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Excision in equivariant fibred G-theory

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This paper provides a generalization of excision theorems in controlled algebra in the context of equivariant G-theory with fibred control and families of bounded actions. It also states and proves several characteristic features of this theory such as existence of the fibred assembly and the fibrewise trivialization.

1. Introduction

The bounded K-theory construction due to Pedersen and Weibel [1985] has been shown to be extremely useful in the analysis of versions of the Novikov conjecture [Carlsson 1995; Carlsson and Goldfarb 2004a; Carlsson and Pedersen 1995; 1998; Ramras et al. 2014]. This conjecture asserts the split injectivity of a natural transformation called the *assembly*. The present paper is the culmination of a series of papers [Carlsson and Goldfarb 2011; 2016; 2019] that extend the techniques sufficient to address the much more difficult *Borel conjecture in algebraic K-theory for a group* Γ, which asserts that the K-theory assembly map $\alpha_\Gamma : B\Gamma_+ \wedge K(\mathbb{Z}) \to K(\mathbb{Z}[\Gamma])$ is an equivalence of spectra. What we have found is that substantial extensions are necessary.

Since the construction of the equivariant fibred G-theory is quite involved and technical, we provide the reader with a discussion of how we arrived at it. Recall that the integral K-theoretic Novikov conjecture asserts that the assembly map

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α_Γ can be identified with a split inclusion on a direct summand of the spectrum $K(\mathbb{Z}[\Gamma])$. Consider the basic geometric situation of a finitely generated group Γ acting properly and freely on $\mathbb{R}^n$. It is shown in [Carlsson 1995; 2005; Carlsson and Goldfarb 2004a; Carlsson and Pedersen 1995; 1998] that a successful strategy for proving the Novikov conjecture proceeds by recognizing the following.

(1) The spectra $B\Gamma_+ \wedge K(\mathbb{Z})$ and $K(\mathbb{Z}[\Gamma])$ can be realized as the fixed point spectra of Γ-actions on certain spectra $h^{\mathrm{lf}}_\Gamma(\mathbb{R}^n, K(\mathbb{Z}))$ and $K^{\mathrm{bdd}}(\mathbb{R}^n; \mathbb{Z})$. The spectrum $h^{\mathrm{lf}}_\Gamma(X, K(\mathbb{Z}))$ is an equivariant version of Borel–Moore homology with coefficients in the spectrum $K(\mathbb{Z})$, and has the property that for proper discontinuous free actions on locally compact spaces,

$$h^{\mathrm{lf}}_\Gamma(X, K(\mathbb{Z}))^\Gamma \cong h^{\mathrm{lf}}(X/\Gamma, K(\mathbb{Z})).$$

The spectrum $K^{\mathrm{bdd}}(\mathbb{R}^n, \mathbb{Z})$ is the bounded K-theory due to Pedersen and Weibel [1985]. It depends on a choice of Γ-invariant metric on $\mathbb{R}^n$.

(2) The assembly map α_Γ is the restriction to the fixed point sets of an equivariant map of spectra

$$\alpha^{\mathrm{amb}} : h^{\mathrm{lf}}(\mathbb{R}^n, K(\mathbb{Z})) \to K^{\mathrm{bdd}}(\mathbb{R}^n; \mathbb{Z}).$$

(3) The Γ-equivariant spectrum $S = h^{\mathrm{lf}}(\mathbb{R}^n, K(\mathbb{Z}))$ has the homotopy invariance property that the canonical map $S^\Gamma \to S^{h\Gamma}$ is an equivalence.

It turns out that α^{amb} can often be proved to be an equivalence by using the excision properties of the functor K^{bdd}. When one can do this for a metric on $\mathbb{R}^n$ which is Γ-invariant, the integral K-theoretic Novikov conjecture follows immediately, using the fact that if $f : X \to Y$ is a map of spaces with Γ-action, and f is an equivalence when regarded as a nonequivariant map, then the map $f^{h\Gamma}$ on homotopy fixed point sets is an equivalence.

Our approach to the K-theoretic Borel conjecture is to use similar techniques to prove that the map $\rho : K(\mathbb{Z}[\Gamma]) \to K^{\mathrm{bdd}}(\mathbb{R}^n, \mathbb{Z})^{h\Gamma} \cong B\Gamma_+ \wedge K(\mathbb{Z})$ can also be identified with an inclusion onto a spectrum summand. This result together with the Novikov conjecture will prove that α_Γ is an equivalence. Of course, as stated, this appears to be difficult since the homotopy fixed point set $W^{h\Gamma}$ of a spectrum with Γ-action is defined as the function spectrum of equivariant maps from $E\Gamma_+$ to W, and therefore is not in any sense finite dimensional or equipped with any reasonable geometric cell structures. However, due to the fact that we are working with stable homotopy theory, there is a proper version of Spanier–Whitehead duality that allows us to obtain a geometric model for the homotopy fixed point spectrum of a spectrum with Γ-action, when Γ is the fundamental group of a $K(\Gamma, 1)$-manifold M. Consider any embedding i of M in $\mathbb{R}^n$ for some n. We let N denote any open tubular neighborhood of $i(M)$. Clearly, $\pi_1(N) \cong \pi_1(M) \cong \Gamma$, and we consider the universal cover $\widetilde{N}$. Using a version of equivariant Spanier–Whitehead duality for free, proper Γ-spaces, it is possible to show that for any spectrum W with Γ-action, there is an equivalence $W^{h\Gamma} \cong h^{\mathrm{lf}}(\widetilde{N}, W)^{\Gamma}$.

To understand how we use this equivalence, we need to describe some of the properties of the construction K^{bdd}. For any commutative ring A, $K^{\mathrm{bdd}}(-, A)$ is a functor on a category of proper metric spaces $\mathfrak{M}$. It is functorial for *coarse* maps $f : X \to Y$ of metric spaces defined as maps that are both *proper*, in the sense that preimages of bounded subsets of Y are bounded in X, and *uniformly expansive*, in the sense that there is a function $c : \mathbb{R} \to \mathbb{R}$ such that if $x, x' \in X$ are any two points with $d(x, x') \leq t$, then $d(f(x), f(x')) \leq c(t)$. We let $\mathfrak{M}^{\infty}$ denote enlargement of the category $\mathfrak{M}$ to pairs (X, d), where the distance function d is permitted to take the value $+\infty$, and refer to objects of $\mathfrak{M}^{\infty}$ as generalized metric spaces. The axioms for a metric space extend naturally, and the functor K^{bdd} also extends to these generalized metric spaces. It is now immediate that the functor can also be extended to the category of simplicial objects in the category $\mathfrak{M}^{\infty}$. For a group Γ, we can consider the category $\mathfrak{M}^{\infty}_{\Gamma}$ of generalized metric spaces with Γ-actions by coarse maps. There is an equivariant version $K^{\mathrm{bdd}}_{\Gamma}$ of the functor K^{bdd} which is defined on $\mathfrak{M}^{\infty}_{\Gamma}$, and which carries a generalized metric space X with Γ-action to a spectrum with Γ-action. It is now possible to prove that we have a sequence of maps

$$\Sigma^n K^{\mathrm{bdd}}_{\Gamma}(\Gamma, A)^{h\Gamma} \simeq h^{\mathrm{lf}}_{\Gamma}(\widetilde{N}, K^{\mathrm{bdd}}_{\Gamma}(\Gamma, A))^{\Gamma} \to K^{\mathrm{bdd}}_{\Gamma}(\Gamma \times \widetilde{N}, A)^{\Gamma},$$

which means that we can now work geometrically in bounded K-theory to construct the splitting map. The main idea is to once again use excision properties to obtain information about $K^{\mathrm{bdd}}_{\Gamma}(\Gamma \times \widetilde{N}, A)^{\Gamma}$.

Remark 1.1. One observation about these properties is that the required excision theorems must be fibrewise in the $\widetilde{N}$-direction. The reason is that we must in an appropriate sense "leave Γ alone" since we are trying to detect $K(\mathbb{Z}[\Gamma])$. Secondly, the excision theorems must be equivariant since the Γ-action is what creates the group ring $\mathbb{Z}[\Gamma]$. The third observation is that the computation should end up with an appropriate suspension of $K(\mathbb{Z}[\Gamma])$, which was the source of the map ρ, so that the composition of all intermediate maps is an equivalence. This latter requirement is the source of very weak geometric conditions on Γ and algebraic conditions on the ground ring R that are needed and will be carried out in a separate paper.

As it stands, we do not have adequate excision properties for $K^{\mathrm{bdd}}_{\Gamma}(\Gamma \times \widetilde{N}, A)^{\Gamma}$ so that all of the three properties hold. However, we can construct further relaxation maps out to another construction we call *fibred homotopy fixed points* in G-theory which does enjoy such properties (Theorems 6.11 and 6.13). There are *three separate "axes" of relaxation* that we need, and it is the constructions of the present paper that will allow us to perform all three.

(1) *Bounded control to fibred bounded control.* We will need to use different notions of control on the morphisms in our category of modules. Of course, one could use the control on the product metric space $\Gamma \times \widetilde{N}$, but that does not possess the right properties. In [Carlsson and Goldfarb 2019] we proved that there is another notion of control which roughly insists that morphisms must be controlled in each fiber (copy of N), but where the bound may vary from fiber to fiber. Fibred control is the analogue of the notion of "parametrized homotopy theory" or "homotopy theory over a base", where X is the base and Y is the fiber; cf. [May and Sigurdsson 2006]. In the equivariant case, where X is equal to a group Γ regarded as a metric space using the word length metric, the fixed points of the Γ-action are analogous to the bundles over a classifying space $B\Gamma$ obtained from Γ-spaces Y by the construction $Y \to E\Gamma \times_{\Gamma} Y$. To realize our results, we will need excision properties holding for coverings of Y.

(2) *Free to nonfree modules.* We will need to enlarge the category of modules we consider. The coarse actions on metric spaces can no longer be assumed to be free, and so the fixed point spectra need to be modeled on a larger module category. In [Carlsson and Goldfarb 2011], we have defined bounded versions of G-theory and developed appropriate techniques for proving excision properties. The paper [Carlsson and Goldfarb 2019] constructs a fibrewise version of that theory, and in this paper we construct the actual equivariant fibrewise excision properties we require. The idea of relaxing the kinds of modules we deal with is very analogous to a situation studied in the context of localization in [Thomason and Trobaugh 1990]. It is well understood that localization theorems for K-theory are much more sensitive and difficult to construct than the corresponding results for G-theory.

(3) *A natural covering metric to left-bounded metric.* Even with the excision properties in place, we must also modify the metric on the $\widetilde{N}$-factor in the product $\Gamma \times \widetilde{N}$. To give intuition about this, we observe that the bundle $E\Gamma \times_{\Gamma} \widetilde{N} \to B\Gamma$ is a topologically trivial $\mathbb{R}^n$-bundle. We would like to have a situation where it is actually a bundle with structure group contained in the bounded automorphisms group of $\widetilde{N}$. That does not in general happen, but it is possible to modify the metric on $\widetilde{N}$ so that it does. The new metric will be smaller than or equal to the original metric. When this metric is used, we are able to work as if the action actually is trivial. This result is also proved in this paper (Theorem 7.4).

Numerous details had to be suppressed in this roughly accurate outline of how the results of this paper are used to prove the K-theoretic Borel conjecture. In the last section, we include a worked out example of the argument for the simple case of the infinite cyclic group. The details for the much more general case of a group with finite decomposition complexity will appear elsewhere.

The main goal of this paper is to prove excision results that incorporate all generalizations (1)–(3) above simultaneously. Because we will only need the excision results where the action on the space Y is bounded in the sense defined above, we will only prove them in that situation. That is, we prove excision theorems (Theorems 6.11 and 6.13) for equivariant G-theory with fibred control of bounded Γ-spaces Y. Additionally we obtain the results suggested above as part of equivariant fibred G-theory.

We will now state the versions of main results that allow us to do so concisely. The full relative versions are stated and proved in the course of the paper.

In what follows we use the following notation for enlargements in products of metric spaces. Given a subset U of $X \times Y$, a number $K \geq 0$, and a function $k : X \to [0, +\infty)$, let $U[K, k]$ be the subset of those points (x, y) for which there is (x', y') in U with $d(x, x') \leq K$ and $d(y, y') \leq k(x')$.

Definition 1.2. An object of the category $\mathbf{G}_X(Y)$ is a module F over a Noetherian ring R together with a filtration, also denoted by F, indexed by the entire power set $\mathcal{P}(X \times Y)$ and subject to a number of conditions:

- $F(X \times Y) = F$, $F(\varnothing) = 0$,
- $F(S)$ is a finitely generated submodule of F for every bounded subset $S \subset X \times Y$,
- F can be equipped with another filtration $\mathcal{F}$ indexed by $\mathcal{P}(Y) \times [0, \infty) \times [0, \infty)^X$ so that the value $\mathcal{F}(C, D, \delta)$ is nested between two submodules

  $$F((X \times C)[D, \delta]) \quad \text{and} \quad F((X \times C)[D + K, \delta + k])$$

  for some $K \geq 0$ and a function k,
- when the submodule $\mathcal{F} = \mathcal{F}(C, D, \delta)$ is given the standard induced (X, Y)-filtration defined by $\mathcal{F}(U) = \mathcal{F} \cap F(U)$, the result has the following property: there is a number $d \geq 0$ and a function $\Delta : X \to [0, +\infty)$ such that
  - for every subset S of X, the associated X-filtered module $\mathcal{F}_X$ satisfies

    $$\mathcal{F}_X(S) \subset \sum_{x \in S} \mathcal{F}_X(x[d]),$$

  - for each pair of subsets U_1 and U_2 of $X \times Y$, the (X, Y)-filtered module $\mathcal{F}$ satisfies both

    $$\mathcal{F}(U_1 \cup U_2) \subset \mathcal{F}(U_1[d, \Delta]) + \mathcal{F}(U_2[d, \Delta])$$

    and

    $$\mathcal{F}(U_1) \cap \mathcal{F}(U_2) \subset \mathcal{F}(U_1[d, \Delta] \cap U_2[d, \Delta]).$$

It should be emphasized that the auxiliary filtration $\mathcal{F}$ is not part of the structure of the object; there is no specific choice of a filtration that is specified.

A morphism $f : F \to G$ in $\mathbf{G}_X(Y)$ is an R-linear homomorphism $F(X \times Y) \to G(X \times Y)$ such that $fF(U) \subset G(U[b, \theta])$ for some number $b \geq 0$ and for some function θ, but for all subsets $U \subset X \times Y$. We will refer to the pair (b, θ) as control data for f.

We specialize to the case of $X = \Gamma$, a finitely generated group with a chosen word metric. Suppose Γ acts on Y via bounded coarse equivalences, in the sense that for every $\gamma \in \Gamma$, there exists an $R_\gamma > 0$ such that $d(y, \gamma(y)) \leq R_\gamma$ for all $y \in Y$. The diagonal action on $\Gamma \times Y$ induces an action on $\mathbf{G}_\Gamma(Y)$.

Definition 1.3. The *fibred homotopy fixed points* is the category $\mathbf{G}^{h\Gamma}(Y)$ with objects which are sets of data $(\{F_\gamma\}, \{\psi_\gamma\})$, where

- F_γ is an object of $\mathbf{G}_\Gamma(Y)$ for each γ in Γ,
- ψ_γ is an isomorphism $F_e \to F_\gamma$ in $\mathbf{G}_\Gamma(Y)$,
- ψ_γ has control data with $b = 0$ and $\theta = +\infty$,
- $\psi_e = \mathrm{id}$,
- $\psi_{\gamma_1\gamma_2} = \gamma_1\psi_{\gamma_2} \circ \psi_{\gamma_1}$ for all γ_1, γ_2 in Γ.

The morphisms $(\{F_\gamma\}, \{\psi_\gamma\}) \to (\{F'_\gamma\}, \{\psi'_\gamma\})$ of $\mathbf{G}^{h\Gamma}(Y)$ are collections of morphisms $\phi_\gamma : F_\gamma \to F'_\gamma$ in $\mathbf{G}_\Gamma(Y)$ such that $\phi_\gamma \circ \psi_\gamma = \psi'_\gamma \circ \phi_e$ for all γ.

Both of these categories can be given exact structures resembling the exact structures in G-theory. We obtain in Section 6 a nonconnective delooping of the equivariant K-theory of $\mathbf{G}_X(Y)$. The fixed points of this theory are modeled by the nonconnective K-theory of $\mathbf{G}^{h\Gamma}(Y)$ (Proposition 6.5). This fixed point spectrum will be denoted $\widetilde{G}^\Gamma(Y)^\Gamma$. As soon as one chooses any subset Y_1 of Y, there is a full subcategory of $\mathbf{G}^{h\Gamma}(Y)$ on objects where all relevant modules are supported on fibred enlargements of $\Gamma \times Y_1$ in $\Gamma \times Y$. This subcategory is invariant under bounded actions, and so we have a nonconnective fixed point spectrum $\widetilde{G}^\Gamma(Y)^\Gamma_{<Y_1}$. Similarly, for two subsets Y_1 and Y_2 of Y, restricting to modules supported on fibred enlargements of both $\Gamma \times Y_1$ and $\Gamma \times Y_2$ gives a spectrum $\widetilde{G}^\Gamma(Y)^\Gamma_{<Y_1,Y_2}$.

We can finally state the absolute version of the main theorem of this paper.

Theorem 1.4 (part of the equivariant fibred excision theorem, Theorem 6.11). *Suppose Y_1 and Y_2 are subsets of a metric space Y on which Γ acts by bounded coarse equivalences, and $Y = Y_1 \cup Y_2$. There is a homotopy pushout diagram of spectra*

$$\begin{array}{ccc} \widetilde{G}^\Gamma(Y)^\Gamma_{<Y_1,Y_2} & \longrightarrow & \widetilde{G}^\Gamma(Y)^\Gamma_{<Y_1} \\ \big\downarrow & & \big\downarrow \\ \widetilde{G}^\Gamma(Y)^\Gamma_{<Y_2} & \longrightarrow & \widetilde{G}^\Gamma(Y)^\Gamma \end{array}$$

where the maps are induced from the exact inclusions.

2. Homotopy fixed points in categories with action

Given an action of a group Γ on a space X, one has the subspace of fixed points X^Γ. This subspace often has geometric significance for the study of X and Γ. A different powerful idea in topology is to model an interesting space or spectrum as the fixed point space or spectrum X^Γ for a specifically designed X with an action by a related group Γ. In either case, there is always the homotopy fixed point spectrum $X^{h\Gamma}$, which is easier to understand than X^Γ, and the canonical reference map $\rho : X^\Gamma \to X^{h\Gamma}$.

Now suppose we have a group action on a category. This automatically produces an action on the nerve and therefore a space. Suppose the category is then fed into a machine such as the algebraic K-theory, and we are interested in the fixed points of the K-theory. Therefore we want to look at the homotopy fixed points. In many important cases it is possible to construct a spatial or categorical description of what we get. Thomason defined the lax limit category whose K-theory turns out to be exactly the homotopy fixed points of the old action.

Definition 2.1. Let $\boldsymbol{E\Gamma}$ be the category with the object set Γ and the unique morphism $\mu : \gamma_1 \to \gamma_2$ for any pair $\gamma_1, \gamma_2 \in \Gamma$. There is a left Γ-action on $\boldsymbol{E\Gamma}$ induced by the left multiplication in Γ. If $\mathcal{C}$ is a category with a left Γ-action, then the category of functors $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ is another category with the Γ-action given on objects by the formulas $\gamma(F)(\gamma') = \gamma F(\gamma^{-1}\gamma')$ and $\gamma(F)(\mu) = \gamma F(\gamma^{-1}\mu)$. It is nonequivariantly equivalent to $\mathcal{C}$.

The category $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ is an interesting and useful object in its own right. There are several manifestations of this, for example in the work of Mona Merling and coauthors [Guillou et al. 2017; Malkiewich and Merling 2019; Merling 2017] or the work of these authors [Carlsson 1995; Carlsson and Goldfarb 2004a; 2004b; 2013]. While in both applications it is crucial to work with the category itself, in this paper we concentrate on approximating the fixed points in $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$.

The following construction has been used by Thomason [1983]. We refer to it as the *homotopy fixed points of a category*, following Merling [2017].

Definition 2.2 (homotopy fixed points). The fixed point subcategory $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})^\Gamma$ of the category of functors $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ consists of equivariant functors and equivariant natural transformations. We denote it by $\mathcal{C}^{h\Gamma}$.

Explicitly, the objects of $\mathcal{C}^{h\Gamma}$ are the pairs (C, ψ) where C is an object of $\mathcal{C}$ and ψ is a function from Γ to the morphisms of $\mathcal{C}$ with $\psi(\gamma) \in \mathrm{Hom}(C, \gamma C)$ that satisfies $\psi(e) = \mathrm{id}$ for the identity group element e, and satisfies the cocycle identity $\psi(\gamma_1\gamma_2) = \gamma_1\psi(\gamma_2)\psi(\gamma_1)$ for all pairs γ_1 and γ_2 in Γ. These conditions imply that $\psi(\gamma)$ is always an isomorphism. The set of morphisms $(C, \psi) \to (C', \psi')$ consists of the morphisms $\phi : C \to C'$ in $\mathcal{C}$ such that the squares

$$\begin{array}{ccc} C & \xrightarrow{\psi(\gamma)} & \gamma C \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\gamma\phi} \\ C' & \xrightarrow{\psi'(\gamma)} & \gamma C' \end{array}$$

commute for all $\gamma \in \Gamma$.

Remark 2.3. As pointed out in [Merling 2017], the homotopy fixed points of a category are not necessarily identical with space level constructions. It is for example not true in general that the nerve of the homotopy fixed point category of a category is the same as the geometric homotopy fixed points of the nerve of the category. It is however true in the case where the category is a discrete Γ-groupoid.

Example 2.4. Let $\mathcal{C}$ denote a category, and equip it with the trivial action by Γ. Then the category $\mathcal{C}^{h\Gamma}$ is the category of representations of Γ in $\mathcal{C}$. In particular, if $\mathcal{C}$ is the category of R-modules for a commutative ring R, then $\mathcal{C}^{h\Gamma}$ may be identified with the category of (left) $R[\Gamma]$-modules.

Example 2.5. Let $F \subseteq E$ denote a Galois field extension, with Galois group G. We consider the skew group ring $\Lambda = E^t[G]$, and consider the category $\mathcal{C}_E$ whose objects are $E^t[G]$-modules and whose morphisms are the E-linear maps. There is a G-action on $\mathcal{C}_E$, which is the identity on objects and which is defined by the group action on the morphisms. In this case, $\mathcal{C}_E^{h\Gamma}$ is equivalent to the category of F-vector spaces.

In Example 2.4, we saw that the group ring of a group Γ with coefficients in a commutative ring R may be realized as the fixed point subcategory of the action of Γ on $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$, where $\mathcal{C}$ denotes the category of all R-modules. In many cases, however, it is important to understand the category of free and finitely generated left $R[\Gamma]$-modules as a fixed point category. This is the case in the papers [Carlsson 2005] and [Carlsson and Goldfarb 2004a], for instance, where the injectivity of the assembly map is proved in a large family of cases. In the case of these two papers, this is achieved by defining a subcategory of $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ by restricting the morphisms $\psi(\gamma)$. The restriction in this case arises by the selection of a subcategory of the category of all R-modules based on the Pedersen–Weibel construction, which is endowed with a filtration and an action of the group Γ. The restricted version of $\mathcal{C}_E^{h\Gamma}$ requires that all of the morphisms $\psi(\gamma)$ have filtration zero. In order to attack the surjectivity problem for the assembly, we are led to the construction of more general forms of restriction of the maps $\psi(\gamma)$. This leads us to the concept of the *relative homotopy fixed points of a category*, which we now define.

Definition 2.6 (relative homotopy fixed points). The category $\mathcal{C}^{h\Gamma}(\mathcal{M})$ is defined using input data consisting of a category $\mathcal{C}$ equipped with an action by a group Γ

and a subcategory $\mathcal{M} \subset \mathcal{C}$ closed under the action of Γ. It is the full subcategory of $\mathcal{C}^{h\Gamma}$ on objects (C, ψ) with the additional condition that $\psi(\gamma)$ is in $\mathcal{M}$ for all elements $\gamma \in \Gamma$.

Example 2.7. Clearly, if $\mathcal{M}$ is the entire category $\mathcal{C}$, the relative homotopy fixed points are the genuine homotopy fixed points.

Example 2.8. In the case where $\mathcal{C}$ is a filtered category, we can consider the situation where $\mathcal{M}$ is the subcategory of the filtration zero morphisms. This is the situation used in [Carlsson 2005] and [Carlsson and Goldfarb 2004a].

We will exploit the relative homotopy fixed points in two applications. The first construction required in [Carlsson 1995] allows us to model the K-theory of a group ring whenever the group has a finite classifying space. It is based on bounded K-theory of the group given a word metric with the isometric action on itself given by the left multiplication. It turns out that the categorical homotopy fixed point construction requires a constraint. We review that construction in Section 3.

3. Bounded K-theory and the K-theory of group rings

Bounded control is the simplest version of a "control condition" that can be imposed in various categories of modules, to which one can apply the algebraic K-theory construction. It was introduced in [Pedersen 1984] and [Pedersen and Weibel 1985] and has become crucial for K-theory computations in geometric topology.

Let X be a metric space and let R be an arbitrary associative ring with unity. We always assume that metric spaces are proper in the sense that closed bounded subsets are compact.

Definition 3.1. The objects of the category of *geometric R-modules over X* are locally finite functions F from points of X to the category of finitely generated free R-modules $\mathbf{Free}_{fg}(R)$. Following Pedersen and Weibel, we denote by F_x the module assigned to the point x of X and denote the object itself by writing down the collection $\{F_x\}$. The *local finiteness* condition requires precisely that for every bounded subset $S \subset X$ the restriction of F to S has finitely many nonzero modules as values.

Let d be the distance function in X. The morphisms $\phi : \{F_x\} \to \{G_x\}$ are collections of R-linear homomorphisms $\phi_{x,x'} : F_x \to G_{x'}$, for all x and x' in X, with the property that $\phi_{x,x'}$ is the zero homomorphism whenever $d(x, x') > D$ for some fixed real number $D = D(\phi) \geq 0$. One says that ϕ is *bounded by D*. The composition of two morphisms $\phi : \{F_x\} \to \{G_x\}$ and $\psi : \{G_x\} \to \{H_x\}$ is given by the formula

$$(\psi \circ \phi)_{x,x'} = \sum_{z \in X} \psi_{z,x'} \circ \phi_{x,z}. \tag{$*$}$$

This sum is finite because of the local finiteness property of G.

We will want to enlarge this category, and so we use instead an equivalent category $\mathcal{B}(X, R)$ that is better for this purpose.

The objects are functors $F : \mathcal{P}(X) \to \mathbf{Free}(R)$ from the power set $\mathcal{P}(X)$ to the category of free modules, both viewed as posets ordered by split inclusions. There are two additional requirements. For every bounded subset C of X the value $F(C)$ has to belong to the subcategory of finitely generated modules $\mathbf{Free}_{fg}(R)$. In the codomain, the values are required to satisfy the equality $F(S) = \bigoplus_{x \in S} F(x)$ for all $S \subset X$. The morphisms in this reformulation are R-linear homomorphisms $\phi : F(X) \to G(X)$ such that the components $\phi_{x,x'} : F(x) \to G(x')$ are zero whenever $d(x, x') > D$ for some D. The composition of two morphisms $\phi : F \to G$ and $\psi : G \to H$ is the usual composition of R-linear homomorphisms; its components are the maps $(\psi \circ \phi)_{x,x'}$ in the formula $(*)$ above.

Definition 3.2. A map $f : X \to Y$ between metric spaces is called *uniformly expansive* if there is a function $\lambda : [0, \infty) \to [0, \infty)$ such that

$$d_X(x_1, x_2) \le r \quad \text{implies} \quad d_Y(f(x_1), f(x_2)) \le \lambda(r).$$

A map f is *proper* if $f^{-1}(S)$ is a bounded subset of X for each bounded subset S of Y. We say f is a *coarse map* if it is uniformly expansive and proper.

Extensively used instances of coarse maps in geometry are quasi-isometries.

It is elementary to check that the geometric R-modules over X is an additive category and that coarse maps between metric spaces induce additive functors. A coarse map f is a *coarse equivalence* if there is a coarse map $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are bounded maps. It follows that an action of a group on a metric space by coarse equivalences induces an additive action on $\mathcal{B}(X, R)$.

We will treat the group Γ equipped with a finite generating set Ω closed under taking inverses as a metric space. The *word-length metric* $d = d_\Omega$ is induced from the condition that $d(\gamma, \gamma\omega) = 1$ whenever $\gamma \in \Gamma$ and $\omega \in \Omega$. It is well-known that varying Ω only changes Γ to a quasi-isometric metric space. The word-length metric makes Γ a proper metric space with a free Γ-action by isometries via left multiplication.

An important observation. A free action of Γ on X by isometries always gives a free action on $\mathcal{C} = \mathcal{B}(X, R)$. In contrast, $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathcal{C})$ with the induced group action does have the subcategory $\mathcal{C}^{h\Gamma}$ of equivariant functors. These homotopy fixed points, however, are not the correct notion for modeling the finitely generated free modules over $R[G]$ and the K-theory of $R[G]$.

Definition 3.3. The category $\mathcal{B}^{\Gamma,0}(X, R)^\Gamma$ is the relative homotopy fixed point spectrum $\mathcal{C}^{h\Gamma}(\mathcal{M})$ with the following data: $\mathcal{C}$ is the category of geometric modules $\mathcal{B}(X, R)$ and $\mathcal{M}$ consists of those morphisms in $\mathcal{C}$ that are bounded by 0.

The additive category $\mathcal{B}^{\Gamma,0}(X, R)^{\Gamma}$ has the associated nonconnective K-theory spectrum $K^{-\infty}(X, R)^{\Gamma}$ constructed as in [Pedersen and Weibel 1985]. There is now the following desired identification.

Theorem 3.4. *Suppose Γ acts on X freely, properly discontinuously by isometries so that the orbit space X/Γ with the orbit metric is bounded.*

It follows that $K^{-\infty}(X, R)^{\Gamma}$ is weakly homotopy equivalent to the nonconnective spectrum $K^{-\infty}(R[\Gamma])$. The stable homotopy groups of the nonconnective spectrum are the Quillen K-groups of $R[G]$ in nonnegative dimensions and the negative K-groups of Bass in negative dimensions.

Proof. The result follows from Corollary VI.8 in [Carlsson 1995]. □

This geometric situation occurs, for example, when Γ acts cocompactly, freely properly discontinuously on a contractible connected Riemannian manifold X or when it acts on itself with a word metric via left multiplication.

4. Fibred homotopy fixed points in K-theory

Let $\mathcal{A}$ be an additive category. Generalizing Definition 3.1, one has the bounded category with coefficients in $\mathcal{A}$.

Notation 4.1. Given a subset S of a metric space and a number $k \geq 0$, $S[k]$ is used for the *k-enlargement* of S defined as the set of all points x with $d(x, S) \leq k$.

Recall that $\mathcal{A}$ is a subcategory of its cocompletion $\mathcal{A}^*$ which is closed under colimits. For example, a construction based on the presheaf category was given in [Kelly 1982, 6.23].

Definition 4.2. $\mathcal{B}(X, \mathcal{A})$ has objects which are covariant functors $F : \mathcal{P}(X) \to \mathcal{A}^*$ from the power set $\mathcal{P}(X)$ to $\mathcal{A}^*$, both ordered by inclusion. Just as in Definition 3.1, there are several requirements:

- $F(x)$ is an object of $\mathcal{A}$ for every point x in X,
- the resulting function $F : X \to \mathcal{A}$ is locally finite, so only finitely many values are nonzero when restricted to any compact subset of X,
- for all subsets $S \subset X$,
$$F(S) = \bigoplus_{x \in S} F(x),$$
- the inclusion $F(S \subset X)$ is onto a direct summand for each subset S.

A morphism in $\mathcal{B}(X, \mathcal{A})$ is a morphism $\phi : F(X) \to G(X)$ in $\mathcal{A}^*$ with a number $D \geq 0$ such that ϕ restricted to $F(S)$ factors through $G(S[D])$ for all $S \subset X$. We say a morphism which admits such a number D is *D-controlled*.

This context, which produces a category isomorphic to $\mathcal{B}(X, R)$ when $\mathcal{A}$ is the category of free finitely generated R-modules, allows us to iterate the bounded control construction as follows.

Definition 4.3 (fibred control for geometric modules). Given two metric spaces X and Y and any ring R, the category $\mathcal{B}_X(Y, R)$, or simply $\mathcal{B}_X(Y)$ when the choice of ring R is clear, is the bounded category $\mathcal{B}(X, \mathcal{A})$ with $\mathcal{A} = \mathcal{B}(Y, R)$.

Among many options for relativizing homotopy fixed points in this setting, there is one of specific interest.

Let $\mathcal{A} = \mathcal{B}(Y, R)$ as before and $\mathcal{A}' = \mathbf{Mod}(R)$ be the category of arbitrary R-modules. There is a *forget control* functor $t : \mathcal{B}(Y, R) \to \mathbf{Mod}(R)$ which only remembers that the objects are R-modules and the morphisms are R-linear homomorphisms. From t we may induce the functor $T : \mathcal{B}(X, \mathcal{A}) \to \mathcal{B}(X, \mathcal{A}')$.

For this construction we assume that Γ acts on X by isometries and so, therefore, on $\mathcal{B}(X, \mathcal{A}')$. On the other hand, we allow the action of Γ on Y to be by coarse equivalences. This can also be used to induce an action on $\mathcal{B}(X, \mathcal{A})$.

Definition 4.4 (fibred homotopy fixed points in bounded K-theory). These are relative homotopy fixed points with the following choice of ingredients:

- the category $\mathcal{C}$ is $\mathcal{B}_X(Y, R)$,
- the subcategory $\mathcal{M}$ consists of all morphisms ϕ such that $T(\phi)$ is a controlled morphism bounded by 0.

Notation 4.5. When X is the group Γ itself with the left multiplication action and the word metric with respect to some choice of a finite set of generators, we obtain a particularly useful case of this construction. We use the special notation $\mathcal{B}^{h\Gamma}(Y)$ for the fibred homotopy fixed points $\mathcal{C}^{h\Gamma}(\mathcal{M})$ and $K_p^{\Gamma}(Y)$ for the nonconnective K-theory spectrum of $\mathcal{B}^{h\Gamma}(Y)$.

5. Summary of bounded G-theory with fibred control

A comprehensive exposition of bounded G-theory with fibred control is available in [Carlsson and Goldfarb 2019]. Compared to K-theory with fibred control from Definition 4.3, G-theory replaces free modules with arbitrary modules over a Noetherian ring R and replaces the split exact sequences with a more general kind of exact sequence. This is a summary of that theory and a number of facts in the form we can refer to in the next section.

Throughout the rest of the paper, R will be a Noetherian ring.

At the basic level, bounded G-theory with fibred control is an analogue of the algebraic K-theory of $\mathcal{B}_X(Y, R)$ locally modeled on finitely generated R-modules. The result is an exact category $\mathbf{B}_X(Y)$ where the exact sequences are not necessarily split but which contains $\mathcal{B}_X(Y)$ as an exact subcategory.

Definition 5.1. Given an R-module F, an (X, Y)*-filtration* of F is a covariant functor $\mathcal{P}(X \times Y) \to \mathcal{I}(F)$ from the power set of the product metric space to the partially ordered family of R-submodules of F, both ordered by inclusion. It is convenient to denote the value of this functor on a subset $U \subset X \times Y$ by $F(U)$ and assume that $F(X \times Y) = F$ and $F(\varnothing) = 0$.

The associated X-filtered R-module F_X is given by $F_X(S) = F(S \times Y)$. Similarly, for each subset $S \subset X$, one has the Y-filtered R-module F^S given by $F^S(T) = F(S \times T)$. In particular, $F^X(T) = F(X \times T)$.

Notation 5.2. We will use the following notation generalizing enlargements in a metric space. Given a subset U of $X \times Y$ and a function $k : X \to [0, +\infty)$, let

$$U[k] = \{(x, y) \in X \times Y \mid \text{there is } (x, y') \in U \text{ with } d(y, y') \le k(x)\}.$$

If in addition we are given a number $K \ge 0$ then

$$U[K, k] = \{(x, y) \in X \times Y \mid \text{ there is } (x', y) \in U[k] \text{ with } d(x, x') \le K\}.$$

For a product set $U = S \times T$, it is more convenient to use the notation $(S, T)[K, k]$ in place of $(S \times T)[K, k]$. We refer to the pair (K, k) in the notation $U[K, k]$ as the *enlargement data*.

Let x_0 be a fixed point in X. Given a monotone function $h : [0, +\infty) \to [0, +\infty)$, there is a function $h_{x_0} : X \to [0, +\infty)$ defined by

$$h_{x_0}(x) = h(d_X(x_0, x)).$$

Definition 5.3. Given two (X, Y)-filtered modules F and G, an R-homomorphism $f : F(X \times Y) \to G(X \times Y)$ is *boundedly controlled* if there are a number $b \ge 0$ and a monotone function $\theta : [0, +\infty) \to [0, +\infty)$ such that

$$fF(U) \subset G(U[b, \theta_{x_0}]) \tag{†}$$

for all subsets $U \subset X \times Y$ and some choice of $x_0 \in X$. It is easy to see that this condition is independent of the choice of x_0. If a homomorphism f is boundedly controlled with respect to some choice of parameters b and θ, we say that f is (b, θ)*-controlled*.

The *unrestricted fibred bounded category* $\mathbf{U}_X(Y)$ has (X, Y)-filtered modules as objects and the boundedly controlled homomorphisms as morphisms.

Theorem 3.1.6 of [Carlsson and Goldfarb 2019] shows that $\mathbf{U}_X(Y)$ is a cocomplete semiabelian category. When Y is the one point space, this construction recovers the controlled category $\mathbf{U}(X, R)$ of X-filtered R-modules used to construct bounded G-theory in [Carlsson and Goldfarb 2011] and [Carlsson and Goldfarb 2019, Chapter 2]. In this case, boundedly controlled homomorphisms are characterized by a single parameter b, so one can specify that by abbreviating the term to simply *b-controlled*. The construction of an X-filtration F_X from a given (X, Y)-filtration in Definition 5.1 allows us to view a (b, θ)-controlled homomorphism in $\mathbf{U}_X(Y)$ as a b-controlled homomorphism in $\mathbf{U}(X, R)$ via the forgetful functor $T : \mathbf{U}_X(Y) \to \mathbf{U}(X, R)$.

We now want to restrict to a subcategory of $\mathbf{U}_X(Y)$ that is full on objects with particular properties. This process consists of two steps that result in a theory with better localization properties.

Definition 5.4. An (X, Y)-filtered module F is called

- *split* or (D, Δ)-*split* if there is a number $D \geq 0$ and a monotone function $\Delta : [0, +\infty) \to [0, +\infty)$ such that

$$F(U_1 \cup U_2) \subset F(U_1[D, \Delta_{x_0}]) + F(U_2[D, \Delta_{x_0}])$$

  for each pair of subsets U_1 and U_2 of $X \times Y$,
- *lean/split* or (D, Δ')-*lean/split* if there is a number $D \geq 0$ and a monotone function $\Delta' : [0, +\infty) \to [0, +\infty)$ such that
  - the X-filtered module F_X is D-*lean*, in the sense that

$$F_X(S) \subset \sum_{x \in S} F_X(x[D])$$

    for every subset S of X, while
  - the (X, Y)-filtered module F is (D, Δ')-split,
- *insular* or (d, δ)-*insular* if there is a number $d \geq 0$ and a monotone function $\delta : [0, +\infty) \to [0, +\infty)$ such that

$$F(U_1) \cap F(U_2) \subset F(U_1[d, \delta_{x_0}] \cap U_2[d, \delta_{x_0}])$$

  for each pair of subsets U_1 and U_2 of $X \times Y$.

There are two subcategories nested in $\mathbf{U}_X(Y)$. The category $\mathbf{LS}_X(Y)$ is the full subcategory of $\mathbf{U}_X(Y)$ on objects F that are lean/split and insular. The category $\mathbf{B}_X(Y)$ is the full subcategory of $\mathbf{LS}_X(Y)$ on objects F such that $F(U)$ is a finitely generated submodule whenever $U \subset X \times Y$ is bounded.

We proceed to define appropriate exact structures in these categories. The admissible monomorphisms are precisely the morphisms isomorphic in $\mathbf{U}_X(Y)$ to the filtrationwise monomorphisms and the admissible epimorphisms are those morphisms isomorphic to the filtrationwise epimorphisms. In other words, the exact structure $\mathcal{E}$ in $\mathbf{U}_X(Y)$ consists of sequences isomorphic to those

$$E^{\cdot} : \quad E' \xrightarrow{i} E \xrightarrow{j} E''$$

which possess filtrationwise restrictions

$$E^{\cdot}(U): \quad E'(U) \xrightarrow{i} E(U) \xrightarrow{j} E''(U)$$

for all subsets $U \subset (X, Y)$, and each $E^{\cdot}(U)$ is an exact sequence of R-modules.

Both $\mathbf{LS}_X(Y)$ and $\mathbf{B}_X(Y)$ are closed under extensions in $\mathbf{U}_X(Y)$. Therefore, they are themselves exact categories, and the inclusion $\mathcal{B}_X(Y) \to \mathbf{B}_X(Y)$ is an exact embedding, as we projected.

There is a useful invariant of a finitely generated group Γ that is defined in terms of the exact category $\mathbf{B}_\Gamma(\text{point})$ in [Carlsson and Goldfarb 2016]. Here Γ can be given the word metric associated to any of the finite generating sets. The left multiplication action gives an action of Γ on $\mathbf{B}_\Gamma(\text{point})$.

Recall that Theorem 3.4 provides an interpretation to the K-theory of a group ring $R[\Gamma]$ in terms of relative homotopy fixed points of the additive category $\mathcal{B}(X, R)$, which can be viewed as $\mathcal{B}_\Gamma(\text{point})$.

Example 5.5 (bounded G-theory of a finitely generated group). In the case where $\mathcal{C}$ is the exact category $\mathbf{B}_\Gamma(\text{point})$ and $\mathcal{M}$ is the subcategory of the filtration zero morphisms, the *bounded G-theory of* Γ is defined to be the nonconnective K-theory of the relative homotopy fixed points $\mathbf{B}_\Gamma(\text{point})^{h\Gamma}$, denoted $G^{-\infty}(R[\Gamma])$.

Notice that this definition makes sense even when the group ring is not Noetherian, unlike the much more restrictive situation with the usual G-theory defined only for Noetherian rings.

Theorem 5.6. *There is an exact subcategory of finitely generated* Γ*-modules for an arbitrary finitely generated group* Γ *such that its relative homotopy fixed points have Quillen K-theory with features similar to G-theory of group rings. In particular, it has a Cartan map from the K-theory of $R[\Gamma]$.*

Proof. The category is equivalent to $\mathbf{B}_\Gamma(\text{point})$. We refer to Sections 2 and 3 of [Carlsson and Goldfarb 2016] for details. The clear resemblance to Definition 3.3 and the identification of $\mathcal{B}^{\Gamma,0}(X, R)^\Gamma$ with $\mathcal{B}_\Gamma(\text{point})^{h\Gamma}$ allow us to induce the Cartan map $K^{-\infty}(R[\Gamma]) \to G^{-\infty}(R[\Gamma])$ from the exact inclusion $\mathcal{B}_\Gamma(\text{point}) \to \mathbf{B}_\Gamma(\text{point})$ above. □

Suppose C is a subset of Y. Let $\mathbf{B}_X(Y)_{<C}$ be the full subcategory of $\mathbf{B}_X(Y)$ on objects F such that

$$F(X, Y) \subset F((X, C)[r, \rho_{x_0}]))$$

for some number $r \geq 0$ and an order preserving function $\rho : [0, +\infty) \to [0, +\infty)$.

Recall that a *Serre subcategory* of an exact category is a full subcategory which is closed under exact extensions and closed under passage to admissible subobjects and admissible quotients. Proposition 3.3.3 of [Carlsson and Goldfarb 2019] verifies that $\mathbf{B}_X(Y)_{<C}$ is a Serre subcategory of $\mathbf{B}_X(Y)$.

The second step in restricting to subcategories with good localization properties is done via introducing a structure called a grading.

Given an arbitrary R-submodule F' of F in $\mathbf{U}_X(Y)$, we can assign to F' the *standard* (X, Y)*-filtration* $F'(U) = F(U) \cap F'$.

Let $\mathcal{M}^{\geq 0}$ be the set of all monotone functions $\delta : [0, +\infty) \to [0, +\infty)$.

Let $\mathcal{P}_X(Y)$ be the subcategory of $\mathcal{P}(X, Y)$ consisting of all subsets of the form $(X, C)[D, \delta_{x_0}]$ for some choices of a subset $C \subset Y$, a number $D \geq 0$, and a function $\delta \in \mathcal{M}^{\geq 0}$.

Definition 5.7. Given an object F of $\mathbf{B}_X(Y)$, a Y*-grading* of F is a functor

$$\mathcal{F} : \mathcal{P}_X(Y) \to \mathcal{I}(F)$$

with the following properties:

- the submodule $\mathcal{F}((X, C)[D, \delta_{x_0}])$, with the standard (X, Y)-filtration induced from F, is an object of $\mathbf{B}_X(Y)$,
- there is an enlargement data (K, k) such that

  $$F((X, C)[D, \delta_{x_0}]) \subset \mathcal{F}((X, C)[D, \delta_{x_0}]) \subset F((X, C)[D + K, \delta_{x_0} + k_{x_0}]),$$

  for all subsets in $\mathcal{P}_X(Y)$.

We say that an object F of $\mathbf{B}_X(Y)$ is Y*-graded* if there exists a Y-grading of F, but the grading itself is not specified, and define $\mathbf{G}_X(Y)$ as the full subcategory of $\mathbf{B}_X(Y)$ on Y-graded filtered modules.

The category $\mathbf{G}_X(Y)$ is of major importance. It is the category to which we will apply the relative homotopy fixed points construction in Definition 6.4. It is given in several stages, so the reader may find it helpful to refer to a compressed Definition 1.2 of this category in the introduction. For some examples of interesting nonprojective objects from $\mathbf{G}_X(Y)$ in an equivariant setting, we refer to Example 4.2 in [Carlsson and Goldfarb 2016]. It is also true that the category $\mathcal{B}_X(Y)$ introduced in Definition 4.3 is contained in $\mathbf{G}_X(Y)$.

Proposition 5.8. *$\mathcal{B}_X(Y)$ is a full exact subcategory of $\mathbf{G}_X(Y)$.*

Proof. The (X, Y)-filtration of the objects in $\mathcal{B}_X(Y)$ is given by

$$F(S) = \bigoplus_{x \in S} F(x).$$

This ensures that for any pair of subsets $T \subset S$ of $X \times Y$ one has

$$F(S) = F(T) \oplus F(S \setminus T).$$

This splitting is the reason the object F possesses the required gradings with all required properties on the nose, with the enlargement data all equal to 0. The structure maps are the boundedly controlled inclusions and projections onto direct summands. □

We summarize some additional results from Section 3.4 of [Carlsson and Goldfarb 2019].

Theorem 5.9. *The subcategory $\mathbf{G}_X(Y)$ is closed under both isomorphisms and exact extensions in $\mathbf{B}_X(Y)$. Therefore, $\mathbf{G}_X(Y)$ is an exact subcategory of $\mathbf{B}_X(Y)$.*

The restriction to Y-gradings in $\mathbf{B}_X(Y)_{<C}$ gives a full exact subcategory $\mathbf{G}_X(Y)_{<C}$ which is a Serre subcategory of $\mathbf{G}_X(Y)$.

Given a graded object F in $\mathbf{G}_X(Y)$, we assume that F is (D, Δ')-lean/split and (d, δ)-insular and is graded by $\mathcal{F}$. For a subset U from the family $\mathcal{P}_X(Y)$, the submodule $\mathcal{F}(U)$ has the following properties:

(1) *$\mathcal{F}(U)$ is graded by $\mathcal{F}_U(T) = \mathcal{F}(U) \cap \mathcal{F}(T)$,*

(2) *$F(U) \subset \mathcal{F}(U) \subset F(U[K, k])$ for some fixed enlargement data (K, k),*

(3) *if $q : F \to H$ is the quotient of the inclusion $i : \mathcal{F}(U) \to F$ in $\mathbf{B}_X(Y)$ and F is (D, Δ')-lean/split, then H is supported on $(X \setminus U)[2D, 2\Delta']$,*

(4) *$H(U[-2D - 2d, -2\Delta' - 2\delta]) = 0$.*

We assume that the reader is familiar with Quillen K-theory of exact categories. This theory can be applied to both $\mathbf{G}_X(Y)$ and $\mathbf{G}_X(Y)_{<C}$. The result can be viewed as spectra $G_X(Y)$ and $G_X(Y)_{<C}$. The stable homotopy groups of these spectra are the Quillen K-groups of the exact categories.

Finally, the main goal of this section is a homotopy fibration

$$G_X(Y)_{<C} \to G_X(Y) \to G_X(Y, C),$$

where $G_X(Y, C)$ is the K-theory of a certain quotient category $\mathbf{G}_X(Y)/\mathbf{G}_X(Y)_{<C}$.

For simplicity we use the notation $\mathbf{G}$ for $\mathbf{G}_X(Y)$ and $\mathbf{C}$ for the Serre subcategory $\mathbf{G}_X(Y)_{<C}$ of $\mathbf{G}$.

There is a class of *weak equivalences* $\Sigma(C)$ in $\mathbf{G}$ which consist of all finite compositions of admissible monomorphisms with cokernels in $\mathbf{C}$ and admissible epimorphisms with kernels in $\mathbf{C}$. We need the class $\Sigma(C)$ to admit calculus of right fractions. This follows from [Schlichting 2004, Lemma 1.13] and the fact that $\mathbf{C}$ in $\mathbf{G}$ is right filtering, in the sense that each morphism $f : F_1 \to F_2$ in $\mathbf{G}$, where F_2 is an object of $\mathbf{C}$, factors through an admissible epimorphism $e : F_1 \to \overline{F}_2$, where $\overline{F}_2$ is in $\mathbf{C}$. The latter fact is [Carlsson and Goldfarb 2019, Lemma 3.5.6].

The category $\mathbf{G}/\mathbf{C}$ is the localization $\mathbf{G}[\Sigma(C)^{-1}]$. From [Carlsson and Goldfarb 2019, Theorem 3.5.8], it is an exact category where the short sequences are isomorphic to images of exact sequences from $\mathbf{G}$.

There is an intrinsic reformulation of the homotopy fibration because the essential full image of the evident inclusion of $\mathbf{G}_X(C)$ in $\mathbf{G}$ is precisely $\mathbf{C}$. This gives a homotopy fibration

$$G_X(C) \to G_X(Y) \to G_X(Y, C).$$

One quick consequence is the ability to relativize the old constructions. If Y' is any subset of Y, one obtains the relative theory $G_X(Y, Y')$.

Another easy application is a nonconnective delooping that applies to all of the theories we have defined. For example in the basic case,

$$G_X^{-\infty}(Y) = \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k G_X(Y \times \mathbb{R}^k).$$

This uses the usual Eilenberg swindle trick and can be seen in Section 4.2 of [Carlsson and Goldfarb 2019].

6. Fibrewise excision in equivariant fibred G-theory

It is well-known that Quillen K-theory of an exact category can be obtained equivalently as Waldhausen's K-theory of bounded chain complexes in the category. The cofibrations are then the chain maps which are the degreewise admissible monomorphisms. The weak equivalences are the chain maps whose mapping cones are homotopy equivalent to acyclic complexes. An exposition with a number of details verified specifically for bounded G-theory can be found in [Carlsson and Goldfarb 2011, Section 4]. The Waldhausen theory setting is crucial in proving the excision theorem in that the approximation theorem [Carlsson and Goldfarb 2011, Theorem 4.5] becomes essential. We indicate passage from an exact category to the derived category of bounded chain complexes by prefixing "ch" in front of the name of the exact category.

We proceed to define the equivariant fibred G-theory. The basic setting consists of

- two proper metric spaces X and Y,
- an arbitrary subset Y' of Y,
- a Γ-action on X by isometries, and
- a bounded action of Γ on Y. This is an action such that for each γ in Γ the set of real numbers $W_\gamma = \{d(x, \gamma(x))\}$ is bounded from above.

Remark 6.1. In a number of situations, we will be specifying subcategories closed under the Γ-action by subsets that are arbitrary, and therefore certainly not closed under the action. This works due to the boundedness of the action. For example, if we have a subset $C \subseteq X$ and define a subcategory as the set of modules supported on some neighborhood of C, then this subcategory is closed under the Γ-action provided the action is bounded. This would definitely not hold were the action not bounded.

Consider the exact category $\mathbf{G}_\Gamma(Y)$ with the induced action by Γ, in the case X is the group Γ with a word metric, acting on itself by isometries via the left multiplication. Since the action on Y is bounded, we have the quotient exact category $\mathbf{G}_\Gamma(Y, Y')$.

Notation 6.2. If Z is another arbitrary subset of Y, it is also useful to consider the full exact subcategory $\mathbf{G}_\Gamma(Y, Y')_{<Z}$, which we denote $\mathbf{G}_\Gamma(Y, Y', Z)$.

Recasting the definition from Definition 4.4, we define the Waldhausen category $\mathcal{G}_{\Gamma,0}(Y, Y', Z)$ to be the full subcategory of $\mathrm{Fun}(\boldsymbol{E\Gamma}, \mathrm{ch}\,\mathbf{G}_\Gamma(Y, Y', Z))$ on those functors that send morphisms in $\boldsymbol{E\Gamma}$ to degreewise 0-controlled homomorphisms of Γ-filtered modules.

Definition 6.3. The *equivariant fibred G-theory* is

$$G^\Gamma(Y, Y', Z) = \Omega K(|wS.\mathcal{G}_{\Gamma,0}(Y, Y', Z)|).$$

This is a functor from the category of triples (Y, Y', Z), where both Y' and Z are subspaces of Y but not necessarily subspaces of each other, and uniformly expansive maps of triples to the category of spectra.

Now we turn to the construction of fibred homotopy fixed points. There is a forget control functor $T : \mathbf{G}_X(Y, Y', Z) \to \mathbf{U}_X(Y, Y', Z)$ sending F to F_X. Since Γ acts on X by isometries, it also acts on $\mathbf{U}_X(Y, Y', Z)$. The combination of this action and a bounded action on Y induces an action on $\mathbf{G}_X(Y, Y', Z)$. With these choices, T is an equivariant functor.

Definition 6.4 (fibred homotopy fixed points in bounded G-theory). This is a special case of relative homotopy fixed points, as defined in Definition 4.4, with the choices of $\mathcal{C}$ and $\mathcal{M}$ as follows:

- the category $\mathcal{C}$ is $\mathbf{G}_X(Y, Y', Z)$,
- the subcategory $\mathcal{M}$ consists of all controlled morphisms ϕ in $\mathcal{C}$ with the property that $T(\phi)$ is bounded by 0 as homomorphisms controlled over X.

Let us recapitulate what this definition entails in the case X is the group Γ with a word metric.

The *fibred homotopy fixed points of a triple* (Y, Y', Z) is the category $\mathbf{G}^{h\Gamma}(Y, Y', Z)$ with objects which are sets of data $(\{F_\gamma\}, \{\psi_\gamma\})$ where

- F_γ is an object of $\mathbf{G}_\Gamma(Y, Y', Z)$ for each γ in Γ,
- ψ_γ is an isomorphism $F_e \to F_\gamma$ in $\mathbf{G}_\Gamma(Y, Y', Z)$,
- ψ_γ is 0-controlled when viewed as a morphism in $\mathbf{U}_\Gamma(Y, Y', Z)$,
- $\psi_e = \mathrm{id}$,
- $\psi_{\gamma_1\gamma_2} = \gamma_1\psi_{\gamma_2} \circ \psi_{\gamma_1}$ for all γ_1, γ_2 in Γ.

A morphism $(\{F_\gamma\}, \{\psi_\gamma\}) \to (\{F'_\gamma\}, \{\psi'_\gamma\})$ is a collection of morphisms $\phi_\gamma : F_\gamma \to F'_\gamma$ in $\mathbf{G}_\Gamma(Y, Y', Z)$ such that the squares

$$\begin{array}{ccc} F_e & \xrightarrow{\psi_\gamma} & F_\gamma \\ {\scriptstyle \phi_e}\downarrow & & \downarrow{\scriptstyle \phi_\gamma} \\ F'_e & \xrightarrow{\psi'_\gamma} & F'_\gamma \end{array}$$

commute for all γ.

The exact structure on $\mathbf{G}^{h\Gamma}(Y, Y', Z)$ is induced from that on $\mathbf{G}_\Gamma(Y, Y', Z)$ as follows. A morphism ϕ in $\mathbf{G}^{h\Gamma}(Y, Y', Z)$ is an admissible monomorphism if $\phi_e : F \to F'$ is an admissible monomorphism in $\mathbf{G}_\Gamma(Y, Y', Z)$. This of course implies that all structure maps ϕ_γ are admissible monomorphisms. Similarly, a morphism ϕ is an admissible epimorphism if $\phi_e : F \to F'$ is an admissible epimorphism. This gives $\mathbf{G}^{h\Gamma}(Y, Y', Z)$ an exact structure.

Since the induced Γ-action on $S.\mathcal{G}_{\Gamma,0}(Y, Y', Z)$ commutes with taking fixed points, we have the following fact.

Proposition 6.5. *The fixed point spectrum $G^\Gamma(Y, Y', Z)^\Gamma$ is equivalent to the K-theory of the relative homotopy fixed point category $\mathbf{G}^{h\Gamma}(Y, Y', Z)$.*

We proceed to consider multiple bounded actions of Γ on Y. Let $\beta(Y)$ be the set of all such actions. Let $\mathcal{F}$ be the functor that assigns to a set Z the partially ordered set of finite subsets of Z.

Definition 6.6. For any S in $\mathcal{F}(\beta(Y))$ we define Y_S as the metric space which is the disjoint union $\bigsqcup_{s \in S} Y_s$, where Y_s are copies of Y with the specified action. The metric on Y_S is induced by the requirement that it restricts to the metric from Y in each Y_s and for the same point y in different components the distance $d(y_s, y_{s'})$ equals 1.

Clearly, the action of Γ on Y_S is bounded.

As a consequence of Proposition 6.5, for each choice of finite subset S of $\beta(Y)$, the spectrum $G^\Gamma(Y_S, Y'_S, Z)^\Gamma$ is the Quillen K-theory spectrum of $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)$, where Z is a subset of Y_S.

Theorem 6.7. *Let C be an arbitrary subset of Y. There is a homotopy fibration*

$$G^\Gamma(Y_S, Y'_S, Z)^\Gamma_{<C} \to G^\Gamma(Y_S, Y'_S, Z)^\Gamma \to G^\Gamma(Y_S, Y'_S, Z)^\Gamma_{>C},$$

where $G^\Gamma(Y_S, Y'_S, Z)^\Gamma_{>C}$ stands for K-theory of the exact quotient $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)_{>C}$ of $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)$ by $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)_{<C}$. In the absolute case, there is an equivalence $\mathbf{G}^{h\Gamma}(Y)_{<C} \simeq \mathbf{G}^{h\Gamma}(C)$, and so there is a homotopy fibration

$$G^\Gamma(C)^\Gamma \to G^\Gamma(Y)^\Gamma \to G^\Gamma(Y)^\Gamma_{>C}.$$

Proof. In view of Remark 6.1, the fact that $\mathbf{G}_\Gamma(Y_S, Y'_S, Z)_{<C}$ is an idempotent complete Serre subcategory of $\mathbf{G}_\Gamma(Y_S, Y'_S, Z)$ implies immediately that $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)^{h\Gamma}_{<C}$ is an idempotent complete Serre subcategory of $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)$. The main technical result of [Schlichting 2004] is a fibration theorem which requires $\mathbf{G}^{h\Gamma}(Y_S, Y'_S, Z)_{<C}$ to satisfy two additional properties: right filtering and right s-filtering. Both of these properties follow directly from the estimates in the proofs of Lemma 3.5.6 and Theorem 3.5.8 in [Carlsson and Goldfarb 2019]. □

Our first application of the fibration is to deloop $G^\Gamma(Y_S, Y'_S, Z)^\Gamma$ and related spectra following the strategy of [Pedersen and Weibel 1985].

Let $\mathbb{R}$, $\mathbb{R}^{\geq 0}$, and $\mathbb{R}^{\leq 0}$ denote the metric spaces of the reals, the nonnegative reals, and the nonpositive reals with the Euclidean metric. Then there is the following map of homotopy fibrations:

$$\begin{array}{ccccc} G^\Gamma(Y_S)^\Gamma & \longrightarrow & G^\Gamma(Y_S \times \mathbb{R}^{\geq 0})^\Gamma & \longrightarrow & G^\Gamma(Y_S \times \mathbb{R}^{\geq 0})^\Gamma_{>Y_S \times 0} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle K(I)} \\ G^\Gamma(Y_S \times \mathbb{R}^{\leq 0})^\Gamma & \longrightarrow & G^\Gamma(Y_S \times \mathbb{R})^\Gamma & \longrightarrow & G^\Gamma(Y_S \times \mathbb{R})^\Gamma_{>Y_S \times \mathbb{R}^{\leq 0}} \end{array}$$

The map $K(I)$ is induced by the inclusion I of the quotient categories.

Theorem 6.8. *$K(I)$ is a weak equivalence of connective spectra.*

Proof. This follows from the approximation theorem applied to I. The first condition of the theorem is evident. To check the second condition, consider a chain complex $F^\cdot$ in $\mathbf{G}^{h\Gamma}(Y_S \times \mathbb{R}^{\geq 0})_{>Y_S \times 0}$. By the nature of the objects and the explanation in Remark 6.1, all maps in $F^\cdot$ and their control features are determined by the values on the objects F^i_e of $\mathbf{G}_\Gamma(Y_S \times \mathbb{R})$. So we can specify $F^\cdot$ by the chain complex

$$F^\cdot_e: \quad 0 \to F^1_e \xrightarrow{\phi_1} F^2_e \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} F^n_e \to 0$$

in $\mathbf{G}_\Gamma(Y_S \times \mathbb{R}^{\geq 0})_{>Y_S \times 0}$. Given a chain complex $G^\cdot$ in $\mathbf{G}^{h\Gamma}(Y_S \times \mathbb{R})_{>Y_S \times \mathbb{R}^{\leq 0}}$, we can apply the same reasoning to $G^\cdot$. Now a chain map $g : F^\cdot \to G^\cdot$ is given uniquely by a chain map $F^\cdot_e \to G^\cdot_e$, where $G^\cdot_e$ is the chain complex

$$0 \to G^1_e \xrightarrow{\psi_1} G^2_e \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{n-1}} G^n_e \to 0$$

in $\mathbf{G}_\Gamma(Y_S \times \mathbb{R})_{>Y_S \times \mathbb{R}^{\leq 0}}$. Also observe that if F^i_e is supported on a neighborhood of $C \subset Y$ then so are all of the F^i_γ. This allows us to transport from [Carlsson and Goldfarb 2019] the rest of the argument for a nonequivariant Lemma 4.2.4. Alternatively, we can refer to the end of the proof of Theorem 6.11, where the details are spelled out in even greater generality. □

The spectra $G^\Gamma(Y_S \times \mathbb{R}^{\geq 0})^\Gamma$ and $G^\Gamma(Y_S \times \mathbb{R}^{\leq 0})^\Gamma$ are contractible as K-theory spectra of flasque categories. This is the standard consequence of the shift functor T in the positive (resp. negative) direction along $\mathbb{R}^{\geq 0}$ (resp. $\mathbb{R}^{\leq 0}$) interpreted as an exact endofunctor. A natural equivalence of functors $1 \oplus T \cong T$ and the additivity theorem give contractibility.

From the map of fibrations, we obtain a map of spectra $G^\Gamma(Y_S)^\Gamma \to \Omega G^\Gamma(Y_S \times \mathbb{R})^\Gamma$ which induces isomorphisms of K-groups in positive dimensions. Iterating this construction for $k \geq 2$ gives weak equivalences

$$\Omega^k G^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma \to \Omega^{k+1} G^\Gamma(Y_S \times \mathbb{R}^{k+1})^\Gamma.$$

Definition 6.9. The nonconnective delooping of algebraic K-theory of the fibred homotopy fixed points is the spectrum

$$\widetilde{G}^\Gamma(Y_S)^\Gamma = \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma.$$

In the case Y is the one point space, $\widetilde{G}^\Gamma(Y)^\Gamma$ coincides with the nonconnective G-theory of the group ring $R[\Gamma]$ defined in [Carlsson and Goldfarb 2016].

The discussion leading up to Definition 6.9 can be repeated verbatim for other Serre subcategory pairs. For example, the subcategory $\mathbf{G}^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma_{<C\times\mathbb{R}^k}$ is evidently a Serre subcategory of $\mathbf{G}^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma$ for any choice of subset $C \subset Y_S$. We define

$$\widetilde{G}^\Gamma(Y_S)^\Gamma_{<C} = \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma_{<C\times\mathbb{R}^k},$$
$$\widetilde{G}^\Gamma(Y_S)^\Gamma_{<C_1,C_2} = \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k)^\Gamma_{<C_1\times\mathbb{R}^k,\, C_2\times\mathbb{R}^k}.$$

Definition 6.10. Let Y', Y_1, and Y_2 be arbitrary subsets of Y such that Y_1 and Y_2 form a covering of Y. There are corresponding subsets Y'_S, $Y_{1,S}$, and $Y_{2,S}$ of Y_S obtained as $Y'_S = \bigsqcup Y'_s$, $Y_{1,S} = \bigsqcup Y'_{1,s}$, and $Y_{2,S} = \bigsqcup Y'_{2,s}$. It is now straightforward to define nonconnective spectra

$$\widetilde{G}^\Gamma(Y_S, Y'_S)^\Gamma = \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k, Y'_S \times \mathbb{R}^k)^\Gamma,$$
$$\widetilde{G}^\Gamma(Y_S, Y'_S)^\Gamma_{<Y_i} = \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k, Y'_S \times \mathbb{R}^k)^\Gamma_{<Y_{i,S}\times\mathbb{R}^k},$$
$$\widetilde{G}^\Gamma(Y_S, Y'_S)^\Gamma_{<Y_1,Y_2} = \underset{\overrightarrow{k>0}}{\text{hocolim}}\ \Omega^k G^\Gamma(Y_S \times \mathbb{R}^k, Y'_S \times \mathbb{R}^k)^\Gamma_{<Y_{1,S}\times\mathbb{R}^k,\, Y_{2,S}\times\mathbb{R}^k}.$$

Theorem 6.11. *Suppose Y_1 and Y_2 are subsets of a metric space Y on which Γ acts by bounded coarse equivalences, and $Y = Y_1 \cup Y_2$. There is a homotopy pushout diagram of spectra*

$$\begin{array}{ccc} \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1,Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1} \\ \downarrow & & \downarrow \\ \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma} \end{array}$$

where the maps of spectra are induced from the exact inclusions. If we define

$$E^{\Gamma}(Y, Y') = \underset{U \in \mathcal{F}(\beta(Y))}{\underrightarrow{\text{hocolim}}}\ \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma},$$

the excision theorem also holds on the level of E^{Γ}. There is a homotopy pushout diagram of spectra

$$\begin{array}{ccc} E^{\Gamma}(Y, Y')_{<Y_1,Y_2} & \longrightarrow & E^{\Gamma}(Y, Y')_{<Y_1} \\ \downarrow & & \downarrow \\ E^{\Gamma}(Y, Y')_{<Y_2} & \longrightarrow & E^{\Gamma}(Y, Y') \end{array}$$

Proof. There is a homotopy pushout

$$\begin{array}{ccc} \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1,Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1} \\ \downarrow & & \downarrow \\ \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma} \end{array}$$

obtained from the map of the fibration sequences

$$\begin{array}{ccccc} \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1,Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1,>Y_2} \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_2} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma} & \longrightarrow & \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{>Y_2} \end{array}$$

both obtained from Theorem 6.7. The map

$$\widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{<Y_1,>Y_2} \to \widetilde{G}^{\Gamma}(Y_S, Y'_S)^{\Gamma}_{>Y_2}$$

induced from the exact inclusion $J : \mathbf{G}^{h\Gamma}(Y_S, Y'_S)_{<Y_1,>Y_2} \to \mathbf{G}^{h\Gamma}(Y_S, Y'_S)_{>Y_2}$ is again an equivalence. It should be instructive to spell out the crucial application of the approximation theorem. Consider a chain complex $F^{\cdot}$ in $\mathbf{G}^{h\Gamma}(Y_S, Y'_S)_{<Y_1,>Y_2}$. All maps in $F^{\cdot}$ and their control features are determined by the values on the objects F^i_e of $\mathbf{G}_{\Gamma}(Y_S, Y'_S)$, so $F^{\cdot}$ can be given by the chain complex

$$F^{\cdot}_e : \quad 0 \to F^1_e \xrightarrow{\phi_1} F^2_e \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} F^n_e \to 0$$

in $\mathbf{G}_\Gamma(Y_S, Y'_S)_{<Y_1,>Y_2}$. Applying the same reasoning to a chain complex $G^\cdot$ in $\mathbf{G}^{h\Gamma}(Y_S, Y'_S)_{>Y_2}$, let $G^\cdot_e$ be the chain complex

$$0 \to G^1_e \xrightarrow{\psi_1} G^2_e \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{n-1}} G^n_e \to 0$$

in $\mathbf{G}_\Gamma(Y_S, Y'_S)_{>Y_2}$. A chain map $g : F^\cdot \to G^\cdot$ can be given by a chain map $g' : F^\cdot_e \to G^\cdot_e$. Since the action is bounded, if F^i_e is supported near a neighborhood of $Y_{1,S} \subset Y_S$ then so are all F^i_γ.

Suppose all F^i_e and G^i_e are (D, Δ')-lean/split and (d, δ)-insular, and there is a number $r \geq 0$ and a monotone function $\rho : [0, +\infty) \to [0, +\infty)$ such that there are containments $F^i_e \subset F^i_e((\Gamma, Y_{1,S})[r, \rho_{x_0}])$ for all $0 \leq i \leq n$. Suppose also that (b, θ) can be used as bounded control data for all maps ϕ_i, ψ_i, and g'_i. Suppose also that (K, k) is an enlargement data for the chosen grading of G_e. We define a submodule F'^i_e as the submodule $\mathcal{G}^i_e((\Gamma, Y_{1,S})[r + 3ib, \rho_{x_0} + 3i\theta_{x_0}])$ in the chosen grading of G_e and define $\xi_i : F'^i_e \to F'^{i+1}_e$ to be the restrictions of ψ_i to F'^i_e. This gives a chain subcomplex (F'^i, ξ_i) of (G^i, ψ_i) in $\mathbf{G}_\Gamma(Y_S, Y'_S)$ with the inclusion $i : F'^i \to G^i$. Notice that we have the induced chain map $\bar{g} : F^\cdot \to F'^\cdot$ in $\mathbf{G}_\Gamma(Y_S, Y'_S)_{<Y_1}$ so that $g = i\,J(\bar{g})$. It remains to prove that the cokernel $C^\cdot$ of i is in $\mathbf{G}_\Gamma(Y_S, Y'_S)_{<Y_2}$. Since

$$F'^i_e \subset G^i_e((\Gamma, Y_{1,S})[r + 3ib + K, \rho_{x_0} + 3i\theta_{x_0} + k_{x_0}]),$$

each C^i is supported on

$$(\Gamma, Y_S \setminus Y_{1,S})[2D + 2d - r - 3ib - K, 2\Delta'_{x_0} + 2\delta_{x_0} - \rho_{x_0} - 3i\theta_{x_0} - k_{x_0}] \\ \subset (\Gamma, Y_{2,S})[2D + 2d, 2\Delta'_{x_0} + 2\delta_{x_0}].$$

This shows that the complex $C^\cdot$ is indeed in $\mathbf{G}_\Gamma(Y_S, Y'_S)_{<Y_2}$. □

Remark 6.12. This observation is warranted as we contrast Theorem 6.11 and its proof with the inability to use other, more standard methods in bounded algebra based on Karoubi filtrations in order to prove similar facts in K-theory. The key idea in the proof is still the commutative diagram from [Cárdenas and Pedersen 1997, Section 8] transported from bounded K-theory to fibred G-theory. Cárdenas and Pedersen use Karoubi quotients and the Karoubi fibrations in order to establish their diagram. One of the crucial points in [Cárdenas and Pedersen 1997] is that the functor I between the Karoubi quotients is an isomorphism of categories. In fibred G-theory the situation is more complicated: I is not necessarily full and, therefore, not an isomorphism of categories. However we can see here, just as in the analogous Theorem 4.4.2 in [Carlsson and Goldfarb 2019], that the approximation theorem suffices to prove that $K(I)$ is nevertheless a weak equivalence.

We make an explicit statement that does not hold in K-theory. Let $V_\Gamma(Y)$ be the K-theory of the fibred homotopy fixed points $\mathcal{B}^{h\Gamma}(Y)$ defined in Notation 4.5.

In these terms, we don't know whether, or under what conditions on Γ and Y, the natural map

$$\underset{\overrightarrow{U \in \mathcal{U}}}{\mathrm{hocolim}}\ V^{\Gamma}(U) \to V^{\Gamma}(Y)$$

is an equivalence in the context of Theorem 6.11. Through indirect ways related to the work on the Borel conjecture, we know that Karoubi filtrations should be impossible to use to compute fibred homotopy fixed points in full generality, because otherwise the outcome would contradict the well-known counterexamples to the integral K-theoretic Borel conjecture in cases of nonregular rings R.

Suppose $\mathcal{U}$ is a finite covering of Y that is closed under intersections and such that the family of all subsets U in $\mathcal{U}$ together with Y' are pairwise coarsely antithetic. The extra conditions in the second statement ensure that the covering is in fact by complete representatives of a covering by "coarse families" in the language introduced in [Carlsson and Goldfarb 2019, Section 4.3].

We define the homotopy colimit

$$\mathcal{E}^{\Gamma}(Y, Y')_{<\mathcal{U}} = \underset{\overrightarrow{U \in \mathcal{U}}}{\mathrm{hocolim}}\ E^{\Gamma}(Y, Y')_{<U}.$$

All of the above discussion can be restricted to full subcategories on objects supported near an arbitrary subset Z of Y_S, so we have the following general statement.

Theorem 6.13 (fibrewise bounded excision). *The natural map*

$$\delta : \mathcal{E}^{\Gamma}(Y, Y', Z)_{<\mathcal{U}} \to E^{\Gamma}(Y, Y', Z),$$

induced by inclusions $\mathbf{G}_{\Gamma}(Y_S, Y'_S, Z)_{<U} \to \mathbf{G}_{\Gamma}(Y_S, Y'_S, Z)$, *is a weak equivalence.*

Proof. Apply Theorem 6.11 inductively to the maximal sets in $\mathcal{U}$. □

7. Other properties of equivariant fibred G-theory

Fibred assembly map. The usual notion of metric assumes only finite values. We will require a *generalized metric* on a set X. It is a function $d : X \times X \to [0, \infty) \cup \{\infty\}$ which is reflexive, symmetric, and satisfies the triangle inequality in the obvious way. The generalized metric space is *proper* if it is a countable disjoint union of metric spaces X_i on each of which the generalized metric d is finite, and all closed metric balls in X are compact. The metric topology on a generalized metric space is defined as usual.

The basic fibred assembly map

$$A(X, Y) : h^{\mathrm{lf}}(X; G^{-\infty}(Y)) \to G^{-\infty}_{X}(Y),$$

for proper generalized metric spaces X and Y, sends the locally finite homology of

X with coefficients in the spectrum $G^{-\infty}(Y)$ to the nonconnective fibred G-theory $G^{-\infty}_X(Y)$ defined in Section 5.

The locally finite homology $h^{\mathrm{lf}}(X; \mathcal{S})$ we use was introduced in [Carlsson 1995, Definition II.5] for any coefficient spectrum $\mathcal{S}$. Let ${}^bS_k X$ be the collection of all locally finite families $\mathcal{F}$ of singular k-simplices in X which are uniformly bounded, in the sense that each family possesses a number N such that the diameter of the image $\mathrm{im}(\sigma)$ is bounded from above by N for all simplices $\sigma \in \mathcal{F}$. For any spectrum $\mathcal{S}$, the theory ${}^bh^{\mathrm{lf}}(X; \mathcal{S})$ is the realization of the simplicial spectrum

$$k \mapsto \underset{C \in {}^bS_k X}{\underrightarrow{\mathrm{hocolim}}}\, h^{\mathrm{lf}}(C, \mathcal{S}).$$

There is an equivalence of spectra ${}^bh^{\mathrm{lf}}(X; \mathcal{S}) \to h^{\mathrm{lf}}(X; \mathcal{S})$, for any proper generalized metric space X, from [Carlsson 1995, Corollary II.21].

A similar theory $J^h(X, \mathcal{A})$ is obtained as the realization of the simplicial spectrum

$$k \mapsto \underset{C \in {}^bS_k X}{\underrightarrow{\mathrm{hocolim}}}\, K^{-\infty}(C, \mathcal{A})$$

by viewing C as a discrete metric space and using the notation $K^{-\infty}(C, \mathcal{A})$ for the nonconnective delooping of the K-theory of $\mathcal{B}(C, \mathcal{A})$ from Definition 4.2. Using the coefficients $\mathcal{A} = \mathbf{B}_C(Y)$, we obtain $J^h(X, \mathcal{A})$, which we denote $J^h(X, Y)$. The proof of [Carlsson 1995, Corollary III.14] gives a weak homotopy equivalence

$$\eta : h^{\mathrm{lf}}(X; G^{-\infty}(Y)) \to J^h(X, Y)$$

of functors from proper locally compact metric spaces and coarse maps to spectra.

We next define a natural transformation

$$\ell : J^h(X, Y) \to G^{-\infty}_X(Y).$$

In the case Y is a point and the coefficients are finitely generated free R-modules, this kind of transformation is defined as part of the proof of Proposition III.20 of [Carlsson 1995]. The definition is entirely in terms of maps between singular simplices in X, so the construction can be generalized to give ℓ as above. For the convenience of the reader, we present the necessary details.

Let us first note that controlled algebra can be used to build equivalent bounded K-theory spectra using the symmetric monoidal category approach which we will find useful in the rest of the paper. For the details we refer to Section 6 of [Carlsson 2005].

Let $\mathcal{D}$ be any collection of singular n-simplices of X and ζ be any point of the standard n-simplex. Define a function $\vartheta_\zeta : \mathcal{D} \to X$ by $\vartheta_\zeta(\sigma) = \sigma(\zeta)$. Since $\mathcal{D}$ is viewed as a discrete metric space, if $\mathcal{D}$ is locally finite then ϑ_ζ is coarse, so we have the induced functor $\mathcal{B}(\mathcal{D}, \mathcal{A}) \to \mathcal{B}(X, \mathcal{A})$ given by

$$\bigoplus_{d\in\mathcal{D}} F_d \to \bigoplus_{x\in X} \bigoplus_{\vartheta_\zeta(d)=x} F_d,$$

which is the identity for each $d \in \mathcal{D}$. Therefore, there is the induced map of spectra

$$K(\vartheta_\zeta, \mathcal{A}) : K(\mathcal{D}, \mathcal{A}) \to K(X, \mathcal{A}).$$

Suppose further that $\mathcal{D} \in {}^bS_k X$ and that N is a bound required to exist for $\mathcal{D}$ in ${}^bS_k X$. If ζ and θ are both points in the standard n-simplex, we have a symmetric monoidal natural transformation $N_\zeta^\theta : K(\vartheta_\zeta, \mathcal{A}) \to K(\vartheta_\theta, \mathcal{A})$ induced from the functors which are identities on objects in the cocompletion of $\mathcal{A}$. Both of those identity morphisms are isomorphisms in $\mathcal{B}(X, \mathcal{A})$ because they and their inverses are bounded by N.

Recall that the standard n-simplex can be viewed as the nerve of the ordered set $\underline{n} = \{0, 1, \ldots, n\}$, with the natural order, viewed as a category. Let $\mathcal{D} \in {}^bS_n X$. We define a functor $l(\mathcal{D}, n) : i\mathcal{B}(\mathcal{D}, \mathcal{A}) \times \underline{n} \to i\mathcal{B}(X, \mathcal{A})$ as follows. On objects, $(l(\mathcal{D}, n)F)_x = \bigoplus_{\vartheta(i)=x} F_d$, where i denotes the vertex of $\Delta^n = N.\underline{n}$ corresponding to i. On morphisms, $l(\mathcal{D}, n)$ is defined by the requirement that the restriction to the subcategory $i\mathcal{B}(\mathcal{D}, \mathcal{A}) \times j$ is the functor induced by θ_j, and that $(\mathrm{id} \times (i \le j))(F)$ is sent to $N_i^j(F)$. This is compatible with the inclusion of elements in ${}^bS_n X$, so we obtain a functor

$$\underrightarrow{\operatorname{colim}}_{\mathcal{D}\in {}^bS_n X} i\mathcal{B}(\mathcal{D}, \mathcal{A}) \times \underline{n} \to i\mathcal{B}(X, \mathcal{A}),$$

and therefore a map

$$\underrightarrow{\operatorname{hocolim}}_{\mathcal{D}\in {}^bS_n X} N.i\mathcal{B}(\mathcal{D}, \mathcal{A}) \times \Delta^n \to N.i\mathcal{B}(X, \mathcal{A}).$$

If $\mathcal{M}$ is a symmetric monoidal category, let the t-th space in $\mathrm{Spt}(\mathcal{M})$ be denoted by $\mathrm{Spt}_t(\mathcal{M})$, and let $\sigma_t : S^1 \wedge \mathrm{Spt}_t(\mathcal{M}) \to \mathrm{Spt}_{t+1}(\mathcal{M})$ be the structure map for $\mathrm{Spt}(\mathcal{M})$. The fact that the natural transformations N_i^j are symmetric monoidal shows in particular that we obtain maps

$$\Lambda_t : \underrightarrow{\operatorname{hocolim}}_{\mathcal{D}\in {}^bS_n X} \mathrm{Spt}_t(i\mathcal{B}(\mathcal{D}, \mathcal{A})) \times \Delta^n \to \mathrm{Spt}_t(i\mathcal{B}(X, \mathcal{A})),$$

so that the diagrams

$$\begin{array}{ccc}
\underrightarrow{\operatorname{hocolim}}_{\mathcal{D}\in {}^bS_n X} (S^1 \wedge \mathrm{Spt}_t(i\mathcal{B}(\mathcal{D}, \mathcal{A}))) \times \Delta^n & \longrightarrow & S^1 \wedge \mathrm{Spt}_t(i\mathcal{B}(X, \mathcal{A})) \\
\Big\downarrow{\scriptstyle \sigma_t \times \mathrm{id}} & & \Big\downarrow{\scriptstyle \sigma_t} \\
\underrightarrow{\operatorname{hocolim}}_{\mathcal{D}\in {}^bS_n X} \mathrm{Spt}_{t+1}(i\mathcal{B}(\mathcal{D}, \mathcal{A})) \times \Delta^n & \xrightarrow{\Lambda_{t+1}} & \mathrm{Spt}_{t+1}(i\mathcal{B}(X, \mathcal{A}))
\end{array}$$

commute. Further, for each t we obtain a map

$$\left| \underline{k} \mapsto \underrightarrow{\text{hocolim}}_{\mathcal{D} \in {}^b S_n X} \text{Spt}_t(i\mathcal{B}(\mathcal{D}, \mathcal{A})) \right| \to \text{Spt}_t(i\mathcal{B}(X, \mathcal{A}))$$

respecting the structure maps in Spt_t. This gives a map $\ell : {}^c\mathcal{J}^h(X; \mathcal{A}) \to K(X, \mathcal{A})$, where ${}^c\mathcal{J}^h(X; \mathcal{A})$ stands for the realization of the simplicial spectrum

$$k \mapsto \underrightarrow{\text{hocolim}}_{C \in {}^b S_k X} K(C, \mathcal{A}).$$

Since ℓ is natural in X and is compatible with delooping, it generalizes to the homotopy natural transformation $\ell_K : J^h(X; \mathcal{A}) \to K^{-\infty}(X, \mathcal{A})$. Composing this with the Cartan natural transformation $K^{-\infty}(X, \mathcal{A}) \to G_X^{-\infty}(Y)$ gives

$$\ell : J^h(X, Y) \to G_X^{-\infty}(Y).$$

Definition 7.1 (fibred assembly map in G-theory). The homotopy natural transformation

$$A(X, Y) : h^{\text{lf}}(X; G^{-\infty}(Y)) \to G_X^{-\infty}(Y)$$

is the composition of η and ℓ.

Remark 7.2. Notice that if we use a different coefficient category $\mathcal{A} = \mathcal{B}_X(Y)$, we obtain a map

$$A_K(X, Y) : h^{\text{lf}}(X; K^{-\infty}(Y)) \to K_X^{-\infty}(Y)$$

as the composition η and ℓ_K. It is called the *fibred assembly map in K-theory.*

Let Γ be a finitely generated group with a word metric associated to some choice of a finite generating set in Γ. We assume that there is a finite $K(\Gamma, 1)$ complex M, and Y is its universal cover. We also assume that X is the universal cover of the normal bundle to an embedding of M in a Euclidean space. This is the situation we described in the introduction.

Recall from Notation 4.5 that the equivariant fibred K-theory spectrum for the pair (Γ, X) is denoted $K_p^\Gamma(X)$. We finish this section by developing $A_K(X, Y)$ into the *twisted assembly map*

$$\alpha_K(X, \Gamma) : h^{\text{lf}}(X; K(R[\Gamma])) \to K_p^\Gamma(X).$$

Given a small additive category $\mathcal{A}$, we already have the fibred assembly map

$$A_K(X, Y) : h^{\text{lf}}(X; K^{-\infty}(Y)) \to K_X^{-\infty}(Y).$$

Next note that the bounded K-theory spectrum $K^{-\infty}(\Gamma)$ can be viewed as the homotopy colimit of a family of nonconnective spectra

$$\underrightarrow{\text{hocolim}}_{d} K[d](Y),$$

where $K[d](Y)$ is the spectrum associated with a Γ-space given by the subspace of the nerve of the category with bounded isomorphisms as morphisms, for which a simplex is included if and only if all the maps which make up the simplex and all the composites which are computed to obtain iterated face maps are bounded by d in Γ. This gives maps $h^{\mathrm{lf}}(TY;\, K[d](Y)) \to K^{-\infty}(Y \times X)$ from which we can induce

$$A^{\times} : \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; h^{\mathrm{lf}}(X;\, K[d](Y)) \to K^{-\infty}(Y \times X).$$

The exact embedding $I : \mathcal{B}(Y \times X) \to \mathcal{B}_Y(X)$ comes from relaxing control on the morphisms. The embedding induces the map of K-theory spectra

$$I_* : K^{-\infty}(Y \times X) \to K_Y^{-\infty}(X)$$

which, in general, is not an equivalence. The composition of I_* with $A^{\times}$ gives

$$A_{\mathrm{ext}}(Y, X) : \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; h^{\mathrm{lf}}(X;\, K[d](Y)) \to K_Y^{-\infty}(X).$$

All of the maps we have defined are equivariant maps of spectra with group actions induced from diagonal action on $Y \times X$. So $A_{\mathrm{ext}}(Y, X)$ is an equivariant map. From the proper Spanier–Whitehead duality theorem (see Section 3 of [Ranicki 1980]), we have an equivalence $\Sigma^{n+1} F(Y, K[d](Y)) \simeq h^{\mathrm{lf}}(X;\, K[d](Y))$. This yields the induced map on the fixed points

$$A^{\Gamma}_{\mathrm{ext}}(Y, X) : \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; \Sigma^{n+1} K[d](Y)^{h\Gamma} \to K_p^{\Gamma}(X)^{\Gamma}.$$

On the other hand, there is a natural equivalence

$$K^{-\infty}(Y)^{h\Gamma} \simeq \underset{\overrightarrow{d}}{\mathrm{hocolim}}\; K[d](Y)^{h\Gamma},$$

because in this case when M is a finite $K(\Gamma, 1)$ the homotopy inverse limit is a finite limit which commutes past a filtered colimit.

Definition 7.3 (twisted assembly map). The result is the desired map

$$\alpha : h^{\mathrm{lf}}(X; K^{-\infty}(Y))^{\Gamma} = \Sigma^{n+1} K^{-\infty}(Y)^{h\Gamma} \to K_p^{\Gamma}(X)^{\Gamma}.$$

Fibrewise trivialization. In this section we want to justify the claim from the introduction that in the new equivariant theory we have built there are fibrewise trivializations. First, we state the desired fact precisely.

Recall the proper metric space Y_S described in Section 6. We assume that the trivial action s_0 is in S and use the notation Y_0 for the space Y with the trivial action.

Theorem 7.4. *The equivariant inclusion of metric spaces $Y_0 \to Y_S$ induces an equivalence $\widetilde{G}^{\Gamma}(Y_0)^{\Gamma} \to \widetilde{G}^{\Gamma}(Y_S)^{\Gamma}$. Therefore, there is an equivalence $\widetilde{G}^{\Gamma}(Y_0) \to E^{\Gamma}(Y)$.*

Before we prove this theorem, we want to emphasize the basic nature of trivializations, a feature transverse to the special object conditions in Definition 1.2, which are important only for excision properties. For clarity, we start with several facts about filtrations of modules and describe the elementary case of trivialization for bounded actions on $\mathbf{B}(Y)$, which is the category $\mathbf{B}_X(Y)$ with X a single point.

Let $\Phi_d : \mathcal{P}(Y) \to \mathcal{P}(Y)$ denote the functor that assigns to a subset of Y its d-neighborhood in Y. We can think of an object of $\mathbf{B}(Y)$ as a pair (F, θ), where θ is a filtration of F as in Definition 5.4. Given two Y-filtrations θ and η, we say θ is contained in η if $\theta(S) \subset \eta(S)$ for all $S \subset X$, and write $\theta \leq \eta$. We say two Y-filtrations θ and η are *similar* if there is a number d such that $\theta \leq \eta \circ \Phi_d$ and $\eta \leq \theta \circ \Phi_d$.

Lemma 7.5. *If θ and η are similar then the objects (F, θ) and (F, η) are isomorphic in $\mathbf{B}(Y)$.*

Proof. The conditions ensure that the identity homomorphism is boundedly controlled in both directions. □

Let $f : X \to Y$ be a coarse map as defined in Definition 3.2. Given an X-filtration θ on an R-module F, we define $f_*(\theta)$ to be the Y-filtration on F given by $f_*(\theta)(U) = \theta(f^{-1}(U))$. Similarly, given a Y-filtration θ on F, we define $f^*(\theta)$ to be the Y-filtration on F given by $f^*(\theta)(U) = \theta(f(U))$. It is easy to see that for a coarse map f these constructions applied to filtrations from $\mathbf{B}(Y)$ give filtrations back in $\mathbf{B}(Y)$. We refer to Propositions 5.2 and 5.3 of [Carlsson and Goldfarb 2011].

Recall the definition of Y_S in Definition 6.6. Let $i : Y \to Y_S$ be the inclusion $y \to (y, s_0)$, an isometric embedding, and let $\pi : Y_S \to Y$ denote the projection, a distance nonincreasing map.

Lemma 7.6. *Let F be any R-module. Then any Y_S-filtration on F is similar to one of the form $i_*\theta$, where θ is a Y-filtration on F.*

Proof. Let $\mathrm{ex} : \mathcal{P}(Y_S) \to \mathcal{P}(Y_S)$ be defined by $\mathrm{ex}(U) = \pi(U) \times S$. It is clear from the definition that $U \subset \mathrm{ex}(U)$. It is also readily checked that $\mathrm{ex}(U) \subset \Phi_1(U)$, which shows that any Y_S-filtration θ on an R-module F is similar to the Y_S-filtration $\theta \circ \mathrm{ex}$. Let $\bar{\theta}$ denote the Y-filtration on F given by $\bar{\theta}(U) = \theta(U \times S)$. Then it is clear that $\theta \circ \mathrm{ex} = \pi^*(\bar{\theta})$. It therefore suffices to show that for any Y-filtration η on F, we have that $i_*\eta$ and $\pi^*\eta$ are similar. But it is clear that $\pi^*\eta \leq i_*\eta \circ \Phi_1$, which gives the result. □

We also have the following useful fact.

Lemma 7.7. *Suppose that we are given two Y-filtrations θ and η on an R-module F, and that $\phi : F \to F$ is bounded as a morphism from (F, θ) to (F, η). Suppose further that θ' and η' are also Y-filtrations, and that θ' and η' are similar to θ*

and η, respectively. Then $\phi : F \to F$ is bounded as a morphism from (F, θ') to (F, η').

Suppose a metric space Y has an action by a discrete group Γ through coarse equivalences. Recall that we say that such *action is bounded* if for each $\gamma \in \Gamma$, there is $b(\gamma) \geq 0$ such that $d(y, \gamma y) \leq b(\gamma)$ for all $y \in Y$. The following is an elementary observation.

Lemma 7.8. *Suppose Y is a proper metric space equipped with a bounded Γ-action by coarse maps. Then, given any Y-filtration θ on an R-module F, and any $\gamma \in \Gamma$, we have that θ and $\gamma_*\theta$ are similar.*

An object of the category $\mathbf{B}^{w\Gamma}(Y)$ is given by data $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \Gamma})$, where

(1) F is an R-module,

(2) θ is a Y-filtration on F,

(3) $\phi_{\gamma,\gamma'}$ is an automorphism of F,

(4) $\phi_{\gamma,\gamma} = \mathrm{id}_F$ and $\phi_{\gamma,\gamma'} \circ \phi_{\gamma',\gamma''} = \phi_{\gamma,\gamma''}$ for all γ, γ', γ'' in Γ,

(5) $\phi_{\gamma,\gamma'}$ is bounded when regarded as a homomorphism $(F, \gamma'_*\theta) \to (F, \gamma_*\theta)$.

Lemmas 7.7 and 7.8 give that condition (5) on $\phi_{\gamma,\gamma'}$ is equivalent to $\phi_{\gamma,\gamma'}$ being bounded as a homomorphism from (F, θ) to (F, θ).

The morphisms $(F, \theta, \{\phi_{\gamma,\gamma'}\}) \to (F', \theta', \{\phi'_{\gamma,\gamma'}\})$ are boundedly controlled homomorphisms $f : F \to F'$ with $\phi'_{\gamma,\gamma'} \circ f = f \circ \phi_{\gamma,\gamma'}$.

Proposition 7.9. *The equivariant inclusion $Y_0 \to Y_S$ induces an equivalence of categories $i\mathbf{B}^{w\Gamma}(Y_0) \to i\mathbf{B}^{w\Gamma}(Y_S)$.*

Proof. The inclusion exhibits $i\mathbf{B}^{w\Gamma}(Y_0)$ as a full subcategory of $i\mathbf{B}^{w\Gamma}(Y_S)$, and it follows that it's enough to prove that every object of $i\mathbf{B}^{w\Gamma}(Y_S)$ is isomorphic to an object of $i\mathbf{B}^{w\Gamma}(Y_0)$. An object of $i\mathbf{B}^{w\Gamma}(Y_0)$ is given by data $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \Gamma})$, where θ is an Y_0-filtration on F, and where ϕ_0 is an automorphism of F which is bounded as a homomorphism from (F, θ) to (F, θ). Note that the transformations by γ's do not occur in this situation because the action of Γ on Y_0 is trivial. The inclusion functor $i\mathbf{B}^{w\Gamma}(Y_0) \hookrightarrow i\mathbf{B}^{w\Gamma}(Y_S)$ is given by

$$(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \Gamma}) \to (F, i_*\theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \Gamma}),$$

so an object $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \Gamma})$ is in the subcategory $i\mathbf{B}^{w\Gamma}(Y_0)$ if and only if θ is of the form $i_*\eta$ for some Y_0-filtration η.

Next, we observe that if $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \Gamma})$ is an object of $i\mathbf{B}^{w\Gamma}(Y_S)$, and if θ' is a Y_S-filtration on F which is similar to θ, then

(a) $(F, \theta', \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \Gamma})$ is also an object of $(F, i_*\theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \Gamma})$, and

(b) $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \Gamma})$ is isomorphic to $(F, \theta', \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma' \in \Gamma})$.

But we have already observed in Lemma 7.6 that every Y_S-filtration on F is equivalent to one of the form $i_*\eta$, for some Y_0-filtration η on F, proving the result. □

Now we are ready to prove Theorem 7.4.

Observe that it suffices to verify that the inclusion $Y_0 \to Y_S$ induces an equivalence of categories $i\mathbf{G}^{h\Gamma}(Y_0) \to i\mathbf{G}^{h\Gamma}(Y_S)$. The equivalence then clearly extends to categories of diagrams of objects in $\mathbf{G}^{h\Gamma}(Y_S)$, and Waldhausen's $S.$-construction used to produce the spectra in Definition 6.3 gives simplicial spaces which in every level are the nerves of categories of isomorphisms of diagrams of cofibrations of objects in $\mathbf{G}^{h\Gamma}(Y_S)$. So Theorem 7.4 follows from the following lemma.

Lemma 7.10. *The inclusion $Y_0 \to Y_S$ induces an equivalence $i\mathbf{G}^{h\Gamma}(Y_0) \to i\mathbf{G}^{h\Gamma}(Y_S)$.*

Proof. The proof of Proposition 7.9 together with the preceding lemmas should be applied verbatim in this case where objects have fibred control over Γ and possess Y-gradings. This is possible due to the facts that $\mathbf{G}^{h\Gamma}(Y)$ is

(a) closed under the required constructions f^* and f_* for a coarse equivalence f, and

(b) is equivalent to the analogue of $\mathbf{B}^{w\Gamma}(Y)$ applied to F in $\mathbf{G}_\Gamma(Y)$.

Fact (a) follows from Proposition 3.4.4 and Lemma 3.4.5 of [Carlsson and Goldfarb 2019]. We now proceed to prove (b).

When Lemmas 7.5 through 7.8 are transported to the fibred setting, the functor Φ_d needs to be interpreted as $\Phi_{(d,b)} : \mathcal{P}(\Gamma \times Y) \to \mathcal{P}(\Gamma \times Y)$ for an enlargement data (d, b). All other features remain the same. We have to state the fibred category to which the proofs apply. An object of this category $\mathbf{G}^{w\Gamma}(Y)$ is given by data $(F, \theta, \{\phi_{\gamma,\gamma'}\}_{\gamma,\gamma'\in\Gamma})$, where

(1) F is an R-module,

(2) θ is a (Γ, Y)-filtration on F which exhibits F as an object of $\mathbf{G}_\Gamma(Y)$,

(3) $\phi_{\gamma,\gamma'}$ is an automorphism of F,

(4) $\phi_{\gamma,\gamma} = \mathrm{id}_F$ and $\phi_{\gamma,\gamma'} \circ \phi_{\gamma',\gamma''} = \phi_{\gamma,\gamma''}$ for all γ, γ', γ'' in Γ,

(5) $\phi_{\gamma,\gamma'}$ is bounded by some enlargement data (B, b) when regarded as a homomorphism $(F, \gamma'_*\theta) \to (F, \gamma_*\theta)$.

Again we remark that (5) is equivalent to $\phi_{\gamma,\gamma'}$ being bounded as a homomorphism from (F, θ) to (F, θ). The morphisms $(F, \theta, \{\phi_{\gamma,\gamma'}\}) \to (F', \theta', \{\phi'_{\gamma,\gamma'}\})$ are boundedly controlled $f : F \to F'$ with $\phi'_{\gamma,\gamma'} \circ f = f \circ \phi_{\gamma,\gamma'}$.

Finally, the only new requirement we need to add in the fibred setting is

(6) ϕ_γ is 0-controlled when viewed as a morphism over Γ.

We can draw the conclusion of Proposition 7.9 that the equivariant inclusion $Y_0 \to Y_S$ induces an equivalence of categories $i\mathbf{G}^{w\Gamma}(Y_0) \to i\mathbf{G}^{w\Gamma}(Y_S)$. On the other hand, it is immediate that when the action of Γ on Y is by bounded coarse equivalences, the category $\mathbf{G}^{w\Gamma}(Y)$ is equivalent to $\mathbf{G}^{h\Gamma}(Y)$ as in Definition 6.4 by way of the rule $\phi_{\gamma,\gamma'} = \psi_{\gamma'} \circ \psi_{\gamma}^{-1}$. □

8. A sample application

We show how the main theorem of this paper fits in a computation of the K-theory of a finitely generated group in terms of group homology. This is done by proving that the Loday assembly map is an equivalence. The basic idea and most of the nontechnical issues can be illustrated in the simple case of $R = \mathbb{Z}$, the ring of integers, and $\Gamma = C$, the infinite cyclic group. In a separate forthcoming paper, we generalize this argument to all groups Γ with finite $K(\Gamma, 1)$ and finite decomposition complexity.

Example 8.1. Let S^1 be the circle viewed as a Riemannian submanifold of $\mathbb{R}^2$. The cyclic group C acts freely and properly discontinuously by translation on the universal cover $\mathbb{R}$ of S^1. Let N be the closure of the total space of the (trivial) normal bundle to the embedding, also embedded in $\mathbb{R}^2$ as a small closed tubular neighborhood of S^1. We denote by Y the universal cover of N. Now Y can be given a metric so that the restriction to the zero section is commensurable with the metric on an orbit of the translation action by C which comes from a fixed word metric on C. It is important to observe that in this example the action of C on Y is bounded. This is a consequence of the fact that C is an abelian group.

As explained in the introduction, we assume that the equivariant assembly map $A_C : h^{\mathrm{lf}}(\mathbb{R}; K^{-\infty}(\mathbb{Z})) \to K^{-\infty}(\mathbb{R}, \mathbb{Z})$ is a weak homotopy equivalence, which is known. This fact shows that the Loday assembly is a split injection with a splitting $\rho : K(\mathbb{R})^C \to K(\mathbb{R})^{hC}$. Our goal is to split ρ.

A proper version of Spanier–Whitehead duality (see, for example, [Ranicki 1980, Section 3]) allows one to view the double suspension $\Sigma^2\rho$ as a fixed point map

$$D : \Sigma^2 K(\mathbb{R})^C \to h^{\mathrm{lf}}(Y; K(\mathbb{R}))^C.$$

We also have the twisted assembly map

$$\alpha : h^{\mathrm{lf}}(Y; K(\mathbb{R})) \to K_p^C(Y)$$

from Definition 7.3. The composition $\alpha^C \circ D$ begins the following sequence:

$$\Sigma^2 K(\mathbb{Z}[C]) \xrightarrow{\alpha^C D} K_p^C(Y)^C \xrightarrow{\kappa} E^C(Y) \simeq \Sigma^2 G(\mathbb{Z}[C]). \qquad (I)$$

The map κ is induced by interpreting split exact sequences as exact sequences of nonfree modules defining G-theory, usually referred to as the Cartan map. Its target is the fibred homotopy fixed point G-theory spectrum (Section 6).

The equivalence $E^C(Y) \simeq \Sigma^2 G(\mathbb{Z}[C])$ is the excision computation from the main theorem. In this application, the normal bundle is trivial and so there is an elementary choice of a coarsely antithetic covering of $Y = \mathbb{R}$ by products of infinite rays in the fiber and in the base. The excision theorem represents $E^C(Y)$ as the homotopy colimit of a diagram of spectra indexed by cells in the standard cellular structure of the square and the face relation. The only nontrivial spectrum $E^C(\text{point})$ corresponds to the initial 2-dimensional cell. Finally, $E^C(\text{point})$ is the spectrum $\widetilde{G}^C(\text{point})^C = G(\mathbb{Z}[C])$, which was introduced in [Carlsson and Goldfarb 2016].

Now we need to explain the relationship between $G(\mathbb{Z}[C])$ and $K(\mathbb{Z}[C])$. It is studied in general under the name *regular coarse coherence* in a separate paper. We summarize it as follows. Suppose Γ is a group with finite decomposition complexity as in [Ramras et al. 2014] and has a finite $K(\Gamma, 1)$. Suppose the ring R is a Noetherian ring of finite global dimension. Under these assumptions, the Cartan map $K(R[\Gamma]) \to G(R[\Gamma])$ is an equivalence. Of course, this holds when $\Gamma = C$ and $R = \mathbb{Z}$. In this particular case, what we need is already contained in [Carlsson and Goldfarb 2004b], written about the smaller class of groups of finite asymptotic dimension.

To conclude the argument, we notice that the domain and the target of the composition in (I) are equivalent, and the first map in the composition is $D = \Sigma^2\rho$. It is important to check, and is done in a forthcoming paper, that the composition of these exhibited maps is indeed an equivalence. The use of the theory E^C is essential for that purpose, as we already observed in Remark 1.1 in the introduction.

The same argument can be used for finitely generated free abelian groups, with the only straightforward change occurring in a larger excision scheme for the computation of $E^\Gamma(Y)$, where Y is similarly a higher-dimensional Euclidean space. The extension to nonabelian groups with finite decomposition complexity requires several new tools and will appear in a separate paper.

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