ANNALS OF K-THEORY

vol. 5 no. 3 2020

Weibel's conjecture for twisted K-theory

Joel Stapleton

msp

A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 5, No. 3, 2020
https://doi.org/10.2140/akt.2020.5.621

Weibel's conjecture for twisted K-theory

Joel Stapleton

We prove Weibel's conjecture for twisted K-theory when twisting by a smooth proper connective dg-algebra. Our main contribution is showing we can kill a negative twisted K-theory class using a projective birational morphism (in the same twisted setting). We extend the vanishing result to relative twisted K-theory of a smooth affine morphism and describe counterexamples to some similar extensions.

1. Introduction

The so-called fundamental theorem for K_1 and K_0 states that for any ring R there is an exact sequence

$$0 \to K_1(R) \to K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t^{\pm}]) \to K_0(R) \to 0.$$

We see K_0 can be defined using K_1. There is an analogous exact sequence, truncated on the right, for K_0. Bass defines $K_{-1}(X)$ as the cokernel of the final morphism. He then iterates the construction to define a theory of negative K-groups [Bass 1968, §XII.7 and §XII.8].

Weibel's conjecture [1980] asks if $K_{-i}(R) = 0$ for $i > \dim R$ when R has finite Krull dimension. Kerz, Strunk, and Tamme [Kerz et al. 2018] have proven Weibel's conjecture for any Noetherian scheme of finite Krull dimension (see the introduction for a historical summary of progress) by establishing pro cdh-descent for algebraic K-theory. Land and Tamme [2019] have shown that a general class of localizing invariants satisfy pro cdh-descent. With this improvement, we extend Weibel's vanishing to some cases of twisted K-theory.

Theorem 1.1. *Let X be a Noetherian d-dimensional scheme and $\mathscr{A}$ a sheaf of smooth proper connective quasicoherent differential graded algebras over X; then $K_{-i}(\mathrm{Perf}(\mathscr{A}))$ vanishes for $i > d$.*

The original goal of this paper was to extend Weibel's conjecture to an Azumaya algebra over a scheme. To an Azumaya algebra $\mathscr{A}$ of rank r^2 on X we can associate a Severi–Brauer variety P of relative dimension $r-1$ over X. Such a variety is

MSC2010: primary 16E20, 19D35; secondary 14F22, 16K50.
Keywords: algebraic K-theory, Brauer groups, excision.

étale-locally isomorphic over X to $\mathbb{P}_X^{r-1}$. Quillen [1973] generalizes the projective bundle formula to Severi–Brauer varieties showing (for $i \geq 0$)

$$K_i(P) \cong \bigoplus_{n=0}^{r-1} K_i(\mathcal{A}^{\otimes n}).$$

At the root of this computation is a semiorthogonal decomposition of $\mathrm{Perf}(P)$. Consequently, the computation lifts to the level of nonconnective K-theory spectra. Statements about the K-theory of Azumaya algebras can generally be extracted through this decomposition. In our case, the dimension of the Severi–Brauer variety jumps and so Weibel's conjecture (for our noncommutative dg-algebra) does not follow from the commutative setting.

We could remedy this by characterizing a class of morphisms to X, which should include Severi–Brauer varieties, and then show the relative K-theory vanishes under $-d-1$. In Remark 4.4, we show that smooth and proper morphisms (in fact, smooth and projective) are not sufficient. We warn the reader that we will use the overloaded words "smooth and proper" in both the scheme and dg-algebra settings.

For dg-algebras and dg-categories, properness and smoothness are module and algebraic finiteness conditions [Toën and Vaquié 2007, Definition 2.4]. Together, the two conditions characterize the dualizable objects in $\mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega})$, whose objects are ω-compactly generated R-linear stable presentable ∞-categories. More surprisingly, the invertible objects of $\mathrm{Mod}_{\mathrm{Mod}_R}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega})$ are exactly the module categories over derived Azumaya algebras [Antieau and Gepner 2014, Theorem 3.15]. So Theorem 1.1 recovers the discrete Azumaya algebra case.

However, any connective derived Azumaya algebra is discrete. After base-changing to a field k, $\mathcal{A}_k \cong H_*\mathcal{A}_k$ is a connective graded k-algebra and $H_*\mathcal{A}_k \otimes_k (H_*\mathcal{A}_k)^{\mathrm{op}}$ is Morita equivalent to k. So $H_*\mathcal{A}_k$ is discrete. The scope of Theorem 1.1 is not wasted as smooth proper connective dg-algebras can be nondiscrete [Raedschelders and Stevenson 2019, §4.3].

The proof of Theorem 1.1 follows [Kerz 2018]. In Section 2, we define and study twisted K-theory. We kill a negative twisted K-theory class using a projective birational morphism in Section 3. Lastly, Section 4 holds the main theorem and we consider some extensions.

Conventions. We make very little use of the language of ∞-categories. For a commutative ring R, there is an equivalence of ∞-categories between the $\mathbb{E}_1$-ring spectra over HR and differential graded algebras over R localized at the quasi-isomorphisms [Lurie 2017, Proposition 7.1.4.6]. For a dg-algebra (or $\mathbb{E}_1$-ring) $\mathcal{A}$, we can consider the ∞-category $\mathrm{RMod}(\mathcal{A})$ of spectra which have a right $\mathcal{A}$-module structure. We will refer to this ∞-category as the derived category of $\mathcal{A}$ and denote it by $D(\mathcal{A})$. The subcategory $\mathrm{Perf}(\mathcal{A})$ consists of all compact objects of $\mathrm{RMod}(\mathcal{A})$, or the right $\mathscr{A}$-modules which corepresent a functor that commutes with filtered colimits. We shall refer to objects of $\mathrm{Perf}(\mathscr{A})$ as perfect complexes over $\mathscr{A}$.

We use $K(-)$ undecorated as nonconnective algebraic K-theory and consider it as a localizing invariant in the sense of Blumberg, Gepner, and Tabuada [Blumberg et al. 2013]. In particular, it is an ∞-functor from $\mathrm{Cat}_\infty^{\mathrm{perf}}$, the ∞-category of idempotent complete small stable infinity categories with exact functors, taking values in Sp, the ∞-category of spectra. For X a quasicompact quasiseparated scheme, $K(\mathrm{Perf}(X))$ is equivalent to the nonconnective K-theory spectrum of Thomason and Trobaugh [1990]. The ∞-category $\mathrm{Cat}_\infty^{\mathrm{perf}}$ has a symmetric monoidal structure which we will denote by $\widehat{\otimes}$. For R an $\mathbb{E}_\infty$-ring spectrum, $\mathrm{Perf}(R)$ is an $\mathbb{E}_\infty$ algebra in $\mathrm{Cat}_\infty^{\mathrm{perf}}$. We will restrict the domain of algebraic K-theory to $\mathrm{Mod}_{\mathrm{Perf}(R)}(\mathrm{Cat}_\infty^{\mathrm{perf}})$.

2. Twisted K-theory

In Grothendieck's original papers [1968a; 1968b; 1968c], he globalizes the notion of a central simple algebra over a field.

Definition 2.1. A locally free sheaf of $\mathscr{O}_X$-algebras $\mathscr{A}$ is a *sheaf of Azumaya algebras* if it is étale-locally isomorphic to $\mathscr{M}_n(\mathscr{O}_X)$ for some n.

An Azumaya algebra is then a PGL_n-torsor over the étale topos of X and so, by Giraud, isomorphism classes are in bijection with $H^1_{\text{ét}}(X, PGL_n)$. The central extension of sheaves of groups in the étale topology

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

leads to an exact sequence of nonabelian cohomology

$$\cdots \to H^1_{\text{ét}}(X, \mathbb{G}_m) \to H^1_{\text{ét}}(X, GL_n) \to H^1_{\text{ét}}(X, PGL_n) \to H^2_{\text{ét}}(X, \mathbb{G}_m).$$

For $d \mid n$ we have a morphism of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & GL_n & \longrightarrow & PGL_n & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & GL_d & \longrightarrow & PGL_d & \longrightarrow & 1 \end{array}$$

with the two right arrows given by block-summing the matrix along the diagonal n/d times. The Brauer group is the filtered colimit of cofibers

$$Br(X) := \varinjlim(\mathrm{cofib}(H^1_{\text{ét}}(X, GL_n) \to H^1_{\text{ét}}(X, PGL_n)))$$

along the partially ordered set of the natural numbers under division. This is the group of Azumaya algebras modulo Morita equivalence with group operation given by tensor product [Grothendieck 1968a]. We have an injection $Br(X) \hookrightarrow H^2_{\text{ét}}(X, \mathbb{G}_m)$ and when X is quasicompact this injection factors through the torsion subgroup. We will call $Br'(X) := H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tor}}$ the cohomological Brauer group. Grothendieck asked if the injection $Br(X) \hookrightarrow Br'(X)$ is an isomorphism.

This map is not generally surjective. Edidin, Hassett, Kresch, and Vistoli [Edidin et al. 2001] give a nonseparated counterexample by connecting the image of the Brauer group to quotient stacks. There are two ways to proceed in addressing the question. The first is to provide a class of schemes for when this holds. In [de Jong 2006], de Jong publishes a proof of O. Gabber that $Br(X) \cong Br'(X)$ when X is equipped with an ample line bundle. Along with reproving Gabber's result for affines, Lieblich [2004] shows that for a regular scheme with dimension less than or equal to 2 there are isomorphisms $Br(X) \cong Br'(X) \cong H^2_{\text{ét}}(X, \mathbb{G}_m)$.

The second perspective is to enlarge the class of objects considered. The Morita equivalence classes of $\mathbb{G}_m$-gerbes over the étale topos of a scheme X are in bijection with $H^2_{\text{ét}}(X, \mathbb{G}_m)$. Lieblich [2004] associates to any Azumaya algebra $\mathscr{A}$ a $\mathbb{G}_m$-gerbe of Morita-theoretic trivializations. Over an étale open $U \to X$, the gerbe gives a groupoid of Morita equivalences from $\mathscr{A}$ to $\mathscr{O}_X$. The gerbe of trivializations represents the boundary class $\delta([\mathscr{A}]) = \alpha \in H^2_{\text{ét}}(X, \mathbb{G}_m)$.

Any class $\alpha \in H^2_{\text{ét}}(X, \mathbb{G}_m)$ is realizable on a Čech cover. We can use this data to build a well defined category of sheaves of $\mathscr{O}_X$-modules which "glue up to α" [Căldăraru 2000, Chapter 1]. Let Mod^{α}_X denote the corresponding derived ∞-category and Perf^{α}_X the full subcategory of compact objects. $K(\text{Perf}^{\alpha}_X)$ is the classical definition of α-twisted algebraic K-theory. Determining when the cohomology class α is represented by an Azumaya algebra reduces to finding a twisted locally free sheaf with trivial determinant on a $\mathbb{G}_m$-gerbe associated to α [Lieblich 2004, §2.2.2]. The endomorphism algebra of the twisted locally free sheaf gives the Azumaya algebra and the twisted module represents the tilt $\text{Mod}^{\alpha}_X \simeq \text{Mod}_{\mathscr{A}}$.

Lieblich also compactifies the moduli of Azumaya algebras. This necessarily includes developing a definition of a derived Azumaya algebra.

Definition 2.2. A *derived Azumaya algebra* over a commutative ring R is a proper dg-algebra $\mathscr{A}$ such that the natural map of R-algebras

$$\mathscr{A} \otimes^{\mathbb{L}}_R \mathscr{A}^{\text{op}} \xrightarrow{\simeq} \mathbb{R}Hom_{D(R)}(\mathscr{A}, \mathscr{A})$$

is a quasi-isomorphism.

After Lieblich, Toën [2012] and (later) Antieau and Gepner [2014] consider the analogous problem posed by Grothendieck in the dg-algebra and $\mathbb{E}_\infty$-algebra settings, respectively. Antieau and Gepner construct an étale sheaf $\boldsymbol{Br}$ in the ∞-topos $\text{Shv}^{\text{ét}}_R$. For any étale sheaf X, we can now associate a Brauer space $\boldsymbol{Br}(X)$. For X a quasicompact quasiseparated scheme, they show $\pi_0(\boldsymbol{Br}(X)) \cong H^1_{\text{ét}}(X, \mathbb{Z}) \times H^2_{\text{ét}}(X, \mathbb{G}_m)$ and every such Brauer class is algebraic. Now for any (possibly nontorsion) $\alpha \in H^2_{\text{ét}}(X, \mathbb{G}_m)$ there is a derived Azumaya algebra $\mathscr{A}$ and an equivalence $\text{Mod}^{\alpha}_X \simeq \text{Mod}_{\mathscr{A}}$ of stable ∞-categories.

This reframes classical twisted K-theory as K-theory with coefficients in a particularly special dg-algebra in $D(X)$. For our purposes, we work with a generalized definition of twisted K-theory which allows "twisting" by any dg-algebra.

Definition 2.3. Let R be a commutative ring. For a dg-algebra $\mathscr{A}$ over R, we define the *$\mathscr{A}$-twisted K-theory* $K^{\mathscr{A}} : \text{Mod}_{\text{Perf}(R)}(\text{Cat}^{\text{perf}}_{\infty}) \to \text{Sp}$ by

$$K^{\mathscr{A}}(\mathscr{C}) := K(\mathscr{C} \mathbin{\widehat{\otimes}}_{\text{Perf}(R)} \text{Perf}(\mathscr{A})).$$

When the dg-algebra "$\mathscr{A}$" is clear, we just write twisted K-theory. If our input to $K^{\mathscr{A}}$ is an R-algebra S, then

$$K^{\mathscr{A}}(S) = K(\text{Perf}(S) \mathbin{\widehat{\otimes}}_{\text{Perf}(R)} \text{Perf}(\mathscr{A})) \simeq K(\text{Perf}(S \otimes_R \mathscr{A})) \simeq K(S \otimes_R \mathscr{A}).$$

Our definition recovers the historical definition of twisted K-theory when $\mathscr{A}$ is a derived Azumaya algebra and we evaluate on the base ring R. The same definition works for a scheme X and $\mathscr{A} \in \text{Alg}_{\mathbb{E}_1}(D_{\text{qc}}(X))$. We will refer to such an $\mathscr{A}$ as *a sheaf of quasicoherent dg-algebras over X*. By [Blumberg et al. 2013, Theorem 9.36], twisted K-theory is a localizing invariant. When X is a quasicompact quasiseparated scheme, Clausen, Mathew, Naumann, and Noel [Clausen et al. 2020, Proposition A.15] establish Nisnevich descent when X is quasicompact quasiseparated.

Definition 2.4. A dg-algebra $\mathscr{A}$ over a ring R is *proper* if it is perfect as a complex over R and *smooth* if it is perfect over $\mathscr{A}^{\text{op}} \otimes_R \mathscr{A}$.

The following is Lemma 2.8 of [Toën and Vaquié 2007] and is an essential property for our proof in Section 3.

Lemma 2.5. *Let $\mathscr{A}$ be a smooth proper dg-algebra over a ring R. Then a complex of $D(\mathscr{A})$ is perfect over $\mathscr{A}$ if and only if it is perfect as an object of $D(R)$.*

The previous definition and lemma both generalize to a sheaf of quasicoherent dg-algebras over a scheme as perfection is a local property. For the remainder of the section, we prove some basic properties of $\mathscr{A}$-twisted K-theory, often assuming $\mathscr{A}$ is connective. We will not use smooth and properness until the later sections.

Proposition 2.6. *Let $\mathscr{A}$, S be connective dg-algebras over R. Then the natural maps induce isomorphisms*

$$K_i^{\mathscr{A}}(S) \cong K_i^{\mathscr{A}}(\pi_0(S)) \cong K_i^{\pi_0(\mathscr{A})}(S) \cong K_i^{\pi_0(\mathscr{A})}(\pi_0(S))$$

for $i \leq 0$.

Proof. We have the following isomorphisms of discrete rings:

$$\pi_0(\mathscr{A} \otimes_R S) \cong \pi_0(\mathscr{A} \otimes_R \pi_0(S)) \cong \pi_0(\pi_0(\mathscr{A}) \otimes_R S) \cong \pi_0(\pi_0(\mathscr{A}) \otimes_R \pi_0(S)).$$

The lemma follows since $K_i(R) \cong K_i(\pi_0(R))$ for $i \leq 0$ [Blumberg et al. 2013, Theorem 9.53]. □

The previous proposition suggests we can work discretely and then transfer the results to the derived setting. This is true to some extent. However, taking π_0 of a connective dg-algebra does not preserve smoothness, which is a necessary property for our proof of Proposition 3.2. We will also need reduction invariance for low-dimensional K-groups.

Proposition 2.7. *Let R be a commutative ring and $\mathscr{A}$ a connective dg-algebra over R. Let S be a commutative ring under R, and let I be a nilpotent ideal of S. Then the induced morphism $K_i^{\mathscr{A}}(S) \xrightarrow{\cong} K_i^{\mathscr{A}}(S/I)$ is an isomorphism for $i \leq 0$.*

Proof. By naturality of the fundamental exact sequence of twisted K-theory (see (†) and the surrounding discussion at the beginning of Section 3), we can restrict the proof to $K_0^{\mathscr{A}}$. By Proposition 2.6, we can assume $\mathscr{A}$ is a discrete algebra. Let $\varphi : S \twoheadrightarrow S/I$ be the surjection. After $- \otimes_R \mathscr{A}$ we have a surjection $(\ker \varphi) \otimes_R \mathscr{A} \twoheadrightarrow \ker(\varphi \otimes_R \mathscr{A})$. The nonunital ring $(\ker \varphi) \otimes_R \mathscr{A}$ is nilpotent. So $\ker(\varphi \otimes_R \mathscr{A})$ is nilpotent as well. The proposition follows from nil-invariance of K_0. □

A Zariski descent spectral sequence argument gives us a global result.

Corollary 2.8. *Let X be a quasicompact quasiseparated scheme of finite Krull dimension d and $\mathscr{A}$ a sheaf of connective quasicoherent dg-algebras over X. The natural morphism $f : X_{\text{red}} \to X$ induces isomorphisms*

$$K_{-i}^{f^*\mathscr{A}}(X_{\text{red}}) \cong K_{-i}^{\mathscr{A}}(X)$$

for $i \geq d$.

Proof. We have descent spectral sequences

$$\begin{aligned} E_2^{p,q} &= H^p_{\text{Zar}}(X, (\pi_q K^{\mathscr{A}})^\sim) &&\Rightarrow \pi_{q-p} K^{\mathscr{A}}(X), \\ E_2^{p,q} &= H^p_{\text{Zar}}(X, f_*(\pi_q K^{f^*(\mathscr{A})})^\sim) &&\Rightarrow \pi_{q-p} K^{f^*\mathscr{A}}(X_{\text{red}}) \end{aligned}$$

both with differential $d_2 = (2, 1)$. We let $F^\sim$ denote the Zariski sheafification of the presheaf F. The spectral sequences agree for $q \leq 0$. By Corollary 3.27 of [Clausen and Mathew 2019], the spectral sequences vanishes for $p > d$. □

In Theorem 4.3, we extend our main theorem across smooth affine morphisms. We will need reduction invariance in this setting.

Definition 2.9. For $f : S \to X$ a morphism of quasicompact quasiseparated schemes and $\mathscr{A}$ a sheaf of quasicoherent dg-algebras over X, the *relative $\mathscr{A}$-twisted K-theory of f* is

$$K^{\mathscr{A}}(f) := \operatorname{fib}(K^{\mathscr{A}}(X) \xrightarrow{f^*} K^{\mathscr{A}}(S)).$$

As defined, $K^{\mathscr{A}}(f)$ is a spectrum. There is an associated presheaf of spectra on the base scheme X given by $U \mapsto K^{\mathscr{A}}(f_{|_U})$. This presheaf sits in a fiber sequence

$$K^{\mathscr{A}}(f) \to K^{\mathscr{A}} \to K_S^{\mathscr{A}}$$

where the presheaf $K_S^{\mathscr{A}}$ is also defined by pullback along f. Both presheaves $K^{\mathscr{A}}$ and $K_S^{\mathscr{A}}$ satisfy Nisnevich descent and so $K^{\mathscr{A}}(f)$ does as well.

Corollary 2.10. *Let $f : S \to X$ be an affine morphism of quasicompact quasiseparated schemes. Suppose X has Krull dimension d and let $\mathscr{A}$ be a sheaf of connective quasicoherent dg-algebras over X. Then the commutative diagram*

$$\begin{array}{ccc} S_{\mathrm{red}} & \xrightarrow{f_{\mathrm{red}}} & X_{\mathrm{red}} \\ \downarrow & & \downarrow g \\ S & \xrightarrow{f} & X \end{array}$$

induces an isomorphism of relative twisted K-theory groups

$$K_{-i}^{g^*\mathscr{A}}(f_{\mathrm{red}}) \cong K_{-i}^{\mathscr{A}}(f)$$

for $i \geq d+1$.

Proof. We have two descent spectral sequences

$$\begin{aligned} E_2^{p,q} &= H^p_{\mathrm{Zar}}(X, (\pi_q K^{\mathscr{A}}(f))^\sim) &&\Rightarrow \pi_{q-p} K^{\mathscr{A}}(f)(X), \\ E_2^{p,q} &= H^p_{\mathrm{Zar}}(X, g_*(\pi_q K^{g^*\mathscr{A}}(f_{\mathrm{red}}))^\sim) &&\Rightarrow \pi_{q-p} K^{g^*\mathscr{A}}(f_{\mathrm{red}})(X_{\mathrm{red}}) \end{aligned}$$

with differential of degree $d = (2, 1)$ and $F^\sim$ the sheafification of the presheaf F. For an open affine $\operatorname{Spec} R \to X$ with pullback $\operatorname{Spec} A \to S$ we examine the morphism of long exact sequences when $q \leq 0$:

$$\begin{array}{ccccccccccc} \cdots \to & \pi_q K^{\mathscr{A}}(R) & \to & \pi_q K^{\mathscr{A}}(A) & \to & \pi_{q-1} K^{\mathscr{A}}(f) & \to & \pi_{q-1} K^{\mathscr{A}}(R) & \to & \pi_{q-1} K^{\mathscr{A}}(A) & \to \cdots \\ & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong & \\ \cdots \to & \pi_q K^{\mathscr{A}}(R_{\mathrm{red}}) & \to & \pi_q K^{\mathscr{A}}(A_{\mathrm{red}}) & \to & \pi_{q-1} K^{\mathscr{A}}(f_{\mathrm{red}}) & \to & \pi_{q-1} K^{\mathscr{A}}(R_{\mathrm{red}}) & \to & \pi_{q-1} K^{\mathscr{A}}(A_{\mathrm{red}}) & \to \cdots \end{array}$$

By the 5-lemma, this induces sheaf isomorphisms

$$g_*(\pi_q K^{g^*\mathscr{A}}(f_{\mathrm{red}}))^\sim \cong (\pi_q K^{\mathscr{A}}(f))^\sim$$

for $q < 0$ and, as in Corollary 2.8, cohomology vanishes for $p > d$. □

We will need proexcision for abstract blow-up squares. Recall that an abstract blow-up square is a pullback square

$$\begin{array}{ccc} D & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array} \tag{$*$}$$

with $Y \to X$ a closed immersion and $\widetilde{X} \to X$ a proper morphism which restricts to an isomorphism of open subschemes $\widetilde{X} \setminus D \to X \setminus Y$. The theorem is stated using the ∞-category of prospectra $\boldsymbol{Pro}(\mathrm{Sp})$, where an object is a small cofiltered diagram, $E : \Lambda \to \mathrm{Sp}$, valued in spectra. We write $\{E_n\}$ for the corresponding prospectrum. If the brackets and index are omitted, then the prospectrum is considered constant. After adjusting equivalence class representatives, we may assume the cofiltered diagram is fixed when working with a finite set of prospectra. Any morphism can then be represented by a natural transformation of diagrams (also known as a level map). We will need no knowledge of the ∞-category beyond the following definition.

Definition 2.11. A square of prospectra

$$\begin{array}{ccc} \{E_n\} & \longrightarrow & \{F_n\} \\ \downarrow & & \downarrow \\ \{X_n\} & \longrightarrow & \{Y_n\} \end{array}$$

is *procartesian* if and only if the induced map on the levelwise fiber prospectra is a weak equivalence [Land and Tamme 2019, Definition 2.27].

The following is Theorem A.8 of [Land and Tamme 2019]. The theorem holds much more generally for any k-connective localizing invariant [Land and Tamme 2019, Definition 2.5]. Twisted K-theory is 1-connective.

Theorem 2.12 [Land and Tamme 2019]. *Given an abstract blow-up square* $(*)$ *of schemes and a sheaf of dg-algebras* $\mathscr{A}$ *on* X*, then the square of prospectra*

$$\begin{array}{ccc} K^{\mathscr{A}}(X) & \longrightarrow & K^{\mathscr{A}}(\widetilde{X}) \\ \downarrow & & \downarrow \\ \{K^{\mathscr{A}}(Y_n)\} & \longrightarrow & \{K^{\mathscr{A}}(D_n)\} \end{array}$$

is procartesian (*where* Y_n *is the infinitesimal thickening of* Y).

The procartesian square of prospectra gives a long exact sequence of progroups

$$\cdots \to \{K^{\mathscr{A}}_{-i+1}(E_n)\} \to K^{\mathscr{A}}_{-i}(X) \to K^{\mathscr{A}}_{-i}(\widetilde{X}) \oplus \{K^{\mathscr{A}}_{-i}(Y_n)\} \to \{K^{\mathscr{A}}_{-i}(E_n)\} \to \cdots$$

which is the key to our induction argument.

3. Blowing up negative twisted K-theory classes

We turn to our main contribution of the existence of a projective birational morphism which kills a given negative twisted K-theory class (when twisting by a smooth proper connective dg-algebra). Let X be a quasicompact quasiseparated scheme and $\mathscr{A}$ a sheaf of quasicoherent dg-algebras on X. We first construct geometric cycles for negative twisted K-theory classes on X using a classical argument of Bass [1968, §XII.7] which works for a general additive invariant. We have an open cover

$$\begin{array}{ccc} X[t^{\pm}] & \xrightarrow{f} & X[t^{-}] \\ \downarrow g & & \downarrow j \\ X[t] & \xrightarrow{k} & \mathbb{P}^1_X \end{array}$$

Since twisted K-theory satisfies Zariski descent, there is an associated Mayer–Vietoris sequence of homotopy groups

$$\cdots \to K^{\mathscr{A}}_{-n}(\mathbb{P}^1_X) \xrightarrow{(j^*k^*)} K^{\mathscr{A}}_{-n}(X[t]) \oplus K^{\mathscr{A}}_{-n}(X[t^{-}]) \xrightarrow{f^*-g^*} K^{\mathscr{A}}_{-n}(X[t^{\pm}]) \xrightarrow{\partial} K^{\mathscr{A}}_{-n-1}(\mathbb{P}^1_X) \to \cdots .$$

As an additive invariant, $K^{\mathscr{A}}(\mathbb{P}^1_X) \simeq K^{\mathscr{A}}(X) \oplus K^{\mathscr{A}}(X)$ splits as a $K^{\mathscr{A}}(X)$-module with generators

$$[\mathscr{O} \otimes_{\mathscr{O}_X} \mathscr{A}] = [\mathscr{A}] \quad \text{and} \quad [\mathscr{O}(1) \otimes_{\mathscr{O}_X} \mathscr{A}] = [\mathscr{A}(1)]$$

corresponding to the Beilinson semiorthogonal decomposition. Adjusting the generators to $[\mathscr{A}]$ and $[\mathscr{A}] - [\mathscr{A}(1)]$, we can identify the map (j^*, k^*) as it is a map of $K^{\mathscr{A}}(X)$-modules. The second generator vanishes under each restriction. This identifies the map as

$$K^{\mathscr{A}}(\mathbb{P}^1_X) \simeq K^{\mathscr{A}}(X)[\mathscr{A}] \oplus K^{\mathscr{A}}(X)([\mathscr{A}] - [\mathscr{A}(1)]) \xrightarrow{\Delta \oplus 0} K^{\mathscr{A}}(X[t]) \oplus K^{\mathscr{A}}(X[t^{-}])$$

with Δ the diagonal map corresponding to pulling back along the projections $X[t] \to X$ and $X[t^{-}] \to X$. As Δ is an embedding the long exact sequence splits as

$$0 \to K^{\mathscr{A}}_{-n}(X) \xrightarrow{\Delta} K^{\mathscr{A}}_{-n}(X[t]) \oplus K^{\mathscr{A}}_{-n}(X[t^{-}]) \xrightarrow{\pm} K^{\mathscr{A}}_{-n}(X[t^{\pm}]) \xrightarrow{\partial} K^{\mathscr{A}}_{-n-1}(X) \to 0. \quad (\dagger)$$

After iterating the complex

$$K^{\mathscr{A}}_{-n}(X[t]) \to K^{\mathscr{A}}_{-n}(X[t^{\pm}]) \twoheadrightarrow K^{\mathscr{A}}_{-n-1}(X),$$

we can piece together a complex

$$K^{\mathscr{A}}_{0}(\mathbb{A}^{n+1}_X) \to K^{\mathscr{A}}_{0}(\mathbb{G}^{n+1}_{m,X}) \twoheadrightarrow K^{\mathscr{A}}_{-n-1}(X).$$

Negative twisted K-theory classes have geometric representations as twisted perfect complexes on $\mathbb{G}^i_{m,X}$. There is even a sufficient geometric criterion implying a given representative is 0; it is the restriction of a twisted perfect complex on $\mathbb{A}^i_X$. Our proof of the main proposition of this section will use these representatives. We first need a lemma about extending finitely generated discrete modules in a twisted setting.

Lemma 3.1. *Let $j : U \to X$ be an open immersion of quasicompact quasiseparated schemes. Let $\mathscr{A}$ be a sheaf of proper connective quasicoherent dg-algebras on X and $j^*\mathscr{A}$ its restriction. Let $\mathscr{N}$ be a discrete $j^*\mathscr{A}$-module which is finitely generated as an $\mathscr{O}_U$-module. Then there exists a discrete $\mathscr{A}$-module $\mathscr{M}$, finitely generated over $\mathscr{O}_X$, such that $j^*\mathscr{M} \cong \mathscr{N}$.*

Proof. Note that $H_{\geq 1}(j^*\mathscr{A})$ necessarily acts trivially on $\mathscr{N}$. So the $j^*\mathscr{A}$-module structure on $\mathscr{N}$ comes from forgetting along the map $j^*\mathscr{A} \to H_0(j^*\mathscr{A})$ and the natural $H_0(j^*\mathscr{A})$-module structure. Under restriction,

$$j^* H_0(\mathscr{A}) \cong H_0(j^*\mathscr{A}).$$

We reduce to when $\mathscr{A}$ is a quasicoherent sheaf of discrete $\mathscr{O}_X$-algebras, finite over the structure sheaf. We have an isomorphism $\mathscr{N} \cong j^* j_* \mathscr{N}$. Write $j_*\mathscr{N}$ as a filtered colimit of its finitely generated $\mathscr{A}$-submodules $j_*\mathscr{N} \cong \varinjlim_\lambda \mathscr{M}_\lambda$. The pullback is exact, so we can write $\mathscr{N} \cong \varinjlim_\lambda j^*\mathscr{M}_\lambda$ as a filtered colimit of finitely generated submodules. As $\mathscr{N}$ is finitely generated itself, this isomorphism factors at some stage and $\mathscr{N} \cong j^*\mathscr{M}_\lambda$. □

Proposition 3.2. *Let X be a reduced scheme which is quasiprojective over a Noetherian affine scheme. Let $\mathscr{A}$ be a sheaf of smooth proper connective quasicoherent dg-algebras on X. Let $\gamma \in K^{\mathscr{A}}_{-i}(X)$ for $i > 0$. Then there is a projective birational morphism $\rho : \widetilde{X} \to X$ so that $\rho^*\gamma = 0 \in K^{\mathscr{A}}_{-i}(\widetilde{X})$.*

Proof. We fix a diagram of schemes over X

$$\begin{array}{ccc} \mathbb{G}^i_{m,X} & \xrightarrow{\quad j \quad} & \mathbb{A}^i_X \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & X & \end{array}$$

For any morphism $f : Y_1 \to Y_2$, we let $\tilde{f} : \mathbb{G}^i_{m,Y_1} \to \mathbb{G}^i_{m,Y_2}$ denote the pullback. Lift γ to a $K^{\mathscr{A}}_0(\mathbb{G}^i_{m,X})$-class $[P_\bullet]$, with $P_\bullet$ some $\pi_1^*\mathscr{A}$-twisted perfect complexes on $\mathbb{G}^i_{m,X}$.

The induction step. We induct on the range of homology of $P_\bullet$. As $\pi_1^*\mathscr{A}$ is a sheaf of proper quasicoherent dg-algebras, $P_\bullet$ is perfect on $\mathbb{G}^i_{m,X}$ by Lemma 2.5. Since $\mathbb{G}^i_{m,X}$ has an ample family of line bundles, we may choose $P_\bullet$ to be strict perfect without changing the quasi-isomorphism class. After some (de)suspension, we may assume $P_\bullet$ is connective as this only alters the K_0-class by ± 1. For the lowest nontrivial differential of $P_\bullet$, d_1, we utilize part (iv) of Lemma 6.5 of [Kerz et al. 2018] (with the morphism $\mathbb{G}^i_{m,X} \to X$) to construct a projective birational morphism $\rho : X_1 \to X$ so that $\operatorname{coker}(\tilde{\rho}^* d_1)$ $(= H_0(\tilde{\rho}^* P_\bullet))$ has tor-dimension ≤ 1 over X_1. Consider the distinguished triangle of $\tilde{\rho}^*\pi_1^*\mathscr{A}$-complexes on $\mathbb{G}^i_{m,X_1}$

$$F_\bullet \to \tilde{\rho}^* P_\bullet \to H_0(\tilde{\rho}^* P_\bullet) \cong \operatorname{coker} \tilde{\rho}^* d_1.$$

In Lemma 3.3 below, we cover the base induction step, when the homology is concentrated in a single degree. Using this, construct a projective birational morphism $\phi : X_2 \to X_1$ such that $L\tilde{\phi}^* H_0(\tilde{\rho}^* P_\bullet)$ is a perfect complex and is the restriction of a perfect complex from $\mathbb{A}^i_{X_2}$. By two out of three, $L\tilde{\phi}^* F_\bullet$ is perfect and $[\tilde{\phi}^*\tilde{\rho}^* P_\bullet] = [L\tilde{\phi}^* F_\bullet] + [L\tilde{\phi}^* H_0(\tilde{\rho}^* P_\bullet)]$ in $K_0^{\mathscr{A}}(\mathbb{G}^i_{m,X_2})$. We then repeat the entire induction step with $L\tilde{\phi}^* F_\bullet$.

We need the induction to terminate, which is the purpose of the first projective birational morphism of each step. Since $\operatorname{coker}(\tilde{\rho}^* d_1)$ has tor-dimension ≤ 1 over X_1, by [Kerz et al. 2018, Lemma 6.5], $L\tilde{\phi}^* \operatorname{coker}(\tilde{\rho}^* d_1) \cong \tilde{\phi}^* \operatorname{coker}(\tilde{\rho}^* d_1)$. This implies $L\tilde{\phi}^* F_\bullet$ will have no homology outside the original range of homology of $P_\bullet$. Since $\tilde{\phi}^* \operatorname{coker}(\tilde{\rho}^* d_1) \cong \operatorname{coker}(\tilde{\phi}^*\tilde{\rho}^* d_1)$, this guarantees $H_0(L\tilde{\phi}^* F_\bullet) = 0$, so the homology of $L\tilde{\phi}^* F_\bullet$ lies in a strictly smaller range than $\tilde{\phi}^*\tilde{\rho}^* P_\bullet$. Proposition 3.2 follows from the next lemma. □

Lemma 3.3. *Let X be a reduced scheme which is quasiprojective over a Noetherian affine scheme. Let $\mathscr{A}$ be a sheaf of smooth proper connective quasicoherent dg-algebras on X. Let $\mathscr{N}$ be a discrete $\pi_1^*\mathscr{A}$-module which is coherent on $\mathbb{G}^i_{m,X}$. Then there exists a birational blow-up $\phi : \widetilde{X} \to X$ so that $\tilde{\phi}^*\mathscr{N}$ is perfect over $\tilde{\phi}^*\pi_1^*\mathscr{A}$ on $\mathbb{G}_{m,\widetilde{X}}$ and is the restriction of a perfect complex over the pullback of $\mathscr{A}$ to $\mathbb{A}^i_{\widetilde{X}}$.*

Proof. Using Lemma 3.1, extend $\mathscr{N}$ from $\mathbb{G}^i_{m,X}$ to a coherent $\pi_2^*\mathscr{A}$-module $\mathscr{M}$ on $\mathbb{A}^i_X$. Using the ample family, choose a resolution in $\mathscr{O}_{\mathbb{A}^i_X}$-modules of the form

$$0 \to \mathscr{K} \to \mathscr{F} \to \mathscr{M} \to 0$$

where $\mathscr{F}$ is a vector bundle and $\mathscr{K}$ is the coherent kernel. As X is reduced, $\mathscr{K}$ is flat over some dense open set U of X. By platification par éclatement [Raynaud and Gruson 1971, Theorem 5.2.2], there is a U-admissible blow-up $\phi : \widetilde{X} \to X$ so that the strict transform of $\mathscr{K}$ along the pullback morphism $p : \mathbb{A}^i_{\widetilde{X}} \to \mathbb{A}^i_X$ is flat over $\widetilde{X}$.

We now show the pullback $p^*\mathscr{M}$ is perfect as a $p^*\pi_2^*\mathscr{A}$-module. Let $j : \mathbb{A}^i_U \to \mathbb{A}^i_{\widetilde{X}}$ be the inclusion of the open set and Z the closed complement. For any sheaf of modules $\mathscr{G}$ on $\mathbb{A}^i_{\widetilde{X}}$, we let $\mathscr{G}_Z$ denote the subsheaf of sections supported on Z. We have a short exact sequence natural in $\mathscr{G}$

$$0 \to \mathscr{G}_Z \to \mathscr{G} \to j^{\mathrm{st}}\mathscr{G} \to 0.$$

We also obtain the following exact sequence of sheaves of abelian groups via pullback:

$$0 \to \mathscr{T}\!or_1^{p^{-1}\mathscr{O}_{\mathbb{A}^i_X}}(p^{-1}\mathscr{M}, \mathscr{O}_{\mathbb{A}^i_{\widetilde{X}}}) \to p^*\mathscr{K} \to p^*\mathscr{F} \to p^*\mathscr{M} \to 0.$$

To make our notation clearer, we set $\mathcal{T} = \mathcal{T}or_1^{p^{-1}\mathcal{O}_{\mathbb{A}^i_X}}(p^{-1}\mathcal{M}, \mathcal{O}_{\mathbb{A}^i_{\widetilde{X}}})$. We flesh both these exact sequences out into a (nonexact) commutative diagram of $p^{-1}\mathcal{O}_{\mathbb{A}^i_X}$-modules

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_Z & \longrightarrow & \mathcal{T} & \longrightarrow & j^{\mathrm{st}}\mathcal{T} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (p^*\mathcal{K})_Z & \longrightarrow & p^*\mathcal{K} & \longrightarrow & j^{\mathrm{st}}p^*\mathcal{K} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (p^*\mathcal{F})_Z & \longrightarrow & p^*\mathcal{F} & \longrightarrow & j^{\mathrm{st}}p^*\mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (p^*\mathcal{M})_Z & \longrightarrow & p^*\mathcal{M} & \longrightarrow & j^{\mathrm{st}}p^*\mathcal{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We observe that every row and the middle column is exact. The first map in the left column is an injection and the last map in the right column is a surjection. Since $p^*\mathcal{F}$ is flat, we have $(p^*\mathcal{F})_Z = 0$. This induces a lifting of the injection

$$\begin{array}{ccc}
\mathcal{T}_Z & \longrightarrow & \mathcal{T} \\
\downarrow & \nearrow & \downarrow \\
(p^*\mathcal{K})_Z & \longrightarrow & p^*\mathcal{K}
\end{array}$$

We finish the proof by showing $j^*\mathcal{T}or_1^{p^{-1}\mathcal{O}_{\mathbb{A}^i_X}}(p^{-1}\mathcal{M}, \mathcal{O}_{\mathbb{A}^i_{\widetilde{X}}}) = 0$. Since $j : \mathbb{A}^i_U \to \mathbb{A}^i_{\widetilde{X}}$ is flat, the sheaf is isomorphic to $\mathcal{T}or_1^{\mathbb{A}^i_U}(j^*p^{-1}\mathcal{M}, j^*\mathcal{O}_{\mathbb{A}^i_{\widetilde{X}}})$ and $j^*\mathcal{O}_{\mathbb{A}^i_{\widetilde{X}}} \cong \mathcal{O}_{\mathbb{A}^i_U}$. Our big diagram can be rewritten as

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_Z & \xrightarrow{\cong} & \mathcal{T} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\cong} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (p^*\mathcal{K})_Z & \longrightarrow & p^*\mathcal{K} & \longrightarrow & j^{\mathrm{st}}p^*\mathcal{K} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & p^*\mathcal{F} & \longrightarrow & j^{\mathrm{st}}p^*\mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (p^*\mathcal{M})_Z & \longrightarrow & p^*\mathcal{M} & \longrightarrow & j^{\mathrm{st}}p^*\mathcal{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and we can glue together to get a flat resolution of $p^*\mathcal{M}$ as an $\mathcal{O}_{\mathbb{A}^i_{\widetilde{X}}}$-module

$$0 \to j^{\mathrm{st}} p^*\mathcal{K} \to p^*\mathcal{F} \to p^*\mathcal{M} \to 0$$

implying globally finite Tor-amplitude. It remains to show the complex is pseudo-coherent. This follows since $\mathbb{A}^i_{\widetilde{X}}$ is Noetherian and $p^*\mathcal{M}$ is coherent. Since $p^*\pi_2^*\mathcal{A}$ is a sheaf of smooth quasicoherent dg-algebras over $\mathcal{O}_{\mathbb{A}^i_{\widetilde{X}}}$, the complex $p^*\mathcal{M}$ is perfect over $p^*\pi_2^*\mathcal{A}$ by Lemma 2.5. By commutativity, $p^*\mathcal{M}$ restricts to $\tilde{\phi}^*\mathcal{N}$ on $\mathbb{G}^i_{m,\widetilde{X}}$. This completes the proof of Proposition 3.2. □

We will need a relative version of Proposition 3.2.

Corollary 3.4. *Let $f : S \to X$ be a smooth quasiprojective morphism of Noetherian schemes with X reduced and quasiprojective over a Noetherian base ring. Let $\mathcal{A}$ be a sheaf of smooth proper connective quasicoherent dg-algebras over X and consider a negative twisted K-theory class $\gamma \in K_i^{\mathcal{A}}(S)$ for $i < 0$. Then there exists a projective birational morphism $\rho : \widetilde{X} \to X$ such that, under the pullback of the pullback morphism, $\rho_S^*\gamma = 0$.*

Proof. We will briefly check that we can run the induction argument in the proof of Proposition 3.2. The assumptions of this corollary are invariant under pullback along projective birational morphisms $\widetilde{X} \to X$. We need to ensure we can select projective birational morphisms to our base X. Lemma 6.5 of [Kerz et al. 2018] is stated in a relative setting. The proof also relies on platification par éclatement. This can still be applied in our relative setting as X is reduced [Kerz and Strunk 2017, Proposition 5]. □

4. Twisted Weibel's conjecture

We now prove Theorem 1.1 and an extension across a smooth affine morphism. We begin with the base induction step for both theorems. Kerz and Strunk [2017] use a sheaf cohomology result of Grothendieck along with a spectral sequence argument to show vanishing for a Zariski sheaf of spectra can be reduced to the setting of local ring.

Proposition 4.1. *Let R be a regular Noetherian ring of Krull dimension d over a local Artinian ring k. Let $\mathcal{A}$ be a smooth proper connective dg-algebra over R; then $K_i^{\mathcal{A}}(R) = 0$ for $i < 0$.*

Proof. By Corollary 2.10, we may assume k is a field. Proposition 5.4 of [Raedschelders and Stevenson 2019] shows that the t-structure on $D(\mathcal{A})$ restricts to a t-structure on $\mathrm{Perf}(\mathcal{A})$, which is observably bounded. The heart is the category of finitely generated modules over $H_0(\mathcal{A})$. As $H_0(\mathcal{A})$ is finite-dimensional over k, this is a Noetherian abelian category. By Theorem 1.2 of Antieau, Gepner, and Heller [Antieau et al. 2019], the negative K-theory vanishes. □

Theorem 1.1. *Let X be a Noetherian scheme of Krull dimension d and $\mathscr{A}$ a sheaf of smooth proper connective quasicoherent dg-algebras on X; then $K^{\mathscr{A}}_{-i}(X)$ vanishes for $i > d$.*

Proof. Proposition 4.1 covers the base case so assume $d > 0$. By the Kerz–Strunk spectral sequence argument and Corollary 2.8, we may assume X is a Noetherian reduced affine scheme.

Choose a negative $K^{\mathscr{A}}$-theory class $\gamma \in K^{\mathscr{A}}_{-i}(X)$ for $i \geq \dim X + 1$. Using Proposition 3.2, construct a projective birational morphism that kills γ and extend it to an abstract blow-up square

$$\begin{array}{ccc} E & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

By [Land and Tamme 2019, Theorem A.8], there is a Mayer–Vietoris exact sequence of progroups

$$\cdots \to \{K^{\mathscr{A}}_{-i+1}(E_n)\} \to K^{\mathscr{A}}_{-i}(X) \to K^{\mathscr{A}}_{-i}(\widetilde{X}) \oplus \{K^{\mathscr{A}}_{-i}(Y_n)\} \to \{K^{\mathscr{A}}_{-i}(E_n)\} \to \cdots .$$

When $i \geq \dim X + 1$, by induction every nonconstant progroup vanishes and $K^{\mathscr{A}}_{-i}(X) \cong K^{\mathscr{A}}_{-i}(\widetilde{X})$ showing $\gamma = 0$. □

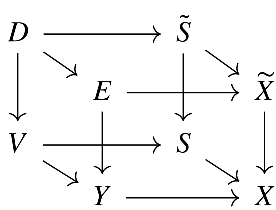
By [Antieau and Gepner 2014, Theorem 3.15], we recover Weibel's vanishing for discrete Azumaya algebras.

Corollary 4.2. *For X a Noetherian d-dimensional scheme and $\mathscr{A}$ a quasicoherent sheaf of discrete Azumaya algebras, then $K^{\mathscr{A}}_{-i}(X) = 0$ for $i > d$.*

The next result nearly covers the K-regularity portion of Weibel's conjecture, but we are missing the boundary case $K^{\mathscr{A}}_{-d}(X) \cong K^{\mathscr{A}}_{-d}(\mathbb{A}^n_X)$.

Theorem 4.3. *Let $f : S \to X$ be a smooth affine morphism of Noetherian schemes and $\mathscr{A}$ a sheaf of smooth proper connective quasicoherent dg-algebras on X. Then $K^{\mathscr{A}}_{-i}(f) = 0$ for $i > \dim X + 1$.*

Proof. The base case is covered by Proposition 4.1 and our reductions are analogous to those in the proof of Theorem 1.1. So assume X is a Noetherian reduced affine scheme of dimension d. Choose $\gamma \in K^{\mathscr{A}}_{-i}(S)$ with $i > d$. Using Corollary 3.4, construct a projective birational morphism $\rho : \widetilde{X} \to X$ that kills γ. We then build a morphism of abstract blow-up squares



By Theorem 2.12, we again get a long exact sequence of progroups corresponding to the back square

$$\cdots \to \{K^{\mathscr{A}}_{-i+1}(D_n)\} \to K^{\mathscr{A}}_{-i}(S) \to K^{\mathscr{A}}_{-i}(\tilde{S}) \oplus \{K^{\mathscr{A}}_{-i}(V_n)\} \to \{K^{\mathscr{A}}_{-i}(D_n)\} \to \cdots.$$

When $i \geq \dim X + 1$, every nonconstant progroup vanishes by induction and we have an isomorphism $K^{\mathscr{A}}_{-i}(S) \cong K^{\mathscr{A}}_{-i}(\tilde{S})$ implying $\gamma = 0$. □

Remark 4.4. The conditions on the morphism in Corollary 3.4 are more general than those of Theorem 4.3. We might hope to generalize Theorem 4.3 to a smooth quasiprojective or smooth projective map of Noetherian schemes. Although the induction step is present, both base cases fail. Consider the descent spectral sequence

$$E_2^{p,q} := H^p(X, \widetilde{K}_q) \Rightarrow K_{q-p}(X) \quad \text{with } d_2 = (2, 1).$$

If $\dim X \leq 3$, then

$$E_3^{2,1} = E_\infty^{2,1} = \operatorname{coker}(H^0(X, \mathbb{Z}) \xrightarrow{d_2} H^2(X, \mathscr{O}_X^*))$$

contributes to $K_{-1}(X)$. The differential is zero as the edge morphism

$$K_0(X) \xrightarrow{\text{rank}} E_\infty^{0,0}$$

identifies $E_\infty^{0,0}$ with the rank component of K_0, implying $E_2^{0,0} = E_\infty^{0,0}$. We now construct a family of examples for schemes X with nontrivial $H^2(X, \mathscr{O}_X^*)$. Let X_{red} be quasiprojective smooth over a field k and form the cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X_{\text{red}} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(k[t]/(t^2)) & \longrightarrow & \operatorname{Spec} k \end{array}$$

The pullback X will be our counterexample. We have an isomorphism

$$\mathscr{O}_X^* \cong g_*(\mathscr{O}_{X_{\text{red}}}^*) \oplus g_*(\mathscr{O}_{X_{\text{red}}})$$

of sheaves of abelian groups on X with $g : X_{\text{red}} \to X$ the pullback of the reduction morphism $\operatorname{Spec} k \to \operatorname{Spec} k[t]/(t^2)$. Locally, $(R[t]/(t^2))^\times$ consists of all elements of the form $u + v \cdot t$ where $u \in R^\times$ and $v \in R$. Sheaf cohomology commutes with coproducts, so this turns into an isomorphism

$$\begin{aligned} H^2(X, \mathscr{O}_X^*) &\cong H^2(X, g_*(\mathscr{O}_{X_{\text{red}}}^*)) \oplus H^2(X, g_*(\mathscr{O}_{X_{\text{red}}})) \\ &\cong H^2(X_{\text{red}}, \mathscr{O}_{X_{\text{red}}}^*) \oplus H^2(X_{\text{red}}, \mathscr{O}_{X_{\text{red}}}). \end{aligned}$$

Now the problem reduces to finding a surface or 3-fold X_{red} with nontrivial degree-2 sheaf cohomology. Take a smooth quartic in $\mathbb{P}_k^3$ for a counterexample which is smooth and proper. Here is a counterexample which is smooth and quasiaffine. Let $(A, \mathfrak{m})$ be a 3-dimensional local ring which is smooth over a field k. Take $X = \operatorname{Spec} A \setminus \{\mathfrak{m}\}$ to be the punctured spectrum. Then $H^2(X, \mathscr{O}_X) \cong H^3_{\mathfrak{m}}(A)$, which is the injective hull of the residue field $A/\mathfrak{m}$.

Acknowledgements

The author is thankful to his advisor, Benjamin Antieau, for the suggested project, patience, and guidance. He also thanks Maximilien Péroux for helpful comments on an earlier draft. The author was partially supported by NSF grant DMS-1552766 and NSF RTG grant DMS-1246844.

References

[Antieau and Gepner 2014] B. Antieau and D. Gepner, "Brauer groups and étale cohomology in derived algebraic geometry", *Geom. Topol.* **18**:2 (2014), 1149–1244. MR Zbl

[Antieau et al. 2019] B. Antieau, D. Gepner, and J. Heller, "K-theoretic obstructions to bounded t-structures", *Invent. Math.* **216**:1 (2019), 241–300. MR Zbl

[Bass 1968] H. Bass, *Algebraic K-theory*, W. A. Benjamin, New York, 1968. MR Zbl

[Blumberg et al. 2013] A. J. Blumberg, D. Gepner, and G. Tabuada, "A universal characterization of higher algebraic K-theory", *Geom. Topol.* **17**:2 (2013), 733–838. MR Zbl

[Căldăraru 2000] A. H. Căldăraru, *Derived categories of twisted sheaves on Calabi–Yau manifolds*, Ph.D. thesis, Cornell University, Ithaca, NY, 2000, https://search.proquest.com/docview/304592237. MR

[Clausen and Mathew 2019] D. Clausen and A. Mathew, "Hyperdescent and étale K-theory", preprint, 2019. arXiv

[Clausen et al. 2020] D. Clausen, A. Mathew, N. Naumann, and J. Noel, "Descent in algebraic K-theory and a conjecture of Ausoni–Rognes", *J. Eur. Math. Soc.* **22**:4 (2020), 1149–1200. MR Zbl

[Edidin et al. 2001] D. Edidin, B. Hassett, A. Kresch, and A. Vistoli, "Brauer groups and quotient stacks", *Amer. J. Math.* **123**:4 (2001), 761–777. MR Zbl

[Grothendieck 1968a] A. Grothendieck, "Le groupe de Brauer, I: Algèbres d'Azumaya et interprétations diverses", pp. 46–66 in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, North-Holland, Amsterdam, 1968. MR Zbl

[Grothendieck 1968b] A. Grothendieck, "Le groupe de Brauer, II: Théorie cohomologique", pp. 67–87 in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, North-Holland, Amsterdam, 1968. MR

[Grothendieck 1968c] A. Grothendieck, "Le groupe de Brauer, III: Exemples et compléments", pp. 88–188 in *Dix exposés sur la cohomologie des schémas*, Adv. Stud. Pure Math. **3**, North-Holland, Amsterdam, 1968. MR Zbl

[de Jong 2006] A. J. de Jong, "A result of Gabber", preprint, 2006, https://www.math.columbia.edu/~dejong/papers/2-gabber.pdf.

[Kerz 2018] M. Kerz, "On negative algebraic K-groups", pp. 181–191 in *Proceedings of the International Congress of Mathematicians* (Rio de Janeiro, 2018), vol. II: Invited lectures, edited by B. Sirakov et al., World Scientific, Hackensack, NJ, 2018. MR

[Kerz and Strunk 2017] M. Kerz and F. Strunk, "On the vanishing of negative homotopy K-theory", *J. Pure Appl. Algebra* **221**:7 (2017), 1641–1644. MR Zbl

[Kerz et al. 2018] M. Kerz, F. Strunk, and G. Tamme, "Algebraic K-theory and descent for blow-ups", *Invent. Math.* **211**:2 (2018), 523–577. MR Zbl

[Land and Tamme 2019] M. Land and G. Tamme, "On the K-theory of pullbacks", *Ann. of Math.* (2) **190**:3 (2019), 877–930. MR Zbl

[Lieblich 2004] M. Lieblich, *Moduli of twisted sheaves and generalized Azumaya algebras*, Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, MA, 2004, https://search.proquest.com/docview/305088000. MR

[Lurie 2017] J. Lurie, "Higher algebra", preprint, 2017, https://www.math.ias.edu/~lurie/papers/HA.pdf.

[Quillen 1973] D. Quillen, "Higher algebraic K-theory, I", pp. 85–147 in *Algebraic K-theory* (Seattle, 1972), vol. I: Higher K-theories, edited by H. Bass, Lecture Notes in Math. **341**, Springer, 1973. MR Zbl

[Raedschelders and Stevenson 2019] T. Raedschelders and G. Stevenson, "The motive of a smooth proper connective DG-algebra is of unit type", preprint, 2019. arXiv

[Raynaud and Gruson 1971] M. Raynaud and L. Gruson, "Critères de platitude et de projectivité: techniques de 'platification' d'un module", *Invent. Math.* **13** (1971), 1–89. MR Zbl

[Thomason and Trobaugh 1990] R. W. Thomason and T. Trobaugh, "Higher algebraic K-theory of schemes and of derived categories", pp. 247–435 in *The Grothendieck Festschrift*, vol. III, edited by P. Cartier et al., Progr. Math. **88**, Birkhäuser, Boston, 1990. MR Zbl

[Toën 2012] B. Toën, "Derived Azumaya algebras and generators for twisted derived categories", *Invent. Math.* **189**:3 (2012), 581–652. MR Zbl

[Toën and Vaquié 2007] B. Toën and M. Vaquié, "Moduli of objects in dg-categories", *Ann. Sci. École Norm. Sup.* (4) **40**:3 (2007), 387–444. MR Zbl

[Weibel 1980] C. A. Weibel, "K-theory and analytic isomorphisms", *Invent. Math.* **61**:2 (1980), 177–197. MR Zbl

Received 4 Feb 2020. Revised 2 Apr 2020. Accepted 20 Apr 2020.

JOEL STAPLETON: jstapl3@uic.edu
Joel Stapleton, University of Illinois at Chicago, Chicago, IL, United States

ANNALS OF K-THEORY

msp.org/akt

EDITORIAL BOARD

Joseph Ayoub	ETH Zürich, Switzerland joseph.ayoub@math.uzh.ch
Paul Balmer	University of California, Los Angeles, USA balmer@math.ucla.edu
Guillermo Cortiñas	Universidad de Buenos Aires and CONICET, Argentina gcorti@dm.uba.ar
Hélène Esnault	Freie Universität Berlin, Germany liveesnault@math.fu-berlin.de
Eric Friedlander	University of Southern California, USA ericmf@usc.edu
Max Karoubi	Institut de Mathématiques de Jussieu – Paris Rive Gauche, France max.karoubi@imj-prg.fr
Moritz Kerz	Universität Regensburg, Germany moritz.kerz@mathematik.uni-regensburg.de
Huaxin Lin	University of Oregon, USA livehlin@uoregon.edu
Alexander Merkurjev	University of California, Los Angeles, USA merkurev@math.ucla.edu
Birgit Richter	Universität Hamburg, Germany birgit.richter@uni-hamburg.de
Jonathan Rosenberg	(Managing Editor) University of Maryland, USA jmr@math.umd.edu
Marco Schlichting	University of Warwick, UK schlichting@warwick.ac.uk
Charles Weibel	(Managing Editor) Rutgers University, USA weibel@math.rutgers.edu
Guoliang Yu	Texas A&M University, USA guoliangyu@math.tamu.edu

PRODUCTION

Silvio Levy	(Scientific Editor) production@msp.org

Annals of K-Theory is a journal of the K-Theory Foundation (ktheoryfoundation.org). The K-Theory Foundation acknowledges the precious support of Foundation Compositio Mathematica, whose help has been instrumental in the launch of the Annals of K-Theory.

See inside back cover or msp.org/akt for submission instructions.

The subscription price for 2020 is US $510/year for the electronic version, and $575/year (+$25, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Annals of K-Theory (ISSN 2379-1681 electronic, 2379-1683 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

AKT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2020 Mathematical Sciences Publishers

ANNALS OF K-THEORY

2020 vol. 5 no. 3