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On the Rost divisibility of henselian discrete valuation fields of cohomological dimension 3

Yong Hu and Zhengyao Wu

Let F be a field, ℓ a prime and D a central division F-algebra of ℓ-power degree. By the Rost kernel of D we mean the subgroup of F^* consisting of elements λ such that the cohomology class $(D)\cup(\lambda)\in H^3(F,\mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ vanishes. In 1985, Suslin conjectured that the Rost kernel is generated by i-th powers of reduced norms from $D^{\otimes i}$ for all $i\geq 1$. Despite known counterexamples, we prove some new special cases of Suslin's conjecture. We assume F is a henselian discrete valuation field with residue field k of characteristic different from ℓ. When D has period ℓ, we show that Suslin's conjecture holds if either k is a 2-local field or the cohomological ℓ-dimension $\mathrm{cd}_\ell(k)$ of k is ≤ 2. When the period is arbitrary, we prove the same result when k itself is a henselian discrete valuation field with $\mathrm{cd}_\ell(k)\leq 2$. In the case $\ell=\mathrm{char}(k)$, an analog is obtained for tamely ramified algebras. We conjecture that Suslin's conjecture holds for all fields of cohomological dimension 3.

1. Introduction

Let F be a field and ℓ a prime number. For simplicity we first assume ℓ is different from the characteristic of F. For an integer $d\geq 1$, we write

$$H^d(F)=H^d(F,\mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1)),$$

the inductive limit of the Galois cohomology groups $H^d(F,\mu_{\ell^r}^{\otimes(d-1)})$, $r\geq 1$. An element $\alpha\in H^2(F)$ may be identified with a Brauer class in the ℓ-primary torsion part of the Brauer group $\mathrm{Br}(F)$. Taking the cup product with α yields a well defined group homomorphism

$$R_\alpha: F^*\to H^3(F),\quad \lambda\mapsto\alpha\cup(\lambda)\,.$$

The kernel of R_α is called the *Rost kernel* of α and denoted by $\mathcal{R}(\alpha)$. Write $\mathrm{Nrd}(\alpha):=\mathrm{Nrd}(D^*)$, where D is the central division F-algebra in the Brauer class

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α and $\mathrm{Nrd}: D^* \to F^*$ denotes the reduced norm map on the nonzero elements of D. It is well known that $\mathrm{Nrd}(\alpha)$ is contained in the Rost kernel of α.

We remark that when $\ell = \mathrm{char}(F)$, the groups $H^d(F) = H^d(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1))$, $d \geq 1$ and the cup product maps $H^d(F) \times F^* \to H^{d+1}(F)$, $(\theta, \lambda) \mapsto \theta \cup (\lambda)$ can still be defined in an appropriate way, and what we said above remains valid (see, e.g., [Kato 1980, §3.2], [Kato 1982], or [Merkurjev 2003, Appendix A]).

In many interesting cases it is known that the Rost kernel consists precisely of the reduced norms. We collect some important results in this direction.

Example 1.1. Suppose as above that the period of $\alpha \in \mathrm{Br}(F)$ is a power of ℓ. In all the following cases we have $\mathcal{R}(\alpha) = \mathrm{Nrd}(\alpha)$.

(1) The (Schur) index of α is ℓ.

This case follows from a well known theorem of Merkurjev and Suslin [Merkurjev and Suslin 1982, Theorem 12.2; Suslin 1984, Theorem 24.4] in the case $\ell \neq \mathrm{char}(F)$ and its p-primary counterpart [Gille 2000, p. 94, Theorem 6] in the case $\ell = \mathrm{char}(F)$.

(2) The *separable ℓ-dimension* $\mathrm{sd}_\ell(F)$ of F is ≤ 2 [Suslin 1984, Theorem 24.8; Gille 2000, p. 94, Theorem 7].

The notion $\mathrm{sd}_\ell(F)$ is introduced by Gille [2000, p. 62] (cf. Remark 5.6). If $\ell \neq \mathrm{char}(F)$, it is the same as the cohomological ℓ-dimension considered in [Serre 1994].

(3) F is a global field.

If $\ell \neq 2$ or if F has no real places (e.g., F is a global function field), this is covered by the previous case. When F is a real number field and $\ell = 2$, this follows from a Hasse principle proved by Kneser [1969, Chapter 5, Theorem 1.a].

(4) F is a 2*-local field*, i.e., a complete discrete valuation field whose residue field k is a nonarchimedean local field (with finite residue field).

This case is a consequence of Kato's two-dimensional local class field theory [1980, p. 657, §3.1, Theorem 3(2)]. The case with $\ell \neq \mathrm{char}(k)$ is also observed in [Parimala et al. 2018, Theorem 1.3].

(5) The following cases are established in recent work of Parimala, Preeti and Suresh [Parimala et al. 2018, Theorems 1.1 and 1.3]:

(a) F is a complete discrete valuation field whose residue field k is a global field of characteristic $\neq \ell$, and if $\ell = 2$, suppose k has no real places.
(If k is a global function field of characteristic ℓ, the same result is obtained in [Hu 2020] under the addition hypothesis that α has period ℓ or is tamely ramified. If k is a number field with real places, see Example 2.3(4.b) for a weaker conclusion.)

(b) F is the function field of an algebraic curve over a nonarchimedean local field K, and ℓ is different from the residue characteristic of K.

(6) The element α has index 4, $\mathrm{char}(F) \neq 2$, and nonsingular quadratic forms of dimension 12 over all finite extensions of F are all isotropic. (For example, F can be an extension of $\mathbb{Q}_2$ of transcendence degree 1, by [Parimala and Suresh 2014, Theorem 4.7].)

In [Preeti and Soman 2015, Theorem 4.1], this result is stated with the extra assumptions that F has characteristic 0 and cohomological dimension ≤ 3. A careful inspection shows that these assumptions are not used in the proof. (Notice however that the cohomological 2-dimension is indeed ≤ 3 under our assumption on the isotropy of 12 dimensional forms.)

We prove a characteristic 2 version in Proposition 2.1.

In general, the equality $\mathcal{R}(\alpha) = \mathrm{Nrd}(\alpha)$ may not hold. In fact, there exists a field F of cohomological dimension 3 in characteristic 0 and a biquaternion algebra over F whose Rost kernel contains more elements than the reduced norms [Colliot-Thélène et al. 2012, Remark 5.1]. A description of the Rost kernel was conjectured by Suslin as follows:

(1.2) Suslin's conjecture [1984, Conjecture 24.6]. If α has period $\mathrm{per}(\alpha) = \ell^n$, then

$$\mathcal{R}(\alpha) = \prod_{i=0}^{n} \mathrm{Nrd}(\ell^i \alpha)^{\ell^i} = \mathrm{Nrd}(\alpha) \cdot \mathrm{Nrd}(\ell\alpha)^{\ell} \cdots \mathrm{Nrd}(\ell^{n-1}\alpha)^{\ell^{n-1}} \cdot (F^*)^{\ell^n}. \quad (1.3)$$

Here, for a multiplicative abelian group A and an integer $m \geq 1$, A^m denotes the subgroup consisting of m-th powers in A. We call the group $\prod_{i=0}^{n} \mathrm{Nrd}(\ell^i \alpha)^{\ell^i}$ the *Suslin group* of α and denote it by $\mathcal{S}(\alpha)$. Suslin's conjecture amounts to saying that $\mathcal{R}(\alpha) = \mathcal{S}(\alpha)$. (In [Merkurjev 1995], $\mathcal{S}(\alpha)$ and $\mathcal{R}(\alpha)$ are denoted by $A(D)$ and $B(D)$, respectively.)

We may also consider the induced map

$$\widetilde{R}_\alpha : F^*/\mathrm{Nrd}(\alpha) \to H^3(F), \quad \lambda \mapsto \alpha \cup (\lambda),$$

which is often called the *Rost invariant* map associated to the semisimple simply connected algebraic group $\mathrm{SL}_1(D)$ (see, e.g., [Merkurjev 2003]). In the special case $\mathrm{per}(\alpha) = \ell$, (1.3) means that the kernel of $\widetilde{R}_\alpha$ consists precisely of the ℓ-divisible elements in the group $F^*/\mathrm{Nrd}(\alpha)$.

We feel that the definition below is now well motivated.

Definition 1.4. Let F be a field, ℓ a prime number and n a positive integer. We say that F is *Rost ℓ^n-divisible* if Suslin's conjecture holds (i.e., the Rost kernel equals the Suslin group) for all $\alpha \in \mathrm{Br}(F)$ whose period divides ℓ^n.

We say that F is *Rost ℓ^∞-divisible* if it is Rost ℓ^n-divisible for all $n \geq 1$. If F is Rost ℓ^∞-divisible for all primes ℓ, we say that F is *Rost divisible*. (In that case, $\mathcal{R}(\alpha) = \mathcal{S}(\alpha)$ for all Brauer classes α over F, by Proposition 3.8.)

Of course, the cases covered by Example 1.1 are all examples of Rost divisibility. On the other hand, in the example where $\mathcal{R}(\alpha) \neq \mathrm{Nrd}(\alpha)$ discussed in [Colliot-Thélène et al. 2012, Remark 5.1], α is the class of a biquaternion algebra and for that α Suslin's conjecture does hold. In fact, in characteristic different from 2, $\mathcal{R}(\alpha)$ and $\mathcal{S}(\alpha)$ are the same for all Brauer classes α of degree at most 4. This was remarked by Suslin immediately after the statement of his conjecture [Suslin 1984, Conjecture 24.6]. His proof, which uses K-cohomology groups of Severi–Brauer varieties, can be found in [Merkurjev 1995, §1]. For biquaternion algebras, an alternative proof is given in [Knus et al. 1995]. In Section 2 we extend this last result to characteristic 2 and derive a number of examples of Rost 2-divisibility.

However, not all fields are Rost 2-divisible and Suslin's conjecture can be false for some α which is a tensor product of three quaternion algebras; see, e.g., [Knus et al. 1995, p. 283]. For an odd prime ℓ, Merkurjev [1995, §2] constructed over a certain field F tensor products of two cyclic algebras of degree ℓ that violate Suslin's conjecture. We notice that in these known counterexamples the cohomological dimension $\mathrm{cd}_\ell(F)$ must be greater than 3 (see Section 6 for some more details). We therefore conjecture that Suslin's conjecture is true whenever $\mathrm{cd}_\ell(F) \leq 3$ if $\mathrm{char}(F) \neq \ell$ (see Conjecture 6.3 for a more precise statement).

The goal of this paper is to provide some evidence to our conjecture.

Theorem 1.5. *Let F be a henselian* (*e.g., complete*) *discrete valuation field with residue field k. Let ℓ be a prime number different from* $\mathrm{char}(k)$. *Suppose that the following properties hold for every finite cyclic extension L/k of degree* 1 *or* ℓ:

(1) (Rost ℓ-divisibility) *L is Rost ℓ-divisible.*

(2) (H^3-corestriction injectivity) *The corestriction map*

$$\mathrm{Cor}_{L/k} : H^3(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to H^3(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

is injective.

Then the field F is Rost ℓ-divisible.

The proof of Theorem 1.5 will be given in Section 4 (page 692). We will see in Remark 6.4 that (2) in this theorem cannot be dropped, although it is not a necessary condition for F to be Rost ℓ-divisible. On the other hand, in Theorem 4.18 we will give a description of the quotient group $\mathcal{R}(\alpha)/\mathcal{S}(\alpha)$ (the Rost kernel modulo the Suslin group) that is valid without the two assumptions of Theorem 1.5.

Theorem 1.5 also has a version in the case $\ell = \mathrm{char}(k)$. But more definitions and technical assumptions have to be introduced for a proper statement of that version. So we postpone it to Section 5 (see Theorem 5.5).

Corollary 1.6. *Let F, k and ℓ be as in Theorem 1.5. Assume that the residue field k satisfies one of the following conditions*:

(1) *k has cohomological ℓ-dimension* $\mathrm{cd}_\ell(k) \le 2$;

(2) *k is a* 2*-local field*;

(3) $k = k_0((x))((y))((z))$, *where k_0 is an algebraically closed field* (*of characteristic* $\neq \ell$).

Then F is Rost ℓ-divisible.

Proof. It is sufficient to show that k has the two properties stated in Corollary 1.6. The case with $\mathrm{cd}_\ell(k) \le 2$ is clear (see Example 1.1(2) for the Rost divisibility). When k is a 2-local field, the Rost divisibility condition is satisfied according to Example 1.1(4) and the corestriction injectivity property holds by [Kato 1980, p. 660, §3.2, Proposition 1]. In the third case, the injectivity of corestriction can be shown in the same way as for 2-local fields. The Rost ℓ-divisibility of k follows from Case 1. □

In the above corollary the field k can also be a number field or the field $\mathbb{R}((x))$. In that situation only the case $\ell = 2$ needs to be treated. For a number field the H^3-corestriction injectivity can be deduced from a result of Tate on the cohomology of global fields (see [Neukirch et al. 2008, Corollary 8.3.12(iii)]). However, we feel that a simpler method in that case is to utilize the theory of quadratic forms (see Example 2.3(4.b)).

We think that whether the field F in Corollary 1.6 is Rost ℓ^∞-divisible is an interesting open problem. In general, we wonder if there is an approach to pass from the Rost ℓ-divisibility to the ℓ^∞-divisibility.

A special case is treated in the following theorem, which we will prove in the second half of Section 4 (see (4.19)–(4.23)).

Theorem 1.7. *Let ℓ be a prime number and $n \in \mathbb{N}^*$. Let k be a henselian discrete valuation field of residue characteristic different from ℓ. Let F be a henselian discrete valuation field with residue field k.*

If $\mathrm{cd}_\ell(k) \le 2$ *and* $\mu_{\ell^n} \subseteq k$ (*i.e., k contains a primitive ℓ^n-th root of unity*), *then F is Rost ℓ^n-divisible.*

For example, the field $\mathbb{C}(x)((y))((z))$ is Rost divisible according to the above theorem.

2. Rost kernel of biquaternion algebras

As a warmup, we extend Suslin's conjecture for biquaternion algebras to the characteristic 2 case.

Recall that by a theorem of Albert, a biquaternion algebra over any field F is the same as a central simple F-algebra of period 2 and degree 4; see, e.g., [Knus et al. 1998, Theorem 16.1].

Proposition 2.1. *Let F be a field and let $\alpha \in \mathrm{Br}(F)$ have period* 2 *and index* 4. *Suppose $\lambda \in F^*$ satisfies $\alpha \cup (\lambda) = 0$ in $H^3(F)$. Then $\lambda \in (F^*)^2 \cdot \mathrm{Nrd}(\alpha)$. If, moreover, the u-invariant of F is less than* 12 (*meaning that every nonsingular quadratic form of dimension* 12 *over F is isotropic*), *then $\lambda \in \mathrm{Nrd}(\alpha)$.*

As was mentioned in the introduction, the proof of the above result in characteristic different from 2 already exists in the literature. The first assertion can be found in [Knus et al. 1995] or [Merkurjev 1995], and the second one follows from [Preeti and Soman 2015, Theorem 4.1] (see also our remarks in Example 1.1(6)).

To complete the proof of Proposition 2.1, we need to use basic facts about quadratic forms and cohomology theories in characteristic 2, which we now recall. The reader may consult [Elman et al. 2008, Chapters 1 and 2] and [Kato 1982] for more details.

Let F be a field of characteristic 2. For any $a, b \in F$, we denote by $[a, b]$ the binary quadratic form $(x, y) \mapsto ax^2 + xy + by^2$. A *quadratic* 1*-fold Pfister form* over F is a quadratic form of the type $[1, b]$ for some $b \in F$. For $n \geq 2$, a *quadratic n-fold Pfister form* is a tensor product of a bilinear $(n-1)$-fold Pfister form with a quadratic 1-fold Pfister form, i.e., a form of the shape

$$[\![b; a_1, \dots, a_{n-1}\rangle\!\rangle := [1, b] \otimes \langle\!\langle a_1, \dots, a_{n-1}\rangle\!\rangle.$$

The group of Witt equivalence classes of nonsingular (even-dimensional) quadratic forms over F is denoted by $I_q(F)$ or $I_q^1(F)$, and for each $n \geq 1$, $I_q^n(F)$ denotes the subgroup of $I_q(F)$ generated by scalar multiples of quadratic n-fold Pfister forms.

For a natural number r, we have the Kato–Milne cohomology group $H_2^{r+1}(F) = H^{r+1}(F, \mathbb{Z}/2\mathbb{Z}(r))$, which can be described using absolute differentials. (A brief review about these groups will be given in (5.1).) For each $n \geq 1$ there is a well defined group homomorphism $e_n : I_q^n(F) \to H_2^n(F)$ with the property that

$$e_n(\varphi) = e_n(\psi) \Longleftrightarrow \varphi \cong \psi \tag{2.2}$$

for all quadratic n-fold Pfister forms φ and ψ over F (cf. [Kato 1982, p. 237, Proposition 3]).

Proof of Proposition 2.1 in characteristic 2. Let $[b, a) \otimes [d, c)$ be a biquaternion algebra representing the Brauer class α, where $[b, a)$ denotes the quaternion F-algebra generated by two elements x, y subject to the relations

$$x^2 - x = b, \quad y^2 = a \quad \text{and} \quad yx = (x+1)y.$$

Then the quadratic form $\phi := [1, b+d]\perp a.[1, b]\perp c.[1, d]$ is an Albert form of α [Knus et al. 1998, (16.4)]. Inside the group $I_q(F)$, we have

$$\phi = [1, b] + a.[1, b] + [1, d] + c.[1, d] = [\![b, a\rangle\!\rangle - [\![d, c\rangle\!\rangle \in I_q^2(F)$$

[Elman et al. 2008, Example 7.23], and for any $\lambda \in F^*$,

$$\phi - \lambda.\phi = \phi \otimes \langle 1, -\lambda\rangle = [\![b, a, \lambda\rangle\!\rangle - [\![d, c, \lambda\rangle\!\rangle \in I_q^3(F).$$

The cohomological invariant $e_3 : I_q^3(F) \to H^3(F, \mathbb{Z}/2\mathbb{Z}(2))$ sends $[\![b, a, \lambda\rangle\!\rangle - [\![d, c, \lambda\rangle\!\rangle$ to the cohomology class $\alpha \cup (\lambda) \in H^3(F, \mathbb{Z}/2\mathbb{Z}(2))$. When $\alpha \cup (\lambda) = 0$, we can deduce from (2.2) and [Knus et al. 1998, (16.6)] that

$$\lambda \in G(\phi) := \{\rho \in F^* \mid \rho.\phi \cong \phi\} = F^{*2} \cdot \mathrm{Nrd}(\alpha).$$

This proves the first assertion.

Now assume further that every nonsingular quadratic form over dimension 12 over F is isotropic. Then for the Albert form ϕ and any $\rho \in F^*$, the form $\langle 1, -\rho\rangle \otimes \phi$ is isotropic and hence ρ is a spinor norm for ϕ. By [Knus et al. 1998, Proposition 16.6], we have $\rho^2 \in \mathrm{Nrd}(\alpha)$. This shows $F^{*2} \subseteq \mathrm{Nrd}(\alpha)$. So the second assertion follows. □

Example 2.3. Proposition 2.1 implies that a field F is Rost 2-divisible if every Brauer class in $\mathrm{Br}(F)[2]$ has index at most 4. The following fields possess this property:

(1) Any field F of characteristic 2 with $[F : F^2] \leq 4$. This is a well known theorem of Albert (cf. [Gille and Szamuely 2017, Lemma 9.1.7]).

(2) $F = k_0((x))((y))((z))$ is an iterated Laurent series field in three variables over a quasifinite field k_0 of characteristic 2.

By a *quasifinite* field we mean a perfect field whose absolute Galois is isomorphic to that of a finite field. That every Brauer class in $\mathrm{Br}(F)[2]$ is a biquaternion algebra is proved in [Aravire and Jacob 1995, Theorem 3.3].

(3) A field extension F of transcendence degree 2 over any finite field [Lieblich 2015].

(4) A complete discrete valuation field F whose residue field k satisfies one of the following hypotheses:

(a) k has characteristic 2 and $[k : k^2] \leq 2$ [Parimala and Suresh 2014, Theorem 2.7].

(b) k has characteristic different from 2 and every 2-torsion Brauer class over k is (the class of) a quaternion algebra.

This case follows from Witt's decomposition of the 2-torsion Brauer subgroup $\mathrm{Br}(F)[2]$ (see (4.4)).

Concrete examples of k include

(i) a global field or a local field,
(ii) any C_2-field,
(iii) the field $\mathbb{R}((x))$,
(iv) a one-variable function field over $\mathbb{R}$,
(v) an iterated Laurent series field $k_0((x))((y))((z))$, where k_0 is a quadratically closed field.

Case (i) is well known. (Notice however that the result here is covered by Example 1.1(4) and (5.a) unless k is a number field with real places.)

For C_2-fields the assertion follows from an easy consideration of the isotropy of Albert forms of biquaternion algebras. For the field $k = \mathbb{R}((x))$, the group $\mathrm{Br}(k)[2]$ can be computed explicitly. The other cases were discussed in [Elman and Lam 1973, p. 126 and Proposition 3.12].

(5) A one-variable function field F over a complete discrete valuation field whose residue field k is a C_1-field of characteristic $\neq 2$.

Indeed, Artin [1982, Theorem 6.2] proved that period coincides with index for 2-primary torsion Brauer classes over any C_2-field. One can apply [Harbater et al. 2009, Corollary 5.6] or [Lieblich 2011, Theorem 6.3] to see that 2-torsion Brauer classes over F have index at most 4. The case with k finite was first proved by Saltman [1997].

(6) A one-variable function field F over a complete discrete valuation field whose residue field k is a perfect field of characteristic 2. This case is established in [Parimala and Suresh 2014, Theorem 3.6].

(7) The field of fractions F of a two-dimensional henselian excellent local domain with finite residue field of characteristic $\neq 2$ [Hu 2013, Theorem 3.4].

Except possibly for case (2), the cases (iii)–(v) in (4.b) and the case of real number fields in (4.b)(i), the fields in the above list have u-invariant ≤ 8, and hence the second conclusion of Proposition 2.1 applies to these fields.

Indeed, case (1) follows from [Mammone et al. 1991, p. 338, Corollary 1]. In case (3), F is a C_3-field. In cases (4.a) and (4.b)(ii), one can combine [Mammone et al. 1991, p. 338, Corollary 1] with Springer's theorem (see [Lam 2005, XI.6.2(7)] if $\mathrm{char}(k) \neq 2$ or [Baeza 1982, Theorem 1.1] and [Mammone et al. 1991, p. 343, Theorem 2] if $\mathrm{char}(k) = 2$). Cases (5), (6) and (7) can be found in [Harbater et al. 2009, Corollary 4.13], [Parimala and Suresh 2014, Theorem 4.7] and [Hu 2013, Theorem 1.2], respectively.

Note that in the above list we have restricted to examples that do not depend on Theorem 1.5 or Corollary 1.6.

3. The Suslin group

In this section, we prove some general properties of the Suslin group of an arbitrary Brauer class and show that the general case can be reduced to the case of prime power degree classes.

(3.1) Let F be a field and $\alpha \in \mathrm{Br}(F)$. By a well known result, called the *norm principle for reduced norms* (see [Gille and Szamuely 2017, Proposition 2.6.8]),

$$\mathrm{Nrd}(\alpha) = \bigcup_{L/F} N_{L/F}(L^*) = \bigcup_{M/F} N_{M/F}(M^*),$$

where L/F runs over finite separable extensions splitting α that can be F-embedded into the central division algebra D_α in the Brauer class α, and M/F runs over finite extensions that split α. As an easy consequence, for all $t \in \mathbb{Z}$ we have

$$\mathrm{Nrd}(\alpha) \subseteq \mathrm{Nrd}(t\alpha) \quad \text{and equality holds if} \quad \gcd(t, \mathrm{per}(\alpha)) = 1. \tag{3.2}$$

Moreover, for all $\alpha_1, \alpha_2 \in \mathrm{Br}(F)$,

$$\gcd(\mathrm{per}(\alpha_1), \mathrm{per}(\alpha_2)) = 1 \quad \Longrightarrow \quad \mathrm{Nrd}(\alpha_1 + \alpha_2) = \mathrm{Nrd}(\alpha_1) \cap \mathrm{Nrd}(\alpha_2). \tag{3.3}$$

Indeed, a field extension M/F splits $\alpha_1 + \alpha_2$ if and only if it splits both α_1 and α_2, since the periods of α_1 and α_2 are coprime. So the norm principle shows that

$$\mathrm{Nrd}(\alpha_1 + \alpha_2) \subseteq \mathrm{Nrd}(\alpha_1) \cap \mathrm{Nrd}(\alpha_2).$$

Conversely, suppose $\lambda \in \mathrm{Nrd}(\alpha_1) \cap \mathrm{Nrd}(\alpha_2)$. Letting D_i be the central division F-algebra in the Brauer class α_i for $i = 1, 2$, by the norm principle we can find a subfield $L_i \subseteq D_i$ such that $\lambda \in N_{L_i/F}(L_i^*)$. Let $L = L_1 \cdot L_2$ be the composite field (inside a fixed algebraic closure of F) and put $d_i = [L_i : F]$. Then we have $\gcd(d_1, d_2) = 1$ since d_i divides a power of $\mathrm{per}(\alpha_i)$. In particular, $[L : F] = d_1 d_2$. Thus,

$$\lambda^{d_2} \in N_{L_1/F}(L_1^{*d_2}) \subseteq N_{L_1/F}(N_{L/L_1}(L^*)) \subseteq N_{L/F}(L^*),$$

and similarly, $\lambda^{d_1} \in N_{L/F}(L^*)$. Since $\gcd(d_1, d_2) = 1$, this implies $\lambda \in N_{L/F}(L^*)$. As $L = L_1 L_2$ is clearly a splitting field of $\alpha_1 + \alpha_2$, we get $\lambda \in \mathrm{Nrd}(\alpha_1 + \alpha_2)$. This proves (3.3).

(3.4) In (1.2) we defined the Suslin group of a Brauer class of prime power period. Now we generalize this definition to arbitrary Brauer classes. For any $\alpha \in \mathrm{Br}(F)$, we define the *Suslin group* $\mathcal{S}(\alpha)$, which in [Merkurjev 1995] was denoted by $A(D)$, as follows:

$$\mathcal{S}(\alpha)=\prod_{i=1}^{\infty}\mathrm{Nrd}(i\alpha)^i$$

$$:= \text{the subgroup of } F^* \text{ generated by } \bigcup_{i\geq 1}\mathrm{Nrd}(i\alpha)^i=\prod_{i=1}^{\mathrm{per}(\alpha)}\mathrm{Nrd}(i\alpha)^i. \quad (3.5)$$

In fact, writing $e=\mathrm{per}(\alpha)$ we have

$$\begin{aligned}\mathcal{S}(\alpha)=\prod_{i=1}^{e}\mathrm{Nrd}(i\alpha)^i&=\prod_{d\,|\,e}\prod_{\substack{1\leq i\leq e\\ \gcd(i,e)=d}}\mathrm{Nrd}(i\alpha)^i\\ &=\prod_{d\,|\,e}\prod_{\substack{1\leq j\leq e/d\\ \gcd(j,e/d)=1}}\mathrm{Nrd}(jd\alpha)^{jd}\\ &=\prod_{d\,|\,e}\prod_{\substack{1\leq j\leq e/d\\ \gcd(j,e/d)=1}}\mathrm{Nrd}(d\alpha)^{jd}\quad(\text{using } (3.2))\\ &=\prod_{d\,|\,e}\mathrm{Nrd}(d\alpha)^d.\end{aligned}$$

Also, from the definition and (3.2) we see easily that for all $t\in\mathbb{N}^*$,

$$\mathcal{S}(t\alpha)^t\subseteq\mathcal{S}(\alpha)\subseteq\mathcal{S}(t\alpha),\quad\text{and}\quad\mathcal{S}(\alpha)=\mathcal{S}(t\alpha)\ \text{if}\ \gcd(t,\mathrm{per}(\alpha))=1.\quad(3.6)$$

We can prove the following analog of (3.3):

Lemma 3.7. *For all* $\alpha_1,\alpha_2\in\mathrm{Br}(F)$, *if* $\gcd(\mathrm{per}(\alpha_1),\mathrm{per}(\alpha_2))=1$, *then*

$$\mathcal{S}(\alpha_1+\alpha_2)=\mathcal{S}(\alpha_1)\cap\mathcal{S}(\alpha_2).$$

Proof. For every $i\geq 1$, $\gcd(\mathrm{per}(i\alpha_1),\mathrm{per}(i\alpha_2))=1$. So by (3.3), $\mathrm{Nrd}(i\alpha_1+i\alpha_2)=\mathrm{Nrd}(i\alpha_1)\cap\mathrm{Nrd}(i\alpha_2)$, and hence

$$\mathrm{Nrd}(i\alpha_1+i\alpha_2)^i\subseteq\mathrm{Nrd}(i\alpha_1)^i\cap\mathrm{Nrd}(i\alpha_2)^i.$$

Together with (3.5) this proves $\mathcal{S}(\alpha_1+\alpha_2)\subseteq\mathcal{S}(\alpha_1)\cap\mathcal{S}(\alpha_2)$.

Conversely, suppose $\lambda\in\mathcal{S}(\alpha_1)\cap\mathcal{S}(\alpha_2)$. Write $e_i=\mathrm{per}(\alpha_i)$ and $\alpha=\alpha_1+\alpha_2$. Using (3.6) we find

$$\lambda^{e_2}\in\mathcal{S}(\alpha_1)^{e_2}=\mathcal{S}(e_2\alpha_1)^{e_2}=\mathcal{S}(e_2\alpha)^{e_2}\subseteq\mathcal{S}(\alpha),$$

and similarly, $\lambda^{e_1}\in\mathcal{S}(\alpha)$. Since $\gcd(e_1,e_2)=1$, a standard argument yields $\lambda\in\mathcal{S}(\alpha)=\mathcal{S}(\alpha_1+\alpha_2)$. This proves the lemma. □

The result below is immediate from Lemma 3.7.

Proposition 3.8. *Let F be a field and N a positive integer. If F is Rost ℓ^n-divisible for every prime power ℓ^n that divides N, then for all $\alpha \in \mathrm{Br}(F)[N]$ we have $\mathcal{R}(\alpha) = \mathcal{S}(\alpha)$, i.e., the Rost kernel and the Suslin group of α coincide.*

Consequently, if F is Rost divisible (i.e., Rost ℓ^∞-divisible for every prime ℓ), then the Rost kernel and the Suslin group are the same for all $\alpha \in \mathrm{Br}(F)$.

This proposition shows that our reformulation of Suslin's conjecture (1.2) is equivalent to the original one given in [Suslin 1984, (24.6)].

Remark 3.9. The Suslin group $\mathcal{S}(\alpha)$ and the Rost kernel $\mathcal{R}(\alpha)$ may be interpreted from a K-cohomological point of view. See, for example, [Merkurjev 1995, (1.6) and (1.10)].

4. Proofs of main results

We prove our main theorems 1.5 and 1.7 in this section.

(4.1) Throughout this section we use the following notation:

(1) F: a henselian discrete valuation field.

(2) $v = v_F$: the normalized discrete valuation on F.

(3) k: the residue field of F.

(4) $\pi \in F$: a fixed uniformizer of F.

(5) ℓ: a prime number different from the characteristic of k.

(6) $\alpha \in \mathrm{Br}(F)[\ell^\infty]$: a Brauer class over F of ℓ-power period.

(7) $H^d(\,\cdot\,) := H^d(\,\cdot\,, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1))$ for all $d \in \mathbb{N}^*$, the cohomology being the Galois cohomology.

(8) For any finite extension L of F, let $\mathcal{O}_L$ be the valuation ring of L and U_L be the group of units in $\mathcal{O}_L$.

An element $\chi_0 \in H^1(k)$ can be determined by a pair $(E_0/k, \bar{\sigma})$, where E_0/k is the cyclic extension and $\bar{\sigma}$ is a generator of the cyclic Galois group $\mathrm{Gal}(E_0/k)$. The correspondence between χ_0 and $(E_0/k, \bar{\sigma})$ is established by requiring that the continuous homomorphism $\chi_0 : \mathrm{Gal}(k_s/k) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell$ has kernel $\mathrm{Gal}(k_s/E)$, k_s denoting a fixed separable closure of k, and that $\bar{\sigma} \in \mathrm{Gal}(E_0/k)$ is the generator which is mapped to the canonical generator of the cyclic group $\mathrm{Im}(\chi_0)$. When the role played by $\bar{\sigma}$ is not explicit in our arguments, we simply write $\chi_0 = (E_0/k)$.

By the *canonical lifting* $\chi \in H^1(F)$ of χ_0 we mean the image of χ_0 under the inflation map $H^1(k) \to H^1(F)$. Explicitly, χ is defined by the pair $(E/F, \sigma)$, where E/F is the unramified extension with residue field extension E_0/k and $\sigma \in \mathrm{Gal}(E/F)$ is the generator corresponding to $\bar{\sigma}$ via the natural isomorphism $\mathrm{Gal}(E/F) \xrightarrow{\sim} \mathrm{Gal}(E_0/k)$. Just as for χ_0, we write $\chi = (E/F)$ for short.

For any element $b \in F^*$, we write

$$(E/F, b) = \chi \cup (b)$$

for the Brauer class given by the cup of $\chi \in H^1(F)$ and $(b) \in H^1(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$.

For each $d \geq 1$, there is a well known homomorphism, called the *residue map*,

$$\partial : H^{d+1}(F) \to H^d(k),$$

whose definition may differ by a sign in different references (cf. [Kato 1980, §1.2; Serre 2003, §7]). Here we are mostly interested in the residue maps defined on H^2 and H^3. We make our choice of sign in such a way that the following formulas hold: for all $\lambda, \mu \in F^*$ and $\chi_0 \in H^1(k)$ with canonical lift $\chi \in H^1(F)$, we have

$$\partial(\chi \cup (\lambda)) = v(\lambda).\chi_0 \in H^1(k) \tag{4.2}$$

and

$$\partial(\chi \cup (\lambda) \cup (\mu)) = \chi_0 \cup (-1)^{v(\lambda)v(\mu)}\overline{\lambda^{v(\mu)}\mu^{-v(\lambda)}} \in H^2(k). \tag{4.3}$$

Moreover, we have an exact sequence

$$0 \to \mathrm{Br}(k)[\ell^\infty] = H^2(k) \xrightarrow{\iota} \mathrm{Br}(F)[\ell^\infty] = H^2(F) \xrightarrow{\partial} H^1(k) \to 0, \tag{4.4}$$

for which the choice of the uniformizer π determines a splitting

$$H^1(k) \to H^2(F), \quad \chi_0 \mapsto \chi \cup (\pi).$$

The map ι can be viewed as the inflation map $H^2(k) \to H^2(F)$, or the composition of the natural map $\mathrm{Br}(\mathcal{O}_F)[\ell^\infty] \to \mathrm{Br}(F)[\ell^\infty]$ with the inverse of the natural isomorphism $\mathrm{Br}(\mathcal{O}_F)[\ell^\infty] \cong \mathrm{Br}(k)[\ell^\infty]$, where $\mathcal{O}_F$ is the valuation ring of F. A Brauer class in the kernel of the residue map $\partial : H^2(F) \to H^1(k)$ is called *unramified.*

For $\alpha \in \mathrm{Br}(F)[\ell^\infty]$, we may write

$$\alpha = \alpha' + (E/F, \pi) \in \mathrm{Br}(F) \quad \text{with } \alpha' \in \mathrm{Br}(F) \text{ unramified,} \tag{4.5}$$

where $(E/F) = \chi$ is the canonical lifting of $\chi_0 := \partial(\alpha) \in H^1(k)$.

For any $\lambda \in F^*$, if we write

$$\lambda = \theta.(-\pi)^r \quad \text{with } r \in \mathbb{Z} \text{ and } \theta \text{ a unit for the valuation of } F, \tag{4.6}$$

then computation of residues shows that

$$\begin{aligned} \alpha \cup (\lambda) \in \ker(\partial : H^3(F) \to H^2(k)) &\iff r\alpha' = (E/F, \theta) \\ &\iff r\alpha = (E/F, (-1)^r\lambda). \end{aligned} \tag{4.7}$$

See [Parimala et al. 2018, Lemma 4.7].

Lemma 4.8. *With notation as above, if $r = v_F(\lambda)$ is coprime to ℓ, then*

$$\partial(\alpha \cup (\lambda)) = 0 \quad \Longrightarrow \quad \lambda \in \mathrm{Nrd}(\alpha).$$

Proof. Let $\ell^n = \mathrm{ind}(\alpha)$ be the index of α. There exist integers $s, c \in \mathbb{Z}$ such that $rs + \ell^n c = 1$. Replacing λ with $\lambda^s \pi^{\ell^n c}$, we may assume $r = v_F(\lambda) = 1$. Then by (4.7),

$$\partial(\alpha \cup (\lambda)) = 0 \quad \Longrightarrow \quad \alpha = (E/F, -\lambda) \quad \Longrightarrow \quad \alpha \text{ is split by } L := F\big(\sqrt[\ell^n]{-\lambda}\big).$$

Hence,

$$(-1)^{\ell^n}\lambda = N_{L/F}\big(\sqrt[\ell^n]{-\lambda}\big) \in N_{L/F}(L^*) \subseteq \mathrm{Nrd}(\alpha),$$

and the lemma is proved. □

Corollary 4.9. *With notation as in* (4.1), *if* $\mathrm{per}(\alpha) = \ell$, *then*

$$\mathcal{R}(\alpha) = \mathcal{S}(\alpha) \cdot (\mathcal{R}(\alpha) \cap U_F) \quad \textit{and} \quad \frac{\mathcal{R}(\alpha) \cap U_F}{\mathcal{S}(\alpha) \cap U_F} \cong \frac{\mathcal{R}(\alpha)}{\mathcal{S}(\alpha)}.$$

Proof. Clearly, it is sufficient to prove $\mathcal{R}(\alpha) \subseteq \mathcal{S}(\alpha) \cdot (\mathcal{R}(\alpha) \cap U_F)$.

Suppose $\lambda \in \mathcal{R}(\alpha)$ and write $r = v_F(\lambda)$. If r is coprime to ℓ, then by Lemma 4.8 we have $\lambda \in \mathrm{Nrd}(\alpha) \subseteq \mathcal{S}(\alpha)$. Otherwise r is a multiple of $\ell = \mathrm{per}(\alpha)$. In particular, this implies $\alpha \cup ((-\pi)^r) = 0$. Using the notation of (4.6), we get $\alpha \cup (\theta) = 0$, i.e., $\theta \in \mathcal{R}(\alpha) \cap U_F$. Hence

$$\lambda = (-\pi)^r.\theta \in (F^*)^\ell \cdot (\mathcal{R}(\alpha) \cap U_F) \subseteq \mathcal{S}(\alpha) \cdot (\mathcal{R}(\alpha) \cap U_F),$$

as desired. □

Lemma 4.10. *With notation as in* (4.5), *let $\overline{\alpha}' \in \mathrm{Br}(k)$ be the canonical image of the unramified class α'.*

(1) *For the unramified Brauer class α', we have*

$$\mathcal{R}(\alpha') = (\mathcal{R}(\alpha') \cap U_F) \cdot (F^*)^{\mathrm{per}(\alpha')}, \quad \mathcal{S}(\alpha') = (\mathcal{S}(\alpha') \cap U_F) \cdot (F^*)^{\mathrm{per}(\alpha')} \tag{4.11}$$

and

$$\mathcal{R}(\alpha') \cap U_F = \{a \in U_F \mid \bar{a} \in \mathcal{R}(\overline{\alpha}')\}, \quad \mathcal{S}(\alpha') \cap U_F = \{a \in U_F \mid \bar{a} \in \mathcal{S}(\overline{\alpha}')\}. \tag{4.12}$$

(2) *If $\ell^n = \mathrm{per}(\alpha)$ and k is Rost ℓ^n-divisible, then $\mathcal{R}(\alpha') = \mathcal{S}(\alpha')$.*

(3) *Letting E_0 denote the residue field of E, we have*

$$\mathcal{R}(\alpha) \cap U_F = \mathcal{R}(\alpha') \cap N_{E/F}(U_E) = \{a \in U_F \mid \bar{a} \in \mathcal{R}(\overline{\alpha}') \cap N_{E_0/k}(E_0^*)\}. \tag{4.13}$$

Proof. (1) Let $\ell^n = \mathrm{per}(\alpha')$ and $\lambda \in F^*$. If $\lambda \in \mathcal{R}(\alpha')$, by computing the residue of the cohomology class $\alpha' \cup (\lambda)$ we see that $r = v_F(\lambda)$ is a multiple of $\ell^n = \mathrm{per}(\alpha')$. Thus, using the notation of (4.6), we get

$$\alpha' \cup (\theta) = \alpha' \cup (\theta.(-\pi)^r) = \alpha' \cup (\lambda) = 0.$$

This means $\theta \in \mathcal{R}(\alpha') \cup U_F$. The first equality in (4.11) is thus proved. Since $(F^*)^{\ell^n}$ is contained in $\mathcal{S}(\alpha')$ by the definition of the Suslin group, this argument also proves the other equality in (4.11).

For any $a \in U_F$, the cohomology class $\alpha' \cup (a)$ is unramified. So its specialization $\bar{\alpha}' \cup (\bar{a})$ vanishes in $H^3(k)$ if and only if $\alpha' \cup (a)$ vanishes in $H^3(F)$. This gives the first equality in (4.12).

To show the second equality in (4.12), let us suppose $\theta \in U_F$. If $\bar{\theta} \in \mathcal{S}(\bar{\alpha}')$, we have

$$\bar{\theta} \in \prod_{i=0}^{n} \mathrm{Nrd}(\ell^{n-i}\bar{\alpha}')^{\ell^{n-i}}.$$

Thus, there exist finite (separable) splitting fields l_i/k of $\ell^{n-i}\bar{\alpha}'$ for $i = 0, 1, \dots, n$ and elements $\bar{u}_i \in l_i^*$ such that

$$\bar{\theta} = \prod_{i=0}^{n} N_{l_i/k}(\bar{u}_i)^{\ell^{n-i}}.$$

Let L_i/F be the unramified extension with residue field l_i/k. Then $\ell^{n-i}\alpha'_{L_i} = 0 \in \mathrm{Br}(L_i)$ since $\ell^{n-i}\bar{\alpha}'_{l_i} = 0$. This implies that $N_{L_i/F}(L_i^*) \subseteq \mathrm{Nrd}(\ell^{n-i}\alpha')$. Lift $\bar{u}_i \in l_i^*$ to a unit $u \in L_i^*$. Then

$$\theta^{-1}.\prod_{i=0}^{n} N_{L_i/F}(u_i)^{\ell^{n-i}} \subseteq U_F^1 := \{a \in F^* \mid v_F(1-a) \geq 1\}.$$

Since $U_F^1 \subseteq (F^*)^{\ell^n} = \mathrm{Nrd}(\ell^n\alpha')^{\ell^n}$ by Hensel's lemma, it follows that

$$\theta \in \prod_{i=0}^{n} \mathrm{Nrd}(\ell^{n-i}\alpha')^{\ell^{n-i}} = \mathcal{S}(\alpha').$$

Now suppose $\theta \in \mathcal{S}(\alpha') \cap U_F$. To finish the proof, we must show $\bar{\theta} \in \mathcal{S}(\bar{\alpha}')$.

Notice that for any unramified Brauer class $\beta \in \mathrm{Br}(F)[\ell^\infty]$, with canonical image $\bar{\beta} \in \mathrm{Br}(k)$, if M/F is a splitting field of β of ℓ-power degree, then the maximal unramified subextension M_0 of M also splits β [Serre 2003, (8.4)]. From this remark and the assumption $\theta \in \mathcal{S}(\alpha')$ we can deduce that

$$\theta = \prod_{i=0}^{n} N_{L_i/F}(u_i)\pi^{t_i},$$

where L_i/F is an unramified extension of ℓ-power degree splitting $\ell^{n-i}\alpha'$, u_i is a unit in L_i and $t_i \in \mathbb{Z}$. Since $\theta \in U_F$, we have

$$\sum_{i=0}^{n} t_i = 0, \quad \text{and hence } \theta = \prod_{i=0}^{n} N_{L_i/F}(u_i).$$

For each i, the residue field l_i of L_i splits $\ell^{n-i}\bar{\alpha}'$. So it follows that

$$\bar{\theta} \in \prod_{i=0}^{n} N_{l_i/k}(\bar{u}_i) = \mathcal{S}(\bar{\alpha}').$$

(2) The Rost ℓ^n-divisibility of k implies that $\mathcal{R}(\bar{\alpha}') = \mathcal{S}(\bar{\alpha}')$. So the result follows immediately from (1).

(3) Note that $[E : F]$ is a power of ℓ and hence invertible in the residue field k. Thus, by Hensel's lemma $U_F^1 = \{a \in U_F \mid \bar{a} = \bar{1}\}$ is contained in $N_{E/F}(U_E)$. This implies

$$U_F \cap N_{E/F}(E^*) = N_{E/F}(U_E) = \{a \in U_F \mid \bar{a} \in N_{E_0/k}(E_0^*)\}.$$

We have seen $\mathcal{R}(\alpha') \cap U_F = \{a \in U_F \mid \bar{a} \in \mathcal{R}(\bar{\alpha}')\}$ in (1). So the second equality in (3) follows.

If $\theta \in N_{E/F}(E^*)$, we have $(E/F, \pi) \cup (\theta) = 0$ since $(E/F) \cup (\theta) = 0$ in $H^2(F)$. Hence, when $\theta \in \mathcal{R}(\alpha') \cap N_{E/F}(U_E)$, we have

$$\alpha \cup (\theta) = \alpha' \cup (\theta) + (E/F, \pi) \cup (\theta) = 0,$$

whence $\theta \in \mathcal{R}(\alpha)$.

Conversely, if $\theta \in \mathcal{R}(\alpha) \cap U_F$, then we have $(E/F, \theta) = 0 \in H^2(F)$ by (4.7). On the one hand, this implies $(E/F, \pi) \cup (\theta) = 0$, so that from $\theta \in \mathcal{R}(\alpha)$ we obtain $\theta \in \mathcal{R}(\alpha')$. On the other hand, it follows that $\theta \in U_F \cap N_{E/F}(E^*) = N_{E/F}(U_E)$. This completes the proof. □

Lemma 4.14. *With notation as in* (4.1), *suppose that $r = v_F(\lambda)$ is a multiple of $\ell^n = \operatorname{per}(\alpha)$. Write $\ell^m = \operatorname{per}(\alpha_E)$. Assume the following properties hold for the residue field E_0 of E*:

(1) *E_0 is Rost ℓ^m-divisible.*

(2) *The corestriction map $H^3(E_0) \to H^3(k)$ is injective.*

Then $\alpha \cup (\lambda) = 0$ implies $\lambda \in \prod_{i=0}^m \operatorname{Nrd}(\ell^{m-i}\alpha)^{\ell^{m-i}} \subseteq \mathcal{S}(\alpha) = \prod_{i=0}^n \operatorname{Nrd}(\ell^{n-i}\alpha)^{\ell^{n-i}}$.

Proof. Replacing λ by $\lambda.\pi^{-r}$, we may assume $r = v_F(\lambda)$ is zero. Then the hypothesis $\alpha \cup (\lambda) = 0$ means $\lambda \in \mathcal{R}(\alpha) \cap U_F$. By (4.13), $\lambda = N_{E/F}(\mu)$ for some unit μ in E. Using the projection formula for corestrictions, we obtain

$$\operatorname{Cor}_{E/F}(\alpha_E \cup \mu) = \alpha \cup N_{E/F}(\mu) = \alpha \cup (\lambda) = 0 \in H^3(F).$$

Note $\alpha_E = \alpha'_E \in \operatorname{Br}(E)$ is unramified. So the cohomology class $\alpha_E \cup (\mu) \in H^3(E)$ is unramified and we have $\operatorname{Cor}_{E_0/k}(\bar{\alpha}_E \cup \bar{\mu}) = 0 \in H^3(k)$. By the corestriction injectivity assumption, $\bar{\alpha}_E \cup (\bar{\mu}) = 0$ and hence $\alpha_E \cup \mu = 0 \in H^3(E)$. Since the residue field E_0 of E is Rost ℓ^m-divisible by assumption, we deduce from Lemma 4.10(2) that $\mu \in \mathcal{S}(\alpha_E) = \prod_{i=0}^m \operatorname{Nrd}(\ell^{m-i}\alpha_E)^{\ell^{m-i}}$. It then follows immediately that $\lambda = N_{E/F}(\mu) \in \prod_{i=0}^m \operatorname{Nrd}(\ell^{m-i}\alpha)^{\ell^{m-i}}$. □

The proof of Theorem 1.5 is now immediate.

Proof of Theorem 1.5. Let $\alpha \in \mathrm{Br}(F)$ be a Brauer class of period ℓ. Assume the two conditions (Rost ℓ-divisibility and H^3-corestriction injectivity) in the theorem hold. In view of Corollary 4.9, we need only to prove the following statement: if $\lambda \in U_F$ and $\alpha \cup (\lambda) = 0 \in H^3(F)$, then $\lambda \in \mathcal{S}(\alpha) = (F^*)^\ell \cdot \mathrm{Nrd}(\alpha)$.

By (4.13), we have $\lambda \in N_{E/F}(E^*)$. If $\alpha_E = 0$, then $N_{E/F}(E^*) \subseteq \mathrm{Nrd}(\alpha)$ and we are done. Otherwise $\mathrm{per}(\alpha_E) = \ell$ and it suffices to apply Lemma 4.14. □

When $\mathrm{per}(\alpha) = \ell$, we have in fact a description of the quotient group $\mathcal{R}(\alpha)/\mathcal{S}(\alpha)$ which holds without assuming the two conditions of Theorem 1.5. To show this, we need the following variant of [Merkurjev 1995, Proposition 2.3]:

Proposition 4.15. *With notation as in* (4.5), *suppose that* $\mathrm{per}(\alpha) = \ell$. *Then*

$$\mathcal{S}(\alpha) \cap U_F = (N_{E/F}\mathrm{Nrd}(\alpha'_E) \cdot (U_F)^\ell) \cap U_F = \{a \in U_F \mid \bar{a} \in N_{E_0/k}(\mathcal{S}(\bar{\alpha}'_{E_0})) \cdot k^{*\ell}\}.$$

Following the same ideas as in [Merkurjev 1995, §2], we will base our proof of Proposition 4.15 (see page 693) on Lemma 4.16 and Proposition 4.17 below.

Lemma 4.16. *Let F be a henselian discrete valuation field, $\pi \in F$ a uniformizer and L/F a cyclic unramified extension of degree m. Let σ be a generator of the Galois group* $\mathrm{Gal}(L/F)$ *and let $\Delta(L/F, \sigma, \pi)$ be the F-algebra generated by L and an element u subject to the following relations*:

$$u^m = \pi \quad \textit{and} \quad au = u\sigma(a) \quad \textit{for all } a \in L.$$

Let $\mathcal{A}$ be an Azumaya algebra over the valuation ring $\mathcal{O}_F$. Write $\mathcal{B} = \mathcal{A} \otimes_{\mathcal{O}_F} \mathcal{O}_L$ and $D = \mathcal{A} \otimes_{\mathcal{O}_F} \Delta(L/F, \sigma, \pi)$.

(1) *Let $\mathcal{B}[u]$ be the $\mathcal{O}_F$-subalgebra of D generated by $\mathcal{B}$ and u. Then*

$$D = \mathcal{B}[u] \otimes_{\mathcal{O}_F} F.$$

(2) *Let $d = \sum_{i=0}^{m-1} d_i u^i \in \mathcal{B}[u]$ with each $d_i \in \mathcal{B}$. Then*

$$\mathrm{Nrd}_D(d) \in \mathcal{O}_F \quad \textit{and} \quad \mathrm{Nrd}_D(d) \equiv N_{L/F}(\mathrm{Nrd}_{\mathcal{B}}(d_0)) \pmod{\pi\mathcal{O}_F}.$$

Proof. Part (1) is clear from the constructions of $\mathcal{B}$ and D. The proof of (2) is an easy adaption of the proof of [Merkurjev 1995, Lemma 2.1]. □

Proposition 4.17. *With notation and hypotheses as in Lemma 4.16, let L_0/k be the residue field extension of L/F and suppose that $m = [L : F]$ is not divisible by* $\mathrm{char}(k)$. *Then*

$$\begin{aligned}(\mathrm{Nrd}(D) \cdot F^{*m}) \cap U_F &= (N_{L/F}\mathrm{Nrd}(\mathcal{A} \otimes L) \cdot (U_F)^m) \cap U_F \\ &= \{a \in U_F \mid \bar{a} \in N_{L_0/k}\mathrm{Nrd}(\mathcal{A} \otimes L_0) \cdot k^{*m}\}.\end{aligned}$$

Proof. Let G_1, G_2 and G_3 be the first, second and third groups in the statement of the proposition.

Note that $D \otimes L$ is Brauer equivalent to $\mathcal{A} \otimes L$ by construction. Hence,

$$N_{L/F}\mathrm{Nrd}(\mathcal{A} \otimes L) = N_{L/F}\mathrm{Nrd}(D \otimes L) \subseteq \mathrm{Nrd}(D).$$

This gives the inclusion $G_2 \subseteq G_1$.

To see $G_3 \subseteq G_2$, suppose $a \in G_3$ and write $\bar{a} = N_{L_0/k}(\beta)\gamma^m$ with $\gamma \in k^*$ and $\beta \in \mathrm{Nrd}(\mathcal{A} \otimes L_0)$. There exists a splitting field M_0/L_0 of $\mathcal{A} \otimes L_0$ such that $\beta = N_{M_0/L_0}(\beta')$ for some $\beta' \in M_0^*$. Let M be the unramified extension of L with residue field M_0. Then M splits $\mathcal{A} \otimes L$, so that $N_{M/L}(M^*) \subseteq \mathrm{Nrd}(\mathcal{A} \otimes L)$. If $b' \in U_M$ is a lifting of β' and $c \in U_F$ is a lifting of γ, then

$$a^{-1} N_{L/F} N_{M/L}(b') c^m \subseteq U_F^1 = \{x \in F^* \mid v_F(x-1) \geq 1\}.$$

By Hensel's lemma we have $U_F^1 \subseteq (F^*)^m$. So we get

$$a \in N_{L/F} N_{M/L}(b') c^m \cdot (F^{*m}) \subseteq N_{L/F}\mathrm{Nrd}(\mathcal{A} \otimes L) \cdot (F^*)^m,$$

proving $a \in G_3$ as desired.

Now suppose $b \in G_1 = (\mathrm{Nrd}(D) \cdot F^{*m}) \cap U_F$. As in the proof of [Merkurjev 1995, Proposition 2.2], we may assume $b = \mathrm{Nrd}(d) g^{-m}$ for some $d = \sum_{i \geq 0} d_i u^i \in \mathcal{B}[u]$ (with $\mathcal{B} = \mathcal{A} \otimes \mathcal{O}_L$ as in Lemma 4.16) and $g \in U_F$. Using Lemma 4.16 (as a substitute for [Merkurjev 1995, Lemma 2.1]), we find that

$$\bar{b} = N_{L_0/k}(\mathrm{Nrd}(\bar{d}_0)) \bar{g}^{-m} \in N_{L_0/k}\mathrm{Nrd}(\mathcal{A} \otimes L_0) \cdot k^{*m},$$

where $\bar{d}_0$ is the canonical image in $\mathcal{A} \otimes L_0$ of $d_0 \in \mathcal{B} = \mathcal{A} \otimes \mathcal{O}_L$. We have thus obtained the inclusion $G_1 \subseteq G_3$. □

Proof of Proposition 4.15. Since we have assumed $\mathrm{per}(\alpha) = \ell$, in the notation of (4.5) we have either $E = F$ or $[E : F] = \ell$. Let H_1, H_2 and H_3 be the first, second and third groups in the statement of the proposition. Note that

$$N_{E_0/k}(\mathcal{S}(\bar{\alpha}'_{E_0})) \cdot k^{*\ell} = N_{E_0/k}\mathrm{Nrd}(\bar{\alpha}'_{E_0}) \cdot k^{*\ell}$$

since both $[E_0 : k]$ and $\mathrm{per}(\bar{\alpha}'_{E_0})$ divide ℓ. So we have

$$H_3 = \{a \in U_F \mid \bar{a} \in N_{E_0/k}\mathrm{Nrd}(\bar{\alpha}'_{E_0}) \cdot k^{*\ell}\}.$$

As in the proof of Proposition 4.17, we have $H_1 \supseteq H_2 \supseteq H_3$. So it suffices to prove $H_1 \subseteq H_3$.

If $[E : F] = \ell$, we take $\mathcal{A}$ to be an Azumaya algebra over $\mathcal{O}_F$ that represents α' and let $L = E$. Then in Lemma 4.16 and Proposition 4.17 the algebra D represents the Brauer class α. Since the group $\mathcal{S}(\alpha)$ is equal to $\mathrm{Nrd}(\alpha)F^{*\ell}$ in the current situation, the result follows from Proposition 4.17.

If $E = F$, then $\alpha = \alpha' = \alpha'_E$ and

$$H_3 = \{a \in U_F \mid \bar{a} \in \mathrm{Nrd}(\overline{\alpha}') \cdot k^{*\ell}\} = \{a \in U_F \mid \bar{a} \in \mathcal{S}(\overline{\alpha}')\}.$$

Now the desired equality $H_1 = H_3$ follows from the second equality in (4.12). □

Theorem 4.18. *With notation as in* (4.5), *suppose that* $\mathrm{per}(\alpha) = \ell$. *Then we have an isomorphism of groups*

$$\frac{\mathcal{R}(\alpha)}{\mathcal{S}(\alpha)} \cong \frac{\mathcal{R}(\overline{\alpha}') \cap N_{E_0/k}(E_0^*)}{N_{E_0/k}(\mathcal{S}(\overline{\alpha}'_{E_0})) \cdot k^{*\ell}},$$

where E_0 denotes the residue field of E and $\overline{\alpha}' \in \mathrm{Br}(k)$ is the canonical image of the unramified Brauer class α'.

Proof. It is easily seen that the natural map

$$\frac{\{a \in U_F \mid \bar{a} \in \mathcal{R}(\overline{\alpha}') \cap N_{E_0/k}(E_0^*)\}}{\{a \in U_F \mid \bar{a} \in N_{E_0/k}(\mathcal{S}(\overline{\alpha}'_{E_0})) \cdot k^{*\ell}\}} \to \frac{\mathcal{R}(\overline{\alpha}') \cap N_{E_0/k}(E_0^*)}{N_{E_0/k}(\mathcal{S}(\overline{\alpha}'_{E_0})) \cdot k^{*\ell}}, \quad a \mapsto \bar{a}$$

is an isomorphism. Thus, combining Corollary 4.9, (4.13) and Proposition 4.15 finishes the proof. □

If the corestriction map $\mathrm{Cor}_{E_0/k} : H^3(E_0) \to H^3(k)$ is injective, then the group $\mathcal{R}(\overline{\alpha}') \cap N_{E_0/k}(E_0^*)$ is equal to $N_{E_0/k}(\mathcal{R}(\overline{\alpha}'_{E_0}))$. If moreover $\mathcal{R}(\overline{\alpha}_{E_0}) = \mathcal{S}(\overline{\alpha}'_{E_0})$, we see immediately from Theorem 4.18 that $\mathcal{R}(\alpha) = \mathcal{S}(\alpha)$. Therefore, Theorem 4.18 is a generalization of Theorem 1.5.

The rest of this section is devoted to the proof of Theorem 1.7.

(4.19) In what follows, we keep the notation and hypotheses of Theorem 1.7. Let $\alpha \in \mathrm{Br}(F)[\ell^n]$ and suppose $\lambda \in F^*$ lies in the Rost kernel of α, i.e.,

$$\alpha \cup (\lambda) = 0 \in H^3(F) = H^3(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)).$$

We fix a uniformizer π of F and decompose α into $\alpha = \alpha' + (E/F, \pi)$, where $\alpha' \in \mathrm{Br}(F)$ is unramified and E/F is a cyclic unramified extension whose degree divides ℓ^n. Note that the residue field extension E_0/k of E/F determines the residue $\partial(\alpha) \in H^1(k)$ of α. Also, we write

$$\lambda = \theta.(-\pi)^{s\ell^m}, \quad \text{with } s \in \mathbb{Z}, m \in \mathbb{N}, \theta \in U_F = \{a \in F^* \mid v_F(a) = 0\}.$$

Our goal is to prove $\lambda \in \mathcal{S}(\alpha)$.

We first explain a few reductions for the proof.

(1) Without loss of generality, we may (and do) assume $\mathrm{per}(\alpha) = \ell^n$.

(2) We may assume $v_F(\lambda) = \ell^m$ with $0 \le m < n$. Indeed, we have already seen that this is true if $\mathrm{per}(\alpha) = \ell^n \mid v_F(\lambda)$ (see Lemma 4.14). Thus, we may

assume $v_F(\lambda) = \ell^m s$ with $0 \le m < n$ and $s \notin \ell\mathbb{Z}$. Choose $a, b \in \mathbb{Z}$ such that $sa + b\ell^{n-m} = 1$. Let $\lambda_1 = \lambda^a.\pi^{b\ell^n}$. Then

$$v_F(\lambda_1) = \ell^m sa + b\ell^n = \ell^m \quad \text{and} \quad \alpha \cup (\lambda_1) = a.\alpha \cup \lambda = 0.$$

If we know $\lambda_1 \in \mathcal{S}(\alpha)$, then we can deduce that $\lambda^a \in \mathcal{S}(\alpha)$. Since the quotient group $F^*/\mathcal{S}(\alpha)$ is ℓ-primary torsion, this implies $\lambda \in \mathcal{S}(\alpha)$ as desired.

(3) In the remainder of this section, we prove $\lambda \in \mathcal{S}(\alpha)$ by induction on m. The case $m = 0$ has already been treated in Lemma 4.8, which is valid even without assuming $\mathrm{cd}_\ell(k) \le 2$ nor $\mu_{\ell^n} \subseteq k$. So we assume $1 \le m < n$ from now on.

(4) We may further assume $\ell^m \alpha' \neq 0$. Note that if $\mathrm{per}(\alpha') \le v_F(\lambda) = \ell^m$, we have

$$\begin{aligned} 0 = \alpha \cup (\lambda) &= \alpha \cup (\theta) + \ell^m \alpha \cup (-\pi) \\ &= \alpha \cup (\theta) + \ell^m \alpha' \cup (-\pi) && \text{since } (E/F, \pi) \cup (-\pi) = 0 \\ &= \alpha \cup (\theta), && \text{since } \ell^m \alpha' = 0 \text{ in this case.} \end{aligned}$$

Thus, by Lemma 4.14 we have $\theta \in \mathcal{S}(\alpha)$. On the other hand, since $\ell^m \alpha \cup (-\pi) = \ell^m \alpha' \cup (-\pi) = 0$, by induction on the period, we obtain $-\pi \in \mathcal{S}(\ell^m \alpha)$, whence $(-\pi)^{\ell^m} \in \mathcal{S}(\ell^m \alpha)^{\ell^m} \subseteq \mathcal{S}(\alpha)$. Hence $\lambda = (-\pi)^{\ell^m} \theta \in \mathcal{S}(\alpha)$, proving what we want.

(5) We may assume $\lambda \notin F^{*\ell}$. Indeed, if $\lambda = \lambda_1^\ell$ for some $\lambda_1 \in F^*$, then $\ell\alpha \cup \lambda_1 = 0$ and $v_F(\lambda_1) = \ell^{m-1} < \ell^m$. By induction on m we have $\lambda_1 \in \mathcal{S}(\ell\alpha)$ and hence $\lambda \in \mathcal{S}(\ell\alpha)^\ell \subseteq \mathcal{S}(\alpha)$.

(4.20) With notation and hypotheses as above, by an *inductive pair* of (α, λ) we mean a degree ℓ extension L/F together with an element $\mu \in L^*$ such that $\lambda = N_{L/F}(\mu)$ and $\mu \in \mathcal{S}(\alpha_L)$. Clearly, if such a pair exists, we have $\lambda \in \mathcal{S}(\alpha)$.

To obtain an inductive pair, our basic strategy is the following: if L/F is an unramified extension of degree ℓ with residue field L_0/k and if $\xi \in U_L$ is an element such that

$$\theta = N_{L/F}(\xi) \quad \text{and} \quad \ell^{m-1}\bar{\alpha}'_{L_0} = (E_0 L_0/L_0, \bar{\xi}) \in \mathrm{Br}(L_0), \tag{4.21}$$

then the pair $(L/F, \mu := (-\pi)^{\ell^{m-1}}\xi)$ is an inductive pair of (α, λ).

To prove this, first observe that the element $\mu = (-\pi)^{\ell^{m-1}}\xi$ clearly satisfies $N_{L/F}(\mu) = \lambda$. Moreover, computing the residue of $\alpha_L \cup \mu$ and taking (4.21) into account we get

$$\begin{aligned} \partial_L(\alpha_L \cup \mu) &= \partial_L(\alpha'_L \cup (\mu)) + \partial_L((EL/L, \pi) \cup (\mu)) \\ &= \ell^{m-1}.\bar{\alpha}'_{L_0} + \partial_L((EL/L, \pi) \cup (\xi)) \\ &= \ell^{m-1}\bar{\alpha}'_{L_0} - (E_0 L_0/L_0, \bar{\xi}) = 0. \end{aligned}$$

Since the residue field L_0 has cohomological ℓ-dimension ≤ 2, it follows that $\alpha_L \cup (\mu) = 0$. Noticing that $v_L(\mu) = \ell^{m-1} < \ell^m$, the induction hypothesis implies $\mu \in \mathcal{S}(\alpha_L)$. This proves our claim.

(4.22) Consider the canonical image $\bar{\theta} \in k$ of $\theta \in U_F$. Suppose that there exists a (separable) degree ℓ extension L_0/k and an element $\xi_0 \in L_0^*$ such that

$$\bar{\theta} = N_{L_0/k}(\xi_0) \quad \text{and} \quad \ell^{m-1}\bar{\alpha}'_{L_0} = (E_0L_0/L_0, \xi_0) \in \mathrm{Br}(L_0).$$

Then we may take L/F to be the unramified extension with residue field L_0/k. Let $\xi_1 \in U_L$ be a lifting of $\xi_0 \in L_0^*$. We have

$$\overline{N_{L/F}(\xi_1)} = N_{L_0/k}(\bar{\xi}_1) = N_{L_0/k}(\xi_0) = \bar{\theta}.$$

Hence, $\theta = N_{L/F}(\xi_1)\rho$ for some $\rho \in U_F^1$. By Hensel's lemma, $\rho = \rho_1^\ell$ for some $\rho_1 \in U_F^1$. Putting $\xi = \xi_1\rho_1$ we obtain

$$\bar{\xi} = \bar{\xi}_1 \cdot \bar{\rho}_1 = \xi_0 \cdot 1 = \xi_0 \quad \text{and} \quad \theta = N_{L/F}(\xi_1)\rho_1^\ell = N_{L/F}(\xi_1\rho_1) = N_{L/F}(\xi).$$

This means that $(L/F, \xi)$ is a pair satisfying (4.21).

Thus, the proof of our main result is reduced to a problem that only involves data over the residue field k: the image $\bar{\theta} \in k$ of $\theta \in U_F$, the residue Brauer class $\bar{\alpha}' \in \mathrm{Br}(k)$ and the cyclic extension E_0/k determined by the residue $\partial(\alpha) \in H^1(k)$ of α.

Notice that the assumption $\alpha \cup \lambda = 0$ at the beginning implies $\ell^m.\bar{\alpha}' = (E_0/k, \bar{\theta})$. Also, we have assumed $\lambda \notin F^{*\ell}$, so that $\bar{\theta} \notin k^{*\ell}$.

Changing notation, we are left to prove the following key lemma:

Lemma 4.23. *Let k be a henselian discrete valuation field with residue field k_0. Assume that* $\mathrm{char}(k_0) \neq \ell$ *and* $\mathrm{cd}_\ell(k) = 2$. *Let $\beta \in \mathrm{Br}(k)[\ell^n]$ be such that $0 \neq \ell^m\beta = (K/k, \theta)$, where $1 \leq m < n$, $\theta \in k^* \setminus k^{*\ell}$ and K/k is a cyclic extension with $[K : k] \mid \ell^n$. Suppose that $\mu_{\ell^n} \subseteq k$. Then there exists a degree ℓ extension L/k and an element $\xi \in L^*$ such that*

$$\theta = N_{L/k}(\xi) \quad \textit{and} \quad \ell^{m-1}\beta_L = (KL/L, \xi) \in \mathrm{Br}(L). \tag{4.24}$$

Proof. If $v_k(\theta) \notin \ell\mathbb{Z}$, then $L := k\big(\sqrt[\ell]{-\theta}\,\big)$ is a degree ℓ extension of k and the element $\xi := -\sqrt[\ell]{-\theta}$ satisfies $N_{L/k}(\xi) = \theta$. The extension L/k is totally ramified, so the residue field L_0 of L is the same as the residue field k_0 of k. In the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H^2(L_0) \arrow[r] \arrow[d, "e(L/k)\mathrm{Cor}"'] & H^2(L) \arrow[r, "\partial"] \arrow[d, "\mathrm{Cor}"] & H^1(L_0) \arrow[r] \arrow[d, "\mathrm{Cor}"] & 0 \\
0 \arrow[r] & H^2(k_0) \arrow[r] & H^2(k) \arrow[r, "\partial"] & H^1(k_0) \arrow[r] & 0
\end{tikzcd}$$

(see [Serre 2003, p. 21, Proposition 8.6]), we have $H^2(L_0) = H^2(k_0) = 0$ since $\mathrm{cd}_\ell(k_0) \le 1$. So, the corestriction map $\mathrm{Cor}_{L/k} : H^2(L) \to H^2(k)$ is an isomorphism. From this we obtain $\ell^{m-1}\beta_L = (KL/L, \xi)$, since

$$\mathrm{Cor}_{L/k}(\ell^{m-1}\beta_L) = \ell^m \beta = (K/k, \theta) = \mathrm{Cor}_{L/k}((KL/L, \xi)).$$

Next let us assume $\ell \mid v_k(\theta)$. We construct an unramified extension L/k of degree ℓ together with an element $\xi \in L^*$ such that

$$\theta = N_{L/k}(\xi) \quad \text{and} \quad \partial_L((KL/L, \xi)) \text{ equals } \partial(\ell^{m-1}\beta)_{L_0} (= \partial_L(\ell^{m-1}\beta_L)). \tag{4.25}$$

This condition is equivalent to (4.24) by the assumption on the cohomological dimension.

We choose a uniformizer x of k and write

$$\theta = \theta_0 x^{\ell s}, \quad \text{where } s \in \mathbb{Z} \text{ and } v_k(\theta_0) = 0.$$

By associating to each cyclic extension M_0/k_0 of ℓ-power degree the Brauer class of the cyclic algebra $(M/k, x)$, M/k denoting the unramified extension with residue field M_0/k_0, we get an inverse of the isomorphism $\partial : H^2(k) \to H^1(k_0)$. In particular, we may write $\beta = (M/k, x)$ for some unramified cyclic extension M/k of degree dividing ℓ^n. By Kummer theory (and using the assumption $\mu_{\ell^n} \subseteq k$), we have

$$M = k(\sqrt[\ell^n]{b}) \quad \text{for some } b \in k^* \text{ with } v_k(b) = 0.$$

(Note that we have $v_k(b) = 0$ since M/k is unramified.) Using a fixed primitive ℓ^n-th root of unity, we may write β in the form of a symbol algebra: $\beta = (b, x)_{\ell^n}$. Similarly, we may write

$$K = k\big(\sqrt[\ell^n]{a}\big) \quad \text{for some } a \in k^* \text{ with } 1 \le v_k(a) = \ell^t \le \ell^n.$$

Suppose L/k is an unramified extension of degree ℓ for which we can find an element $\bar{\xi}_0$ in the residue field L_0 such that $\bar{\theta}_0 = N_{L_0/k_0}(\bar{\xi}_0)$. Then there exists a lifting $\xi_0 \in U_L$ of $\bar{\xi}_0$ with $N_{L/k}(\xi_0) = \theta_0$. (The basic idea for the proof of this statement has been discussed in (4.22).) Thus, $\theta = \theta_0 x^{\ell s}$ is the norm of $\xi := \xi_0 x^s \in L$. Computation now shows

$$\partial_L((KL/L, \xi)) = (-1)^{v(a)s} \bar{\xi}_0^{-v(a)} \bar{a}_0^{-s} \in L_0^*/L_0^{*\ell^n},$$
$$\partial(\ell^{m-1}\beta)_{L_0} = \bar{b}^{\ell^{m-1}} \in L_0^*/L_0^{*\ell^n},$$

where $a_0 := x^{v(a)}/a$. (Here we use a primitive root of unity to identity $H^1(L_0)$ with L^*/ℓ^n.) Therefore, to have condition (4.25) satisfied, it suffices to find a degree ℓ extension L_0/k_0 and $\bar{\xi}_0 \in L_0^*$ with $N_{L_0/k_0}(\bar{\xi}_0) = \bar{\theta}_0$ such that

$$(-1)^{v(a)s} \bar{b}^{\ell^{m-1}} \cdot \bar{\xi}_0^{v(a)} \bar{a}_0^s \in L_0^{*\ell^n}. \tag{4.26}$$

Notice that the assumption $\ell^m\beta=(K/k,\theta)=(a,\theta)_{\ell^n}$ yields

$$\bar{b}^{\ell^m}=\partial(\ell^m\beta)=(-1)^{v(a)s\ell}\bar{\theta}_0^{-v(a)}\bar{a}_0^{-s\ell}\in k_0^*/k_0^{*\ell^n},$$

i.e.,

$$(-1)^{v(a)s\ell}\bar{b}^{\ell^m}\cdot\bar{\theta}_0^{v(a)}\bar{a}_0^{s\ell}\in k_0^{*\ell^n}. \tag{4.27}$$

If $\ell^t=v(a)=1$ (i.e., $t=0$), this contradicts the assumption $\theta_0x^{\ell s}=\theta\notin k^{*\ell}$. So we have $t\geq 1$ and $\ell\mid v(a)=\ell^t$. In particular, $(-1)^{v(a)s}\in k_0^{*\ell^n}$. Hence, we may ignore the (-1)-powers in (4.26) and (4.27).

From (4.27) we get

$$\text{the order of } \bar{b}^{\ell^m}\bar{a}_0^{\ell s} \text{ in } k_0^*/\ell^n=\text{the order of } \bar{\theta}_0^{\ell^t} \text{ in } k_0^*/k_0^{*\ell^n}=\ell^{n-t}$$

since θ_0 is not in $k_0^{*\ell}$ (and k_0 contains enough roots of unity). It follows that $\bar{b}^{\ell^m}\cdot\bar{a}_0^{\ell s}$ is in $k_0^{*\ell^t}\setminus k_0^{*\ell^{t+1}}$, and hence

$$\bar{b}^{\ell^{m-1}}\bar{a}_0^s=\bar{b}_0^{\ell^{t-1}}\quad\text{for some } \bar{b}_0\in k_0^*\setminus k_0^{*\ell}.$$

Now set

$$L_0:=k_0\big(\sqrt[\ell]{\bar{b}_0}\big),\quad \bar{c}_0=\sqrt[\ell]{\bar{b}_0}\in L_0^*\quad\text{and}\quad \bar{c}=\bar{c}_0^{\ell^{t-1}}.$$

Then $\bar{c}^\ell=\bar{b}_0^{\ell^{t-1}}=\bar{b}^{\ell^{m-1}}\bar{a}_0^s$. Since $\mathrm{cd}_\ell(k_0)\leq 1$, the norm map $N:L_0^*\to k_0^*$ is surjective. So we can find $\bar{\xi}_1\in L_0^*$ such that $N(\bar{\xi}_1)=\bar{\theta}_0$. Then

$$N(\bar{c}\bar{\xi}_1^{\ell^{t-1}})=N(\bar{c}_0)^{\ell^{t-1}}\bar{\theta}_0^{\ell^{t-1}}=\pm\bar{b}_0^{\ell^{t-1}}\bar{\theta}_0^{\ell^{t-1}}=\pm\bar{b}^{\ell^{m-1}}\bar{a}_0^s\cdot\bar{\theta}_0^{\ell^{t-1}}\in k_0^{*\ell^{n-1}}$$

by (4.27). (Here if $\ell=2$, then $-1\in k_0^{*\ell^{n-1}}$ since we assumed $\mu_{\ell^n}\subseteq k_0$.) Writing $T=\ker(N:L_0^*\to k_0^*)$, we can deduce from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "\ell^{n-1}"'] & L_0^* \arrow[r, "N"] \arrow[d, "\ell^{n-1}"] & k_0^* \arrow[r] \arrow[d, "\ell^{n-1}"] & 0 \\
0 \arrow[r] & T \arrow[r] \arrow[d] & L_0^* \arrow[r, "N"] \arrow[d] & k_0^* \arrow[r] \arrow[d] & 0 \\
 & T/\ell^{n-1} \arrow[r] & L_0^*/\ell^{n-1} \arrow[r, "N"] & k_0^*/\ell^{n-1} \arrow[r] & 0
\end{tikzcd}$$

that

$$\bar{c}.\bar{\xi}_1^{\ell^{t-1}}.\bar{\rho}\in L_0^{*\ell^{n-1}}\quad\text{for some } \bar{\rho}\in T. \tag{4.28}$$

Note that $\bar{c}=\bar{c}_0^{\ell^{t-1}}$. Thus, (4.28) implies $\bar{\rho}=\bar{\rho}_0^{\ell^{t-1}}$ for some $\bar{\rho}_0\in L_0^*$. Since $N(\bar{\rho}_0)^{\ell^{t-1}}=N(\rho)=1$, we have $N(\bar{\rho}_0)=\eta^{-\ell}=N(\eta^{-1})$ for some $\eta\in\mu_{\ell^t}\subseteq k_0^*$. Putting $\bar{\rho}_1=\bar{\rho}_0.\eta$ and $\bar{\xi}_0=\bar{\xi}_1.\bar{\rho}_1$, we get

$$N(\bar{\rho}_1)=1\quad\text{and}\quad N(\bar{\xi}_0)=N(\bar{\xi}_1)=\bar{\theta}_0.$$

Finally,

$$\begin{aligned}\bar{b}^{\ell^{m-1}}.\bar{a}_0^s.\bar{\xi}_0^{\ell^t} &= \bar{c}^{\ell}.\bar{\xi}_0^{\ell^t} = \big(\bar{c}.\bar{\xi}_0^{\ell^{t-1}}\big)^{\ell} = \big(\bar{c}.\bar{\xi}_1^{\ell^{t-1}}.\bar{\rho}_1^{\ell^{t-1}}\big)^{\ell} \\ &= \big(\bar{c}.\bar{\xi}_1^{\ell^{t-1}}.\bar{\rho}_0^{\ell^{t-1}}.\eta^{\ell^{t-1}}\big)^{\ell} = \big(\bar{c}.\bar{\xi}_1^{\ell^{t-1}}.\bar{\rho}\big)^{\ell}\cdot\eta^{\ell^t} \\ &= \big(\bar{c}.\bar{\xi}_1^{\ell^{t-1}}.\bar{\rho}\big)^{\ell}, \quad \text{since } \eta\in\mu_{\ell^t}.\end{aligned}$$

The last term belongs to $L_0^{*\ell^n}$, by (4.28). We have thus obtained a pair $(L_0/k_0,\bar{\xi}_0)$ satisfying (4.26). So the proof is finished. □

5. Tame classes of period equal to residue characteristic

Our aim in this section is to state and prove a variant of Theorem 1.5 for Brauer classes of prime period p, when p is the characteristic of the residue field k.

(5.1) Let us first recall a few facts about the Kato–Milne cohomology.

Let k be a field of characteristic $p>0$. Let $r\in\mathbb{N}$. The Kato–Milne cohomology group $H^{r+1}(k,\mathbb{Q}_p/\mathbb{Z}_p(r))$ was originally defined by using technical theories such as Bloch's groups [Bloch 1977] and de Rham–Witt theory [Illusie 1979; Kato 1980; Milne 1976]. A simpler definition via Galois cohomology of Milnor K-groups of the separable closure is explained in [Merkurjev 2003, Appendix A].

We note that $H^1(k,\mathbb{Q}_p/\mathbb{Z}_p)$ is nothing but the p-primary torsion part of the usual Galois cohomology group $H^1(k,\mathbb{Q}/\mathbb{Z})$ (with the Galois action on $\mathbb{Q}/\mathbb{Z}$ being trivial), and that $H^2(k,\mathbb{Q}_p/\mathbb{Z}_p(1))$ may be identified with the p-primary torsion part $\mathrm{Br}(k)[p^\infty]$ of the Brauer group $\mathrm{Br}(k)$.

Most useful in this paper is the p-torsion part of $H^{r+1}(k,\mathbb{Q}_p/\mathbb{Z}_p(r))$, which we denote by $H^{r+1}(k,\mathbb{Z}/p\mathbb{Z}(r))$ or $H_p^{r+1}(k)$ and which we can describe easily using differential forms. Let Ω_k^r denote the r-th exterior power (over k) of the space of absolute differential forms $\Omega_k:=\Omega_{k/\mathbb{Z}}$ (with the convention $\Omega_k^0=k$). Let B_k^r be the image of the exterior differential map $\mathrm{d}:\Omega_k^{r-1}\to\Omega_k^r$ if $r\ge 1$ and put $B_k^0=0$. We have the Artin–Schreier map

$$\wp:\Omega_k^r\to\Omega_k^r/B_k^r,\quad b\frac{\mathrm{d}a_1}{a_1}\wedge\cdots\wedge\frac{\mathrm{d}a_r}{a_r}\mapsto(b^p-b)\frac{\mathrm{d}a_1}{a_1}\wedge\cdots\wedge\frac{\mathrm{d}a_r}{a_r},$$

and we may define

$$H_p^{r+1}(k)=H^{r+1}(k,\mathbb{Z}/p\mathbb{Z}(r)):=\mathrm{coker}(\wp:\Omega_k^r\to\Omega_k^r/B_k^r).$$

Under the natural identification $H_p^2(k)=\mathrm{Br}(k)[p]$, the differential form $b\frac{\mathrm{d}a}{a}$ corresponds to the Brauer class of the *symbol p-algebra* $[b,a)_p$, the k-algebra generated by two elements x,y subject to the relations

$$x^p-x=b,\quad y^p=a\quad\text{and}\quad yx=(x+1)y.$$

The cup product

$$\cup : H_p^2(k) \times (k^*/k^{*p}) \to H_p^3(k) \subseteq H^3(k, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

satisfies

$$\left(b\frac{\mathrm{d}a}{a}\right) \cup (\lambda) = b\frac{\mathrm{d}a}{a} \wedge \frac{\mathrm{d}\lambda}{\lambda}$$

for all $b\frac{\mathrm{d}a}{a} \in H_p^2(k)$ and $\lambda \in k^*$.

(5.2) Now let F be a henselian excellent discrete valuation field with residue field k of characteristic $\mathrm{char}(k) = p > 0$. (Here F may have characteristic 0 or p, and by saying F is excellent we mean that its valuation ring is excellent.) We define

$$H_p^{r+1}(F)_{\mathrm{tr}} := \ker\big(H^{r+1}(F, \mathbb{Z}/p\mathbb{Z}(r)) \to H^{r+1}(F_{\mathrm{nr}}, \mathbb{Z}/p\mathbb{Z}(r))\big),$$

where F_{nr} denotes the maximal unramified extension of F. There is a natural inflation map

$$\mathrm{Inf} : H_p^{r+1}(k) \to H_p^{r+1}(F)$$

(see [Kato 1980, p. 659, §3.2, Definition 2]) such that

$$\mathrm{Inf}\Big(b\frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_r}{a_r}\Big) = \tilde{b} \cup (\tilde{a}_1) \cup \cdots \cup (\tilde{a}_r),$$

where $\tilde{a}_i$ is any lifting of a_i, and $\tilde{b} \in H^1(F, \mathbb{Z}/p\mathbb{Z})$ denotes the canonical lifting (i.e., inflation) of the character $\chi_b \in H^1(k, \mathbb{Z}/p)$ corresponding to the Artin–Schreier extension $k[T]/(T^p - T - b)$ of k. The choice of a uniformizer $\pi \in F$ defines a homomorphism

$$h_\pi : H_p^r(k) \to H_p^{r+1}(F), \quad w \mapsto \mathrm{Inf}(w) \cup (\pi)$$

(see [Kato 1980, p. 659, §3.2, Lemma 3]). The images of Inf and h_π are both contained in $H_p^{r+1}(F)_{\mathrm{tr}}$. It is proved in [Kato 1982, p. 219, Theorem 3] that the above two maps induce an isomorphism

$$\mathrm{Inf} \oplus h_\pi : H_p^{r+1}(k) \oplus H_p^r(k) \xrightarrow{\sim} H_p^{r+1}(F)_{\mathrm{tr}}.$$

We can thus define a *residue map* $\partial : H_p^{r+1}(F)_{\mathrm{tr}} \to H_p^r(k)$, which fits into the split exact sequence

$$0 \to H_p^{r+1}(k) \xrightarrow{\mathrm{Inf}} H_p^{r+1}(F)_{\mathrm{tr}} \xrightarrow{\partial} H_p^r(k) \to 0. \tag{5.3}$$

Moreover, we have analogs of the formulas (4.2) and (4.3) for the residue maps.

Let us specialize to the case $r = 1$. Then $H_p^{r+1}(k) = H_p^2(k)$ is the p-torsion subgroup $\mathrm{Br}(k)[p]$ of $\mathrm{Br}(k)$, and similarly for $H_p^2(F)$. So in this case (5.3) becomes

$$0 \to \mathrm{Br}(k)[p] \xrightarrow{\mathrm{Inf}} \mathrm{Br}_{\mathrm{tr}}(F)[p] \xrightarrow{\partial} H^1(k, \mathbb{Z}/p) \to 0, \tag{5.4}$$

where $\mathrm{Br}_{\mathrm{tr}}(F)[p] := \ker(\mathrm{Br}(F)[p] \to \mathrm{Br}(F_{\mathrm{nr}})[p])$ is identified with

$$H^2_p(F)_{\mathrm{tr}} = \ker(H^2_p(F) \to H^2_p(F_{\mathrm{nr}})).$$

We remark that (5.4) can be extended to p-primary torsion subgroups and pieced together with (4.4) to give the exact sequence

$$0 \to \mathrm{Br}(k) \to \mathrm{Br}_{\mathrm{tr}}(F) \to H^1(k, \mathbb{Q}/\mathbb{Z}) \to 0,$$

where

$$\mathrm{Br}_{\mathrm{tr}}(F) := \ker(\mathrm{Br}(F) \to \mathrm{Br}(F_{\mathrm{nr}}))$$

is referred to as the *tame* or *tamely ramified* part of $\mathrm{Br}(F)$. (Perhaps a more appropriate name for this subgroup is the *inertially split* part of $\mathrm{Br}(F)$, according to the terminology of [Jacob and Wadsworth 1990, §5] and [Wadsworth 2002, §3]. In fact, the tame part and the inertially split part can be defined for any henselian valued field. Fortunately, these two notions make no difference in our situation, i.e., for discrete valuation fields.)

Now we state our main result in this section. For the construction and main properties of the corestriction maps needed below, we refer the reader to [Kato 1980, p. 658].

Theorem 5.5. *Let F be a henselian excellent discrete valuation field with residue field k of characteristic* $\mathrm{char}(k) = p > 0$. *Suppose that the following properties hold for every finite cyclic extension L/k of degree* 1 *or p*:

(1) (period equals index) *Every Brauer class of period p over L has index p.*

(2) (H^3-corestriction injectivity) *The corestriction map*

$$\mathrm{Cor}_{L/k} : H^3_p(L) \to H^3_p(k)$$

is injective.

Then for every $\alpha \in \mathrm{Br}_{\mathrm{tr}}(F)[p]$, one has

$$\{\lambda \in F^* \mid \alpha \cup (\lambda) = 0 \in H^3_p(F)\} = F^{*p} \cdot \mathrm{Nrd}(\alpha).$$

Proof. The basic strategy is identical to that of the proof of Theorem 1.5. A slight difference is that we now assume the period-index condition to strengthen the Rost-divisibility hypothesis. This is needed to overcome the difficulty that the method of lifting p-th powers using Hensel's lemma fails.

The tameness assumption on α allows us to consider its residue $\partial(\alpha) \in H^1(k, \mathbb{Z}/p)$, which we represent by a finite cyclic extension E_0/k of degree 1 or p. We write E/F for the unramified extension with residue field extension E_0/k. We fix a uniformizer $\pi \in F$ and use the split exact sequence (5.4) to write

$$\alpha = \alpha' + (E/F, \pi) \quad \text{with } \alpha' \in \mathrm{Br}(F)[p] \text{ unramified,}$$

as an analog of (4.5). (Here we call a Brauer class $\beta \in \mathrm{Br}(F)[p]$ *unramified* if $\beta \in \mathrm{Br}_{\mathrm{tr}}(F)[p]$ and if $\partial(\beta)=0$. Equivalently, an unramified element of $\mathrm{Br}(F)[p]$ is an element in the image of the inflation map $\mathrm{Br}(k)[p] \to \mathrm{Br}_{\mathrm{tr}}(F)[p]$.)

Let $\lambda \in F^*$ be such that $\alpha \cup (\lambda) = 0 \in H^3_p(F)$. We want to show $\lambda \in F^{*p} \cdot \mathrm{Nrd}(\alpha)$. Replacing λ with $\lambda^s \pi^{pc}$ for suitable integers $s, c \in \mathbb{Z}$ if necessary, we may assume that $v_F(\lambda) = 0$ or 1.

If $v_F(\lambda) = 1$, computing the residue of $\alpha \cup (\lambda)$ we see that $\alpha = (E/F, -\lambda)$ (see (4.7)). Thus α is split over $L := F\big(\sqrt[p]{-\lambda}\big)$ and

$$(-1)^p\lambda = N_{L/F}\big(\sqrt[p]{-\lambda}\big) \in N_{L/F}(L^*) \subseteq \mathrm{Nrd}(\alpha).$$

This shows $\lambda \in F^{*p} \cdot \mathrm{Nrd}(\alpha)$, as desired.

Now assume $v_F(\lambda) = 0$. As in the proof of Lemma 4.14, we have $\lambda = N_{E/F}(\mu)$ for some unit μ in E such that $\alpha_E \cup (\mu) = 0 \in H^3_p(E)$. Since $\alpha_E = \alpha'_E$ is unramified, the period-index assumption implies that α_E has index p or 1. By [Gille 2000, p. 94, Theorem 6], μ is a reduced norm for α_E and it follows that $\lambda \in N_{E/F}(\mu) \in \mathrm{Nrd}(\alpha)$. This completes the proof. $\square$

Remark 5.6. For a field k of characteristic p, there are two useful variants of the cohomological p-dimension $\mathrm{cd}_p(k)$: the *separable p-dimension* $\mathrm{sd}_p(k)$ [Gille 2000, p. 62] and *Kato's p-dimension* $\dim_p(k)$ [Kato 1982, p. 220]. They are defined as follows:

$$\mathrm{sd}_p(k) := \inf\{r \in \mathbb{N} \mid H^{r+1}_p(k') = 0 \text{ for all finite separable extensions } k'/k\},$$
$$\dim_p(k) := \inf\{r \in \mathbb{N} \mid [k:k^p] \le p^r \text{ and } H^{r+1}_p(k') = 0 \text{ for all finite extensions } k'/k\}.$$

(In characteristic different from p, both the separable p-dimension and Kato's p-dimension are defined to be the same as the cohomological p-dimension.)

It is easy to see that $\mathrm{sd}_p(k) \le \dim_p(k)$. Moreover, it can be shown that

$$\log_p[k:k^p] \le \dim_p(k) \le \log_p[k:k^p] + 1.$$

Therefore, we have the implications

$$\dim_p(k) \le 1 \quad\Longrightarrow\quad [k:k^p] \le p \quad\Longrightarrow\quad \dim_p(k) \le 2 \quad\Longrightarrow\quad \mathrm{sd}_p(k) \le 2.$$

Notice that condition (2) in Theorem 5.5 is true when $\mathrm{sd}_p(k) \le 2$, and (1) holds if $[k:k^p] \le p$ by a theorem of Albert (see [Gille and Szamuely 2017, Lemma 9.1.7]).

If in Theorem 5.5 we assume $\dim_p(k) \le 1$, then condition (1) also holds, but the theorem is not new. In fact, even more is true in that case. This is because we have $\dim_p(F) \le 2$ by [Kato and Kuzumaki 1986, Theorem 3.1]. In particular, $\mathrm{sd}_p(F) \le 2$. So, by Example 1.1(2), F is Rost p^∞-divisible.

We now show that the Rost p-divisibility of F is still true when we only assume $[k:k^p] \le p$. This is based on the following result of Kato.

Proposition 5.7 [Kato 1979, p. 337, §3, Lemma 5]. *Let F be a henselian excellent discrete valuation field with residue field k of characteristic p. Assume $[k:k^p]=p$. Then for every $\alpha\in \mathrm{Br}(F)[p]$ that is not in $\mathrm{Br}_{\mathrm{tr}}(F)[p]$, one has $\mathrm{ind}(\alpha)=p$.*

Corollary 5.8. *Let F be a henselian excellent discrete valuation field with residue field k of characteristic p. If $[k:k^p]\le p$ (e.g., $k=k_0(x)$ or $k_0((x))$ for some perfect field k_0), then F is Rost p-divisible.*

Proof. Combine Theorem 5.5, Remark 5.6 and Proposition 5.7. □

Example 5.9. We give some further examples to which Theorem 5.5 applies, i.e., examples where the residue field k satisfies the two conditions of the theorem.

We first note that by [Kato and Kuzumaki 1986, p. 234, Proposition 2(2)] (or [Arason and Baeza 2010]), if k is a C_2-field of characteristic p, then $\dim_p(k)\le 2$, so that condition (2) of Theorem 5.5 holds by Remark 5.6. If $p=2$, condition (1) is true for any C_2-field k of characteristic 2. (The period-index condition for 2-torsion Brauer classes can be shown easily by considering the Albert forms of biquaternion algebras.)

For general p, the field k can be any of the following fields:

(1) k is a field of transcendence degree 2 over an algebraically closed field (of characteristic p). In this case the period-index condition follows from [Lieblich 2008, Theorem 4.2.2.3]. The field k is C_2 by the Tsen–Lang theorem.

(2) $k=k_0((x))$, where k_0 is a C_1-field of characteristic p. Here k is C_2 by Greenberg's theorem [1966]. The "period equals index" property holds for wildly ramified classes in $\mathrm{Br}(k)[p]$, by Proposition 5.7. For tame classes this follows from the split exact sequence (5.4) and the fact that $\mathrm{Br}(k_0)=0$.

(3) $k=\mathbb{F}((x))((y))$, where $\mathbb{F}$ is a finite field of characteristic p. The period-index condition follows from [Aravire and Jacob 1995, Corollary 3.5]. The injectivity of the corestriction map is proved in [Kato 1980, p. 660, §3.2, Proposition 1].

6. A modified version of Suslin's conjecture

Our main results (Theorems 1.5 and 1.7) provide new examples of fields of cohomological dimension 3 that satisfy Suslin's conjecture (1.2). On the other hand, there exist counterexamples to Suslin's conjecture, as we have said in the introduction. We now take a closer look at the known counterexamples.

(6.1) Let ℓ be a prime. Let k be a field of characteristic different from ℓ and suppose that k contains a primitive ℓ-th root of unity. Let $F=k(t_1,\dots,t_n)$ be a purely transcendental extension in n variables over k. Let $a_1,\dots,a_n\in k^*$, $M_i=k\big(\sqrt[\ell]{a_i}\,\big)$ and $L=M_1\cdots M_n=k\big(\sqrt[\ell]{a_1},\dots,\sqrt[\ell]{a_n}\,\big)$. Let $\alpha\in\mathrm{Br}(F)$ be the Brauer class of the tensor product $\bigotimes_{i=1}^{n}(a_i, t_i)$. By [Merkurjev 1995, Proposition 2.5], a necessary condition for $\mathcal{R}(\alpha) = \mathcal{S}(\alpha)$ is

$$\bigcap_{i=1}^{n} N_{M_i/k}(M_i^*) = k^{*\ell} \cdot N_{L/k}(L^*). \tag{6.2}$$

For any odd prime ℓ and $n = 2$, Merkurjev [1995, §2] constructed examples where (6.2) fails, whence an example with $\mathcal{R}(\alpha) \neq \mathcal{S}(\alpha)$.

If $\ell = 2$ and $n = 3$, (6.2) is encoded in the property $P_1(3)$ of fields defined and studied in [Tignol 1981] and [Shapiro et al. 1982] (see also [Gille 1997, p. 741, Proposition 3]). Thus, for a field k that does not possess this property (see [Shapiro et al. 1982, §5]) one can find counterexamples to (6.2).

Note, however, a simple observation: (6.2) holds trivially if the field k satisfies $\mathrm{cd}_\ell(k) \leq 1$. Therefore, in the above counterexamples we must have $\mathrm{cd}_\ell(k) > 1$ and hence $\mathrm{cd}_\ell(F) > 3$.

The above discussions together with the main results of this paper lead us to propose the following modification of Suslin's conjecture:

Conjecture 6.3. *Let ℓ be a prime and F a field with separable ℓ-dimension*

$$\mathrm{sd}_\ell(F) \leq 3$$

(see Remark 5.6). *Then F is Rost ℓ^∞-divisible.*

As a related question, one may ask whether every C_3-field is Rost divisible.

When $\ell \neq \mathrm{char}(F)$, Conjecture 6.3 is true if for every Severi–Brauer variety X associated to an ℓ-power degree division algebra over F, the Chow groups $\mathrm{CH}^d(X)$, $d \geq 1$ are all torsion free [Merkurjev 1995, Proposition 1.11]. (In fact, it suffices to assume $\mathrm{CH}^d(X)$ is torsion free for $d \geq 3$.)

As we have mentioned in the introduction, in cohomological dimension 3 it may happen that the Rost kernel is strictly larger than the group of reduced norms. For example, it is shown in the proof of [Merkurjev 1991, Theorem 4] that there exists a field k of characteristic 0 and of cohomological dimension 2 such that some biquaternion algebra D_0 over k is a division algebra. If $F = k((t))$ and $\alpha \in \mathrm{Br}(F)$ is the Brauer class of $D_0 \otimes_k F$, then F has cohomological dimension 3, and the element $t^2 \in F^*$ lies in the Rost kernel of α but is not a reduced norm for α [Colliot-Thélène et al. 2012, Remark 5.1].

Remark 6.4. The rational function field $k = \mathbb{C}(x, y)$ does not have the property $P_1(3)$ [Shapiro et al. 1982, Corollary 5.6], so one can find quadratic extensions $M_i = k(\sqrt{a_i}\,)$, $i = 1, 2, 3$ such that (6.2) fails. Similar to [Merkurjev 1995, Proposition 2.5], for the field $K = k((t_1))((t_2))((t_3))$ and the Brauer class α of $\bigotimes_{i=1}^{3}(a_i, t_i)$, one has $\mathcal{R}(\alpha) \neq \mathcal{S}(\alpha)$ [Knus et al. 1995, p. 283].

So, the field $K = k((t_1))((t_2))((t_3))$, with $k = \mathbb{C}(x, y)$, is not Rost 2-divisible. This shows that Theorem 1.5 can be false if we drop the H^3-corestriction injectivity assumption, for otherwise it would imply by induction that the field K here is Rost 2-divisible.

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References

[Arason and Baeza 2010] J. K. Arason and R. Baeza, "La dimension cohomologique des corps de type C_r en caractéristique p", *C. R. Math. Acad. Sci. Paris* **348**:3-4 (2010), 125–126. MR Zbl

[Aravire and Jacob 1995] R. Aravire and B. Jacob, "p-algebras over maximally complete fields", pp. 27–49 in *K-theory and algebraic geometry: connections with quadratic forms and division algebras* (Santa Barbara, CA, 1992), edited by B. Jacob and A. Rosenberg, Proc. Sympos. Pure Math. **58**, Amer. Math. Soc., Providence, RI, 1995. MR Zbl

[Artin 1982] M. Artin, "Brauer–Severi varieties", pp. 194–210 in *Brauer groups in ring theory and algebraic geometry* (Wilrijk, 1981), edited by F. M. J. van Oystaeyen and A. H. M. J. Verschoren, Lecture Notes in Math. **917**, Springer, 1982. MR Zbl

[Baeza 1982] R. Baeza, "Comparing u-invariants of fields of characteristic 2", *Bol. Soc. Brasil. Mat.* **13**:1 (1982), 105–114. MR Zbl

[Bloch 1977] S. Bloch, "Algebraic K-theory and crystalline cohomology", *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 187–268. MR Zbl

[Colliot-Thélène et al. 2012] J.-L. Colliot-Thélène, R. Parimala, and V. Suresh, "Patching and local-global principles for homogeneous spaces over function fields of p-adic curves", *Comment. Math. Helv.* **87**:4 (2012), 1011–1033. MR Zbl

[Elman and Lam 1973] R. Elman and T. Y. Lam, "Quadratic forms and the u-invariant, II", *Invent. Math.* **21** (1973), 125–137. MR Zbl

[Elman et al. 2008] R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, AMS Colloquium Publications **56**, Amer. Math. Soc., Providence, RI, 2008. MR Zbl

[Gille 1997] P. Gille, "Examples of non-rational varieties of adjoint groups", *J. Algebra* **193**:2 (1997), 728–747. MR Zbl

[Gille 2000] P. Gille, "Invariants cohomologiques de Rost en caractéristique positive", *K-Theory* **21**:1 (2000), 57–100. MR Zbl

[Gille and Szamuely 2017] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, 2nd ed., Cambridge Studies in Advanced Mathematics **165**, Cambridge University Press, 2017. MR Zbl

[Greenberg 1966] M. J. Greenberg, "Rational points in Henselian discrete valuation rings", *Inst. Hautes Études Sci. Publ. Math.* **31** (1966), 59–64. MR Zbl

[Harbater et al. 2009] D. Harbater, J. Hartmann, and D. Krashen, "Applications of patching to quadratic forms and central simple algebras", *Invent. Math.* **178**:2 (2009), 231–263. MR Zbl

[Hu 2013] Y. Hu, "Division algebras and quadratic forms over fraction fields of two-dimensional henselian domains", *Algebra Number Theory* **7**:8 (2013), 1919–1952. MR Zbl

[Hu 2020] Y. Hu, "Reduced norms of division algebras over complete discrete valuation fields of local-global type", *J. Algebra Appl.* **19**:11 (2020), 2050217, 16. MR Zbl

[Illusie 1979] L. Illusie, "Complexe de de Rham–Witt et cohomologie cristalline", *Ann. Sci. École Norm. Sup.* (4) **12**:4 (1979), 501–661. MR Zbl

[Jacob and Wadsworth 1990] B. Jacob and A. Wadsworth, "Division algebras over Henselian fields", *J. Algebra* **128**:1 (1990), 126–179. MR Zbl

[Kato 1979] K. Kato, "A generalization of local class field theory by using K-groups, I", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **26**:2 (1979), 303–376. MR Zbl

[Kato 1980] K. Kato, "A generalization of local class field theory by using K-groups, II", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27**:3 (1980), 603–683. MR Zbl

[Kato 1982] K. Kato, "Galois cohomology of complete discrete valuation fields", pp. 215–238 in *Algebraic K-theory, II* (Oberwolfach, 1980), edited by R. K. Dennis, Lecture Notes in Math. **967**, Springer, 1982. MR Zbl

[Kato and Kuzumaki 1986] K. Kato and T. Kuzumaki, "The dimension of fields and algebraic K-theory", *J. Number Theory* **24**:2 (1986), 229–244. MR Zbl

[Kneser 1969] M. Kneser, *Lectures on Galois cohomology of classical groups*, Lectures on Math. **47**, Tata Institute of Fundamental Research, Bombay, 1969. MR Zbl

[Knus et al. 1995] M.-A. Knus, T. Y. Lam, D. B. Shapiro, and J.-P. Tignol, "Discriminants of involutions on biquaternion algebras", pp. 279–303 in *K-theory and algebraic geometry: connections with quadratic forms and division algebras* (Santa Barbara, CA, 1992), edited by B. Jacob and A. Rosenberg, Proc. Sympos. Pure Math. **58**, Amer. Math. Soc., Providence, RI, 1995. MR Zbl

[Knus et al. 1998] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, AMS Colloquium Publications **44**, Amer. Math. Soc., Providence, RI, 1998. MR Zbl

[Lam 2005] T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Math. **67**, Amer. Math. Soc., Providence, RI, 2005. MR Zbl

[Lieblich 2008] M. Lieblich, "Twisted sheaves and the period-index problem", *Compos. Math.* **144**:1 (2008), 1–31. MR Zbl

[Lieblich 2011] M. Lieblich, "Period and index in the Brauer group of an arithmetic surface", *J. Reine Angew. Math.* **659** (2011), 1–41. MR Zbl

[Lieblich 2015] M. Lieblich, "The period-index problem for fields of transcendence degree 2", *Ann. of Math.* (2) **182**:2 (2015), 391–427. MR Zbl

[Mammone et al. 1991] P. Mammone, R. Moresi, and A. R. Wadsworth, "u-invariants of fields of characteristic 2", *Math. Z.* **208**:3 (1991), 335–347. MR Zbl

[Merkurjev 1991] A. S. Merkurjev, "Simple algebras and quadratic forms", *Izv. Akad. Nauk SSSR Ser. Mat.* **55**:1 (1991), 218–224. In Russian; translated in *Math. USSR-Izv.* **38**:1 (1992), 215–221. MR Zbl

[Merkurjev 1995] A. S. Merkurjev, "Certain K-cohomology groups of Severi–Brauer varieties", pp. 319–331 in *K-theory and algebraic geometry: connections with quadratic forms and division algebras* (Santa Barbara, CA, 1992), edited by B. Jacob and A. Rosenberg, Proc. Sympos. Pure Math. **58**, Amer. Math. Soc., Providence, RI, 1995. MR Zbl

[Merkurjev 2003] A. Merkurjev, "Rost invariants of simply connected algebraic groups", pp. 101–158 in *Cohomological invariants in Galois cohomology*, Univ. Lecture Ser. **28**, Amer. Math. Soc., Providence, RI, 2003. MR

[Merkurjev and Suslin 1982] A. S. Merkurjev and A. A. Suslin, "K-cohomology of Severi–Brauer varieties and the norm residue homomorphism", *Izv. Akad. Nauk SSSR Ser. Mat.* **46**:5 (1982), 1011–1046, 1135–1136. In Russian; translated in *Math. USSR-Izv.* **21**:2 (1983), 307–340. MR Zbl

[Milne 1976] J. S. Milne, "Duality in the flat cohomology of a surface", *Ann. Sci. École Norm. Sup.* (4) **9**:2 (1976), 171–201. MR Zbl

[Neukirch et al. 2008] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of number fields*, 2nd ed., Grundlehren der Math. Wissenschaften **323**, Springer, 2008. MR Zbl

[Parimala and Suresh 2014] R. Parimala and V. Suresh, "Period-index and u-invariant questions for function fields over complete discretely valued fields", *Invent. Math.* **197**:1 (2014), 215–235. MR Zbl

[Parimala et al. 2018] R. Parimala, R. Preeti, and V. Suresh, "Local-global principle for reduced norms over function fields of p-adic curves", *Compos. Math.* **154**:2 (2018), 410–458. MR Zbl

[Preeti and Soman 2015] R. Preeti and A. Soman, "Adjoint groups over $\mathbb{Q}_p(X)$ and R-equivalence", *J. Pure Appl. Algebra* **219**:9 (2015), 4254–4264. MR Zbl

[Saltman 1997] D. J. Saltman, "Division algebras over p-adic curves", *J. Ramanujan Math. Soc.* **12**:1 (1997), 25–47. MR Zbl

[Serre 1994] J.-P. Serre, *Cohomologie galoisienne*, 5th ed., Lecture Notes in Math. **5**, Springer, 1994. MR Zbl

[Serre 2003] J.-P. Serre, "Cohomological invariants, Witt invariants, and trace forms", pp. 1–100 in *Cohomological invariants in Galois cohomology*, Univ. Lecture Ser. **28**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Shapiro et al. 1982] D. B. Shapiro, J.-P. Tignol, and A. R. Wadsworth, "Witt rings and Brauer groups under multiquadratic extensions, II", *J. Algebra* **78**:1 (1982), 58–90. MR Zbl

[Suslin 1984] A. A. Suslin, "Algebraic K-theory and the norm residue homomorphism", pp. 115–207 in *Current problems in mathematics*, vol. 25, edited by R. V. Gamkrelidze, Akad. Nauk SSSR, Moscow, 1984. In Russian; translated in *J. Sov. Math.* **30** (1985), 2556–2611. MR Zbl

[Tignol 1981] J.-P. Tignol, "Corps à involution neutralisés par une extension abélienne élémentaire", pp. 1–34 in *Groupe de Brauer* (Les Plans-sur-Bex, 1980), edited by M. Kervaire and M. Ojanguren, Lecture Notes in Math. **844**, Springer, 1981. MR Zbl

[Wadsworth 2002] A. R. Wadsworth, "Valuation theory on finite dimensional division algebras", pp. 385–449 in *Valuation theory and its applications* (Saskatoon, 1999), vol. I, edited by F.-V. Kuhlmann et al., Fields Inst. Commun. **32**, Amer. Math. Soc., Providence, RI, 2002. MR Zbl

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On the norm and multiplication principles for norm varieties

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Let p be a prime, and suppose that F is a field of characteristic zero which is p-special (that is, every finite field extension of F has dimension a power of p). Let $\alpha \in \mathcal{K}_n^M(F)/p$ be a nonzero symbol and X/F a norm variety for α. We show that X has a $\mathcal{K}_m^M$-norm principle for any m, extending the known $\mathcal{K}_1^M$-norm principle. As a corollary we get an improved description of the kernel of multiplication by a symbol. We also give a new proof for the norm principle for division algebras over p-special fields by proving a decomposition theorem for polynomials over F-central division algebras. Finally, for $p = n = m = 2$ we show that the known $\mathcal{K}_1^M$-multiplication principle cannot be extended to a $\mathcal{K}_2^M$-multiplication principle for X.

1. Introduction

Let D be a finite dimensional F-central division algebra. Then D has the reduced norm homomorphism $\mathrm{Nrd} : D \to F$. The norm principle for D states that the image of the reduced norm is an invariant of the class of D in $\mathrm{Br}(F)$, that is, $\mathrm{Nrd}(D) = \mathrm{Nrd}(\mathrm{M}_k(D))$ for any $k \in \mathbb{N}$. Equivalently, $\mathrm{N}_{K/F}(K) \subseteq \mathrm{Nrd}(D)$ for any finite separable field extension K/F splitting D. The multiplication principle states that for any two maximal subfields $K_1, K_2 \subset D$ and elements $k_1 \in K_1, k_2 \in K_2$, there is a third maximal subfield $K_3 \subset D$ and an element $k_3 \in K_3$ such that $\mathrm{Nrd}(k_1)\,\mathrm{Nrd}(k_2) = \mathrm{Nrd}(k_3)$, reflecting the fact that the reduced norm is multiplicative with respect to the multiplication of D.

The above can be rephrased as follows: Let D be a central division algebra over F of index n and let $X = \mathrm{SB}(D)$ be the Severi–Brauer variety of D. Let $A_0(X, \mathcal{K}_1^M)$ be the group of $\mathcal{K}_1^M$-zero cycles on X. It is generated by elements $[x, \lambda]$, where x is a closed point of X and $\lambda \in F(x)$. Let $A_0^n(X, \mathcal{K}_1^M)$ be the subgroup generated

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by elements $[x, \lambda]$, where x is of degree at most n (that is, $[F(x):F] \leq n$). There is a well defined norm homomorphism

$$\mathrm{N} : A_0(X, \mathcal{K}_1^M) \to A_0(\operatorname{spec}(F), \mathcal{K}_1^M) = F^\times,$$

defined on generators by $\mathrm{N}([x, \lambda]) = \mathrm{N}_{F(x)/F}(\lambda)$. The above norm principle can be restated as $\mathrm{N}(A_0(X, \mathcal{K}_1^M)) = \mathrm{N}(A_0^n(X, \mathcal{K}_1^M))$, that is, for any closed point $x \in X$ and $\lambda \in F(x)$ there is a finite number of closed points $x_1, \ldots, x_t \in X$ of degree at most n and $\lambda_i \in F(x_i)$ such that $\mathrm{N}_{F(x)/F}(\lambda) = \prod_{i=1}^t \mathrm{N}_{F(x_i)/F}(\lambda_i)$.

The multiplication principle says that for any two closed points $x_1, x_2 \in X$ of degree at most n and $\lambda_1 \in F(x_1)$, $\lambda_2 \in F(x_2)$, there is a third closed point $x_3 \in X$ of degree at most n and $\lambda_3 \in F(x_3)$ such that $\mathrm{N}_{F(x_1)/F}(\lambda_1)\,\mathrm{N}_{F(x_2)/F}(\lambda_2) = \mathrm{N}_{F(x_3)/F}(\lambda_3)$.

This is generalized as follows. Let p be a fixed prime, F a field of characteristic zero which is p-special, and $\alpha \in \mathcal{K}_n^M(F)/p$ a nonzero symbol $\alpha = a_1 \cdots a_n$, where $a_i \in \mathcal{K}_1^M(F)/p$. A crucial part of the proof of the Bloch–Kato conjecture is the fact that symbols have (at least in characteristic zero) p-generic splitting varieties, generalizing Severi–Brauer varieties, introduced by Rost; see [Haesemeyer and Weibel 2009; Rost 2002]. Rost then generalized the norm principle for division algebras (more specifically symbol algebras) to a norm principle for the group of reduced $\mathcal{K}_1^M$-zero cycles:

$$\bar{A}_0(X, \mathcal{K}_1^M) = \operatorname{coker}\bigl(A_0(X \times X, \mathcal{K}_1^M) \xrightarrow{(\mathrm{pr}_1)_* - (\mathrm{pr}_2)_*} A_0(X, \mathcal{K}_1^M)\bigr).$$

This principle states that $\mathrm{N}(\bar{A}_0(X, \mathcal{K}_1^M)) = \mathrm{N}(\bar{A}_0^p(X, \mathcal{K}_1^M))$, that is, the image of the norm on $\bar{A}_0(X, \mathcal{K}_1^M)$ is the same as the image of the norm restricted to the subgroup $\bar{A}_0^p(X, \mathcal{K}_1^M)$ generated by elements $[x, \lambda]$, where $x \in X$ is closed of degree at most p.

The multiplication principle is also generalized and states that the product of two generators of $\bar{A}_0^p(X, \mathcal{K}_1^M)$ is a generator; equivalently, for any two closed points $x_1, x_2 \in X$ of degree at most p and $\lambda_1 \in F(x_1)$, $\lambda_2 \in F(x_2)$, there is a third closed point $x_3 \in X$ of degree at most p and $\lambda_3 \in F(x_3)$ such that

$$\mathrm{N}_{F(x_1)/F}(\lambda_1)\,\mathrm{N}_{F(x_2)/F}(\lambda_2) = \mathrm{N}_{F(x_3)/F}(\lambda_3).$$

Together, the above norm and multiplication principles state the following: Let X be a norm variety for a nonzero symbol $\alpha \in \mathcal{K}_n^M(F)/p$, $x \in X$ any closed point (of arbitrary finite degree) and $\lambda \in F(x)$. Then there is a closed point $y \in X$ of degree at most p and $\gamma \in F(y)$ such that $\mathrm{N}_{F(x)/F}(\lambda) = \mathrm{N}_{F(y)/F}(\gamma)$.

In this work we show that the norm principle can be extended to higher $\mathcal{K}$-cohomology groups, but the multiplication principle does not extend. As an application we recall that using these varieties one can give the following exact sequence describing the kernel of multiplication by a symbol, taken from [Merkurjev and Suslin 2010]; see also [Weibel and Zakharevich 2017] for a similar description.

Theorem 1. *Let F be a field of characteristic prime to p and $\theta \in H^n_{\text{ét}}(F, \mu_p^{\otimes n})$ a symbol, where μ_p denotes the Galois module of all p-th roots of unity. Then for an arbitrary $k \in \mathbb{N}$, there is an exact sequence*

$$\coprod_L H^k_{\text{ét}}(L, \mu_p^{\otimes k}) \xrightarrow{\sum \mathrm{N}_{L/F}} H^k_{\text{ét}}(F, \mu_p^{\otimes k}) \xrightarrow{\cdot\theta} H^{k+n}_{\text{ét}}(F, \mu_p^{\otimes k+n}) \xrightarrow{\prod_E \mathrm{res}_{E/F}} \prod_E H^{k+n}_{\text{ét}}(E, \mu_p^{\otimes k+n}),$$

where the coproduct is taken over all finite splitting field extensions L/F for θ and the product is taken over all splitting field extensions E/F.

As a result of the higher norm principle we can add that the coproduct is taken over all splitting fields L/F such that p^2 does not divide $[L : F]$, as is the case for $p = 2$; see [Orlov et al. 2007] for details.

The work is organized as follows. In Section 3 we prove a generalized norm principle. In Section 4 we give a purely algebraic proof of the main theorem used for the proof of the norm principle in [Haesemeyer and Weibel 2009] for the case of division algebras, resulting in a new proof for the norm principle for division algebras over p-special fields. To this end we prove that if F is p-special (with no restriction on the characteristic) and D is an F-central division algebra, then any polynomial in $D[\lambda]$ of degree less than p splits into linear factors (see Theorem 14). In Section 5 we show that (at least for $p = n = m = 2$) there is no generalized multiplication principle.

2. Background and notations

Let p be a fixed prime, and suppose that F is a field of characteristic zero which is p-special — that is, for any field extension K/F we have that $[K : F]$ is a p-th power, or equivalently (for perfect fields), the absolute Galois group of F is a pro-p group.

Let $a_1, \dots, a_n$ be in $\mathcal{K}_1^M(F)/p \cong F^\times/(F^\times)^p$ and $\alpha = a_1 \cdots a_n$ be a nontrivial symbol in $\mathcal{K}_n^M(F)/p$, where $\mathcal{K}_n^M(F)$ is the n-th Milnor $\mathcal{K}$ group of F. In a work by Rost, it was shown that there exists a "p-generic splitting variety" over F of dimension $p^{n-1} - 1$ for α, namely a smooth, irreducible, projective variety X of dimension $p^{n-1} - 1$, such that for any field extension L/F, α_L vanishes in $K_n^M(L)/p$ if and only if $X(L') \neq \varnothing$, where L'/L is a field extension of dimension prime to p. Such a variety is called a norm variety for α. For a detailed construction of such X we refer the reader to [Haesemeyer and Weibel 2009; Suslin and Joukhovitski 2006].

As an example, in the case $n = 2$, X can be the Severi–Brauer variety of the central simple algebra associated to α by the norm residue map.

Definition 2. Let X/F be a smooth irreducible projective variety of dimension d and let n be an integer. The group of $\mathcal{K}_m^M$-zero cycles, $A_0(X, \mathcal{K}_m^M)$, is defined as

$$A_0(X, \mathcal{K}_m^M) = \operatorname{coker}\Bigg(\coprod_{\operatorname{codim}(x)=d-1} \mathcal{K}_{m+1}^M(F(x)) \to \coprod_{\operatorname{codim}(x)=d} \mathcal{K}_m^M(F(x)) \Bigg).$$

It is generated by elements $[x, \alpha]$, where x is a closed point of X (marking its index in the coproduct) and $\alpha \in \mathcal{K}_m^M(F(x))$. Also define the subgroup

$$A_0^p(X, \mathcal{K}_m^M) = \big\langle [x, \alpha] \mid x \text{ is closed of degree at most } p \big\rangle.$$

Remark 3. There is a well defined norm map $\mathrm{N} : A_0(X, \mathcal{K}_m^M) \to \mathcal{K}_m^M(F)$ induced by the usual norm on fields: $\mathrm{N}([x, \alpha]) = \operatorname{Cor}_{F(x)/F}(\alpha)$.

As we are going to be interested in norms of elements, we make the following definitions.

Definition 4. For X/F as above define the group

$$\widetilde{A}_0(X, \mathcal{K}_m^M) = A_0(X, \mathcal{K}_m^M)/\operatorname{Ker}(\mathrm{N})$$

and its subgroup,

$$\widetilde{A}_0^p(X, \mathcal{K}_m^M)) = A_0^p(X, \mathcal{K}_m^M))/\operatorname{Ker}(\mathrm{N}).$$

Definition 5. We say that X has a $\mathcal{K}_m^M$-norm principle if $\widetilde{A}_0(X, \mathcal{K}_m^M)) = \widetilde{A}_0^p(X, \mathcal{K}_m^M))$, or equivalently, $\mathrm{N}(A_0(X, \mathcal{K}_m^M)) = \mathrm{N}(A_0^p(X, \mathcal{K}_m^M))$.

Definition 6. We say that X has a $\mathcal{K}_m^M$-multiplication principle if every element in $\widetilde{A}_0^p(X, \mathcal{K}_m^M)$ is a single generator $[x, \beta]$, or equivalently, the norm of every element in $A_0^p(X, \mathcal{K}_m^M)$ can be obtained as the norm of just one generator $[x, \beta]$.

Let $\alpha \in \mathcal{K}_n^M(F)/p$ be a nontrivial symbol. For every m, we have the morphism of multiplication by α:

$$\mathcal{K}_m^M(F)/p \xrightarrow{\cdot\alpha} \mathcal{K}_{n+m}^M(F)/p.$$

Let X be a norm variety for α. Then by Theorem 1, for every m, the kernel $\operatorname{Ker}_m(\alpha)$ of this morphism can be described by the exact sequence

$$\widetilde{A}_0(X, \mathcal{K}_m^M) \xrightarrow{\pi\circ\mathrm{N}} \mathcal{K}_m^M(F)/p \xrightarrow{\cdot\alpha} \mathcal{K}_{n+m}^M(F)/p,$$

where $\pi : \mathcal{K}_m^M(F) \to \mathcal{K}_m^M(F)/p$ is the natural projection. If X has a $\mathcal{K}_m^M$-norm principle we get that the sequence

$$\widetilde{A}_0^p(X, \mathcal{K}_m^M) \xrightarrow{\pi\circ\mathrm{N}} \mathcal{K}_m^M(F) \xrightarrow{\cdot\alpha} \mathcal{K}_{n+m}^M(F)$$

is exact, giving a better description of the kernel of multiplication by α.

Beyond proving the norm principle for X, we would like to give a "nice" generating set for the kernel $\operatorname{Ker}_m(\alpha)$. To this end we make the following definitions.

Definition 7. A basic element of Ker(α) in $\mathcal{K}_m^M(F)$ is an element of the form $a_1 \cdots a_{m-1} \cdot a_m$, where $a_1, \ldots, a_{m-1} \in \mathcal{K}_1^M(F)/p$ and $a_m \in \mathrm{N}_{L/F}(L^\times)/p$, where L is a splitting field for α of degree (at most) p.

Definition 8. For a symbol α define $\mathrm{BKer}_m(\alpha)$ to be the subgroup of $\mathrm{Ker}_m(\alpha)$ generated by all basic elements of $\mathrm{Ker}_m(\alpha)$.

Remark 9. A description of $\mathrm{Ker}_m(\alpha)$ was given in [Orlov et al. 2007] for the case $p = 2$, where it was proved that $\mathrm{BKer}_m(\alpha) = \mathrm{Ker}_m(\alpha)$ for all m. Also, by the norm and multiplication principles for $\bar{A}_0(X, \mathcal{K}_1^M)$ one has $\mathrm{BKer}_1(\alpha) = \mathrm{Ker}_1(\alpha)$.

We prove that over p-special fields (of characteristic zero), $\mathrm{BKer}_m(\alpha) = \mathrm{Ker}_m(\alpha)$ for a symbol $\alpha \in \mathcal{K}_n^M(F)$ for arbitrary n and m.

3. Norm principle

In this section we prove the higher norm principle for the norm variety of a symbol α. Recall the following well known lemma.

Lemma 10 [Gille and Szamuely 2006, p. 195, Corollary 7.2.10]. *Let F be a field of characteristic prime to p which is p-special, and K/F be a field extension of degree p. Then $\mathcal{K}_n^M(K) = \sum \mathcal{K}_{n-1}^M(F)\mathcal{K}_1^M(K)$.*

Also recall the following theorem taken from [Haesemeyer and Weibel 2009] (which is the main ingredient in the proof of the norm principle).

Theorem 11 [Haesemeyer and Weibel 2009, Theorem 9.6]. *Let F be a p-special field of characteristic zero, and E/F a field extension with $[E : F] = p$. Write $E = F[\epsilon]$ with $\epsilon^p \in F$. For a nontrivial symbol $\alpha \in \mathcal{K}_n^M(F)/p$ suppose that $\alpha_E \neq 0$ and that X is a norm variety for α. For $[x, \alpha] \in \bar{A}_0(E)$, where $x \in X_E$ is of degree at most p (over E), there exist points $x_i \in X$ of degree p over F, $t_i \in F$ and $b_i \in F(x_i)$ such that $\mathrm{N}_{E(x)/E}(\alpha) = \prod \mathrm{N}_{E(x_i)/E}(b_i + t_i\epsilon)$.*

We are ready to prove the higher norm principle:

Theorem 12. *Let F be a p special field of characteristic zero, and X a norm variety for a nontrivial symbol $\alpha \in \mathcal{K}_n^M(F)/p$. Then X has a $\mathcal{K}_m^M$-norm principle for any m. Moreover, $\mathrm{Ker}_m(\alpha) = \mathrm{BKer}_m(\alpha)$ for any m.*

Proof. In order to prove the higher norm principle, we have to show that

$$\mathrm{N}(A_0(X, \mathcal{K}_m^M)) = \mathrm{N}(A_0^p(X, \mathcal{K}_m^M)).$$

It is clear that it is enough to show

$$\mathrm{N}(A_0(X, \mathcal{K}_m^M)) \subseteq \mathrm{N}(A_0^p(X, \mathcal{K}_m^M)).$$

Let $[x, \gamma] \in A_0(X, \mathcal{K}_m^M)$, so that $x \in X$ is a closed point and $\gamma \in \mathcal{K}_m^M(F(x))$. Since F is p-special, x is of degree p^t for some $t \geq 1$. If $t = 1$ there is nothing to prove, so we assume $t > 1$. Pick subfields $F \subseteq L \subseteq K \subseteq F(x)$ such that $[F(x) : K] = p$, $[K : L] = p$ and K is not a splitting field of α (if K is a splitting field, then $\mathrm{cor}_{F(x)/F}(\gamma) = \mathrm{cor}_{K/F}(\mathrm{cor}_{F(x)/K}(\gamma))$ and we are done by induction). Write $F(x) = K(x')$ for a closed point $x' \in X_K$. By Lemma 10, we may write $\mathrm{cor}_{F(x)/F}(\gamma) = \mathrm{cor}_{F(x)/F}\big(\sum \gamma_i \cdot b_i\big)$ for some $\gamma_i \in \mathcal{K}^M_{m-1}(K)$, $b_i \in F(x)$. By Theorem 11, there are closed points $x_{ij} \in X_L$ of degree p and $\beta_{i,j} \in L(x_{i,j})$ such that $\mathrm{cor}_{K(x')/K}(b_i) = \sum_j \mathrm{cor}_{K(x_{i,j})/K}(\beta_{i,j})$. Now compute

$$\begin{aligned}
\mathrm{N}([x,\gamma]) &= \mathrm{cor}_{F(x)/F}(\gamma)\\
&= \mathrm{cor}_{F(x)/F}\Big(\sum_i \gamma_i \cdot b_i\Big)\\
&= \mathrm{cor}_{K/F}\Big(\sum_i \gamma_i \cdot \mathrm{cor}_{K(x')/K}(b_i)\Big)\\
&= \mathrm{cor}_{K/F}\Big(\sum_i \gamma_i \cdot \sum_j \mathrm{cor}_{K(x_{i,j})/K}(\beta_{i,j})\Big)\\
&= \mathrm{cor}_{K/F}\Big(\sum_{i,j} \gamma_i \cdot \mathrm{cor}_{K(x_{i,j})/K}(\beta_{i,j})\Big)\\
&= \sum_{i,j} \mathrm{cor}_{K/F}\,\mathrm{cor}_{K(x_{i,j})/K}(\gamma_i \cdot \beta_{i,j})\\
&= \sum_{i,j} \mathrm{cor}_{K(x_{i,j})/F}(\gamma_i \cdot \beta_{i,j})\\
&= \sum_{i,j} \mathrm{cor}_{L(x_{i,j})/F}\,\mathrm{cor}_{K(x_{i,j})/L(x_{i,j})}(\gamma_i \cdot \beta_{i,j})\\
&= \mathrm{N}\Big(\sum_{i,j} [x_{i,j}, \mathrm{cor}_{K(x_{i,j})/L(x_{i,j})}(\gamma_i \cdot \beta_{i,j})]\Big).
\end{aligned}$$

Notice that $[L(x_{i,j}) : F] = p^{t-1}$, and clearly $L(x_{i,j})$ splits α, so we are done by induction on t. We proved that $\mathrm{N}(A_0(X, \mathcal{K}^M_m)) = \mathrm{N}(A_0^p(X, \mathcal{K}^M_m))$, so X has a $\mathcal{K}^M_m$-norm principle. The last statement follows from Lemma 10 and the proof thus far. □

Corollary 13. *Let F be a field of characteristic prime to p and $\theta \in H^n_{\text{ét}}(F, \mu_p^{\otimes n})$ a symbol, where μ_p denotes the Galois module of all p-th roots of unity. Then for an arbitrary $k \in \mathbb{N}$, there is an exact sequence*

$$\coprod_L H^k_{\text{ét}}(L, \mu_p^{\otimes k}) \xrightarrow{\sum \mathrm{N}_{L/F}} H^k_{\text{ét}}(F, \mu_p^{\otimes k}) \xrightarrow{\cdot\theta} H^{k+n}_{\text{ét}}(F, \mu_p^{\otimes k+n}) \xrightarrow{\prod_E \mathrm{res}_{E/F}} \prod_E H^{k+n}_{\text{ét}}(E, \mu_p^{\otimes k+n}),$$

where the coproduct is taken over all splitting field extensions L/F for θ of degree not divisible by p^2, and the product is taken over all splitting field extensions E/F.

Proof. This is just applying Theorem 12 to Theorem 1. □

4. Norm principle for division algebras

In this section we give a purely algebraic proof of a variant of Theorem 11 for any division algebra and not just a symbol. When $n = 2$, we have that the m torsion part of the Brauer group ${}_m\mathrm{Br}(F)$ is isomorphic to $\mathcal{K}_2^M(F)/m$. So we may consider symbols as division algebras and their norm varieties as the Severi–Brauer varieties.

Polynomials over division rings. We first recall some known facts concerning polynomials over division algebras. For a more thorough reference we point the reader to [Jacobson 1996, Chapter 1; 1943, Chapter 3; Haile and Rowen 1995]. Let D be an F-central division algebra (i.e., its center is F) and $R = D[\lambda]$ the ring of (left) polynomials over D (where λ is central in R). Let $c \in D$ be a central element. Then there is a well defined substitution homomorphism $\varphi : R \to D$ defined by $\lambda \mapsto c$. In particular, if we can decompose $f(\lambda) = g(\lambda)h(\lambda)$, then $f(c) = g(c)h(c)$.

The polynomial ring $R = D[\lambda]$ is a left (and also right) Euclidean domain; that is, for any $f(\lambda), g(\lambda) \in R$ there are $q(\lambda), r(\lambda) \in R$ such that $f(\lambda) = q(\lambda)g(\lambda) + r(\lambda)$ and $\deg r(\lambda) < g(\lambda)$ or $r(\lambda) = 0$. As a consequence we get that every left (and right) ideal is principal, so R is a (left and right) principle ideal domain (PID). For left ideals of R, $Rf \subseteq Rg$ if and only if $f = hg$ for some $h \in R$. Thus Rf is maximal if and only if f is irreducible, which happens if and only if R/Rf is simple as a (left) module over R.

The two-sided ideals of R are all of the form $Rf = fR$, where $f \in F[\lambda]$. For a left ideal Rf, the maximal two-sided ideal contained in Rf (called the bound of Rf) is the annihilator $I = \mathrm{ann}(R/Rf)$. Note that if Rf is a maximal left ideal (f is irreducible) and $I \neq 0$ then I is a maximal two-sided ideal. Write $I = Rg \neq 0$ (where $g \in F[\lambda]$). Then $R/I \cong D \otimes (F[\lambda]/F[\lambda]g)$, where $F[\lambda]/F[\lambda]g$ is a simple $F[\lambda]$-module of dimension $\deg(g)$ — that is, a field extension of degree $\deg(g)$.

Norm principle for division algebras. For this subsection we assume our base field F is p-special (no restriction on $\mathrm{char}(F)$), and D is any F-central division algebra. We discuss further polynomials over D, and then prove our version of Theorem 11.

Theorem 14. *Let D be an F-central division algebra* (*which by assumption has index p^t for some t*)*. The only irreducible polynomials over D are of degree a power of p.*

Proof. Let $f \in R = D[\lambda]$ be a polynomial of degree d such that d is not a power of p. Recall from the previous subsection that f is irreducible if and only if $M = R/Rf$ is a simple R-module.

If M is a simple R-module then it is also a simple module over $S = R/\operatorname{ann}(M)$. The ideal $\operatorname{ann}(M)$ is nonzero, since f divides its reduced norm, which is nonzero (see [Haile and Rowen 1995]). This can also be seen by comparing the dimension over F of M and R. As we assume f is irreducible we have that $\operatorname{ann}(M)$ is a maximal two-sided ideal, and since R is a principle ideal domain we can write $\operatorname{ann}(M) = \langle g \rangle$ such that $g \in F[\lambda]$ (see the previous subsection). This implies that $S \cong D \otimes_F E$, where $E = F[\lambda]/\langle g \rangle$ is a field extension of F. As F is p-special we have that $\dim_F(E) = p^s$ for some s, so $\dim_F D \otimes_F E = p^{2t+s}$.

If indeed M were a simple module, it would be a simple module of $D \otimes E$, which is either $D \otimes_F E$ (if it is a division algebra) or E^{p^t} (if $D \otimes E \cong \mathrm{M}_{p^t}(E)$) or any other possibility in between. Either way, we get that the dimension of a simple module is a power of p but $\dim_F(M) = d \dim_F(D) = dp^{2t}$ is not a power of p. Hence M is not a simple module, which forces f to be reducible. □

The next corollary is now immediate.

Corollary 15. *For a division algebra D over a p-special field, every polynomial over D of degree less than p splits into linear factors.*

The factorization of polynomials of degree less then p over D enables us to give the following purely algebraic proof of the crucial Theorem 11 (which works for any division algebra, not just symbols).

Corollary 16. *Let D be a division algebra over a p-special field F, and let $F \subset E$ be a field extension of dimension p such that $D_E = D \otimes_F E$ is a division algebra. Then for every $d \in D_E$ there are $d_0, \dots, d_{p-1} \in D$ and $\epsilon \in E$ such that*

$$d = d_0 \prod_{i=1}^{p} (\epsilon - d_i).$$

Proof. Since F is p-special we may write $E = F[\epsilon \mid \epsilon^p \in F]$, and so the extension of D can be written as $D_E = D + D\epsilon + D\epsilon^2 + \cdots + D\epsilon^{p-1}$. Thus, any element $d \in D_E$ is of the form $d = d'_0 + d'_1\epsilon + \cdots + d'_{p-1}\epsilon^{p-1}$, where $d'_i \in D$. Looking at the polynomial $f(\lambda) = d'_0 + d'_1\lambda + \cdots + d'_{p-1}\lambda^{p-1} \in D[\lambda]$, we have that $f(\epsilon) = d$. By Corollary 15, $f(\lambda)$ splits to linear factors $f(\lambda) = d_0(\lambda - d_1) \cdot (\lambda - d_2) \cdots (\lambda - d_{p-1})$ in $D[\lambda]$, and since ϵ is central we get that $d = f(\epsilon) = d_0(\epsilon - d_1) \cdot (\epsilon - d_2) \cdots (\epsilon - d_{p-1})$. □

Corollary 17. *Suppose F is p-special and D is an F-central division algebra of degree d. Let $E = F[\epsilon \mid \epsilon^p \in F]$ be a field extension of degree p such that D_E is a division algebra. For every element $d \in D_E$, there are maximal subfields $E_i \subseteq D$*

and elements $d_i \in D$ such that

$$\mathrm{N}_{D_E/E}(d) = \mathrm{N}_{E_0 \otimes_F E/E}(d_0) \prod \mathrm{N}_{E_i \otimes_F E/E}(\epsilon - d_i),$$

where $\mathrm{N}_{D_E/E}$ is the reduced norm for D_E.

Proof. The proof follows from the well known fact that for maximal subfields of D the field norm coincides with the reduced norm. In particular, write $d = d_0(\epsilon - d_1) \cdot (\epsilon - d_2) \cdots (\epsilon - d_{p-1})$ as in Corollary 16. Now define E_i to be any maximal subfield of D containing d_i and apply the reduced norm on both sides of the factorization of d to get the required result. □

We use this factorization to get a direct proof of the norm principle for division algebras.

Theorem 18 (norm principle for division algebras). *Let F be a p-special field and D an F-central division algebra of index $d = p^n$. Let E/F be a finite dimensional splitting field for D and let $e \in E$. Then there is a maximal subfield K of D and $k \in K$ such that $\mathrm{N}_{E/F}(e) = \mathrm{N}_{K/F}(k)$.*

Proof. We proceed by induction on the index $\mathrm{ind}(D) = p^n$. The case of $n = 0$ is trivial. We now assume the theorem for $\mathrm{ind}(D) \leq p^k$ and prove it for $\mathrm{ind}(D) = p^{k+1}$. Let E/F be a splitting field for D of degree r, noting that since D is division we have $r \geq \mathrm{ind}(D)$. We proceed by induction on r. The case $r = d$ follows from the fact that in this case E embeds in D as a maximal subfield. We now assume the theorem for $r \leq \mathrm{ind}(D) + s$ and prove it for $r = \mathrm{ind}(D) + s + 1$. As F is p-special we can find a subfield $F \subset E_1 \subset E$ such that E_1 is of degree p over F. Consider $D_{E_1} = D \otimes E_1$. First assume that $D_{E_1} \cong \mathrm{M}_p(D')$ for an E_1-central division algebra D' of index p^k. Then by induction on $\mathrm{ind}(D)$ we get that there is a maximal subfield $T \subset D'$ and $t \in T$ such that $\mathrm{N}_{T/E_1}(t) = \mathrm{N}_{E/E_1}(e)$, implying $\mathrm{N}_{T/F}(t) = \mathrm{N}_{E/F}(e)$. But now, considering T over F, we see that T splits D and $[T : F] = p^{k+1} = \mathrm{ind}(D)$, so T embeds in D as a maximal subfield and we are done.

Now assume D_{E_1} is division. Notice that E splits D_{E_1} and is of lesser degree (over E_1). Thus by induction there exist a maximal subfield $T \subset D_{E_1}$ and $t \in T$ such that $\mathrm{N}_{T/E_1}(t) = \mathrm{N}_{E/E_1}(e)$, implying $\mathrm{N}_{T/F}(t) = \mathrm{N}_{E/F}(e)$. Writing $E_1 = F[\epsilon \mid \epsilon^p \in F]$ and using Corollary 17 we get maximal subfields $K_i \subset D$ and elements $d_i \in K_i$ such that

$$\begin{aligned}
\mathrm{N}_{T/F}(t) &= \mathrm{N}_{E_1/F}(\mathrm{N}_{T/E_1}(t)) \\
&= \mathrm{N}_{E_1/F}(\mathrm{N}_{K_0 \otimes_F E_1/E_1}(d_0)) \prod \mathrm{N}_{E_1/F}(\mathrm{N}_{K_i \otimes_F E_1/E_1}(\epsilon - d_i)) \\
&= \mathrm{N}_{K_0 \otimes_F E_1/F}(d_0) \prod \mathrm{N}_{K_i \otimes_F E_1/F}(\epsilon - d_i) \\
&= \mathrm{N}_{K_0/F}(\mathrm{N}_{K_0 \otimes_F E_1/K_0}(d_0)) \prod \mathrm{N}_{K_i/F}(\mathrm{N}_{K_i \otimes_F E_1/K_i}(\epsilon - d_i)).
\end{aligned}$$

Thus $\mathrm{N}_{T/F}(t)$ is a product of norms from the maximal subfields $K_i \subseteq D$. Using the fact that the reduced norm is multiplicative and coincides with the field norm for maximal subfields we see that $\mathrm{N}_{T/F}(t) = \mathrm{N}_{K/F}(d)$, where K is any maximal subfield of D containing d and $d = \mathrm{N}_{E/F}(t) \prod \mathrm{N}_{K_i \otimes_F E_1/K_i}(\epsilon - d_i)$. $\square$

Remark 19. Using a noncommutative analog of the determinant, the Dieudonné determinant, one can show that over any field the image of the reduced norm of a central simple algebra is an invariant of its class in the Brauer group of F; see [Pierce 1982, 16.5]. The above gives a simple proof of this result for p-special fields.

5. Multiplication principle

In this section we prove that for $p = n = m = 2$ there is no generalized multiplication principle (see Definition 6). We start by quoting the following lemma.

Lemma 20 [Matzri 2019, Lemma 4.1]. *Let F be a p-special field of characteristic prime to p. Let $\alpha \in \mathcal{K}_n^M(F)/p$ be a symbol and $b \in \mathcal{K}_1^M(F)/p$. Then $\alpha \cdot b = 0$ if and only if there exist $s_i \in \mathcal{K}_1^M(F)$, $i = 1, \ldots, n$, and a presentation $\alpha = s_1 \cdots s_n$ such that $s_n \cdot b = 0$.*

The proof uses both norm and multiplication principles. Since we already have a generalized norm principle, if there were a generalized multiplication principle we would be able to prove a generalization of Lemma 20.

Lemma 21. *Assume the generalized multiplication principle holds. Let $\alpha \in \mathcal{K}_n^M/p$ and $\beta \in \mathcal{K}_m^M/p$ be symbols. Then, $\alpha \cdot \beta = 0$ if and only if there are presentations $\alpha = a \cdot \alpha'$ and $\beta = b \cdot \beta'$, where $a, b \in \mathcal{K}_1^M(F)$ are such that $a \cdot b = 0$ (that is, b is a norm from the field extension $F[\sqrt[p]{a}]$).*

Proof. The "only if" part is clear. For the other direction, assume that $\alpha \in \mathcal{K}_n^M/p$, $\beta \in cK_m/p$ are symbols such that $\alpha \cdot \beta = 0$. By Theorem 12,

$$\beta = \sum_{[K_i:F]=p} \mathrm{N}_{K_i/F}(\beta_i)$$

for some splitting fields K_i and elements $\beta_i \in K_i$. By the generalized multiplication principle this is equal to $\sum_{i=1}^{M} \mathrm{N}_{K/F}(\gamma_i \cdot k_i)$, where K is a (single) splitting field for α of degree p, $k_i \in K$ and $\gamma_i \in F$. Thus, as α splits over $K = F[\sqrt[p]{b} : b \in F]$ we can write $\alpha = \alpha' \cdot b$. Now by the projection formula and the description of β we have that $\beta \cdot b = 0$, and so by Lemma 20 we can decompose $\beta = \beta' \cdot a$ such that $a \cdot b = 0$. $\square$

We now show that, at least for $p = n = m = 2$, the generalization of Lemma 20 implied in the last lemma is false. To this end we use the theory of quadratic forms and valuations.

Lemma 22. *Let F be a field of characteristic $\neq 2$, with a valuation $\nu : F \to \Gamma$, where Γ is a well ordered abelian group. Assume that for $a_1, \dots, a_n \in F^\times$, the images of $\nu(a_1), \dots, \nu(a_n)$ in $\Gamma/2\Gamma$ are pairwise different. Then the quadratic form $\varphi = \langle a_1, \dots, a_n\rangle$ is anisotropic.*

Proof. Write $\varphi(v) = \alpha_1^2 a_1 + \cdots + \alpha_n^2 a_n$. Notice that for $i \neq j$ we have that $\nu(\alpha_i^2 a_i) \neq \nu(\alpha_j^2 a_j)$ (for otherwise $\nu(a_i)$ would be equivalent to $\nu(a_j)$ in $\Gamma/2\Gamma$). Thus, $\nu(\varphi(v)) = \nu(\alpha_i^2 a_i)$ for some i such that $\alpha_i \neq 0$. Now,

$$\nu(\alpha_i^2 a_i) = 2\nu(\alpha_i) + \nu(a_i) \neq \infty,$$

so $\varphi(v) \neq 0$ unless $v = 0$. □

Now we can give a counterexample (suggested by Stephen Scully and communicated to us by Stefan Gille)

Proposition 23. *Let $F = \mathbb{Q}(x, y, z)$ and $\alpha = \langle\langle x, y\rangle\rangle$, $\beta = \langle\langle z, -x + yz\rangle\rangle$. Then $\alpha \perp \beta'$ is anisotropic, where β' is the pure subform of β.*

Proof. Take the (x, y, z)-adic valuation of F, with values in the discrete group $\Gamma = \mathbb{Z}^3$ ordered lexicographically from left to right. By the previous lemma, the quadratic forms α, β and $\phi = \alpha \perp \langle -z, x - yz\rangle$ are anisotropic, and so $\alpha \perp \beta' = \phi \perp \langle -xz + yz^2\rangle$ is anisotropic if and only if $-xz + yz^2$ is not a value of ϕ. Assume $\alpha_0^2(-xz+yz^2) = \alpha_1^2 - \alpha_2^2 x - \alpha_3^2 y - \alpha_4^2 z + \alpha_5^2 xy + \alpha_6^2(x - yz)$ for some $\alpha_i \in F$, where $\alpha_0 \neq 0$. Multiplying by a common denominator, we can assume $\alpha_i \in \mathbb{Q}[x, y, z]$. We can rewrite the equation as

$$z(-\alpha_0^2 x + \alpha_4^2 + \alpha_6^2 y) = -\alpha_0^2 yz^2 + \alpha_1^2 - \alpha_2^2 x - \alpha_3^2 y + \alpha_5^2 xy + \alpha_6^2 x.$$

Compering even and odd z-degree, we get that $-\alpha_0^2 x + \alpha_4^2 + \alpha_6^2 y = 0$, which contradicts the fact that α is anisotropic. □

Remark 24. Note that by Springer's theorem, the above example works even if we take prime to 2 closure of F.

Corollary 25. *There is no generalized multiplication principle for the case $p = m = n = 2$.*

Proof. Assume that the generalized multiplication principle holds. By the Milnor conjecture, consider the quadratic Pfister forms $\alpha = \langle\langle x, y\rangle\rangle$ and $\beta = \langle\langle z, -x + yz\rangle\rangle$ in I^2/I^3 over a prime to p closure of $F = \mathbb{Q}(x, y, z)$. We notice that

$$\alpha \cdot \beta = \langle\langle x, y, z, -x + yz\rangle\rangle \cong \langle\langle x, y, -yz, -x + yz\rangle\rangle \cong \langle\langle y, xyz, x - yz, -x + yz\rangle\rangle$$

is hyperbolic. Thus, by Lemma 21 there is a presentation $\beta = \langle\langle b\rangle\rangle \cdot \langle\langle t\rangle\rangle$ such that $\alpha \cdot \langle\langle b\rangle\rangle$ is hyperbolic. Since b is an entry of β if and only if b is a value of the pure subform β', and b is also a value of α, we get that $\alpha \perp \beta'$ is isotropic, in contradiction to Proposition 23. □

We note that even for an odd prime p and $F = \mathbb{Q}(\rho, x, y, z)$, where ρ is a primitive root of unity, we can define $\alpha = (x, y)$ and $\beta = (z, -x + yz)$ in $\mathcal{K}_2^M(F)/p$ and still get that $\alpha \cdot \beta = 0$. We conjecture that this should give a counterexample for the generalized multiplication principle for the case $n = m = 2$, p an odd prime. In order to prove it one would need to show there is no presentation $\alpha = a \cdot b$ such that $b \cdot \beta = 0$. It seems that considering α, β as symbol algebras in the Brauer group of F and using valuation theory one could show such a presentation is not possible, but we could not make it work. For example, if one can show that $\mathrm{Im}(\mathrm{Nrd}_\alpha) \cap \mathrm{Im}(\mathrm{Nrd}_\beta) = F^p$, it would imply the needed condition, but again, we could not do it.

References

[Gille and Szamuely 2006] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics **101**, Cambridge Univ. Press, 2006. MR Zbl

[Haesemeyer and Weibel 2009] C. Haesemeyer and C. Weibel, "Norm varieties and the chain lemma (after Markus Rost)", pp. 95–130 in *Algebraic topology*, edited by N. A. Baas et al., Abel Symp. **4**, Springer, 2009. MR Zbl

[Haile and Rowen 1995] D. Haile and L. H. Rowen, "Factorizations of polynomials over division algebras", *Algebra Colloq.* **2**:2 (1995), 145–156. MR Zbl

[Jacobson 1943] N. Jacobson, *The theory of rings*, AMS Mathematical Surveys **II**, Amer. Math. Soc., Providence, RI, 1943. MR Zbl

[Jacobson 1996] N. Jacobson, *Finite-dimensional division algebras over fields*, Springer, 1996. MR Zbl

[Matzri 2019] E. Matzri, "Higher triple Massey products and symbols", *J. Algebra* **527** (2019), 136–146. MR Zbl

[Merkurjev and Suslin 2010] A. Merkurjev and A. Suslin, "Motivic cohomology of the simplicial motive of a Rost variety", *J. Pure Appl. Algebra* **214**:11 (2010), 2017–2026. MR Zbl

[Orlov et al. 2007] D. Orlov, A. Vishik, and V. Voevodsky, "An exact sequence for $K_*^M/2$ with applications to quadratic forms", *Ann. of Math.* (2) **165**:1 (2007), 1–13. MR Zbl

[Pierce 1982] R. S. Pierce, *Associative algebras*, Graduate Texts in Mathematics **88**, Springer, 1982. MR Zbl

[Rost 2002] M. Rost, "Norm varieties and algebraic cobordism", pp. 77–85 in *Proceedings of the International Congress of Mathematicians* (Beijing, 2002), vol. II, edited by T. Li, Higher Ed. Press, Beijing, 2002. MR Zbl

[Suslin and Joukhovitski 2006] A. Suslin and S. Joukhovitski, "Norm varieties", *J. Pure Appl. Algebra* **206**:1-2 (2006), 245–276. MR Zbl

[Weibel and Zakharevich 2017] C. Weibel and I. Zakharevich, "Principal ideals in mod-ℓ Milnor K-theory", *J. Homotopy Relat. Struct.* **12**:4 (2017), 1033–1049. MR Zbl

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Excision in equivariant fibred G-theory

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This paper provides a generalization of excision theorems in controlled algebra in the context of equivariant G-theory with fibred control and families of bounded actions. It also states and proves several characteristic features of this theory such as existence of the fibred assembly and the fibrewise trivialization.

1. Introduction

The bounded K-theory construction due to Pedersen and Weibel [1985] has been shown to be extremely useful in the analysis of versions of the Novikov conjecture [Carlsson 1995; Carlsson and Goldfarb 2004a; Carlsson and Pedersen 1995; 1998; Ramras et al. 2014]. This conjecture asserts the split injectivity of a natural transformation called the *assembly*. The present paper is the culmination of a series of papers [Carlsson and Goldfarb 2011; 2016; 2019] that extend the techniques sufficient to address the much more difficult *Borel conjecture in algebraic K-theory for a group* Γ, which asserts that the K-theory assembly map $\alpha_\Gamma : B\Gamma_+ \wedge K(\mathbb{Z}) \to K(\mathbb{Z}[\Gamma])$ is an equivalence of spectra. What we have found is that substantial extensions are necessary.

Since the construction of the equivariant fibred G-theory is quite involved and technical, we provide the reader with a discussion of how we arrived at it. Recall that the integral K-theoretic Novikov conjecture asserts that the assembly map

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α_Γ can be identified with a split inclusion on a direct summand of the spectrum $K(\mathbb{Z}[\Gamma])$. Consider the basic geometric situation of a finitely generated group Γ acting properly and freely on $\mathbb{R}^n$. It is shown in [Carlsson 1995; 2005; Carlsson and Goldfarb 2004a; Carlsson and Pedersen 1995; 1998] that a successful strategy for proving the Novikov conjecture proceeds by recognizing the following.

(1) The spectra $B\Gamma_+ \wedge K(\mathbb{Z})$ and $K(\mathbb{Z}[\Gamma])$ can be realized as the fixed point spectra of Γ-actions on certain spectra $h^{\mathrm{lf}}_\Gamma(\mathbb{R}^n, K(\mathbb{Z}))$ and $K^{\mathrm{bdd}}(\mathbb{R}^n; \mathbb{Z})$. The spectrum $h^{\mathrm{lf}}_\Gamma(X, K(\mathbb{Z}))$ is an equivariant version of Borel–Moore homology with coefficients in the spectrum $K(\mathbb{Z})$, and has the property that for proper discontinuous free actions on locally compact spaces,
$$h^{\mathrm{lf}}_\Gamma(X, K(\mathbb{Z}))^\Gamma \cong h^{\mathrm{lf}}(X/\Gamma, K(\mathbb{Z})).$$
The spectrum $K^{\mathrm{bdd}}(\mathbb{R}^n, \mathbb{Z})$ is the bounded K-theory due to Pedersen and Weibel [1985]. It depends on a choice of Γ-invariant metric on $\mathbb{R}^n$.

(2) The assembly map α_Γ is the restriction to the fixed point sets of an equivariant map of spectra
$$\alpha^{\mathrm{amb}} : h^{\mathrm{lf}}(\mathbb{R}^n, K(\mathbb{Z})) \to K^{\mathrm{bdd}}(\mathbb{R}^n; \mathbb{Z}).$$

(3) The Γ-equivariant spectrum $S = h^{\mathrm{lf}}(\mathbb{R}^n, K(\mathbb{Z}))$ has the homotopy invariance property that the canonical map $S^\Gamma \to S^{h\Gamma}$ is an equivalence.

It turns out that α^{amb} can often be proved to be an equivalence by using the excision properties of the functor K^{bdd}. When one can do this for a metric on $\mathbb{R}^n$ which is Γ-invariant, the integral K-theoretic Novikov conjecture follows immediately, using the fact that if $f : X \to Y$ is a map of spaces with Γ-action, and f is an equivalence when regarded as a nonequivariant map, then the map $f^{h\Gamma}$ on homotopy fixed point sets is an equivalence.

Our approach to the K-theoretic Borel conjecture is to use similar techniques to prove that the map $\rho : K(\mathbb{Z}[\Gamma]) \to K^{\mathrm{bdd}}(\mathbb{R}^n, \mathbb{Z})^{h\Gamma} \cong B\Gamma_+ \wedge K(\mathbb{Z})$ can also be identified with an inclusion onto a spectrum summand. This result together with the Novikov conjecture will prove that α_Γ is an equivalence. Of course, as stated, this appears to be difficult since the homotopy fixed point set $W^{h\Gamma}$ of a spectrum with Γ-action is defined as the function spectrum of equivariant maps from $E\Gamma_+$ to W, and therefore is not in any sense finite dimensional or equipped with any reasonable geometric cell structures. However, due to the fact that we are working with stable homotopy theory, there is a proper version of Spanier–Whitehead duality that allows us to obtain a geometric model for the homotopy fixed point spectrum of a spectrum with Γ-action, when Γ is the fundamental group of a $K(\Gamma, 1)$-manifold M. Consider any embedding i of M in $\mathbb{R}^n$ for some n. We let N denote any open tubular neighborhood of $i(M)$. Clearly, $\pi_1(N) \cong \pi_1(M) \cong \Gamma$, and we consider the universal cover $\widetilde{N}$. Using a version of equivariant Spanier–Whitehead duality for free, proper Γ-spaces, it is possible to show that for any spectrum W with Γ-action, there is an equivalence $W^{h\Gamma} \cong h^{\mathrm{lf}}(\widetilde{N}, W)^{\Gamma}$.

To understand how we use this equivalence, we need to describe some of the properties of the construction K^{bdd}. For any commutative ring A, $K^{\mathrm{bdd}}(-, A)$ is a functor on a category of proper metric spaces $\mathfrak{M}$. It is functorial for *coarse* maps $f : X \to Y$ of metric spaces defined as maps that are both *proper*, in the sense that preimages of bounded subsets of Y are bounded in X, and *uniformly expansive*, in the sense that there is a function $c : \mathbb{R} \to \mathbb{R}$ such that if $x, x' \in X$ are any two points with $d(x, x') \leq t$, then $d(f(x), f(x')) \leq c(t)$. We let $\mathfrak{M}^{\infty}$ denote enlargement of the category $\mathfrak{M}$ to pairs (X, d), where the distance function d is permitted to take the value $+\infty$, and refer to objects of $\mathfrak{M}^{\infty}$ as generalized metric spaces. The axioms for a metric space extend naturally, and the functor K^{bdd} also extends to these generalized metric spaces. It is now immediate that the functor can also be extended to the category of simplicial objects in the category $\mathfrak{M}^{\infty}$. For a group Γ, we can consider the category $\mathfrak{M}^{\infty}_{\Gamma}$ of generalized metric spaces with Γ-actions by coarse maps. There is an equivariant version $K^{\mathrm{bdd}}_{\Gamma}$ of the functor K^{bdd} which is defined on $\mathfrak{M}^{\infty}_{\Gamma}$, and which carries a generalized metric space X with Γ-action to a spectrum with Γ-action. It is now possible to prove that we have a sequence of maps

$$\Sigma^n K^{\mathrm{bdd}}_{\Gamma}(\Gamma, A)^{h\Gamma} \simeq h^{\mathrm{lf}}_{\Gamma}(\widetilde{N}, K^{\mathrm{bdd}}_{\Gamma}(\Gamma, A))^{\Gamma} \to K^{\mathrm{bdd}}_{\Gamma}(\Gamma \times \widetilde{N}, A)^{\Gamma},$$

which means that we can now work geometrically in bounded K-theory to construct the splitting map. The main idea is to once again use excision properties to obtain information about $K^{\mathrm{bdd}}_{\Gamma}(\Gamma \times \widetilde{N}, A)^{\Gamma}$.

Remark 1.1. One observation about these properties is that the required excision theorems must be fibrewise in the $\widetilde{N}$-direction. The reason is that we must in an appropriate sense "leave Γ alone" since we are trying to detect $K(\mathbb{Z}[\Gamma])$. Secondly, the excision theorems must be equivariant since the Γ-action is what creates the group ring $\mathbb{Z}[\Gamma]$. The third observation is that the computation should end up with an appropriate suspension of $K(\mathbb{Z}[\Gamma])$, which was the source of the map ρ, so that the composition of all intermediate maps is an equivalence. This latter requirement is the source of very weak geometric conditions on Γ and algebraic conditions on the ground ring R that are needed and will be carried out in a separate paper.

As it stands, we do not have adequate excision properties for $K^{\mathrm{bdd}}_{\Gamma}(\Gamma \times \widetilde{N}, A)^{\Gamma}$ so that all of the three properties hold. However, we can construct further relaxation maps out to another construction we call *fibred homotopy fixed points* in G-theory which does enjoy such properties (Theorems 6.11 and 6.13). There are *three separate "axes" of relaxation* that we need, and it is the constructions of the present paper that will allow us to perform all three.

(1) *Bounded control to fibred bounded control.* We will need to use different notions of control on the morphisms in our category of modules. Of course, one could use the control on the product metric space $\Gamma \times \widetilde{N}$, but that does not possess the right properties. In [Carlsson and Goldfarb 2019] we proved that there is another notion of control which roughly insists that morphisms must be controlled in each fiber (copy of N), but where the bound may vary from fiber to fiber. Fibred control is the analogue of the notion of "parametrized homotopy theory" or "homotopy theory over a base", where X is the base and Y is the fiber; cf. [May and Sigurdsson 2006]. In the equivariant case, where X is equal to a group Γ regarded as a metric space using the word length metric, the fixed points of the Γ-action are analogous to the bundles over a classifying space $B\Gamma$ obtained from Γ-spaces Y by the construction $Y \to E\Gamma \times_\Gamma Y$. To realize our results, we will need excision properties holding for coverings of Y.

(2) *Free to nonfree modules.* We will need to enlarge the category of modules we consider. The coarse actions on metric spaces can no longer be assumed to be free, and so the fixed point spectra need to be modeled on a larger module category. In [Carlsson and Goldfarb 2011], we have defined bounded versions of G-theory and developed appropriate techniques for proving excision properties. The paper [Carlsson and Goldfarb 2019] constructs a fibrewise version of that theory, and in this paper we construct the actual equivariant fibrewise excision properties we require. The idea of relaxing the kinds of modules we deal with is very analogous to a situation studied in the context of localization in [Thomason and Trobaugh 1990]. It is well understood that localization theorems for K-theory are much more sensitive and difficult to construct than the corresponding results for G-theory.

(3) *A natural covering metric to left-bounded metric.* Even with the excision properties in place, we must also modify the metric on the $\widetilde{N}$-factor in the product $\Gamma \times \widetilde{N}$. To give intuition about this, we observe that the bundle $E\Gamma \times_\Gamma \widetilde{N} \to B\Gamma$ is a topologically trivial $\mathbb{R}^n$-bundle. We would like to have a situation where it is actually a bundle with structure group contained in the bounded automorphisms group of $\widetilde{N}$. That does not in general happen, but it is possible to modify the metric on $\widetilde{N}$ so that it does. The new metric will be smaller than or equal to the original metric. When this metric is used, we are able to work as if the action actually is trivial. This result is also proved in this paper (Theorem 7.4).

Numerous details had to be suppressed in this roughly accurate outline of how the results of this paper are used to prove the K-theoretic Borel conjecture. In the last section, we include a worked out example of the argument for the simple case of the infinite cyclic group. The details for the much more general case of a group with finite decomposition complexity will appear elsewhere.

The main goal of this paper is to prove excision results that incorporate all generalizations (1)–(3) above simultaneously. Because we will only need the excision results where the action on the space Y is bounded in the sense defined above, we will only prove them in that situation. That is, we prove excision theorems (Theorems 6.11 and 6.13) for equivariant G-theory with fibred control of bounded Γ-spaces Y. Additionally we obtain the results suggested above as part of equivariant fibred G-theory.

We will now state the versions of main results that allow us to do so concisely. The full relative versions are stated and proved in the course of the paper.

In what follows we use the following notation for enlargements in products of metric spaces. Given a subset U of $X \times Y$, a number $K \geq 0$, and a function $k : X \to [0, +\infty)$, let $U[K, k]$ be the subset of those points (x, y) for which there is (x', y') in U with $d(x, x') \leq K$ and $d(y, y') \leq k(x')$.

Definition 1.2. An object of the category $\mathbf{G}_X(Y)$ is a module F over a Noetherian ring R together with a filtration, also denoted by F, indexed by the entire power set $\mathcal{P}(X \times Y)$ and subject to a number of conditions:

- $F(X \times Y) = F$, $F(\varnothing) = 0$,
- $F(S)$ is a finitely generated submodule of F for every bounded subset $S \subset X \times Y$,
- F can be equipped with another filtration $\mathcal{F}$ indexed by $\mathcal{P}(Y) \times [0, \infty) \times [0, \infty)^X$ so that the value $\mathcal{F}(C, D, \delta)$ is nested between two submodules
$$F((X \times C)[D, \delta]) \quad \text{and} \quad F((X \times C)[D + K, \delta + k])$$
for some $K \geq 0$ and a function k,
- when the submodule $\mathcal{F} = \mathcal{F}(C, D, \delta)$ is given the standard induced (X, Y)-filtration defined by $\mathcal{F}(U) = \mathcal{F} \cap F(U)$, the result has the following property: there is a number $d \geq 0$ and a function $\Delta : X \to [0, +\infty)$ such that
  - for every subset S of X, the associated X-filtered module $\mathcal{F}_X$ satisfies
  $$\mathcal{F}_X(S) \subset \sum_{x \in S} \mathcal{F}_X(x[d]),$$
  - for each pair of subsets U_1 and U_2 of $X \times Y$, the (X, Y)-filtered module $\mathcal{F}$ satisfies both
  $$\mathcal{F}(U_1 \cup U_2) \subset \mathcal{F}(U_1[d, \Delta]) + \mathcal{F}(U_2[d, \Delta])$$
  and
  $$\mathcal{F}(U_1) \cap \mathcal{F}(U_2) \subset \mathcal{F}(U_1[d, \Delta] \cap U_2[d, \Delta]).$$

It should be emphasized that the auxiliary filtration $\mathcal{F}$ is not part of the structure of the object; there is no specific choice of a filtration that is specified.

A morphism $f : F \to G$ in $\mathbf{G}_X(Y)$ is an R-linear homomorphism $F(X \times Y) \to G(X \times Y)$ such that $fF(U) \subset G(U[b, \theta])$ for some number $b \geq 0$ and for some function θ, but for all subsets $U \subset X \times Y$. We will refer to the pair (b, θ) as control data for f.

We specialize to the case of $X = \Gamma$, a finitely generated group with a chosen word metric. Suppose Γ acts on Y via bounded coarse equivalences, in the sense that for every $\gamma \in \Gamma$, there exists an $R_\gamma > 0$ such that $d(y, \gamma(y)) \leq R_\gamma$ for all $y \in Y$. The diagonal action on $\Gamma \times Y$ induces an action on $\mathbf{G}_\Gamma(Y)$.

Definition 1.3. The *fibred homotopy fixed points* is the category $\mathbf{G}^{h\Gamma}(Y)$ with objects which are sets of data $(\{F_\gamma\}, \{\psi_\gamma\})$, where

- F_γ is an object of $\mathbf{G}_\Gamma(Y)$ for each γ in Γ,
- ψ_γ is an isomorphism $F_e \to F_\gamma$ in $\mathbf{G}_\Gamma(Y)$,
- ψ_γ has control data with $b = 0$ and $\theta = +\infty$,
- $\psi_e = \mathrm{id}$,
- $\psi_{\gamma_1\gamma_2} = \gamma_1\psi_{\gamma_2} \circ \psi_{\gamma_1}$ for all γ_1, γ_2 in Γ.

The morphisms $(\{F_\gamma\}, \{\psi_\gamma\}) \to (\{F'_\gamma\}, \{\psi'_\gamma\})$ of $\mathbf{G}^{h\Gamma}(Y)$ are collections of morphisms $\phi_\gamma : F_\gamma \to F'_\gamma$ in $\mathbf{G}_\Gamma(Y)$ such that $\phi_\gamma \circ \psi_\gamma = \psi'_\gamma \circ \phi_e$ for all γ.

Both of these categories can be given exact structures resembling the exact structures in G-theory. We obtain in Section 6 a nonconnective delooping of the equivariant K-theory of $\mathbf{G}_X(Y)$. The fixed points of this theory are modeled by the nonconnective K-theory of $\mathbf{G}^{h\Gamma}(Y)$ (Proposition 6.5). This fixed point spectrum will be denoted $\widetilde{G}^\Gamma(Y)^\Gamma$. As soon as one chooses any subset Y_1 of Y, there is a full subcategory of $\mathbf{G}^{h\Gamma}(Y)$ on objects where all relevant modules are supported on fibred enlargements of $\Gamma \times Y_1$ in $\Gamma \times Y$. This subcategory is invariant under bounded actions, and so we have a nonconnective fixed point spectrum $\widetilde{G}^\Gamma(Y)^\Gamma_{<Y_1}$. Similarly, for two subsets Y_1 and Y_2 of Y, restricting to modules supported on fibred enlargements of both $\Gamma \times Y_1$ and $\Gamma \times Y_2$ gives a spectrum $\widetilde{G}^\Gamma(Y)^\Gamma_{<Y_1,Y_2}$.

We can finally state the absolute version of the main theorem of this paper.

Theorem 1.4 (part of the equivariant fibred excision theorem, Theorem 6.11). *Suppose Y_1 and Y_2 are subsets of a metric space Y on which Γ acts by bounded coarse equivalences, and $Y = Y_1 \cup Y_2$. There is a homotopy pushout diagram of spectra*

$$\begin{array}{ccc}
\widetilde{G}^\Gamma(Y)^\Gamma_{<Y_1,Y_2} & \longrightarrow & \widetilde{G}^\Gamma(Y)^\Gamma_{<Y_1} \\
\big\downarrow & & \big\downarrow \\
\widetilde{G}^\Gamma(Y)^\Gamma_{<Y_2} & \longrightarrow & \widetilde{G}^\Gamma(Y)^\Gamma
\end{array}$$

where the maps are induced from the exact inclusions.

2. Homotopy fixed points in categories with action

Given an action of a group Γ on a space X, one has the subspace of fixed points X^{Γ}. This subspace often has geometric significance for the study of X and Γ. A different powerful idea in topology is to model an interesting space or spectrum as the fixed point space or spectrum X^{Γ} for a specifically designed X with an action by a related group Γ. In either case, there is always the homotopy fixed point spectrum $X^{h\Gamma}$, which is easier to understand than X^{Γ}, and the canonical reference map $\rho : X^{\Gamma} \to X^{h\Gamma}$.

Now suppose we have a group action on a category. This automatically produces an action on the nerve and therefore a space. Suppose the category is then fed into a machine such as the algebraic K-theory, and we are interested in the fixed points of the K-theory. Therefore we want to look at the homotopy fixed points. In many important cases it is possible to construct a spatial or categorical description of what we get. Thomason defined the lax limit category whose K-theory turns out to be exactly the homotopy fixed points of the old action.

Definition 2.1. Let $\boldsymbol{E\Gamma}$ be the category with the object set Γ and the unique morphism $\mu : \gamma_1 \to \gamma_2$ for any pair $\gamma_1, \gamma_2 \in \Gamma$. There is a left Γ-action on $\boldsymbol{E\Gamma}$ induced by the left multiplication in Γ. If $\mathcal{C}$ is a category with a left Γ-action, then the category of functors $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ is another category with the Γ-action given on objects by the formulas $\gamma(F)(\gamma') = \gamma F(\gamma^{-1}\gamma')$ and $\gamma(F)(\mu) = \gamma F(\gamma^{-1}\mu)$. It is nonequivariantly equivalent to $\mathcal{C}$.

The category $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ is an interesting and useful object in its own right. There are several manifestations of this, for example in the work of Mona Merling and coauthors [Guillou et al. 2017; Malkiewich and Merling 2019; Merling 2017] or the work of these authors [Carlsson 1995; Carlsson and Goldfarb 2004a; 2004b; 2013]. While in both applications it is crucial to work with the category itself, in this paper we concentrate on approximating the fixed points in $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$.

The following construction has been used by Thomason [1983]. We refer to it as the *homotopy fixed points of a category*, following Merling [2017].

Definition 2.2 (homotopy fixed points). The fixed point subcategory $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})^{\Gamma}$ of the category of functors $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ consists of equivariant functors and equivariant natural transformations. We denote it by $\mathcal{C}^{h\Gamma}$.

Explicitly, the objects of $\mathcal{C}^{h\Gamma}$ are the pairs (C, ψ) where C is an object of $\mathcal{C}$ and ψ is a function from Γ to the morphisms of $\mathcal{C}$ with $\psi(\gamma) \in \mathrm{Hom}(C, \gamma C)$ that satisfies $\psi(e) = \mathrm{id}$ for the identity group element e, and satisfies the cocycle identity $\psi(\gamma_1\gamma_2) = \gamma_1\psi(\gamma_2)\psi(\gamma_1)$ for all pairs γ_1 and γ_2 in Γ. These conditions imply that $\psi(\gamma)$ is always an isomorphism. The set of morphisms $(C, \psi) \to (C', \psi')$ consists of the morphisms $\phi : C \to C'$ in $\mathcal{C}$ such that the squares

$$\begin{array}{ccc} C & \xrightarrow{\psi(\gamma)} & \gamma C \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\gamma\phi} \\ C' & \xrightarrow{\psi'(\gamma)} & \gamma C' \end{array}$$

commute for all $\gamma \in \Gamma$.

Remark 2.3. As pointed out in [Merling 2017], the homotopy fixed points of a category are not necessarily identical with space level constructions. It is for example not true in general that the nerve of the homotopy fixed point category of a category is the same as the geometric homotopy fixed points of the nerve of the category. It is however true in the case where the category is a discrete Γ-groupoid.

Example 2.4. Let $\mathcal{C}$ denote a category, and equip it with the trivial action by Γ. Then the category $\mathcal{C}^{h\Gamma}$ is the category of representations of Γ in $\mathcal{C}$. In particular, if $\mathcal{C}$ is the category of R-modules for a commutative ring R, then $\mathcal{C}^{h\Gamma}$ may be identified with the category of (left) $R[\Gamma]$-modules.

Example 2.5. Let $F \subseteq E$ denote a Galois field extension, with Galois group G. We consider the skew group ring $\Lambda = E^t[G]$, and consider the category $\mathcal{C}_E$ whose objects are $E^t[G]$-modules and whose morphisms are the E-linear maps. There is a G-action on $\mathcal{C}_E$, which is the identity on objects and which is defined by the group action on the morphisms. In this case, $\mathcal{C}_E^{h\Gamma}$ is equivalent to the category of F-vector spaces.

In Example 2.4, we saw that the group ring of a group Γ with coefficients in a commutative ring R may be realized as the fixed point subcategory of the action of Γ on $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$, where $\mathcal{C}$ denotes the category of all R-modules. In many cases, however, it is important to understand the category of free and finitely generated left $R[\Gamma]$-modules as a fixed point category. This is the case in the papers [Carlsson 2005] and [Carlsson and Goldfarb 2004a], for instance, where the injectivity of the assembly map is proved in a large family of cases. In the case of these two papers, this is achieved by defining a subcategory of $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ by restricting the morphisms $\psi(\gamma)$. The restriction in this case arises by the selection of a subcategory of the category of all R-modules based on the Pedersen–Weibel construction, which is endowed with a filtration and an action of the group Γ. The restricted version of $\mathcal{C}_E^{h\Gamma}$ requires that all of the morphisms $\psi(\gamma)$ have filtration zero. In order to attack the surjectivity problem for the assembly, we are led to the construction of more general forms of restriction of the maps $\psi(\gamma)$. This leads us to the concept of the *relative homotopy fixed points of a category*, which we now define.

Definition 2.6 (relative homotopy fixed points). The category $\mathcal{C}^{h\Gamma}(\mathcal{M})$ is defined using input data consisting of a category $\mathcal{C}$ equipped with an action by a group Γ

and a subcategory $\mathcal{M} \subset \mathcal{C}$ closed under the action of Γ. It is the full subcategory of $\mathcal{C}^{h\Gamma}$ on objects (C, ψ) with the additional condition that $\psi(\gamma)$ is in $\mathcal{M}$ for all elements $\gamma \in \Gamma$.

Example 2.7. Clearly, if $\mathcal{M}$ is the entire category $\mathcal{C}$, the relative homotopy fixed points are the genuine homotopy fixed points.

Example 2.8. In the case where $\mathcal{C}$ is a filtered category, we can consider the situation where $\mathcal{M}$ is the subcategory of the filtration zero morphisms. This is the situation used in [Carlsson 2005] and [Carlsson and Goldfarb 2004a].

We will exploit the relative homotopy fixed points in two applications. The first construction required in [Carlsson 1995] allows us to model the K-theory of a group ring whenever the group has a finite classifying space. It is based on bounded K-theory of the group given a word metric with the isometric action on itself given by the left multiplication. It turns out that the categorical homotopy fixed point construction requires a constraint. We review that construction in Section 3.

3. Bounded K-theory and the K-theory of group rings

Bounded control is the simplest version of a "control condition" that can be imposed in various categories of modules, to which one can apply the algebraic K-theory construction. It was introduced in [Pedersen 1984] and [Pedersen and Weibel 1985] and has become crucial for K-theory computations in geometric topology.

Let X be a metric space and let R be an arbitrary associative ring with unity. We always assume that metric spaces are proper in the sense that closed bounded subsets are compact.

Definition 3.1. The objects of the category of *geometric R-modules over X* are locally finite functions F from points of X to the category of finitely generated free R-modules $\mathbf{Free}_{fg}(R)$. Following Pedersen and Weibel, we denote by F_x the module assigned to the point x of X and denote the object itself by writing down the collection $\{F_x\}$. The *local finiteness* condition requires precisely that for every bounded subset $S \subset X$ the restriction of F to S has finitely many nonzero modules as values.

Let d be the distance function in X. The morphisms $\phi : \{F_x\} \to \{G_x\}$ are collections of R-linear homomorphisms $\phi_{x,x'} : F_x \to G_{x'}$, for all x and x' in X, with the property that $\phi_{x,x'}$ is the zero homomorphism whenever $d(x, x') > D$ for some fixed real number $D = D(\phi) \geq 0$. One says that ϕ is *bounded by D*. The composition of two morphisms $\phi : \{F_x\} \to \{G_x\}$ and $\psi : \{G_x\} \to \{H_x\}$ is given by the formula

$$(\psi \circ \phi)_{x,x'} = \sum_{z \in X} \psi_{z,x'} \circ \phi_{x,z}. \tag{$*$}$$

This sum is finite because of the local finiteness property of G.

We will want to enlarge this category, and so we use instead an equivalent category $\mathcal{B}(X, R)$ that is better for this purpose.

The objects are functors $F : \mathcal{P}(X) \to \mathbf{Free}(R)$ from the power set $\mathcal{P}(X)$ to the category of free modules, both viewed as posets ordered by split inclusions. There are two additional requirements. For every bounded subset C of X the value $F(C)$ has to belong to the subcategory of finitely generated modules $\mathbf{Free}_{fg}(R)$. In the codomain, the values are required to satisfy the equality $F(S) = \bigoplus_{x \in S} F(x)$ for all $S \subset X$. The morphisms in this reformulation are R-linear homomorphisms $\phi : F(X) \to G(X)$ such that the components $\phi_{x,x'} : F(x) \to G(x')$ are zero whenever $d(x, x') > D$ for some D. The composition of two morphisms $\phi : F \to G$ and $\psi : G \to H$ is the usual composition of R-linear homomorphisms; its components are the maps $(\psi \circ \phi)_{x,x'}$ in the formula $(*)$ above.

Definition 3.2. A map $f : X \to Y$ between metric spaces is called *uniformly expansive* if there is a function $\lambda : [0, \infty) \to [0, \infty)$ such that

$$d_X(x_1, x_2) \le r \quad \text{implies} \quad d_Y(f(x_1), f(x_2)) \le \lambda(r).$$

A map f is *proper* if $f^{-1}(S)$ is a bounded subset of X for each bounded subset S of Y. We say f is a *coarse map* if it is uniformly expansive and proper.

Extensively used instances of coarse maps in geometry are quasi-isometries.

It is elementary to check that the geometric R-modules over X is an additive category and that coarse maps between metric spaces induce additive functors. A coarse map f is a *coarse equivalence* if there is a coarse map $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are bounded maps. It follows that an action of a group on a metric space by coarse equivalences induces an additive action on $\mathcal{B}(X, R)$.

We will treat the group Γ equipped with a finite generating set Ω closed under taking inverses as a metric space. The *word-length metric* $d = d_\Omega$ is induced from the condition that $d(\gamma, \gamma\omega) = 1$ whenever $\gamma \in \Gamma$ and $\omega \in \Omega$. It is well-known that varying Ω only changes Γ to a quasi-isometric metric space. The word-length metric makes Γ a proper metric space with a free Γ-action by isometries via left multiplication.

An important observation. A free action of Γ on X by isometries always gives a free action on $\mathcal{C} = \mathcal{B}(X, R)$. In contrast, $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ with the induced group action does have the subcategory $\mathcal{C}^{h\Gamma}$ of equivariant functors. These homotopy fixed points, however, are not the correct notion for modeling the finitely generated free modules over $R[G]$ and the K-theory of $R[G]$.

Definition 3.3. The category $\mathcal{B}^{\Gamma,0}(X, R)^\Gamma$ is the relative homotopy fixed point spectrum $\mathcal{C}^{h\Gamma}(\mathcal{M})$ with the following data: $\mathcal{C}$ is the category of geometric modules $\mathcal{B}(X, R)$ and $\mathcal{M}$ consists of those morphisms in $\mathcal{C}$ that are bounded by 0.

The additive category $\mathcal{B}^{\Gamma,0}(X, R)^{\Gamma}$ has the associated nonconnective K-theory spectrum $K^{-\infty}(X, R)^{\Gamma}$ constructed as in [Pedersen and Weibel 1985]. There is now the following desired identification.

Theorem 3.4. *Suppose Γ acts on X freely, properly discontinuously by isometries so that the orbit space X/Γ with the orbit metric is bounded.*

It follows that $K^{-\infty}(X, R)^{\Gamma}$ is weakly homotopy equivalent to the nonconnective spectrum $K^{-\infty}(R[\Gamma])$. The stable homotopy groups of the nonconnective spectrum are the Quillen K-groups of $R[G]$ in nonnegative dimensions and the negative K-groups of Bass in negative dimensions.

Proof. The result follows from Corollary VI.8 in [Carlsson 1995]. □

This geometric situation occurs, for example, when Γ acts cocompactly, freely properly discontinuously on a contractible connected Riemannian manifold X or when it acts on itself with a word metric via left multiplication.

4. Fibred homotopy fixed points in K-theory

Let $\mathcal{A}$ be an additive category. Generalizing Definition 3.1, one has the bounded category with coefficients in $\mathcal{A}$.

Notation 4.1. Given a subset S of a metric space and a number $k \geq 0$, $S[k]$ is used for the *k-enlargement* of S defined as the set of all points x with $d(x, S) \leq k$.

Recall that $\mathcal{A}$ is a subcategory of its cocompletion $\mathcal{A}^*$ which is closed under colimits. For example, a construction based on the presheaf category was given in [Kelly 1982, 6.23].

Definition 4.2. $\mathcal{B}(X, \mathcal{A})$ has objects which are covariant functors $F : \mathcal{P}(X) \to \mathcal{A}^*$ from the power set $\mathcal{P}(X)$ to $\mathcal{A}^*$, both ordered by inclusion. Just as in Definition 3.1, there are several requirements:

- $F(x)$ is an object of $\mathcal{A}$ for every point x in X,
- the resulting function $F : X \to \mathcal{A}$ is locally finite, so only finitely many values are nonzero when restricted to any compact subset of X,
- for all subsets $S \subset X$,
$$F(S) = \bigoplus_{x \in S} F(x),$$
- the inclusion $F(S \subset X)$ is onto a direct summand for each subset S.

A morphism in $\mathcal{B}(X, \mathcal{A})$ is a morphism $\phi : F(X) \to G(X)$ in $\mathcal{A}^*$ with a number $D \geq 0$ such that ϕ restricted to $F(S)$ factors through $G(S[D])$ for all $S \subset X$. We say a morphism which admits such a number D is *D-controlled*.

This context, which produces a category isomorphic to $\mathcal{B}(X, R)$ when $\mathcal{A}$ is the category of free finitely generated R-modules, allows us to iterate the bounded control construction as follows.

Definition 4.3 (fibred control for geometric modules). Given two metric spaces X and Y and any ring R, the category $\mathcal{B}_X(Y, R)$, or simply $\mathcal{B}_X(Y)$ when the choice of ring R is clear, is the bounded category $\mathcal{B}(X, \mathcal{A})$ with $\mathcal{A} = \mathcal{B}(Y, R)$.

Among many options for relativizing homotopy fixed points in this setting, there is one of specific interest.

Let $\mathcal{A} = \mathcal{B}(Y, R)$ as before and $\mathcal{A}' = \mathbf{Mod}(R)$ be the category of arbitrary R-modules. There is a *forget control* functor $t : \mathcal{B}(Y, R) \to \mathbf{Mod}(R)$ which only remembers that the objects are R-modules and the morphisms are R-linear homomorphisms. From t we may induce the functor $T : \mathcal{B}(X, \mathcal{A}) \to \mathcal{B}(X, \mathcal{A}')$.

For this construction we assume that Γ acts on X by isometries and so, therefore, on $\mathcal{B}(X, \mathcal{A}')$. On the other hand, we allow the action of Γ on Y to be by coarse equivalences. This can also be used to induce an action on $\mathcal{B}(X, \mathcal{A})$.

Definition 4.4 (fibred homotopy fixed points in bounded K-theory). These are relative homotopy fixed points with the following choice of ingredients:

- the category $\mathcal{C}$ is $\mathcal{B}_X(Y, R)$,
- the subcategory $\mathcal{M}$ consists of all morphisms ϕ such that $T(\phi)$ is a controlled morphism bounded by 0.

Notation 4.5. When X is the group Γ itself with the left multiplication action and the word metric with respect to some choice of a finite set of generators, we obtain a particularly useful case of this construction. We use the special notation $\mathcal{B}^{h\Gamma}(Y)$ for the fibred homotopy fixed points $\mathcal{C}^{h\Gamma}(\mathcal{M})$ and $K_p^{\Gamma}(Y)$ for the nonconnective K-theory spectrum of $\mathcal{B}^{h\Gamma}(Y)$.

5. Summary of bounded G-theory with fibred control

A comprehensive exposition of bounded G-theory with fibred control is available in [Carlsson and Goldfarb 2019]. Compared to K-theory with fibred control from Definition 4.3, G-theory replaces free modules with arbitrary modules over a Noetherian ring R and replaces the split exact sequences with a more general kind of exact sequence. This is a summary of that theory and a number of facts in the form we can refer to in the next section.

Throughout the rest of the paper, R will be a Noetherian ring.

At the basic level, bounded G-theory with fibred control is an analogue of the algebraic K-theory of $\mathcal{B}_X(Y, R)$ locally modeled on finitely generated R-modules. The result is an exact category $\mathbf{B}_X(Y)$ where the exact sequences are not necessarily split but which contains $\mathcal{B}_X(Y)$ as an exact subcategory.

Definition 5.1. Given an R-module F, an (X, Y)-*filtration* of F is a covariant functor $\mathcal{P}(X \times Y) \to \mathcal{I}(F)$ from the power set of the product metric space to the partially ordered family of R-submodules of F, both ordered by inclusion. It is convenient to denote the value of this functor on a subset $U \subset X \times Y$ by $F(U)$ and assume that $F(X \times Y) = F$ and $F(\varnothing) = 0$.

The associated X-filtered R-module F_X is given by $F_X(S) = F(S \times Y)$. Similarly, for each subset $S \subset X$, one has the Y-filtered R-module F^S given by $F^S(T) = F(S \times T)$. In particular, $F^X(T) = F(X \times T)$.

Notation 5.2. We will use the following notation generalizing enlargements in a metric space. Given a subset U of $X \times Y$ and a function $k : X \to [0, +\infty)$, let

$$U[k] = \{(x, y) \in X \times Y \mid \text{there is } (x, y') \in U \text{ with } d(y, y') \le k(x)\}.$$

If in addition we are given a number $K \ge 0$ then

$$U[K, k] = \{(x, y) \in X \times Y \mid \text{ there is } (x', y) \in U[k] \text{ with } d(x, x') \le K\}.$$

For a product set $U = S \times T$, it is more convenient to use the notation $(S, T)[K, k]$ in place of $(S \times T)[K, k]$. We refer to the pair (K, k) in the notation $U[K, k]$ as the *enlargement data*.

Let x_0 be a fixed point in X. Given a monotone function $h : [0, +\infty) \to [0, +\infty)$, there is a function $h_{x_0} : X \to [0, +\infty)$ defined by

$$h_{x_0}(x) = h(d_X(x_0, x)).$$

Definition 5.3. Given two (X, Y)-filtered modules F and G, an R-homomorphism $f : F(X \times Y) \to G(X \times Y)$ is *boundedly controlled* if there are a number $b \ge 0$ and a monotone function $\theta : [0, +\infty) \to [0, +\infty)$ such that

$$f F(U) \subset G(U[b, \theta_{x_0}]) \tag{†}$$

for all subsets $U \subset X \times Y$ and some choice of $x_0 \in X$. It is easy to see that this condition is independent of the choice of x_0. If a homomorphism f is boundedly controlled with respect to some choice of parameters b and θ, we say that f is (b, θ)-*controlled*.

The *unrestricted fibred bounded category* $\mathbf{U}_X(Y)$ has (X, Y)-filtered modules as objects and the boundedly controlled homomorphisms as morphisms.

Theorem 3.1.6 of [Carlsson and Goldfarb 2019] shows that $\mathbf{U}_X(Y)$ is a cocomplete semiabelian category. When Y is the one point space, this construction recovers the controlled category $\mathbf{U}(X, R)$ of X-filtered R-modules used to construct bounded G-theory in [Carlsson and Goldfarb 2011] and [Carlsson and Goldfarb 2019, Chapter 2]. In this case, boundedly controlled homomorphisms are characterized by a single parameter b, so one can specify that by abbreviating the term to simply *b-controlled*. The construction of an X-filtration F_X from a given (X, Y)-filtration in Definition 5.1 allows us to view a (b, θ)-controlled homomorphism in $\mathbf{U}_X(Y)$ as a b-controlled homomorphism in $\mathbf{U}(X, R)$ via the forgetful functor $T : \mathbf{U}_X(Y) \to \mathbf{U}(X, R)$.

We now want to restrict to a subcategory of $\mathbf{U}_X(Y)$ that is full on objects with particular properties. This process consists of two steps that result in a theory with better localization properties.

Definition 5.4. An (X, Y)-filtered module F is called

- *split* or (D, Δ)-*split* if there is a number $D \geq 0$ and a monotone function $\Delta : [0, +\infty) \to [0, +\infty)$ such that
$$F(U_1 \cup U_2) \subset F(U_1[D, \Delta_{x_0}]) + F(U_2[D, \Delta_{x_0}])$$
for each pair of subsets U_1 and U_2 of $X \times Y$,
- *lean/split* or (D, Δ')-*lean/split* if there is a number $D \geq 0$ and a monotone function $\Delta' : [0, +\infty) \to [0, +\infty)$ such that
  - the X-filtered module F_X is *D-lean*, in the sense that
  $$F_X(S) \subset \sum_{x \in S} F_X(x[D])$$
  for every subset S of X, while
  - the (X, Y)-filtered module F is (D, Δ')-split,
- *insular* or (d, δ)-*insular* if there is a number $d \geq 0$ and a monotone function $\delta : [0, +\infty) \to [0, +\infty)$ such that
$$F(U_1) \cap F(U_2) \subset F(U_1[d, \delta_{x_0}] \cap U_2[d, \delta_{x_0}])$$
for each pair of subsets U_1 and U_2 of $X \times Y$.

There are two subcategories nested in $\mathbf{U}_X(Y)$. The category $\mathbf{LS}_X(Y)$ is the full subcategory of $\mathbf{U}_X(Y)$ on objects F that are lean/split and insular. The category $\mathbf{B}_X(Y)$ is the full subcategory of $\mathbf{LS}_X(Y)$ on objects F such that $F(U)$ is a finitely generated submodule whenever $U \subset X \times Y$ is bounded.

We proceed to define appropriate exact structures in these categories. The admissible monomorphisms are precisely the morphisms isomorphic in $\mathbf{U}_X(Y)$ to the filtrationwise monomorphisms and the admissible epimorphisms are those morphisms isomorphic to the filtrationwise epimorphisms. In other words, the exact structure $\mathcal{E}$ in $\mathbf{U}_X(Y)$ consists of sequences isomorphic to those

$$E^{\boldsymbol{\cdot}} : \quad E' \xrightarrow{i} E \xrightarrow{j} E''$$

which possess filtrationwise restrictions

$$E^{\bullet}(U): \quad E'(U) \xrightarrow{i} E(U) \xrightarrow{j} E''(U)$$

for all subsets $U \subset (X, Y)$, and each $E^{\bullet}(U)$ is an exact sequence of R-modules.

Both $\mathbf{LS}_X(Y)$ and $\mathbf{B}_X(Y)$ are closed under extensions in $\mathbf{U}_X(Y)$. Therefore, they are themselves exact categories, and the inclusion $\mathcal{B}_X(Y) \to \mathbf{B}_X(Y)$ is an exact embedding, as we projected.

There is a useful invariant of a finitely generated group Γ that is defined in terms of the exact category $\mathbf{B}_\Gamma(\text{point})$ in [Carlsson and Goldfarb 2016]. Here Γ can be given the word metric associated to any of the finite generating sets. The left multiplication action gives an action of Γ on $\mathbf{B}_\Gamma(\text{point})$.

Recall that Theorem 3.4 provides an interpretation to the K-theory of a group ring $R[\Gamma]$ in terms of relative homotopy fixed points of the additive category $\mathcal{B}(X, R)$, which can be viewed as $\mathcal{B}_\Gamma(\text{point})$.

Example 5.5 (bounded G-theory of a finitely generated group). In the case where $\mathcal{C}$ is the exact category $\mathbf{B}_\Gamma(\text{point})$ and $\mathcal{M}$ is the subcategory of the filtration zero morphisms, the *bounded G-theory of* Γ is defined to be the nonconnective K-theory of the relative homotopy fixed points $\mathbf{B}_\Gamma(\text{point})^{h\Gamma}$, denoted $G^{-\infty}(R[\Gamma])$.

Notice that this definition makes sense even when the group ring is not Noetherian, unlike the much more restrictive situation with the usual G-theory defined only for Noetherian rings.

Theorem 5.6. *There is an exact subcategory of finitely generated Γ-modules for an arbitrary finitely generated group Γ such that its relative homotopy fixed points have Quillen K-theory with features similar to G-theory of group rings. In particular, it has a Cartan map from the K-theory of $R[\Gamma]$.*

Proof. The category is equivalent to $\mathbf{B}_\Gamma(\text{point})$. We refer to Sections 2 and 3 of [Carlsson and Goldfarb 2016] for details. The clear resemblance to Definition 3.3 and the identification of $\mathcal{B}^{\Gamma,0}(X, R)^\Gamma$ with $\mathcal{B}_\Gamma(\text{point})^{h\Gamma}$ allow us to induce the Cartan map $K^{-\infty}(R[\Gamma]) \to G^{-\infty}(R[\Gamma])$ from the exact inclusion $\mathcal{B}_\Gamma(\text{point}) \to \mathbf{B}_\Gamma(\text{point})$ above. $\square$

Suppose C is a subset of Y. Let $\mathbf{B}_X(Y)_{<C}$ be the full subcategory of $\mathbf{B}_X(Y)$ on objects F such that

$$F(X, Y) \subset F((X, C)[r, \rho_{x_0}]))$$

for some number $r \geq 0$ and an order preserving function $\rho : [0, +\infty) \to [0, +\infty)$.

Recall that a *Serre subcategory* of an exact category is a full subcategory which is closed under exact extensions and closed under passage to admissible subobjects and admissible quotients. Proposition 3.3.3 of [Carlsson and Goldfarb 2019] verifies that $\mathbf{B}_X(Y)_{<C}$ is a Serre subcategory of $\mathbf{B}_X(Y)$.

The second step in restricting to subcategories with good localization properties is done via introducing a structure called a grading.

Given an arbitrary R-submodule F' of F in $\mathbf{U}_X(Y)$, we can assign to F' the *standard* (X, Y)*-filtration* $F'(U) = F(U) \cap F'$.

Let $\mathcal{M}^{\geq 0}$ be the set of all monotone functions $\delta : [0, +\infty) \to [0, +\infty)$.

Let $\mathcal{P}_X(Y)$ be the subcategory of $\mathcal{P}(X, Y)$ consisting of all subsets of the form $(X, C)[D, \delta_{x_0}]$ for some choices of a subset $C \subset Y$, a number $D \geq 0$, and a function $\delta \in \mathcal{M}^{\geq 0}$.

Definition 5.7. Given an object F of $\mathbf{B}_X(Y)$, a Y*-grading* of F is a functor

$$\mathcal{F} : \mathcal{P}_X(Y) \to \mathcal{I}(F)$$

with the following properties:

- the submodule $\mathcal{F}((X, C)[D, \delta_{x_0}])$, with the standard (X, Y)-filtration induced from F, is an object of $\mathbf{B}_X(Y)$,
- there is an enlargement data (K, k) such that

  $$F((X, C)[D, \delta_{x_0}]) \subset \mathcal{F}((X, C)[D, \delta_{x_0}]) \subset F((X, C)[D + K, \delta_{x_0} + k_{x_0}]),$$

  for all subsets in $\mathcal{P}_X(Y)$.

We say that an object F of $\mathbf{B}_X(Y)$ is Y*-graded* if there exists a Y-grading of F, but the grading itself is not specified, and define $\mathbf{G}_X(Y)$ as the full subcategory of $\mathbf{B}_X(Y)$ on Y-graded filtered modules.

The category $\mathbf{G}_X(Y)$ is of major importance. It is the category to which we will apply the relative homotopy fixed points construction in Definition 6.4. It is given in several stages, so the reader may find it helpful to refer to a compressed Definition 1.2 of this category in the introduction. For some examples of interesting nonprojective objects from $\mathbf{G}_X(Y)$ in an equivariant setting, we refer to Example 4.2 in [Carlsson and Goldfarb 2016]. It is also true that the category $\mathcal{B}_X(Y)$ introduced in Definition 4.3 is contained in $\mathbf{G}_X(Y)$.

Proposition 5.8. *$\mathcal{B}_X(Y)$ is a full exact subcategory of $\mathbf{G}_X(Y)$.*

Proof. The (X, Y)-filtration of the objects in $\mathcal{B}_X(Y)$ is given by

$$F(S) = \bigoplus_{x \in S} F(x).$$

This ensures that for any pair of subsets $T \subset S$ of $X \times Y$ one has

$$F(S) = F(T) \oplus F(S \setminus T).$$

This splitting is the reason the object F possesses the required gradings with all required properties on the nose, with the enlargement data all equal to 0. The structure maps are the boundedly controlled inclusions and projections onto direct summands. $\square$

We summarize some additional results from Section 3.4 of [Carlsson and Goldfarb 2019].

Theorem 5.9. *The subcategory $\mathbf{G}_X(Y)$ is closed under both isomorphisms and exact extensions in $\mathbf{B}_X(Y)$. Therefore, $\mathbf{G}_X(Y)$ is an exact subcategory of $\mathbf{B}_X(Y)$.*

The restriction to Y-gradings in $\mathbf{B}_X(Y)_{<C}$ gives a full exact subcategory $\mathbf{G}_X(Y)_{<C}$ which is a Serre subcategory of $\mathbf{G}_X(Y)$.

Given a graded object F in $\mathbf{G}_X(Y)$, we assume that F is (D, Δ')-lean/split and (d, δ)-insular and is graded by $\mathcal{F}$. For a subset U from the family $\mathcal{P}_X(Y)$, the submodule $\mathcal{F}(U)$ has the following properties:

(1) *$\mathcal{F}(U)$ is graded by $\mathcal{F}_U(T) = \mathcal{F}(U) \cap \mathcal{F}(T)$,*

(2) *$F(U) \subset \mathcal{F}(U) \subset F(U[K, k])$ for some fixed enlargement data (K, k),*

(3) *if $q : F \to H$ is the quotient of the inclusion $i : \mathcal{F}(U) \to F$ in $\mathbf{B}_X(Y)$ and F is (D, Δ')-lean/split, then H is supported on $(X \setminus U)[2D, 2\Delta']$,*

(4) *$H(U[-2D - 2d, -2\Delta' - 2\delta]) = 0$.*

We assume that the reader is familiar with Quillen K-theory of exact categories. This theory can be applied to both $\mathbf{G}_X(Y)$ and $\mathbf{G}_X(Y)_{<C}$. The result can be viewed as spectra $G_X(Y)$ and $G_X(Y)_{<C}$. The stable homotopy groups of these spectra are the Quillen K-groups of the exact categories.

Finally, the main goal of this section is a homotopy fibration

$$G_X(Y)_{<C} \to G_X(Y) \to G_X(Y, C),$$

where $G_X(Y, C)$ is the K-theory of a certain quotient category $\mathbf{G}_X(Y)/\,\mathbf{G}_X(Y)_{<C}$.

For simplicity we use the notation $\mathbf{G}$ for $\mathbf{G}_X(Y)$ and $\mathbf{C}$ for the Serre subcategory $\mathbf{G}_X(Y)_{<C}$ of $\mathbf{G}$.

There is a class of *weak equivalences* $\Sigma(C)$ in $\mathbf{G}$ which consist of all finite compositions of admissible monomorphisms with cokernels in $\mathbf{C}$ and admissible epimorphisms with kernels in $\mathbf{C}$. We need the class $\Sigma(C)$ to admit calculus of right fractions. This follows from [Schlichting 2004, Lemma 1.13] and the fact that $\mathbf{C}$ in $\mathbf{G}$ is right filtering, in the sense that each morphism $f : F_1 \to F_2$ in $\mathbf{G}$, where F_2 is an object of $\mathbf{C}$, factors through an admissible epimorphism $e : F_1 \to \overline{F}_2$, where $\overline{F}_2$ is in $\mathbf{C}$. The latter fact is [Carlsson and Goldfarb 2019, Lemma 3.5.6].

The category $\mathbf{G}/\mathbf{C}$ is the localization $\mathbf{G}[\Sigma(C)^{-1}]$. From [Carlsson and Goldfarb 2019, Theorem 3.5.8], it is an exact category where the short sequences are isomorphic to images of exact sequences from $\mathbf{G}$.

There is an intrinsic reformulation of the homotopy fibration because the essential full image of the evident inclusion of $\mathbf{G}_X(C)$ in $\mathbf{G}$ is precisely $\mathbf{C}$. This gives a

homotopy fibration
$$G_X(C) \to G_X(Y) \to G_X(Y, C).$$

One quick consequence is the ability to relativize the old constructions. If Y' is any subset of Y, one obtains the relative theory $G_X(Y, Y')$.

Another easy application is a nonconnective delooping that applies to all of the theories we have defined. For example in the basic case,
$$G_X^{-\infty}(Y) = \underset{\substack{\longrightarrow \\ k>0}}{\text{hocolim}}\ \Omega^k G_X(Y \times \mathbb{R}^k).$$
This uses the usual Eilenberg swindle trick and can be seen in Section 4.2 of [Carlsson and Goldfarb 2019].

6. Fibrewise excision in equivariant fibred G-theory

It is well-known that Quillen K-theory of an exact category can be obtained equivalently as Waldhausen's K-theory of bounded chain complexes in the category. The cofibrations are then the chain maps which are the degreewise admissible monomorphisms. The weak equivalences are the chain maps whose mapping cones are homotopy equivalent to acyclic complexes. An exposition with a number of details verified specifically for bounded G-theory can be found in [Carlsson and Goldfarb 2011, Section 4]. The Waldhausen theory setting is crucial in proving the excision theorem in that the approximation theorem [Carlsson and Goldfarb 2011, Theorem 4.5] becomes essential. We indicate passage from an exact category to the derived category of bounded chain complexes by prefixing "ch" in front of the name of the exact category.

We proceed to define the equivariant fibred G-theory. The basic setting consists of

- two proper metric spaces X and Y,
- an arbitrary subset Y' of Y,
- a Γ-action on X by isometries, and
- a bounded action of Γ on Y. This is an action such that for each γ in Γ the set of real numbers $W_\gamma = \{d(x, \gamma(x))\}$ is bounded from above.

Remark 6.1. In a number of situations, we will be specifying subcategories closed under the Γ-action by subsets that are arbitrary, and therefore certainly not closed under the action. This works due to the boundedness of the action. For example, if we have a subset $C \subseteq X$ and define a subcategory as the set of modules supported on some neighborhood of C, then this subcategory is closed under the Γ-action provided the action is bounded. This would definitely not hold were the action not bounded.

Consider the exact category $\mathbf{G}_\Gamma(Y)$ with the induced action by Γ, in the case X is the group Γ with a word metric, acting on itself by isometries via the left multiplication. Since the action on Y is bounded, we have the quotient exact category $\mathbf{G}_\Gamma(Y, Y')$.

Notation 6.2. If Z is another arbitrary subset of Y, it is also useful to consider the full exact subcategory $\mathbf{G}_\Gamma(Y, Y')_{<Z}$, which we denote $\mathbf{G}_\Gamma(Y, Y', Z)$.

Recasting the definition from Definition 4.4, we define the Waldhausen category $\mathcal{G}_{\Gamma,0}(Y, Y', Z)$ to be the full subcategory of $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathrm{ch}\,\mathbf{G}_\Gamma(Y, Y', Z))$ on those functors that send morphisms in $\boldsymbol{E\Gamma}$ to degreewise 0-controlled homomorphisms of Γ-filtered modules.

Definition 6.3. The *equivariant fibred G-theory* is

$$G^\Gamma(Y, Y', Z) = \Omega K(|wS.\mathcal{G}_{\Gamma,0}(Y, Y', Z)|).$$

This is a functor from the category of triples (Y, Y', Z), where both Y' and Z are subspaces of Y but not necessarily subspaces of each other, and uniformly expansive maps of triples to the category of spectra.

Now we turn to the construction of fibred homotopy fixed points. There is a forget control functor $T : \mathbf{G}_X(Y, Y', Z) \to \mathbf{U}_X(Y, Y', Z)$ sending F to F_X. Since Γ acts on X by isometries, it also acts on $\mathbf{U}_X(Y, Y', Z)$. The combination of this action and a bounded action on Y induces an action on $\mathbf{G}_X(Y, Y', Z)$. With these choices, T is an equivariant functor.

Definition 6.4 (fibred homotopy fixed points in bounded G-theory). This is a special case of relative homotopy fixed points, as defined in Definition 4.4, with the choices of $\mathcal{C}$ and $\mathcal{M}$ as follows:

- the category $\mathcal{C}$ is $\mathbf{G}_X(Y, Y', Z)$,
- the subcategory $\mathcal{M}$ consists of all controlled morphisms ϕ in $\mathcal{C}$ with the property that $T(\phi)$ is bounded by 0 as homomorphisms controlled over X.

Let us recapitulate what this definition entails in the case X is the group Γ with a word metric.

The *fibred homotopy fixed points of a triple* (Y, Y', Z) is the category $\mathbf{G}^{h\Gamma}(Y, Y', Z)$ with objects which are sets of data $(\{F_\gamma\}, \{\psi_\gamma\})$ where

- F_γ is an object of $\mathbf{G}_\Gamma(Y, Y', Z)$ for each γ in Γ,
- ψ_γ is an isomorphism $F_e \to F_\gamma$ in $\mathbf{G}_\Gamma(Y, Y', Z)$,
- ψ_γ is 0-controlled when viewed as a morphism in $\mathbf{U}_\Gamma(Y, Y', Z)$,
- $\psi_e = \mathrm{id}$,
- $\psi_{\gamma_1\gamma_2} = \gamma_1\psi_{\gamma_2} \circ \psi_{\gamma_1}$ for all γ_1, γ_2 in Γ.

A morphism $(\{F_\gamma\}, \{\psi_\gamma\}) \to (\{F'_\gamma\}, \{\psi'_\gamma\})$ is a collection of morphisms $\phi_\gamma : F_\gamma \to F'_\gamma$ in $\mathbf{G}_\Gamma(Y, Y', Z)$ such that the squares

$$\begin{tikzcd} F_e \arrow[r, "\psi_\gamma"] \arrow[d, "\phi_e"'] & F_\gamma \arrow[d, "\phi_\gamma"] \\ F'_e \arrow[r, "\psi'_\gamma"] & F'_\gamma \end{tikzcd}$$

commute for all γ.

The exact structure on $\mathbf{G}^{h\Gamma}(Y, Y', Z)$ is induced from that on $\mathbf{G}_\Gamma(Y, Y', Z)$ as follows. A morphism ϕ in $\mathbf{G}^{h\Gamma}(Y, Y', Z)$ is an admissible monomorphism if $\phi_e : F \to F'$ is an admissible monomorphism in $\mathbf{G}_\Gamma(Y, Y', Z)$. This of course implies that all structure maps ϕ_γ are admissible monomorphisms. Similarly, a morphism ϕ is an admissible epimorphism if $\phi_e : F \to F'$ is an admissible epimorphism. This gives $\mathbf{G}^{h\Gamma}(Y, Y', Z)$ an exact structure.

Since the induced Γ-action on $S.\mathcal{G}_{\Gamma,0}(Y, Y', Z)$ commutes with taking fixed points, we have the following fact.

Proposition 6.5. *The fixed point spectrum $G^\Gamma(Y, Y', Z)^\Gamma$ is equivalent to the K-theory of the relative homotopy fixed point category $\mathbf{G}^{h\Gamma}(Y, Y', Z)$.*

We proceed to consider multiple bounded actions of Γ on Y. Let $\beta(Y)$ be the set of all such actions. Let $\mathcal{F}$ be the functor that assigns to a set Z the partially ordered set of finite subsets of Z.

Definition 6.6. For any S in $\mathcal{F}(\beta(Y))$ we define Y_S as the metric space which is the disjoint union $\bigsqcup_{s \in S} Y_s$, where Y_s are copies of Y with the specified action. The metric on Y_S is induced by the requirement that it restricts to the metric from Y in each Y_s and for the same point y in different components the distance $d(y_s, y_{s'})$ equals 1.

Clearly, the action of Γ on Y_S is bounded.

As a consequence of Proposition 6.5, for each choice of finite subset S of $\beta(Y)$, the spectrum $G^\Gamma(Y_S, Y'_S, Z)^\Gamma$ is the Quillen K-theory spectrum of $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)$, where Z is a subset of Y_S.

Theorem 6.7. *Let C be an arbitrary subset of Y. There is a homotopy fibration*

$$G^\Gamma(Y_S, Y'_S, Z)^\Gamma_{<C} \to G^\Gamma(Y_S, Y'_S, Z)^\Gamma \to G^\Gamma(Y_S, Y'_S, Z)^\Gamma_{>C},$$

where $G^\Gamma(Y_S, Y'_S, Z)^\Gamma_{>C}$ stands for K-theory of the exact quotient $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)_{>C}$ of $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)$ by $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)_{<C}$. In the absolute case, there is an equivalence $\mathbf{G}^{h\Gamma}(Y)_{<C} \simeq \mathbf{G}^{h\Gamma}(C)$, and so there is a homotopy fibration

$$G^\Gamma(C)^\Gamma \to G^\Gamma(Y)^\Gamma \to G^\Gamma(Y)^\Gamma_{>C}.$$

Proof. In view of Remark 6.1, the fact that $\mathbf{G}_\Gamma(Y_S, Y'_S, Z)_{<C}$ is an idempotent complete Serre subcategory of $\mathbf{G}_\Gamma(Y_S, Y'_S, Z)$ implies immediately that $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)^{h\Gamma}_{<C}$ is an idempotent complete Serre subcategory of $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)$. The main technical result of [Schlichting 2004] is a fibration theorem which requires $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)_{<C}$ to satisfy two additional properties: right filtering and right s-filtering. Both of these properties follow directly from the estimates in the proofs of Lemma 3.5.6 and Theorem 3.5.8 in [Carlsson and Goldfarb 2019]. □

Our first application of the fibration is to deloop $G^\Gamma(Y_S, Y'_S, Z)^\Gamma$ and related spectra following the strategy of [Pedersen and Weibel 1985].

Let $\mathbb{R}$, $\mathbb{R}^{\geq 0}$, and $\mathbb{R}^{\leq 0}$ denote the metric spaces of the reals, the nonnegative reals, and the nonpositive reals with the Euclidean metric. Then there is the following map of homotopy fibrations:

$$\begin{tikzcd} G^\Gamma(Y_S)^\Gamma \arrow[r] \arrow[d] & G^\Gamma(Y_S \times \mathbb{R}^{\geq 0})^\Gamma \arrow[r] \arrow[d] & G^\Gamma(Y_S \times \mathbb{R}^{\geq 0})^\Gamma_{>Y_S \times 0} \arrow[d, "K(I)"] \\ G^\Gamma(Y_S \times \mathbb{R}^{\leq 0})^\Gamma \arrow[r] & G^\Gamma(Y_S \times \mathbb{R})^\Gamma \arrow[r] & G^\Gamma(Y_S \times \mathbb{R})^\Gamma_{>Y_S \times \mathbb{R}^{\leq 0}} \end{tikzcd}$$

The map $K(I)$ is induced by the inclusion I of the quotient categories.

Theorem 6.8. *$K(I)$ is a weak equivalence of connective spectra.*

Proof. This follows from the approximation theorem applied to I. The first condition of the theorem is evident. To check the second condition, consider a chain complex $F^\cdot$ in $\mathbf{G}^{h\Gamma}(Y_S \times \mathbb{R}^{\geq 0})_{>Y_S \times 0}$. By the nature of the objects and the explanation in Remark 6.1, all maps in $F^\cdot$ and their control features are determined by the values on the objects F^i_e of $\mathbf{G}_\Gamma(Y_S \times \mathbb{R})$. So we can specify $F^\cdot$ by the chain complex

$$F^\cdot_e : \quad 0 \to F^1_e \xrightarrow{\phi_1} F^2_e \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} F^n_e \to 0$$

in $\mathbf{G}_\Gamma(Y_S \times \mathbb{R}^{\geq 0})_{>Y_S \times 0}$. Given a chain complex $G^\cdot$ in $\mathbf{G}^{h\Gamma}(Y_S \times \mathbb{R})_{>Y_S \times \mathbb{R}^{\leq 0}}$, we can apply the same reasoning to $G^\cdot$. Now a chain map $g : F^\cdot \to G^\cdot$ is given uniquely by a chain map $F^\cdot_e \to G^\cdot_e$, where $G^\cdot_e$ is the chain complex

$$0 \to G^1_e \xrightarrow{\psi_1} G^2_e \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{n-1}} G^n_e \to 0$$

in $\mathbf{G}_\Gamma(Y_S \times \mathbb{R})_{>Y_S \times \mathbb{R}^{\leq 0}}$. Also observe that if F^i_e is supported on a neighborhood of $C \subset Y$ then so are all of the F^i_γ. This allows us to transport from [Carlsson and Goldfarb 2019] the rest of the argument for a nonequivariant Lemma 4.2.4. Alternatively, we can refer to the end of the proof of Theorem 6.11, where the details are spelled out in even greater generality. □

The spectra $G^\Gamma(Y_S \times \mathbb{R}^{\geq 0})^\Gamma$ and $G^\Gamma(Y_S \times \mathbb{R}^{\leq 0})^\Gamma$ are contractible as K-theory spectra of flasque categories. This is the standard consequence of the shift functor T in the positive (resp. negative) direction along $\mathbb{R}^{\geq 0}$ (resp. $\mathbb{R}^{\leq 0}$) interpreted as an exact endofunctor. A natural equivalence of functors $1 \oplus T \cong T$ and the additivity theorem give contractibility.

From the map of fibrations, we obtain a map of spectra $G^\Gamma(Y_S)^\Gamma \to \Omega G^\Gamma(Y_S \times \mathbb{R})^\Gamma$ which induces isomorphisms of K-groups in positive dimensions. Iterating this construction for $k \geq 2$ gives weak equivalences

$$\Omega^k G^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma \to \Omega^{k+1} G^\Gamma(Y_S \times \mathbb{R}^{k+1})^\Gamma.$$

Definition 6.9. The nonconnective delooping of algebraic K-theory of the fibred homotopy fixed points is the spectrum

$$\widetilde{G}^\Gamma(Y_S)^\Gamma = \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma.$$

In the case Y is the one point space, $\widetilde{G}^\Gamma(Y)^\Gamma$ coincides with the nonconnective G-theory of the group ring $R[\Gamma]$ defined in [Carlsson and Goldfarb 2016].

The discussion leading up to Definition 6.9 can be repeated verbatim for other Serre subcategory pairs. For example, the subcategory $\mathbf{G}^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma_{<C\times\mathbb{R}^k}$ is evidently a Serre subcategory of $\mathbf{G}^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma$ for any choice of subset $C \subset Y_S$. We define

$$\widetilde{G}^\Gamma(Y_S)^\Gamma_{<C} = \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma_{<C\times\mathbb{R}^k},$$

$$\widetilde{G}^\Gamma(Y_S)^\Gamma_{<C_1,C_2} = \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma_{<C_1\times\mathbb{R}^k,\, C_2\times\mathbb{R}^k}.$$

Definition 6.10. Let Y', Y_1, and Y_2 be arbitrary subsets of Y such that Y_1 and Y_2 form a covering of Y. There are corresponding subsets Y'_S, $Y_{1,S}$, and $Y_{2,S}$ of Y_S obtained as $Y'_S = \bigsqcup Y'_s$, $Y_{1,S} = \bigsqcup Y'_{1,s}$, and $Y_{2,S} = \bigsqcup Y'_{2,s}$. It is now straightforward to define nonconnective spectra

$$\widetilde{G}^\Gamma(Y_S, Y'_S)^\Gamma = \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k, Y'_S \times \mathbb{R}^k)^\Gamma,$$

$$\widetilde{G}^\Gamma(Y_S, Y'_S)^\Gamma_{<Y_i} = \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k, Y'_S \times \mathbb{R}^k)^\Gamma_{<Y_{i,S}\times\mathbb{R}^k},$$

$$\widetilde{G}^\Gamma(Y_S, Y'_S)^\Gamma_{<Y_1,Y_2} = \underset{\overrightarrow{k>0}}{\operatorname{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k, Y'_S \times \mathbb{R}^k)^\Gamma_{<Y_{1,S}\times\mathbb{R}^k,\, Y_{2,S}\times\mathbb{R}^k}.$$

Theorem 6.11. *Suppose Y_1 and Y_2 are subsets of a metric space Y on which Γ acts by bounded coarse equivalences, and $Y = Y_1 \cup Y_2$. There is a homotopy pushout diagram of spectra*

$$\begin{array}{ccc}
\widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1,Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1} \\
\downarrow & & \downarrow \\
\widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}
\end{array}$$

where the maps of spectra are induced from the exact inclusions. If we define

$$E^{\Gamma}(Y, Y') = \underset{U \in \mathcal{F}(\beta(Y))}{\underrightarrow{\text{hocolim}}}\ \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma},$$

the excision theorem also holds on the level of E^{Γ}. There is a homotopy pushout diagram of spectra

$$\begin{array}{ccc}
E^{\Gamma}(Y, Y')_{<Y_1,Y_2} & \longrightarrow & E^{\Gamma}(Y, Y')_{<Y_1} \\
\downarrow & & \downarrow \\
E^{\Gamma}(Y, Y')_{<Y_2} & \longrightarrow & E^{\Gamma}(Y, Y')
\end{array}$$

Proof. There is a homotopy pushout

$$\begin{array}{ccc}
\widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1,Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1} \\
\downarrow & & \downarrow \\
\widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}
\end{array}$$

obtained from the map of the fibration sequences

$$\begin{array}{ccccc}
\widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1,Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1,>Y_2} \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{>Y_2}
\end{array}$$

both obtained from Theorem 6.7. The map

$$\widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1,>Y_2} \to \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{>Y_2}$$

induced from the exact inclusion $J : \mathbf{G}^{h\Gamma}(Y_S, Y'_S)_{<Y_1,>Y_2} \to \mathbf{G}^{h\Gamma}(Y_S, Y'_S)_{>Y_2}$ is again an equivalence. It should be instructive to spell out the crucial application of the approximation theorem. Consider a chain complex $F^{\cdot}$ in $\mathbf{G}^{h\Gamma}(Y_S, Y'_S)_{<Y_1,>Y_2}$. All maps in $F^{\cdot}$ and their control features are determined by the values on the objects F^i_e of $\mathbf{G}_{\Gamma}(Y_S, Y'_S)$, so $F^{\cdot}$ can be given by the chain complex

$$F^{\cdot}_e : \quad 0 \to F^1_e \xrightarrow{\phi_1} F^2_e \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} F^n_e \to 0$$

in $\mathbf{G}_\Gamma(Y_S, Y'_S)_{<Y_1,>Y_2}$. Applying the same reasoning to a chain complex $G^\cdot$ in $\mathbf{G}^{h\Gamma}(Y_S, Y'_S)_{>Y_2}$, let $G^\cdot_e$ be the chain complex

$$0 \to G_e^1 \xrightarrow{\psi_1} G_e^2 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{n-1}} G_e^n \to 0$$

in $\mathbf{G}_\Gamma(Y_S, Y'_S)_{>Y_2}$. A chain map $g : F^\cdot \to G^\cdot$ can be given by a chain map $g' : F^\cdot_e \to G^\cdot_e$. Since the action is bounded, if F^i_e is supported near a neighborhood of $Y_{1,S} \subset Y_S$ then so are all F^i_γ.

Suppose all F^i_e and G^i_e are (D, Δ')-lean/split and (d, δ)-insular, and there is a number $r \geq 0$ and a monotone function $\rho : [0, +\infty) \to [0, +\infty)$ such that there are containments $F^i_e \subset F^i_e((\Gamma, Y_{1,S})[r, \rho_{x_0}])$ for all $0 \leq i \leq n$. Suppose also that (b, θ) can be used as bounded control data for all maps ϕ_i, ψ_i, and g'_i. Suppose also that (K, k) is an enlargement data for the chosen grading of G_e. We define a submodule F'^i_e as the submodule $\mathcal{G}^i_e((\Gamma, Y_{1,S})[r + 3ib, \rho_{x_0} + 3i\theta_{x_0}])$ in the chosen grading of G_e and define $\xi_i : F'^i_e \to F'^{i+1}_e$ to be the restrictions of ψ_i to F'^i_e. This gives a chain subcomplex (F'^i, ξ_i) of (G^i, ψ_i) in $\mathbf{G}_\Gamma(Y_S, Y'_S)$ with the inclusion $i : F'^i \to G^i$. Notice that we have the induced chain map $\bar{g} : F^\cdot \to F'^\cdot$ in $\mathbf{G}_\Gamma(Y_S, Y'_S)_{<Y_1}$ so that $g = iJ(\bar{g})$. It remains to prove that the cokernel $C^\cdot$ of i is in $\mathbf{G}_\Gamma(Y_S, Y'_S)_{<Y_2}$. Since

$$F'^i_e \subset G^i_e((\Gamma, Y_{1,S})[r + 3ib + K, \rho_{x_0} + 3i\theta_{x_0} + k_{x_0}]),$$

each C^i is supported on

$$\begin{aligned}(\Gamma, Y_S \setminus Y_{1,S})[2D + 2d - r - 3ib - K, 2\Delta'_{x_0} + 2\delta_{x_0} - \rho_{x_0} - 3i\theta_{x_0} - k_{x_0}]& \\ \subset (\Gamma, Y_{2,S})[2D + 2d, 2\Delta'_{x_0} + 2\delta_{x_0}]&.\end{aligned}$$

This shows that the complex $C^\cdot$ is indeed in $\mathbf{G}_\Gamma(Y_S, Y'_S)_{<Y_2}$. □

Remark 6.12. This observation is warranted as we contrast Theorem 6.11 and its proof with the inability to use other, more standard methods in bounded algebra based on Karoubi filtrations in order to prove similar facts in K-theory. The key idea in the proof is still the commutative diagram from [Cárdenas and Pedersen 1997, Section 8] transported from bounded K-theory to fibred G-theory. Cárdenas and Pedersen use Karoubi quotients and the Karoubi fibrations in order to establish their diagram. One of the crucial points in [Cárdenas and Pedersen 1997] is that the functor I between the Karoubi quotients is an isomorphism of categories. In fibred G-theory the situation is more complicated: I is not necessarily full and, therefore, not an isomorphism of categories. However we can see here, just as in the analogous Theorem 4.4.2 in [Carlsson and Goldfarb 2019], that the approximation theorem suffices to prove that $K(I)$ is nevertheless a weak equivalence.

We make an explicit statement that does not hold in K-theory. Let $V_\Gamma(Y)$ be the K-theory of the fibred homotopy fixed points $\mathcal{B}^{h\Gamma}(Y)$ defined in Notation 4.5.

In these terms, we don't know whether, or under what conditions on Γ and Y, the natural map

$$\underset{\overrightarrow{U \in \mathcal{U}}}{\text{hocolim}}\; V^\Gamma(U) \to V^\Gamma(Y)$$

is an equivalence in the context of Theorem 6.11. Through indirect ways related to the work on the Borel conjecture, we know that Karoubi filtrations should be impossible to use to compute fibred homotopy fixed points in full generality, because otherwise the outcome would contradict the well-known counterexamples to the integral K-theoretic Borel conjecture in cases of nonregular rings R.

Suppose $\mathcal{U}$ is a finite covering of Y that is closed under intersections and such that the family of all subsets U in $\mathcal{U}$ together with Y' are pairwise coarsely antithetic. The extra conditions in the second statement ensure that the covering is in fact by complete representatives of a covering by "coarse families" in the language introduced in [Carlsson and Goldfarb 2019, Section 4.3].

We define the homotopy colimit

$$\mathcal{E}^\Gamma(Y, Y')_{<\mathcal{U}} = \underset{\overrightarrow{U \in \mathcal{U}}}{\text{hocolim}}\; E^\Gamma(Y, Y')_{<U}.$$

All of the above discussion can be restricted to full subcategories on objects supported near an arbitrary subset Z of Y_S, so we have the following general statement.

Theorem 6.13 (fibrewise bounded excision). *The natural map*

$$\delta : \mathcal{E}^\Gamma(Y, Y', Z)_{<\mathcal{U}} \to E^\Gamma(Y, Y', Z),$$

induced by inclusions $\mathbf{G}_\Gamma(Y_S, Y'_S, Z)_{<U} \to \mathbf{G}_\Gamma(Y_S, Y'_S, Z)$, *is a weak equivalence.*

Proof. Apply Theorem 6.11 inductively to the maximal sets in $\mathcal{U}$. □

7. Other properties of equivariant fibred G-theory

Fibred assembly map. The usual notion of metric assumes only finite values. We will require a *generalized metric* on a set X. It is a function $d : X \times X \to [0, \infty) \cup \{\infty\}$ which is reflexive, symmetric, and satisfies the triangle inequality in the obvious way. The generalized metric space is *proper* if it is a countable disjoint union of metric spaces X_i on each of which the generalized metric d is finite, and all closed metric balls in X are compact. The metric topology on a generalized metric space is defined as usual.

The basic fibred assembly map

$$A(X, Y) : h^{\mathrm{lf}}(X; G^{-\infty}(Y)) \to G_X^{-\infty}(Y),$$

for proper generalized metric spaces X and Y, sends the locally finite homology of

X with coefficients in the spectrum $G^{-\infty}(Y)$ to the nonconnective fibred G-theory $G_X^{-\infty}(Y)$ defined in Section 5.

The locally finite homology $h^{\mathrm{lf}}(X; \mathcal{S})$ we use was introduced in [Carlsson 1995, Definition II.5] for any coefficient spectrum $\mathcal{S}$. Let ${}^bS_k X$ be the collection of all locally finite families $\mathcal{F}$ of singular k-simplices in X which are uniformly bounded, in the sense that each family possesses a number N such that the diameter of the image $\mathrm{im}(\sigma)$ is bounded from above by N for all simplices $\sigma \in \mathcal{F}$. For any spectrum $\mathcal{S}$, the theory ${}^bh^{\mathrm{lf}}(X; \mathcal{S})$ is the realization of the simplicial spectrum

$$k \mapsto \underset{C \in {}^bS_k X}{\underrightarrow{\mathrm{hocolim}}}\, h^{\mathrm{lf}}(C, \mathcal{S}).$$

There is an equivalence of spectra ${}^bh^{\mathrm{lf}}(X; \mathcal{S}) \to h^{\mathrm{lf}}(X; \mathcal{S})$, for any proper generalized metric space X, from [Carlsson 1995, Corollary II.21].

A similar theory $J^h(X, \mathcal{A})$ is obtained as the realization of the simplicial spectrum

$$k \mapsto \underset{C \in {}^bS_k X}{\underrightarrow{\mathrm{hocolim}}}\, K^{-\infty}(C, \mathcal{A})$$

by viewing C as a discrete metric space and using the notation $K^{-\infty}(C, \mathcal{A})$ for the nonconnective delooping of the K-theory of $\mathcal{B}(C, \mathcal{A})$ from Definition 4.2. Using the coefficients $\mathcal{A} = \mathbf{B}_C(Y)$, we obtain $J^h(X, \mathcal{A})$, which we denote $J^h(X, Y)$. The proof of [Carlsson 1995, Corollary III.14] gives a weak homotopy equivalence

$$\eta : h^{\mathrm{lf}}(X; G^{-\infty}(Y)) \to J^h(X, Y)$$

of functors from proper locally compact metric spaces and coarse maps to spectra.

We next define a natural transformation

$$\ell : J^h(X, Y) \to G_X^{-\infty}(Y).$$

In the case Y is a point and the coefficients are finitely generated free R-modules, this kind of transformation is defined as part of the proof of Proposition III.20 of [Carlsson 1995]. The definition is entirely in terms of maps between singular simplices in X, so the construction can be generalized to give ℓ as above. For the convenience of the reader, we present the necessary details.

Let us first note that controlled algebra can be used to build equivalent bounded K-theory spectra using the symmetric monoidal category approach which we will find useful in the rest of the paper. For the details we refer to Section 6 of [Carlsson 2005].

Let $\mathcal{D}$ be any collection of singular n-simplices of X and ζ be any point of the standard n-simplex. Define a function $\vartheta_\zeta : \mathcal{D} \to X$ by $\vartheta_\zeta(\sigma) = \sigma(\zeta)$. Since $\mathcal{D}$ is viewed as a discrete metric space, if $\mathcal{D}$ is locally finite then ϑ_ζ is coarse, so we have the induced functor $\mathcal{B}(\mathcal{D}, \mathcal{A}) \to \mathcal{B}(X, \mathcal{A})$ given by

$$\bigoplus_{d\in\mathcal{D}} F_d \to \bigoplus_{x\in X} \bigoplus_{\vartheta_\zeta(d)=x} F_d,$$

which is the identity for each $d \in \mathcal{D}$. Therefore, there is the induced map of spectra

$$K(\vartheta_\zeta, \mathcal{A}) : K(\mathcal{D}, \mathcal{A}) \to K(X, \mathcal{A}).$$

Suppose further that $\mathcal{D} \in {}^bS_kX$ and that N is a bound required to exist for $\mathcal{D}$ in ${}^bS_kX$. If ζ and θ are both points in the standard n-simplex, we have a symmetric monoidal natural transformation $N_\zeta^\theta : K(\vartheta_\zeta, \mathcal{A}) \to K(\vartheta_\theta, \mathcal{A})$ induced from the functors which are identities on objects in the cocompletion of $\mathcal{A}$. Both of those identity morphisms are isomorphisms in $\mathcal{B}(X, \mathcal{A})$ because they and their inverses are bounded by N.

Recall that the standard n-simplex can be viewed as the nerve of the ordered set $\underline{n} = \{0, 1, \dots, n\}$, with the natural order, viewed as a category. Let $\mathcal{D} \in {}^bS_nX$. We define a functor $l(\mathcal{D}, n) : i\mathcal{B}(\mathcal{D}, \mathcal{A}) \times \underline{n} \to i\mathcal{B}(X, \mathcal{A})$ as follows. On objects, $(l(\mathcal{D}, n)F)_x = \bigoplus_{\vartheta(i)=x} F_d$, where i denotes the vertex of $\Delta^n = N.\underline{n}$ corresponding to i. On morphisms, $l(\mathcal{D}, n)$ is defined by the requirement that the restriction to the subcategory $i\mathcal{B}(\mathcal{D}, \mathcal{A}) \times j$ is the functor induced by θ_j, and that $(\mathrm{id} \times (i \le j))(F)$ is sent to $N_i^j(F)$. This is compatible with the inclusion of elements in ${}^bS_nX$, so we obtain a functor

$$\underrightarrow{\operatorname{colim}}_{\mathcal{D}\in {}^bS_nX}\; i\mathcal{B}(\mathcal{D}, \mathcal{A}) \times \underline{n} \to i\mathcal{B}(X, \mathcal{A}),$$

and therefore a map

$$\underrightarrow{\operatorname{hocolim}}_{\mathcal{D}\in {}^bS_nX}\; N.i\mathcal{B}(\mathcal{D}, \mathcal{A}) \times \Delta^n \to N.i\mathcal{B}(X, \mathcal{A}).$$

If $\mathcal{M}$ is a symmetric monoidal category, let the t-th space in $\mathrm{Spt}(\mathcal{M})$ be denoted by $\mathrm{Spt}_t(\mathcal{M})$, and let $\sigma_t : S^1 \wedge \mathrm{Spt}_t(\mathcal{M}) \to \mathrm{Spt}_{t+1}(\mathcal{M})$ be the structure map for $\mathrm{Spt}(\mathcal{M})$. The fact that the natural transformations N_i^j are symmetric monoidal shows in particular that we obtain maps

$$\Lambda_t : \underrightarrow{\operatorname{hocolim}}_{\mathcal{D}\in {}^bS_nX}\; \mathrm{Spt}_t(i\mathcal{B}(\mathcal{D}, \mathcal{A})) \times \Delta^n \to \mathrm{Spt}_t(i\mathcal{B}(X, \mathcal{A})),$$

so that the diagrams

$$\begin{array}{ccc}
\underrightarrow{\operatorname{hocolim}}_{\mathcal{D}\in {}^bS_nX}\,(S^1 \wedge \mathrm{Spt}_t(i\mathcal{B}(\mathcal{D}, \mathcal{A}))) \times \Delta^n & \longrightarrow & S^1 \wedge \mathrm{Spt}_t(i\mathcal{B}(X, \mathcal{A})) \\
\Big\downarrow{\scriptstyle \sigma_t \times \mathrm{id}} & & \Big\downarrow{\scriptstyle \sigma_t} \\
\underrightarrow{\operatorname{hocolim}}_{\mathcal{D}\in {}^bS_nX}\, \mathrm{Spt}_{t+1}(i\mathcal{B}(\mathcal{D}, \mathcal{A})) \times \Delta^n & \xrightarrow{\Lambda_{t+1}} & \mathrm{Spt}_{t+1}(i\mathcal{B}(X, \mathcal{A}))
\end{array}$$

commute. Further, for each t we obtain a map

$$\left|\underline{k} \mapsto \underset{\substack{\longrightarrow \\ \mathcal{D}\in {}^bS_n X}}{\text{hocolim}}\ \mathrm{Spt}_t(i\mathcal{B}(\mathcal{D},\mathcal{A}))\right| \to \mathrm{Spt}_t(i\mathcal{B}(X,\mathcal{A}))$$

respecting the structure maps in Spt_t. This gives a map $\ell : {}^c\mathcal{J}^h(X;\mathcal{A}) \to K(X,\mathcal{A})$, where ${}^c\mathcal{J}^h(X;\mathcal{A})$ stands for the realization of the simplicial spectrum

$$k \mapsto \underset{\substack{\longrightarrow \\ C\in {}^bS_k X}}{\text{hocolim}}\ K(C,\mathcal{A}).$$

Since ℓ is natural in X and is compatible with delooping, it generalizes to the homotopy natural transformation $\ell_K : J^h(X;\mathcal{A}) \to K^{-\infty}(X,\mathcal{A})$. Composing this with the Cartan natural transformation $K^{-\infty}(X,\mathcal{A}) \to G_X^{-\infty}(Y)$ gives

$$\ell : J^h(X,Y) \to G_X^{-\infty}(Y).$$

Definition 7.1 (fibred assembly map in G-theory). The homotopy natural transformation

$$A(X,Y) : h^{\mathrm{lf}}(X; G^{-\infty}(Y)) \to G_X^{-\infty}(Y)$$

is the composition of η and ℓ.

Remark 7.2. Notice that if we use a different coefficient category $\mathcal{A} = \mathcal{B}_X(Y)$, we obtain a map

$$A_K(X,Y) : h^{\mathrm{lf}}(X; K^{-\infty}(Y)) \to K_X^{-\infty}(Y)$$

as the composition η and ℓ_K. It is called the *fibred assembly map in K-theory*.

Let Γ be a finitely generated group with a word metric associated to some choice of a finite generating set in Γ. We assume that there is a finite $K(\Gamma,1)$ complex M, and Y is its universal cover. We also assume that X is the universal cover of the normal bundle to an embedding of M in a Euclidean space. This is the situation we described in the introduction.

Recall from Notation 4.5 that the equivariant fibred K-theory spectrum for the pair (Γ, X) is denoted $K_p^\Gamma(X)$. We finish this section by developing $A_K(X,Y)$ into the *twisted assembly map*

$$\alpha_K(X,\Gamma) : h^{\mathrm{lf}}(X; K(R[\Gamma])) \to K_p^\Gamma(X).$$

Given a small additive category $\mathcal{A}$, we already have the fibred assembly map

$$A_K(X,Y) : h^{\mathrm{lf}}(X; K^{-\infty}(Y)) \to K_X^{-\infty}(Y).$$

Next note that the bounded K-theory spectrum $K^{-\infty}(\Gamma)$ can be viewed as the homotopy colimit of a family of nonconnective spectra

$$\underset{\substack{\longrightarrow \\ d}}{\text{hocolim}}\ K[d](Y),$$

where $K[d](Y)$ is the spectrum associated with a Γ-space given by the subspace of the nerve of the category with bounded isomorphisms as morphisms, for which a simplex is included if and only if all the maps which make up the simplex and all the composites which are computed to obtain iterated face maps are bounded by d in Γ. This gives maps $h^{\mathrm{lf}}(TY; K[d](Y)) \to K^{-\infty}(Y \times X)$ from which we can induce

$$A^{\times} : \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; h^{\mathrm{lf}}(X; K[d](Y)) \to K^{-\infty}(Y \times X).$$

The exact embedding $I : \mathcal{B}(Y \times X) \to \mathcal{B}_Y(X)$ comes from relaxing control on the morphisms. The embedding induces the map of K-theory spectra

$$I_* : K^{-\infty}(Y \times X) \to K_Y^{-\infty}(X)$$

which, in general, is not an equivalence. The composition of I_* with $A^{\times}$ gives

$$A_{\mathrm{ext}}(Y, X) : \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; h^{\mathrm{lf}}(X; K[d](Y)) \to K_Y^{-\infty}(X).$$

All of the maps we have defined are equivariant maps of spectra with group actions induced from diagonal action on $Y \times X$. So $A_{\mathrm{ext}}(Y, X)$ is an equivariant map. From the proper Spanier–Whitehead duality theorem (see Section 3 of [Ranicki 1980]), we have an equivalence $\Sigma^{n+1} F(Y, K[d](Y)) \simeq h^{\mathrm{lf}}(X; K[d](Y))$. This yields the induced map on the fixed points

$$A^{\Gamma}_{\mathrm{ext}}(Y, X) : \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; \Sigma^{n+1} K[d](Y)^{h\Gamma} \to K^{\Gamma}_p(X)^{\Gamma}.$$

On the other hand, there is a natural equivalence

$$K^{-\infty}(Y)^{h\Gamma} \simeq \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; K[d](Y)^{h\Gamma},$$

because in this case when M is a finite $K(\Gamma, 1)$ the homotopy inverse limit is a finite limit which commutes past a filtered colimit.

Definition 7.3 (twisted assembly map). The result is the desired map

$$\alpha : h^{\mathrm{lf}}(X; K^{-\infty}(Y))^{\Gamma} = \Sigma^{n+1} K^{-\infty}(Y)^{h\Gamma} \to K^{\Gamma}_p(X)^{\Gamma}.$$

Fibrewise trivialization. In this section we want to justify the claim from the introduction that in the new equivariant theory we have built there are fibrewise trivializations. First, we state the desired fact precisely.

Recall the proper metric space Y_S described in Section 6. We assume that the trivial action s_0 is in S and use the notation Y_0 for the space Y with the trivial action.

Theorem 7.4. *The equivariant inclusion of metric spaces $Y_0 \to Y_S$ induces an equivalence $\widetilde{G}^{\Gamma}(Y_0)^{\Gamma} \to \widetilde{G}^{\Gamma}(Y_S)^{\Gamma}$. Therefore, there is an equivalence $\widetilde{G}^{\Gamma}(Y_0) \to E^{\Gamma}(Y)$.*

Before we prove this theorem, we want to emphasize the basic nature of trivializations, a feature transverse to the special object conditions in Definition 1.2, which are important only for excision properties. For clarity, we start with several facts about filtrations of modules and describe the elementary case of trivialization for bounded actions on $\mathbf{B}(Y)$, which is the category $\mathbf{B}_X(Y)$ with X a single point.

Let $\Phi_d : \mathcal{P}(Y) \to \mathcal{P}(Y)$ denote the functor that assigns to a subset of Y its d-neighborhood in Y. We can think of an object of $\mathbf{B}(Y)$ as a pair (F, θ), where θ is a filtration of F as in Definition 5.4. Given two Y-filtrations θ and η, we say θ is contained in η if $\theta(S) \subset \eta(S)$ for all $S \subset X$, and write $\theta \leq \eta$. We say two Y-filtrations θ and η are *similar* if there is a number d such that $\theta \leq \eta \circ \Phi_d$ and $\eta \leq \theta \circ \Phi_d$.

Lemma 7.5. *If θ and η are similar then the objects (F, θ) and (F, η) are isomorphic in $\mathbf{B}(Y)$.*

Proof. The conditions ensure that the identity homomorphism is boundedly controlled in both directions. □

Let $f : X \to Y$ be a coarse map as defined in Definition 3.2. Given an X-filtration θ on an R-module F, we define $f_*(\theta)$ to be the Y-filtration on F given by $f_*(\theta)(U) = \theta(f^{-1}(U))$. Similarly, given a Y-filtration θ on F, we define $f^*(\theta)$ to be the Y-filtration on F given by $f^*(\theta)(U) = \theta(f(U))$. It is easy to see that for a coarse map f these constructions applied to filtrations from $\mathbf{B}(Y)$ give filtrations back in $\mathbf{B}(Y)$. We refer to Propositions 5.2 and 5.3 of [Carlsson and Goldfarb 2011].

Recall the definition of Y_S in Definition 6.6. Let $i : Y \to Y_S$ be the inclusion $y \to (y, s_0)$, an isometric embedding, and let $\pi : Y_S \to Y$ denote the projection, a distance nonincreasing map.

Lemma 7.6. *Let F be any R-module. Then any Y_S-filtration on F is similar to one of the form $i_*\theta$, where θ is a Y-filtration on F.*

Proof. Let $\mathrm{ex} : \mathcal{P}(Y_S) \to \mathcal{P}(Y_S)$ be defined by $\mathrm{ex}(U) = \pi(U) \times S$. It is clear from the definition that $U \subset \mathrm{ex}(U)$. It is also readily checked that $\mathrm{ex}(U) \subset \Phi_1(U)$, which shows that any Y_S-filtration θ on an R-module F is similar to the Y_S-filtration $\theta \circ \mathrm{ex}$. Let $\bar{\theta}$ denote the Y-filtration on F given by $\bar{\theta}(U) = \theta(U \times S)$. Then it is clear that $\theta \circ \mathrm{ex} = \pi^*(\bar{\theta})$. It therefore suffices to show that for any Y-filtration η on F, we have that $i_*\eta$ and $\pi^*\eta$ are similar. But it is clear that $\pi^*\eta \leq i_*\eta \circ \Phi_1$, which gives the result. □

We also have the following useful fact.

Lemma 7.7. *Suppose that we are given two Y-filtrations θ and η on an R-module F, and that $\phi : F \to F$ is bounded as a morphism from (F, θ) to (F, η). Suppose further that θ' and η' are also Y-filtrations, and that θ' and η' are similar to θ*

and η, respectively. Then $\phi : F \to F$ is bounded as a morphism from (F, θ') to (F, η').

Suppose a metric space Y has an action by a discrete group Γ through coarse equivalences. Recall that we say that such *action is bounded* if for each $\gamma \in \Gamma$, there is $b(\gamma) \geq 0$ such that $d(y, \gamma y) \leq b(\gamma)$ for all $y \in Y$. The following is an elementary observation.

Lemma 7.8. *Suppose Y is a proper metric space equipped with a bounded Γ-action by coarse maps. Then, given any Y-filtration θ on an R-module F, and any $\gamma \in \Gamma$, we have that θ and $\gamma_*\theta$ are similar.*

An object of the category $\mathbf{B}^{w\Gamma}(Y)$ is given by data $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\Gamma})$, where

(1) F is an R-module,

(2) θ is a Y-filtration on F,

(3) $\phi_{\gamma,\gamma'}$ is an automorphism of F,

(4) $\phi_{\gamma,\gamma} = \mathrm{id}_F$ and $\phi_{\gamma,\gamma'} \circ \phi_{\gamma',\gamma''} = \phi_{\gamma,\gamma''}$ for all γ, γ', γ'' in Γ,

(5) $\phi_{\gamma,\gamma'}$ is bounded when regarded as a homomorphism $(F, \gamma'_*\theta) \to (F, \gamma_*\theta)$.

Lemmas 7.7 and 7.8 give that condition (5) on $\phi_{\gamma,\gamma'}$ is equivalent to $\phi_{\gamma,\gamma'}$ being bounded as a homomorphism from (F, θ) to (F, θ).

The morphisms $(F, \theta, \{\phi_{\gamma,\gamma'}\}) \to (F', \theta', \{\phi'_{\gamma,\gamma'}\})$ are boundedly controlled homomorphisms $f : F \to F'$ with $\phi'_{\gamma,\gamma'} \circ f = f \circ \phi_{\gamma,\gamma'}$.

Proposition 7.9. *The equivariant inclusion $Y_0 \to Y_S$ induces an equivalence of categories $i\mathbf{B}^{w\Gamma}(Y_0) \to i\mathbf{B}^{w\Gamma}(Y_S)$.*

Proof. The inclusion exhibits $i\mathbf{B}^{w\Gamma}(Y_0)$ as a full subcategory of $i\mathbf{B}^{w\Gamma}(Y_S)$, and it follows that it's enough to prove that every object of $i\mathbf{B}^{w\Gamma}(Y_S)$ is isomorphic to an object of $i\mathbf{B}^{w\Gamma}(Y_0)$. An object of $i\mathbf{B}^{w\Gamma}(Y_0)$ is given by data $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\Gamma})$, where θ is an Y_0-filtration on F, and where ϕ_0 is an automorphism of F which is bounded as a homomorphism from (F, θ) to (F, θ). Note that the transformations by γ's do not occur in this situation because the action of Γ on Y_0 is trivial. The inclusion functor $i\mathbf{B}^{w\Gamma}(Y_0) \hookrightarrow i\mathbf{B}^{w\Gamma}(Y_S)$ is given by

$$(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\Gamma}) \to (F, i_*\theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\Gamma}),$$

so an object $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\Gamma})$ is in the subcategory $i\mathbf{B}^{w\Gamma}(Y_0)$ if and only if θ is of the form $i_*\eta$ for some Y_0-filtration η.

Next, we observe that if $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\Gamma})$ is an object of $i\mathbf{B}^{w\Gamma}(Y_S)$, and if θ' is a Y_S-filtration on F which is similar to θ, then

(a) $(F, \theta', \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\Gamma})$ is also an object of $(F, i_*\theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\Gamma})$, and

(b) $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\Gamma})$ is isomorphic to $(F, \theta', \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\Gamma})$.

But we have already observed in Lemma 7.6 that every Y_S-filtration on F is equivalent to one of the form $i_*\eta$, for some Y_0-filtration η on F, proving the result. $\square$

Now we are ready to prove Theorem 7.4.

Observe that it suffices to verify that the inclusion $Y_0 \to Y_S$ induces an equivalence of categories $i\mathbf{G}^{h\Gamma}(Y_0) \to i\mathbf{G}^{h\Gamma}(Y_S)$. The equivalence then clearly extends to categories of diagrams of objects in $\mathbf{G}^{h\Gamma}(Y_S)$, and Waldhausen's $S.$-construction used to produce the spectra in Definition 6.3 gives simplicial spaces which in every level are the nerves of categories of isomorphisms of diagrams of cofibrations of objects in $\mathbf{G}^{h\Gamma}(Y_S)$. So Theorem 7.4 follows from the following lemma.

Lemma 7.10. *The inclusion $Y_0 \to Y_S$ induces an equivalence $i\mathbf{G}^{h\Gamma}(Y_0) \to i\mathbf{G}^{h\Gamma}(Y_S)$.*

Proof. The proof of Proposition 7.9 together with the preceding lemmas should be applied verbatim in this case where objects have fibred control over Γ and possess Y-gradings. This is possible due to the facts that $\mathbf{G}^{h\Gamma}(Y)$ is

(a) closed under the required constructions f^* and f_* for a coarse equivalence f, and

(b) is equivalent to the analogue of $\mathbf{B}^{w\Gamma}(Y)$ applied to F in $\mathbf{G}_\Gamma(Y)$.

Fact (a) follows from Proposition 3.4.4 and Lemma 3.4.5 of [Carlsson and Goldfarb 2019]. We now proceed to prove (b).

When Lemmas 7.5 through 7.8 are transported to the fibred setting, the functor Φ_d needs to be interpreted as $\Phi_{(d,b)} : \mathcal{P}(\Gamma \times Y) \to \mathcal{P}(\Gamma \times Y)$ for an enlargement data (d, b). All other features remain the same. We have to state the fibred category to which the proofs apply. An object of this category $\mathbf{G}^{w\Gamma}(Y)$ is given by data $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \Gamma})$, where

(1) F is an R-module,

(2) θ is a (Γ, Y)-filtration on F which exhibits F as an object of $\mathbf{G}_\Gamma(Y)$,

(3) $\phi_{\gamma,\gamma'}$ is an automorphism of F,

(4) $\phi_{\gamma,\gamma} = \mathrm{id}_F$ and $\phi_{\gamma,\gamma'} \circ \phi_{\gamma',\gamma''} = \phi_{\gamma,\gamma''}$ for all γ, γ', γ'' in Γ,

(5) $\phi_{\gamma,\gamma'}$ is bounded by some enlargement data (B, b) when regarded as a homomorphism $(F, \gamma'_*\theta) \to (F, \gamma_*\theta)$.

Again we remark that (5) is equivalent to $\phi_{\gamma,\gamma'}$ being bounded as a homomorphism from (F, θ) to (F, θ). The morphisms $(F, \theta, \{\phi_{\gamma,\gamma'}\}) \to (F', \theta', \{\phi'_{\gamma,\gamma'}\})$ are boundedly controlled $f : F \to F'$ with $\phi'_{\gamma,\gamma'} \circ f = f \circ \phi_{\gamma,\gamma'}$.

Finally, the only new requirement we need to add in the fibred setting is

(6) ϕ_γ is 0-controlled when viewed as a morphism over Γ.

We can draw the conclusion of Proposition 7.9 that the equivariant inclusion $Y_0 \to Y_S$ induces an equivalence of categories $i\mathbf{G}^{w\Gamma}(Y_0) \to i\mathbf{G}^{w\Gamma}(Y_S)$. On the other hand, it is immediate that when the action of Γ on Y is by bounded coarse equivalences, the category $\mathbf{G}^{w\Gamma}(Y)$ is equivalent to $\mathbf{G}^{h\Gamma}(Y)$ as in Definition 6.4 by way of the rule $\phi_{\gamma,\gamma'} = \psi_{\gamma'} \circ \psi_{\gamma}^{-1}$. □

8. A sample application

We show how the main theorem of this paper fits in a computation of the K-theory of a finitely generated group in terms of group homology. This is done by proving that the Loday assembly map is an equivalence. The basic idea and most of the nontechnical issues can be illustrated in the simple case of $R = \mathbb{Z}$, the ring of integers, and $\Gamma = C$, the infinite cyclic group. In a separate forthcoming paper, we generalize this argument to all groups Γ with finite $K(\Gamma, 1)$ and finite decomposition complexity.

Example 8.1. Let S^1 be the circle viewed as a Riemannian submanifold of $\mathbb{R}^2$. The cyclic group C acts freely and properly discontinuously by translation on the universal cover $\mathbb{R}$ of S^1. Let N be the closure of the total space of the (trivial) normal bundle to the embedding, also embedded in $\mathbb{R}^2$ as a small closed tubular neighborhood of S^1. We denote by Y the universal cover of N. Now Y can be given a metric so that the restriction to the zero section is commensurable with the metric on an orbit of the translation action by C which comes from a fixed word metric on C. It is important to observe that in this example the action of C on Y is bounded. This is a consequence of the fact that C is an abelian group.

As explained in the introduction, we assume that the equivariant assembly map $A_C : h^{\mathrm{lf}}(\mathbb{R}; K^{-\infty}(\mathbb{Z})) \to K^{-\infty}(\mathbb{R}, \mathbb{Z})$ is a weak homotopy equivalence, which is known. This fact shows that the Loday assembly is a split injection with a splitting $\rho : K(\mathbb{R})^C \to K(\mathbb{R})^{hC}$. Our goal is to split ρ.

A proper version of Spanier–Whitehead duality (see, for example, [Ranicki 1980, Section 3]) allows one to view the double suspension $\Sigma^2\rho$ as a fixed point map

$$D : \Sigma^2 K(\mathbb{R})^C \to h^{\mathrm{lf}}(Y; K(\mathbb{R}))^C.$$

We also have the twisted assembly map

$$\alpha : h^{\mathrm{lf}}(Y; K(\mathbb{R})) \to K_p^C(Y)$$

from Definition 7.3. The composition $\alpha^C \circ D$ begins the following sequence:

$$\Sigma^2 K(\mathbb{Z}[C]) \xrightarrow{\alpha^C D} K_p^C(Y)^C \xrightarrow{\kappa} E^C(Y) \simeq \Sigma^2 G(\mathbb{Z}[C]). \qquad (I)$$

The map κ is induced by interpreting split exact sequences as exact sequences of nonfree modules defining G-theory, usually referred to as the Cartan map. Its target is the fibred homotopy fixed point G-theory spectrum (Section 6).

The equivalence $E^C(Y) \simeq \Sigma^2 G(\mathbb{Z}[C])$ is the excision computation from the main theorem. In this application, the normal bundle is trivial and so there is an elementary choice of a coarsely antithetic covering of $Y = \mathbb{R}$ by products of infinite rays in the fiber and in the base. The excision theorem represents $E^C(Y)$ as the homotopy colimit of a diagram of spectra indexed by cells in the standard cellular structure of the square and the face relation. The only nontrivial spectrum $E^C(\text{point})$ corresponds to the initial 2-dimensional cell. Finally, $E^C(\text{point})$ is the spectrum $\widetilde{G}^C(\text{point})^C = G(\mathbb{Z}[C])$, which was introduced in [Carlsson and Goldfarb 2016].

Now we need to explain the relationship between $G(\mathbb{Z}[C])$ and $K(\mathbb{Z}[C])$. It is studied in general under the name *regular coarse coherence* in a separate paper. We summarize it as follows. Suppose Γ is a group with finite decomposition complexity as in [Ramras et al. 2014] and has a finite $K(\Gamma, 1)$. Suppose the ring R is a Noetherian ring of finite global dimension. Under these assumptions, the Cartan map $K(R[\Gamma]) \to G(R[\Gamma])$ is an equivalence. Of course, this holds when $\Gamma = C$ and $R = \mathbb{Z}$. In this particular case, what we need is already contained in [Carlsson and Goldfarb 2004b], written about the smaller class of groups of finite asymptotic dimension.

To conclude the argument, we notice that the domain and the target of the composition in (I) are equivalent, and the first map in the composition is $D = \Sigma^2 \rho$. It is important to check, and is done in a forthcoming paper, that the composition of these exhibited maps is indeed an equivalence. The use of the theory E^C is essential for that purpose, as we already observed in Remark 1.1 in the introduction.

The same argument can be used for finitely generated free abelian groups, with the only straightforward change occurring in a larger excision scheme for the computation of $E^\Gamma(Y)$, where Y is similarly a higher-dimensional Euclidean space. The extension to nonabelian groups with finite decomposition complexity requires several new tools and will appear in a separate paper.

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References

[Cárdenas and Pedersen 1997] M. Cárdenas and E. K. Pedersen, "On the Karoubi filtration of a category", *K-Theory* **12**:2 (1997), 165–191. MR Zbl

[Carlsson 1995] G. Carlsson, "Bounded K-theory and the assembly map in algebraic K-theory", pp. 5–127 in *Novikov conjectures, index theorems and rigidity* (Oberwolfach, 1993), vol. 2, edited by S. C. Ferry et al., London Math. Soc. Lecture Note Ser. **227**, Cambridge Univ. Press, 1995. MR Zbl

[Carlsson 2005] G. Carlsson, "Deloopings in algebraic K-theory", pp. 3–37 in *Handbook of K-theory*, vol. 1, edited by E. M. Friedlander and D. R. Grayson, Springer, 2005. MR Zbl

[Carlsson and Goldfarb 2004a] G. Carlsson and B. Goldfarb, "The integral K-theoretic Novikov conjecture for groups with finite asymptotic dimension", *Invent. Math.* **157**:2 (2004), 405–418. MR Zbl

[Carlsson and Goldfarb 2004b] G. Carlsson and B. Goldfarb, "On homological coherence of discrete groups", *J. Algebra* **276**:2 (2004), 502–514. MR Zbl

[Carlsson and Goldfarb 2011] G. Carlsson and B. Goldfarb, "Controlled algebraic G-theory, I", *J. Homotopy Relat. Struct.* **6**:1 (2011), 119–159. MR Zbl

[Carlsson and Goldfarb 2013] G. Carlsson and B. Goldfarb, "Algebraic K-theory of geometric groups", preprint, 2013. arXiv

[Carlsson and Goldfarb 2016] G. Carlsson and B. Goldfarb, "On modules over infinite group rings", *Internat. J. Algebra Comput.* **26**:3 (2016), 451–466. MR Zbl

[Carlsson and Goldfarb 2019] G. Carlsson and B. Goldfarb, "Bounded G-theory with fibred control", *J. Pure Appl. Algebra* **223**:12 (2019), 5360–5395. MR Zbl

[Carlsson and Pedersen 1995] G. Carlsson and E. K. Pedersen, "Controlled algebra and the Novikov conjectures for K- and L-theory", *Topology* **34**:3 (1995), 731–758. MR Zbl

[Carlsson and Pedersen 1998] G. Carlsson and E. K. Pedersen, "Čech homology and the Novikov conjectures for K- and L-theory", *Math. Scand.* **82**:1 (1998), 5–47. MR Zbl

[Guillou et al. 2017] B. J. Guillou, J. P. May, and M. Merling, "Categorical models for equivariant classifying spaces", *Algebr. Geom. Topol.* **17**:5 (2017), 2565–2602. MR Zbl

[Kelly 1982] G. M. Kelly, *Basic concepts of enriched category theory*, London Math. Soc. Lecture Note Ser. **64**, Cambridge Univ. Press, 1982. MR Zbl

[Malkiewich and Merling 2019] C. Malkiewich and M. Merling, "Equivariant A-theory", *Doc. Math.* **24** (2019), 815–855. MR Zbl

[May and Sigurdsson 2006] J. P. May and J. Sigurdsson, *Parametrized homotopy theory*, Mathematical Surveys and Monographs **132**, Amer. Math. Soc., Providence, RI, 2006. MR Zbl

[Merling 2017] M. Merling, "Equivariant algebraic K-theory of G-rings", *Math. Z.* **285**:3-4 (2017), 1205–1248. MR Zbl

[Pedersen 1984] E. K. Pedersen, "On the K_{-i}-functors", *J. Algebra* **90**:2 (1984), 461–475. MR Zbl

[Pedersen and Weibel 1985] E. K. Pedersen and C. A. Weibel, "A nonconnective delooping of algebraic K-theory", pp. 166–181 in *Algebraic and geometric topology* (New Brunswick, NJ, 1983), edited by A. Ranicki et al., Lecture Notes in Math. **1126**, Springer, 1985. MR Zbl

[Ramras et al. 2014] D. A. Ramras, R. Tessera, and G. Yu, "Finite decomposition complexity and the integral Novikov conjecture for higher algebraic K-theory", *J. Reine Angew. Math.* **694** (2014), 129–178. MR Zbl

[Ranicki 1980] A. Ranicki, "The algebraic theory of surgery, II: Applications to topology", *Proc. London Math. Soc.* (3) **40**:2 (1980), 193–283. MR Zbl

[Schlichting 2004] M. Schlichting, "Delooping the K-theory of exact categories", *Topology* **43**:5 (2004), 1089–1103. MR Zbl

[Thomason 1983] R. W. Thomason, “The homotopy limit problem”, pp. 407–419 in *Proceedings of the Northwestern Homotopy Theory Conference* (Evanston, IL, 1982), edited by H. R. Miller and S. B. Priddy, Contemp. Math. **19**, Amer. Math. Soc., Providence, RI, 1983. MR Zbl

[Thomason and Trobaugh 1990] R. W. Thomason and T. Trobaugh, “Higher algebraic K-theory of schemes and of derived categories”, pp. 247–435 in *The Grothendieck Festschrift*, vol. III, edited by P. Cartier et al., Progr. Math. **88**, Birkhäuser, Boston, 1990. MR Zbl

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Zero-cycles with modulus and relative K-theory

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Let D be an effective Cartier divisor on a regular quasiprojective scheme X of dimension $d \geq 1$ over a field. For an integer $n \geq 0$, we construct a cycle class map from the higher Chow groups with modulus $\{\mathrm{CH}^{n+d}(X \mid mD, n)\}_{m \geq 1}$ to the relative K-groups $\{K_n(X, mD)\}_{m \geq 1}$ in the category of pro-abelian groups. We show that this induces a proisomorphism between the additive higher Chow groups of relative 0-cycles and the reduced algebraic K-groups of truncated polynomial rings over a regular semilocal ring which is essentially of finite type over a characteristic zero field.

1. Introduction

The story of Chow groups with modulus began with the discovery of additive higher 0-cycles by Bloch and Esnault [2003a; 2003b]. Their hope was that these additive 0-cycle groups would serve as a guide in developing a theory of motivic cohomology with modulus which could describe the algebraic K-theory of nonreduced schemes. Recall that Bloch's original higher Chow groups (equivalently,

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Voevodsky's motivic cohomology) overlook the difference between nonreduced and reduced schemes.

Motivated by the work of Bloch and Esnault, a theory of motivic cohomology with modulus was proposed by Binda and Saito [2019] in the name of "higher Chow groups with modulus" (recalled in Section 2E). The expectation was that one would be able to describe relative algebraic K-theory in terms of these Chow groups. The theory of Chow groups with modulus generalized the theory of additive higher Chow groups defined by Bloch–Esnault and further studied by Rülling [2007], Krishna–Levine [2008] and Park [2009]. It also generalized the theory of 0-cycles with modulus of Kerz–Saito [2016] and the higher Chow groups of Bloch [1986].

Recall that one way to study the algebraic K-theory of a nonreduced (or any singular) scheme is to embed it as a closed subscheme of a smooth scheme and study the resulting relative K-theory. Since there are motivic cohomology groups which can completely describe the algebraic K-theory of a smooth scheme, what one needs is a theory of motivic cohomology to describe the relative K-theory.

The expectation that the higher Chow groups with modulus should be the candidate for the motivic cohomology to describe the relative K-theory has generated a lot of interest in them in past several years. In a recent work, Iwasa and Kai [2019], constructed a theory of Chern classes from the relative K-theory to a variant of the higher Chow groups with modulus. In [Iwasa and Kai 2018], they proved a Riemann–Roch type theorem showing that the relative group K_0-group of an affine modulus pair is rationally isomorphic to a direct sum of Chow groups with modulus. An integral version of this isomorphism for all modulus pairs in dimension up to two was earlier proven by Binda and Krishna [2018]. They also constructed a cycle class map for relative K_0-group in all dimensions.

The above results suggest strong connection between cycles with modulus and relative K-theory. However, an explicit construction of cycle class maps in full generality or Atiyah–Hirzebruch-type spectral sequences, which may directly connect Chow groups with modulus to relative algebraic K-theory, remains a challenging problem today.

1A. ***Main results and consequences.*** The objective of this paper is to investigate the original question of Bloch and Esnault [2003a] in this subject. Namely, can 0-cycles with modulus explicitly describe relative K-theory in terms of algebraic cycles? We provide an answer to this question in this paper. We prove two results. The first is that there is indeed a direct connection between 0-cycles with modulus and relative K-theory in terms of an explicit cycle class map. The second is that in many cases of interest, these 0-cycles with modulus are strong enough to completely describe the relative algebraic K-theory. More precisely, we prove the following. The terms and notations used in the statements of these results are explained in the body of the text. In particular, we refer to Section 2C for relative K-theory and Section 2E for higher Chow groups with modulus.

Theorem 1.1. *Let X be a regular quasiprojective variety of pure dimension $d \geq 1$ over a field k and let $D \subset X$ be an effective Cartier divisor. Let $n \geq 0$ be an integer. Then there is a cycle class map*

$$\mathrm{cyc}_{X \mid D} \colon \{\mathrm{CH}^{n+d}(X \mid mD, n)\}_m \to \{K_n(X, mD)\}_m \tag{1.1}$$

between pro-abelian groups. This map is covariant functorial for proper morphisms, and contravariant functorial for flat morphisms of relative dimension zero.

For those interested in the precise variation in the modulus in the definition of $\mathrm{cyc}_{X \mid D}$, we actually prove that for every pair of integers $m \geq 1$ and $n \geq 0$, there exists a cycle class map $\mathrm{CH}^{n+d}(X \mid (n+1)mD, n) \to K_n(X, mD)$ such that going to pro-abelian groups, we get the cycle class map of Theorem 1.1. For a general divisor $D \subset X$, we do not expect that the cycle class map that we construct in Theorem 1.1 will exist without increasing the modulus. However, if we use rational coefficients, then the usage of pro-abelian groups can indeed be avoided, as the following result shows. In this paper, we use this improved version in the proof of Theorem 1.3.

Theorem 1.2. *Let X be a regular quasiprojective variety of pure dimension $d \geq 1$ over a field k and let $D \subset X$ be an effective Cartier divisor. Let $n \geq 0$ be an integer. Then there is a cycle class map*

$$\mathrm{cyc}_{X \mid D} \colon \mathrm{CH}^{n+d}(X \mid D, n)_{\mathbb{Q}} \to K_n(X, D)_{\mathbb{Q}}. \tag{1.2}$$

This map is covariant functorial for proper morphisms, and contravariant functorial for flat morphisms of relative dimension zero. Furthermore, it coincides with the map (1.1) *on the generators of* $\mathrm{CH}^{n+d}(X \mid D, n)$.

We now address as to why the cycle class maps of Theorems 1.1 and 1.2 should be nontrivial and what we expect of these maps. Recall that the relative K-theory $K_n(X, mD)$ has Adams operations (e.g., see [Levine 1997] for their construction). From our construction, we expect the map (1.2) to be injective in the pro-setting, with image $\{K_n(X, mD)^{(d+n)}_{\mathbb{Q}}\}_m$. Here, $K_n(X, mD)^{(d+n)}_{\mathbb{Q}}$ is $(d+n)$-th eigenspace of the Adams operations. When $D = \varnothing$, the cycle class map $\mathrm{cyc}_X := \mathrm{cyc}_{X \mid \varnothing}$ is not new and it was constructed by Levine [1994] by a different method. He also showed that in this special case, cyc_X is indeed injective with image $K_n(X)^{(d+n)}_{\mathbb{Q}}$.

When $X = \mathrm{Spec}(k)$ and $D = \varnothing$, the cycle class map cyc_X coincides with Totaro's map $\mathrm{CH}^n(k, n) \to K^M_n(k) \to K_n(k)$ [Totaro 1992]. Totaro showed that the map $\mathrm{CH}^n(k, n) \to K^M_n(k)$ is an isomorphism and one knows that the canonical map $K^M_n(k)_{\mathbb{Q}} \to K_n(k)^{(n)}_{\mathbb{Q}}$ is an isomorphism. The remaining part of this paper is devoted to showing that $\mathrm{cyc}_{X \mid D}$ is in fact an isomorphism with integral coefficients for the modulus pair $(\mathbb{A}^1_R, \{0\})$, where R is a regular semilocal ring essentially of finite type over a characteristic zero field.

We make some further remarks on the past works on the cycle class map for 0-cycles with modulus. Following Levine's strategy, Binda [2018] showed that there is a cycle class map to relative K-theory provided one makes the following changes: replace the higher Chow group with modulus by a variant of it (which imposes a stronger version of the modulus condition, originally introduced in [Krishna and Park 2012b]), assume that D_{red} is a strict normal crossing divisor, and assume rational coefficients. Theorem 1.1 imposes none of these conditions. If $D \subset X$ is a regular divisor, a cycle class map was defined in [Krishna and Pelaez 2018, Theorem 1.5] using the stable $\mathbb{A}^1$-homotopy theory.

We now describe our results about the cycle class map of Theorem 1.1 for the modulus pair $(\mathbb{A}^1_R, \{0\})$. Recall that in case of the higher K-theory of a smooth scheme X, the cycle-class map $\mathrm{CH}^{n+d}(X, n) \to K_n(X)$ from the 0-cycle group can not be expected to describe all of $K_n(X)$ (even with rational coefficients). However, we show in our next result that the cycle class map of Theorem 1.1 is indeed enough to describe all of the (integral) relative K-theory of nilpotent extensions of smooth schemes, if we work in the category of pro-abelian groups instead of the usual category of abelian groups. This demonstrates a remarkable feature of relative K-theory which is absent in the usual K-theory.

Before we state the precise result, recall that the additive higher Chow groups are special cases of higher Chow groups with modulus. More precisely, for an equidimensional scheme X, the additive higher Chow group $\mathrm{TCH}^p(X, n+1; m)$ is same thing as the higher Chow group with modulus $\mathrm{CH}^p(X \times \mathbb{A}^1_k \mid X \times (m+1)\{0\}, n)$ for $m, n, p \geq 0$. To understand the reason for the shift in the value of n, we need to recall that the additive higher Chow groups are supposed to compute the relative K-theory of truncated polynomial extensions and one knows that the connecting homomorphism $\partial\colon K_{n+1}(X[t]/(t^{m+1}), (x)) \to K_n(X \times \mathbb{A}^1_k, X \times (m+1)\{0\})$ is an isomorphism when X is regular. Under this dichotomy, we shall use the notation cyc_X for $\mathrm{cyc}_{\mathbb{A}^1_X \mid (X \times \{0\})}$ whenever we use the language of additive higher Chow groups. In particular, for a ring R, we shall write cyc_R for $\mathrm{cyc}_{\mathbb{A}^1_R \mid \{0\}}$ while using additive higher Chow groups.

Let R now be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Recall that there is a canonical map $K^M_*(R) \to K_*(R)$ from the Milnor to the Quillen K-theory of R. For $n \geq 1$, the group $\mathrm{TCH}^n(R, n; m)$ is not a 0-cycle group if $\dim(R) \geq 1$. Hence, Theorem 1.1 does not give us a cycle class map for this group. However, using this theorem for fields and various other deductions, we can in fact prove an improved version of Theorem 1.1. Namely, we can avoid the usage of pro-abelian groups for the existence of the cycle class map with integral coefficients.

Theorem 1.3. *Let R be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Let $m \geq 0$ and $n \geq 1$ be two integers. Then the following hold.*

(1) *There exists a cycle class map*

$$\mathrm{cyc}_R^M \colon \mathrm{TCH}^n(R, n; m) \to K_n^M(R[x]/(x^{m+1}), (x)).$$

(2) *The composite map*

$$\mathrm{cyc}_R \colon \mathrm{TCH}^n(R, n; m) \xrightarrow{\mathrm{cyc}_R^M} K_n^M(R[x]/(x^{m+1}), (x)) \to K_n(R[x]/(x^{m+1}), (x))$$

coincides with the map of Theorem 1.1 when R is a field.

(3) cyc_R^M *and* cyc_R *are natural in R.*

(4) cyc_R^M *is an isomorphism.*

(5) *The map*

$$\mathrm{cyc}_R \colon \{\mathrm{TCH}^n(R, n; m)\}_m \to \{K_n(R[x]/(x^{m+1}), (x))\}_m$$

of pro-abelian groups is an isomorphism.

In other words, Theorem 1.3 (5) says that the relative K-theory of truncated polynomial rings can indeed be completely described by the relative 0-cycles over R (the cycles in $\mathrm{TCH}^n(R, n; m)$ have relative dimension zero over R). This shows that the additive Chow groups defined by Bloch–Esnault [2003a] and [Rülling 2007] are indeed the relative K-groups, at least in characteristic zero. This was perhaps the main target of the introduction of additive higher Chow groups by Bloch and Esnault.

By the works of several authors (see [Elbaz-Vincent and Müller-Stach 2002] and [Kerz 2009] for regular semilocal rings and [Nesterenko and Suslin 1989] and [Totaro 1992] for fields), it is now well known that the motivic cohomology of a regular semilocal ring in the equal bidegree (the Milnor range) coincides with its Milnor K-theory. Theorem 1.3 (4) says that this isomorphism also holds for truncated polynomial rings over such rings. This provides a concrete evidence that if one could extend Voevodsky's theory of motives to the theory of "non-$\mathbb{A}^1$-invariant" motives over so-called fat points (infinitesimal extensions of spectra of fields), then the underlying motivic cohomology groups must be the additive higher Chow groups (see [Krishna and Park 2012a]).

It should be remarked that the objective of Theorem 1.3 is not to compute the relative K-groups. There are already known computations of these by many authors (e.g., see [Goodwillie 1985; Hesselholt 2008]). Instead, the above result addresses the question whether these relative (Milnor or Quillen) K-groups could be described by additive 0-cycles.

Theorem 1.3 has following consequences. The first corollary below is in fact part of our proof of Theorem 1.3.

Corollary 1.4. *Let R be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Let $n \geq 0$ be an integer. Then the canonical map*

$$\{K_n^M(R[x]/(x^m), (x))\}_m \to \{K_n(R[x]/(x^m), (x))\}_m$$

of pro-abelian groups is an isomorphism. In particular,

$$\{K_n(R[x]/(x^m), (x))^{(p)}\}_m = 0$$

for $p \neq n$.

Let R be any regular semilocal ring containing $\mathbb{Q}$. Then the Néron–Popescu desingularization theorem says that R is a direct limit of regular semilocal rings $\{R_i\}$, where each R_i is essentially of finite type over $\mathbb{Q}$ (see [Swan 1998, Theorem 1.1]). One knows from [Rülling 2007, Lemma 1.17] that if each

$$\{K_n^M(R_i[x]/(x^m), (x))\}_{n,m\geq 1}$$

is a restricted Witt complex over R_i, then $\{K_n^M(R[x]/(x^m), (x))\}_{n,m\geq 1}$ is a restricted Witt-complex over $\varinjlim_{i\geq 1} R_i = R$ (see [Rülling 2007, Definition 1.14] for the definition of a restricted Witt-complex). On the other hand, it was shown in [Krishna and Park 2016, Theorem 1.2] that each collection $\{\mathrm{TCH}^n(R_i, n; m)\}_{n,m\geq 1}$ is a restricted Witt-complex over R_i. We therefore obtain our next consequence of Theorem 1.3.

Corollary 1.5. *Let R be a regular semilocal ring containing $\mathbb{Q}$. Then the relative Milnor K-theory $\{K_n^M(R[x]/(x^m), (x))\}_{n,m\geq 1}$ is a restricted Witt-complex over R.*

Bloch [1977, Chapter II] had shown (without using the terminology of Witt-complex) that if R is a regular local ring containing a field of characteristic $p > 2$, then the subgroup of the relative Quillen K-theory of truncated polynomial rings over R, generated by Milnor symbols (the symbolic K-theory in the language of Bloch), has the structure of a restricted Witt-complex. The above corollary extends the result of Bloch to characteristic zero.

The last consequence of Theorem 1.3 is the following. Park and Ünver [2018] proposed a definition of motivic cohomology of truncated polynomial ring $k[x]/(x^m)$ over a field. They showed that these motivic cohomology in the Milnor range coincide with the Milnor K-theory of $k[x]/(x^m)$ when k is a characteristic zero field. Theorem 1.3 implies that the Milnor range (relative) motivic cohomology of Park–Ünver coincides with the additive higher Chow groups.

1B. ***Comments and questions.*** We make a couple of remarks related to the above results.

(1) Since Theorem 1.1 is characteristic-free, one would expect the same to be true for Theorem 1.3 and Corollary 1.4 as well. Our remark is that Theorem 1.3 and Corollary 1.4 are indeed true in all characteristics $\neq 2$. Since the techniques of our proofs in positive characteristics are different from the present paper, they are presented in [Gupta and Krishna 2019].

(2) Our second remark is actually a question. Recall that Chow groups with modulus are supposed to be the motivic cohomology to describe the relative K-theory, just as Bloch's higher Chow groups describe K-theory. Analogous to Bloch's Chow groups, the ones with modulus exist in all bidegrees. However, as we explained earlier, Theorem 1.3 says that the 0-cycles groups with modulus are often enough to describe all of relative K-theory in the setting of pro-abelian groups. One can therefore ask the following.

Question 1.6. Let R be a regular semilocal ring essentially of finite type over a perfect field. Let n, $p \geq 1$ be two integers such that $n \neq p$. Is $\{\mathrm{TCH}^p(R, n; m)\}_m = 0$?

Note that this question is consistent with the second part of Corollary 1.4. Note also that it is already shown in [Krishna and Park 2020b] that the answer to this question is yes when $p > n$. So the open case is when $p < n$. We hope to address this question in a future work. Reader may recall that when $p < n/2$, the additive version of the deeper Beilinson–Soulé vanishing conjecture says that $\mathrm{TCH}^p(R, n; m)$ should vanish for every $m \geq 1$.

1C. ***An outline of the paper.*** We end this section with a brief outline of the layout of this text. In Sections 2 and 3, we set up our notations, recollect the main objects of study and prove some intermediate results. In Section 4, we define the cycle class map on the group of generators of 0-cycles. Our definition of the cycle class map is *a priori* completely different from the one in [Binda 2018; Levine 1994]. The novelty of the new construction is that it is very explicit in nature and, therefore, it becomes possible to check that it factors through the rational equivalence. We also prove in this section that the cycle class map is natural for suitable proper and flat morphisms. One can check that this map does coincide with more abstractly defined maps of [Binda 2018; Levine 1994] on generators. But we do not discuss this in this paper (see however Section 4D for a sketch of this).

We break the proof of Theorem 1.1 into two steps. In Section 5, we prove it for a very specific type of curves using the results of Section 2. This is the technical part of the proof of Theorem 1.1. It turns out that the general case can be reduced to the above type using the results of Section 2D. This is done in Section 6. The idea that we have to increase the modulus for factoring the cycle class map through the rational equivalence is already evident in the technical results of Section 3B.

Sections 7 and 8 constitute the heart of the proof of Theorem 1.3. In Section 7, we provide some strong relations between the additive 0-cycles, relative Milnor K-theory and the big de Rham–Witt complex. In particular, we show that it suffices to know the image of certain very specific 0-cycles under the cycle class map in order to show that it factors through the relative Milnor K-theory of a truncated polynomial ring (see Lemma 7.3). In Section 8, we give an explicit description of the relative Milnor K-theory in terms of the module of Kähler differentials (see Lemma 8.4). This allows us to establish the isomorphism between the additive higher Chow groups of 0-cycles and the relative Milnor K-theory.

To pass to Quillen K-theory, we prove a vanishing theorem (see Proposition 9.5) using some results of [Krishna 2010]. This allows us to show that the additive 0-cycle groups for fields are isomorphic to the relative K-theory in the setting of pro-abelian groups. In Section 10, we extend the results of Section 9 to regular semilocal rings using the main results of [Krishna and Park 2020a]. The last section is the appendix which contains some auxiliary results on the relation between Milnor and Quillen K-theory of fields. These results are used in the main proofs.

2. The relative K-theory and cycles with modulus

Here we fix our notations and prove some basic results in relative algebraic K-theory. We shall also recall the definition of the higher Chow groups with modulus.

2A. ***Notations.*** We shall in general work with schemes over an arbitrary base field k. We shall specify further conditions on k as and when it is required. We let $\mathbf{Sch}_k$ denote the category of separated finite type schemes over k. Recall that $X \in \mathbf{Sch}_k$ is called regular if $\mathcal{O}_{X,x}$ is a regular local ring for all points $x \in X$. We let $\mathbf{Sm}_k$ denote the full subcategory of $\mathbf{Sch}_k$ consisting of regular schemes. For $X, Y \in \mathbf{Sch}_k$, we shall denote the product $X \times_k Y$ simply by $X \times Y$. For any point $x \in X$, we shall let $k(x)$ denote the residue field of x. For a reduced scheme $X \in \mathbf{Sch}_k$, we shall let X^N denote the normalization of X. For $p \geq 0$, we shall denote the set of codimension p points of a scheme X by $X^{(p)}$. For an affine scheme $X \in \mathbf{Sch}_k$, we shall let $k[X]$ denote the coordinate ring of X.

We shall let $\overline{\square}$ denote the projective space $\mathbb{P}^1_k = \mathrm{Proj}(k[Y_0, Y_1])$ and let $\square = \overline{\square} \setminus \{1\}$. We shall let $\mathbb{A}^n_k = \mathrm{Spec}(k[y_1, \dots, y_n])$ be the open subset of $\overline{\square}^n$, where $(y_1, \dots, y_n)$ denotes the coordinate system of $\overline{\square}^n$ with $y_j = Y_1^j / Y_0^j$. Given a rational map $f : X \dashrightarrow \overline{\square}^n$ in $\mathbf{Sch}_k$ and a point $x \in X$ lying in the domain of definition of f, we shall let $f_i(x) = (y_i \circ f)(x)$, where $y_i : \overline{\square}^n \to \overline{\square}$ is the i-th projection. For any $1 \leq i \leq n$ and $t \in \overline{\square}(k)$, we let $F^t_{n,i}$ denote the closed subscheme of $\overline{\square}^n$ given by $\{y_i = t\}$. We let $F^t_n = \sum_{i=1}^n F^t_{n,i}$.

By a closed pair (X, D) in $\boldsymbol{Sch}_k$, we shall mean a closed immersion $D \hookrightarrow X$ in $\boldsymbol{Sch}_k$, where X is reduced and D is an effective Cartier divisor on X. We shall write $X \setminus D$ as X^o. We shall say that (X, D) is a modulus pair if $X^o \in \boldsymbol{Sm}_k$. If (X, D) is a closed pair, we shall let $mD \subset X$ be the closed subscheme defined by the sheaf of ideals $\mathcal{I}_D^m$, where D is defined by the sheaf of ideals $\mathcal{I}_D$.

All rings in this text will be commutative and Noetherian. For such a ring R and an integer $m \geq 0$, we shall let $R_m = R[t]/(t^{m+1})$ denote the truncated polynomial algebra over R. We shall write $\mathrm{Spec}(R[t_1, \ldots, t_n])$ as $\mathbb{A}^n_R$. The tensor product $M \otimes_{\mathbb{Z}} N$ will be denoted simply as $M \otimes N$. Tensor products over other bases will be explicitly indicated.

2B. ***The category of pro-objects.*** By a pro-object in a category $\mathcal{C}$, we shall mean a sequence of objects $\{A_m\}_{m \geq 0}$ together with a map $\alpha^A_m \colon A_{m+1} \to A_m$ for each $m \geq 0$. We shall write this object often as $\{A_m\}$. We let $\mathrm{Pro}(\mathcal{C})$ denote the category of pro-objects in $\mathcal{C}$ with the morphism set given by

$$\mathrm{Hom}_{\mathrm{pro}(\mathcal{C})}(\{A_m\}, \{B_m\}) = \varprojlim_n \varinjlim_m \mathrm{Hom}_{\mathcal{C}}(A_m, B_n). \tag{2.1}$$

In particular, giving a morphism f as above is equivalent to finding a function $\lambda : \mathbb{N} \to \mathbb{N}$, a map $f_n : A_{\lambda(n)} \to B_n$ for each $n \geq 0$ such that for each $n' \geq n$, there exists $l \geq \lambda(n), \lambda(n')$ so that the diagram

$$\begin{tikzcd} A_l \arrow[r] \arrow[dr] & A_{\lambda(n')} \arrow[r, "f_{n'}"] & B_{n'} \arrow[d] \\ & A_{\lambda(n)} \arrow[r, "f_n"] & B_n \end{tikzcd} \tag{2.2}$$

is commutative, where the unmarked arrows are the structure maps of $\{A_m\}$ and $\{B_m\}$. We shall say that f is strict if λ is the identity function. If $\mathcal{C}$ admits all sequential limits, we shall denote the limit of $\{A_m\}$ by $\varprojlim_m A_m \in \mathcal{C}$. If $\mathcal{C}$ is an abelian category, then so is $\mathrm{Pro}(\mathcal{C})$. We refer the reader to [Artin and Mazur 1986, Appendix 4] for further details about $\mathrm{Pro}(\mathcal{C})$.

2C. ***The relative algebraic K-theory.*** Given a closed pair (X, D) in $\boldsymbol{Sch}_k$, we let $K(X, D)$ be the homotopy fiber of the restriction map between the Thomason–Trobaugh nonconnective algebraic K-theory spectra $K(X) \to K(D)$. We shall let $K_i(X)$ denote the homotopy groups of $K(X)$ for $i \in \mathbb{Z}$. We similarly define $K_i(X, D)$. We shall let $K^D(X)$ denote the homotopy fiber of the restriction map $K(X) \to K(X \setminus D)$. Note that $K^D(X)$ does not depend on the subscheme structure of D but $K(X, D)$ does. Note also that if $D' \subset X$ is another closed subscheme such that $D \cap D' = \varnothing$, then $K^D(X)$ is canonically homotopy equivalent to the homotopy fiber $K^D(X, D')$ of the restriction map $K(X, D') \to K(X \setminus D, D')$.

If (X, D) is a closed pair, we have the canonical restriction map $K(X, (m+1)D) \to K(X, mD)$. In particular, this gives rise a pro-spectrum $\{K(X, mD)\}$ and a levelwise homotopy fiber sequence of pro-spectra

$$\{K(X, mD)\} \to K(X) \to \{K(mD)\}. \tag{2.3}$$

If $X = \mathrm{Spec}(R)$ is affine and $D = V(I)$, we shall often write $K(X, mD)$ as $K(R, I^m)$ and $K(X)$ as $K(R)$. For a ring R, we shall let $\widetilde{K}(R_m)$ denote the reduced K-theory of R_m, defined as the homotopy fiber of the augmentation map $K(R_m) \to K(R)$. Observe that there exists a canonical decomposition $K(R_m) \cong \widetilde{K}(R_m) \times K(R)$.

Suppose that R is a regular semilocal ring. Let $f(t) \in R[t]$ be a polynomial such that $f(0) \in R^\times$ and let $Z = V(f(t)) \subset \mathbb{A}^1_R$ be the closed subscheme defined by $f(t)$. Since $Z \cap \{0\} = \varnothing$, the composite map $K^Z(\mathbb{A}^1_R) \to K(\mathbb{A}^1_R) \to K((m+1)\{0\})$ is null-homotopic for all $m \geq 0$. Hence, there is a factorization $K^Z(\mathbb{A}^1_R) \to K(\mathbb{A}^1_R, (m+1)\{0\}) \to K(\mathbb{A}^1_R)$. Let $[\mathcal{O}_Z]$ denote the fundamental class of Z in $K^Z_0(\mathbb{A}^1_R)$ (see [Thomason and Trobaugh 1990, Exercise 5.7]). Note that Z may not be reduced or irreducible. Let α_Z denote the image of $[\mathcal{O}_Z]$ under the map $K^Z_0(\mathbb{A}^1_R) \to K_0(\mathbb{A}^1_R, (m+1)\{0\})$. Let $\partial_n : \widetilde{K}_n(R_m) \to K_{n-1}(\mathbb{A}^1_R, (m+1)\{0\})$ denote the connecting homomorphism obtained by considering the long exact homotopy groups sequence associated to (2.3). The homotopy invariance of K-theory on $\boldsymbol{Sm}_k$ shows that this map is an isomorphism. For $g(t) \in R[t]$, let $\overline{g(t)}$ denote its image in R_m.

Lemma 2.1. *Given $Z = V((f(t)))$ as above, we have*

$$\alpha_Z = \partial_1(\overline{(f(0))^{-1} f(t)}).$$

Proof. Since $f(0) \in R^\times$, we note that $Z = V((f(0))^{-1} f(t))$. We let $g(t) = (f(0))^{-1} f(t)$ so that $g(0) = 1$ and therefore $\overline{g(t)} \in \widetilde{K}_1(R_m)$. We let $\Lambda = \{(a, b) \in R[t] \times R[t] \mid a - b \in (t^{m+1})\}$ be the double of $R[t]$ along the ideal (t^{m+1}) as in [Milnor 1971, Chapter 2]. Let $p_1 : \Lambda \to R[t]$ be the first projection. Then recall from [Milnor 1971, Chapter 6] that $K_0(R[t], (t^{m+1})) \cong \mathrm{Ker}((p_1)_\# : K_0(\Lambda) \to K_0(R[t]))$ and [Milnor 1971, Chapter 3] shows that $\partial_1(u) = [M(u)] - [\Lambda] \in K_0(R[t], (t^{m+1}))$, where $M(u)$ is the rank one projective Λ-module given by $M(u) = \{(x, y) \in R[t] \times R[t] \mid u\bar{x} = \bar{y} \text{ in } R_m\}$ for any $u \in R_m^\times$.

Let $u = \overline{g(t)}$ and let $M = M(\overline{g(t)})$. Let $p_2 : M \to R[t]/(g(t))$ denote the composition of the second projection $M \to R[t]$ with the surjection $R[t] \twoheadrightarrow R[t]/(g(t))$. It is then easy to see that the sequence

$$0 \to \Lambda \xrightarrow{\theta} M \xrightarrow{p_2} R[t]/(g(t)) \to 0$$

is a short exact sequence of Λ-modules if we let $\theta((a, b)) = \overline{(a, bg(t))} \in M$. In particular, we get $[\mathcal{O}_Z] = [V((g(t)))] = [M] - [\Lambda] = \partial_1(\overline{g(t)})$. This proves the lemma. □

2D. ***The projection formula for relative K-theory.*** Let (X, D) be a modulus pair in $\boldsymbol{Sch}_k$ and let S_X be the double of X along D. Recall from [Binda and Krishna 2018, §2.1] that S_X is the pushout $X \amalg_D X$ of the diagram of schemes $X \hookleftarrow D \hookrightarrow X$. On each affine open subset $U \subset X$, the double is the spectrum S_U of the ring $\{(a, b) \in \mathcal{O}_U(U) \times \mathcal{O}_U(U) \mid a - b \in \mathcal{I}_D(U)\}$, where $\mathcal{I}_D \subset \mathcal{O}_X$ is the ideal sheaf of D. We have two inclusions $\iota_\pm \colon X \hookrightarrow S_X$ and a projection $p \colon S_X \to X$ such that $p \circ \iota_\pm = \mathrm{id}_X$. In particular, there is a canonical decomposition $K(S_X) \cong K(S_X, X_-) \times K(X)$. There is an inclusion of modulus pairs $(X, D) \hookrightarrow (S_X, X_-)$, with respect to the embedding $X_+ \hookrightarrow S_X$. This yields the pull-back map $\iota_+^* \colon K(S_X, X_-) \to K(X, D)$.

We now let $u \colon Z \hookrightarrow X$ be a closed immersion such that $Z \cap D = \varnothing$. This gives rise to a closed embedding $Z \hookrightarrow X \xrightarrow{\iota_+} S_X$ such that $Z \cap D = Z \cap X_- = \varnothing$. Since $Z \cap D = \varnothing$, the push-forward map (which exists because $Z \subset X_{\mathrm{reg}}$) $u_* \colon K(Z) \to K(X)$ composed with the restriction $K(X) \to K(D)$ is null-homotopic. Hence, there is a canonical factorization $K(Z) \to K(X, D) \to K(X)$ of the push-forward map. We shall denote the map $K(Z) \to K(X, D)$ also by u_*. It is clearly functorial in (X, D) and Z. Recall also that $K(Z)$ and $K(X, D)$ are module spectra over the ring spectrum $K(X)$ (e.g., see [Thomason and Trobaugh 1990, Chapter 3]). We shall need to know the following result about the map u_* in the proof of Lemma 9.2.

Lemma 2.2. *The push-forward map $u_* \colon K_*(Z) \to K_*(X, D)$ is $K_*(X)$-linear.*

Proof. Since $Z \subset S_X \setminus X_-$, we also have the push-forward map $v_* \colon K(Z) \to K(S_X, X_-)$, where we let $v = \iota_+ \circ u$. Suppose we know that

$$u_* = \iota_+^* \circ v_* \colon K(Z) \xrightarrow{v_*} K(S_X, X_-) \xrightarrow{\iota_+^*} K(X, D)$$

and the lemma holds for v_*. Then for any $\alpha \in K_*(X)$ and $\beta \in K_*(Z)$, we get

$$u_*(u^*(\alpha)\beta) = \iota_+^*(v_*(v^* p^*(\alpha)\beta)) = \iota_+^*(p^*(\alpha) v_*(\beta)) = (p \circ \iota_+)^*(\alpha) u_*(\beta) = \alpha u_*(\beta).$$

We thus have to show the following.

(1) The lemma holds for the inclusion $Z \hookrightarrow S_X$, and

(2) $u_* = \iota_+^* \circ v_*$.

To prove (1), we can use that the map $K_*(S_X, X_-) \to K_*(S_X)$ is a split inclusion (as we saw above). Using this and the fact that $K(S_X, X_-) \to K(S_X)$ is $K(S_X)$-linear, it suffices to prove (1) for the composite push-forward map $v_* \colon K(Z) \to K(S_X)$. But we already saw above that $K(Z)$ is a module spectrum over $K(S_X)$.

We now prove (2). By the definition of the push-forward maps to the relative K-theory, we have factorizations

$$\begin{tikzcd} K(Z) \arrow[r] \arrow[d, equal] & K^Z(S_X, X_-) \arrow[r] \arrow[d, "\iota_+^*"] & K(S_X, X_-) \arrow[d, "\iota_+^*"] \\ K(Z) \arrow[r] & K^Z(X, D) \arrow[r] & K(X, D), \end{tikzcd} \tag{2.4}$$

such that the square on the right is commutative and the top (resp. bottom) composite arrow is v_* (resp. u_*). Hence, it suffices to show that the left square is commutative.

For showing this, we use the diagram

$$\begin{tikzcd} K(Z) \arrow[r] \arrow[d, equal] & K^Z(S_X, X_-) \arrow[r, "\cong"] \arrow[d, "\iota_+^*"] & K^Z(S_X \setminus X_-) \arrow[d, "\iota_+^*"] \\ K(Z) \arrow[r] & K^Z(X, D) \arrow[r, "\cong"] & K^Z(X \setminus D), \end{tikzcd} \tag{2.5}$$

where the horizontal arrows on the right are the restriction maps induced by the open immersions of modulus pairs. In particular, the square on the right is commutative. The right horizontal arrows are homotopy equivalences by the excision theorem. Hence, it suffices to show that the composite square in (2.5) commutes.

To see this, we note that the composite horizontal arrows in (2.5) have the factorizations:

$$\begin{tikzcd} K(Z) \arrow[r] \arrow[d, equal] & G(Z) \arrow[r] \arrow[d, equal] & K^Z(S_X \setminus X_-) \arrow[d, "\iota_+^*", "\cong"'] \\ K(Z) \arrow[r] & G(Z) \arrow[r] & K^Z(X \setminus D), \end{tikzcd} \tag{2.6}$$

where $G(Z)$ is the K-theory of pseudocoherent complexes on Z [Thomason and Trobaugh 1990, Chapter 3] and $K(Z) \to G(Z)$ is the canonical map. We are now done because the square on the right in (2.6) clearly commutes. $\square$

2E. ***The 0-cycles with modulus.*** Let k be a field and let (X, D) be an equidimensional closed pair in $\boldsymbol{Sch}_k$ of dimension $d \geq 1$. We recall the definition of the higher Chow groups with modulus from [Binda and Saito 2019] or [Krishna and Park 2017a]. For any integers $n, p \geq 0$, we let $\underline{z}^p(X \mid D, n)$ denote the free abelian group on the set of integral closed subschemes of $X \times \square^n$ of codimension p satisfying the following.

(1) Z intersects $X \times F$ properly for each face $F \subset \square^n$.

(2) If $\bar{Z}$ is the closure of Z in $X \times \overline{\square}^n$ and $\nu : \bar{Z}^N \to X \times \overline{\square}^n$ is the canonical map from the normalization of $\bar{Z}$, then the inequality (called the modulus condition)

$$\nu^*(D \times \overline{\square}^n) \leq \nu^*(X \times F_n^1)$$

holds in the set of Weil divisors on $\bar{Z}^N$.

An element of the group $\underline{z}^p(X \mid D, n)$ will be called an admissible cycle. It is known that $\{n \mapsto \underline{z}^p(X \mid D, n)\}$ is a cubical abelian group (see [Krishna and Levine 2008, §1]). We denote this by $\underline{z}^p(X \mid D, *)$. We let

$$z^p(X \mid D, *) = \frac{\underline{z}^p(X \mid D, *)}{\underline{z}^p_{\mathrm{degn}}(X \mid D, *)},$$

where $\underline{z}^p_{\mathrm{degn}}(X \mid D, *)$ is the degenerate part of the cubical abelian group $\underline{z}^p(X \mid D, *)$. For $n \geq 0$, we let

$$\mathrm{CH}^p(X \mid D, n) = H_n(z^p(X \mid D, *))$$

and call them the higher Chow groups with modulus of (X, D). The direct sum

$$\mathrm{CH}_0(X \mid D, *) := \bigoplus_{n \geq 0} \mathrm{CH}^{d+n}(X \mid D, n) = \bigoplus_{n \geq 0} \mathrm{CH}_{-n}(X \mid D, n) \tag{2.7}$$

is called the *higher Chow group of 0-cycles with modulus*. The subject of this paper is to study the relation between $\mathrm{CH}_0(X \mid D, *)$ and the relative K-theory $K_*(X, D)$.

We recall for the reader that the groups $\mathrm{CH}^p(X \mid D, *)$ satisfy the flat pull-back and the proper push-forward properties under certain conditions. We refer the reader to [Binda and Saito 2019] or [Krishna and Park 2017a] for these and other properties of the Chow groups with modulus.

3. The Milnor K-theory

Recall that for a semilocal ring R, the Milnor K-group $K_i^M(R)$ is defined to be the i-th graded piece of the graded Milnor K-theory $\mathbb{Z}$-algebra $K_*^M(R)$. The latter is defined to be the quotient of the tensor algebra $T_*(R^\times)$ by the two-sided graded ideal generated by homogeneous elements $\{a \otimes (1-a) \mid a, 1-a \in R^\times\}$. The image of an element $a_1 \otimes \cdots \otimes a_n \in T_n(R^\times)$ in $K_n^M(R)$ is denoted by the Milnor symbol $\underline{a} = \{a_1, \ldots, a_n\}$. If $I \subset R$ is an ideal, the relative Milnor K-theory $K_i^M(R, I)$ is defined to be the kernel of the natural surjection $K_n^M(R) \to K_n^M(R/I)$. It follows from [Kato and Saito 1986, Lemma 1.3.1] that $K_n^M(R, I)$ is generated by Milnor symbols $\{a_1, \ldots, a_n\}$, where $a_i \in \mathrm{Ker}(R^\times \to (R/I)^\times)$ for some $1 \leq i \leq n$, provided R is a finite product of local rings.

The product structures on the Milnor and Quillen K-theories yield a natural graded ring homomorphism $\psi_R\colon K_*^M(R) \to K_*(R)$. If $I \subset R$ is an ideal, we have a natural isomorphism $K_1^M(R, I) \cong \widehat{K}_1(R, I)$, where $\widehat{K}_*(R, I)$ is the group $\mathrm{Ker}(K_*(R) \to K_*(R/I))$. Using the module structure on $\widehat{K}_*(R, I)$ over $K_*(R)$ and the ring homomorphism $K_*^M(R) \to K_*(R)$, we obtain a natural graded $K_*^M(R)$-linear map $\psi_{R\,|\,I}\colon K_*^M(R, I) \to \widehat{K}_*(R, I)$. The cup product on Milnor K-theory yields maps $K_n^M(R) \otimes K_{n'}^M(R, I) \to K_{n+n'}^M(R, I)$. In the sequel, we shall loosely denote the image of this map also by $K_n^M(R)K_{n'}^M(R, I)$ (e.g., see Lemma 3.3).

3A. ***The improved Milnor K-theory.*** If R is a semilocal ring whose residue fields are not infinite, then the Milnor K-theory $K_*^M(R)$ does not have good properties. For example, the Gersten conjecture does not hold even if R is a regular local ring containing a field. If R is a finite product of local rings containing a field, Kerz [2010] defined an improved version of Milnor K-theory, which is denoted as $\widehat{K}_*^M(R)$. This is a graded commutative ring and there is natural map of graded commutative rings $\eta^R : K_*^M(R) \to \widehat{K}_*^M(R)$. For an ideal $I \subset R$, we let $\widehat{K}_*^M(R, I) = \mathrm{Ker}(\widehat{K}_*^M(R) \to \widehat{K}_*^M(R/I))$. We thus have a natural map $K_*^M(R, I) \to \widehat{K}_*^M(R, I)$. We state some basic facts about $\widehat{K}_*^M(R)$ in the following result and refer the reader to [Kerz 2010] for proofs.

Proposition 3.1. *Let R be a finite product of local rings containing a field. Then the map $\eta^R : K_*^M(R) \to \widehat{K}_*^M(R)$ has the following properties*:

(1) *η^R is surjective.*

(2) *η_n^R is an isomorphism for all $n \geq 0$ if R is a field.*

(3) *η_n^R is an isomorphism for $n \leq 1$.*

(4) *η_n^R is an isomorphism for all n if each residue fields of R are infinite.*

(5) *The natural map $K_n^M(R) \to K_n(R)$ factors through η_n^R.*

(6) *The map $\widehat{K}_2^M(R) \to K_2(R)$ is an isomorphism.*

(7) *The Gersten conjecture holds for $\widehat{K}_n^M(R)$.*

We now let R be a regular semilocal ring (not necessarily a product of local rings) containing a field. Let F denote the total quotient ring (a product of fields) of R. Recall from [Kato 1986, §1] that there is a (Gersten) complex of abelian groups

$$K_n^M(F) \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=1} K_{n-1}^M(k(\mathfrak{p})) \to \cdots \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=n-1} K_1^M(k(\mathfrak{p})) \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=n} K_0^M(k(\mathfrak{p})). \tag{3.1}$$

We let $\widehat{K}^M_n(R)$ denote the kernel of the boundary map

$$\partial : K^M_n(F) \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=1} K^M_{n-1}(k(\mathfrak{p}))$$

in (3.1). For any $X \in \boldsymbol{Sch}_k$, the improved Milnor K-theory Zariski sheaf $\widehat{\mathcal{K}}^M_{n,X}$ was defined in [Kerz 2010] whose stalk at a point $x \in X$ is $\widehat{K}^M_n(\mathcal{O}_{X,x})$. As (3.1) gives rise to a resolution of $\widehat{K}^M_n(R_{\mathfrak{p}})$ for every prime ideal $\mathfrak{p} \subset R$ by Proposition 3.1 (7), it follows that $\widehat{K}^M_n(R)$ coincides with the group of global sections of the sheaf $\widehat{\mathcal{K}}^M_{n,X}$ on $X = \mathrm{Spec}(R)$.

Since the composite map $K^M_n(R) \to K^M_n(F) \to K^M_{n-1}(k(\mathfrak{p}))$ is well known to be zero for every height one prime ideal $\mathfrak{p} \subset R$, it follows from the definition of $\widehat{K}^M_n(R)$ and the Gersten resolution of Quillen K-theory that there are natural maps

$$K^M_n(R) \to \widehat{K}^M_n(R) \xrightarrow{\psi_R} K_n(R). \tag{3.2}$$

Suppose now that R is a regular semilocal integral domain of dimension one containing a field and $I \subset R$ is an ideal of height one. Then R/I is a finite product of Artinian local rings. In particular, the improved Milnor K-theory $\widehat{K}^M_*(R/I)$ is defined. We can write $R/I = \prod_{i=1}^r R_{\mathfrak{m}_i}/IR_{\mathfrak{m}_i}$, where $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ are the minimal primes of I. We thus have the canonical maps

$$\widehat{K}^M_n(R) \hookrightarrow \prod_{i=1}^r \widehat{K}^M_n(R_{\mathfrak{m}_i}) \twoheadrightarrow \prod_{i=1}^r \widehat{K}^M_n(R_{\mathfrak{m}_i}/IR_{\mathfrak{m}_i}) \xleftarrow{\cong} \widehat{K}^M_n(R/I), \tag{3.3}$$

where the first arrow is induced from the definition of $\widehat{K}^M_n(R)$ and the Gersten resolutions of the improved Milnor K-theory of the localizations of R. We define the relative improved Milnor K-group $\widehat{K}^M_n(R, I)$ as the kernel of the composite map. Note that this agrees with the relative improved Milnor K-groups defined earlier if R is a product of local rings.

Note that (3.3) also shows that the diagram

$$\begin{tikzcd} K^M_n(R) \arrow[r] \arrow[d] & \widehat{K}^M_n(R) \arrow[r] \arrow[d] & K_n(R) \arrow[d] \\ K^M_n(R/I) \arrow[r] & \widehat{K}^M_n(R/I) \arrow[r] & K_n(R/I) \end{tikzcd} \tag{3.4}$$

commutes. We therefore get the canonical maps of relative K-theories

$$K^M_*(R, I) \to \widehat{K}^M_*(R, I) \to \widehat{K}_*(R, I), \tag{3.5}$$

where recall that $\widehat{K}_*(R, I) = \mathrm{Ker}(K_*(R) \to K_*(R/I))$.

3B. ***Some results on Milnor-K-theory.*** We shall need few results on the Milnor K-theory of discrete valuation rings. For a discrete valuation ring R with field of fractions F, we shall let $\mathrm{ord} : F^\times \to \mathbb{Z}$ denote the valuation map. We begin with the following elementary computation in Milnor K-theory. We shall use the additive notation for the group operation of the Milnor K-theory.

Lemma 3.2. *Let R be a semilocal integral domain with field of fractions F. Let a, b, s, t be nonzero elements of R such that $1+as, 1+bt \neq 0$. Then we have the following identity in $K_2^M(F)$.*

$$\{1+as,\, 1+bt\} = \begin{cases} -\left\{1+\dfrac{ab}{1+as}st,\, -as(1+bt)\right\} & \text{if } 1+(1+bt)\,as \neq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.6}$$

Proof. Suppose first that $1+(1+bt)as = 0$. Then we have

$$\{1+as,\, 1+bt\} = \{1+as,\, (-as)^{-1}\} = -\{1+as,\, -as\} = 0.$$

Otherwise, we write

$$\begin{aligned}
&\left\{1+\frac{ab}{1+as}st,\, -as(1+bt)\right\} \\
&\quad = \left\{1+\frac{ab}{1+as}st,\, -as\right\} + \left\{1+\frac{ab}{1+as}st,\, (1+bt)\right\} \\
&\quad = \left\{1+\frac{ab}{1+as}st,\, -as\right\} \\
&\qquad\qquad + \{1+(1+bt)as,\, 1+bt\} - \{1+as,\, 1+bt\} \\
&\quad = \{1+(1+bt)as,\, -as\} - \{1+as,\, -as\} \\
&\qquad\qquad + \{1+(1+bt)as,\, 1+bt\} - \{1+as,\, 1+bt\} \\
&\quad = \{1+(1+bt)as,\, -(1+bt)as\} \\
&\qquad\qquad - \{1+as,\, -as\} - \{1+as,\, 1+bt\} \\
&\quad = \{1-u,\, u\} - \{1-v,\, v\} - \{1+as,\, 1+bt\} \\
&\quad = -\{1+as,\, 1+bt\},
\end{aligned} \tag{3.7}$$

where we let $u = -(1+bt)as$ and $v = -as$. This proves the lemma. $\square$

Lemma 3.3. *Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and field of fractions F. For $m, n \geq 1$, let $K_n^M(F, m)$ denote the subgroup of $K_n^M(F)$ generated by Milnor symbols $\{y_1, \ldots, y_n\} \in K_n^M(F)$ such that $\sum_{i=1}^n \mathrm{ord}(y_i - 1) \geq m$. Then for any $n \geq 0$, we have*

$$K_{n+1}^M(F, m) \subseteq (1+\mathfrak{m}^m) K_n^M(F).$$

Proof. Note that for $n = 0$, we actually have $K_1^M(F, m) = 1 + \mathfrak{m}^m$ and this is obvious from the definition of $K_1^M(F, m)$. We shall prove $n \geq 1$ case by induction on n. We let π denote a uniformizing parameter of R. We can write $y_i = 1 + u_i \pi^{m_i}$ for some $u_i \in R^\times$ and $m_i \in \mathbb{Z}$ for $1 \leq i \leq n$. We first observe that if $m_i \geq m$ for some $i \geq 1$, then $y_i = 1 + u_i \pi^{m_i} \in 1 + \mathfrak{m}^m$ and we are done.

We now assume that $n = 1$. In this case, if some $m_i \leq 0$, then we must have that some $m_j \geq m$ and we are done as above. We can therefore assume that $0 < m_1, m_2 < m$. In this case, Lemma 3.2 says that $\{y_1, y_2\}$ is either zero or it is $-\{1 + u_1 u_2 y_1^{-1} \pi^{m_1+m_2}, -u_1 y_2 \pi^{m_1}\}$. Since $m_1 \geq 0$, we see that $y_1^{-1} \in R^\times$. In particular, $1 + u_1 u_2 y_1^{-1} \pi^{m_1+m_2} \in 1 + \mathfrak{m}^{m_1+m_2}$. We therefore get $\{y_1, y_2\} \in (1 + \mathfrak{m}^m) K_1^M(F)$.

If $n \geq 2$, we have must have $m_i \geq 0$ for some $1 \leq i \leq n$ as $m \geq 0$. Since the permutation of coordinates of a Milnor symbol only changes its sign in the Milnor K-group, we can assume that $m_1 \geq 0$ so that $y_1 \in R^\times$. We can now write $\{y_1, \ldots, y_n\} = \{y_1, y_2\} \cdot \{y_3, \ldots, y_n\}$. We have seen before that the term $\{y_1, y_2\}$ is either zero or we have

$$\begin{aligned} \{y_1, \ldots, y_n\} &= \{1 + u_1 \pi^{m_1}, 1 + u_2 \pi^{m_2}\} \cdot \{y_3, \ldots, y_n\} \\ &= -\{1 + u_1 u_2 y_1^{-1} \pi^{m_1+m_2}, -u_1 \pi^{m_1} y_2\} \cdot \{y_3, \ldots, y_n\} \\ &= \{-u_1 y_2 \pi^{m_1}\} \cdot \{1 + u_1 u_2 y_1^{-1} \pi^{m_1+m_2}, y_3, \ldots, y_n\}. \end{aligned}$$

Since $y_1 \in R^\times$, it follows that $y_2' := 1 + u_1 u_2 y_1^{-1} \pi^{m_1+m_2} \in 1 + \mathfrak{m}^{m_1+m_2}$. In particular, we see that $\operatorname{ord}(y_2' - 1) + \sum_{i=3}^n \operatorname{ord}(y_i - 1) \geq \sum_{i=1}^n m_i \geq m$. Hence, the induction hypothesis implies that $\{y_2', y_3, \ldots, y_n\} \in (1 + \mathfrak{m}^m) K_{n-1}^M(F)$. This implies that $\{y_1, \ldots, y_n\} = \{-u_1 y_2 \pi^{m_1}\} \cdot \{y_2', y_3, \ldots, y_n\} \in (1 + \mathfrak{m}^m) K_n^M(F)$. $\square$

Lemma 3.4. *Let R be a discrete valuation ring containing a field. Let $\mathfrak{m}$ and F denote the maximal ideal and the field of fractions of R, respectively. Then the following hold for every integer $n \geq 0$:*

(1) $(1 + \mathfrak{m}) K_n^M(F) \subseteq \widehat{K}_{n+1}^M(R)$.

(2) $(1 + \mathfrak{m}^{m+n}) K_n^M(F) \subseteq (1 + \mathfrak{m}^m) \widehat{K}_n^M(R)$ *for all* $m \geq 1$.

Proof. We shall prove the lemma by induction on n. As the base case $n = 0$ trivially follows, we shall assume that $n \geq 1$. Suppose we show that

$$(1 + \mathfrak{m}) K_1^M(F) \subseteq (1 + \mathfrak{m}) \widehat{K}_1^M(R) \subset \widehat{K}_2^M(R). \tag{3.8}$$

We will then have

$$\begin{aligned} (1 + \mathfrak{m}) K_n^M(F) &\subseteq (1 + \mathfrak{m}) \widehat{K}_1^M(R) K_{n-1}^M(F) \\ &= \widehat{K}_1^M(R)(1 + \mathfrak{m}) K_{n-1}^M(F) \subseteq^1 \widehat{K}_1^M(R) \widehat{K}_n^M(R) \subseteq \widehat{K}_{n+1}^M(R), \end{aligned}$$

where $\subseteq^1$ holds by induction on n. This will prove (1).

Similarly, suppose we show for all $m \geq 1$ that

$$(1+\mathfrak{m}^{m+1})K_1^M(F) \subseteq (1+\mathfrak{m}^m)\widehat{K}_1^M(R) \subset \widehat{K}_2^M(R). \tag{3.9}$$

Then for any $m \geq 1$ and $n \geq 2$, we will have

$$\begin{aligned}(1+\mathfrak{m}^{m+n})K_n^M(F) &\subseteq (1+\mathfrak{m}^{m+n-1})\widehat{K}_1^M(R)K_{n-1}^M(F)\\ &= \widehat{K}_1^M(R)(1+\mathfrak{m}^{m+n-1})K_{n-1}^M(F) \subseteq^1 \widehat{K}_1^M(R)(1+\mathfrak{m}^m)\widehat{K}_{n-1}^M(R)\\ &\subseteq (1+\mathfrak{m}^m)\widehat{K}_n^M(R),\end{aligned}$$

where $\subseteq^1$ holds by induction on n. This will prove (2). We are therefore left with showing (3.8) and (3.9) in order to prove the lemma.

To prove (3.8), we let π be a uniformizing parameter of R. For $j \in \mathbb{Z}$ and $u, v \in R^\times$, we then have

$$\begin{aligned}\{1+u\pi, v\pi^j\} &= \{1+u\pi, v\} + \{1+u\pi, \pi^j\}\\ &= \{1+u\pi, v\} + j\{1+u\pi, \pi\}\\ &= \{1+u\pi, v\} - j\{1+u\pi, -u\},\end{aligned}$$

where the last equality holds because $\{1+u\pi, -u\pi\} = 0$. It follows that $\{1+u\pi, v\pi^j\} \in (1+\mathfrak{m})\widehat{K}_1^M(R)$. If $i \geq 2$, then $\{1+u\pi^i, v\pi^j\} \in (1+\mathfrak{m}^{i-1})\widehat{K}_1^M(R) \subseteq (1+\mathfrak{m})\widehat{K}_1^M(R)$ by (3.9). It remains therefore to prove (3.9).

We now fix $m \geq 1$, $j \in \mathbb{Z}$, $a \in R$ and $u \in R^\times$. We consider the element $\{1+a\pi^{m+1}, u\pi^j\} \in (1+\mathfrak{m}^{m+1})K_1^M(F)$. We set

$$t = -a\pi^m, \;\; v' = (1+t(-1-\pi))^{-1} \quad \text{and} \quad v'' = -1-\pi.$$

With these notations, it is clear that $1+v't,\ 1+v''t \in (1+\mathfrak{m}^m)$ and $(1+v't)(1+v''t) = 1-\pi t$. In $K_2^M(F)$, we now compute

$$\begin{aligned}\{1+a\pi^{m+1}, u\pi^j\} &= \{1+a\pi^{m+1}, u\} + j\{1+a\pi^{m+1}, \pi\}\\ &= \{1+a\pi^{m+1}, u\} + j\{1+(a\pi^m)\pi, \pi\}\\ &= \{1+a\pi^{m+1}, u\} - j\{1+(a\pi^m)\pi, -(a\pi^m)\}\\ &= \{1+a\pi^{m+1}, u\} - j\{1-\pi t, t\}\\ &= \{1+a\pi^{m+1}, u\} - j\{(1+v't)(1+v''t), t\}\\ &= \{1+a\pi^{m+1}, u\} - j\{1+v't, t\} - j\{1+v''t, t\}\\ &= \{1+a\pi^{m+1}, u\} + j\{1+v't, -v'\} + j\{1+v''t, -v''\}\\ &\in (1+\mathfrak{m}^m)\widehat{K}_1^M(R).\end{aligned}$$

This proves (3.9) and completes the proof of the lemma. □

Let us now assume that R is a regular semilocal integral domain of dimension one containing a field. Let $\{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ be the set of maximal ideals of R and let F be the field of fractions of R. If $\underline{m} = (m_1, \dots, m_r)$ is an r-tuple of positive integers, we shall write the relative improved Milnor K-theory $\widehat{K}^M_*(R, \mathfrak{m}_1^{m_1} \cdots \mathfrak{m}_r^{m_r})$ (see (3.3)) as $\widehat{K}^M_*(R, \underline{m})$. For $n \geq 1$, we let

$$K^M_n(F, \underline{m}, \mathrm{sum}) = \bigcap_{i=1}^{r} (1 + \mathfrak{m}_i^{m_i} R_{\mathfrak{m}_i}) K^M_{n-1}(F).$$

We let $\underline{m+n} = (m_1 + n, \dots, m_r + n)$.

Lemma 3.5. *For every integer $n \geq 1$, we have $K^M_n(F, \underline{m+n}, \mathrm{sum}) \subseteq \widehat{K}^M_n(R, \underline{m})$.*

Proof. If R is local, then Lemma 3.4 says that

$$K^M_n(F, m+n, \mathrm{sum}) \subseteq (1 + \mathfrak{m}^m)\widehat{K}^M_{n-1}(R) \subseteq \widehat{K}^M_n(R, \mathfrak{m}^m).$$

In particular, the lemma holds if R is local.

In general, let $y \in K^M_n(F, \underline{m+n}, \mathrm{sum})$. Then it is clear that

$$y \in (1 + \mathfrak{m}^{m_i+n} R_{\mathfrak{m}_i}) K^M_{n-1}(F)$$

for each $1 \leq i \leq r$. It follows by Lemma 3.4 that $y \in (1 + \mathfrak{m}_i^m R_{\mathfrak{m}_i})\widehat{K}^M_{n-1}(R_{\mathfrak{m}_i}) \subseteq \widehat{K}^M_n(R_{\mathfrak{m}_i})$ for each i. We conclude from (3.1) and Proposition 3.1 that $y \in \widehat{K}^M_n(R)$.

We now consider the diagram

$$\begin{tikzcd}
\widehat{K}^M_n(R) \arrow[rr] \arrow[dr, "\pi"] \arrow[dd, "\Delta_R"'] & & K^M_n(F) \arrow[dd, "\Delta_F"] \\
& \widehat{K}^M_n(R/\mathfrak{m}_1^{m_1} \cdots \mathfrak{m}_r^{m_r}) \arrow[dd, "\phi"] & \\
\bigoplus_{i=1}^r \widehat{K}^M_n(R_{\mathfrak{m}_i}) \arrow[rr] \arrow[dr, "\bigoplus_i \pi_i"'] & & \bigoplus_{i=1}^r K^M_n(F) \\
& \bigoplus_{i=1}^r \widehat{K}^M_n(R_{\mathfrak{m}_i}/\mathfrak{m}_i^{m_i}), &
\end{tikzcd} \tag{3.10}$$

where ϕ is the last in the sequence of arrows in (3.3).

This diagram is clearly commutative. It follows from the case of local rings shown above that $\left(\bigoplus_i \pi_i\right) \circ \Delta_R(y) = 0$. Since ϕ is an isomorphism, we conclude that $\pi(y) = 0$, which is what we wanted to show. $\square$

Lemma 3.6. *With notations as in Lemma 3.5, we have*

$$K^M_n(F, \underline{m}, \mathrm{sum})_{\mathbb{Q}} \subseteq \widehat{K}^M_n(R, \underline{m})_{\mathbb{Q}}.$$

Proof. The reduction from the semi-local ring R to it being a local ring (dvr) goes through exactly as in the proof of Lemma 3.5 without any change. So the proof of the lemma is eventually reduced to showing the following improved version of Lemma 3.4 (2) for every pair of integers $n \geq 0$ and $m \geq 1$:

$$(1+\mathfrak{m}^m)K_n^M(F)_{\mathbb{Q}} \subseteq (1+\mathfrak{m}^m)\widehat{K}_n^M(R)_{\mathbb{Q}}. \tag{3.11}$$

This inclusion is obvious for $n=0$. To prove this for $n \geq 1$, an easy induction (see (3.9) in the proof of Lemma 3.4) reduces to the case $n=1$. We now let π be a uniformizing parameter of R and let $\{1+a\pi^m, u\pi^j\} \in (1+\mathfrak{m}^m)K_1^M(F)_{\mathbb{Q}}$, where $u \in R^\times$, $a \in R$ and $j \in \mathbb{Z}$. If $a=0$, there is nothing to show and so we can write $a = u_0\pi^i$ with $u_0 \in R^\times$ and $i \geq 0$. We then get the following in $K_2^M(F)_{\mathbb{Q}}$:

$$\begin{aligned}
\{1+a\pi^m, u\pi^j\} &= \{1+u_0\pi^{i+m}, u\pi^j\} \\
&= \{1+u_0\pi^{i+m}, u\} + j\{1+u_0\pi^{i+m}, \pi\} \\
&= \{1+u_0\pi^{i+m}, u\} + \frac{j}{i+m}\{1+u_0\pi^{i+m}, \pi^{i+m}\} \\
&= \{1+u_0\pi^{i+m}, u\} - \frac{j}{i+m}\{1+u_0\pi^{i+m}, -u_0\}.
\end{aligned}$$

Since the last term clearly belongs to $(1+\mathfrak{m}^m)\widehat{K}_1^M(R)_{\mathbb{Q}}$, we conclude the proof of (3.11) and hence of the lemma. $\square$

4. The cycle class map

In this section, we shall define the cycle class map on the group of 0-cycles with modulus and prove a very special case of Theorem 1.1. The final proof of this theorem will be done by the end of the next section. We fix an arbitrary field k.

4A. ***The map*** $\mathbf{cyc}_{X\,|\,D}$ ***on generators.*** Let (X, D) be a modulus pair in $\boldsymbol{Sch}_k$ of dimension $d \geq 1$. Let $n \geq 0$ be an integer. We begin by defining the cycle class map $\mathrm{cyc}_{X\,|\,D}$ on the group $z^{d+n}(X \mid D, n)$. Let $Z \in X \times \square^n$ be an admissible closed point. Since Z is a closed point, we have that $Z = \mathrm{Spec}(k(Z))$. We let $p_{\overline{\square}^n}\colon X \times \overline{\square}^n \to \overline{\square}^n$ and $p_X\colon X \times \overline{\square}^n \to X$ denote the projection maps. We let $f\colon Z \to X$ denote the projection map. It is clear that f is a finite map and its image is a closed point $x \in X$ which does not lie in D. We thus have a factorization $Z \to \mathrm{Spec}(k(x)) \to X^o \to X$ of f. The latter is actually a map of modulus pairs $f\colon (Z, \varnothing) \to (X, D)$. Hence, it induces the proper push-forward $f_*\colon \mathrm{CH}_q(Z, *) \to \mathrm{CH}_q(X \mid D, *)$, where $\mathrm{CH}_q(Z, *)$ are Bloch's higher Chow groups of Z [Bloch 1986].

Now, the closed point $Z \in X \times \square^n$ defines a unique $k(Z)$-rational point (which we also denote by Z) in $\square^n_Z$ such that the composite projection map $Z \hookrightarrow \square^n_Z \to Z$ is identity. Furthermore, $[Z] \in z^{d+n}(X \mid D, n)$ is the image of $[Z] \in z^n(Z, n)$ under the push-forward map f_*. Since $Z \hookrightarrow \square^n_Z$ does not meet any face of $\square^n$, it follows that $y_i(Z) \in k(Z)^\times$ for every $1 \le i \le n$, where $y_i : \overline{\square}^n_Z \to \overline{\square}_Z$ is the projection to the i-th factor. In particular, $\{y_1(Z), \dots, y_n(Z)\}$ is a well-defined element of $K^M_n(k(Z))$. We let

$$\mathrm{cyc}^M_Z([Z]) = \{y_1(Z), \dots, y_n(Z)\} \in K^M_n(k(Z)), \tag{4.1}$$

$$\mathrm{cyc}_Z([Z]) = \psi_Z \circ \mathrm{cyc}^M_Z([Z]) \in K_n(Z), \tag{4.2}$$

where recall that $\psi_Z : K^M_*(k(Z)) \to K_*(k(Z)) = K_*(Z)$ is the canonical map from the Milnor to the Quillen K-theory.

We next recall from Section 2D that as $x = f(Z) \in X^o$ (which is regular), the finite map f defines a map of spectra $f_* : K(Z) \to K(X, D)$ such that the composite map $K(Z) \to K(X, D) \to K(X)$ is the usual push-forward map. The same holds for the inclusion $\iota^x : \mathrm{Spec}(k(x)) \hookrightarrow X$. We let

$$\mathrm{cyc}_{X \mid D}([Z]) = f_* \circ \mathrm{cyc}_Z([Z]) \in K_n(X, D). \tag{4.3}$$

Extending this linearly, we obtain our cycle class map

$$\mathrm{cyc}_{X \mid D} : z^{d+n}(X \mid D, n) \to K_n(X, D). \tag{4.4}$$

If Z is an admissible closed point as above and $x = f(Z)$, then we have a commutative diagram

$$\begin{tikzcd} K^M_n(k(Z)) \arrow[r, "\psi_Z"] \arrow[d, "N_{Z/x}"'] & K_n(k(Z)) \arrow[d, "T_{k(Z)/k}"] \arrow[dr, "f_*"] & \\ K^M_n(k(x)) \arrow[r, "\psi_x"] & K_n(k(x)) \arrow[r, "\iota^x_*"] & K_n(X, D), \end{tikzcd} \tag{4.5}$$

where $N_{Z/x}$ is the Norm map between the Milnor K-theory of fields [Bass and Tate 1973] (see also [Kerz 2009]) and the right vertical arrow is the transfer (push-forward) map between the Quillen K-theory of fields. The square on the left commutes by Lemma A.3. We can therefore write

$$\begin{aligned} \mathrm{cyc}_{X \mid D}([Z]) &= f_* \circ \mathrm{cyc}_Z([Z]) \\ &= f_* \circ \psi_Z \circ \mathrm{cyc}^M_Z([Z]) \\ &= f_* \circ \psi_Z(\{y_1(Z), \dots, y_n(Z)\}) \\ &= \iota^x_* \circ \psi_x \circ N_{Z/\{x\}}(\{y_1(Z), \dots, y_n(Z)\}). \end{aligned}$$

It is clear from the definition that for any integer $m \geq 1$, there is a commutative diagram

$$\begin{tikzcd} z^{d+n}(X \mid (m+1)D, n) \arrow[r, "\mathrm{cyc}_{X \mid D}"] \arrow[d] & K_n(X, (m+1)D) \arrow[d] \\ z^{d+n}(X \mid mD, n) \arrow[r, "\mathrm{cyc}_{X \mid D}"] & K_n(X, mD), \end{tikzcd} \tag{4.6}$$

where the vertical arrows are the canonical restriction maps. We therefore have a strict map of pro-abelian groups

$$\mathrm{cyc}_{X \mid D} \colon \{z^{d+n}(X \mid mD, n)\}_m \to \{K_n(X, mD)\}_m. \tag{4.7}$$

We next prove that the map $\mathrm{cyc}_{X \mid D}$ is covariant with respect to proper morphisms of modulus pairs and contravariant for the flat morphisms of modulus pairs which are of relative dimension 0. Note that these are the only general cases where the functoriality of the cycle class map makes sense.

4B. *Naturality for flat morphisms.* Let (Y, E) and (X, D) be modulus pairs. Let $h : Y \to X$ be a flat morphisms of relative dimension 0 such that $E = h^*(D)$. Recall from [Krishna and Park 2017a, Proposition 2.12] that we have a pull-back map $h^* : z^{d+n}(X \mid D, n) \to z^{d+n}(Y \mid E, n)$ such that $h^*([Z]) = [W]$, where $W = (h \times \mathrm{id}_{\square^n})^{-1}(Z)$ and Z a closed point in $X \setminus D \times \square^n$.

Lemma 4.1. *With notations as above*, *the following diagram commutes*:

$$\begin{tikzcd} z^{d+n}(X \mid D, n) \arrow[r, "\mathrm{cyc}_{X \mid D}"] \arrow[d, "h^*"] & K_n(X, D) \arrow[d, "h^*"] \\ z^{d+n}(Y \mid E, n) \arrow[r, "\mathrm{cyc}_{Y \mid E}"] & K_n(Y, E). \end{tikzcd} \tag{4.8}$$

Proof. Let Z be a closed point in $z^{d+n}(X \mid D, n)$ and let

$$[W] = \sum_{i=1}^{r} m_i [W_i] \in z^{d+n}(Y \mid E, n),$$

where W_i are irreducible components of the inverse image scheme W with multiplicities m_i. Let $f^Z : Z \to X$, $f^W : W \to Y$ and $f^{W_i} : W_i \to Y$ denote the respective projections. Let $y(Z) = \mathrm{cyc}^M_Z([Z])$ and $y(W_i) = \mathrm{cyc}^M_{W_i}([W_i])$ as in (4.1). We then have to show that

$$h^* \circ f^Z_* \circ \psi_Z(y(Z)) = \sum_{i=1}^{r} m_i (f^{W_i}_* \circ \psi_{W_i}(y(W_i))). \tag{4.9}$$

Consider the following diagram:

$$
\begin{tikzcd}
K(Z) \arrow[r] \arrow[d, "h^*"] & G(Z) \arrow[r] \arrow[d, "h^*"] & K^Z(X \setminus D) \arrow[d, "h^*"] & K^Z(X, D) \arrow[l, "\cong"'] \arrow[r] \arrow[d, "h^*"] & K(X, D) \arrow[d, "h^*"] \\
K(W) \arrow[r] & G(W) \arrow[r] & K^W(Y \setminus E) & K^W(Y, E) \arrow[l, "\cong"'] \arrow[r] & K(Y, E).
\end{tikzcd}
\tag{4.10}
$$

Since h is flat, it follows that all the squares in (4.10) commute. Indeed, since the canonical map $K(-) \to G(-)$ respects flat pull-back, the left-most square in (4.10) commutes. The middle-left square commutes by [Thomason and Trobaugh 1990, Proposition 3.18] and the middle-right square commutes because each map is a pull-back map. Lastly, the right-most square in (4.10) commutes by the definition of the left arrow in the square. As discussed in Section 2D, the composition of the top horizontal arrows is the push forward map f_*^Z and the composition of the top horizontal arrows is the push forward map f_*^W.

It then suffices to show that

$$
f_*^W \circ h^* \circ \psi_Z(y(Z)) = \sum_{i=1}^{r} m_i(f_*^{W_i} \circ \psi_{W_i}(y(W_i))). \tag{4.11}
$$

Since $W = (h \times \mathrm{id}_{\square^n})^{-1}(Z)$, we have $y(W_i) = y(Z)$ for each $1 \leq i \leq r$ under the injective map $k(Z)^\times \hookrightarrow k(W_i)^\times$. It then follows that $\psi_{W_i}(y(W_i)) = h_i^* \psi_Z(y(Z)) \in K_n(W_i)$, where $h_i : W_i \to Z$ is the induced map. Note that $h_i = h \circ g_i$, where $g_i : W_i \hookrightarrow W$ denotes the inclusion of the irreducible component W_i into W. We are therefore reduced to show that

$$
f_*^W \circ h^* \circ \psi_Z(y(Z)) = \sum_{i=1}^{r} m_i(f_*^{W_i} \circ g_i^* \circ h^* \circ \psi_Z(y(Z))). \tag{4.12}
$$

We shall actually show that for all $a \in K_n(W)$, we have

$$
f_*^W(a) = \sum_{i=1}^{r} m_i \; f_*^{W_i} \circ g_i^*(a). \tag{4.13}
$$

Observe that the equality (4.12) follows from (4.13) with $a = h^* \circ \psi_Z(y(Z))$. To show (4.13), consider the diagram:

$$
\begin{tikzcd}
\coprod_{i=1}^r G(W_i) \arrow[r, "\cong"] \arrow[dr, "(g_{i*})_i"] & G(W_{\mathrm{red}}) \arrow[r] \arrow[d, "g_*"] & K^W(Y \setminus E) \arrow[d, equal] & K^W(Y, E) \arrow[l, "\cong"'] \arrow[r] \arrow[d, equal] & K(Y, E) \arrow[d, equal] \\
K(W) \arrow[r] & G(W) \arrow[r] & K^W(Y \setminus E) & K^W(Y, E) \arrow[l, "\cong"'] \arrow[r] & K(Y, E).
\end{tikzcd}
\tag{4.14}
$$

Since $W_{\mathrm{red}} = \coprod_i W_i$, it follows that the push-forward map $\coprod_{i=1}^r G(W_i) \to G(W_{\mathrm{red}})$ is an isomorphism and the left triangle in (4.14) commutes. Observe that the left-most square commutes because all arrow in the square are (compatible) push-forward maps. As before, the composition of the top horizontal arrows on $G(W_i)$ is the push-forward map $f_*^{W_i}$. Let $b \in G_n(W)$ be the image of $a \in K_n(W)$ under the map $K_n(W) \to G_n(W)$ induced by the bottom left arrow in (4.14). The equality (4.13) then follows if we show that

$$b = g_*\left(\sum_{i=1}^r m_i g_i^*(b)\right) \in G_n(W). \tag{4.15}$$

The equality (4.15) however follows from the following calculation:

$$\begin{aligned} g_*\left(\sum_{i=1}^r m_i g_i^*(b)\right) &= \sum_{i=1}^r m_i g_{i*} \circ g_i^*(b) \\ &=^1 \sum_{i=1}^r m_i g_{i*}(g_i^*(b)[\mathcal{O}_{W_i}]) \\ &=^2 b\left(\sum_{i=1}^r m_i \ g_{i*}([\mathcal{O}_{W_i}])\right) \\ &=^3 b[\mathcal{O}_W] = b, \end{aligned}$$

where $=^1$ follows because for each i, we have $[\mathcal{O}_{W_i}] = 1 \in G_0(W_i)$, $=^2$ follows from the projection formula [Thomason and Trobaugh 1990, Proposition 3.17] for G-theory because $a \in K_n(W)$ and $=^3$ follows as $1 = [\mathcal{O}_W] = \sum_{i=1}^r m_i \ g_{i*}([\mathcal{O}_{W_i}]) \in G_0(W)$. This completes the proof of the lemma. $\square$

4C. *Naturality for proper morphisms.* Let (Y, E) and (X, D) be modulus pairs such that X and Y are regular schemes over k of pure dimension d_X and d_Y. Let $h : Y \to X$ be a proper morphisms such that $E = h^*(D)$. By [Krishna and Park 2017a, Proposition 2.10], we have a proper push-forward map $h_* : z^{d_Y+n}(Y \mid E, n) \to z^{d_X+n}(X \mid D, n)$ such that $h_*([z]) = [k(z) : k(w)][w]$, where $w = (h \times \mathrm{id}_{\square^n})(z)$ and z is a closed point in $Y \setminus E \times \square^n$. The existence of the push-forward map $h_* : K(Y, E) \to K(X, D)$ follows from the following lemma.

Lemma 4.2. *The map h induces a proper push-forward map $h_*\colon K(Y, E) \to K(X, D)$ of relative K-theory spectra.*

Proof. Since Y is regular, we can assume with out loss of generality that Y is integral. Since X is regular, the map h has finite tor-dimension. By [Thomason and Trobaugh 1990, 3.16.4], we have a push-forward map $f_*\colon K(Y) \to K(X)$. Observe that if $E = \varnothing$, then we have a push-forward map $f_*\colon K(Y) \to K(X, D)$.

We can therefore assume that $E \neq \varnothing$. To prove the lemma, it suffices to show that Y and D are tor-independent over X. This will in particular imply that $E \to D$ also has finite tor-dimension. For tor-independence, we note that D is an effective Cartier divisor. Hence, the only possible nontrivial tor term can be $\mathrm{Tor}^1_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_D)$. But this is same as the $\mathcal{I}_D$-torsion subsheaf of $\mathcal{O}_Y$. Since Y is integral, this torsion subsheaf is nonzero if and only if the ideal $\mathcal{I}_E$ is zero. But this can not happen as E is a proper divisor on Y. This finishes the proof. $\square$

Remark 4.3. Observe that Lemma 4.2 is true for a general integral scheme (may not be regular) Y over a regular scheme X.

We now prove that the map $\mathrm{cyc}_{X \mid D}$ in (4.4) commutes with the push-forward map.

Lemma 4.4. *For a cycle $\alpha \in z^{d_Y+n}(Y \mid E, n)$, we have* $\mathrm{cyc}_{X \mid D} \circ h_*(\alpha) = h_* \circ \mathrm{cyc}_{Y \mid E}(\alpha)$.

Proof. We can assume α is represented by a closed point $z \in z^{d_Y+n}(Y \mid E, n)$. We set $w = (h \times \mathrm{id}_{\square^n})(z) \in X \times \square^n$ and $x = p_X(w)$. The compatibility between norm maps in the Milnor K-theory of fields and push-forward maps in the Quillen K-theory (see Lemma A.3) yields a commutative diagram

$$\begin{tikzcd}
K_n^M(k(z)) \arrow[r, "N_{z/w}"] \arrow[d, "\psi_z"] \arrow[dd, bend right=60, "\iota_{z*}"'] & K_n^M(k(w)) \arrow[d, "\psi_w"'] \arrow[dd, bend left=60, "\iota_{w*}"] \\
K_n(k(z)) \arrow[r, "h_*"'] \arrow[d, "p_*"] & K_n(k(w)) \arrow[d, "p_*"] \\
K_n(Y, E) \arrow[r, "h_*"'] & K_n(X, D).
\end{tikzcd} \tag{4.16}$$

Using this commutative diagram, we get

$$\begin{aligned}
\mathrm{cyc}_{X \mid D} \circ h_*([z]) &= \mathrm{cyc}_{X \mid D}([k(z) : k(w)][w]) \\
&= \iota_{w*}(\{y_1(w), \dots, y_n(w)\}^{[k(z):k(w)]}) \\
&=^1 \iota_{w*} \circ \iota_{z/w_*} \circ \iota^*_{z/w}(\{y_1(w), \dots, y_n(w)\}) \\
&=^2 \iota_{w*} \circ N_{z/w}(\{y_1(z), \dots, y_n(z)\}) \\
&= h_* \circ \iota_{z*}(\{y_1(z), \dots, y_n(z)\}) \\
&= h_* \circ \mathrm{cyc}_{Y \mid E}([z]).
\end{aligned}$$

In this set of equalities, recall our notation (preceding Lemma 5.2) that $y_i(z)$ is the image of z under the i-th projection $\mathrm{Spec}(k(z)) \to \square_{k(z)}$. In particular, $y_i(z) \in k(z)^\times$ for all $1 \leq i \leq n$. The coordinates $y_i(w) \in k(w)^\times$ have similar meaning.

The map $\iota_{z/w} : \mathrm{Spec}(k(z)) \to \mathrm{Spec}(k(w))$ is the projection. The equality $=^1$ is a consequence of the projection formula for the Milnor K-theory associated to the resulting inclusion $k(w) \hookrightarrow k(z)$. The equality $=^2$ follows from the fact that $y_i(z) = y_i(w) = \iota^*_{z/w}(y_i(w))$ via the inclusion $k(w) \hookrightarrow k(z)$ for all $1 \le i \le n$. This proves the lemma. $\square$

We end this section with some comments below on $\mathrm{cyc}_{X \mid D}$ in the nonmodulus case.

4D. ***Agreement with Levine's map.*** We had mentioned in Section 1 that for Bloch's higher Chow groups of 0-cycles, a cycle class map to the ordinary K-theory of a regular variety was constructed by Levine [1994] with rational coefficients. Binda [2018] constructed such a map in the modulus setting and his map is identical to that of Levine by definition. It is not hard to check that the map $\mathrm{cyc}_{X \mid D}$ coincides with Levine's map when $D = \varnothing$ (note that $z^{d+n}(X \mid D, n) \subseteq z^{d+n}(X, n)$). In particular, it turns out that Levine's cycle class map for the ordinary 0-cycles exists with integral coefficients. We give a sketch of this agreement and leave the details for the reader.

Let $n \ge 0$ be an integer. Since X is regular, the homotopy invariance implies that the multirelative K-theory exact sequence (see [Levine 1994, §1]) yields an isomorphism

$$\theta_n : K_n(X) \xrightarrow{\cong} K_0(X \times \square^n, \partial\square^n), \tag{4.17}$$

where $K(X \times \square^n, \partial\square^n)$ is the iterated multirelative K-theory of $X \times \square^n$ relative to all codimension one faces.

Let $Z \subset X \times \square^n$ be a closed point. Then Levine's cycle class map $\mathrm{cyc}^L_X([Z])$ is the image of $1 \in K_0(Z)$ under the composition

$$K_0(Z) \cong K^Z_0(X \times \square^n, \partial\square^n) \to K_0(X \times \square^n, \partial\square^n) \xleftarrow{\theta_n} K_n(X). \tag{4.18}$$

Let $f : Z \to X$ denote the projection map (which is finite). We then have a finite map of multiclosed pairs $f : (Z \times \square^n, \partial\square^n) \to (X \times \square^n, \partial\square^n)$. Since the relative K-theory fiber sequence commutes with finite push-forward for regular schemes (see Lemma 4.2), and since $\mathrm{cyc}^L_X([Z])$ is the image of $[Z] \in K_0(Z \times \square^n, \partial\square^n)$ under the push-forward map f_*, we can assume that $X = \mathrm{Spec}(k)$ and $f : Z \to \mathrm{Spec}(k)$ is the identity map.

When $n = 0$, the agreement of $\mathrm{cyc}_k([Z])$ and $\mathrm{cyc}^L_k([Z])$ is immediate. When $n \ge 1$ and if we follow our notation of (4.1), then we see that $y_i(Z) = a_i \in k^\times$ for each $1 \le i \le n$. In particular, we have $\mathrm{cyc}_k([Z]) = \{a_1, \dots, a_n\} \in K_n(k)$. One therefore has to show that if $z = (a_1, \dots, a_n) \in (k^\times)^n$ is a k-rational point in $\square^n$, then the class $[k(z)] \in K_0(\square^n, \partial\square^n)$ coincides with $\theta_n(\{a_1, \dots, a_n\})$. But this is an elementary exercise in K-theory using repeated application of relative K-theory exact sequence. For $n = 1$, it already follows from a straightforward generalization of Lemma 2.1 (where we replace (t^{m+1}) by any ideal of $R[t]$) with identical proof. We leave it to the reader to check the details for $n \geq 2$.

5. The case of regular curves

The goal now is to show that $\mathrm{cyc}_{X\,|\,D}$ kills the rational equivalence if we allow the modulus to vary along $\{mD\}_{m\geq 1}$. In this section, we shall prove a very special case of this. The proof of Theorem 1.1 will be reduced to this case in the next section. We consider the following situation. Let $n \geq 0$ be an integer. We let X be a regular connected curve over k and let $D \subset X$ be an effective Cartier divisor. Let $\overline{W} \subset X \times \overline{\square}^{n+1}$ be a closed subscheme such that the following hold:

(1) The composite map $\overline{W} \xrightarrow{\nu} X \times \overline{\square}^{n+1} \xrightarrow{p_X} X$ is an isomorphism.

(2) $W = \overline{W} \cap (X \times \square^{n+1})$ is an admissible cycle on $X \times \square^{n+1}$ with modulus $(n+1)D$. That is, $[W] \in z^{n+1}(X \mid (n+1)D, n+1)$.

Let

$$\partial = \sum_{i=1}^{n+1} (-1)^i (\partial_i^{\infty} - \partial_i^0) : z^{n+1}(X \mid E, n+1) \to z^{n+1}(X \mid E, n)$$

be the boundary map in the cycle complex with modulus for an effective divisor $E \subset X$. We want to prove the following result in this subsection.

Proposition 5.1. *The class* $\mathrm{cyc}_{X\,|\,D}(\partial W)$ *dies in* $K_n(X, D)$ *under the cycle class map*

$$z^{n+1}(X \mid D, n) \xrightarrow{\mathrm{cyc}_{X\,|\,D}} K_n(X, D).$$

We shall prove this proposition in several steps. We begin with the following description of the cycle class map on various boundaries of W. Let F denote the function field of X. Let g denote the Milnor symbol $\{g_1, \dots, g_{n+1}\} \in K^M_{n+1}(F)$, where $g_i : \overline{W} \to \overline{\square}$ is the i-th projection for $1 \leq i \leq n+1$. Note that this symbol is well-defined because no g_i can be identically zero by the admissibility of W. For any closed point $z \in \overline{W}$, let $\mathrm{ord}_z : F^{\times} \to \mathbb{Z}$ denote the valuation associated to the discrete valuation ring $\mathcal{O}_{\overline{W},z}$. We let $\partial^M_z : K^M_{i+1}(F) \to K^M_i(k(z))$ denote a boundary map in the Gersten complex (3.1). We let $\mathfrak{m}_z$ denote the maximal ideal of the local ring $\mathcal{O}_{\overline{W},z}$. A symbol $\{a_1, \dots, \widehat{a_i}, \dots, a_n\}$ will mean the one obtained from $\{a_1, \dots, a_n\}$ by omitting a_i. For any point $z \in X \times \square^{n+1}$, let $y_i : \square^{n+1}_{k(z)} \to \square_{k(z)}$ denote the projection map to i-th factor. Let $f^z : \mathrm{Spec}(k(z)) \to X$ denote the projection to X.

Lemma 5.2. *For $1 \le i \le n+1$, we have*

$$\mathrm{cyc}_{X\,|\,D}(\partial_i^0 W) = \sum_{z \in \partial_i^0 W} \mathrm{ord}_z(g_i) f_*^z \circ \psi_z(\{y_1(z), \dots, \widehat{y_i(z)}, \dots, y_{n+1}(z)\}),$$
$$\mathrm{cyc}_{X\,|\,D}(\partial_i^\infty W) = \sum_{z \in \partial_i^\infty W} \mathrm{ord}_z(1/g_i) f_*^z \circ \psi_z(\{y_1(z), \dots, \widehat{y_i(z)}, \dots, y_{n+1}(z)\}).$$

Proof. We should first observe that the admissibility of W implies that if $z \in \partial_i^t W$ for $t \in \{0, \infty\}$, then we must have $y_j(z) \neq 0$ for all $j \neq i$. In particular, the element $\{y_1(z), \dots, \widehat{y_i(z)}, \dots, y_{n+1}(z)\} \in K_n^M(k(z))$ is well defined. By the definition of $\mathrm{cyc}_{X\,|\,D}$, it suffices to show that for $1 \le i \le n+1$, we have

$$\partial_i^0 W = \sum_{z \in \partial_i^0 W} \mathrm{ord}_z(g_i)[z] \quad \text{and} \quad \partial_i^\infty W = \sum_{z \in \partial_i^\infty W} \mathrm{ord}_z(1/g_i)[z]. \tag{5.1}$$

But this is an immediate consequence of the definition of the intersection product of an integral cycle with the faces of $X \times \square^n$. □

Lemma 5.3. *Suppose that $n \ge 1$ and $\partial_X^M(g) \in \bigoplus_{z \in X^o} K_n^M(k(z))$ under the Milnor boundary map $\partial_X^M : K_{n+1}^M(F) \to \bigoplus_{z \in X^{(1)}} K_n^M(k(z))$. Then*

$$\sum_{z \in X^o} f_*^z \circ \psi_z \circ \partial_X^M(g) = 0. \tag{5.2}$$

Proof. Suppose first that $D = \varnothing$. In this case, we have a diagram

$$\begin{tikzcd} K_{n+1}^M(F) \arrow[r, "\partial_X^M"] \arrow[d, "\psi_F"'] & \bigoplus_{z \in X^{(1)}} K_n^M(k(z)) \arrow[d, "(\psi_z)_z"] & \\ K_{n+1}(F) \arrow[r, "\partial_X^Q"] & \bigoplus_{z \in X^{(1)}} K_n(k(z)) \arrow[r, "(f_*^z)_z"] & K_n(X). \end{tikzcd} \tag{5.3}$$

The Gersten complex for Milnor K-theory canonically maps to the Gersten complex for the Quillen K-theory by Lemma A.1. In particular, the square in the above diagram is commutative. The bottom row is exact by Quillen's localization sequence and a limit argument. The lemma follows immediately from this diagram.

We now let $D = m_1 x_1 + \cdots + m_r x_r$, where $x_1, \dots, x_r$ are distinct closed points of X and $m_1, \dots, m_r$ are positive integers. Let $A = \mathcal{O}_{X,D}$ be the semilocal ring of X at D and let I denote the ideal of D inside $\mathrm{Spec}(A)$. The localization and relativization sequences give us the commutative diagram of homotopy fiber sequences

$$\begin{tikzcd}
\amalg_{z\in X^o} K(k(z)) \arrow[r, equal] \arrow[d] & \amalg_{z\in X^o} K(k(z)) \arrow[d] & \\
K(X,D) \arrow[r] \arrow[d] & K(X) \arrow[r] \arrow[d] & K(D) \arrow[d, equal] \\
K(A,I) \arrow[r] & K(A) \arrow[r] & K(D).
\end{tikzcd} \tag{5.4}$$

The associated homotopy groups long exact sequences yield the commutative diagram of exact sequences of abelian groups

$$\begin{tikzcd}
K_{n+1}(A,I) \arrow[r, "\partial^Q_A"] \arrow[d] & \bigoplus_{z\in X^o} K_n(k(z)) \arrow[r, "(f^z_*)_z"] \arrow[d, equal] & K_n(X,D) \arrow[d] \\
K_{n+1}(A) \arrow[r, "\partial^Q_A"] \arrow[d, "u^*"'] & \bigoplus_{z\in X^o} K_n(k(z)) \arrow[r, "(f^z_*)_z"] & K_n(X) \\
K_{n+1}(D), & &
\end{tikzcd} \tag{5.5}$$

where $u\colon D \hookrightarrow \operatorname{Spec}(A)$ is the inclusion map. It follows from this diagram that there is an exact sequence

$$\widehat{K}_{n+1}(A,I) \xrightarrow{\partial^Q_A} \bigoplus_{z\in X^o} K_n(k(z)) \xrightarrow{(f^z_*)_z} K_n(X,D). \tag{5.6}$$

In order to compare this with the Milnor K-theory, we consider the diagram

$$\begin{tikzcd}
\widehat{K}^M_{n+1}(A) \arrow[rr, "\partial^M_A"] \arrow[dr] \arrow[dd, "\psi_A"'] & & \bigoplus_{z\in X^o} K^M_n(k(z)) \arrow[dr, hook] \arrow[dd, "(\psi_z)_z"] & \\
& K^M_{n+1}(F) \arrow[rr, "\partial^M_X"] \arrow[dd, "\psi_F"] & & \bigoplus_{z\in X} K^M_n(k(z)) \arrow[dd, "(\psi_z)_z"] \\
K_{n+1}(A) \arrow[rr, "\partial^Q_A"] \arrow[dr] & & \bigoplus_{z\in X^o} K_n(k(z)) \arrow[dr, hook] & \\
& K_{n+1}(F) \arrow[rr, "\partial^Q_X"] & & \bigoplus_{z\in X} K_n(k(z)).
\end{tikzcd} \tag{5.7}$$

The map ψ_A comes from (3.2). By the definition of $\widehat{K}^M_{n+1}(A)$, we know that the composite map

$$\widehat{K}^M_{n+1}(A) \to K^M_{n+1}(F) \xrightarrow{\partial^M_z} K^M_n(k(z))$$

is zero for all closed points $z \in D$. Hence, the composite

$$\widehat{K}^M_{n+1}(A) \to K^M_{n+1}(F) \xrightarrow{\partial^M_X} \bigoplus_{z \in X} K^M_n(k(z))$$

factors through the map denoted by ∂^M_A in the above diagram. In particular, the top face of (5.7) commutes. Exactly the same reason shows that the bottom face also commutes. Furthermore, ∂^Q_A is same as the boundary map in the bottom row of (5.5). The left and the right faces clearly commute and so does the front face by Lemma A.1. A diagram chase shows that the back face of (5.7) commutes too.

In order to show (5.2), we consider our final diagram

$$\begin{tikzcd}
 & \widehat{K}^M_{n+1}(A) \arrow[dd, "\psi_A"] \arrow[dr, "\partial^M_A"] & & \\
\widehat{K}^M_{n+1}(A, I) \arrow[ur, hook] \arrow[rr, "\partial^M_A"] \arrow[dd, "\psi_{(A,I)}"'] & & \bigoplus_{z \in X^o} K^M_n(k(z)) \arrow[dd, "(\psi_z)_z"] & \\
 & K_{n+1}(A) \arrow[dr, "\partial^Q_A"] & & \\
\widehat{K}_{n+1}(A, I) \arrow[ur, hook] \arrow[rr, "\partial^Q_A"] & & \bigoplus_{z \in X^o} K_n(k(z)) \arrow[r, "(f^z_*)_z"] & K_n(X, D).
\end{tikzcd} \tag{5.8}$$

The left face of this diagram commutes by (3.5) and we just showed above that the right face commutes. In particular, the bottom face also commutes. Furthermore, the bottom row is same as the exact sequence (5.6). Our assertion will therefore follow if we can show that $g \in \widehat{K}^M_{n+1}(A, I)$ provided $W \in z^{n+1}(X \mid (n+1)D, n+1)$.

Suppose now that $W \in z^{n+1}(X \mid (n+1)D, n+1)$. By the definition of the modulus condition (see Section 2E), it means that $\sum_{j=1}^{n+1} \mathrm{ord}_{x_i}(g_j - 1) \geq (n+1)m_i$ for each $1 \leq i \leq r$. Since $n, m_i \geq 1$, we must have $(n+1)m_i \geq m_i + n$ for each $1 \leq i \leq r$. This implies that $g = \{g_1, \dots, g_{n+1}\} \in K^M_{n+1}(F, m_i + n)$ for every $1 \leq i \leq r$ in the notations of Lemma 3.3. If we now apply Lemma 3.3 with $R = A_{\mathfrak{m}_i}$, it follows that $g \in (1 + \mathfrak{m}_i^{m_i+n})K^M_n(F)$ for every $1 \leq i \leq r$. In other words, g lies in the intersection $\bigcap_{i=1}^r (1 + \mathfrak{m}_i^{m_i+n})K^M_n(F)$ as an element of $K^M_{n+1}(F)$. It follows from Lemma 3.5 that $g \in \widehat{K}^M_{n+1}(A, \underline{m}) = \widehat{K}^M_{n+1}(A, I)$. □

Proof of Proposition 5.1. We assume first that $n = 0$. In this case, we will show the stronger assertion that $\mathrm{cyc}_{X \mid D}(\partial W)$ dies in $K_n(X, D)$ if $W \in z^{n+1}(X \mid D, n+1)$. Let A be the semilocal ring and $I \subset A$ the ideal as in the proof of Lemma 5.3. By

(5.4), we have an exact sequence

$$K_1(A, I) \xrightarrow{\partial_A^Q} \bigoplus_{z \in X^o} K_0(k(z)) \xrightarrow{(f_*^z)_z} K_0(X, D) \to 0. \tag{5.9}$$

Comparing this with the exact sequence

$$K_1(A) \xrightarrow{\partial_A^Q} \bigoplus_{z \in X^o} K_0(k(z)) \xrightarrow{(f_*^z)_z} K_0(X),$$

we see that we can replace $K_1(A, I)$ by $\widehat{K}_1(A, I)$ in (5.9). But then, it is same as the exact sequence

$$(1+I)^\times \xrightarrow{\partial_A^M} z^1(X \mid D, 0) \xrightarrow{\mathrm{cyc}_{X \mid D}} K_0(X, D) \to 0.$$

Moreover, one knows that $\mathrm{Coker}(\partial_A^M) \cong \mathrm{CH}^1(X \mid D, 0)$ (e.g, see [Krishna 2015, §2]). We therefore showed that $\mathrm{cyc}_{X \mid D} : \mathrm{CH}^1(X \mid D, 0) \to K_0(X, D)$ is actually an isomorphism.

We now assume for the remaining part of the proof that $n \geq 1$. As before, let $g_i : \overline{W} \to \overline{\square}$ denote the projections and let $f = p_X \circ \nu : \overline{W} \to X$ be the projection to X. We also recall the element $g = \{g_1, \ldots, g_{n+1}\} \in K_{n+1}^M(F)$. Our task is to show that

$$\mathrm{cyc}_{X \mid D}(\partial W) = \sum_{i=1}^{n+1} (-1)^i \, \mathrm{cyc}_{X \mid D}(\partial_i^\infty W - \partial_i^0 W) = 0 \quad \text{in} \quad K_n(X, D) \tag{5.10}$$

if W satisfies the modulus condition for $(n+1)D$.

Our idea is to compute the cycle class of ∂W in terms of the cycle class of the Milnor boundary $\partial_X^M(g)$. In order to do this, we consider in general a closed point $z \in \overline{W}$. If $z \in \overline{W} \cap F_n^1$, then we must have $g_i(z) = 1$ for some $1 \leq i \leq n+1$. This means that $g_i - 1 \in \mathfrak{m}_z$. Lemma 3.4 then implies that $\partial_z^M(g) = 0$. If $z \in W \setminus \partial_i^{\{0,\infty\}} W$ for all $1 \leq i \leq n+1$, then we must have $g_i \in \mathcal{O}_{\overline{W},z}^\times$ for all $1 \leq i \leq n+1$ and hence $\partial_z^M(g) = 0$. If $z \in \partial_i^0 W$ for some $1 \leq i \leq n+1$, then $z \notin \partial_j^t W$ unless $(t, j) = (0, i)$. This implies that $g_j \in \mathcal{O}_{\overline{W},z}^\times$ for all $j \neq i$. Furthermore, the image of g_j under the map $\mathcal{O}_{\overline{W},z}^\times \twoheadrightarrow k(z)^\times$ is simply $y_j(z)$. By the definition of the boundary map in the Gersten complex for the Milnor K-theory (e.g., see [Bass and Tate 1973]), we therefore have

$$\partial_z^M(g) = (-1)^i \mathrm{ord}_z(g_i) \{y_1(z), \ldots, \widehat{y_i(z)}, \ldots, y_{n+1}(z)\} \in K_n^M(k(z)).$$

We have the same expression for $\partial_z^M(g)$ if $z \in \partial_i^\infty W$ for some $1 \leq i \leq n+1$.

If we identify $\overline{W}$ with X via f so that $W \subseteq X^o = X \setminus D$, it follows from the above computation of $\{\partial^M_z(g) \mid z \in \overline{W}^{(1)}\}$ and the comparison of (5.10) and Lemma 5.2 that the two things hold. Namely,

(1) The image of g under the Milnor boundary $K^M_{n+1}(F) \xrightarrow{\partial^M_X} \bigoplus_{z \in X} K^M_n(k(z))$ lies in the subgroup $\bigoplus_{z \in \partial W} K^M_n(k(z)) \subset \bigoplus_{z \in X^o} K^M_n(k(z))$.

(2) The element $\partial^M_X(g)$ maps to $\mathrm{cyc}_{X \mid D}(\partial W)$ under the composition of maps

$$\bigoplus_{z \in X^o} K^M_n(k(z)) \xrightarrow{(\psi_z)_z} \bigoplus_{z \in X^o} K_n(k(z)) \xrightarrow{(f^z_*)_z} K_n(X, D). \tag{5.11}$$

The proposition is therefore reduced to showing that $\sum_{z \in X^o} f^z_* \circ \psi_z \circ \partial^M_X(g) = 0$ if W lies in $z^{n+1}(X \mid (n+1)D, n+1)$. But this follows at once from Lemma 5.3. $\square$

6. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1 using the case of regular curves. So let k be any field. Let X be a regular quasiprojective variety of pure dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. We fix an integer $n \geq 0$. In Section 4A, we constructed the cycle class map

$$\mathrm{cyc}_{X \mid D} \colon z^{d+n}(X \mid D, n) \to K_n(X, D).$$

The naturality statements in Theorem 1.1 follow from Lemma 4.4 and Lemma 4.1. To prove Theorem 1.1, it therefore suffices to show the following.

Proposition 6.1. *Let $W \subset z^{d+n}(X \mid (n+1)D, n+1)$ be an integral cycle. Then the image of W under the composition*

$$z^{d+n}(X \mid (n+1)D, n+1) \hookrightarrow z^{d+n}(X \mid D, n+1) \xrightarrow{\partial} z^{d+n}(X \mid D, n) \xrightarrow{\mathrm{cyc}_{X \mid D}} K_n(X, D)$$

is zero.

We shall prove this proposition in several steps. Let $\overline{W} \subset X \times \overline{\square}^{n+1}$ be the closure of W and let $\nu \colon \overline{W}^N \to X \times \overline{\square}^{n+1}$ be the map induced on the normalization of $\overline{W}$. We begin with a direct proof of one easy case of the proposition as a motivating step.

Lemma 6.2. *Suppose that W lies over a closed point of X. Then the assertion of Proposition 6.1 holds.*

Proof. In this case, the modulus condition implies that such a closed point must lie in X^o. In other words, there is a closed point $x \in X^o$ such that

$$W \in z^n(\mathrm{Spec}(k(x)), n+1) \subset z^{d+n}(X \mid D, n+1).$$

Using the commutative diagram (see the construction of $\mathrm{cyc}_{X\,|\,D}$ in Section 4A)

$$\begin{tikzcd} z^n(\mathrm{Spec}(k(x)), n+1) \arrow[r, "\partial"] \arrow[d] & z^n(\mathrm{Spec}(k(x)), n) \arrow[r, "\mathrm{cyc}_x"] \arrow[d] & K_n(k(x)) \arrow[d] \\ z^{d+n}(X \mid D, n+1) \arrow[r, "\partial"] & z^{d+n}(X \mid D, n) \arrow[r, "\mathrm{cyc}_{X\,|\,D}"] & K_n(X, D), \end{tikzcd} \tag{6.1}$$

it suffices to show that $\mathrm{cyc}_x(\partial W) = 0$ in $K_n(k(x))$. We can thus assume that $X = \mathrm{Spec}(k)$ and $D = \varnothing$.

We let F denote the function field of W and let $g = \{g_1, \dots, g_{n+1}\} \in K^M_{n+1}(F)$ denote the Milnor symbol given by the projection maps $g_i : \overline{W} \to \overline{\square}$. Following the proof of Proposition 5.1, our assertion is equivalent to showing that the composition

$$K^M_{n+1}(F) \xrightarrow{\partial^M_{\overline{W}}} \bigoplus_{z \in \overline{W}^{(1)}} K^M_n(k(z)) \to \bigoplus_{z \in \overline{W}^{(1)}} K_n(k(z)) \xrightarrow{(f^z_*)_z} K_n(k)$$

kills g. Arguing as in (5.3), it suffices to show that the composite map

$$K_{n+1}(F) \xrightarrow{\partial^Q_{\overline{W}}} \bigoplus_{z \in \overline{W}^{(1)}} K_n(k(z)) \xrightarrow{(f^z_*)_z} K_n(k)$$

kills g. Since this map is same as the composite map

$$K_{n+1}(F) \xrightarrow{\partial^Q_{\overline{W}^N}} \bigoplus_{z \in (\overline{W}^N)^{(1)}} K_n(k(z)) \xrightarrow{(f^z_*)_z} K_n(k),$$

we need to show that g dies under this map. But this is the well-known Weil reciprocity theorem in algebraic K-theory (e.g., see [Weibel 2013, Chapter IV, Theorem 6.12.1]). $\square$

We now proceed to the proof of the general case of Proposition 6.1. In view of Lemma 6.2, we can assume that the projection map $f : \overline{W}^N \to X$ is finite.

This gives rise to a Cartesian square

$$\begin{tikzcd} \overline{W}^N \arrow[dr, hook, "\phi"] \arrow[drr, bend left] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\ & \overline{W}^N \times \overline{\square}^{n+1} \arrow[r] \arrow[d, "p'"] & X \times \overline{\square}^{n+1} \arrow[d, "p_X"] \\ & \overline{W}^N \arrow[r, "f"'] & X. \end{tikzcd} \tag{6.2}$$

Note that ϕ is a closed immersion. Using the finiteness of f and admissibility of W, it is evident that $W^N = \phi(\overline{W}^N) \cap (\overline{W}^N \times \square^{n+1})$ is an admissible cycle on $\overline{W}^N \times \square^{n+1}$. In other words, it intersects the faces of $\overline{W}^N \times \square^{n+1}$ properly, and satisfies the modulus $(n+1)E$, if we let $E = f^*(D) \subsetneq \overline{W}^N$. Notice that a consequence of the modulus condition for W is that E is a proper Cartier divisor on $\overline{W}^N$. Since f is finite and $E = f^*(D)$, we have a push-forward map $f_* \colon z^{n+1}(\overline{W}^N \mid (n+1)E, *) \to z^{n+d}(X \mid (n+1)D, *)$ (see [Krishna and Park 2017a, Proposition 2.10]). Since $(f \times \mathrm{id}_{\square^n})$ takes W^N to W and since $W^N \to W$ is the normalization map, we see that $f_*([W^N]) = [W] \in z^{n+d}(X \mid (n+1)D, n+1)$. In particular, we get

$$f_*(\partial W^N) = \partial(f_*([W^N])) = \partial W. \tag{6.3}$$

Proof of Proposition 6.1. In view of Lemma 6.2, we can assume that $f \colon \overline{W}^N \to X$ is finite. Using (6.3) and Lemma 4.4, we have that

$$\mathrm{cyc}_{X \mid D}(\partial W) = \mathrm{cyc}_{X \mid D} \circ f_*(\partial W^N) = f_* \circ \mathrm{cyc}_{\overline{W}^N \mid E}(\partial W^N).$$

We can therefore assume that X is a regular curve and $W \in z^{n+1}(X \mid (n+1)D, n+1)$ is an integral cycle such that the map $p_X : \overline{W} \to X$ is an isomorphism. We can now apply Proposition 5.1 to finish the proof. $\square$

Remark 6.3. We remark that throughout the proof of Theorem 1.1, it is only in Lemma 4.2 where we need to assume that X is regular everywhere (see Remark 4.3). One would like to believe that for a proper map of modulus pairs $f \colon (Y, f^*(D)) \to (X, D)$, there exists a push-forward map $f_* \colon K_*(Y, f^*(D)) \to K(X, D)$. But we do not know how to prove it.

6A. ***The cycle class map with rational coefficients.*** If we work with rational coefficients, we can prove the following improved version of Theorem 1.1. This may not be very useful in positive characteristic. However, one expects it to have many consequences in characteristic zero. The reason for this is that the relative algebraic K-groups of nilpotent ideals are known be $\mathbb{Q}$-vector spaces in characteristic zero. The proofs of Theorem 1.3 and its corollaries in this paper are crucially based on this improved version.

Theorem 6.4. *Let X be a regular quasiprojective variety of pure dimension $d \geq 1$ over a field k and let $D \subset X$ be an effective Cartier divisor. Let $n \geq 0$ be an integer. Then there is a cycle class map*

$$\mathrm{cyc}_{X \mid D} \colon \mathrm{CH}^{n+d}(X \mid D, n)_{\mathbb{Q}} \to K_n(X, D)_{\mathbb{Q}}. \tag{6.4}$$

Proof. We shall only indicate where do we use rational coefficients in the proof of Theorem 1.1 to achieve this improvement as rest of the proof is just a repetition. Since we work with rational coefficients, we shall ignore the subscript $A_{\mathbb{Q}}$ in an abelian group A in this proof and treat A as a $\mathbb{Q}$-vector space.

As we did before, we need to prove Proposition 6.1 with $W \in z^{d+n}(X \mid D, n+1)$. We can again reduce the proof of this proposition to the case when X is a regular curve and $W \in z^{n+1}(X \mid D, n+1)$ is an integral cycle such that the map $p_X : \overline{W} \to X$ is an isomorphism. We thus have to prove Proposition 5.1 with $W \in z^{n+1}(X \mid D, n+1)$. In turn, this is reduced to proving Lemma 5.3 when $W \in z^{n+1}(X \mid D, n+1)$. However, a close inspection shows that the proof of Lemma 5.3 works in the present case too with no change until its last step where we need to use Lemma 3.6 instead of Lemma 3.5. □

6B. ***Chow groups and K-theory with compact support.*** Let X be a quasiprojective scheme of pure dimension d over a field k and let $\overline{X}$ be a proper compactification of X such that $\overline{X} \setminus X$ is supported on an effective Cartier divisor D. Recall from [Binda and Saito 2019, Lemma 2.9] that $\mathrm{CH}^p(X, n)_c := \varprojlim_{m \geq 1} \mathrm{CH}^p(\overline{X} \mid mD, n)$ is independent of the choice of $\overline{X}$ and is called the higher Chow group of X with compact support. One can similarly define the algebraic K-theory with compact support by $K_n(X)_c := \varprojlim_{m \geq 1} K_n(\overline{X}, mD)$. It follows from [Kerz et al. 2018, Theorem A] that this is also independent of the choice of $\overline{X}$. As a consequence of Theorem 1.1, we get

Corollary 6.5. *Let X be a regular quasiprojective scheme of pure dimension d over a field admitting resolution of singularities. Then there exists a cycle class map*

$$\mathrm{cyc}_X : \mathrm{CH}^{n+d}(X, n)_c \to K_n(X)_c.$$

We remark that even if $\mathrm{CH}^{n+d}(X, n)_c$ is defined without resolution of singularities, this condition is needed in Corollary 6.5 because the usage of Theorem 1.1 requires that X admits regular compactifications (see Remark 6.3).

7. Milnor K-theory, 0-cycles and de Rham–Witt complex

Our next goal is to show that the cycle class map of Theorem 1.1 completely describes the relative K-theory of truncated polynomial rings in terms of additive 0-cycles in characteristic zero. We shall give a precise formulation of our main result for fields in Section 9 and for semilocal rings in Section 10. In this section, we prove some results on the connection between *a priori* three different objects: the additive 0-cycles, the relative Milnor K-theory and the de Rham-Witt forms. These results will form one of the two keys steps in showing that the cycle class map cyc_k (see (7.2)) factors through the Milnor K-theory in characteristic zero.

7A. ***The additive 0-cycles.*** To set up the notations, let k be a field of any characteristic. Let $m \geq 0$ be an integer. Recall that the additive higher Chow groups $\mathrm{TCH}^p(X, *; m)$ of $X \in \boldsymbol{Sch}_k$ with modulus m are defined so that there are canonical isomorphisms $\mathrm{Tz}^p(X, n+1; m) \cong z^p(X \times \mathbb{A}^1_k \mid X \times (m+1)\{0\}, n)$ and

$$\mathrm{TCH}^p(X, n+1; m) \xrightarrow{\cong} \mathrm{CH}^p(X \times \mathbb{A}^1_k \mid X \times (m+1)\{0\}, n), \tag{7.1}$$

where the term on the right are the Chow groups with modulus defined in Section 2E. Using a similar isomorphism between the relative K-groups, Theorem 1.1 provides a commutative diagram of pro-abelian groups

$$\begin{tikzcd} \{\mathrm{TCH}^{d+n+1}(X, n+1; m)\}_m \arrow[r, "\mathrm{cyc}_X"] \arrow[d, "\cong"'] & \{K_{n+1}(X[t]/(t^{m+1}), (t))\}_m \arrow[d, "\partial"', "\cong"] \\ \{\mathrm{CH}^{d+1+n}(X \times \mathbb{A}^1_k \mid X \times (m+1)\{0\}, n)\}_m \arrow[r, "\mathrm{cyc}_{\mathbb{A}^1_X \mid X \times \{0\}}"] & \{K_n(X \times \mathbb{A}^1_k, X \times (m+1)\{0\})\}_m \end{tikzcd} \tag{7.2}$$

for an equidimensional regular scheme X of dimension d and integer $n \geq 0$.

7B. ***Connection with de Rham–Witt complex.*** Let k be a field with $\mathrm{char}(k) \neq 2$. Let R be a regular semilocal ring which is essentially of finite type over k. Let $m, n \geq 1$ be two integers. Let $\mathbb{W}_m\Omega^*_R$ be the big de Rham–Witt complex of Hesselholt and Madsen (see [Rülling 2007, §1]). We shall let $\underline{a} = (a_1, \ldots, a_m)$ denote a general element of $\mathbb{W}_m(R)$. Recall from [Rülling 2007, Appendix] that there is an isomorphism of abelian groups $\gamma\colon \mathbb{W}(R) \xrightarrow{\cong} (1 + tR[[t]])^\times$ (with respect to addition in $\mathbb{W}(R)$ and multiplication in $R[[t]]$) such that $\gamma(\underline{a}) = \gamma((a_1, \ldots)) = \prod_{i=1}^\infty (1 - a_i t^i)$. This map sends $\mathrm{Ker}(\mathbb{W}(R) \twoheadrightarrow \mathbb{W}_m(R))$ isomorphically onto the subgroup $(1 + t^{m+1}k[[t]])^\times$ and hence there is a canonical isomorphism of abelian groups

$$\gamma_m\colon \mathbb{W}_m(R) \xrightarrow{\cong} \frac{(1 + tR[[t]])^\times}{(1 + t^{m+1}R[[t]])^\times}. \tag{7.3}$$

Under this isomorphism, the Verschiebung map $V_r\colon \mathbb{W}_m(R) \to \mathbb{W}_{mr+r-1}(R)$ corresponds to the map on the unit groups induced by the R-algebra homomorphism $R[[t]]/(t^{m+1}) \to R[[t]]/(t^{rm+r})$ under which $t \mapsto t^r$. Recall also that there is a restriction ring homomorphism $\xi_0^R\colon \mathbb{W}_{m+1}(R) \twoheadrightarrow \mathbb{W}_m(R)$ as part of the Witt-complex structure. We shall often use the notation V_r also for the composition

$$\mathbb{W}_m(R) \xrightarrow{V_r} \mathbb{W}_{mr+r-1}(R) \overset{(\xi_0^R)^{r-1}}{\twoheadrightarrow} \mathbb{W}_{mr}(R).$$

With this interpretation of the Verschiebung map, every element $\underline{a} \in \mathbb{W}_m(R)$ has a unique presentation

$$\underline{a} = \sum_{i=1}^m V_i([a_i]_{\lfloor m/i \rfloor}), \tag{7.4}$$

where for a real number $x \in \mathbb{R}_{\geq 0}$, one writes $\lfloor x \rfloor$ for the greatest integer not bigger than x and $[\ ]_{\lfloor x \rfloor}\colon R \to \mathbb{W}_{\lfloor x \rfloor}(R)$ for the Teichmüller map $[a]_{\lfloor x \rfloor} = (a, 0, \ldots, 0)$.

It was shown in [Krishna and Park 2016, Theorem 7.10] that $\{\mathrm{TCH}^*(R, *; m)\}_{m\ge 1}$ is a pro-differential graded algebra which has the structure of a restricted Witt-complex over R in the sense of [Rülling 2007, Definition 1.14]. Using the universal property of $\{\mathbb{W}_m\Omega^*_R\}_{m\ge 1}$ as the universal restricted Witt-complex over R, one gets a functorial morphism of restricted Witt-complexes

$$\tau^R_{n,m} \colon \mathbb{W}_m\Omega^{n-1}_R \to \mathrm{TCH}^n(R, n; m). \tag{7.5}$$

It was shown in [Krishna and Park 2020a, Theorem 1.0.2] that this map is an isomorphism. When R is a field, this isomorphism was shown earlier by [Rülling 2007]. We shall use this isomorphism throughout the remaining part of this paper and consequently, will usually make no distinction between the source and target of this map.

7C. ***Connection with Milnor K-theory.*** Continuing with the above notations, we have another set of maps

$$\begin{tikzcd}[column sep=small] & \mathrm{TCH}^1(R, 1; m) \otimes \mathrm{CH}^{n-1}(R, n-1) \arrow[dr, "\psi^R_{n,m}"] & \\ \mathbb{W}_m(R) \otimes K^M_{n-1}(R) \arrow[ur, "\tau^R_{1,m}\otimes\nu^R_{n-1}"] & & \mathrm{TCH}^n(R, n; m). \end{tikzcd} \tag{7.6}$$

Here, $\nu^R_n \colon K^M_n(R) \to \mathrm{CH}^n(R, n)$ is the semilocal ring analog of the Milnor-Chow homomorphism of Totaro [1992]. It takes a Milnor symbol $\{b_1, \dots, b_{n-1}\}$ to the graph of the function $(b_1, \dots, b_{n-1}) \colon \mathrm{Spec}(R) \to \square^{n-1}$. A combination of the main results [Elbaz-Vincent and Müller-Stach 2002; Kerz 2009] implies that this map is an isomorphism. The map $\psi^R_{n,m}$ is given by the action of higher Chow groups on the additive higher Chow groups, shown in [Krishna and Levine 2008]. It takes cycles $\alpha \in \mathrm{TCH}^i(R, n; m)$ and $\beta \in \mathrm{CH}^j(R, n')$ to $\Delta^*_R(\alpha \times \beta) \in \mathrm{TCH}^{i+j}(R, n+n'; m)$, where

$$\Delta_R \colon \mathrm{Spec}(R) \times \mathbb{A}^1_k \times \square^{n+n'-1} \to \mathrm{Spec}(R) \times \mathrm{Spec}(R) \times \mathbb{A}^1_k \times \square^{n+n'-1}$$

is the diagonal on $\mathrm{Spec}(R)$ and identity on $\mathbb{A}^1_k \times \square^{n+n'-1}$.

Lemma 7.1. *For $\underline{a} = (a_1, \dots, a_m) \in \mathbb{W}_m(R)$ and $\underline{b} = \{b_1, \dots, b_{n-1}\} \in K^M_{n-1}(R)$, one has $\psi^R_{n,m} \circ (\tau^R_{1,m} \otimes \nu^R_{n-1})(\underline{a} \otimes \underline{b}) = [Z]$, where*

$$Z \subset \mathrm{Spec}(R) \times \mathbb{A}^1_k \times \square^{n-1} \cong \mathbb{A}^1_R \times_R \square^{n-1}_R$$

is the closed subscheme given by

$$Z = \left\{ (t, y_1, \dots, y_n) \mid \prod_{i=1}^{m} (1 - a_i t^i) = y_1 - b_1 = \cdots = y_{n-1} - b_{n-1} = 0 \right\}. \tag{7.7}$$

Proof. We let $f(t) = \prod_{i=1}^{m}(1 - a_i t^i)$. In view of the description of $\psi^R_{n,m}$, we only have to show that $\tau^R_{1,m}(\underline{a}) = V(f(t))$. But this is a part of the definition of the restricted Witt-complex structure on $\{\mathrm{TCH}^*(R, *; m)\}_{m \geq 1}$ over R (see [Krishna and Park 2016, Proposition 7.6]). $\square$

Let $\phi^R_{n,m} : \mathbb{W}_m(R) \otimes K^M_{n-1}(R) \to \mathbb{W}_m \Omega^{n-1}_R$ be the unique map such that $\tau^R_{n,m} \circ \phi^R_{n,m} = \psi^R_{n,m} \circ (\tau^R_{1,m} \otimes \nu^R_{n-1})$. The following lemma describes the map $\phi^R_{n,m}$.

Lemma 7.2. *For any $\underline{a} \in \mathbb{W}_m(R)$ and $\underline{b} = \{b_1, \dots, b_{n-1}\} \in K^M_{n-1}(R)$, we have*

$$\phi^R_{n,m}(\underline{a} \otimes \underline{b}) = \underline{a} d \log([b_1]) \wedge \cdots \wedge d \log([b_{n-1}]).$$

Proof. For an ideal $I = (f_1, \dots, f_r) \subset R[t, y_1, \dots, y_{n-1}]$, we let $Z(f_1, \dots, f_r)$ denote the closed subscheme of $\mathrm{Spec}(R[t, y_1, \dots, y_{n-1}])$ defined by I. We let $\underline{a} \in \mathbb{W}_m(R)$ and $\underline{b} = \{b_1, \dots, b_{n-1}\} \in K^M_{n-1}(R)$. We write $b = b_1 \cdots b_{n-1} \in R^\times$. Then we have by (7.4),

$$\tau^R_{n,m}(\underline{a} d \log([b_1]) \wedge \cdots \wedge d \log([b_{n-1}])) \\ = \sum_{i=1}^{m} \tau^R_{n,m}(V_i([a_i]_{\lfloor m/i \rfloor}) d \log([b_1]) \wedge \cdots \wedge d \log([b_{n-1}])). \quad (7.8)$$

We now recall that $\tau^R_{n,m}$ is a part of the morphism of restricted Witt-complexes. In particular, we have for each $1 \leq i \leq m$,

$$\begin{aligned} &\tau^R_{n,m}(V_i([a_i]_{\lfloor m/i \rfloor}) d \log([b_1]) \wedge \cdots \wedge d \log([b_{n-1}])) \\ &\quad = V_i(Z(1 - a_i t)) \left(\prod_{j=1}^{n-1} Z(1 - b_i^{-1} t) \right) \\ &\qquad\qquad \times (d(Z(1 - b_1 t)) \wedge \cdots \wedge d(Z(1 - b_{n-1} t))) \\ &\quad =^1 Z(1 - a_i t^i) \left(\prod_{j=1}^{n-1} Z(1 - b_i^{-1} t) \right) \\ &\qquad\qquad \times (d(Z(1 - b_1 t)) \wedge \cdots \wedge d(Z(1 - b_{n-1} t))), \\ &\quad =^2 Z(1 - a_i t^i) Z(1 - b^{-1} t)(d(Z(1 - b_1 t)) \wedge \cdots \wedge d(Z(1 - b_{n-1} t))), \end{aligned} \quad (7.9)$$

where $=^1$ follows from the fact the Verschiebung map on the additive higher Chow groups is induced by the pull-back through the power map $\pi_r : \mathbb{A}^1_R \to \mathbb{A}^1_R$, given by $\pi_r(t) = t^r$ (see [Krishna and Park 2016, §6]). The equality $=^2$ follows from the fact that the product in $\mathrm{TCH}^1(R, 1; m)$ is induced by the multiplication map $\mu : \mathbb{A}^1_R \times_R \mathbb{A}^1_R \to \mathbb{A}^1_R$ (see [Krishna and Park 2016, §6]).

Since the differential of the additive higher Chow groups is induced by the antidiagonal map $(t, \underline{y}) \mapsto (t, t^{-1}, \underline{y})$, we see that $d(Z(1 - b_i t)) = Z(1 - b_i t, y_i - b_i)$.

In particular, we get

$$d(Z(1-b_1t)) \wedge \cdots \wedge d(Z(1-b_{n-1}t)) = Z(1-bt, y_1-b_1, \dots, y_{n-1}-b_{n-1}).$$

As $Z(1-b^{-1}t) \cdot Z(1-bt) = Z(1-t) = \tau^R_{1,m}(1)$ is the identity element for the differential graded algebra structure on $\mathrm{TCH}^*(R, *; m)$, we therefore get

$$\begin{aligned}\tau^R_{n,m}(V_i([a_i]_{\lfloor m/i \rfloor})d\log([b_1]) \wedge \cdots \wedge d\log([b_{n-1}])) \\ = Z(1-a_it^i, y_1-b_1, \dots, y_{n-1}-b_{n-1}).\end{aligned}$$

Combining this with (7.8), we get

$$\begin{aligned}\tau^R_{n,m}(\underline{a}d\log([b_1]) \wedge \cdots \wedge d\log([b_{n-1}])) \\ = Z\biggl(\prod_{i=1}^{m}(1-a_it^i), y_1-b_1, \dots, y_{n-1}-b_{n-1}\biggr). \quad (7.10)\end{aligned}$$

Since $\tau^R_{n,m}$ is an isomorphism, we now conclude the proof by applying Lemma 7.1. $\square$

7D. ***Additive 0-cycles in characteristic zero.*** We shall assume in this subsection that the base field k has characteristic zero. As above, we let R be a regular semilocal ring which is essentially of finite type over k and $m, n \geq 1$ two integers. For any $\underline{b} = \{b_1, \dots, b_{n-1}\} \in K^M_{n-1}(R)$, we let $b = b_1 \cdots b_{n-1} \in R^\times$. Under our assumption on $\mathrm{char}(k)$, we can prove the following result which is the first key step for showing that the cycle class map cyc_k (see (7.2)) factors through the Milnor K-theory in characteristic zero.

Lemma 7.3. *The map*

$$\psi^R_{n,m} \circ (\tau^R_{1,m} \otimes \nu^R_{n-1}) \colon \mathbb{W}_m(R) \otimes K^M_{n-1}(R) \to \mathrm{TCH}^n(R, n; m)$$

is surjective. In particular, $\psi^R_{n,m}$ is surjective.

Proof. Let $\phi^R_{n,m} \colon \mathbb{W}_m(R) \otimes K^M_{n-1}(R) \to \mathbb{W}_m\Omega^{n-1}_R$ be the unique map such that $\tau^R_{n,m} \circ \phi^R_{n,m} = \psi^R_{n,m} \circ (\tau^R_{1,m} \otimes \nu^R_{n-1})$. The lemma is then equivalent to showing that $\phi^R_{n,m}$ is surjective.

For $\underline{a} \in \mathbb{W}_m(R)$ and $\underline{b} = \{b_1, \dots, b_{n-1}\} \in K^M_{n-1}(R)$, it follows from Lemma 7.2 that $\phi^R_{n,m}(\underline{a} \otimes \underline{b}) = \underline{a}d\log([b_1]) \wedge \cdots \wedge d\log([b_{n-1}])$. We shall now use that the ground field has characteristic zero. Let $p > m$ be a prime. It then follows from [Rülling 2007, Theorem 1.11, Remark 1.12] that there is a canonical (Ghost) isomorphism

$$\zeta^R_{r,m} \colon \mathbb{W}_m\Omega^r_R \xrightarrow{\cong} \prod_{i=1}^{m} \Omega^r_R \quad (7.11)$$

such that

$$\zeta^R_{r,m}(x\,dy_1\cdots dy_r) = \left(\frac{1}{j^r}F_j(x)dF_j(y_1)\cdots dF_j(y_r)\right)_{1\le j\le m},$$

where $x, y_i \in \mathbb{W}_m(R)$ and $F_j(y_i)$ means its restriction to $\mathbb{W}_1\Omega^r_R$ via the restriction map of the de Rham–Witt complex. In particular, $\zeta^R_{0,m}(x) = \left(\sum_{d\mid j} dx_d^{j/d}\right)_{1\le j\le m}$ is the classical Ghost map, where $x = (x_1, \dots, x_m) \in \mathbb{W}_m(R)$.

It follows that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{W}_m(R)\otimes K^M_{n-1}(R) \arrow[r, "\phi^R_{n,m}"] \arrow[d, "\zeta^R_{0,m}\otimes\mathrm{id}"', "\cong"] & \mathbb{W}_m\Omega^{n-1}_R \arrow[d, "\cong"', "\zeta^R_{n-1,m}"] \\
\left(\prod_{i=1}^m R\right)\otimes K^M_{n-1}(R) \arrow[r] & \prod_{i=1}^m \Omega^{n-1}_R,
\end{tikzcd} \tag{7.12}$$

where the bottom arrow is defined componentwise so that $(a_i)_i \otimes \{b_1, \dots, b_{n-1}\}$ maps to $(1/(i^{n-1})a_i d\log(b_1)\wedge\cdots\wedge d\log(b_{n-1}))_i$. Since we are working with characteristic zero field, it then suffices to show that the map $R\otimes K^M_{n-1}(R)\to\Omega^{n-1}_R$, given by $a\otimes \underline{b}\mapsto ad\log(b_1)\wedge\cdots\wedge d\log(b_{n-1})$, is surjective. By an iterative procedure, it suffices to prove this surjectivity when $n = 2$.

We now let $a, b \in R$. By Lemma 7.4 below, we can write $b = b_1 + b_2$, where $b_1, b_2 \in R^\times$. We then get $adb = adb_1 + adb_2 = ab_1d\log(b_1) + ab_2d\log(b_2)$. Since Ω^1_R is generated by the universal derivations of the elements of R as an R-module, we are done. □

Lemma 7.4. *Let R be a semilocal ring which contains an infinite field k. Then every element $a \in R$ can be written as $a = u_1 + u_2$, where $u_1 \in k^\times$ and $u_2 \in R^\times$.*

Proof. Let $M = \{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ denote the set of all maximal ideals of R. Fix $a \in R$. Suppose that there exists $u \in k^\times \subseteq R^\times$ such that $a + u \in R^\times$. Then we are done. Otherwise, every element $u \in k^\times$ has the property that $a + u \in \mathfrak{m}_i$ for some i. Since k is infinite and M is finite, there are two distinct elements $u_1, u_2 \in k^\times$ such that $a + u_1$ and $a + u_2$ both belong to a maximal ideal $\mathfrak{m}_j$. Then we get $u_1 - u_2 \in \mathfrak{m}_j$. But $u_1 \ne u_2$ in k implies that $u_1 - u_2 \in k^\times$ and this forces $\mathfrak{m}_j = R$, a contradiction. We conclude that there must exist $u \in k^\times$ such that $a + u \in R^\times$. □

8. The relative Milnor K-theory

In this section, we shall prove our second key step (see Lemma 8.4) to show the factorization of the cycle class map through the relative Milnor K-theory and prove the isomorphism of the resulting map.

8A. ***Recollection of relative Hochschild and cyclic homology.*** In the next two sections, we shall use Hochschild, André–Quillen and cyclic homology of commutative rings as our tools. We refer to [Loday 1998] for their definitions and some properties that we shall use. While using a specific result from [Loday 1998], we shall mention the exact reference.

Let R be a commutative ring. For an integer $m \geq 0$, recall from Section 2A that the truncated polynomial algebra $R[t]/(t^{m+1})$ is denoted by R_m. Throughout our discussion of truncated polynomial algebras, we shall make no distinction between the variable $t \in R[t]$ and its image in R_m.

If $k \subset R$ is a subring, we shall use the notation $HH_*(R \mid k)$ for the Hochschild homology of R over k. Similarly, $D_*^{(q)}(R \mid k)$ for $q \geq 0$ and $HC_*(R \mid k)$ will denote the André–Quillen and cyclic homology of R over k, respectively. When $k = \mathbb{Z}$, we shall write $HH_*(R \mid k)$ simply as $HH_*(R)$. Similar notations will be used for $D_*^{(q)}(R \mid \mathbb{Z})$ and $HC_*(R \mid \mathbb{Z})$. Note that $HH_*(R) \cong HH_*(R \mid \mathbb{Q})$, $D_*^{(q)}(R) \cong D_*^{(q)}(R \mid \mathbb{Q})$ and $HC_*(R) \cong HC_*(R \mid \mathbb{Q})$ if R contains $\mathbb{Q}$. We also have $\Omega^q_R := \Omega^q_{R/\mathbb{Z}} \cong \Omega^q_{R/\mathbb{Q}}$ for $q \geq 0$.

Recall that for an ideal $I \subset R$, the relative Hochschild homology $HH_*((R, I) \mid k)$ is defined as the homology of the complex $\operatorname{Ker}(HH(R \mid k) \twoheadrightarrow HH(R/I \mid k))$, where $k \subset R$ is a subring and $HH(R \mid k)$ is the Hochschild complex of R over k. The relative cyclic homology $HC_*((R, I) \mid k)$ is defined to be the homology of the complex $\operatorname{Ker}(CC(R \mid k) \twoheadrightarrow CC(R/I \mid k))$, where $CC(R \mid k)$ is the total cyclic complex of R over k. We refer to [Loday 1998, 1.1.16, 2.1.15] for these definitions. One defines $D_*^{(q)}((R, I) \mid k)$ similarly. If R is a commutative ring, we shall write $HH_*((R_m, (t)) \mid \mathbb{Z})$, $D_*^{(q)}((R_m, (t)) \mid \mathbb{Z})$ and $HC_*((R_m, (t)) \mid \mathbb{Z})$ simply as $\widetilde{HH}_*(R_m)$, $\widetilde{D}_*^{(q)}(R_m)$ and $\widetilde{HC}_*(R_m)$, respectively. We let $\widetilde{\Omega}^n_{R_m} = \operatorname{Ker}(\Omega^n_{R_m} \twoheadrightarrow \Omega^n_R)$. Recall from Section 2C that $\widetilde{K}_*(R_m)$ denotes the relative K-theory $K_*(R_m, (t))$. Suppose now that R contains $\mathbb{Q}$. For $x \in tR_m$, we shall write $\exp(x) = \sum_{i \geq 0} x^i/i!$ and $\log(1 + x) = \sum_{i \geq 1} (-1)^{i-1} x^i/i$. Note that these are finite sums and define homomorphisms

$$tR_m \xrightarrow{\exp} \widetilde{K}_1(R_m) \xrightarrow{\log} tR_m, \tag{8.1}$$

which are inverses to each other.

8B. ***Relative Milnor K-theory of truncated polynomial rings.*** Let R be a semilocal ring and let $m \geq 0$, $n \geq 1$ be two integers. We shall write the relative Milnor K-groups (see Section 3) $K_*^M(R_m, (t))$ as $\widetilde{K}_*^M(R_m)$. Since $\widetilde{K}_*(R_m)$ is same as the kernel of the augmentation map $K_*(R_m) \twoheadrightarrow K_*(R)$, we have the canonical map $\psi_{R_m} : \widetilde{K}_*^M(R_m) \to \widetilde{K}_*(R_m)$.

Let us now assume that R is a semilocal ring containing $\mathbb{Q}$. Recall that there is a Dennis trace map $\operatorname{tr}^R_{m,n} : K_n(R_m) \to HH_n(R_m)$ which restricts to the dlog map on the Milnor K-theory (e.g., see [Geller and Weibel 1994, Example 2.1]).

Equivalently, there is a commutative diagram

$$\begin{tikzcd} K_n^M(R_m) \arrow[r, "d\log"] \arrow[d, "\psi_{R_m,n}"'] & \Omega^n_{R_m} \arrow[d, "\epsilon_n"] \\ K_n(R_m) \arrow[r, "\mathrm{tr}^R_{m,n}"] & HH_n(R_m), \end{tikzcd} \tag{8.2}$$

where ϵ_n is the canonical antisymmetrization map from Kähler differentials to Hochschild homology (see [Loday 1998, §1.3.4]).

A very well known result of [Goodwillie 1985] says that the relativization of the Dennis trace map with respect to a nilpotent ideal factors through a trace map $\widetilde{K}_n(R_m) \to \widetilde{HC}_{n-1}(R_m)$ such that its composition with the canonical Connes' periodicity map $B\colon \widetilde{HC}_{n-1}(R_m) \to \widetilde{HH}_n(R_m)$ is the relative Dennis trace map. This factorization is easily seen on $\widetilde{K}_1(R_m)$ via the chain of maps

$$\widetilde{K}_1(R_m) \xrightarrow{\log} tR_m \cong \widetilde{HC}_0(R_m) \xrightarrow{d} \widetilde{HH}_1(R_m) \cong \widetilde{\Omega}^1_{R_m}. \tag{8.3}$$

Recall that Connes' periodicity map B on $HC_0(R_m)$ coincides with the differential $d\colon R_m \to \Omega^1_{R_m}$ under the isomorphisms $R_m \cong HC_0(R_m)$ and $\Omega^1_{R_m} \cong HH_1(R_m)$. Goodwillie showed that his factorization $\widetilde{K}_n(R_m) \to \widetilde{HC}_{n-1}(R_m)$ is an isomorphism of $\mathbb{Q}$-vector spaces. We shall denote this Goodwillie's isomorphism also by $\mathrm{tr}^R_{m,n}$.

Going further, Cathelineau [1990, Theorem 1] showed that the K-group $\widetilde{K}_n(R_m)$ and the relative cyclic homology group $\widetilde{HC}_{n-1}(R_m)$ are λ-rings. Furthermore, Goodwillie's map is an isomorphism of λ-rings, thanks to [Cortiñas and Weibel 2009, Theorem 6.5.1]. In particular, it induces an isomorphism between the Adams graded pieces $\mathrm{tr}^R_{m,n}\colon \widetilde{K}^{(q)}_n(R_m) \xrightarrow{\cong} \widetilde{HC}^{(q-1)}_{n-1}(R_m)$ for every $1 \le q \le n$. As a corollary of this isomorphism and [Loday 1998, Theorem 4.6.8], we get the following:

Lemma 8.1. *The Dennis trace map induces an isomorphism of $\mathbb{Q}$-vector spaces*

$$\mathrm{tr}^R_{m,n}\colon \widetilde{K}^{(n)}_n(R_m) \xrightarrow{\cong} \frac{\widetilde{\Omega}^{n-1}_{R_m}}{d\widetilde{\Omega}^{n-2}_{R_m}}.$$

In order to relate these groups with the Milnor K-theory, we first observe that as the map $K_1^M(R_m) \to K_1(R_m)$ is an isomorphism, it follows from the properties of γ-filtration associated to the λ-ring structure on K-theory that the canonical map $K_n^M(R_m) \to K_n(R_m)$ factors through $K_n^M(R_m) \to F^n_\gamma K_n(R_m)$, where $F^\bullet_\gamma K_n(R_m)$ denotes the γ-filtration. If we consider the induced map on the relative K-groups, it follows that the canonical map $\widetilde{K}^M_n(R_m) \to \widetilde{K}_n(R_m)$ factors as

$$\psi_{R_m,n}\colon \widetilde{K}^M_n(R_m) \to \widetilde{K}^{(n)}_n(R_m) = F^n_\gamma \widetilde{K}_n(R_m) \hookrightarrow \widetilde{K}_n(R_m).$$

It follows from [Stienstra 1985, Theorem 12.3] that $\mathrm{Ker}(\psi_{R_m,n})$ is a torsion group. On the other hand, it follows from [Gorchinskiĭ and Tyurin 2018, Proposition 5.4] that $\widetilde{K}^M_n(R_m)$ is a $\mathbb{Q}$-vector space. If we now apply Soulé's [1985, Théoréme 2] computation of $F^n_\gamma \widetilde{K}_n(R_m)$, we conclude that the map $\psi_{R_m,n}$ is in fact an isomorphism. We have thus shown the following.

Lemma 8.2. *The maps*

$$\widetilde{K}^M_n(R_m) \xrightarrow{\psi_{R_m,n}} \widetilde{K}^{(n)}_n(R_m) \xrightarrow{\mathrm{tr}^R_{m,n}} \frac{\widetilde{\Omega}^{n-1}_{R_m}}{d\widetilde{\Omega}^{n-2}_{R_m}}$$

are all isomorphisms of $\mathbb{Q}$*-vector spaces.*

One knows from [Geller and Weibel 1989, 3.4.4] that the map

$$d = B\colon \widetilde{\Omega}^{n-1}_{R_m}/d\widetilde{\Omega}^{n-2}_{R_m} \to \widetilde{\Omega}^n_{R_m}$$

is injective. Using the fact that $\mathrm{tr}^R_{m,n}\colon \widetilde{K}^M_n(R_m) \to \widetilde{\Omega}^n_{R_m}$ is multiplicative (e.g., see [Kantorovitz 1999, Property 1.3] or [Loday 1998, 8.4.12]) and it is the usual logarithm on $\widetilde{K}^M_1(R_m)$ (see (8.3)), it follows from (8.2) that modulo $d\widetilde{\Omega}^{n-2}_{R_m}$, the composite map $\mathrm{tr}^R_{n,m} \circ \psi_{R_m,n}$ is the dlog map:

$$\begin{aligned} &d\log(\{1 - tf(t), b_1, \dots, b_{n-1}\}) \\ &\qquad = \log(1 - tf(t))\, d\log(b_1) \wedge \cdots \wedge d\log(b_{n-1}). \end{aligned} \tag{8.4}$$

8C. ***More refined structure on $\widetilde{K}^M_n(R_m)$.*** We shall further simplify the presentation of $\widetilde{K}^M_n(R_m)$ in the next result. This will be our second key step in factoring the cycle class map through the relative Milnor K-theory and showing that the resulting map is an isomorphism. For this, we assume $m \geq 1$ and look at the diagram

$$\begin{tikzcd}
tR_m \otimes K^M_{n-1}(R) \arrow[r] \arrow[d, "\mathrm{id}\otimes d\log"'] & \widetilde{K}^M_n(R_m) \arrow[dd, "\mathrm{tr}^R_{m,n}\circ\psi_{R_m,n}"] \\
tR_m \otimes \Omega^{n-1}_R \arrow[d, two heads] & \\
tR_m \otimes_R \Omega^{n-1}_R \arrow[r, "\theta^{n-1}_{R_m}"] \arrow[uur, dotted] & \widetilde{\Omega}^{n-1}_{R_m}/d\widetilde{\Omega}^{n-2}_{R_m},
\end{tikzcd} \tag{8.5}$$

where the top horizontal arrow is the product

$$a \otimes \{b_1, \dots, b_{n-1}\} \mapsto \{\exp(a), b_1, \dots, b_{n-1}\}.$$

The map $\theta^{n-1}_{R_m}$ in (8.5) is the composition of the canonical map $tR_m \otimes_R \Omega^{n-1}_R \to \widetilde{\Omega}^{n-1}_{R_m}$ (sending $t^i \otimes \omega$ to $t^i\omega$) with the surjection $\widetilde{\Omega}^{n-1}_{R_m} \twoheadrightarrow \widetilde{\Omega}^{n-1}_{R_m}/d\widetilde{\Omega}^{n-2}_{R_m}$. Using (8.4), it is easy to check that (8.5) is commutative. It follows from Lemma 8.2 that $\theta^{n-1}_{R_m}$ factors through a unique map $\widetilde{\theta}^n_{R_m} : tR_m \otimes_R \Omega^{n-1}_R \to \widetilde{K}^M_n(R_m)$. We want to show that this map is an isomorphism. Equivalently, $\theta^{n-1}_{R_m}$ an isomorphism. Since the proof of this is a bit long, we prove that it is surjective and injective in separate lemmas.

Lemma 8.3. *The map $\theta^n_{R_m}$ is surjective for all $n \geq 0$.*

Proof. This is obvious for $n = 0$ and so we assume $n \geq 1$. We now consider the exact sequence

$$R_m \otimes_R \Omega^1_R \to \Omega^1_{R_m} \to \Omega^1_{R_m/R} \to 0.$$

We claim that the first arrow in this sequence is split injective. For this, we consider the map $d' : R_m \to R_m \otimes_R \Omega^1_R$, given by $d'\big(\sum_{i=0}^m a_i t^i\big) = \sum_{i=0}^m t^i \otimes d(a_i)$. The computation

$$\begin{aligned} d'\Big(\Big(\sum_i a_i t^i\Big)\Big(\sum_j b_j t^j\Big)\Big) \\ &= d'\Big(\sum_{i,j} a_i b_j t^{i+j}\Big) \\ &= \sum_{i,j} t^{i+j} d(a_i b_j) \\ &= \sum_{i,j} b_j t^{i+j} \otimes d(a_i) + \sum_{i,j} a_i t^{i+j} \otimes d(b_j) \\ &= \Big(\sum_j b_j t^j\Big) \otimes d'\Big(\sum_i a_i t^i\Big) + \Big(\sum_i a_i t^i\Big) \otimes d'\Big(\sum_j b_j t^j\Big) \end{aligned}$$

shows that d' is a $\mathbb{Q}$-linear derivation on the R_m-module $\Omega^1_R \otimes_R R_m$. Hence, it induces an R_m-linear map $u : \Omega^1_{R_m} \to R_m \otimes_R \Omega^1_{R_m}$. Moreover, it is clear that the composite $R_m \otimes_R \Omega^1_R \to \Omega^1_{R_m} \xrightarrow{u} R_m \otimes_R \Omega^1_R$ is identity. This proves the claim. We thus get a direct sum decomposition of R_m-modules $\Omega^1_{R_m} = (R_m \otimes_R \Omega^1_R) \oplus \Omega^1_{R_m/R}$. As $\Omega^n_{R_m/R} = 0$ for $n \geq 2$, we get $\Omega^n_{R_m} = (R_m \otimes \Omega^n_R) \oplus (\Omega^1_{R_m/R} \otimes_R \Omega^{n-1}_R)$ for any $n \geq 1$. This implies that

$$\widetilde{\Omega}^n_{R_m} = (tR_m \otimes_R \Omega^n_R) \oplus (\Omega^1_{R_m/R} \otimes_R \Omega^{n-1}_R). \tag{8.6}$$

The other thing we need to observe is that the exact sequence

$$(t^{m+1})/(t^{2m+2}) \xrightarrow{d} \Omega^1_{R[t]/R} \otimes_{R[t]} R_m \to \Omega^1_{R_m/R} \to 0$$

implies that $\Omega^1_{R_m/R} \cong R_m dt/(t^m)dt$. In particular, $\Omega^1_{R_m/R} \otimes_R \Omega^{n-1}_R$ is generated as an R-module by elements of the form $\big(\sum_{i=0}^{m-1} a_i t^i dt\big) \otimes \omega$, where $a_i \in R$.

We now let $\omega \in \Omega^{n-1}_R$. We then get

$$d\left(\sum_{i=0}^{m-1} \frac{a_i}{i+1} t^{i+1} \otimes \omega\right)$$
$$= \sum_{i=0}^{m-1} \frac{a_i}{i+1} t^{i+1} \otimes d\omega + \sum_{i=0}^{m-1} \frac{da_i}{i+1} t^{i+1} \otimes \omega + \sum_{i=0}^{m-1} a_i t^i dt \otimes \omega$$
$$= \sum_{i=0}^{m-1} t^{i+1} \otimes \frac{a_i d\omega + da_i \wedge \omega}{i+1} + \sum_{i=0}^{m-1} a_i t^i dt \otimes \omega. \tag{8.7}$$

It follows from this that the composite map $tR_m \otimes_R \Omega^n_R \hookrightarrow \widetilde{\Omega}^n_{R_m} \to \widetilde{\Omega}^n_{R_m}/d\widetilde{\Omega}^{n-1}_{R_m}$ is surjective. $\square$

Lemma 8.4. *For $m, n \geq 1$, the map*

$$\widetilde{\theta}^n_{R_m} : tR_m \otimes_R \Omega^{n-1}_R \to \widetilde{K}^M_n(R_m);$$
$$\widetilde{\theta}^n_{R_m}(a \otimes d\log(b_1) \wedge \cdots \wedge d\log(b_{n-1})) = \{\exp(a), b_1, \ldots, b_{n-1}\},$$

is an isomorphism.

Proof. In view of Lemmas 8.2 and 8.3, the assertion that $\widetilde{\theta}^n_{R_m}$ is an isomorphism for all $m, n \geq 1$ is equivalent to showing that the map $\theta^n_{R_m} : tR_m \otimes_R \Omega^n_R \to \widetilde{\Omega}^n_{R_m}/d\widetilde{\Omega}^{n-1}_{R_m}$ (see (8.5)) is injective for all $n \geq 0$ and $m \geq 1$. We can again assume that $n \geq 1$. We shall prove this by induction on $m \geq 1$. Assume first that $m = 1$. In this case, we want to show that the map $tR \otimes_R \Omega^n_R \to \widetilde{\Omega}^n_{R_1}/d\widetilde{\Omega}^{n-1}_{R_1}$ is injective. To show this, we should first observe that every element of $tR \otimes_R \Omega^{n-1}_R$ must be of the form $t \otimes \omega$ and every element of $Rdt \otimes_R \Omega^{n-2}_R$ must be of the form $dt \otimes \omega'$. In this case, we get

$$d(t \otimes \omega + dt \otimes \omega') = dt \wedge \omega + td\omega - dt \wedge d\omega' = t \otimes d\omega + dt \otimes (\omega - d\omega'). \tag{8.8}$$

If $d(t\otimes\omega + dt\otimes\omega')$ has to lie in $Rt \otimes_R \Omega^n_R$, then we must have $dt \otimes (\omega - d\omega') = 0$. But this implies that $\omega = d\omega'$. Putting this in (8.8), we get $t \otimes d\omega = 0$. This shows that $d\widetilde{\Omega}^{n-1}_{R_1} \cap (Rt \otimes_R \Omega^n_R) = 0$. But this is equivalent to the desired injectivity.

To prove the $m \geq 2$ case by induction, we shall need the following

Claim. The restriction map $\mathrm{Ker}(\widetilde{\Omega}^{n-1}_{R_{m+1}} \xrightarrow{d} \widetilde{\Omega}^n_{R_{m+1}}) \to \mathrm{Ker}(\widetilde{\Omega}^{n-1}_{R_m} \xrightarrow{d} \widetilde{\Omega}^n_{R_m})$ is surjective for every $m \geq 1$.

To prove the claim, recall from (8.6) that

$$\widetilde{\Omega}^{n-1}_{R_m} \cong (tR_m \otimes_R \Omega^{n-1}_R) \oplus (R_m dt/(t^m)\, dt \otimes_R \Omega^{n-1}_R)$$
$$\cong \left(\bigoplus_{i=0}^{m-1} Rt^{i+1} \otimes_R \Omega^{n-1}_R\right) \oplus \left(\bigoplus_{i=0}^{m-1} Rt^i\, dt \otimes_R \Omega^{n-2}_R\right). \tag{8.9}$$

Suppose that $\omega \in \widetilde{\Omega}^{n-1}_{R_m}$ is such that $d\omega = 0$. It follows from (8.9) that we can write $\omega = \sum_{i=0}^{m-1}(i+1)^{-1}t^{i+1}\omega_i + \sum_{i=0}^{m-1} t^i dt \wedge \omega'_i$. Therefore, we have

$$\begin{aligned} d\omega &= \sum_{i=0}^{m-1}(i+1)^{-1}t^{i+1}\, d\omega_i + \sum_{i=0}^{m-1} t^i\, dt \wedge d\omega_i - \sum_{i=0}^{m-1} t^i\, dt \wedge d\omega'_i \\ &= \sum_{i=0}^{m-1}(i+1)^{-1}t^{i+1}\, d\omega_i + \sum_{i=0}^{m-1} t^i\, dt \wedge (\omega_i - d\omega'_i). \end{aligned} \tag{8.10}$$

Since the left-hand term is zero, it implies from the decomposition (8.9) (for $\widetilde{\Omega}^n_{R_m}$) that $d\omega_i = 0 = \omega_i - d\omega'_i$ for $0 \le i \le m-1$. If we now let $\omega' = \sum_{i=0}^{m-1}(i+1)^{-1}t^{i+1}\omega'_i \in \widetilde{\Omega}^{n-2}_{R_m}$, we get

$$d\omega' = \sum_{i=0}^{m-1}(i+1)^{-1}t^{i+1}d\omega'_i + \sum_{i=0}^{m-1} t^i dt \wedge \omega'_i = \omega. \tag{8.11}$$

The claim now follows from the surjectivity of the map $\widetilde{\Omega}^{n-2}_{R_{m+1}} \to \widetilde{\Omega}^{n-2}_{R_m}$. Indeed, this implies that some $\widehat{\omega} \in \widetilde{\Omega}^{n-2}_{R_{m+1}}$ maps onto ω' and therefore $d\widehat{\omega} \in \widetilde{\Omega}^{n-1}_{R_{m+1}}$ maps onto ω. Since $d\widehat{\omega}$ is clearly a closed form, we are done.

In order to use the above claim, we let $F^n_m = (Rt^m \otimes_R \Omega^n_R) \oplus (Rt^{m-1}dt \otimes_R \Omega^{n-1}_R)$. We then have an exact sequence of R_{m+1}-modules

$$0 \to F^n_{m+1} \to \widetilde{\Omega}^n_{R_{m+1}} \to \widetilde{\Omega}^n_{R_m} \to 0. \tag{8.12}$$

Letting $m \geq 1$ and taking the quotient of this short exact sequence by the similar exact sequence for $n-1$ via the differential map, we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Rt^{m+1} \otimes_R \Omega^n_R \arrow[r] \arrow[d] & tR_{m+1} \otimes_R \Omega^n_R \arrow[r] \arrow[d, "\theta^n_{R_{m+1}}"] & tR_m \otimes_R \Omega^n_R \arrow[r] \arrow[d, "\theta^n_{R_m}"] & 0 \\ 0 \arrow[r] & F^n_{m+1}/dF^{n-1}_{m+1} \arrow[r] & \widetilde{\Omega}^n_{R_{m+1}}/d\widetilde{\Omega}^{n-1}_{R_{m+1}} \arrow[r] & \widetilde{\Omega}^n_{R_m}/d\widetilde{\Omega}^{n-1}_{R_m} \arrow[r] & 0. \end{tikzcd} \tag{8.13}$$

The top row is clearly exact and the above claim precisely says that the bottom sequence is exact. The right vertical arrow is injective by induction. It suffices therefore to show that the map $Rt^{m+1} \otimes_R \Omega^n_R \to F^n_{m+1}/dF^{n-1}_{m+1}$ is injective. The proof of this is almost identical to that of $m = 1$ case. Indeed, it is easy to check that every element of F^{n-1}_{m+1} must be of the form $t^{m+1} \otimes \omega + t^m dt \otimes \omega'$. We therefore get

$$\begin{aligned} d(t^{m+1} \otimes \omega + t^m dt \otimes \omega') &= t^{m+1}d\omega + (m+1)t^m dt \wedge \omega - t^m dt \wedge d\omega' \\ &= t^{m+1}d\omega + t^m dt \wedge ((m+1)\omega - d\omega') \\ &= t^{m+1}d\omega + t^m dt \otimes ((m+1)\omega - d\omega'). \end{aligned} \tag{8.14}$$

If $d(t^{m+1} \otimes \omega + t^m dt \otimes \omega')$ lies in $Rt^{m+1} \otimes_R \Omega^n_R$, then we must have $t^m dt \otimes ((m+1)\omega - d\omega'_i) = 0$. But this implies that $\omega = d((m+1)^{-1}\omega')$. Putting this in (8.14), we get $t^{m+1} \otimes d\omega = 0$. This shows that $dF^{n-1}_{m+1} \cap (Rt^{m+1} \otimes_R \Omega^n_R) = 0$. Equivalently, the left vertical arrow in (8.10) is injective. This proves that $\theta^n_{R_m}$ is injective for all $m \geq 1$ and completes the proof of the lemma. □

We shall also need the following related result later on.

Lemma 8.5. *Let R be as above. Let $n \geq 0$ and $m \geq 1$ be integers. Then the map $d\colon tR_m \otimes_R \Omega^n_R \to \widetilde{\Omega}^{n+1}_{R_m}$ is injective.*

Proof. Suppose $\omega = \sum_{i=0}^{m-1} t^{i+1} \otimes \omega_i$ and $d(\omega) = 0$. That is, $\sum_{i=0}^{m-1} t^{i+1} \otimes d\omega_i + \sum_{i=0}^{m-1} (i+1)t^i dt \otimes \omega_i = 0$. But this implies by (8.6) that $\sum_{i=0}^{m-1} (i+1)t^i dt \otimes \omega_i = 0$. Since $\Omega^1_{R_m/R} \otimes_R \Omega^n_R \cong \Omega^n_R dt \oplus \cdots \oplus \Omega^n_R t^{m-1} dt \cong (\Omega^n_R)^m$ as an R-module, we must have $\omega_i = 0$ for each i. In particular, we have $\omega = 0$. □

9. The cycle class map in characteristic zero

In this section, we shall show that the cycle class map for the additive 0-cycles completely describes the relative K-theory of the truncated polynomial rings over a characteristic zero field in terms of additive 0-cycles. This was perhaps the main target for the introduction of the additive higher Chow groups by Bloch and Esnault [2003a]. We formulate our precise result as follows.

Let k be a characteristic zero field. Let R be a regular semilocal ring which is essentially of finite type over k. Let $m \geq 0, n \geq 1$ be two integers. Recall from Section 8B that the canonical map from the Milnor to the Quillen K-theory induces a map $\psi_{R_m,n}\colon \widetilde{K}^M_n(R_m) \to \widetilde{K}_n(R_m)$. Since this map is clearly compatible with change in $m \geq 0$, we have a strict map of pro-abelian groups

$$\psi_{R,n}\colon \{\widetilde{K}^M_n(R_m)\}_m \to \{\widetilde{K}_n(R_m)\}_m. \tag{9.1}$$

In this section, we shall restrict our attention to the case when R is the base field k itself. We shall prove a general result for regular semilocal rings in the next section. In the case of the field k, every integer $n \geq 1$ has associated to it a diagram of pro-abelian groups:

$$\begin{tikzcd} & \{\widetilde{K}^M_n(k_m)\}_m \arrow[d, "\psi_k"] \\ \{\mathrm{TCH}^n(k, n; m)\}_m \arrow[ur, dotted] \arrow[r, "\mathrm{cyc}_k"] & \{\widetilde{K}_n(k_m)\}_m. \end{tikzcd} \tag{9.2}$$

The following is our main result.

Theorem 9.1. *If k is a field of characteristic zero, then all maps in* (9.2) *are isomorphisms.*

The proof of this theorem will be done by combining the results of Section 7 and Section 8 with a series of new steps.

9A. *Factorization of* $\mathbf{cyc}_k$ *into Milnor K-theory.* We follow two step strategy for proving Theorem 9.1. We shall first show that cyc_k factors through the Milnor K-theory and the resulting map is an isomorphism. The second step will be to compare the Milnor and Quillen K-groups in the pro-setting. Apart from showing factorization through the Milnor K-theory, the following result also improves Theorem 1.1 in that it tells us that for additive higher Chow groups of 0-cycles, the cycle class map is a strict morphism of pro-abelian groups (see Section 2B).

Lemma 9.2. *Let $m \geq 0, n \geq 1$ be two integers. Then the following hold.*

(1) *The map* $\mathrm{cyc}_k \colon \mathrm{Tz}^n(k, n; m) \to \widetilde{K}_n(k_m)$ *descends to a group homomorphism*
$$\mathrm{cyc}_k \colon \mathrm{TCH}^n(k, n; m) \to \widetilde{K}_n(k_m).$$

(2) *The map* cyc_k *has a factorization*
$$\mathrm{TCH}^n(k, n; m) \xrightarrow{\mathrm{cyc}_k^M} \widetilde{K}_n^M(k_m) \xrightarrow{\psi_{k_m,n}} \widetilde{K}_n(k_m).$$

Proof. Since $\mathrm{char}(k) = 0$, we know by the main result of [Rülling 2007] (see also [Krishna and Park 2016, Theorem 1.2]) that each $\mathrm{TCH}^n(k, n; m)$ is a k-vector space. Similarly, $\widetilde{K}_n(k_m)$ is a $\mathbb{Q}$-vector space because it is isomorphic to $\widetilde{HC}_{n-1}(k_m)$ by [Goodwillie 1985]. The first part of the lemma therefore follows directly from Theorem 6.4.

We shall now prove the second part. Since $\psi_{k_m,n}$ is injective for each $m \geq 0, n \geq 1$ by Lemma 8.2, we only need to show that cyc_k takes a set of generators of the group $\mathrm{TCH}^n(k, n; m)$ to $\widetilde{K}_n^M(k_m)$.

We first assume that $n = 1$. Let $z \in \mathbb{A}^1_k$ be a closed point. We can write $z = \mathrm{Spec}(k[t]/(f(t)))$, where $f(t)$ is an irreducible polynomial. The modulus condition for z implies that $f(0) \in k^\times$. If we let $g(t) = (f(0))^{-1} f(t)$ and let $\overline{g(t)}$ denote the image of $g(t)$ in k_m, then we see that $\overline{g(t)} \in \widetilde{K}_1^M(k_m)$. By the definition of the cycle class map in (4.3), we have that $\mathrm{cyc}_k(z) = [k(z)] \in K_0(\mathbb{A}^1_k, (m+1)\{0\})$. But Lemma 2.1 says that $[k(z)] = \partial(\overline{g(t)})$ under the isomorphism $\widetilde{K}_1^M(k_m) = \widetilde{K}_1(k_m) \xrightarrow{\partial} K_0(k[t], (t^{m+1}))$. So we are done.

Suppose now that $n \geq 2$. In this case, Lemma 7.3 says that $\mathrm{TCH}^n(k, n; m)$ is generated by closed points $z \in \mathbb{A}^1_k \times \square^{n-1}$ which lie in $\mathbb{A}^1_k \times \mathbb{G}_m^{n-1} \subset \mathbb{A}^n_k$. Furthermore, $z \in \mathbb{A}^n_k$ is defined by an ideal $I \subset k[t, y_1, \ldots, y_{n-1}]$ of the type $I = (f(t), y_1 - b_1, \ldots, y_{n-1} - b_{n-1})$, where $b_i \in k^\times$ for each $1 \leq i \leq n-1$. Since z is a closed point, $f(t)$ must be an irreducible polynomial in $k[t]$. Moreover, $f(t)$ defines an element of $\mathbb{W}_m(k) \cong \widetilde{K}_1^M(k_m)$. In particular, $f(0) \in k^\times$.

We next note that the push-forward map $K_*(k(z)) \to K_*(k(z'))$ is $K_*(k)$-linear, where $z' = \mathrm{Spec}(k[t]/(f(t)))$ (see [Thomason and Trobaugh 1990, Chapter 3]). The map $K_*(k(z')) \to K_*(k[t], (t^{m+1}))$ is $K_*(k)$-linear by Lemma 2.2. It follows that the composition $K_*(k(z)) \to K_*(k[t], (t^{m+1}))$ is $K_*(k)$-linear.

It follows therefore from the definition of the cycle class map in (4.3) that under the map $\mathrm{cyc}_k \colon \mathrm{Tz}^n(k, n; m) \to K_{n-1}(k[t], (t^{m+1}))$, we have

$$\begin{aligned}&\mathrm{cyc}_k([z])\\ &\quad = \{b_1, \ldots, b_{n-1}\} \cdot [k(z')] \in K^M_{n-1}(k) \cdot K_0(k[t], (t^{m+1})) \subseteq K_{n-1}(k[t], (t^{m+1})),\end{aligned}$$

where $\{b_1, \ldots, b_{n-1}\} \in K^M_{n-1}(k)$. We let $g(t) = (f(0))^{-1} f(t)$ and let $\overline{g(t)}$ be the image of $g(t)$ in k_m via the surjection $k[t] \twoheadrightarrow k_m$.

Since $\partial \colon \widetilde{K}_n(k_m) \xrightarrow{\cong} K_{n-1}(k[t], (t^{m+1}))$ is $K_*(k)$-linear, we see that

$$\{b_1, \ldots, b_{n-1}\} \cdot [k(z')] = \{b_1, \ldots, b_{n-1}\} \cdot \partial(\overline{g(t)}) = \partial(\{b_1, \ldots, b_{n-1}\} \cdot \overline{g(t)}).$$

We are now done because

$$\{b_1, \ldots, b_{n-1}\} \cdot \overline{g(t)} \in K^M_{n-1}(k) \cdot \widetilde{K}^M_1(k_m) \subseteq \widetilde{K}^M_n(k_m).$$

We have thus shown that $\mathrm{cyc}_k([z])$ lies in the image of $\widetilde{K}^M_n(k_m)$ under the map ∂. □

9B. ***The main result for*** $\mathbf{cyc}^M_k$**.** Let k be a characteristic zero field as before. We shall now show that the cycle class map cyc^M_k that we obtained in Lemma 9.2 is an isomorphism. Since the map $\tau^k_{n,m} \colon \mathbb{W}_m\Omega^{n-1}_k \to \mathrm{TCH}^n(k, n; m)$ is an isomorphism, we shall make no distinction between $\mathbb{W}_m\Omega^{n-1}_k$ and $\mathrm{TCH}^n(k, n; m)$ throughout our discussion of the proof of Theorem 9.1. Furthermore, we shall denote $\mathrm{cyc}_k \circ \tau^k_{n,m}$ also by cyc_k in what follows.

Theorem 9.3. *For every pair of integers $m \geq 0$, $n \geq 1$, the map*

$$\mathrm{cyc}^M_k \colon \mathrm{TCH}^n(k, n; m) \to \widetilde{K}^M_n(k_m)$$

is an isomorphism.

Proof. When $m = 0$, the group on the right of cyc_k is zero by definition and the one on the left is zero by [Krishna and Park 2017b, Theorem 6.3]. We can therefore assume that $m \geq 1$.

We can replace $\mathrm{TCH}^n(k, n; m)$ by $\mathbb{W}_m\Omega^{n-1}_k$. Accordingly, we can identify cyc^M_k with $\mathrm{cyc}^M_k \circ \tau^k_{n,m}$. Let $\eta^k_m \colon \Omega^{n-1}_k \to t^m k_m \otimes_k \Omega^{n-1}_k$ denote the (k-linear) map $\eta^k_m(a\omega) = -at^m \otimes \omega$. This is clearly an isomorphism of k-vector spaces.

We shall prove the theorem by induction on $m \geq 1$. Suppose first that $m = 1$. In this case, it follows from Lemmas 7.1 and 7.2 that

$$\begin{aligned}
&\mathrm{cyc}_k^M(ad\log(b_1) \wedge \cdots \wedge d\log(b_{n-1})) \\
&\qquad = \mathrm{cyc}_k^M(Z(1-at, y_1 - b_1, \ldots, y_{n-1} - b_{n-1})) \\
&\qquad = \{1-at, b_1, \ldots, b_{n-1}\} \\
&\qquad = \widetilde{\theta}^n_{k_1}((-at) \otimes d\log(b_1) \wedge \cdots \wedge d\log(b_{n-1})) \\
&\qquad = \widetilde{\theta}^n_{k_1} \circ \eta^k_1(ad\log(b_1) \wedge \cdots \wedge d\log(b_{n-1})).
\end{aligned}$$

It follows from this that $\mathrm{cyc}_k^M = \widetilde{\theta}^n_{k_1} \circ \eta^k_1$. We now apply Lemma 8.4 to conclude that cyc_k^M is an isomorphism.

Suppose now that $m \geq 2$. Let F^n_m denote the kernel of the restriction map $\widetilde{K}^M_n(k_m) \twoheadrightarrow \widetilde{K}^M_n(k_{m-1})$. It is easy to see that the isomorphism $\widetilde{\theta}^n_{k_m}$ of Lemma 8.4 commutes with the restriction map $k_m \twoheadrightarrow k_{m-1}$ for all $m \geq 1$. It follows therefore from Lemma 8.4 and the snake lemma that $\widetilde{\theta}^n_{k_{m+1}}$ restricts to an isomorphism

$$\widetilde{\theta}^n_{k_{m+1}} : \Omega^{n-1}_k \otimes_k t^{m+1}k_{m+1} \xrightarrow{\cong} F^n_{m+1}; \tag{9.3}$$
$$\widetilde{\theta}^n_{k_{m+1}}((d\log(b_1) \wedge \cdots \wedge d\log(b_{n-1})) \otimes at^{m+1}) = \{1 + at^{m+1}, b_1, \ldots, b_{n-1}\}.$$

We now consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{n-1}_k \arrow[r, "V_{m+1}"] \arrow[d, "\widetilde{\theta}^n_{k_{m+1}} \circ \eta^k_{m+1}"'] & \mathbb{W}_{m+1}\Omega^{n-1}_k \arrow[r] \arrow[d, "\mathrm{cyc}_k^M"] & \mathbb{W}_m\Omega^{n-1}_k \arrow[r] \arrow[d, "\mathrm{cyc}_k^M"] & 0 \\
0 \arrow[r] & F^n_{m+1} \arrow[r] & \widetilde{K}^M_n(k_{m+1}) \arrow[r] & \widetilde{K}^M_n(k_m) \arrow[r] & 0,
\end{tikzcd} \tag{9.4}$$

where the horizontal arrows on the right in both rows are the restriction maps. In particular, the square on the right is commutative.

We next note that $V_{m+1}(ad\log(b_1) \wedge \cdots \wedge d\log(b_{n-1})) = V_{m+1}(a)d\log(b_1) \wedge \cdots \wedge d\log(b_{n-1})$ (see [Rülling and Saito 2018, Proposition 4.4]). Indeed, by iteration, it is enough to check this for $n = 2$. Furthermore, we can check it in $\mathrm{TCH}^2(k, 2; m)$. Now, we have

$$\begin{aligned}
V_{m+1}(ad\log(b_1)) &= V_{m+1}(Z(1-ab_1^{-1}t)d(Z(1-b_1t))) \\
&= V_{m+1}(Z(1-at, y_1 - b_1)) \\
&= Z(1-at^{m+1}, y_1 - b_1),
\end{aligned} \tag{9.5}$$

where the last two equalities follow from the definitions of differential and Verschiebung on the additive higher Chow groups (see the proof of Lemma 7.2). On the other hand, we have

$$\begin{aligned} V_{m+1}(a)d\log(b_1) &= V_{m+1}(a)b_1^{-1}db_1 \\ &= (V_{m+1}(Z(1-at))Z(1-b_1^{-1}t)d(Z(1-b_1t)) \\ &= Z(1-at^{m+1})Z(1-b_1^{-1}t)d(Z(1-b_1t)) \\ &= Z(1-at^{m+1})Z(1-b_1^{-1}t)Z(1-b_1t,\, y_1-b_1) \\ &= Z(1-at^{m+1})Z(1-t,\, y_1-b_1) \\ &= Z(1-at^{m+1},\, y_1-b_1), \end{aligned} \tag{9.6}$$

where the equalities again follow from various definitions; see the proof of Lemma 7.2. A combination of (9.5) and (9.6) proves what we had claimed. Since the map $k\otimes K^M_{n-1}(k)\to\Omega^{n-1}_k$ (given by $a\otimes\underline{b}\mapsto ad\log(b_1)\wedge\cdots\wedge d\log(b_{n-1})$) is surjective, it follows from (7.10) and (9.3) that the left square in (9.4) is also commutative.

The top row of (9.4) is well known to be exact in characteristic zero (e.g., see [Rülling and Saito 2018, Remark 4.2]). The bottom row is exact by the definition of F^n_{m+1} and the fact that the map $\widetilde{K}^M_n(k_{m+1})\to\widetilde{K}^M_n(k_m)$ is surjective. The left vertical arrow in (9.4) is an isomorphism by (9.3) and the right vertical arrow is an isomorphism by induction. We conclude that the middle vertical arrow is also an isomorphism. This completes the proof of the theorem. □

Remark 9.4. The reader can check that the proof of Theorem 9.3 continues to work for any regular semilocal ring R containing k as long as cyc_R is defined in a way it is for fields. In more detail, the isomorphism η^R_m makes sense. Lemmas 7.1 and 7.2 as well as (9.3) are all valid for R. The final step (9.4) also holds for R. This observation will be used in the next section in extending Theorem 9.1 to regular semilocal rings.

9C. *The final step.* We shall now prove our final step in the proof of Theorem 9.1. Namely, we shall show that the reduced Milnor and Quillen K-theories of the truncated polynomial rings are pro-isomorphic. This will finish the proof of Theorem 9.1. The desired pro-isomorphism is based on the following general result about pro-vanishing of relative cyclic homology.

Let R be a ring containing $\mathbb{Q}$. Recall from [Loday 1998, §4.5, 4.6] that for integers $m\ge 0$ and $n\ge 1$, there are functorial λ-decompositions $\widetilde{HH}_n(R_m)=\bigoplus_{i=1}^n\widetilde{HH}^{(i)}_n(R_m)$ and $\widetilde{HC}_n(R_m)=\bigoplus_{i=1}^n\widetilde{HC}^{(i)}_n(R_m)$. Moreover, there are isomorphisms

$$\widetilde{HH}^{(n)}_n(R_m)\cong\widetilde{\Omega}^n_{R_m}\quad\text{and}\quad\widetilde{HC}^{(n)}_n(R_m)\cong\frac{\widetilde{\Omega}^n_{R_m}}{d\widetilde{\Omega}^{n-1}_{R_m}} \tag{9.7}$$

such that the canonical map $\widetilde{HH}_n^{(n)}(R_m) \to \widetilde{HC}_n^{(n)}(R_m)$ is the quotient map $\widetilde{\Omega}^n_{R_m} \to \widetilde{\Omega}^n_{R_m}/d\widetilde{\Omega}^{n-1}_{R_m}$. We also have the functorial isomorphisms

$$\widetilde{HH}_n^{(i)}(R_m) \cong \widetilde{D}^{(i)}_{n-i}(R_m). \tag{9.8}$$

The key result is the following.

Proposition 9.5. *Let R be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Let $0 < i < n$ be two integers. Then*

$$\{\widetilde{HC}_n^{(i)}(R_m)\}_m = 0.$$

Proof. First we show that $\{\widetilde{HH}_n^{(i)}(R_m)\}_m = 0$. By (9.8), this is equivalent to showing that $\{\widetilde{D}^{(i)}_{n-i}(R_m)\}_m = 0$. To show this latter vanishing, we let $A = R[t^2, t^3] \subset R[t]$ be the monomial subalgebra generated by $\{t^2, t^3\}$. We then know that the inclusion $A \hookrightarrow R[t]$ is the normalization homomorphism whose conductor ideal is $I = (t^2, t^3) \subset A$ such that $IR[t] = (t^2)$. Since $R[t]$ is regular, we know that $D^{(i)}_{n-i}(R[t]) = 0$ for $0 < i < n$ by [Loday 1998, Theorem 3.5.6]. We now apply part (ii) of [Krishna 2010, Proposition 5.2] with $A = R[t^2, t^3]$ and $B = R[t]$ to conclude that $\{D^{(i)}_{n-i}(R_{2m-1})\}_m = \{D^{(i)}_{n-i}(R[t]/(t^{2m}))\}_m = 0$. But this is same as saying that $\{D^{(i)}_{n-i}(R_m)\}_m = 0$. In particular, we get $\{\widetilde{D}^{(i)}_{n-i}(R_m)\}_m = 0$.

To prove the result for cyclic homology, we first assume $n \geq 2$ and $i = n-1$. Connes' periodicity exact sequence (see [Loday 1998, Theorem 4.6.9]) gives an exact sequence

$$\widetilde{HH}_n^{(n-1)}(R_m) \xrightarrow{I} \widetilde{HC}_n^{(n-1)}(R_m) \xrightarrow{S} \frac{\widetilde{\Omega}^{n-2}_{R_m}}{d\widetilde{\Omega}^{n-3}_{R_m}} \xrightarrow{B} \widetilde{\Omega}^{n-1}_{R_m} \xrightarrow{I} \frac{\widetilde{\Omega}^{n-1}_{R_m}}{d\widetilde{\Omega}^{n-2}_{R_m}}.$$

We have shown that the first term in this exact sequence vanishes. It suffices therefore to show that the map B is injective. But B is same as the differential map $d\colon \widetilde{\Omega}^{n-2}_{R_m}/d\widetilde{\Omega}^{n-3}_{R_m} \to \widetilde{\Omega}^{n-1}_{R_m}$ by [Loday 1998, Corollary 2.3.5]. By Lemma 8.4, we therefore have to show that the map $d\colon \Omega^{n-2}_R \otimes_R tR_m \to \widetilde{\Omega}^{n-1}_{R_m}$ is injective. But this follows from Lemma 8.5.

We prove the general case by induction on $n \geq 2$. The case $n = 2$ is just proven above. When $n \geq 3$, we can assume that $1 \leq i \leq n-2$ by what we have shown above. We now again use the periodicity exact sequence to get an exact sequence of pro-abelian groups:

$$\{\widetilde{HH}_n^{(i)}(R_m)\}_m \xrightarrow{I} \{\widetilde{HC}_n^{(i)}(R_m)\}_m \xrightarrow{S} \{\widetilde{HC}_{n-2}^{(i-1)}(R_m)\}_m.$$

Since $i \leq n-2$, we get $i-1 \leq n-3$. Hence, the induction hypothesis implies that the last term of this exact sequence is zero. We have shown above that the first term is zero. We conclude that the middle term is zero as well. $\square$

Corollary 9.6. *Let R be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Let $n \geq 0$ be an integer. Then the canonical map*

$$\psi_{R,n} \colon \{\widetilde{K}^M_n(R_m)\}_m \to \{\widetilde{K}_n(R_m)\}_m$$

is an isomorphism of pro-abelian groups.

Proof. The case $n \leq 2$ is well known (e.g, see [Kerz 2010, Proposition 2]) and in fact an isomorphism at every level $m \geq 0$. We can therefore assume that $n \geq 3$. We have seen in Section 8B that Goodwillie provides an isomorphism of $\mathbb{Q}$-vector spaces

$$\mathrm{tr}^R_{m,n} \colon \widetilde{K}_n(R_m) \xrightarrow{\cong} \widetilde{HC}_{n-1}(R_m) \cong \bigoplus_{i=1}^{n-1} \widetilde{HC}^{(i)}_{n-1}(R_m).$$

Moreover, Lemmas 8.1 and 8.2 say that the map $\psi_{R_m,n} \colon \widetilde{K}^M_n(R_m) \to \widetilde{K}_n(R_m)$ is injective and $\mathrm{tr}^R_{m,n}$ maps $\widetilde{K}^M_n(R_m)$ isomorphically onto $\widetilde{HC}^{(n-1)}_{n-1}(R_m)$. We therefore have to show that $\{\widetilde{HC}^{(i)}_{n-1}(R_m)\}_m = 0$ for $1 \leq i \leq n-2$. But this follows from Proposition 9.5. □

Proof of Theorem 9.1. The proof is a combination of Lemma 9.2, Theorem 9.3 and Corollary 9.6. □

10. The cycle class map for semilocal rings

In this section, we shall define the cycle class map for relative 0-cycles over regular semilocal rings and prove an extension of Theorem 9.1 for such rings. Let k be a characteristic zero field and let R be a regular semilocal ring which is essentially of finite type over k. We shall let F denote the total quotient ring of R. Note that R is a product of regular semilocal integral domains and F is the product of their fields of fractions. Since all our proofs for regular semilocal integral domains directly generalize to finite products of such rings, we shall assume throughout that R is an integral domain. We shall let $\pi : \mathrm{Spec}(F) \to \mathrm{Spec}(R)$ denote the inclusion of generic point. We shall often write $X = \mathrm{Spec}(R)$ and $\eta = \mathrm{Spec}(F)$. We shall let Σ denote the set of all maximal ideals of R.

10A. ***The sfs cycles.*** Let $m \geq 0$ and $n \geq 1$ be two integers. Recall from Section 2E and Section 7A that $\mathrm{TCH}^n(R, n; m)$ is the defined as the middle homology of the complex $\mathrm{Tz}^n(R, n+1; m) \xrightarrow{\partial} \mathrm{Tz}^n(R, n; m) \xrightarrow{\partial} \mathrm{Tz}^n(R, n-1; m)$. A cycle in $\mathrm{Tz}^n(R, n; m)$ has relative dimension zero over R. For this reason, $\mathrm{TCH}^n(R, n; m)$ is often called the additive higher Chow group of relative 0-cycles on R. When R is a field, it coincides with the one used in the statement of Theorem 9.1.

Since $\mathrm{TCH}^n(R, n; m)$ does not consist of 0-cycles if $\dim(R) \geq 1$, we can not directly apply Theorem 1.1 to define a cycle class map for $\mathrm{TCH}^n(R, n; m)$. We have to use a different approach for constructing the cycle class map. We shall use the main results of [Krishna and Park 2020b] and the case of fields to construct a cycle class map in this case. We shall show later in this section that this map is an isomorphism. We shall say that an extension of regular semilocal rings $R_1 \subset R_2$ is simple if there is an irreducible monic polynomial $f \in R_1[t]$ such that $R_2 = R_1[t]/(f(t))$.

Let $Z \subset X \times \mathbb{A}^1_k \times \square^{n-1}$ be an irreducible admissible relative 0-cycle. Recall from [Krishna and Park 2020a, Definition 2.5.2, Proposition 2.5.3] that Z is called an sfs-cycle if the following hold:

(1) Z intersects $\Sigma \times \mathbb{A}^1_k \times F$ properly for all faces $F \subset \square^{n-1}$.

(2) The projection $Z \to X$ is finite and surjective.

(3) Z meets no face of $X \times \mathbb{A}^1_k \times \square^{n-1}$.

(4) Z is closed in $X \times \mathbb{A}^1_k \times \mathbb{A}^{n-1}_k = \operatorname{Spec}(R[t, y_1, \ldots, y_{n-1}])$ (by (2) above) and there is a sequence of simple extensions of regular semilocal rings

$$R = R_{-1} \subset R_0 \subset \cdots \subset R_{n-1} = k[Z]$$

such that $R_0 = R[t]/(f_0(t))$ and $R_i = R_{i-1}[y_i]/(f_i(y_i))$ for $1 \le i \le n-1$.

Note that an sfs-cycle has no boundary by (3) above. We let $\mathrm{Tz}^n_{\mathrm{sfs}}(R, n; m) \subset \mathrm{Tz}^n(R, n; m)$ be the free abelian group on integral sfs-cycles and define

$$\mathrm{TCH}^n_{\mathrm{sfs}}(R, n; m) = \frac{\mathrm{Tz}^n_{\mathrm{sfs}}(R, n; m)}{\partial(\mathrm{Tz}^n(R, n+1; m)) \cap \mathrm{Tz}^n_{\mathrm{sfs}}(R, n; m)}. \tag{10.1}$$

We shall use the following result from [Krishna and Park 2020b, Theorem 1.1].

Proposition 10.1. *The canonical map* $\mathrm{TCH}^n_{\mathrm{sfs}}(R, n; m) \to \mathrm{TCH}^n(R, n; m)$ *is an isomorphism.*

10B. ***The cycle class map.*** By Proposition 10.1, it suffices to define the cycle class map on $\mathrm{TCH}^n_{\mathrm{sfs}}(R, n; m)$. We can now repeat the construction of Section 4 word by word to get our map. So let $Z \subset X \times \mathbb{A}^1_k \times \square^{n-1}$ be an irreducible sfs-cycle and let $R' = k[Z]$. Let $f : Z \to X \times \mathbb{A}^1_k$ be the projection map. Let $g_i : Z \to \square$ denote the i-th projection. Then the sfs property implies that each g_i defines an element of $R'^{\times}$, and this in turn gives a unique element $\mathrm{cyc}^M_{R'}([Z]) = \{g_1, \ldots, g_{n-1}\} \in K^M_{n-1}(R')$. We let $\mathrm{cyc}_{R'}([Z])$ be its image in $K_{n-1}(R')$ under the map $K^M_{n-1}(R') \to K_{n-1}(R')$. Since Z does not meet $X \times \{0\}$, we see that the finite map f defines a push-forward map of spectra $f_* : K(R') \to K(R[t], (t^{m+1}))$. We let $\mathrm{cyc}_R([Z]) = f_*(\mathrm{cyc}_{R'}([Z])) \in K_{n-1}(R[t], (t^{m+1}))$. We extend this definition linearly to get a cycle map $\mathrm{cyc}_R : \mathrm{Tz}^n_{\mathrm{sfs}}(R, n; m) \to K_{n-1}(R[t], (t^{m+1}))$. We can now prove our first result of this section.

Theorem 10.2. *The assignment* $[Z] \mapsto \mathrm{cyc}_R([Z])$ *defines a cycle class map*

$$\mathrm{cyc}_R \colon \mathrm{TCH}^n(R, n; m) \to K_{n-1}(R[t], (t^{m+1})),$$

which is functorial in R.

Proof. Let F be the fraction field of R. We consider the diagram

$$\begin{tikzcd} \partial^{-1}(\mathrm{Tz}^n_{\mathrm{sfs}}(R, n; m)) \arrow[r, "\partial"] \arrow[d, "\pi^*"'] & \mathrm{Tz}^n_{\mathrm{sfs}}(R, n; m) \arrow[r, "\mathrm{cyc}_R"] \arrow[d, "\pi^*"] & K_{n-1}(R[t], (t^{m+1})) \arrow[d, "\pi^*"] \\ \mathrm{Tz}^n(F, n+1; m) \arrow[r, "\partial"] & \mathrm{Tz}^n(F, n; m) \arrow[r, "\mathrm{cyc}_F"] & K_{n-1}(F[t], (t^{m+1})). \end{tikzcd} \tag{10.2}$$

Assume first that this diagram is commutative. Then Theorem 9.3 says that $\mathrm{cyc}_F \circ \partial \circ \pi^*$ is zero. Equivalently, $\pi^* \circ \mathrm{cyc}_R \circ \partial = 0$. We will be therefore done if we know that the right vertical map π^* between the relative K-groups is injective. To show this, we can replace these relative K-groups by the relative cyclic homology groups by [Goodwillie 1985]. These relative cyclic homology groups in turn can be replaced by the Hochschild homology $HH_*(R)$ and $HH_*(F)$ by [Hesselholt 2005, Proposition 8.1]. Since R is regular, we can go further and replace $HH_*(R)$ and $HH_*(F)$ by Ω^*_R and Ω^*_F, respectively, by the famous Hochschild–Kostant–Rosenberg theorem. We therefore have to show that the map $\Omega^*_R \to \Omega^*_F$ is injective. But this follows from Lemma 10.3.

To show that cyc_R is natural for homomorphisms of regular semilocal rings $R \to R'$, we first observe that (10.2) shows that cyc_R is functorial for the inclusion $R \hookrightarrow F$. Since the right-most vertical arrow in (10.2) is injective, we can replace R and R' by their fraction fields to check the naturality of cyc_R in general. In this case, the naturality of cyc_R follows from Theorem 1.1. It remains now to show that (10.2) is commutative.

The left square is known to be commutative by the flat pull-back property of additive cycle complex. To show that the right square commutes, let $Z \subset \mathbb{A}^1_R \times_R \square^{n-1}_R$ be an irreducible sfs-cycle and let $R_{n-1} = k[Z]$. Let R_0 be the coordinate ring of the image of Z in $\mathbb{A}^1_R$ as in the definition of the sfs-cycles. By definition of sfs-cycles, we have the commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\pi"'] & R_0 \arrow[r, "f_0"] \arrow[d, "\pi_0"] & R_{n-1} \arrow[d, "\pi_{n-1}"] \\ F \arrow[r] & F_0 \arrow[r, "f_0"] & F_{n-1}, \end{tikzcd} \tag{10.3}$$

where each term in the bottom row is the quotient field of the corresponding term on the top row. Note also that all horizontal arrows are finite maps of regular semilocal integral domains. In particular, we have $F_0 = R_0 \otimes_R F$ and $F_{n-1} = R_{n-1} \otimes_R F = R_{n-1} \otimes_{R_0} F_0$.

We let $f\colon Z \to \mathbb{A}^1_R$ be the projection and let $p\colon \operatorname{Spec}(R_0) \hookrightarrow \mathbb{A}^1_R$ be the inclusion. We denote the projection $\operatorname{Spec}(F_{n-1}) \to \mathbb{A}^1_F$ and inclusion $\operatorname{Spec}(F_0) \hookrightarrow \mathbb{A}^1_F$ also by f and p, respectively. Note that f is a finite map which has factorization $Z \to \mathbb{A}^1_R \setminus \{0\} \subset \mathbb{A}^1_R$.

We let $\alpha = \operatorname{cyc}^M_{R_{n-1}}([Z]) = \{g_1, \ldots, g_{n-1}\} \in K^M_{n-1}(R_{n-1})$. We then have, by definition, $\operatorname{cyc}_R([Z]) = f_* \circ \psi_{R_{n-1}}(\alpha)$ and $\operatorname{cyc}_F(\pi^*([Z])) = f_* \circ \psi_{F_{n-1}} \circ \pi^*_{n-1}(\alpha)$. Using (10.3), we can write these as $\operatorname{cyc}_R([Z]) = p_* \circ f_{0*} \circ \psi_{R_{n-1}}(\alpha)$ and

$$\operatorname{cyc}_F(\pi^*([Z])) = p_* \circ f_{0*} \circ \psi_{F_{n-1}} \circ \pi^*_{n-1}(\alpha).$$

Since $\pi^*_{n-1} \circ \psi_{R_{n-1}} = \psi_{F_{n-1}} \circ \pi^*_{n-1}$, we only have to show that the diagram

$$\begin{array}{ccccc} K(R_{n-1}) & \xrightarrow{f_{0*}} & K(R_0) & \xrightarrow{p_*} & K(R[t], (t^{m+1})) \\ \downarrow{\scriptstyle \pi^*_{n-1}} & & \downarrow{\scriptstyle \pi^*_0} & & \downarrow{\scriptstyle \pi^*} \\ K(F_{n-1}) & \xrightarrow{f_{0*}} & K(F_0) & \xrightarrow{p_*} & K(F[t], (t^{m+1})) \end{array} \tag{10.4}$$

commutes.

The left square commutes by [Thomason and Trobaugh 1990, Proposition 3.18] since $F_{n-1} = R_{n-1} \otimes_{R_0} F_0$ and f_0 is finite. To see that the right square commutes, note that we can replace $K(R_0)$ by $K^{Z_0}(\mathbb{A}^1_R)$ and $K(F_0)$ by $K^{\eta_0}(\mathbb{A}^1_F)$, where $Z_0 = \operatorname{Spec}(R_0)$ and $\eta_0 = \operatorname{Spec}(F_0)$ (see Section 2C). We can do this because $R[t]$ and R_0 are regular. We are now done because the diagram

$$\begin{array}{ccc} K(\mathbb{A}^1_R, (m+1)\{0\}) & \longrightarrow & K(\mathbb{A}^1_R \setminus Z_0, (m+1)\{0\}) \\ \downarrow & & \downarrow \\ K(\mathbb{A}^1_F, (m+1)\{0\}) & \longrightarrow & K(\mathbb{A}^1_F \setminus \eta_0, (m+1)\{0\}) \end{array} \tag{10.5}$$

of restriction maps is commutative and the right square in (10.4) is gotten by taking the homotopy fibers of the two rows of (10.5). We have now shown that both squares in (10.2) are commutative. This also shows that cyc_R is compatible with the inclusion $R \hookrightarrow F$. The proof of the theorem is complete. $\square$

Throughout the rest of our discussion, we shall identify $\operatorname{TCH}^n(R, n; m)$ with $\mathbb{W}_m\Omega^{n-1}_R$ (by (7.5)) and $K_{n-1}(R[t], (t^{m+1}))$ with $\widetilde{K}_n(R_m)$ (via the connecting homomorphism).

10C. ***Factorization through Milnor K-theory.*** We shall now show that cyc_R factors through the relative Milnor K-theory. The proof is identical to the case of fields and we shall only sketch it. We shall reduce the proof to the case of fields using the following result.

Lemma 10.3. *For $n \geq 0$ and $m \geq 1$, the map $\pi^*\colon \mathbb{W}_m\Omega^n_R \to \mathbb{W}_m\Omega^n_F$ is injective. In particular, the map $\pi^*\colon \widetilde{K}^M_n(R_m) \to \widetilde{K}^M_n(F_m)$ is injective for all $m \geq 0$.*

Proof. Since $\mathbb{W}_m\Omega^n_R \cong (\Omega^n_R)^m$ (and also for F), we need to show that $\Omega^n_R \to \Omega^n_F$ is injective to prove the first assertion of the lemma. Since $\Omega^n_F \cong \Omega^n_R \otimes_R F$, it suffices to show that Ω^1_R is a free R-module. Since R is regular, we have $D_1(R \mid k) = 0$ and $\Omega^1_{R/k}$ is a free R-module. The Jacobi–Zariski exact sequence (see [Loday 1998, 3.5.5]) therefore tells us that $\Omega^1_R \cong (\Omega^1_k \otimes_k R) \oplus \Omega^1_{R/k}$. This proves the first part.

For the second part, there is nothing to prove when $m = 0$. When $m \geq 1$, Lemma 8.4 reduces to showing that the map $\Omega^n_R \otimes_R tR_m \to \Omega^n_F \otimes_F tF_m$ is injective for all $n \geq 0$. Since $\Omega^n_F \otimes_F tF_m \cong \Omega^n_F \otimes_R tR_m$ and tR_m is a free R-module, the problem is reduced to showing that $\Omega^n_R \to \Omega^n_F$ is injective. We can now use the first part of the lemma. □

Our second main result of this section is the following. This generalizes the main results of [Elbaz-Vincent and Müller-Stach 2002; Nesterenko and Suslin 1989; Totaro 1992] to truncated polynomial rings.

Theorem 10.4. *Let R and $m \geq 0, n \geq 1$ be as above. Then the cycle class map* cyc_R *has a factorization*

$$\mathrm{TCH}^n(R, n; m) \xrightarrow{\mathrm{cyc}^M_R} \widetilde{K}^M_n(R_m) \xrightarrow{\psi_{R_{m+1},n}} \widetilde{K}_n(R_m).$$

Furthermore, cyc^M_R *is natural in R and is an isomorphism.*

Proof. We shall use Proposition 10.1, which allows us to repeat the proof of the field case Lemma 9.2 word by word. When $n = 1$, any sfs irreducible cycle $Z \subset \mathbb{A}^1_R$ is of the form $Z = V(f(t))$, where $f(t)$ is an irreducible polynomial such that $f(0) \in R^\times$. We now repeat the argument of the field case and use Lemma 2.1 to finish the proof. The $n \geq 2$ case follows from Lemma 7.3 and the proof is identical to the case of fields. To prove that cyc^M_R is an isomorphism, we again repeat the case of fields and use Remark 9.4. The naturality of cyc^M_R follows from Theorem 10.2 since $\psi_{R_{m+1},n}$ is injective. □

Finally, we are now ready to prove Theorem 1.3 (5). We restate it again for reader's convenience.

Theorem 10.5. *Let R be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Let $n \geq 1$ be an integer. Then the cycle class map*

$$\mathrm{cyc}_R\colon \{\mathrm{TCH}^n(R, n; m)\}_m \to \{\widetilde{K}_n(R_m)\}_m$$

is an isomorphism of the pro-abelian groups.

Proof. Combine Theorem 10.4 and Corollary 9.6. □

11. Appendix: Milnor vs. Quillen K-theory

In this section, we collect some results on the compatibility of various maps between Milnor and Quillen K-theories of fields. They are used in the proofs of the main results of this paper. We expect these results to be known to experts but could not find their written proofs in the literature.

Let k be a field and let X be a regular scheme which is essentially of finite type over k. Let $x, y \in X$ be two points in X of codimensions p and $p-1$, respectively, such that $x \in \overline{\{y\}}$. Let

$$Y = \overline{\{y\}}, \quad F = k(y), \quad A = \mathcal{O}_{Y,x} \quad \text{and} \quad l = k(x).$$

Lemma A.1. *For any $n \geq 1$, the diagram*

$$\begin{tikzcd} K_n^M(F) \arrow[r, "\partial^M"] \arrow[d] & K_{n-1}^M(l) \arrow[d] \\ K_n(F) \arrow[r, "\partial^Q"] & K_{n-1}(l) \end{tikzcd} \tag{A.1}$$

is commutative.

Proof. Let B denote the normalization of A and let S denote the set of maximal ideals of B. Note that B is semilocal so that S is finite. Since the localization sequence for Quillen K-theory of coherent sheaves is functorial for proper push-forward, we have a commutative diagram

$$\begin{tikzcd} G_n(B) \arrow[r] \arrow[d] & K_n(F) \arrow[r, "\partial^Q"] \arrow[d, equal] & \bigoplus_{z\in S} K_{n-1}(k(z)) \arrow[d, "\sum_z T_{k(z)/l}"] \\ G_n(A) \arrow[r] & K_n(F) \arrow[r, "\partial^Q"] & K_{n-1}(l), \end{tikzcd} \tag{A.2}$$

where $T_{k(z)/l}$ is our notation for the finite push-forward $K_*(k(z)) \to K_*(l)$ and $G_*(\text{-})$ is Quillen K-theory of coherent sheaves functor (for proper morphisms). Note that $K_*(B) \xrightarrow{\cong} G_*(B)$.

On the other hand, the boundary map in Milnor K-theory also has the property that the diagram

$$\begin{tikzcd} K_n^M(F) \arrow[r, "\partial^M"] \arrow[d, equal] & \bigoplus_{z\in S} K_{n-1}^M(k(z)) \arrow[d, "\sum_z N_{k(z)/l}"] \\ K_n^M(F) \arrow[r, "\partial^M"] & K_{n-1}^M(l) \end{tikzcd} \tag{A.3}$$

commutes, where the right vertical arrow is the sum of the Norm maps in Milnor K-theory of fields (see [Bass and Tate 1973; Kato 1980]). The lemma therefore follows if we prove Lemmas A.2 and A.3 below. □

Lemma A.2. *Let $z \in S$ be a closed point as above and let R denote the discrete valuation ring of F associated to z. Then the diagram*

$$\begin{tikzcd} K_n^M(F) \arrow[r, "\partial^M"] \arrow[d] & K_{n-1}^M(k(z)) \arrow[d] \\ K_n(F) \arrow[r, "\partial^Q"] & K_{n-1}(k(z)) \end{tikzcd} \tag{A.4}$$

is commutative for every $n \geq 1$.

Proof. It is well-known and elementary to see (using the Steinberg relations) that $K_*^M(F)$ is generated by $K_1^M(F)$ as an $K_*^M(R)$-module. Furthermore, ∂^M is $K_*^M(R)$-linear (see [Bass and Tate 1973, §4, Proposition 4.5]). Since the localization sequence such as the one on the top of (A.2) (with B replaced by R) is $K_*(R)$-linear, it follows that all arrows in (A.4) are $K_*^M(R)$-linear. It therefore suffices to prove the lemma for $n = 1$. But in this case, both ∂^M and ∂^Q are simply the valuation map of F corresponding to z. □

Lemma A.3. *Let $k \hookrightarrow k'$ be a finite extension of fields and $n \geq 0$ an integer. Then we have a commutative diagram*

$$\begin{tikzcd} K_n^M(k') \arrow[r, "N_{k'/k}"] \arrow[d] & K_n^M(k) \arrow[d] \\ K_n(k') \arrow[r, "T_{k'/k}"] & K_n(k). \end{tikzcd} \tag{A.5}$$

Proof. Assume first that $k \hookrightarrow k'$ is a simple extension so that $k' = k[t]/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset k[t]$. Let v_∞ be the valuation of $k(t)$ associated to the point $\infty \in \mathbb{P}_k^1$. Its valuation ring $R_\infty \subset k(t)$ has uniformizing parameter t^{-1}. In this case, we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & K_{n+1}^M(k) \arrow[r] \arrow[d] & K_{n+1}^M(k(t)) \arrow[r, "\partial^M=(\partial_v)_v"] \arrow[d] & \bigoplus_{v\neq v_\infty} K_n^M(k(v)) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K_{n+1}(k) \arrow[r] & K_{n+1}(k(t)) \arrow[r, "\partial^Q=(\partial_v)_v"] & \bigoplus_{v\neq v_\infty} K_n(k(v)) \arrow[r] & 0. \end{tikzcd} \tag{A.6}$$

The horizontal arrows on the left in both rows are induced by the inclusion $k \subset k(t)$. The top row is Milnor's exact sequence (see [Weibel 2013, Chapter III, Theorem 7.4]). The bottom row is the localization sequence in Quillen K-theory (using the isomorphism $K_*(k) \xrightarrow{\cong} K_*(k[t])$) and is known to be exact (see [Weibel 2013, Chapter V, Corollary 6.7.1]). The right square commutes by Lemma A.2.

On the other hand, we have another diagram

$$\begin{tikzcd} & K_n^M(k') \arrow[d, hook] \arrow[dr, "N_{k'/k}"] & \\ K_{n+1}^M(k(t)) \arrow[r, two heads, "\partial^M=(\partial_v)_v"] \arrow[d] & \bigoplus_{v\neq v_\infty} K_n^M(k(v)) \arrow[r, "\sum_v N_{k(v)/k}"] \arrow[d] & K_n^M(k) \arrow[d] \\ K_{n+1}(k(t)) \arrow[r, two heads, "\partial^Q=(\partial_v)_v"] & \bigoplus_{v\neq v_\infty} K_n(k(v)) \arrow[r, "\sum_v T_{k(v)/k}"] & K_n(k) \\ & K_n(k'). \arrow[u, hook] \arrow[ur, "T_{k'/k}"'] & \end{tikzcd} \tag{A.7}$$

By the definition of the norm $N_{k'/k}$ in Milnor K-theory, the composition of the horizontal arrows on the top is the map $(-1)\partial_\infty^M\colon K_{n+1}^M(k(t)) \to K_n^M(k)$ (see [Weibel 2013, Chapter III, Definition 7.5]). Similarly, the composite of the horizontal arrows on the bottom is the map $(-1)\partial_\infty^Q\colon K_{n+1}(k(t)) \to K_n(k)$ (see [Weibel 2013, Chapter V, 6.12.1]). Note that both of these assertions are another way of stating the Weil reciprocity formulas for the Milnor and Quillen K-theories.

Since the left horizontal arrows in both rows are surjective, we are reduced to showing therefore that the diagram

$$\begin{tikzcd} K_{n+1}^M(k(t)) \arrow[r, "\partial_\infty^M"] \arrow[d] & K_n^M(k) \arrow[d] \\ K_{n+1}(k(t)) \arrow[r, "\partial_\infty^Q"] & K_n(k) \end{tikzcd} \tag{A.8}$$

commutes. But this follows from Lemma A.2. This proves the lemma for simple extensions.

In general, we can write $k' = k(x_1, \ldots, x_r)$. Since the norm maps in Milnor K-theory and the push-forward maps in Quillen K-theory satisfy the transitivity property, and since $k \hookrightarrow k'$ is a composite of simple extensions, the proof of the lemma follows. $\square$

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References

[Artin and Mazur 1986] M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Mathematics **100**, Springer, 1986. MR Zbl

[Bass and Tate 1973] H. Bass and J. Tate, "The Milnor ring of a global field", pp. 349–446 in *Algebraic K-theory, II: "Classical" algebraic K-theory and connections with arithmetic* (Seattle, WA, 1972), edited by H. Bass, Lecture Notes in Math. **342**, 1973. MR Zbl

[Binda 2018] F. Binda, "A cycle class map from Chow groups with modulus to relative K-theory", *Doc. Math.* **23** (2018), 407–444. MR Zbl

[Binda and Krishna 2018] F. Binda and A. Krishna, "Zero cycles with modulus and zero cycles on singular varieties", *Compos. Math.* **154**:1 (2018), 120–187. MR Zbl

[Binda and Saito 2019] F. Binda and S. Saito, "Relative cycles with moduli and regulator maps", *J. Inst. Math. Jussieu* **18**:6 (2019), 1233–1293. MR Zbl

[Bloch 1977] S. Bloch, "Algebraic K-theory and crystalline cohomology", *Inst. Hautes Études Sci. Publ. Math.* 47 (1977), 187–268. MR Zbl

[Bloch 1986] S. Bloch, "Algebraic cycles and higher K-theory", *Adv. Math.* **61**:3 (1986), 267–304. MR Zbl

[Bloch and Esnault 2003a] S. Bloch and H. Esnault, "The additive dilogarithm", *Doc. Math.* extra vol. in Kato's honor (2003), 131–155. MR Zbl

[Bloch and Esnault 2003b] S. Bloch and H. Esnault, "An additive version of higher Chow groups", *Ann. Sci. École Norm. Sup.* (4) **36**:3 (2003), 463–477. MR Zbl

[Cathelineau 1990] J.-L. Cathelineau, "λ-structures in algebraic K-theory and cyclic homology", *K-Theory* **4**:6 (1990), 591–606. MR Zbl

[Cortiñas and Weibel 2009] G. Cortiñas and C. Weibel, "Relative Chern characters for nilpotent ideals", pp. 61–82 in *Algebraic topology*, edited by N. A. Baas et al., Abel Symp. **4**, Springer, 2009. MR Zbl

[Elbaz-Vincent and Müller-Stach 2002] P. Elbaz-Vincent and S. Müller-Stach, "Milnor K-theory of rings, higher Chow groups and applications", *Invent. Math.* **148**:1 (2002), 177–206. MR Zbl

[Geller and Weibel 1989] S. Geller and C. Weibel, "$K(A, B, I)$ II", *K-Theory* **2**:6 (1989), 753–760. MR

[Geller and Weibel 1994] S. C. Geller and C. A. Weibel, "Hodge decompositions of Loday symbols in K-theory and cyclic homology", *K-Theory* **8**:6 (1994), 587–632. MR Zbl

[Goodwillie 1985] T. G. Goodwillie, "On the general linear group and Hochschild homology", *Ann. of Math.* (2) **121**:2 (1985), 383–407. MR Zbl

[Gorchinskiĭ and Tyurin 2018] S. O. Gorchinskiĭ and D. N. Tyurin, "Relative Milnor K-groups and differential forms of split nilpotent extensions", *Izv. Ross. Akad. Nauk Ser. Mat.* **82**:5 (2018), 23–60. MR Zbl

[Gupta and Krishna 2019] R. Gupta and A. Krishna, "Relative K-theory via 0-cycles in finite characteristic", 2019. arXiv

[Hesselholt 2005] L. Hesselholt, "K-theory of truncated polynomial algebras", pp. 71–110 in *Handbook of K-theory*, edited by E. M. Friedlander and D. R. Grayson, Springer, 2005. MR Zbl

[Hesselholt 2008] L. Hesselholt, “The tower of K-theory of truncated polynomial algebras”, *J. Topol.* **1**:1 (2008), 87–114. MR Zbl

[Iwasa and Kai 2018] R. Iwasa and W. Kai, “Isomorphisms up to bounded torsion between relative K_0-groups and Chow groups with modulus”, 2018. To appear in *J. Inst. Math. Jussieu*. arXiv

[Iwasa and Kai 2019] R. Iwasa and W. Kai, “Chern classes with modulus”, *Nagoya Math. J.* **236** (2019), 84–133. MR Zbl

[Kantorovitz 1999] M. R. Kantorovitz, “Adams operations and the Dennis trace map”, *J. Pure Appl. Algebra* **144**:1 (1999), 21–27. MR Zbl

[Kato 1980] K. Kato, “A generalization of local class field theory by using K-groups II”, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27**:3 (1980), 603–683. MR Zbl

[Kato 1986] K. Kato, “Milnor K-theory and the Chow group of zero cycles”, pp. 241–253 in *Applications of algebraic K-theory to algebraic geometry and number theory* (Boulder, CO, 1983), edited by S. J. Bloch et al., Contemp. Math. **55**, Amer. Math. Soc., Providence, RI, 1986. MR Zbl

[Kato and Saito 1986] K. Kato and S. Saito, “Global class field theory of arithmetic schemes”, pp. 255–331 in *Applications of algebraic K-theory to algebraic geometry and number theory* (Boulder, CO, 1983), edited by S. J. Bloch et al., Contemp. Math. **55**, Amer. Math. Soc., Providence, RI, 1986. MR Zbl

[Kerz 2009] M. Kerz, “The Gersten conjecture for Milnor K-theory”, *Invent. Math.* **175**:1 (2009), 1–33. MR Zbl

[Kerz 2010] M. Kerz, “Milnor K-theory of local rings with finite residue fields”, *J. Algebraic Geom.* **19**:1 (2010), 173–191. MR Zbl

[Kerz and Saito 2016] M. Kerz and S. Saito, “Chow group of 0-cycles with modulus and higher-dimensional class field theory”, *Duke Math. J.* **165**:15 (2016), 2811–2897. MR Zbl

[Kerz et al. 2018] M. Kerz, F. Strunk, and G. Tamme, “Algebraic K-theory and descent for blow-ups”, *Invent. Math.* **211**:2 (2018), 523–577. MR Zbl

[Krishna 2010] A. Krishna, “An Artin–Rees theorem in K-theory and applications to zero cycles”, *J. Algebraic Geom.* **19**:3 (2010), 555–598. MR Zbl

[Krishna 2015] A. Krishna, “On 0-cycles with modulus”, *Algebra Number Theory* **9**:10 (2015), 2397–2415. MR Zbl

[Krishna and Levine 2008] A. Krishna and M. Levine, “Additive higher Chow groups of schemes”, *J. Reine Angew. Math.* **619** (2008), 75–140. MR Zbl

[Krishna and Park 2012a] A. Krishna and J. Park, “Mixed motives over $k[t]/(t^{m+1})$”, *J. Inst. Math. Jussieu* **11**:3 (2012), 611–657. MR Zbl

[Krishna and Park 2012b] A. Krishna and J. Park, “Moving lemma for additive higher Chow groups”, *Algebra Number Theory* **6**:2 (2012), 293–326. MR Zbl

[Krishna and Park 2016] A. Krishna and J. Park, “On additive higher Chow groups of affine schemes”, *Doc. Math.* **21** (2016), 49–89. MR Zbl

[Krishna and Park 2017a] A. Krishna and J. Park, “A module structure and a vanishing theorem for cycles with modulus”, *Math. Res. Lett.* **24**:4 (2017), 1147–1176. MR Zbl

[Krishna and Park 2017b] A. Krishna and J. Park, “A moving lemma for cycles with very ample modulus”, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **17**:4 (2017), 1521–1549. MR Zbl

[Krishna and Park 2020a] A. Krishna and J. Park, “de Rham–Witt sheaves via algebraic cycles”, 2020. With an appendix by K. Rülling. Zbl arXiv

[Krishna and Park 2020b] A. Krishna and J. Park, “A moving lemma for relative 0-cycles”, *Algebra Number Theory* **14**:4 (2020), 991–1054. MR Zbl

[Krishna and Pelaez 2018] A. Krishna and P. Pelaez, “Motivic spectral sequence for relative homotopy K-theory”, 2018. To appear in *Ann. Sc. Norm. Super. Pisa Cl. Sci.* arXiv

[Levine 1994] M. Levine, “Bloch’s higher Chow groups revisited”, pp. 10, 235–320 in *K-theory* (Strasbourg, 1992), Astérisque **226**, Société Mathématique de France, Paris, 1994. MR Zbl

[Levine 1997] M. Levine, “Lambda-operations, K-theory and motivic cohomology”, pp. 131–184 in *Algebraic K-theory* (Toronto, 1996), edited by V. P. Snaith, Fields Inst. Commun. **16**, Amer. Math. Soc., Providence, RI, 1997. MR Zbl

[Loday 1998] J.-L. Loday, *Cyclic homology*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **301**, Springer, 1998. MR Zbl

[Milnor 1971] J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies **72**, Princeton University Press, 1971. MR Zbl

[Nesterenko and Suslin 1989] Y. P. Nesterenko and A. A. Suslin, “Homology of the general linear group over a local ring, and Milnor’s K-theory”, *Izv. Akad. Nauk SSSR Ser. Mat.* **53**:1 (1989), 121–146. In Russian. MR

[Park 2009] J. Park, “Regulators on additive higher Chow groups”, *Amer. J. Math.* **131**:1 (2009), 257–276. MR Zbl

[Park and Ünver 2018] J. Park and S. Ünver, “Motivic cohomology of fat points in Milnor range”, *Doc. Math.* **23** (2018), 759–798. MR Zbl

[Rülling 2007] K. Rülling, “The generalized de Rham–Witt complex over a field is a complex of zero-cycles”, *J. Algebraic Geom.* **16**:1 (2007), 109–169. MR Zbl

[Rülling and Saito 2018] K. Rülling and S. Saito, “Higher Chow groups with modulus and relative Milnor K-theory”, *Trans. Amer. Math. Soc.* **370**:2 (2018), 987–1043. MR Zbl

[Soulé 1985] C. Soulé, “Opérations en K-théorie algébrique”, *Canad. J. Math.* **37**:3 (1985), 488–550. MR Zbl

[Stienstra 1985] J. Stienstra, “Cartier-Dieudonné theory for Chow groups”, *J. Reine Angew. Math.* **355** (1985), 1–66. MR Zbl

[Swan 1998] R. G. Swan, “Néron–Popescu desingularization”, pp. 135–192 in *Algebra and geometry* (Taipei, 1995), edited by M.-c. Kang, Lect. Algebra Geom. **2**, Int. Press, Cambridge, MA, 1998. MR Zbl

[Thomason and Trobaugh 1990] R. W. Thomason and T. Trobaugh, “Higher algebraic K-theory of schemes and of derived categories”, pp. 247–435 in *The Grothendieck Festschrift III*, edited by P. Cartier et al., Progr. Math. **88**, Birkhäuser, Boston, 1990. MR Zbl

[Totaro 1992] B. Totaro, “Milnor K-theory is the simplest part of algebraic K-theory”, *K-Theory* **6**:2 (1992), 177–189. MR Zbl

[Weibel 2013] C. A. Weibel, *The K-book: An introduction to algebraic K-theory*, Graduate Studies in Mathematics **145**, American Mathematical Society, Providence, RI, 2013. MR Zbl

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Format. Authors are encouraged to use LaTeX and the standard amsart class, but submissions in other varieties of TeX, and exceptionally in other formats, are acceptable. Initial uploads should normally be in PDF format; after the refereeing process we will ask you to submit all source material.

References. Bibliographical references should be complete, including article titles and page ranges. All references in the bibliography should be cited in the text. The use of BibTeX is preferred but not required. Tags will be converted to the house format, however, for submission you may use the format of your choice. Links will be provided to all literature with known web locations and authors are encouraged to provide their own links in addition to those supplied in the editorial process.

Figures. Figures must be of publication quality. After acceptance, you will need to submit the original source files in vector graphics format for all diagrams in your manuscript: vector EPS or vector PDF files are the most useful.

Most drawing and graphing packages — Mathematica, Adobe Illustrator, Corel Draw, MATLAB, etc. — allow the user to save files in one of these formats. Make sure that what you are saving is vector graphics and not a bitmap. If you need help, please write to graphics@msp.org with as many details as you can about how your graphics were generated.

Bundle your figure files into a single archive (using zip, tar, rar or other format of your choice) and upload on the link you been provided at acceptance time. Each figure should be captioned and numbered so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text (“the curve looks like this:”). It is acceptable to submit a manuscript with all figures at the end, if their placement is specified in the text by means of comments such as “Place Figure 1 here”. The same considerations apply to tables.

White Space. Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal’s preferred fonts and layout.

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