Algebra & Number Theory

Volume 1

2007

No. 2

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mathematical sciences publishers

Surfaces over a p-adic field with infinite torsion in the Chow group of 0-cycles

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We give an example of a projective smooth surface X over a p-adic field K such that for any prime ℓ different from p, the ℓ-primary torsion subgroup of $\mathrm{CH}_0(X)$, the Chow group of 0-cycles on X, is infinite. A key step in the proof is disproving a variant of the Bloch–Kato conjecture which characterizes the image of an ℓ-adic regulator map from a higher Chow group to a continuous étale cohomology of X by using p-adic Hodge theory. With the aid of the theory of mixed Hodge modules, we reduce the problem to showing the exactness of the de Rham complex associated to a variation of Hodge structure, which is proved by the infinitesimal method in Hodge theory. Another key ingredient is the injectivity result on the cycle class map for Chow group of 1-cycles on a proper smooth model of X over the ring of integers in K, due to K. Sato and the second author.

1. Introduction

Let X be a smooth projective variety over a base field K and let $\mathrm{CH}^m(X)$ be the Chow group of algebraic cycles of codimension m on X modulo rational equivalence. In case K is a number field, there is a folklore conjecture that $\mathrm{CH}^m(X)$ is finitely generated, which in particular implies that its torsion part $\mathrm{CH}^m(X)_{\mathrm{tor}}$ is finite. The finiteness question has been intensively studied by many authors, particularly for the case $m = 2$ and $m = \dim(X)$; see the nice surveys [Otsubo 2001; Colliot-Thélène 1995].

When K is a p-adic field (namely the completion of a number field at a finite place), Rosenschon and Srinivas [2007] have constructed the first example where $\mathrm{CH}^m(X)_{\mathrm{tor}}$ is infinite. They prove that there exists a smooth projective fourfold X over a p-adic field such that the ℓ-torsion subgroup $\mathrm{CH}_1(X)[\ell]$ (see Notation on p. 166) of $\mathrm{CH}_1(X)$, the Chow group of 1-cycles on X, is infinite for each $\ell \in \{5, 7, 11, 13, 17\}$.

This paper gives an example of a projective smooth surface X over a p-adic field such that for any prime ℓ different from p, the ℓ-primary torsion subgroup

MSC2000: primary 14C25; secondary 14G20, 14C30.

Keywords: Chow group, torsion 0-cycles on surface.

$\mathrm{CH}_0(X)\{\ell\}$ (see Notation on p. 166) of $\mathrm{CH}_0(X)$, the Chow group of 0-cycles on X, is infinite. Here we note that for X as above, $\mathrm{CH}_0(X)\{\ell\}$ is known to always be of finite cotype over $\mathbb{Z}_\ell$, namely the direct sum of a finite group and a finite number of copies of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. This fact follows from Bloch's exact sequence (2-3). Thus our example presents infinite phenomena of different nature from the example in [Rosenschon and Srinivas 2007]. Another noteworthy point is that the phenomena discovered in our example happen rather *generically*.

To make it more precise, we prepare a notion of "generic surfaces" in $\mathbb{P}^3$. Let

$$M \subset \mathbb{P}\big(H^0(\mathbb{P}^3_{\mathbb{Q}}, \mathbb{O}_{\mathbb{P}}(d)\big) \cong \mathbb{P}_{\mathbb{Q}}^{(d+3)(d+2)(d+1)/6-1}$$

be the moduli space over $\mathbb{Q}$ of the nonsingular surfaces in $\mathbb{P}^3_{\mathbb{Q}}$ (the subscript $\mathbb{Q}$ indicates the base field), and let

$$f : \mathscr{X} \longrightarrow M$$

be the universal family over M. For $X \subset \mathbb{P}^3_K$, a nonsingular surface of degree d defined over a field K of characteristic zero, there is a morphism $t : \operatorname{Spec} K \to M$ such that $X \cong \mathscr{X} \times_M \operatorname{Spec} K$. We call X *generic* if t is dominant, that is, t factors through the generic point of M. In other words, X is generic if it is defined by an equation

$$F = \sum_I a_I z^I, \quad (a_I \in K)$$

satisfying the following condition:

(∗) $a_I \neq 0$ for all I and $\{a_I/a_{I_0}\}_{I \neq I_0}$ are algebraically independent over $\mathbb{Q}$ where $I_0 = (1, 0, 0, 0)$.

Here $[z_0 : z_1 : z_2 : z_3]$ is the homogeneous coordinate of $\mathbb{P}^3$, $I = (i_0, \ldots, i_3)$ are multiindices and $z^I = z_0^{i_0} \cdots z_3^{i_3}$.

The main theorem is

Theorem 1.1. *Let K be a finite extension of $\mathbb{Q}_p$ and $X \subset \mathbb{P}^3_K$ a nonsingular surface of degree $d \geq 5$. Suppose that X is generic and has a projective smooth model $X_{\mathbb{O}_K} \subset \mathbb{P}^3_{\mathbb{O}_K}$ over the ring $\mathbb{O}_K$ of integers in K. Let r be the Picard number* (*that is the rank of the Néron–Severi group*) *of the smooth special fiber of $X_{\mathbb{O}_K}$. Then we have*

$$\mathrm{CH}_0(X)\{\ell\} \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus r-1} \oplus (\textit{finite group})$$

for $\ell \neq p$.

One can construct a surface with infinite torsion in the Chow group of 0-cycles in the following way. Let k be the residue field of K. Let Y be a smooth surface of degree $d \geq 5$ in $\mathbb{P}^3_k$ defined by an equation $\sum_I c_I z^I$ ($c_I \in k$) such that the Picard number $r \geq 2$. There exist such surfaces for each d. (For example if $(p, d) = 1$, one may choose a Fermat type surface defined by $z_0^d - z_1^d + z_2^d - z_3^d$. Then the intersection of Y with the hyperplane $H \subset \mathbb{P}^3_k$ defined by $z_0 - z_1$ is not irreducible, so $r \geq 2$.) Take any lifting $\tilde{c}_I \in O_K$ and choose $a_I \in O_K$ with $\mathrm{ord}(a_I) > 0$ for each index I such that $\{a_I\}_I$ are algebraically independent over $\mathbb{Q}(\tilde{c}_I)$, the subfield of K generated over $\mathbb{Q}$ by $\tilde{c}_I$ for all I. Let $X \subset \mathbb{P}^3_K$ be the surface defined by the equation $\sum_I \tilde{c}_I z^I + \sum_I a_I z^I$. Then it is clear that X is generic and has a smooth projective model over O_K whose the special fiber is Y . Since Y has the Picard number $r \geq 2$, $\mathrm{CH}_0(X)$ has an infinite torsion subgroup by Theorem 1.1. It is proved in [Raskind 1989] that if the special fiber satisfies the Tate conjecture for divisors, the geometric Picard number is congruent to d modulo 2. Thus if d is even, $\mathrm{CH}_0(X)$ has an infinite torsion subgroup after a suitable unramified base change. Theorem 1.1 may be compared with the finiteness results [Colliot-Thélène and Raskind 1991] and [Raskind 1989] on $\mathrm{CH}_0(X)_{\mathrm{tor}}$ for a surface X over a p-adic field under the assumption that $H^2(X, \mathbb{O}_X) = 0$ or, more generally, that the rank of the Néron–Severi group does not change by reduction. For a nonsingular surface $X \subset \mathbb{P}^3_K$ of degree $d \geq 1$, the last condition is satisfied if $d \leq 3$. Hence Theorem 1.1 leaves us an interesting open question whether there is an example of a nonsingular surface $X \subset \mathbb{P}^3_K$ of degree 4 for which $\mathrm{CH}_0(X)\{\ell\}$ is infinite.

A distinguished role is played in the proof of Theorem 1.1 by the ℓ-adic regulator map

$$\rho_X : \mathrm{CH}^2(X, 1) \otimes \mathbb{Q}_\ell \longrightarrow H^1_{\mathrm{cont}}\big(\mathrm{Spec}(K), H^2(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big) \quad (X_{\overline{K}} = X \times_K \overline{K})$$

from higher Chow group to continuous étale cohomology [Jannsen 1988], where $\overline{K}$ is an algebraic closure of K and ℓ is a prime different from $\mathrm{ch}(K)$. It is known that the image of ρ_X is contained in the subspace

$$H^1_g(\mathrm{Spec}(K), V) \subset H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V) \quad \big(V = H^2(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big)$$

introduced by Bloch and Kato [1990]. If $\ell \neq p$ this is obvious since $H^1_g = H^1$ by definition. For $\ell = p$ this is a consequence of a fundamental result in p-adic Hodge theory, which confirms that every representation of $G_K = \mathrm{Gal}(\overline{K}/K)$ arising from the cohomology of a variety over K is a de Rham representation; see the discussion after [Bloch and Kato 1990, (3.7.4)].

When K is a number field or a p-adic field, it is proved in [Saito and Sato 2006a] that in case the image of ρ_X coincides with $H^1_g(\mathrm{Spec}(K), V)$, $\mathrm{CH}^2(X)\{\ell\}$ is finite. Bloch and Kato conjecture that it should be always the case if K is a number field.

The first key step in the proof of Theorem 1.1 is to disprove the variant of the Bloch–Kato conjecture for a generic surface $X \subset \mathbb{P}^3_K$ over a p-adic field K (see Theorem 3.6). In terms of Galois representations of $G_K = \mathrm{Gal}(\overline{K}/K)$, our result implies the existence of a 1-extension of $\mathbb{Q}_\ell$-vector spaces with continuous G_K-

action

$$0 \to H^2\big(X_{\overline{K}}, \mathbb{Q}_\ell(2)\big) \to E \to \mathbb{Q}_l \to 0, \tag{1-1}$$

such that E is a de Rham representation of G_K but that there is no 1-extension of motives over K,

$$0 \to h^2(X)(2) \to M \to h\big(\mathrm{Spec}(K)\big) \to 0,$$

which gives rise to (1-1) under the realization functor. The rough idea of the proof of the first key result is to relate the ℓ-adic regulator map to an analytic regulator map by using the comparison theorem for étale and analytic cohomology and then to show that the analytic regulator map is the zero map. With the aid of the theory of mixed Hodge modules [Saito 1990], this is reduced to showing the exactness of the de Rham complex associated to a variation of Hodge structure, which is proved by the infinitesimal method in Hodge theory. This is done in Section 3 after in Section 2, we review some basic facts on the cycle class map for higher Chow groups.

Another key ingredient is the injectivity result on the cycle class map for the Chow group of 1-cycles on a proper smooth model of X over the ring O_K of integers in K due to Sato and the second author [Saito and Sato 2006b]. It plays an essential role in deducing the main result, Theorem 1.1 from the first key result, which is done in Section 4.

Finally, in the Appendix, we will apply our method to produce an example of a curve C over a p-adic field such that $SK_1(C)_{\mathrm{tor}}$ is infinite.

Notation. For an abelian group M, we denote by $M[n]$ (respectively M/n) the kernel (respectively cokernel) of multiplication n. For a prime number ℓ we put

$$M\{\ell\} := \bigcup_n M[\ell^n], \qquad M_{\mathrm{tor}} := \bigoplus_\ell M\{\ell\}.$$

For a nonsingular variety X over a field, $\mathrm{CH}^j(X, i)$ denotes Bloch's higher Chow groups. We write $\mathrm{CH}^j(X) := \mathrm{CH}^j(X, 0)$ for the (usual) Chow groups.

2. Review of the cycle class map and ℓ-adic regulator

In this section X denotes a smooth variety over a field K and n denotes a positive integer prime to $\mathrm{ch}(K)$.

By [Geisser and Levine 2001] we have the cycle class map

$$c^{i,j}_{\text{ét}} : \mathrm{CH}^i(X, j, \mathbb{Z}/n\mathbb{Z}) \to H^{2i-j}_{\text{ét}}\big(X, \mathbb{Z}/n\mathbb{Z}(i)\big),$$

where the right hand side is the étale cohomology of X with coefficients $\mu_n^{\otimes i}$, Tate twist of the sheaf of n-th roots of unity. The left hand side is Bloch's higher Chow group with finite coefficient which fits into the exact sequence

$$0 \to \mathrm{CH}^i(X, j)/n \to \mathrm{CH}^i(X, j, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^i(X, j-1)[n] \to 0. \tag{2-1}$$

In this paper we are only concerned with the map

$$c_{\text{ét}} = c_{\text{ét}}^{2,1} : CH^2(X, 1, \mathbb{Z}/n\mathbb{Z}) \to H^3_{\text{ét}}\big(X, \mathbb{Z}/n\mathbb{Z}(2)\big). \tag{2-2}$$

By [Bloch and Ogus 1974] it is injective and its image is equal to

$$NH^3_{\text{ét}}(X, \mathbb{Z}/n\mathbb{Z}(2)\big) = \mathrm{Ker}\Big(H^3_{\text{ét}}\big(X, \mathbb{Z}/n\mathbb{Z}(2)\big) \to H^3_{\text{ét}}(\mathrm{Spec}(K(X)\big), \mathbb{Z}/n\mathbb{Z}(2)\big)\Big),$$

where $K(X)$ is the function field of X. In view of (2-1) it implies an exact sequence

$$0 \longrightarrow \mathrm{CH}^2(X, 1)/n \xrightarrow{c_{\text{ét}}} NH^3_{\text{ét}}\big(X, \mathbb{Z}/n\mathbb{Z}(2)\big) \longrightarrow \mathrm{CH}^2(X)[n] \longrightarrow 0. \tag{2-3}$$

We also need the cycle map to the continuous étale cohomology group

$$c_{\text{cont}} : \mathrm{CH}^2(X, 1) \longrightarrow H^3_{\text{cont}}\big(X, \mathbb{Z}_\ell(2)\big)$$

(see [Jannsen 1988]), where ℓ is a prime different from $\mathrm{ch}(K)$. In case K is a p-adic field, we have

$$H^3_{\text{cont}}\big(X, \mathbb{Z}_\ell(2)\big) = \varprojlim_n H^3_{\text{ét}}\big(X, \mathbb{Z}/\ell^n\mathbb{Z}(2)\big)$$

and c_{cont} is induced by $c_{\text{ét}}$ by passing to the limit. We have the Hochschild–Serre spectral sequence

$$E_2^{i,j} = H^i_{\text{cont}}\big(\mathrm{Spec}(K), H^j(X_{\overline{K}}, \mathbb{Z}_\ell(2))\big) \Rightarrow H^{i+j}_{\text{cont}}(X, \mathbb{Z}_\ell(2)). \tag{2-4}$$

If K is finitely generated over the prime subfield and X is proper smooth over K, the Weil conjecture proved by Deligne implies that

$$H^0\big(\mathrm{Spec}(K), H^3(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big) = 0.$$

The same conclusion holds if K is a p-adic field and X is proper smooth having good reduction over K. (If $\ell \neq p$ this follows from the proper smooth base change theorem for étale cohomology. If $\ell = p$ one uses comparison theorems between p-adic étale and crystalline cohomology and the Weil conjecture for crystalline cohomology) Thus we get under these assumptions the map

$$\rho_X : \mathrm{CH}^2(X, 1) \longrightarrow H^1_{\text{cont}}\big(\mathrm{Spec}(K), H^2(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big) \tag{2-5}$$

as the composite of $c_{\text{cont}} \otimes \mathbb{Q}_\ell$ and an edge homomorphism

$$H^3_{\text{cont}}(X, \mathbb{Q}_\ell(2)) \to H^1_{\text{cont}}\big(\mathrm{Spec}(K), H^2(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big).$$

For later use, we need an alternative definition of cycle class maps. For an integer $i \geq 1$, we denote by $\mathcal{K}_i$ the sheaf on X_{Zar}, the Zariski site on X, associated to the presheaf $U \mapsto K_i(U)$. By [Landsburg 1991, 2.5], we have canonical isomorphisms

$$\mathrm{CH}^2(X,1) \simeq H^1_{\mathrm{Zar}}(X,\mathcal{K}_2), \quad \mathrm{CH}^2(X,1,\mathbb{Z}/n\mathbb{Z}) \simeq H^1_{\mathrm{Zar}}(X,\mathcal{K}_2/n). \tag{2-6}$$

Let $\epsilon^{\text{ét}} : X_{\text{ét}} \to X_{\mathrm{Zar}}$ be the natural map of sites and put

$$\mathcal{H}^i_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(r)) = R^i\epsilon^{\text{ét}}_*\mu_n^{\otimes r}.$$

The universal Chern classes in the cohomology groups of the simplicial classifying space for GL_n $(n \geq 1)$ give rise to higher Chern class maps on algebraic K-theory; see [Gillet 1981; Schneider 1988]. It gives rise to a map of sheaves

$$\mathcal{K}_i/n \longrightarrow \mathcal{H}^i_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(i)). \tag{2-7}$$

By [Merkurjev and Suslin 1982] it is an isomorphism for $i = 2$ and induces an isomorphism

$$H^1_{\mathrm{Zar}}(X,\mathcal{K}_2/n) \xrightarrow{\cong} H^1_{\mathrm{Zar}}\big(X,\mathcal{H}^2_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(2))\big). \tag{2-8}$$

By the spectral sequence

$$E_2^{pq} = H^p_{\mathrm{Zar}}\big(X,\mathcal{H}^q_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(2))\big) \Longrightarrow H^{p+q}_{\text{ét}}(X,\mathbb{Z}/n\mathbb{Z}(2)),$$

together with the fact

$$H^p_{\mathrm{Zar}}\big(X,\mathcal{H}^q_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(2))\big) = 0$$

for $p > q$ shown by Bloch and Ogus [1974], we get an injective map

$$H^1_{\mathrm{Zar}}\big(X,\mathcal{H}^2_{\text{ét}}(\mathbb{Z}/n\mathbb{Z}(2))\big) \longrightarrow H^3_{\text{ét}}(X,\mathbb{Z}/n\mathbb{Z}(2)).$$

Again by the Bloch–Ogus theory the image of the above map coincides with the coniveau filtration $NH^3_{\text{ét}}(X,\mathbb{Z}/n\mathbb{Z}(2))$. Combined with (2-6) and (2-8) we thus get the map

$$\begin{aligned} c_{\text{ét}} : \mathrm{CH}^2(X,1,\mathbb{Z}/n\mathbb{Z}) &\xrightarrow{\cong} H^1_{\mathrm{Zar}}(X,\mathcal{K}_2/n) \\ &\xrightarrow{\cong} NH^3_{\text{ét}}\big(X,\mathbb{Z}/n\mathbb{Z}(2)\big) \xrightarrow{\subset} H^3_{\text{ét}}\big(X,\mathbb{Z}/n\mathbb{Z}(2)\big). \end{aligned}$$

This agrees with the map (2-2); see [Colliot-Thélène et al. 1983, Proposition 1].

Now we work over the base field $K = \mathbb{C}$. Let X_{an} be the site on the underlying analytic space $X(\mathbb{C})$ endowed with the ordinary topology. Let $\epsilon^{\mathrm{an}} : X_{\mathrm{an}} \to X_{\mathrm{Zar}}$ be the natural map of sites and put

$$H^i_{\mathrm{an}}(\mathbb{Z}(r)) = R^i\epsilon^{\mathrm{an}}_*\mathbb{Z}(r) \quad \big(\mathbb{Z}(r) = \big(2\pi\sqrt{-1}\big)^r\mathbb{Z}\big).$$

The higher Chern class map then gives a map of sheaves

$$\mathcal{K}_i \longrightarrow \mathcal{H}^i_{\mathrm{an}}(\mathbb{Z}(i)). \tag{2-9}$$

By the same argument as before, it induces a map

$$c_{\mathrm{an}} : \mathrm{CH}^2(X, 1) \xrightarrow{\cong} H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \longrightarrow H^3_{\mathrm{an}}(X(\mathbb{C}), \mathbb{Z}(2)).$$

Lemma 2.1. *The image of c_{an} is contained in $F^2H^3_{\mathrm{an}}(X(\mathbb{C}), \mathbb{C})$, the Hodge filtration defined in [Deligne 1971]. In particular if X is complete, the image is the torsion.*

Proof. Let $\mathcal{H}^r_D(\mathbb{Z}(i))$ be the sheaf on X_{Zar} associated to a presheaf

$$U \mapsto H^r_D(U, \mathbb{Z}(i))$$

where $H^\bullet_D$ denotes Deligne–Beilinson cohomology; see [Esnault and Viehweg 1988, 2.9]. Higher Chern class maps to Deligne–Beilinson cohomology give rise to the map $K_2 \to \mathcal{H}^2_D(\mathbb{Z}(2))$ and c_{an} factors as in the commutative diagram

$$\begin{tikzcd}
H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \arrow[r] & H^1_{\mathrm{Zar}}\big(X, \mathcal{H}^2_D(\mathbb{Z}(2))\big) \arrow[r] \arrow[d, "a"'] & H^1_{\mathrm{Zar}}\big(X, \mathcal{H}^2_{\mathrm{an}}(\mathbb{Z}(2))\big) \arrow[d] \\
 & H^3_D(X, \mathbb{Z}(2)) \arrow[r, "b"] & H^3_{\mathrm{an}}(X(\mathbb{C}), \mathbb{Z}(2)).
\end{tikzcd}$$

Here the map a is induced from the spectral sequence

$$E_2^{pq} = H^p_{\mathrm{Zar}}\big(X, \mathcal{H}^q_D(\mathbb{Z}(2))\big) \Longrightarrow H^{p+q}_D(X, \mathbb{Z}(2))$$

in view of the fact that $H^p_{\mathrm{Zar}}(X, \mathcal{H}^1_D(\mathbb{Z}(2))) = 0$ for all $p > 0$, since $\mathcal{H}^1_D(\mathbb{Z}(2)) \cong \mathbb{C}/\mathbb{Z}(2)$ (constant sheaf). Since the image of b is contained in $F^2H^3_{\mathrm{an}}(X(\mathbb{C}), \mathbb{C})$ (see [Esnault and Viehweg 1988, 2.10]), so is the image of c_{an}. □

Lemma 2.2. *We have the diagram*

$$\begin{tikzcd}
\mathrm{CH}^2(X, 1) \arrow[r, "c_{\mathrm{an}}"] \arrow[d] & H^3_{\mathrm{an}}\big(X(\mathbb{C}), \mathbb{Z}(2)\big) \arrow[d] \\
\mathrm{CH}^2(X, 1, \mathbb{Z}/n\mathbb{Z}) \arrow[r, "c_{\acute{e}t}"] & H^3_{\acute{e}t}\big(X, \mathbb{Z}/n\mathbb{Z}(2)\big).
\end{tikzcd}$$

Here the right vertical map is the composite

$$H^3_{\mathrm{an}}\big(X(\mathbb{C}), \mathbb{Z}(2)\big) \to H^3_{\mathrm{an}}\big(X(\mathbb{C}), \mathbb{Z}(2) \otimes \mathbb{Z}/n\mathbb{Z}\big) \xrightarrow{\cong} H^3_{\acute{e}t}\big(X, \mathbb{Z}/n\mathbb{Z}(2)\big)$$

and the isomorphism comes from the comparison isomorphism between étale cohomology and ordinary cohomology (SGA $4\,{}^1\!/_2$ = [Deligne 1977], Arcata, 3.5) *together with the isomorphism*

$$\mathbb{Z}(1)\otimes\mathbb{Z}/n\mathbb{Z}\simeq(\epsilon^{\mathrm{an}})^{*}\mu_n$$

given by the exponential map.

Proof. This follows from the compatibility of (2-7) and (2-9), namely the commutativity of the diagram

$$\begin{tikzcd} \mathcal{K}_i \arrow[r] \arrow[d] & \mathcal{H}^i_{\mathrm{an}}\big(\mathbb{Z}(i)\big) \arrow[d] \\ \mathcal{K}_i/n \arrow[r] & \mathcal{H}^i_{\mathrm{\acute{e}t}}\big(\mathbb{Z}/n\mathbb{Z}(i)\big), \end{tikzcd}$$

and it follows from the compatibility of the universal Chern classes [Gillet 1981; Schneider 1988]. □

3. Counterexample to the Bloch–Kato conjecture over p-adic field

In this section K denotes a p-adic field and let X be a proper smooth surface over K. We fix a prime ℓ (possibly $\ell=p$) and consider the map (2-5)

$$\rho_X:\mathrm{CH}^2(X,1)\longrightarrow H^1_{\mathrm{cont}}(\mathrm{Spec}(K),V)\quad\big(V=H^2_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbb{Q}_\ell(2))\big).\tag{3-1}$$

Define the primitive part $\widetilde{V}$ of V by

$$\widetilde{V}:=H^2_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbb{Q}_\ell(2))/V_0,\qquad V_0=[H_X]\otimes\mathbb{Q}_\ell(1),\tag{3-2}$$

where $[H_X]\in H^2_{\mathrm{cont}}(X_{\overline{K}},\mathbb{Q}_\ell(1))$ is the cohomology class of a hyperplane section. With the notation

$$\widetilde{V}\simeq\mathrm{Ker}\big(H^2_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbb{Q}_\ell(2))\xrightarrow{\cup[H_X]}H^4_{\mathrm{\acute{e}t}}(X_{\overline{K}},\mathbb{Q}_\ell(3))\big),$$

we get a decomposition as G_K-modules:

$$V=\widetilde{V}\oplus V_0.\tag{3-3}$$

Let $\widetilde{\rho}:\mathrm{CH}^2(X,1)\longrightarrow H^1_{\mathrm{cont}}(\mathrm{Spec}(K),\widetilde{V})$ be the induced map.

Theorem 3.1. *Let $X\subset\mathbb{P}^3_K$ be a generic smooth surface of degree $d\geq 5$. Then $\widetilde{\rho}$ is the zero map for arbitrary ℓ.*

Remark 3.2. (1) This is an analogue of [Voisin 1995, 1.6], where she worked on Deligne–Beilinson cohomology.

(2) Bloch and Kato [1990] considers regulator maps such as (3-1) for a smooth projective variety over a number field and conjectures that its image coincides with H^1_g. We will see later (see Theorem 3.6) that the variant of the conjecture over a p-adic field is false in general.

(3) The construction of a counterexample mentioned in (2) hinges on the assumption that the surface $X \subset \mathbb{P}^3_K$ is generic. One may still ask whether the image of l-adic regulator map coincides H^1_g for a proper smooth variety X over a p-adic field when X is defined over a number field.

Proof. Let $f : \mathscr{X} \to M$ be as in the introduction and let $t : \mathrm{Spec}(K) \to M$ be a dominant morphism such that $X \simeq \mathscr{X} \times_M \mathrm{Spec}(K)$. For a morphism $S \to M$ of smooth schemes over $\mathbb{Q}$, let $f_S : X_S = \mathscr{X} \times_M S \to S$ be the base change of f. The same construction of (2-5) gives rise to the regulator map

$$\rho_S : \mathrm{CH}^2(X_S, 1) \to H^1_{\mathrm{cont}}(S, V_S),$$

where $V_S = R^2(f_S)_*\mathbb{Q}_l(2)$ is a smooth $\mathbb{Q}_l$-sheaf on S. Define the primitive part of V_S,

$$\widetilde{V}_S = R^2(f_S)_*\mathbb{Q}_l(2)/[H] \otimes \mathbb{Q}_l(1),$$

where $[H] \in H^0(S, R^2(f_S)_*\mathbb{Q}_l(1))$ is the class of a hyperplane section. Let

$$\widetilde{\rho}_S : \mathrm{CH}^2(X_S, 1) \to H^1_{\mathrm{cont}}(S, \widetilde{V}_S)$$

be the induced map. Note that

$$\mathrm{CH}^2(X, 1) = \varinjlim_S \mathrm{CH}^2(X_S, 1),$$

where $S \to M$ ranges over the smooth morphisms which factor $t : \mathrm{Spec}(K) \to M$. We have for such S the commutative diagram

$$\begin{array}{ccc}
\mathrm{CH}^2(X_S, 1) & \xrightarrow{\widetilde{\rho}_S} & H^1_{\mathrm{cont}}(S, \widetilde{V}_S) \\
\downarrow & & \downarrow \\
\mathrm{CH}^2(X, 1) & \xrightarrow{\widetilde{\rho}} & H^1_{\mathrm{cont}}(\mathrm{Spec}(K), \widetilde{V}).
\end{array}$$

Thus it suffices to show

$$H^1_{\mathrm{cont}}(S, \widetilde{V}_S) = 0.$$

Without loss of generality we suppose S is an affine smooth variety over a finite extension L of $\mathbb{Q}$.

Claim 3.3. *Assume $d \geq 4$. The natural map*

$$H^1_{\mathrm{cont}}(S, \widetilde{V}_S) \longrightarrow H^1_{\acute{e}t}(S_{\overline{\mathbb{Q}}}, \widetilde{V}_S) \quad (S_{\overline{\mathbb{Q}}} := S \times_L \mathrm{Spec}(\overline{\mathbb{Q}}))$$

is injective.

Indeed, by the Hochschild–Serre spectral sequence, it is enough to see

$$H^0_{\acute{e}t}(S_{\overline{\mathbb{Q}}}, \widetilde{V}_S) = 0,$$

which follows from [Asakura and Saito 2006b, Theorem 6.1(2)].

By SGA $4\,{}^{1}\!/_{2}$, Arcata, Cor. (3.3) and (3.5.1), we have

$$H^1_{\text{ét}}(S_{\overline{\mathbb{Q}}}, \widetilde{V}_S) \cong H^1_{\text{ét}}(S_{\mathbb{C}}, \widetilde{V}_S) \simeq H^1_{\text{an}}(S(\mathbb{C}), \widetilde{V}^{\text{an}}_S) \otimes \mathbb{Q}_l \quad (S_{\mathbb{C}} := S \times_L \operatorname{Spec}(\mathbb{C})),$$

where $\widetilde{V}^{\text{an}}_S$ is the primitive part of $V^{\text{an}}_S = R^2(f^{\text{an}}_S)_*\mathbb{Q}(2)$ with $f^{\text{an}}_S : (X_{S_{\mathbb{C}}})_{\text{an}} \to (S_{\mathbb{C}})_{\text{an}}$, the natural map of sites. By definition $\widetilde{V}^{\text{an}}_S$ is a local system on $S(\mathbb{C})$ whose fiber over $s \in S(\mathbb{C})$ is the primitive part of $H^2_{\text{an}}(X_s(\mathbb{C}), \mathbb{Q}(2))$ for X_s, the fiber of $X_S \to S$ over s. Due to Lemma 2.2, it suffices to show the triviality of the image of the map

$$\widetilde{\rho}^{\text{an}}_S : \mathrm{CH}^2(X_{S_{\mathbb{C}}}, 1) \longrightarrow H^1_{\text{an}}(S(\mathbb{C}), \widetilde{V}^{\text{an}}_S)$$

which is induced from

$$c_{\text{an}} : \mathrm{CH}^2(X_{S_{\mathbb{C}}}, 1) \longrightarrow H^3_{\text{an}}(X_S(\mathbb{C}), \mathbb{Q}(2))$$

by using the natural map

$$H^3_{\text{an}}(X_S(\mathbb{C}), \mathbb{Q}(2)) \to H^1_{\text{an}}(S(\mathbb{C}), V^{\text{an}}_S)$$

arising from the Leray spectral sequence for $f^{\text{an}}_S : (X_{S_{\mathbb{C}}})_{\text{an}} \to (S_{\mathbb{C}})_{\text{an}}$ and the vanishing $R^3(f^{\text{an}}_S)_*\mathbb{Q}(2) = 0$.

Claim 3.4. *The image of $\widetilde{\rho}^{\text{an}}_S$ is contained in the Hodge filtration*

$$F^2 H^1_{\text{an}}(S(\mathbb{C}), \widetilde{V}^{\text{an}}_S \otimes \mathbb{C})$$

defined by the theory of Hodge modules [Saito 1990, §4].

This follows from the functoriality of Hodge filtrations and Lemma 2.1.

It is quite complicated to describe the Hodge filtration on $H^1_{\text{an}}(S(\mathbb{C}), \widetilde{V}^{\text{an}}_S \otimes \mathbb{C})$ precisely. However, all that we need is the following property:

Claim 3.5. *For integers $m, p \geq 0$ there is a natural injective map*

$$F^p H^m_{\text{an}}(S(\mathbb{C}), \widetilde{V}^{\text{an}}_S \otimes \mathbb{C}) \to H^m_{\text{Zar}}(S_{\mathbb{C}}, G^p DR(\widetilde{V}^{\text{an}}_S))$$

where $G^p DR(\widetilde{V}^{\text{an}}_S)$ is the complex of Zariski sheaves on $S_{\mathbb{C}}$

$$\begin{aligned} F^p H^2_{\text{dR}}(X_S/S)_{\text{prim}} \otimes \mathcal{O}_{S_{\mathbb{C}}} \to F^{p-1} H^2_{\text{dR}}(X_S/S)_{\text{prim}} \otimes \Omega^1_{S_{\mathbb{C}}/\mathbb{C}} \to \\ \cdots \to F^{p-r} H^2_{\text{dR}}(X_S/S)_{\text{prim}} \otimes \Omega^r_{S_{\mathbb{C}}/\mathbb{C}} \to F^{p-r} H^2_{\text{dR}}(X_S/S)_{\text{prim}} \otimes \Omega^{r+1}_{S_{\mathbb{C}}/\mathbb{C}} \to \cdots . \end{aligned}$$

Here $H^\bullet_{\text{dR}}(X_S/S)$ denotes the de Rham cohomology of X_S/S, and $H^\bullet_{\text{dR}}(X_S/S)_{\text{prim}}$ is its primitive part defined by the same way as before, and the maps are induced from the Gauss–Manin connection thanks to Griffiths transversality.

This follows from [Asakura 2002, Lemma 4.2]. We note that its proof hinges on the theory of mixed Hodge modules. The key points are Deligne's comparison theorem [1970, §6] for algebraic and analytic cohomology of a vector bundle with integrable connection with regular singularities and the degeneration of Hodge spectral sequence for cohomology with coefficients; see [Saito 1990, (4.1.3)].

By the above claims we are reduced to showing the exactness at the middle term of the complex

$$F^2 H^2_{\mathrm{dR}}(X_S/S)_{\mathrm{prim}} \otimes \mathbb{O}_{S_{\mathbb{C}}} \longrightarrow F^1 H^2_{\mathrm{dR}}(X_S/S)_{\mathrm{prim}} \otimes \Omega^1_{S_{\mathbb{C}}/\mathbb{C}} \longrightarrow H^2_{\mathrm{dR}}(X_S/S)_{\mathrm{prim}} \otimes \Omega^2_{S_{\mathbb{C}}/\mathbb{C}}. \tag{3-4}$$

This is proved by the infinitesimal method in Hodge theory. We sketch the proof. Let $f : X_S \to S$ be the natural morphism. The assertion follows from the exactness at the middle term of the complex

$$f_*\Omega^2_{X_S/S} \otimes \mathbb{O}_{S_{\mathbb{C}}} \longrightarrow (R^1 f_*\Omega^1_{X_S/S})_{\mathrm{prim}} \otimes \Omega^1_{S_{\mathbb{C}}/\mathbb{C}} \longrightarrow R^2 f_*\mathbb{O}_{X_S} \otimes \Omega^2_{S_{\mathbb{C}}/\mathbb{C}} \tag{3-5}$$

and the injectivity of the complex

$$f_*\Omega^2_{X_S/S} \otimes \Omega^1_{S_{\mathbb{C}}/\mathbb{C}} \longrightarrow (R^1 f_*\Omega^1_{X_S/S})_{\mathrm{prim}} \otimes \Omega^2_{S_{\mathbb{C}}/\mathbb{C}}. \tag{3-6}$$

These complexes are induced by the complex (3-4) by Griffiths transversality. If

$$S = M \subset \mathbb{P}\big(H^0(\mathbb{P}^3_{\mathbb{Q}}, \mathbb{O}_{\mathbb{P}}(d))\big),$$

these assertions are proved as follows. Let $P = \mathbb{C}[z_0, z_1, z_2, z_3]$, and $P^n \subset P$ be the subspace of the homogeneous polynomials of degree n. Take a point $x \in M(\mathbb{C})$ and choose $F \in P^d$ which defines the surface corresponding to x. Let $R = \mathbb{C}[z_0, z_1, z_2, z_3]/(\partial F/\partial z_0, \cdots, \partial F/\partial z_3)$ be the Jacobian ring and $R^n \subset R$ be the image of P^n in R. Then the fibers over x of (3-5) and (3-6) are identified with the Koszul complexes

$$R^{d-4} \longrightarrow R^{2d-4} \otimes (R^d)^* \longrightarrow R^{3d-4} \otimes \overset{2}{\wedge}(R^d)^*, \tag{3-7}$$

$$R^{d-4} \otimes (R^d)^* \longrightarrow R^{2d-4} \otimes \overset{2}{\wedge}(R^d)^* \tag{3-8}$$

where $(R^d)^*$ denotes the dual space of R and the maps are induced from the multiplication $R \otimes R \to R$. Then the Donagi symmetrizer lemma [Green 1994, p. 76] implies that (3-7) is exact at the middle term if $d \geq 5$ and (3-8) is injective if $d \geq 3$, which proves the desired assertion in case $S = M$. The assertion in case S is dominant over M is reduced to the case $S = M$ by an easy argument; see [Asakura and Saito 2006a, §9]. This completes the proof of Theorem 3.1. □

Let $\mathbb{O}_K \subset K$ be the ring of integers and k be the residue field. In order to construct an example where the image of the regulator map

$$\rho_X : \mathrm{CH}^2(X, 1) \xrightarrow{\rho_X} H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V) \quad \big(V = H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(2))\big)$$

is not equal to $H^1_g(\mathrm{Spec}(K), V)$, we now take a proper smooth surface X having good reduction over K so that X has a proper smooth model $X_{\mathbb{O}_K}$ over $\mathrm{Spec}(\mathbb{O}_K)$. We denote the special fiber by Y. By [Langer and Saito 1996, p. 341, diagram below 5.7], there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^2(X,1) & \xrightarrow{\tilde{\rho}} & H^1_g(\mathrm{Spec}(K), V) \\ \downarrow{\scriptstyle \partial} & & \downarrow \\ \mathrm{CH}^1(Y) & \xrightarrow{\alpha} & H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V)/H^1_f(\mathrm{Spec}(K), V) \end{array} \tag{3-9}$$

where $H^1_f \subset H^1_g \subset H^1_{\mathrm{cont}}$ are the subspaces introduced by Bloch and Kato [1990] and ∂ is a boundary map in localization sequence for higher Chow groups.

Theorem 3.6. *Let $X \subset \mathbb{P}^3_K$ be a generic smooth surface of degree $d \geq 5$. Assume that X has a projective smooth model $X_{\mathbb{O}_K} \subset \mathbb{P}^3_{\mathbb{O}_K}$ over $\mathbb{O}_K$ and let $Y \subset \mathbb{P}^3_k$ be its special fiber.*

(1) *The image of $\partial \otimes \mathbb{Q}$ is contained in the subspace of $\mathrm{CH}^1(Y) \otimes \mathbb{Q}$ generated by the class $[H_Y]$ of a hyperplane section of Y.*

(2) *Let r be the Picard number of Y. Then*

$$\dim_{\mathbb{Q}_\ell}\big(H^1_g(\mathrm{Spec}(K), V)/\mathrm{Image}(\rho_X)\big) \geq r - 1.$$

Proof. Recall $V = \widetilde{V} \oplus V_0$, a decomposition as G_K-modules; see (3-3). Let $W \subset CH^2(X, 1)$ be the image of $\mathbb{Z}\cdot[H_X]\otimes K^\times$ under the product map $\mathrm{CH}^1(X)\otimes K^\times \to \mathrm{CH}^2(X, 1)$. Then it is easy to see ρ_X induces an isomorphism

$$W \otimes \mathbb{Q}_\ell \simeq H^1_g(\mathrm{Spec}(K), V_0) = H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V_0)$$

and that $\partial(W) = \mathbb{Z}\cdot[H_Y] \subset \mathrm{CH}^1(Y)$. Hence (1) follows from Theorem 3.1 together with injectivity of α in (3-9), proved by [Langer and Saito 1996, Lemma 5–7].

As for (2) we first note from [Bloch and Kato 1990, 3.9] that

$$\dim_{\mathbb{Q}_\ell}\big(H^1_{\mathrm{cont}}(\mathrm{Spec}(K), V_0)/H^1_f(\mathrm{Spec}(K), V_0)\big) = 1.$$

Moreover the same argument (except using the Tate conjecture) in the last part of [Langer and Saito 1996, §5] shows

$$\dim_{\mathbb{Q}_\ell}(\mathrm{CH}^1(Y) \otimes \mathbb{Q}_\ell) \leq \dim_{\mathbb{Q}_\ell}(H^1_g(\mathrm{Spec}(K), V)/H^1_f(\mathrm{Spec}(K), V)).$$

Hence (2) follows from (1). □

Remark 3.7. Let the assumption be as in Theorem 3.6. Then

$$\dim_{\mathbb{Q}_\ell}\big(H^1_g(\mathrm{Spec}(K), V)/\mathrm{Image}(\rho_X)\big) \geq \begin{cases} r-1, & \ell \neq p, \\ r-1+(h^{0,2}+h^{1,1}-1)[K:\mathbb{Q}_p], & \ell = p, \end{cases}$$

where $h^{p,q} := \dim_K H^q(X, \Omega^p_{X/K})$ denotes the Hodge number. Moreover the equality holds if and only if the Tate conjecture for divisors on Y holds. This follows from Theorem 3.1 and the computation of $\dim_{\mathbb{Q}_\ell} H^1_g(\mathrm{Spec}(K), V)$ using [Bloch and Kato 1990, 3.8 and 3.8.4]. The details are omitted.

4. Proof of Theorem 1.1

Let K be a p-adic field and $\mathbb{O}_K \subset K$ the ring of integers and k the residue field. Let us consider schemes

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d] & X_{\mathbb{O}_K} \arrow[d] & Y \arrow[l, "i"'] \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(\mathbb{O}_K) & \mathrm{Spec}(k) \arrow[l] \end{tikzcd}$$

where all vertical arrows are projective and smooth of relative dimension 2 and the diagrams are Cartesian. We have a boundary map in localization sequence for higher Chow groups with finite coefficients

$$\partial : \mathrm{CH}^2(X, 1, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{CH}^1(Y)/n.$$

For a prime number ℓ, it induces

$$\partial_\ell : \mathrm{CH}^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathrm{CH}^1(Y) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

where $\mathrm{CH}^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) := \varinjlim_n \mathrm{CH}^2(X, 1, \mathbb{Z}/\ell^n\mathbb{Z})$.

Theorem 4.1. *For $\ell \neq p := \mathrm{ch}(k)$, ∂_ℓ is surjective and has finite kernel. Hence we have*

$$\mathrm{CH}^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{\oplus r} + (\textit{finite group}),$$

where r is the rank of $\mathrm{CH}^1(Y)$.

Theorem 1.1 is an immediate consequence of Theorem 3.6(1), Theorem 4.1, and the exact sequence (2-1)

$$0 \to \mathrm{CH}^2(X, 1) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \mathrm{CH}^2(X, 1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathrm{CH}^2(X)\{\ell\} \to 0.$$

Proof of Theorem 4.1. Write $\Lambda = \mathbb{Q}_\ell/\mathbb{Z}_\ell$. We have a commutative diagram

$$\begin{tikzcd} \mathrm{CH}^2(X, 1, \Lambda) \arrow[r, "\partial"] \arrow[d, "c_1"] & \mathrm{CH}^1(Y) \otimes \Lambda \arrow[r, "i_*"] \arrow[d, "c_2"] & \mathrm{CH}^2(X_{\mathbb{O}_K}) \otimes \Lambda \arrow[r, "j^*"] \arrow[d, "c_3"] & \mathrm{CH}^2(X) \otimes \Lambda \arrow[d, "c_4"] \\ H^3_{\text{ét}}(X, \Lambda(2)) \arrow[r, "\partial_{\text{ét}}"] & H^2_{\text{ét}}(Y, \Lambda(1)) \arrow[r, "i_*"] & H^4_{\text{ét}}(X_{\mathbb{O}_K}, \Lambda(2)) \arrow[r, "j^*_{\text{ét}}"] & H^4_{\text{ét}}(X, \Lambda(2)). \end{tikzcd}$$

Here the upper exact sequence arises from the localization theory for higher Chow groups with finite coefficient, as in [Levine 2001, Theorem 1.7], and the lower from the localization theory for étale cohomology together with absolute purity [Fujiwara 2002]. The vertical maps are étale cycle class maps. By Equation (2-3), c_1 is injective. Since

$$\mathrm{CH}^1(Y) = H^1(Y, \mathbb{G}_m),$$

c_2 is injective by the Kummer theory. It is shown in [Saito and Sato 2006b] that c_3 is an isomorphism. Hence the diagram reduces the proof of Theorem 4.1 to showing that $\mathrm{Ker}(\partial_{\text{ét}})$ and $\mathrm{Ker}(j^*_{\text{ét}})$ are finite. This is an easy consequence of the proper base change theorem for étale cohomology and the Weil conjecture [Deligne 1980]. For the former we also use an exact sequence

$$H^3_{\text{ét}}(X_{\mathbb{O}_K}, \Lambda(2)) \to H^3_{\text{ét}}(X, \Lambda(2)) \xrightarrow{\partial_{\text{ét}}} H^2_{\text{ét}}(Y, \Lambda(1)). \qquad \square$$

Appendix. SK_1 of curves over p-adic fields

Let C be a proper smooth curve over a field K and consider $\mathrm{CH}^2(C, 1)$. By [Landsburg 1991, 2.5], we have an isomorphism

$$\mathrm{CH}^2(C, 1) \simeq H^1_{\mathrm{Zar}}(C, \mathcal{K}_2) \simeq SK_1(C).$$

By definition

$$SK_1(C) = \mathrm{Coker}\Big(K_2(K(C)) \xrightarrow{\delta} \bigoplus_{x\in C_0} K(x)^\times\Big),$$

where $K(C)$ is the function field of C, C_0 is the set of the closed points of C, and $K(x)$ is the residue field of $x \in C_0$, and δ is given by the tame symbols. The norm maps $K(x)^\times \to K^\times$ for $x \in C_0$ induce

$$N_{C/K} : SK_1(C) \to K^\times.$$

We write $V(C) = \mathrm{Ker}(N_{C/K})$.

When K is a p-adic field, it is known by class field theory for curves over a local field [Saito 1985] that $V(C)$ is a direct sum of its maximal divisible subgroup and a finite group. An interesting question is whether the divisible subgroup is uniquely divisible, or equivalently whether $SK_1(C)_{\mathrm{tor}}$ is finite. In case the genus $g(C) = 1$, confirmative results have been obtained in [Sato 1985; Asakura 2006]. The purpose of this section is to show that the method in the previous sections gives rise to an example of a curve C of $g(C) \geq 2$ such that $SK_1(C)_{\mathrm{tor}}$ is infinite.

Let C be as in the beginning of this section and let n be a positive integer prime to $\mathrm{ch}(K)$. We have the cycle class map

$$c_{\text{ét}} : \mathrm{CH}^2(C, 2, \mathbb{Z}/n\mathbb{Z}) \to H^2_{\text{ét}}(C, \mathbb{Z}/n\mathbb{Z}(2)).$$

The main result of [Merkurjev and Suslin 1982] implies that the above map is an isomorphism. In view of the exact sequence (compare (2-1))

$$0 \to \mathrm{CH}^2(C,2)/n \to \mathrm{CH}^2(C,2,\mathbb{Z}/n\mathbb{Z}) \to SK_1(C)[n] \to 0,$$

we get the exact sequence

$$0 \to \mathrm{CH}^2(C,2)/n \to H^2_{\text{ét}}(C,\mathbb{Z}/n\mathbb{Z}(2)) \to SK_1(C)[n] \to 0; \tag{A-2}$$

see [Suslin 1985, 23.4]. We will also use cycle class map to continuous étale cohomology

$$c_{\text{cont}} : \mathrm{CH}^2(C,2)\otimes\mathbb{Q}_\ell \to H^2_{\text{cont}}(C,\mathbb{Q}_\ell(2))$$

where ℓ is any prime number different from $\mathrm{ch}(K)$. When K is a p-adic field, one easily shows

$$H^2_{\text{cont}}(C,\mathbb{Q}_\ell(2)) \simeq H^1_{\text{cont}}\big(\mathrm{Spec}(K), H^1_{\text{ét}}(C_{\overline{K}},\mathbb{Q}_\ell(2))\big) \tag{A-3}$$

by using the Hochschild–Serre spectral sequence (2-4). Hence we get the map

$$\rho_C : \mathrm{CH}^2(C,2)\otimes\mathbb{Q}_\ell \to H^1_{\text{cont}}\big(\mathrm{Spec}(K), H^1_{\text{ét}}(C_{\overline{K}},\mathbb{Q}_\ell(2))\big). \tag{A-4}$$

Note that ρ_C is trivial if C has good reduction and $\ell \neq p$, since the group on the right hand side is trivial. The last fact is a consequence of the proper smooth base change theorem for étale cohomology and the weight argument.

Let M_g be the moduli space of tricanonically embedded projective nonsingular curves of genus $g \geq 2$ over the base field $\mathbb{Q}$ (compare [Deligne and Mumford 1969]), and let $f : \mathscr{C} \to M_g$ be the universal family.

Definition A.2. Let C be a proper smooth curve over a field K of characteristic zero. We say C is generic if there is a dominant morphism $\mathrm{Spec}(K) \to M_g$ such that $C \cong \mathscr{C} \times_{M_g} \mathrm{Spec}(K)$.

Theorem A.3. *Let K be a p-adic field and let C be a generic curve of genus $g \geq 2$ over K. Then ρ_C is the zero map for all ℓ. We have an isomorphism*

$$SK_1(C)_{\text{tor}} \cong H^2_{\acute{e}t}(C,\mathbb{Q}/\mathbb{Z}(2)) \quad \big(:= \varinjlim_n H^2_{\acute{e}t}(C,\mathbb{Z}/n\mathbb{Z}(2))\big).$$

Remark A.4. Theorem A.3 is comparable with the main result of [Green and Griffiths 2002] where they worked on Deligne–Beilinson cohomology.

Proof. The second assertion follows easily from the first in view of Equation (A-2). The first assertion is shown by the same method as the proof of Theorem 3.1, with the following fact from [Green and Griffiths 2002, §3] noted. Let $S \to M_g$ be a dominant smooth morphism, and put $f : C_S := \mathscr{C} \times_{M_g} S \to S$, then the map

$$f_*\Omega^1_{C_S/S} \longrightarrow R^1 f_*\mathscr{O}_{C_S} \otimes \Omega^1_{S/\mathbb{Q}}$$

induced from the Gauss–Manin connection is injective. □

Corollary A.5. *Let C be as in Theorem A.3. Assume the Jacobian variety $J(C)$ has semistable reduction over K. Let $\mathcal{J}$ be the Néron model of J with $\mathcal{J}_s$, its special fiber. Let r be the dimension of the maximal split torus in $\mathcal{J}_s$. For a prime ℓ, we have*

$$SK_1(C)\{\ell\} \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{r_\ell} \oplus (\textit{finite group}),$$

where $r_\ell = r$ for $\ell \neq p$ and $r_p = r + 2g[K : \mathbb{Q}_p]$.

For example, $SK_1(C)\{\ell\}$ is infinite for any ℓ if C is a Mumford curve (a proper smooth curve with semistable reduction over K such that the irreducible components are isomorphic to $\mathbb{P}^1_k$ and intersect each other at k-rational points, where k is the residue field of K), which is generic in the sense of Definition A.2.

Corollary A.5 follows from Theorem A.3 and the next result:

Lemma A.6. *Let C be proper smooth curve over a p-adic field K. Assume $J(C)$ has semistable reduction over K and let r_ℓ be as above. Then*

$$\dim_{\mathbb{Q}_\ell} H^2_{\text{cont}}(C, \mathbb{Q}_\ell(2)) = \dim_{\mathbb{Q}_\ell} H^1_{\text{cont}}(\operatorname{Spec}(K), V) = r_\ell. \ \big(V = H^1_{ét}(C_{\overline{K}}, \mathbb{Q}_\ell(2))\big).$$

Proof. The first equality follows from (A-3). By [Jannsen 1989, p. 354–355, Th. 5 and Cor. 7], we have

$$H^0_{\text{cont}}(\operatorname{Spec}(K), V) = 0, \qquad \dim_{\mathbb{Q}_\ell} H^2_{\text{cont}}(\operatorname{Spec}(K), V) = r.$$

Lemma A.6 now follows from the computation of Euler–Poincaré characteristic given in [Serre 1965, II 5.7]. □

Remark A.7. Using [Bloch and Kato 1990, 3.8.4] and the $\operatorname{Gal}(\overline{K}/K)$-module structure of the Tate module of an abelian variety over K (see [Grothendieck 1972, exposé IX]), one can show that

$$H^1_{\text{cont}}(\operatorname{Spec}(K), V) = H^1_g(\operatorname{Spec}(K), V).$$

Hence, if C is a generic curve of genus greater than or equal to 2, then the map ρ_C in Equation (A-4) does not surject onto H^1_g if $r_\ell \geq 1$. This gives another counterexample to a variant of the Bloch–Kato conjecture for p-adic fields.

Acknowledgements

The authors are grateful to Dr. Kanetomo Sato for stimulating discussions and helpful comments. The authors thank the referee for constructive comments which have improved the presentation of our paper very much.

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Communicated by Jean-Louis Colliot-Thélène
Received 2007-01-30 Revised 2007-08-15 Accepted 2007-09-15

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