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Joint moments of derivatives of characteristic polynomials

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We investigate the joint moments of the $2k$-th power of the characteristic polynomial of random unitary matrices with the $2h$-th power of the derivative of this same polynomial. We prove that for a fixed h, the moments are given by rational functions of k, up to a well-known factor that already arises when $h = 0$.

We fully describe the denominator in those rational functions (this had already been done by Hughes experimentally), and define the numerators through various formulas, mostly sums over partitions.

We also use this to formulate conjectures on joint moments of the zeta function and its derivatives, or even the same questions for the Hardy function, if we use a "real" version of characteristic polynomials.

Our methods should easily be applied to other similar problems, for instance with higher derivatives of characteristic polynomials.

More data and computer programs are available as expanded content.

1. Introduction

Our central object of study is the characteristic polynomial

$$Z_U(\theta) := \prod_{j=1}^{N} (1 - e^{\mathrm{i}(\theta_j - \theta)})$$

of an $N \times N$ random unitary matrix U with eigenangles θ_j, and specifically the *joint moments* of the powers of this polynomial and its derivative. Our results, which we state forthwith, are motivated by questions in number theory (page 33), and obtain by techniques from representation theory and algebraic combinatorics (see Section 1).

We define

$$V_U(\theta) := e^{\mathrm{i}N(\theta+\pi)/2} e^{-\mathrm{i}\sum_{j=1}^{N} \theta_j/2} Z_U(\theta). \tag{1}$$

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It is easily checked that for real θ, $V_U(\theta)$ is real and $|V_U(\theta)| = |Z_U(\theta)|$.

In this paper, we will investigate the averages (with respect to Haar measure)

$$|\mathcal{M}|_N(2k,r) := \left\langle |Z_U(0)|^{2k} \left| \frac{Z_U'(0)}{Z_U(0)} \right|^r \right\rangle_{\mathrm{U}(N)},$$

$$(\mathcal{M})_N(2k,r) := \left\langle |Z_U(0)|^{2k} \left(\frac{Z_U'(0)}{Z_U(0)} \right)^r \right\rangle_{\mathrm{U}(N)},$$

$$|\mathcal{V}|_N(2k,r) := \left\langle |V_U(0)|^{2k} \left| \frac{V_U'(0)}{V_U(0)} \right|^r \right\rangle_{\mathrm{U}(N)}$$

and their asymptotics

$$|\mathcal{M}|(2k,r) := \lim_{N\to\infty} |\mathcal{M}|_N(2k,r)/N^{k^2+r},$$

$$(\mathcal{M})(2k,r) := \lim_{N\to\infty} (\mathcal{M})_N(2k,r)/N^{k^2+r},$$

$$|\mathcal{V}|(2k,r) := \lim_{N\to\infty} |\mathcal{V}|_N(2k,r)/N^{k^2+r}.$$

As they involve both the characteristic polynomials and their derivatives, we call these averages joint moments. It is easy to show (by expanding the Haar measure explicitly) that the joint moments at finite N only make sense when $2k - r > -1$. For the asymptotics, the normalization by N^{k^2+r} is due to Hughes [2005] (and proved in this paper anyway).

This and related problems have been looked at by Conrey et al. [2006], Hughes [2001; 2005], Hughes et al. [2000], Forrester and Witte [2006a] and Mezzadri [2003]. However, much mystery remains, in particular for the dependency in r when $r \in \mathbb{R} \setminus \mathbb{N}$.

While $r \in \mathbb{R} \setminus \mathbb{N}$ remains out of reach, we offer in this paper an alternative approach that uncovers some of the structure in those averages.

Theorem 1.1. *For $r \in \mathbb{N}$ and $k \in \mathbb{C}$, the moments $(\mathcal{M})(2k,r)$ are essentially given by rational functions, that is, as meromorphic functions of k we have*

$$(\mathcal{M})(2k,r) = \left(-\frac{\mathrm{i}}{2}\right)^r \frac{G(k+1)^2}{G(2k+1)} \frac{X_r(2k)}{Y_r(2k)}, \tag{2}$$

where X_r and Y_r are even monic polynomials with integer coefficients and with $\deg X_r = \deg Y_r$ and G is the Barnes G-function [Hughes et al. 2000, Appendix].

Moreover

$$Y_r(u) = \prod_{\substack{1 \le a \le r-1 \\ a \text{ odd}}} (u^2 - a^2)^{\alpha_a(r)},$$

with the $\alpha_a(\cdot)$ given by

$$\alpha_a(r) = \left\lfloor \frac{-a+\sqrt{a^2+4r}}{2} \right\rfloor.$$

We derive from this a similar result (Theorem 6.1, page 55) for $|\mathcal{M}|(2k, 2h)$ and $|\mathcal{V}|(2k, 2h)$ (for h an integer). Finally, we have explicit expressions for $(\mathcal{M})(2k, r)$ given in Theorem 5.11, page 53 and Theorem 8.2, page 62 which allow us to compute the $X_r(u)$s, as given in Table 2, page 56, and additional data (available in Section 7).

Motivation. Ever since the works by Keating and Snaith [2000a; 2000b], the Riemann ζ-function can be (conjecturally but quantitatively) better understood through the modeling by characteristic polynomials of unitary matrices. The classical example concerns moments. Let

$$g(k) := \frac{G(k+1)^2}{G(2k+1)},$$
$$a(k) := \prod_{p \text{ prime}} \left(1-\frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\,\Gamma(k)}\right)^2 p^{-m}.$$

Then one can prove (fairly immediately, using the Selberg integral) that

$$|M|(2k, 0) = g(k), \tag{3}$$

which according to the Keating–Snaith philosophy leads to the following conjecture (for $k > -1/2$):

$$\frac{1}{T}\int_0^T \left|\zeta(\tfrac{1}{2}+\mathrm{i}t)\right|^{2k} \mathrm{d}t \sim_T g(k)\,a(k)\left(\log\frac{T}{2\pi}\right)^{k^2}. \tag{4}$$

The main point is thus that $a(k)$ is obtained by looking at primes, while $g(k)$ is guessed at from the random matrix side.

Observe also that Equations (3) and then (4) can be analytically continued in k.

Many of the authors cited above have now shown that this philosophy should be extended to the derivatives of characteristic polynomials.

In particular, $|\mathcal{M}|(2k, r)$ should show up as the RMT factor of[1]

$$\mathcal{I}(2k, r) := \lim_{T\to\infty} T^{-1}\left(\log\frac{T}{2\pi}\right)^{-k^2-r} \int_0^T \left|\zeta(\tfrac{1}{2}+\mathrm{i}t)\right|^{2k-r} \left|\zeta'(\tfrac{1}{2}+\mathrm{i}t)\right|^{r} \mathrm{d}t,$$

[1]It is a conjecture of Hall [2004] and Hughes [2001] that this is the appropriate normalization with respect to T.

k	h	$a(k)$	$\lvert\mathcal{M}\rvert(2k,2h)$	$\mathcal{I}(2k,2h)$	
1	1	1	$\left(\frac{1}{2^2}\right)\frac{1^2}{1}\frac{(2\times1)^2}{(2^2-1^2)^1}$	$\frac{1}{3}$	[Ingham 1928]
2	1	$\frac{6}{\pi^2}$	$\left(\frac{1}{2^2}\right)\frac{1^2}{12}\frac{(2\times2)^2}{(4^2-1^2)^1}$	$\frac{2}{15\pi^2}$	[Conrey 1988]
2	2	$\frac{6}{\pi^2}$	$\left(\frac{1}{2^4}\right)\frac{1^2}{12}\frac{(2\times2)^4-8(2\times2)^2-6}{(4^2-1^2)^1(4^2-3^2)^1}$	$\frac{61}{1680\pi^2}$	[Conrey 1988]
k	h	$a(k)$	$\lvert\mathcal{V}\rvert(2k,2h)$	$\mathcal{J}(2k,2h)$	
1	1	1	$\left(\frac{2!}{1!2^3}\right)\frac{1^2}{1}\frac{1}{(2^2-1^2)^1}$	$\frac{1}{12}$	[Ingham 1928]; see also [Hughes 2005]
2	1	$\frac{6}{\pi^2}$	$\left(\frac{2!}{1!2^3}\right)\frac{1^2}{12}\frac{1}{(4^2-1^2)^1}$	$\frac{1}{120\pi^2}$	[Conrey 1988]; see also [Hall 2002b]
2	2	$\frac{6}{\pi^2}$	$\left(\frac{4!}{2!2^6}\right)\frac{1^2}{12}\frac{1}{(4^2-1^2)^1(4^2-3^2)^1}$	$\frac{1}{1120\pi^2}$	[Conrey 1988]; see also [Hall 2002b]

Table 1. Summary of results on $\mathcal{I}(2k,2h)$ and $\mathcal{J}(2k,2h)$ when $h \neq 0$. The values for $\lvert\mathcal{M}\rvert(2k,2h)$ and $\lvert\mathcal{V}\rvert(2k,2h)$ are as obtained from Theorem 6.1. The fifth column equals the product of the third and the fourth. The last column gives the source where the result in the fifth column was first published.

and similarly $\lvert\mathcal{V}\rvert(2k,r)$ is needed for

$$\mathcal{J}(2k,r) := \lim_{T\to\infty} T^{-1}\left(\log\frac{T}{2\pi}\right)^{-k^2-r}\int_0^T \left|\mathfrak{Z}(\tfrac{1}{2}+\mathrm{i}t)\right|^{2k-r}\left|\mathfrak{Z}'(\tfrac{1}{2}+\mathrm{i}t)\right|^{r}\mathrm{d}t,$$

where $\mathfrak{Z}$ is Hardy's function (the relationship of $\mathfrak{Z}$ to ζ is analogous to the relationship of V_U to Z_U, that is, when $t \in \mathbb{R}$, $\mathfrak{Z}(\frac{1}{2}+\mathrm{i}t) \in \mathbb{R}$ and $\pm\mathfrak{Z}(\frac{1}{2}+\mathrm{i}t) = \left|\zeta(\frac{1}{2}+\mathrm{i}t)\right|$). More precisely, it is expected that

$$\mathcal{I}(2k,r) = a(k)\ \lvert\mathcal{M}\rvert(2k,r) \quad\text{and}\quad \mathcal{J}(2k,r) = a(k)\ \lvert\mathcal{V}\rvert(2k,r).$$

Thus Theorems 1.1 and 6.1 give us a conjectural handle on the moments of ζ and $\mathfrak{Z}$.

One can compute some small cases (for integer k and r) and show that they agree with previous Number Theory (proved) results. This had already been done before and is repeated in Table 1.

However, while Keating and Snaith obtained a full conjecture for $\mathcal{I}(2k,0)$ and $\mathcal{J}(2k,0)$ by computing $\lvert\mathcal{M}\rvert(2k,0)$ and $\lvert\mathcal{V}\rvert(2k,0)$, for the case of joint moments this goal remains elusive. All the available formulas for $|\mathcal{M}|(2k, r)$ or $|\mathcal{V}|(2k, r)$ are rather inadequate. In particular, those formulas are limited to $r := 2h$ (h an integer), they are hard to compute for large values of k and h, they obscure some of the structure in the results, and finally they cannot be analytically continued in h.

Analytic continuation would be important, because Conrey and Ghosh [1989] have proved (assuming the Riemann Hypothesis) that

$$\mathcal{J}(2, 1) = \frac{e^2 - 5}{4\pi}$$

and hence effectively conjectured[2]

$$|\mathcal{V}|(2, 1) = \frac{e^2 - 5}{4\pi}$$

as well since $a(1) = 1$. In order to get this, we would need to have a sufficiently nice formula for $|\mathcal{V}|(2k, 2h)$ that would allow for the analytic continuation in h. We have simply been unable to do this but have no doubt that our results should be helpful for that goal (see the connection with Noumi's work below).

On the other hand, the formulas obtained in Theorem 5.11, page 53 allow for much more effective computation than possible before, and we can compute longer tables for the different moments (see Section 7).

This numerical data is useful as well, as Hall has devised (around 2002) a method that uses $\mathcal{J}(2k, 2h)$ for all $0 \le h \le k$ to produce a lower bound $\Lambda(k)$ on

$$\Lambda := \limsup_{n\to\infty} \frac{t_{n+1} - t_n}{2\pi/\log t_n},$$

where the t_n is the n-th positive real zero of $\zeta(1/2 + \imath t)$. It is probably good to insist that this method does not depend on the Riemann Hypothesis, but only on values for moments! At the time of writing [Hall 2004], Hall only had the information he needed for k up to 2 (conjecturally, up to 6). In Section 7, we present our conjectural data for $\mathcal{J}(2k, 2h)$ as a direct function of k for h up to 15 (also available online at [Dehaye 2007a] for h up to 30). For a fixed h, various conjectural formulas are also given in this paper for $\mathcal{J}(2k, 2h)$ as a function of k. This, combined with Hall's method, should lead to more (conjectural) lower bounds on Λ. It is widely believed that $\Lambda = \infty$ so potentially we could also see if Hall's method has any hope to reach that, assuming only information on the $\mathcal{J}(2k, 2h)$, but not on the Riemann Hypothesis. In other words, it would also inform us on the relationship between moment conjectures, the Riemann Hypothesis and the conjecture $\Lambda = \infty$. We leave this to a further paper.

[2] This is completely backwards from the usual flow of conjectures *from* random matrix theory *to* number theory, and possibly an unique instance of a reversal of this type.

Finally, Noumi [2004] investigates the relationship between Painlevé equations and expressions similar to one of the expressions we obtain for $(\mathcal{M})(2k, r)$, in Theorem 8.2. Connections of this sort have been uncovered before (see [Forrester and Witte 2006a; 2006b] and works of Borodin), but an approach through Noumi's ideas would be original. One of our goals then would be to obtain analytic continuation for $(\mathcal{M})(2k, r)$ in r, which would again allow to compute $|\mathcal{V}|(2, 1)$. We also leave this for further study.

Our techniques are quite disconnected from the original motivation, so we discuss them separately.

Techniques. As mentioned, our techniques lie mostly in representation theory and algebraic combinatorics. We look at the characteristic polynomials or the derivatives as symmetric functions of the eigenvalues of U, and express them in that way. We eventually express those symmetric functions in the most natural basis to use, the Schur functions. This basis is particularly suitable since those functions are also (irreducible) characters of unitary groups $\mathrm{U}(N)$. We find ourselves integrating irreducible characters over their support (groups), which is very enviable!

To express all the different functions in this basis of the Schur functions, we use ideas present in [Bump and Gamburd 2006] and the author's thesis [Dehaye 2006]. We will introduce those ideas as we need them.

For a more thorough discussion of why a similar approach should always be attempted and other examples of its applications, please see the author's thesis and the results in [Dehaye 2007b].

Once we have a concise expression for the various moments, we still have to evaluate it. This will involve sums over partitions of values of the Schur functions. After reparametrizing those sums over the Frobenius coordinates of the partitions, the results of El-Samra and King were immediately useful to obtain the Schur values, and the results of Borodin to handle the combinatorics of the sums. We then obtain a very big sum for the moments (Theorem 5.9), but that can directly be evaluated on computer (and thus checked against small N results). After taking asymptotics, our results start simplifying into Theorem 5.11, enough to prove Theorem 1.1 on the general shape of those moments. However, the best expression is probably obtained once we use Macdonald's ninth variation of the Schur functions (Theorem 8.2).

Organization of this paper. In Section 2, we introduce all the nonstandard notation we will be using. In Section 3, we present the basic relations satisfied by the integrands

$$|Z_U(0)|^{2k}\left|\frac{Z_U'(0)}{Z_U(0)}\right|^r, \quad |Z_U(0)|^{2k}\left(\frac{Z_U'(0)}{Z_U(0)}\right)^r, \quad \text{and} \quad |V_U(0)|^{2k}\left|\frac{V_U'(0)}{V_U(0)}\right|^r.$$

The bulk of this paper is contained in Sections 4 and 5. In Section 4, we reexpress the integrands as a sum in the Schur basis, in a way similar to Bump and Gamburd (via the Dual Cauchy Identity). In Section 5, we engage in a long computation to evaluate the result obtained in the previous section, mostly using the results of El-Samra and King, and Borodin. Section 6 merely serves to tie what has been done in Sections 4 and 5 into the proof of Theorem 1.1. In Section 7 we present the data we are now able to compute, and particularly discuss the position of the roots of $|\mathcal{V}|(2k, 2h)$ starting on page 57. Section 8 describes two attempts to simplify our results further, one using Macdonald's ninth variation of the Schur functions, and the other imitating a proof of the Cauchy identity.

2. Notation

We let $\mathbb{N}_+$ be the set $\mathbb{N} \setminus 0$. To avoid confusion with the index i, we set $\mathrm{i}^2 = -1$.

We use $\mathbf{v}$ for a generic *vector* (of integers) $(v_1, \dots, v_d)$, and $\vec{\mathbf{v}}$ for a strictly decreasing sequence of integers $v_1 > v_2 > \cdots > v_d$, which we call a *Frobenius sequence*. Frobenius sequences are thus a special type of vectors.

Sequences of weakly decreasing positive integers amount to partitions, and we stick with classical notation for those, $\lambda = (\lambda_1, \dots, \lambda_{l(\lambda)})$, which defines $l(\lambda)$. We also freely change our point of view to Young tableaux when discussing partitions. We denote by λ^t the conjugate of a partition λ of $|\lambda|$. Define two sequences $p_i := \lambda_i - i$, $q_i := \lambda_i^t - i$. They are strictly decreasing; λ_i and λ_i^t are eventually 0, and hence $p_i = -i$ and $q_i = -i$ eventually. There exists d such that $p_d \geq 0 > p_{d+1}$ and $q_d \geq 0 > q_{d+1}$. We call d the *rank* of λ. The vectors $\vec{\mathbf{p}} = (p_1, \dots, p_d)$ and $\vec{\mathbf{q}} = (q_1, \dots, q_d)$ are Frobenius sequences, and we call $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ the *Frobenius coordinates* of the partition λ. We write $\lambda = \{\vec{\mathbf{p}} : \vec{\mathbf{q}}\}$.

Given $\mathbf{p}$, we define $\sigma_{\mathbf{p}} \in \mathcal{S}_d$ such that $\mathrm{sort}(\mathbf{p}) := (p_{\sigma_{\mathbf{p}}(i)})$ is strictly decreasing (and hence a Frobenius sequence). This is thus not defined if $p_i = p_j$ while $i \neq j$. We set $\mathrm{sgn}(\mathbf{p}) := \mathrm{sgn}(\sigma_{\mathbf{p}})$, with the added convention that $\mathrm{sgn}(\mathbf{p}) := 0$ if $\sigma_{\mathbf{p}}$ is not defined.

If λ and μ are partitions, $\lambda \cup \mu$ is the partition obtained by taking the union of their parts. The partition $\langle X^Y \rangle$ has a $Y \times X$ rectangle for Young tableau.

We also use the notation $\left[1^R\right]$ for R copies of 1, used as argument to a (Schur) function.

3. Basic relations among the integrands

We logarithmically differentiate Equation (1) to obtain

$$\frac{V_U'(\theta)}{V_U(\theta)} = \frac{\mathrm{i}N}{2} + \frac{Z_U'(\theta)}{Z_U(\theta)}$$

and hence, when θ is real,

$$\left|\frac{Z_U'(\theta)}{Z_U(\theta)}\right|^2 = \left|\frac{V_U'(\theta)}{V_U(\theta)}\right|^2 + \frac{N^2}{4} = \left(\frac{V_U'(\theta)}{V_U(\theta)}\right)^2 + \frac{N^2}{4} = \left(\frac{Z_U'(\theta)}{Z_U(\theta)}\right)^2 + \mathrm{i}N\left(\frac{Z_U'(\theta)}{Z_U(\theta)}\right).$$

These basic relations give

$$|\mathcal{M}|_N\,(2k,2h) = \sum_{j=0}^{h} (\mathrm{i}N)^{h-j}\binom{h}{j}(\mathcal{M})_N(2k,h+j),$$

$$|\mathcal{M}|\,(2k,2h) = \sum_{j=0}^{h} \mathrm{i}^{h-j}\binom{h}{j}(\mathcal{M})(2k,h+j), \tag{5}$$

$$|\mathcal{V}|_N\,(2k,2h) = \sum_{j=0}^{h} \binom{h}{j}\left(\frac{-N^2}{4}\right)^{h-j}|\mathcal{M}|_N(2k,2j),$$

$$|\mathcal{V}|\,(2k,2h) = \sum_{j=0}^{h} \binom{h}{j}\left(\frac{-1}{4}\right)^{h-j}|\mathcal{M}|(2k,2j). \tag{6}$$

These formulas are initially valid only when h is a nonnegative integer, but the right-hand sides can be analytically continued by plugging in noninteger h and extending the sum to infinity.[3] Thus we see that computing $(\mathcal{M})_N(2k,r)$ would get us most of the way to $|\mathcal{M}|_N(2k,2h)$ or $|\mathcal{V}|_N(2k,2h)$, and we now focus on the integrand $|Z_U(0)|^{2k}\,(Z_U'(0)/Z_U(0))^r$.

4. Derivation into the Schur basis

The goal here is to follow ideas similar of Bump and Gamburd [2006] in order to prove Proposition 4.3, page 40. One of their main tools was the dual Cauchy identity. We encourage the reader to look at their first proposition and corollary for the unitary group, since this is all we really exploit from that paper.

Lemma 4.1 (Dual Cauchy identity). *If $\{x_i\}$ and $\{y_j\}$ are finite sets of variables,*

$$\prod_{i,j}(1+x_iy_j) = \sum_{\lambda} s_{\lambda^t}(x_i)s_\lambda(y_j),$$

where the sum is over all partitions λ and s_λ is the Schur polynomial.

[3] Getting the correct analytic continuation can be tricky. The relation

$$|\mathcal{V}|\,(2k,2h) = \sum_{j=0}^{2h}\binom{2h}{j}\left(\frac{\mathrm{i}}{2}\right)^j(\mathcal{M})(2k,2h-j)$$

is also valid for integers h, but here the right-hand side does not analytically continue in h to the left-hand side, since we exploit $|V_U'(\theta)/V_U(\theta)|^{2h} = (V_U'(\theta)/V_U(\theta))^{2h}$, where h must be an integer.

Apply this lemma setting $\{x_j := e^{\mathrm{i}\theta_j} \mid j \in [1, \ldots, N]\}$ to be the set of eigenvalues of U, and $\{y_j := 1 \mid j \in [1, \ldots, 2k]\}$. We chose the notation $s_\lambda(U) := s_\lambda(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_N})$. This gives

$$\sum_\lambda \overline{s_{\lambda^t}(U)}\, s_\lambda([1^{2k}]) = \det(\mathrm{Id} + \overline{U})^{2k} = \overline{\det U}^k\, |\det(\mathrm{Id}+U)|^{2k}$$
$$= \overline{s_{\langle k^N\rangle}(U)}\, |\det(\mathrm{Id}+U)|^{2k}$$

or (replacing U by $-U$)

$$|Z_U(0)|^{2k} = |\det(\mathrm{Id}-U)|^{2k} = (-1)^{kN} s_{\langle k^N\rangle}(U) \sum_\lambda (-1)^{|\lambda|} \overline{s_{\lambda^t}(U)} s_\lambda([1^{2k}]).$$

We can also reexpress

$$\frac{Z'_U(0)}{Z_U(0)} = \sum_{j=1}^N \frac{\mathrm{i}e^{\mathrm{i}\theta_j}}{1-e^{\mathrm{i}\theta_j}} = \sum_{j=1}^N \mathrm{i} \lim_{z\to 1^-} \sum_{m=1}^\infty z^m e^{\mathrm{i}m\theta_j} = \mathrm{i} \lim_{z\to 1^-} \sum_{m=1}^\infty z^m p_m(U), \qquad (7)$$

where $p_m(x_1, \ldots, x_N)$ is the m-th power sum $x_1^m + \cdots + x_N^m$ and we have used the same convention as for $s_\lambda(U)$ of inputting the eigenvalues. We will use the same convention soon for the power sums $p_\lambda := \prod_i p_{\lambda_i}$.

In practice, we want the reader to just ignore the variable z and set it to 1. This will be justified *a posteriori*.

Putting everything together, we thus get for $|Z_U(0)|^{2k}\, (Z'_U(0)/Z_U(0))^r$

$$(-1)^{kN} s_{\langle k^N\rangle}(U) \left(\mathrm{i} \sum_{m=1}^\infty p_m(U) \right)^r \sum_\lambda (-1)^{|\lambda|} s_\lambda([1^{2k}])\, \overline{s_{\lambda^t}(U)}. \qquad (8)$$

At this point, we will soon want to use the fact that the s_λs are the characters of unitary groups.

Indeed, if $U \in \mathrm{U}(N)$ then when $l(\lambda) > N$, we have[4] $s_\lambda(U) \equiv 0$, but when $l(\lambda), l(\mu) \le N$, we have

$$\left\langle s_\lambda(U) \overline{s_\mu(U)} \right\rangle_{\mathrm{U}(N)} = \delta_{\lambda\mu};$$

that is, for large enough N, s_λ is an irreducible character of $\mathrm{U}(N)$. This orthogonality is obviously good for our purposes, but the only obstacle is the need to express $s_{\langle k^N\rangle}(U)\big(\sum_{m=1}^\infty p_m(U)\big)^r$ exclusively in terms of the Schur functions. This can be done and will require the Murnaghan–Nakayama rule.

Let a ribbon be a connected Young skew-tableau not containing any 2×2-block. If a ribbon contains m blocks, it is called a m-ribbon. A first approximation to one version of the M–N rule says that $s_\lambda p_m$ is given by a signed sum of s_μs, where μ runs through all partitions obtained by adding a m-ribbon to λ.

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[4] This is a consequence of the fact that $s_\lambda(x_1, \ldots, x_n) \equiv 0$ if $l(\lambda) > n$.

If we average Expression (8) over $\mathrm{U}(N)$, we could thus see λ as running through all partitions obtained by adding r ribbons to the rectangle $\langle N^k \rangle$ (this uses the fact that this lax version of the M–N rule is invariant under transpositions, since we have yet to discuss the signs). There are more conditions, however. We also need $l(\lambda^t) \leq N$ (since otherwise $s_{\lambda^t}(U) \equiv 0$, as in note 4), and we need $l(\lambda) \leq 2k$ (since otherwise $s_\lambda([1^{2k}]) = 0$, again just as in that note). In other words, λ contains $\langle N^k \rangle$ but is contained in $\langle N^{2k} \rangle$. There are only finitely many (ways to obtain) such partitions, which will make the sum over λs finite, and thus only finitely many sets of lengths of the r ribbons will contribute. This justifies *a posteriori* setting z to 1 in (7), but *only when we can apply the dominated convergence theorem*. This will only occur if we know of a bound on the integrand independent of z that is itself integrable. We can pick $|Z_U(0)|^{2k} \left| Z'_U(0)/Z_U(0) \right|^r$ whenever this is integrable, that is, only when $2k - r > -1$.

We now state a more precise version of the M–N rule.

Theorem 4.2 (Murnaghan–Nakayama). *Let λ be a partition and $\boldsymbol{\rho}$ be a vector with $|\lambda| = \sum_i \boldsymbol{\rho}_i$. If $\chi^\lambda_{\boldsymbol{\rho}}$ is the value of the irreducible character of $\mathcal{S}_{|\lambda|}$ associated to λ on the conjugacy class of cycle-type* $\operatorname{sort}(\boldsymbol{\rho})$, *then*

$$p_{\boldsymbol{\rho}} = \sum_\lambda \chi^\lambda_{\boldsymbol{\rho}} s_\lambda \tag{9}$$

and (*more importantly*)

$$\chi^\lambda_{\boldsymbol{\rho}} = \sum_S (-1)^{\operatorname{ht}(S)}$$

summed over all sequences of partitions $S = (\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(r)})$ such that $r := l(\lambda)$, $0 = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \lambda$, and such that each $\lambda^{(i)} - \lambda^{(i-1)}$ is a ribbon of length $\boldsymbol{\rho}_i$, and $\operatorname{ht}(S) = \sum_i \operatorname{ht}(\lambda^{(i)} - \lambda^{(i-1)})$.

We have not defined the height ht of a ribbon, but rather than doing so or detailing the computation here, we only expose the idea. Equation (9) tells us that $(\sum_m p_m)^r$ can be computed using the character values of symmetric groups, which can be evaluated by summing over sequences of partitions $(\lambda^{(0)}, \dots, \lambda^{(r)})$. For each such sequence, the sequence $(\tilde\lambda^{(0)}, \dots, \tilde\lambda^{(r)})$, with $\tilde\lambda^{(i)} := \langle N^k \rangle \cup \lambda^{(i)}$, would be associated with the combinatorics of the expansion of the product in (8). Indeed the combinatorics of ribbon is unchanged under translations (down by k) as long as the partitions are kept within a rectangle (actually, a horizontally bounded region).

If the computation is explicitly carried out, we get the following result.

Proposition 4.3. *If $2k - r > -1$, we have*

$$(\mathcal{M})_N(2k, r) = (-\mathrm{i})^r \sum_{\boldsymbol{\mu} \in \mathbb{N}_+^r} \sum_{\substack{\lambda \text{within} \\ k \times N}} \chi^\lambda_{\boldsymbol{\mu}} \, s_{\langle N^k \rangle \cup \lambda}([1^{2k}]), \tag{10}$$

with the understanding that $\chi^{\lambda}_{\boldsymbol{\mu}} = 0$ *if* $|\lambda| \neq \sum_i \boldsymbol{\mu}_i$.

For this result, we have preferred to index all the partitions containing $\langle N^k \rangle$ but contained in $\langle N^{2k} \rangle$ as $\langle N^k \rangle \cup \lambda$, for $\lambda \subset \langle N^k \rangle$.

We are now left with the task of evaluating the right-hand side of (10), which will turn out to be a tedious process.

5. Main computation

We are left with two problems. The first one is due to the characters of the symmetric group. Those are of course desperately hard to evaluate directly and individually. We are helped here because we will actually only evaluate something close to

$$\sum_{\boldsymbol{\mu} \in \mathbb{N}^l_+} \chi^{\lambda}_{\boldsymbol{\mu}}$$

for given λ. This amounts to computing the sum of values of the character χ^{λ} over the permutations with l cycles. The second issue is evaluating $s_{\langle N^k \rangle \cup \lambda}([1^{2k}])$. The author had previously used the Weyl Dimension Formula to do this (see [Dehaye 2006]). A formula giving that dimension in terms of the Frobenius coordinates of λ is probably better adapted for our purposes.

In addition, both "problems" combine extremely well, in that both expressions should involve a sign, which turns out to be the same.

We will then sum our terms over all partitions, expressed in Frobenius coordinates. This amounts to summing over possible ranks ($1 \leq d$) and then pairs of Frobenius sequences of length d.

The value of the Schur function in Frobenius coordinates.

Dimension formula in Frobenius coordinates. El Samra and King [1979] use the notation $D_R\{p:q\}$ for $s_{\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}}([1^R])$.

Assume $\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}$ has d Frobenius coordinates. They prove that

$$\begin{aligned} s_{\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}}([1^R]) &= \left| \frac{(R+p_i)!}{(R-q_j-1)!\,p_i!\,q_j!\,(p_i+q_j+1)} \right|_{d\times d} \\ &= \prod_{i=1}^{d} \frac{(R+p_i)!}{(R-q_i-1)!\,p_i!\,q_i!} \prod_{1\leq i<j\leq d} (p_i-p_j)(q_i-q_j) \prod_{i,j=1}^{d} \frac{1}{p_i+q_j+1} \end{aligned} \tag{11}$$

where the first expression is also known as the reduced determinantal form (see [Foulkes 1951], as cited in [El Samra and King 1979]).

It is a consequence of Cauchy's Lemma that the two expressions in (11) are equivalent.

Lemma 5.1 (Cauchy).

$$\left|\frac{1}{p_i+q_j+1}\right|_{d\times d} = \prod_{1\le i<j\le d}(p_i-p_j)(q_i-q_j)\prod_{i,j=1}^{d}\frac{1}{p_i+q_j+1}.$$

Observe that Formula (11) is positive (as it should, given that it is also a dimension) because the p_i and q_i are strictly decreasing.

However, the right-hand side of (11) still makes sense if we plug in the unsorted vectors $\mathbf{p}$, $\mathbf{q}$ (with even the possibility of $i\neq j$ but $p_i=p_j$). Hence this can be used to define $s_{\{\mathbf{p}\,:\,\mathbf{q}\}}([1^R])$ as well, which is then skew-symmetric in both the p_is and the q_is separately. This can be written

$$s_{\{\mathbf{p}\,:\,\mathbf{q}\}}([1^R]) = \operatorname{sgn}(\mathbf{p})\operatorname{sgn}(\mathbf{q})s_{\{\operatorname{sort}(\mathbf{p})\,:\,\operatorname{sort}(\mathbf{q})\}}([1^R]). \tag{12}$$

Observe that Formula (12) is still valid when sort($\mathbf{p}$) or sort($\mathbf{q}$) is not defined (this happens when two of the entries of $\mathbf{p}$ or $\mathbf{q}$ are equal) thanks to $\operatorname{sgn}(\mathbf{p})\operatorname{sgn}(\mathbf{q})=0$! (See the conventions in Section 2.)

Finally, it is helpful to remark that Formula (11) for $s_{\{\mathbf{p}\,:\,\mathbf{q}\}}([1^R])$ can be seen as a product indexed by the sets $\mathbf{p}\cup\mathbf{q}$ and pairs in the set $(\mathbf{p}\times\mathbf{p})\cup(\mathbf{q}\times\mathbf{q})\cup(\mathbf{p}\times\mathbf{q})$.

Evaluation of $s_{\langle N^k\rangle\cup\lambda}([1^{2k}])$. We take $\lambda=\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}$ to have d Frobenius coordinates.

In total analogy with (12), we first extend the definition of $s_{\langle N^k\rangle\cup\lambda}$ and set

$$s_{\langle N^k\rangle\cup\{\mathbf{p}\,:\,\mathbf{q}\}} := \operatorname{sgn}(\mathbf{p})\operatorname{sgn}(\mathbf{q})\,s_{\langle N^k\rangle\cup\{\operatorname{sort}(\mathbf{p})\,:\,\operatorname{sort}(\mathbf{q})\}},$$

with the understanding (as before) that the value of the right-hand side is taken as 0 if $p_i=p_j$ or $q_i=q_j$ for some $i\neq j$. Again, this is skew-symmetric in the p_is and separately in the q_is.

We have the following lemma.

Lemma 5.2. *Let* $\mathbf{p}$, $\mathbf{q}$ *be vectors with* d *coordinates. Then*

$$\begin{aligned} &s_{\langle N^k\rangle\cup\{\mathbf{p}\,:\,\mathbf{q}\}}([1^{2k}]) \\ &\qquad= s_{\langle N^k\rangle}([1^{2k}])\left(\prod_{i=1}^{d}\frac{(N-p_i)^{(k)}(k-q_i)^{(k)}}{(p_i+k+1)^{(k)}(N+q_i+1)^{(k)}}\right)s_{\{\mathbf{p}\,:\,\mathbf{q}\}}([1^{2k}]). \end{aligned} \tag{13}$$

Proof. By skew-symmetry, we really only have to check this for $\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}$. If we want to use (11), we should look at the Frobenius coordinates of $\langle N^k\rangle\cup\lambda$. This would be rather unpleasant (particularly because the number of Frobenius coordinates would change for fixed N and k according to the λ considered).

We look instead at

$$\vec{\mathbf{x}} := (N+k-1, \ldots, N),$$
$$\vec{\mathbf{y}} := (2k-1, \ldots, k),$$
$$\vec{\boldsymbol{\alpha}} := \vec{\mathbf{x}} \cup \vec{\mathbf{p}} \quad \text{(sorted)}, \quad \text{and}$$
$$\vec{\boldsymbol{\beta}} := \vec{\mathbf{y}} \cup \vec{\mathbf{q}} \quad \text{(sorted)}.$$

Then $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\beta}}$ are strictly decreasing, so those are Frobenius coordinates. The partition corresponding to those coordinates is obtained geometrically by sticking a $\langle k^{2k}\rangle$ block to the left of $\langle N^k\rangle \cup \lambda$, or equivalently, shifting $\langle N^k\rangle \cup \lambda$ by k spots to the right, while considering $\lambda = (\lambda_1, \ldots, \lambda_k)$ to have exactly k parts (with some possibly empty).

Because of this, we have (as in [Bump and Gamburd 2006, page 6])

$$s_{\{\vec{\boldsymbol{\alpha}}\,:\,\vec{\boldsymbol{\beta}}\}}([1^{2k}]) = e_{2k}^k([1^{2k}])\, s_{\langle N^k\rangle\cup\lambda}([1^{2k}]) = s_{\langle N^k\rangle\cup\lambda}([1^{2k}]).$$

Additionally, $\{\vec{\mathbf{x}} : \vec{\mathbf{y}}\}$ are the Frobenius coordinates of $\langle (N+k)^k\rangle \cup \langle k^k\rangle$. Hence, for the same reason as above, we have

$$s_{\{\vec{\mathbf{x}}\,:\,\vec{\mathbf{y}}\}}([1^{2k}]) = s_{\langle (N+k)^k\rangle\cup\langle k^k\rangle}([1^{2k}]) = e_{2k}^k([1^{2k}])\, s_{\langle N^k\rangle}([1^{2k}]) = s_{\langle N^k\rangle}([1^{2k}]).$$

When evaluating the product described in (11) using the $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\beta}}$ coordinates, we have a big product taken over the sets $\vec{\boldsymbol{\alpha}}$, $\vec{\boldsymbol{\beta}}$, $\vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{\alpha}}$, $\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{\beta}}$ and $\vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{\beta}}$. We expand those index sets using $\vec{\boldsymbol{\alpha}} = \vec{\mathbf{x}} \cup \vec{\mathbf{p}}$ and $\vec{\boldsymbol{\beta}} = \vec{\mathbf{y}} \cup \vec{\mathbf{q}}$.

One can see that the products indexed by $\vec{\mathbf{p}}$, $\vec{\mathbf{q}}$, $\vec{\mathbf{p}} \times \vec{\mathbf{p}}$, $\vec{\mathbf{q}} \times \vec{\mathbf{q}}$ and $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$ together give

$$s_{\{\vec{\mathbf{p}}\,:\,\vec{\mathbf{q}}\}}([1^{2k}]) = s_\lambda([1^{2k}]).$$

Similarly, the products indexed by $\vec{\mathbf{x}}$, $\vec{\mathbf{y}}$, $\vec{\mathbf{x}} \times \vec{\mathbf{x}}$, $\vec{\mathbf{y}} \times \vec{\mathbf{y}}$ and $\vec{\mathbf{x}} \times \vec{\mathbf{y}}$ give

$$s_{\{\vec{\mathbf{x}}\,:\,\vec{\mathbf{y}}\}}([1^{2k}]) = s_{\langle N^k\rangle}([1^{2k}]).$$

We are left with only "cross-products" to evaluate, for the index sets $\vec{\mathbf{x}} \times \vec{\mathbf{p}}$, $\vec{\mathbf{x}} \times \vec{\mathbf{q}}$, $\vec{\mathbf{y}} \times \vec{\mathbf{p}}$ and $\vec{\mathbf{y}} \times \vec{\mathbf{q}}$. The definitions of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ now give the result. □

Sums of characters over conjugacy classes with same number of cycles. Assume $f(\{\mathbf{p} : \mathbf{q}\})$ is a function of pairs of vectors of the same length (say d). One can set $f(\lambda) := f(\{\vec{\mathbf{p}} : \vec{\mathbf{q}}\})$, where $\lambda = \{\vec{\mathbf{p}} : \vec{\mathbf{q}}\}$.

The goal in this section is to evaluate sums of characters of the general form

$$\sum_{\mu\in\mathbb{N}_+^l} \chi_\mu^\lambda f(\lambda).$$

We will eventually take $f(\lambda) = s_{\langle N^k\rangle\cup\lambda}([1^{2k}])$ but there is no reason to limit ourselves in that way for a while.

We rely on a few results of Borodin, that give a slightly different version of the Murnaghan–Nakayama rule.

Definitions. This is based on [Borodin 2000, around page 15] and [Borodin and Olshanski 1998, around page 6]. The relevant definitions (not included here) are *fragment*, the different *block* types, the *filling numbers*, *filled structure*, *sign of a structure*.

Theorem 5.5 is almost in Borodin's work, and his definitions are used in Proposition 5.6. Both of those results are used for Theorem 5.9, which can be read without looking at Borodin's papers.

However, the first condition to have a fragment needs clarification in both papers, that is, we change

> "(1) there is exactly one hook block that precedes the others"

to

> "(1) there is exactly one hook block in each fragment. That hook block precedes any other block in the fragment".

We also would like to correct a statement in [Borodin 2000], in that linear horizontal or vertical blocks are *positive*, not just *nonnegative* integers (in agreement with the other cited paper of Borodin [Borodin and Olshanski 1998]).

We can highlight one of the definitions. Any filled structure T with d fragments produces a set of pairs

$$\{(p_1, q_1), \dots, (p_d, q_d)\}$$

which consists of the filling p- and q- numbers of the fragments.

The sign of T is defined as follows.

$$\operatorname{sgn}(T) = \operatorname{sgn}(\mathbf{p}) \operatorname{sgn}(\mathbf{q}) (-1)^{\sum q_i + v(T)},$$

where, as a reminder, the sgn inside the formula is 0 if $p_i = p_j$ (respectively, $q_i = q_j$) for $i \neq j$.

Simplified Murnaghan–Nakayama rule. Although we haven't defined anything, we state Proposition 4.3, taken from the first paper of Borodin.

Proposition 5.3. *For any two partitions λ and ρ with $|\lambda| = |\rho|$, we have*

$$\chi_\rho^\lambda = \sum_T \operatorname{sgn} T,$$

where the sum is taken over all filled structures of cardinality $\rho = (\rho_1, \dots, \rho_l)$ such that the sequences $(p_1, \dots, p_d)$ and $(q_1, \dots, q_d)$ of filling p-numbers and q-numbers of the structure T coincide, up to a permutation, with the Frobenius p-coordinates and q-coordinates of the partition λ, that is, $\lambda = \{\operatorname{sort}(\mathbf{p}) : \operatorname{sort}(\mathbf{q})\}$.

The proof of this proposition is quite simple. Back to the original presentation of the Murnaghan–Nakayama rule in terms of hooks, Borodin analyzes what happens to Frobenius coordinates when subtracting hooks/ribbons. Each such subtraction corresponds to a block. There are three cases to distinguish: the hook/ribbon can be above or below the "Frobenius diagonal" or even overlap it. Those cases correspond respectively to linear horizontal blocks, linear vertical blocks, and hook blocks.

This proposition, as stated in Borodin's work, is slightly restrictive: there is no need for ρ to be a partition. Let $\boldsymbol{\rho} = (\rho_1, \dots, \rho_l)$ be a vector of positive integers and define (just as in Theorem 4.2) $\chi^\lambda_{\boldsymbol{\rho}} := \chi^\lambda_{\mathrm{sort}(\boldsymbol{\rho})}$. Then, by summing over all vectors $\boldsymbol{\rho}$, we get:

Proposition 5.4. *For any partition* λ,

$$\sum_{\boldsymbol{\rho}\in\mathbb{N}_+^l} \chi^\lambda_{\boldsymbol{\rho}} = \sum_T \operatorname{sgn} T,$$

where the sum is taken over all filled structures T of l blocks and with filling p-numbers $(p_1, \dots, p_d)$ and q-numbers $(q_1, \dots, q_d)$ such that $\lambda = \{\mathrm{sort}(\mathbf{p}) : \mathrm{sort}(\mathbf{q})\}$.

Observe that d, the rank of λ, has to be less than or equal to l in order to have a structure.

We now state the main theorem we will use, which originates in Borodin's work.

Theorem 5.5. *Assume f is skew-symmetric within its two vector entries (separately), that is* $f\big(\{\mathrm{sort}(\mathbf{p}) : \mathrm{sort}(\mathbf{q})\}\big) = \operatorname{sgn}(\mathbf{p}) \operatorname{sgn}(\mathbf{q}) f\big(\{\mathbf{p} : \mathbf{q}\}\big)$. *Then*

$$\sum_{\substack{\lambda \text{within} \\ k\times N}} \sum_{\boldsymbol{\rho}\in\mathbb{N}_+^l} \chi^\lambda_{\boldsymbol{\rho}} f(\lambda) = \sum_{d=1}^{l} \sum_{\substack{\mathbf{p}\in[0,N-1]^d \\ \mathbf{q}\in[0,k-1]^d}} f\big(\{\mathbf{p}:\mathbf{q}\}\big) \sum_{T(\mathbf{p},\mathbf{q})} (-1)^{\sum q_i + v(T)},$$

where $T(\mathbf{p}, \mathbf{q})$ goes trough all filled structures of d fragments, l blocks, $v(T)$ vertical blocks with filling p-numbers $(p_1, \dots, p_d)$ and q-numbers $(q_1, \dots, q_d)$.

Proof. We start by summing Proposition 5.4 over λs fitting inside a $k \times N$ box:

$$\begin{aligned}\sum_{\substack{\lambda \text{within} \\ k\times N}} \sum_{\boldsymbol{\rho}\in\mathbb{N}_+^l} \chi^\lambda_{\boldsymbol{\rho}} f(\lambda) &= \sum_{\substack{\lambda \text{within} \\ k\times N}} \sum_{T(\mathbf{p},\mathbf{q})} (-1)^{\sum q_i + v(T)} \operatorname{sgn}(\mathbf{p}) \operatorname{sgn}(\mathbf{q})\, f(\lambda) \\ &= \sum_{\substack{\lambda \text{within} \\ k\times N}} \sum_{T(\mathbf{p},\mathbf{q})} (-1)^{\sum q_i + v(T)} f\big(\{\mathbf{p}:\mathbf{q}\}\big),\end{aligned}$$

where the second sum in each right-hand side is over all filled structures $T(\mathbf{p}, \mathbf{q})$ of l blocks and d fragments such that the sequences of filling p-numbers $(p_1, \dots, p_d)$ and q-numbers $(q_1, \dots, q_d)$ of the structure coincide, up to two permutations, with the sequences of Frobenius p-coordinates and q-coordinates of the partition $\lambda = \{\mathrm{sort}\,(\mathbf{p}) : \mathrm{sort}\,(\mathbf{q})\}$. Note that d changes with λ.

We then obtain the final result by seeing the double sum over λ then permuted Frobenius coordinates of λ as a sum over *all* vectors of appropriate lengths.

We should not be concerned about vectors having two identical coordinates (say $p_i = p_j$), since the corresponding term on the right-hand side vanishes by skew-symmetry of f. □

Counting structures. We now need to compute the sum

$$\sum_{T\,(\mathbf{p},\mathbf{q})} (-1)^{\sum q_i + v(T)},$$

which is taken over the structures described above, that is, for given l, d, $\mathbf{p}$, $\mathbf{q}$, v. It would help to know how many structures there are for each choice of those parameters. We prove the following proposition.

Proposition 5.6. *There are exactly* $\#T\,(l, d, \mathbf{p}, \mathbf{q}, v) :=$

$$\sum_{\substack{\mathbf{s},\mathbf{t}\in\mathbb{N}^d \\ \sum t_i = v \\ d+\sum s_i + t_i = l}} \left[\frac{(s_d + t_d + \cdots + s_1 + t_1 + d)!}{\prod s_i! \prod t_i! \prod_{i=1}^{d} (d + 1 - i + \sum_{j=i}^{d} s_j + t_j)}\right] \times \left[\prod_{i}^{d} \binom{p_i}{s_i}\binom{q_i}{t_i}\right] \quad (14)$$

structures with d fragments, l blocks, filling numbers $\mathbf{p} = (p_1, \ldots, p_d)$ and $\mathbf{q} = (q_1, \ldots, q_d)$ and v vertical blocks. The indices in the sum s_i (respectively, t_i) count horizontal (respectively, vertical) blocks in the i-th fragment.

Proof. This is a purely combinatorial problem. Given the number of vertical blocks on each fragment, we essentially have a partial order on blocks that we want to extend to form a linear order (across fragments). Part of the rules in the initial partial order say that the hook-block in the i-th fragment precedes any other block in that fragment. We then need to fill the structure, that is, to choose filling numbers for each block.

We can reverse this process.

- We first choose the numbers of horizontal and vertical blocks s_i and t_i on the i-th fragment. We have the conditions that $\sum t_i = v(T)$ and $d + \sum s_i + t_i = l$ (that is, there are l blocks in total, d hook, s_i horizontal in the i-th fragment and t_i horizontal in the i-th fragment).
- Starting from the d-th fragment, we decide where to insert the horizontal and vertical blocks of the i-th fragment in the partial order that is established so far on the set of fragments from the $(i + 1)$-th to the d-th one.
- We decide how to cut up the i-th fragment into filled blocks, respecting the number of horizontal/vertical blocks decided upon earlier.

The equality in the statement is intended to reflect clearly the layering described above: the sum corresponds to the first layer, while the other two layers correspond to one square-bracketed factor each.

Observe that the relation $s_d+t_d+\cdots+s_1+t_1+d=l$ could be used to simplify the numerator in this expression.

The only hard part is to derive for the second step

$$\frac{(s_d+t_d+\cdots+s_1+t_1+d)!}{\prod s_i!\prod t_i!\prod_{i=1}^{d}(d+1-i+\sum_{j=i}^{d}s_j+t_j)} = \frac{(s_d+t_d+\cdots+s_1+t_1+d)!}{\prod s_i!\prod t_i!\begin{pmatrix}(s_d+t_d+1)\times\\(s_d+t_d+s_{d-1}+t_{d-1}+2)\times\\\cdots\times\\(s_d+t_d+s_{d-1}+t_{d-1}+\cdots+s_1+t_1+d)\end{pmatrix}}.$$

This is obtained by simplifying

$$\prod_{i=0}^{d-1}\binom{i+\sum_{j=d-i}^{d}s_j+t_j}{s_{d-i}+t_{d-i}}\binom{s_{d-i}+t_{d-i}}{s_{d-i}},$$

where the i-th factor in the $\prod_{i=0}^{d-1}$-product counts the number of ways of choosing the linear order on the blocks of the $(d-i)$-th fragment, as we know that the linear order restricted on the blocks of the fragments $d-i+1$ to d.

The first binomial factor intersperses the set of blocks of the $(d-i)$-th fragment among the blocks of fragments $d-i+1$ to d, while the second factor decides which blocks are horizontal and which are vertical. □

We wish to insist on the fact that the summand in (14) is not symmetric in the p_is or the q_is, because the factor in the denominator,

$$\prod_{i=1}^{d}\Bigl(d+1-i+\sum_{j=i}^{d}s_j+t_j\Bigr),$$

is not symmetric in the s_js or the t_js. For instance, s_d appears d times while s_1 appears only once.

Sum of determinants. We now aim to put together all the results obtained so far in this section, but we first need a quick lemma.

Lemma 5.7. *Let* $\mathbf{s}$ *and* $\mathbf{t}$ *be vectors of integers. Then*

$$\sum_{\sigma,\tau\in\mathcal{S}_d}\frac{(\operatorname{sgn}\sigma\ \operatorname{sgn}\tau)}{\prod_{i=1}^{d}(d+1-i+\sum_{j=i}^{d}s_{\sigma(j)}+t_{\tau(j)})} = \prod_{1\le i<j\le d}(s_i-s_j)(t_i-t_j)\prod_{1\le i,j\le d}\frac{1}{1+s_i+t_j}.$$

Proof. The proof proceeds as for the classical computation for the Vandermonde determinant: the left-hand side is skew-symmetric in $\mathbf{s}$ and $\mathbf{t}$ separately, and has obvious poles as prescribed on the right-hand side (when $s_{i_0}+t_{j_0}=-1$), and the degrees on the right-hand side are appropriate. Up to a constant of proportionality, both sides are thus the same. This constant is shown to be 1 by looking at the rates of decrease when s_1 goes to infinity. □

Proposition 5.8. *Assume f is skew-symmetric within its two vector entries (separately), that is, $f\big(\{\mathrm{sort}(\mathbf{p}):\mathrm{sort}(\mathbf{q})\}\big) = \mathrm{sgn}(\mathbf{p})\,\mathrm{sgn}(\mathbf{q})\, f\big(\{\mathbf{p}:\mathbf{q}\}\big)$. Then*

$$\sum_{\boldsymbol{\mu}\in\mathbb{N}^l_+} \sum_{\substack{\lambda \text{within}\\ k\times N}} \chi^\lambda_{\boldsymbol{\mu}} f(\lambda) = l! \sum_{d=1}^{l} \sum_{\substack{\mathbf{p}\in[0,N-1]^d\\ \mathbf{q}\in[0,k-1]^d\\ v}} f\big(\{\mathbf{p}:\mathbf{q}\}\big)(-1)^{\sum q_i+v}$$

$$\sum_{\substack{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d\\ \sum t_i=v\\ d+\sum s_i+t_i=l}} \prod_i^d \binom{p_i}{s_i}\binom{q_i}{t_i} \prod_i \frac{1}{s_i!\,t_i!} \prod_{1\le i<j\le d} (s_i-s_j)(t_i-t_j) \prod_{1\le i,j\le d} \frac{1}{1+s_i+t_j}.$$

Proof. We first combine (10) with Theorem 5.5:

$$\begin{aligned}
&\sum_{\boldsymbol{\mu}\in\mathbb{N}^l_+} \sum_{\substack{\lambda \text{within}\\ k\times N}} \chi^\lambda_{\boldsymbol{\mu}} f(\lambda)\\
&\quad = l! \sum_{d=1}^{l} \sum_{\substack{\mathbf{p}\in[0,N-1]^d\\ \mathbf{q}\in[0,k-1]^d\\ v}} f\big(\{\mathbf{p}:\mathbf{q}\}\big)(-1)^{\sum q_i+v}\\
&\qquad\qquad \times \sum_{\substack{\mathbf{s},\mathbf{t}\in\mathbb{N}^d\\ \sum t_i=v\\ d+\sum s_i+t_i=l}} \frac{\prod_i^d \binom{p_i}{s_i}\binom{q_i}{t_i}}{\prod s_i!\prod t_i!\prod_{i=1}^d (d+1-i+\sum_{j=i}^d s_j+t_j)}\\
&\quad = l! \sum_{d=1}^{l} \sum_{\substack{\mathbf{p}\in[0,N-1]^d\\ \mathbf{q}\in[0,k-1]^d\\ v}} f\big(\{\mathrm{sort}(\mathbf{p}):\mathrm{sort}(\mathbf{q})\}\big)(-1)^{\sum q_i+v}\\
&\qquad\qquad \times \sum_{\substack{\mathbf{s},\mathbf{t}\in\mathbb{N}^d\\ \sum t_i=v\\ d+\sum s_i+t_i=l}} \frac{\mathrm{sgn}(\mathbf{p})\,\mathrm{sgn}(\mathbf{q})\prod_i^d \binom{p_i}{s_i}\binom{q_i}{t_i}}{\prod s_i!\prod t_i!\prod_{i=1}^d (d+1-i+\sum_{j=i}^d s_j+t_j)}\\
&\quad = l! \sum_{d=1}^{l} \sum_{\substack{\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\\ \text{within } k\times N\\ v}} f\big(\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\big)(-1)^{\sum q_i+v}\\
&\qquad \sum_{\substack{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d\\ \sum t_i=v\\ d+\sum s_i+t_i=l}} \sum_{\sigma,\tau\in\mathcal{S}_d} \frac{\Big[\sum_{\pi,\theta\in\mathcal{S}_d}\mathrm{sgn}(\pi)\,\mathrm{sgn}(\theta)\prod_i^d \binom{p_{\pi(i)}}{s_{\sigma(i)}}\binom{q_{\theta(i)}}{t_{\tau(i)}}\Big]}{\prod s_{\sigma(i)}!\prod t_{\tau(i)}!\prod_{i=1}^d (d+1-i+\sum_{j=i}^d s_{\sigma(j)}+t_{\tau(j)})}.
\end{aligned}$$

Now it is crucial that for fixed $\vec{\mathbf{p}}, \vec{\mathbf{q}}, \vec{\mathbf{s}}, \vec{\mathbf{t}}$, the sign of this last numerator (bracketed) will depend on the parity of σ and τ. Hence we obtain for the preceding expression

$$l!\sum_{d=1}^{l}\sum_{\substack{\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\\ \text{within } k\times N\\ v}} f\big(\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\big)(-1)^{\sum q_i+v}\sum_{\substack{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d\\ \sum t_i=v\\ d+\sum s_i+t_i=l}}\sum_{\pi,\theta\in\mathcal{S}_d}\operatorname{sgn}(\pi)\operatorname{sgn}(\theta)\prod_i^d\binom{p_{\pi(i)}}{s_i}\binom{q_{\theta(i)}}{t_i}$$
$$\times\sum_{\sigma,\tau\in\mathcal{S}_d}\frac{\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)}{\prod s_{\sigma(i)}!\prod t_{\tau(i)}!\prod_{i=1}^d(d+1-i+\sum_{j=i}^d s_{\sigma(j)}+t_{\tau(j)})}$$
$$=l!\sum_{d=1}^{l}\sum_{\substack{\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\\ \text{within } k\times N\\ v}} f\big(\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}\big)(-1)^{\sum q_i+v}$$
$$\times\sum_{\substack{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d\\ \sum t_i=v\\ d+\sum s_i+t_i=l}}\sum_{\sigma,\tau\in\mathcal{S}_d}\frac{\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_i^d\binom{p_i}{s_{\sigma(i)}}\binom{q_i}{t_{\tau(i)}}}{\prod s_{\sigma(i)}!\prod t_{\tau(i)}!\prod_{i=1}^d(d+1-i+\sum_{j=i}^d s_{\sigma(j)}+t_{\tau(j)})}.$$

The last line is now perfectly set for the substitution using Lemma 5.7. After changing the range of summation on $\{\vec{\mathbf{p}}:\vec{\mathbf{q}}\}$ within $k\times N$ to $\mathbf{p}\in[0,N-1]^d$, $\mathbf{q}\in[0,k-1]^d$, we obtain the announced result.

Admittedly, this is not very enlightening. It is thus worth highlighting what happens: the sums we deal with initially are sums over partitions. By using Frobenius coordinates, and sorting the partitions by their rank d, we are expressing the main sum into a sum over d of multisums in d variables. We thus now have sums over two sets of d strictly decreasing variables (the sets $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$) of different ways of building up this partition (the data encoded in $\mathbf{s}$ and $\mathbf{t}$). Using skew-symmetry, we can unsort the variables $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ to $\mathbf{p}$ and $\mathbf{q}$ and decide instead to sort the variables according to "building blocks", that is, switch from $\mathbf{s}$ and $\mathbf{t}$ to $\vec{\mathbf{s}}$ and $\vec{\mathbf{t}}$. □

Putting everything together. We combine all the information obtained so far, and simultaneously clear the restriction $d+\sum s_i+t_i=l$ in (14) by encoding all the moments at once into an exponential generating function.

Theorem 5.9. *For a fixed $k\in\mathbb{N}$, the two series $\sum_{r>0}(\mathcal{M})_N(2k,r)\dfrac{(\mathrm{i}z)^r}{r!}$ and*

$$s_{\langle N^k\rangle}([1^{2k}])\sum_{d=1}^{\infty}\sum_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d}\left|\frac{z^{1+s_i+t_j}}{s_i!t_j!(1+s_i+t_j)}\right|_{d\times d}$$
$$\times\sum_{\substack{\mathbf{p}\in[0,N-1]^d\\ \mathbf{q}\in[0,k-1]^d}}\left|k\frac{\binom{p_i}{s_i}\binom{q_j}{t_j}\binom{k+p_i}{p_i}\binom{k-1}{q_j}(N-p_i)^{(k)}(-1)^{q_j}}{(N+q_j+1)^{(k)}(1+p_i+q_j)}\right|_{d\times d} \tag{15}$$

have equal z^r coefficients for $r < 2k+1$.

Proof. A first necessary remark is that as a formal power series, the second series is well defined: the sum to obtain the r-th coefficient in that series reduces to a finite sum (because $s_i \le p_i$ and $t_j \le q_j$).

We know from (12) that

$$s_{\langle N^k\rangle \cup \{\mathbf{p}:\mathbf{q}\}}([1^{2k}])$$

is skew-symmetric in $\mathbf{p}$ and $\mathbf{q}$ (separately). Hence we can combine the relations (10), (11) and (13) with Proposition 5.8 to obtain a huge sum. The main statement then follows from the recombinations of the main product into determinants, using Cauchy's Lemma (5.1). □

Remarks on Theorem 5.9.

- This is a hypergeometric multisum (at least for fixed d), when we expand the determinants using Cauchy's Lemma. However, not even small d's seem tractable on computer.

- A definite advantage of this formula is that it can be tested at finite N (by expanding the integral defining $(M_N)(2k,r)$ symbolically using the Haar measure). This is helpful to confirm the results obtained so far.

- We wish to insist on the idea behind this theorem: initially we had a combinatorial problem on structures — see (14) — that had no symmetry for its summands in the s_is or t_is. We have exploited some skew-symmetry in the as and bs in (13) to change this. In particular, we have now switched from a sum over $\vec{\mathbf{p}}, \vec{\mathbf{q}}, \mathbf{s}, \mathbf{t}$ to a sum over $\mathbf{p}, \mathbf{q}, \vec{\mathbf{s}}, \vec{\mathbf{t}}$. We have also simplified the denominator in (14).

- As a consequence of the previous point, we can now assume that the s_is are all different. The same is true for the t_i.

- This has useful consequences, especially for computational purposes. It is interesting to compute a bound on r such that partitions with d fragments will have a nonzero contribution to the final sum in $(\mathcal{M})_N(2k,r)$. We have $r \ge d + \sum s_i + t_i$, and the s_is (respectively, t_is) should be all different. We can take them to be $0, 1, \ldots, d-1$. We thus have $r \ge d + 2\frac{d(d-1)}{2} = d^2$.

We now define

$$H^{N,k,s,t} := s!t! \sum_{\substack{p\in[0,N-1]\\ q\in[0,k-1]}} \frac{k(N-p)^{(k)}(-1)^q}{(N+q+1)^{(k)}(1+p+q)} \binom{k+p}{p}\binom{k-1}{q}\binom{p}{s}\binom{q}{t},$$

where the right-hand side is taken to be similar to the entries in one of the determinants in (15).

I have not been able to obtain a much better expression for this with Mathematica. Normally, the package MultiSum [Wegschaider 2004] should be able to deal with multiple hypergeometric series, but this particular one is too complicated. We will thus focus on an easier problem from now on, the problem of asymptotics (that is, we switch from $(\mathcal{M})_N(2k,r)$ to $(\mathcal{M})(2k,r)$).

Asymptotics. We need to compute asymptotics for $H^{N,k,s,t}$ more precisely.

Proposition 5.10. *For a fixed integer $k \geq 1$, when $k > t$,*

$$H^{k,s,t} := \lim_{N\to\infty} \frac{H^{N,k,s,t}}{N^{1+s+t}} = k \sum_{i=0}^{k-t-1} \frac{\Gamma(k+i)\Gamma(s+i+t+1)}{\Gamma(i+1)\Gamma(k+s+t+i+2)}$$
$$= \frac{1}{1+s+t} \frac{\prod_{i=k}^{2k-1}(i-t)}{\prod_{j=k+1}^{2k}(j+s)} = \frac{1}{1+s+t} \frac{\Gamma(2k-t)\Gamma(k+s+1)}{\Gamma(k-t)\Gamma(2k+s+1)}. \tag{16}$$

(This last expression is well defined since $k > t$.)

Proof. Define

$$\tilde{H}^{N,k,s,t} := t! \sum_{\substack{p\in[0,N-1]\\ q\in[0,k-1]}} \frac{k(N-p)^k(-1)^q}{(N+q+1)^k(1+p+q)} \frac{p^k}{k!} \binom{k-1}{q} p^s \binom{q}{t},$$

that is to say, $H^{N,k,s,t}$ stripped of some of its terms of obviously lower order in p, N and q combined. We do this because we want to compute the leading order of $H^{N,k,s,t}$ and there will be lots of cancellations due to the sum over q, as shown by (18) below.

Thus we wish to compute

$$\lim_{N\to\infty} \frac{\tilde{H}^{N,k,s,t}}{N^{1+s+t}} = \lim_{N\to\infty} \frac{H^{N,k,s,t}}{N^{1+s+t}}.$$

The proof of the second equality in the proposition follows from two basic identities on formal series:

$$(1-rX+r^2X^2-\cdots)^k(1-sX+s^2X^2-\cdots) = \sum_j (-1)^j \sum_{i=0}^{j} \binom{k+i-1}{i} r^i s^{j-i} X^j \tag{17}$$

and

$$\sum_{\substack{0\leq j\leq k-1\\ 0\leq q\leq k-1}} (-1)^q \binom{k-1}{q} q^j X^j = (-1)^{k+1}(k-1)!\, X^{k-1}. \tag{18}$$

We expand the definition of $\tilde{H}^{N,k,s,t}$ as a power series in q. Identity (17) indicates that we should only look at the coefficient of q^{k-1}, which we obtain from (18), setting $r := 1/N,\ s := 1/(p+1)$. We then let N tend to infinity, so the sum over p becomes a Riemann sum. Its limit is a β-integral, and thus a β-function appears, which can be expanded into a product of Γ-functions, giving the desired equality.

The last equality in the statement of the proposition is immediate and is the only one to require the bound $k > t$.

For the third equality in the statement,[5] we define

$$\begin{aligned} H_a^{k,s,t} &:= k \sum_{i=0}^{\infty} \frac{\Gamma(k+a+i)\Gamma(s+a+i+t+1)}{\Gamma(a+i+1)\Gamma(k+s+t+a+i+2)} \\ &= \frac{k\Gamma(a+k)\Gamma(a+s+t+1)}{\Gamma(a+1)\Gamma(a+k+s+t+2)}\, {}_3F_2\left({1, a+k, a+s+t+1 \atop a+1, a+k+s+t+2}; 1\right), \end{aligned}$$

where the second equality follows form the definition of ${}_3F_2$. Then

$$H^{k,s,t} = H_0^{k,s,t} - H_{k-t}^{k,s,t}. \tag{19}$$

Since

$${}_3F_2\left({1, c, d \atop e, c+d-e+2}; 1\right) = \frac{c+d-e+1}{(c-e+1)(d-e+1)}\left(1-e+\frac{\Gamma(c+d-e+1)\Gamma(e)}{\Gamma(c)\Gamma(d)}\right),$$

(see [Mat 2001], for example), we have

$$H_a^{k,s,t} = \frac{1}{1+s+t}\left(1-\frac{a\Gamma(a+k)\Gamma(a+s+t+1)}{\Gamma(a+1)\Gamma(a+k+s+t+1)}\right),$$

which yields the desired equality thanks to (19). □

Let $G(\cdot)$ be the Barnes G-function [Hughes et al. 2000, Appendix]. It is a quick consequence of the Weyl dimension formula [Bump and Gamburd 2006, (18)] that

$$s_{\langle N^k \rangle}([1^{2k}]) \sim_N \frac{G(k+1)^2}{G(2k+1)} N^{k^2}.$$

We use the previous proposition to give a relatively concise expression for $(\mathcal{M})(2k, r)$.

Theorem 5.11. *For a fixed $k \in \mathbb{N}$, the two series* $\sum_{r>0} (\mathcal{M})(2k, r) \frac{(\mathrm{i}z)^r}{r!}$ *and*

$$\frac{G(k+1)^2}{G(2k+1)} \sum_{d=1}^{\infty} \sum_{\vec{\mathbf{s}},\vec{\mathbf{t}} \in \mathbb{N}^d} \left| \frac{1}{s_i! t_j! (1+s_i+t_j)} \right|_{d\times d} \left| \frac{H^{k,s_i,t_j}}{s_i! t_j!} \right|_{d\times d} z^{d+\sum(s_i+t_i)} \tag{20}$$

have equal coefficients of z^r for $r < 2k+1$. For a fixed r, the coefficients of z^r for low values of k can be meromorphically continued into each other. The series (20)

[5] This equality was first proved using Mathematica. Paul Abbott observed that the hypergeometric function that appears is Saalschützian and extracted the following proof by tracing Mathematica's output.

can also be written as

$$\frac{G(k+1)^2}{G(2k+1)} \sum_{d=1}^{\infty} \sum_{\vec{\mathbf{s}},\vec{\mathbf{t}} \in \mathbb{N}^d} \left| \frac{1}{s_i!\,t_j!\,(1+s_i+t_j)} \right|^2_{d \times d} \left(\prod_{i,j=1}^{d} \frac{\Gamma(2k-t_j)\,\Gamma(k+s_i+1)}{\Gamma(k-t_j)\,\Gamma(2k+s_i+1)} \right) z^{d+\sum s_i+t_i} \tag{21}$$

and

$$\frac{G(k+1)^2}{G(2k+1)} \sum_{\substack{\lambda=\{\vec{\mathbf{s}}:\vec{\mathbf{t}}\}\\ \operatorname{rank}\lambda=d}} s_\lambda([1^k]) \left| \frac{\dfrac{\Gamma(2k-t_j)}{\Gamma(2k+s_i+1)}}{s_i!\,t_j!\,(1+s_i+t_j)} \right|_{d \times d} z^{|\lambda|}. \tag{22}$$

Furthermore, by using Cauchy's Lemma, one can switch to an expression involving products instead of determinants (that is, a hypergeometric expression).

Proof. For (20), we proceed by substitution into (15), and looking at terms of order N^{k^2+r}. Again, Cauchy's Lemma is used repeatedly to reorganize determinants.

To obtain (21) or (22), we reorganized yet again the determinants using Cauchy's Lemma into a form corresponding to (11). We also summed over the partitions λ instead of summing first over their rank d then their Frobenius coordinates $\vec{\mathbf{s}}$, $\vec{\mathbf{t}}$.

For a fixed r, both sides indeed admit meromorphic continuations in k, which are equal by Carlson's Theorem [Andrews et al. 1999, Theorem 2.8.1, p. 110]. (See also [Dehaye 2006, p. 86].) Indeed, the left-hand side is shown to admit a meromorphic continuation in k using a Pochhammer contour. The meromorphic continuation on the right-hand side is already written in (21), if we admit that what is meant there is the value of the meromorphic continuation in k *evaluated at* k. The difference of the two sides satisfies the hypotheses in Carlson's Theorem, in that its value is 0 at integers, it is of exponential type, and the type $< \pi$ along the axes parallel to the imaginary axis. similar statements are shown in [Dehaye 2006].

It is probably good to insist that the meromorphic continuation of

$$\frac{\Gamma(2k-t_j)\,\Gamma(k+s_i+1)}{\Gamma(k-t_j)\,\Gamma(2k+s_i+1)}$$

to the left has to be taken very carefully and cannot be obtained by just plugging in values of k, once $k \le t$. We will discuss similar issues in Section 9. □

We now aim to replace the determinant left in (22) by a friendlier expression, a rational function of k.

6. General shape of $(\mathcal{M})(2k, r)$, $|\mathcal{M}|(2k, 2h)$ and $|\mathcal{V}|(2k, 2h)$

Proof of Theorem 1.1. By (20), we know that (for fixed r and as meromorphic functions of k)

$$\frac{\mathrm{i}^r}{r!}(\mathcal{M})(2k,r) = \frac{G(k+1)^2}{G(2k+1)} \sum_{\substack{1\le d \\ \vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d \\ d+\sum(s_i+t_i)=r}} C(d,\vec{\mathbf{s}},\vec{\mathbf{t}}) \left|H^{k,s_i,t_j}\right|_{d\times d},$$

with $C(d,\vec{\mathbf{s}},\vec{\mathbf{t}}) \in \mathbb{Q}$, while for s and t fixed (and nonnegative, of course), Equation (16) indicates that $H^{k,s,t}$ is a rational function of k:

$$H^{k,s,t} = \frac{1}{1+s+t} \prod_{i=-t}^{s} \frac{k+i}{2k+i}. \tag{23}$$

This already shows that we have a rational function of k and that the numerator and denominator have the same degree. Equations (23) and (20) together, along with the fact that $H^{k,s,t} = H^{-k,t,s}$, a consequence of (16), explain why X_r is even.

In order to determine the Y_rs a bit better, we need to investigate possible denominators in terms of $\left|H^{k,s_i,t_j}\right|_{d\times d}$. If a is positive, $\left|H^{k,s_i,t_j}\right|_{d\times d}$ will have a factor $(2k+a)^{\alpha_a(r)}$ in its denominator if and only if a is odd (because there is cancellation in (23)) and all of $s_1, \dots, s_{\alpha_a}$ are greater than a. For this to happen, we need

$$r = d + \sum s_i + \sum t_i \ge \alpha_a(r) + \sum_{i=1}^{\alpha_a(r)} (a+i-1) + \sum_{i=1}^{\alpha_a(r)} (i-1), \tag{24}$$

where the inequality is obtained by taking as small as possible values for d, that is $\alpha_a(r)$, for the s_is (while requiring them to be different and greater or equal to a) and for the t_is (all different). We turn this inequality around and get

$$\alpha_a(r) \le \left\lfloor \frac{-a+\sqrt{a^2+4r}}{2} \right\rfloor.$$

The case of negative a is the same, exchanging the roles played by $\vec{\mathbf{s}}$ and $\vec{\mathbf{t}}$.

Finally, the constant $D(r)$ ensuring that both X_r and Y_r are monic can be found, thanks to (20) and (23), taking $\lim_{k\to\infty}$:

$$D(r) = \sum_{\substack{1\le d \\ \vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d \\ d+\sum(s_i+t_i)=r}} \left|\frac{1}{s_i!\,t_j!\,(1+s_i+t_j)}\right|_{d\times d}^2 \frac{1}{2^{d+\sum(s_i+t_i)}} = \frac{1}{r!\,2^r}, \tag{25}$$

where the last equality is left to the reader.

Actually, this last equality is enough to also guarantee that $X_r(u)$ and $Y_r(u)$ both have integer coefficients: just substitute for $H^{k,s,t}$ in (20) the value

$$H^{k,s,t} = \frac{1}{1+s+t} \prod_{i=-t}^{s} \frac{1}{2k+i} \left(k \sum_{i=0}^{s+t} h_i k^i \right)$$

for the appropriate (integer) h_is (in particular, $h_{s+t} = 1$).

This proves (2), at least for large k.

Meromorphic continuation has already been obtained in Theorem 5.11. □

Theorem 6.1. *For $h \in \mathbb{N}$, there are polynomials $\tilde{X}_{2h}$, $\hat{X}_{2h}$, with integer coefficients and* $\deg \hat{X}_{2h} = \deg X_{2h} > \deg \tilde{X}_{2h}$ *such that as meromorphic functions of k,*

$$|\mathcal{M}|(2k, 2h) = \hat{C}(h) \frac{G(k+1)^2}{G(2k+1)} \frac{\hat{X}_{2h}(2k)}{Y_{2h}(2k)},$$

$$|\mathcal{V}|(2k, 2h) = \tilde{C}(h) \frac{G(k+1)^2}{G(2k+1)} \frac{\tilde{X}_{2h}(2k)}{Y_{2h}(2k)},$$

where $Y_r(u)$ is as defined in Theorem 1.1.

Moreover (*but this is conjectural*), *the numerators are additionally monic polynomials*[6] *when* $\hat{C}(h) = 1/2^{2h}$, $\tilde{C}(h) = (2h)!/(h!2^{3h})$, *and* $\deg X_{2h} - \deg \tilde{X}_{2h} = 2h$.

Proof. For fixed integer r and large integer k, most of this follows immediately from (5) and (6), combined with Theorem 1.1.

The fact that $\deg \tilde{X}_{2h} < \deg X_{2h}$ for instance is a consequence of

$$(\mathcal{M})\,(2k, r) \sim_k \left(-\frac{\mathrm{i}}{2}\right)^r \frac{G(k+1)^2}{G(2k+1)},$$

which we use in the equation from note 3 (page 38):

$$\sum_{j=0}^{2h} \binom{2h}{j} \left(\frac{\mathrm{i}}{2}\right)^j \left(-\frac{\mathrm{i}}{2}\right)^{2h-j} = 0.$$

We can similarly show that *if it exists*, $\hat{C}(h) = 1/2^{2h}$. The constant $\tilde{C}(h)$ is more mysterious, and involves the lower order terms in k of (23).

The meromorphic continuation is obtained as in the proof of Theorem 1.1. □

Remark. Unfortunately, within their degree restrictions, the $X_r(u)$, $\tilde{X}_{2h}(u)$ and $\hat{X}_{2h}(u)$ polynomials still look utterly random. We merely have an expression for them as a sum of determinants of rank $d \le \sqrt{r}$ (respectively, $2h$). This expression is relatively quick and allows at least to compute a few of those polynomials.

7. Computational data

The polynomials $X_r(u)$, $\tilde{X}_{2h}(u)$ and $\hat{X}_{2h}(u)$. We present our data for $(\mathcal{M})(2k, r)$ in Table 2, followed by the data for $|\mathcal{M}|(2k, 2h)$ in Table 3 and finally those for $|\mathcal{V}|(2k, 2h)$ in Table 4. Everything extends numerical results previously published,

[6] This is the normalization we will keep later, when discussing data about those polynomials.

r	$X_r(u)$
1	1
2	u^2-2
3	u^2-4
4	u^4-16u^2+66
5	u^4-20u^2+114
6	$u^8-51u^6+864u^4-5554u^2+4860$
7	$u^8-57u^6+1134u^4-8758u^2+8520$
8	$u^{10}-113u^8+4620u^6-86332u^4+682844u^2-765660$
9	$u^{10}-121u^8+5460u^6-115564u^4+1053964u^2-1457820$
10	$u^{14}-220u^{12}+18897u^{10}-831010u^8+20196928u^6-260164440u^4+1428629724u^2-2060092440$
11	$u^{14}-230u^{12}+20997u^{10}-996820u^8+26447168u^6-374214600u^4+2270621484u^2-3994446960$
12	$u^{18}-363u^{16}+52929u^{14}-4083011u^{12}+183649422u^{10}-4906031274u^8+73323636100u^6$ $-512994314412u^4+1371835414728u^2-927651213720$
13	$u^{18}-375u^{16}+57141u^{14}-4663655u^{12}+224398746u^{10}-6467410170u^8+105010072036u^6$ $-806857605660u^4+2461218471576u^2-1755890884440$
14	$u^{22}-582u^{20}+141344u^{18}-18977780u^{16}+1571817537u^{14}-84339778978u^{12}$ $+2962887441370u^{10}-66386724069396u^8+884603961264548u^6-6212383525692744u^4$ $+19176051246319080u^2-13863690471430800$
15	$u^{22}-596u^{20}+149296u^{18}-20838716u^{16}+1807941481u^{14}-102286957136u^{12}$ $+3809004157906u^{10}-90891702433976u^8+1298188100828836u^6-9917808021410976u^4$ $+33986748108863880u^2-25682708695644000$
16	$u^{24}-836u^{22}+295486u^{20}-58491716u^{18}+7245863641u^{16}-593291868896u^{14}$ $+32861804018536u^{12}-1227084273320096u^{10}+29900504376591736u^8$ $-444180655702337856u^6+3616035044845449600u^4-13500165816324763200u^2$ $+10671545982659562000$
17	$u^{24}-852u^{22}+308606u^{20}-62999492u^{18}+8101703961u^{16}-692989945072u^{14}$ $+40321523165416u^{12}-1589469869122752u^{10}+41098203910503416u^8$ $-652694167393180032u^6+5757854141711318400u^4-23590053001525406400u^2$ $+19761261673907754000$
18	$u^{30}-1216u^{28}+641547u^{26}-195081042u^{24}+38335269063u^{22}-5171814422892u^{20}$ $+495753742037253u^{18}-34353739684203042u^{16}+1726507702228490928u^{14}$ $-62290017635596811632u^{12}+1575250938092261972152u^{10}-26886933063310680515376u^8$ $+293595553738705518511056u^6-1882598606626601433513600u^4$ $+5855125431247144869877200u^2-4699357338080820827412000$
19	$u^{30}-1234u^{28}+663111u^{26}-206226048u^{24}+41629109007u^{22}-5794171874298u^{20}$ $+575320671855777u^{18}-41443936954862628u^{16}+2171988993390059952u^{14}$ $-81956498940701768368u^{12}+2174685878160406187416u^{10}-39111313358222167862304u^8$ $+452360970074645727302736u^6-3084756281794829726025120u^4$ $+10210913321050424344698000u^2-8861284072193198189544000$
20	$u^{34}-1615u^{32}+1140143u^{30}-467224385u^{28}+124593557421u^{26}-22981261798995u^{24}$ $+3040237566735165u^{22}-294611133821587635u^{20}+21107532245623967310u^{18}$ $-1116405478738744697410u^{16}+43058312795636550000904u^{14}$ $-1183070247664529791035320u^{12}+22374172979188549647921632u^{10}$ $-277662183945403036368852000u^8+2095071747708073688848702224u^6$ $-8269151494407104768839910640u^4+12529695816553717113566335200u^2$ $-6099189940914050054558484000$

Table 2. The first polynomials $X_r(u)$ as the numerators in $(\mathcal{M})(u,r)$.

$2h$	$\hat{X}_{2h}(u)$
2	u^2
4	$u^4 - 8u^2 - 6$
6	$u^8 - 33u^6 + 198u^4 + 74u^2 - 360$
8	$u^{10} - 81u^8 + 1740u^6 - 8284u^4 - 7716u^2 + 34020$
10	$u^{14} - 170u^{12} + 9597u^{10} - 215560u^8 + 1846928u^6 - 4247400u^4 - 12317076u^2 + 42366240$
12	$u^{18} - 291u^{16} + 30177u^{14} - 1379507u^{12} + 28177518u^{10} - 236602818u^8 + 604630084u^6 + 1570591476u^4 - 10008266040u^2 + 7829929800$
14	$u^{22} - 484u^{20} + 90384u^{18} - 8378492u^{16} + 415889897u^{14} - 11196067680u^{12} + 157699171570u^{10} - 1023611526808u^8 + 1699483809828u^6 + 11589901952544u^4 - 62361799232760u^2 + 44754182272800$
16	$u^{24} - 708u^{22} + 198590u^{20} - 28525892u^{18} + 2275085529u^{16} - 102837376096u^{14} + 2598141390568u^{12} - 34807690054560u^{10} + 213458763180152u^8 - 261862022455104u^6 - 3402805264433280u^4 + 19256263380043200u^2 - 11718802173078000$
18	$u^{30} - 1054u^{28} + 460431u^{26} - 109299828u^{24} + 15577804767u^{22} - 1394331670638u^{20} + 79872695247657u^{18} - 2932723486507728u^{16} + 68022586503825552u^{14} - 962308385613255088u^{12} + 7682283932820069016u^{10} - 26475220331016986304u^8 - 59889950570120914224u^6 + 976582356673028315040u^4 - 3441287004848413282800u^2 + 1366282646437284576000$
20	$u^{34} - 1415u^{32} + 840943u^{30} - 275540385u^{28} + 55049482221u^{26} - 7022476724835u^{24} + 584090828573565u^{22} - 31869278744265555u^{20} + 1134427249824868110u^{18} - 25880772100948222330u^{16} + 365485578445889268104u^{14} - 2970099871666499086840u^{12} + 10773785732163438366432u^{10} + 24904735536575464181280u^8 - 474478390713139651278576u^6 + 1993984711160163968152080u^4 - 1770512318771949573760800u^2 + 214967318998766249916000$

Table 3. The first polynomials $\hat{X}_{2h}(u)$ as the numerators in $|\mathcal{M}|(u, 2h)$.

for instance in [Hall 2002a; 2004] (but those rely on [Hughes 2005]) or [Conrey et al. 2006] (which is limited to $k = h$).

Extensions of those tables up to $r = 2h = 60$ are available on this article's web page or (possibly further extended) at [Dehaye 2007a].

To obtain those tables, we have implemented (21), which is the most computationally accessible version of the formulas available in Theorem 5.11. A Magma implementation of this algorithm is also available as expanded content.

The roots of $\tilde{X}_{2h}(u)$. It has been suggested before, based on limited numerical data, that the polynomials $\tilde{X}_{2h}(u)$ have only real roots. In fact we list in Table 6 the number of real roots and degree for each such polynomial. One quickly observes that $\tilde{X}_{42}(u)$ (of course!) is actually the first polynomial to break the initial fluke and have nonreal roots; see Table 5 on the next page. (It is not clear if this is related to a similar observation on the last line of [Hall 2002a] and throughout [Hall 2004].) This polynomial has four nonreal roots ($\pm 18.8631835 \pm 0.0090603\mathrm{i}$) that show up at once, since they would have to come in pairs of conjugate pairs by evenness of $\tilde{X}_{2h}(u)$. One could wonder why nonreal roots show up so late.

$2h$	$\tilde{X}_{2h}(u)$
2	1
4	1
6	u^2-9
8	u^2-33
10	u^4-90u^2+1497
12	$u^6-171u^4+6867u^2-27177$
14	$u^8-316u^6+30702u^4-982572u^2+6973305$
16	$u^8-484u^6+76902u^4-4461348u^2+67692705$
18	$u^{12}-766u^{10}+215847u^8-27766980u^6+1653656895u^4-41530140126u^2+337968054585$
20	$u^{14}-1055u^{12}+421093u^{10}-79486155u^8+7242179715u^6-290444510205u^4+4099101803991u^2-8381907513945$
22	$u^{16}-1496u^{14}+892108u^{12}-272180808u^{10}+45430344630u^8-4121412379560u^6+189676636728876u^4-3674923533427896u^2+14539253947899345$
24	$u^{18}-1961u^{16}+1566628u^{14}-658984788u^{12}+157743552510u^{10}-21750520014270u^8+1678578114026196u^6-67707100461703716u^4+1235110338400818825u^2-6787336148294472225$
26	$u^{20}-2610u^{18}+2860437u^{16}-1718473240u^{14}+620475009522u^{12}-139083336332460u^{10}+19348398203611266u^8-1624490941247619480u^6+77190294570345945549u^4-1813095317449668401010u^2+15009483262024846096425$
28	$u^{22}-3243u^{20}+4462647u^{18}-3407674501u^{16}+1586340567882u^{14}-466277764083726u^{12}+86845227411024846u^{10}-10042821279688179978u^8+688582088681764130469u^6-25698037955845496067927u^4+444470604942195922015755u^2-2654155080367803900605025$
30	$u^{28}-4190u^{26}+7631083u^{24}-7953124300u^{22}+5258554468937u^{20}-2313326757869890u^{18}+691451285514065259u^{16}-141062107217586416040u^{14}+19477099336547993586171u^{12}-1781103872658227723795970u^{10}+103764470143371018680338137u^8-3607131084573924222894990540u^6+66647887693999747894954784187u^4-515514421669410774166185623070u^2+658183121944091618062137174225$

Table 4. The first polynomials $\tilde{X}_{2h}(u)$ as the numerators in $|\mathcal{V}|(u, 2h)$.

$$
\begin{aligned}
u^{44} &- 12302u^{42} + 69239935u^{40} - 236610412148u^{38} + 549459541784707u^{36} \\
&- 919748248913270486u^{34} + 1148989069656897835213u^{32} \\
&- 1094474723973849448826480u^{30} + 805533314533281755701371226u^{28} \\
&- 461541928967718110253944237052u^{26} + 206514429127544387915748094513446u^{24} \\
&- 72119441118339869972121541587076920u^{22} \\
&+ 19577196457693502603026719624834404502u^{20} \\
&- 4099121776759328236737053383626986012604u^{18} \\
&+ 654170727960937096861203148250462720819850u^{16} \\
&- 78212503734767115379758317319774926243800176u^{14} \\
&+ 6836980008003428572296900814856434321006155189u^{12} \\
&- 422028250886223501142365592098345343850710857462u^{10} \\
&+ 17476800084974190439148752639441918166326024419531u^8 \\
&- 448540393629268182677088044978029477583305447285620u^6 \\
&+ 6253526937210642323596984565394593401672539709730775u^4 \\
&- 37013087756228993438266827460643377628945508512487 50u^2 \\
&+ 3621605245660957150164210097394163569047273383876 5625.
\end{aligned}
$$

Table 5. The first polynomial X_r with nonreal roots occurs for $r = 42$.

h	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\deg(\tilde{X}_{2h})$	0	0	2	2	4	6	8	8	12	14	16	18	20	22	28
# real roots	0	0	2	2	4	6	8	8	12	14	16	18	20	22	28
h	16	17	18	19	20	**21**	22	**23**	**24**	25	**26**	**27**	**28**	**29**	**30**
$\deg(\tilde{X}_{2h})$	28	30	34	36	38	44	46	48	50	54	56	62	64	66	72
# real roots	28	30	34	36	38	40	46	44	46	54	52	58	60	62	68

Table 6. The degree and the number of real roots of $\tilde{X}_{2h}$. The hs for which there are nonreal roots are highlighted.

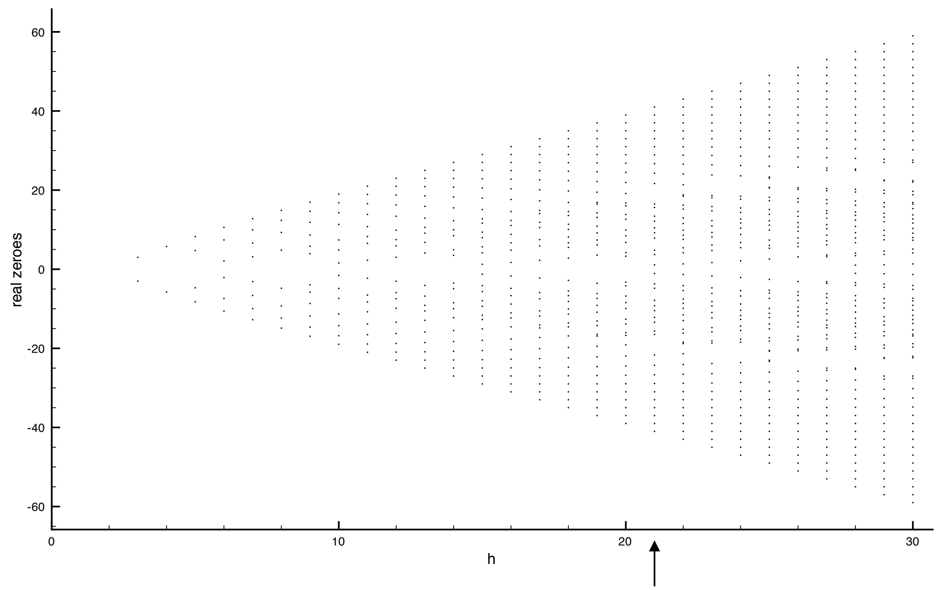

Figure 1. The roots of $\tilde{X}_{2h}(u)$. The line corresponding to $\tilde{X}_{42}(u)$, where the first real zeros go missing, has been indicated.

Fact. The polynomials $\tilde{X}_{2h}(u)$ tend to have many, *but not all*, of their roots real. For instance, for high h, $\tilde{X}_{2h}(u)$ has one root very close by to every odd integer between h and $2h$.

We first present graphical clues for this fact in Figure 1, which depicts the position of the real roots for $h = 1$ to $h = 30$. It thus omits the complex roots.

We now explain the fact. It helps at this point to remember that $\tilde{X}_{2h}(u)$ is obtained by summing various $X_r(u)$ for $r \leq 2h$, which are themselves obtained from (20), for instance. Furthermore, the summand in that equation associated to

$d, \mathbf{s}, \mathbf{t}$ (with $r = d + \sum_i s_i + t_i$) will have poles (as a function of $u = 2k$) at the odd integers a such that $-s_1 \le a \le t_1$ (this uses Lemma 5.1 to expand the determinant in H^{k,s_i,t_j}s). For each pole a, there are a few summands where this pole comes with multiplicity exactly $\alpha_a(r)$, but for most others the multiplicity is lower; see (24). So if we sum all of those terms, and multiply by $Y_{2h}(u)$ (the common denominator) to obtain $\tilde{X}_{2h}(u)$, a vast majority of terms factor a $(u-a)$ out. We thus have an expression of the form

$$\tilde{X}_{2h}(u) = (u-a)P_1(u) + P_2(u),$$

where the coefficients of $P_1(u)$ are expected to be much bigger than the coefficients of $P_2(u)$ (simply because much more terms are summed to obtain $P_1(u)$ than $P_2(u)$). Hence, we should expect $\tilde{X}_{2h}(u)$ to change sign when u travels along the real axis from below a to above a (because $|P_1(a)| > |P_2(a)|$ and $(u-a)$ changes sign) and we know that a root will be around $u = a$. This is especially true if $a > r/2$, because the restrictions impose then $s_1 > a > s_2$, and as a consequence $\alpha_r(a) = 1$ and the phenomenon described just now is accentuated. We present in Table 7 some numerical data associated to this phenomenon.

It is obvious from Figure 1 that a lot is yet to be understood about the polynomials $\tilde{X}_{2h}(u)$. For instance, it is not clear if asymptotically in h there is a positive proportion of real roots.

8. Alternative expressions

Using Macdonald's ninth variation of the Schur functions. Define, as in [Nakagawa et al. 2001] and [Noumi 2004], and similarly to [Macdonald 1992],

$$\tilde{s}_{\lambda}^{(R)} := \left| \tilde{h}_{\lambda_i - i + j}^{(R-j+1)} \right|_{l(\lambda)\times l(\lambda)}, \tag{26}$$

with

$$\tilde{h}_k^{(R)} := \frac{(R-1)!}{(R+k-1)!\,k!}.$$

We first prove that this variation of the Schur functions satisfies a *Giambelli identity*.

Proposition 8.1. *Let λ be a partition and $\{\vec{\mathbf{s}} : \vec{\mathbf{t}}\}$ its Frobenius coordinates, of rank d. Then,*

$$\tilde{s}_{\lambda}^{(R)} = \left| \tilde{s}_{(s_i|t_j)}^{(R)} \right|_{d\times d} = \left| \frac{\Gamma(R-t_j)/\Gamma(R+s_i+1)}{s_i!\,t_j!\,(1+s_i+t_j)} \right|_{d\times d}.$$

Note how this provides a second determinantal expression for this variation of Schur functions, but with a matrix of different rank.

Proof. We intend to use Exercise 3.21 of [Macdonald 1998], but to show that the exercise applies, we need to prove that

h	largest root of $\tilde{X}_{2h}(u)$	difference with $2h-1$	log. difference
1	no root	no root	no root
2	no root	no root	no root
3	3.0000000000000000000	2.000000000	0.69315
4	5.7445626465380286598	1.255437354	0.22748
5	8.2448923938491831987	0.7551076062	−0.28090
6	10.568920444013080343	0.4310795560	−0.84146
7	12.769459455674733521	0.2305405443	−1.4673
8	14.886048429155973920	0.1139515708	−2.1720
9	16.948550444560344620	0.05144955544	−2.9672
10	18.978943770872905688	0.02105622913	−3.8606
11	20.992206162055831068	0.007793837944	−4.8544
12	22.997383184072186530	0.002616815928	−5.9458
13	24.999198051064882757	0.0008019489351	−7.1285
14	26.999774030173017860	0.0002259698270	−8.3951
15	28.999941044846106152	$5.895515389 \times 10^{-5}$	−9.7388
16	30.999985671005722891	$1.432899428 \times 10^{-5}$	−11.153
17	32.999996738730003824	$3.261269996 \times 10^{-6}$	−12.633
18	34.999999301847217917	$6.981527821 \times 10^{-7}$	−14.175
19	36.999999858891343014	$1.411086570 \times 10^{-7}$	−15.774
20	38.999999972983353984	$2.701664602 \times 10^{-8}$	−17.427
21	40.999999995085836086	$4.914163914 \times 10^{-9}$	−19.131
22	42.999999999148595422	$8.514045781 \times 10^{-10}$	−20.884
23	44.999999999859167358	$1.408326421 \times 10^{-10}$	−22.683
24	46.999999999977712180	$2.228782021 \times 10^{-11}$	−24.527
25	48.999999999996618870	$3.381129731 \times 10^{-12}$	−26.413
26	50.999999999999507453	$4.925468142 \times 10^{-13}$	−28.339
27	52.999999999999930988	$6.901186254 \times 10^{-14}$	−30.304
28	54.999999999999990686	$9.313971788 \times 10^{-15}$	−32.307
29	56.999999999999998787	$1.212486889 \times 10^{-15}$	−34.346
30	58.999999999999999847	$1.524414999 \times 10^{-16}$	−36.420

Table 7. The largest root of $\tilde{X}_{2h}(u)$.

$$\tilde{s}^{(R)}_{(p|q)} := \det \begin{pmatrix} \tilde{h}^{(R)}_{p+1} & \tilde{h}^{(R-1)}_{p+2} & \cdots & \cdots & \cdots & \tilde{h}^{(R-q)}_{p+q+1} \\ 1 & \tilde{h}^{(R-1)}_{1} & \tilde{h}^{(R-2)}_{2} & \cdots & \cdots & \tilde{h}^{(R-q)}_{q} \\ 0 & 1 & \tilde{h}^{(R-2)}_{1} & \tilde{h}^{(R-3)}_{2} & \cdots & \tilde{h}^{(R-q)}_{q-1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \tilde{h}^{(R-q)}_{1} \end{pmatrix}$$

$$= \frac{1}{p!q!(1+p+q)} \frac{\Gamma(R-q)}{\Gamma(R+p+1)}.$$

where the matrix has dimensions $(q+1)\times(q+1)$. This can be shown by expanding the determinant along the last column to obtain

$$\tilde{s}^{(R)}_{(p|q)} = (-1)^q \tilde{h}^{(R-q)}_{p+q+1} + \sum_{i=1}^{q} (-1)^{i+1} \tilde{h}^{(R-q)}_{i} \tilde{s}^{(R)}_{(p|q-i)}.$$

Subtract the left-hand side from the right-hand side, proceed by induction on q, and factor out

$$\frac{\Gamma(R-q)}{\Gamma(R+p+1)}.$$

The result then follows from the following equalities, for p and q positive integers,

$$\begin{aligned}
&\frac{(-1)^q}{(p+q+1)!} - \sum_{i=1}^{q} \frac{(-1)^i}{i!\,p!\,(q-i)!\,(p+q-i+1)} - \frac{1}{p!\,q!\,(1+p+q)} \\
&\quad = \frac{(-1)^q}{(p+q+1)!} + \frac{p+q+q \quad {}_2F_1\left(\begin{smallmatrix}1-q & -p-q \\ 1-p-q\end{smallmatrix}; 1\right)}{(p+q)(p+q+1)\,p!\,q!} - \frac{1}{p!\,q!\,(1+p+q)} \\
&\quad = \frac{(-1)^q}{(p+q+1)!} + \frac{q \quad {}_2F_1\left(\begin{smallmatrix}1-q & -p-q \\ 1-p-q\end{smallmatrix}; 1\right)}{(p+q)(p+q+1)\,p!\,q!} \\
&\quad = \frac{(-1)^q}{(p+q+1)!} + \frac{q}{(p+q)(p+q+1)\,p!\,q!}\,\frac{(q-1)!}{(1-p-q)^{(q+1)}} \\
&\quad = 0,
\end{aligned}$$

the last one being a consequence of Gauss's hypergeometric theorem.

The theorem now results directly from Exercise 3.21 in Macdonald's book. □

In essence, this proposition allows us to switch from a Giambelli-type expression to a Jacobi–Trudi expression. It immediately leads to a simplified version of Theorem 5.11:

Theorem 8.2. *With $G(\cdot)$ the Barnes G-function, and $\tilde{s}_\lambda$ defined as in (26),*

$$\sum_{r>0} (\mathcal{M})(2k,r)\frac{(\mathrm{i}z)^r}{r!} = \frac{G(k+1)^2}{G(2k+1)} \sum_{\lambda} \tilde{s}^{(2k)}_{\lambda} s_\lambda([1^k]) z^{|\lambda|},$$

in the sense that their coefficients of z^r are equal for fixed r and large enough k so the coefficient on the left-hand side is defined.

Imitating the Cauchy identity. We can also give an alternative for the expression in (22), proceeding as in Gessel's theorem in its lead up to the Cauchy identity; see [Tracy and Widom 2001]. This uses Theorem 8.2.

Theorem 8.3.

$$\frac{G(k+1)^2}{G(2k+1)} \sum_{r>0} (\mathcal{M})(2k,r)\frac{(\mathrm{i}z)^r}{r!}$$
$$= \lim_{n\to\infty} \det\Big(\big(h_{j-i}([1^k])\big)_{n\times\infty} \cdot \big(\tilde{h}_{i-j}^{(2k-n+j)} z^{i-j}\big)_{\infty\times n}\Big)$$
$$= \lim_{n\to\infty} \det\Big(\big(h_{j-i}([1^k]) z^{j-i}\big)_{n\times\infty} \cdot \big(\tilde{h}_{i-j}^{(2k-n+j)}\big)_{\infty\times n}\Big)$$
$$= \lim_{n\to\infty} \det\Big(\sum_{l\geq 0} h_{l-i}([1^k]) \tilde{h}_{l-j}^{(2k-n+j)} z^{l-j}\Big)_{n\times n}$$
$$= \lim_{n\to\infty} \det\Big(\sum_{l\geq 0} \binom{l-i+k-1}{k-1} \frac{(2k-n+j-1)!}{(l-j)!\,(2k-n+l-1)!} z^{l-j}\Big)_{n\times n},$$

in the sense that their coefficients of z^r are equal for fixed r and large enough k so the coefficient on the left-hand side is defined. The factorials on the last line should really be evaluated in groups, *to give* 0 *if $l < j$*, *and*

$$\frac{\Gamma(2k-n+j)}{\Gamma(2k-n+l)(l-j)!}$$

otherwise.

This can be truncated significantly when we are only after

$$\sum_{0<r\leq S} (\mathcal{M})(2k,r)\frac{(\mathrm{i}z)^r}{r!}$$

for a finite S, that is, when we are computing the head of the sequence of polynomials: we can drop the limit in n and settle for a sufficiently big n instead, and then cut the matrices in their infinite directions as well.

In Gessel's Theorem, in order to get to the other side of the Cauchy identity, one would then observe that the matrix on the last line is Toeplitz, and then use Szegö's theorem. Of course, that fails here because the matrix on the last line is not Toeplitz.

9. The result of Conrey and Ghosh

As explained on page 35, Conrey and Ghosh's theorem [1989] that $\mathcal{J}(2,1) = (e^2 - 5)/4\pi$ immediately leads to a conjecture that $|\mathcal{V}|(2,1) = (e^2-5)/4\pi$ as well. Our main concern is that we only know $|\mathcal{V}|(2k,2h)$ for integer h through (5) and (6) (while we would need $h = 1/2$).

We offer in Figure 2 one way to circumvent this problem. The idea is to compute for each fixed integer h the values of the meromorphic continuation in k of

$(\mathcal{M})(2k, 2h)$ at $k = 1$, that is, at the crosses. This should be enough to know through (5) any value of the form $|\mathcal{M}|\,(2, 2h)$, which could then finally be used to meromorphically continue $|\mathcal{V}|\,(2, 2h)$ to $h = 1/2$.

Getting the meromorphic continuation of (21) to $k = 1$ is quite subtle.

Proposition 9.1. *Define* $(\mathcal{M})(2, r)$ *as the meromorphic continuation of* $(\mathcal{M})(2k, r)$ *in* k*, evaluated at* $k = 1$*. Then the exponential generating series of* $(\mathcal{M})(2, r)$ *is given by*

$$\sum_{r>0} (\mathcal{M})(2, r)\frac{(\mathrm{i}z)^r}{r!} = \sum_{d=1}^{\infty} \sum_{\vec{\mathbf{s}},\vec{\mathbf{t}}\in\mathbb{N}^d} \left|\frac{1}{s_i!t_j!(1+s_i+t_j)}\right|^2_{d\times d} \Big(\prod_{i,j=1}^{d} \frac{1}{2^{\nu(t_j)}}\frac{1-t_j}{2+s_j}\Big) z^{d+\sum s_i+t_i}, \quad (27)$$

where $\nu(0) = 0$ *when* $t = 0$, $\nu(t) = 1$ *when* $t \geq 2$*. The value* $\nu(1)$ *is free to choose.*

Proof. When looking for the analytic continuation in k, most of the formulas we have found so far are misleading. For instance, in light of the remark in note 4

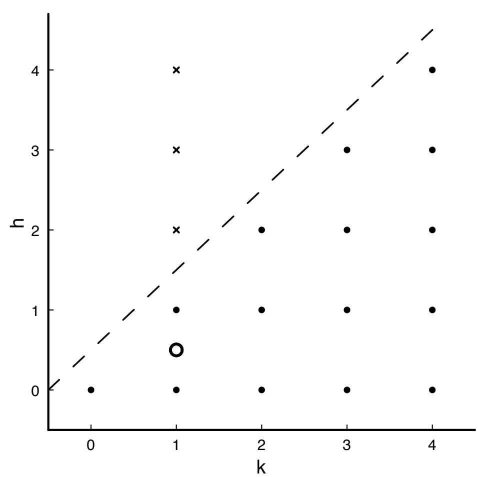

Figure 2. The real part of the situation in the Conrey–Ghosh case. The circle at $(1, 1/2)$ indicates the point for which the value of $\mathcal{J}(2k, 2h)$ is coveted. The dots indicate the locations where (21) applies, and the crosses indicate the points to which that expression is meromorphically continued (for a fixed h, that is, horizontally) thanks to (27). Note that for fixed integer h, this continuation hits a pole when crossing the dashed line (and many more before reaching $k = 1$, as h increases: see Figure 1).

(page 39), one could think that the sums over partitions λ in (22) or Theorem 8.2 immediately reduce when $k = 1$ to sums over partitions λ of length 1, that is, the partitions indexed by a single variable. However, in those cases, the other factor in the summands, for instance $\tilde{s}_\lambda^{(2k)}$ in Theorem 8.2, might actually be undefined if we take $k = 1$ (in that particular case, when $l(\lambda) \geq 3$, or equivalently when $t_1 \geq 2$ if $\lambda = \{\vec{\mathbf{s}} : \vec{\mathbf{t}}\}$).

We can get a better intuition through Expression (21), which we use as a basis of our proof. We are clearly required to find the meromorphic continuation to $k = 1$ and for fixed $s, t \geq 0 \in \mathbb{N}$ of

$$\frac{\Gamma(2k-t)}{\Gamma(k-t)} \frac{\Gamma(k+s+1)}{\Gamma(2k+s+1)}.$$

The second factor is certainly not a problem and immediately gives $1/(s+2)$. For the first factor, we have to look at $\lim_{k\to 1}(\Gamma(2k-t)/\Gamma(k-t))$ for $t \geq 0$. Pick any integer a such that $1 + a - t \geq 0$. Then, using the functional equation for Γ, we have

$$\lim_{k\to 1} \frac{\Gamma(2k-t)}{\Gamma(k-t)} = \lim_{k\to 1} \frac{\Gamma(2k+a-t)}{\Gamma(k+a-t+1)} \cdot \frac{(k-t)(k-t+1)\cdots(k-1)\cdots(k+a-t)}{(2k-t)\cdots(2k-2)\cdots(2k+a-t-1)}.$$

Note that the terms $(k-1)/(2k-2)$ only appear if $t \geq 2$. In that case we get

$$\lim_{k\to 1} \frac{\Gamma(2k-t)}{\Gamma(k-t)} = (1-t) \lim_{k\to 1} \frac{\Gamma(2k+a-t)}{\Gamma(k+a-t+1)} \cdot \frac{k-1}{2k-2} = \frac{1}{2}(1-t),$$

and in the case $t \leq 2$ the factor of 2 is missing. □

One can also check that the values recovered using Proposition 9.1 agree with the values obtained using $X_r(2)/Y_r(2)$ and thus Theorem 1.1.

For completeness, we give the beginning of the sequence of $X_r(2)$s, for $r = 1$ to 15:

$$1, 2, 0, 18, 50, -6540, -11760, 852180, 1228500, 590126040, 558613440,$$
$$-39273224760, 455842787400, 5775116644337040, 14904865051876800.$$

Unfortunately, we fall short of actually finding the full meromorphic continuation of $(\mathcal{M})(2, r)$ and have to leave this for a further paper.

10. Conclusion

Our initial goal was to compute the $(\mathcal{M})$ $(2k, r)$, $|\mathcal{M}|$ $(2k, 2h)$ and $|\mathcal{V}|(2k, 2h)$ more effectively than previously done.

We feel that we have achieved this goal, since we have been able to shed some light (for instance in Theorem 1.1) on the structure of the results. This structure (rational functions with known denominators) underlines tables already available in

[Hughes 2005] or [Conrey et al. 2006]. We have also been able to use these results to obtain better algorithms to compute those rational functions, thereby extending the data that was available. As a corollary we have shown that for large(r) h the roots (in k) of $|\mathcal{V}|(2k, 2h)$ cease to all be real, a fluke only for the small-h cases available previously.

However, we have not obtained a formula for *all* $|\mathcal{V}|(2k, r)$. In particular, we cannot recover the value of $|\mathcal{V}|(2, 1)$, which can be conjectured from Conrey and Ghosh's result for $\mathcal{J}(2, 1)$.

Those methods should also give more general moments, for instance for expressions of the form

$$\left\langle |Z_U(\theta_1)|^{2k} \left| \frac{Z'_U(\theta_2)}{Z_U(\theta_2)} \right|^r \right\rangle_{\mathrm{U}(N)}$$

or

$$\left\langle |Z_U(\theta_1)|^{2k} \left| \frac{Z''_U(\theta_2)}{Z_U(\theta_2)} \right|^r \right\rangle_{\mathrm{U}(N)}. \tag{28}$$

An expression for those two extensions in the shape of (10) would definitely be available (for instance, in the case of Expression (28), we would most likely have to compute the equivalent of (10) by summing over $\vec{\boldsymbol{\mu}} \in (2\mathbb{N}_+)^r$). However, the second part of the computation, the part covered here by Proposition 5.6, would probably be significantly worsened.

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