{R}$ by the formula

$$\rho^{\mathrm{ep}}(X, S, \gamma, f, D, n) = \rho^{\mathrm{ep}}(\sigma_1, \sigma_2; D, f, \gamma) + n.$$

Then the following theorem is the analog of Theorem 3.15 is proved in a similar way.

Theorem 5.8. *The end-periodic rho invariant $\rho^{\mathrm{ep}}(X, S, \gamma, f, \sigma_1, \sigma_2) + n$ associated to the $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycle (M, S, γ, f, D, n) depends only on the equivalence class of (M, S, γ, f, D, n) in $S_1^{\mathrm{ep}}(\sigma_1, \sigma_2)$. Hence there is a well-defined group homomorphism*

$$\rho^{\mathrm{ep}} : S_1^{\mathrm{ep}}(\sigma_1, \sigma_2) \to \mathbb{R}.$$

Furthermore, the following diagram commutes:

$$\begin{tikzcd} S_1^{\mathrm{ep}}(\sigma_1, \sigma_2) \arrow[rr, leftrightarrow, "\sim"] \arrow[dr, "\rho^{\mathrm{ep}}"'] & & S_1(\sigma_1, \sigma_2) \arrow[dl, "\rho"] \\ & \mathbb{R} & \end{tikzcd}$$

Here the maps

$$S_1^{\mathrm{ep}}(\sigma_1, \sigma_2) \leftrightarrow S_1(\sigma_1, \sigma_2)$$

are the analog of the maps in K-homologies given earlier.

Also, Higson and Roe establish a commuting diagram of short exact sequences; see [Higson and Roe 2010], the paragraph below Definition 8.6,

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "="] & S_1(\sigma_1, \sigma_2) \arrow[r] \arrow[d, "\rho"] & K_1(B\pi) \arrow[r] \arrow[d, "\rho(\sigma_1,\sigma_2)"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[r] & 0. \end{tikzcd} \tag{5.1}$$

By Theorems 5.8 and 3.15, we deduce that there is a commuting diagram of short exact sequences,

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "="] & S_1^{\mathrm{ep}}(\sigma_1, \sigma_2) \arrow[r] \arrow[d, "\rho^{\mathrm{ep}}"] & K_1^{\mathrm{ep}}(B\pi) \arrow[r] \arrow[d, "\rho^{\mathrm{ep}}(\sigma_1,\sigma_2)"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \arrow[r] & 0. \end{tikzcd} \tag{5.2}$$

This tells us when the $\mathbb{R}/\mathbb{Z}$-index theorem can be refined to an $\mathbb{R}$-index theorem.

6. Applications to positive scalar curvature

Using the above isomorphisms of K-homologies and cobordism theories, we can immediately transfer results on positive scalar curvature from the odd-dimensional case to the even-dimensional case in which a primitive 1-form is given.

6.1. ***Odd-dimensional results in the literature.*** First we will state the odd-dimensional results that we will be generalising to the even-dimensional case using our isomorphisms. The first ones are obstructions to positive scalar curvature.

Theorem 6.1 (Weinberger 1988; Higson and Roe 2010, Theorem 6.9). *Let (M, S, f) be an odd K-cycle for $B\pi$, where M is an odd dimensional spin manifold with a Riemannian metric of positive scalar curvature, and S is the bundle of spinors on M. Then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated rho invariant $\rho(\sigma_1, \sigma_2\,; M, S, f)$ is a rational number.*

Theorem 6.2 (Higson and Roe 2010, Remark 6.10). *Let (M, S, f) be an odd K-cycle for $B\pi$, where M is an odd dimensional spin manifold with a Riemannian metric of positive scalar curvature, and S is the bundle of spinors on M. If the maximal Baum–Connes map for π is injective, then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated rho invariant $\rho(\sigma_1, \sigma_2\,; M, S, f)$ is an integer.*

Remarks 6.3. The maximal Baum–Connes map for π is injective whenever for instance π is a torsion-free linear discrete group [Guentner et al. 2005].

Theorem 6.4 (Higson and Roe 2010, Theorem 1.1; Keswani 2000). *Let (M, S, f) be an odd K-cycle for $B\pi$, where M is an odd dimensional spin manifold with a Riemannian metric of positive scalar curvature, and S is the bundle of spinors on M. If the maximal Baum–Connes conjecture holds for π, then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated rho invariant $\rho(\sigma_1, \sigma_2;$ $M, S, f)$ is zero.*

Remarks 6.5. The maximal Baum–Connes conjecture holds for π whenever π is K-amenable.

We now turn to a result on the number of path components of the moduli space of PSC metrics modulo diffeomorphism, $\mathfrak{M}^+(M)$. Denote for a group π, the representation ring $R(\pi)$ consisting of formal differences of finite dimensional unitary representations, and let $R_0(\pi)$ be those formal differences with virtual dimension zero (an element of $R_0(\pi)$ can be thought of as an ordered pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$). Following Botvinnik and Gilkey [1995], introduce the subgroups

$$R_0^{\pm}(\pi) = \{\alpha \in R_0(\pi) : \operatorname{tr}(\alpha(\lambda)) = \pm \operatorname{tr}(\alpha(\lambda^{-1})) \text{ for all } \lambda \in \pi\}$$

and define

$$r_m(\pi) = \begin{cases} \operatorname{rank}_{\mathbb{Z}} R_0^+(\pi) & \text{if } m = 3 \mod 4, \\ \operatorname{rank}_{\mathbb{Z}} R_0^-(\pi) & \text{if } m = 1 \mod 4. \end{cases}$$

The following is a result of Botvinnik and Gilkey on the number of path components of the moduli space of PSC metrics modulo diffeomorphism.

Theorem 6.6 (Botvinnik and Gilkey 1995, Theorem 0.3). *Let M be a compact connected spin manifold of odd dimension $m \geq 5$ admitting a metric of positive scalar curvature. Suppose that $\pi = \pi_1(M)$ is finite and nontrivial, and that $r_m(\pi) > 0$. Then the moduli space of PSC metrics modulo diffeomorphism $\mathfrak{M}^+(M)$ has infinitely many path components.*

Their proof involves finding a countably indexed family of metrics g_i of positive scalar curvature on M so that $\rho(M, g_i) \neq \rho(M, g_j)$ for $i \neq j$. If these metrics were homotopic through PSC metrics, then they would lie in the same PSC bordism class and hence have equal rho invariants. We will extend this result to the even-dimensional case under the additional hypothesis of "psc-adaptability"; see Definition 6.11.

6.2. *Our even-dimensional results.* In the following theorems, we assume that Y is a submanifold of X that is Poincaré dual to a primitive class $\gamma \in H^1(X, \mathbb{Z})$ such that the scalar curvature of Y in the induced metric is positive. In fact, the theorems even hold under the weaker hypothesis that the induced metric on Y is conformal to a metric of positive scalar curvature. By a theorem of [Schoen and Yau 1979], if $\dim(X) = n \leq 7$, then every homology class in $H_{n-1}(X, \mathbb{Z})$ has a representative that is a smooth, orientable minimal hypersurface. It follows that if X is spin with positive scalar curvature, then a Poincaré dual to a primitive class $\gamma \in H^1(X, \mathbb{Z})$ can be chosen to be a smooth, spin minimal hypersurface Y, such that the induced metric on Y is conformal to one of positive scalar curvature. So this weaker assumption is automatically true when $\dim(X) = n \leq 7$.

The following is our even-dimensional analog of Theorem 6.1.

Theorem 6.7. *Let (X, S, γ, f) be an odd K^{ep}-cycle for $B\pi$, where X is an even-dimensional spin manifold with a Riemannian metric of positive scalar curvature, S is the bundle of spinors on X and γ a primitive class in $H^1(X, \mathbb{Z})$ such that there is a Poincaré dual submanifold Y whose scalar curvature in the induced metric is positive. Then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated end-periodic rho invariant $\rho^{\text{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f)$ is a rational number.*

Proof. The odd K^{ep}-cycle for $B\pi$, (X, S, γ, f) determines an odd K-cycle for $B\pi$, (Y, S^+, f) where Y is a Poincaré dual submanifold for γ having positive scalar curvature, and is given the induced spin structure from X. By Theorem 6.1,

$\rho(\sigma_1, \sigma_2\,; Y, S^+, f) \in \mathbb{Q}$. By Theorem 3.15 it follows that $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f) \in \mathbb{Q}$ as claimed. □

Next is our even-dimensional analog of Theorem 6.2, and is argued as above.

Theorem 6.8. *Let (X, S, γ, f) be an odd K^{ep}-cycle for $B\pi$, where X is an even-dimensional spin manifold with a Riemannian metric of positive scalar curvature, S is the bundle of spinors on X and γ a primitive class in $H^1(X, \mathbb{Z})$ such that there is a Poincaré dual submanifold Y whose scalar curvature in the induced metric is positive. If the maximal Baum–Connes map for π is injective, then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated end-periodic rho invariant $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f)$ is an integer.*

Proof. As for Theorem 6.7. □

Here is the even-dimensional analog of Theorem 6.4.

Theorem 6.9. *Let (X, S, γ, f) be an odd K^{ep}-cycle for $B\pi$, where X is an even-dimensional spin manifold with a Riemannian metric of positive scalar curvature, S is the bundle of spinors on X and γ a primitive class in $H^1(X, \mathbb{Z})$ such that there is a Poincaré dual submanifold Y whose scalar curvature in the induced metric is positive. If the maximal Baum–Connes conjecture holds for π, then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated end-periodic rho invariant $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f)$ is zero.*

Proof. The odd K^{ep}-cycle for $B\pi$, (X, S, γ, f) determines an odd K-cycle (Y, S^+, f) for $B\pi$, where Y is a Poincaré dual submanifold for γ having positive scalar curvature, and is endowed with the induced spin structure. By Theorem 6.4,

$$\rho(\sigma_1, \sigma_2; Y, S^+, f) = 0.$$

By 4.17 it follows that $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f) = 0$. □

Example 6.10. Although ρ-invariants are difficult to compute, nevertheless thanks to many authors, there is now a decent set of computations that are available. We can use these to compute end-periodic rho invariants, which we will show in a simple example. Consider $Y = S^1$ with the trivial spin structure. Then unitary characters σ_1, σ_2 of the fundamental group of S^1 can be identified with real numbers, and a computation (see [Gilkey 1984b, p. 82]) says that the rho invariant of the spin Dirac operator is $\rho(S^1, \sigma_1, \sigma_2) = \sigma_1 - \sigma_2 \mod \mathbb{Z}$. In particular, $\rho(S^1, \sigma_1, \sigma_2)$ can take on any real value mod $\mathbb{Z}$. Let W be a spin cobordism from S^1 to S^1, and Σ be the compact spin Riemann surface (whose genus is ≥ 1) obtained as a result of gluing the two boundary components of W. Then S^1 is a codimension one submanifold of Σ that represents a generator a of $\pi_1(\Sigma)$. We can extend the characters σ_1, σ_2 of $a\mathbb{Z}$ to all of $\pi_1(\Sigma)$ by declaring them to be trivial on the other generators. Then by Theorem 3.15, it follows that $\rho^{\mathrm{ep}}(\Sigma, \gamma, \sigma_1, \sigma_2) = \sigma_1 - \sigma_2$

mod $\mathbb{Z}$, can take on any real value mod $\mathbb{Z}$, where γ is the degree one cohomology class on Σ which is Poincaré dual to S^1. We conclude by Theorem 6.7 that the Riemann surface Σ does not admit a PSC metric. This of course can also be proved by the Gauss–Bonnet theorem and is well-known.

The construction generalises easily to any odd dimensional spin manifold Y with nonzero rho invariant $\rho(Y, \sigma_1, \sigma_2) \neq 0 \bmod \mathbb{Z}$. We conclude by Theorem 3.15 that the resulting even dimensional spin manifold X constructed from a spin cobordism from Y to itself, has nonzero end-periodic rho invariant $\rho^{\mathrm{ep}}(X, \gamma, \sigma_1, \sigma_2) \neq 0 \bmod \mathbb{Z}$, where γ is the degree one cohomology class on X which is Poincaré dual to the submanifold Y. In particular, such an X does not admit a PSC metric. Examples of Y include odd-dimensional lens spaces $L(p; \vec{q})$, where it is shown in [Gilkey 1984a, Theorem 2.5, part (c)], that for any spin structure on $L(p; \vec{q})$, there is a representation σ of $\pi_1(L(p; \vec{q}))$ such that $\rho(L(p; \vec{q}), \mathrm{Id}, \sigma) \neq 0 \in \mathbb{Q}/\mathbb{Z}$. Explicitly, for 3 dimensional lens spaces $L(p, q)$, consider the representation $\sigma : \pi_1(L(p; \vec{q})) \to U(1)$ taking the generator $t \in \pi_1(L(p; q))$ to the unit complex number $\exp(2\pi\sqrt{-1}/p)$. Then $\rho(L(p; q), \mathrm{Id}, \sigma) = -(d/2p)(p+1) \neq 0 \in \mathbb{Q}/\mathbb{Z}$ where d is a certain integer relatively prime to $48p$. Then $\rho^{\mathrm{ep}}(X, \gamma, \mathrm{Id}, \sigma) \neq 0 \in \mathbb{Q}/\mathbb{Z}$. These results confirm Theorem 6.7 in these examples.

6.3. ***Size of the space of components of positive scalar curvature metrics.*** Hitchin [1974] proved the first results on the size of the space of components of the space of Riemannian metrics of positive scalar curvature metrics on a compact spin manifold, when nonempty. This sparked much interest in the topic and results by Botvinnik–Gilkey, Piazza–Schick and many others.

We now extend Theorem 6.6 to the even-dimensional case. We would like to say something like "Given an even-dimensional manifold X with PSC having a submanifold Y of PSC Poincaré dual to a primitive one-form γ, if $\mathfrak{M}^+(Y)$ has infinitely many path components then so does $\mathfrak{M}^+(X)$." The argument would involve using a countable family of PSC metrics on Y with distinct rho invariants to find a countable such family on X. There are complications however, since given an arbitrary PSC metric on Y, there is not necessarily a PSC metric on X whose restriction to Y is the given metric. Because we are already assuming that there is at least one PSC metric on X which restricts to a metric of PSC on Y, there are no obstructions from topology preventing this from being the case.

Definition 6.11. Let X be a compact even-dimensional manifold, and $\gamma \in H^1(X, \mathbb{Z})$ a primitive cohomology class with accompanying Poincaré dual submanifold Y. Suppose that there is at least one PSC metric on X which restricts to a PSC metric on Y. We say that X is *psc-adaptable with respect to Y* if for *every* PSC metric g_Y on Y, there is a PSC metric g_X on X that is a product metric $dt^2 + g_Y$ in a tubular neighbourhood of Y.

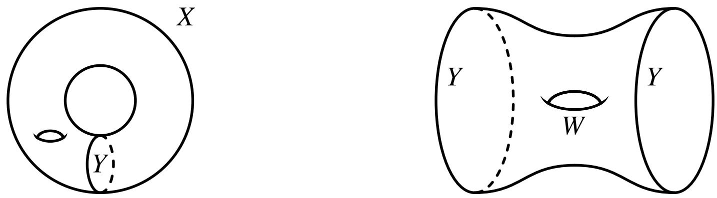

Figure 5. Pushing a PSC metric across W using the Miyazaki–Rosenberg construction.

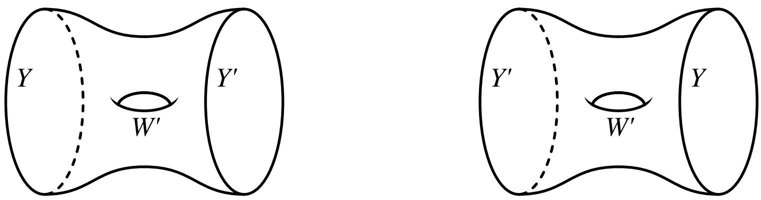

Figure 6. Obtaining a psc-adaptable manifold through a symmetric bordism.

Some notes and comments on the notion of psc-adaptability. Let X and Y be as in the above definition, and take an arbitrary PSC metric g_Y on Y. Cutting X open along Y, we obtain a self cobordism W of Y; see Figure 5. Under suitable assumptions on the topology of X and Y, a construction of Miyazaki [1984] and Rosenberg [1986] (using the theory of Gromov and Lawson [1980a] and Schoen and Yau [1979]) enables one to *push* the PSC metric on Y across the bordism (pictured on the right in the figure) to get a PSC metric on W restricting to metrics of PSC on each boundary component. One might then try to glue the manifold back together to obtain a PSC metric on X which restricts to the given metric g_Y on Y. The problem is that one doesn't know whether the new psc metric on Y is isotopic to the original. Hence the concept of psc-adaptability which hypothesizes that this is true. It is the case when the bordism is *symmetric* for instance. That is, starting with a bordism W' from Y to Y', we get a bordism from Y to itself by thinking of W' as a bordism from Y' to Y and gluing to the original bordism; see Figure 6.

Then one can use the Miyazaki–Rosenberg construction starting with the PSC metric g on Y to get another PSC metric on Y' halfway through, and then reverse the Miyazaki–Rosenberg construction from the PSC metric on the halfway Y' to get a PSC metric g' on Y on the other end, which isotopic to the original PSC metric g on Y. Considering a small cylinder over Y, and using the fact that isotopy implies concordance, see [Gromov and Lawson 1980a, Lemma 3], we can further push g' to g. Since the metrics agree on either end, the bordisms can be glued together.

Mrowka, Ruberman and Saveliev also note a class of psc-adaptable manifolds — those of the form $(S^1 \times Y)\#M$ where Y and M are manifolds of positive scalar curvature; see [Mrowka et al. 2016, Theorem 9.2]. The end-periodic bordism groups provide a more natural framework for their proof of the following:

Theorem 6.12 (Mrowka et al. 2016, Theorem 9.2). *Let X be a compact even-dimensional spin manifold of dimension ≥ 6 admitting a metric of positive scalar curvature. Suppose there is a submanifold $Y \subset X$ of PSC that is Poincaré dual to a primitive cohomology class $\gamma \in H^1(X, \mathbb{Z})$, such that $\pi = \pi_1(Y)$ is finite and nontrivial. Further assume that the classifying map $f : Y \to B\pi$ of the universal cover extends to X, and that X is psc-adaptable with respect to Y. If $r_m(\pi_1(Y)) > 0$, then $\pi_0(\mathfrak{M}^+(X))$ is infinite, where $\mathfrak{M}^+(X)$ denotes the quotient of the space of positive scalar curvature metrics by the diffeomorphism group.*

Proof. In the terminology of Section 4, we have an $\Omega_m^{\mathrm{ep,spin},+}(B\pi)$-cycle $(X, g, \sigma, \gamma, f)$, with associated $\Omega_m^{\mathrm{spin},+}(B\pi)$-cycle (Y, g, σ, f). Botvinnik and Gilkey [1995] construct a representation $\alpha : \pi \to U(N)$ of π and a countable family of metrics g_i on Y with

$$\rho(\alpha, 1; Y, g_i, \sigma, f) \neq \rho(\alpha, 1; Y, g_j, \sigma, f)$$

for $i \neq j$, where $1 : \pi \to U(N)$ is the trivial representation. Our assumption of psc-adaptability and Theorem 4.15 imply there is a countable family of metrics g_i on X with

$$\rho^{\mathrm{ep}}(\alpha, 1; X, g_i, \sigma, \gamma, f) \neq \rho(\alpha, 1; X, g_j, \sigma, \gamma, f)$$

for $i \neq j$. But [Mrowka et al. 2016, Theorem 9.1] says that homotopic metrics of PSC on X should have the same rho invariants. $\square$

7. Vanishing of end-periodic rho using the representation variety

In this section we give a proof of the vanishing of the end-periodic rho invariant of the twisted Dirac operator with coefficients in a flat Hermitian vector bundle on a compact even-dimensional Riemannian spin manifold X of positive scalar curvature using the representation variety of $\pi_1(X)$ instead.

Let $\iota\colon Y \hookrightarrow X$ be a codimension one submanifold of X which is Poincaré dual to a generator $\gamma \in H^1(X, \mathbb{Z})$.

Let $\mathfrak{R} = \mathrm{Hom}(\pi, U(N))$ denote the representation variety of $\pi = \pi_1(Y)$, and $\widetilde{\mathfrak{R}}$ denote the representation variety of $\pi_1(X)$. We now construct a generalisation of the Poincaré vector bundle $\mathcal{P}$ over $B\pi \times \mathfrak{R}$. Let $E\pi \to B\pi$ be a principal π-bundle over the space $B\pi$ with contractible total space $E\pi$. Let $h\colon Y \to B\pi$ be a continuous map classifying the universal π-covering of Y. We construct a tautological rank N Hermitian vector bundle $\mathcal{P}$ over $B\pi \times \mathfrak{R}$ as follows: consider the action of π on $E\pi \times \mathfrak{R} \times \mathbb{C}^N$ given by

$$E\pi \times \mathfrak{R} \times \mathbb{C}^N \times \pi \to E\pi \times \mathfrak{R} \times \mathbb{C}^N, \quad ((q, \sigma, v), \tau) \to (q\tau, \sigma, \sigma(\tau^{-1})v).$$

Define the universal rank N Hermitian vector bundle $\mathcal{P}$ over $B\pi \times \mathfrak{R}$ to be the quotient $(E\pi \times \mathfrak{R} \times \mathbb{C}^N)/\pi$. Then $\mathcal{P}$ has the property that the restriction $\mathcal{P}|_{B\pi \times \{\sigma\}}$ is the flat Hermitian vector bundle over $B\pi$ defined by σ. Let I denote the closed unit interval $[0, 1]$ and $\beta : I \to \mathfrak{R}$ be a smooth path in $\mathfrak{R}$ joining the unitary representation α to the trivial representation. Define $E = (f \times \beta)^*\mathcal{P} \to Y \times I$ to be the Hermitian vector bundle over $Y \times I$, where $f : Y \to B\pi$ is the classifying map of the universal cover of Y. Let $E_t \to Y \times \{t\}$ denote the restriction of E to $Y \times \{t\}$. Then E_t is the flat unitary Hermitian vector bundle over Y determined by the unitary representation $\beta(t)$ of π. Thus E has a natural flat unitary connection, whose restriction on each E_t, $t \in I$ is the flat unitary connection, which can be extended to a full $U(n)$-connection ∇^E on $Y \times I$, which amounts to giving an action of $\partial/\partial t$, or equivalently of identifying E with a bundle pulled back from Y. With such a choice of connection, it follows that the curvature of E is a multiple of dt, and so the only nonzero component of the Chern character form $\mathrm{ch}(\nabla^E) - N$ is the first Chern form $\alpha_t \wedge dt$ in dimension 2, where α_t is a closed 1-form on Y, whose cohomology class $\alpha = [\alpha_t] \in H^1(Y, \mathbb{R}) = H^1(B\pi, \mathbb{R})$ is independent of $t \in I$.

Theorem 7.1 (PSC and vanishing of end-periodic rho). *Let (X, g) be a compact spin manifold of even dimension, and let $\iota : Y \hookrightarrow X$ be a codimension one submanifold of X which is Poincaré dual to a primitive class $\gamma \in H^1(X, \mathbb{Z})$. Suppose that*

(1) *g is a Riemannian metric of positive scalar curvature*;

(2) *the restriction $g|_Y$ is also a metric of positive scalar curvature.*

Let $\tilde{\alpha} : \tilde{\pi} \to U(N)$ be a unitary representation of $\tilde{\pi} = \pi_1(X)$, and $\alpha : \pi \to U(N)$ be the unitary representation of $\pi = \pi_1(Y)$ defined by $\tilde{\alpha} \circ \iota_*$. Assume that α can be connected by a smooth path $\beta : I \to \mathfrak{R}$ to the trivial representation in the representation space $\mathfrak{R}$.

Then $\rho^{\mathrm{ep}}(X, S, \gamma, g; \tilde{\alpha}, 1) = 0$, where the flat hermitian bundle $E_{\tilde{\alpha}}$ is determined by $\tilde{\alpha}$.

Proof. As observed above, the unitary connection ∇^E induced on E has curvature which is a multiple of dt, so that the Chern character form $\mathrm{ch}(\nabla^E) = N + \alpha_t \wedge dt$, where $\alpha_t \wedge dt$ is the first Chern form of the connection on E and t is the variable on the interval I. It follows that $\mathrm{ch}(E) = N + \alpha \wedge dt$ where $\alpha \in H^1(Y, \mathbb{R})$ is the cohomology class of α_t. Consider the integrand $\int_{Y \times I} \widehat{A}(Y \times I)\,\mathrm{ch}(E)$. Since $\widehat{A}(Y \times I) = \widehat{A}(Y)$, where $\widehat{A}(Y)$ is the A-hat characteristic class of Y. From the

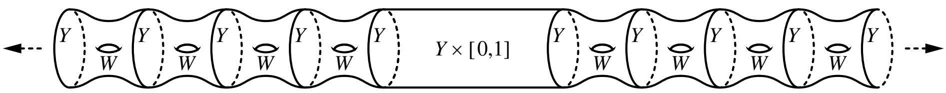

Figure 7. End-periodic manifold with 2 ends.

discussion above

$$\int_{Y\times I} \widehat{A}(Y)\,\mathrm{ch}(E) = \int_Y \widehat{A}(Y)\alpha \int_I dt.$$

Since (Y, g) is a spin Riemannian manifold of positive scalar curvature, it follows from [Gromov and Lawson 1980b, Theorem 2.1] that $\int_Y \widehat{A}(Y) f^*(x) = 0$ for all $x \in H^1(B\pi, \mathbb{R}) = H^1(Y, \mathbb{R})$.

Therefore we conclude that $\int_{Y\times I} \widehat{A}(Y)\,\mathrm{ch}(E) = 0$.

Consider the manifold $Y \times I$. It can be made into an end-periodic manifold with two ends as follows. Let W be the fundamental segment obtained by cutting X open along Y, and W_k be isometric copies of W. Then we can attach $X_1 = \bigcup_{k\geq 0} W_k$ to one boundary component of $Y \times I$ and $X_0 = \bigcup_{k<0} W_k$ to the other boundary component. Call the resulting end-periodic manifold Z_∞ (see the Figure 7). It is clear that Z_∞ is diffeomorphic to $\widetilde{X}$, the cyclic Galois cover of X corresponding to γ. Let $f_0 = -f$ and $f_1 = f$ for a choice of real-valued function f on Z_∞ such that $\gamma = [df]$.

The flat hermitian bundle $E_{\tilde{\alpha}}$ over X induces a flat hermitian bundle $p^*(E_{\tilde{\alpha}})$ over $\widetilde{X}$, where $p\colon \widetilde{X} \to X$ is the projection. The restriction of $p^*(E_{\tilde{\alpha}})$ to the subset X_1 is denoted by E_1. Let E_0 denote the trivial bundle over X_0. We use the smooth path γ to define the bundle E over $Y \times I$ which has the property that the restriction of $\widetilde{E}$ to the boundary components agree with E_0 and E_1, thereby defining a global vector bundle $\widetilde{E}$ over Z_∞.

We can apply Theorem C in [Mrowka et al. 2016] to see that

$$\begin{aligned}\mathrm{index}(D^{\pm}_{\widetilde{E}}(Z_\infty)) = \int_{Y\times I} &\widehat{A}(Y\times I)\,\mathrm{ch}(E)\\ &- \int_Y \omega + \int_X df\wedge\omega - \tfrac{1}{2}(h_1 + \eta_{ep}(X, E_{\tilde{\alpha}}, \gamma, g))\\ &+ \int_Y \omega - \int_X df\wedge\omega - \tfrac{1}{2}((h_0 - \eta_{ep}(X, E_{id}, \gamma, g))\end{aligned}$$

Since g and $g|_Y$ are metrics of positive scalar curvature by hypothesis, it follows that $\mathrm{index}(D^{+}_{E}(Z_\infty)) = 0$ by [Mrowka et al. 2016, Lemma 8.1] and that $\int_{Y\times I} \widehat{A}(Y\times I)\,\mathrm{ch}(E) = 0$ by the earlier argument. Therefore $\rho^{ep}(X, S, \gamma, g; \tilde{\alpha}, 1) = 0$ as claimed. $\square$

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References

[Atiyah 1976] M. F. Atiyah, "Elliptic operators, discrete groups and von Neumann algebras", pp. 43–72 in *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan* (Orsay, 1974), Astérisque **32-33**, Société Mathématique de France, Paris, 1976. MR Zbl

[Atiyah et al. 1975a] M. F. Atiyah, V. K. Patodi, and I. M. Singer, "Spectral asymmetry and Riemannian geometry I", *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 43–69. MR Zbl

[Atiyah et al. 1975b] M. F. Atiyah, V. K. Patodi, and I. M. Singer, "Spectral asymmetry and Riemannian geometry II", *Math. Proc. Cambridge Philos. Soc.* **78**:3 (1975), 405–432. MR Zbl

[Atiyah et al. 1976] M. F. Atiyah, V. K. Patodi, and I. M. Singer, "Spectral asymmetry and Riemannian geometry III", *Math. Proc. Cambridge Philos. Soc.* **79**:1 (1976), 71–99. MR Zbl

[Baum and Douglas 1982] P. Baum and R. G. Douglas, "K-homology and index theory", pp. 117–173 in *Operator algebras and applications I* (Kingston, Ontario, 1980), edited by R. Kadison, Proc. Sympos. Pure Math. **38**, Amer. Math. Soc., Providence, RI, 1982. MR Zbl

[Benameur and Mathai 2013] M. T. Benameur and V. Mathai, "Conformal invariants of twisted Dirac operators and positive scalar curvature", *J. Geom. Phys.* **70** (2013), 39–47. Corrigendum in **76** (2014), 263–264. Zbl

[Benameur and Mathai 2014] M. T. Benameur and V. Mathai, "Index type invariants for twisted signature complexes and homotopy invariance", *Math. Proc. Cambridge Philos. Soc.* **156**:3 (2014), 473–503. MR Zbl

[Benameur and Mathai 2015] M. T. Benameur and V. Mathai, "Spectral sections, twisted rho invariants and positive scalar curvature", *J. Noncommut. Geom.* **9**:3 (2015), 821–850. MR Zbl

[Botvinnik and Gilkey 1995] B. Botvinnik and P. B. Gilkey, "The eta invariant and metrics of positive scalar curvature", *Math. Ann.* **302**:3 (1995), 507–517. MR Zbl

[Deeley and Goffeng 2016] R. J. Deeley and M. Goffeng, "Realizing the analytic surgery group of Higson and Roe geometrically part II: relative η-invariants", *Math. Ann.* **366**:3-4 (2016), 1319–1363. MR Zbl

[Gilkey 1984a] P. B. Gilkey, "The eta invariant and the K-theory of odd-dimensional spherical space forms", *Invent. Math.* **76**:3 (1984), 421–453. MR Zbl

[Gilkey 1984b] P. B. Gilkey, *Invariance theory, the heat equation, and the Atiyah–Singer index theorem*, Mathematics Lecture Series **11**, Publish or Perish, Wilmington, DE, 1984. MR Zbl

[Gromov and Lawson 1980a] M. Gromov and H. B. Lawson, Jr., "The classification of simply connected manifolds of positive scalar curvature", *Ann. of Math.* (2) **111**:3 (1980), 423–434. MR Zbl

[Gromov and Lawson 1980b] M. Gromov and H. B. Lawson, Jr., "Spin and scalar curvature in the presence of a fundamental group I", *Ann. of Math.* (2) **111**:2 (1980), 209–230. MR Zbl

[Gromov and Lawson 1983] M. Gromov and H. B. Lawson, "Positive scalar curvature and the Dirac operator on complete Riemannian manifolds", *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* **58**:1 (1983), 83–196. Zbl

[Guentner et al. 2005] E. Guentner, N. Higson, and S. Weinberger, "The Novikov conjecture for linear groups", *Publ. Math. Inst. Hautes Études Sci.* 101 (2005), 243–268. MR Zbl

[Higson and Roe 2010] N. Higson and J. Roe, "K-homology, assembly and rigidity theorems for relative eta invariants", *Pure Appl. Math. Q.* **6**:2 (2010), 555–601. MR Zbl

[Hitchin 1974] N. Hitchin, "Harmonic spinors", *Adv. Math.* **14** (1974), 1–55. MR Zbl

[Keswani 1999] N. Keswani, "Geometric K-homology and controlled paths", *New York J. Math.* **5** (1999), 53–81. MR Zbl

[Keswani 2000] N. Keswani, "Relative eta-invariants and C^*-algebra K-theory", *Topology* **39**:5 (2000), 957–983. MR Zbl

[Mathai 1992a] V. Mathai, "Nonnegative scalar curvature", *Ann. Global Anal. Geom.* **10**:2 (1992), 103–123. MR Zbl

[Mathai 1992b] V. Mathai, "Spectral flow, eta invariants, and von Neumann algebras", *J. Funct. Anal.* **109**:2 (1992), 442–456. MR Zbl

[Melrose 1993] R. B. Melrose, *The Atiyah–Patodi–Singer index theorem*, Research Notes in Mathematics **4**, A K Peters, Wellesley, MA, 1993. MR Zbl

[Miyazaki 1984] T. Miyazaki, "On the existence of positive scalar curvature metrics on non-simply-connected manifolds", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **30**:3 (1984), 549–561. MR Zbl

[Mrowka et al. 2011] T. Mrowka, D. Ruberman, and N. Saveliev, "Seiberg–Witten equations, end-periodic Dirac operators, and a lift of Rohlin's invariant", *J. Differential Geom.* **88**:2 (2011), 333–377. MR Zbl

[Mrowka et al. 2016] T. Mrowka, D. Ruberman, and N. Saveliev, "An index theorem for end-periodic operators", *Compos. Math.* **152**:2 (2016), 399–444. MR Zbl

[Piazza and Schick 2007a] P. Piazza and T. Schick, "Bordism, rho-invariants and the Baum-Connes conjecture", *J. Noncommut. Geom.* **1**:1 (2007), 27–111. MR Zbl

[Piazza and Schick 2007b] P. Piazza and T. Schick, "Groups with torsion, bordism and rho invariants", *Pacific J. Math.* **232**:2 (2007), 355–378. MR Zbl

[Rosenberg 1986] J. Rosenberg, "C^*-algebras, positive scalar curvature and the Novikov conjecture II", pp. 341–374 in *Geometric methods in operator algebras* (Kyoto, 1983), edited by H. Araki and E. G. Effros, Pitman Res. Notes Math. Ser. **123**, Longman Sci. Tech., Harlow, England, 1986. MR Zbl

[Schoen and Yau 1979] R. Schoen and S. T. Yau, "On the structure of manifolds with positive scalar curvature", *Manuscripta Math.* **28**:1-3 (1979), 159–183. MR Zbl

[Taubes 1987] C. H. Taubes, "Gauge theory on asymptotically periodic 4-manifolds", *J. Differential Geom.* **25**:3 (1987), 363–430. MR Zbl

[Weinberger 1988] S. Weinberger, "Homotopy invariance of η-invariants", *Proc. Nat. Acad. Sci. U.S.A.* **85**:15 (1988), 5362–5363. MR Zbl

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